6 Appendix

Lemma 22 *Let f be a nonnegative Lebesgue-integrable function with a Lipschitz continuous derivative. Then f and f' are bounded.*

Proof.

(i) First we show that f is bounded.
Let C be the Lipschitz constant of f' and $A := \int f(u)du$. Let $\alpha := \sqrt[3]{8A^2C}$, $\beta := -\sqrt[3]{8AC^2}$ and $\gamma := \sqrt[3]{\frac{8A}{C}}$.
We will show that $\sup_{y \in \mathbb{R}} f(y) < \alpha$.
The idea of the proof is as follows: assuming that $f(y_0) = \alpha$ for some $y_0 \in \mathbb{R}$, we show that f has to lie above some function g on $[y_0, y_0 + \gamma]$, since otherwise we have $f(y_0+\gamma) < 0$ because of the Lipschitz continuity of f'. On the other hand is $\int_{y_0}^{y_0+\gamma} g(u)du > A$ implying $\int f(u)du > A$ which is a contradiction to the assumptions.

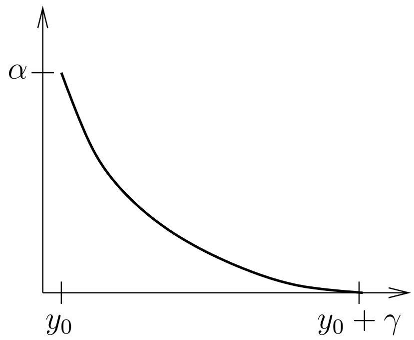

Figure 38: $g(y)$

Define, for all $y \in [y_0, y_0 + \gamma]$,

$$g(y) := \alpha + \beta(y - y_0) + \frac{C}{2}(y - y_0)^2.$$

Notice that $g(y_0) = \alpha$, $g'(y_0) = \beta$ and

$$\begin{aligned} g(y_0 + \gamma) &= \alpha + \beta\gamma + \frac{C}{2}\gamma^2 \\ &= \sqrt[3]{8A^2C} - \sqrt[3]{64A^2C} + \sqrt[3]{8A^2C} \\ &= 0. \end{aligned}$$

Also

$$\int_{y_0}^{y_0+\gamma} g(u)du = \int_{y_0}^{y_0+\gamma} \left(\alpha + \beta(u - y_0) + \frac{C}{2}(u - y_0)^2\right) du$$

$$\begin{aligned}
&= \int_0^\gamma \left(\alpha + \beta u + \frac{C}{2}u^2\right) du \\
&= \alpha\gamma + \frac{1}{2}\beta\gamma^2 + \frac{C}{6}\gamma^3 \\
&= 4A - 4A + \frac{4}{3}A \\
&> A.
\end{aligned}$$

Assume $f(y_0) = \alpha$. Then $f(y) \geq g(y)$ for all $y \in [y_0, y_0 + \gamma]$. This is seen as follows: If there is some $y_1 \in (y_0, y_0 + \gamma)$ with $f(y_1) < g(y_1)$ then there is $y_0 \leq y_2 < y_1$ with $f(y_2) \leq g(y_2)$ and $f'(y_2) < g'(y_2)$.

But then is

$$\begin{aligned}
f(y_0 + \gamma) &= f(y_2) + \int_{y_2}^{y_0+\gamma} f'(u)du \\
&\leq f(y_2) + \int_{y_2}^{y_0+\gamma} \left(f'(y_2) + C(u - y_2)\right) du \\
&< f(y_2) + \int_{y_2}^{y_0+\gamma} \left(g'(y_2) + C(u - y_2)\right) du \\
&\leq g(y_2) + \int_{y_2}^{y_0+\gamma} \left(g'(y_2) + C(u - y_2)\right) du \\
&= g(y_2) + \int_{y_2}^{y_0+\gamma} \left(\beta + C(y_2 - y_0) + C(u - y_2)\right) du \\
&= g(y_2) + \int_{y_2}^{y_0+\gamma} \left(\beta + C(u - y_0)\right) du \\
&= g(y_2) + \int_{y_2}^{y_0+\gamma} g'(u)du \\
&= g(y_0 + \gamma) \\
&= 0
\end{aligned}$$

which contradicts the assumption $f(y) \geq 0$ for all $y \in I\!R$. It follows that $f(y) \geq g(y)$ on $[y_0, y_0 + \gamma]$. This implies $\int_{y_0}^{y_0+\gamma} f(u)du > A$ and hence $\int f(u)du > A$. Since this also contradicts the assumptions, there is no $y_0 \in I\!R$ with $f(y_0) = \alpha$. Hence $f(y) < \alpha$ for all $y \in I\!R$.

(ii) Now we show that f' is bounded.
Let $B = \sup_{y \in I\!R} f(y)$. Then, because of the Lipschitz continuity of f', we have for all $y \in I\!R$,

$$f(y + \gamma) = f(y) + \int_0^\gamma f'(y + u)du$$

$$
\begin{aligned}
&\le \quad f(y) + \int_0^\gamma (f'(y) + Cu)du \\
&= \quad f(y) + f'(y)\gamma + \frac{1}{2}C\gamma^2,
\end{aligned}
$$

and, because of $0 \le f \le B$,

$$f'(y) \ge -\frac{B}{\gamma} - \frac{1}{2}C\gamma.$$

Likewise

$$f\,(y+\gamma) \ge f(y) + f'(y)\gamma - \frac{1}{2}C\gamma^2,$$

and hence

$$f'(y) \le \frac{B}{\gamma} + \frac{1}{2}C\gamma.$$

□

The following lemma is a generalization of Minkowski's inequality.

Lemma 23 *Let $f : I\!R^2 \longrightarrow I\!R$ be Lebesgue integrable and $p > 1$. Then*

$$\left(\int \left|\int f(x,u)du\right|^p dx\right)^{\frac{1}{p}} \le \int \left(\int |f(x,u)|^p \, dx\right)^{\frac{1}{p}} du.$$

Proof.

(i) Let first $f(x,\cdot)$ be a step function for every $x \in I\!R$, i.e. $f(x,u) := \sum_{k=1}^n f_k(x)1\!\!1_{A_k}(u)$, where $\cup_{k=1}^n A_k = I\!R$ and $A_k \cap A_j = \emptyset$ for $k \neq j$, $k, j \in \{1, \ldots, n\}$. Then, for all $y \in I\!R$,

$$\int f(x,u)du = \sum_{k=1}^n f_k(x)\lambda(A_k),$$

where λ is the Lebesgue measure. Further

$$
\begin{aligned}
&\left(\int \left|\int f(x,u)du\right|^p dx\right)^{\frac{1}{p}} \\
= \quad &\left(\int \left|\sum_{k=1}^n f_k(x)\lambda(A_k)\right|^p dx\right)^{\frac{1}{p}} \\
\le \quad &\sum_{k=1}^n \left(\int |f_k(x)\lambda(A_k)|^p \, dx\right)^{\frac{1}{p}} \quad \text{by Minkowski's inequality}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \left(\int |f_k(x)|^p \, dx \right)^{\frac{1}{p}} \lambda(A_k) \\
&= \int \sum_{k=1}^{n} \left(\int |f_k(x)|^p \, dx \right)^{\frac{1}{p}} \mathbb{1}_{A_k}(u) \, du \\
&= \int \left(\int \left| \sum_{k=1}^{n} f_k(x) \mathbb{1}_{A_k}(u) \right|^p dx \right)^{\frac{1}{p}} du \\
&= \int \left(\int |f(x,u)|^p \, dx \right)^{\frac{1}{p}} du,
\end{aligned}
$$

where the second last equality is true because the A_k are disjoint.

(ii) Let now $f(x,u)$ be a positive Lebesgue integrable function. Then there is, for all fixed $x \in I\!R$, a sequence of isotone step functions $(f_n(x,u))_{n\in I\!N}$ with $f_n(x,u) := \sum_{k=1}^{K_n} f_{n,k}(x)\mathbb{1}_{A_k}(u)$ and $f(x,u) = \sup_{n\in I\!N} f_n(x,u)$. Then, by definition,

$$\int f(x,u)du := \sup_{n\in I\!N} \int f_n(x,u)du$$

for all $x \in I\!R$ and

$$
\begin{aligned}
&\left(\int \left| \int f(x,u)du \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int \left| \sup_{n\in I\!N} \int f_n(x,u)du \right|^p dx \right)^{\frac{1}{p}} \\
&= \sup_{n\in I\!N} \left(\int \left| \int f_n(x,u)du \right|^p dx \right)^{\frac{1}{p}} \\
&\le \sup_{n\in I\!N} \int \left(\int |f_n(x,u)|^p \, dx \right)^{\frac{1}{p}} du \qquad \text{by (i)} \\
&= \int \left(\int \left| \sup_{n\in I\!N} f_n(x,u) \right|^p dx \right)^{\frac{1}{p}} du \\
&= \int \left(\int |f(x,u)|^p \, dx \right)^{\frac{1}{p}} du.
\end{aligned}
$$

iii) Let finally $f(x,u)$ be an arbitrary Lebesgue integrable function. Then

$$
\begin{aligned}
&\left(\int \left| \int f(x,u)du \right|^p dx \right)^{\frac{1}{p}} \\
&\le \left(\int \left(\int |f(x,u)|du \right)^p dx \right)^{\frac{1}{p}}
\end{aligned}
$$

$$\leq \int \left(\int |f(x,u)|^p \, dx \right)^{\frac{1}{p}} du \qquad \text{by (ii).}$$

□

Lemma 24 *Assume that $L : I\!R \longrightarrow I\!R$ has two Lipschitz continuous derivatives, is nonnegative, symmetric, supported by $(-g,g)$ for some $g \in I\!R^+$ and strongly unimodal on its support, i.e. L' is positive on $(-g,0)$. Let further L'' have a finite number of zeros on $(-g,g)$. Consider an arbitrary distribution function G and let $h'_G(y) := \int L'(y-u)dG(u)$. Then*

$$h'_G(y) = \int L''(u)G(y-u)du.$$

Proof.
Let $k_i := -g + \frac{2gi}{n}$ for $i \in \{0,\dots,n\}$. Then

$$\begin{aligned}
& \int L'(y-u)dG(u) \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{n} \min_{v\in[y-k_i,y-k_{i-1}]} L'(y-v)\left(G(y-k_{i-1}) - G(y-k_i)\right) \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{n} \min_{v\in[k_{i-1},k_i]} L'(v)\left(G(y-k_{i-1}) - G(y-k_i)\right) \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{n-1} G(y-k_i) \left(\min_{v\in[k_i,k_{i+1}]} L'(v) - \min_{v\in[k_{i-1},k_i]} L'(v) \right) \\
& +G(y+g)L'(-g) - G(y-g)L'(g)) \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{n-1} G(y-k_i) \frac{n}{2g} \left(\min_{v\in[k_i,k_{i+1}]} L'(v) - \min_{v\in[k_{i-1},k_i]} L'(v) \right) \cdot \frac{2g}{n} \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{n-1} G(y-k_i) L''(k_i) \cdot \frac{2g}{n} \qquad (12) \\
= \; & \lim_{n\to\infty} \sum_{n=0}^{n-1} G(y-k_i) \min_{v\in[k_i,k_{i+1}]} L''(v) \cdot \frac{2g}{n} \qquad (13) \\
= \; & \int L''(u)G(y-u)du.
\end{aligned}$$

(12) is seen as follows: If $L''(k_i) > 0$ then, for sufficient large n,

$$\min_{v\in[k_i,k_{i+1}]} L'(v) - \min_{v\in[k_{i-1},k_i]} L'(v) = L'(k_i) - L'(k_{i-1}),$$

and with the mean value theorem and the Lipschitz continuity of L' and L'', we get

$$L'(k_i) - L'(k_{i-1}) = \frac{2g}{n} L''(k_i) + o\left(\frac{1}{n}\right).$$

For $L'' < 0$ we get a similar result. Since

$$\min_{v \in [k_i, k_{i+1}]} L'(v) - \min_{v \in [k_{i-1}, k_i]} L'(v) = O\left(\frac{1}{n}\right),$$

the finite number of cases with $L''(k_i) = 0$ is negligible.
(13) follows from the Lipschitz continuity of L'' and the fact that $L''(-g) = 0$. □

Lemma 25 *Let $\psi(u)$ be a monotone, odd and bounded function having a bounded and Lipschitz continuous derivative $\psi'(u)$ and let G be an arbitrary distribution function. Let further $B := \lim_{y\to\infty} \psi(y)$. Define $h'_G(y - m(x)) := \int \psi(u - y + m(x))dG(u)$. Then*

$$h'_G(y - m(x)) = B - \int \psi'(u - y + m(x))G(u)du. \tag{14}$$

Proof.
As in Lemma 24, it can be shown that, for arbitrary large l and $k_i := -l + \frac{2li}{n}$, $i = 0, \dots, n$,

$$\begin{aligned}
& \int_{-l}^{l} \psi(u - y)dG(u) \\
= \; & \lim_{n\to\infty} \sum_{k=1}^{n} \psi(k_{i-1}) \left(G(k_i - y) - G(k_{i-1} - y)\right) \\
= \; & \lim_{n\to\infty} \sum_{k=1}^{n} G(k_i - y) \left(\psi(k_{i-1}) - \psi(k_i)\right) \\
& -\psi\left(k_0\right) G\left(k_0 - y\right) + \psi\left(k_n\right) G\left(k_n - y\right) \\
= \; & \lim_{n\to\infty} \sum_{k=1}^{n} G(k_i - y) \left(\psi(k_{i-1}) - \psi(k_i)\right) \\
& -\psi\left(-l\right) G\left(-l - y\right) + \psi\left(l\right) G\left(l - y\right) \\
= \; & -\lim_{n\to\infty} \sum_{k=1}^{n} G(k_i - y)\psi'(k_i) \cdot \frac{2l}{n} \\
& -\psi\left(-l\right) G\left(-l - y\right) + \psi\left(l\right) G\left(l - y\right) \\
= \; & -\lim_{n\to\infty} \sum_{k=1}^{n} G(k_i - y) \min_{u \in [k_{i-1}, k_i]} \psi'(u) \cdot \frac{2l}{n} \\
& -\psi\left(-l\right) G\left(-l - y\right) + \psi\left(l\right) G\left(l - y\right) \\
= \; & -\int_{-l}^{l} \psi'(u)G(u - y)du - \psi\left(-l\right) G\left(-l - y\right) + \psi\left(l\right) G\left(l - y\right)
\end{aligned}$$

As l becomes large, the last two terms of the sum converge to 0 resp. B, and, with $y - m(x)$ instead of y, the claim follows. □

Lemma 26 *Let*

$$q_\varepsilon(y) := \begin{cases} a\left(1 - \left(y - c - \frac{1}{2b}\right)^2 b^2\right)^2 & \text{if } y \in \left[c - \frac{1}{2b}, c + \frac{3}{2b}\right] \\ 0 & \text{else,} \end{cases}$$

where $a := \sqrt{\frac{5\delta}{8\varepsilon}}$ *and* $b := \sqrt{\frac{32\delta}{45\varepsilon}}$. *Then*

(i) $q_\varepsilon(y)$ *is continuously differentiable on* $\mathbb{R}$

(ii) $q_\varepsilon(y)$ *is Lipschitz continuous in* $\mathbb{R}$

(iii) $q'_\varepsilon(c) = \frac{\delta}{\varepsilon}$

(iv) $\int q_\varepsilon(u)du = 1$.

Proof.

(i) It is sufficient to regard the points $c - 1/(2b)$ and $c + 3/(2b)$, because $q_\varepsilon(y)$ is a polynomial on $\mathbb{R} \setminus \{c - 1/(2b), c + 3/(2b)\}$.

$$q_\varepsilon\left(c - \frac{1}{2b}\right) = a\left(1 - \left(c - \frac{1}{2b} - c - \frac{1}{2b}\right)^2 b^2\right)^2 = 0$$

and

$$q_\varepsilon\left(c + \frac{3}{2b}\right) = a\left(1 - \left(c + \frac{3}{2b} - c - \frac{1}{2b}\right)^2 b^2\right)^2 = 0.$$

Also, since

$$\begin{aligned} q'_\varepsilon(y) &= 2a\left(1 - \left(y - c - \frac{1}{2b}\right)^2 b^2\right)\left(-2\left(y - c - \frac{1}{2b}\right)b^2\right) \\ &= -4ab^2\left(1 - \left(y - c - \frac{1}{2b}\right)^2 b^2\right)\left(y - c - \frac{1}{2b}\right), \end{aligned}$$

we have

$$q'_\varepsilon\left(c - \frac{1}{2b}\right) = -4ab^2\left(1 - \left(-\frac{2}{2b}\right)^2 b^2\right)\left(c - \frac{1}{2b} - c - \frac{1}{2b}\right) = 0$$

and

$$q'_\varepsilon\left(c + \frac{3}{2b}\right) = -4ab^2\left(1 - \left(\frac{2}{2b}\right)^2 b^2\right)\left(c + \frac{3}{2b} - c - \frac{1}{2b}\right) = 0.$$

(ii) The Lipschitz continuity follows from the fact that $q_\varepsilon(y)$ has a bounded derivative since $q'_\varepsilon(y)$ is continuous and has bounded support.

(iii)

$$\begin{aligned} q'_{ep}(c) &= -4ab^2\left(1-\left(c-c-\frac{1}{2b}\right)^2 b^2\right)\left(c-c-\frac{1}{2b}\right) \\ &= \frac{3}{2}ab \\ &= \frac{\delta}{\varepsilon} \end{aligned}$$

(iv) Set

$$e := c + \frac{1}{2b}.$$

Then

$$\begin{aligned} q_\varepsilon(y) &= a\left(1-(y-e)^2 b^2\right)^2 \\ &= a\left(1-y^2b^2+2b^2ey-e^2b^2\right)^2 \\ &= ab^4y^4-4ab^4ey^3+\left(4ab^4e^2-2ab^2+2ab^4e^2\right)y^2 \\ &\quad +\left(4ab^2e-4ab^4e^3\right)y+a-2ab^2e^2+ab^4e^4 \\ &= ab^4y^4-4ab^4ey^3+\left(-2ab^2+6ab^4e^2\right)y^2 \\ &\quad +\left(4ab^2e-4ab^4e^3\right)y+a-2ab^2e^2+ab^4e^4. \end{aligned}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} q_\varepsilon(u)du &= \int_{e-\frac{1}{b}}^{e+\frac{1}{b}} q_\varepsilon(u)du \\ &= \frac{1}{5}ab^4u^5-ab^4eu^4+\left(-\frac{2}{3}ab^2+2ab^4e^2\right)u^3 \\ &\quad \left.+\left(2ab^2e-2ab^4e^3\right)u^2+\left(a-2ab^2e^2+ab^4e^4\right)u\right|_{e-\frac{1}{b}}^{e+\frac{1}{b}} \\ &= \frac{1}{5}ab^4\left(10\frac{e^4}{b}+20\frac{e^2}{b^3}+2\frac{1}{b^5}\right)-ab^4e\left(8\frac{e^3}{b}+8\frac{e}{b^3}\right) \\ &\quad +\left(-\frac{2}{3}ab^2+2ab^4e^2\right)\left(6\frac{e^2}{b}+2\frac{1}{b^3}\right) \\ &\quad +\left(2ab^2e-2ab^4e^3\right)\left(4\frac{e}{b}\right)+\left(a-2ab^2e^2+ab^4e^4\right)\left(2\frac{1}{b}\right) \\ &= 2ab^3e^4+4abe^2+\frac{2}{5}\frac{a}{b}-8ab^3e^4-8abe^2 \end{aligned}$$

$$
\begin{aligned}
& -4abe^2 - \frac{4}{3}\frac{a}{b} + 12ab^3e^4 + 4abe^2 + 8abe^2 - 8ab^3e^4 \\
& +2\frac{a}{b} - 4abe^2 + 2ab^3e^4 \\
= \quad & \frac{16}{15}\frac{a}{b} \\
= \quad & 1.
\end{aligned}
$$

□