Index theorem on fibred boundary manifolds

Von der Fakultät für Mathematik und Naturwissenschaften

der Carl von Ossietzky Universität Oldenburg

zur Erlangung des Grades und Titels

Doktor der Naturwissenschaften (Dr. rer. nat.)

angenommene Dissertation von

Herrn Álvaro Sánchez Hernández

geboren am 24. Februar 1998 in Salamanca, Spanien

Oldenburg 2025

Gutachter:

Prof. Dr. Boris Vertman

Institut für Mathematik

Carl von Ossietzky Universität Oldenburg

Prof. Dr. Pierre Albin

Department of Mathematics

University of Illinois at Urbana-Champaign

Tag der Disputation:

17. Februar 2026

Abstract

The aim of this thesis is twofold:
On the one hand, it is **expositive**, in that it strives to provide a broad overview of index theory in singular spaces through blow-up methods. To this effect, it starts by introducing the machinery necessary to understand the statement of the Atiyah-Singer and Atiyah-Patodi-Singer results, with focus on those tools that will later be useful in the singular setting, together with some historical remarks. Then, it describes the class of singular manifolds that respond to this methodology and their related geometrical and analytical constructions relevant to the procedure, such as the heat blow-up space and Getzler rescaling.
On the other hand, it presents the current stand of the author's work tackling the non-fully elliptic **local index problem on manifolds with fibred boundary**, which also go by the name of ϕ-manifolds or fibred cusp spaces. This is a non-Fredholm extension of the work in [LMP07]. The *entrée* is provided by scattering manifolds, the case of trivial fibres, which already presents most of the crucial difficulties of the general setting.
This work stands clearly on the shoulder of giants and avoids technicalities when reasonable. It is therefore teeming with references which should help the reader dive further in the subtopic of their choice.

Zusammenfassung

Diese Doktorarbeit hat zwei Hauptziele:
Einerseits ist sie **expositorisch**, indem sie versucht eine Übersicht der Indextheorie in singulären Räumen durch Blow-up Methoden zu verschaffen. Zu diesem Zweck werden zunächst die Konzepte vorgestellt, die notwendig sind, um die Sätze von Atiyah-Singer and Atiyah-Patodi-Singer zu verstehen. Der Fokus liegt hierbei auf den Werkzeugen, die im singulären Kontext anwendbar sind. Die singulären Mannigfaltigkeiten, bei denen Methodik relevant ist werden danach eingeführt, zusammen mit den für das Verfahren wesentlichen geometrischen und analytischen Konstruktionen, wie beispielsweise dem Heat Blow-up Raum und der Getzlers Reskalierung.
Andererseits wird der aktuelle Stand der Arbeit des Autors im Bereich des "non-fully elliptic" **lokalen Indexsatzes auf Mannigfaltigkeiten mit gefasertem Rand** (auch ϕ-Mannifaltigkeiten oder fibred cusp Räume) präsentiert. Dies ist eine Nicht-Fredholm Erweiterung von [LMP07]. Der "scattering" Fall (triviale Fasern) ist die Vorspeise, in der die wesentlichen Schwierigkeiten des allgemeinen Falls bereits zu finden sind.
Diese Arbeit ruht auf den Schultern von Giganten und vermeidet, soweit möglich, technische Details. Sie enthält daher zahlreiche Verweise, die dem Leser helfen sollten, weitere Informationen zu konkreten Themen und Aussagen zu entdecken.

Acknowledgements

There is a plethora of people who in one way or another have enabled and influenced this work. Here is my attempt at a list of the main actors I am indebted to:

- First, I would like to thank my supervisors Boris Vertman and Daniel Grieser. They provided me with the position in Oldenburg, support and advice during this path, and generously introduced me to their mathematical community. We also had several helpful conversations beyond the purely mathematical plane and shared some beautiful musical moments. Ich freue mich auf die nächsten Mathecamps!
- I am deeply thankful to Pierre Albin, who carried me through many moments of stagnation and taught me plenty about the intuition and technicalities behind the theory I am about to present. This thesis would not have come to fruition without our Zoom chats.
- I was also lucky to get in contact with Jayson Liu, who recently finished his PhD focusing in a problem very much related to mine, and whose answers where critical for me to understand the nitty-gritty of this approach.
- My most sincere thanks to Cipriana Anghel and Oussama Benabide, who I met in my first conference and with whom I have kept in contact ever since. They have been a support, motivation and learning group, with whom I am looking forward to keep discussing in the future, and also good friends.
- Fortunately, I got help from a myriad of great mathematicians, in longer or shorter conversations all of which opened my eyes to very interesting perspectives into the problem/field that went above and beyond what I was able to extract from the literature. Here I would like to at least mention Fréderic Rochon, Richard Melrose, Jesse Gell-Redman and Sergiu Moroianu.
- The community of researchers in this or adjacent fields welcomed me with open arms and this also motivated me to keep at it. There are many people here to be named, so let me just mention the hospitality of the Topology/Geometry group in Göttingen, the Geometry group in Hamburg, the Analysis and the Differential Geometry groups in Hannover, and the Geometry group in Kiel.
- The fact that I am here would not be a reality if it were not for the amazing people I met along my university journey in Madrid, Bonn and Oldenburg. Special shot-out a los matfis y matinfor de la complu, my covid guys in Bonn, Julian für seine Unterstützung, und Alex für 3 Jahre gemeinsamen Alltags. También quiero agradecer a mi tutor de TFG, Luis Giraldo Suárez, su amabilidad y apoyo.
- Last but not least, my friends (many of whom I already mentioned) and family, who push me to be better and hold me in my lowest moments, and are definitely the most critical link in the chain. I do not name them, but they know who they are. A ellos les dedico esta tesis, en concreto a mis padres y a Linnéa.

Notation

Along the text, we use colours to distinguish different types of informations in figures or formulas. In particular, since we deal with statements connecting Analysis and Geometry/Topology (see §2.1), we use the color purple to signify that the term in question is of analytical nature, and teal for the geometrical/topological terms. This can be exemplified by the Atiyah-Singer local index formula for Dirac operators in closed manifolds (5):

$$\text{ind}\,(\not\partial) = \int_X \hat{\mathrm{A}}(TX)Ch(E)$$

For the usual Definition, Lemma, Proposition, Remark, Theorem and Conjecture environments, we avoid using italics for formatting and readability reasons. Therefore, we signal the end of the environment with a white square box □ at the **left of the page**. This is in contrast with the same square box at the **right of the page**, which acts as a QED symbol, marking the end of a proof.
The symbol § is short for "Section" and formulas/equations are referenced with a number between parentheses, like (27). All written references to statements, formulas, figures and citations are hyperlinked.
The rest of the notations are standard or introduced in due time in the corresponding sections.

Contents

1 Introduction

In their seminal work [MM98], Mazzeo and Melrose single out the class of singular **manifolds with fibred boundary/fibred cusp structures** (we use the prefix ϕ- for short) at infinity and construct an adapted calculus of pseudodifferential operators on them.
One starts with a non-compact Riemannian manifold M where the non-compact end(s) is **modelled on the infinite end of a cone** with link Y, times a closed manifold (relevant examples to be found in [GSHV25, §2.2]). These manifolds can be compactified to produce a compact manifold with boundary X whose boundary is the total space of a fibre bundle $\phi : \partial X \to Y$. The geometric structure of a ϕ-manifold is then determined by the choice of a boundary defining function x, with respect to which the metric in the interior of a collar neighborhood of the boundary $[0, \delta)_x \times \partial X$ takes the form:

$$g_\phi\big|_{(0,\delta)_x \times \partial X} = \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_{\partial X/Y}$$

where g_Y is a Riemannian metric on the closed manifold Y, and $g_{\partial X/Y}$ is a symmetric 2-tensor on ∂X which is positive definite on the fibrewise directions $T\partial X/Y$ (we use both Z and $\partial X/Y$ to denote fibrewise structures interchangeably). A usual assumption is that the boundary is connected, i.e. the manifold has just one such end, but the general case is treated similarly.
This is one of several classes of **singular manifolds with bounded geometry** or conformal versions of them that have been studied in recent years in the geometric microlocal analysis literature (more concretely, using Melrose's blow-up approach), such as cylindrical ends [Mel93], fibred hyperbolic cusps [Vai01], edges [Maz91] [Alb07], wedges [AGR16] [AGR23], incomplete cusp edges [Liu25], etc.. The two degenerate extremes of ϕ-geometries correspond to **cusp structures** (when the base Y is a point) and **scattering/asymptotically conical ends** (when the fibre is a point).
The motivation for this writing is the study of the **index theory for Dirac-type operators** in this geometrical setting. In particular, we would like to extend the local index formula of [LMP07] by **dropping the assumption of full ellipticity**, that is, Fredholmness of the operator.
We begin however by **introducing the subject of index theory** anew in order to provide an outlook on how the ideas related to this method have evolved so as to tackle the singular setting. This is intended as an expository overview that highlights the main aspects and underscores the intuitions the author considers relevant to the topic.
In **§2** we embed the index problem within the family of **comparison theorems** in Analysis, Geometry and Topology, whose study saw great progress in the second third of the 20th century. We first present some of the most representative results and introduce the concepts necessary to formulate and comprehend the statement of **Atiyah and Singer** (Theorem 2.12) on closed manifolds. We then describe the main ideas of **the heat kernel method**, since this lies at the heart of index theory in the singular setting, and explain the **Atiyah-Patodi-Singer** extension to manifolds with boundary. In particular, we stress the importance of understanding the asymptotics of the heat kernel of the square of the Dirac operator and emphasize the **McKean-Singer formula**:

$$\lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\partial\!\!\!/^2}\right) = \lim_{t\to 0} {}^R\mathrm{Str}\left(e^{-t\partial\!\!\!/^2}\right) + \int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\partial\!\!\!/^2}\right) dt$$

as the source of local index formulas. The superscript "R" symbolizes Riesz renormalization (§4.2.1), which is needed in many singular settings because the heat kernel is no longer trace class.
§3 is concerned with outlining the **blow-up approach** to investigating the heat kernel, developed in [Mel93] and advanced after. Guided by elementary examples, **§3.1** walks the reader through the theoretical framework supporting the methodology, i.e. the analytic study of manifolds with corners using compactifications, blow-ups and polyhomogeneous conormal distributions. These

ideas are applied to the study of the heat kernel in **§3.2**, where the notion of **heat blow-up space** is introduced: this space is obtained from $\mathbb{R}_+ \times X^2$ (where the heat kernel of an operator in the manifold X a priori lives) by a sequence of blow-ups, along which the heat kernel lifts to be a **polyhomogeneous conormal** distribution, and thus yields an operational understanding of its asymptotics.

The literature tends to treat each singular setting separately, even though the process followed is usually similar. Therefore, after illustrating the method in the closed case, we spell out the steps followed in the heat kernel contruction, in particular focusing on the principles underlying heat blow-up spaces. **Conjecture 3.22** proposes a rule of thumb, surely known to experts, on **how to find the heat blow-up space for an exact klm-metric** with $k \geq l \geq m$ and $k \geq 0$. This setting is similar to the ϕ-case in that we consider a compact manifold with boundary X, a fibre bundle $\phi : \partial X \to Y$ and a boundary defining function x, so that the metric in the interior of a collar neighborhood of ∂X takes the form:

$$g_{klm}\big|_{(0,\delta)_x \times \partial X} = \frac{dx^2}{x^{2k}} + \frac{\phi^* g_Y}{x^{2l}} + \frac{g_{\partial X/Y}}{x^{2m}}, \quad k, l, m \in \mathbb{Z}$$

This encompasses all singular settings treated so far with this machinery.

Conjecture 1.1 (Conjecture 3.22)**.** The heat blow-up space for an elliptic, positive differential operator of order r associated to a klm-metric in a compact manifold with boundary X is the space obtained from $\mathbb{R}_+ \times X^2$ by the minimal sequence of blow-ups that allow us to reach the coordinates:

$$\xi = \frac{x - x'}{(x')^k t^{1/r}}, \quad \eta_i = \frac{y_i - y_i'}{(x')^l t^{1/r}}, \quad \mu_j = \frac{z_j - z_j'}{(x')^m t^{1/r}}, \quad x', \quad y_i', \quad z_j', \quad \tau = \begin{cases} t^{1/r} & m \geq 0 \\ t^{1/r}(x')^m & m \leq 0 \end{cases}$$

around the last blown-up face, which corresponds to the blow-up of the (lift of the) p-submanifold $\{0\} \times \mathrm{diag}_X$. The resulting boundary hypersurface is denoted tf.

□

We describe the blow-up sequence leading to the heat blow-up space in each case and write down the model problems for Laplace-type operators in the last two blown-up faces as a first step in the proof of the conjecture in that particular case, which is the most interesting due to its relevance to index and spectral theory.

We conclude that section by recalling a general **composition** result of [Alb07] using the standard **triple space** construction (Proposition 3.27).

The most important chapter is §4. Here, we enumerate the main classes of singular settings that have received a similar treatment and the geometric intuition behind them (§4.1.1), to then formalize this discussion under the umbrella term "klm-metrics" mentioned above (4.1.2). With an eye on index theory, we say some words about the geometric-analytical structures related to these model metrics, like calculi of pseudodifferential operators, the form of Dirac-type operators and curvature asymptotics. Again, this has been done in several concrete settings and we generalize it to cover klm-metrics. This should serve as a head start into their index theory.

It is worth pointing out that, even though we focus on these fibred boundary structures (klm-metrics), many of the constructions are readily generalizable to iterated fibration structures (see e.g. Remark 3.24 and Example 3.25).

In order to recover the correct terms in the heat kernel asymptotics that contribute to the index formula, the **rescaling technique of Getzler** [Get86] is adapted to each setting. In the ϕ-case, this rescaling is especially interesting when trying to recover the contribution of the **integral term** in the McKean-Singer formula. In §4.2.1, we rewrite this term for a klm-metric in a more malleable form taking into account the required renormalization of the supertrace:

$$\partial_t \, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = \frac{1}{2}\int_{\partial X}\left[\mathrm{str}_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\right]_{lb+mf} dy'dz'$$

In general, this means we need to perform a rescaling, not of the heat kernel, but of $cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}$, unless the leading order term is the one contributing, as was the case so far in the literature. The ϕ-case (and the scattering too) does require such a rescaling, and we give some ideas on how to tackle it in §4.5. The geometrical framework supporting this rescaling is delineated in §4.2.2 and exemplified within the closed manifold setting (Example 4.20), whose analysis extends to the **contribution from the tf face** in every heat blow-up space to produce the well-known Atiyah-Singer integrand as (perhaps part of the) short time asymptotics:

$$ {}^R\!\!\int_X \hat{\mathrm{A}}(X)Ch(E)$$

for a Dirac operator associated to the Hermitian Clifford module $E \to X$, as discussed in [Alb07]. Once this general perspective has been developed, we come back to the problem in hand, namely the index of ϕ-Dirac-type operators. Most of the difficulties arising in the analysis have to do with the base directions Y, so we decide to treat the **scattering case**, of trivial fibres, separately to begin with. In §4.3 we particularize the constructions in §4.1.2 and the heat blow-up space of §3.2.1 to the asymptotically conical case.
We show that the heat kernel respects a "certain symbol calculus of Clifford algebras", i.e. it rescales with respect to a geometrical Getzler rescaling analogous to the one in [AGR23] and [Liu25]:

Theorem 1.2 (Theorem 4.22)**.** The heat kernel of the square of a Dirac-type operator in a scattering manifold belongs to a subcalculus $\Psi_{\mathcal{G}}^{-2,0}(X;E)$ of the scattering heat calculus, with coefficients in a bundle $\mathcal{G}$ (related to $\mathrm{End}(E)$) of rescaled sections. These respect a Clifford structure on X at $t \to 0$ and a Clifford structure on the boundary ∂X at $x \to 0$. The rescaled leading order terms at the respective faces are:

$$N^{\mathcal{G}}_{\mathrm{tf},-2}(h)dx'dy' = \frac{1}{(4\pi)^{\frac{n}{2}}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(-\frac{1}{4}\left\langle\frac{R}{2}\coth\frac{R}{2}\xi,\xi\right\rangle\right)$$

$$N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)dy' = e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2R_{\partial X}/2}{\sinh(\tau^2R_{\partial X}/2)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2R_{\partial X}}{2}\coth\left(\frac{\tau^2R_{\partial X}}{2}\right)u,u\right\rangle\right)\right]e^{L}$$

□

The contributions to the McKean-Singer formula correspond to the four faces in the heat blow-up space intersecting the lifted diagonal: tf, ϕf, bf_0 and zf.
The face tf is the only contribution to the short time limit and its behavior is, as above:

$$\lim_{t\to 0} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = {}^R\!\!\int_X \hat{\mathrm{A}}(X)Ch(E)$$

The large time limit corresponds to the faces zf and bf_0, as we show using a commutation result for the limit and the supertrace. Mimicking [Sim93], **the zf contribution** produces the renormalized index (Proposition 4.23):

$${}^R\mathrm{ind}\left(\eth^+\right) := {}^R\mathrm{Str}\left(\Pi_{\ker\eth}\right)$$

where $\Pi_{\ker\eth}$ denotes the projection onto the kernel. When the operator is Fredholm, i.e. fully elliptic, the renormalization is not needed and this is the usual Fredholm index. This is, for example, not the case for the spinor-Dirac operator.

The face $\mathbf{bf}_0$ contributes a term of the form:

$$\int_{\partial X} {}^R\!\!\int_0^\infty \mathrm{str}_p\left(N_{\mathrm{bf}_0,n}\left(e^{-t\eth^2}\right)\right)(T,y')\frac{dT}{T^2}dy'$$

and we discuss our tries on interpreting it in §4.5. As indicated before, we also present there an idea on how to treat the integral term of the McKean-Singer formula corresponding to the ϕf boundary hypersurface. These contributions are gathered in:

Theorem 1.3 (Theorem 4.25)**.** The renormalized index of a (perhaps non-Fredholm) scattering Dirac operator satisfies:

$$\begin{aligned} {}^R\mathrm{ind}\left(\eth^+\right) &= {}^R\!\!\int_X \hat{\mathrm{A}}(X)Ch(E) \\ &+ \frac{1}{2}\int_{\partial X}\left[\mathrm{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b dy' - \int_{\partial X} {}^R\!\!\int_0^\infty \mathrm{str}_p\left(N_{\mathrm{bf}_0,n}\left(e^{-t\eth^2}\right)\right)\frac{dT}{T^2}dy' \end{aligned}$$

□

In spite of the computations in the fibred boundary case being more cumbersome, there are not many substantial differences with the scattering approach. In §4.4 we describe these differences and obtain similar behaviors for the tf and zf faces. The heat kernel rescales analogously:

Theorem 1.4 (Theorem 4.27)**.** In the ϕ-case, the heat kernel of $\eth^2$ belongs to a calculus $\Psi_{\mathcal{G}}^{-2,0}(X;E)$ of rescaled sections, whose behavior as $t \to 0$ is compatible with $Cl({}^\phi T^*X)$, and at $x \to 0$, with $Cl\left(\frac{1}{x}\phi^*T^*Y\right)$. The rescaled normal operators are:

$$N^{\mathcal{G}}_{\mathrm{tf},-2}(h)dx'dy'dz' = \frac{1}{(4\pi)^{\frac{n}{2}}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(-\frac{1}{4}\left\langle\frac{R}{2}\coth\frac{R}{2}\xi,\xi\right\rangle\right)$$

$$N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)dy'dz' = e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2R_Y/2}{\sinh(\tau^2R_Y/2)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2R_Y}{2}\coth\left(\frac{\tau^2R_Y}{2}\right)u,u\right\rangle\right)\right]e^Le^{-\tau^2\mathbb{B}^2}$$

where $\mathbb{B}$ represents the Bismut superconnection (33) associated to the fibre bundle $\phi:\partial X\to Y$.

□

The characterization of Fredholmness via full ellipticity tells us that the index problem is Fredholm whenever an induced vertical family of Dirac operators $\left(\eth_{\partial X/Y,y}\right)_{y\in Y}$ has trivial kernel. To treat the non-Fredholm case, we can proceed as in [GTV22] or [KR22] (similar to [Vai01]) and work with the family-index-theorem-like **assumption**:

$$\mathcal{K} := \ker\eth_{\partial X/Y}\to Y \quad \text{is a vector bundle}$$

Then, the action of the operator can be split into a **fibrewise harmonic part** (that behaves like a scattering operator) and **its orthogonal complement** (which acts fully elliptic).

In this context, the face bf_0 is replaced by $\phi\mathrm{f}_0$, but the fully elliptic part has no contribution coming from this face to the index formula (cf. [LMP07]), so the $\phi\mathrm{f}_0$ contribution is effectively a scattering-type term already polyhomogeneous in bf_0. Consequently, this can be treated as in the scattering case if the role of $\eth_{\partial X}$ is taken by **the operator** (40):

$$\eth_Y^{\mathcal{K}} := \eth_Y + \frac{3}{4}\sum_{i=1}^b k\left(\tilde{U}_i\right)cl\left(\frac{\tilde{U}_i^\flat}{x}\right) - \frac{1}{4}\sum_{ijp} g_{\partial X/Y}\left([\tilde{U}_i,V_p],V_j\right)cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(V_j\right)cl\left(V_p\right) - \frac{x}{2}cl\left(T^\oplus\right)$$

a boundary operator acting on the bundle $\mathcal{K}$, analogous to Vaillant's D_Y [Vai01]. Here, $k\left(\tilde{U}_i\right)$ is the mean curvature (34) of the fibre bundle ϕ in the direction of the vector field $\tilde{U}_i$, and $T^\oplus$ is the

torsion (35) of the connection $\nabla^{\oplus}$ respecting the splitting in horizontal and vertical sections in ∂X (32).
As suggested by [LMP07], a Bismut-Cheeger degeneration [BC89] happens in the ϕf face, so we define the Bismut superconnection and **Bismut-Cheeger η-form** to propose an interpretation of this term in §4.5.
Up to the (to date) inconclusive ideas of §4.5, which we wish to build upon to elucidate the geometric meaning of the ϕf and $\mathrm{bf}_0/\phi\mathrm{f}_0$ terms, we end up with the **index formula**:

Theorem 1.5 (Theorem 4.28)**.** A Dirac operator on a Hermitian Clifford module $E \to X$ over a (0-)ϕ-manifold, for which the kernel of its induced vertical family forms a vector bundle over the base Y, satisfies the formula:

$$
{}^{R}\mathrm{ind}\left(\eth^{+}\right) = {}^{R}\!\!\int_X \hat{\mathrm{A}}(X)Ch(E) + \frac{1}{2}\int_{\partial X}\left[\mathrm{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b dy'dz'
$$

$$
- \int_{\partial X} {}^{R}\!\!\int_0^{\infty} \mathrm{str}_p\left(N_{\mathrm{bf}_0,b+1}\left(e^{-t\eth^2|_{ac\ T^*Y^+\otimes\mathcal{K}}}\right)\right)\frac{dT}{T^2}dy'dz'
$$

□

Upon further investigation, we expect the last two terms to behave as in the conformal setting of fibred hyperbolic cusps [Vai01]:

$$
-\frac{1}{2}\int_Y \hat{\mathrm{A}}(Y)\bar{\eta}(\partial X/Y) - \frac{1}{2}\eta\left(\eth_Y^{\mathcal{K}}\right)
$$

In particular, the second to last term should vanish in the asymptotically conical setting.
The last section **§5** sketches some future work ideas of interest to the author within this topic and mentions the standard scalar curvature result found in analogous settings as a consequence of the index formula, Proposition 5.1.
This writing was produced roughly in parallel with [GSHV25], where an overview of the index and spectral theory of the singular models conformal to ϕ-manifolds can be found.

Difficulties in relation to other settings

As pointed out above, we discuss in §4.5 our attempts to interpret the terms coming from the ϕf and $\phi\mathrm{f}_0/\mathrm{bf}_0$ faces, which we currently lack for a conclusive index statement.
What differs from the literature **at ϕf** (Remark 4.21) is that a rescaling is needed in the integral term of the McKean-Singer formula, that is, we need to Getzler rescale $cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}$ to recover the $\rho_{\phi\mathrm{f}}^b$ coefficient. As explained in §4.3 and §4.4 , even though $e^{-t\eth^2}$ rescales in the b base directions, $\eth$ does not. Moreover, $\eth$ has contributions of different orders in $\rho_{\phi\mathrm{f}}$ and some of the terms in $e^{-t\eth^2}$ pairing with them to produce the $\rho_{\phi\mathrm{f}}^b$ contribution are not computable with a Getzler rescaling of this sort. Thus, in §4.5 we resort to trying to relate their supertrace with that of the heat kernel of a different operator which does Getzler rescale in the appropriate manner. Motivated by [BF86], our candidate operator is $\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth$, which rescales in the b base directions **and** the normal direction, as was the rescaling of tff in [Vai01]. As of yet, we are not able to argue why its supertrace should be equal to that of the original term, which would solve the problem.
For $\phi\mathrm{f}_0$ and bf_0 we are unable to find a form of the solution of the model problem at the face whose supertrace is interpretable as an eta invariant. This is in particular because the boundary (resp. base) directions appear coupled to the normal direction in the heat equation in the term $x^2\left(\eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}\right)$ (24) (41) and the lifted heat equation at the face takes on a complicated form (31) which makes it hard to express the kernel as a distribution containing a factor similar to $\eth_{\partial X}e^{-t\eth_{\partial X}^2}$, as would correspond to the expected eta invariant at this face.

2 The Atiyah-Singer index theorem

> *A group of blind men heard that a strange animal, called an elephant, had been brought to the town, but none of them were aware of its shape and form. Out of curiosity, they said: "We must inspect and know it by touch, of which we are capable". So, they sought it out, and when they found it they groped about it. The first person, whose hand landed on the trunk, said, "This being is like a thick snake". For another one whose hand reached its ear, it seemed like a kind of fan. As for another person, whose hand was upon its leg, said, the elephant is a pillar like a tree-trunk. The blind man who placed his hand upon its side said the elephant, "is a wall". Another who felt its tail, described it as a rope. The last felt its tusk, stating the elephant is that which is hard, smooth and like a spear.*
>
> THE PARABLE OF THE BLIND MEN AND AN ELEPHANT

The Atiyah-Singer index theorem is considered as one of the major advances in understanding the connection between geometry, topology and analysis in the second half of the 20th century. We will now briefly explain the motivation behind this paramount result to, at the same time, set up the stage for our posterior discussion.

2.1 Comparison theorems

The index theorem fits within the context of the so-called *comparison theorems*[1]. These are results about manifolds that establish relationships between analytically defined objects (solutions of a differential equation on the manifold, quantities constructed from the spectrum of an operator [spectral invariants], etc.) and geometric/topological objects (number of "holes" or "handles" of a closed orientable surface, cohomology classes, sometimes of vector bundles, like characteristic classes constructed from the curvature, etc.). They usually take the shape of a formula or equality:

$$\textit{Analytical side} = \textit{Geometric/topological side}$$

2.1.1 Hodge theorem

The first such result we would like to introduce is the **Hodge theorem**, which can be used in combination with the **de Rham theorem**. Our setting consists of a compact smooth n-manifold X, from which we are interested on the (real) vector spaces $\Omega^k(X) = \Gamma(\Lambda^k T^*X)$ of smooth differential forms of degree k of X for all degrees: they form a cochain complex with the exterior derivative d:

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega^n(X) \xrightarrow{d_n} 0$$

meaning $d_{k+1} \circ d_k = 0$ at any step. The de Rham theorem claims that the cohomology of this complex is the same as the singular cohomology of the manifold seen as a topological space:

$$H^k_{dR}(X) := \frac{\ker d_k}{\operatorname{im} d_{k-1}} \cong H^k_{sing}(X, \mathbb{R})$$

If the manifold is equipped with a Riemannian metric, this induces a metric on differential forms of each degree. This scalar product allows us to define the adjoint operator d^* of the exterior derivative via

$$\langle d^*\alpha, \beta \rangle = \langle \alpha, d\beta \rangle$$

[1] I first heard this nomenclature in a talk by Ursula Ludwig, and I found it very fitting.

The associated chain complex is

$$0 \longrightarrow \Omega^n(X) \xrightarrow{d_n^*} \Omega^{n-1}(X) \xrightarrow{d_{n-1}^*} \dots \xrightarrow{d_1^*} \Omega^0(X) \xrightarrow{d_0^*} 0$$

so again, $d_k^* \circ d_{k-1}^* = 0$. With this information, we can define the **Hodge Laplacian** (or Laplace-de Rham operator) as

$$\Delta := (d + d^*)^2 = d^* d + dd^*$$

which acts on k-forms as $\Delta^{(k)} = d_{k+1}^* d_k + d_{k-1} d_k^*$ (as one can imagine, here $d_{-1} = 0 = d_{n+1}^*$ is meant). The solutions of the (differential) equation

$$\Delta^{(k)} \omega = 0, \quad \omega \in \Omega^k(X)$$

are called **harmonic k-forms** and we can "count" them if we compute the dimension of $\ker \Delta^{(k)}$. The Hodge theorem states that each cohomology class of the manifold has a unique harmonic representative (the one with minimal L^2-norm). So there is an isomorphism

$$\ker \Delta^{(k)} \cong H_{dR}^k(X)$$

and therefore counting the solutions of the above equation is the same as computing the dimension of the corresponding cohomology.
A somewhat similar game can be played with Kähler manifolds and holomorphic bundles, where de Rham gets replaced by Dolbeault and the exterior derivative by the "del-bar" operator

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

in complex coordinates $z = x + iy$.

2.1.2 Gauß-Bonnet theorem

The Hodge theorem allows us to understand other fundamental results in geometry as comparison theorems. The more accessible one is the **Gauß-Bonnet theorem** for (real) compact oriented smooth surfaces (with boundary). It states a connection between curvatures of the manifold and its **Euler characteristic**, an alternating sum of dimensions of cohomology of different degrees (or the sum vertices - edges + faces in a triangulation of the manifold). Thus, it counts (in an alternating fashion) solutions of differential equations (harmonic forms) by relating them to how much the surface deviates from being planar or its boundary "straight".
To be more exact, consider a compact 2-dimensional Riemannian manifold X with boundary ∂X. Then one can construct its second fundamental form by (Whitney-)embedding the manifold in Euclidean space and computing its eigenvalues, the **principal curvatures**. The product of them is the **Gaussian curvature** K. Turns out the Gaussian curvature is precisely half the scalar curvature of the manifold, and thus independent of the embedding, that is, intrinsic to the manifold. The boundary of the surface ∂X is a 1-dimensional manifold, a curve, and we can compute its **geodesic curvature** k_g also with the help of an embedding. It measures how far the curve differs from a geodesic, so from being "straight".
The Euler characteristic of a general topological space is defined as the alternating sum of **Betti numbers**, i.e. of dimensions of (singular) homology of different degrees:

$$\chi(X) = \dim H_0(X; R) - \dim H_1(X; R) + \dim H_2(X; R) - \dim H_3(X; R) + \dots$$

where R is any coefficient ring with respect to which we take the orientation (think $R = \mathbb{R}$ or $\mathbb{Z}$).

In the case of a compact n-manifold without boundary, we have **Poincaré duality** $H_k(X;R) \cong H^{n-k}(X;R)$ and we can also use $H_k(X;R) = 0$ for all $k > n$ to write the Euler characteristic as

$$\chi(X) = (-1)^n \sum_{k=0}^{n} (-1)^k \dim H^k(X;R)$$

For surfaces, we will just have three terms ($n = 2$). To treat the case with boundary, one should employ **Lefschetz duality**.
With these ingredients, the statement of the Gauß-Bonnet theorem[2] is condensed in the following formula:

$$2\pi\chi(X) = \int_X K dvol_X + \int_{\partial X} k_g dvol_{\partial X}$$

Notice how the addition of boundary adds an extra term (*defect*), the last integral.
We can make this into a comparison theorem (we do it just in the boundaryless case) by using the Hodge theorem:

$$\dim\ker\Delta^{(0)} - \dim\ker\Delta^{(1)} + \dim\ker\Delta^{(2)} = \frac{1}{2\pi}\int_X K dvol_X$$

Now the left hand side does not just contain the kernel of an operator, but an alternating sum of kernels: we "count solutions in an alternating fashion". Furthermore, it may come as a surprise that the right hand side is "doomed" to always be an integer!

2.1.3 Riemann-Roch theorem

We can tinker with surfaces a bit more and obtain our last comparison theorem before we move on to the index theorem. We now want to equip our compact oriented surfaces without boundary (treating the boundary case in this setting generally turns out way harder, see Example 2.14) with a complex structure: our local charts are modelled on $\mathbb{C}$ instead of $\mathbb{R}^2$, and chart transitions are holomorphic. When we think of surfaces equipped in this way, i.e. as complex curves, we refer to them as **Riemann surfaces**. It is perhaps important to note, that any compact orientable surface is realizable as a Riemann surface, meaning there is a way to equip it with a complex structure.
A new topological invariant comes into play now: the **genus** of the surface g, which counts the number of "handles" one has to attach to a sphere to arrive at a surface homeomorphic to the one in hand. The genus is a complete invariant for compact Riemann surfaces, in the sense that it classifies them up to homeomorphism; in other words, classes modulo homeomorphism are distinguished by their genera.
The genus also carries relevant information about the complex structure. A Riemann surface is the simplest case of a Kähler manifold, where the k-th cohomology (think in terms of forms, like de Rham) decomposes as

$$H^k_{dR}(X;\mathbb{C}) = \sum_{p+q=k} H^{p,q}_{Dol}(X;\mathbb{C})$$

where p counts the holomorphic and q the anti-holomorphic differentials appearing in the k-th form (this is Dolbeault cohomology).

[2]Since we are looking for bridges between analysis and topology/geometry, we purposefully overlook many interesting aspects of some of these theorems. For example, a finer look at the Gauß-Bonnet theorem in the boundaryless case states that a local geometric quantity measuring how the surface *curves* itself (K) is closely related to the global topological quantity counting the *number of holes* ($\chi(X)$), which a priori is far from evident.

It is a consequence of Hodge theory on complex projective varieties that the dimensions of cohomology satisfy

$$\dim H^{1,0}(X;\mathbb{C}) = \dim H^{0,1}(X;\mathbb{C}) = g \quad \text{and} \quad \dim H^{0,0}(X;\mathbb{C}) = \dim H^{1,1}(X;\mathbb{C}) = 1$$

so the genus is also the (complex) dimension of the space of holomorphic 1-forms in X. Knowledge of this should make the statement of the theorem to come less striking. This also fits nicely with the fact that the Euler characteristic of a surface of genus g is $\chi(X) = 2 - 2g$ [3].

An important actor in the result is the notion of **divisor**, which can be defined as a (formal) finite linear combination of points on the surface with integer coefficients. We can assign a divisor to each meromorphic function (or form) by considering its zeroes and poles as points, multiplying each of them with an integer corresponding to their order (positive for zeroes and negative for poles) and taking their (formal) linear combination.

As an example, if the meromorphic function f has a zero at p_1 of order 3 and poles at p_2 of order 1 and at p_3 of order 2, then its divisor (f) takes the form $3p_1 - p_2 - 2p_3$. There is not so much freedom in choosing the examples: it can be shown that the **degree** of the divisor $\deg((f))$ for (f) a global meromorphic function, that is, the sum of the integer coefficients appearing in the linear combinations, is always 0.

We can think of the rest of the points not appearing in the expression as having coefficient 0. One can add or substract different divisors by adding or substracting their linear combinations with this idea in mind.

There is an equivalence relation on the space of divisors called **linear equivalence**: two divisors are linear equivalent if they differ by a divisor of the form (f), where f is a non-zero meromorphic function (such divisors are called **principal**). Thus, degree is well defined within an equivalence class.

$$(g) \sim (h) \iff \exists f : (g) = (h) + (f)$$

If we consider two global holomorphic 1-forms α and β, their quotient is a non-zero global meromorphic function $f = \frac{\alpha}{\beta}$. Note that $(fg) = (f) + (g)$. This means:

$$(\alpha) = (f\beta) = (f) + (\beta) \implies (\alpha) \sim (\beta)$$

so all global meromorphic 1-forms have the same divisor, the so-called **canonical divisor** K, up to linear equivalence.

For a divisor D, the space of interest $\mathcal{L}(D)$ is that of meromorphic functions f such that the coefficients of $(f) + D$ are non-negative: if D has a pole at a point, f should have a zero at that point of at least that order; if D has a zero at a point, f is allowed to have a pole of order less than the order of the zero of D. This is a vector space and the dimension of this vector space satisfies the following formula:

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(K - D) = \deg(D) - g + 1$$

It is called **Riemann-Roch theorem**, since Riemann found an inequality not involving the second term on the left and his student Roch (who sadly passed away at a young age) improved it to an equality by finding said term.

[3] Recall that when computing cohomology over a ring, the dimension is taken with respect to said ring. When we work with Riemann surfaces, we are interested in their cohomology over $\mathbb{C}$ and thus the dimensions in the formula for the Euler characteristic are the complex dimensions of the cohomology vector spaces, and not the real dimensions. In a Kähler manifold we can alternatively use the relationship between de Rham and Dolbeault cohomology we just described.

As our coloring shows, this also fits the bill of comparison theorem, since the left hand side is occupied with counting meromorphic functions of a certain type (so in some sense, solutions of $\bar{\partial} f = 0$) and the right hand side is purely topological (only the number of handles and a marking of points with integers attached to them appears).

2.2 The topological face of index theory

In the period between 1945 and 1955, a young Hirzebruch was able to extend such results to a more general class of manifolds. These developments where conducive to the later formulation (1962) and proof (1963) of the index theorem by Atiyah and Singer.
His first statement goes by the name of **Hirzebruch-Riemann-Roch theorem** and claims that for any projective complex manifold X (not only Riemann surfaces) with a holomorphic vector bundle $V \to X$ over it, the following equality holds:

$$\chi(X, V) = \mathrm{Todd}(X)\mathrm{ch}(V)[X]$$

We will explain what the elements in the right hand side of the formula mean in the next section. We can already say that the notation $[X]$ means "evaluation on the fundamental class", that is, cap product of the cohomology class with the orientation homology class of top degree, as done in Poincaré duality:

$$H^k(X; R) \xrightarrow{\cong} H_{n-k}(X; R), \quad \omega \mapsto \omega \frown [X]$$

$$\alpha \in H^n(X; R) \implies \alpha[X] \coloneqq \alpha \frown [X] = \int_X \alpha$$

so we are effectively just integrating[4] the top degree part of the form $\mathrm{Todd}(X)\mathrm{ch}(V)$ on the manifold (and integration depends on the choice of volume form, which corresponds to the fundamental class):

$$\chi(X, V) = \int_X \mathrm{Todd}(X)\mathrm{ch}(V)$$

The left is the alternating sum of dimensions of (sheaf) cohomologies for the bundle V and thus an integer. This is an interesting fact, since the elements in the right are a priori rational numbers. It is a similar situation as what happened in the Gauß-Bonnet theorem: we can deduce the integrality of a quantity that a priori does not necessarily need to be an integer via the formula.
With the help of some computations it can be shown that the case X being a Riemann surface and $V = \mathcal{L}(D)$ the bundle over it corresponds exactly to the Riemann-Roch theorem, hence the name. In particular, $\deg D$ comes roughly from integrating the Chern character of the sheaf and the integral of the Todd class is half the Euler characteristic of the surface, so $1 - g$.
The other result is **Hirzebruch signature theorem** for closed oriented smooth $4m$-manifolds:

$$\mathrm{Sign}(X) = L(X)[X]$$

[4]Beware of the following important distinction we do not make here: the cap product is already defined at the level of topological spaces, e.g. in singular cohomology. When we talk about integrating a cohomology class, we mean we also have a differentiable structure and the de Rham cohomology that comes with it, and we use de Rham's theorem to find a differential form representing the topological cohomology class, which we can then integrate. This is actually a relevant difference and we will shortly mention it when we talk about how the index theorem moved historically from the study of **global topological invariants** to that of **local geometric invariants** (and so one usually finds the expression *a local index theorem/formula* in the literature when the second type is meant).

Recall the cup product operation on cohomology classes, that adds the degrees of the classes involved:

$$H^k(X;R) \otimes H^l(X;R) \longrightarrow H^{k+l}(X;R), \quad \alpha \otimes \beta \mapsto \alpha \smile \beta$$

If we take cup product of middle degree cohomology and then cap product with the fundamental class (as we just did above, the isomorphism of Poincaré duality), we obtain a nondegenerate symmetric bilinear pairing

$$H^{2m}(X;\mathbb{R}) \otimes H^{2m}(X;\mathbb{R}) \xrightarrow{\smile} H^{4m}(X;\mathbb{R}) \xrightarrow{\frown[X]} \mathbb{R}$$

$$\alpha \otimes \beta \mapsto \alpha \smile \beta \mapsto (\alpha \smile \beta)[X] = \int_X (\alpha \smile \beta)$$

The signature of this symmetric bilinear form is $\mathrm{Sign}(X)$ (number of positive eigenvalues minus number of negative ones; or in a diagonal representation, number of positive entries minus number of negative ones). Thus, it is always an integer, and so must $\int_X L(X)$ be.
These are not the only results showing integrality of certain characteristic classes: trying to deeply understand the integrality of the Â-genus on spin manifolds ((1) shows how this is not evident at all) is precisely what led Atiyah and Singer to their discovery. Even thought their methods are first based on cobordism (like Hirzebruch used in his theorems) and the development of a K-theory in the algebraic topological context (here Grothendieck plays an important role in the history with his K-theory in algebraic geometry and Bott with the periodicity theorem, which we will not mention further), we will not concentrate ourselves on these topics in the present discussion. For a more detailed discussion of the history of the index theorem see [Fre21a] and the corresponding talk [Fre21b]. A standard reference on vector bundles and K-theory is [Hat17]. Let us now understand what the characteristic classes appearing in the formulas (and the Â-genus) are and then set up the framework to formulate the index theorem for elliptic operators on compact manifolds without boundary.

2.2.1 What are characteristic classes?

Suppose we have two vector bundles $E, F \to X$ over a smooth manifold. A natural question to ask is whether they are the "same" (isomorphic in some way, e.g. as topological spaces, that is, homeomorphic, so that the homeomorphism commutes with the projections to X). This question is of course in general hard to answer, as is the case when trying to distinguish topological spaces. A usual approach from algebraic topology to tackle such questions is the introduction of **invariants**. These are quantities that remain unchanged under isomorphism. Hopefully they are also not too hard to compute, so that one can actually put them into use for telling spaces apart.
Characteristic classes are an instance of such a philosophy: they are topological invariants of vector bundles over a smooth manifold [5]. They roughly inform us about whether and how much the bundle is twisted, meaning how much it differs from having a product structure. They also count linearly independent sections of it. On top of that, they are usually computable: we will see some of them can be written as polynomials on the curvature form. If we find a characteristic class

[5] The reason why we can expect vector bundles to produce topological invariants is because they are rigid, i.e. if we consider a deformation given by a path of vector bundles $E^t \to X$, $0 \leq t \leq 1$, then we can produce a vector bundle over the cylinder $E \to X \times [0,1]_t$ with fibres $E_{(x,t)} = (E^t)_x$. Because of the contractibility of $[0,1]$, we can horizontally transport elements in $(E^0)_x$ by horizontal parallel transport to elements in $(E^1)_x$ and this produces an isomorphism of vector bundles: all the E^t are "the same" vector bundle.

that the bundles E and F do not share, we can conclude that they are not isomorphic. Otherwise, we have to keep looking.

Characteristic classes assign cohomology classes (of the base space) to bundles. They were developed in parallel to some of the advances we mentioned in the last section. If we want to be very precise, we can employ the language of category theory: if we have a topological space X and a topological group G we can consider the set $\mathrm{Bun}_G(X)$ of all principal G-bundles over X (we care about vector bundles mostly, which are not principal bundles[6], but the underlying structure for them is similar as we will see in a second). The assignment Bun_G that takes a space X and outputs the set $\mathrm{Bun}_G(X)$ induces a contravariant functor from the category of topological spaces to the category of sets, taking continuous functions to pullbacks:

$$\mathrm{Bun}_G : \mathrm{Top} \longrightarrow \mathrm{Set}$$
$$X \mapsto \mathrm{Bun}_G(X)$$
$$(f : X \to Y) \mapsto (f^* : \mathrm{Bun}_G(Y) \to \mathrm{Bun}_G(X), \quad E \mapsto f^*E)$$
$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

A cohomology theory H^* is also a contravariant functor of this kind, since it assigns a set of cohomology groups of various degrees to a topological space:

$$H^* : \mathrm{Top} \longrightarrow \mathrm{Set}$$
$$X \mapsto H^*(X)$$
$$(f : X \to Y) \mapsto (f^* : H^*(Y) \to H^*(X), \quad \alpha \mapsto f^*\alpha)$$

Of course cohomology theories can be augmented from sets to rings with the cup product. However, we only want here to take the set structure into account because then we can talk about natural transformations from Bun_G to H^*. Such a natural transformation c is what we call a **characteristic class**. This means that for every topological space X, the natural transformation produces a morphism c_X that associates a cohomology class to each bundle over X that respects pullbacks (of bundles and in cohomology):

$$\begin{array}{ccc} \mathrm{Bun}_G(Y) & \xrightarrow{\ c_Y\ } & H^*(Y) \\ {\scriptstyle f^*}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle f^*} \\ \mathrm{Bun}_G(X) & \xrightarrow{\ c_X\ } & H^*(X) \end{array}$$

$$\begin{array}{ccc} E & \longmapsto & c_Y(E) \\ \big\downarrow & & \big\downarrow \\ f^*E & \longmapsto & c_X(f^*E) = f^*(c_Y(E)) \end{array}$$

[6] The fibres of principal bundles are never vector spaces, since there is no distinguished 0 element as in vector bundles. One could however "forget" the zero element, e.g. work with affine spaces instead of vector spaces. It is the case that affine bundles are principal $\mathbb{R}^n$-bundles. The caveat is then, that even though vector bundles have a Lie group as structure group, the way it acts on the fibres is not transitive, as in principal bundles. This distinction is relevant, since principal bundles are trivial precisely when they have a (smooth) global section, and the zero section is a (smooth) global section of any vector bundle.

(we tend to omit the notation c_X in favour of just c if it is clear what the base space for the bundle is).
All in all, in order to construct a characteristic class, we need to find a cohomology theory of our base manifold we can work with and a way to assign cohomology classes in it to our bundles (preferably in a computable way). We will see this more clearly with some examples.
In a way, characteristic classes are an attempt to use the following structure in algebraic topology: one can check that the (contravariant) functor Bun_G satisfies the requisites of **Brown's representability theorem** within the localization of the homotopy category of pointed CW complexes by weak homotopy (this is the appropriate category to work with topological spaces and continuous functions up to homotopy equivalence). What this means is that there exists a topological space BG corresponding to the group G, called the **classifying space**, and for each G-bundle $E \to X$ we can produce a (homotopy class of) map(s) $f_E : X \to BG$. The map f_E goes by the name of **classifying map**. This is, in fact, a 1-1 correspondence:

$$\text{Bun}_G(X) \cong [X, BG], \quad (E \to X) \mapsto (f_E : X \to BG)$$

There is another relevant space for each G, the total space EG of the universal bundle over BG, and these actors can be combined into the following pullback diagram:

$$\begin{array}{ccc} E & \xrightarrow{f_E^*} & EG \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f_E} & BG \end{array}$$

which works for any G-bundle $E \to X$ and is a way of describing the relationship between E and f_E, in particular gives us a way to recover E from f_E.
So in some sense, maps from X to BG "classify" all G-bundles over X. If we want to study whether a bundle is trivial, we could try to investigate whether its associated classifying map is nullhomotopic. As usual, the question of whether a map is nullhomotopic is hard to answer, but we can weaken it by "transporting it to the (co)homology world", and ask whether the map induces a trivial map f_E^* on (co)homology (recall that nullhomotopic maps induce the trivial map on (co)homology, but the converse is not true in general). Characteristic classes inform us precisely of the structure of this induced map.
In the case of real vector bundles, the formal structure is analogous. The contravariant functor

$$\text{Vect}_n : \text{Top} \longrightarrow \text{Set}$$

that takes a topological space X and assigns it the set of (isomorphism classes of) rank n vector bundles over it has exactly the same behavior as Bun_G:

1. It is (Brown-)representable with classifying space $Gr(n, \mathbb{R}^\infty)$ the infinite Grasmannian of n-dimensional linear subspaces of $\mathbb{R}^\infty$.

$$\text{Vect}_n(X) \cong [X, Gr(n, \mathbb{R}^\infty)]$$

The role of EG is taken in this case by the tautological bundle $\gamma_{n,\infty} \to Gr(n, \mathbb{R}^\infty)$, which is the bundle whose fibre at a point $W \in Gr(n, \mathbb{R}^\infty)$ is precisely W. Since all $W \in Gr(n, \mathbb{R}^\infty)$ are n-dimensional linear subspaces, they are modelled on $\mathbb{R}^n$, so it is a rank n vector bundle over the infinite n-Grasmannian.

This means that for any vector bundle $E \to X$ of rank n, the following is a pullback diagram:

$$\begin{array}{ccc} E & \xrightarrow{f_E^*} & \gamma_{n,\infty} \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f_E} & Gr(n,\mathbb{R}^\infty) \end{array}$$

2. A **characteristic class for (real) vector bundles** will be a natural transformation from Vect_n to a certain cohomology theory H^*: for every topological space X a morphism c_X assigning cohomology classes to rank n vector bundles over X, such that

$$\begin{array}{ccc} \mathrm{Vect}_n(Y) & \xrightarrow{c_Y} & H^*(Y) \\ f^* \downarrow & \circlearrowleft & \downarrow f^* \\ \mathrm{Vect}_n(X) & \xrightarrow{c_X} & H^*(X) \end{array}$$

$$\begin{array}{ccc} E & \longmapsto & c_Y(E) \\ \downarrow & & \downarrow \\ f^*E & \longmapsto & c_X(f^*E) = f^*(c_Y(E)) \end{array}$$

By point one, every rank n real vector bundle $E \to X$ can be constructed as $f^*\gamma_{n,\infty}$ with $f : X \to Gr(n,\mathbb{R}^\infty)$ a (homotopy class of) map(s) from the base to the classifying space ($f = f_E$). Thus, they are all pullbacks of the tautological bundle.
Adding point two (naturality) to that, we realize that

$$c(E) = c_X(E) = c_X(f^*\gamma_{n,\infty}) = f^*(c_{Gr(n,\mathbb{R}^\infty)}(\gamma_{n,\infty})) = f^*(c(\gamma_{n,\infty}))$$

which shows that knowledge of the characteristic classes of the tautological bundle allows us to construct the characteristic classes for the rest of the bundles by pullback.
So our last step in order to construct a characteristic class for rank n real vector bundles is to choose a nice cohomology theory on the infinite n-Grasmannian. As a topologist, one might be first drawn to singular cohomology with integer coefficients, but the ring structure of such cohomology is a bit more complicated than in the case of $\mathbb{Z}_2$ coefficients, where it is just a polynomial ring. This motivates the definition of the **Stiefel-Whitney classes** ω_i, which are chosen so that

$$H^*(Gr(n,\mathbb{R}^\infty);\mathbb{Z}_2) \cong \mathbb{Z}_2[\omega_1(\gamma_{n,\infty}),\ldots,\omega_n(\gamma_{n,\infty})]$$

Moreover, $\omega_0(E)$ is the generator 1 in H^0 for any bundle $E \to X$ and ω_k vanishes if $k > n = \mathrm{rank}(E)$ [7].
Pontrjagin classes are obtained by working with the cohomology ring $H^*(Gr(n,\mathbb{R}^\infty);\mathbb{Z})$. They come in degrees multiples of 4.
A similar procedure can be carried out with complex vector bundles and integral cohomology to obtain the **Chern classes** (change $Gr(n,\mathbb{R}^\infty)$ to $Gr(n,\mathbb{C}^\infty)$). However, as we mentioned above, we can also construct characteristic classes through differential geometry (and obtain the same natural transformations as through the topological way). This procedure is known as Chern-Weil theory.

[7] If X is a manifold, its cohomology vanishes for degree above its dimension, meaning $\omega_k = 0$ for $k > \dim(X)$ also applies.

Before the curvature makes its stellar appearance, we close this segment by stating the axiomatic definition of Stiefel-Whitney and Chern classes. The construction above should serve as a motivation for the choice of axioms with which these objects are introduced. It also "shows" existence of natural transformations that satisfy those axioms and hints at their uniqueness. The axioms are a neat way of working formally with the classes, since they recollect their most important properties. We will not prove that the axiomatic and the constructive approaches are equivalent.

Even though we discussed what happens with vector bundles of dimension n, our notation ω_i did not reflect this dimension. This is a slight abuse of notation motivated by the fact that we can define the characteristic class for each dimension n in a manner independent of it: the only ways the dimension affects the construction is through the classifying space and universal bundle over it (the infinite n-Grasmannian and the tautological bundle) and the fact that the classes with index greater than n vanish. We can therefore articulate the whole framework in a more general manner on $\mathrm{Vect}(X) = \bigsqcup_{n\geq 1} \mathrm{Vect}_n(X)$: the coproduct of those categories. $\mathrm{Vect}(X)$ is the category of all isomorphism classes of (real) vector bundles over X, where morphisms are only allowed between bundles of the same rank and are precisely those in each $\mathrm{Vect}_n(X)$ [8].

Definition 2.1. The **Stiefel-Whitney classes** are the unique sequence $\omega_0, \omega_1, \ldots$ of "functions" that assign to each real vector bundle $E \to X$ a class $\omega_i(E) \in H^i(X; \mathbb{Z}_2)$ with the properties:

1. They only depend on the isomorphism class of E. In other words, they are morphisms
$$\omega_i : \mathrm{Vect}(X) \longrightarrow H^i(X; \mathbb{Z}_2)$$
2. For any $f : X \to Y$,
$$\omega_i(f^*E) = f^*(\omega_i(E))$$
meaning they commute with pullbacks, i.e. they are natural transformations if we see $\mathrm{Vect}(X)$ and $H^i(X; \mathbb{Z}_2)$ as functors $\mathrm{Top} \to \mathrm{Set}$.
3. $\omega_0(E) = 1$ and $\omega_k(E) = 0$ for $k > \dim E$.
4. If we bunch up all natural transformations into the formal sum $\omega(E) = \omega_0(E) + \omega_1(E) + \ldots$, called the **total Stiefel-Whitney class**,
$$\omega : \mathrm{Vect}(X) \longrightarrow H^*(X; \mathbb{Z}_2)$$
then this class respects the monoidal structure of vector bundles (with $\oplus$) and cohomology groups (with $\smile$) over X, which translates into
$$\omega(E \oplus F) = \omega(E) \smile \omega(F)$$
5. They are non-trivial: it is demanded that $\omega_1(\gamma_{1,\infty})$ is a generator of $H^1(Gr(1, \mathbb{R}^\infty); \mathbb{Z}_2)$.

□

We do the same with complex vector bundles $\mathrm{Vect}^{\mathbb{C}}(X)$:

Definition 2.2. The **Chern classes** are the unique sequence $c_0, c_1, \ldots$ of "functions" that assign to each complex vector bundle $E \to X$ a class $c_i(E) \in H^{2i}(X; \mathbb{Z})$ with the properties:

[8] The base space X does not appear in the notation either, and one could strive for more generality by considering the coproduct of categories $\mathrm{Vect}(X)$ for all topological spaces X. Because the main goal is to distinguish vector bundles, we could first tell them apart by their base space and then by their rank, before even starting with characteristic classes. However, in our axiomatic definition, it is useful to consider all ranks at the same time: this allows us to talk about *total* classes and how they interact with the monoid operations (see point 4. in the definitions).

1. They only depend on the isomorphism class of E. In other words, they are morphisms

$$c_i : \mathrm{Vect}^{\mathbb{C}}(X) \longrightarrow H^{2i}(X;\mathbb{Z})$$

2. For any $f : X \to Y$,

$$c_i(f^*E) = f^*(c_i(E))$$

meaning they commute with pullbacks, i.e. they are natural transformations if we see $\mathrm{Vect}^{\mathbb{C}}(X)$ and $H^{2i}(X;\mathbb{Z})$ as functors Top $\to$ Set.

3. $c_0(E) = 1$ and $c_k(E) = 0$ for $k > \dim E$.

4. If we bunch up all natural transformations into the formal sum $c(E) = c_0(E) + c_1(E) + \dots$, called the **total Chern class**,

$$c : \mathrm{Vect}^{\mathbb{C}}(X) \longrightarrow H^{2*}(X;\mathbb{Z})$$

then this class respects the monoidal structure of vector bundles (with $\oplus$) and cohomology groups (with $\smile$) over X, which translates into

$$c(E \oplus F) = c(E) \smile c(F)$$

5. They are non-trivial: they are normalised by demanding that $c_1(\gamma^{\mathbb{C}}_{1,\infty})$ is a generator of $H^2(Gr(2,\mathbb{C}^\infty);\mathbb{Z})$ (to be chosen in advance).

□

Remark 2.3. The fact that the Stiefel-Whitney classes have coefficients in $\mathbb{Z}_2$ makes them more rigid than Chern classes. If we focus on the last condition, one could maybe ask if there is a different normalization that works for any of both cases. It turns out that given any integer $k \in \mathbb{Z}\backslash\{0\}$, the classes $\tilde{c}_i := k^i c_i$ also satistfy conditions $1. - 4.$ and are non trivial, but the same cannot be done with the ω_i to generate new classes because of the modulo 2 coefficients.
This is why the last condition is only a non-triviality constraint for Stiefel-Whitney classes and both a non-triviality *and* normalization condition for Chern classes. That is, there is only one possible generator to be chosen for the Stiefel-Whitney classes but more than one for the Chern classes.

□

2.2.2 Chern-Weil theory

As we mentioned before, characteristic classes try to measure how much a bundle is twisted, i.e. how much it differs from the trivial bundle. One could look for a measure of the twisting by computing the curvature of the bundle, if there is enough structure so as to have a notion of *curvature*. So a priori it is plausible that by measuring twisting, characteristic classes are also measuring the way the bundle curves itself in space. Chern approached the construction of characteristic classes through this lens: we will see how his definition explicitly uses the notion of **curvature form** of a bundle. The resulting theory eventually showed that the natural transformations obtained in this way are equivalent to the ones given by the axiomatic and classifying space points of view.
When we talk about curvature we intuitively think about how much a line deviates from being a straight line or how much a surface differs from a plane. There is a way of measuring such discrepancies by embedding our manifolds within larger dimensional spaces and looking at the

behaviour of their normal directions at each point. Curvature however turns out to be an intrinsic quantity, i.e. independent of the embedding, and thus we should be able to formulate it without reference to normal directions.
A way one generalizes the notion of curvature is by looking at how local objects defined on the tangent space of a manifold at a point behave while being transported along the manifold (think of a local frame being carried along a curve in the manifold). The curving of the space is then represented by the transported objects "deforming". This suggests that to define a notion of curvature, one needs a procedure by which to transport these local informations from the tangent space at a point to the tangent space at another in a smooth way, i.e. a way of *connecting* the tangent spaces. One needs a **connection**.

Definition 2.4. A **connection** (or covariant derivative) ∇ on a vector bundle $E \to X$ over a smooth manifold is a mapping

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$$

such that:

1. It is $\mathbb{R}$-linear
2. It is $\mathcal{C}^\infty(X)$-linear on vector fields:
$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \quad X, Y \in \Gamma(TX), \quad f, g \in \mathcal{C}^\infty(X)$$
3. For every $f \in C^\infty(X)$ and $s \in \Gamma(E)$ the following Leibniz-type rule is satisfied:
$$\nabla(fs) = df \otimes s + f\nabla s$$
In other words, it is a derivation.

□
Such a connection extends uniquely to E-valued forms to produce a so-called **exterior covariant derivative** [9] (notice how it raises form degrees by 1)

$$\nabla : \Gamma(\Lambda^k T^*X \otimes E) \longrightarrow \Gamma(\Lambda^{k+1} T^*X \otimes E)$$

if we require that a similar Leibniz-type condition holds also in this setting:

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg \alpha} \alpha \wedge \nabla s$$

In an open subset $U \subset X$ where the vector bundle trivialises, we can construct a local smooth frame of sections $(e_1, \ldots, e_n)$, where $e_i \in \Gamma(E\big|_U)$. Their restriction at each point $x \in U$ is a basis of E_x. We can use this basis to write local sections as

$$s\big|_U = \sum_{i=1}^{n} s^i e_i, \quad s^i \in C^\infty(U)$$

Thanks to the Leibniz rule, this means that knowing how the connection acts on the local frame is enough to determine it completely. We can store this information into a matrix of 1-forms which we will call the (local) **connection form** of ∇.

[9] In an abuse of notation, we employ ∇ both for the usual covariant derivative and for its extension to E-valued forms. In the literature one might find the notation d^∇ for the latter.

Since $\nabla e_i \in \Gamma(T^*U \otimes E\big|_U)$ it has a local expresion of the form

$$\nabla e_i = \sum_{i=1}^{n} \omega_i^j \otimes e_j, \quad \omega_i^j \in \Gamma(T^*U)$$

The matrix

$$\omega = \begin{pmatrix} \omega_1^1 & \dots & \omega_n^1 \\ \vdots & \ddots & \vdots \\ \omega_1^n & \dots & \omega_n^n \end{pmatrix} \in \Gamma(T^*U \otimes \mathrm{End}(E\big|_U)), \quad \mathrm{End}(E\big|_U) = E^*\big|_U \otimes E\big|_U$$

is called (local) **connection form** and summarises the information about the action of the connection ∇ on the bundle E, since

$$\nabla s\big|_U = \sum_{j=1}^{n} \left(ds^j + \sum_{i=1}^{n} \omega_i^j s^i \right) \otimes e_j \mathrel{\widehat{=}} d \begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix} + \begin{pmatrix} \omega_1^1 & \dots & \omega_n^1 \\ \vdots & \ddots & \vdots \\ \omega_1^n & \dots & \omega_n^n \end{pmatrix} \begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix} = ds\big|_U + \omega s\big|_U$$

which is sometimes written $\nabla\big|_U = d + \omega$. This means that the space of connections is locally modelled on $\Gamma(T^*U \otimes \mathrm{End}(E\big|_U))$, since a connection is determined by specifying ω, i.e. how much it differs from the exterior derivative d.
The **curvature** Ω of a connection ∇ is defined in the same way as the Riemann curvature tensor:

$$\Omega(V, W)(s) = \nabla_V \nabla_W s - \nabla_W \nabla_V s - \nabla_{[V,W]}, \quad V, W \in \Gamma(TX), \quad s \in \Gamma(E)$$

It is an endomorphism valued 2-form

$$\Omega = (\Omega_1 \otimes \Omega_2) \in \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}(E))$$

whose action on E-valued forms corresponds to the square of the connection:

$$\nabla^2 : \Gamma(\Lambda^k T^*X \otimes E) \longrightarrow \Gamma(\Lambda^{k+2} T^*X \otimes E), \quad \nabla^2(\alpha \otimes s) = \Omega \wedge (\alpha \otimes s) = (\Omega_1 \wedge \alpha) \otimes \Omega_2(s)$$

The local expression of the curvature form Ω relates to the local expression of the connection through

$$\Omega = d\omega + \omega \wedge \omega$$

The curvature form is a nicer object in the sense that it behaves tensorially under change of frame, whereas the connection form does not. As in the case of the connection form, we can think of the curvature form as a matrix of 2-forms. In the case of a Hermitian vector bundle[10], Chern computed the characteristic polynomial of this matrix for the case of a Hermitian connection (analogue of Levi-Civita connection in the Riemannian case: it is compatible with the Hermitian product):

$$\det \left(I - \frac{\Omega}{2\pi i} t \right) = \sum_k c_k(E) t^k$$

The coefficients $c_k(E)$ are differential forms of degree $2k$, since the entries of Ω are 2-forms, sometimes called **Chern forms**. They can be interpreted as representants of cohomology classes à la

[10]Being Hermitian ensures the Chern polynomial, i.e. the polynomial on (the forms representing) the Chern classes, has real coefficients.

de Rham: they produce precisely the Chern classes we introduced in the previous section, which justifies the abuse of notation. This is the starting point of Chern-Weil theory [11].
In particular, the total Chern class can be recovered by setting $t = 1$:

$$\det\left(I - \frac{\Omega}{2\pi i}\right) = c_0(E) + c_1(E) + \cdots = c(E)$$

Furthermore, using that t is just a parameter to generate the polynomial, we can Taylor expand the left hand side for small t. Using the identity $\det(\exp(A)) = \exp(\mathrm{tr}(A))$ for A a square matrix over $\mathbb{C}$, we can rewrite it as

$$\det\left(I - \frac{\Omega}{2\pi i}t\right) = \exp\left(\mathrm{tr}\left(\ln\left(I - \frac{\Omega}{2\pi i}t\right)\right)\right)$$

Now, from the Taylor expansion for $\ln(x+1)$ follows:

$$\ln(I - A) = \sum_{k=1}^{\infty}\left(-\frac{A^k}{k}\right) = -A - \frac{A^2}{2} - \frac{A^3}{3} - \dots$$

Taking the trace:

$$\mathrm{tr}\left(\ln\left(I - \frac{\Omega}{2\pi i}t\right)\right) = \sum_{k=1}^{\infty} -\left(\frac{\mathrm{tr}(\Omega^k)}{k}\right)\left(\frac{1}{2\pi i}\right)^k t^k = -\frac{\mathrm{tr}(\Omega)}{2\pi i}t + \frac{\mathrm{tr}(\Omega^2)}{8\pi^2}t^2 + \frac{\mathrm{tr}(\Omega^3)}{24\pi^3 i}t^3 - \dots$$

And finally the expansion of the exponential for small argument yields:

$$\begin{aligned}
\exp\left(\mathrm{tr}\left(\ln\left(I - \frac{\Omega}{2\pi i}t\right)\right)\right) &= 1 + \mathrm{tr}\left(\ln\left(I - \frac{\Omega}{2\pi i}t\right)\right) + \frac{1}{2}\left(\mathrm{tr}\left(\ln\left(I - \frac{\Omega}{2\pi i}t\right)\right)\right)^2 + \dots \\
= 1 - \frac{\mathrm{tr}(\Omega)}{2\pi i}t + \frac{\mathrm{tr}(\Omega^2)}{8\pi^2}t^2 + \frac{\mathrm{tr}(\Omega^3)}{24\pi^3 i}t^3 &+ \frac{1}{2}\left(-\frac{\mathrm{tr}(\Omega)}{2\pi i}t + \frac{\mathrm{tr}(\Omega^2)}{8\pi^2}t^2\right)^2 + \frac{1}{6}\left(-\frac{\mathrm{tr}(\Omega)}{2\pi i}t\right)^3 + O(t^4) \\
= 1 - \frac{\mathrm{tr}(\Omega)}{2\pi i}t &+ \frac{\mathrm{tr}(\Omega^2) - \mathrm{tr}(\Omega)^2}{8\pi^2}t^2 + \frac{2\mathrm{tr}(\Omega^3) - 3\mathrm{tr}(\Omega^2)\mathrm{tr}(\Omega) + \mathrm{tr}(\Omega)^3}{48\pi^3 i}t^3 + O(t^4)
\end{aligned}$$

From this expression, one can directly read off the first four Chern forms

$$\begin{aligned}
c_0(E) &= 1 \\
c_1(E) &= -\frac{\mathrm{tr}(\Omega)}{2\pi i} \\
c_2(E) &= \frac{\mathrm{tr}(\Omega^2) - \mathrm{tr}(\Omega)^2}{8\pi^2} \\
c_3(E) &= \frac{2\mathrm{tr}(\Omega^3) - 3\mathrm{tr}(\Omega^2)\mathrm{tr}(\Omega) + \mathrm{tr}(\Omega)^3}{48\pi^3 i}
\end{aligned}$$

So given a bundle with a connection, we can proceed algorithmically in each local trivialisation by computing the connection form, then the curvature form and finally the Chern forms with these formulas, attaining representatives of the Chern classes.

[11] The fundamental pillar sustaining Chern-Weil theory is that the cohomology classes so constructed are independent of the curvature form, i.e. of the choice of connection.

2.2.3 The meaning behind *some* characteristic classes

There is a reason why we focused on Stiefel-Whitney and Chern classes: many other relevant characteristic classes are polynomials on them. One should compare this with how symmetric polynomials are polynomials on elementary symmetric polynomials. Underlying all this theory is a technique called the **splitting principle**, which aims to simplify the computations of characteristic classes by relating general vector bundles with direct sums of line bundles. The splitting principle claims that for every vector bundle $E \to X$, where X is paracompact, there exists a morphism $p : Fl(E) \to X$ from the "flag bundle" associated to E so that the following is a pullback diagram:

$$\begin{array}{ccc} \bigoplus_{i=1}^n L_i & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow \\ Fl(E) & \xrightarrow{\;p\;} & X \end{array}$$

for certain line bundles L_i. Furthermore, p^* is injective on cohomology[12]. This means we can translate the computations of characteristic classes of bundles to direct sums of line bundles. So in some sense, it is enough to understand these.

The total Chern (resp. Stiefel-Whitney) class of a line bundle has the simple form $c(L) = 1 + c_1(L)$, using property 3. in their axiomatic definition. By property 4.,

$$c(L_1 \oplus \cdots \oplus L_n) = c(L_1) \smile \cdots \smile c(L_n) = (1 + c_1(L_1)) \cdots (1 + c_n(L_n))$$

where we omit $\smile$ in the last step, since it is clear this is the product we equipped the cohomology ring with.

If we write x_i instead of $c_1(L_i)$ we can use the following property of the elementary symmetric polynomials on n-variables $\sigma_i(x_1, \dots, x_n)$:

$$(t + x_1) \cdots (t + x_n) = t^n + \sigma_1(x_1, \dots, x_n)t^{n-1} + \cdots + \sigma_n(x_1, \dots, x_n)$$

Any symmetric polynomial (a polynomial invariant by interchange of any of the variables) can be written in a unique way as a polynomial on these elementary symmetric polynomials σ_k, which are defined as:

$$\sigma_k(x_1, \dots, x_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k}$$

Setting $t = 1$, this means that the pullback by p^* of the Chern class $c(E) = c_0(E) + c_1(E) + \cdots + c_n(E)$ to the direct sum of line bundles over the flag bundle is of the form:

$$p^*c(E) = c(p^*E) = c(L_1 \oplus \cdots \oplus L_n) = 1 + \sigma_1(c_1(L_1), \dots, c_1(L_n)) + \cdots + \sigma_n(c_1(L_1), \dots, c_1(L_n))$$

where the $c_1(L_i)$ are named **Chern roots** of E. In particular

$$p^*c_k(E) = \sigma_k(c_1(L_1, \dots, c_1(L_n))$$

All in all, the splitting principle induces an embedding $H^{2k}(X; \mathbb{Z}) \subset H^{2k}(Fl(E), \mathbb{Z})$ through which Chern classes are mapped to elementary symmetric polynomials on the Chern roots.

This is a machine to produce new characteristic classes: we can start with a symmetric polynomial and express it as a polynomial on the elementary symmetric polynomials σ_k, to then replace these by Chern classes $c_k(E)$ (resp. Stiefel-Whitney classes).

[12] p^* is also injective on K-theory.

The first important example was used as a tool to study homotopy groups of spheres. By point 4. in their definition, Chern classes take direct sums of bundles to cup products in cohomology. In the space of vector bundles, not only the direct sum operation is natural, but also the tensor product. We want to look for a combination of Chern classes that is compatible with both operations.
The abelian monoid structure in $(\mathrm{Vect}(X), \oplus)$ can be augmented to an abelian group structure by means of the so-called **Grothendieck completion**, which is the base of K-theory. The idea behind it is the same as how one would formally construct the positive rational numbers as quotients of natural numbers: $(\mathbb{N}, \cdot)$ is an abelian monoid, but it does not contain inverses; by adding their inverses we obtain $(\mathbb{Q}_{>0}, \cdot)$, an abelian group. This completion is "natural" in the sense that it can be seen as a functor from the category of abelian monoids to that of abelian groups which is left adjoint to the forgetful functor in the opposite direction ("if we get rid of inverses in an abelian group, we obtain an abelian monoid")

$$\mathrm{AbMon} \underset{Forg}{\overset{Comp}{\rightleftarrows}} \mathrm{AbGrp}, \quad \bot \quad \text{i.e.}$$

$$\hom_{\mathrm{AbGrp}}(Comp(A), G) \cong \hom_{\mathrm{AbMon}}(A, Forg(G))$$

so when we augment a monoid A to a group, every map $Comp(A) \to G$ from the augmentation to another group comes with a corresponding map $A \to Forg(G)$ of the underlying monoids, and viceversa.
Applying this procedure to $(\mathrm{Vect}^{\mathbb{C}}(X), \oplus)$ yields[13] the K-theory of X, denoted $K^0(X)$, so we have an abelian group of bundles. We can make this into a ring by adding the tensor product operation $\otimes$. We want to compare this ring structure $(K^0(X), \oplus, \otimes)$ on bundles with the one in cohomology $(H^*(X; \mathbb{Q}), +, \smile)$. It is not without meaning that one employs rational coefficients, we will see why in a minute.
The aim is to construct a characteristic class Ch, the **Chern character**, such that

$$Ch(E \oplus F) = Ch(E) + Ch(F)$$
$$Ch(E \otimes F) = Ch(E) \smile Ch(F)$$

We will generate it as a differential form using Chern forms, and as a form it will satisfy

$$Ch(E \otimes 1 + 1 \otimes F) = Ch(E) \wedge Ch(F)$$

The name "character" comes from representation theory of groups: given a group G and representations $\rho, \sigma : G \to \mathrm{GL}(V)$ onto a finite dimensional $\mathbb{F}$-vector space V, their characters $\chi_\rho, \chi_\sigma : G \to \mathbb{F}$ are defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$ (resp. χ_σ) and satisfy

$$\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$$
$$\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$$

By the splitting principle, it is enough to define Ch for line bundles. For line bundles, the first Chern class is a complete invariant, in the sense that there is a group isomorphism

$$(\mathrm{Vect}_1^{\mathbb{C}}(X), \otimes) \longrightarrow (H^2(X; \mathbb{Z}), +), \quad L \mapsto c_1(L)$$

In particular $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. We can exploit the property $e^{x+y} = e^x e^y$ of the exponential function to obtain the desired ring homomorphism:

[13]The same could be done for real vector bundles but, since we focus our discussion on Chern classes, we will stay within the complex realm for now.

Definition 2.5. The Chern character of a line bundle $L \to X$ is the cohomology class

$$Ch(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \frac{c_1(L)^3}{3!} + \dots \in H^*(X;\mathbb{Q})$$

□

The fact that this is a ring homomorphism follows from the discussion above, more concretely:

$$Ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = e^{c_1(L_1)}e^{c_1(L_2)} = Ch(L_1)Ch(L_2)$$

For a general bundle $E \to X$ that pulls back by p^* to a direct sum $L_1 \oplus \dots \oplus L_n \to Fl(E)$:

$$\begin{aligned} p^*Ch(E) &= Ch(L_1 \oplus \dots \oplus L_n) = Ch(L_1) + \dots + Ch(L_n) = e^{c_1(L_1)} + \dots + e^{c_1(L_n)} \\ &= n + \frac{c_1(L_1) + \dots + c_1(L_n)}{1!} + \frac{c_1(L_1)^2 + \dots + c_1(L_n)^2}{2!} + \dots \end{aligned}$$

The polynomials $s_k(x_1, \dots, x_n) = x_1^k + \dots + x_n^k$ for $k \geq 1$ are called **Newton polynomials** and are symmetric. Thus, they can be written in terms of the elementary symmetric polynomials $\sigma_l(x_1, \dots, x_n)$. For example:

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \end{aligned}$$

This means we can rewrite each term in $Ch(E)$ in terms of the $\sigma_k(c_1(L_1), \dots, c_1(L_n))$, i.e. in terms of the Chern classes $c_k(E)$:

$$\begin{aligned} Ch(E) &\mathrel{\hat{=}} p_* \left(n + \tfrac{c_1(L_1)+\dots+c_1(L_n)}{1!} + \tfrac{c_1(L_1)^2+\dots+c_1(L_n)^2}{2!} + \tfrac{c_1(L_1)^3+\dots+c_1(L_n)^3}{3!} + \dots \right) \\ &= \dim(E) + c_1(E) + \tfrac{c_1(E)^2 - 2c_2(E)}{2} + \tfrac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \dots \end{aligned}$$

where we use the fact that the number n of line bundles in the direct sum of p^*E is precisely equal to $\dim(E)$. The Chern character is hence defined as a polynomial on the Chern classes and can be checked to have the desired properties via the splitting principle.
The Chern character induces a ring homomorphism

$$Ch : (K^0(X), \oplus, \otimes) \longrightarrow (H^{2*}(X;\mathbb{Q}), +, \smile)$$

The reason why rational coefficients are chosen is because the resulting map

$$K^0(X) \otimes \mathbb{Q} \longrightarrow H^{2*}(X;\mathbb{Q})$$

is an isomorphism on finite cell complexes. This does not hold true in integral cohomology[14]. In other words, the Chern character is a complete invariant for the K-theory of finite cell complexes. With that we have understood one of the ingredients in the Hirzebruch-Riemann-Roch theorem, the other being the **Todd class** (or Todd genus). We will introduce it together with its cousin the **Â-genus**, since both are multiplicative and appear in similar instances in the context of index theorems. We will also introduce another *genus*, the L-genus appearing in the signature theorem. So what is a "genus"? It is another machine to construct characteristic classes which have the nice multiplicative property:

$$\text{genus}(E \oplus F) = \text{genus}(E) \smile \text{genus}(F)$$

[14]For finite cell complexes, both $K^0(X)$ and $H^*(X;\mathbb{Z})$ are finitely generated abelian groups which differ only by their torsion subgroups. Tensoring both by $\mathbb{Q}$ gets rid of the torsion and produces the isomorphism.

as the Chern/Stiefel-Whitney classes had[15].
The process starts by constructing a symmetric polynomial and applying the philosophy we mentioned above. Consider a formal power series $f(x)$ with constant term 1. For variables $\{x_i\}_{i\geq 1}$ we can look at the product $\prod_i f(x_i t)$. The term that goes with t^j is a symmetric polynomial in the x_i's, homogeneous of weigth j: we can write it in terms of the elementary symmetric polynomials $\sigma_k(x_1, x_2, \dots)$ as a function $K_j(\sigma_1, \sigma_2, \dots)$, i.e. we treat the σ_i's as variables. The sequence $(K_1, K_2, \dots)$ is a **multiplicative sequence**, meaning that if

$$1 + \sum_{i\geq 1} a_i x^i = \left(1 + \sum_{j\geq 1} b_j x^j\right)\left(1 + \sum_{k\geq 1} c_k x^k\right)$$

then

$$\sum_{i\geq 1} K_i(a_1, a_2, \dots) x^i = \left(\sum_{j\geq 1} K_j(b_1, b_2, \dots) x^j\right)\left(\sum_{k\geq 1} K_k(c_1, c_2, \dots) x^k\right)$$

This suggests we can interpret multiplicative sequences as characteristic classes with the above multiplicative property if the role of the x_i's (resp. σ_i's) is taken by the Chern roots (resp. Chern classes) associated to the vector bundle.

Definition 2.6. The **genus** [16] of a real vector bundle $E \to X$ associated to a power series $f(x)$ is given by

$$\Phi(E) = K(p_1(E), p_2(E), \dots) \in H^{4*}(X; \mathbb{Q})$$

where the role of the elementary symmetric functions is taken by the Pontrjagin classes. The power series is named **characteristic power series** of Φ.
For a complex bundle one can take the Chern classes in place of the Pontrjagin classes and land in $H^{2*}(X; \mathbb{Z})$.

□
The formal power series[17]

$$\frac{\sqrt{x}}{\tanh(\sqrt{x})} = \sum_{i\geq 0} \frac{2^{2i} B_{2i}}{(2i)!} x^i = 1 + \frac{x}{3} - \frac{x^2}{45} + \dots, \qquad B_{2i} = 4i(-1)^{i+1} \int_0^\infty \frac{t^{2i-1}}{e^{2\pi t} - 1} dt$$

with the Pontrjagin classes produces the L**-genus**

$$\begin{aligned} L(p_1(E), p_2(E), \dots) &= L(\sigma_1(x_1, x_2, \dots), \sigma_2(x_1, x_2, \dots), \dots)\big|_{\sigma_k = p_k(E)} \\ &= \prod_{j\geq 1}\left(\sum_{i\geq 0} \frac{2^{2i} B_{2i}}{(2i)!} (x_j)^i\right)\Bigg|_{\sigma_k = p_k(E)} \end{aligned}$$

[15]We did not say much about Pontrjagin classes for the sake of brevity, but they behave in a similar way modulo 2-torsion: $p(E \oplus F) = p(E) \smile p(F) \mod 2$. This is a consequence of the fact that

$$p_i(E) = c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X; \mathbb{Z})$$

and that Chern classes only come in even orders. They can be constructed à la Chern-Weil and depend only on the square of the curvature form Ω, which agrees with them coming only in degrees multiples of 4 in cohomology

$$p(E) = I - \frac{\mathrm{tr}(\Omega^2)}{8\pi^2} + \frac{\mathrm{tr}(\Omega^2)^2 - 2\mathrm{tr}(\Omega^4)}{128\pi^4} - \dots$$

They are the building blocks of the Â- and L-genera.

[16]One can also understand genus as a ring homomorphism, like the Chern character, but instead of it being defined on the K-theory, it is defined on the cobordism ring. In other words, if two manifolds are cobordant, i.e. there exists a manifold of one dimension higher whose boundary is the disjoint union of them, then their genera are equal.

[17]The B_{2i} are the even Bernoulli numbers.

$$
\begin{aligned}
L_0(E) &= 1 \\
L_1(E) &= \frac{p_1(E)}{3} \\
L_2(E) &= \frac{7p_2(E) - p_1(E)^2}{45}
\end{aligned}
$$

and so on.
In the signature theorem, $L(X) := L(TX)$ is meant. Seeing the expresions for $L_i(X)$, it is the more surprising that the L-genus of a smooth manifold always integrates to an integer (we call the evaluation of a characteristic class on the fundamental class a **characteristic number**). In particular, this integrality is an obstruction for a manifold to admit a smooth structure!
The **Todd class** (or genus) corresponds to the characteristic power series

$$
\frac{x}{e^x - 1} = \sum_{i \geq 0} \frac{B_i}{i!} x^i = -\frac{x}{2} + \sum_{i \geq 0} \frac{B_{2i}}{(2i)!} x^{2i}
$$

where the last equality follows from computing

$$
\frac{x}{e^x - 1} - \frac{-x}{e^{-x} - 1} = -x \implies \frac{x}{e^x - 1} + \frac{x}{2} \quad \text{is even}
$$

Consequently,

$$
\mathrm{Todd}(E) = \prod_{j \geq 1} \left(-\frac{x_j}{2} + \sum_{i \geq 0} \frac{B_{2i}}{(2i)!} (x_j)^{2i} \right) \Bigg|_{\sigma_k = c_k(E)} \in H^{2*}(X; \mathbb{Q})
$$

$$
\begin{aligned}
\mathrm{Todd}_0(E) &= 1 \\
\mathrm{Todd}_1(E) &= \frac{c_1(E)}{2} \\
\mathrm{Todd}_2(E) &= \frac{c_2(E) + c_1(E)^2}{12}
\end{aligned}
$$

and so on.
The n-th Todd class of (the tangent bundle of) the n-th projective space is always 1.
Other authors prefer to take the Chern-Weil route and compute the Todd class from the curvature form:

$$
\mathrm{Todd}(E) = \det\left(\frac{\Omega}{e^{\Omega} - 1} \right) = \exp\left(\mathrm{tr}\left(\ln\left(\frac{\Omega}{e^{\Omega} - 1} \right) \right) \right) \in H^{2*}(X; \mathbb{C})
$$

This is equivalent to the definition above if we were to normalize the curvature form via

$$
\Omega \mapsto \frac{\Omega}{2\pi i}
$$

We did not do it as an excuse to make the reader aware of the (and stress the existence of) different normalizations in the literature [BGV04, p. 47].
Finally, the power series

$$
\frac{\frac{\sqrt{x}}{2}}{\sinh\left(\frac{\sqrt{x}}{2}\right)} = 1 - \frac{x}{24} + \frac{7x^2}{5760} - \ldots
$$

generates the **Â-genus** with the Pontrjagin classes

$$\begin{aligned}\hat{A}_0(E) &= 1\\ \hat{A}_1(E) &= -\frac{p_1(E)}{24}\\ \hat{A}_2(E) &= \frac{-4p_2(E)+7p_1(E)^2}{5760}\end{aligned}$$

and so on, or à la Chern-Weil, up to normalization[18]

$$\begin{aligned}\hat{A}(E) = \det{}^{1/2}\left(\frac{\frac{\Omega}{2}}{\sinh\left(\frac{\Omega}{2}\right)}\right) &= \exp\left(\operatorname{tr}\left(\frac{1}{2}\ln\left(\frac{\frac{\Omega}{2}}{\sinh\left(\frac{\Omega}{2}\right)}\right)\right)\right)\\ &= 1-\frac{\operatorname{tr}(\Omega^2)}{48}+\frac{4\operatorname{tr}(\Omega^4)+5\operatorname{tr}(\Omega^2)^2}{23040}-\cdots \in H^{4*}(X;\mathbb{R})\end{aligned} \tag{1}$$

where the coefficients are real without the previous normalization of the curvature form, since only even powers of Ω appear, so after normalizing only even powers of i (with the normalization it is rational).
We called the Todd- and Â-genera "cousins". The justification is the following relation for almost complex manifolds:

$$\operatorname{Todd}(TX) = e^{\frac{c_1(TX)}{2}}\hat{A}(TX)$$

In particular, they are equal when the first Chern class vanishes.
The (local) index theorem will show us that the Â-genus is an integer for **spin manifolds**, since it correspond to the index of the (spinor) Dirac operator. The (local) index theorem was in fact nothing else than a product of trying to understand why this genus is an integer.
To close this section, we want to mention that the Chern and Stiefel-Whitney classes have a very nice and "direct" interpretability, which actually motivated the first steps in the development of this theory. The meaning of other classes could be grasped for example from the content of the theorems of Hirzebruch and Atiyah-Singer.
The Stiefel-Whitney classes partly measure how many everywhere linearly independent sections the bundle admits. If the bundle $E \to X$ has rank n:

$$E \quad \text{admits} \quad n-i+1 \quad \text{everywhere linearly independent sections} \implies \omega_i(E) = 0$$

except for $i = 0$, for which $\omega_0(E) = 1$ is always satisfied (by property 3.) to enable property 4. of the axiomatic definition.
Consequently, if E admits j everywhere linearly independent sections, then $\omega_{n-j+1}(E) = \cdots = \omega_n(E) = 0$; however, the converse does not hold. As a result, it is said that Stiefel-Whitney classes are a *primary obstruction* to the existence of everywhere linearly independent sections, but there could be *secondary obstructions*.
An important example to keep in mind are even spheres, for which the Stiefel-Whitney classes of their tangent bundles $T\mathbb{S}^{2n}$ all vanish (except $\omega_0(T\mathbb{S}^{2n})$). Even so, the hairy ball theorem tells us that they do not admit an everywhere linearly independent section, i.e. a nowhere vanishing vector field.

[18] This is the *geometers' normalization* [BGV04, p. 143]; when we later study the local index formula for Dirac operators, the Â-genus associated to the bundle TX will appear and we will consider the *topologists' normalization*, which corresponds to multiplying the terms of differential form degree k by a factor of $(2\pi i)^{-\frac{k}{2}}$ and is a purely aesthetical choice that ensures Chern classes are integral and the index formula acquires the concise form (5).

In particular, the manifold is orientable if and only if $\omega_1(TX) = 0 \in H^1(X;\mathbb{Z}_2)$, and is *spin*[19] (introduced later) precisely when it is orientable and $\omega_2(TX) = 0 \in H^2(X;\mathbb{Z}_2)$ [Pat, p. 110-111]. Thus, all spheres are spin.
The idea behind why orientability can be read off from a class in $H^1(X;\mathbb{Z}_2)$ is the following: to detect orientability, we can travel around loops in our base manifold and look at whether the fibres preserve or reverse their orientation after traversing the loop. Preserving or reversing orientation is a continuous property and thus it depends only on the homotopy class of the loop, i.e. we can construct a map $\pi_1(X) \to \mathbb{Z}_2$ which takes the value 0 if the orientation of the fibres are preserved and 1 if they are reversed. This map is a homomorphism, since the products of homotopy classes correspond to traversing loops one after the other.
A group homomorphism into an abelian group factors through the abelianization of the source, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \mathbb{Z}_2 \\ & \searrow \quad \nearrow & \\ & H_1(X;\mathbb{Z}) & \end{array}$$

since the fundamental group abelianizes to the first homology group. So the information about the constructed map is purely contained in the pointed arrow. Now, the bundle is orientable when the orientation of the fibres is always preserved, in which case the morphism has image 0. On the other hand, homomorphisms $H_1(X;\mathbb{Z}) \to \mathbb{Z}_2$ are identified with elements in $H^1(X;\mathbb{Z}_2)$, e.g. by the universal coefficient theorem. This element (our $\omega_1(E)$) is zero exactly when the map is 0, hence the result.
The vanishing of the k-Chern class is also an obstruction to the existence of $n-k+1$ everywhere linearly independent sections of a complex bundle. The top Chern class is sometimes called **Euler class**[20] and its non-vanishing indicates the lack of existence of a nowhere zero global section.
The following section will elucidate what the Todd- and Â-genera, together with the Chern character, might represent.

2.3 The analytical face of index theory

To talk about the index theorem, we first need to introduce the analytical notion of Fredholm index. Once that is done, we proceed to formulate the groundbreaking result of Atiyah and Singer, a statement they unveiled while *navigating the universes* of K-theory and cobordism theory.
There is however another way to *palpate the elephant* of index theory, which we will be most interested in because it naturally allows us to extend the statement to several classes of manifolds, in particular our beloved singular manifolds, in that it provides us with a straightforward roadmap on how to approach the problem. This perspective goes by the name of **heat kernel method**, since it derives the index formula purely from the understanding of certain solutions to heat equations. It moreover provides explicit local expressions of differential forms representing the characteristic classes appearing in the topological statement, i.e. *local index formulas*. However, this fine understanding of the corresponding heat kernel asymptotics turns out to be quite challenging and in most cases forces us to restrict our scope to Dirac type operators.

[19] A complex manifold X satisfies

$$\omega_{2i}(TX) = c_i(TX) \mod 2$$

which means it only has even degree non-trivial Stiefel-Whitney classes: it admits a spin structure if and only if $c_1(TX)$ is even.

[20] The Euler class is defined also for real vector bundles and independently of our other characteristic classes. In the real case, it is related to the top Stiefel-Whitney class, though it carries more information about orientations.

2.3.1 Elliptic operators and the index theorem

As already hinted at, one of the main objects of our study later on is going to be the Dirac operator. It is an example of a first order **elliptic differential operator** acting on sections of vector bundles over the manifold.

Definition 2.7. A differential operator P of order $m \geq 0$ in an n-dimensional manifold X is a linear map

$$P : C^\infty(X) \longrightarrow C^\infty(X)$$

that locally in $U \subset X$ admits a coordinate expresion of the form

$$P\big|_U = \sum_{|\alpha| \leq m} p_\alpha D^\alpha, \quad p_\alpha \in C^\infty(U) \tag{2}$$

where[21] for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ (such that $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^n \alpha_i$),

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

What is meant is that for any $u \in C^\infty(U)$, the operator maps it to

$$P\big|_U u = \sum_{|\alpha| \leq m} p_\alpha D^\alpha u \in C^\infty(U)$$

The space of differential operators on a manifold is a non-commutative filtered associative Lie algebra (with composition as product and commutator as Lie bracket), denoted $\text{Diff}^*(X)$. The filtration is given by the order of the operators, in which case $\text{Diff}^m(X)$ contains those of order $\leq m$.
We obtain a similar structure if we consider operators acting between sections of bundles. For bundles $E \to X$ and $F \to X$ these are linear maps

$$P : \Gamma(E) \longrightarrow \Gamma(F)$$

whose local description in a $U \subset X$ over which the bundles trivialize looks the same as (2), but now the p_α are not smooth functions, but matrices of smooth functions of size $\text{rk}(F) \times \text{rk}(E)$, i.e. $p_\alpha \in \Gamma(\hom(E\big|_U, F\big|_U))$. Taking derivative of sections is done as follows:

$$D^\alpha s\big|_U = D^\alpha \left(\sum_{k=1}^n s^k e_k\right) = \sum_{k=1}^n \left(D^\alpha s^k\right) e_k, \quad s^k \in C^\infty(U)$$

Their filtered algebra is denoted $\text{Diff}^*(X; E, F)$ or simply $\text{Diff}^*(E, F)$.
We are mostly interested in the case $E = F$, for which the p_α's are square matrices.

□

An equivalent way to define differential operators between bundles over X that does not explicitly use a choice of coordinates from which we construct the derivatives D_i is just to consider compositions of vector fields with coefficients in $\hom(E, F)$. In other words, $\text{Diff}^*(E, F)$ is the **universal enveloping algebra** of $\mathcal{V}(X)$ (the space of vector fields over X) with $\hom(E, F)$ coefficients. In particular, $\text{Diff}^*(X)$ is the universal enveloping algebra over $\mathcal{V}(X)$.

[21] The Fourier transform is a fundamental tool to understand (pseudo)-differential operators, and is a reason why we choose to work with such a normalization of the derivatives in the D^α notation.

Spelled out, the algebraic structure of the algebra of differential operators is constructed in the following way: start with the tensor algebra

$$T(\mathcal{V}(X)) = C^\infty(X) \oplus \mathcal{V}(X) \oplus (\mathcal{V}(X) \otimes \mathcal{V}(X)) \oplus (\mathcal{V}(X) \otimes \mathcal{V}(X) \otimes \mathcal{V}(X)) \oplus \dots$$

and consider within it the equivalence relation given by the Lie algebra structure on $\mathcal{V}(X)$, i.e. the Lie bracket of vector fields,

$$V \otimes W - W \otimes V \sim [V, W] \in \mathcal{V}(X), \quad V, W \in \mathcal{V}(X)$$

The universal enveloping algebra of $\mathcal{V}(X)$ is

$$U(\mathcal{V}(X)) = T(\mathcal{V}(X)) /_{\sim}$$

which corresponds to quotienting by the (two-sided) ideal generated by elements of the form $VW - WV - [V, W]$. This invariant description as universal enveloping algebra readily generalises to our later cases when we construct differential operators adapted to specific classes of metrics, since we just need to repeat the construction above with the Lie subalgebra of vector fields that are uniformly bounded with respect to the metric.
From a filtered algebra $\mathcal{A} = \bigcup_{n\geq 0} \mathcal{A}_n$ one can always construct a graded algebra by quotienting:

$$\text{gr}\ \mathcal{A} = \bigoplus_{n\geq 0} \text{gr}_n\ \mathcal{A} = \bigoplus_{n\geq 0} \left(\mathcal{A}_n \big/ \mathcal{A}_{n-1} \right)$$

For the algebra of differential operators, the graded algebra singles out the terms of each order $|\alpha|$. Throughout the whole theory, we are mostly interested in the highest order terms of differential operators, which transform as a symmetric tensor due to the commutation properties of derivatives. In fact, "projecting" on the highest order part, the so-called **principal symbol map**, induces an isomorphism:

$$\sigma_m : \text{gr}_m\ \text{Diff}^*(E, F) \longrightarrow \Gamma(S^m(T^*X) \otimes \text{hom}(E, F))$$

where $S^m(T^*X)$ is the m-th symmetric power of the cotangent bundle of X, which also means we can construct the highest order part of a differential operator of order m from any symmetric tensor of this form.
In local coordinates, we could first take the **total symbol**[22], which corresponds to replacing the

[22]The total symbol of a differential operator in a manifold is not invariantly defined, since the x_i and the $\frac{\partial}{\partial x_i}$ do not commute:

$$\left[x_i, \frac{\partial}{\partial x_i}\right] u = x_i \frac{\partial u}{\partial x_i} - \frac{\partial}{\partial x_i}(x_i u) = -u \implies \left[x_i, \frac{\partial}{\partial x_i}\right] = -Id$$

However, they commute up to lower order terms, which is why taking the highest order term makes the principal symbol map well defined. We can computationally illustrate this with an example: consider the operator $P = \frac{d^2}{dx^2}$ with symbol ξ^2 and a diffeomorphism $x = f(y)$. Then the transformed operator can be obtained applying the chain and product rule:

$$(Pg)(x) = g''(x) = g''(f(y)) = (g'' \circ f)(y), \quad (g \circ f)''(y) = ((g' \circ f) \cdot f')'(y) = \left((g'' \circ f) \cdot (f')^2 + (g' \circ f) \cdot f''\right)(y)$$
$$\implies g'' \circ f = \frac{1}{(f')^2}\left((g \circ f)'' + (g' \circ f) \cdot f''\right) \implies P = \frac{1}{(f'(y))^2}\left(\frac{d^2}{dy^2} + f''(y)\frac{d}{dy}\right), \quad \text{while } dx = f'(y)dy$$

so the symbol of the first summand is exactly ξ^2, but the second summand adds lower order terms. The key is the need to apply the product rule when differentiating more than one time (and the commutation rule above is another way of saying that in this case the derivative of the smooth function f' during this product rule contributes a new term).

Even though the total symbol is in general not invariant under non-affine changes of coordinates, there are situations where one can fix a canonical choice, e.g. in a Riemmanian manifold with Riemann normal coordinates, the one obtained working on those coordinates.

derivatives with cotangent variables:

$$\sigma(P)(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)\xi^\alpha, \quad (x,\xi) \in T^*X$$

The **principal symbol** is then

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$$

which is a $\hom(E_x, F_x)$-valued degree m homogeneous polynomial on $\xi \in T_x^*X$ [23]. It can also be defined independent of local coordinates by its **action on wave packets** e^{itf} (since derivating these k times produces a factor t^k in front, so allows to distinguish contributions of different derivation orders) as

$$\sigma_m(P)(x,\xi) = \lim_{t\to\infty} t^{-m}(e^{-itf}Pe^{itf})(x) \in \hom(E_x, F_x), \quad f \in C^\infty(X) \quad \text{so that} \quad df(x) = \xi$$

One can also understand the principal part of an (differential) operator at a point $x_0 \in X$ as the constant coefficient operator one obtains by **"zooming in"** (a similar idea lies behind the definition of other constructions we carry out later, like that of *normal operator*). We illustrate this for $X = \mathbb{R}^n$, but applies generally by working on appropriate charts (e.g. the exponential map composed with a homothety). Start by focusing on a small neigborhood $B_\varepsilon(x_0)$ around the point and consider the "zoom-in" map given by

$$\delta_\varepsilon : B_\varepsilon(x_0) \longrightarrow B_1(0), x \mapsto \frac{x - x_0}{\varepsilon}$$

with inverse the zoom-out $\delta_\varepsilon^{-1}(y) = x_0 + \varepsilon y$.

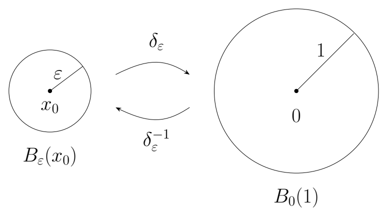


Figure 1: "Zooming in" as a way of recovering the principal part of a differential operator at a point $x_0 \in X$.

The idea is to first zoom-in on the ε-ball, apply the operator on this radius 1 enlargement of it, and then zoom-out, and see what happens as ε shrinks to 0, i.e. as we localize the action of the

[23]In other words, for each $(x,\xi) \in T^*X$, $\sigma_m(P)(x,\xi)$ gives a homomorphism $E_x \to F_x$. Since, for the usual projection $\pi : T^*X \to X$ and a bundle $E \to X$, we have $(\pi^*E)_{(x,\xi)} = E_{\pi(x,\xi)} = E_x$, then

$$\sigma_m(P) : \pi^*E \to \pi^*F$$

operator to x_0. Following the computations[24]:

$$\left[(\delta_\varepsilon^{-1} \circ P \circ \delta_\varepsilon)(f)\right](x) = \delta_\varepsilon \left(\sum_{|\alpha| \le m} p_\alpha(x) D^\alpha \left(f\left(\frac{x - x_0}{\varepsilon} \right) \right) \right)$$

$$= \delta_\varepsilon \left(\sum_{|\alpha| \le m} p_\alpha(x) \varepsilon^{-|\alpha|} \left(D^\alpha f\right) \left(\frac{x - x_0}{\varepsilon} \right) \right) = \sum_{|\alpha| \le m} p_\alpha(x_0 + \varepsilon x) \varepsilon^{-|\alpha|} \left(D^\alpha f\right)(x)$$

$$\implies \varepsilon^m \delta_\varepsilon P \delta_\varepsilon^{-1} \underset{\varepsilon \to 0}{\longrightarrow} \sum_{|\alpha| = m} p_\alpha(x_0) D^\alpha$$

leads us to the order m part of the differential operator $P \in \mathrm{Diff}^m(X)$ with constant coefficients $p_\alpha(x_0)$. The principal symbol is then just its Fourier transform.

Definition 2.8. An operator P of order m is called **elliptic** if its principal symbol $\sigma_m(P)$ is invertible over the open set $\{(x, \xi) \in T^*X \mid \xi \neq 0\} \subset T^*X$

□

Elliptic operators are relevant in the theory because, over compact manifolds without boundary, they "are" (extend to) Fredholm (operators).

Definition 2.9. A linear bounded operator $P : X \to Y$ between Banach spaces is **Fredholm** if and only if it is invertible modulo compact operators: there exists a bounded linear operator $S : Y \to X$ taking the role of its "quasi-inverse", i.e. $Id_X - ST$ and $Id_Y - TS$ are compact operators.
Equivalently, $\dim \ker T < \infty$, $\dim \mathrm{coker}\ T < \infty$ ($\mathrm{coker}\ T := Y/\mathrm{im}\ T$) and T has closed image, which allows us to define the following quantity called (Fredholm) **index**:

$$\mathrm{ind}\ T = \dim \ker T - \dim \mathrm{coker}\ T$$

□

If the Banach spaces are equipped with an inner product (i.e. they are Hilbert spaces) we can talk about the (Hermitian) adjoint T^* of the operator T:

$$\langle T^*(y), x \rangle = \langle y, T(x) \rangle, \quad x \in X,\ y \in Y$$

When $K \subset Y$ is a closed subset of a Hilbert space, then $Y/K \cong K^\perp$, and thus

$$\mathrm{coker}\ T := Y/T(X) = T(X)^\perp = \ker T^*$$

in which case the formula for the index takes the form

$$\mathrm{ind}\ T = \dim \ker T - \dim \ker T^*$$

This is the usual formulation we will apply in the following developments.
Within the context of comparison theorems, as its name suggests, the index theorem will relate the index of a Fredholm operator with purely geometrical/topological quantities. So the question is: in which sense can elliptic differential operators be seen as Fredholm[25]?
The first step is to endow the space $\Gamma(E)$ with a norm: this will be provided by the exterior covariant derivative ∇.

[24] One could argue the local form of Getzler rescaling (§4.2, [BGV04, p. 140-141]) is a refined version of this, where the map δ_ε would take into account the differential form degree.

[25] If we allow our operators to act on smooth sections, we run into trouble: the space $\Gamma(E)$ of smooth sections of E is not necessarily Banach (but a Fréchet space), since it is not complete (think about how the absolute value function can be constructed as limit of smooth functions). Thus, one needs to be slightly more careful about how to translate this operator theory into manifolds. We will avoid here a complete description of the minutiae in favour of clarity.

Definition 2.10. The k**-th Sobolev norm** on $\Gamma(E)$ is given by

$$||u||_k^2 := \sum_{j=0}^{k} \int_X |\nabla^j u|^2 dvol_X$$

We denote $H^k(X)$ the **Sobolev space** obtained by completing $\Gamma(E)$ with this norm.

□

More generally, we can produce non-integer Sobolev spaces also in the usual way by first defining bundle-valued distributions. We introduce the space[26] $\mathcal{D}(E)$ of compactly supported smooth sections of E, also called "test sections". A sequence $\phi_n \in \mathcal{D}(E)$ converges if there is a compact set containing the supports of all the ϕ_n's and they converge in C^k for every $k \geq 0$ in that compact set.

Definition 2.11. A **distribution** $u \in \mathcal{D}'(E)$ in E is a continuous linear functional

$$u : \mathcal{D}(E) \longrightarrow \mathbb{R}$$

where continuity means respecting convergence of sequences $\phi_n \to \phi$, i.e. $u(\phi_n) \to u(\phi)$.

□

One can also define the **Schwartz space**

$$\mathcal{S}(E) = \{u \in \Gamma(E) \mid u \equiv 0 \text{ at } \partial X\} = \{u \in \Gamma(E) \mid \sup |x^\alpha (D^\beta u)(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n\}$$

of rapidly decaying sections on E ($u \equiv 0$ means u vanishes to all orders/to infinite order) and its dual, the tempered distributions $\mathcal{S}'(E)$. In the case of manifolds with boundary we will focus on the spaces $\dot{\Gamma}(E)$ of sections vanishing to infinite order at the boundary and its dual, the space $\Gamma^{-\infty}(E)$ of extendible distributions.

As in the Euclidean case, we can define non-integer Sobolev spaces H^s as the completion of Schwartz spaces under the Sobolev norm:

$$||u||_s^2 := \int \left(1 + |\eta|^{2s}\right) |\hat{u}(\eta)|^2 d\eta, \quad s \in \mathbb{R}, \quad u \in \mathcal{S}(E)$$

where $\hat{u}$ is the Fourier transform of u.

Such definitions lead to a Sobolev embedding theorem and allow us to extend differential operators $P : \Gamma(E) \to \Gamma(F)$ of order m to bounded linear operators

$$P : H^s(E) \longrightarrow H^{s-m}(F), \quad \forall s \in \mathbb{R}$$

When the operators are elliptic on compact manifolds without boundary, the resulting extensions are Fredholm and the index is independent of the extension, i.e. of the choice of $s \in \mathbb{R}$. In particular, we might zero in on $s = m$ to have as codomain the standard L^2-completion $H^0(E) = L^2(E)$ of $\Gamma(E)$.

So now given an elliptic differential operator on a compact manifold without boundary it is clear what we mean when we say it is Fredholm and by its index.

To understand the topological side of the index theorem we just need some last ingredients (cf. [Lan05, §1.2-2.3]): suppose we have a complex $E^\bullet$ of vector bundles over the compact manifold X

$$0 \to E^0 \xrightarrow{d_0} E^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} E^n \to 0$$

[26] In the case of compact manifolds without boundary, this is just $\Gamma(E)$, but for manifolds with boundary the distinction is relevant.

so that the restriction of the complex to the fibre over each $x \in X$ is exact except for a compact subset in X. Then we can associate to the complex a class in $K_0(X)$. We use this construction in two instances:

- Given the projection $\pi : T^*X \to X$, the principal symbol map of an m-th order differential operator induces a complex
$$0 \to \pi^*E \xrightarrow{\sigma_m(P)} \pi^*F \to 0$$
and ellipticity means the complex is exact outside the 0-section in T^*X, which corresponds to the compact set $X \hookrightarrow T^*X$. As a result, one can interpret the principal symbol of an elliptic operator as a class in $K_0(T^*X)$ (effectively representing $[\pi^*E] - [\pi^*F]$). The same can be said switching T^*X by its complexification $T^*_{\mathbb{C}}X := T^*X \otimes_{\mathbb{R}} \mathbb{C}$.

- For each $(x, \xi) \in T^*X$ we can construct a complex using the algebra of differential forms
$$0 \to \left(\Lambda^0 T^*X\right)_x \xrightarrow{\xi\wedge} \left(\Lambda^1 T^*X\right)_x \xrightarrow{\xi\wedge} \dots \xrightarrow{\xi\wedge} \left(\Lambda^n T^*X\right)_x \to 0$$
since $\xi \wedge \xi = 0$. This is exact except if $\xi = 0$. Noticing that $\left(\pi^*\Lambda^k T^*X\right)_{(x,\xi)} = \left(\Lambda^k T^*X\right)_x$, this produces a complex
$$0 \to \pi^*\Lambda^0 T^*X \xrightarrow{\xi\wedge} \pi^*\Lambda^1 T^*X \xrightarrow{\xi\wedge} \dots \xrightarrow{\xi\wedge} \pi^*\Lambda^n T^*X \to 0$$
exact away from the 0-section, so it corresponds to a class in $K_0(T^*X)$. We can work analogously with $T^*_{\mathbb{C}}X$ instead, obtaining $\lambda \in K_0(T^*_{\mathbb{C}}X)$ [27]. In this complexified setting, a fundamental theorem in K-theory states that
$$\phi : K_0(X) \to K_0(T^*_{\mathbb{C}}X), \quad [E] \mapsto \lambda \cdot [\pi^*E]$$
is an isomorphism.

  On a different note, the following map is also an isomorphism:
$$\varphi : H^k(X; R) \to H^{k+n}_c\left(T^*_{\mathbb{C}}X; R\right), \quad [\alpha] \mapsto [\pi^*\alpha] \smile u$$
where H_c denotes compactly supported cohomology, R is any ring and $u \in H^n_c\left(T^*_{\mathbb{C}}X; R\right) \cong H^n\left(T^*_{\mathbb{C}}X, T^*_{\mathbb{C}}X/X; R\right) \cong \mathbb{Z}$ is the generator whose restriction to each fibre corresponds to the chosen orientation given by the complex structure[28].

  Both ϕ and φ are sometimes called **Thom isomorphisms**[29] and can be constructed in the same way for any complex vector bundle $V \to X$.

Furthermore, given a (Whitney) embedding $i : X \hookrightarrow \mathbb{R}^m$, there exists a (wrong-way/shriek) map $i_! : K_0(T^*_{\mathbb{C}}X) \to K_0(T^*_{\mathbb{C}}\mathbb{R}^m)$. It allows us to define the **topological index** map
$$\text{top-ind} : K_0(T^*_{\mathbb{C}}X) \xrightarrow{i_!} K_0(T^*_{\mathbb{C}}\mathbb{R}^m) \cong K_0(\mathbb{R}^{2m}) \cong K_0(\text{pt}) \overset{\text{rank}}{\cong} \mathbb{Z}$$
independent of the choice of embedding.
We now know enough to understand the result of Atiyah and Singer in two of its formulations (respectively: K-theory [AS68] in 1968; cohomological, proved with cobordism [AS63] in 1963):

[27] This λ is called Thom class and has to do with Clifford multiplication (see [Kot10, p. 2]), a relevant actor in the sequel.

[28] $T^*_{\mathbb{C}}X/X$ is made sense of by identifying X with the 0-section

[29] The failure of the Thom isomorphisms to commute with the Chern character is related to the Todd class:
$$\varphi\left((-1)^n \text{Todd}^{-1}(\overline{T^*_{\mathbb{C}}X}) \smile Ch(E)\right) = Ch(\phi(E))$$
This actually explains its appearance in the index theorem. Note that $\overline{T^*_{\mathbb{C}}X} = T^*_{\mathbb{C}}X$.

Theorem 2.12. Consider vector bundles $E, F \to X$ over a compact manifold without boundary and an elliptic operator $P \in \mathrm{Diff}^m(E, F)$. Then it extends to a Fredholm operator $P : H^s(E) \to H^{s-m}(F)$, its symbol corresponds to a class $[\sigma_m(P)] \in K_0(T^*_{\mathbb{C}}X)$ and its Fredholm index satisfies:

$$\mathrm{ind}(P) = \text{top-ind}([\sigma_m(P)])$$

In terms of cohomology, the statement takes the form:

$$\mathrm{ind}(P) = (-1)^n \left(Ch([\sigma_m(P)]) \smile \mathrm{Todd}(T^*_{\mathbb{C}}X)\right)[T^*_{\mathbb{C}}X]$$

and we can rewrite this for oriented X as:

$$\mathrm{ind}(P) = (-1)^{\frac{n(n+1)}{2}} \int_X \varphi^{-1} \left(Ch([\sigma_m(P)]) \smile \mathrm{Todd}(T^*_{\mathbb{C}}X)\right)$$

□

2.3.2 Heat kernel and Fredholm index

Besides the K-theoretical and cobordism formulations, the index theorem has been studied through many perspectives; this allows one to adapt the point of view to one's specific purpose. In the case of Dirac-type operators on singular manifolds with certain geometrical ends[30], the so-called **heat kernel method** has been sucessfully applied to similar contexts, so we choose it as our route guide. We will now briefly present it and explain its philosophy. For more details, a classical reference is [LM89, §3]. Apart from the methods mentioned above, another interesting approach (also local, like the one we are about to develop) is the stochastic one, found in [Hsu02, §7.6], which exploits the relationship between the heat kernel and the transition density of Brownian motion[31].
It will be advantageous in our discourse to follow the spirit of the **Schwartz kernel theorem** and study operators by looking at their integral action via kernels. A general formulation would consider a continuous linear operator $A : \mathcal{D}(X) \to \mathcal{D}'(Y)$ mapping (a small space of) test functions to (a large space of) distributions, and its (integral) kernel $K_A \in \mathcal{D}'(X \times Y)$ defined through:

$$(Au)(x) = \int_X K_A(x, x')u(x')dx'$$

The Schwartz kernel theorem then asserts that there is a one-to-one correspondence between such operators and such distributions representing their (integral) kernels. Since we are interested in the study of singularities, for us this means we can analyze the action of operators by understanding the asymptotical behaviour of their (Schwartz) kernels as they approach the singular loci. Nevertheless, we do not need to go so far so as to take advantage of this perspective: it will prove useful even in the compact case without boundary.
Given an elliptic differential operator P, we will look at the **heat operator** related to P:

$$(e^{-tP}u)(x) = \int_X h_P(t, x, x')u(x')dx'$$

whose kernel $h_P(t, x, x')$ (the **heat kernel**) is the fundamental solution to the heat equation:

$$\begin{cases} (\partial_t + P)\, h_P(t, x, x') = 0, \quad t > 0 \\ \lim_{t\to 0} h_P(t, x, x') = \delta(x - x') \end{cases}$$

[30] E.g. cylindrical [Mel93], hyperbolic-cuspidal [Vai01], iterated wedge [AGR23], etc. See §4.1.
[31] This connection was also in the mind of Getzler when he interpreted the result of his rescaling [Get86, Theorem A.1].

Example 2.13. A fundamental example that pops up every once and again in the theory is the uncomplicated Euclidean heat kernel, i.e. the fundamental solution of the heat equation in $\mathbb{R}^n$ with the standard metric [32]:

$$\begin{cases} (\partial_t + \Delta_{\mathbb{R}^n})\, h_{\mathbb{R}^n}(t,x,x') = 0 \\ \lim_{t\to 0} h_{\mathbb{R}^n}(t,x,x') = \delta(x-x') \end{cases}$$

The explicit expression can be computed by Fourier transform (which is in general a good technique to try on these problems):

$$h_{\mathbb{R}^n}(t,x,x') = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{||x-x'||^2}{4t}}$$

□

Looking at the formula in Theorem 2.12 it is not at all clear at first sight where the heat kernel could play a role. Notice however that if $Q : \Gamma(E) \to \Gamma(E)$ is non-negative self-adjoint elliptic for some vector bundle $E \to X$ over a closed manifold, then it is also compact and thus admits a complete basis of eigenfunctions, which by the non-negativity satisfy:

$$Qu_k = \lambda_k u_k, \quad \lambda_k \geq 0$$

The action of its heat operator is thus given by

$$e^{-tQ} u_k = e^{-t\lambda_k} u_k \xrightarrow[t\to\infty]{} \begin{cases} 0, & \lambda_k > 0 \\ u_k, & \lambda_k = 0 \end{cases}$$

meaning only the elements in the kernel survive this large time limit, so it functions as a projection:

$$\lim_{t\to\infty} e^{-tQ} = \Pi_{\ker Q}$$

The trace of a projection operator gives out the dimension of the space in which we are projecting, therefore:

$$\mathrm{Tr}\left(\lim_{t\to\infty} e^{-tQ}\right) = \dim\ker Q$$

so we can already guess how to fit this within the left hand side of the index formula. We can rewrite this expression in terms of heat kernels using **Lidskii's formula** for trace class operators [Mel93, Prop. 4.55]:

$$\mathrm{Tr}\,(A) = \int_X K_A(x,x)dx$$

In particular, only the values of the kernel at the diagonal $\mathrm{diag}_X \subset X^2$ contribute to the trace of the operator!

Now come back to our elliptic operator $P \in \mathrm{Diff}(E,F)$ and assume the vector bundles have a hermitian structure so that we can define the adjoint $P^* : \Gamma(F) \to \Gamma(E)$. Then, the operators $P^*P : \Gamma(E) \to \Gamma(E)$ and $PP^* : \Gamma(F) \to \Gamma(F)$ are non-negative self-adjoint elliptic, as can be seen from

$$\langle P^*Pu, u\rangle = \langle Pu, Pu\rangle = ||Pu||^2 \geq 0, \quad \langle PP^*u, u\rangle = ||P^*u||^2 \geq 0$$

[32] As the paramount example of elliptic operator of second order, when we do not specify the operator whose heat equation we are solving, it is usually the Laplacian or a Laplace-type operator what we refer to. We will mostly solve the heat equation for a Dirac-type operator squared, which is precisely a Laplace-type operator.

Intuitively, the heat kernel of the Laplace-Beltrami operator in a manifold X models the diffusion of heat in X, i.e. $h(t,x,x')$ represents the heat density at a point $x' \in X$ at time t, assuming all the heat was concentrated at a point $x \in X$ at time $t = 0$. Equivalently, it represents the probability that a Brownian motion (a continuous random walk, i.e. the infinitesimal free random movement of a particle) starting at $x \in X$ is at the position $x' \in X$ at time t. If one wants to model jumps, then one could swap the Laplacian Δ for one of its fractional cousins Δ^s, $s \in (0,1)$.

Furthermore, if $P^*Pu = \lambda u$ for $\lambda \neq 0$, then $\lambda Pu = P\lambda u = PP^*Pu$, so

$$P\big|_{E_\lambda} : E_\lambda \to F_\lambda$$

is an isomorphism for $E_\lambda = \{u \in \Gamma(E) : P^*Pu = \lambda u\}$ and $F_\lambda = \{u \in \Gamma(F) : PP^*u = \lambda u\}$ with inverse $\frac{1}{\lambda}P^*\big|_{F_\lambda}$, in particular $\ker P^*P = \ker P$.
Finally, by [LM89, III.(5.14)] the series

$$\operatorname{Tr}\left(e^{-tP^*P}\right) = \sum_{\lambda_k \geq 0} e^{-t\lambda_k^{P^*P}}$$

converges absolutely (analogous for PP^*) and

$$\begin{aligned}\operatorname{Tr}\left(e^{-tP^*P}\right) - \operatorname{Tr}\left(e^{-tPP^*}\right) &= \sum_{\lambda_k \geq 0} e^{-t\lambda_k^{P^*P}} - \sum_{\lambda_k \geq 0} e^{-t\lambda_k^{PP^*}} \\ &= \sum_{\lambda_k > 0} \underbrace{\left(e^{-t\lambda_k^{P^*P}} - e^{-t\lambda_k^{PP^*}}\right)}_{=0 \text{ since } E_\lambda \cong F_\lambda} + \dim\ker P^*P - \dim\ker PP^*\end{aligned}$$

so it is independent of time! We can write it more compactly by introducing the operator

$$\hat{P} = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix} : \Gamma(E) \oplus \Gamma(F) \longrightarrow \Gamma(E) \oplus \Gamma(F)$$

and the *supertrace*[33] of its heat operator

$$\operatorname{Str}\left(e^{-t\hat{P}^2}\right) := \operatorname{Tr}\left(e^{-tP^*P}\right) - \operatorname{Tr}\left(e^{-tPP^*}\right) \overset{\text{Lidskii}}{=} \int_X \left(h_{P^*P}(t,x,x) - h_{PP^*}(t,x,x)\right) dx$$

We will sometimes denote this difference of kernels by $\operatorname{str}\left(h_{\hat{P}^2}\right)(t,x)$ or simply $\operatorname{str}(h)(t,x)$ if the associated operator is clear.
Putting all together:

$$\begin{aligned}\operatorname{ind}(P) = \dim\ker P - \dim\ker P^* &= \dim\ker P^*P - \dim\ker PP^* \\ &= \operatorname{Tr}\left(e^{-tP^*P}\right) - \operatorname{Tr}\left(e^{-tPP^*}\right) = \operatorname{Str}\left(e^{-t\hat{P}^2}\right) \\ &= \operatorname{Tr}\left(\lim_{t\to\infty} e^{-tP^*P}\right) - \operatorname{Tr}\left(\lim_{t\to\infty} e^{-tPP^*}\right) = \lim_{t\to\infty} \operatorname{Str}\left(e^{-t\hat{P}^2}\right)\end{aligned}$$

With the last row we just want to remark that in more general settings, the (analytical) Fredholm index term will come from the large time limit of the supertrace, i.e. from the behaviour of (the supertrace of) the heat kernel at the diagonal for *infinite time*. In fact, we can a priori (just the fundamental theorem of calculus) write:

$$\lim_{t\to\infty} \operatorname{Str}\left(e^{-t\hat{P}^2}\right) = \lim_{t\to 0} \operatorname{Str}\left(e^{-t\hat{P}^2}\right) + \int_0^\infty \partial_t \operatorname{Str}\left(e^{-t\hat{P}^2}\right) dt$$

[33] Here we define the supertrace just as a difference of traces in order to introduce the McKean-Singer formula right after. This is an oversimplification: as we will see in the context of Dirac operators in §3.1.4 and §4.2, a key property of the supertrace is that *it is blind* to terms lower than the top order in the Clifford filtrations, which turns out to be fundamental in the proof of the local index formula and one of the grounds why this analysis restricts to Dirac-type operators.

which we will refer to as the **McKean-Singer formula** [34]. This formula is the core of the heat kernel method that permits us to look for local index formulas in more general settings: nothing stops us from studying the heat kernel on the context of choice, and if we are able to interpret the asymptotics of these supertraces, the hope is to obtain a (generalized) index formula.
We just saw that, in the closed case, the supertrace is independent of time, and thus the integral term in the McKean-Singer formula vanishes. In general, we will check that the heuristics work in the following way:

- The large time limit gives us (at least) a term like the (analytical) Fredholm index, or a generalized version of it.
- The short time limit produces (at least) an integrand like the one we will now compute for the closed case, quite similar to the topological index we are already acquainted with.
- The integral term (and the rest of the contributions that are not of the form found in the closed case but appear in the 0 and ∞ time asymptotics) will come as an effect of the existence of boundary, since our modus operandi will always consist in resolving the singular space into a compact manifold with boundary (where the singular information is *remembered* by the metric on the interior). Such contributions are sometimes collected under the umbrella term *boundary defect*. However, not all the boundary contributions need come from the integral term (for example, it vanishes on incomplete settings).

For a proof of the index theorem in the closed case, we *just* need to compute $\mathrm{Str}\left(e^{-t\hat{P}^2}\right)$ at some t. It turns out the short time limit $t \to 0$ gives us the best chance to obtain an expression resembling the right hand side of Theorem 2.12. Yet to interpret this contribution we will have to restrict the class of operators we work with[35].
Consider a **generalized Laplacian** or **Laplace-type operator**, i.e. an operator $L \in \mathrm{Diff}^2(E)$ such that $\sigma_2(L)(x,\xi) = |\xi|^2$, or equivalently an operator which differs from the Laplacian of the metric by terms of order ≤ 1. The asymptotics of its heat kernel in that regime are illustrated in the work of Minakshisundaram and Pleijel [MP49]:

$$h(t,x,x) \underset{t\to 0}{\sim} t^{-\frac{n}{2}} \sum_{k=0}^{\infty} A_k(x) t^k, \quad A_k \in \Gamma(\hom(E)) \tag{3}$$

The A_k are completely determined by the metric and its derivatives. A similar formula in a neighborhood of the diagonal is given in [BGV04, Theorem 2.30]. In particular,

$$\int_X \lim_{t\to 0}\left(t^{\frac{n}{2}} h(t,x,x)\right) dx = \int_X A_0(x)dx = \frac{\mathrm{rk}(E)\mathrm{vol}(X)}{(4\pi)^{\frac{n}{2}}}$$

is Weyl's theorem [BGV04, Theorem 2.31]. What's more, for the Laplace-Beltrami operator the first two terms are computed in [MS67]:

$$\int_X t^{\frac{n}{2}} h(t,x,x)dx \underset{t\to 0}{\sim} \mathrm{vol}(X) + \frac{t}{6}\int_X \mathrm{scal}_g(x)dx + O(t^2)$$

so they conclude that *one can hear the volume and the scalar curvature* of a closed Riemannian manifold[36].
Yet the higher order A_k's are increasingly harder to compute, even though one has explicit recursive expressions for them, cf. [Gil84, Theorem 4.8.16] and [BGV04, Theorem 2.26].

[34] In some contexts, the words "McKean-Singer formula" are used to point out the fact that the supertrace of the heat kernel in a compact manifold without boundary is independent of time, i.e. the third term in the formula

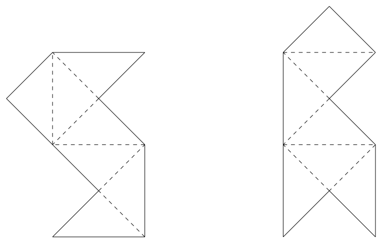


Figure 2: Non-isometric isospectral domains [GWW92].

In the McKean-Singer formula for an elliptic operator P such that P^*P and PP^* are generalized Laplacians, the Minakshisundaram-Pleijel expansion translates to:

$$\operatorname{ind}(P) = \lim_{t\to 0} \operatorname{Str}\left(e^{-t\hat{P}^2}\right) = \lim_{t\to 0} \int_X \sum_{k=0}^{\infty} \left(A_k^{P^*P}(x) - A_k^{PP^*}(x)\right) t^{-\frac{n}{2}+k} dx$$

The limit thus exists if there are cancellations for $k < \frac{n}{2}$; this was first explicitly checked by Patodi [Pat71] and later proved by Gilkey [Gil73], i.e. we get:

$$\operatorname{ind}(P) = \int_X \left(A_{n/2}^{P^*P}(x) - A_{n/2}^{PP^*}(x)\right) dx \tag{4}$$

For general dimension n, as we mentioned, it is hard to identify explicitly what the A_k's represent. In the case of the Dirac operator (defined in the next section), a rescaling procedure developed by Getzler [Get86], inspired by *physical* proofs of the index theorem using supersymmetry (which also explains the cancellations of lower order terms) in quantum mechanics, allows us to explicitly

vanishes.

[35]This is a clear drawback of the heat kernel method: even though it can be a priori applied generally, the asymptotics of the supertrace can be very hard to interpret, and most of the work focuses on those of generalized Laplacians, which are nonetheless a very relevant class of geometric operators.

[36]Observe how the McKean-Singer formula contains terms dependent on the spectra of the operators P^*P and PP^*. Consequently, both sides of the index formula are determined by the eigenvalues of these operators. In our posterior discussion, the role of P will be taken by a Dirac operator and thus these two compositions will correspond to Laplace-type operators. In a manifold, the quantities that can be computed out of the spectrum of the Laplacians of the metric are called spectral invariants. In particular, the terms appearing in the index theorem for Dirac operators will all be spectral invariants. Other relevant examples are the ζ-function and Ray-Singer's analytic torsion.

In line with the renowned article [Kac66], spectral invariants are *audible* properties of the manifold in the following sense: the vibration of a string or a membrane (or any other object) can be decomposed into modes with different frequencies. The timbre, the characteristic sound of an object, is dependent on the relative intensity each of the modes contributes to the produced sound. These modes are usually called harmonics, the same name the solutions to the Helmholtz equation (the eigenvalue problem for the Laplace operator)

$$\left(\Delta + \frac{\omega^2}{c^2}\right) u = 0$$

receive. This is because sound is described by the three dimensional wave equation and the solutions with constant frequencies ω are obtained by Fourier transform (in t) as solutions of the Helmholtz equation. This is probably also the reason why one calls the solutions to Laplace's equation $\Delta u = 0$ harmonic functions, since the word *harmonic* comes from the greek "to fit together", likely refering to how the Pythagorean school understood music as a law of perfect proportions in nature (as in "the music of the spheres"). Motivated by this connection, one could in principle hear the eigenvalues of the Laplacian if one was able to analyse the vibrations of the object sufficiently precisely, hence any quantity constructed by them, i.e. any spectral invariant.

In spite of that, it turns out *one cannot hear the shape of a drum*, since there are membranes that share the spectrum of their Laplacian but are not isometric [GWW92]. So we need more than the auditive system to distinguish geometric spaces.

compute the $A_{n/2}$'s, relating them to the heat kernel of a harmonic oscillator. The resulting index formula is:

$$\mathrm{ind}\,(\not{\partial}) = \int_X \hat{\mathrm{A}}(TX)Ch(E) \tag{5}$$

Since we employ this heat kernel approach, we will also restrict ourselves to Dirac-type operators and our local[37] index formulas will resemble this one. We will return to Getzler rescaling in Section 4.2 since it is also a pillar in our argumentation.

2.4 The Atiyah-Patodi-Singer index theorem

Let us conclude this chapter by sketching the result of Atiyah, Patodi and Singer [APS75] where they extend the index theorem to compact manifolds with boundary, giving rise to *secondary invariants* in their local index formula.
As in the case of the Atiyah-Singer index theorem, this is a generalization of other classical results, like the Gauß-Bonnet theorem on compact manifolds with boundary (recall §2.1.2). Note that the extra term obtained by the addition of boundary, the **boundary defect**, vanishes whenever the geodesic curvature of the boundary is zero, i.e. when the manifold has a collar neigborhood where the metric exhibits a product structure.
When studying the signature defect, i.e. the extra term appearing in the signature theorem after the addition of boundary (even if the boundary has a product-type neigborhood), they realized that in order to set up a Fredholm problem for the signature operator, *local boundary conditions* (like Dirichlet or von Neumann) were not enough[38]. They therefore introduce *global boundary conditions*, i.e. those that at each point in the boundary not only depend on the value of the function and finitely many of its derivatives at that point, and involve e.g. integrals over the boundary. Their choice (and variations of it) is now referred to as **APS boundary conditions**. Let us depict the simplest ocurrence of such a behaviour:

Example 2.14. Consider the Dolbeault operator $\bar{\partial}$ acting on the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{C}$. The unit disk is a compact manifold with boundary $\partial\overline{\mathbb{D}} = \mathbb{S}^1$ the circle. The kernel of $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y) = \partial_{\bar{z}}$ is the space of holomorphic functions, in particular all polynomials on the complex variable z, so it is infinite dimensional even if we just consider its restriction to $\mathbb{S}^1$. By a standard integration by parts argument, we can compute $\bar{\partial}^* = -\frac{1}{2}(\partial_x - i\partial_y) = -\partial_z$, with domain ensuring that the corresponding boundary integral vanishes. The kernel of this operator acting on smooth functions is the space of anti-holomorphic functions, which contains all polynomials on $\bar{z}$ and is thus infinite dimensional as well.
If we chose to impose Dirichlet boundary conditions, the kernel of $\bar{\partial}$ would become empty due to the maximum modulus principle. Moreover, the dual condition to vanishing at the boundary is no condition at all on the smooth functions in the domain of $\bar{\partial}^*$. This means $\ker \bar{\partial}^*$ (equivalently $\mathrm{coker}\,\bar{\partial}$) is infinite dimensional, so there is no Fredholmness.
However, we can alternatively write $f \in \mathcal{C}^\infty(\mathbb{S}^1)$ as a Fourier series

$$f(e^{ix}) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}dx$$

[37] As mentioned, a distinction is usually made between index theorems of the form of Theorem 2.12, written in terms of cohomology classes, i.e. global topological invariants, and those where the integrands have local differential form expressions representing these classes, like the one we just saw. We focus on the latter.

[38] There is, in fact, a topological obstruction: the non-vanishing of the restriction of the elliptic symbol class of the operator to $K^0(T^*_{\partial X}X)$ [Fre21a, §5.2]. This is not the case for e.g. the Gauß-Bonnet operator, for which local boundary conditions can also be used to set up a Fredholm problem.

and ask for all the Fourier coefficients c_n strictly greater than a certain $a \in \mathbb{R}$ to vanish. This is a global boundary condition, since the c_n's are integrals over the whole $\mathbb{S}^1$. Holomorphic functions have Fourier series where the negative c_n's vanish (and the rest are given by their Taylor coefficients), whereas anti-holomorphic ones lack the positive c_n's ($\bar{z}^k = z^{-k}$ if $z \in \mathbb{S}^1$). Moreover, the complementary boundary condition corresponding to $\bar{\partial}^*$ that makes the boundary integral be zero is the vanishing of the Fourier coefficients smaller or equal than a (cf. [APS75, (2.4)]). Putting all together, the kernel and cokernel of $\bar{\partial}$ with this global boundary condition become finite dimensional, which leads to a Fredholm problem with index:

$$\ker \bar{\partial} = \begin{cases} \lfloor a \rfloor + 1 & a \geq 0 \\ 0 & a < 0 \end{cases}, \quad \ker \bar{\partial}^* = \begin{cases} 0 & a \geq 0 \\ -\lfloor a \rfloor & a < 0 \end{cases} \implies \operatorname{ind}(\bar{\partial}) = \begin{cases} \lfloor a \rfloor + 1 & a \geq 0 \\ \lfloor a \rfloor & a < 0 \end{cases}$$

□

A similar phenomenon happens in the context studied by Atiyah, Patodi and Singer. If the compact manifold X with boundary $Y = \partial X$ has a collar neighborhood $U \supset Y$ of the boundary where the metric takes the form

$$g\big|_U = du^2 + dy^2$$

the Dirac operator in U admits the local expression (see §3.1.4)

$$\partial\!\!\!/\big|_U = cl(du)\partial_u + \partial\!\!\!/_Y$$

where $\partial\!\!\!/_Y$ is the induced operator on the boundary.
To obtain a Fredholm problem, the global APS boundary condition is imposed, which chooses $a \in \mathbb{R} \setminus \operatorname{Spec}\left(\partial\!\!\!/_Y\right)$ and restricts to spinors that at the boundary lie in the eigenspaces of $\partial\!\!\!/_Y$ with eigenvalue smaller than a (any such choice of a yields a Fredholm problem and a resulting index formula). The index of this Dirac operator is related to that of the corresponding operator induced by attaching an infinite half-cylinder at the boundary with L^2-domain [APS75, Prop. 3.11]. A heat kernel argument justifies:

Theorem 2.15. In a compact manifold with boundary X with a product structure near the boundary and Dirac operator $\partial\!\!\!/$ acting on sections of a Clifford module $E \to X$ subject to APS boundary conditions, the following index formula holds:

$$\operatorname{ind}(\partial\!\!\!/) = \int_X \hat{A}(TX)Ch(E) \underbrace{-\frac{\eta_Y}{2} - \frac{\dim\ker \partial\!\!\!/_Y}{2}}_{\text{boundary defect}}$$

where $\partial\!\!\!/_Y$ is the induced operator on the boundary $Y = \partial X$ and η_Y is the **eta invariant** given by the value at zero of the analytic continuation of the eta function:

$$\eta(z) = \sum_{\lambda \in \operatorname{Spec}\left(\partial\!\!\!/_Y\right) \setminus \{0\}} \operatorname{sign}(\lambda)\lambda^{-z}. \quad \eta_Y = \eta(0)$$

that converges for $\operatorname{Re}(z)$ sufficiently large and whose value at 0 morally reflects the spectral asymmetry of $\partial\!\!\!/_Y$, i.e. the difference between its number of positive eigenvalues and its number of negative eigenvalues.
In particular, the signature defect is given by $-\eta_{\partial X}$.

□

Remark 2.16. In [APS75, Theorem 3.10], a formula for a general first order elliptic differential operator P of product form near the boundary is given, provided one does not need the explicit expression of the short time contribution of the heat kernel appearing in the right-hand side of (4). The case of interest for us, and the one we aim to extend, is that of Dirac operators, for which this term is computed in the same manner as the Atiyah-Singer integrand.

□

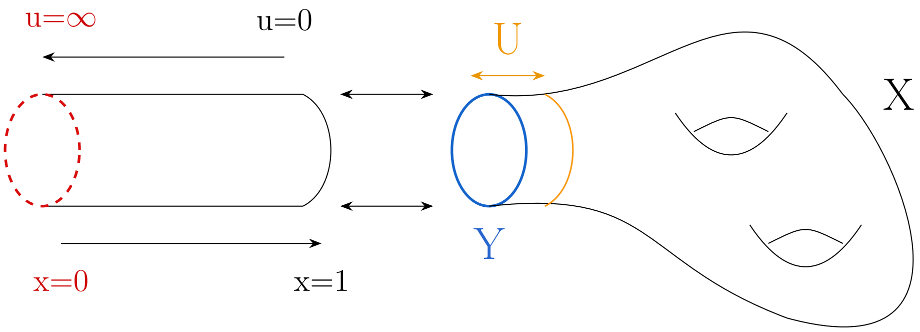


Figure 3: The Atiyah-Patodi-Singer index theorem [APS75] can be obtained by attaching an infinite half-cylinder to the boundary of a compact manifold and restricting to sections that are square integrable along the cylinder. The Melrose approach [Mel93] is to start with a manifold with (non-compact) cylindrical ends and compactify those by introducing appropriate boundary defining functions; check Example 4.1 below. Compare [APS75, (3.24)] and [Mel93, p. 12].

The η-invariant is an example of a spectral invariant with an structure similar to an L-function in number theory, which is also the case of the ζ-*function of an operator* P:

$$\zeta_P(z) = \sum_{\lambda \in \mathrm{Spec}(P)\setminus\{0\}} \lambda^{-z} = \mathrm{Tr}(P^{-z}) = \frac{1}{\Gamma(z)} \int_0^\infty t^z \,\mathrm{Tr}\left(e^{-tP}\right) \frac{dt}{t}$$

also defined by analytic continuation from $\mathrm{Re}(z)$ large[39].
It is also an instance of a **secondary invariant**: in contrast with the **primary invariants** that appear in the closed case and consist of classes on the manifold X, these come up when the boundary $Y = \partial X$ is present and are classes on it, i.e. one dimension lower. They tend to be related to the primary invariants modulo 1 due to the characteristic integrality that the terms in the boundaryless

[39]This is precisely the Mellin transform of the trace of the heat kernel P. It is a spectral invariant of the manifold if we choose P to be a *geometric differential operator*, usually a Laplacian Δ. It is a building block for other relevant invariants like the Ray-Singer analytic torsion:

$$\ln\left(T(X,E)\right) := \frac{1}{2}\sum_{k=0}^{n}(-1)^k k \zeta'_{\Delta^k}(0), \quad \Delta^k \curvearrowright \Gamma(\Lambda^k T^*X \otimes E)$$

or the Casimir energy $\zeta_\Delta(-\frac{1}{2})$ [BVW88]. Note that $e^{-\zeta'_{\Delta^k}(0)}$ is the regularized determinant of Δ^k, i.e. morally the product of its non-zero eigenvalues:

$$\zeta(z) = \sum_{\lambda\neq 0}\lambda^{-z} \implies \zeta'(z) = -\sum_{\lambda\neq 0}\lambda^{-z}\ln(\lambda) \implies e^{-\zeta'(0)} = \prod_{\lambda\neq 0} e^{\lambda^{-z}\ln(\lambda)}$$

which formally tends to $\prod_{\lambda\neq 0}\lambda$ as $z \to 0$.

comparison theorems show:

$$\frac{1}{2\pi}\int_X K dvol_X \equiv -\frac{1}{2\pi}\int_Y k_g dvol_Y \pmod 1 \qquad \text{(Gauß-Bonnet)}$$

$$\int_X L(X) \equiv \eta_Y \pmod 1 \qquad \text{(Signature)}$$

$$\int_X \hat{A}(TX)Ch(E) \equiv \frac{\eta_Y + \dim\ker \partial\!\!\!/_Y}{2} \pmod 1 \qquad \text{(Atiyah-Patodi-Singer)}$$

This relationship is extended to other Chern-Weil constructions via the so-called Chern-Simons invariants [CS74].

The last interesting fact about η_Y is that it is a global object[40], meaning it cannot be obtained by integrating a differential form over Y; this contrasts with the local nature of the Â-genus contribution and can be tested on lens spaces [APS76, Prop. 2.12]. In the sequel, we will witness how index theory in singular spaces reveals new secondary global invariants.

[40] The same can be said about the Bismut-Cheeger η-form (§4.4), which is global on the fibres.

3 Blow-up analysis in index theory

In this section, we would like to introduce a more geometric way of understanding the asymptotics of heat kernels, which is a specific application of the blow-up analysis commenced by Melrose. We will show how it can be adapted not only to the closed case, but also for manifolds with boundary à la Atiyah-Patodi-Singer. This will set the stage for our subsequent efforts to generalize the method.

3.1 Background

When we are seeking to understand the (asymptotic) behaviour of distributions defined on a certain space, **blowing up** is a procedure whose goal is to simplify or interpret such information. From an analytical point of view, blow-ups are just changes of coordinates, i.e. diffeomorphisms in certain subsets of the space which are then extended or completed. Geometrically, blowing up a *p-submanifold*[41] means replacing it with its (interior) spherical normal bundle. From a philosophical perspective, they are a tradeoff: by blowing up, the distributions subject of study become well-defined or their asymptotics are expressed in an operational way, while the geometry of their space of definition becomes more complex (but in a controllable way).

3.1.1 Model cases

Let us try to elucidate these vague statements with an example:

Example 3.1. Consider the distribution $f(z,t) = e^{-\frac{z}{t}}$ for $z, t > 0$ and let us read its asymptotic behaviour for $z, t \to 0$. This function should serve as a toy version of the Euclidean heat kernel. In the space $\mathbb{R}^2_+ = [0,\infty)_z \times [0,\infty)_t$, its form is clear away from the origin $(0,0)$.

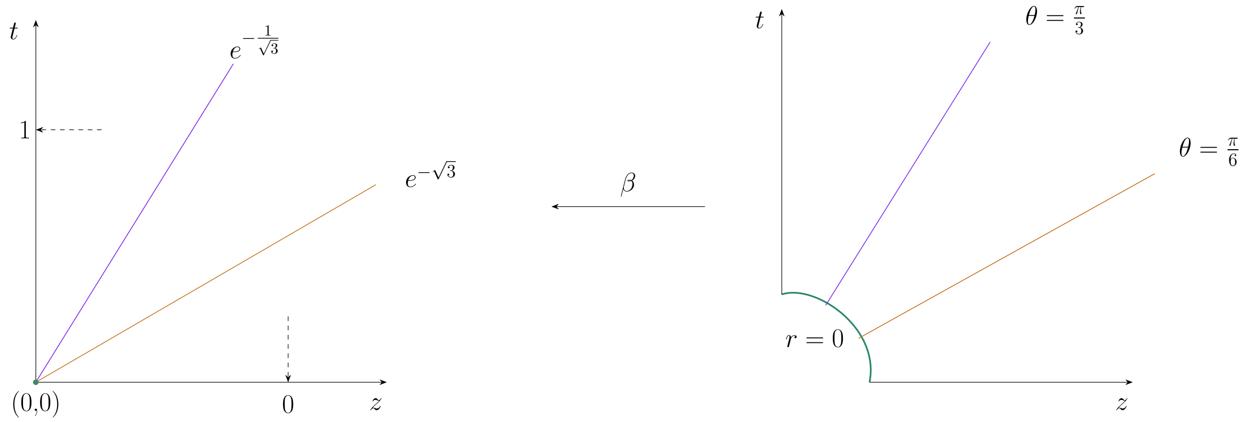


Figure 4: Blow-up of $\mathbb{R}^2_+$ at the origin to resolve $e^{-\frac{z}{t}}$: by introducing polar coordinates near the origin, it "blows-up" into a quarter circumference corresponding to $\beta^{-1}(0,0) = \{(0,\theta) : \theta \in \{0, \frac{\pi}{2}]\}$. The blow-down map β is the extended version of p that maps $[0,\infty) \times [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2_+$. $f(z,t)$ lifts up to a function $\beta^* f(r,\theta)$ which is defined up to (i.e. including) $r = 0$.

In particular, the function is constant along sets $\{\frac{z}{t} = k\}$ for $k \in \mathbb{R}_+$, i.e. along rays emanating from the origin. This might suggest that polar coordinates are *natural* coordinates for f. In them, it becomes $f(r,\theta) = e^{-\text{cotan}(\theta)}$. The change of coordinates from cartesian (z,t) to polar (r,θ) is a diffeomorphism $p : (0,\infty) \times [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2_+ \backslash \{(0,0)\}$ away from the origin. Furthermore, the *natural* extension of the domain $(0,\infty) \times [0, \frac{\pi}{2}]$ is its closure $[0,\infty) \times [0, \frac{\pi}{2}]$ which differs from the closure $\mathbb{R}^2_+$ of $\mathbb{R}^2_+ \backslash \{(0,0)\}$ by the fact that the origin now unfolds to produce a quarter circumference

[41] This is the class of submanifolds for which blow-ups are defined.

corresponding to $r = 0$. We call this *the blow-up of the origin in* $\mathbb{R}^2_+$. The blow-down map has the global expression $\beta(r, \theta) = (r\cos\theta, r\sin\theta)$.
Even more at hand given the form of f would be to choose projective coordinates. However, these only give local charts $\left(z, \frac{t}{z}\right)$ away from the vertical axis $\{z = 0\}$, and $\left(\frac{z}{t}, t\right)$ away from the horizontal axis $\{t = 0\}$. Writing $\xi = \frac{t}{z}$ and $\eta = \frac{z}{t}$, the blow-down map takes the local forms $\beta(z, \xi) = (z, z\xi)$ and $\beta(\eta, t) = (\eta t, t)$. We find the computations with projective coordinates more convenient and clearer, and thus choose to use them in the upcoming discussion.

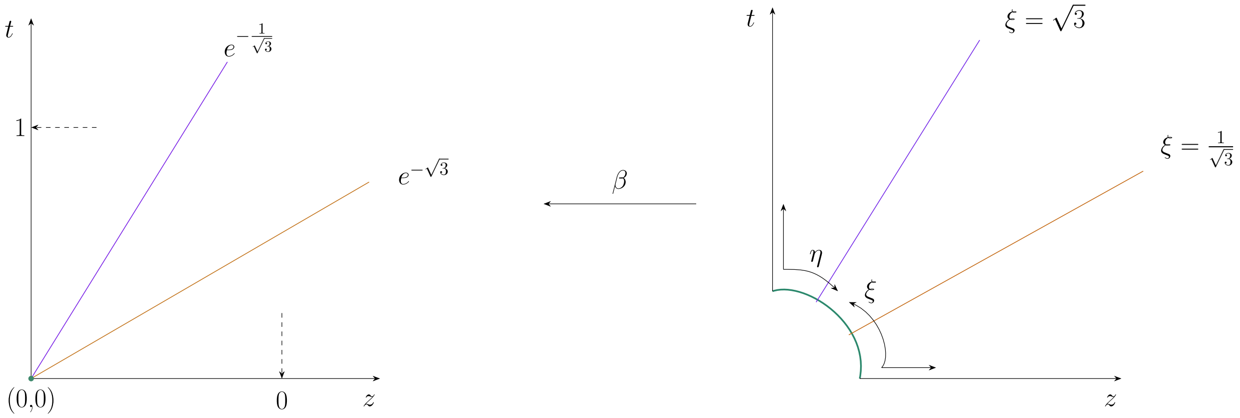


Figure 5: Blow-up of $\mathbb{R}^2_+$ *at the origin using local projective coordinates. Now the lift* $\beta^* f$ *takes the form* $e^{-\eta}$ *or* $e^{-\frac{1}{\xi}}$ *depending on the local chart at the point of the blow-up space chosen, but both expressions are readily extendible to the whole space. The new face is precisely (diffeomorphic to) the radial compactification of* $(0, \infty)_\xi$ *or* $(0, \infty)_\eta$ *(which we could denote by* $[0, \infty]$*).*

Observe the following rule of thumb: to obtain the coordinate $\frac{z}{t}$ we blew up the (p-)submanifold $\{z = 0,\ t = 0\}$.
To solidify our understanding, let us remark again that:

- Blowing-up the origin replaced it with its interior spherical normal bundle (a quarter circunference). Because this fact holds generally, blow-ups can be understood as the local introduction of polar coordinates (in pinched neighborhoods of the p-submanifold).
- We are usually going to blow-up manifolds with corners. The blown up spaces will still be manifolds with corners. In the example, we start with a manifold with two boundary hypersurfaces meeting at a corner and obtain one with three boundary hypersurfaces and two corners, each shared by the new middle face (often denominated *front face*) with one of the others.
- The goal is that the (class of) distribution(s) we are interested in lifts to a "nice" (usually *polyhomogeneous conormal*) distribution from which we can understand the asymptotics at all (desired) boundary regimes. The function $e^{-\eta}$ for $\eta \in [0, \infty]$ is "nicer" than $e^{-\frac{z}{t}}$ for $(z, t) \in \mathbb{R}^2_+$.

□
It is not hard to apply what we learnt in the example above to the Euclidean heat kernel, which models the heat kernel locally around a point in a compact manifold without boundary (or in an interior point of a compact manifold with boundary).

Example 3.2. Recall the form of the heat kernel in $\mathbb{R}^n$ with the standard metric:

$$h_{\mathbb{R}^n}(t, x, x') = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-x'|^2}{4t}} = C_n t^{-\frac{n}{2}} e^{-\frac{|x-x'|^2}{4t}}$$

(we are not interested in the constant C_n for now). If we wish to study the behaviour of this kernel when $t \to 0$, we might want to have an asymptotic expansion in certain coordinates that elucidate the different behaviours when $x - x' \to 0$ and $t \to 0$ simultaneously (analogous to $z, t \to 0$ in the previous example).
Since the expansion is in half integer powers of t and moreover there is a mismatch in the powers in the exponent ($|x - x'|^2$ vs t), it might seem reasonable to choose $\tau := \sqrt{t}$ rather than t as a "natural" coordinate. Then the space $[0, \infty)_t \times X^2$ where the distribution $h_{\mathbb{R}^n}$ lives virtually does not change, up to the scale in the t direction: it becomes $[0, \infty)_\tau \times X^2$. The kernel lifts to:

$$C_n \tau^{-n} e^{-\frac{1}{4}\left(\frac{|x-x'|}{\tau}\right)^2}$$

Here we can again take projective (would also work with polar/spherical, see [Mel93, p. 253]) coordinates $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_i = \frac{x_i - x_i'}{\tau} = \frac{x_i - x_i'}{\sqrt{t}}$ so that the lifted distribution looks like

$$C_n \tau^{-n} e^{-\frac{|\xi|^2}{4}}$$

with $\xi \in \mathbb{R}^n$ now. Our rule of thumb tells us that to obtain the coordinate $\xi = \frac{x-x'}{\sqrt{t}}$ geometrically, we have to blow up the p-submanifold $\{x - x' = 0,\ \sqrt{t} = 0\}$, the diagonal at time 0. This resolves the behaviour at $t \to 0$ by distinguishing the different directions of approaching the problematic subset $\{x - x' = 0,\ t = 0\} = \{0\} \times \mathrm{diag}_X$ similar to the previous example. The difference is that now the "z"-coordinate can be positive or negative and is n-dimensional. For every point in $\{0\} \times \mathrm{diag}_X$ we only have to distinguish the approaching directions transversal to $\{0\} \times \mathrm{diag}_X$. Thus, for each $1 \leq i \leq n$ we have two copies of the picture from Example 3.1: one for positive $x_i - x_i'$ and one when negative, producing a half-circunference front face from the corresponding two quarter-circunferences (because of the square in the argument of the exponential, the lift is well-defined in this whole half-circunference).
Notice that in each of these sections transversal to $\{0\} \times \mathrm{diag}_X$, the front face is modelled by the radial compactification[42] of $\mathbb{R}^n_\xi$ and the lift $e^{-\frac{|\xi|^2}{4}}$ vanishes to infinite order as $|\xi| \to \pm\infty$, i.e. in the boundary of this radial compactification, that is, in its intersection with the blue face in the bottom right of Figure 6.
We could have also maintained the coordinate t and introduced the coordinates ξ_i as described above, since the only asymptotic we need to resolve is for $t \to 0$. In this way, we would obtain an asymptotic expansion with non-integer powers of the *boundary defining* function t (which is allowed) and the ξ_i would be a quotient of the original coordinates raised to different powers (1 for $x - x'$ and $\frac{1}{2}$ for t): blow-ups introducing such coordinates are sometimes called *quasi-homogeneous* [Beh21, §1.3] or *weighted* blow-ups [CR23, Def. 4.1] (in this case, it is a *parabolic* blow-up, since the powers of the coordinates are in relation 2:1).

□

In §2.3.2 we discussed how one can try to obtain local index formulas by understanding the supertrace of the heat kernel at different time regimes via the McKean-Singer formula. By Lidskii's theorem, this amounts to computing the kernel at the diagonal. Since the asymptotics are usually

[42] The radial compactification of a finite dimensional vector space of dimension n, i.e. isomorphic to $\mathbb{R}^n$, can be defined in the following way: first embed diffeomorphically $\mathbb{R}^n$ as the open northern hemisphere of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. This can be done by first embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as the hyperplane $\{x_{n+1} = 1\}$ and then taking its stereographic projection $s : \mathbb{S}^n_+ \to \{x_{n+1} = 1\}$ to the origin of the sphere $\mathbb{S}^n$ of radius 1 centered at $0 \in \mathbb{R}^{n+1}$. Since s is a diffeomorphism, the differentiable structure of $\mathbb{R}^n$ corresponds to that of $\mathbb{R}^{n+1}$ restricted to the open northern hemisphere, i.e. $f \in \mathcal{C}^\infty(\mathbb{R}^n) \iff s^* f \in \mathcal{C}^\infty(\mathbb{S}^n_+)$. Now just add the equatorial line $\mathbb{S}^n \cap \{x_{n+1} = 0\}$ and thus obtain a compact n-dimensional submanifold of $\mathbb{R}^{n+1}$. Compare [Mel06, L1.2]

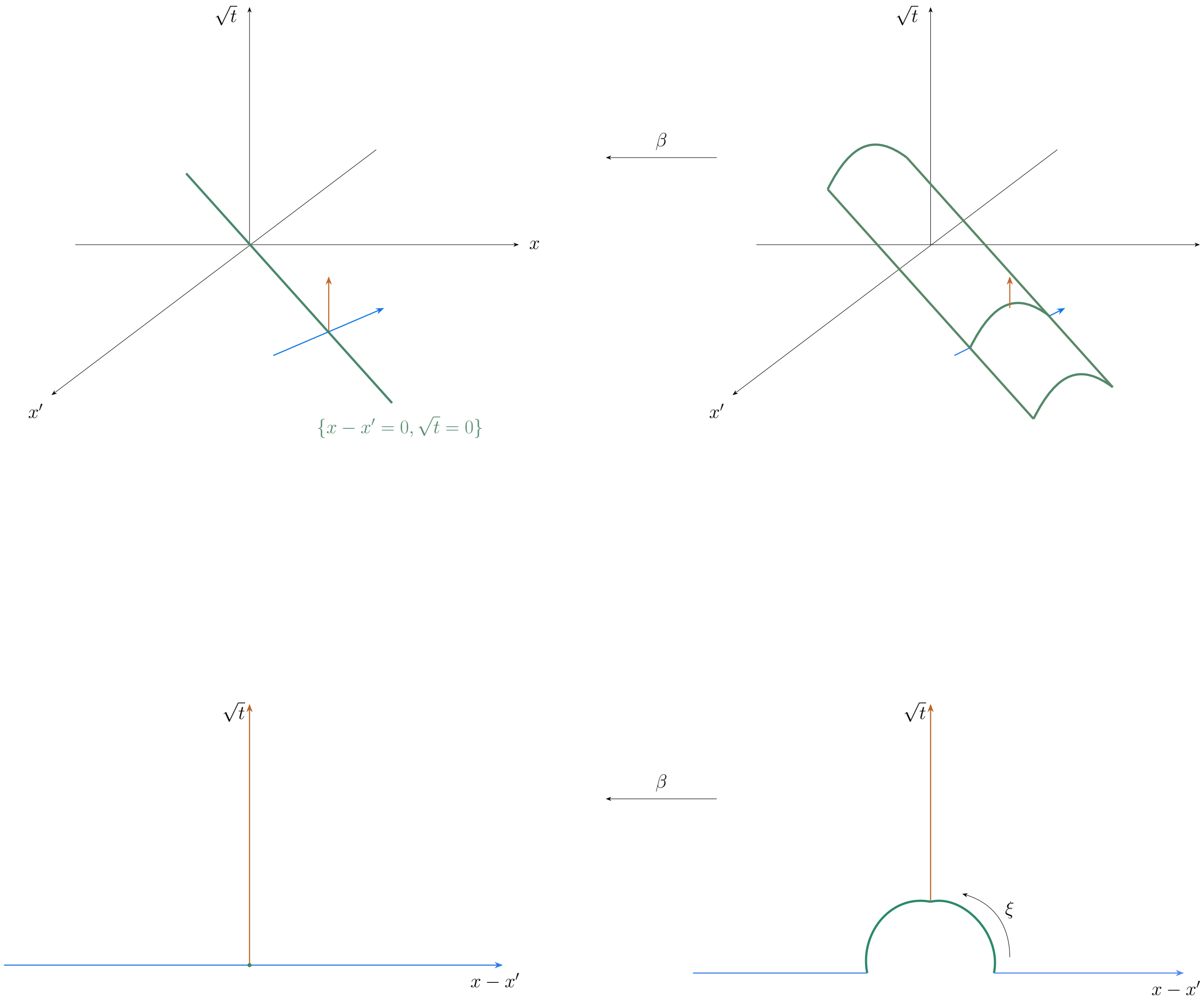


Figure 6: Blow-up space resolving $h_{\mathbb{R}^n}$ as $t \to 0$ (the same procedure resolves the asymptotics of the heat kernel in closed manifolds). We blow up the diagonal at time 0 (green) obtaining a half-cylinder face at $t = 0$. The kernel vanishes to infinite order at the other boundary hypersurface, which is also at $t = 0$. Note that the time direction is parametrized by $\sqrt{t}$. To resolve the heat kernel one distinguishes the different directions of approach of $x - x', t \to 0$. This means that in each section transversal to $\{0\} \times diag_X$ we have a situation similar to Example 3.1, and thus the lift vanishes at the blue face (to infinite order).

hard to read off when considering the kernel as a distribution in $\mathbb{R}_+ \times X^2$ as we just saw, it is useful to blow-up this space until we obtain one where the heat kernel lifts to a distribution with "nice" asymptotics (ones from which we can calculate the contributions in the McKean-Singer formula or perform other more general computations). The class of distributions one usually hopes to land on is that of **polyhomogeneous conormal** distributions and the space with the least amount of blow-ups ensuring this regularity for the heat kernel receives the name of **heat blow-up space**[43]. Thus, we are interested on the form of the lifted heat kernel at the *lifted diagonal* on the heat blow-up space.

[43]Usually the heat blow-up space is constructed to resolve the heat kernel for finite times, i.e. without providing information about the $t \to \infty$ asymptotics, which are also needed in the McKean-Singer formula. We will see that one can also compactify this space e.g. by introducing the coordinate $\kappa = \frac{1}{t}$ and pulling apart the different regimes as $\kappa \to 0$ in a similar manner as is done with $t \to 0$. What has been usually done is to use this blow-up formulation for the contributions $t \to 0$ and $t \in (0, \infty)$ and the asymptotics of the resolvent of the Laplacian or Dirac operator for the $t \to \infty$ term: more on that later in Remark 3.21.

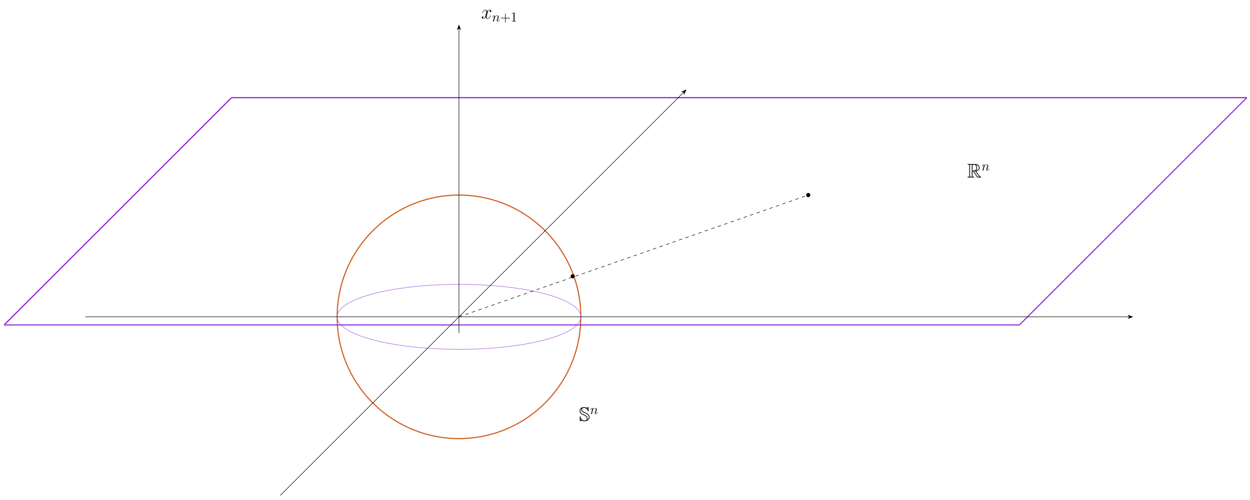


Figure 7: Radial compactification of $\mathbb{R}^n$ by stereographically projecting into the northern hemisphere and extending to the equator with the differential structure as an embedded submanifold of $\mathbb{R}^{n+1}$. In brown, the unit sphere in $\mathbb{R}^{n+1}$; in purple, the hyperplane $\{x_{n+1} = 0\}$.

Now, every Riemannian manifold admits a Levi-Civita connection, which is a symmetric affine connection, and for every point in a differentiable manifold with such a connection, one can choose normal coordinates, i.e. coordinates where the Christoffel symbols of the connections vanish at the point. This means that for any point $p \in X$ of a compact manifold without boundary (or any interior point of a compact manifold with boundary) we can choose normal coordinates $(x_1, \ldots, x_n)$ in a neighbourhood of it so that the metric at that point is the Euclidean metric $g(p) = dx_1^2 + \cdots + dx_n^2$. So a first approximation of the heat kernel in an interior point of a compact manifold should be the Euclidean heat kernel. In fact, the blow-up space resolving the heat kernel we just saw turns out to be analogous to the heat blow-up space for compact manifolds without boundary.

3.1.2 Manifolds with corners and polyhomogeneous conormal distributions

Choose your corner, pick away at it carefully, intensively and to the best of your ability and that way you might change the world.

Charles Eames

Let us dive into the technical framework we mentioned along the way (check [Gri00] for a more detailed introduction):

Definition 3.3. A **manifold with corners** is a manifold locally modelled on $\mathbb{R}_+^k \times \mathbb{R}^n$, where $\mathbb{R}_+ = [0, \infty)$ (with different (k, n) depending on the point, but $k + n$ stays constant and is the dimension of the manifold).
Those points locally modelled in $\mathbb{R}^{n+k}$ belong to the interior $\mathring{X}$ and those modelled on $\mathbb{R}_+^m \times \mathbb{R}^{n+k-m}$, $1 \leq m \leq k$ belong to the interior of a boundary face of codimension m (usually when talking about the face one refers to its closure, which is in general also a manifold with corners). The boundary faces of codimension 1 are called **boundary hypersurfaces**.
Given an embedded (not just immersed [Gri00, Fig. 2]) boundary hypersurface H one can always find $\rho_H \in \mathcal{C}^\infty(X)$ so that $\rho_H \geq 0$, $\{\rho_H = 0\} = H$ and $d\rho_H(x) \neq 0$ for every $x \in H$. We call ρ_H a **boundary defining function** for H (although at times in the praxis the second condition is relaxed to $\{\rho_H = 0\} \supset H$).

As in the case of manifolds with boundary, smooth functions on X are those which extend to be smooth on a bigger manifold on which we can embed X; equivalently, their derivatives of any order are bounded in bounded subsets of the interior.

□

Note that manifolds without boundary and with boundary are also manifolds with corners. The category of manifolds with corners is appropiate for blow-up constructions, since, as witnessed in the examples, blow-ups of manifolds with corners are also manifolds with corners.

Consider a boundary hypersurface H with local coordinates[44] y and boundary defining function x. We are interested in functions which in a neighborhood $[0,\varepsilon)_x \times H$ of it have expansions with terms like $a_{z,k}(y)x^z \log^k x$ with $z \in \mathbb{C}$ and $a_{z,k} \in \mathcal{C}^\infty(H)$ (think of Laurent expansions but with complex powers and logarithms). We can say a lot about their integrals (Melrose's *pushforward theorem*) and thus express the asymptotics when approaching the boundary hypersurface in an usable way, in particular for computing supertraces, which are integrals along the diagonal. Moreover, they appear naturally when integrating smooth functions[45]. More precisely:

Definition 3.4. An **index set** $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}_0$ is a discrete set so that every left half-plane in the $\mathbb{C}$ coordinate contains finitely many points of $\mathcal{E}$, i.e. for all $r \in \mathbb{R}$, $|\{(z,k) \in \mathcal{E} : \mathrm{Re}(z) < r\}| < \infty$. For technical reasons[46], we also assume that if $(z,k) \in \mathcal{E}$ and $k > 0$, then $(z, k-1) \in \mathcal{E}$.

The index set encodes the information of which terms appear in an asymptotic expansion, and the condition ensures that there is a leading order term: there is a smallest real power $\min_{(z,k)\in\mathcal{E}} \mathrm{Re}(z)$ (even when it is allowed that several terms share $\mathrm{Re}(z)$ with different $\mathrm{Im}(z)$), and each x^z term can only be combined with finitely many logarithm terms (for all $z \in \mathbb{C}$, $|\{k \in \mathbb{N}_0 : (z,k) \in \mathcal{E}\}| < \infty$).

A smooth function u on the interior of a manifold with corners is called **polyhomogeneous conormal** with respect to an index set $\mathcal{E}$ if when approaching (i.e. in a collar neighborhood of) each boundary hypersurface it has an **asymptotic expansion** of the form

$$u(x,y) \underset{x\to 0}{\sim} \sum_{(z,k)\in\mathcal{E}} a_{z,k}(y)x^z \log^k x, \quad a_{z,k} \in \mathcal{C}^\infty(H)$$

where x is the boundary defining function of the corresponding hypersurface H and y are local coordinates for H. By this we mean that for every $r \in \mathbb{R}$, the truncated series is a good approximation to order x^r of the function around H:

$$|u_r(x,y) - u(x,y)| \le C_r x^r, \quad \text{where} \quad u_r(x,y) = \sum_{\substack{(z,k)\in\mathcal{E} \\ \mathrm{Re}(z)\le r}} a_{z,k}(y)x^z \log^k x$$

[44] One tipically uses x_i's to denote boundary defining functions or more generally coordinates which are non-negative ($x_i \ge 0$), and y_j's for those without that restriction (usually coordinates in a boundary face).

[45] There is a more fundamental reason for why polyhomogeneous conormality is the "right" notion of regularity for geometric analysis on manifolds with corners, related to the fact that *the Mellin transform is the Fourier transform on the locally compact abelian group* $(\mathbb{R}_{>0}, \cdot)$ *with Haar measure* $\frac{dx}{x}$. In particular, polyhomogeneous conormal functions correspond by Mellin transform to meromorphic functions with rapid vertical decay, where the index set determines the location and order of the poles. This should be compared with how one can read from the decay of its Fourier transform whether a real-valued function is smooth.

[46] One reason for it is that we want to describe how the asymptotics behave under integration using index sets, and the integral of a $x^z \log^k x$ term against a b-density can be computed by parts as

$$\int x^z \log^k x \frac{dx}{x} = \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{j! z^{k-j+1}} x^z \log^j x + C$$

and the same estimates hold for $(x\partial_x)^j(\partial_y)^\alpha u$ for $(j,\alpha)\in\mathbb{N}_0^{1+\dim H}$, so for his differentials along vector fields tangent to H (this is the *conormal* in the name[47]). The space of polyhomogeneous conormal distributions in X with respect to $\mathcal{E}$ is denoted $\mathcal{A}^{\mathcal{E}}_{phg}(X)$, where $\mathcal{E}$ contains the information about the index set $\mathcal{E}_H$ for each boundary hypersurface H.

□

Example 3.5. The function $\sin\left(\frac{1}{x}\right)$ is not conormal in $\mathbb{R}_+=[0,\infty)$ as $x\to 0$ (i.e. approaching $H=\{0\}\subset\mathbb{R}_+$), since derivating $x\partial_x\sin\left(\frac{1}{x}\right)=-\frac{1}{x}\cos\left(\frac{1}{x}\right)$ brings the estimates one power down. On the other hand, $\cos\log x$ is polyhomogeneous conormal in $\mathbb{R}_+$ as $x\to 0$ (with respect to $\mathcal{E}=\{(i,0),(-i,0)\}$), since it equals $\frac{e^{i\log x}+e^{-i\log x}}{2}=\frac{x^i+x^{-i}}{2}$.
The Euclidean heat kernel $C_n t^{-\frac{n}{2}}e^{-\frac{|x-x'|^2}{4t}}$ is not polyhomogeneous conormal as $t\to 0$ for the same reason as for $\sin\left(\frac{1}{x}\right)$, but its lift $C_n t^{-\frac{n}{2}}e^{-\frac{|\xi|^2}{4}}$ to the heat blow-up space is, with index set $\mathcal{E}=\{\left(-\frac{n}{2},0\right)\}$.

□

The key is that we can control how the asymptotics vary after integrating a polyhomogenenous conormal function whenever the integral corresponds to a pushforward along a certain type of mapping known as b-fibration. An intuition behind it can be found in [GG01].

Definition 3.6. Let X and Z be manifolds with corners and denote the set of their boundary hypersurfaces by $\mathcal{M}_1(X)$ and $\mathcal{M}_1(Z)$, respectively.
The map $f:X\to Z$ is a (interior)[48] ***b*-map** if at each point $z\in Z$ it admits a product form in coordinates, i.e.

$$f^*(\rho_{H'})=a_{H'}\prod_{H\in\mathcal{M}_1(X)}\rho_H^{e(H,H')},\quad a_{H'}\in\mathcal{C}^\infty(X),\quad e(H,H')\in\mathbb{N}_0$$

Note that the preimage of a boundary hypersurface H' under a b-map is a union of boundary hypersurfaces (those for which $e(H,H')\neq 0$). b-maps are closed under composition and their pullback preserves polyhomogeneity.
It is called ***b*-normal** if it does not increase the codimension after mapping; in other words, it does not map boundary hypersurfaces into corners, or more precisely codim $f(H)\leq$ codim H (so for each H, all $e(H,H')$ are zero except at most one).
It is a ***b*-submersion** if the image of the differential df acting on vector fields tangent to the boundary faces of X spans the space of vector fields tangent to the boundary faces of Z (i.e. it is surjective restricted to vector fields tangent to the boundary in source and image). A b-normal map is thus a b-submersion if $f\big|_{\mathring{H}}:\mathring{H}\to f(\mathring{H})$ is a fibration.
A (interior) b-map that is also b-normal and a b-submersion is called ***b*-fibration**.

□

[47]A distribution $u\in\mathbb{R}_+$ is **conormal** of order $t\in\mathbb{R}$ if it is smooth in the interior and $(x\partial_x)^k u=O(x^t)$ as $x\to 0$ for any $k\in\mathbb{N}_0$.
In a manifold with corners X with boundary hypersurfaces H_i, $1\leq i\leq m$ and corresponding boundary defining functions x_i, a function u is conormal of order $t\in\mathbb{R}^m$ if it is smooth on the interior and $(x_i\partial_{x_i})^k\partial_{y_j}^l u=O(\prod_{i=1}^m x_i^{t_i})=:O(x^t)$ as $x\to 0$ for any boundary defining function x_i or other coordinate direction y_j.
Since the vector fields $x_i\partial_{x_i},\partial_{y_j}$ are tangential to the boundary hypersurfaces, their universal enveloping algebra is called algebra of b-differential operators $\mathrm{Diff}_b(X)$ (for *boundary*) and thus the space of conormal distributions of order $t\in\mathbb{R}$ is the subset of $\{u\in\mathcal{C}^\infty(\mathring{X}):u=O(x^t)\}$ that is closed under the action of $\mathrm{Diff}_b(X)$, i.e. $Pu=O(x^t)$ for any $P\in\mathrm{Diff}_b(X)$.
[48]If some of the $\rho_{H'}$ are allowed to have identically zero pullback, then we talk about *boundary* b-maps.

Example 3.7. The blow-down map in Example 3.1 is not b-normal. The local expression for the map around the bottom corner in the blow-up space is $\beta(z,\xi) = (z, z\xi)$ so it is a b-map, but the front face (of codimension 1) is mapped into a corner (of codimension 2). The same happens with pretty much any blow-down map.
The most relevant examples of b-fibrations will be given by the so-called *triples spaces* that allow us to explicitly determine the asymptotics of a composition of operators whose kernels are polyhomogeneous conormal (§3.2.2).

□
With that we can formulate the pullback and pushforward theorems of Melrose:

Proposition 3.8. [Mel92] Let $f : X \to Z$ be a b-map and $u \in \mathcal{A}^{\mathcal{E}}_{phg}(Z)$. Then $f^*u \in \mathcal{A}^{\mathcal{F}}_{phg}(X)$ with

$$\mathcal{F}_H = \{(n + \sum e(H,H')z_{H'}, \sum k_{H'}) : n \in \mathbb{N}_0\}$$

where the sum is only over the $(z_{H'}, k_{H'}) \in \mathcal{E}_{H'}$ for which $H' \in \mathcal{M}_1(Z)$ with $e(H,H') > 0$.
Let ${}^b\Omega(X) \to X$ be the b-density bundle over X, which is locally spanned[49] by

$$\left| \frac{dx_1}{x_1} \cdots \frac{dx_m}{x_m} dy_{m+1} \cdots dy_n \right|$$

for boundary defining functions x_i for the boundary hypersurfaces H_i, $1 \le i \le m$.
If f is moreover a b-fibration and $v \in \mathcal{A}^{\mathcal{E}}_{phg}(X; {}^b\Omega(X))$, and for each face $H \in \mathcal{M}_1(X)$ satisfying $e(H,H') = 0$ for all $H' \in \mathcal{M}_1(Z)$ the elements in each index set $\mathcal{E}_H$ all have first component with strictly positive real part[50] (denoted $\mathcal{E}_H > 0$), then $f_*v \in \mathcal{A}^{\mathcal{F}}_{phg}(Z; {}^b\Omega(Z))$ with

$$\mathcal{F}_{H'} = \overline{\bigcup_{e(H,H')\neq 0}} \left(\bigcup_{(z,k)\in\mathcal{E}_H} \left(\frac{z}{e(H,H')}, k \right) \right)$$

where the **extended union** of index sets is defined as

$$\mathcal{E}\overline{\cup}\mathcal{F} = \mathcal{E} \cup \mathcal{F} \cup \{(z, k + k' + 1) : (z,k) \in \mathcal{E},\ (z,k') \in \mathcal{F}\}$$

so every time the complex power of an element in both sets coincides, the logarithm power increases one more than expected.

□
In short and up to exponents $e(H,H')$: under pullback, the asymptotic behaviour at a face is precisely that of the faces it projects to; under pushforward, the asymptotics at a face are the extended union of those at the faces that fibre over it.
We can put all of this to use in an example from Grieser, where one also sees how to interpret an integral as a pushforward:

Example 3.9. What regularity does the function $u(x) = \int_0^1 \sqrt{x^2 + y^2}dy$ have at $x = 0$?

[49] We use the absolute value notation to distinguish it from the volume form, since it exists also for non-orientable manifolds. In any case, integration against it corresponds to the usual Lebesgue integration, as is the case with the volume form [BGV04, p. 29].

[50] This makes sure that the integral of the function with respect to the b-density, which corresponds to the pushforward, exists. For example, if we integrate against $\left|\frac{dy}{y}\right|$, this makes sure the function behaves like y^α with $\alpha > 0$ when $y \to 0$ (note that $y^{\alpha-1}$ is then integrable around $y = 0$).

For each x, the function is obtained by integrating $\sqrt{x^2+y^2}$ along vertical lines, i.e. we can consider the projection map $f : \mathbb{R}^2_+ \to \mathbb{R}_+$, $f(x,y) = x$ which satisfies $f^{-1}(x) = \{(x,y) : y \in \mathbb{R}_+\}$. Now, observe that the b-density $\mu = \chi_{\{0\leq x\leq 1\}} y\sqrt{x^2+y^2}\frac{dx}{x}\frac{dy}{y}$ has a push-forward

$$(f_*\mu)(x) = \left(\int_0^1 y\sqrt{x^2+y^2}\frac{dy}{y}\right)\frac{dx}{x} = u(x)\frac{dx}{x}$$

To apply the pushforward theorem to a b-density it needs to be polyhomogeneous. Ours is not at the origin, so we blow it up. For a coordinate $\eta = \frac{x}{y}$, the lift looks like $y\sqrt{(\eta y)^2+y^2}\frac{d(\eta y)}{\eta y}\frac{dy}{y} = y^2\sqrt{\eta^2+1}\frac{d\eta}{\eta}\frac{dy}{y}$ away from the horizontal axis at $y=0$. The composition of the blow-down map with f is a b-fibration since it has the local form $(x, \frac{y}{x}) \to x$ away from the vertical axis, and this (and the front face) maps to the boundary hypersurface $0 \in \mathbb{R}_+$. Its local form away from the horizontal is $(\eta, y) \mapsto \eta y$, so both these faces have exponents $e(\{0\}, H) = 1$ with respect to the 0 in $\mathbb{R}_+$. The other has exponent 0.
The lift of the density $y^2\sqrt{\eta^2+1}\frac{d\eta}{\eta}\frac{dy}{y}$ away from the horizontal axis is smooth in both variables, with only a term in y^2 as $y \to 0$ and even powers η^{2l} as $\eta \to 0$ (by Taylor), so the index set at the front face is $\{(2,0)\}$ and at the vertical axis it is $\{(2l,0) : l \in \mathbb{N}_0\}$. The lift away from the front face has the original form $y\sqrt{x^2+y^2}$ with boundary defining function y for the horizontal axis, so at this face it has leading term $(1,0)$, which has real part greater than 0.

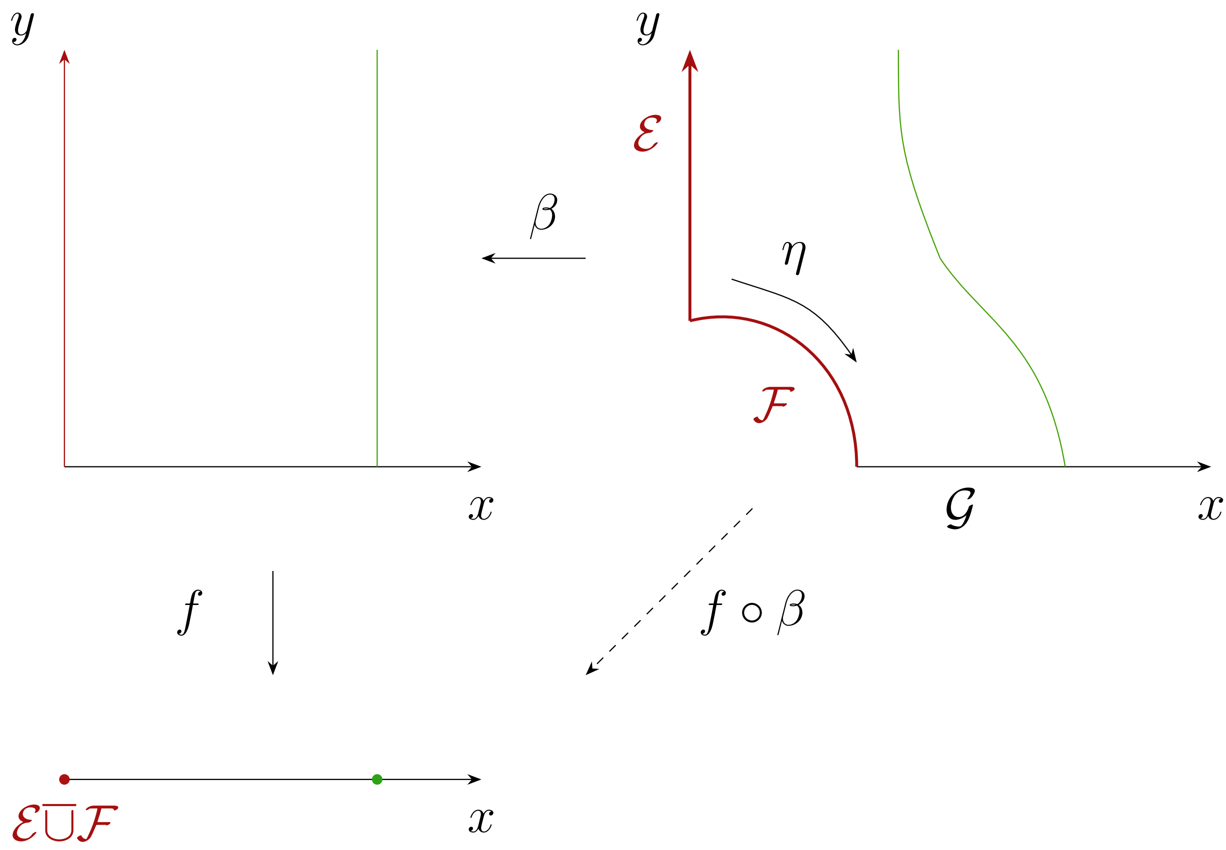


Figure 8: Applying the pushforward theorem to Example 3.9, where we attempt to determine the regularity at $x = 0$ of a function obtained by integrating along the fibres of $(x,y) \in \mathbb{R}^2_+ \mapsto x \in \mathbb{R}_+$. For this to work, the map $f \circ \beta$ has to be a b-fibration and all points in $\mathcal{G}$ need to have strictly positive real part.

The pushforward theorem tells us that $f_*\mu$ (i.e. our desired u) has the index set

$$\{(2,0)\}\overline{\cup}\{(2l,0) : l \in \mathbb{N}_0\} = \{(2l,0)) : l \in \mathbb{N}_0\} \cup \{(2,1)\}$$

meaning $u(x)$ has an expansion at $x \to 0$ with terms in

$$x^0,\ x^2 \log x,\ x^2,\ x^4,\ x^6,\ \dots$$

and this shows that integrating a function smooth in the interior can very well generate logarithms in the asymptotics.

□

3.1.3 Blow-ups of manifolds with corners

Even though we used it extensively already, let us for the sake of completeness introduce the general notion of blow-up[51] (cf. [Mel96]).
Only a certain class of submanifolds produces a well-behaved blow-up:

Definition 3.10. A submanifold Z of a manifold with corners X is a **p-submanifold** ("p" stands for "product") if for each point $z \in Z$ where X is locally modelled on $\mathbb{R}^k_+ \times \mathbb{R}^n$ (with z being the origin), there exists a coordinate subspace i.e. a subset

$$L = \{(x, y) \in \mathbb{R}^k_+ \times \mathbb{R}^n : x_j = \cdots = x_k = 0,\ y_l = \cdots = y_n = 0\}$$

(up to reordering of the coordinates (x, y)) so that $Z = X \cap L$ locally around z.

□
Note how the subsets we blew-up until now are all p-submanifolds:

- In Example 3.1, we blew-up $\{0\} = \{(z, t) \in \mathbb{R}^2_+ : z = 0,\ t = 0\} \subset \mathbb{R}^2_+$
- In Example 3.2, we blew-up $\{(t, x, x') \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n : x - x' = 0,\ t = 0\} \subset \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ where $x - x' = 0$ means $x_i - x'_i = 0$ for all $1 \le i \le n$. This is (globally) a coordinate subspace e.g. using the coordinates $(x_i + x'_i, x_i - x'_i)_{1 \le i \le n}$.

Definition 3.11. The **blow-up** $[X; Z]$ of a closed p-submanifold $Z \subset X$ in a manifold with corners is a new manifold with corners whose underlying set is given by

$$S^+NZ \sqcup (X \setminus Z)$$

with the **front face** S^+NZ being the interior spherical normal bundle of Z, where the interior normal bundle is given by T^+X/TZ (T^+X is the inward pointing part of the tangent space) and the spherical version is as usual obtained by restricting to norm 1 vectors, i.e. taking away the origin and quotienting by the $\mathbb{R}_+$-action that translates along each ray emanating from the origin:

$$\left(S^+NZ\right)_z = [(T^+_z X/T_z Z) \setminus \{0\}] \Big/ \mathbb{R}_+$$

It comes equipped with a **blow-down map** $[X; Z] \to X$, which corresponds to the bundle projection in the front face $S^+NZ \to Z$ and is just the identity outside it $X \setminus Z \to X \setminus Z$. This suggests a canonical way to provide it with a smooth structure to make it a manifold with corners, namely take the same as in $X \setminus Z$ away from the front face and then extend it. This extension can always be performed and is unique (thanks to the p-submanifold condition, making it locally look around the front face like the blow-up of a vector subspace inside a vector space [Mel96, §5.3]).

[51] If the reader is familiar with the blow-up construction in algebraic geometry, they will notice that the idea (and execution) are pretty much the same, but the context there restricts to complex varieties or schemes, while here we treat real manifolds with corners; the projectivization of the normal bundle corresponds to the (interior, since we have corners) spherical normal bundle, and the exceptional divisor corresponds to the front face.

It is known [Hir64] that an algebraic variety over a field of characteristic zero can be resolved by a finite sequence of blow-ups. The general case is open. One should keep in mind, however, that blowing-up is not the only way in which algebraic geometers resolve singularities.

□

Remark 3.12. For some problems, we are perhaps interested in blowing-up several p-submanifolds $Z_1, \dots, Z_m \subset X$. We can proceed iteratively but two caveats need to be taken into account:

- After blowing-up e.g. Z_1, it might well be the case that $Z_2 \not\subset [X; Z_1]$. We therefore need to define the **lift/pullback of a set** $W \subset X$ along a blow-down map $\beta : [X; Z] \to X$. We have two options:

  1. If $W \subset Z$, then the pullback along β is given by $\beta^*(W) := \beta^{-1}(W)$
  2. If $W = \overline{W \setminus Z}$, then $\beta^*(W) := \overline{\beta^{-1}(W \setminus Z)}$

  otherwise the pullback is not well-defined.

  If the set was a (embedded) submanifold, the $\mathcal{C}^\infty$-structure is still inherited from the ambient space (i.e. it is still a (embedded) submanifold) but not necessarily a p-submanifold. To ensure this last property one usually considers families $\{Z_i\}_{1 \le im}$ of p-submanifolds of X which are **clean**, meaning that for each point in the intersection of several of them, the Z_i's appearing in this intersection can be simultaneously expressed as coordinate subspaces $L_i \cap X$ within the same coordinate system around the shared point. In particular, disjoint p-submanifolds form clean families.

  For example (Figure 9), the diagonal $\{(x, x) : x \in \mathbb{R}_+\} \subset \mathbb{R}^2_+$ in the first quadrant is not a p-submanifold, since the first quadrant is already in the model form $\mathbb{R}^2_+$ and the diagonal is not one of its coordinate subspaces. However, after blowing up the origin, its lift becomes one: $\{\xi = 1\}$ for $\xi = \frac{y}{x}$ coordinate away from the vertical axis (so in a neigborhood of the lift). On the other hand, consider the graph $W = \{(x, x^3) : x \in \mathbb{R}\}$ of $f(x) = x^3$ as a submanifold of $\mathbb{R}^2$. Then it is a p-submanifold if we change coordinates via the diffeomorphism $f(x, y) = (x, y - x^3)$, represented globally as the coordinate subspace given by the vanishing of the second coordinate. It intersects the p-submanifold (in the usual coordinates) $Z = \{y = 0\}$ tangentially at the origin. They do not intersect cleanly, i.e. they do not form a clean family, and so the lift $\beta^*(W)$ along $\beta : [X; Z] \to X$ is not a p-submanifold.

- Even if $\{Z_i\}_{1 \le i \le m}$ is a clean family of p-submanifolds in X, the order in which we blow them up matters, although in some cases different orders yield diffeomorphic manifolds with corners (see [Beh21, §3.1-3.2]). In our main focus, the heat blow-up spaces, the order of blow-up cannot be permuted.

□

3.1.4 Clifford structures

For lack of a better place and the urgency of investigating their kernels in the following section, let us introduce now what we mean precisely by **Dirac-type operators**. We follow [BGV04, §3], where one can look up more details.
Intuitively these are operators whose square is a Laplace-type operator as defined before. Their first appearance dates back to [Dir28], hence their name. The idea is to determine the algebraic conditions that the coefficients of a first order differential operator should have in order for its square to be the Laplacian (start with the $\mathbb{R}^n$ case: $\Delta = \sum_i D_i^2 = -\sum_i \partial_i^2$); if the operator has the form $\partial\!\!\!/ = \sum_i c_i \partial_{x_i}$ for some constant coefficients $c_i \in \mathbb{R}$ then its square is $\partial\!\!\!/^2 = \sum_i c_i^2 \partial_i^2 + \sum_{i<j} (c_i c_j + c_j c_i) \partial_i \partial_j$ and we obtain $\partial\!\!\!/^2 = \Delta$ by imposing

$$c_i c_j + c_j c_i = -2\delta_{ij} \quad \forall i, j$$

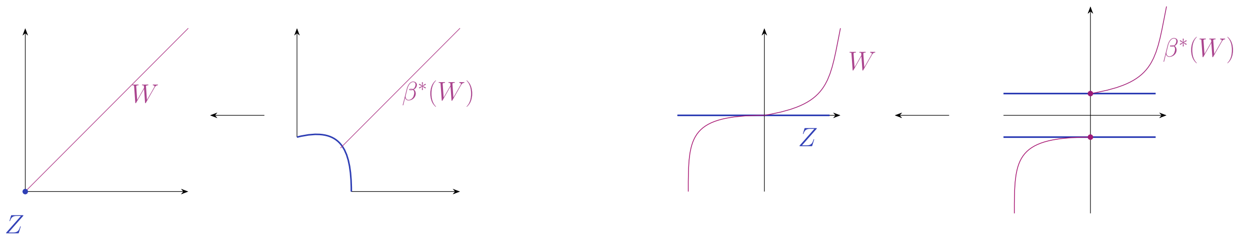


Figure 9: On the left, the diagonal W becomes a p-submanifold when pulled back after the blow-up of the origin Z in $\mathbb{R}^2_+$. On the right, the pullback of the p-submanifold W, which does not intersect Z cleanly, is not a p-submanifold any more. In this space we also see how blowing-up a hyperplane within $\mathbb{R}^n$ splits it into two disjoint halves. If this disjointness is confusing, consider a similar blow-up as in Figure 6 for $X = \mathbb{R}$, but instead of blowing-up the diagonal at time 0, blow-up the horizontal axis at time 0, i.e. $Z = \{(0, x, 0) : x \in \mathbb{R}\}$ (the blow-up looks the same but shifted by an angle of $\frac{\pi}{4}$ radians). This blown up p-submanifold does not intersect the p-submanifold $W = \{(0, x, x^3) : x \in \mathbb{R}\} \subset \mathbb{R}_+ \times \mathbb{R}^2$ cleanly, and the pullback of W is not a p-submanifold. In fact if we look at the blow-up space $[\mathbb{R}_+ \times \mathbb{R}^2; Z]$ and $\beta^*(W)$ sitting on it from above, the picture looks like the one we just drew, where the disjoint space between copies of Z is now the interior of the front face, i.e. the breadth of the half-cylinder that appears after blowing-up.

We can geometrize this condition by considering a non-constant coefficient first order differential operator with local expression $\partial\!\!\!/ = \sum_i c_i(x)\partial_i$ with $c_i \in \mathcal{C}^\infty(U)$ and ask for its square to have the same symbol as a Laplace-type operator, i.e.

$$\sigma_2\left(\underbrace{\sum_i c_i^2\partial_i^2 + \sum_{i<j}\left[c_ic_j + c_jc_i\right]\partial_i\partial_j}_{\text{second order operator}} + \underbrace{\sum_i c_i(\partial_i c_i)\partial_i + \sum_{i,j}\left[c_i(\partial_i c_j)\partial_j + c_j(\partial_j c_i)\partial_i\right]}_{\text{first order operator}}\right)(x,\xi) = -\sum_i \xi_i^2$$

This can be rewritten as a local condition on the cotangent space at the point: for $\xi, \eta \in T^*_xX$ this means

$$(c_i(x)c_j(x) + c_j(x)c_i(x)\xi_i\eta_j) = -2\xi_i\eta_j\delta_{ij}$$

so writing $\langle c(x), \xi\rangle = \sum_i c_i(x)\xi_i$ we obtain

$$\langle c(x), \xi\rangle\langle c(x), \eta\rangle + \langle c(x), \eta\rangle\langle c(x), \xi\rangle = -2g(\xi, \eta)$$

by extending the Riemannian metric g to the cotangent bundle (in particular to T^*_xX)..
This is a condition on a vector space over each point, so we can abstract it to a geometric structure and then construct the associated Dirac operators on manifolds admitting such structures. In terms of linear algebra:

Definition 3.13. A **Clifford algebra** $Cl(V, Q)$ over a real vector space V with a quadratic form Q (not necessarily a scalar product) is the algebra generated by V with relations $uv + vu = -2Q(u, v)$ for all $u, v \in V$, i.e.

$$Cl(V, Q) = T(V) \big/ \mathcal{I}_Q$$

with $\mathcal{I}_Q$ the ideal generated by $\{u \otimes v + v \otimes u + 2Q(u, v) : u, v \in V\}$. The quotient respects the natural $\mathbb{Z}_2$-grading by order parity (since the ideal contains only elements of order 2 or 0 in $T(V)$) and thus the Clifford algebra has a $\mathbb{Z}_2$-grading by "counting modulo 2 how many vectors multiply in a term" (before or after applying the relations).

In a Riemannian manifold X, the bundle $Cl(X)$ or $Cl(T^*X)$ whose fibre over a point is $Cl(X)_x = Cl(T_x^*X)$ the Clifford algebra over the cotangent space is called **Clifford bundle**. It carries a $\mathbb{Z}_2$-grading with it.
More generally, we can consider **Clifford modules**, i.e. the bundle version of a module over the Clifford algebra: a $\mathbb{Z}_2$-graded vector bundle with a graded action of $Cl(X)$. We talk about **twisted Clifford modules** when they are obtained by tensor product $E \otimes F$ of a Clifford module E with an arbitrary vector bundle F with the original action in the first factor and trivial in the second.
Note that the Levi-Civita connection ∇ on X extends to $Cl(X)$ naturally. If a (super)connection ∇^E in the Clifford module E satisfies

$$[\nabla^E, cl(\alpha)] = cl(\nabla\alpha), \quad \alpha \in Cl(X)$$

then it is called **Clifford (super)connection**.
Finally, a **Dirac(-type) operator** on a $\mathbb{Z}_2$-graded vector bundle E is an odd first order differential operator with square a Laplace-type operator:

$$\partial\!\!\!/ : \Gamma(E^\pm) \longrightarrow \Gamma(E^\mp)$$

□
We have been hinting at the connection between Dirac operators and Clifford structures, which can be summarized in:

- Whenever we have a Dirac operator on a $\mathbb{Z}_2$-graded vector bundle E, we can make E into a Clifford module with the action

  $$cl(df)s := [\partial\!\!\!/, f]s, \quad s \in \Gamma(E), \quad f \in \mathcal{C}^\infty(X)$$

  which satisfies the Clifford relations.

- Conversely, given a Clifford module, if we can write its action in the form above for some operator, then it is a Dirac operator. One can check that the local formula

  $$\partial\!\!\!/ = \sum_i cl(dx_i)\partial_{x_i}$$

  always works (but it is not the only solution).

The most relevant examples we have to keep in mind are the following:

Example 3.14. The most important Clifford module over the Clifford algebra $Cl(V)$ is the exterior algebra ΛV with action:

$$cl(v)\alpha = \varepsilon(v)\alpha - \iota(v)\alpha$$

where we specified the notations $\varepsilon(v)$ for exterior product with v and $\iota(v)$ for contraction with the dual covector $Q(v, \cdot)$ of v. If Q is a scalar product, ε and ι are adjoint operators and we consequently say that ΛV is a self-adjoint Clifford module.
As a result, using the identity $1 \in \Lambda^0 V$ we can construct a map $\sigma : Cl(V) \to \Lambda V, \quad \alpha \mapsto cl(\alpha)1$, with inverse the **quantization map** $cl : \Lambda V \to Cl(V), \quad v_1 \wedge \cdots \wedge v_k \mapsto cl(v_1)\cdots cl(v_k)$, an isomorphism as $\mathbb{Z}_2$-graded $O(V)$-modules (but not as algebras). However, the graded algebra of $Cl(V)$ is isomorphic to ΛV and the difference between them only resides in the "quantized" product given by the Clifford relations.
Given a basis $e_1, \ldots, e_n$ of V, we can construct an element of maximal degree $cl(e_1 \wedge \cdots \wedge e_n) = cl(e_1)\cdots cl(e_n) \in Cl^n(V)$ which satisfies

$$(cl(e_1)\cdots cl(e_n))^2 = (-1)^{(n-1)+\cdots+1}(cl(e_1))^2\cdots(cl(e_n))^2 = (-1)^{\frac{n(n-1)}{2}}(-1)^n = (-1)^{\frac{n(n+1)}{2}}$$

so the **chirality operator**

$$\Gamma := \begin{cases} i^{\frac{n}{2}} cl(e_1)\cdots cl(e_n), & n \text{ even} \\ i^{\frac{n+1}{2}} cl(e_1)\cdots cl(e_n), & n \text{ odd} \end{cases}$$

has square $\Gamma^2 = 1$ and for all $v \in V$, $\Gamma v = (-1)^{n+1} v\Gamma$. Note that Γ has an imaginary number as coefficient, so in general lives in $Cl(V) \otimes \mathbb{C}$, unless $n \equiv 3 \mod 4$ or $n \in 4\mathbb{N}$. Its induced Clifford action for $n \in 2\mathbb{N}$ is an involution and thus allows us to introduce a $\mathbb{Z}_2$-grading in every complex Clifford module E (or real ones in case $n \in 4\mathbb{N}$) by

$$E^{\pm} = \{e \in E : \Gamma e = \pm e\}$$

In the odd case there is no such canonical grading, since Γ belongs to the center as we saw.
The structure of Γ is fundamental because it is embedded in our definition of supertrace and consequently marks a difference of approaches to index theory between even and odd dimensional manifolds, as we will see. In particular, our Atiyah-Singer-like approach produces a vanishing index in the case of odd dimensional manifolds, a fact that can also be explained by the form of the Â-genus. An alternative to carry out index theory on non-compact odd-dimensional manifolds is the Callias index theorem [Cal78, Theorem 2].
If V is even-dimensional and oriented, there is a special Clifford module S called **spinor module**[52] with the property:

$$Cl(V) \otimes \mathbb{C} \cong \mathrm{End}(S)$$

They are constructed using a polarization of $V \otimes \mathbb{C}$. If we exponentiate the degree 2 part $Cl^2(V) \subset Cl(V)$ of the Clifford algebra, we obtain a group $\mathrm{Spin}(V) \subset Cl^+(V)$ called **spin group** of V, and S^+ and S^- are representations of it called *chiral/half-spinor* representations. $\mathrm{Spin}(V)$ is also the unique double cover of $SO(V)$.
One can talk about spinor representations independently of the parity of the dimension of V (as the simplest representations of $\mathrm{Spin}(V)$ which do not descend to representations of $SO(V)$); however, while the spin representation is reducible in the even case ($S = S^+ \oplus S^-$ by the chirality grading), it is irreducible in the odd case, where in contrast $Cl(V) \otimes \mathbb{C} \cong \mathrm{End}(S) \oplus \mathrm{End}(S^*)$ holds.
Lastly, there is a unique way to define the supertrace in $Cl(V)$ for even-dimensional V:

$$\mathrm{Str}(\alpha) := \mathrm{Tr}_S(\Gamma\alpha) = \begin{cases} \mathrm{Tr}_{S^+}(\alpha) - \mathrm{Tr}_{S^-}(\alpha), & \alpha \in Cl^+(V) \\ 0, & \alpha \in Cl^-(V) \end{cases}$$

□
Of course, this linear algebra can be transferred to the level of Clifford bundles. In particular, given a Clifford module E with Clifford connection ∇^E, we can always associate it a Dirac operator using the quantization map [53]:

$$\partial\!\!\!/ : \Gamma(E) \xrightarrow{cl} \Gamma(T^*X \otimes E) \xrightarrow{\nabla^E} \Gamma(E), \quad \text{locally:} \quad \partial\!\!\!/ = \sum_i cl(dx_i)\nabla^E_{\partial_{x_i}}$$

There is a topological obstruction to constructing a bundle of spinors, the bundle version of the spinor module: the manifold has to be spin.

[52] The name hints at a deep connection between mathematics and physics coming from the fact that the sections of the spinor bundle, the spinors, have a role in describing the spin of particles.

[53] We make use of the notation $\partial\!\!\!/$ for the Dirac operator often used in the literature. This is motivated by Feynman's slash notation in QFT, according to which we should rather write the Dirac operator as $\nabla\!\!\!\!/$. The notation $\partial\!\!\!/$ then makes sense as the (Clifford) quantization of the trivial connection in Euclidean space, i.e. for the case $\nabla_{\partial_{x_i}} = \partial_{x_i}$.

Definition 3.15. An oriented manifold is **spin** if it admits a **spin-structure**, i.e. a Spin(n)-principal bundle Spin(X) such that

$$T^*X \cong \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} \mathbb{R}^n$$

The frame bundle is doubly covered by the spin-structure, thus only orientable ($GL^+(n, \mathbb{R})$ reduction) Riemannian ($SO(n, \mathbb{R})$-reduction) manifolds can be spin. As we saw at the end of §2.2.3, the topological obstruction to its existence is the non-vanishing of the second Stiefel-Whitney class $w_2 \in H^2(X; \mathbb{Z}_2)$. Spin structures are in general not unique.
For $\dim X = n$ even, the **spinor bundle** is the associated bundle $\mathcal{S} = \mathrm{Spin}(X) \times_{\mathrm{Spin}(n)} S$, where S is the spinor module in $\mathbb{R}^n$.

□
The spinor bundle is a Clifford module whose Levi-Civita connection is a Clifford connection, with local expression:

$$\nabla^{\mathcal{S}}_{\partial_{x_i}} s = \partial_{x_i} s + \frac{1}{4} \sum_{j,k} g(\nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{x_k}) cl(dx_j) cl(dx_k) s, \quad s \in \Gamma(\mathcal{S}) \tag{6}$$

Any other Clifford module is obtained by twisting it [BGV04, Prop. 3.35], i.e. is of the form $E \cong \mathcal{S} \otimes \mathrm{Hom}_{Cl(X)}(\mathcal{S}, E)$, where the sub-index means "those homomorphisms commuting with $Cl(X)$". It admits a pointwise supertrace for operators $A \in \mathrm{End}(E)$ at each fibre $\mathrm{End}(E)_x$ of the form

$$\mathrm{str}_x(A) = \mathrm{Tr}_{E_x^+}(A_x) - \mathrm{Tr}_{E_x^-}(A_x)$$

which can be integrated to a global supertrace (much in the style of Lidskii's theorem). The kernel K_P of a differential operator $P \in \mathrm{Diff}(E, F)$ is a section of the bundle $\mathrm{HOM}(E, F) = E^* \boxtimes F \to X^2$ whose fibre over a point is $(E^* \boxtimes F)_{(x,y)} = \mathrm{Hom}(E_x, F_y)$. If $E = F$, this is denoted $\mathrm{END}(E)$. Over the diagonal $\mathrm{diag}_X \subset X^2$ we have an isomorphism $\mathrm{HOM}(E, F) \cong \mathrm{Hom}(E, F)$, since both bundles share fibres over points $(x, x) \in X^2$. The resulting (global) supertrace for a differential operator $P \in \mathrm{Diff}(E)$ over a Clifford module looks like:

$$\mathrm{Str}(P) = \int_X \mathrm{str}\left(K_P(x, x)\right) dx$$

which is a graded version of Lidskii's formula where the Clifford superspace grading is taken into account via the supertrace.
When we talk about *the* **spin-Dirac operator** $\partial\!\!\!/^{\mathcal{S}}$, we mean the Dirac operator associated to $\mathcal{S}$ with the Levi-Civita connection $\nabla^{\mathcal{S}}$. Its index will be our main object of study: once we obtain an index formula for it, the case of twisted Dirac operators (from twisted Clifford modules) follows without much complications: the crux of the analysis lies in the spinor part of the bundle and boils down to finding a nice form for the heat kernel of its square from which the supertrace formula above can be computed.
We will use extensively the fact that $Cl(X) \otimes \mathbb{C} \cong \mathrm{End}(\mathcal{S})$, in particular in the geometric description of the Getzler rescaling argument. Even if not specified, our Clifford structures can be assumed to be complex all along, and the manifolds, even-dimensional.
Due to the structural richness we make use of, we pretty much only study (twisted) spin-Dirac operators on even-dimensional spin manifolds directly with our method. Notwithstanding, there is no reason why the explicit results we obtain here for that class of operators could not be related to index formulas for other operators via different arguments.
We conclude this section by saying some words about the analytical side of the index formula for a Dirac operator in the context we just sketched and providing some examples. If the manifold X is

compact, a symmetric Dirac operator is essentially self-adjoint ([BGV04, Prop. 3.37 2.], which bases off of [BGV04, Prop. 2.33]). For *our* singular (non-compact) manifolds, the self-adjointness will be no issue in the complete case [Gaf55, Theorem], whereas in the incomplete one we might need to strengthen the assumptions and adapt the argumentations to the specific geometry (sometimes arguing as in the compact case, by constructing the heat kernel of the corresponding Laplace-type operator; others perhaps by directly crafting a parametrix of $\eth$ [AGR16, §3], with a *geometric Witt assumption*, etc.). More on that later.
But if $\eth$ is self-adjoint[54], this means its index is 0, so what do we exactly mean by computing its index, i.e. what does $\mathrm{ind}(\eth)$ mean in the literature? The point is that here the operator $\eth$ splits due to the $\mathbb{Z}_2$-grading of the corresponding Clifford module into

$$\eth = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix} : \Gamma(E) = \Gamma(E^+) \oplus \Gamma(E^-) \longrightarrow \Gamma(E^+) \oplus \Gamma(E^-)$$

and the self-adjointness of $\eth$ translates into $\left(\eth^+\right)^* = \eth^-$.
Comparing it to our discussion in § 2.3.1, $\eth$ takes the role of $\hat{P}$, $\eth^+$ that of P and $\eth^-$ that of P^*. That is, we are actually interested in the index of $\eth^+$. From a geometric point of view we can consider the superspace $\ker \eth = \ker \eth^+ \oplus \ker \eth^-$ whose *superdimension* $\dim\ker \eth^+ - \dim\ker \eth^- = \mathrm{ind}(\eth^+)$ we are trying to compute; this should remove the ambiguity in the notation of the left hand side of the index formula.

Example 3.16. Dirac operators are not as artificial as they might seem given their definition. In fact, they play very much in unison with the geometry of the manifold and therefore are usually included in the class of *geometric differential operators*.
Going all the way back to 2.1.1, we encounter the very first example: notice that we expressed the Hodge Laplacian as a square $\Delta = (d + d^*)^2$: the operator $d + d^*$ is a *square root of a Laplace-type operator*. It moreover maps forms of even degree into odd degree and viceversa

$$d + d^* : \Lambda^k T^*X \otimes \mathbb{C} \longrightarrow \left(\Lambda^{k+1} T^*X \otimes \mathbb{C}\right) \oplus \left(\Lambda^{k-1} T^*X \otimes \mathbb{C}\right)$$

and is an odd first order differential operator if we choose the grading of $\Lambda T^*X \otimes \mathbb{C}$ by parity of degree

$$(\Lambda T^*X \otimes \mathbb{C})^+ = \sum_{i\geq 0} \Lambda^{2i} T^*X \otimes \mathbb{C}, \quad (\Lambda T^*X \otimes \mathbb{C})^- = \sum_{i\geq 0} \Lambda^{2i+1} T^*X \otimes \mathbb{C}$$

This Dirac operator is called the **Gauß-Bonnet operator** because its index theorem is the Gauß-Bonnet theorem. Because the quantization map is an isomorphism of modules, we have the following identification where the spinor bundle $\mathcal{S}$ appears:

$$\Lambda T^*X \otimes \mathbb{C} \cong Cl(X) \otimes \mathbb{C} \cong \mathrm{End}(\mathcal{S}) \cong \mathcal{S} \otimes \mathcal{S}^*$$

The grading by parity on $\Lambda T^*X \otimes \mathbb{C}$ descends, in particular, into a grading $\mathcal{S}^* = (\mathcal{S}^*)^+ \oplus (\mathcal{S}^*)^-$. However, comparing the previous formula with the usual tensor decomposition of a Clifford module into $\mathcal{S}$ and the twisting part $\mathrm{Hom}_{Cl(X)}(E, \mathcal{S})$ explicitly describes $E = \Lambda T^*X \otimes \mathbb{C}$ as the Clifford module with twisting $\mathrm{Hom}_{Cl(X)}(E, \mathcal{S}) = \mathcal{S}^*$. As a result, we could leave the twisting part ungraded and still obtain a Clifford module, since the grading need only occur in the spinor factor. The operator $d + d^*$ is also a Dirac operator with this new grading; since a change of grading means a different formula for the supertrace, it leads us to a different index formula, which turns out to be

[54] We assume all over that our Clifford modules are Hermitian so that we can talk about adjoints of operators acting on them.

the signature theorem. Therefore we call the operator $d+d^*$ on $\Lambda T^*X\otimes\mathbb{C}$ with this new grading the **signature operator**. The grading actually has a geometrical meaning: it is the one induced by the Hodge star operator $*:\Lambda^kT^*X\otimes\mathbb{C}\longrightarrow\Lambda^{n-k}T^*X\otimes\mathbb{C}$ (which shares action on $\Lambda T^*X\otimes\mathbb{C}$ with the chirality operator, up to a power of i).
Following the thread of Hodge theory, we can play a similar game in a Kähler manifold X with the **Dolbeault operator** $\bar\partial$ whose kernel are the holomorphic forms. Given a holomorphic vector bundle $V\to X$, the bundle of anti-holomorphic differential forms $\Lambda^{0,1}T^*X\otimes V$ is Clifford with the action $cl(\alpha)=\varepsilon(\alpha^{1,0})-\iota(\alpha^{0,1})$, for $\alpha\in\Lambda T^*X$, and $\bar\partial+\bar\partial^*$ is a Dirac operator whose index formula is the Hirzebruch-Riemann-Roch theorem.

□

3.2 Heat blow-up spaces

3.2.1 Resolution of the heat kernel

With that we set up the stage and can concentrate now on the blow-up spaces for which our beloved heat kernels will lift to polyhomogeneous conormal distributions (let's call them *heat blow-up spaces*). We motivate the development with the heat kernel for a Laplace-type operator in a closed manifold and then move onto more exotic territories. The complete argument for the compact and asymptotically cylindrical cases within this blow-up language can be found in [Mel93, §7] (a good explanation of a similar argument can be found in [Gri04]). An analogous procedure is carried out for the edge case in [Alb07], for the asymptotically conical case in [She13], for the hyperbolic fibred cusp case in [Vai01, §4], for the fibred boundary case in [TV22] and for the incomplete cusp edge case in [Liu25, §4-5]. The compact case is also analyzed by local computations in [BGV04, §2], but the blow-up picture might be useful in visually condensing the information about the behaviour of the kernel on the several regimes of interest, which is notably more involved e.g. when we travel to the realm of singular spaces whose boundaries are modelled on fibre bundles.
Let us first consider a compact manifold without boundary X and a Laplace-type operator Δ, although the same argument works for arbitrary elliptic differential operators of order 2 with positive symbol. We want to solve:

$$\begin{cases}(\partial_t+\Delta)h(t,x,x')=0, & t>0\\ \lim_{t\to0}h(t,x,x')=\delta(x-x')\end{cases}$$

where the kernel h is a distribution in $\mathbb{R}_+\times X^2$.
As mentioned before, if we were to take normal coordinates at a point, the metric at the point will have the same expression as the Euclidean metric in $\mathbb{R}^n$ and thus a good approximation for the heat kernel would be the density described in Example 3.2. Furthermore, if the manifold has constant metric $g=g_{ij}dx_idx_j$, the Laplace-Beltrami operator takes the local form

$$\Delta=-\sum_{ij}g^{ij}\partial_{x_i}\partial_{x_j}$$

with g^{ij} representing the coefficients of the inverse of the metric with respect to the local frame given by the ∂_{x_i}'s (remember we choose the sign of the operator so that the symbol is positive and consequently our heat equation is $\partial_t+\Delta$ and not $\partial_t-\Delta$, like [Mel93] and [BGV04], unlike [Gri04]). Then the heat kernel should be modelled on

$$h(t,x,x')=(4\pi t)^{-\frac{n}{2}}\,e^{-\frac{|x-x'|_g^2}{4t}}$$

where $|x|_g^2 = \sum_{ij} g_{ij} x_i x_j$. This suggests that for a general smooth metric, when we zoom in around a point x', the heat kernel should be well approximated by that of the constant metric $g_{ij}(x')dx_i dx_j$. Indeed, denoting now $|x|^2 = |x|^2_{g(x')} = \sum_{ij} g_{ij}(x') x_i x_j$ for clarity:

$$\partial_t \left(\underbrace{(4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-x'|^2}{4t}}}_{h_{g(x')}} \right) = \left(-\frac{n}{2t} + \frac{|x-x'|^2}{4t^2} \right) h_{g(x')}$$

$$\partial_{x_i}\partial_{x_j} \left((4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-x'|^2}{4t}} \right) = \left(-\frac{1}{2t} g_{ij}(x') + \frac{1}{4t^2} \left(\sum_k g_{ik}(x_k - x'_k) \right) \left(\sum_k g_{jk}(x_k - x'_k) \right) \right) h_{g(x')}$$

$$\begin{aligned} \implies -\sum_{ij} g^{ij}(x) \partial_{x_i}\partial_{x_j} \left(h_{g(x')} \right) &= -\sum_{ij} \left[g^{ij}(x') + \eta^{ij}(z)(x-x') \right] \partial_{x_i}\partial_{x_j} \left(h_{g(x')} \right) \\ &= \left(\frac{n}{2t} - \frac{1}{4t^2} \sum_{kl} g_{kl}(x_k - x'_k)(x_l - x'_l) \right. \end{aligned}$$

$$\left. + \frac{1}{2t} \sum_{ij} \left[\eta^{ij}(z)(x-x') \right] g_{ij}(x') - \frac{1}{4t^2} \sum_{ijkl} \left[\eta^{ij}(z)(x-x') \right] g_{kj}(x') g_{li}(x')(x_k - x'_k)(x_l - x'_l) \right) h_{g(x')}$$

where we used the Taylor expansion $g^{ij}(x) = g^{ij}(x') + \eta^{ij}(z)(x-x')$ for $z \in \{(1-t)x + tx' : t \in (0,1)\}$ (η^{ij} is the gradient of g^{ij} for each pair i,j), the fact that $g^{ij} g_{kj} = \delta^i_j$ and $g_{ij} = g_{ji}$. In particular, the rest term $R(t,x,x') = (\partial_t + \Delta) h_{g(x')}(t,x,x')$ is given by the last line, where the gradient of the metric appears. In fact, recall that we argued $\sqrt{t}$ and $\xi = \frac{x-x'}{\sqrt{t}}$ where natural coordinates for the kernel around $t = 0$ (cf. Example 3.2). This still holds here, since

$$h_{g(x')} = (4\pi)^{-\frac{n}{2}} \left(\sqrt{t} \right)^{-n} e^{-\frac{1}{4} \sum_{kl} g_{kl}(x') \xi_k \xi_l}$$

i.e. of the form $\left(\sqrt{t}\right)^{-n} \times \mathcal{C}^\infty$-function in ξ. In these coordinates, the gradient contribution to the rest term is

$$\eta^{ij}(z)(x-x') = \eta^{ij}(z)\sqrt{t}\xi \implies R = \left(\sqrt{t}\right)^{-(n+1)} \times \mathcal{C}^\infty\text{- function of } \xi$$

so the time power is $\frac{1}{2}$ more negative than for $h_{g(x')}$, i.e. $h_{g(x')}$ is indeed an approximation to the heat kernel to leading order in $\sqrt{t}$[55].

To improve it, we could use Duhamel's principle: we look for a correction term $\hat{h} = t^{-\frac{n+1}{2}} k$ with k smooth in the variables $\sqrt{t}$, ξ and x', following the heuristic above (above we swapped x for ξ as variable of the function and got smoothness). We would like to compute the form of k that ensures that the new rest $(\partial_t + \Delta)(h_{g(x')} + \hat{h})$ goes like $\left(\sqrt{t}\right)^{-(n+2)} \times \mathcal{C}^\infty \left([0,\infty)_{\sqrt{t}} \times \mathbb{R}^n_\xi \times U_{x'} \right)$, where $U_{x'} \subset X$ should be thought of as a neigborhood of the point x' around which we were approximating (x' is now treated as a variable). Then, based on the following calculation, we can determine $\hat{h}$ by Duhamel's principle (or by Fourier transform):

$$\begin{aligned} (\partial_t + \Delta) \left(h_{g(x')} + \hat{h} \right) = R + \left(-\frac{n+1}{2t} k + (\partial_t + \Delta) k \right) t^{-\frac{n+1}{2}} = t^{-\frac{n+2}{2}} \hat{r} \\ \implies (\partial_t + \Delta) k = -t^{\frac{n+1}{2}} R = -t^{\frac{n+1}{2}} (\partial_t + \Delta) h_{g(x')} \end{aligned}$$

[55]Even though this procedure shows the kernel has an asymptotic in powers of $\sqrt{t}$, to the trace (i.e. at the diagonal $x = x'$) only even powers contribute (cf. (3) (19) [Gri04, §2.4.3] [Mel93, Lemma 7.16]), so the corresponding asymptotic is in powers of t.

for some $\hat{r} \in \mathcal{C}^\infty\left([0,\infty)_{\sqrt{t}} \times \mathbb{R}^n_\xi \times U_{x'}\right)$. We could a priori iteratively follow this procedure to improve the approximation to arbitrary order in $\sqrt{t}$. Nevertheless, to obtain an approximation to order $t^{-\infty}$ (i.e. to order t^{-N} for all $N \in \mathbb{N}$), we would need to sum an infinite series, so it is not clear whether the process converges. We will solve this by a composition trick. This process of "solving away" the series expansion on t around $t \to 0$ is usually referred to as (heat) **parametrix construction** (if there are more asymptotics involved, then *parametrix* might designate the solution obtained by patching up the approximations at each face).
Let us now abstract the main properties of the distributions that just appeared which allowed them to be solved away at $t = 0$ and to add up to a good approximation of the heat kernel due to their regularity properties. We would like to construct a **(heat) calculus** of pseudodifferential operators $\Psi^\alpha_H(X)$ out of them and then figure out a way to talk about their composition. From the heat kernel heuristic and the remarks above it should not be surprising that we choose functions with the following properties (cf. [Gri04, Def. 2.1]):

- An operator $A \in \Psi^\alpha_H(X)$ is of the form $\left(\sqrt{t}\right)^{k(\alpha)} \mathcal{C}^\infty\left([0,\infty)_{\sqrt{t}} \times \mathbb{R}^n_\xi \times X\right)$, where the power $k(\alpha)$ is chosen to ensure the composition law $\Psi^\alpha_H \cdot \Psi^\beta_H \subset \Psi^{\alpha+\beta}_H$ (also the heat kernel, which is the "heat inverse" of a second order differential operator should have order -2).

  If we denote by HX^2 the heat blow-up space $[\mathbb{R}_+ \times X^2; \{0\} \times \mathrm{diag}_X]$[56], this condition can be translated into $A \in \left(\sqrt{t}\right)^{k(\alpha)} \mathcal{C}^\infty\left(HX^2\right)$ (to be more precise, the lift $\beta^* A$ along $\beta : HX^2 \to \mathbb{R}_+ \times X^2$ satisfies such condition; this is a common abuse of notation). This should again underline the relationship between the heat blow-up space and the regularity of the heat kernel that motivated its construction.

- The exponential part of the heat kernel is not just a generic smooth function in ξ, but has a time decay $O(t^\infty)$ as $t \to 0$ away from the diagonal, i.e. for $x \neq x'$, and an exponential decay $O(|\xi|^{-\infty})$ as $|\xi| \to \infty$. This also applies for its derivatives. Thus we ask for $A(t, x, x') = \left(\sqrt{t}\right)^{k(\alpha)} a(\sqrt{t}, \xi, x')$ such that:

$$\partial^\gamma_{t,x,x'} A = O(t^\infty) \quad \text{as} \quad t \to 0, \quad \partial^\gamma_{\sqrt{t},\xi,x'} a = O(|\xi|^{-\infty}) \quad \text{as} \quad |\xi| \to \infty$$

  for all multiindices γ. If we look again at the lift $\beta^* A$ within the heat space, this condition can be summarized in $\beta^* A$ vanishing to infinite order at the blue face in Fig. 6, that is, at $\beta^*(\{t = 0\})$.

  If we call this face tb and the front face $\beta^*(\{0\} \times \mathrm{diag}_X)$ of the blow-up tf (as in [Mel93]), then the limit $|\xi| \to \infty$ corresponds to sliding down tf towards its intersection with tb, and

[56] By diag_X we mean the set
$$\{(x, x') \in X^2 : x - x' = 0\}$$
which is a p-submanifold precisely when X has no boundary. HX^2 has the same form as $H\mathbb{R}^n$ in the top right of Fig. 6, but changing the $\mathbb{R}^n_x \times \mathbb{R}^n_{x'}$-directions for $X_x \times X_{x'}$-directions.

we can summarize all conditions above[57][58] in

$$A \in \Psi_H^\alpha(X) \iff \beta^* A = \rho_{\mathrm{tf}}^{k(\alpha)} \rho_{\mathrm{tb}}^\infty \mathcal{C}^\infty\left(HX^2\right)$$

By the Schwartz kernel theorem, the kernels of standard (not heat) pseudodifferential operators $A \in \Psi^\alpha(X)$ and $B \in \Psi^\beta(X)$ induce operators:

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X), \quad u \mapsto (Au)(x) = \int_X A(x,x')u(x')dx''$$

(analogous for B; notice the abuse of notation by denoting the operator and its kernel by the same symbol) and the kernel of their composition is given by

$$(A \circ B)(x,x') = \int_X A(x,x'')B(x'',x')dx''$$

Written in terms of pullbacks and pushforwards, this corresponds to[59]

$$\begin{array}{ccccc} X_x \times X_{x''} & \xleftarrow{\pi_L} & X_x \times X_{x'} \times X_{x''} & \xrightarrow{\pi_R} & X_{x''} \times X_{x'} \\ & & \downarrow{\scriptstyle \pi_C} & & \\ & & X_x \times X_{x'} & & \end{array}$$

$$A \circ B = (\pi_C)_* \left(\pi_L^* A \cdot \pi_R^* B \cdot \mu\right), \quad \mu = dx''$$

(this is the triple space formalism one refers to when applying the pushforward theorem to composition of operators (in which case, it is useful to write each of the terms as b-densities); "triple space" refers to the X^3 in the middle, the others are "double spaces". One can generalize these ideas to describe a wide variety of calculi [Mel23]).
The elements $A \in \Psi_H^\alpha(X)$ also define operators

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(\mathbb{R}_+ \times X), \quad u \mapsto (Au)(t,x) = \int_X A(t,x,x')u(x')dx'$$

[57] Since the pullback of a half-density from $\mathbb{R}_+ \times X^2$ to the heat blow-up space differs from the half-density in HX^2 by powers of t, the exponent $k(\alpha)$ appearing in the definition of $\Psi_H^\alpha(X)$ through $\beta^* A$ is not the same as the one through A. We will compute the one for $\beta^* A$, but the difference with that of A can be recovered from:

$$\beta^*|dtdxdx'| = 2\left(\sqrt{t}\right)^{n+1}|d(\sqrt{t})d\xi dx'|$$

which also explains the disparity between [Mel93, (7.11)] and [Gri04, (9)]:

$$t^{-\frac{n+2}{2}-\alpha}\beta^*|dtdxdx'|^{\frac{1}{2}} = 2\left(\sqrt{t}\right)^{-\frac{n+3}{2}-2\alpha}|d(\sqrt{t})d\xi dx'|^{\frac{1}{2}}$$

[58] Notice that for $k(\alpha)$ small enough, the integral (7) does not converge in t', so there is an extra condition we have to impose in those elements of the calculus to have it well-defined. What is usually done is to ask for the integral along each of the fibres of $\mathrm{tf} \to X$ to vanish; see the *mean value condition* in [DM12, (1.3)] [Vai01, (72)] [Gri04, Lemma 2.8 (b)] [Alb07, p. 13]. Compare this with the initial condition (10).

[59] Here instead of having to write explicitly the density μ, one could consider Ψ^α as a space of half-densities, i.e.

$$A = A(x,x'')|dxdx''|^{\frac{1}{2}}, B = B(x'',x')|dx''dx'|^{\frac{1}{2}}$$

Then, the induced operators map half-densities $u(x)|dx|^{\frac{1}{2}}$ into half-densities and composition of half-densities produces a half-density, and one need not keep track of the subjacent density μ when applying pushforward. We choose not to follow this treatment here, so we need to integrate with respect to $2\left(\sqrt{t}\right)^{n+1}|d(\sqrt{t})d\xi dx'|$ when working in HX^2.

but their composition is not well-defined by this action, since the domain and range are different spaces. However, we can rather consider the action by *convolution*:

$$A : \mathcal{C}^\infty(\mathbb{R}_+ \times X) \longrightarrow \mathcal{C}^\infty(\mathbb{R}_+ \times X), u \mapsto (Au)(t,x) = \int_X \int_0^t A(t-t', x, x')u(t', x')dt'dx' \tag{7}$$

and now the composition law reads

$$(A \circ B)(t, x, x') = \int_X \int_0^t A(t-t', x, x'')B(t', x'', x')dt'dx''$$

$$\begin{array}{ccccc} (\mathbb{R}_+)_{t-t'} \times X_x \times X_{x''} & \xleftarrow{\pi_L} & (\mathbb{R}_+)_{t-t'} \times (\mathbb{R}_+)_{t'} \times X_x \times X_{x'} \times X_{x''} & \xrightarrow{\pi_R} & (\mathbb{R}_+)_{t'} \times X_{x''} \times X_{x'} \\ & & \big\downarrow \pi_C & & \\ & & (\mathbb{R}_+)_t \times X_x \times X_{x'} & & \end{array}$$

$$A \circ B = (\pi_C)_* \left(\pi_L^* A \cdot \pi_R^* B \cdot \mu\right), \quad \mu = dt'dx''$$

Now, one can explicitly compute how the heat order behaves under composition either by integrating explicitly ([Gri04, Prop. 2.6]) or by applying the pushforward theorem (since at the end of the day we are interested in the t-asymptotic as $t \to 0$). The latter approach corresponds to the triple space construction in the next section. The result is that $\Psi_H^\alpha(X) \cdot \Psi_H^\beta(X) \subset \Psi_H^{\alpha+\beta}(X)$ if we choose $k(\alpha) = -n - 2 - \alpha$ as the time power of $\beta^* A$ for $A \in \Psi_H^\alpha(X)$. Notice that $h_{g(x')} \in \Psi_H^{-2}(X)$.
Observe that if we consider the zoom-in of Fig. 1 again, applied to the operator Δ, then we would recover the constant coefficient principal part

$$\varepsilon^2 \delta_\varepsilon \Delta \delta_\varepsilon^{-1} \underset{\varepsilon \to 0}{\longrightarrow} -\sum_{ij} g^{ij}(x_0)\partial_{x_i}\partial_{x_j}$$

In particular, the solution of the "zoomed-in" heat equation at x_0 is exactly given by $h_{g(x_0)}$. The same way as $\lim_{\varepsilon\to 0}\left(\varepsilon^m \delta_\varepsilon P \delta_\varepsilon^{-1}\right)$ represents the spatial zoom-in of $P \in \mathrm{Diff}^m(X)$, we can also perform an analogous operation to recover the leading term in the time asymptotic of a distribution in $\Psi_H^\alpha(X)$ (these leading order terms usually go by the name of **normal operators**). We do this by simply computing

$$N_\alpha(A) = \left((\rho_{\mathrm{tf}})^{n+2+\alpha} \beta^* A\right)\Big|_{\mathrm{tf}}, \quad A \in \Psi_H^\alpha(X) \tag{8}$$

(think $\rho_{\mathrm{tf}} = \sqrt{t}$). The key caveat is that this quantity is defined invariantly, i.e. independent of the choice of coordinates [Gri04, Lemma 2.3]. By cutting off at the leading term in the t-expansion, we obtain a function living in $\mathrm{tf} \subset HX^2$ with infinite order decay as $|\xi| \to \infty$. Since tf was obtained as the blow-up of the p-submanifold $\{0\} \times \mathrm{diag}_X$, we have a fibration

$$\mathrm{tf} \longrightarrow \{0\} \times \mathrm{diag}_X \cong X$$

In terms of local coordinates, this fibration looks like $(\xi, x') \mapsto (x')$, since the coordinate $\sqrt{t}$ is transversal to the face (remember that $\rho_{\mathrm{tf}} \propto \sqrt{t}$). This in turn indicates that the variables ξ parametrize the $\mathbb{R}^n$-fibres of $\mathrm{tf} \to X$ and one can in fact identify tf with the radial compactification of TX along the fibre directions (the boundary added through this compactification corresponds to $\mathrm{tf} \cap \mathrm{tb}$). Therefore, $N_\alpha(A)$ can be understood as an invariantly defined smooth function on TX decaying rapidly along the fibres (compare this with the principal symbol seen as a smooth function on T^*X), which we denote by $N_\alpha(A) \in \mathcal{S}_{\mathrm{fib}}(TX)$ (cf. [Vai01, §4.3]) as a reference to

Schwartz spaces. Notice that an operator $B \in \Psi_H^{\alpha-1}(X)$ has no term of order $\left(\sqrt{t}\right)^{-n-2-\alpha}$ but only higher order terms, so there is a short exact sequence:

$$0 \longrightarrow \Psi_H^{\alpha-1}(X) \longrightarrow \Psi_H^{\alpha}(X) \longrightarrow \mathcal{S}_{\text{fib}}(TX) \longrightarrow 0 \tag{9}$$

since a function in $\mathcal{S}_{\text{fib}}(TX)$ can always be realized as the leading part of an operator in $\Psi_H^{\alpha}(X)$.
Once these ideas are introduced, the philosophy is the following: we started by approximating locally the heat kernel $h(t,x,x')$ by $h_{g(x')}$, which produced an error term of higher order in $\sqrt{t}$. Suppose we had not been inspired to make that choice; the systematic approach would have been to consider the normal operator of $t(\partial_t+\Delta)h(t,x,x')$ at the face tf[60] corresponding to leading order in $\sqrt{t}$, i.e. [Mel93, (7.52)]:

$$N_{-2}\left(t(\partial_t+\Delta)h\right) = \left(-\frac{1}{2}\left(\xi\partial_\xi + n\right) + \sigma_2(\Delta)\right) N_{-2}(h)(x,\xi)$$

(note $\xi\partial_\xi$ is the radial vector on the fibres of $\text{tf} \to X$). Since we work locally, we can e.g. choose Riemannian normal coordinates around x' so that the equation to solve (to get the first approximation) becomes

$$\left(-\frac{1}{2}\left(\xi\partial_\xi + n\right) + \sum_{i=1}^{n} \partial_{x_i}^2\right) N_{-2}(h)(x,\xi) = 0$$

We also need to reinterpret the initial condition in the blow-up space, which amounts to performing the corresponding coordinate change [Gri04, Lemma 2.8][Mel93, (7.42)] and produces:

$$\int_{\text{tf}/X} N_{-2}(h)(x,\xi)d\xi = 1 \tag{10}$$

Notice that for each $x \in X$, this is a condition on the fibre of x along $\text{tf} \to X$, which as mentioned is modelled after the radial compactification of T_xX (so this can be seen as an integral over T_xX with respect to the measure coming from $g(x)$ as scalar product on T_xX). Therefore, we can solve the problem on each fibre by e.g. Fourier transform [Mel93, p. 268] and recover our first order approximation of h. This will be an operator h_0 of order -2 in the heat calculus whose leading order normal operator is given by $N_{-2}(h_0) = N_{-2}(h)$, i.e. the solution of the equation above. Such operator exists by (9). Note that this kernel vanishes to infinite order at tb, i.e. as $|\xi| \to \infty$.
Once the leading order problem at the face is solved, we get a remainder term that we wish to improve by adding more terms to our approximation, each of them improving the solution by an order on $\sqrt{t}$. This is usually referred to as "solving away the Taylor expansion of the kernel at the face" or "iterating away the error terms". The terms we add are solutions of equations obtained by taking normal operators of higher orders. For example, the next term h_1 in the expansion is chosen to satisfy:

$$t(\partial_t+\Delta)(h_0+h_1) = R_1 \in \Psi_H^{-4}(X)$$

If we denote the original remainder by R_0:

$$t(\partial_t+\Delta)h_0 = R_0$$

then

$$N_{-2}\left(t(\partial_t+\Delta)h_0\right) = \left(-\frac{1}{2}\left(\xi\partial_\xi+n\right) + \sum_{i=1}^{n}\partial_{x_i}^2\right) N_{-2}(h) = 0 \implies N_{-2}(R_0) = 0 \overset{(9)}{\implies} R_0 \in \Psi_H^{-3}(X)$$

[60] Multiplying the operator by t turns out to be useful in the calculations; in particular, with the methodology of the b-calculus in mind, this allows us to work with vector fields that are tangent to the faces lying at $\{t=0\}$. Observe that this ensures that the operator appearing in the model heat problem at tf is smooth; see the lifts at the end of the statement of Conjecture 3.22.

Similary, we obtain $R_1 \in \Psi_H^{-4}(X)$ by looking for

$$N_{-3}\left(t(\partial_t+\Delta)(h_0+h_1)\right)=0 \implies N_{-3}(R_0)+N_{-3}(t(\partial_t+\Delta)h_1)=0$$

We can explicitly compute the second normal operator and obtain an inhomogeneous heat equation

$$\left(-\frac{1}{2}\left(\xi\partial_\xi+n-1\right)+\sum_{i=1}^n \partial_{x_i}^2\right)N_{-3}(h_1)=-N_{-3}(R_0)$$

with explicit solution [Mel93, (7.62)] (e.g. again by Fourier transform or by applying Duhamel's principle). The iterated version of the equation is:

$$\left(-\frac{1}{2}\left(\xi\partial_\xi+n-\alpha\right)+\sum_{i=1}^n \partial_{x_i}^2\right)N_{-2-\alpha}(h_\alpha)=-N_{-2-\alpha}(R_{\alpha-1})$$

The improved solution is constructed by taking the asymptotic sum of the h_i's (this can be done by Borel's lemma [Mel93, Lemma 5.24], uniquely up to a distribution vanishing to infinite order at tf). The corresponding remainder lives in $\Psi_H^{-\infty}(X)$. The approximation $h_{(1)}$ obtained at the end of this process goes by the name of **initial heat parametrix**.
To get rid of the remainder, we conclude with a Volterra series argument ([Gri04, §3] shows how this could have been done directly after computing h_0, but requires the proof of a stronger Volterra invertibility result for those bigger rest terms). This is where considering the operators as acting by convolution (7) and the composition theorem §3.2.2 find their telos. As a result, we obtain the exact heat kernel.
Indeed, we just reached the point

$$t(\partial_t+\Delta)h_{(1)}=R\in\Psi_H^{-\infty}(X)$$

which in terms of convolution action reads

$$(\partial_t+\Delta)h_{(1)}*=id+\frac{1}{t}R*$$

Inverting formally the right hand side produces

$$\left(id+\frac{1}{t}R*\right)^{-1}=id-\frac{1}{t}R*+\frac{1}{t}R*\frac{1}{t}R*-\ldots=id+\underbrace{\sum_{j=1}^\infty\left(-\frac{1}{t}R*\right)^j}_{=:S*}$$

One can directly estimate each term to ensure convergence of the series defining S [Mel93, Prop. 7.17], which moreover belongs to $\Psi_H^{-\infty}(X)$ because R does and the composition theorem[61]. The exact heat kernel is finally given by

$$h*=h_{(1)}*\left(id+\frac{1}{t}R*\right)^{-1}=h_{(1)}*(id+S*)=h_{(1)}*+h_{(1)}*S*$$

In particular, $h\in\Psi_H^{-2}(X)$ and its leading order behaviour at tf is given by the normal operator $N_{-2}(h_0)$.
So how much of this reasoning still holds when treating singular settings? It turns out the same routine succeeds; here is a recipe for contriving the heat space for Laplace-type operators:

[61] So actually, we just needed a composition result for elements in $\Psi_H^{-\infty}(X)$. This is not enough when the geometry of X gets more complicated (e.g. singular).

1. Find the heat blow-up space. This follows a specific heuristic which we will discuss below.
2. Construct a heat calculus (and the corresponding notion of normal operators) that keeps track of the orders of vanishing at each face. For aesthetics and to get a better composition formula, the identity should be of order 0 and the heat kernel of order -2 at tf[62].
3. Develop a composition statement within the aforementioned calculus. This amounts to finding a heat triple space and applying the pullback and pushforward theorems.
4. Compute the normal operators at each face of the terms appearing in the heat equation.
5. Solve the heat equation to first order at each face, which has an explicit expression once we know the normal operators. We start with the face tf and initial condition analogous to (10). The solution in that face provides an initial condition for the previously blown up front face that intersects tf, and so on: iteratively, the solution in one face defines the initial condition on the next in a cascading process.
6. Construct the initial heat parametrix by improving the error to infinite order vanishing at tf by "solving away the Taylor series" at this face. This can also be done at other faces but is not always needed.
7. Invert the remainder via a Volterra series argument using the composition theorem carried out in Step 3.

Some realizations of this programme to be found in the literature are:

- Closed and b-case [Mel93, §7]
- Conical ends [Moo96]
- Hyperbolic fibred cusps [Vai01, §4]
- Edge case [Alb07, §3-5]
- Asymptotically conical ends [She13, Appendix A]
- Fibred boundary manifolds [TV22]
- Iterated wedge metrics [AGR23, §3-4]
- Incomplete cusp edge spaces [Liu25, §4-5]

Steps 2-7 are mainly of computational nature (they require local coordinate calculations, integrations and pullbacks), whereas Step 1 elicits the question: How does one find the appropriate heat blow-up space? The answer is, at this point perhaps unsurprisingly, motivated by the closed case and a general principle can be derived from the samples in the literature, which we try to outline here.

Example 3.17. Consider first the case of cylindrical ends, which comes from compactifying a half-cylinder at infinity (Example 4.1). Start by looking at $M = \mathbb{R}_+ = [0,\infty)_u$ with metric $g = du^2$ and compactify at $u \to \infty$ by choosing the coordinate $x = e^{-u}$. Then M becomes $(0,1]_x$ and compactifies to $X = [0,1]_x$. The Euclidean kernel

$$h_M(t,u,u') = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(u-u')^2}{4t}}$$

[62] We will now see that all heat spaces manifest a face similar to the face tf appearing in the closed case, i.e. related to the blow-up of the diagonal $\mathrm{diag}_X \subset X^2$ at time 0.

takes the form

$$h_X(t,x,x') = \frac{1}{\sqrt{4\pi t}} e^{-\frac{\left(\ln\left(\frac{x}{x'}\right)\right)^2}{4t}}$$

Expanding in Taylor series around the diagonal $\frac{x}{x'} \sim 1$:

$$\ln\left(\frac{x}{x'}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\left(\frac{x}{x'}-1\right)^j = \left(\frac{x}{x'}-1\right) - \frac{1}{2}\left(\frac{x}{x'}-1\right)^2 + \dots$$

so to leading order with respect to the distance to the diagonal

$$h_X(t,x,x') \sim \frac{1}{\sqrt{4\pi t}} e^{-\frac{\left(\frac{x}{x'}-1\right)^2}{4t}}$$

Recall (Example 3.2) that the "natural" coordinates ensuring that the Euclidean kernel has "nice" asymptotics where motivated by the prefactor $t^{-\frac{n}{2}}$ (suggests the coordinate $\sqrt{t}$) and the argument of the exponential $\left(\frac{1}{2}\frac{u-u'}{\sqrt{t}}\right)^2$ (suggesting $\xi = \frac{u-u'}{\sqrt{t}}$). We complete the coordinate system by preserving the original coordinate x'. The same reasoning applied to the b-case yields:

$$\xi = \frac{\frac{x}{x'}-1}{\sqrt{t}}, \quad x', \quad \sqrt{t}$$

as appropriate coordinates. The rule of thumb for blow-ups then tells us that we first rescale the time axis from t to $\sqrt{t}$, then introduce the coordinate $s = \frac{x}{x'}$ by blowing up $\{x=0,\ x'=0\} = \mathbb{R}_+ \times (\partial X)^2 \subset \mathbb{R}_+ \times X^2$, and finally introduce the coordinate $\xi = \frac{s-1}{\sqrt{t}} = \frac{x-x'}{x'\sqrt{t}}$ by (parabolically) blowing up $\{s-1=0,\ \sqrt{t}=0\} \mathrel{\widehat{=}} \{0\} \times \mathrm{diag}_X$.

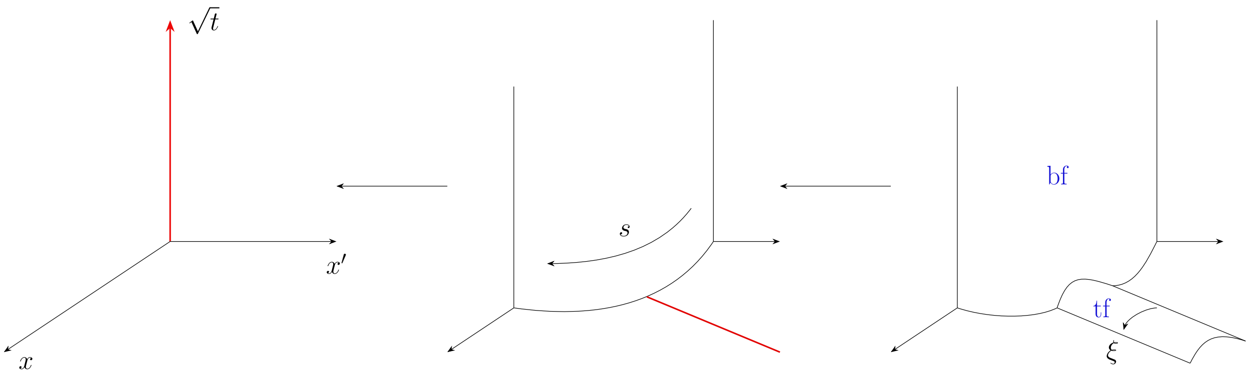


Figure 10: Blow-up sequence that gives rise to the heat blow-up space for manifolds with asymptotically cylindrical ends [Mel93, Fig. 13]. In blue, the name of the boundary hypersurfaces intersecting the diagonal: the "time face" tf and the "boundary/b-face" bf. There are 3 other boundary hypersurfaces, namely the ones to the left and right of bf (lb and rb, respectively) and the one surrounding tf (tb), and the heat kernel vanishes to infinite order in all 3 of them.

The order of blow-up is important: $\{0\} \times \mathrm{diag}_X$ only became a p-submanifold after we blew up $\mathbb{R}_+ \times (\partial X)^2$. If we denote the b-double space X_b^2 [Mel93, §4.2], the b-heat blow-up space is obtained by:

$$HX_b^2 = [\mathbb{R}_+ \times X_b^2; \{0\} \times \mathrm{diag}_X]$$

The b-heat kernel is constructed by first solving away the Taylor series at tf and then at bf, and afterwards applying the same Volterra series argument [Mel93, Theorem 7.24].

The same coordinate system could have been deduced by fixing x' and considering the expansion of $\ln x$ for $x \to x'$:

$$\ln x = \ln x' + \frac{1}{x'}(x - x') - \frac{1}{(x')^2}(x - x')^2 + \dots$$

so the argument of the exponent looks again like

$$\frac{1}{4}\left(\frac{\ln x - \ln x'}{\sqrt{t}}\right)^2 \sim \frac{1}{4}\left(\underbrace{\frac{(x - x')}{x'\sqrt{t}}}_{=\xi} + \frac{1}{\sqrt{t}}O\left((x - x')^2\right)\right)^2$$

What's more, if we take the whole expansion of the argument of the exponential, it is a polynomial on ξ and $\sqrt{t}$:

$$\frac{1}{4}\left(\frac{\ln x - \ln x'}{\sqrt{t}}\right)^2 \sim \frac{1}{4}\left(\frac{1}{\sqrt{t}}\sum_{j=1}^{\infty}\frac{(-1)^j}{j(x')^j}(x - x')^j\right)^2 = \frac{1}{4}\left(\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\xi^j(\sqrt{t})^{j-1}\right)^2$$

□

Remark 3.18. As noted in Example 4.1, there is another way to compactify a cylinder into a compact manifold with boundary, i.e. by introducing the coordinate $r = \frac{1}{u+1}$. This changes the heat space (and the domain of the operator), but the same kind of analysis can be carried out and yields an analogous result.
In the example above we treated a cylinder with trivial cross section (a point), but the general case does not contain new ideas.

□

Example 3.19. Another example directly related to the Euclidean kernel up to a compactification is that of manifolds with scattering ends (Example 4.2). The model case is the infinite end of a cone, compactified from $\mathbb{R}^n$ by working on polar coordinates (r, y) and introducing a coordinate inverting the radius $x = \frac{1}{r}$. This means points in $\mathbb{R}^n$ can be written in the form

$$p = |p|y = ry = \frac{y}{x}, \quad y \in \mathbb{S}^{n-1}$$

and we are interested on what happens as $x \to 0$ (we now write $p \in \mathbb{R}^n$ instead of u because later we want to use u to denote one of the coordinates in the blow-up space).
The argument of the exponential of the Euclidean heat kernel takes the form:

$$\frac{1}{4}\left(\frac{\frac{y}{x} - \frac{y'}{x'}}{\sqrt{t}}\right)^2 = \frac{1}{4}\left(-y'\frac{x - x'}{(x')^2\sqrt{t}} + \frac{y - y'}{x'\sqrt{t}} + \dots\right)^2$$

which comes from Taylor in (x, y) approaching fixed (x', y'):

$$\frac{y}{x} = \frac{y'}{x'} - \frac{y'}{(x')^2}(x-x') + \frac{1}{x'}(y-y') + \dots = \frac{y'}{x'} + \sum_{j=1}^{\infty}\frac{(-1)^j}{j}\frac{y'}{(x')^{j+1}}(x-x')^j + \sum_{j=0}^{\infty}\frac{(-1)^j}{(x')^{j+1}}(x-x')^j(y-y')$$

Along with the coordinates $(x', y', \sqrt{t})$, this suggests introducing:

$$\xi = \frac{x - x'}{(x')^2\sqrt{t}}, \quad \eta = \frac{y - y'}{x'\sqrt{t}} = \left(\frac{y_1 - y_1'}{x'\sqrt{t}}, \dots, \frac{y_{n-1} - y_{n-1}'}{x'\sqrt{t}}\right)$$

for which the argument becomes a polynomial

$$\frac{1}{4}\left(\sum_{j=1}^{\infty}\frac{(-1)^j}{j}y'\xi^j\left(x'\sqrt{t}\right)^{j-1}+\sum_{j=0}^{\infty}(-1)^j\xi^j\eta\left(x'\sqrt{t}\right)^j\right)^2$$

The *minimal* sequence of blow-ups achieving this corresponds to (swapping t by $\sqrt{t}$ and):

1. Introducing $s=\frac{x}{x'}$ by performing the b-blow up of $\mathbb{R}_+\times(\partial X)^2$
2. Introducing $S=\frac{x-x'}{(x')^2}=\frac{s-1}{x'}$ and $u=\frac{y-y'}{x'}$ by blowing up $\{s-1=0,\ x'=0,\ y-y'=0\}$, which is kind of a diagonal at the boundary for all times.
3. Introducing ξ and η by blowing up (the lift of) $\{0\}\times \mathrm{diag}_X$, that is $\{S=0,\ u=0,\ \sqrt{t}=0\}$.

Again, denoting the scattering double space by X^2_{ac}, we just performed $HX^2_{ac}=[\mathbb{R}_+\times X^2_{ac};\{0\}\times \mathrm{diag}_X]$.

□

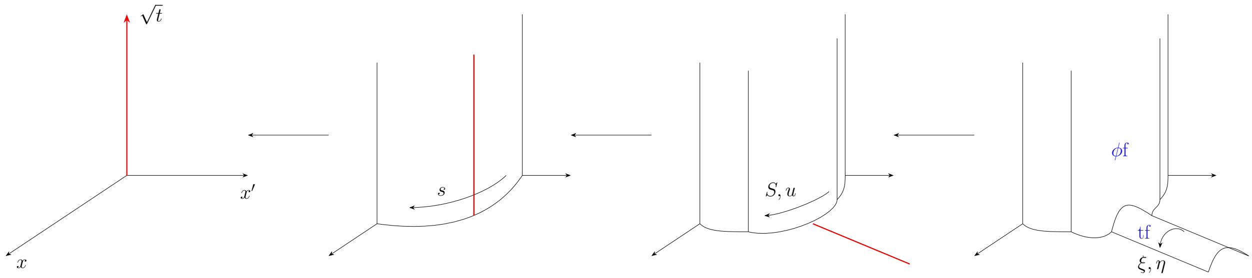


Figure 11: Blow-up sequence that gives rise to the heat blow-up space for manifolds with scattering/asymptotically conical ends. In blue, the name of the boundary hypersurfaces intersecting the diagonal: the "time face" tf and the "fibre boundary face" ϕf, which derives its name from the heat blow-up space for manifolds with fibred boundary/fibred cusp ends, so-called ϕ-manifolds (the scattering case corresponds to trivial fibres of the boundary fibre bundle in that context). There are 4 other boundary hypersurfaces: bf, lb, rb and tb.

Example 3.20. For an example in the context of an incomplete singularity (Example 4.2), consider the tip of a model cone, which corresponds to introducing polar coordinates in $\mathbb{R}^n$ and adding the singular stratum [63] corresponding to radius 0 to be a sphere $\mathbb{S}^{n-1}$. For consistency with the previous cases, denote the radial coordinate by $x\geq 0$ and the angular coordinate by $y\in\mathbb{S}^{n-1}$ (of course, the analysis extends to cones with any other closed manifold as link).
Proceeding as before, the argument of the exponential is

$$\frac{1}{4}\left(\frac{xy-x'y'}{\sqrt{t}}\right)^2=\frac{1}{4}\left(y'\frac{x-x'}{\sqrt{t}}+\frac{x'(y-y')}{\sqrt{t}}+\frac{(x-x')(y-y')}{\sqrt{t}}\right)^2$$

[63] I.e. "blowing up the tip of the cone", so that the metric extends to this stratum to be degenerate, that is, points in the stratum are at distance 0 from each other. For $n=2$, the picture is the same as the blow-up of the origin in $\mathbb{R}^2$ (Example 3.1) if we were to have a full circle and all four quadrants instead of a quarter circle and the top right quadrant.

so natural coordinates are[64]

$$\xi = \frac{x - x'}{\sqrt{t}}, \quad \eta = \frac{x'(y - y')}{\sqrt{t}}$$

To reach this coordinates with blow-ups we need to rethink them as

$$\xi = \frac{\frac{x}{x'} - 1}{\frac{\sqrt{t}}{x'}}, \quad \eta = \frac{y - y'}{\frac{\sqrt{t}}{x'}}$$

so the coordinate $\tau = \frac{\sqrt{t}}{x'}$ should also be obtained in the process. In summary, we need to end up with

$$\xi = \frac{s - 1}{\tau}, \quad \eta = \frac{y - y'}{\tau}, \quad x', \quad y', \quad \tau = \frac{\sqrt{t}}{x'}$$

This is reached by first blowing up the "corner" $\{x = 0,\ x' = 0,\ \sqrt{t} = 0\} = \{0\} \times (\partial X)^2$, which introduces s and τ, and then blowing up our beloved $\{s = 1,\ y = y',\ \tau = 0\} \mathrel{\widehat{=}} \{0\} \times \mathrm{diag}_X$, the lift of the time 0 diagonal. About the argument of the exponential:

$$\frac{1}{4}\left(y'\xi + \eta + \xi\eta\tau\right)^2$$

and the prefactor $t^{-\frac{n}{2}}$ is now $(\tau x')^{-n}$.

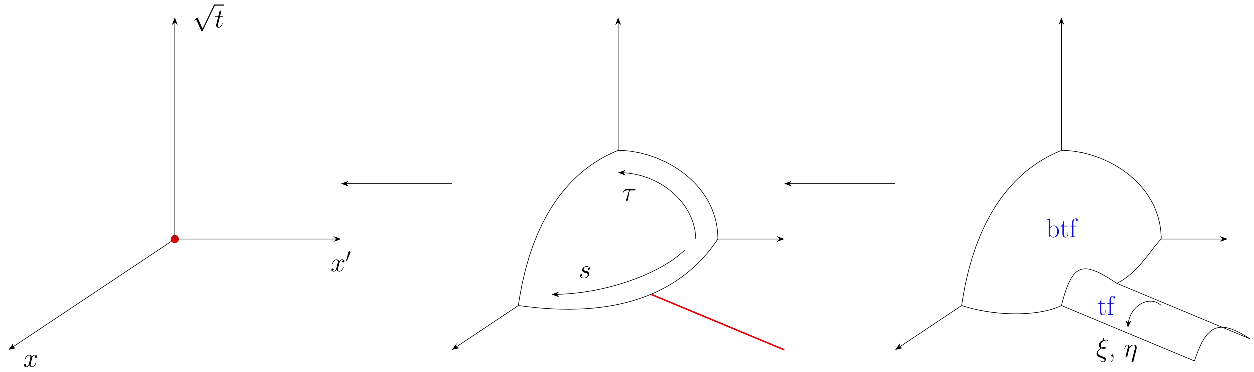


Figure 12: Blow-up sequence that gives rise to the heat blow-up space for manifolds with conical ends. In blue, the name of the boundary hypersurfaces intersecting the diagonal: the "time face" tf and the "boundary-time face" btf, lying at $\sqrt{t} = x = x' = 0$. We again have the 3 other boundary hypersurfaces lb, rb and tb.

□

To unearth a pattern, compare side-by-side the form of the metric and the coordinates on the always present (last front face) tf of the heat blow-up space (we leave out u', x' and y'):

- Closed case

$$g = du^2 \quad \longleftrightarrow \quad \frac{u - u'}{\sqrt{t}}, \quad \sqrt{t}$$

[64]We could also think that natural coordinates for this problem are

$$\frac{x - x'}{\sqrt{t}}, \quad \frac{y - y'}{\sqrt{t}}, \quad x', \quad y', \quad \sqrt{t}$$

but these are not feasible from a blow-up sequence, since they would need to come as a result of the blow-up of $\{0\} \times \mathrm{diag}_X \subset \mathbb{R}_+ \times X^2$, which is not a p-submanifold.

- b-case

$$g = \left(\frac{dx}{x}\right)^2 + dy^2 \quad \longleftrightarrow \quad \frac{x - x'}{x'\sqrt{t}}, \quad \frac{y - y'}{\sqrt{t}}, \quad \sqrt{t}$$

- ac-case

$$g = \left(\frac{dx}{x^2}\right)^2 + \left(\frac{dy}{x}\right)^2 \quad \longleftrightarrow \quad \frac{x - x'}{(x')^2\sqrt{t}}, \quad \frac{y - y'}{x'\sqrt{t}}, \quad \sqrt{t}$$

- Cone case

$$g = dx^2 + (xdy)^2 \quad \longleftrightarrow \quad \frac{x - x'}{\sqrt{t}}, \quad \frac{x'(y - y')}{\sqrt{t}}, \quad \tau = \frac{\sqrt{t}}{x'}$$

Notice how in all cases the coordinates correspond to the square roots of each of the summands appearing in (think e.g. $p = (x, y)$)

$$\frac{|p - p'|^2_{g(p')}}{t}, \quad \text{where} \quad |p - p'|^2_{g(p')} = \sum_{ij} g_{ij}(p')(p_i - p'_i)(p_j - p'_j), \quad g = g_{ij}(p')dx_i dx_j$$

which is nothing but the argument of the exponential in the Euclidean setting (times 4). This is not surprising, since an interior point of a compact manifold has a heat kernel that behaves "in a manner independent of the manifold having boundary or not" (since the whole construction is local), and the last blow-up (tf) is carried out to resolve the behavior when $|p - p'|_g \to 0$ and $t \to 0$ simultaneously, independent of whether p, p' are on the interior or not. Furthermore, we saw that the heat kernel in the closed case was well approximated by the Euclidean kernel related to a constant coefficient metric at the point p' (we called it x' there). Notice that when some of the metric coefficients g_{ij} contain a positive power $x^{2\alpha}$ of the boundary defining function, then in order to reach the desired coordinate by blow-up, we need to introduce the coordinate $\tau = \frac{\sqrt{t}}{(x')^\alpha}$ at some point along the process.
The heat blow-up space is the consequence of carrying out the minimal blow-up sequence to arrive at these coordinates. This principle also holds for the rest of the literature cited along this manuscript. Such disquisitions lead us to conjecture that such is a general principle behind the construction of heat blow-up spaces.

Remark 3.21. Notice these constructions do not take into consideration the behaviour of the heat kernel at large times, i.e. when $t \to \infty$. This is because the nature of large time asymptotics is global, while short time asymptotics (and parametrix methods) are local in essence.
To understand the heat kernel asymptotics in that regime, the literature usually opts to study the low energy limit of the resolvent of Dirac- [Vai01, §3] or Laplace-type [Mel93, §7.7-7.8] [GH08] [She13, §4] [GTV22] operators, and exploit a suitable (Borel) functional calculus correspondence between resolvent and heat kernel. This part of the heat kernel method is where self-adjointness of the operator is crucial. At the heart of the correspondence between the heat kernel and the resolvent lies Stone's formula for the spectral measure associated to a self-adjoint operator P:

$$dE_P(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left((P - (\lambda + i\varepsilon))^{-1} - (P - (\lambda - i\varepsilon))^{-1} \right) d\lambda$$

If the resolvent of the Dirac operator $\eth$ is known, the heat kernel of its square $e^{-t\eth^2}$ can be computed via

$$e^{-t\eth^2} = \int_{\mathbb{R}} e^{-t\lambda^2} dE_{\eth}(\lambda)$$

as done in [Vai01, p. 62]. The subjacent idea comes from complex analytical integration along contours in $\mathbb{C}$, where the parameter of the resolvent a priori lives (here, because of self-adjointness,

we know it to lie on $\mathbb{R}$). The formula above thus corresponds to integration along a contour surrounding the real line, hence the spectrum.
In the case of the Laplacian we have more information: the spectrum lies on the positive half of the real axis and thus the contour of integration can be adapted to circumvent this ray, see Figure 13. The computation to make is:

$$e^{-t\Delta} = \int_\Gamma e^{-t\lambda}(\Delta - \lambda)^{-1} d\lambda$$

for each fixed $t > 0$.
One option is to choose the contour as in [She13, Figure 4] and [LV25, Figure 5], where it lies outside of the ball of radius $\frac{1}{t}$ so that the exponential $e^{-t\lambda}$ is bounded. Another useful formula with the same contour is [LV25, (2.3)]. To deduce polyhomogeneity from this contour integral, we need to work in the space where the resolvent is polyhomogeneous when $\lambda \to 0$ along rays different from $\mathbb{R}_+ \subset \mathbb{C}$, where the spectrum lies. This is done for asymptotically conical manifolds in [She13, §2] and for fibred cusp metrics in [LV25, §2].
There are however many choices of contours that would also avoid the spectrum and might still ensure control over the integral terms, like the ones mentioned in [Mel93, Figure 14]. In particular, one could work like in the Dirac case and write the integral with respect to the spectral measure. This might require sharper knowledge of the resolvent asymptotics but at the same time provides finer information about the heat kernel because it detects cancellations in the asymptotics coming from the argument of the limit appearing in the definition of the spectral measure [GHS10, after Theorem 1.1] [She13, after Figure 6].

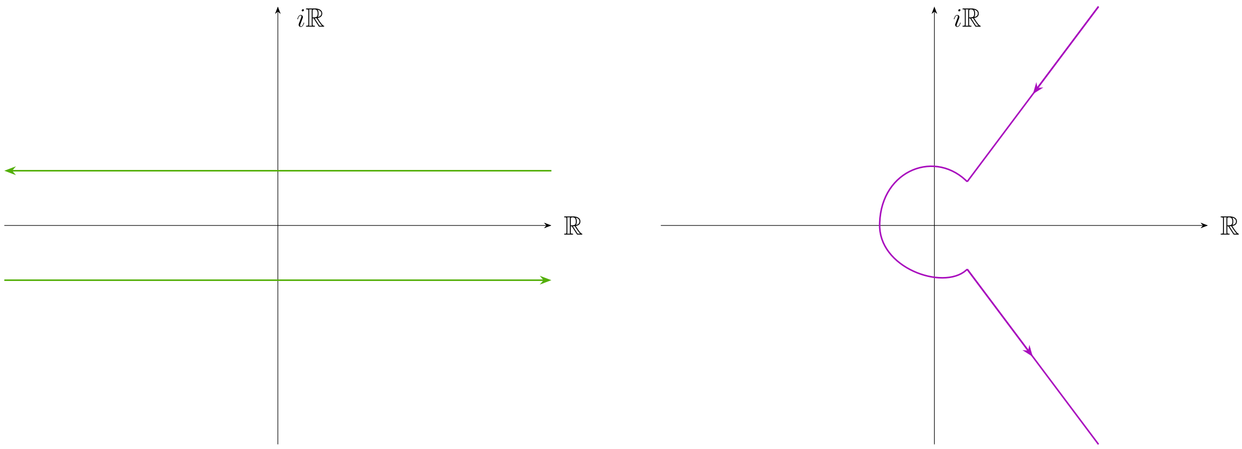


Figure 13: Integration contours for functional calculus integrals relating the heat kernel with the resolvent of the Dirac operator (green) and Laplacian (magenta). [Vai01, p. 62] analyses what happens when the contour on the left figure approaches the real axis from both sides, thus integrating against the spectral measure. We choose the circular sector in the second figure to have radius $\frac{1}{t}$ so that the integral along it has a bounded contribution from the exponential [She13, §2] [LV25, Theorem 2.5].

□
Based on that observation, the main claim of this section is:

Conjecture 3.22. Let X be a compact manifold with boundary and metric in a collar neighborhood of the boundary of the form (13). Let $h = x^{-2\alpha_0} g_{klm}$ be the reference metric in its conformal class (Definition 4.11).
Assuming $k \geq l \geq m$ and $k \geq 0$, the heat kernel of an elliptic, positive differential operator $P \in x^{-\alpha_0 r}\mathrm{Diff}^r_h(X)$ lifts to a polyhomogeneous distribution in the heat blow-up space HX^2_{klm}, which

is depicted in all cases in the following Figures. This is constructed with the minimal amount of blow-ups so that the last front face corresponding to the blow-up of the lift of $\{0\}\times \mathrm{diag}_X \subset \mathbb{R}_+\times X^2$ has local coordinates:

$$\xi = \frac{x-x'}{(x')^k t^{1/r}}, \quad \eta_i = \frac{y_i - y_i'}{(x')^l t^{1/r}}, \quad \mu_j = \frac{z_j - z_j'}{(x')^m t^{1/r}}, \quad x', \quad y_i', \quad z_j', \quad \tau = \begin{cases} t^{1/r} & m \geq 0 \\ t^{1/r}(x')^m & m \leq 0 \end{cases}$$

(this front face is the half-cylinder protruding in the bottom right of all figures).
As pointed out by Albin, in case $r = 2$[65], this is equivalent to saying that we start with $\mathbb{R}_+ \times X^2$ and perform the least amount of blow-ups that ensure $\frac{|p-p'|_g^2}{t}$ is polyhomogeneous. Notice that if $p = (x,y,z)$ and $p' = (x',y',z')$, and we approximate g by $g(x')$, then

$$\frac{|p-p'|_g^2}{t} \sim \left(\frac{x-x'}{(x')^k\sqrt{t}}\right)^2 + \left(\frac{y-y'}{(x')^l\sqrt{t}}\right)^2 + \left(\frac{z-z'}{(x')^m\sqrt{t}}\right)^2 = \xi^2 + \eta^2 + \mu^2$$

i.e. in each slice $\{x'\}\times \partial X \subset \mathring{\mathcal{U}}$, we make the argument of the exponential in the Euclidean heat kernel with respect to the metric $g(x')$ polyhomogeneous.
This can be invariantly formulated in a similar way as double spaces for the small calculi are defined, as noted by D. Grieser. Consider the fibration

$$\mathrm{tf} \to \{0\}\times \mathrm{diag}_X =: \mathrm{diag}_{X,0} \overset{d}{\cong} X, \quad (\xi,\eta,\mu,x',y',z',\tau) \mapsto (x',y',z')$$

Then there is a short exact sequence of vector bundles:

$$0 \longrightarrow {}^b N\mathrm{tf} \longrightarrow {}^b THX^2_{klm}\Big|_{\mathrm{tf}} \overset{\pi}{\longrightarrow} T\mathrm{tf} \longrightarrow 0$$

Denoting $\beta : HX^2_{klm} \longrightarrow \mathbb{R}_+ \times X^2$ the blow-down map, the condition is:

$$\beta^*\left(\left(t^{1/r}\, d^*\left({}^{klm}TX\right)\right) \oplus {}^b N \underbrace{\left(\{0\}\times X^2\right)}_{\subset \mathbb{R}_+\times X^2}\right) \quad \text{spans} \quad \pi^*\left(T\mathrm{tf}/X\right)$$

In short: the vector bundle spanned by

$$t^{1/r}x^k\partial_x, \quad t^{1/r}x^l\partial_y, \quad t^{1/r}x^m\partial_z, \quad t\partial_t$$

over $\mathrm{diag}_{X,0}$ lifts to smooth vector fields

$$\left((x')^{k-1}t^{1/r}\xi + 1\right)^k \partial_\xi, \quad \left((x')^{k-1}t^{1/r}\xi + 1\right)^l \partial_\eta,$$
$$\left((x')^{k-1}t^{1/r}\xi + 1\right)^m \partial_\mu, \quad \frac{1}{r}\left(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta - \mu\partial_\mu\right)$$

in case $k \geq 1$ (note that $\xi\partial_\xi + \eta\partial_\eta + \mu\partial_\mu$ is the radial vector field on the fibres tf/X), or

$$\partial_\xi, \quad \frac{\left(t^{1/r}\xi + x'\right)^l}{(x')^l}\partial_\eta, \quad \frac{\left(t^{1/r}\xi + x'\right)^m}{(x')^m}\partial_\mu, \quad \frac{1}{r}\left(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta - \mu\partial_\mu\right)$$

in case $k = 0$, which are tangent to the fibres of $\mathrm{tf} \to \mathrm{diag}_{X,0}$ (just take $t^{1/r} = 0$) and span the vector bundle locally generated by

$$\partial_\xi, \quad \partial_\eta, \quad \partial_\mu, \quad \tau\partial_\tau$$

In other words, $\mathring{\mathrm{tf}} \cong t^{1/r}\left({}^{klm}TX\right)$ and tf identifies with its radial compactification.
In particular, for the corresponding reference metric in each case, $HX_h^2 = [\mathbb{R}_+ \times X_h^2; \{0\}\times \mathrm{diag}_X]$.

[65]For general even r, the same can be said about polyhomogeneity of $\frac{|p-p'|^r}{t}$.

□

Observe that the conditions on (k,l,m) exclude only the Grushin plane from the examples appearing in Section 4.1. The question of how to treat $k \leq -1$ and whether this case carries any practical relevance remains unknown to the author.

The invariant formulation hints at the reasoning behind this construction: the heat equation for the operator P (locally $\sum_{q+|\alpha|+|\gamma|\leq r} p_{q,\alpha,\gamma}(x,y,z)\left(x^k D_x\right)^q \left(x^l D_y\right)^\alpha (x^m D_z)^\gamma$)

$$\begin{cases} t\left(\partial_t + P\right) h(t,x,x',y,y',z,z') = 0, & t>0 \\ \lim_{t\to 0} h(t,x,x',y,y',z,z') = \delta(x-x')\delta(y-y')\delta(z-z') \end{cases}$$

lifts to the face tf of HX^2_{klm} as the equation

$$\left(\frac{1}{r}\left(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta - \mu\partial_\mu\right) + i^r \sum_{q+|\alpha|+|\gamma|=r} p_{q,\alpha,\gamma}(x',y',z')\partial_\xi^q\partial_\eta^\alpha\partial_\mu^\gamma + O\left(t^{1/r}\right)\right) h = 0$$

with initial condition analogous to (10).

In particular, if P is a Laplace-type operator ($r=2$), we get to leading order in t[66]:

$$\left(\frac{1}{2}\left(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta - \mu\partial_\mu\right) - \Delta_{\xi,\eta,\mu,x',y',z'}\right) N_{\mathrm{tf}}(h) = 0$$

where $\Delta_{\xi,\eta,\mu,x',y',z'}$ is the corresponding Euclidean Laplace-type operator (with only second order part) in the coordinates

$$(\xi,\eta,\mu) \in \mathbb{R}\times T_{y'}Y \times T_{(y',z')}\partial X/Y \cong \mathbb{R}\times\mathbb{R}^b\times\mathbb{R}^f$$

with respect to the metric $g_Y(y')$ in $T_{y'}Y$ and $g_{\partial X/Y}(y',z')$ in $T_{(y',z')}\partial X/Y$, and away from ff the corresponding Euclidean Laplace-type operator on $T_{(x',y',z')}X$ with metric $g_{klm}(x',y',z')$.

A similar equation is solved e.g. in [Mel93, Lemma 7.16] or [TV22, §6.2-6.3] by the very same iterative argument in powers of $t^{1/2}$ and Volterra series we sketched in the start of this section (or to first order and then through a composition theorem as in [Gri04]). The leading order contribution is given by the Euclidean heat kernel:

$$N_{\mathrm{tf}}(h) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{\xi^2}{4}} e^{-\frac{1}{4}|\eta|^2_{g_Y(y')}} e^{-\frac{1}{4}|\mu|^2_{g_{\partial X/Y}(y',z')}}$$

In the front face obtained through the second-to-last blow-up, which we will call ff for now, we can also describe the corresponding lifts by distinguishing two cases:

- If $m \geq 1$, the coordinates are:

$$S = \frac{x-x'}{(x')^k}, \quad u = \frac{y-y'}{(x')^l}, \quad w = \frac{z-z'}{(x')^m}, \quad x', \quad y', \quad z', \quad \tau = t^{1/r} \tag{11}$$

  with lifts of vector fields given by

$$\beta^* x^k\partial_x = \left((x')^{k-1}S+1\right)^k \partial_S, \quad \beta^* x^l\partial_y = \left((x')^{k-1}S+1\right)^l \partial_u$$

[66]Similarly, if we just know the operator to have positive symbol, we can always find local coordinates at the point that make the $p_{q,\alpha,\gamma}(x',y',z')$ equal to those of a Laplace-type operator (cf. [Mel93, (7.55)]). This is why Conjecture 3.22 is formulated for general positive operators.

$$\beta^* x^m \partial_z = \left((x')^{k-1}S+1\right)^m \partial_w, \quad \beta^* t\partial_t = \frac{1}{r}\tau\partial_\tau$$

Thus, the heat equation lifts to

$$\left(\frac{1}{r}\tau\partial_\tau + (i\tau)^r \sum_{q+|\alpha|+|\gamma|\leq r} p_{q,\alpha,\gamma}\left(\left((x')^{k-1}S+1\right)x', (x')^l u + y', (x')^m w + z'\right)\cdot\right.$$
$$\left.\cdot\left((x')^{k-1}S+1\right)^{kq+l|\alpha|+m|\gamma|}\partial_S^q\partial_u^\alpha\partial_w^\gamma\right)h$$
$$= \left(\frac{1}{r}\tau\partial_\tau + (i\tau)^r \sum_{q+|\alpha|+|\gamma|\leq r} p_{q,\alpha,\gamma}(0,y',z')\partial_S^q\partial_u^\alpha\partial_w^\gamma + O(x')\right)h = 0$$

Solving this corresponds to solving

$$\left(\partial_t + i^r \sum_{q+|\alpha|+|\gamma|\leq r} p_{q,\alpha,\gamma}(0,y',z')\partial_S^q\partial_u^\alpha\partial_w^\gamma + O(x')\right)h = 0$$

by changing back to the coordinate t[67]. When solved iteratively in powers of x' (or another choice of boundary defining function for ff), this corresponds to an "usual" heat equation on the Euclidean fibres $\mathbb{R}_S \times \mathbb{R}_u^b \times \mathbb{R}_w^f \cong \mathbb{R} \times T_{y'}Y \times T_{(y',z')}\partial X/Y$ of the induced fibration ff $\to \partial X$ given by $(S,u,w,y',z',\tau) \mapsto (y',z')$, again with respect to the metric $g_Y(y')$ in $T_{y'}Y$ and $g_{\partial X/Y}(y',z')$ in $T_{(y',z')}\partial X/Y$.

For a Laplace-type operator whose lower order terms vanish as $x \to 0$, i.e. $p_{q,\alpha,\gamma} = O(x)$ except for $|(q,\alpha,\gamma)| = 2$, we obtain the model equation:

$$\left(\partial_t - \Delta_{S,u,w,y',z'}\right) N_{\mathrm{ff}}(h) = 0$$

and initial condition (note that $\tau(\xi,\eta,\mu) = (S,u,w)$; compare e.g. [Vai01, p. 76]):

$$N_{\mathrm{ff}}(h)\big|_{\mathrm{tf}\cap\mathrm{ff}} = N_{\mathrm{tf}}(h)\big|_{\mathrm{tf}\cap\mathrm{ff}}$$

whose solution in powers of x' has leading term given by the Euclidean kernel:

$$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi\tau^2)^{\frac{n}{2}}} e^{-\frac{S^2}{4\tau^2}} e^{-\frac{1}{4\tau^2}|u|^2_{g_Y(y')}} e^{-\frac{1}{4\tau^2}|w|^2_{g_{\partial X/Y}(y',z')}}$$

- When $m \leq 0$, we get:

$$S = \frac{x-x'}{(x')^{k-m}}, \quad u = \frac{y-y'}{(x')^{l-m}}, \quad z, \quad x', \quad y', \quad z', \quad \tau = t^{1/r}(x')^m \tag{12}$$

$$\beta^* x^k\partial_x = \left((x')^{k-m-1}S+1\right)^k (x')^m\partial_S, \quad \beta^* x^l\partial_y = \left((x')^{k-m-1}S+1\right)^l (x')^m\partial_u$$

$$\beta^* x^m\partial_z = \left((x')^{k-m-1}S+1\right)^m (x')^m\partial_z, \quad \beta^* t\partial_t = \frac{1}{r}\tau\partial_\tau$$

[67]Since for $m \geq 1$ the face ff does not lie at the lift of $t = 0$, so multiplying by t in this case does not deliver a problem tangential to the face, thus it is not needed or helpful.

(that is, we only localize in the base and normal directions) and the equation

$$\left(\frac{1}{r}\tau\partial_\tau + (i\tau)^r (x')^{-rm} \sum_{q+|\alpha|+|\gamma|\le r} p_{q,\alpha,\gamma}\left(\left((x')^{k-m-1}S+1\right)x', (x')^{l-m}u + y', z\right) \cdot \right.$$
$$\left. \cdot \left((x')^{k-m-1}S+1\right)^{qk+l|\alpha|+m|\gamma|} (x')^{m(q+|\alpha|+|\gamma|)}\partial_S^q\partial_u^\alpha\partial_z^\gamma\right)h = 0$$

and by introducing $T = \tau^r$ this corresponds to the equation

$$\left(\partial_T + i^r \sum_{q+|\alpha|+|\gamma|=r} p_{q,\alpha,\gamma}(0,y',z)\partial_S^q\partial_u^\alpha\partial_z^\gamma\right) N_{\mathrm{ff}}(h) = 0$$

if $m \le -1$, and

$$\left(\partial_T + i^r \sum_{q+|\alpha|+|\gamma|\le r} p_{q,\alpha,\gamma}(0,y',z)\partial_S^q\partial_u^\alpha\partial_z^\gamma\right) N_{\mathrm{ff}}(h) = 0$$

if $m = 0$. The initial condition again reads:

$$N_{\mathrm{ff}}(h)\big|_{\mathrm{tf}\cap\mathrm{ff}} = N_{\mathrm{tf}}(h)\big|_{\mathrm{tf}\cap\mathrm{ff}}$$

In the case $m \le -1$ or if the lower order terms vanish as $x \to 0$, once again only the leading order part participates in the construction of the normal operator and we obtain $\left(\partial_T - \Delta_{S,u,y'} - \sigma_2\left(\Delta_{\partial X/Y,y'}\right)\right) h = 0$ (in the notation of [Mel93, Lemma 7.16] for σ), with solution (cf. [Vai01, Theorem 4.11 (b)])

$$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi T)^{\frac{b+1}{2}}} e^{-\frac{S^2}{4T}} e^{-\frac{1}{4T}|u|^2_{g_Y(y')}} e^{-T\sigma_2\left(\Delta_{\partial X/Y,y'}\right)}$$

where the last factor is the heat kernel of the vertical second order operator $\sigma_2\left(\Delta_{\partial X/Y,y'}\right)$ which acts on the fibre $\phi^{-1}(y')$.

For $m = 0$ we can allow non vanishing lower order terms purely in the vertical directions, which lead us to the operator $\Delta_{S,u,y'} + \Delta_{\partial X/Y,y'}$ as in [TV22, (6.2)][68] [Alb07, (5.7)] and still produce quite a similar solution:

$$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi T)^{\frac{b+1}{2}}} e^{-\frac{S^2}{4T}} e^{-\frac{1}{4T}|u|^2_{g_Y(y')}} e^{-T\Delta_{\partial X/Y,y'}}$$

In particular, within $m \le 0$, in case the vertical family $\{\Delta_{\partial X/Y,y'}\}_{y'\in Y}$ has trivial kernel, the heat kernel at ff vanishes exponentially as $T \to \infty$, i.e. as we approach the intersection of ff with the next blown-up face which intersects the lifted diagonal. In that case, the heat kernel would vanish to infinite order at that third face assuming it does not lie in the $t \to \infty$ regime[69] (see e.g. [Liu25, p. 21]).

[68] This holds not only for the Hodge Laplacian, but also for the square of the spin-Dirac operator, see Footnote 108.

[69] In particular, that face would not contribute to the local index formula. This hints at the role of the vertical family $\Delta_{\partial X/Y}$ in determining Fredholmness of the associated operator (cf. full ellipticity for ϕ-operators [MM98] [Vai01] [GTV22] [TV22] and (geometric) Witt condition/assumption in incomplete settings [AGR16] [AGR23] [Liu25]) and how that relates to the large time asymptotics of the supertrace contributing just a Fredholm index term.

Example 3.23. Now we carry out the blow-up sequences that give rise to each of the heat blow-up spaces mentioned in the statement, thus constructively showing that there is a minimal way to obtain the desired coordinates at tf. We will try to avoid repeating constructions already present in the literature and cite them instead.
Bear in mind that $k \geq l \geq m$ and $k \geq 0$. Let us keep track of which cases we have treated using capital letters and a discrete subset in $\mathbb{Z}^2$ for the allowed points (l, m) for each fixed $k \geq 0$ (Figure 14).
Let us first think about $l = m$ (so think about the boundary as a whole and suppress the fibre in the notation). If $k = 0$, then $l = 0$ is the closed case of Figure 6 and $l = -1$ corresponds to the cone case of Figure 12. If $l \leq -2$ this is an incomplete cuspidal singularity and [Liu25, §4] with trivial base of the boundary fibre bundle covers it (this does not affect his picture/procedure). If $k = 1$ and $l = 0$, we are in the b-case of Figure 10. If $k \geq 1$ and $k \geq l+2$, we distinguish two cases:

- (A) If $l > 0$, first blow up $\mathbb{R}_+ \times (\partial X)^2$ to obtain the coordinate $s = \frac{x}{x'}$. Then, perform a quasi-homogeneous blow-up of order $k - l - 1$ of $\{s = 1,\ x' = 0\}$, producing $\hat{s} = \frac{x - x'}{(x')^{k-l}}$. Then, one of order l of the lifted diagonal $\{\hat{s} = 0, x' = 0, y - y' = 0\}$, getting $S = \frac{x-x'}{(x')^k}$ and $u = \frac{y-y'}{(x')^l}$. Finally, blow $\{0\} \times \mathrm{diag}_X$ up.

  If $l = 0$, we are in the cusp calculus and the sequence is the same as before without the order l blow-up.

- (B) Otherwise, if $l \leq -1$, start by blowing up $\mathbb{R}_+ \times (\partial X)^2$ producing $s = \frac{x}{x'}$, and then $\{s = 1,\ x' = 0\}$ to order $k - 1$ so as to obtain $\hat{s} = \frac{x-x'}{(x')^k}$. After that, an order $-l$ blow-up of $\{\hat{s} = 0,\ x' = 0, ,\ \sqrt{t} = 0\}$ to obtain $S = \frac{x-x'}{(x')^{k-l}}$ and $\tau = \frac{\sqrt{t}}{(x')^{-l}}$. Finally, $\{0\} \times \mathrm{diag}_X$ gives us:

  $$\xi = \frac{S}{\tau} = \frac{x - x'}{(x')^k \sqrt{t}}, \quad \eta = \frac{y - y'}{\tau} = \frac{y - y'}{(x')^l \sqrt{t}}, \quad x', \quad y', \quad \tau = \sqrt{t}(x')^l$$

  If $k = 1$, as is the case for isolated hyperbolic cusp singularities, we just need to skip the order $k - 1$ blow-up, i.e. we are in [Vai01, §4] for trivial base of the boundary fibre bundle $Y = \{*\}$.

(C) For $k = l + 1 \geq 2$ (if $k = 2$ these are scattering metrics [She13, Theorem 3]), the procedure of the first point above without the order $k - l - 1$ blow-up would be the correct approach.
(D) For the case $k = l = m > 0$ ($k = 1$ corresponds to 0-metrics exemplified by the upper half-plane model of hyperbolic space), the sequence can be schematically written as:

1. Blow up $\mathbb{R}_+ \times \mathrm{diag}_{\partial X} = \{x = x' = 0,\ y = y'\}$

$$s = \frac{x}{x'}, \quad Y = \frac{y - y'}{x'}, \quad x', \quad y', \quad \sqrt{t}$$

2. If $k > 1$, blow up $\{s = 1,\ x' = 0,\ Y = 0\}$ to order $k - 1$ (else skip)

$$S = \frac{x - x'}{(x')^k}, \quad u = \frac{y - y'}{(x')^k}, \quad x', \quad y', \quad \sqrt{t}$$

3. Blow up $\{0\} \times \mathrm{diag}_X$

$$\xi = \frac{x - x'}{(x')^k \sqrt{t}}, \quad \eta = \frac{y - y'}{(x')^k \sqrt{t}}, \quad x', \quad y', \quad \sqrt{t}$$

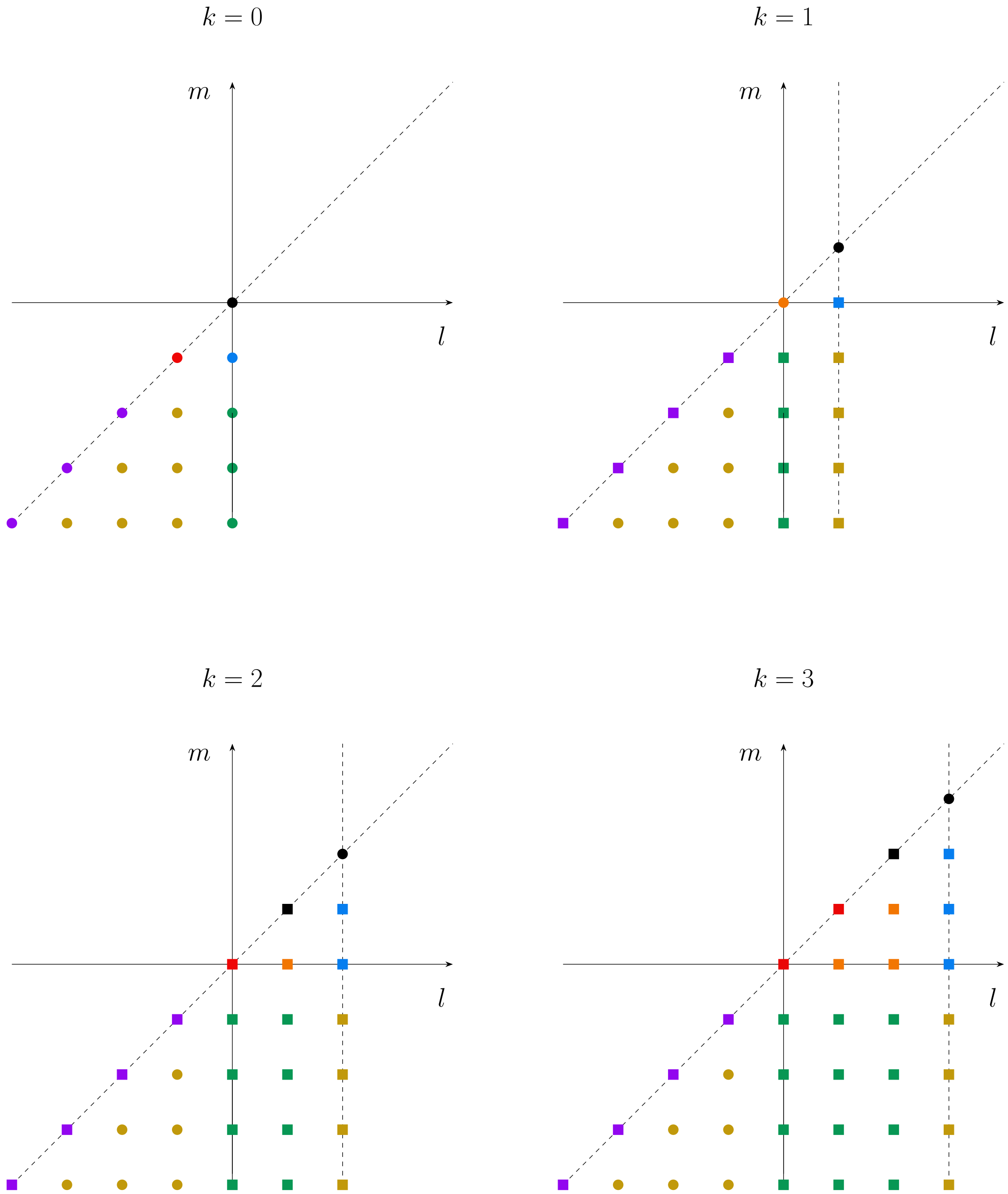


Figure 14: Represented are the discrete subsets of points $(l, m) \in \mathbb{Z}$ for each $k \in \{0, 1, 2, 3\}$ that satisfy the assumptions of Conjecture 3.22, i.e. $k \geq l \geq m$. The discrete subset slices for $k \geq 4$ are analogous to e.g. $k = 2$ and $k = 3$. The diagonal dashed line delimits $l = m$ and the vertical one, $k = l$. The color legend is the following: red circle - cone, purple circles - incomplete cusp, blue circle - wedge, green circles - incomplete cusp edge, black circles - closed ($k = 0$) and (D) ($k \geq 1$), gold circles - (J) ($k = 0$) and (I) ($k \geq 1$), orange circle - cylindrical ends, red squares - (A) , purple squares - (B) , black squares - (C) , blue squares - (E) , orange squares - (F) , gold squares - (G) , green squares - (H) .

If one understands those, the solution for $l > m$ is very much in reach and (also) splits into three families. Each member of the family is embodied in one of the three heat blow-up spaces for fibred cusp metrics ($c \in \{0, 1, 2\}$ in [GSHV25, Figures 4-6]). Let us again write the blow-up sequences and the coordinates obtained in each step algorithmically:

- If $m \geq 0$, the blow-up sequence "does not see" the time direction until the very last blow-up, and thus can be obtained from the corresponding double space used for defining the calculus of pseudodifferential operators adapted to the metric. This is in particular the case for reference metrics.

  (E) If $k = l$:

  1. Blow up $\{x = x' = 0,\ y = y'\}$

  $$s = \frac{x}{x'}, \quad Y = \frac{y - y'}{x'}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  2. If $k > m + 1$, blow up $\{s = 1,\ x' = 0,\ Y = 0\}$ to order $k - m - 1$ (else skip this step)

  $$\hat{s} = \frac{x - x'}{(x')^{k-m}}, \quad Y = \frac{y - y'}{(x')^{k-m}}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  3. If $m > 0$, blow up $\mathbb{R}_+ \times \mathrm{diag}_X = \{\hat{s} = 0,\ x' = 0,\ Y = 0,\ z = z'\}$ to order m (else skip)

  $$S = \frac{x - x'}{(x')^k}, \quad u = \frac{y - y'}{(x')^k}, \quad w = \frac{z - z'}{(x')^m}, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  With this we obtain $\mathbb{R}_+ \times X^2_{g_{klm}}$.

  4. Blow up $\{0\} \times \mathrm{diag}_X$

  $$\xi = \frac{x - x'}{(x')^k \sqrt{t}}, \quad \eta = \frac{y - y'}{(x')^k \sqrt{t}}, \quad \mu = \frac{z - z'}{(x')^m \sqrt{t}}, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  (F) If $k \geq l + 1$:

  1. Blow up $\{x = x' = 0\}$

  $$s = \frac{x}{x'}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  2. If $k \geq l + 2$, blow up $\{s = 1,\ x' = 0\}$ to order $k - l - 1$ (else skip):

  $$\hat{s} = \frac{x - x'}{(x')^{k-l}}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  3. Blow up $\{\hat{s} = 0,\ x' = 0,\ y = y'\}$ to order $l - m$

  $$\tilde{s} = \frac{x - x'}{(x')^{k-m}}, \quad Y = \frac{y - y'}{(x')^{l-m}}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  4. If $m > 0$, blow up $\{\tilde{s} = 0,\ x' = 0,\ Y = 0,\ z = z'\}$ to order m (else skip)

  $$S = \frac{x - x'}{(x')^k}, \quad u = \frac{y - y'}{(x')^l}, \quad w = \frac{z - z'}{(x')^m}, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  With this we obtain $\mathbb{R}_+ \times X^2_{g_{klm}}$

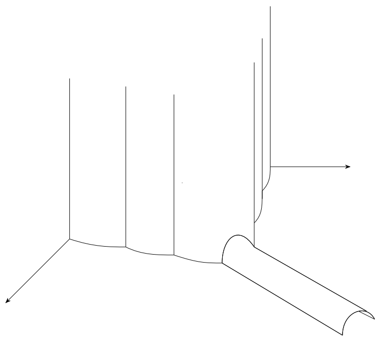


Figure 15: (F) Heat blow-up space for $k \geq l+2 \geq 3$ and $m = 0$ (if $k = l+1$ we have one less of the vertical blow-ups, like [GSHV25, Figure 4], and if $m > 0$ we have one more.

5. Blow up $\{0\} \times \text{diag}_X$

$$\xi = \frac{x - x'}{(x')^k \sqrt{t}}, \quad \eta = \frac{y - y'}{(x')^l \sqrt{t}}, \quad \mu = \frac{z - z'}{\sqrt{t}}, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

- The cases $l \geq 0 > m$ and $k \geq 1 > l+1 > m+1$ are a combination of the other two. They require blow-ups that do not involve time to start with and then blow-ups of faces in the $t = 0$ regime.

  If $k = l = 0$, $m = -1$ we are in the wedge case [AGR23] and $m \leq -2$ is the incomplete cusp edge case [Liu25].

  (G) If $k = l \geq 1$

  1. Blow up $\{x = x' = 0, \ y = y'\}$

  $$s = \frac{x}{x'}, \quad Y = \frac{y - y'}{x'}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  2. If $k \geq 1$, blow up $\{s = 1, \ x' = 0, \ Y = 0\}$ to order $k - 1$ (else skip this step)

  $$\hat{s} = \frac{x - x'}{(x')^k}, \quad \hat{Y} = \frac{y - y'}{(x')^k}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

  3. Blow up the corner $\{\hat{s} = 0, \ \hat{Y} = 0, \ x' = 0, \ \sqrt{t} = 0\}$ to order $-m$

  $$S = \frac{x - x'}{(x')^{k-m}}, \quad u = \frac{y - y'}{(x')^{k-m}}, \quad z, \quad x', \quad y', \quad z', \quad \tau = \frac{\sqrt{t}}{(x')^{-m}}$$

  4. Blow up $\{0\} \times \text{diag}_X$

  $$\xi = \frac{S}{\tau} = \frac{x - x'}{(x')^k \sqrt{t}}, \quad \eta = \frac{u}{\tau} = \frac{y - y'}{(x')^k \sqrt{t}}, \quad \mu = \frac{z - z'}{\tau} = \frac{(z - z')}{(x')^m \sqrt{t}}, \quad x', \quad y', \quad z', \quad \tau$$

  (H) If $k \geq l + 1 \geq 1$

  1. Blow up $\{x = x' = 0\}$

  $$s = \frac{x}{x'}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

2. If $k \geq l+2$, blow up $\{s=1,\ x'=0\}$ to order $k-l-1$ (else skip):

$$\hat{s} = \frac{x-x'}{(x')^{k-l}}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

3. If $l \geq 1$, blow up $\{\hat{s}=0,\ x'=0,\ y=y'\}$ to order l (else skip)

$$\tilde{s} = \frac{x-x'}{(x')^{k}}, \quad Y = \frac{y-y'}{(x')^{l}}, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

Until here we have done the same as for $k \geq l+1 > m = 0$; now we change it up a bit.

4. Blow up the corner $\{\tilde{s}=0,\ Y=0,\ x'=0,\ \sqrt{t}=0\}$ to order $-m$

$$S = \frac{x-x'}{(x')^{k-m}}, \quad u = \frac{y-y'}{(x')^{l-m}}, \quad z, \quad x', \quad y', \quad z', \quad \tau = \frac{\sqrt{t}}{(x')^{-m}}$$

5. Blow up $\{0\} \times \mathrm{diag}_X$

$$\xi = \frac{S}{\tau} = \frac{x-x'}{(x')^{k}\sqrt{t}}, \quad \eta = \frac{u}{\tau} = \frac{y-y'}{(x')^{l}\sqrt{t}}, \quad \mu = \frac{z-z'}{\tau} = \frac{(z-z')}{(x')^{m}\sqrt{t}}, \quad x', \quad y', \quad z', \quad \tau$$

Both skips happen when constructing the heat blow-up space for generalized hyperbolic cusp or d-metrics [Vai01, §4].

(I) If $k \geq 1$ and $0 > l > m$:

1. Blow up $\{x = x' = 0\}$

$$s = \frac{x}{x'}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

2. If $k > 1$, blow up $\{s=1,\ x'=0\}$ to order $k-1$ (else skip):

$$\hat{s} = \frac{x-x'}{(x')^{k}}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \sqrt{t}$$

3. Blow up $\{\hat{s}=0,\ x'=0,\ \sqrt{t}=0\}$ to order $-l$

$$\tilde{s} = \frac{x-x'}{(x')^{k-l}}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad T = \frac{\sqrt{t}}{(x')^{-l}}$$

4. Blow up $\{\tilde{s}=0,\ y=y',\ x'=0,\ T=0\}$ to order $l-m$

$$S = \frac{x-x'}{(x')^{k-m}}, \quad u = \frac{y-y'}{(x')^{l-m}}, \quad z, \quad x', \quad y', \quad z', \quad \tau = \frac{\sqrt{t}}{(x')^{-m}}$$

5. Blow up $\{0\} \times \mathrm{diag}_X$

$$\xi = \frac{S}{\tau} = \frac{x-x'}{(x')^{k}\sqrt{t}}, \quad \eta = \frac{u}{\tau} = \frac{y-y'}{(x')^{l}\sqrt{t}}, \quad \mu = \frac{z-z'}{\tau} = \frac{(z-z')}{(x')^{m}\sqrt{t}}, \quad x', \quad y', \quad z', \quad \tau$$

- (J) The other extreme case is $k = 0 > l > m$, where all blow-ups happen at $t = 0$. To this class belongs the incomplete fibred cusp model [GSHV25, Figure 6].

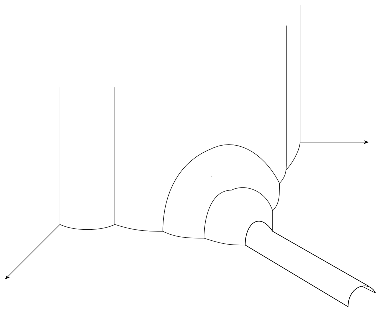


Figure 16: (I) Heat blow-up space for $k > 1$ and $m < l < 0$ (if $k = 1$ we just get one less of the vertical blow-ups)

1. Blow up the corner $\{x = x' = 0,\ \sqrt{t} = 0\}$

$$s = \frac{x}{x'}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad T = \frac{\sqrt{t}}{x'}$$

2. If $l < -1$, blow up $\{s = 1,\ x' = 0,\ T = 0\}$ to order $-l - 1$ (else skip):

$$\hat{s} = \frac{x - x'}{(x')^{-l}}, \quad y, \quad z, \quad x', \quad y', \quad z', \quad \hat{T} = \frac{\sqrt{t}}{(x')^{-l}}$$

3. Blow up $\{\hat{s} = 0,\ x' = 0,\ y = y',\ \hat{T} = 0\}$ to order $l - m$

$$S = \frac{x - x'}{(x')^{-m}}, \quad u = \frac{y - y'}{(x')^{l-m}}, \quad z, \quad x', \quad y', \quad z', \quad \tau = \frac{\sqrt{t}}{(x')^{-m}}$$

4. Blow up $\{0\} \times \mathrm{diag}_X$

$$\xi = \frac{S}{\tau} = \frac{x - x'}{\sqrt{t}}, \quad \eta = \frac{u}{\tau} = \frac{y - y'}{(x')^{l}\sqrt{t}}, \quad \mu = \frac{z - z'}{\tau} = \frac{(z - z')}{(x')^{m}\sqrt{t}}, \quad x', \quad y', \quad z', \quad \tau$$

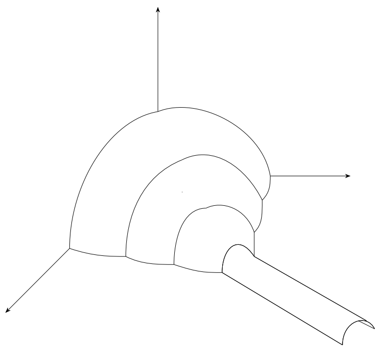


Figure 17: (J) Heat blow-up space for $k = 0$ and $-1 > l > m$ (if $l = -1$ we can do with one blow-up less at the "corner", like [GSHV25, Figure 6]

□

Remark 3.24. The conjecture can be straightforwardly extended to the case of iterated fibration structures[70], i.e. if the metric locally looks like (up to higher order terms):

$$g = \frac{dx^2}{x^{2k}} + \frac{dy_1^2}{x^{2l_1}} + \frac{dy_2^2}{x^{2l_2}} + \cdots + \frac{dy_q^2}{x^{2l_q}}$$

with $k \geq l_1 > l_2 > \cdots > l_q$ and $k \geq 0$ (negative l_i's are allowed), corresponding to a tower of fibre bundles[71] (that extends to a collar neighborhood of the boundary)

$$\partial X \longrightarrow Y_{q-1} \longrightarrow Y_{q-2} \longrightarrow \cdots \longrightarrow Y_1$$

Here, y_1 are (lifts of) coordinates in Y_1 and y_j (lifts of) coordinates in the fibres of $Y_j \xrightarrow{\phi_j} Y_{j-1}$ for $2 \leq j \leq q$ (denoting $Y_q = \partial X$). For the case $l_q \geq 0$, these are called MICE in [Alb07], but observe that we impose the extra restriction $l_j \geq l_{j+1}$.
The coordinates to be obtained at tf, the last front face of the blow-up, would be:

$$\xi = \frac{x - x'}{(x')^k\, t^{1/r}}, \quad \eta_j = \frac{y_j - y_j'}{(x')^{l_j}\, t^{1/r}}, \quad x', \quad y_j', \quad \tau = \begin{cases} t^{1/r} & l_q \geq 0 \\ t^{1/r}(x')^{l_q} & l_q \leq 0 \end{cases}$$

This can be reached by blow-up sequences analogous to the ones presented, where the second to last front face has coordinates

$$S = \frac{x - x'}{(x')^k}, \quad u_j = \frac{y_j - y_j'}{(x')^{l_j}}, \quad x', \quad y_j', \quad \tau = t^{1/r}$$

if $l_q \geq 0$, or

$$S = \frac{x - x'}{(x')^{k-l_q}}, \quad u_j = \frac{y_j - y_j'}{(x')^{l_j - l_q}}, \quad y_q, \quad x', \quad y_j', \quad y_q', \quad \tau = t^{1/r}(x')^{l_q}, \quad \text{for} \quad j \neq q$$

if $l_q \leq 0$. The computations of the lifts of the heat equation are also analogous, interchanging m by l_q in the distinctions made, as shown in the following examples.

□

Example 3.25. Let us outline the finite time heat kernel construction for elliptic second order differential operators of positive symbol with respect to an iterated fibration structure (suppose $q \geq 3$, which was not treated above), under some assumptions. We can once more restrict to Laplace-type operators by choosing appropriate coordinates at the point.
We can mimic the analysis in §4.1.2 and study differential operators as universal enveloping algebras of the corresponding Lie algebroids within reference metrics, i.e. operators take the local form:

$$P = \sum_{i+|\alpha_1|+\cdots+|\alpha_q| \leq r} p_{i,\alpha_1,\ldots,\alpha_q}(x, y_1, \ldots, y_q)(x^k D_x)^i (x^{l_1} D_{y_1})^{\alpha_1} \cdots (x^{l_q} D_{y_q})^{\alpha_q}$$

Pseudodifferential operators can also be understood by describing their kernels in the blow-up space associated to the pertinent reference metric.

[70]This term is sometimes used in the literature to refer to the same iterated constructions but in the context of manifolds with corners, where extra compatibility conditions at the intersection of hypersurfaces are usually required [ALMP12].

[71]More generally, we can consider metrics g_{Y_j} in each factor and impose that $(Y_j, g_{Y_j}) \xrightarrow{\phi_j} (Y_{j-1}, g_{Y_{j-1}})$ is a Riemannian submersion (cf. §4.1.2).

The model problem at the face tf is given by the equation:

$$\left(\frac{1}{2}\left(\tau\partial_\tau - \xi\partial_\xi - \sum_{j=1}^{q}\eta_j\partial_{\eta_j}\right) - \Delta_{\xi,\eta_1,\ldots,\eta_q,x',y'_1,\ldots,y'_q}\right)N_{\mathrm{tf}}(h) = 0$$

where $\Delta_{\xi,\eta_1,\ldots,\eta_q,x',y'_1,\ldots,y'_q}$ is the Euclidean Laplacian on

$$(\xi,\eta_1,\ldots,\eta_q) \in \mathbb{R}\times T_{y'_1}Y_1 \times T_{(y'_1,y'_2)}Y_2/Y_1 \times \cdots \times T_{(y'_1,\ldots,y'_q)}\partial X/Y_{q-1}$$

with respect to the metric $g_{Y_1}(y'_1)$ and the fibre metrics $g_{Y_j/Y_{j-1}}(y'_1,\ldots,y'_j)$ in each factor $TY_j/Y_{j-1} =: \ker(\phi_j)_*$ $(Y_q = \partial X)$. The solution follows the same patters explained before (in particular, the kernel has leading order $-n = -\dim X$ at tf) and satisfies:

$$N_{\mathrm{tf}}(h) = \frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{\xi^2}{4}}e^{-\frac{1}{4}|\eta_1|^2_{g_{Y_1}(y'_1)}}\cdots e^{-\frac{1}{4}|\eta_q|^2_{g_{\partial X/Y_{q-1}}(y'_1,\ldots,y'_q)}}$$

We can distinguish three scenarios, once again embodied by the three c-ϕ-metrics with $c\in\{0,1,2\}$:

- **$l_q \geq 0$ and $p_{i,\alpha_1,\ldots,\alpha_q} = O(x)$ for $|(q,\alpha_1,\ldots,\alpha_q)| \leq 1$**

  In this case, the heat blow-up space is constructed similarly to (F) (Figure 15). The model problem at the front face is:

  $$\left(\partial_t - \Delta_{S,u_1,\ldots,u_q,y'_1,\ldots,y'_q}\right)N_{\mathrm{ff}}(h) = 0$$

  where $\Delta_{S,u_1,\ldots,u_q,y'_1,\ldots,y'_q}$ is the Euclidean Laplace operator on $\mathbb{R}_S\times T_{y'_1}Y_1\times\cdots\times T_{(y'_1,\ldots,y'_q)}\partial X/Y_{q-1}$ (except if $l_q = 0$, for which we can allow lower order terms in the vertical directions $\partial X/Y_{q-1}$ not to vanish and obtain for this last step of the fibre bundle the family of operators on the fibres $\Delta_{\partial X/Y_{q-1},(y'_1,\ldots,y'_{q-1})}$). For $l_q > 0$, the solution has leading order contribution:

  $$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi\tau^2)^{\frac{n}{2}}}e^{-\frac{S^2}{4\tau^2}}e^{-\frac{1}{4\tau^2}|u_1|_{g_{Y_1}(y'_1)}}\cdots e^{-\frac{1}{4\tau^2}|u_q|_{g_{\partial X/Y_{q-1}}(y'_1,\ldots,y'_q)}}$$

  and for $l_q = 0$:

  $$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi\tau^2)^{\frac{n-f_q}{2}}}e^{-\frac{S^2}{4\tau^2}}e^{-\frac{1}{4\tau^2}|u_1|_{g_{Y_1}(y'_1)}}\cdots e^{-\frac{1}{4\tau^2}|u_{q-1}|_{g_{Y_{q-1}/Y_{q-2}}(y'_1,\ldots,y'_{q-1})}}e^{-\tau^2\Delta_{\partial X/Y_{q-1},(y'_1,\ldots,y'_{q-1})}}$$

  where $f_q = \dim\partial X/Y_{q-1}$.

- **$k > 0 > l_q$ and the vertical family $\{\Delta_{\partial X/Y_{q-1},(y'_1,\ldots,y'_{q-1})}\}_{(y'_1,\ldots,y'_{q-1})\in Y_{q-1}}$ has trivial kernel**

  This construction is similar to (I) and Figure 16. The number of new faces appearing during the blow-up process in the lift of $t = 0$ corresponds to $\#\{j : l_j < 0\} + 1$ (one is always tf). The model problem at ff is:

  $$\left(\frac{1}{2}\tau\partial_\tau - \tau^2\Delta_{S,u_1,\ldots,u_{q-1},y'_1,\ldots,y'_{q-1}} - \tau^2\sigma_2\left(\Delta_{\partial X/Y_{q-1},(y'_1,\ldots,y'_{q-1})}\right)\right)N_{\mathrm{ff}}(h) = 0$$

  with solution:

  $$N_{\mathrm{ff}}(h) = \frac{1}{(4\pi\tau^2)^{\frac{n-f_q}{2}}}e^{-\frac{S^2}{4\tau^2}}e^{-\frac{1}{4\tau^2}|u_1|_{g_{Y_1}(y'_1)}}\cdots e^{-\frac{1}{4\tau^2}|u_{q-1}|_{g_{Y_{q-1}/Y_{q-2}}(y'_1,\ldots,y'_{q-1})}}e^{-\tau^2\sigma_2\left(\Delta_{\partial X/Y_{q-1},(y'_1,\ldots,y'_{q-1})}\right)}$$

  The trivial vertical kernel condition ensures that this vanishes as $\tau\to\infty$ and thus the kernel vanishes to infinite order at all other faces.

- **$k = 0$ and the vertical family $\{\Delta_{\partial X/Y_{q-1},(y_1',\dots,y_{q-1}')}\}_{(y_1',\dots,y_{q-1}')\in Y_{q-1}}$ has trivial kernel**

  This resembles (J) and Figure 17. The model problem and its solution are the same as in the previous section.

Note that within those assumptions, the heat kernel vanishes to infinite order at all faces except tf and ff. In particular, this describes the Laplace-type heat kernel for incomplete fibred cusps (2-ϕ-metrics), when the vertical family has trivial kernel (equiv. $x^4\Delta$ is fully elliptic, i.e. Fredholm on ϕ-Sobolev spaces), although this was already clear from the discussion before Example 3.23.

□

Remark 3.26. In the study of "families" of heat kernels e.g. if we consider how the heat kernel behaves under an adiabatic limit with respect to the degeneration parameter $\varepsilon \to 0$ or other parameter-dependent settings (key words are "semiclassical/surgery calculi"), the heuristic described above for finding the heat kernel does not work due to the asymmetry resulting from the introduction of the parameter. This is the case in [ARS14, §6], [AQ20, Figures 2-5] and [ARS22, §5].

□

3.2.2 Heat calculus and composition

In order to establish a composition theorem for the $\Psi_H^\alpha(X)$ calculus, we need to make sure that in the triple space diagram π_L and π_R are b-maps and π_C is a b-fibration. Well, they are not as of yet, but we can blow up the factors correspondingly to ensure this. Focus first on the time directions and write $s = t - t'$ and $s' = t'$ for simplicity (cf. [Alb07, §4]):

$$\begin{array}{ccccc} (\mathbb{R}_+)_s & \xleftarrow{\pi_L} & (\mathbb{R}_+)_s \times (\mathbb{R}_+)_{s'} & \xrightarrow{\pi_R} & (\mathbb{R}_+)_{s'} \\ & & \downarrow{\scriptstyle \pi_C} & & \\ & & (\mathbb{R}_+)_{s+s'} & & \end{array}$$

Notice that π_L and π_R are projections, thus b-maps, but π_C is not a b-fibration (it is not even a b-map), because the corner of codimension 2 in $(\mathbb{R}_+)_s \times (\mathbb{R}_+)_{s'}$ is precisely mapped into the boundary of $(\mathbb{R}_+)_{s+s'}$. We can solve this by our well-known blow-up of the corner, and now the whole front face will map into the boundary in $(\mathbb{R}_+)_{s+s'}$. As a result, $\pi_C \circ \beta$ is a b-fibration and $\pi_L \circ \beta$ and $\pi_R \circ \beta$ are (still) b-maps (even b-fibrations).
If, as was the case in (E) and (F) , the metric g (with the general form of Remark 3.24) has a heat blow-up space of the form

$$HX_g^2 = [\mathbb{R}_+ \times X_g^2; \{0\} \times \mathrm{diag}_X]$$

then it is straightforward to construct the heat triple space that allows us to prove a composition theorem by applying the pushforward theorem, namely as summarized in the following proposition already present in the literature:

Proposition 3.27. [Alb07, after Theorem 4.2] For a metric of an iterated fibration structure with $l_q \geq 0$, the heat blow-up space has the simple form above and the composition laws for the

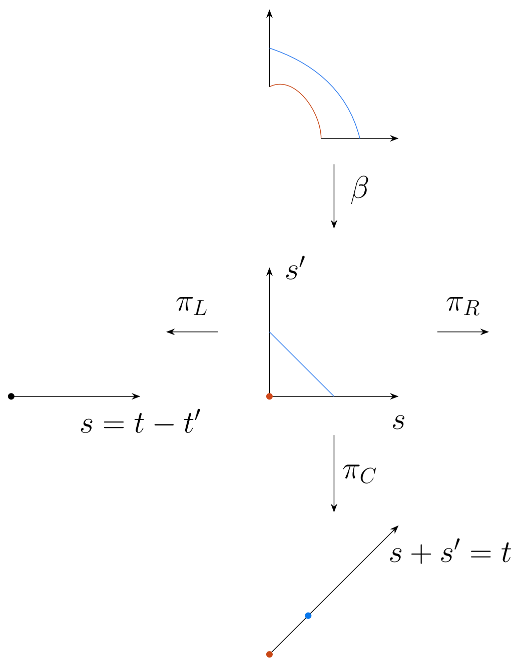
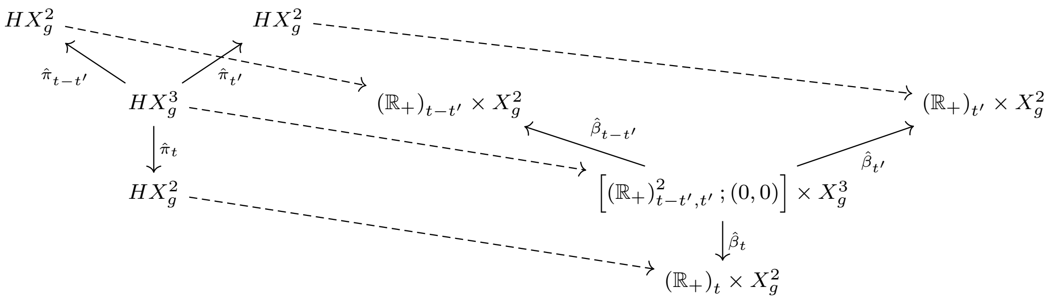


Figure 18: Blow-up analysis of the time convolution for a composition of heat operators via Melrose's push-forward theorem.

corresponding heat calculus can be derived from the triple space construction



where $\hat{\pi}_{t-t'}$ and $\hat{\pi}_{t'}$ are b-maps and $\hat{\pi}_t$ is a b-fibration, and

$$HX_g^3 = \left[\left[(\mathbb{R}_+)^2_{t-t',t'}\,;(0,0)\right] \times X_g^3\,; D_0^3\,;\ \hat{\beta}_{t-t'}^{-1}\left(\{0\} \times \mathrm{diag}_X\right), \hat{\beta}_{t'}^{-1}\left(\{0\} \times \mathrm{diag}_X\right), \hat{\beta}_t^{-1}\left(\{0\} \times \mathrm{diag}_X\right)\right]$$

where D_0^3 is the lift of $\{p = p' = p'',\ t-t' = t' = 0\}$ along $\left[(\mathbb{R}_+)^2_{t-t',t'}\,;(0,0)\right] \times X_g^3 \to (\mathbb{R}_+)^2_{t-t',t'} \times X^3$

□

The corresponding composition formula is written in [Alb07, Theorem 4.3].
The reason why this construction works is the following: denote for simplicity

$$W = \left[(\mathbb{R}_+)^2_{t-t',t'}\,;(0,0)\right] \times X_g^3$$
$$A = D_0^3, \quad B_1 = \hat{\beta}_{t-t'}^{-1}\left(\{0\} \times \mathrm{diag}_X\right), \quad B_2 = \hat{\beta}_{t'}^{-1}\left(\{0\} \times \mathrm{diag}_X\right), \quad B_3 = \hat{\beta}_t^{-1}\left(\{0\} \times \mathrm{diag}_X\right)$$

so that $HX_g^3 = [W; A; B_1, B_2, B_3]$.
First, $\{A, B_1, B_2, B_3\}$ is a clean family of p-submanifolds, i.e. we can find common coordinates to write all of them simultaneously as coordinate subspaces of W.
Second, using standard commutation rules for sequences of blow-ups in a clean family, we can deduce the following:

1. Since $A = B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3$ and $A \subsetneq B_i$, the order of blow-up of the B_i's after having blown up A is irrelevant.

2. Since $A \subset B_i$, we can commute them.

Thus, we have blow-downs and diffeomorphisms:

$$[W; A; B_1, B_2, B_3] \to [W; A, B_i] \cong [W; B_i, A] \to [W; B_i]$$

and this can be blown down into the double space corresponding to the chosen i using one of $\hat{\beta}_{t-t'}, \hat{\beta}_{t'}, \hat{\beta}_t$ and again [Alb07, (4.4)]:

$$[W; B_i] = \left[\left[(\mathbb{R}_+)^2_{t-t',t'}\,; (0,0)\right] \times X^3_g\;;\; \hat{\beta}_i^{-1}\left(\{0\} \times \mathrm{diag}_X\right)\right] \xrightarrow{\hat{\beta}_i} \left[\mathbb{R}_+ \times X^2_g; \{0\} \times \mathrm{diag}_X\right] = HX^2_g$$

Third, these blow-down maps are actually b-fibrations, e.g. by argumenting as in [Alb07, (4.4)].

Remark 3.28. As mentioned above, the special property that makes the construction work for such metrics is that all the blow-ups performed to construct the heat blow-up space except the last one, that of $\{0\} \times \mathrm{diag}_X$, do not involve the time coordinate. Thus, the result holds for any metric for which we can construct a double space describing its characteristic operators. By the discussion above or Section 4.1, we know how to carry the double space construction for metrics precisely satisfying the condition $k \geq l_1 > \cdots > l_q \geq 0$, where the non-negativity of the coefficients is demanded so that the space of characteristic vector fields forms a Lie subalgebra of the algebra of vector fields on X, allowing us to define differential operators as the corresponding universal enveloping algebra (as in §4.1).

□

It would be interesting to generally formulate the heat triple space construction for the metrics considered in the previous section, perhaps again by formulating the correct heuristic (the literature provides here also several instances, e.g. [Vai01, §4.2] and [GRS19, Appendix B]). In current work of the author with C. Anghel and D. Grieser, questions indirectly related to heat triple spaces are being investigated and some light might thus be shed on the matter.

4 Differential geometry of fibred boundary manifolds

The term 'geometry', for instance, refers to a pattern of processing within our brains related to our spatial and visual senses, more than it refers to a separate content area of mathematics. One illustration of this is the concept of correlation between two measurements on a set, which is formally nearly identical with the concept of cosine of the angle between two vectors. The content is almost the same (for correlation, you first project to a hyperplane before measuring the cosine of the angle) , but the human psychology is very different. Each mode of thinking has its own power, and ideally, people harness both modes of thought to work together. However, in formalized expositions, this psychological difference vanishes.
In the same way, any idea in mathematics can be thought about in many different ways, with competing advantages. When mathematics is explained, formalized and written down, there is a strong tendency to favor symbolic modes of thought at the expense of everything else, because symbols are easier to write and more standardized than other modes of reasoning. But when mathematics loses its connection to our minds, it dissolves into a haze.

William Thurston's Foreword on Hubbard's Teichmüller theory and applications, Volume 1

In the past years, Melrose's philosophy for the study of (non-linear) partial differential equations and their relationship to the geometry of singular spaces has seen a plethora of developments, extensions and applications. Most of the work is however very much concerned with specific structures and it is still not totally clear what the full scope of the methods are. At the very least, we are lacking a formal framework delimiting exactly which problems can be treated with such techniques. Having said that, the experts already have a pretty good understanding of what the limits of the theory could be and a source for a formalism describing the relevant classes of spaces and operators is to be expected in the near future. Watch out in particular for the future volumes by Albin, Grieser and Hintz (check also [Hin24]).
In order to avoid getting ahead of ourselves before this formalism is set in stone, we would just like to give some general ideas into what kind of metrics we are interested in, due to the feasibility of a blow-up analysis and e.g. the construction of a Fredholm theory on them. These will all be non-compact manifolds with bounded geometry or conformal models to them, where the relevant geometric structures are smooth subalgebras of the algebra of vector fields $\Gamma(TX)$[72], resp. weighted versions of those.

4.1 Manifolds with asymptotical geometrical ends

We start with a non-compact manifold M and want to study some geometric-analytical problem on it, such as a local index formula, its L^2-cohomology, etc. through a blow-up analysis à la Melrose. For that, it is convenient to compactify the manifold into a compact manifold with boundary, for example by introducing a function on the manifold that near the non-compact ends tends to 0 (as we did in Fig. 3 with x) and then adding the $\{x = 0\}$ stratum (assuming this can be done in a natural way). It is of course not clear at first sight in how many ways one can do this (or if there is a way at all), and that is where the future formalism will come in handy. Within the compact manifold with boundary, we will blow-up to resolve the "singularities", both in the geometry and in the domains of the kernels of the relevant classes of (pseudodifferential) operators, until

[72]More precisely, Lie algebroids.

we obtain "nice" behaviour (usually polyhomogeneous conormal distributions, polyhomogeneous metrics, etc.). When the proper resolutions (blow-ups) are found, it is expected that we are able to carry out several constructions that were possible in the closed case, but naturally with more layers of complexity, and thus attack the corresponding problems with analogous microlocal techniques as e.g. the b-calculus [Mel93].

4.1.1 Motivating examples

The spaces we have in mind when trying to spell out this generalized picture are the following:

Example 4.1. Consider the half-cylinder $M = \mathbb{R}_+ \times \mathbb{S}^1$ (cf. Fig. 3) with the product metric

$$g = du^2 + g_{\mathbb{S}^1} = du^2 + dy^2, \quad u \geq 0$$

(we could swap $\mathbb{S}^1$ by any closed manifold of arbitrary dimension, even a point, and the discussion would not be affected, since here we are essentially just compactifying the real line at one end). This has a non-compact end at $u \to \infty$, so we would like to compactify it to carry out a blow-up analysis. There is however more than one choice:

- As we did before, we could consider the boundary defining function $x = e^{-u}$. This means $M \cong (0,1]_x \times \mathbb{S}^1$ with metric
$$g_b = \frac{dx^2}{x^2} + dy^2$$
and the compactification $X = [0,1] \times \mathbb{S}^1$ lies at hand. Note that the metric degenerates at the boundary $\{x = 0\}$, i.e. it is only a metric on the whole space if we restrict to vector fields spanned by
$$x\partial_x, \quad \partial_y$$
Denoting by ${}^bTX$ the vector bundle whose sections correspond to the $\mathcal{C}^\infty$-span of $x\partial_x$ and ∂_y (such a vector bundle exists since we specified its sections: this is an instance of the geometric rescaling in [Mel93, Example 8.2], consequence of the Serre-Swan theorem on finitely generated projective $\mathcal{C}^\infty(X)$-modules; cf. Example 4.18), we can at least say that g is a metric on ${}^bTX$, including at the boundary. With the Lie bracket of vector fields, its sections correspond to the subalgebra of vector fields tangent to the boundary:
$$\mathcal{V}_b = \{V \in \Gamma(TX) : V\big|_{\partial X} \in T\partial X\}$$
There are several other geometric-analytical structures related to this class of **asymptotically cylindrical/b-metrics** (compact manifolds that around the boundary have a metric of this local form up to higher order terms), such as a b-cotangent bundle, a calculus of b-pseudodifferential operators[73] with a b-blow-up double and triple space construction, etc. [Mel93]. So, in some sense, there is a whole "b-category" and one can for example interpret the Atiyah-Patodi-Singer index theorem as the Atiyah-Singer index theorem in this category.

[73] The space of b-differential operators $\mathrm{Diff}_b(X)$ can be constructed as the enveloping algebra of $\mathcal{V}_b$, meaning the operators have local expressions:

$$P(x,y) = \sum_{k+|\alpha|\leq m} p_{k,\alpha}(x,y)(xD_x)^k D_y^\alpha, \quad P \in \mathrm{Diff}_b^m(X)$$

where $\alpha \in \mathbb{N}_0^{\dim(\partial X)}$ is a multiindex representing the order of derivation in the y-directions. This enveloping algebra can be *microlocalized* to an algebra of b-pseudodifferential operators $\Psi_b(X)$ in analogy with the standard closed case. The case of operators acting on bundles follows similar principles. This construction can be carried out for other Lie subalgebra structures or Lie algebroids in general, such as $\mathcal{V}_c$.

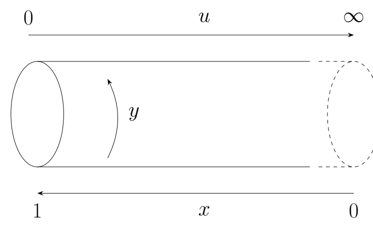


Figure 19: Model cylindrical end (b-metric).

- Another option is to consider $r = \frac{1}{u+1}$[74], producing $M \cong (0,1]_r \times \mathbb{S}^1$ and

  $$g_c = \frac{dr^2}{r^4} + dy^2$$

  The metric again degenerates (now *faster*) at the boundary of the compactification $X = [0,1]_r \times \mathbb{S}^1$ and the $\mathcal{C}^\infty$-span of

  $$r^2\partial_r,\ \partial_y$$

  is the space of sections of the bundle ${}^cTX$ of c-/cusp-vector fields, which inherit the Lie subalgebra structure

  $$\mathcal{V}_c(X) = \{V \in \Gamma(TX) : Vx \in x^2\mathcal{C}^\infty(X)\} \subset \mathcal{V}_b(X)$$

  and the metric is non-degenerate (so an usual Riemannian metric) precisely when acting on these vector fields. Notice that this definition now depends on the choice of boundary defining function r, so this has to be specified when studying a **cusp/c-structure**.

  The denomination "cusp" probably comes from the fact that this model is conformal to a metric of the form $dx^2 + x^4dy^2$, which is a particular case of a **metric horn**

  $$g_{\text{horn}} = dx^2 + x^{2a}dy^2, \quad a > 1$$

  and these degenerate as $x \to 0$ to a singularity which has an appearance comparable to that of a cuspidal curve $\{y^2 = x^3\} \subset \mathbb{R}^2$ $(a = \frac{3}{2})$, i.e. if one flows into the singularity, one needs to completely reverse the direction to get out, as opposed to a polyhedral or conical singularity; see Fig. 20.

Considering different compactifications means a priori obtaining different $\mathcal{C}^\infty$-structures by extension to the boundary, although they might turn out to be equivalent, i.e. what we understand to be smooth around the boundary is a priori different in both cases (for the b-case smoothness means you have an expansion in e^{-nu} for $n \in \mathbb{N}$ as $u \to \infty$, while for the c-case it means an expansion in u^{-n}). In terms of operators[75], this translates into choosing different closed extensions to the respective Sobolev spaces. See, the b-metric has an associated b-density bundle which in

[74] We are essentially considering $\frac{1}{u}$ as boundary defining function, but we do not want to lose the compactness at the other end $u = 0$; in practice, one would simply focus on a smaller neighborhood of the infinite cylindrical end and choose local coordinates there for the compactification.

[75] With an eye on index theory, other key structures also change, like the underlying Clifford algebra (since it is obtained as the quantization by the Clifford relations of the respective cotangent bundle adapted to the Lie algebroid and metric structures, although they are isomorphic. E.g. $Cl({}^bTX)$ is generated by $cl\left(\frac{dx}{x}\right)$ and $cl(dy)$, as opposed to $Cl({}^cTX)$, which is generated by $cl\left(\frac{dx}{x^2}\right)$ and $cl(dy)$), even if $cl\left(\frac{dx}{x}\right) \mapsto cl\left(\frac{dx}{x^2}\right)$ defines an isomorphism between them) or the renormalized integrals in the computation of the supertrace in the McKean-Singer formula, which are dependent on the class of boundary defining functions one considers, i.e. in the compactification.

local coordinates takes the form $\left|\frac{dx}{x}dy\right|$ and, accordingly, the "natural" L^2 domain of sections of a bundle $E \to X$ to be considered is

$$L^2_b(E) = \{u \in \Gamma(E) : \int_X |u(x,y)|^2 \frac{dx}{x} dy < \infty\}$$

(compare Def. 2.10 and proceed likewise for higher order Sobolev spaces). The corresponding structures on the c-metric contain in contrast the density $\left|\frac{dx}{x^2}dy\right|$, which results in a smaller L^2-domain:

$$L^2_c(E) = x^{\frac{1}{2}} L^2_b(E) = \{u \in \Gamma(E) : u = x^{\frac{1}{2}} v,\ v \in L^2_b(E)\} \subset L^2_b(E)$$

since higher powers of x make the function vanish to higher order at the boundary, hence better behaved under integration around $x = 0$.
In both cases, the geometric setting corresponds to a characteristic Lie subalgebra of vector fields. One can generalize this parallelism using the language of Lie manifolds and Lie algebroids [Nis16] (the cusp case g_a for $a \in \mathbb{N}_{\geq 2}$ has been treated recently in [BdPW23]), where the double space constructions solving the singular behaviour of operators are associated to the integration of the Lie algebroids (like ${}^bTX$) to Lie groupoids (like [a subset of] X^2_b). This offers a slightly different, rich perspective into this genre of problems. In particular, one can also treat codimension 2 spaces (which corresponds to complex codimension 1, thus important in complex/algebraic geometry), as depicted in [Wen25].

□

Example 4.2. The metrics

$$\frac{dx^2}{x^2} + dy^2, \quad \frac{dx^2}{x^4} + dy^2$$

in the previous example are both complete. Consider the "simplest" non-compact manifold: $\mathbb{R}^{n+1}$ with the Euclidean metric

$$g_{\mathbb{R}^{n+1}} = du_1^2 + \cdots + du_{n+1}^2$$

and introduce polar coordinates

$$g_{\text{cone}} = dr^2 + r^2 g_{\mathbb{S}^n} = dr^2 + r^2 dy^2$$

making it a cone with link $\mathbb{S}^n$. Around $r \to 0$, the tip of the cone (originally the origin) becomes a **conical singularity**: the cone is not a manifold at the tip and the new metric g degenerates (this is related to the fact that polar coordinates are diffeomorphic to cartesian coordinates exactly away from the origin; Fig. 4 is the quadrant version). The manifold $(0,\infty)_r \times \mathbb{S}^n \cong \mathbb{R}^{n+1} \setminus \{0\}$ with the metric g is a non-compact manifold with a conical singularity at $r \to 0$. This manifold is incomplete, since the singularity can be reached in finite time (so there is no geodesic completeness): the length of a path of constant hyperspherical coordinate y_0 from (r_0, y_0) towards the singularity is given by[76]

$$\gamma : (0, r_0] \to (0,\infty)_r \times \mathbb{S}^n, \quad t \mapsto (r_0 - t, y_0); \quad l(\gamma) = \int_0^{r_0} ||\gamma'(t)||_{\gamma(t)} dt = \int_0^{r_0} dt = r_0 < \infty$$

[76] Notice that for metrics where the term in the boundary defining function x comes with a power of itself:

$$g = \frac{dx^2}{x^{2a}} + \dots, \quad a \in \mathbb{R}$$

and that we compactify to a compact manifold with boundary by adding the $\{x = 0\}$ stratum ($a = 1$ in the b-case and $a = 2$ in the c-case), we can flow towards the singularity fastest along a path in the x-direction, which would have length:

$$\gamma : [0, x_0) \to X, \quad t \mapsto (x_0 - t, y_0); \quad l(\gamma) = \int_0^{x_0} \frac{dt}{t^a} < \infty \iff a < 1$$

so the metrics we consider are incomplete if their local expression has such a term with $a < 1$ and are otherwise complete.

In particular, there is no way to extend the manifold smoothly into the singularity preserving the $\mathcal{C}^\infty$-structure.
The already mentioned metric horns also correspond to compact manifolds with boundary with an incomplete metric in the interior:

$$g_{\text{horn}} = dx^2 + x^{2a} dy^2, \quad a > 1$$

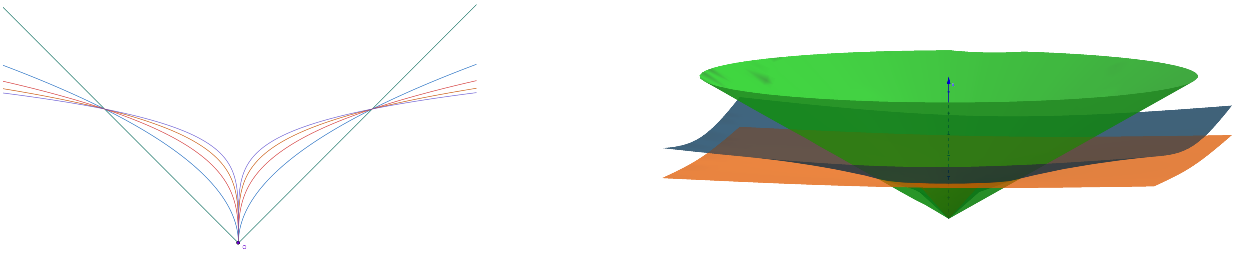


Figure 20: Slices (left) of model surfaces with cone and metric horn singularities [Hoh25]: represented are the graphs of $f(x) = |x|^{\frac{1}{a}}$ for $a \in \{1, 2, 3, 4, 5\}$. The surfaces of revolution (right, for $a \in \{1, 2, 4\}$) obtained by rotating these around the vertical line through the origin O correspond to the spaces $(\mathbb{R}_+)_x \times (\mathbb{S}^1)_y$ with incomplete metrics $g^a = dx^2 + x^{2a} dy^2$, where x is the vertical direction and y the horizontal. O is the singularity at $\{x = 0\}$. Green ($a = 1$) is the conical singularity, the rest are metric horns. Blue ($a = 2$) is sometimes called cusp, a conformal model to the c-metric: $g^2 = x^4 g_c$ (see also [GSHV25, Fig. 1]).

In these cases, the metric coefficients do not diverge (meaning the boundary is not infinitely away) but rather degenerate as $x \to 0$, so we lose positive definiteness. We can think of the compactified model through a blow-up of the boundary singularity with degenerate metric: in the example, the boundary would be pictured as a copy of $\mathbb{S}^n$ where the distance between any two points is zero. Again by a rescaling process, the metric induces a vector bundle of characteristic vector fields (we restrict here to the case of $a \in \mathbb{N}$):

$$\Gamma({}^{g^a}TX) = \mathcal{C}^\infty\text{-span}\{\partial_x, \frac{1}{x^a}\partial_y\}, \quad g^a = dx^2 + x^{2a} dy^2$$

and notice that the existence of terms with coefficients with positive powers of x in the metric (the term $x^{2a} dy^2$) makes sure the bundle ${}^{g^a}TX$ is not a subbundle of TX and does not admit a Lie algebra structure with respect to the bracket of vector fields:

$$[\partial_x, \frac{1}{x^a}\partial_y] = -\frac{a}{x^{a+1}}\partial_y \notin \Gamma({}^{g_a}TX)$$

Other geometric structures can however still be carried out in analogy to the previous cases or one can use a conformal model to define them; in particular, if we consider the complete metrics:

$$g_a = \frac{dx^2}{x^{2a}} + dy^2 = x^{-2a} g^a$$

with characteristic Lie subalgebras of vector fields ${}^{g_a}TX$ spanned by $x^a\partial_x$ and ∂_y, then $\text{Diff}^m_{g^a}(X)$ is not well defined as an universal enveloping algebra, but $\text{Diff}^m_{g_a}(X)$ is and the relevant differential operators constructed from the metric g^a lie in $x^{-ma}\text{Diff}^m_{g_a}(X)$.
The most straightforward example of this is the Laplacian at a model conical singularity (i.e. the Euclidean Laplacian in polar coordinates):

$$\Delta_{g^1} = \partial_x^2 + \frac{1}{x}\partial_x + \frac{1}{x^2}\partial_y^2 = \frac{1}{x^2}\left((x\partial_x)^2 + \partial_y^2\right) \in x^{-2}\text{Diff}^2_{g_1}(X)$$

Because polar coordinates are no diffeomorphism at the origin, we could now restrict ourselves to $\mathbb{R}^{n+1}$ away from a neighborhood of the origin. In that subset, the Euclidean metric pulls back diffeomorphically via polar coordinates to the same expression as above but, for e.g. $r \geq r_0 > 0$, the change of coordinates no longer artificially induces an incomplete singularity. We could use this to perform a radial compactification of $\mathbb{R}^{n+1}$ by inverting the radial coordinate $x = \frac{1}{r}$ and compactifying at $r \to \infty$, i.e. $x \to 0$. The resulting metric takes the form

$$g_{ac} = \frac{dx^2}{x^4} + \frac{g_{\mathbb{S}^n}}{x^2} = \frac{dx^2}{x^4} + \frac{dy^2}{x^2}$$

and since it models the infinite end of a cone, it goes by the name of **asymptotically conical/scattering metric**. Locally:

$$\Gamma({}^{ac}TX) = \langle x^2\partial_x, x\partial_y \rangle, \quad P = \sum_{k+|\alpha| \leq m} p_{k,\alpha}(x,y)(x^2 D_x)^k (x D_y)^\alpha \in \mathrm{Diff}^m_{ac}(X)$$

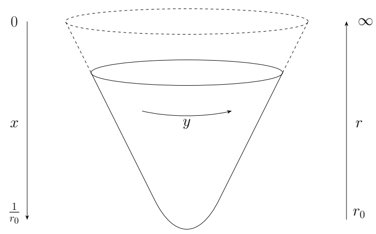


Figure 21: Model asymptotically conical end (scattering metric)

There are also two model ends that appear naturally in the context of hyperbolic geometry, concretely relating to Riemann surfaces. These are **funnels**

$$g_0 = \frac{dx^2}{x^2} + \frac{dy^2}{x^2}$$

(this model is usually referred to as 0-metric or hyperbolic metric, since it is the natural metric of constant curvature -1 in the upper half-plane modelling hyperbolic space $\mathbb{H}^n$, if x is the vertical coordinate) and **hyperbolic cusps**[77]

$$g_{hc} = \frac{dx^2}{x^2} + x^2 dy^2$$

both around $x = 0$. They are complete metrics and appear naturally as geometric models of points in the ideal boundary. In particular, hyperbolic cusps can be obtained by pinching geodesics of a closed Riemann surface [Ang24].

□

Example 4.3. For an example where the direction transverse to the boundary grows slower than the directions tangent to the boundary when approaching it, consider the (half) **Grushin plane**[78]:

$$g = dx^2 + \frac{dy^2}{x^2}, \quad x > 0$$

[77] With respect to the hyperbolic upper half-plane model, these correspond to strips escaping to $x \to \infty$. To write the metric with respect to a boundary defining function, we perform thus the coordinate change $x \mapsto \frac{1}{x}$.

[78] For a calculus adapted to this model, see [BQ24].

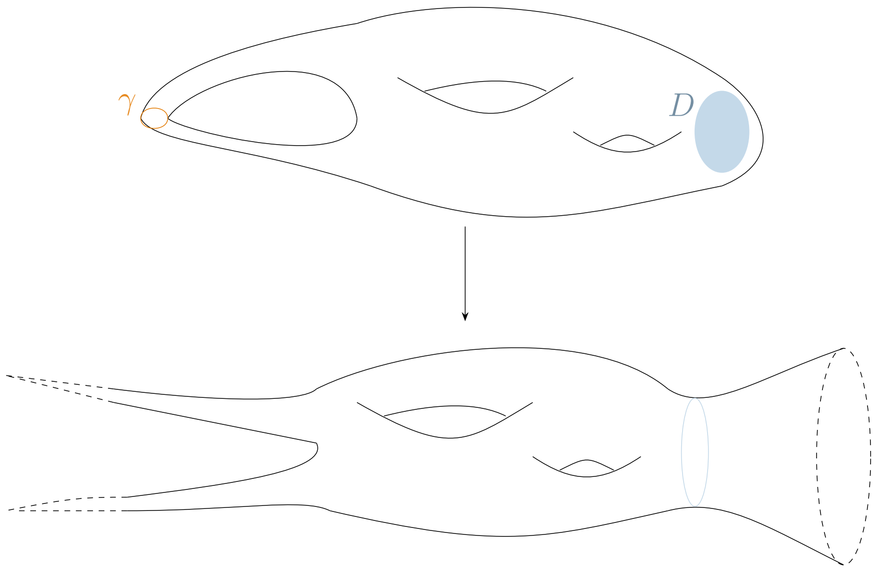


Figure 22: From a closed Riemann surface we can obtain two hyperbolic cusps by pinching a geodesic γ and we can attach a funnel to the boundary of a disk D if we take away its interior. Around the hyperbolic cusps, the surface is still of finite area (though infinite length), but not around the funnels.

(this can be generalized to $\frac{dy^2}{x^{2\alpha}}$ for $\alpha > 0$). Its metric completion (by adding $\{x = 0\}$) is a sub-Riemannian manifold, i.e. a metric space with the Carnot-Carathéodory metric given by

$$d_{CC}(p, q) = \inf_{\gamma\,:\,(\gamma(0),\gamma(1))=(p,q)} \int_0^1 \sqrt{(\dot{x}(t))^2 + \frac{(\dot{y}(t))^2}{(x(t))^2}} dt, \quad \gamma(t) = (x(t), y(t))$$

for horizontal curves γ, i.e. $\dot{y}(t) = 0$ for all $t \in [0, 1]$ such that $x(t) = 0$.

□

Example 4.4. It could also be the case that after compactifying, the manifold has disparate geometric behaviours along different directions at the boundary (in particular, we can combine elements from the previous examples). We will restrict to the case where the boundary ∂X is the total space of a fibre bundle

$$Z \longrightarrow \partial X \xrightarrow{\phi} Y$$

and fix the notation $b = \dim Y$, $f = \dim Z$, so that $n = b + f + 1$. Since the boundary of a compact manifold is compact, the base Y and fibre Z are compact too. Moreover, by taking X with boundary and without corners, ∂X, Y and Z all lack boundary. Nonetheless, this description can be extended to treat iterated fibre bundles [Alb07, §1.2] or other iterated constructions on manifolds with corners [AGR23, §1.1] [KR24].
The most relevant model families one encounters are (cf. [GSHV25, §2]):

- **Edge metrics** [Maz91]

$$g_e = \frac{dx^2}{x^2} + \frac{\phi^* g_Y}{x^2} + g_Z$$

  and their related **wedge metrics** $g_w = x^2 g_e$ (also known as incomplete edge). They generalize b-, 0- and cone metrics. One variant of these are **incomplete cusp edge** spaces [Liu25]

$$g_{ice} = dx^2 + \phi^* g_Y + x^{2a} g_Z, \quad a > 1$$

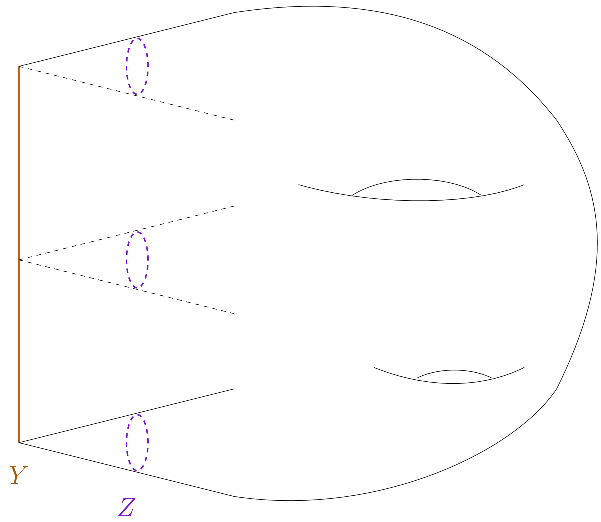


Figure 23: Sketch of a manifold with wedge ends: the boundary $\partial X = Y \times Z$ has a collar neighborhood modelled on a family of cones parametrized by Y. Compare with the incomplete edge cusp case [Liu25, Figure 1].

generalizing metric horns. An iterated version for the case $a = 3$ corresponds to the completion of the Weil-Petersson metric in Teichmüller space [Wol05, Corollary 4].

- **Fibred cusp metrics** [MM98] (sometimes called fibred boundary or ϕ-metrics, referencing the fibre bundle $\phi : \partial X \to Y$)
$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z$$
This models e.g. the radial compactification of $\mathbb{R}^n \times Z$ as the infinite end of a "fat" cone, where Z is closed. To distinguish it from their also relevant conformal cousins, the **generalized hyperbolic cusp/d-metric** $g_d = x^2 g_\phi$ [Vai01] and the **incomplete fibred cusp metric** $x^4 g_\phi$, we follow the terminology of [GSHV25]: a c-ϕ-metric is a metric of the form $x^{2c} g_\phi$ for $c \in \{0, 1, 2\}$. They generalize cusp, hyperbolic cusp and asymptotically conical metrics.

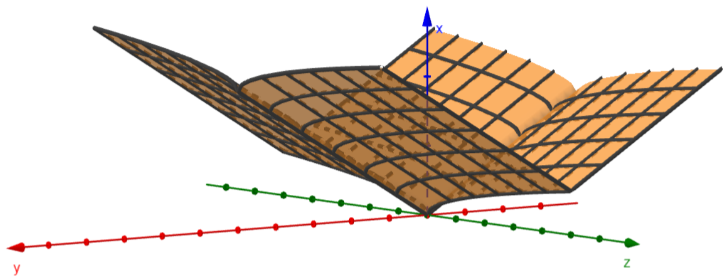


Figure 24: Illustration of a model incomplete fibred cusp close to the singular tip [Hoh25]. The manifold X is given by the closed upper half-space bounded by the orange surface. In the base Y direction with local coordinate y, the space degenerates like a cone. In the fibre Z direction with coordinate z, it resembles a cuspidal curve. The way it is depicted, the space corresponds to the collar neigborhood $[0, \varepsilon) \times [0, 1]_y \times [0, 1]_z$, so both the fibre and the base have boundaries and the resulting X is thus a manifold with corners, a case out of our scope. For a treatment of the case where the fibres are allowed to have boundary, see the concept of ϕ-BC-manifolds in [FGS23, §2.1].

□

We will mostly focus our attention on fibred cusp spaces i.e. c-ϕ-metrics. A review on their spectral and index theory can be found in [GSHV25].

4.1.2 Common principles

We can attempt to simultaneously treat all these cases by considering metrics on a compact manifold with boundary X obtained after compactifying a non-compact M through the introduction of a boundary defining function x and adding the $\{x = 0\}$ stratum. ∂X is assumed to be the total space of a fibre bundle as above and the metric on the interior has the asymptotic form[79]

$$g_{klm} = \frac{dx^2}{x^{2k}} + \frac{\phi^* g_Y}{x^{2l}} + \frac{g_{\partial X/Y}}{x^{2m}} \tag{13}$$

in a neighborhood of the boundary ($\partial X/Y$ is just another way to denote the fibres Z). Such a metric and the resulting characteristic geometric structures (we can again talk about doing analysis in the "category of klm-metrics") are usually dependent on the choice of the boundary defining function, so we assume it is fixed (this is not the case e.g. in the b- or ac-metrics). We recall that a boundary defining function satisfies the following conditions:

- $x \in \mathcal{C}^\infty(X)$
- $\{x = 0\} = \partial X$
- $dx \neq 0$ at ∂X

The choice of x corresponds to the existence of a collar neighborhood of the boundary $\mathcal{U} \cong [0, \delta)_x \times (\partial X)_{y,z}$, since dx trivializes the conormal bundle to the boundary $T_pX/T_p\partial X$ for each $p \in \partial X$. It is in $\mathcal{U}$ where the metric takes the local form above. We can consider local coordinates (y, z) at a point in ∂X, where the y correspond to lifts of coordinates on Y along ϕ and the z restrict to coordinates on the fibres. There is an associated boundary metric in all these models given by

$$g_{\partial X} = \phi^* g_Y + g_{\partial X/Y}$$

Along the construction, (Y, g_Y) can be any closed Riemannian manifold and $g_{\partial X/Y}$ any symmetric 2-tensor whose pushforward under ϕ vanishes. The metric $g_{\partial X}$ is a submersion metric for ∂X, i.e.

$$(\partial X, g_{\partial X}) \xrightarrow{\phi} (Y, g_Y)$$

is a Riemannian submersion. This corresponds to the following: the subbundle of the tangent bundle of vector fields vertical with respect to ϕ is invariantly defined without the metric structures as $T\partial X/Y := \ker(\phi_*) \subset TX$. We now can choose a subbundle $H\partial X$ of horizontal vector fields to get a decomposition:

$$T\partial X = H\partial X \oplus T\partial X/Y$$

This choice can be made so that the decomposition is orthogonal with respect to the metric $g_{\partial X}$, meaning $\phi^* g_Y$ is a metric on $H\partial X$ and vanishes on $T\partial X/Y$, and $g_{\partial X/Y}$ is a metric on $T\partial X/Y$ and vanishes on $H\partial X$. Then we have a Riemannian submersion by identifying $H\partial X$ with TY via

[79] We will restrict ourselves to exact metrics, but many of the results we obtain do not see higher order terms in the metric. For example, the index theorem could be extended to metrics with polyhomogeneous error term by employing a standard transgression argument as in [Liu25, Theorem 9.10]. On a different note, all the constructions developed here should be once more generalizable to iterated fibration structures.

pullback along ϕ, so we write $H\partial X = \phi^* TY$. There are moreover maps projecting on the vertical or horizontal factor of the tangent bundle over the boundary:

$$v : \Gamma(\partial X, T\partial X) \longrightarrow \Gamma(X, T\partial X/Y), \quad h : \Gamma(X, T\partial X) \longrightarrow \Gamma(X, \phi^* TY)$$

Note that in general the pullback of vector fields is not necessarily well-defined, so this procedure allows us to single out a horizontal distribution to which we pull back the vector fields from Y. In contrast, the pullback of forms is well-defined from the start.
Once these choices are made, the metric g_{klm} is properly defined.
Around a point $q \in Y$, we can consider an orthogonal basis of vector fields of T_qY and pull it back to an orthogonal basis of horizontal vector fields $\{U_i\big|_p\}_{i=1}^b$ of $\phi^* T_pY \subset T_pX$, where $\phi(p) = q$, and so $p \in \partial X$. We multiply each of them by x^l to adapt them to the metric and still preserve the orthogonality with respect to g_{klm} [80]. Finally, we complete to a basis of ${}^{klm}T_pX$ by adding the normal direction $x^k \partial_x\big|_p$ and the vertical directions. We multiply each of the vertical vectors so obtained by x^m to adapt them to the metric. From this we can obtain an orthogonal basis of the collar neighborhood of the boundary by trivially extending (along the directions normal to the boundary) these vector fields to generate the sections[81]

$$x^k \partial_x, \quad x^l \tilde{U}_i, \quad x^m V_j$$

where the $x^l \tilde{U}_i$ are the extensions of horizontal vector fields $x^l U_i$ and the V_j the vertical ones. Locally:

$$\tilde{U}_i = \sum_{j=1}^{b} a_{ij} \partial_{y_j} + \sum_{j=1}^{f} b_{ij} \partial_{z_j}$$

for $a_{ij}, b_{ij} \in \mathcal{C}^\infty(X), b_{ij}\big|_{\partial X} = 0$, since given a metric, it is not always possible to find orthogonal coordinates with respect to it, e.g. so that their associated coordinate vector fields are orthogonal, so we cannot make the ∂_{y_i} and ∂_{z_j} orthogonal in the whole collar neighborhood.
Not only can we restrict ourselves to exact metrics, but also to spin-Dirac operators and recover most of the interesting aspects of the geometric model in the context of local index theory. The Clifford

[80]This is clear when $k, l, m \geq 0$, i.e. when the characteristic vector fields lie in a subbundle of TX, otherwise we can consider the formal rescaling [Mel93] of the bundle TX with respect to the orthogonal decomposition

$$TX \cong \langle \partial_x \rangle \oplus \phi^* TY \oplus T\partial X/Y$$

into

$${}^{klm}TX \cong \langle x^k \partial_x \rangle \oplus x^l \phi^* TY \oplus x^m T\partial X/Y$$

to single out the maximal vector fields adapted to the metric, even if they are singular as sections in TX. Another approach would be to first go to the conformal reference metric $h = x^{-2\alpha_0} g_{klm} \in \mathcal{G}_{(k-l),(l-m)}$ (Definition 4.11), construct the corresponding space of characteristic vector fields ${}^hTX$ there as a subset of $\Gamma(TX)$ and then rescale that bundle back by $x^{-2\alpha_0}$, i.e. ${}^{klm}TX = x^{-2\alpha_0}\left({}^hTX\right)$.

[81]Note that

$$[x^k \partial_x, x^l \tilde{U}_i] = l x^{k+l-1} \tilde{U}_i, \quad [x^k \partial_x, x^m V_j] = x^{k+m-1} V_j$$

Moreover, if $l < 0$ (analogous $k < 0, m < 0$):

$$0 = [x^k \partial_x, \tilde{U}_i] \neq x^{-l}[x^k \partial_x, x^l \tilde{U}_i] = x^{k-1} \tilde{U}_i$$

This means that ${}^{klm}TX$ is not closed under Lie brackets unless $k \geq 1$ or $l = m = 0$, and the Lie bracket is not $\mathcal{C}^\infty$-linear if one of k, l or m are negative. Thus, ${}^{klm}TX$ forms a Lie algebra iff

$$k, l, m \geq 0, \quad (k, l, m) \notin \{(0,1,1), (0,1,0), (0,0,1)\}$$

This also means that there is always a conformal model which generates a Lie algebra, so we can always do our geometric analysis with x-weighted versions of it; see Definition 4.11.

algebra structure associated to the metric can be obtained by quantization of the characteristic cotangent bundle ${}^{klm}T^*X$, orthogonally spanned by

$$\frac{dx}{x^k}, \quad \frac{\tilde{U}_i^\flat}{x^l}, \quad \frac{V_j^\flat}{x^m}, \quad \text{with the help of the musical isomorphisms } \Gamma(TX) \underset{\sharp}{\overset{\flat}{\rightleftarrows}} \Gamma(T^*X)$$

which induce the usual identifications $Cl({}^{klm}T^*X) \cong Cl(T^*X)$ and $Cl(T^*Y) \cong Cl(\phi^*T^*Y) \subset Cl\left(T^*X\big|_{\partial X}\right)$ via

$$cl_{klm}\left(x^{-k}dx\right) \cong cl\left(dx\right), \quad cl_{klm}\left(x^{-l}\tilde{U}_i^\flat\right) \cong cl\left(\tilde{U}_i^\flat\right) \cong cl_Y\left(U_i^\flat\right), \quad cl_{klm}\left(x^{-m}V_j^\flat\right) \cong cl\left(V_j^\flat\right)$$

The corresponding spin-Dirac operator (related to the spinor bundle $\mathcal{S}$ associated to the metric structure g_{klm}) exhibits the local form

$$\eth_{\mathcal{S}} = cl_{klm}\left(x^{-k}dx\right)\nabla^{\mathcal{S}}_{x^k\partial_x} + \sum_{i=1}^{b} cl_{klm}\left(x^{-l}\tilde{U}_i^\flat\right)\nabla^{\mathcal{S}}_{x^l\tilde{U}_i} + \sum_{j=1}^{f} cl_{klm}\left(x^{-m}V_j^\flat\right)\nabla^{\mathcal{S}}_{x^mV_j}$$

with spinor connection

$$\nabla^{\mathcal{S}}_W = W + \frac{1}{4}\sum_{p,q} g(\nabla_W\partial_{w_p}, \partial_{w_q})cl_{klm}\left(w_p^\flat\right)cl_{klm}\left(w_q^\flat\right), \quad W \in \Gamma({}^{klm}TX),\ w_p \in \{x^k\partial_x,\ x^l\tilde{U}_i,\ x^mV_j\}$$

where ∇ denotes the Levi-Civita connection for the metric g_{klm}. We are thus interested in the asymptotics of this connection when approaching the boundary. We can compute them by using the Koszul formula (from now on, we will suppress the symbol $\mathcal{S}$ and the subscripts klm to avoid notational overload and instead use g when it is clear what structure we mean):

$$\begin{aligned} 2g(\nabla_{W_1}W_2, W_3) &= W_1(g(W_2,W_3)) + W_2(g(W_1,W_3)) - W_3(g(W_1,W_2)) \\ &\quad + g([W_1,W_2],W_3) - g([W_1,W_3],W_2) - g([W_2,W_3],W_1) \end{aligned}$$

In the last terms appearing in the formula, it will be important to remember that $[x^l\tilde{U}, x^l\tilde{U}']$ can be horizontal and/or vertical, and $[x^l\tilde{U}, x^mV]$ and $[x^mV, x^mV']$ are vertical.
To describe these asymptotics we need to additionally define two auxiliary tensors:

1. The **second fundamental form** of the fibre bundle S^ϕ:

$$\begin{aligned} S^\phi : \Gamma(X, T\partial X/Y \otimes T\partial X/Y) &\longrightarrow \Gamma(X, \phi^*TY) \\ S^\phi(V,V') &= h(\nabla_V V') \\ \text{so that} \quad g_{\partial X}(\nabla_V V', \tilde{U}) &= \phi^*g_Y(S^\phi(V,V'), \tilde{U}) \end{aligned}$$

2. The **curvature** of the fibre bundle R^ϕ:

$$\begin{aligned} R^\phi : \Gamma(X, \phi^*TY \otimes \phi^*TY) &\longrightarrow \Gamma(X, T\partial X/Y) \\ R^\phi(\tilde{U}, \tilde{U}') &= v[\tilde{U}, \tilde{U}'] \\ \text{so that} \quad g_{\partial X}([\tilde{U}, \tilde{U}'], V) &= g_{\partial X/Y}(R^\phi(\tilde{U}, \tilde{U}'), V) \end{aligned}$$

$g(\nabla_{W_i} W_j, W_p)$	$x^m V_p$	$x^l \tilde{U}_p$
$\nabla_{V_i} x^m V_j$	$g_{\partial X/Y}(\nabla^{\partial X/Y}_{V_i} V_j, V_p)$	$x^{l-m}\phi^* g_Y(S^\phi(V_i,V_j),\tilde{U}_p)$
$\nabla_{\tilde{U}_i} x^m V_j$	$\phi^* g_Y(S^\phi(V_j,V_p),\tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i, V_p], V_j)$	$-\frac{x^{l-m}}{2} g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_p),V_j)$
$\nabla_{V_i} x^l \tilde{U}_j$	$-x^{l-m}\phi^* g_Y(S^\phi(V_i,V_p),\tilde{U}_j)$	$-\frac{x^{2(l-m)}}{2} g_{\partial X/Y}(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i)$
$\nabla_{\tilde{U}_i} x^l \tilde{U}_j$	$\frac{x^{l-m}}{2} g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p)$	$g_Y(\nabla^Y_{U_i} U_j, U_p)$

In this context, we obtain the following asymptotics for ∇:

$$g(\nabla_{\tilde{U}_i} x^l\tilde{U}_j, x^k\partial_x) = -g(\nabla_{\tilde{U}_i} x^k\partial_x, x^l\tilde{U}_j) = l x^{k-l-1} g_Y(U_i,U_j) = l x^{k-l-1}\delta_{ij}$$
$$g(\nabla_{V_i} x^m V_j, x^k\partial_x) = -g(\nabla_{V_i} x^k\partial_x, x^m V_j) = m x^{k-m-1} g_{\partial X/Y}(V_i,V_j) = m x^{k-m-1}\delta_{ij}$$

A priori, the Levi-Civita connection of a metric g is a *g-connection*, meaning a map

$$\nabla : \Gamma(X, {}^gTX) \longrightarrow \Gamma(X, {}^gT^*X \otimes {}^gTX)$$

However, since we applied ∇ in the table to elements in TX, it will be a *true connection* precisely if none of the terms diverge. Thus the following proposition, proved by the calculations leading to the table above:

Proposition 4.5. The Levi-Civita connection of a metric

$$g = x^{-2k}dx^2 + x^{-2l}\phi^* g_Y + x^{-2m} g_{\partial X/Y}$$

is a true connection if and only if the following three conditions are met:

- $l \geq m$
- $k \geq l+1$ or $l = 0$
- $k \geq m+1$ or $m = 0$

□

Proof. The first condition ensures the terms in the table with powers x^{l-m} are non-singular. The second condition does the same for the term

$$g(\nabla_{\tilde{U}_i} x^l\tilde{U}_j, x^k\partial_x) = g(\nabla_{\tilde{U}_i} x^k\partial_x, x^l\tilde{U}_j) = l x^{k-l-1} g_Y(U_i,U_j)$$

The third condition takes care of

$$g(\nabla_{V_i} x^m V_j, x^k\partial_x) = g(\nabla_{V_i} x^k\partial_x, x^m V_j) = m x^{k-m-1} g_{\partial X/Y}(V_i,V_j)$$

□

Example 4.6. Note that for an edge metric (corresponding to $k = l = 1, m = 0$), the Levi-Civita connection is not a true connection. Nevertheless, it is a real one in the case of a wedge metric ($k = l = 0, m = -1$).

□

Remark 4.7. To have a different behaviour at the base of the fibre bundle compared to the fibres, their asymptotics should be different, meaning $l \neq m$. Otherwise we could have just considered them within one term of our metric, something like

$$g = x^{-2k} dx^2 + x^{-2l} g_{\partial X}$$

which would correspond to the analysis above in case either the fibres or the base are trivial.

□

We can also use this to compute the explicit asymptotics of the Riemann curvature from the definition $R(W_1, W_2)W_3 = \nabla_{W_1}\nabla_{W_2}W_3 - \nabla_{W_2}\nabla_{W_1}W_3 - \nabla_{[W_1,W_2]}W_3$ and obtain the following standard results (cf. [Vai01, Prop. 1.5(e), Prop. 1.14], [AGR23, Prop. 1.5], [Liu25, Lemma 2.2]):

Proposition 4.8. Denote the horizontal-normal and vertical-normal projections by:

$$h_+ : \Gamma({}^gTX) \longrightarrow \left\langle x^k \partial_x \right\rangle \oplus x^l \phi^* TY, \quad v_+ : \Gamma({}^gTX) \longrightarrow \left\langle x^k \partial_x \right\rangle \oplus x^m T\partial X/Y$$

and extend h and v to also act on g-vector fields on the collar neighborhood of the boundary where the splitting

$${}^gTX \cong \left\langle x^k \partial_x \right\rangle \oplus x^l \phi^* TY \oplus x^m T\partial X/Y$$

is present. Let $N, W_0 \in \Gamma(TX)$, where W_0 is moreover tangent to the fibres of ϕ to order $l-m$ (i.e. $W_0 = vW_0 + O(x^{l-m})$), and $W_1, W_2 \in \Gamma({}^gTX)$. Then:

1. If $l > m$ and $l = 0$, then the connection and the Riemannian curvature preserves the splitting:

   $$\left[\left\langle x^k \partial_x \right\rangle \oplus x^m T\partial X/Y\right] \oplus x^l \phi^* TY$$

   If still $l > m$ but rather either $m = 0$ or $k - m - 1 > 0$, then the splitting preserved is:

   $$\left[\left\langle x^k \partial_x \right\rangle \oplus x^l \phi^* TY\right] \oplus x^m T\partial X/Y$$

2. $g\left(R\left(N, W\right) W_1, W_2\right) = g\left(R\left(N, vW\right) v_+ W_1, v_+ W_2\right)$
   $+ (l-m) x^{l-m-1} N(x) \left(\phi^* g_Y \left(S^\phi\left(vW_0, vW_1\right), hW_2\right) - \phi^* g_Y \left(S^\phi\left(vW_0, vW_2\right), hW_1\right)\right) + O(x^{l-m})$

3. $$\begin{aligned} g\left(\left(\nabla_N R\right)\left(N, W_0\right) hW_1, hW_2\right) = {} & N(x) x^{l-m-1} g\left(R\left(N, hW_0\right) hW_1, hW_2\right) \\ & + x^{l-m} N\left(g\left(R(N, hW_0) hW_1, hW_2\right)\right) \\ & - (l-m)(2l-2m-1)\left(N(x)\right)^2 x^{2l-2m-2} g_{\partial X/Y}\left(R^\phi\left(hW_1, hW_2\right), vW_0\right) \\ & + O(x^{2l-2m-1}) \end{aligned}$$

4. The scalar curvature $\mathrm{scal}_g = \sum_{ij} g\left(R(e_i, e_j)e_j, e_i\right)$ satisfies:

$$\begin{aligned} \mathrm{scal}_g = {} & 2\sum_i^b g\left(R\left(x^k\partial_x, x^l \tilde{U}_i\right) x^l \tilde{U}_i, x^k \partial_x\right) + 2\sum_i^f g\left(R\left(x^k \partial_x, x^m V_i\right) x^m V_i, x^k \partial_x\right) \\ & + \sum_{ij}^b g\left(R\left(x^l \tilde{U}_i, x^l \tilde{U}_j\right) x^l \tilde{U}_j, x^l \tilde{U}_i\right) + \sum_{ij}^f g\left(R\left(x^m V_i, x^m V_j\right) x^m V_j, x^m V_i\right) \\ & + 2\sum_i^b \sum_j^f g\left(R\left(x^l \tilde{U}_i, x^m V_j\right) x^m V_j, x^l \tilde{U}_i\right) \end{aligned}$$

where:

$$\sum_{i}^{b} g\left(R\left(x^k\partial_x, x^l\tilde{U}_i\right)x^l\tilde{U}_i, x^k\partial_x\right) = bl(k-l-1)x^{2k-2}$$

$$\sum_{i}^{f} g\left(R\left(x^k\partial_x, x^m V_i\right)x^m V_i, x^k\partial_x\right) = fm(k-m-1)x^{2k-2}$$

$$\sum_{ij}^{b} g\left(R\left(x^l\tilde{U}_i, x^l\tilde{U}_j\right)x^l\tilde{U}_j, x^l\tilde{U}_i\right) = \sum_{ij}^{b}\Bigg(-l^2x^{2k-1}$$

$$+ x^{2l}\sum_{p}^{b}\left(g_Y\left(\nabla^Y_{U_j}U_j, U_p\right)g_Y\left(\nabla^Y_{U_i}U_p, U_i\right) - g_Y\left(\nabla^Y_{U_i}U_j, U_p\right)g_Y\left(\nabla^Y_{U_j}U_p, U_i\right)\right)$$

$$-\frac{x^{4l-2m}}{2}\sum_{p}^{f}\left(g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_i, \tilde{U}_j\right), V_p\right)g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_j, \tilde{U}_i\right), V_p\right)\right)$$

$$- x^{2l}g_Y\left(\nabla^Y_{h^\phi[\tilde{U}_i,\tilde{U}_j]}U_j, U_i\right) + \frac{x^{4l-2m}}{2}g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_j, \tilde{U}_i\right), v[\tilde{U}_i, \tilde{U}_j]\right)\Bigg)$$

$$\sum_{ij}^{f} g\left(R\left(x^m V_i, x^m V_j\right)x^m V_j, x^m V_i\right) = \sum_{ij}^{f}\Bigg(-m^2x^{2k-1}$$

$$+ x^{2m}\sum_{p}^{f}\Bigg(g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_j}V_j, V_p\right)g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_p, V_i\right)$$

$$- g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_j, V_p\right)g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_j}V_p, V_i\right)\Bigg)$$

$$+ x^{2l}\sum_{p}^{b}\Bigg(\phi^*g_Y\left(S^\phi(V_j, V_j), \tilde{U}_p\right)\phi^*g_Y\left(S^\phi(V_i, V_i), \tilde{U}_p\right)$$

$$- \phi^*g_Y\left(S^\phi(V_i, V_j), \tilde{U}_p\right)\phi^*g_Y\left(S^\phi(V_j, V_i), \tilde{U}_p\right)\Bigg)$$

$$- x^{2m}g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{[V_i,V_j]}V_j, V_i\right)\Bigg)$$

$$\sum_{i}^{b}\sum_{j}^{f} g\left(R\left(x^l\tilde{U}_i, x^m V_j\right)x^m V_j, x^l\tilde{U}_i\right) = \sum_{i}^{b}\sum_{j}^{f}\Bigg(-lmx^{2k-1}$$

$$- x^{2l}\sum_{p}^{f}\left(\left(\phi^*g_Y\left(S^\phi(V_j, V_p), \tilde{U}_i\right) - g_{\partial X/Y}\left([\tilde{U}_i, V_p], V_j\right)\right)\phi^*g_Y\left(S^\phi(V_j, V_p), \tilde{U}_i\right)\right)$$

$$+ x^{2l}\sum_{p}^{b}\Bigg(\phi^*g_Y\left(S^\phi(V_j, V_j), \tilde{U}_p\right)g_Y\left(\nabla^Y_{U_i}U_p, U_i\right)$$

$$-\frac{x^{2l-2m}}{4}g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_i, \tilde{U}_p\right), V_j\right)g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_p, \tilde{U}_i\right), V_j\right)\Bigg)$$

$$- x^{2l}\phi^*g_Y\left(S^\phi([\tilde{U}_i, V_j], V_j), \tilde{U}_i\right)\Bigg)$$

Here, $h^\phi[\tilde{U}_i, \tilde{U}_j]$ is the vector field in Y which lifts to the horizontal $h[\tilde{U}_i, \tilde{U}_j]$.

□

Proof. 1. follows from the connection asymptotics, which show that vertical vectors are mapped to $mx^{k-m-1}\left\langle x^k\partial_x\right\rangle \oplus x^{l-m}\left(x^l\phi^*TY\right)\oplus x^mT\partial X/Y$, horizontal vectors to $lx^{k-l-1}\left\langle x^k\partial_x\right\rangle \oplus x^l\phi^*TY \oplus x^{l-m}\left(x^mT\partial X/Y\right)$ and normal vectors to $x^{2k}\left\langle x^k\partial_x\right\rangle \oplus lx^{k-l-1}\left(x^l\phi^*TY\right)\oplus mx^{k-m-1}\left(x^mT\partial X/Y\right)$.
2., 3. and 4. are just (long) computations. For 2. and 4. we can explicitly write all the terms and use that R^ϕ is antisymmetric by definition. Then we just need to apply the resulting expressions to 3., where we moreover use the customary:

$$\begin{aligned} &g\left(\left(\nabla_N R\right)(N, W_0)\, hW_1, hW_2\right) = g\left(\nabla_N\left(R(N,W_0)hW_1\right), hW_2\right) - g\left(R\left(N, W_0\right)\nabla_N hW_1, hW_2\right)\\ &= Ng\left(R(N,W_0)hW_1, hW_2\right) - g\left(R\left(N,W_0\right)hW_1, \nabla_N hW_2\right) - g\left(R\left(N,W_0\right)\nabla_N hW_1, hW_2\right) \end{aligned}$$

since the Levi-Civita connection ∇ is a metric connection. Here, the second and third terms are $(l-m)O(x^{2l-2m-1})$ when interacting with the vertical part of W_0, and produce the $O(x^{l-m-1})$ and $O(x^{l-m})$ contributions when interacting with the $O(x^{l-m})$-horizontal part of W_0. The first term gives the rest. □

Remark 4.9. The fact that the connection preserves the splitting and the form of the asymptotics of the curvature tensor is useful to perform Getzler rescaling at the faces lying in the $x = x' = 0$ regime of the (compactified) heat blow-up space, since at them $N = N(x)\partial_x$ is a valid choice of transverse vector field, and to investigate through which vectors the connection acts on the rescaled bundle. This is all based on [Mel93, Lemma 8.10] (it is similar to e.g. [Vai01, Lemma 5.2, Lemma 5.8] or [AGR23, Lemma 5.1]) and also provides a different approach into computing the rescaled normal operator, namely through [Mel93, (8.36)]. We decide to work however with the rescaling maps δ_τ and $\delta_{x'}$, although the result should be the same.
The calculation concerning the scalar curvature is used in the Lichnerowicz formula for the square of the Dirac operator, see e.g. (15).

□

A useful choice[82] are radial coordinates [BGV04, Propositions 1.17, 1.18 and 3.43] with respect to Y, through which we can write

$$g_Y(\nabla^Y_{U_i}U_j, U_p) = -\frac{1}{2}\sum_{q=1}^{b} g_Y\left(R_Y\left(U_i, y_q\partial_{y_q}\right)U_j, U_p\right) \tag{14}$$

From the connection asymptotics one can read off directly the action of the spinor connection:

$$\begin{aligned} \nabla^{\mathcal{S}}_{\partial_x} &= \partial_x\\ \nabla^{\mathcal{S}}_{\tilde{U}_i} &= \sum_{j=1}^{b} a_{ij}\partial_{y_j} + \sum_{j=1}^{f} b_{ij}\partial_{z_j} + \frac{lx^{k-l-1}}{2} cl_g\left(x^{-l}\tilde{U}_i^\flat\right) cl_g\left(x^{-k}dx\right)\\ &+ \frac{1}{4}\sum_{j,p=1}^{f}\left(\phi^* g_Y\left(S^\phi(V_j,V_p), \tilde{U}_i\right) - g_{\partial X/Y}\left([\tilde{U}_i, V_p], V_j\right)\right) cl_g\left(x^{-m}V_j^\flat\right) cl_g\left(x^{-m}V_p^\flat\right) \end{aligned}$$

[82]Since they allow us to interpret the corresponding heat kernel contribution as the Mehler kernel of the harmonic oscillator on the curvature of the base R_Y and thus produce the Â-genus of Y term. This is analogous to the local treatment of the closed case in [BGV04, §4.3] for all of the X directions, which brings about the Â-genus of X and can be used in the local index formula for any metric to understand the tf contribution; see Example 4.20.

$$+\frac{x^{l-m}}{4}\sum_{j=1}^{b}\sum_{p=1}^{f} g_{\partial X/Y}\left(R^{\phi}(\tilde{U}_i,\tilde{U}_j),V_p\right) cl_g\left(x^{-l}\tilde{U}_j^{\flat}\right) cl_g\left(x^{-m}V_p^{\flat}\right)$$

$$-\frac{1}{8}\sum_{j,p,q=1}^{b} g_Y\left(R_Y(U_i,y_q\partial_{y_q})U_j,U_p\right) cl_g\left(x^{-l}\tilde{U}_j^{\flat}\right) cl_g\left(x^{-l}\tilde{U}_p^{\flat}\right)$$

$$\nabla^{\mathcal{S}}_{\tilde{V}_i} = \underbrace{V_i + \frac{1}{4}\sum_{j,p=1}^{f} g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_j,V_p\right) cl_g\left(x^{-m}V_j^{\flat}\right) cl_g\left(x^{-m}V_p^{\flat}\right)}_{\nabla^{\mathcal{S}_{\partial X/Y}}_{V_i}}$$

$$+\frac{mx^{k-m-1}}{2} cl_g\left(x^{-m}V_i^{\flat}\right) cl_g\left(x^{-k}dx\right)$$

$$+\frac{x^{l-m}}{2}\sum_{j=1}^{f}\sum_{p=1}^{b}\phi^* g_Y\left(S^{\phi}(V_i,V_j),\tilde{U}_p\right) cl_g\left(x^{-m}V_j^{\flat}\right) cl_g\left(x^{-l}\tilde{U}_p^{\flat}\right)$$

$$-\frac{x^{2(l-m)}}{8}\sum_{j,p=1}^{b} g_{\partial X/Y}\left(R^{\phi}(\tilde{U}_j,\tilde{U}_p),V_i\right) cl_g\left(x^{-l}\tilde{U}_j^{\flat}\right) cl_g\left(x^{-l}\tilde{U}_p^{\flat}\right)$$

Remark 4.10. Since we know the form of the coordinates in the second-to-last blown up face $\mathrm{ff} \subset HX_g^2$ for metrics g of the type considered in Conjecture 3.22, which always lies in the lift of the boundaries $\{x = x' = 0\}$, we can lift the connection asymptotics (from the left X factor via β_x, see §4.2) to this face and obtain an explicit expression for the lift of the corresponding spinor Dirac operator. Indeed, in the case $m \geq 1$, using (11) for $r = 2$:

$$\beta_x^*\nabla^{\mathcal{S}}_{x^k\partial_x} = \left((x')^{k-1}S+1\right)^k \partial_S$$

$$\beta_x^*\nabla^{\mathcal{S}}_{x^l\tilde{U}_i} = \sum_{j=1}^{b} a_{ij}\left((x')^{k-1}S+1\right)^l \partial_{u_j} + \sum_{i=1}^{f} b_{ij}\left((x')^{k-1}S+1\right)^l (x')^{l-m}\,\partial_{w_j}$$

$$+\frac{l\left((x')^{k-1}S+1\right)^{k-1}(x')^{k-1}}{2} cl_g\left(x^{-l}\tilde{U}_i^{\flat}\right) cl_g\left(x^{-k}dx\right)$$

$$+\frac{\left((x')^{k-1}S+1\right)^l (x')^l}{4}\sum_{j,p=1}^{f}\left(\phi^* g_Y\left(S^{\phi}(V_j,V_p),\tilde{U}_i\right) - g_{\partial X/Y}\left([\tilde{U}_i,V_p],V_j\right)\right) cl_g\left(x^{-m}V_j^{\flat}\right) cl_g\left(x^{-m}V_p^{\flat}\right)$$

$$+\frac{\left((x')^{k-1}S+1\right)^{2l-m}(x')^{2l-m}}{4}\sum_{j=1}^{b}\sum_{p=1}^{f} g_{\partial X/Y}\left(R^{\phi}(\tilde{U}_i,\tilde{U}_j),V_p\right) cl_g\left(x^{-l}\tilde{U}_j^{\flat}\right) cl_g\left(x^{-m}V_p^{\flat}\right)$$

$$-\frac{\left((x')^{k-1}S+1\right)^l (x')^l}{8}\sum_{j,p,q=1}^{b} g_Y\left(R_Y(U_i,\left((x')^l u_q + y_q'\right)\partial_{y_q})U_j,U_p\right) cl_g\left(x^{-l}\tilde{U}_j^{\flat}\right) cl_g\left(x^{-l}\tilde{U}_p^{\flat}\right)$$

$$\beta_x^*\nabla^{\mathcal{S}}_{x^m\tilde{V}_i} = +\left((x')^{k-1}S+1\right)^m (x')^m\left[\underbrace{V_i + \frac{1}{4}\sum_{j,p=1}^{f} g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_j,V_p\right) cl_g\left(x^{-m}V_j^{\flat}\right) cl_g\left(x^{-m}V_p^{\flat}\right)}_{\nabla^{\mathcal{S}_{\partial X/Y}}_{V_i}}\right.$$

$$+\frac{m\left((x')^{k-1}S+1\right)^{k-1}(x')^{k-1}}{2}cl_g\left(x^{-m}V_i^\flat\right)cl_g\left(x^{-k}dx\right)$$
$$+\frac{\left((x')^{k-1}S+1\right)^{l}(x')^{l}}{2}\sum_{j=1}^{f}\sum_{p=1}^{b}\phi^*g_Y\left(S^\phi(V_i,V_j),\tilde{U}_p\right)cl_g\left(x^{-m}V_j^\flat\right)cl_g\left(x^{-l}\tilde{U}_p^\flat\right)$$
$$-\frac{\left((x')^{k-1}S+1\right)^{2l-m}(x')^{2l-m}}{8}\sum_{j,p=1}^{b}g_{\partial X/Y}\left(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i\right)cl_g\left(x^{-l}\tilde{U}_j^\flat\right)cl_g\left(x^{-l}\tilde{U}_p^\flat\right)$$

On the other hand, for $m\leq 0$, with the help of (12):

$$\beta_x^*\nabla^{\mathcal{S}}_{x^k\partial_x}=\left((x')^{k-m-1}S+1\right)^k(x')^m\,\partial_S$$
$$\beta_x^*\nabla^{\mathcal{S}}_{x^l\tilde{U}_i}=\sum_{j=1}^{b}a_{ij}\left((x')^{k-m-1}S+1\right)^l(x')^m\,\partial_{u_j}+\sum_{j=1}^{f}b_{ij}\left((x')^{k-m-1}S+1\right)^l(x')^{l-m}\,\partial_{z_j}$$
$$+\frac{l\left((x')^{k-m-1}S+1\right)^{k-1}(x')^{k-1}}{2}cl_g\left(x^{-l}\tilde{U}_i^\flat\right)cl_g\left(x^{-k}dx\right)$$
$$+\frac{\left((x')^{k-m-1}S+1\right)^{l}(x')^{l}}{4}\sum_{j,p=1}^{f}\left(\phi^*g_Y\left(S^\phi(V_j,V_p),\tilde{U}_i\right)-g_{\partial X/Y}\left([\tilde{U}_i,V_p],V_j\right)\right)cl_g\left(x^{-m}V_j^\flat\right)cl_g\left(x^{-m}V_p^\flat\right)$$
$$+\frac{\left((x')^{k-m-1}S+1\right)^{2l-m}(x')^{2l-m}}{4}\sum_{j=1}^{b}\sum_{p=1}^{f}g_{\partial X/Y}\left(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p\right)cl_g\left(x^{-l}\tilde{U}_j^\flat\right)cl_g\left(x^{-m}V_p^\flat\right)$$
$$-\frac{\left((x')^{k-m-1}S+1\right)^{l}(x')^{l}}{8}\sum_{j,p,q=1}^{b}g_Y\left(R_Y(U_i,\left((x')^{l-m}u_q+y_q'\right)\partial_{y_q})U_j,U_p\right)cl_g\left(x^{-l}\tilde{U}_j^\flat\right)cl_g\left(x^{-l}\tilde{U}_p^\flat\right)$$

$$\beta_x^*\nabla^{\mathcal{S}}_{x^m\tilde{V}_i}=\left((x')^{k-m-1}S+1\right)^m(x')^m\left[\underbrace{V_i+\frac{1}{4}\sum_{j,p=1}^{f}g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_j,V_p\right)cl_g\left(x^{-m}V_j^\flat\right)cl_g\left(x^{-m}V_p^\flat\right)}_{\nabla^{\mathcal{S}_{\partial X/Y}}_{V_i}}\right]$$
$$+\frac{m\left((x')^{k-m-1}S+1\right)^{k-1}(x')^{k-1}}{2}cl_g\left(x^{-m}V_i^\flat\right)cl_g\left(x^{-k}dx\right)$$
$$+\frac{\left((x')^{k-m-1}S+1\right)^{l}(x')^{l}}{2}\sum_{j=1}^{f}\sum_{p=1}^{b}\phi^*g_Y\left(S^\phi(V_i,V_j),\tilde{U}_p\right)cl_g\left(x^{-m}V_j^\flat\right)cl_g\left(x^{-l}\tilde{U}_p^\flat\right)$$
$$-\frac{\left((x')^{k-m-1}S+1\right)^{2l-m}(x')^{2l-m}}{8}\sum_{j,p=1}^{b}g_{\partial X/Y}\left(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i\right)cl_g\left(x^{-l}\tilde{U}_j^\flat\right)cl_g\left(x^{-l}\tilde{U}_p^\flat\right)$$

From this, it is straightforward to compute the local expression for the Dirac operator. As we moreover know the $x\to 0$ asymptotics of the scalar curvature scal_g, we can get $\eth^2$ using Lichnerowicz's formula [Mel93, (8.77)]:

$$\beta_x^*\eth^2=-\sum_{i=1}^{n}\left(\left(\beta_x^*\nabla^{\mathcal{S}}_{e_i}\right)^2-\beta_x^*\nabla^{\mathcal{S}}_{(\nabla_{e_i}e_i)}\right)+\frac{\beta_x^*\mathrm{scal}_g}{4}\tag{15}$$

Here $\nabla^{\mathcal{S}}$ is the spinor connection and ∇ is the Levi-Civita connection of the metric g. We can use it to arrive at a local expression for the heat equation at the face ff, and choosing $\rho_{\mathrm{ff}}=x'$ one can calculate the normal operators of various orders to try solve away the Taylor series at ff.

□

In order to study differential operators related to the metric g_{klm} ("geometric differential operators"), it is often convenient to translate the problem to a metric conformal to g_{klm} whose associated vector fields form a Lie subalgebra of the algebra of vector fields:

Definition 4.11. In a conformal class of metrics $\mathcal{G}_{(k-l),(l-m)} := \{x^{-2\alpha} g_{klm}\}_{\alpha\in\mathbb{Z}}$, we will single out a metric $h = x^{-2\alpha_0} g_{klm} \in \mathcal{G}_{(k-l),(l-m)}$, which we call the **reference metric** for that class, satisfying[81]:

$$\alpha_0 = \begin{cases} \min\{\alpha \in \mathbb{Z} : \left(x^\alpha \Gamma\left({}^{klm}TX\right),\ [\cdot,\cdot]\right) \text{ is a Lie algebra}\} = \max\{1-k, -l, -m\} & \text{if} \quad k \neq l \quad \text{or} \quad k \neq m \\ 1-k & \text{if} \quad k = l = m \end{cases}$$

□

Remark 4.12. Since we just look for the Lie algebra structure on vector fields, we could have chosen any metric which blows up faster than the reference metric as $x \to 0$, i.e. the reason why we select the minimum is to have a canonical choice in each conformal class.
Remember that if $l = m$ this kind of analysis is overkill, since one could consider it as a case of trivial base or trivial fibres. In particular, if $k = l = m$ we decide to work with 0-metrics as reference metric (this explains the case distinction in the definition) instead of g_{000}, since there is already existing literature and this ensures, for example, that the reference metric blows up faster than $\frac{dx}{x}$ along the x-directions as $x \to 0$, so it is complete. Note that here g_{000} represents a manifold with boundary, not a closed manifold as in the previous section.

□

From now on we will assume we work with a fixed g_{klm} metric, which will be the representative of its conformal class $\mathcal{G}_{(k-l),(l-m)}$, and denote its reference metric h (without extra indices for simplicity), with algebra of vector fields $\Gamma\left({}^hTX\right)$. The point is that the Lie algebra structure allows us to define h-differential operators as the universal enveloping algebra of $\Gamma({}^hTX)$, i.e. by composing vector fields in $\Gamma({}^hTX)$ with $\mathcal{C}^\infty(\mathcal{U})$-coefficients. Locally:

$$P \in \mathrm{Diff}^r_h(X) \implies P = \sum_{q+|\alpha|+|\gamma|\leq r} p_{q,\alpha,\gamma}(x,y,z) \left(x^{k+\alpha_0} D_x\right)^q \left(x^{l+\alpha_0} D_x\right)^\alpha \left(x^{m+\alpha_0} D_z\right)^\gamma$$

In particular, differential operators which are "morally" of order r associated to g_{klm} (the Dirac operator in g_{klm} is morally of order 1 and the Laplacian, of order 2) can then be seen as elements in $x^{-\alpha_0 r}\mathrm{Diff}^r_h(X)$ if $\alpha_0 > 0$, or just as elements in $\mathrm{Diff}^r_h(X)$ if $\alpha_0 \leq 0$.
Denote $(k_0, l_0, m_0) = (k+\alpha_0, l+\alpha_0, m+\alpha_0)$, i.e. $h = g_{k_0 l_0 m_0}$. With this local expression, ellipticity can be defined straighforwardly by introducing the natural notion of h-principal symbol:

Definition 4.13. For a reference metric $h = g_{k_0 l_0 m_0}$, the h-**principal symbol** $\sigma_h(P)$ of an h-differential operator $P \in \mathrm{Diff}^r_h(X)$ with the local form above is the r-homogeneous polynomial

$$\sigma_h(P)(\xi,\eta,\mu) = \sum_{q+|\alpha|+|\gamma|=r} p_{q,\alpha,\gamma}(x,y,z)\xi^q \eta^\alpha \mu^\gamma$$

on elements $\xi x^{-k_0} dx + \eta x^{-l_0} dy + \mu x^{-m_0} dz \in \Gamma\left({}^hT^*X\right)$ of the h-cotangent bundle. It fits into a short exact sequence

$$0 \longrightarrow \mathrm{Diff}^{r-1}_h(X) \hookrightarrow \mathrm{Diff}^r_h(X) \xrightarrow{\sigma_h} P^r({}^hT^*X) \longrightarrow 0$$

The operator is called h-**elliptic** if $\sigma_h(P)$ is invertible outside the zero section, i.e. for all $(\xi,\eta,\mu) \in \Gamma({}^hTX) \setminus \{(0,0,0)\}$.

□

Furthermore, we can microlocalize this graded algebra[83] by finding the appropriate double space, in the philosophy of Melrose. This consists of a sequence of blow-ups of X^2 resolving the kernels of h-differential operators so that they lift to be polyhomogeneous conormal with respect to the lifted diagonal, smoothly up to the front face of the last blow-up.

In the front face of the last blow-up defining the double space X_h^2 we look for coordinates of the following form:

- If $(k_0, l_0, m_0) = (1, 0, 0)$ we are in the b-case [Mel93]:

  $$s = \frac{x}{x'}, \quad x', \quad y, \quad y'$$

  It can be obtained by $X_b^2 = [X^2; \partial X \times \partial X]$

- If $(k_0, l_0, m_0) = (1, 1, 1)$ this is the 0-case:

  $$s = \frac{x}{x'}, \quad x', \quad u = \frac{y - y'}{x'}, \quad y'$$

  and it comes via $X_0^2 = [X^2; \{x = x' = 0,\ y - y' = 0\}]$. The manifold blown up is the boundary diagonal $\mathrm{diag}_{\partial X} = \partial X \times_{\partial X} \partial X \subset X \times X$ (think of this as a pull-back diagram).

- If $(k_0, l_0, m_0) = (1, 1, 0)$ this is the edge case [Maz91]:

  $$s = \frac{x}{x'}, \quad x', \quad u = \frac{y - y'}{x'}, \quad y', \quad z, \quad z'$$

  and it can be achieved by the blow-up $X_e^2 = [X^2; \{x = x' = 0,\ y = y'\}]$. Note this is the same as above but with the extra fibre coordinates; the p-submanifold blown up is the fibre diagonal $\mathrm{diag}_\phi = \partial X \times_\phi \partial X$. In these terms, the previous case corresponds to $\phi = id$.

- If $k_0 \geq 2$ and $l_0 = m_0 = 0$ this is a cusp-like case (cusp is $k_0 = 2$):

  $$S = \frac{x - x'}{(x')^{k_0}}, \quad x', \quad y, \quad y'$$

  It comes from two blow-ups (the second quasi-homogeneous of order $k_0 - 1$): $X_{c,k_0}^2 = [X_b^2; \{s = 1,\ x = x' = 0\}]$.

- If $k_0 \geq 2$ and $l_0 = m_0 \geq 1$ this is an ac-like case (ac is $k_0 = 2$, $l_0 = m_0 = 1$):

  $$S = \frac{x - x'}{(x')^{k_0}}, \quad x', \quad u = \frac{y - y'}{(x')^{l_0}}, \quad y'$$

  If $k_0 = l_0 + 1$, this is achieved by two blow-ups, the second quasi-homogeneous of order l_0: $X_{ac,(l_0+1,l_0)}^2 = [X_b^2; \{s = 1,\ x = x' = 0,\ y - y' = 0\}]$. If instead $k_0 \geq l_0 + 2$, we need to perform a blow-up of order $k_0 - l_0 - 1$ in between the two: $X_{ac,(k_0,l_0)}^2 = [X_{c,k_0-l_0-1}^2; \{\tilde{S} = 1,\ x = x' = 0,\ y - y' = 0\}]$, where $\tilde{S} = \frac{x - x'}{(x')^{k_0 - l_0 - 1}}$.

[83] Introducing (quasi-)inverses in order to solve differential equations by parametrix constructions, trying to upgrade it into a ring. We have already carried out a completion (very loosely akin to this one) with other mathematical objects at the start of this manuscript: when we talked about K-theory, which upgrades the monoid structure of vector bundles into a group one.

- If $k_0 \geq 2$, $l_0 \geq 1$ and $m_0 = 0$ this is a (0-)ϕ-like case (ϕ is g_{210}) and is treated exactly as the previous, up to adding the coordinates z, z'. For a geometrical description in the ϕ-case, see [GSHV25, Figure 2].

These choices of coordinates ensure that the lifts of vector fields in $\Gamma({}^hTX)$ are transversal to the lifted diagonal and form a Lie subalgebra that is tangent to the front face and spans its tangent bundle. This can be checked explicitly by noting that under coordinates:

$$S = \frac{x - x'}{(x')^{k_0}}. \quad x', \quad u = \frac{y - y'}{(x')^{l_0}}, \quad y', \quad z, \quad z'$$

(the computations also work for s in case $k_0 = 1$), the lifts satisfy:

$$\beta^* x^{k_0}\partial_x = ((x')^{k_0-1}S + 1)^{k_0}\partial_S, \quad \beta^* x^{l_0}\partial_y = ((x')^{k_0-1}S + 1)^{l_0}\partial_u$$

which restricted to the front face $\mathrm{ff} \subset \{x' = 0\}$ produce the basis vectors ∂_S and ∂_u of $T\mathrm{ff}/\mathrm{diag}_\phi$ for the fibration[84]

$$\mathrm{ff} \longrightarrow \partial X \times_\phi \partial X = \mathrm{diag}_\phi, \quad (S, u, y', z, z') \mapsto (y', z, z')$$

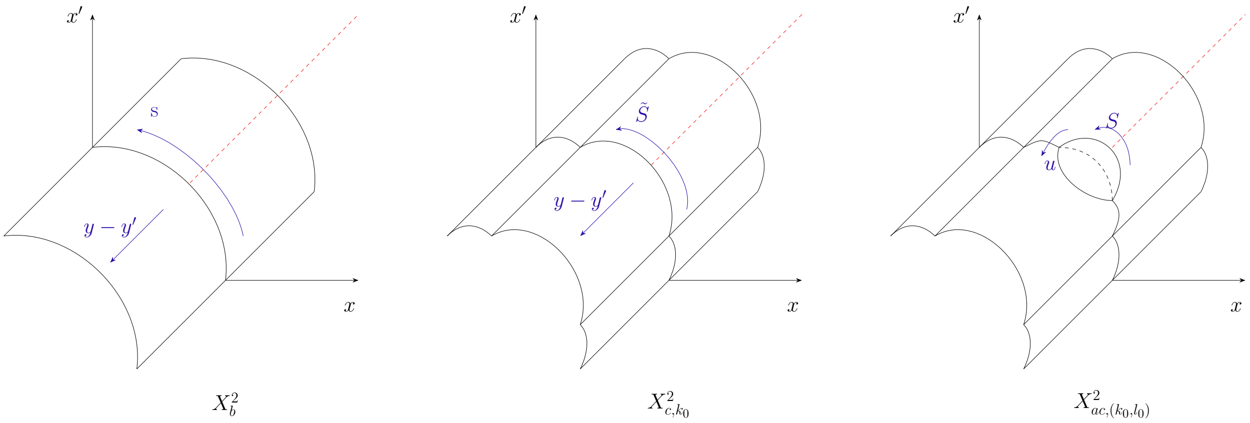


Figure 25: Constructing the double space for a reference metric for which $k_0 \geq l_0 + 2 \geq 1$. We start with the usual b-blow-up, then a quasi-homogeneous cusp-like blow-up (producing X^2_{c,k_0}) and finally the fibre diagonal is blown up. The relevant coordinates on the faces are painted blue and the lifted diagonal is added in red. Compare with Figure 26.

In particular, this means that differential operators in $\mathrm{Diff}^r_h(X)$ lift to X^2_h as:

$$\beta^* P = \sum_{q+|\alpha|+|\gamma| \leq r} \left((x')^{k_0-1}S + 1\right)^{k_0 q + l_0|\alpha| + m_0|\gamma|} p_{q,\alpha,\gamma}\left((x')^{k_0}S + x', (x')^{l_0}u + y', (x')^{m_0}w + z'\right) D^q_S D^\alpha_u D^\gamma_w$$

with normal/indicial operator at the front face:

$$\beta^* P\big|_{\mathrm{ff}} = \sum_{q+|\alpha|+|\gamma| \leq r} p_{q,\alpha,\gamma}(0, y', z') D^q_S D^\alpha_u D^\gamma_w$$

[84]In the b-case (resp. cusp case), the coordinate y is left untouched and we have a corresponding fibration

$$\mathrm{ff} \longrightarrow \partial X \times \partial X, \quad (s, x', y, y') \mapsto (y, y')$$

(resp. S instead of s) so that $\beta^* x^{k_0}\partial_x$ restricts to ff as ∂_s (resp. ∂_S) and so spans the tangent fibres $T\mathrm{ff}/(\partial X \times \partial X)$.

In analogy with [Mel93, §4.14, 4.15], if we studied Fredholmness of h-operators via a parametrix construction, ellipticity would not be enough, since the non-vanishing of the normal operator is an obstruction to compactness of the rest term. Thus, one needs to ask for more than invertibility of the principal symbol, namely invertibility of a normal family which can be seen as a boundary symbol (in the b-case, this is the Mellin transform of the indicial operator [Mel93, Theorem 5.40][85]). Note that the normal operator takes into account all orders of the operator and not only the leading term. The **normal/indical family** is defined by Fourier/Mellin transform of the normal operator in the directions blown up in the double space (i.e. those whose metric contributions have powers of x):

- In the b- and (generalized) cusp-cases, the indicial family is given by

$$N(P)(y) = \sum_{q+|\alpha|\leq r} p_{q,\alpha}(0,y) D_S^q D_y^\alpha \implies \widehat{N(P)}(y;\xi) = \sum_{q+|\alpha|\leq r} p_{q,\alpha}(0,y)\xi^q D_y^\alpha$$

  This is a family of differential operators in the boundary, parametrized by ξ.

- In the rest (give or take the z-directions depending on whether we have trivial fibres or not), the normal family is:

$$N(P)(y',z) = \sum_{q+|\alpha|\leq r} p_{q,\alpha}(0,y',z) D_S^q D_u^\alpha D_z^\gamma \implies \widehat{N(P)}(y',z;\xi,\eta) = \sum_{q+|\alpha|\leq r} p_{q,\alpha}(0,y',z)\xi^q \eta^\alpha D_z^\gamma \tag{16}$$

  In this case, the normal operator is a family of differential operators acting on $\mathbb{R}\times\mathbb{R}^b\times Z$ parametrized by the base $y'\in Y$ and the normal family is a family of differential operators acting on the fibres Z, parametrized by $(y',\xi,\eta)\in Y\times\mathbb{R}\times\mathbb{R}^b$.

The cases known in the literature suggest that an elliptic operator is Fredholm[86] if the normal family is invertible for all cotangent coordinates (including the zero section). In particular, many relevant geometric differential operators are not Fredholm in such Sobolev domains (see §4.3).
The definition of the calculi of pseudodifferential operators is straightforward once the double space construction is carried out:

Definition 4.14. The **small calculus of h-pseudodifferential operators** $\Psi_h^r(X)$ consists of distributions on X_h^2 conormal with respect to the lifted diagonal, smoothly up to the front face ff and vanishing to infinite order at all other boundary faces.
The **full calculus** $\Psi_h^{r,\mathcal{E}}(X)$ extends the small calculus by allowing general polyhomogenous conormal (cf. §3.1.2) behavior at all other faces, given by the data of the index set $\mathcal{E}$:

$$\Psi_h^{r,\mathcal{E}}(X) = \Psi_h^r(X) + \mathcal{A}_{phg}^{\mathcal{E}}(X_h^2)$$

□
The small calculus contains the parametrices (i.e. "quasi-inverses") of elliptic differential operators, while the full calculus is closed under composition.
To underscore afresh the virtue of reference metrics: when dealing with geometric differential operators with respect to g_{klm}, we can rescale them to make them belong to the algebra of differential

[85] Moreover, he shows that if we let the operator act on weighted Sobolev spaces, the indical family is invertible for all weights outside (the imaginary part of) a discrete set which he denotes $\mathrm{Spec}_b(P)$. This is in contrast with e.g. the ac-case, where Fredholmness in a Sobolev space means Fredholmness in all Sobolev spaces [Kot09, §2.2]

[86] If we consider its action on (weighted) Sobolev spaces with respect to the density bundle associated with g. That said, this is not the only reasonable choice (see e.g. the split Sobolev spaces in [GH14, Def. 7], which come with a different characterization of Fredholmness [GH14, Theorem 13]).

operators of the corresponding reference metric. In that setting, their spectral and microlocal study can be performed thanks to the corresponding geometric-analytical structures e.g. this pseudodifferential operator technology. Once the analysis is carried out, we can just rescale back to the original metric. Such a streamlined approach is quite often seen in the literature.
For instance, if we care about local index theory, for constructions like the resolvent (recall Remark 3.21) it is useful to transport the parametrix problem to a reference metric and solve it in that setting, as done in [Vai01, §3] or [AGR23, (1.4), §4.3], but also to show Fredholmness [Liu25, §7].

4.2 Getzler rescaling

As already pointed out at the end of Section 2.3.2 and can be read from the orders at each face in the blow-up spaces constructed in §3.2.1, it is not necessarily the coefficient of the leading order term in the asymptotic expansion of the heat kernel at each face the one that contributes to the McKean-Singer formula, but rather higher order terms related to the dimension of the manifold X or of the base Y of the fibre bundle $\phi : \partial X \to Y$ (see e.g. in Figures 27 and 30 how $n = \dim X$ and $b = \dim Y$ affect what the leading order of the heat kernel is at various faces). Computing such coefficients is not at all straightforward[87]. The usual approach is to employ a variant of the rescaling technique of Getzler [Get86].
On top of that, we find another issue in several singular settings: the heat kernel is a priori no longer a trace class operator, that is, the Lidskii integral defining its supertrace diverges. The remedy: we *through away* (in a specific way) the terms in the integral that diverge; formally, this is achieved by the use of **renormalized integrals**.

4.2.1 Renormalized integrals and the integral term in the McKean-Singer formula

Definition 4.15. [Alb07, §6.1] For a polyhomogeneous conormal density $\nu \in \mathcal{A}^{\mathcal{E}}_{phg}(X; {}^b\Omega(X))$ on a manifold with boundary X and a choice of boundary defining function ρ, the **renormalized integral** of ν is defined as

$$ {}^R\!\!\int_X \nu := \underset{z=0}{\mathrm{FP}} \int_X \rho^z \nu $$

where FP refers to the "finite part" procedure in complex integration. By the condition that polyhomogeneous expansions have powers with real part bounded below, it is clear that taking z (with real part) big enough makes the integral on the right converge, so the operation is well-defined by meromorphic continuation from a certain right half-plane in $\mathbb{C}_z$.
In general, it depends on the choice of boundary defining function ρ. By fixing a density μ, we could also talk about the renormalized integral of a polyhomogeneous conormal distribution $f \in \mathcal{A}^{\mathcal{E}}_{phg}(X)$ analogously.
The same can be done on a manifold with corners by splitting the integrals in each face with the corresponding choices of boundary defining functions.

□

Remark 4.16. The renormalization approach above often goes by the name of Riesz regularization, in contrast with Hadamard's, which is defined as:

$$ {}^H\!\!\int_X \nu := \underset{\varepsilon=0}{\mathrm{FP}} \int_{\{x \geq \varepsilon\}} \nu $$

[87] There are inductive formulas for this, but if we aim to obtain expressions for arbitrary dimension of X (or Y), they become untractable. For instance, [Mel93, (7.62)] or [BGV04, Theorem 2.26] give explicit iterative formulas which are however hardly identifiable as a seed to a geometric/topological invariant.

Both approaches do not always produce the same result and we choose to stick with Riesz'. More on them in [Alb09, §2.2].

□

In particular, Lidskii's formula can be renormalized for non-trace class operators:

$$ {}^R\mathrm{Tr}\,(A) = {}^R\!\!\int_X K_A(x,x) $$

and the formula for the supertrace is also clear.

Once we have shown that the kernel of the heat operator $e^{-t\eth^2}$ is a polyhomogeneous conormal distribution in $HX^2_{g_{klm}}$, we can compute its supertrace ${}^R\mathrm{Str}\left(e^{-t\eth^2}\right)$ as the renormalized integral of this kernel over the lifted diagonal in this heat blow-up space. This means we have to make a choice of boundary defining function at each of the faces intersecting the lifted diagonal and we only require a very fine understanding of the kernel at them. This choice does not appear in the local index formula we obtain at the end of the day, and so the method should be independent of it.

We are hence lead to an adapted ("generalized") form of the McKean-Singer formula that we can use in any of these settings:

$$ \lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = \lim_{t\to 0} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) + \int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) dt \tag{17} $$

In particular, for a g_{klm} metric, we can perform formal manipulations on the argument of the integral term and re-express it as:

$$ \begin{aligned} \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) &= \underset{z=0}{FP}\left(\frac{1}{2}\mathrm{Str}\left(cl_g\left(d(x')^z\right)\eth e^{-t\eth^2}\right)\right) \\ &= \underset{z=0}{FP}\left(\frac{z}{2}(x')^{z+k-1}\mathrm{Str}\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\right) \\ &= \underset{z=0}{FP}\left(\frac{z}{2}\int_{\partial X}\int_0^\delta (x')^{z+k-1}\mathrm{str}_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\frac{dx'}{(x')^k}\frac{dy'}{(x')^{lb}}\frac{dz'}{(x')^{mf}}\right) \\ &= \frac{1}{2}\int_{\partial X}\left[\mathrm{str}_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\right]_{lb+mf} dy'dz' \end{aligned} \tag{18} $$

This transforms an integral over a collar neighborhood of the boundary[88] into an integral over the faces of the heat blow-up space in the $t\in(0,\infty)$ regime (so has the flavour of a Stokes' theorem argument). Since the $t\to 0$ contribution can also be recovered from the associated model problems at the corresponding faces, this means that the "finite time" contributions to the local index formula all come from the faces in HX^2_g intersecting the lifted diagonal[89].

[88] There is no contribution from the rest of the manifold, as occurred in the closed case. To see this, we could use cutoff functions to split the heat problem into the collar contribution and one away from it. We would then construct the corresponding boundary parametrix and interior parametrix and patch them together. The interior parametrix corresponds to a smoothing operator away from the boundary and therefore similar arguments (e.g. [BGV04, Theorem 3.50]) as in the closed case justify that its supertrace is independent of time.

[89] For the large time limit we could look at the compactified heat blow-up spaces obtained by "gluing" at $t\to\infty$ the blow-up spaces where the low energy resolvent is polyhomogeneous (see e.g. Figures 27 and 30). This limit is of global nature and can be studied like in Remark 3.21. The crux is that, in the finite time regime $t\in[0,\infty)$, the heat kernel vanishes to infinite order away from the faces that intersect the lifted diagonal. Thus, a parametrix of the heat equation can be constructed in a neighborhood of those faces by solving only their model problems before restricting to the diagonal for the supertrace. However, for $t\to\infty$ more faces have non-trivial asymptotics and could affect the expansion in the neighborhood of the lifted diagonal that contributes to the left-hand side of the McKean-Singer formula.

For the first equality, there is an usual trick which can be used before worrying about convergence by choosing z big enough:

$$\begin{aligned}\partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) &= \underset{z=0}{FP}\left(\partial_t \mathrm{Str}\left((x')^z e^{-t\eth^2}\right)\right) = \underset{z=0}{FP}\left(-\frac{1}{2}\mathrm{Str}\left((x')^z\left[\eth, \eth e^{-t\eth^2}\right]\right)\right) \\ &= \underset{z=0}{FP}\left(\frac{1}{2}\mathrm{Str}\left(z(x')^{z+k-1} cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\right)\end{aligned}$$

where we applied the same reasoning as [BGV04, Theorem 3.50]:

$$\begin{aligned}\partial_t \mathrm{Str}\left((x')^z e^{-t\eth^2}\right) &= -\mathrm{Str}\left((x')^z \eth^2 e^{-t\eth^2}\right) = -\frac{1}{2}\mathrm{Str}\left((x')^z\left[\eth, \eth e^{-t\eth^2}\right]\right) \\ (x')^z\left[\eth, \eth e^{-t\eth^2}\right] &= \left[\eth, (x')^z \eth e^{-t\eth^2}\right] - \left[\eth, (x')^z\right]\eth e^{-t\eth^2}\end{aligned}$$

For z large enough, $(x')^z \eth e^{-t\eth^2}$ is smooth and therefore the first term on the right vanishes [BGV04, Lemma 3.49]. For the second equality,

$$\left[\eth, (x')^z\right] = cl_g\left(d((x')^z)\right) = z(x')^{z+k-1} cl_g\left(\frac{dx}{x^k}\right)$$

by writing the Clifford action with respect to the base element in the normal direction of $\Gamma({}^{klm}T^*X)$. The third and fourth equality correspond to writing down and computing the supertrace integral in a collar neighborhood $[0,\delta)_{x'} \times \partial X$ of the boundary:

$$\begin{aligned}\partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) &= \underset{z=0}{FP}\left(\frac{z}{2}\mathrm{Str}\left((x')^{z+k-1} cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\right) \\ &= \underset{z=0}{FP}\left(\frac{z}{2}\int_{\partial X}\int_0^\delta (x')^{z+k-1}\, \mathrm{str}_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\frac{dx'}{(x')^k}\frac{dy'}{(x')^{lb}}\frac{dz'}{(x')^{mf}}\right) \\ &= \underset{z=0}{FP}\left(\frac{z}{2}\int_{\partial X}\int_0^\delta (x')^{z-lb-mf}\, \mathrm{str}_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right)\frac{dx'}{x'}dy'dz'\right)\end{aligned}$$

Because the heat kernel is polyhomogeneous conormal in HX^2_g, it has an asymptotic expansion in all the faces in the regime $t \in (0,\infty)$, for all of which x' is a valid choice of boundary defining function (we fix this choice from now on; it corresponds to the lift of the boundary defining function from the right X factor along $HX^2_g \to (\mathbb{R}_+)_t \times X_x \times X_{x'}$). The same holds for the Schwartz kernel of the lift of $\eth$ to HX^2_g. Thus, the kernel of $\eth e^{-t\eth^2}$ has an expansion in powers of (x') and thus its pointwise supertrace does too:

$$str_p\left(cl_g\left(\frac{dx}{x^k}\right)\eth e^{-t\eth^2}\right) \underset{x'\to 0}{\sim} \sum_\alpha \psi_\alpha (x')^\alpha$$

Plugging in this expansion, we can deal with the finite part computation:

$$\begin{aligned}\underset{z=0}{FP}\left(\frac{z}{2}\int_0^\delta (x')^{z-lb-mf}\sum_\alpha \psi_\alpha (x')^\alpha \frac{dx'}{x'}\right) &= \underset{z=0}{FP}\left(\frac{z}{2}\sum_\alpha \psi_\alpha \int_0^\delta (x')^{z-lb-mf+\alpha}\frac{dx'}{x'}\right) \\ &= \frac{1}{2}\underset{z=0}{FP}\left(\sum_\alpha \frac{z\,\delta^{z-lb-mf+\alpha}\,\psi_\alpha}{z-lb-mf+\alpha}\right) = \frac{\psi_{lb+mf}}{2}\end{aligned}$$

because only the term with denominator z can survive the finite part computation with z in the numerator.

All in all, in the regime $t \in (0,\infty)$ in HX^2_g, the index formula only sees terms of order $lb + mf$ at the corresponding face[90] and thus, the rescaling should be adapted to this fact.

Example 4.17. In the asymptotically conical and 0-fibred cusp cases, $l = 1$ and $f = 0$ (resp. $m = 0$). Thus, in the face of the heat blow-up space lying in the $t \in (0,\infty)$ regime (in this case, ϕf), we care about the b-th term in the expansion at that face (i.e. the term with $\rho^b_{\phi \mathrm{f}} \propto (x')^b$). However, the heat kernel has leading order 0 there. In any case, since the argument of the pointwise supertrace contains the composition of kernels $\eth e^{-t\eth^2}$, it is not enough to compute the b-th term of the heat kernel.

□

4.2.2 Getzler rescaling and the tf contribution

We will now describe Getzler rescaling in detail in the geometrically invariant formulation of [Mel93, §8.1-8.4]. This geometric rescaling is also the theoretical framework supporting the forging of the rescaled bundles ${}^{klm}TX$, among others. We will again motivate the construction through examples.

Example 4.18. For a compact manifold with boundary, the bundle $T\partial X \to \partial X$ of vector fields tangent to the boundary is a subbundle of $TX\big|_{\partial X} \to \partial X$. The subspace

$$\mathcal{V}_b = \{W \in \mathcal{C}^\infty(X, TX) : W\big|_{\partial X} \in \mathcal{C}^\infty(\partial X, T\partial X)\} \subset \mathcal{C}^\infty(X, TX) = \Gamma(TX)$$

is a finitely generated projective $\mathcal{C}^\infty(X)$-module and so, the Serre-Swan theorem [Mel93, Proposition 8.1] ensures that there exists a vector bundle ${}^bTX$ with an induced $\mathcal{C}^\infty$-structure and a bundle map

$$\iota_b : {}^bTX \longrightarrow TX, \quad \text{such that} \quad \iota_b^* \mathcal{V}_b = \mathcal{C}^\infty(X, {}^bTX)$$

and ι_b is an isomorphism over the interior $\mathring{X}$ (compare this with the definition of a Lie algebroid with anchor map ι; this construction can however be carried for spaces of sections of arbitrary vector bundles, not necessarily for the tangent bundle).
Other interesting examples are the subspace

$$\mathcal{V}_{ac} = \{W \in \mathcal{V}_b : W\big|_{\partial X} = 0,\ Wx = O(x^2)\} \subset \Gamma(TX)$$

and, if the boundary is the total space of a fibre bundle $\phi : \partial X \to Y$ as considered above, the subspace

$$\mathcal{V}_\phi = \{W \in \mathcal{V}_b : W\big|_{\partial X} \text{ tangent to the fibres, } Wx = O(x^2)\} \subset \Gamma(TX)$$

where x is a choice of boundary defining function[91]. These produce bundles:

$$\iota_\star : {}^\star TX \longrightarrow TX, \quad \iota_\star^* \mathcal{V}_\star = \mathcal{C}^\infty(X, {}^\star TX), \quad \star \in \{ac,\ \phi\}$$

The same can be done for any reference metric h (some will be dependent on the choice of boundary defining function, as with $\mathcal{V}_\phi$) and the expression above would hold for $\star = h$ (or more generally, for an iterated fibration structure whose characteristic vector fields are a Lie subalgebra of the algebra of vector fields in X, e.g. if $l_q \geq 0$).

[90] The heat blow-up space constructions in §3.2.1 show that there is always only one face in this regime intersecting the lifted diagonal.

[91] $\mathcal{V}_{ac}$ is independent of this choice. In contrast, for $\mathcal{V}_\phi$ to remain invariant under a change of boundary defining function $\bar{x} = ax$ with smooth $a > 0$, a has to be constant along the fibres (to preserve $Wx = O(x^2)$), so the resulting geometric structures rely on this choice of equivalence class of boundary defining functions.

□

This way of applying the Serre-Swan theorem works for any manifold with boundary X, even non-compact, and any subbundle $\mathcal{F} \subset \mathcal{E}\big|_{\partial X}$ of a vector bundle $\mathcal{E} \to X$ restricted to the boundary. We can generalize the procedure in two ways: on the one hand, we allow a filtration at the boundary, i.e. an increasing sequence of subbundles of $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_j \subset \mathcal{E}\big|_{\partial X}$; on the other hand, we consider a manifold with corners X and different filtrations at different boundary hypersurfaces of X. The general (technical) statement would be:

Proposition 4.19. [Mel93, §8.2] Let W be a manifold with corners with boundary hypersurfaces H_i and a vector bundle $\mathcal{E} \to W$. Suppose that at each face there is a *jet filtration* (in the sense of [Mel93, (8.9)])

$$\mathcal{F}_0^{H_i} \subset \mathcal{F}_1^{H_i} \subset \cdots \subset \mathcal{F}_{j_i}^{H_i} \subset \mathcal{E}\big|_{H_i}$$

(this could be trivial, i.e. $j_i = 0$ and $\mathcal{F}_0^{H_i} = \mathcal{E}\big|_{H_i}$, meaning we do not need to have filtrations at each face). Denote by $\tilde{\mathcal{F}}_k^{H_i}$ an extension of $\mathcal{F}_k^{H_i}$ to a neighborhood of H_i, i.e. a representative of the equivalence class of subbundles of $\mathcal{E}$ around H_i which has the same jet structure at H_i as $\mathcal{F}$. Then, there is a smooth vector bundle ${}^{\mathcal{F}}\mathcal{E} \to W$ with bundle map $\iota_{\mathcal{F}} : {}^{\mathcal{F}}\mathcal{E} \to \mathcal{E}$ restricting to an isomorphism ${}^{\mathcal{F}}\mathcal{E}\big|_{\mathring{W}} \cong \mathcal{E}\big|_{\mathring{W}}$ such that

$$\mathcal{C}^\infty(W; {}^{\mathcal{F}}\mathcal{E}) = \iota_{\mathcal{F}}^* \left(\{u \in \Gamma(\mathcal{E}) : u \in \sum_{k=0}^{j_i} \rho_{H_i}^k \mathcal{C}^\infty(W; \tilde{\mathcal{F}}_k^{H_i}) + \rho_{H_i}^{k+1}\Gamma(\mathcal{E})\} \right)$$

In other words, there is a bundle whose sections are those sections of $\mathcal{E}$ whose expansion at a face H_i respects the filtration $\mathcal{F}^{H_i}$.
In addition, the rescaled bundle has a *canonical* form when restricted to H_i:

$${}^{\mathcal{F}}\mathcal{E}\big|_{H_i} \cong \bigoplus_{k=0}^{j_i+1} (N^*H_i)^k \otimes \left(\mathcal{F}_k^{H_i}/\mathcal{F}_{k-1}^{H_i}\right), \quad \mathcal{F}_{j_i+1}^{H_i} := \mathcal{E}\big|_{H_i}, \quad \mathcal{F}_{-1}^{H_i} = \{0\}$$

where N^*H_i denotes the conormal bundle to H_i induced by the *choice* of boundary defining function ρ_{H_i} (equiv. of direction transversal to H_i). Because of this choice, the isomorphism map is not canonical (cf. [Vai01, p. 82]).
Finally, sections vanishing to infinite order at a face in the original bundle preserve this property after rescaling.

□

The proposition is effectively shown by iterating the argument in [Mel93, Prop. 8.4] for each boundary hypersurface H_i in W. There are two caveats to this construction:

- One can achieve the jet condition by extending the filtration to a neighborhood of the face H_i. In practice, we do this by parallel transport along a vector field transversal to the face, i.e. we require a connection on $\mathcal{E}$[92]. The case of interest to us is the heat blow-up space

[92]In fact, a true connection [Mel93, first paragraph in §8.3]. Since one of the building blocks of the "natural connection" on HX_g^2 is the Levi-Civita connection of the metric g, Proposition 4.5 raises the question of whether this geometrical rescaling can be carried out for metrics where this connection is not a true connection, even if we can construct their heat blow-up spaces (e.g. for edge metrics).

$W = HX_g^2$. Denote the associated blow-down and projections by

$$\begin{array}{ccccc} & & HX_g^2 & & \\ & \swarrow^{\beta_x} & \downarrow \beta & \searrow^{\beta_{x'}} & \\ X & \xleftarrow[\pi_x]{} & \mathbb{R} \times X^2 & \xrightarrow[\pi_{x'}]{} & X \\ & & \downarrow \pi_t & & \\ & & \mathbb{R} & & \end{array}$$

with $\pi_t \circ \beta = \beta_t$ and $\beta_{xx'} = \pi_{xx'} \circ \beta = (\pi_x \times \pi_{x'}) \circ \beta = \beta_x \times \beta_{x'} : HX_g^2 \to X^2$. We would like to rescale the bundle $\mathcal{E} = \beta_{xx'}^* \mathrm{END}(E) \to HX_g^2$. Here, $\mathrm{END}(E) = \mathrm{HOM}(E, E) \to X^2$ is the bundle where the heat kernel $e^{-t\eth^2}$ at a fixed time t of the square of a Dirac operator associated to a Clifford module $E \to X$ lives, so its polyhomogeneous conormal lift for all times $t \geq 0$ lives in $\beta_{xx'}^* \mathrm{END}(E)$. The "natural" connection is (cf. [Vai01, p. 80]):

$$\nabla = \beta_x^* \nabla^E \otimes \beta_{x'}^* \nabla^{E^*} \otimes \beta_t^* (dt \otimes \partial_t) : \mathcal{C}^\infty(HX_g^2; \beta_{xx'}^* \mathrm{END}(E)) \longrightarrow \mathcal{C}^\infty(HX_g^2; T^* \left(HX_g^2\right) \otimes \beta_{xx'}^* \mathrm{END}(E))$$

where ∇^E is the Clifford connection in E used to define the Dirac operator (see §3.1.4). As a result, ∇ carries (and lifts to HX_g^2) the information about the metric g (reflected in the Clifford algebra $Cl({}^gTX)$) and the bundle E[93].

- According to [Mel93, bottom of p. 302], the jet filtration depends on ∇ and on which transversal direction we choose to extend the filtration via parallel transport, up to positive smooth function coefficients. Once the jet filtration or the extension of the filtration to the boundary is obtained, the resulting rescaled bundle is uniquely defined. This means we need to make yet another choice in our analysis, namely that of the transversal direction (cf. [Mel93, (8.41), (8.42)] [Vai01, p. 91] [AR09, after Theorem 1.4] [AGR23, §5.1]), although once more this choice will not echo on the index formula.

The reason why Proposition 4.19 is interesting to us is the following: recall the discussion in §2.3.2 about the heat kernel method applied to closed manifolds (so keep the corresponding heat blow-up space in mind, i.e. Figure 6). In there, we remarked that a major difficulty in effectively employing the McKean-Singer formula for a general elliptic differential operator P was the fact that the terms in the $t \to 0$ asymptotic contributing to the supertrace were difficult to compute explicitly for arbitrary $n = \dim X$. This is where the work of Getzler comes in: based on ideas of Witten and Álvarez-Gaumé [AG83], who study certain operators (Dirac-type) as charges of supersymmetric quantum mechanical systems, he is able to compute local expressions for the n-th term in the expansion in powers of $\sqrt{t}$ of the heat kernel as $t \to 0$ using its Feynman-Kac representation [Get86].

The point is the following: first, taking the limit $t \to 0$ localizes the problem, i.e. the kernel at a point depends only on the metric and the connection in a small neighborhood of it, and so it is sufficient to consider a Dirac operator in $\mathbb{R}^n$ (once again: our heat blow-up spaces look the same

[93] After the rescaling, the connection does not necessarily act in the rescaled bundle along all original directions. To determine which directions still function, we can apply [Mel93, Lemma 8.10], and to compute how they act, we can use [Mel93, (8.36)]. In the context of g_{klm} metrics appearing in the literature, the rescaling at the ff face is usually done only in the base directions (and perhaps the normal direction), and one can use Proposition 4.8 2. and 3. to show that the connection acts on the rescaled bundle along the directions tangent to the fibres of $\mathrm{ff} \to Y$ (when there is such a fibration). This is also the case in the contexts of Sections 4.3 (with $Y = \partial X$) and 4.4, although we compute the rescaled connections with a different method. Check [Vai01, Lemma 5.2, Prop. 5.6] or [AGR23, Prop. 1.5, Lemma 5.1].

for closed manifolds as in the Euclidean setting). More importantly, a Dirac operator is not just *any* elliptic differential operator, but rather carries a certain structure with it: Getzler calls this the *symbol calculus for Clifford algebras.* In other words, for a Dirac operator we not only have the Minakshisundaram-Pleijel expansion (3); we know more about each of the A_k's, namely that they lie in different orders within the Clifford algebra filtration [BGV04, Theorem 4.1.1]:

$$h_{\eth^2}(t,x,x) \underset{t\to 0}{\sim} (4\pi t)^{-n/2} \sum_{i=0}^{\infty} k_i(x) t^i, \quad k_i \in Cl^{2i}(X) \otimes \mathrm{End}_{Cl(X)}(E) \tag{19}$$

i.e. the $t \to 0$ expansion of the heat kernel of $\eth^2$ respects the filtration $Cl^k(TX)$[94]. Since we are interested in k_n, we just need a computational tool that gets rid of all the terms that have lower than top order in the Clifford filtration: the supertrace.
That is: we use the fact that Dirac operators go hand in hand with Clifford structures, and when we want to obtain the term of a certain order in the expansion at a face we just need to use a tool that extracts the corresponding order in the Clifford filtration, i.e. a supertrace.

Example 4.20. Let us apply Getzler rescaling's argument to the closed setting, both in its geometrical and local forms (we use this as a chance to introduce and motivate the latter).
We care about the contribution from the $t \to 0$ asymptotic, i.e. the one coming from the cylindrical face tf in the heat blow-up space of Figure 6. Inspired by (19), the strategy is to obtain a Clifford filtration of endomorphisms of E at tf, extend it by Prop. 4.19 to a bundle in the whole space and compute the supertrace to get rid of all negative powers of t in the expansion (which go with lower than top Clifford filtration degree) and recover the Atiyah-Singer integrand for Dirac operators (5).
If we had not known that $k_i \in Cl^{2i}(X)$ beforehand, we would have needed to construct a calculus of operators with sections on the rescaled bundle and show that the heat kernel belongs to it, i.e. that its expansion is of the desired form (this would just be a minimally refined version of what we did in Sections 3.2.1 and 3.2.2, see [Vai01, §5.2]).
For starters, let us set up the filtration at tf. We consider a Dirac operator associated to a Clifford module $E \to X$. Recall from Section 3.1.4 that any such bundle can be identified with

$$E \cong \mathcal{S} \otimes \mathrm{Hom}_{Cl(X)}(\mathcal{S}, E)$$

and its bundle of endomorphisms $\mathrm{End}(E) \to X$ can be decomposed as [BGV04, p. 2]

$$\mathrm{End}(E) \cong Cl(X) \otimes \mathrm{End}_{Cl(X)}(E)$$

(from here on we work with the complexified Clifford algebra but do not include that in the notation for simplicity). Note that $\mathrm{End}(E) \cong E \otimes E^* \cong \mathcal{S} \otimes \mathcal{S}^* \otimes \mathrm{End}_{Cl(X)}(E)$ since $\mathrm{End}(\mathcal{S}) \cong Cl(X)$.
The heat kernel of $\eth^2$ is however for each time t a section of the bundle $\mathrm{END}(E) \to X^2$, but over the diagonal $\mathrm{diag}_X \subset X^2$ we have once again the identification $\mathrm{END}(E)\big|_{\mathrm{diag}_X} \cong \mathrm{End}(E)$ through $\mathrm{diag}_X \cong X$. Thus:

$$\mathrm{END}(E)\big|_{\mathrm{diag}_X} \cong Cl(X) \otimes \mathrm{End}_{Cl(X)}(E)$$

Luckily, the face tf fibres over $\{0\} \times \mathrm{diag}_X \cong \mathrm{diag}_X$, so we can pull this identification (fibrewise) back to the whole face:

$$\left(\beta^*_{xx'} \mathrm{END}(E)\big|_{\mathrm{tf}}\right)_{(\xi,x')} \cong \left(Cl(X) \otimes \mathrm{End}_{Cl(X)}(E)\right)_{x'}$$

[94] As we argued in §3.2.1, the expansion is a priori on powers of $\sqrt{t}$, but one can show that the heat kernel lies in an *even calculus* and only the powers of even order contribute to the expansion, i.e. it is actually an asymptotic expansion on t. This does not hold generally in the boundary case.

The filtration by Clifford degree is in this manner lifted to tf:

$$\mathcal{F}_k^{\text{tf}} := \left(\beta_{xx'}^* \text{END}(E)\big|_{\text{tf}}\right)^k \cong Cl^k(X) \otimes \text{End}_{Cl(X)}(E) \subset \beta_{xx'}^* \text{END}(E)\big|_{\text{tf}} =: \mathcal{E}\big|_{\text{tf}}$$

As explained before, we extend this filtration to a neighborhood of tf by parallel transport along a transverse vector field using the connection described above. The natural choice of vector field seems to be ∂_τ for $\tau = \sqrt{t}$. Applying Proposition 4.19, we get a rescaled bundle, which we denote $\mathcal{G}$ so that

$$\mathcal{G}\big|_{\text{tf}} \cong \bigoplus_{k=0}^{n} (N^*\text{tf})^k \otimes \Big(Cl^k(X)/Cl^{k-1}(X)\Big) \otimes \text{End}_{Cl(X)}(E) \tag{20}$$

where $N^*\text{tf}$ can be trivialized by $d\tau$. With that trivialization, since $Cl^k(X)/Cl^{k-1}(X) \cong \Lambda^k T^*X$, the rescaled bundle restricted to tf consists of differential forms ΛT^*X with $\text{End}_{Cl(X)}(E)$-coefficients. Thus, the connection acts on them as an exterior derivative (cf. [Vai01, p. 91]). Sections of $\mathcal{G}$ have a characteristic form in a neighborhood of tf:

$$u = u_0 + u_1\tau + \cdots + u_n\tau^n, \quad \Longrightarrow \quad u\big|_{\mathcal{G},\text{tf}} = u_0 + \sum_{k=1}^{n} [u_k]_{\left(Cl^k(X)/Cl^{k-1}(X)\right)\otimes \text{End}_{Cl(X)(E)}} \tag{21}$$

that is, in the term with τ^k the part of order $\leq k-1$ in the filtration is thrown away. Notice how these sections carry much more information as those of $\mathcal{E} = \beta_{xx'}^*\text{END}(E)$, where only u_0 would survive when restricting at tf. Consequently, we use $u\big|_{\mathcal{G},\text{tf}}$ to denote the restriction of u to tf seen as a section in $\mathcal{G}$ and not in $\mathcal{E}$.

Since Clifford degree counts covectors that appear multiplying in each (differential form) term, Getzler rescaling essentially ensures that the number of covectors is preserved by parallel transport (and independent of the path). Because of the form (19) of the heat kernel, each covector comes with a $\sqrt{t}$, so taking the supertrace means in particular throwing away terms with less than n covectors (and the evenness on the $\sqrt{t}$ powers means that covectors come in pairs).

The form of the rescaled sections at tf suggests that to recover the desired contribution we need to solve the rescaled model problem of the heat equation at tf and then apply the supertrace to kill off all terms u_k for $k < n$.

To do this, first refine the heat calculus so that it only contains kernels with coefficients in the rescaled bundle [Mel93, §8.6]:

$$\Psi_{\mathcal{G}}^{\alpha}(X;E) = \rho_{\text{tf}}^{-n-2-\alpha}\rho_{\text{tb}}^{\infty}\, \mathcal{C}^\infty(HX^2;\mathcal{G}) \subset \Psi_H^\alpha(X;E)$$

Composition of operators in $\Psi_{\mathcal{G}}(X;E)$ can be deduced from that of the original heat calculus and its triple space construction (§3.2.2) by writing the space of sections in a clever way [Vai01, Prop. 5.9]:

$$\begin{aligned}\Gamma(\mathcal{G}) &= \iota_{\mathcal{F}}^* \left(\{u \in \Gamma(\mathcal{E}) : \nabla_{\partial_\tau}^k u\Big|_{\text{tf}} \in Cl^k(X) \otimes \text{End}_{Cl(X)}(E)\}\right) \\ &= \iota_{\mathcal{F}}^* \left(\{u \in \Gamma(\mathcal{E}) : \nabla^k u\Big|_{\text{tf}} \in T\left(HX^2\right) \otimes Cl^k(X) \otimes \text{End}_{Cl(X)}(E)\}\right)\end{aligned}$$

and using the compatibility of the connection with pullback and pushforward (integration), and the corresponding Leibniz product rule [Vai01, Theorem 5.12]. Alternatively, the composition $A \circ B$ of the operators A and B, with respective kernels $u \in \Psi_{\mathcal{G}}^\alpha(X;E)$ and $v \in \Psi_{\mathcal{G}}^\beta(X;E)$ has a kernel

$$(\pi_t)_* \left(\pi_{t-t'}^* u \cdot \pi_{t'}^* v \cdot \mu\right) \mathrel{\widehat{=}} \int_X \int_0^t u(t-t',x,x'')v(t',x'',x')dx''dt'$$

with respect to a pre-fixed density in the heat blow-up space $\mu \in \Omega(HX^2)$. To see that this kernel also lies in the rescaled bundle (and thus the rescaled calculus is indeed a calculus) we can write their local expansions around tf and compute the time integral:

$$\int_0^t \sum_{k=0}^n u_k(t-t')^{k/2} \sum_{l=0}^n v_l (t')^{l/2} dt' \underset{s=\frac{t'}{t}}{=} \sum_{k,l=0}^n u_k v_l \, t^{\frac{k+l}{2}+1} \int_0^1 (1-s)^{\frac{k}{2}} s^{\frac{l}{2}} ds$$
$$= \sum_{k,l=0}^n B\left(\frac{k}{2}+1, \frac{l}{2}+1\right) u_k v_l \tau^{k+l+2}$$

where B is the beta function satisfying $B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$. Since $u_k \in Cl^k(X) \otimes \mathrm{End}_{Cl(X)}(E)$ and $v_l \in Cl^l(X) \otimes \mathrm{End}_{Cl(X)}(E)$, this shows the coefficient of τ^α belongs to $Cl^{\alpha-2}(X) \subset Cl^\alpha(X)$ as claimed.

Associated to $\Psi_{\mathcal{G}}(X;E)$ there is a notion of rescaled normal operator [Mel93, §8.7], which is analogous to (8), except the restriction is now as a $\mathcal{G}$-section:

$$N_\alpha^{\mathcal{G}}(A) = \left((\rho_{\mathrm{tf}})^{n+2+\alpha} \beta^* A\right)\Big|_{\mathcal{G},\mathrm{tf}}, \quad A \in \Psi_{\mathcal{G}}^\alpha(X;E) \tag{22}$$

As expected (9), this comes with an associated short exact sequence:

$$0 \longrightarrow \Psi_{\mathcal{G}}^{\alpha-1}(X) \longrightarrow \Psi_{\mathcal{G}}^{\alpha}(X) \longrightarrow \mathcal{S}\left(\Lambda T^*X \otimes \mathrm{End}_{Cl(X)}(E)\right) \longrightarrow 0$$

by the identifications and trivializations above. Taking the 0-th order part of the differential form bundle recovers the usual restriction map to tf, thus fitting with (21) and (9).

Using the rescaled normal operator, we can now proceed as in §3.2.1, i.e. we can solve the parametrix problem for the rescaled heat equation by solving away the power series at tf, thus showing that the heat kernel belongs to the rescaled calculus and hence has the asymptotic properties desired (19). We can do this using Lichnerowicz formula [BGV04, Theorem 3.52] and [Mel93, (8.36)] as in e.g. [Mel93, Prop. 8.21], [Vai01, §5.2] or [AGR23, §5.1], or proceed in local fashion as [BGV04, Prop. 4.20] or [Liu25, §9]. We illustrate here the second approach, which already appeared in a similar form in J. Liu's thesis, for the spinor Dirac operator (i.e. $E = \mathcal{S}$).

As pointed out previously, it is useful to solve the local problem around a point $u' \in X$ by choosing Riemannian normal coordinates u_i at the point and extending the orthonormal basis ∂_{u_i} by radial parallel transport, cf. [BGV04, §4.3]. The spinor bundle can be trivialized analogously (this essentially reduces the problem to $\mathbb{R}^n$). The spinor connection takes the form (6):

$$\nabla^{\mathcal{S}}_{\partial_{u_i}} = \partial_{u_i} + \frac{1}{4}\sum_{j,p=1}^n g(\nabla_{\partial_{u_i}}\partial_{u_j}, \partial_{u_p}) cl(du_j) cl(du_p)$$
$$= \partial_{u_i} - \frac{1}{8}\sum_{j,p,q=1}^n g\left(R(\partial_{u_i}, \partial_{u_q}) u_q \partial_{u_j}, \partial_{u_p}\right) cl(du_j) cl(du_p) + O(|u|^2)$$

argumenting as in (14). Now choose the coordinate chart so that the point we are zooming in is the coordinate origin, i.e. $u' = 0$ and consider the lift of the connection to the heat blow-up space around tf:

$$\beta^*\left(\sqrt{t}\nabla^{\mathcal{S}}_{\partial_{u_i}}\right) = \partial_{\xi_i} - \frac{\tau^2}{8}\sum_{j,p,q=1}^n g\left(R(\partial_{u_i}, \partial_{u_q})\xi_q \partial_{u_j}, \partial_{u_p}\right) cl(du_j) cl(du_p) + O(\tau^3), \quad \xi = \frac{u-u'}{\tau}, \quad \tau = \sqrt{t}$$

Now compute the rescaled version of this operator at tf. By looking at (21) and the ensuing discussion, we can deduce that the rescaling "pairs up" each (Clifford) covector with a τ and maps them into a differential form. In local coordinates, this can be expressed by a map [BGV04, p. 140-141, 157]:

$$\delta_\tau : \Lambda^k T^* X \longrightarrow \Lambda^k T^* X, \quad \delta_\tau \omega = \tau^{-k}\omega$$

so that δ_τ is computationally related to $\big|_{\mathcal{G},\mathrm{tf}}$ by taking the limit $\tau \to 0$ (notice that we again incur in a choice when considering the map δ_τ as our local rescaling tool: any $\delta_{\rho_{\mathrm{tf}}}$ would a priori have worked). We say that operators "rescale" if conjugating them by δ_τ does not make them singular as $\tau \to 0$; the operator above rescales:

$$\begin{aligned}
&\delta_\tau \circ \left(\beta^*\left(\sqrt{t}\nabla^{\mathcal{S}}_{\partial_{u_i}}\right)\right) \circ \delta_\tau^{-1} = \\
&\partial_{\xi_i} - \frac{\tau^2}{8}\sum_{j,p,q=1}^{n} g\left(R(\partial_{u_i},\partial_{u_q})\xi_q\partial_{u_j},\partial_{u_p}\right)\left(\frac{\varepsilon(du_j)}{\tau} - \tau\iota(\partial_{u_j})\right)\left(\frac{\varepsilon(du_p)}{\tau} - \tau\iota(\partial_{u_p})\right) + O(\tau)
\end{aligned}$$

since the $O(\tau^3)$ error terms have at most order 2 as differential forms (they are curvature terms). The rescaled heat equation satisfied by the rescaled heat kernel $\delta_\tau h$ at tf is:

$$\left[\delta_\tau \circ \beta^*\left(t\partial_t + t\partial\!\!\!/^2\right) \circ \delta_\tau^{-1}\right]\delta_\tau h = 0$$

and can be derived from the Lichnerowicz formula for the spinor Dirac operator (15):

$$\begin{aligned}
&\delta_\tau \circ \left(\beta^*\left(t\left(\nabla^{\mathcal{S}}_{\partial_{u_i}}\right)^2\right)\right) \circ \delta_\tau^{-1} = \left(\delta_\tau \circ \beta^*\left(\sqrt{t}\nabla^{\mathcal{S}}_{\partial_{u_i}}\right) \circ \delta_\tau^{-1}\right)^2 \\
&= \left(\partial_{\xi_i} - \frac{1}{8}\sum_{j,p,q=1}^{n} g\left(R(\partial_{u_i},\partial_{u_q})\xi_q\partial_{u_j},\partial_{u_p}\right)\varepsilon(du_j)\varepsilon(du_p)\right)^2 + O(\tau) \\
&\delta_\tau \circ \left(\beta^*\left(t\nabla^{\mathcal{S}}_{\nabla_{\partial_{u_i}}\partial_{u_i}}\right)\right) \circ \delta_\tau^{-1} = \tau\left[\delta_\tau \circ \left(\beta^*\left(\sqrt{t}\nabla^{\mathcal{S}}_{\nabla_{\partial_{u_i}}\partial_{u_i}}\right)\right) \circ \delta_\tau^{-1}\right] = O(\tau) \\
&\delta_\tau \circ \left(\beta^*\left(t\frac{\mathrm{scal}}{4}\right)\right) \circ \delta_\tau^{-1} = \beta^*\left(t\frac{\mathrm{scal}}{4}\right) = O(\tau^2) \\
&\delta_\tau \circ \left(\beta^*\left(t\partial_t\right)\right) \circ \delta_\tau^{-1} = \delta_\tau \circ \left(\frac{1}{2}\tau\partial_\tau - \sum_{i=1}^{n}\frac{1}{2}\xi_i\partial_{\xi_i}\right) \circ \delta_\tau^{-1} = \frac{1}{2}\left(\tau\partial_\tau + \mathbf{N} - \sum_{i=1}^{n}\xi_i\partial_{\xi_i}\right)
\end{aligned}$$

where $\mathbf{N}$ is the number operator counting the differential form order: $\mathbf{N}\omega = k\omega$ for $\omega \in \Lambda^k T^* X$. The model problem at the face is obtained by taking the corresponding rescaled normal operator. Since the heat kernel belongs to $\Psi^{-2}_{\mathcal{G}}(X;\mathcal{S})$ (its asymptotic at tf begins with τ^{-n} as we know from the analysis in §3.2.1), then by (22):

$$\frac{1}{2}\left(-n + \mathbf{N} - \sum_{i=1}^{n}\left[\xi_i\partial_{\xi_i} - \left(\partial_{\xi_i} - \frac{1}{8}\sum_{j,p,q=1}^{n} g\left(R(\partial_{u_i},\partial_{u_q})\xi_q\partial_{u_j},\partial_{u_p}\right)\varepsilon(du_j)\varepsilon(du_p)\right)^2\right]\right) N^{\mathcal{G}}_{-2}(h) = 0$$

The initial condition was written in terms of the original calculus (10), so it can be translated to the rescaled one by taking the part of 0-th differential form degree:

$$\int_{\mathrm{tf}/X}\left[N^{\mathcal{G}}_{-2}(h)(x,\xi)\right]_0 d\xi = 1$$

This is the heat equation satisfied by a harmonic oscillator, so the solution[95] is given by Mehler's kernel [BGV04, §4.2, Theorem 4.21]:

$$N^{\mathcal{G}}_{-2}(h)du' = \frac{1}{(4\pi)^{\frac{n}{2}}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(-\frac{1}{4}\left\langle\frac{R}{2}\coth\frac{R}{2}\xi,\xi\right\rangle\right), \quad \left[N^{\mathcal{G}}_{-2}(h)\right]_0 = \frac{1}{(4\pi)^{\frac{n}{2}}}e^{\frac{|\xi|^2}{4}}$$

An iterative argument can be employed to construct the full rescaled parametrix. At the end of the day, for the index formula we just need to take the supertrace of this local expression (recall that by taking the supertrace we restrict to the lifted diagonal in HX^2_g at tf, i.e. $\xi = 0$):

$$\begin{aligned}\lim_{t\to 0}\operatorname{Str}\left(e^{-t\eth^2}\right) &= \int_X \operatorname{str}_{u'}\left(N^{\mathcal{G}}_{-2}(h)\big|_{\xi=0}\right)du' = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_X \operatorname{str}_{u'}\left(\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\right)\\ &= \left(\frac{-2i}{4\pi}\right)^{\frac{n}{2}}\int_X\left[\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\right]_n = \int_X \hat{\mathrm{A}}(X)\end{aligned}$$

with the topologists normalization of the Â-genus (Footnote 18). This is in essence Getzler's rescaling argument bringing us to (5). The case of a twisted Dirac operator is very similar [BGV04, §4]: the heat equation has an extra term from the twisting curvature $F^{E/\mathcal{S}}$ of the bundle, the Mehler formula sees that term as an extra factor $e^{-F^{E/\mathcal{S}}}$ and it is dealt with by means of the relative supertrace, producing the Chern character $Ch(E) = \operatorname{str}_{E/\mathcal{S}}\left(e^{-F^{E/\mathcal{S}}}\right)$.

□

This rescaling procedure is not only relevant for the closed case: as we have seen, the (non-compactified) heat blow-up space of a g_{klm} metric (or even of an iterated fibration structure), where the heat kernel of $\eth^2$ lifts to a polyhomogeneous conormal distribution, always has a face tf, which contributes to the short time limit in the McKean-Singer formula and fibres over $\operatorname{diag}_X \cong X$. Moreover, a point in $\mathring{\mathrm{tf}}$ is in the fibre of a point $(0, p', p') \in \mathbb{R}_+ \times X^2$ with $p' \in \mathring{X}$, and by the same local considerations, the heat kernel at a neighborhood of that point (away from the boundary) has the same structure as in the closed case. Consequently, the heat kernel at the face tf always presents an expansion starting in ρ_{tf}^{-n} and we can Getzler rescale this contribution in exactly the same away as above (e.g. by choosing ∂_τ as transverse direction or τ as boundary defining function for tf), obtaining precisely the term we just did (if you want, it follows by continuity, see [Mel93, §8.11]). Its integrability in the whole face can be deduced in some contexts ([Vai01, Lemma 5.27a], [AGR23, before §6]) from the vanishing of the supertrace at the intersections of boundary hypersurfaces, since the corresponding kernel restriction is not of maximal Clifford degree. In those cases, this density is integrable; otherwise, the integral has to be renormalized (like in [BS18, (1)]).
Note however that the short time limit of the supertrace could contain non-trivial contributions from other faces, as is the case in all heat blow-up spaces where the second-to-last front face ff lies also at the preimage of $t = 0$ (equiv. when the metric is not of the form of Prop. 3.27).
In particular, for the b-case (Figure 10), [Mel93] shows that Getzler rescaling at tf is enough, since the contribution from bf comes from the leading order term in the kernel. This is in contrast with what most of the cases treated in the literature show, namely that the face ff tends to require its own rescaling ([Vai01, p. 80-83], [AGR23, §5.1], [Liu25, §9], among others).
To know whether or not a Getzler rescaling is called for, one should compute which terms in the polyhomogeneous expansions at the faces participate in the McKean-Singer formula (for $t \to 0$

[95] To be completely precise, the following is the solution as endomorphism-valued density. We did fix a density beforehand and our calculus is not of endomorphism-valued densities but just endomorphisms, as opposed to e.g. [Mel93, (8.58)] or [Vai01, p. 86]; however, the Mehler solution is more cleanly expressed as a density, compare [BGV04, Theorem 4.2]. We attempt to remedy this by writing the density du' in the left hand side.

this can be done as in [Liu25, Lemma 8.1, (8.13)]; for $t \in (0, \infty)$, resort to (18)). If the orders participating depend on the dimension of the manifold X and/or of the base Y of the fibre bundle at the boundary, then we need to Getzler rescale those directions[96]. If however the order contributing from a face is independent of these dimensions or parameters, the corresponding contribution is already given by the heat parametrix construction (if the contributing order is the leading order) or by applying Duhamel's principle a fixed number of times (if we e.g. need the second or fourth order independent of dimensions/parameters). Both situations can be mixed (cf. [Liu25, (8.8)]): imagine the kernel at a face has a smooth expansion starting at order 0 and the order that contributes to the McKean-Singer formula is $\dim Y + 2$; then we can obtain it by a Getzler rescaling in the Y-directions and two applications of Duhamel (or any other recursive procedure to obtain heat terms of ever growing order). Of course, this all assumes that the heat kernel has an asymptotic expansion at the face that respects those Clifford filtrations, i.e. that the number of covectors and the power of the boundary defining function in each term are related. Otherwise, the heat kernel will not belong to the desired rescaled calculus and a different approach is required.

The local computation using an (or several) analog of δ_τ does not formally need a geometrical rescaling of the bundle and subsequent constructions and can be applied for a purely computational approach, as in [BGV04] and [Liu25]. For ease of computations, it seems helpful to work with projective coordinates in the heat blow-up space (as we have been doing all along) and rescale with respect to (powers of) τ and/or x'. If one nonetheless decides to take the geometrical route, the analogous seemingly useful choice is to extend the filtrations along and trivialize the normal bundles to the faces using the transverse vector fields ∂_τ and $\partial_{x'}$, where appropriate. In the literature, these are characterized by being tangent to the other faces being rescaled, which simplifies certain operations (cf. [AGR23, §5.1]).

That being said, a word of caution: in the $t \in (0, \infty)$ regime, we do not only care about the heat kernel under rescaling, but about the whole $cl\left(\frac{dx}{x^k}\right) \eth e^{-t\eth^2}$, as elucidated in (18). Therefore, the rescaling to be undertaken cannot be inferred so readily from the heat kernel asymptotics. This is in particular the case for scattering and 0-fibred cusp metrics, our focus in the following sections. It has never been treated in the literature before and requires a bit more work, as we shall soon expound.

The strategy to treat the local index theorem on manifolds with asymptotical geometrical ends with a heat kernel and Getzler rescaling approach could be summarized in the following:

1. Construct the heat kernel following the recipe in §3.2.1.
2. From the leading order in the heat kernel expansion at each face and taking into account (18), deduce the terms of which order contribute to the (generalized) McKean-Singer formula (17).
3. Construct the corresponding Getzler rescaling at the faces requiring it: pull-back the filtrations using the identifications of $\mathrm{END}(E)$ and the desired Clifford algebra over the diagonal, and extend them using Prop. 4.19 to a rescaled bundle.
4. Set up a rescaled heat calculus, a composition theorem for rescaled operators and define rescaled normal operators.

[96] When rescaling the Y directions, it is sometimes useful to include also the direction normal to the boundary in the rescaling (hyperbolic fibred cusps [Vai01]), and sometimes not (iterated wedges [AGR23] and incomplete edge cusps [Liu25]). If the metric has higher order degeneracy (like the parameter a in the cusp case, the incomplete cusp case and the incomplete edge cusp case in our notation of §4.1), then the rescaling should also take into account this order [Liu25, before (9.9)]

5. Solve the rescaled heat equation within this calculus by using the rescaled normal operators to construct a rescaled parametrix iteratively, thus showing that the expansion of the kernel at each face respects the Clifford filtrations and obtaining the relevant contributions to the McKean-Singer formula as rescaled normal operators of the solution. One does not need to solve again for the faces not rescaled: the solution is the same as in Step 1.

6. Compute the supertraces and interpret the integrands as geometric invariants.

In the next two sections we will apply these general considerations to two settings: asymptotically conical/scattering metrics and its generalization into 0-fibred cusp metrics/fibred boundary manifolds. The first contains already many of the difficulties that appear in the second, hence serves as a good illustration of our approach. For examples of manifolds carrying these metrics or generalized versions of them, check e.g. [GSHV25, §2].
To date and to the author's knowledge, the only work achieving an explicit local index formula for Dirac-type operators in this setting is [LMP07]. Here, a fibred boundary metric g_ϕ is studied by relating it through an adiabatic deformation argument to the conformal hyperbolic fibred cusp case $x^2 g_\phi$, which is studied with similar methods as ours in [Vai01]. The adiabatic limit construction requires them however to restrict their scope to fully elliptic operators. In fact, [LMP07] asks for a heat kernel approach for g_ϕ, not only to extend the result to non-fully elliptic operators (which we will see are a relevant class), but also because the constructions that need to be developed in the process are intrinsically interesting in the context of the spectral theory on these manifolds:

> *"[...] While it is likely that the index formula for this operator can be obtained by methods similar to those employed in [[Vai01]] for d-metrics, that proof is very long and difficult, and it is a reasonable goal to obtain this formula as a consequence either of that theorem or of the APS-theorem. [...] It should be possible, and would still be of genuine interest, to prove the Φ-index theorem directly using heat equation methods. In particular, one would hope to obtain another derivation of the fundamental Bismut-Cheeger result in the course of this."*

Fortunately, we do not start empty-handed: Step 1 above (which comprises the whole procedure of §3.2.1) has already been examined in [She13] for the ac-case and [GTV22], [TV22] and [LV25, §2] for the ϕ-case[97], so in essence we just need to Getzler rescale for the $t \in [0, \infty)$ regime and compute the explicit contributions at $t \to \infty$. The formula we finally obtain can be sanity-checked against [LMP07, (1.4)] in the fully elliptic subcase (and its ac-restriction for trivial fibres [Mel95, Theorem 6.4]). Expected is an expression analogous to the conformal [Vai01, Theorem 5.29].

4.3 Asymptotically conical/scattering manifolds

We start by considering the simplest case of a fibred boundary metric, which is an asymptotically conical (also called scattering[98]) metric, i.e. the case of trivial fibres in the boundary fibre bundle.

[97] Although we speak during most of the work of the study of the low energy resolvent in [GTV22], it is crucial to mention that their result for Hodge Laplacians was reached at approximately the same time through a different method for squares of Dirac operators in the work of Kottke and Rochon [KR22, Theorem 7.4, Corollary 7.13], which in particular shows that the resolvents of the Hodge Laplacian and the square of a Dirac operator have similar low energy asymptotics (the difference in the statements arising from the choice of densities).

[98] The denomination *scattering* [Mel95] comes from the fact that in the field of scattering theory, the mathematical formulation of the interaction of physical systems via e.g. particle collision, the states being studied are idealized to come from the distant past and after the interaction "fly away" to the distant future. The information of the interaction is summed up in the so-called S-matrix, which connects the asymptotically non-interacting states enter-

We restrict ourselves to **exact** asymptotically conical metrics for now.
We recall that this corresponds to a compact manifold X with boundary with a Riemannian metric on the interior $\mathring{X}$ which in the interior $\mathring{\mathcal{U}}$ of a collar neighborhood $\mathcal{U} \cong [0,\delta)_x \times \partial X$ of the boundary takes the local form

$$g_{ac} = \frac{dx^2}{x^4} + \frac{g_{\partial X}}{x^2}$$

for a boundary defining function x.
We have a bundle ${}^{ac}TX$ (independent of the choice of x) which can be constructed as a rescaling of the bundle TX (Example 4.18) and whose sections are locally $\mathcal{C}^\infty(X)$-spanned by

$$x^2\partial_x, \quad x\partial_{y_i}$$

The bracket of vector fields confers $\mathcal{V}_{ac} \,\widehat{=}\, \Gamma\left({}^{ac}TX\right)$ the structure of a Lie algebra, whose dual ${}^{ac}T^*X$ is locally spanned by

$$\frac{dx}{x^2}, \quad \frac{dy_i}{x}$$

We can define the space of ac-differential operators as the universal enveloping algebra of $\mathcal{V}_{ac}$ i.e. locally at the collar neighborhood by composition of ac-vector fields with $\mathcal{C}^\infty(\mathcal{U})$ coefficients[99]. Locally

$$P \in \mathrm{Diff}^m_{ac}(X) \implies P = \sum_{i+|\alpha|\leq m} p_{i,\alpha}(x,y)\left(x^2D_x\right)^i (xD_y)^\alpha, \quad p_{i,\alpha} \in \mathcal{C}^\infty(\mathcal{U})$$

There is a corresponding ac-principal symbol which is the homogeneous polynomial on the cotangent coordinates $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}^{n-1}$, representing $\xi\frac{dx}{x^2} + \eta\frac{dy}{x} \in \Gamma({}^{ac}T^*X)$, of the form

$$\sigma_{ac}(P)(x,y;\xi,\eta) = \sum_{i+|\alpha|=m} p_{i,\alpha}(x,y)\xi^i\eta^\alpha$$

The operator P is ac-elliptic if $\sigma_{ac}(P)$ is invertible for $(\xi,\eta) \neq (0,0)$. One gets the usual short exact sequence:

$$0 \longrightarrow \mathrm{Diff}^{m-1}_{ac}(X) \hookrightarrow \mathrm{Diff}^m_{ac}(X) \xrightarrow{\sigma_{ac}} P^m({}^{ac}T^*X) \longrightarrow 0$$

where $P^m({}^{ac}T^*X)$ denotes homogeneous polynomials of degree m on ${}^{ac}T^*X$.
This can be microlocalized to define ac-pseudodifferential operators (compare [GTV22, §4]. For that, the relevant double space is

$$X^2_{ac} = [X^2; \partial X \times \partial X, \mathrm{diag}_{\partial X}]$$

where

$$\mathrm{diag}_{\partial X} = \{s = 1, y - y' = 0, x' = 0\}$$

see Figure 26.

ing the scaterring process with the asymptotically non-interacting products of the interaction that leave away. Since asymptotically conical manifolds model Euclidean space at infinity (under radial compactification, i.e. by considering equivalence classes of rays under bounded distance), it is not totally surprising that the operators involved in scattering, mapping far away states in the past to far away states in the future, behave in this distant limit like the geometric operators related to the Riemannian structure of asymptotically conical manifolds.

[99] We can also define the space of differential operators $\mathrm{Diff}^m_{ac}(X;E,F)$ acting between sections of the bundles $E,F \to X$ in an analogous manner, so that the $p_{i,\alpha}$ become $\hom(E,F)$-valued smooth functions on $\mathcal{U}$. The rest of the constructions are extended similarly.

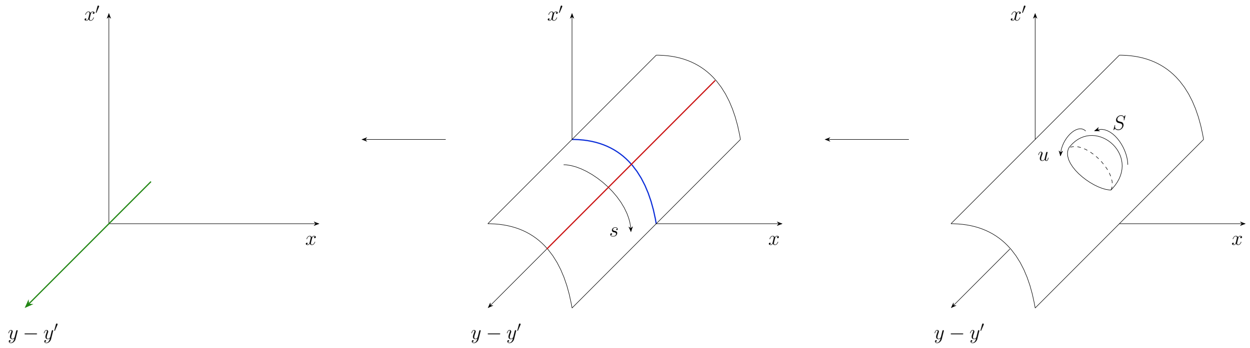


Figure 26: The ac-double space X^2_{ac}. first, $\partial X \times \partial X \subset X^2$ (green line) is blown up, which introduces the coordinate $s = \frac{x}{x'}$ in the resulting front face and gives rise to the b-double space X^2_b. Then, the boundary diagonal $\mathrm{diag}_{\partial X} \subset X^2_b$, i.e. the intersection of $\{s = 1, x' = 0\}$ (the red line) and $\{y - y' = 0, x' = 0\}$ (blue line), is blown up. This separates the second order behaviour in x of approach along paths towards the origin in X^2 [GSHV25, Figure 2] and introduces the coordinates $S = \frac{s-1}{x'} = \frac{x-x'}{(x')^2}$ and $u = \frac{y-y'}{x'}$. We will call the front face of this blow-up ϕf.

- The small calculus $\Psi^m_{ac}(X)$ of ac-pseudodifferential operators consists of distributions on X^2 whose lift to X^2_{ac} is conormal with respect to the lifted diagonal smoothly up to the front face ϕf and vanish to infinite order at all other faces.
- The full calculus $\Psi^{m,\mathcal{E}}_{ac}(X)$ allows more generally for polyhomogeneous behaviour at the faces as long as the smoothness in ϕf is preserved, i.e. $\mathcal{E}_{\phi\mathrm{f}} \geq 0$.

The Fredholm theory for these operators when acting on weighted Sobolev spaces [100]

$$x^s H^m_{ac}(X) := \{u = x^s v : Pv \in L^2_b(X) \quad \forall P \in \Psi^m_{ac}(X)\}$$

(where $L^2_b(X)$ denotes L^2-sections in X with respect to the b-volume density $\frac{dx}{x}dy$) is studied in [MM98] (specialized to the case of the fibres of ϕ being a point). They show that Fredholmness between weighted Sobolev spaces is equivalent to **full ellipticity**, i.e. the invertibility of a family of vertical boundary operators over the base, the normal family (in the sense of (16)), **including at the zero section**. Since ac-operators correspond to ϕ-operators in the case of trivial fibres, the normal family in this context is just a boundary symbol (cf. [Kot09, §2.2] and [Mel95, §6.4, 6.5]):

$$\sigma_{ac}(P)(y;\xi,\eta) := \sum_{i+|\alpha|\leq m} p_{i,\alpha}(0,y)\xi^i\eta^\alpha$$

In particular, Laplace-type operators are **not fully elliptic** unless they have a 0-order part. This can be seen for the Hodge Laplacian on $\Lambda\,{}^{ac}T^*X$, whose boundary symbol is [GTV22, (5.8)]:

$$\sigma_{ac}(\Delta_{ac})(y;\xi,\eta) = \xi^2 + |\eta|^2$$

and vanishes at $(\xi,\eta) = (0,0)$ (though, for example, $\Delta_{ac} + 1$ is fully elliptic and thus Fredholm). This highlights the interest in index theory for non-fully elliptic (equivalently non-Fredholm) elliptic operators, which we tackle here.
Consider now a Clifford module $E \to X$ with respect to the Clifford algebra $Cl({}^{ac}T^*X)$ (with action $cl(\cdot)$) quantized from the exterior algebra $\Lambda\,{}^{ac}T^*X$ using the Clifford relations and the

[100]This should be compared with the b-case, where Fredholmness does not happen for all weights simultaneously: one needs to avoid weights related to the spectrum of the indicial operator (the boundary operator); see e.g. [Mel93, In. 29].

extension of the metric g_{ac} to ac-differential forms by duality with ${}^{ac}TX$. Equip the bundle E with a Clifford connection ∇^E with respect to the Levi-Civita connection of g_{ac}. The associated Dirac operator $\eth = cl \circ \nabla^E$ is expressed locally as:

$$\eth = cl\left(\frac{dx}{x^2}\right)\nabla^E_{x^2\partial_x} + \sum_{i=1}^{n-1} cl\left(\frac{dy_i}{x}\right)\nabla^E_{x\partial_{y_i}}$$

Let us for simplicity assume $E = \mathcal{S}$ is the spinor bundle and denote $g = g_{ac}$. Write $b = n-1$ for easier comparison with the ϕ-case below. Then, using the connection asymptotics coming from the Koszul formula:

$$g\left(\nabla_{\partial_{y_i}} x\partial_{y_j}, x^2\partial_x\right) = -g\left(\nabla_{\partial_{y_i}} x^2\partial_x, x\partial_{y_j}\right) = g_{\partial X}(\partial_{y_i}, \partial_{y_j}) = \delta_{ij}$$
$$g\left(\nabla_{\partial_{y_i}} x\partial_{y_j}, x\partial_{y_p}\right) = g_{\partial X}\left(\nabla^{\partial X}_{\partial_{y_i}}\partial_{y_j}, \partial_{y_p}\right) = -\frac{1}{2}\sum_{q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i}, \partial_{y_q})y_q\partial_{y_j}, \partial_{y_p}\right)$$

where the last equality in the second line is justified by the choice of radial coordinates in the closed manifold ∂X and the last equality in the first line is a consequence of $\{x^2\partial_x, x\partial_y\}$ being orthonormal and coming from a rescaling of a lift of an orthonormal basis $\{\partial_{y_i}\}_{i=1}^b$ of $\Gamma(T\partial X)$ (we abuse notation and denote the y_i coordinates in ∂X and their lifts to $\mathcal{U}$ by the same letter). The spinor connection in terms of the orthonormal basis of $\Gamma(TX)$ looks in a neighborhood of a point in the boundary like:

$$\begin{aligned}
\nabla^{\mathcal{S}}_{\partial_x} &= x^2\partial_x \\
\nabla^{\mathcal{S}}_{\partial_{y_i}} &= \partial_{y_i} + \frac{1}{2}cl\left(\frac{dy_i}{x}\right)cl\left(\frac{dx}{x^2}\right) \\
&\quad - \frac{1}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i}, \partial_{y_q})y_q\partial_{y_j}, \partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)
\end{aligned}$$

so the spin-Dirac operator takes the form:

$$\begin{aligned}
\eth &= cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x - \frac{xb}{2}\right) \\
&\quad + \sum_{i=1}^{b} cl\left(\frac{dy_i}{x}\right)\left(x\partial_{y_i} - \frac{x}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i}, \partial_{y_q})y_q\partial_{y_j}, \partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)\right)
\end{aligned}$$

Notice that the last summand is x times a Dirac operator on ∂X, so we can shorten the formula by:

$$\eth = cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x - \frac{xb}{2}\right) + x\eth_{\partial X} \tag{23}$$

in particular

$$xcl\left(\frac{dx}{x^2}\right)\eth = -x^3\partial_x + x^2\frac{b}{2} + x^2 cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}$$

The square of this operator is given by:

$$\eth^2 = -x^4\partial_x^2 + x^2\eth^2_{\partial X} + (b-2)x^3\partial_x + x^2 cl\left(\frac{dx}{x^2}\right)\eth_{\partial X} + x^2\frac{b}{2}\left(1 - \frac{b}{2}\right) \tag{24}$$

In particular, the boundary symbol (normal "family") of these operators is:

$$\sigma_{ac}(\eth)(y';\xi,\eta) = cl\left(\frac{dx}{x^2}\right) i\xi + \sum_{i=1}^{b} cl\left(\frac{dy_i}{x}\right) i\eta$$
$$\sigma_{ac}(\eth^2)(y';\xi,\eta) = \xi^2 + |\eta|^2$$

(do not confuse these ξ and η with coordinates on tf) so neither of them is fully elliptic (take $(\xi,\eta) = (0,0)$) and thus they are not Fredholm[101] (observe that $\eth^2$ has the same normal family as the Hodge Laplacian). In particular, their local "index" formula cannot have a Fredholm index term in it, but rather a generalization which we will introduce later: the *renormalized index* (this is essentially the Witten index, as found in [GS88]).
Afterwards we will need:

$$\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth = -x^4\partial_x^2 + x^2\eth^2_{\partial X} + (b-1)x^3\partial_x - x^2\frac{b^2}{4} \tag{25}$$

We are interested in computing the heat kernel of this operator to plug it in the generalized McKean-Singer formula:

$$\lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = \lim_{t\to 0} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) + \int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) dt$$

We follow the heuristic that the Laplace-type operator $\eth^2$ has a polyhomogeneous heat kernel in the same blow-up space as the Laplace-Beltrami operator associated to (X, g_{ac}), i.e. in the (compactified) heat blow-up space of [She13, Figure 1] (see Figure 27), with the same index sets at the faces[102] (we only care about those intersecting the lifted diagonal):

$$\mathcal{E}_{\mathrm{tf}} \geq -n, \quad \mathcal{E}_{\phi\mathrm{f}} \geq 0, \quad \mathcal{E}_{\mathrm{bf}_0} \geq n, \quad \mathcal{E}_{\mathrm{zf}} \geq n$$

Denoting the **compactified heat blow-up space** by $\overline{HX}^2_{ac}$, we can rewrite this as:

$$h \in \rho_{\mathrm{tf}}^{-n}\rho_{\phi\mathrm{f}}^{0}\rho_{\mathrm{bf}_0}^{n}\rho_{\mathrm{zf}}^{n}\rho_{\mathrm{tb}}^{\infty}\rho_{\mathrm{bf}}^{\infty}\rho_{\mathrm{lb}}^{\infty}\rho_{\mathrm{rb}}^{\infty}\rho_{\mathrm{lb}_0}^{n}\rho_{\mathrm{rb}_0}^{n}\mathcal{C}^\infty(\overline{HX}^2_{ac})$$

We recall what the geometry at each of the faces is:

- Around $\mathring{\mathrm{tf}}$ and its intersection with ϕf, we have coordinates

$$\xi = \frac{x-x'}{(x')^2\sqrt{t}}, \quad \eta = \frac{y-y'}{x'\sqrt{t}}, \quad x', \quad y', \quad \tau = \sqrt{t}$$

  We will choose the boundary defining function $\rho_{\mathrm{tf}} = \tau$ in the sequel.

  The face tf comes from the blow-up of the time 0 lifted diagonal and so it fibres over X:

$$\mathrm{tf} \longrightarrow \mathrm{diag}_{X,0} \cong X, \quad (\xi,\eta,x',y') \mapsto (x',y')$$

  with fibres tf$/X$ modelled on the radial compactification of $\mathbb{R}_\xi \times \mathbb{R}^b_\eta$.

[101] For the case of a Dirac operator induced from a general Clifford module, the twisting contributes a 0-th order term [KR22, (2.20)] which might bring the operator back to being fully elliptic if it is non-vanishing.

[102] This heat blow-up space is shared by Laplace-type operators. Compare e.g. [TV22, Figure 4] for the small time behaviour and [GTV22, Figure 8] for the large time behavior of the Hodge Laplacian in the ϕ-case, and restrict to punctual fibres. Moreover, see the discussion in §3.2.1

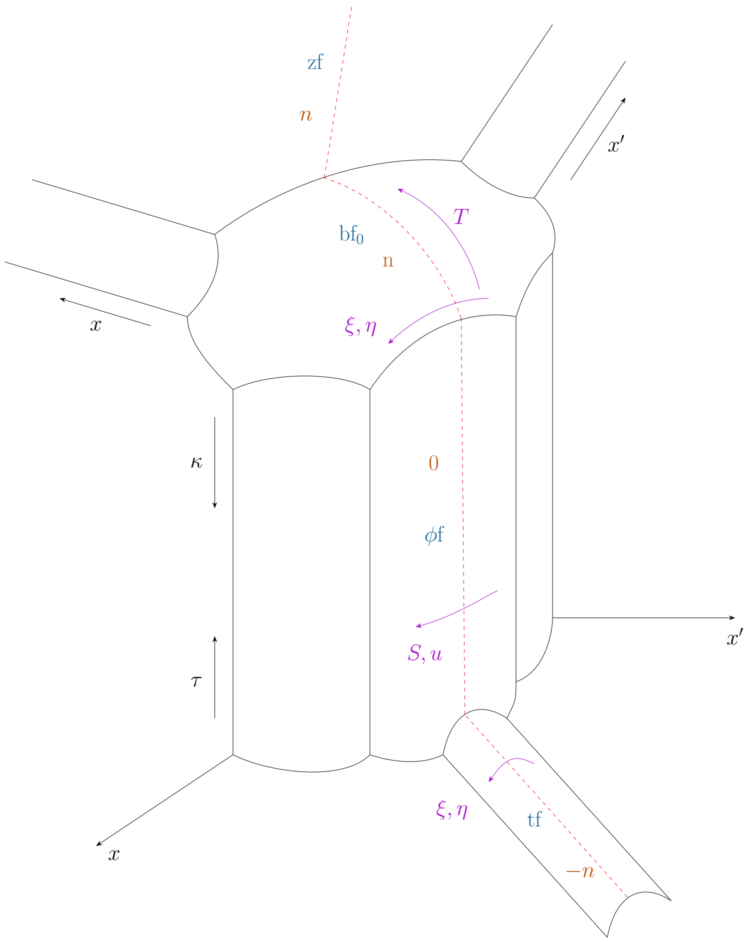


Figure 27: Compactified heat blow-up space for ac-metrics [She13, Figure 1]. The intersection of the lifted diagonal with the boundary hypersurfaces is represented with a red dashed line, the leading orders of the kernel at each of the named (in blue) faces in orange and the coordinates in the fibres at each face in magenta.

The vector fields

$$\sqrt{t}x^2\partial_x, \quad \sqrt{t}x\partial_y, \quad t\partial_t$$

on $\mathbb{R}_+ \times X^2$ pull back along the blow-down map β to become

$$(x'\sqrt{t}\xi + 1)^2\partial_\xi, \quad (x'\sqrt{t}\xi + 1)\partial_\eta, \quad \frac{1}{2}\left(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta\right)$$

i.e. they are tangent to the fibres tf/X and span ${}^bT\mathrm{tf}/X$. These lifts show that we can identify tf with the radial compactification of the rescaled bundle $\sqrt{t}\,({}^{ac}TX)$, which is isomorphic to ${}^{ac}TX$.

- Around $\mathring{\phi}$f, we have coordinates

$$S = \frac{x - x'}{(x')^2}, \quad u = \frac{y - y'}{x'}, \quad x', \quad y', \quad \tau = \sqrt{t}$$

Let us take $\rho_{\phi\mathrm{f}} = x'$ and notice that ϕf fibres over ∂X:

$$\phi\mathrm{f} \longrightarrow \overline{\mathbb{R}}_+ \times \mathrm{diag}_{\partial X} \longrightarrow \mathrm{diag}_{\partial X} \cong \partial X, \quad (S, u, y', \tau) \mapsto y'$$

with fibres modelled on the radial compactification of $\mathbb{R}_S \times \mathbb{R}^b_u \times (0, \infty)_\tau$.

The vector fields

$$x^2\partial_x, \quad x\partial_y, \quad t\partial_t$$

lift to

$$(x'S+1)^2\partial_S, \quad (x'S+1)\partial_u, \quad \frac{1}{2}\tau\partial_\tau$$

and so we have an identification of ϕf with the radial compactification of $(0,\infty)_\tau \times {}^{ac}T_{\partial X}X$.

- Around $\mathring{\text{bf}}_0$ and its intersection with ϕf, we have coordinates

$$\xi = \frac{x-x'}{(x')^2\sqrt{t}} = \kappa\frac{x-x'}{(x')^2}, \quad \eta = \frac{y-y'}{x'\sqrt{t}} = \kappa\frac{y-y'}{x'}, \quad T = x'\sqrt{t} = \frac{x'}{\kappa}, \quad y', \quad \kappa = \frac{1}{\sqrt{t}}$$

and vector field lifts (partly mimicking the calculations at tf):

$$\beta^* x^2\partial_x = (T\xi+1)^2\partial_\xi, \quad \beta^* x\partial_y = (T\xi+1)\partial_\eta, \quad \beta^* t\partial_t = \frac{1}{2}\left(T\partial_T - \kappa\partial_\kappa\right)$$

Since we want to compute the supertrace of the heat kernel, Lidskii's formula tells us that only the faces intersecting the lifted diagonal are going to contribute to the McKean-Singer formula; that is why we focus on them. By the work of Sher [She13], we know the leading order of the heat kernel asymptotics at each face and can use it to deduce what rescaling to perform (Figure 27).
At tf we have order $-n = -\dim X$, so we would like to rescale with respect to the filtration induced from $Cl({}^{ac}T^*X)$ to recover the n-th term with the supertrace, which is the coefficient of $\rho_{\text{tf}}^{-n+\dim X} = \rho_{\text{tf}}^0$ and thus contributes to the limit $t \to 0$ of the McKean-Singer formula. As we discussed in the previous section, since a point in $\mathring{\text{tf}}$ fibres over a point in $\mathring{X}$, the same argument as in Example 4.20 can be applied, i.e. the heat kernel does in fact respect this Clifford filtration structure as desired (recall that ${}^{ac}T^*X \cong T^*X$ over $\mathring{X}$). Since this is the only face at $t \to 0$ in $\overline{HX}^2_{ac}$, we conclude:

$$\lim_{t\to 0} {}^R\text{Str}\left(e^{-t\eth^2}\right) = {}^R\!\!\int_X \hat{\text{A}}(X)Ch(E)$$

for a (twisted) Dirac operator associated to a (Hermitian) Clifford module $E \to X$. We will explicitly describe this part of the rescaling in detail below, once we establish which other rescalings are necessary.
At ϕf, the order of the heat kernel is 0 and the face belongs to the $t \in (0,\infty)$ regime. Thus, the McKean-Singer contribution to look at is the integral term (cf. (18) and Example 4.17):

$$\int_0^\infty \partial_t\, {}^R\text{Str}\left(e^{-t\eth^2}\right)dt = \frac{1}{2}\int_0^\infty \int_{\partial X}\left[\text{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b dy'dt \tag{26}$$

In other words, we need the coefficient of $\rho_{\phi\text{f}}^b$ in the Schwartz kernel of $\eth e^{-t\eth^2}$ and we know the expansion of $e^{-t\eth^2}$ at the face starts with $\rho_{\phi\text{f}}^0$. Choosing again the coordinate chart lifted from the closed manifold ∂X, so that the point we focus on corresponds to the origin of coordinates, translates into $y' = 0$ in our local expressions. Then, lifting the (spinor) Dirac operator to a neighborhood of the face:

$$\beta^*\eth = cl\left(\frac{dx}{x^2}\right)\left(\left(x'S+1\right)^2\partial_S - \frac{\left(x'S+1\right)x'b}{2}\right)$$
$$+ \left(x'S+1\right)\sum_{i=1}^b cl\left(\frac{dy_i}{x}\right)\left(\partial_{u_i} - \frac{(x')^2}{8}\sum_{j,p,q=1}^b g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)\right)$$

and choosing $\rho_{\phi \mathrm{f}} = x'$, we see that a rescaling is also needed here to get to the coefficient of $(x')^b$. The local rescaling procedure should be carried with the help of a mapping $\delta_{x'}$ that pairs up powers of x' and Clifford terms, just as δ_τ did with τ instead of x'. And because the target term is of order $b = \dim \partial X$, we could attempt to lift the Clifford structure $Cl\left(\frac{1}{x}T^*\partial X\right)$[103] along the fibration $\phi\mathrm{f} \to \partial X$. However, the Dirac operator (resp. its product with $cl\left(\frac{dx}{x^2}\right)$) has terms of the form:

$$\left(x'S+1\right) cl\left(\frac{dy_i}{x}\right)\partial_{u_i} \quad \text{resp.} \quad \left(x'S+1\right) cl\left(\frac{dx}{x^2}\right) cl\left(\frac{dy_i}{x}\right)\partial_{u_i}$$

where both boundary and normal Clifford terms appear unpaired with powers of x'; that is, this operator does not rescale in that way, i.e. would not belong to the corresponding rescaled calculus. It is also not enough to just rescale the heat kernel to order b, since the unrescaled Dirac operator has terms e.g. in (x') which would be paired with the $(x')^{b-1}$-coefficient of $e^{-t\eth^2}$, and it is not clear how to compute such a term with a rescaling.

In §4.5, we present some work in progress with the aim of dealing with this issue; before that, let us check how the heat kernel rescales at this face. Remember that this could be deduced from how the heat equation rescales, as in Example 4.20. From our previous calculations or alternatively via the Lichnerowicz formula, we can compute the lift of the square of the Dirac operator:

$$\begin{aligned}\beta^*\eth^2 =& -\left(x'S+1\right)^4\partial_S^2 + (b-2)\left(x'S+1\right)^3 x'\partial_S + \left(x'S+1\right)^2 (x')^2\frac{b}{2}\left(1-\frac{b}{2}\right)\\ &-\left(x'S+1\right)^2\sum_{i=1}^{b}\left(\partial_{u_i} - \frac{(x')^2}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right) cl\left(\frac{dy_j}{x}\right) cl\left(\frac{dy_p}{x}\right)\right)^2\\ &+\left(x'S+1\right)^2 x' cl\left(\frac{dx}{x^2}\right)\sum_{i=1}^{b} cl\left(\frac{dy_i}{x}\right)\left(\partial_{u_i} - \frac{(x')^2}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right) cl\left(\frac{dy_j}{x}\right) cl\left(\frac{dy_p}{x}\right)\right)\end{aligned} \tag{27}$$

In this case, each boundary Clifford term comes paired with an x', so we could set up a rescaling from the filtration in $Cl\left(\frac{1}{x}T^*\partial X\right)$. This does not deal with a rescaling of $\eth e^{-t\eth^2}$ but motivates the construction carried out after this remark.

Remark 4.21. Trying to always keep in mind how to extend this to the 0-ϕ-case for the next section, it is interesting to take a look at what was done in the conformal case $x^2 g_\phi$ in the literature. The face corresponding to ϕf (in terms of the nature of its contributions to the index formula due to the similarity of their model problems) is denoted by tff in [Vai01, Figure 5]. This face lies in the $t \to 0$ regime and thus it is clearer which kind of rescaling is needed.

Taking into account their choice of densities and their definition of the heat calculus [Vai01, p. 66-67], based on the orders of the kernel at each face computed in [Vai01, Theorem 4.11], the author realizes that they need to perform a rescaling of order $b+1$, i.e. in the base directions and the normal direction [Vai01, Lemma 5.1(b)].

Inspired by this, we might expect a similar procedure to work for us, but we have three key differences:

- Our corresponding front face is ϕf, which lies in $t \in (0,\infty)$ and not $t \to 0$, thus contributing to the integral term in the McKean-Singer formula and requiring a rescaling adapted to $cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}$, for which there is no approach to date in the literature.

[103] We assume the Clifford structures $Cl\left(\frac{1}{x}T^*\partial X\right)$ and $Cl({}^{ac}T^*X)\big|_{\partial X}$ are compatible, i.e. $Cl({}^{ac}T^*X)$ is the algebra spanned by the generators of $Cl\left(\frac{1}{x}T^*\partial X\right)$ together with $cl\left(\frac{dx}{x^2}\right)$ in the collar neighborhood $[0,\delta)_x \times \partial X$. The corresponding compatibility in the fibred boundary setting is required among $Cl({}^{\phi}T^*X)\Big|_{\partial X}$, $Cl\left(\frac{1}{x}\phi^*T^*Y\right)$ and $Cl(\partial X/Y)$ (and similarly for any g_{klm} metric).

- We need the b-th term in the expansion of that composition, not the $(b+1)$-th. Thus, a similar rescaling would not work even if ϕf belonged to the $t \to 0$ regime.
- $\beta^* \eth^2$ would not even rescale with respect to the $b+1$ normal+base Clifford filtrations. This is due to the appearance of the terms:

$$x' cl \left(\frac{dx}{x^2} \right) cl \left(\frac{dy_i}{x} \right) \partial_{u_i}$$

This phenomenon is also new in the literature, since it comes from g_{klm} metrics with $l \neq 0$[104], which have not been treated explicitly with these methods, to the author's knowledge. We can trace this back to the connection asymptotics in §4.1, specifically to:

$$g(\nabla_{\tilde{U}_i} x^l \tilde{U}_j, x^k \partial_x) = -g(\nabla_{\tilde{U}_i} x^k \partial_x, x^l \tilde{U}_j) = l x^{k-l-1} g_Y(U_i, U_j)$$

which at the level of the spinor connection e.g. for $m \leq 0$ is reflected in

$$\frac{l \left((x')^{k-m-1} S + 1 \right)^{k-1} (x')^{k-1} b}{2} cl_g \left(x^{-l} \tilde{U}_i^\flat \right) cl_g \left(x^{-k} dx \right)$$

and at the level of $\eth^2$ interacts (among others) with the term

$$\sum_{j=1}^{b} a_{ij} \left((x')^{k-m-1} S + 1 \right)^l (x')^m \partial_{u_j}$$

via $-\left(\nabla^{\mathcal{S}}_{x^l \tilde{U}_i} \right)^2$ in the Lichnerowicz formula to produce

$$-\sum_{j=1}^{b} a_{ij}\, l \left((x')^{k-m-1} S + 1 \right)^{k+l-1} (x')^{k+m-1} b\, cl_g \left(x^{-l} \tilde{U}_i^\flat \right) cl_g \left(x^{-k} dx \right) \partial_{u_j}$$

Other terms in the spinor connection asymptotics suggest at least a horizontal rescaling $\delta_{x'} \omega = (x')^{-l} \omega$ for $\omega \in x^{-l} \phi^* T^* Y$, and this term would only tolerate such a transformation if $k+m-1 \geq l$. If, on top of that, we rescaled also the normal direction with the same weight, then $k+m-1 \geq 2l$ would be required, which excludes several cases.

□

We now want to build a rescaling of the bundle $\mathcal{E} = \beta^*_{xx'} \mathrm{END}(E)$ that describes the Clifford structure of the heat kernel at the faces tf and ϕf. As mentioned, we will proceed as in Example 4.20.

[104] In the asymptotically conical case, the boundary directions could be treated as fibre instead of base directions, i.e. as z-like coordinates. In that case, the connection asymptotics

$$g(\nabla_{V_i} x^m V_j, x^k \partial_x) = -g(\nabla_{V_i} x^k \partial_x, x^m V_j) = m x^{k-m-1} g_{\partial X/Y}(V_i, V_j)$$

appear (if $m \neq 0$). This is already covered in the literature and therefore the ac-case is not really new in this particular point, but the fibred boundary case is. At the level of Getzler rescaling, the base and fibre directions are distinct in that the second-to-last front face ff $\subset HX^2_g$ fibres only over Y and not over Z, unless $m \geq 1$, which has not been looked at (since it seems to have no clear geometrical significance); thus, one can only lift $Cl(Y)$ (perhaps adding the normal direction). Intuitively, blowing-up $\{v = v', [...]\}$, Clifford rescaling in the v-directions, zooming-in/localizing in the v-directions and obtaining a local form in the v-directions (the Â-genus) representing the geometric invariant in the index formula seem to go all together.

From the fibrations

$$\mathrm{tf} \longrightarrow \mathrm{diag}_X \cong X, \quad \phi\mathrm{f} \longrightarrow \mathrm{diag}_{\partial X} \cong \partial X$$

we can lift the identifications

$$\begin{aligned} \mathrm{END}(E)\big|_{\mathrm{diag}_X} &\cong \mathrm{End}(E) \cong Cl({}^{ac}T^*X) \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E) \\ \mathrm{END}(E)\big|_{\mathrm{diag}_{\partial X}} &\cong \mathrm{End}(E\big|_{\partial X}) \cong Cl\left(\frac{1}{x}T^*\partial X\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}T^*\partial X\right)}(E) \end{aligned}$$

to filtrations at the faces tf and ϕf, which lie at the preimage of $\{0\} \times \mathrm{diag}_X$ resp. $\mathbb{R}_+ \times \mathrm{diag}_{\partial X}$ along $\beta : HX^2_{ac} \to \mathbb{R}_+ \times X^2$ (we could do the same with $\overline{HX}^2_{ac}$):

$$\begin{aligned} \left(\beta^*_{xx'}\mathrm{END}(E)\big|_{\mathrm{tf}}\right)^k_{(\xi,\eta,x',y')} &\cong \left(Cl^k({}^{ac}T^*X) \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E)\right)_{(x',y')} \\ &\implies \quad \mathcal{F}^{\mathrm{tf}}_k \cong Cl^k({}^{ac}T^*X) \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E) \\ \left(\beta^*_{xx'}\mathrm{END}(E)\big|_{\phi\mathrm{f}}\right)^l_{(S,u,y',\tau)} &\cong \left(Cl^l\left(\frac{1}{x}T^*\partial X\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}T^*\partial X\right)}(E)\right)_{(y')} \\ &\implies \quad \mathcal{F}^{\phi\mathrm{f}}_l \cong Cl^l\left(\frac{1}{x}T^*\partial X\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}T^*\partial X\right)}(E) \end{aligned}$$

We extend these filtrations to a neighborhood of the faces by parallel transport along the transverse vector fields (of our choice) ∂_τ and $\partial_{x'}$, respectively. These have the property that they are tangent to the other face. Consequently, an application of Proposition 4.19 produces[105] a rescaled bundle $\mathcal{G} \to HX^2_{ac}$ (or $\overline{HX}^2_{ac}$) with sections (cf. [Vai01, Prop. 5.9]):

$$\begin{aligned} \Gamma(\mathcal{G}) = \iota^*_{\mathcal{F}}\Bigg(\{u \in \Gamma(\mathcal{E}) :& \nabla^k u\Big|_{\mathrm{tf}} \in T\left(HX^2\right) \otimes Cl^k({}^{c}T^*X) \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E), \quad \text{and} \\ & \nabla^l u\Big|_{\phi\mathrm{f}} \in T\left(HX^2\right) \otimes Cl^l\left(\frac{\tau}{x}T^*\partial X\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}T^*\partial X\right)}(E)\}\Bigg) \end{aligned}$$

Locally, the rescaled bundle is spanned by sections

$$\rho_{\mathrm{tf}}\frac{dx}{x^2}, \quad \rho_{\mathrm{tf}}\rho_{\phi\mathrm{f}}\frac{dy_i}{x}$$

with $\mathrm{End}(E)$-coefficients (think $\rho_{\mathrm{tf}} = \tau$ and $\rho_{\phi\mathrm{f}} = x'$). Its restriction at the faces is given by (cf. [Vai01, (93)]):

$$\begin{aligned} \mathcal{G}\big|_{\mathrm{tf}} &\cong \bigoplus_{k=0}^{n} (N^*\mathrm{tf})^k \otimes \underbrace{\left(Cl^k({}^{c}T^*X)/Cl^{k-1}({}^{c}T^*X)\right)}_{\cong \Lambda^k({}^{c}T^*X)} \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E) \\ \mathcal{G}\big|_{\phi\mathrm{f}} &\cong \bigoplus_{l=0}^{b} (N^*\phi\mathrm{f})^l \otimes \underbrace{\left(Cl^l\left(\frac{\tau}{x}T^*\partial X\right)\Big/Cl^{l-1}\left(\frac{\tau}{x}T^*\partial X\right)\right)}_{\cong \Lambda^l\left(\frac{\tau}{x}T^*\partial X\right)} \otimes \mathrm{End}_{Cl\left(\frac{1}{x}T^*\partial X\right)}(E) \end{aligned}$$

[105] We rescale iteratively, i.e. first at tf and then at ϕf or vice versa. For example, if we first rescale at ϕf, this affects the filtration at tf, which becomes:

$$Cl^k({}^{c}T^*X) \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E)$$

spanned by the cusp covectors $\frac{dx}{x^2}$ and dy_i (with $\mathrm{End}(E)$-coefficients). The rescaling of the once rescaled bundle with this filtration at tf produces $\mathcal{G}$. We become the same rescaled bundle if we started at tf.

$$\mathcal{G}\big|_{\mathrm{tf}\cap\phi\mathrm{f}} \cong \bigoplus_{k=0}^{n} (N^*\mathrm{tf})^k \otimes \left[\bigoplus_{j+l=k} \Lambda^j \left({}^cT^*X\right) \otimes (N^*\phi\mathrm{f})^l \otimes \Lambda^l \left(\frac{\tau}{x} T^*\partial X\right)\right] \otimes \mathrm{End}_{Cl({}^{ac}T^*X)}(E)$$

The associated rescaled calculus $\Psi_{\mathcal{G}}(X;E)$ follows the same composition rules as the standard ac-heat calculus, deduced from the same triple space construction. Rescaled normal operators can be defined as:

$$N^{\mathcal{G}}_{\mathrm{tf},\alpha}(A) = \left((\rho_{\mathrm{tf}})^{n+2+\alpha}\beta^* A\right)\big|_{\mathcal{G},\mathrm{tf}}$$
$$N^{\mathcal{G}}_{\phi\mathrm{f},\gamma}(A) = \left((\rho_{\phi\mathrm{f}})^{\gamma}\beta^* A\right)\big|_{\mathcal{G},\phi\mathrm{f}}$$

for $A \in \Psi^{\alpha,\gamma}_{\mathcal{G}}(X;E)$. In particular, we can show:

Theorem 4.22. Consider a Dirac operator $\eth$ associated to a Hermitian Clifford module $E \to X$ over a compact manifold with boundary with an asymptotically conical metric. Then, its heat kernel $h = e^{-t\eth^2}$ belongs to the rescaled calculus $\Psi^{-2,0}_{\mathcal{G}}(X;E)$, with rescaled normal operators at the faces tf and ϕf of the form:

$$N^{\mathcal{G}}_{\mathrm{tf},-2}(h)dx'dy' = \frac{1}{(4\pi)^{\frac{n}{2}}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(-\frac{1}{4}\left\langle \frac{R}{2}\coth\frac{R}{2}\xi,\xi\right\rangle\right)$$
$$N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)dy' = e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2 R_{\partial X}/2}{\sinh(\tau^2 R_{\partial X}/2)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2 R_{\partial X}}{2}\coth\left(\frac{\tau^2 R_{\partial X}}{2}\right)u,u\right\rangle\right)\right]e^{L}$$

where

$$L = -\frac{1}{2}\sum_{i=1}^{b} u_i\, cl\left(\frac{dx}{x^2}\right)\varepsilon\left(\frac{dy_i}{x}\right)$$

□

Proof. This is deducible from the non-rescaled heat problem and by solving the rescaled model problems at tf (Example 4.20) and ϕf. For the latter, from (27) and with the local rescaling map $\delta_{x'}\omega = (x')^{-k}\omega$ for $\omega \in \Lambda^k\left(\frac{1}{x}T^*\partial X\right)$, we calculate:

$$\delta_{x'} \circ \left(\beta^*\left(t\partial_t + t\eth^2\right)\right) \circ \delta_{x'}^{-1} = \frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2\sum_{i=1}^{b}\Bigg(\partial_{u_i} - \frac{1}{2}cl\left(\frac{dx}{x^2}\right)\varepsilon\left(\frac{dy_i}{x}\right)$$
$$-\frac{1}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right)\varepsilon\left(\frac{dy_j}{x}\right)\varepsilon\left(\frac{dy_p}{x}\right)\Bigg)^2 + O(x')$$

so the equation to solve ($x' \to 0$) is:

$$\Bigg(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2\sum_{i=1}^{b}\Bigg(\partial_{u_i} - \frac{1}{2}cl\left(\frac{dx}{x^2}\right)\varepsilon\left(\frac{dy_i}{x}\right)$$
$$-\frac{1}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right)\varepsilon\left(\frac{dy_j}{x}\right)\varepsilon\left(\frac{dy_p}{x}\right)\Bigg)^2\Bigg)N^{\mathcal{G}}_{\phi\mathrm{f},0}(h) = 0 \tag{28}$$

(with initial condition given by the restriction at tf$\cap\phi$f of the solution at tf). This can be formally solved iteratively as usual. To compute the leading order term notice that without (cf. Remark 4.21 point 3):

$$-\frac{1}{2}cl\left(\frac{dx}{x^2}\right)\varepsilon\left(\frac{dy_i}{x}\right)$$

it would be decomposable into a second derivative on $\mathbb{R}_S$ and a harmonic oscillator [BGV04, Prop. 4.20] on ∂X:

$$H_{\partial X} \coloneqq \sum_{i=1}^{b}\left(\partial_{u_i} - \frac{1}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right)\varepsilon\left(\frac{dy_j}{x}\right)\varepsilon\left(\frac{dy_p}{x}\right)\right)^2$$

We can deal with this extra term by conjugation with the exponential operator:

$$e^L \coloneqq \exp\left(-\frac{1}{2}\sum_{i=1}^{b} u_i\, cl\left(\frac{dx}{x^2}\right)\varepsilon\left(\frac{dy_i}{x}\right)\right)$$

which interacts with the ∂_{u_i} so that the operator in (28) becomes:

$$\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - e^{-L}\tau^2 H_{\partial X}e^L = e^{-L}\left(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2 H_{\partial X}\right)e^L \tag{29}$$

meaning the solution to the rescaled problem is the conjugated solution to:

$$\left(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2 H_{\partial X}\right)h'(S,u,y',\tau) = 0$$

which is again given by Mehler's formula [BGV04, Theorem 4.21]. Putting all together we obtain for $N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)$ the following expression:

$$e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2 R_{\partial X}/2}{\sinh\left(\tau^2 R_{\partial X}/2\right)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2 R_{\partial X}}{2}\coth\left(\frac{\tau^2 R_{\partial X}}{2}\right)u,u\right\rangle\right)\right]e^L \tag{30}$$

□

We reiterate that this does not solve our index problem, but might be useful in other contexts.
Let us now deal with the large time asymptotics, where we have to distinguish the contributions from bf_0 and zf. For the latter, as proposed by Grieser and Vertman, we will adapt the argument in [Sim93] to the context of a ϕ-manifold (and use its restriction to trivial fibres here). The statement we look for is:

Proposition 4.23. [Sim93, Theorem] Let M be a non-compact manifold with 0-fibred cusp ends, i.e. a Riemannian manifold that compactifies to a compact manifold X with boundary, together with a 0-ϕ-metric in a neighborhood of the boundary (which is a metric at the boundary only when restricted to sections of ${}^{\phi}TX$). Recall that such a compactification comes with a choice of boundary defining function and $\mathcal{C}^\infty$-structure at the boundary.
We consider a Laplace-type operator (think Hodge-Laplace operator acting on ϕ-differential forms, as in [GTV22]). Let Δ constitute its unique self-adjoint extension (cf. [GTV22, §5.3.1]). We denote by $h(t,p,p')$ its heat kernel. Δ has continuous spectrum up to 0 ([Mel95, §6.6, Footnote 38], [GH08, §1]), i.e. the bottom of its spectrum is 0. Whether it has non-trivial kernel depends on the specific operator and is studied in [HHM04].
Then for all $p,p' \in M$,

$$\lim_{t\to\infty} h(t,p,p') = \Pi_{\ker\Delta}$$

where $\Pi_{\ker\Delta}$ is the projection onto the L^2-kernel of Δ.

□

Proof. To prove the result we make use of the following lemma [Sim93, Lemma]:

> *If A is a selfadjoint operator and $f(x,t)$ a measurable function on $\sigma(A) \times [0,\infty]$ so that for fixed x holds that $f(x,\cdot)$ is monotone decreasing and there exists $f(x,\infty) := \inf_t f(x,t) = \lim_{t\to\infty} f(x,t)$, then $s - \lim_{t\to\infty} f(A,t) = f(A,\infty)$.*

Here, $s - \lim$ denotes the limit in the strong topology, meaning that

$$\forall u \in \mathcal{H}: \quad \left|f(A,\infty)u - f(A,t)u\right| \xrightarrow[t\to\infty]{} 0$$

where $\mathcal{H}$ is the Hilbert space where A acts. We apply it to $f(x,t) = e^{-tx}$, with corresponding limit

$$f(x,\infty) = \left\{\begin{matrix} 1, & x = 0 \\ 0, & x \neq 0 \end{matrix}\right\} = \chi_{\{0\}}(x)$$

Then, the spectral measure point of view of the spectral theorem delivers Stone's formula, which together with the lemma above result in:

$$e^{-t\Delta} = f(\Delta,t) = \int_{\mathbb{R}} f(x,t)dE_\Delta(x) \xrightarrow[t\to\infty]{} f(\Delta,\infty) = \int_{\mathbb{R}} f(x,\infty)dE_\Delta(x) = \int_{\{0\}} dE_\Delta(x) = \Pi_{\ker\Delta}$$

meaning $f(\Delta,t) \to \Pi_{\ker\Delta}$ as L^2-operators. We get the pointwise convergence thanks to elliptic regularity by looking at the function

$$g_p := e^{-\Delta}\delta_p : p' \mapsto \int_M h(1,p',p'')\delta_p(p'')dp'' = h(1,p',p) = h(1,p,p') = e^{-\Delta}(p,p')$$

This function is on L^2 by the Riesz representation theorem, since it is the one representing the functional

$$\alpha \in \underbrace{L^2(\Lambda T^*M)}_{\langle\alpha,\beta\rangle = \int_M \alpha\wedge*\beta} \mapsto \langle\alpha, e^{-\Delta}\delta_p\rangle = \int_{M^2} \alpha(p')h(1,p',p'')\delta_p(p'')dp'dp'' = \int_M \alpha(p')h(1,p',p)dp' = \left(e^{-\Delta}\alpha\right)(p)$$

and this is bounded on L^2 from the fact that

$$\left|(e^{-\Delta}\alpha)\big(p\big)\right| \leq C(p) \underbrace{||\alpha||_{L^2}}_{\int_M \alpha\wedge*\alpha}, \quad \text{with} \quad C(p) = \sqrt{\int_M |h(1,p,p')^2|\, dp'} < \infty$$

the finiteness following from the heat kernel construction in [TV22]. More concretely, since we are working on the non-compact model M, points on it correspond to points in $\mathring{X}$. Thus, the fibre over which the kernel integrates in HX^2_ϕ is at time $\tau = 1$ resp. away from the boundary, i.e. it lies away from tf and tb resp. away from ϕf and bf. This fibre then only touches the boundary of HX^2_ϕ at the side face lf, where the heat kernel by construction vanishes to infinite order, so the integral is bounded.
We can use the semigroup property $h(t+s,p,p') = \int_M h(t,p,p'')h(s,p'',p')dp''$ to show

$$h(t+2,p,p') = \int_M^2 \underbrace{h(1,p,p'')}_{e^{-\Delta}(p,p'')=g_p(p'')} \; h(t,p'',p''') \; \underbrace{h(1,p''',p')}_{e^{-\Delta}(p''',p')=g_{p'}(p''')} \; dp''dp''' = \langle g_p, e^{-t\Delta}g_{p'}\rangle$$

$$\xrightarrow[t\to\infty]{} \langle g_p, \Pi_{\ker\Delta}g_{p'}\rangle = \langle e^{-\Delta}\delta_p, \Pi_{\ker\Delta}e^{-\Delta}\delta_{p'}\rangle = \langle \Pi_{\ker\Delta}e^{-\Delta}\delta_p, \Pi_{\ker\Delta}e^{-\Delta}\delta_{p'}\rangle$$

$$= \langle \Pi_{\ker\Delta}\delta_p, \Pi_{\ker\Delta}\delta_{p'}\rangle = \langle \delta_p, \Pi_{\ker\Delta}\delta_{p'}\rangle = \Pi_{\ker\Delta}(p,p')$$

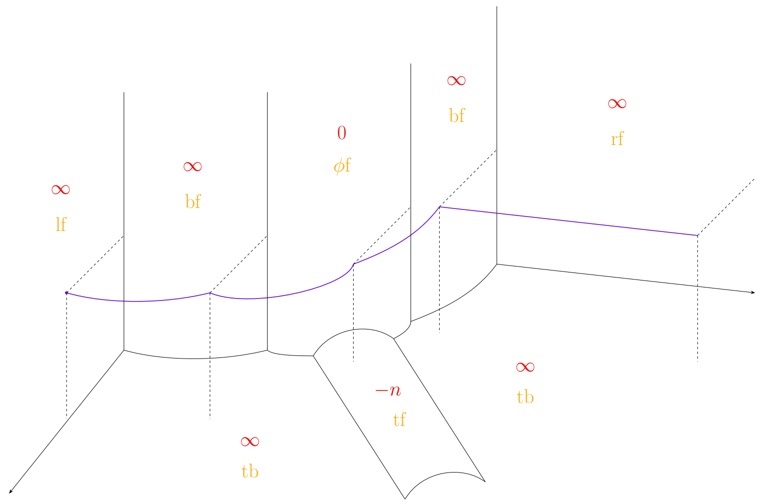


Figure 28: Fibre (purple) in HX^2_ϕ over which we pushforward to integrate $h(1,p,p')$. In orange the names of the faces and in red the order of the heat kernel at them. Compare with Figure 30.

where we used $\Pi_{\ker\Delta} = \Pi^2_{\ker\Delta} = \Pi^*_{\ker\Delta}$ and $\Pi_{\ker\Delta} e^{-\Delta} = \Pi_{\ker\Delta}$.

□

This settles what the form of $\lim_{t\to\infty} e^{-t\not{\partial}^2}$ is. However, in the McKean-Singer formula we need to compute the supertrace before taking the limit. Thus, we now look for an explicit expression of the correction term arising when trying to commute supertrace and limit. This term corresponds to the contribution of bf_0.
We will describe now an argument of Vertman which can be also applied to the ϕ-case by swapping bf_0 for $\phi\mathrm{f}_0$. Focus on a neighborhood of the intersection of the lifted diagonal with the face bf_0 in the heat blow-up space [She13, Figure 5].

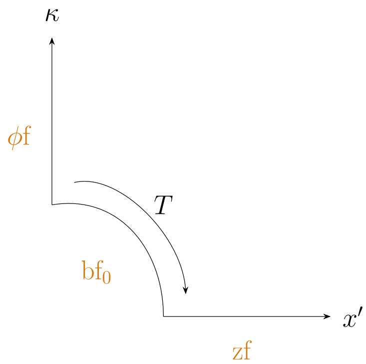


Figure 29: Cross section $HX^2_{ac} \cap \beta^*(\mathbb{R}_+ \times diag_X)$ over which the supertrace is computed.

We would like to compute the correction term (Corr) in:

$$\lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\not{\partial}^2}\right) = \lim_{t\to\infty} {}^R\!\!\int_X \mathrm{str}_p\left(h(t,p,p')\right)\frac{dx'}{(x')^2}\frac{dy'}{(x')^b} = \lim_{\kappa\to 0}\int_{\partial X} {}^R\!\!\int_0^\delta \underbrace{\mathrm{str}_p(h)\left(\frac{1}{\kappa^2},x',y'\right)(x')^{-(b+1)}\frac{dx'}{x'}}_{=:f_{y'}(\kappa,x')}dy'$$

$$\overset{\partial X \text{ closed}}{=} \int_{\partial X}\lim_{\kappa\to 0} {}^R\!\!\int_0^\delta f_{y'}(\kappa,x')\frac{dx'}{x'}dy' \overset{!}{=} \underbrace{\int_{\partial X} {}^R\!\!\int_0^\delta \lim_{\kappa\to 0} f_{y'}(\kappa,x')\frac{dx'}{x'}dy'}_{{}^R\mathrm{Str}\left(\lim_{t\to\infty}\left(e^{-t\not{\partial}^2}\right)\right)} + \int_{\partial X}\mathrm{Corr}\,dy'$$

This can be recovered from [LV13, Lemma 3.3], which shows that an α-homogeneous function $g \in \mathcal{C}^\infty\left(\mathbb{R}^2_+ \setminus \{(0,0)\}; \mathbb{C}\right)$ satisfies:

$$\underset{b\to\infty}{\mathrm{LIM}} {}^R\!\!\int_c^\infty g(a,b)da = {}^R\!\!\int_c^\infty \underset{b\to\infty}{\mathrm{LIM}}\, g(a,b)da + \begin{cases} {}^R\!\!\int_0^\infty g(a,1)da & \alpha = -1 \\ 0 & \alpha \neq -1 \end{cases}$$

for $c \geq 0$[106]. We are rather interested in integrals and limits around 0, so we can perform a change of coordinates $a = \frac{1}{x'}$ and $b = \frac{1}{\kappa}$, obtaining:

$$\underset{\kappa\to 0}{\mathrm{LIM}} {}^R\!\!\int_0^{1/c} g\left(\frac{1}{x'}, \frac{1}{\kappa}\right) \frac{dx'}{(x')^2} = {}^R\!\!\int_0^{1/c} \underset{\kappa\to 0}{\mathrm{LIM}}\, g\left(\frac{1}{x'}, \frac{1}{\kappa}\right) \frac{dx'}{(x')^2} + \begin{cases} {}^R\!\!\int_0^\infty g\left(\frac{1}{x'}, 1\right) \frac{dx'}{(x')^2} & \alpha = -1 \\ 0 & \alpha \neq -1 \end{cases}$$

By choosing $f_{y'}(\kappa, x') \mathrel{\hat{=}} g\left(\frac{1}{x'}, \frac{1}{\kappa}\right)\frac{1}{x'}$, we obtain an expression like the one above. The heat kernel expansions we work with are smooth at the corresponding faces and have no leading order logarithm terms, so for us the regularised limit is a real limit, i.e. $\underset{\kappa\to 0}{\mathrm{LIM}} \mathrel{\hat{=}} \lim_{\kappa\to 0}$. To figure out whether the connection term vanishes, notice that g being (-1)-homogeneous corresponds to f being 0-homogeneous:

$$f_{y'}(\lambda\kappa, \lambda x') = g\left(\frac{1}{\lambda x'}, \frac{1}{\lambda\kappa}\right)\frac{1}{\lambda x'} = \left(\frac{1}{\lambda}\right)^{-1} g\left(\frac{1}{x'}, \frac{1}{\kappa}\right)\frac{1}{\lambda x'} = f_{y'}(\kappa, x')$$

Since the heat kernel has leading order $b+1$ at bf_0 and zf, and κ is a boundary defining function for both faces, this means:

$$f_{y'}(\kappa, x') = \mathrm{str}_p(h)(x')^{-(b+1)} \sim h_0(x', y')\kappa^{b+1}(x')^{-(b+1)}$$

The form of the leading order coefficient $h_0(x', y')$ is (heuristically) given by writing the solution at ϕf (e.g. [TV22, (6.3)] with trivial fibres) in the bf_0 coordinates. The arguments of the exponentials become:

$$\frac{S^2}{\tau^2} \mathrel{\hat{=}} \frac{\xi^2}{\kappa^2}\kappa^2 = \xi^2, \quad \frac{u^2}{\tau^2} \mathrel{\hat{=}} \frac{\eta^2}{\kappa^2}\kappa^2 = \eta^2$$

which are unaffected by the λ-homothety in x' and κ (this also works with their rescaled forms, since, in the harmonic oscillator term, $\coth(w) \sim \frac{1}{w}$ for w small). All in all, $f_{y'}$ turns out to be 0-homogeneous, so there is indeed a correction term to be computed:

$$\lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = {}^R\mathrm{Str}\left(\lim_{t\to\infty}\left(e^{-t\eth^2}\right)\right) + \int_{\partial X} {}^R\!\!\int_0^\infty f_{y'}(1, r)dr dy'$$

Remark 4.24. In the fibred boundary case, the existence of fibres pertains to the extra $e^{-t\Delta_Z}$ factor in the heat kernel. Paralleling the analysis in [GTV22, §7], we can decompose the problem in fibrewise harmonic and fibrewise non-harmonic sections, i.e. on those seen by Δ_Z and those in its kernel.
Notice that for functions in the kernel, the operator $e^{-t\Delta_Z}$ behaves like the identity: "$e^{-t\cdot 0} = 1$". Thus, the heat kernel behaves on these sections as an asymptotically conical operator and the

[106] Observe that the correction term comes from the change of coordinates $a \mapsto \frac{a}{b}$ in the proof of the lemma, so even though a is written as dummy variable, the projective coordinate $\frac{a}{b}$ is the one against which we are effectively integrating. After we make the change of coordinates to take the limit and integration to be around 0, this becomes $\frac{\kappa}{x'} = \frac{1}{T} =: r$, which is the coordinate parametrizing the front face of the blow-up in Figure 29 starting with value 0 at the intersection with zf and increasing to ∞ at the intersection with ϕf. This is the reason why the correction term corresponds to the contribution at bf_0.

discussion above applies. This only occurs if we allow the operator to be non-fully elliptic, since we saw this equates to having non-trivial kernel in the fibre directions.
On the contrary, when restricting to elements that are not harmonic in Z, the operator behaves fully elliptic. Moreover, the term $e^{-t\Delta}$ contributes with the usual $t^{-\frac{f}{2}} = \kappa^f$ heat kernel pre-factor to the leading order term and thus the corresponding function $f_{y',z'}(\kappa, x')$ is now f-homogeneous. This means this fully elliptic part does not contribute to the correction term and thus is invisible to the local index formula. This is similar to what happens in [Vai01, p. 97] at the face hbf, where only the base directions acting on fibrewise harmonic forms contribute with an eta invariant.
For this breakdown of the operator to work technically, the precise assumption is that the kernel of the vertical family $\{\Delta_{Z_y}\}_{y\in Y}$ (resp. $\{\eth_{Z_y}\}_{y\in Y}$)) forms a vector bundle over the base Y (denoted $\mathcal{H}$ in [GTV22, after Assumption 1.3] and $\mathcal{K}$ in [Vai01, (39)]), as happens in the treatment of the families index theorem [BGV04, §9]. We come back to this in the next sections.

□

The contribution from zf is clear from Proposition 4.23 and gives rise to a renormalized index in the notation of [Alb07, §6.1.2]:

$$^R\mathrm{Str}\left(\lim_{t\to\infty}\left(e^{-t\eth^2}\right)\right) = {}^R\mathrm{Str}\left(\Pi_{\ker\eth^2}\right) = {}^R\mathrm{Str}\left(\Pi_{\ker\eth}\right) =: {}^R\mathrm{ind}\left(\eth^+\right)$$

in accordance with the classical $\ker\eth = \ker\eth^+ \oplus \ker\eth^-$ and $\mathrm{ind}\,\eth = \dim\ker\eth^+ - \dim\ker\eth^-$ [BGV04, Def. 3.46] (in which case we could write ${}^R\mathrm{ind}\left(\eth\right)$ instead to refer to the same contribution).
For the bf_0 term we have:

$$\int_{\partial X}{}^R\!\!\int_0^\infty f_{y'}(1,r)dr dy' = \int_{\partial X}{}^R\!\!\int_0^\infty f_{y'}\left(1,\frac{1}{T}\right)\frac{dT}{T^2}dy' = \int_{\partial X}{}^R\!\!\int_0^\infty \underbrace{\mathrm{str}_p\left(\frac{1}{T^n}\sum_{j\geq 0} h_j^{\mathrm{bf}_0} T^{-j}\right)T^{b+1}}_{\mathrm{str}_p\left(N_{\mathrm{bf}_0,n}(h)\right)(T,y')}\frac{dT}{T^2}dy'$$

where we used the expansion of the heat kernel $h \sim \rho_{\mathrm{bf}_0}^n \sum_{j\geq 0} h_j^{\mathrm{bf}_0}\rho_{\mathrm{bf}_0}^j$ around bf_0 with the choice $\rho_{\mathrm{bf}_0} = x'$ restricted to $x' = T^{-1}$ (since we did not rescale at this face, the normal operator is the same as in the stardard heat calculus). So once more we need to solve the model problem at this face. Since it lies in the $x' = 0$ regime, we have to make use of the connection asymptotics. The final lifts take the form (again, with coordinates in ∂X so that $y' = 0$):

$$\begin{aligned}
\beta^*\left(t\partial_t\right) &= \frac{1}{2}\left(T\partial_T - \kappa\partial_\kappa\right)\\
\beta^*\left(\sqrt{t}\eth\right) &= cl\left(\frac{dx}{x^2}\right)\left(\left(T\xi+1\right)^2\partial_\xi - \frac{T\left(T\xi+1\right)b}{2}\right)\\
&+(T\xi+1)\sum_{i=1}^b cl\left(\frac{dy_i}{x}\right)\left(\partial_{\eta_i} - \frac{T^2}{8}\sum_{j,p,q=1}^b g_{\partial X}\left(R_{\partial X}\left(\partial_{y_i},\partial_{y_q}\right)\eta_q\partial_{y_j},\partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)\right)\\
\beta^*\left(t\eth^2\right) &= -\left(T\xi+1\right)^4\partial_\xi^2 + (b-2)T\left(T\xi+1\right)^3\partial_\xi - T^2\left(T\xi+1\right)^2\frac{b}{2}\left(1-\frac{b}{2}\right)\\
&+T(T\xi+1)^2 cl\left(\frac{dx}{x^2}\right)\sum_{i=1}^b cl\left(\frac{dy_i}{x}\right)\left(\partial_{\eta_i} - \frac{T^2}{8}\sum_{j,p,q=1}^b g_{\partial X}\left(R_{\partial X}\left(\partial_{y_i},\partial_{y_q}\right)\eta_q\partial_{y_j},\partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)\right)\\
&-(T\xi+1)^2\sum_{i=1}^b\left(\partial_{\eta_i} - \frac{T^2}{8}\sum_{j,p,q=1}^b g_{\partial X}\left(R_{\partial X}\left(\partial_{y_i},\partial_{y_q}\right)\eta_q\partial_{y_j},\partial_{y_p}\right)cl\left(\frac{dy_j}{x}\right)cl\left(\frac{dy_p}{x}\right)\right)^2
\end{aligned} \tag{31}$$

This basically tells us that all the terms contribute to leading order, since there is no κ or x' dependency to be seen (everything was absorbed by the face parameter T). Our strategy to solve this equation will be to come back to the original, non-lifted form (24), try to make modifications to simplify and solve the problem in those coordinates, and finally lift the solution to bf_0. This is done in §4.5.
We summarize the whole discussion in the following statement:

Theorem 4.25. Let $E \to X$ be a Hermitian Clifford module over a compact manifold with boundary with a metric which differs by polyhomogeneous higher order terms from the exact asymptotically conical metric

$$g_{ac} = \frac{dx^2}{x^4} + \frac{g_{\partial X}}{x^2}$$

on $\mathring{\mathcal{U}}$, where x is a fixed choice of boundary defining function and $\mathcal{U} \cong [0, \varepsilon)_x \times \partial X \subset X$ is a collar neigbhorhood of the boundary so that $\mathcal{U} \cap \mathring{X} = \mathring{\mathcal{U}}$. Equip E with a Clifford connection ∇^E and construct the associated Dirac operator $\eth = cl \circ \nabla^E$. Then, the McKean-Singer formula allows us to compute the renormalized index of $\eth^+$ as:

$$\begin{aligned} {}^R\mathrm{ind}\left(\eth^+\right) &= {}^R\!\!\int_X \hat{\mathrm{A}}(X) Ch(E) \\ &+ \frac{1}{2} \int_{\partial X} \left[\mathrm{str}_p \left(cl\left(\frac{dx}{x^2}\right) \eth e^{-t\eth^2} \right) \right]_b dy' - \int_{\partial X} {}^R\!\!\int_0^\infty \mathrm{str}_p \left(N_{\mathrm{bf}_0, n}\left(e^{-t\eth^2}\right)\right) \frac{dT}{T^2} dy' \end{aligned}$$

□

The asymptotically conical structure of the metric appears in this formula through the term $\hat{\mathrm{A}}(X)$, which is constructed à la Chern-Weil from the Riemannian curvature R^E of g_{ac}, part of the curvature of the connection ∇^E by the standard splitting of [BGV04, Prop. 3.43].
Assuming that the third term vanishes and the last corresponds to an eta invariant, as postulated in §4.5, this formula would be in the nature of the Atiyah-Patodi-Singer result (Theorem 2.15) and the index formula for cylindrical ends [Mel93, (9.5)], although in our case all the analytical contributions might be hidden in the rescaled index term. Moreover, if $\eth^+$ is fully elliptic, then the large time supertrace would contribute (only) the Fredholm index of $\eth^+$ and there would be no η-term, so we would recover the Atiyah-Singer relation (5) as in [Mel95, Theorem 6.4].

4.4 Manifolds with fibred cusps

It's like a new toaster that comes with a 16-page manual. If you already understand toasters and if the toaster looks like previous toasters you've encountered, you might just plug it in and see if it works, rather than first reading all the details in the manual.

On proof and progress in mathematics, *William P. Thurston*

We would like to generalize the study of scattering manifolds to the setting of **fibred cusps** by mimicking and adapting the arguments made above. In the previous section, we formulated our analysis in a way so that it could be extended to the case of 0-ϕ-metrics after only minor modifications. We now describe which these are and the ensuing results.
Recall briefly that a compact manifold with a boundary fibre bundle has an exact 0-ϕ-metric (in the notation of [GSHV25]) if in a collar neigborhood of the boundary the metric tensor takes the form:

$$g_\phi = \frac{dx^2}{x^2} + \frac{\phi^* g_Y}{x^2} + g_Z, \quad \phi : \partial X \longrightarrow Y$$

This assumes a choice of boundary defining function x and thus the geometric structures constructed adapted to it all depend a priori on this selection. From now on, we will denote 0-ϕ-structures by the symbol ϕ for simplicity, since we will say little about c-ϕ-metrics for $c \neq 0$. These structures all follow from the developments in §4.1 for $(k, l, m) = (2, 1, 0)$. In particular, this is a reference metric (Def. 4.11) for its conformal class and therefore in the analysis of the spectral and index theory of 1-ϕ-manifolds of [Vai01] already several constructions in the ϕ-category are carried out. The seminal paper is however [MM98], which in particular establishes the equivalence between Fredholmness of elliptic ϕ-differential operators and triviality of the kernel of the vertical normal family (full ellipticity) [MM98, Theorem 1]. Notice how most aspects of the ϕ-Fredholm and ΨDO theory are already portrayed in the scattering case of the previous section. See also [GTV22, §4]. [MM98, Problem 1] is solved in [LMP07] and our goal here is to extend this **local index formula** to the class of **non-fully elliptic** (Dirac-type) operators under the standard assumption that the kernels of the vertical family form a vector bundle. The previous section actually deals with the most relevant hurdles, i.e. the inclusion of fibres does not affect the analysis greatly.

In this setting, the Lie subalgebra of ϕ-vector fields $\mathcal{V}_\phi \subset \mathcal{V} = \mathcal{C}^\infty(X, TX)$ can be defined via the following property:

$$\mathcal{V}_\phi = \{V \in \mathcal{V} \quad | \quad V(x) \in x^2\mathcal{C}^\infty(X) \quad \text{and} \quad V_p \quad \text{is tangent to} \quad \phi^{-1}(\phi(p)) \quad \forall p \in \partial X\}$$

Note that the second condition could be reformulated as: *the restriction of the vector field to the boundary is tangent to the fibres of the fibre bundle ϕ, that is, "vertical"*.

Choose local coordinates $\{\mathbf{y}_i\}_{i=1}^b$ for Y and pull them back via ϕ to ∂X: we will denote the resulting local coordinates $y_i = \phi^*\mathbf{y}_i = \mathbf{y}_i \circ \phi \quad \forall 1 \leq i \leq b$. We can complete these *horizontal* coordinates to a basis of ∂X by adding *vertical* coordinates $\{\mathbf{z}_j\}_{j=1}^f$ that represent the fibre directions. Thus, in the collar neigborhood of the boundary we have coordinates $(x, y_1, \dots, y_b, z_1, \dots, z_f)$.

With these local coordinates, the space of ϕ-vector fields $\mathcal{V}_\phi$ is locally spanned at a point $p = (x, y, z) \in \partial X$ by the generating sections:

$$x^2\partial_x, \quad x\partial_y, \quad \partial_z$$

Here, y is short for $(y_1, \dots, y_b)$. Similarly, $x\partial_y$ represents all the $\{x\partial_{y_i}\}_{i=1}^b$ (analogous for z and ∂_z).

Note that, away from the boundary, where x vanishes, $\mathcal{V}_\phi$ can be identified with $\mathcal{V}$, since its defining properties only affect its behavior around the boundary, via $(x^2\partial_x, x\partial_y, \partial_z) \leftrightarrow (\partial_x, \partial_y, \partial_z)$. In short: $\mathcal{V}_\phi\big|_{\mathring{X}} \cong \mathcal{V}\big|_{\mathring{X}}$.

The local description of $\mathcal{V}_\phi$ via generating sections above shows that this space is a finitely generated projective module over $\mathcal{C}^\infty(X)$ and the Serre-Swan theorem tells us that it then has to be the space of sections of a smooth vector bundle over X, which we will denote ${}^\phi TX$, the ϕ-tangent bundle[107]. Its dual, the ϕ-cotangent bundle ${}^\phi T^*X$, is locally "spanned" around the boundary by the sections:

$$\frac{dx}{x^2}, \quad \frac{dy}{x}, \quad dz$$

This is also isomorphic to the usual cotangent bundle in the interior, but it becomes "singular" at the boundary.

We also construct orthogonal bases for $\Gamma({}^\phi TX)$ and its dual in the manner of §4.1, i.e. by lifting an orthonormal basis from Y, completing with the normal and fibre directions and extending them trivially to the collar neighborhood. The resulting sections are likewise denoted:

$$x^2\partial_x, \quad x\tilde{U}_i, \quad V_j$$

[107] The precise statement relates the sections of the ϕ-tangent bundle and $\mathcal{V}_\phi$ via the anchor map, similarly to Example 4.18. The same is true for the ϕ-cotangent bundle in relation to the cotangent bundle of X.

It is again useful to do our local computations in coordinates whose Y-part is centered at the origin $y' = 0$, and to choose the orthonormal vectors U_i in the $\Gamma(TY)$ basis to be extended from an orthonormal frame at $T_{\phi(p)}Y$ by radial parallel transport.
Since we want to apply the heat kernel method through the (regularized) McKean-Singer formula, the main ingredient we rely on is the heat kernel analysis of [TV22] (short time), [GTV22] and [LV25, §2] (large time), which is summarized in the (compactified) heat blow-up space $\overline{HX}^2_\phi$ of Figure 30.

Remark 4.26. In [LV25, §2], the authors do not need sharp bounds on the index sets at the faces for their application to analytic torsion. However, one can apply the same argument as [She13, Theorem 1] to the class of ϕ-manifolds to obtain $\mathcal{E}_{\text{zf}} \geq n$ for the heat kernel.
This is based off of the fact that ϕ-manifolds are of bounded geometry, i.e. of bounded sectional curvature (more generally of bounded Riemann curvature tensor and positive injectivity radius): since X is a compact manifold, the unboundedness can only come from the boundary; moreover, as we will see below (consequence of Proposition 4.8 4.), the sectional curvature vanishes when approaching the boundary in the normal and horizontal directions, and along the fibres it stays bounded because they are modelled on a closed manifold Z. Thus, the work of [CLY81] applies to the non-compact manifold M whose compactification gives rise to X, i.e. to the large time limit of the heat kernel on X away from the boundary. That is, at the face zf we have the leading order $t^{-\frac{n}{2}} = \kappa^n$, where $\kappa = \rho_{\text{zf}}$ is a valid choice of boundary defining function. This is our grain of sand to Figure 30.
On top of that, the face zf (i.e. the face in the compactified heat blow-up space that corresponds to the lift of $\{\infty\} \times X^2$ or equivalently the lift of $\{0\} \times X^2$ to the resolvent blow-up space related to spectral parameter $\lambda = 0$) is, much alike the face tf (with its leading order $t^{-\frac{n}{2}}$), always present in the compactified heat blow-up space. Hence, if a manifold has bounded sectional curvature, the theorem of Cheng-Li-Yau goes through to produce leading order at least n at that face. In particular, reference metrics and consequently most of the complete examples presented in §4.1 make the cut and have therefore similar asymptotics.
Be that as it may, this is also irrelevant for our index pursuits, since we apply the Simon + limit-supertrace commutation trick introduced in last section to study the large time limit.

□

The coordinates in each of the relevant faces are:

- Around $\mathring{\text{tf}}$ and its intersection with ϕf:
$$\xi = \frac{x - x'}{(x')^2\sqrt{t}}, \quad \eta = \frac{y - y'}{x'\sqrt{t}}, \quad \mu = \frac{z - z'}{\sqrt{t}}, \quad x', \quad y', \quad z', \quad \tau = \sqrt{t}$$
The "natural" choice is again $\rho_{\text{tf}} = \tau$. There is a fibration:
$$\text{tf} \longrightarrow \{0\} \times \text{diag}_X \cong X, \quad (\xi, \eta, \mu, x', y', z') \mapsto (x', y', z')$$
with fibres tf/X modelled on the radial compactification of $\mathbb{R}_\xi \times \mathbb{R}^b_\eta \times \mathbb{R}^f_\mu$.
The vector fields
$$\sqrt{t}x^2\partial_x, \quad \sqrt{t}x\partial_y, \quad \sqrt{t}\partial_z, \quad t\partial_t$$
on $\mathbb{R}_+ \times X^2$ pull back along the blow-down map β to become
$$(x'\sqrt{t}\xi + 1)^2\partial_\xi, \quad (x'\sqrt{t}\xi + 1)\partial_\eta, \quad \partial_\mu, \quad \frac{1}{2}(\tau\partial_\tau - \xi\partial_\xi - \eta\partial_\eta - \mu\partial_\mu)$$
which are tangent to the fibres tf/X and span ${}^bT\text{tf}/X$, so we identify tf with the radial compactification of the rescaled bundle $\sqrt{t}\left({}^\phi TX\right) \cong {}^\phi TX$.

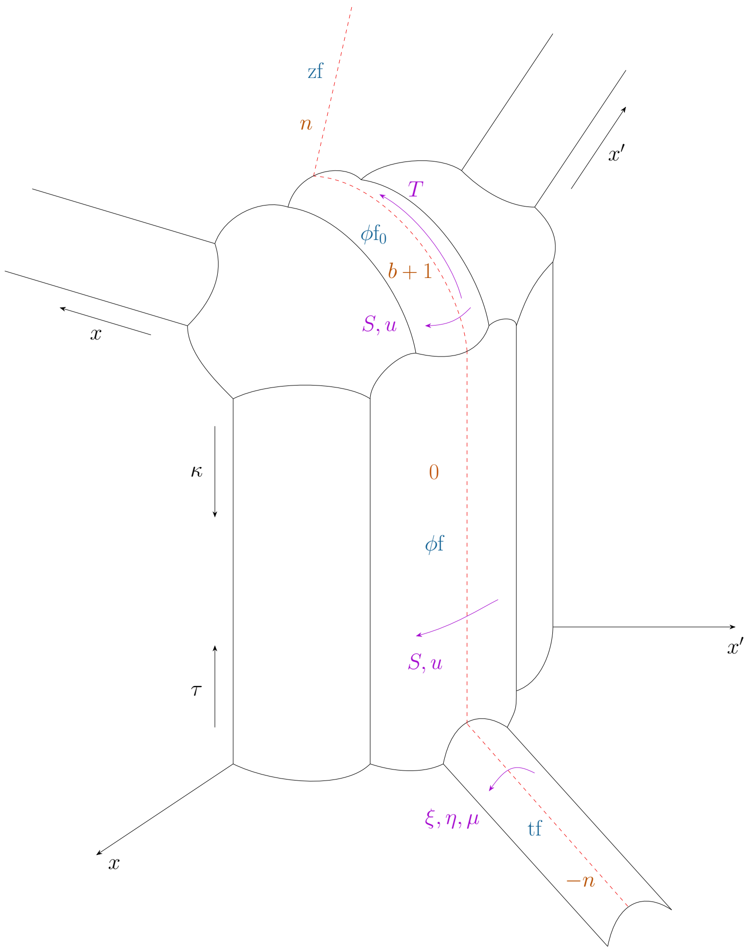


Figure 30: Compactified heat blow-up space for 0-ϕ-metrics (compare [GTV22, Figure 9] and [TV22, Figure 4]). In red the intersection of the lifted diagonal with the boundary hypersurfaces, in orange the leading order of the heat kernel at each face [LV25, §2, §3], which is named in blue. In magenta the fibrewise coordinates.

- Around $\mathring{\phi}$f:

$$S = \frac{x - x'}{(x')^2}, \quad u = \frac{y - y'}{x'}, \quad z, \quad x', \quad y', \quad z', \quad \tau = \sqrt{t}$$

We choose $\rho_{\phi f} = x'$. Notice that ϕf fibres over the base directions:

$$\phi f \longrightarrow \overline{\mathbb{R}}_+ \times \text{diag}_\phi \longrightarrow \text{diag}_\phi \cong \partial X \times_\phi \partial X \longrightarrow Y, \quad (S, U, z, y', z', \tau) \mapsto (z, y', z') \mapsto y'$$

with fibres modelled on the radial compactification of $\mathbb{R}_S \times \mathbb{R}^b_u \times Z^2 \times (0, \infty)_\tau$. The fibre product $\partial X \times_\phi \partial X$ refers to the usual pullback on the category of smooth manifolds

$$\begin{array}{ccc} \partial X \times_Y \partial X & \dashrightarrow & \partial X \\ \downarrow & & \downarrow \phi \\ \partial X & \xrightarrow[\phi]{} & Y \end{array}$$

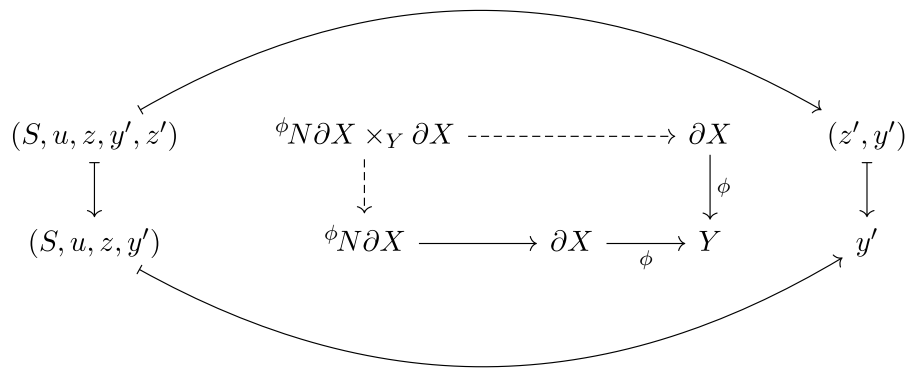
The lifts of

$$x^2 \partial_x, \quad x\partial_y, \quad \partial_z, \quad t\partial_t$$

are given by

$$(x'S+1)^2\partial_S, \quad (x'S+1)\partial_u, \quad \partial_z, \quad \frac{1}{2}\tau\partial_\tau$$

and so there is an identification of ϕf with the radial compactification of $(0,\infty)_\tau \times {}^\phi N\partial X \times_Y \partial X$, where the notation is borrowed from [Vai01, §1.1, 4.1]. That is, ${}^\phi N\partial X \to \partial X$ corresponds to the bundle whose sections are non-vertical ϕ-vector fields, i.e. ${}^\phi TX\big|_{\partial X} \cong T\partial X/Y \oplus {}^\phi N\partial X$. The corresponding pullback diagram is:



- In the case of $\mathring{\phi}\mathrm{f}_0$ and its intersection with ϕf, the coordinates are:

$$S = \frac{x-x'}{(x')^2}, \quad u = \frac{y-y'}{x'}, \quad z, \quad T = \frac{x'}{\kappa} = x'\sqrt{t}, \quad y', \quad z', \quad \kappa = \frac{1}{\sqrt{t}}$$

and the vector fields lift to:

$$\beta^* x^2\partial_x = (T\kappa S+1)^2\partial_S, \quad \beta^* x\partial_y = (T\kappa S+1)\partial_\eta, \quad \beta^*\partial_z = \partial_z, \quad \beta^* t\partial_t = \frac{1}{2}\left(T\partial_T - \kappa\partial_\kappa\right)$$

so there is a fibration

$$\phi\mathrm{f}_0 \longrightarrow \partial X \times_\phi \partial X \longrightarrow Y, \quad (S,u,z,T,y',z') \mapsto (z,y',z') \mapsto y'$$

and an identification of $\phi\mathrm{f}_0$ with the radial compactification of $(0,\infty)_T \times {}^\phi N\partial X \times_Y \partial X$, as was the case for ϕf.

Let us focus again on the spinor Dirac operator $\eth$ induced by the Clifford algebra $Cl({}^\phi T^*X)$ with action $cl_\phi(\cdot) =: cl(\cdot)$. Getzler rescaling in this case is inspired by the scattering case; due to the form of the metric (and comparison of heat blow-up spaces), the role of ∂X in the ac-case will now be taken by Y.
The connection asymptotics look like (see table below too):

$$g_\phi(\nabla_{\tilde{U}_i} x\tilde{U}_j, x^2\partial_x) = -g_\phi(\nabla_{\tilde{U}_i} x^2\partial_x, x\tilde{U}_j) = g_Y(U_i,U_j) = \delta_{ij}$$

Accordingly, when studying the faces ϕf and $\phi\mathrm{f}_0$, we will resort to the following expressions for the

$g_\phi(\nabla_{W_i}W_j, W_p)$	V_p	$x\tilde{U}_p$
$\nabla_{V_i}V_j$	$g_{\partial X/Y}(\nabla^{\partial X/Y}_{V_i}V_j, V_p)$	$x\phi^* g_Y(S^\phi(V_i,V_j),\tilde{U}_p)$
$\nabla_{\tilde{U}_i}V_j$	$\phi^* g_Y(S^\phi(V_j,V_p),\tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)$	$-\frac{x}{2}g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_p),V_j)$
$\nabla_{V_i}x\tilde{U}_j$	$-x\phi^* g_Y(S^\phi(V_i,V_p),\tilde{U}_j)$	$-\frac{x^2}{2}g_{\partial X/Y}(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i)$
$\nabla_{\tilde{U}_i}x\tilde{U}_j$	$\frac{x}{2}g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p)$	$g_Y(\nabla^Y_{U_i}U_j,U_p)$

spinor connection:

$$\nabla^{\mathcal{S}}_{\partial_x} = \partial_x$$

$$\begin{aligned}
\nabla^{\mathcal{S}}_{\tilde{U}_i} &= \sum_j^b a_{ij}\partial_{y_j} + \sum_j^f b_{ij}\partial_{z_j} + \frac{1}{2}cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(\frac{dx}{x^2}\right) \\
&\quad + \frac{1}{4}\sum_{jp}^f \Big(\phi^* g_Y(S^\phi(V_j,V_p),\tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)\Big)cl(V_j)cl(V_p) \\
&\quad + \frac{x}{4}\sum_{jp} g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl(V_p) - \frac{1}{8}\sum_{jpq}^b g_Y(R_Y(U_i,y_q\partial_{y_q})U_j,U_p)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl\left(\frac{\tilde{U}_p^\flat}{x}\right)
\end{aligned}$$

$$\nabla^{\mathcal{S}}_{V_i} = \nabla^{\mathcal{S}_{\partial X/Y}}_{V_i} + \frac{x}{2}\sum_{jp}\phi^* g_Y(S^\phi(V_i,V_j),\tilde{U}_p)cl(V_j)cl\left(\frac{\tilde{U}_p^\flat}{x}\right) - \frac{x^2}{8}\sum_{jp}^b g_{\partial X/Y}(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl\left(\frac{\tilde{U}_p^\flat}{x}\right)$$

using

$$g_Y(\nabla^Y_{U_i}U_j,U_p) = -\frac{1}{2}\sum_q^b g_Y(R_Y(U_i,\partial_{y_q})y_qU_j,U_p) = -\frac{1}{2}\sum_q^b g_Y(R_Y(U_i,y_q\partial_{y_q})U_j,U_p)$$

$$\nabla^{\mathcal{S}_{\partial X/Y}}_{V_i} = V_i + \frac{1}{4}\sum_{jp}^f g_{\partial X/Y}(\nabla^{\partial X/Y}_{V_i}V_j,V_p)cl(V_j)cl(V_p), \quad \tilde{U}_i = \sum_j^b a_{ij}\partial_{y_j} + \sum_j^f b_{ij}\partial_{z_j}$$

Recall that $\sum_{q=1}^b y_q\partial_{y_q}$ is the radial vector field in Y, so the corresponding term plays a similar role as e.g. [Mel93, (8.64)] or the terms with radial vector fields in [AGR23, Lemma 5.2], as noted previously.
The spin Dirac operator then takes the form:

$$\begin{aligned}
\eth &= cl\left(\frac{dx}{x^2}\right)\nabla^{\mathcal{S}}_{x^2\partial_x} + \sum_{i=1}^b cl\left(\frac{\tilde{U}_i^\flat}{x}\right)\nabla^{\mathcal{S}}_{x\tilde{U}_i} + \sum_{i=1}^f cl(V_i)\nabla^{\mathcal{S}}_{V_i} \\
&= cl\left(\frac{dx}{x^2}\right)x^2\partial_x \\
&\quad + x\sum_{i=1}^b cl\left(\frac{\tilde{U}_i^\flat}{x}\right)\left(\sum_j^b a_{ij}\partial_{y_j} + \sum_j^f b_{ij}\partial_{z_j}\right) - \frac{xb}{2}cl\left(\frac{dx}{x^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{x}{4}\sum_{ijp}\Big(\phi^*g_Y(S^\phi(V_j,V_p),\tilde{U}_i)-g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)\Big)\,cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl(V_j)cl(V_p)\\
&+\frac{x^2}{4}\sum_{ijp}g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p)cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl(V_k)\\
&-\frac{x}{8}\sum_{ijpq}^{b}g_Y(R_Y(U_i,y_q\partial_{y_q})U_j,U_p)cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\\
&+\sum_{i=1}^{f}cl(V_i)\nabla^{\mathcal{S}_{\partial X/Y}}_{V_i}+\frac{x}{2}\sum_{ijp}\phi^*g_Y(S^\phi(V_i,V_j),\tilde{U}_p)cl(V_i)cl(V_j)cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\\
&-\frac{x^2}{8}\sum_{ijp}g_{\partial X/Y}(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i)cl(V_i)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl\left(\frac{\tilde{U}_p^\flat}{x}\right)
\end{aligned}
$$

with normal family:

$$
\widehat{N_\phi(\eth)}(y',z;\xi,\eta)=icl\left(\frac{dx}{x^2}\right)\xi+i\sum_{jp}^{b}cl\left(\frac{\tilde{U}_j^\flat}{x}\right)a_{jp}\eta_p+\sum_{j}^{f}cl(V_j)\nabla^{\mathcal{S}_{\partial X/Y}}_{V_j}
$$

by restricting to $(\xi,\eta)=(0,0)$, we can see that this operator is fully elliptic precisely when the vertical operator $\eth_{\partial X/Y}:=\sum_j^f cl(V_j)\nabla^{\mathcal{S}_{\partial X/Y}}_{V_j}$ has trivial kernel, which we do not assume [108].
In this case, it seems clearly advantageous to compute $\eth^2$ using Lichnerowicz formula (15) instead of brute forcing it. So it is time we computed the sectional curvature of g_ϕ (this is just a special case of Proposition 4.8 4.):

$$
\mathrm{scal}_\phi=\sum_{ij}g_\phi\left(R\left(e_i,e_j\right)e_j,e_i\right),\quad e_i,e_j\in\{x^2\partial_x,x\tilde{U}_i,V_j\},\quad R\left(e_i,e_j\right)=\nabla_{e_i}\nabla_{e_j}-\nabla_{e_j}\nabla_{e_i}-\nabla_{[e_i,e_j]}
$$

By the connection asymptotics, the Levi-Civita connection is a true connection when acting on ϕ-vector fields. Moreover, $\nabla_{x^2\partial_x}(xA)=x^2A$ when A is a ϕ-tensor. Using $[x^2\partial_x,x\tilde{U}_i]=x^2\tilde{U}_i$, that $[\tilde{U}_i,\tilde{U}_j]$ is horizontal or vertical, that $[\tilde{U}_i,V_j]$ is vertical and by collecting the powers of x in the front e.g. like $\nabla_{x\tilde{U}_i}\nabla_{x\tilde{U}_j}=x^2\nabla_{\tilde{U}_i}\nabla_{\tilde{U}_j}$ we get:

$$
\begin{aligned}
\mathrm{scal}_\phi&=\sum_{ij}g_\phi\left(R(V_i,V_j)V_j,V_i\right)+\sum_{ij}g_\phi\left(R(\tilde{U}_i,V_j)V_j,\tilde{U}_i\right)+O(x^2)\\
&=\sum_{ij}g_{\partial X/Y}\left(R(V_i,V_j)V_j,V_i\right)+O(x^2)=\mathrm{scal}_{\partial X/Y}+O(x^2)
\end{aligned}
$$

[108]Most of the terms in the complete explicit form of $\eth^2$ close to $x=0$ (cf. (37)) are 0-th order contributions and $O(x)$ before lifting, so its normal family is rather uncomplicated:

$$
\widehat{N_\phi(\eth^2)}(y',z;\xi,\eta)=\xi^2+|\eta|^2+\eth^2_{\partial X/Y}
$$

This is again analogous to the Hodge Laplacian [GTV22, (5.8)] and thus fully elliptic exactly when $\ker\eth^2_{\partial X/Y}=\ker\eth_{\partial X/Y}=\{0\}$.

since (remember that hW denotes the horizontal part of W, i.e. its terms in $\Gamma\left(x\phi^*TY\right)$):

$$\begin{aligned}
h\left(\nabla_{x\tilde{U}_i}\nabla_{V_j}V_j\right) &= xh\left(\nabla_{\tilde{U}_i}\nabla_{V_j}V_j\right)\\
&= xh\left(\nabla_{\tilde{U}_i}\left(\sum_p^f g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_j}V_j,V_p\right)V_p + x\sum_p^b \phi^* g_Y\left(S^\phi\left(V_i,V_j\right),\tilde{U}_p\right)x\tilde{U}_p\right)\right)\\
&= -\frac{x^2}{2}\sum_p^f\sum_q^b g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_j}V_j,V_p\right)\left(g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_i,\tilde{U}_q\right),V_p\right)\right)x\tilde{U}_q\\
&+ x^2\sum_{pq}^b \phi^* g_Y\left(S^\phi\left(V_i,V_j\right),\tilde{U}_p\right)\left(g_Y\left(\nabla^Y_{U_i}U_p,U_q\right)\right)x\tilde{U}_q \in x^2\left(\Gamma\left(x\phi^*TY\right)\right)\\
h\left(\nabla_{V_j}\nabla_{x\tilde{U}_i}V_j\right) &= xh\left(\nabla_{V_j}\nabla_{\tilde{U}_i}V_j\right)\\
&= xh\Bigg(\nabla_{V_j}\Bigg(\sum_p^f\left(\phi^* g_Y\left(S^\phi\left(V_j,V_p\right),\tilde{U}_i\right) - g_{\partial X/Y}\left([\tilde{U}_i,V_p],V_j\right)\right)V_p\\
&- \frac{x}{2}\sum_p^b g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_i,\tilde{U}_p\right),V_j\right)x\tilde{U}_p\Bigg)\Bigg)\\
&= x^2\sum_p^f\sum_q^b\left(\phi^* g_Y\left(S^\phi\left(V_j,V_p\right),\tilde{U}_i\right) - g_{\partial X/Y}\left([\tilde{U}_i,V_p],V_j\right)\right)\phi^* g_Y\left(S^\phi\left(V_j,V_p\right),\tilde{U}_q\right)x\tilde{U}_q\\
&- \frac{x^4}{4}\sum_{pq}^b g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_i,\tilde{U}_p\right),V_j\right)g_{\partial X/Y}\left(R^\phi\left(\tilde{U}_p,\tilde{U}_q\right),V_j\right)x\tilde{U}_q \in x^2\left(\Gamma\left(x\phi^*TY\right)\right)\\
h\left(\nabla_{[x\tilde{U}_i,V_j]}V_j\right) &= xh\left(\nabla_{[\tilde{U}_i,V_j]}V_j\right) = x^2\sum_p^b \phi^* g_Y\left(S^\phi\left([\tilde{U}_i,V_j],V_j\right),\tilde{U}_p\right)x\tilde{U}_p \in x^2\left(\Gamma\left(x\phi^*TY\right)\right)
\end{aligned}$$

In particular, the scalar curvature of a scattering metric is $O(x^2)$.
We will compute the rough Laplacian term in the Lichnerowicz formula by first lifting the spinor connection to the corresponding face, Getzler rescaling inspired by the orders in the scattering case and applying the rescaled normal operator, thus dealing in the process with terms only contributing to leading order.
The face tf is treated again in the same manner as before, i.e. we lift the Clifford structure $Cl\left({}^\phi T^*X\right)\otimes \mathrm{End}_{Cl\left({}^\phi T^*X\right)}(E)\to X\cong\{0\}\times \mathrm{diag}_X$ along the fibration tf $\to X$ to obtain:

$$\mathcal{F}^{\mathrm{tf}}_k := \left(\beta^*_{xx'}\left(\mathrm{END}(E)\right)\big|_{\mathrm{tf}}\right)^k \cong Cl^k\left({}^\phi T^*X\right)\otimes \mathrm{End}_{Cl\left({}^\phi T^*X\right)}(E)$$

Lifting the spinor connection (i.e. just for $E\cong\mathcal{S}$, but the twisted case is analogous with the usual extra curvature term) to ϕf gives:

$$\begin{aligned}
\beta^*\nabla^{\mathcal{S}}_{x^2\partial_x} &= (x'S+1)^2\partial_S\\
\beta^*\nabla^{\mathcal{S}}_{x\tilde{U}_i} &= (x'S+1)\left(\sum_j^b a_{ij}\partial_{u_j} + \sum_j^f x' b_{ij}\partial_{z_j}\right) + \frac{(x'S+1)x'}{2}cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(\frac{dx}{x^2}\right)\\
&+ \frac{(x'S+1)x'}{4}\sum_{jp}^f\left(\phi^* g_Y(S^\phi(V_j,V_p),\tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)\right)cl(V_j)cl(V_p)\\
&+ \frac{(x'S+1)^2(x')^2}{4}\sum_{jp} g_{\partial X/Y}(R^\phi(\tilde{U}_i,\tilde{U}_j),V_p)cl\left(\frac{\tilde{U}_j^\flat}{x}\right)cl(V_p)
\end{aligned}$$

$$- \frac{(x'S+1)(x')^2}{8} \sum_{jpq}^{b} g_Y(R_Y(U_i, u_q\partial_{y_q})U_j, U_p) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)$$

$$\beta^*\nabla^{\mathcal{S}}_{V_i} = \nabla^{\mathcal{S}_{\partial X/Y}}_{V_i} + \frac{(x'S+1)x'}{2} \sum_{jp} \phi^* g_Y(S^\phi(V_i, V_j), \tilde{U}_p) cl(V_j) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)$$

$$- \frac{(x'S+1)^2(x')^2}{8} \sum_{jp}^{b} g_{\partial X/Y}(R^\phi(\tilde{U}_j, \tilde{U}_p), V_i) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)$$

Again, the connection does not rescale in all normal and base directions because of the second term in the second line, coming from the fact that the ϕ-metric is a g_{klm} metric with $l \neq 0$ and $k = l+1$. It does however rescale in the base directions, so writing $\delta_{x'}\omega = (x')^{-k}\omega$ for $\omega \in \Gamma\left(\frac{1}{x}\phi^*T^*Y\right)$:

$$\delta_{x'} \circ \beta^*\nabla^{\mathcal{S}}_{x^2\partial_x} \circ \delta_{x'}^{-1} = \partial_S + O(x')$$

$$\delta_{x'} \circ \beta^*\nabla^{\mathcal{S}}_{x\tilde{U}_i} \circ \delta_{x'}^{-1} = \sum_{j}^{b} a_{ij}\partial_{u_j} + \frac{1}{2}\varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right) cl\left(\frac{dx}{x^2}\right)$$

$$- \frac{1}{8}\sum_{jpq}^{b} g_Y(R_Y(U_i, u_q\partial_{y_q})U_j, U_p)\varepsilon\left(\frac{\tilde{U}_j^\flat}{x}\right)\varepsilon\left(\frac{\tilde{U}_p^\flat}{x}\right) + O(x')$$

$$\delta_{x'} \circ \beta^*\nabla^{\mathcal{S}}_{V_i} \circ \delta_{x'}^{-1} = \nabla^{\mathcal{S}_{\partial X/Y}}_{V_i} + \frac{1}{2}\sum_{jp} \phi^* g_Y(S^\phi(V_i, V_j), \tilde{U}_p) cl(V_j)\varepsilon\left(\frac{\tilde{U}_p^\flat}{x}\right)$$

$$- \frac{1}{8}\sum_{jp}^{b} g_{\partial X/Y}(R^\phi(\tilde{U}_j, \tilde{U}_p), V_i)\varepsilon\left(\frac{\tilde{U}_j^\flat}{x}\right)\varepsilon\left(\frac{\tilde{U}_p^\flat}{x}\right) + O(x')$$

Therefore, we would like to lift the Clifford module over the base $Cl\left(T^*Y\right) \otimes \mathrm{End}_{Cl(T^*Y)}(E) \to Y$ to its isomorphic rescaled form $Cl\left(\frac{1}{x}\phi^*T^*Y\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}\phi^*T^*Y\right)}(E) \to \mathrm{diag}_\phi$ along $\mathrm{diag}_\phi \cong \partial X \times_Y \partial X \to Y$ and from there to ϕf, yielding:

$$\mathcal{F}_k^{\phi\mathrm{f}} := \left(\beta^*_{xx'}\left(\mathrm{END}(E)\right)\big|_{\phi\mathrm{f}}\right)^k \cong Cl^k\left(\frac{1}{x}\phi^*T^*Y\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}\phi^*TY\right)}(E)$$

We extend these filtrations away from the faces by ∂_τ and $\partial_{x'}$, respectively, and apply Proposition 4.19 to the bundle $\mathcal{E} = \beta^*_{xx'}\left(\mathrm{END}(E)\right)$ to obtain a rescaled bundle $\mathcal{G}$ spanned by the local sections:

$$\rho_{\mathrm{tf}}\frac{dx}{x^2}, \quad \rho_{\mathrm{tf}}\rho_{\phi\mathrm{f}}\frac{dy}{x}, \quad \rho_{\mathrm{tf}}dz$$

and whose restriction at the rescaled faces is:

$$\mathcal{G}\big|_{\mathrm{tf}} \cong \bigoplus_{k=0}^{n} (N^*\mathrm{tf})^k \otimes \underbrace{\left(Cl^k({}^cT^*X)/Cl^{k-1}({}^cT^*X)\right)}_{\cong\Lambda^k({}^cT^*X)} \otimes \mathrm{End}_{Cl({}^\phi T^*X)}(E)$$

$$\mathcal{G}\big|_{\phi\mathrm{f}} \cong \bigoplus_{l=0}^{b} (N^*\phi\mathrm{f})^l \otimes \underbrace{\left(Cl^l\left(\frac{\tau}{x}\phi^*T^*Y\right) \Big/ Cl^{l-1}\left(\frac{\tau}{x}\phi^*T^*Y\right)\right)}_{\cong\Lambda^l\left(\frac{\tau}{x}\phi^*T^*Y\right)} \otimes \mathrm{End}_{Cl\left(\frac{1}{x}\phi^*T^*Y\right)}(E)$$

$$\mathcal{G}\big|_{\mathrm{tf}\cap\phi\mathrm{f}} \cong \bigoplus_{k=0}^{n} (N^*\mathrm{tf})^k \otimes \left[\bigoplus_{j+l=k} \Lambda^j\left({}^cT^*X\right) \otimes (N^*\phi\mathrm{f})^l \otimes \Lambda^l\left(\frac{\tau}{x}\phi^*T^*Y\right)\right] \otimes \mathrm{End}_{Cl({}^\phi T^*X)}(E)$$

Reproducing Example 4.20 gives

$$\lim_{t\to 0} \mathrm{Str}\left(e^{-t\eth^2}\right) = {}^{R}\!\!\int_X \hat{\mathrm{A}}(X) Ch(E)$$

from the splitting $R^E + F^{E/\mathcal{S}}$ [BGV04, Prop. 3.43] of the curvature of the Clifford connection ∇^E, being R^E the Riemannian curvature of g_ϕ, used to construct the Â-genus, and $F^{E/\mathcal{S}}$ the part giving rise to the Chern character.
As announced, to recover the model problem at ϕf in the rescaled calculus $\Psi_{\mathcal{G}}^{\alpha,\gamma}(X;E)$ in the case $E=\mathcal{S}$ we apply (the rescaled) Lichnerowicz formula (15), with the observations:

$$\delta_{x'} \circ \left(-\sum_i \left(\beta^*\nabla^{\mathcal{S}}_{e_i}\right)^2\right) \circ \delta_{x'}^{-1} = -\sum_i \left(\delta_{x'} \circ \beta^*\nabla^{\mathcal{S}}_{e_i} \circ \delta_{x'}^{-1}\right)^2$$

$$\left(\delta_{x'} \circ \left(\sum_i \beta^*\nabla^{\mathcal{S}}_{\beta^*\nabla_{e_i}e_i}\right) \circ \delta_{x'}^{-1}\right) = \left(\delta_{x'} \circ \left(\sum_{i=1}^{f} \beta^*\nabla^{\mathcal{S}}_{\nabla_{V_i}V_i}\right) \circ \delta_{x'}^{-1}\right) + O(x')$$

$$= \sum_{i,j=1}^{f} g_{\partial X/Y}\left(\nabla^{\partial X/Y}_{V_i}V_i, V_j\right)\left(\delta_{x'} \circ \left(\beta^*\nabla^{\mathcal{S}}_{V_j}\right) \circ \delta_{x'}^{-1}\right) + O(x')$$

$$\delta_{x'} \circ \beta^*\mathrm{scal}_\phi \circ \delta_{x'}^{-1} = \beta^*\mathrm{scal}_\phi = \beta^*\mathrm{scal}_{\partial X/Y} + O((x')^2)$$

for $e_i \in \{x^2\partial_x, x\tilde{U}_i, V_j\}$, coming from the properties of the rescaling and the asymptotics laid down above. The equation reads[109]:

$$\left(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2 e^{-L}H_Y e^L + \tau^2\mathbb{B}^2\right) N^{\mathcal{G}}_{\phi\mathrm{f},0}(h) = 0$$

with the notation of (29) and the Bismut superconnection $\mathbb{B}$ to be introduced right now. One can show $h \in \Psi_{\mathcal{G}}^{-2,0}(X;E)$ and the normal operator in the spinor case at ϕf is the solution to the above problem:

$$e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2R_Y/2}{\sinh(\tau^2R_Y/2)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2R_Y}{2}\coth\left(\frac{\tau^2R_Y}{2}\right)u,u\right\rangle\right)\right]e^Le^{-\tau^2\mathbb{B}^2}$$

compare (30). This and the ensuing discussion concerning the Bismut superconnection justify the analogue of Theorem 4.22:

Theorem 4.27. The heat kernel of the square of a Dirac operator associated to a Hermitian Clifford module $E \to X$ over a ϕ-manifold belongs to the rescaled calculus $\Psi_{\mathcal{G}}^{-2,0}(X;E)$, with rescaled normal operators:

$$N^{\mathcal{G}}_{\mathrm{tf},-2}(h)dx'dy'dz' = \frac{1}{(4\pi)^{\frac{n}{2}}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(-\frac{1}{4}\left\langle\frac{R}{2}\coth\frac{R}{2}\xi,\xi\right\rangle\right)$$

$$N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)dy'dz' = e^{-L}\left[\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}\frac{1}{(4\pi\tau^2)^{\frac{b}{2}}}\det{}^{1/2}\left(\frac{\tau^2R_Y/2}{\sinh(\tau^2R_Y/2)}\right)\exp\left(-\frac{1}{4\tau^2}\left\langle\frac{\tau^2R_Y}{2}\coth\left(\frac{\tau^2R_Y}{2}\right)u,u\right\rangle\right)\right]e^Le^{-\tau^2\mathbb{B}^2}$$

□

[109] The orthonormal basis $\{U_i\}_{i=1}^b$ of $\Gamma(TY)$ around $\phi(p)$ can be chosen so that $U_i\big|_{\phi(p)} = \partial_{y_i}\big|_{\phi(p)}$, i.e. $a_{ij}(p) = \delta_{ij}$ holds.

In the language of (19), this could be re-expressed as:

$$h(t,x,x,y,y,z,z) \underset{t\to 0}{\sim} t^{-\frac{n}{2}} \sum_{i=0}^{\infty} k_i(x,y,z)t^{\frac{i}{2}}, \quad k_i \in Cl^i({}^{\phi}T^*X) \otimes \mathrm{End}_{Cl({}^{\phi}T^*X)}(E) \quad \text{away from } x=0$$

$$h(t,x,x,y,y,z,z) \underset{x\to 0}{\sim} \sum_{i=0}^{\infty} \hat{k}_i(y,z,t^{\frac{1}{2}})x^i, \quad \hat{k}_i \in Cl^i\left(\frac{1}{x}\phi^*T^*Y\right) \otimes \mathrm{End}_{Cl\left(\frac{1}{x}\phi^*T^*Y\right)}(E) \quad \text{away from } t=0$$

(and there is also a compatible joint expansion when approaching $\mathrm{tf} \cap \phi\mathrm{f}$).
Let us introduce the **Bismut superconnection** $\mathbb{B}$ and the **Bismut-Cheeger eta form** $\hat{\eta}$ [BC89], which is constructed from a transgression of the former. We define this one first in the style of [BGV04, §10] (practically the same is done in [AGR23, §1.4, Def. 6.2] or [Liu25, §9]). Consider an odd dimensional manifold M which is the total space of a fibre bundle

$$F \longrightarrow M \stackrel{\pi}{\longrightarrow} B$$

(in our later application, ∂X will take the role of M). The vertical bundle $T(M/B)$ is canonically defined as the subbundle of TM whose sections vanish under pushforward along π; however, the horizontal one is not. Suppose we have a metric on the vertical bundle $g_{M/B}$ and a metric on the base g_B. We can choose and fix a splitting $TM = T(M/B) \oplus T_H M$ so that

$$(M, g_{M/B} + \pi^* g_B) \stackrel{\pi}{\longrightarrow} (B, g_B)$$

is a Riemannian submersion, i.e. $T_H M \cong \pi^* TB$ and $T_H M = (T(M/B))^{\perp}$. This induces the vertical and horizontal projections:

$$v : TM \longrightarrow T(M/B), \quad h : TM \longrightarrow \pi^* TB$$

The splitting of TM induces a corresponding splitting in T^*M between horizontal and vertical forms (until now, this is similar to what we did with the fibre bundle ϕ).
There are two relevant connections with respect to this structure in M. One is the Levi-Civita connection ∇^g of the metric $g = g_{M/B} + \pi^* g_B$, which is torsion-free. The other, is the non-torsion-free connection respecting the above splitting of TM:

$$\nabla^{\oplus} = (v \circ \nabla^g \circ v) \oplus \left(\pi^* \nabla^B \circ h\right) =: \nabla^{M/B} \oplus \nabla^H \tag{32}$$

where ∇^B is the Levi-Civita connection for the metric g_B on B.
The difference between both connections $\omega = \nabla^g - \nabla^{\oplus}$ is given by

$$\begin{aligned} \omega(W_i)(W_j, W_p) = S^{\pi}(vW_i, vW_j)(W_p) - S^{\pi}(vW_i, vW_p)(W_j) + \frac{1}{2} g_{M/B}\left(R^{\pi}(hW_i, hW_j), vW_p\right) \\ - \frac{1}{2} g_{M/B}\left(R^{\pi}(hW_i, hW_p), vW_j\right) - \frac{1}{2} g_{M/B}\left(R^{\pi}(hW_j, hW_p), vW_i\right) \end{aligned}$$

and has to do with the curvature R^{π} and second fundamental form S^{π} of a fibre bundle as defined in §4.1 (compared to [BGV04, Definition 10.5], we choose a different sign for both R^{π} and S^{π}, since we follow the convention in [AR09, (1.5), (1.6)]).
Now, introduce a small parameter $\varepsilon \in (0, 1]$ and consider the family of metrics[110]

$$g_{\varepsilon} = \frac{\pi^* g_B}{\varepsilon^2} + g_{M/B}$$

[110] Note how a similar picture takes place on each x-slice $\{x_0\} \times \partial X \subset [0, \delta)_x \times \partial X = \mathcal{U} \subset X$ in a collar neighborhood of the boundary for a manifold X with exact 0-ϕ-metric:

$$(g_{\partial X})_x = \frac{\phi^* g_Y}{x^2} + g_{\partial X/Y}$$

since in this slice, the variable x becomes just a degeneration parameter as $x \to 0$.

The corresponding metric on cotangent directions will be given by:

$$g^{\varepsilon} = \varepsilon^2 g_B + g_{M/B}$$

where we reuse the notation g_B and $g_{M/B}$ to refer to the induced metrics on cotangent sections. In the degeneration process $\varepsilon \to 0$, the horizontal directions blow-up, so the base points get "infinitely far away" and the fibres become "separated from each other". Such degeneration processes are usually called adiabatic limits, since they intuitively relate to the adiabatic theorem in quantum mechanics. They were introduced in [Wit85]. The adiabatic theorem in quantum mechanics states that if the state of the system corresponds to an eigenvector of the Hamiltonian for which the eigenvalue is separated by a gap *sufficiently large* from the rest of the spectrum and the state evolves *sufficiently slowly*, then the system will stay represented by the corresponding eigenvector of the evolved Hamiltonian (due to the eigenvalue being isolated, this is well-defined). It is in contrast with fast evolution, where the system "does not have time to adapt" and thus the final state is a combination of eigenvectors of the new Hamiltonian that sum up to produce the initial eigenvector of the original Hamiltonian[111]. The link to our setting is that for ε small enough, a Dirac operator associated to the metric g_{ε} has eigenvalues separated by *large enough* gaps and its Dirac equation can be separated using the quantum mechanical adiabatic approximation.

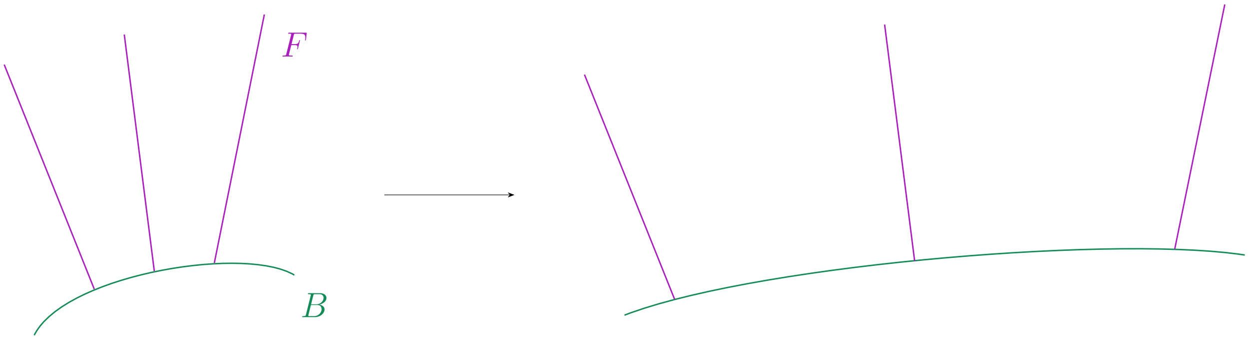


Figure 31: Intuitive depiction of the adiabatic degeneration process (M, g_{ε}), $\varepsilon \to 0$: the fibres preserve their length, but the base directions blow-up until the different fibres become "disconnected" in the limit. In other words, we "localize" in the base directions.

In particular, in the limit $\varepsilon \to 0$, the metric on the cotangent bundle degenerates to $g^0 = g_{M/B}$. meaning horizontal forms collapse to a point (they are at distance 0). The Clifford structure on M, which comes from a quantization of the space of differential forms ΛT^*M, inherits the splitting of T^*M into horizontal and vertical Clifford directions. Since the Clifford algebra in (M, g_{ε}) is defined upon the relation

$$cl(\alpha)cl(\beta) + cl(\beta)cl(\alpha) = -2g^{\varepsilon}(\alpha, \beta), \quad \alpha, \beta \in \Gamma(T^*M)$$

this means that the horizontal Clifford structure in the limit $\varepsilon \to 0$ is the Clifford algebra of a vector space with vanishing metric, i.e. just the exterior algebra itself $\Lambda T^*_H M \cong \pi^* \Lambda T^* B$ (compare this with how Getzler rescaling transforms the Clifford filtration at a face into the grading of an exterior bundle of differential forms, see 20). Thus, in this adiabatic limit the relevant bundles where our superconnections live are vertical Clifford modules.

[111] Not to confuse with adiabatic processes in thermodynamics, where no heat is transferred to the environment, and for that, the process needs to be fast enough; in quantum mechanics, the adiabatic processes are, in contrast, slow evolutions.

Consider thus a bundle $E \to M$ that is a $Cl(M/B)$-module[112] with Hermitian connection ∇^E which is Clifford with respect to the vertical connection $\nabla^{M/B}$:

$$\left[\nabla^E, cl(\alpha)\right] = cl(\nabla^{M/B}\alpha), \quad \alpha \in \Gamma(T^*(M/B))$$

Construct out of it the bundle $\mathbb{E} = \pi^*\Lambda T^*B \otimes E$ whose splitting is respected by a degenerate Clifford action cl_0[113] corresponding to the degenerate Clifford module $Cl_0(M)$ of (M, g_0):

$$cl_0(W)(\alpha\otimes\beta) = \left(\varepsilon(hW^\flat)\alpha\right)\otimes\beta + (-1)^{\deg\alpha}\alpha\otimes\left(cl_E(vW^\flat)\beta\right), \quad W = hW + vW \in T_HM \oplus T(M/B)$$

where $\varepsilon(\cdot)$ denotes exterior multiplication and $\alpha \otimes \beta \in \mathbb{E}$.
The relevant connection in this bundle is

$$\nabla^{\mathbb{E}} := \pi^*\nabla^B \otimes id + id \otimes \nabla^E + \frac{1}{2}cl_0(\omega)$$

which is Clifford with respect to ∇^{g_0} and cl_0 [BGV04, Proposition 10.10]. Here, $\omega = \nabla^g - \nabla^\oplus$ as above and

$$cl_0(\omega) = \frac{1}{2}\sum_{ijk}\omega(e_i)(e_j, e_k)e^i \otimes cl_0(e^j)cl_0(e^k)$$

on orthonormal frames e_* of TM and e^* of T^*M.
With this, we can define the **Bismut superconnection** as

$$\mathbb{B} := cl_0 \circ \nabla^{\mathbb{E}} \tag{33}$$

i.e. the Dirac operator associated to $(\mathbb{E}, \nabla^{\mathbb{E}})$. This can be seen as a superconnection on the vector bundle $\pi_*E \to B$, hence its name. This is an infinite dimensional superbundle with fibres given by the Fréchet spaces $(\pi_*E)_b = \mathcal{C}^\infty\left(\pi^{-1}(B), E\big|_{\pi^{-1}(b)}\right)$, that is $\mathcal{C}^\infty(B, \pi_*E) = \mathcal{C}^\infty(M, E)$.
In an orthogonal frame for M split into horizontal sections $\tilde{U}_i$ (lifted from B) and vertical sections V_j, the Bismut superconnection takes the local form

$$\mathbb{B} = \sum_j^f cl(V_j)\nabla^E_{V_j} + \sum_i^b \varepsilon(\tilde{U}_i)\left(\nabla^E_{\tilde{U}_i} - \frac{1}{2}k(\tilde{U}_i)\right) + \frac{1}{4}\sum_{i<p}^b\sum_j^f g_{M/B}\left(R^\pi\left(\tilde{U}_i, \tilde{U}_p\right), V_j\right)\varepsilon(\tilde{U}_i)\varepsilon(\tilde{U}_p)cl(V_j)$$

where $k(\cdot)$ is the **mean curvature** of the fibre bundle given by the trace of the second fundamental form:

$$k(\tilde{U}_i) = \sum_j S^\pi(V_j, V_j)(\tilde{U}_i) \tag{34}$$

As pointed out in [Liu25, p. 42-43] and originally described in [BC90], the third term has to do with the torsion of the connection $\nabla^\oplus$ on horizontal vectors and so can be denoted by

$$cl_0\left(T^\oplus\right) = -\frac{1}{4}\sum_{i<p}^b\sum_j^f g_{M/B}\left(R^\pi\left(\tilde{U}_i, \tilde{U}_p\right), V_j\right)\varepsilon(\tilde{U}_i)\varepsilon(\tilde{U}_p)cl(V_j) \tag{35}$$

Finally, $\sum_j^f cl(V_j)\nabla^E_{V_j}$ is just the vertical Dirac operator associated to E (in the spinor case, to $\mathcal{S}_{M/B}$), which we will denote by $\partial\!\!\!/_{M/B}$.

[112]This means $E\big|_{\pi^{-1}(b)} \to \pi^{-1}(b)$ is a Clifford module for each $b \in B$ with respect to the vertical Clifford directions in M.

[113]This is e.g. analogous to m_d in [Vai01, p. 92] if you take away their normal direction.

Being a Dirac operator, the Bismut superconnection has its very own Lichnerowicz formula [BGV04, Theorem 10.17]:

$$\begin{aligned}\mathbb{B}^2 &= \Delta^{M/B} + \frac{\mathrm{scal}_{M/B}}{4} + \frac{1}{2}\sum_{ij} F^{E/S}(e_i,e_j)cl_0(e_i)cl_0(e_j)\\ &= -\sum_i \left(\nabla^{\mathbb{E}}_{e_i}\right)^2 + \sum_i \nabla^{\mathbb{E}}_{\nabla_{e_i}e_i} + \frac{\mathrm{scal}_{M/B}}{4} + \frac{1}{2}\sum_{ij} F^{E/S}(e_i,e_j)cl_0(e_i)cl_0(e_j)\end{aligned}$$

where $\Delta^{M/B}$ denotes the vertical family of rough Laplacians acting on (π^*E)-sections. The last contribution is the customary twisting curvature term (we did not see it until now since we were only working with the spinor Dirac operator).
In particular, the Bismut superconnection associated to the fibre bundle $\phi : \partial X \to Y$ and the rescaled Clifford module $\mathbb{E} \cong \phi^*(\Lambda T^*Y) \otimes \mathcal{S}_{\partial X/Y}$ (isomorphic to $\mathcal{G}\big|_{\phi\mathrm{f}}$ for $E = \mathcal{S}$) satisfies the Lichnerowicz formula:

$$\begin{aligned}\mathbb{B}^2 &= -\sum_i \left(\nabla^{\mathbb{E}}_{V_i}\right)^2 + \sum_i \nabla^{\mathbb{E}}_{\nabla_{V_i}V_i} + \frac{\mathrm{scal}_{\partial X/Y}}{4}\\ &\overset{!}{=} -\sum_{i=1}^{f}\left(\delta_{x'}\circ\beta^*\nabla^{\mathcal{S}}_{V_i}\circ\delta_{x'}^{-1}\right)^2 + \left(\delta_{x'}\circ\left(\sum_{i=1}^{f}\beta^*\nabla^{\mathcal{S}}_{\nabla_{V_i}V_i}\right)\circ\delta_{x'}^{-1}\right) + \frac{\mathrm{scal}_{\partial X/Y}}{4}\end{aligned}$$

due to the nature of our rescaling at ϕf. This is the reason why $\mathbb{B}^2$ deals with all the vertical terms in the rescaled model problem at ϕf and underscores why the scattering case contained the main difficulties[114].
Let us come back to general $M \to B$ to define the Bismut-Cheeger eta form. We can rescale in the horizontal directions via $\delta^B_\lambda\alpha = \lambda^{-\frac{j}{2}}\alpha$ for $\alpha \in \pi^*\Lambda^j T^*B$ and with that introduce the rescaled Bismut superconnection:

$$\begin{aligned}\mathbb{B}_\lambda &:= \lambda^{\frac{1}{2}}\delta^B_\lambda \circ \mathbb{B} \circ \left(\delta^B_\lambda\right)^{-1} = \lambda^{\frac{1}{2}}\mathbb{B}_0 + \mathbb{B}_1 + \lambda^{-\frac{1}{2}}\mathbb{B}_2\\ &= \sqrt{\lambda}\,\eth_{M/B} + \sum_i^b \varepsilon(\tilde{U}_i)\left(\nabla^E_{\tilde{U}_i} - \frac{1}{2}k(\tilde{U}_i)\right) - \frac{1}{4\sqrt{\lambda}}cl_0\left(T^{\oplus}\right)\end{aligned}$$

If M is odd-dimensional, the **Bismut-Cheeger eta form** associated to the fibre bundle $M \to B$ is one of the (convergent [BGV04, Theorem 10.32]) integrals:

$$\hat{\eta}(M/B) = \begin{cases}\int_0^\infty \mathrm{Tr}^{\mathrm{even}}_{M/B}\left(\frac{\partial\mathbb{B}_\lambda}{\partial\lambda}e^{-\mathbb{B}^2_\lambda}\right)d\lambda & \dim(B)\text{ even}\\ \int_0^\infty \mathrm{Str}_{M/B}\left(\frac{\partial\mathbb{B}_\lambda}{\partial\lambda}e^{-\mathbb{B}^2_\lambda}\right)d\lambda & \dim(B)\text{ odd}\end{cases}$$

where the superscript "even" means taking trace only on the endomorphism-valued coefficients of the differential form parts of even degree. Another definition adding an extra dimension is found in [Vai01, (117)] and [AGR23, p. 301].
This is the geometers' normalization. The topologists' one, usually called "normalized eta form" is $\bar{\eta}(M/B) = \sum_j (2\pi i)^{\lfloor\frac{j}{2}\rfloor}\left[\hat{\eta}(M/B)\right]_j$.
In local coordinates:

$$\frac{\partial\mathbb{B}_\lambda}{\partial\lambda}e^{-\mathbb{B}^2_\lambda} = \left(\eth_{M/B} + \frac{cl_0\left(T^{\oplus}\right)}{4\lambda}\right)\frac{1}{2\sqrt{\lambda}}e^{-\mathbb{B}^2_\lambda}$$

[114] This is not an isolated case, see [Vai01, Corollary 5.21(b)], [AGR23, Lemma 5.3] or [Liu25, Lemma 9.5]. Intuitively this happens because all these metrics are conformal to a metric which, to leading order and taking away the dx term, is the metric of an adiabatic degeneration $x^{-2\alpha}\phi^*g_Y + g_{\partial X/Y}$ with $\alpha \in \mathbb{N}$.

The reason why we introduce the Bismut-Cheeger eta form is because it will appear in the contribution from the face ϕf to the index theorem (corresponding to the second term on the right of [LMP07, (1.4)]). This will be discussed in §4.5.
Observe that the notations introduced above simplify the expression of the (non-lifted) Dirac operator:

$$\begin{aligned}\eth &= cl\left(\frac{dx}{x^2}\right)x^2\partial_x + x\eth_Y - \frac{xb}{2}cl\left(\frac{dx}{x^2}\right) + \frac{3x}{4}\sum_{i=1}^{b} k\left(\tilde{U}_i\right)cl\left(\frac{\tilde{U}_i^\flat}{x}\right) \\ &- \frac{x}{4}\sum_{ijp} g_{\partial X/Y}\left([\tilde{U}_i, V_p], V_j\right)cl\left(\frac{\tilde{U}_i^\flat}{x}\right)cl\left(V_j\right)cl\left(V_p\right) - \frac{x^2}{2}cl\left(T^{\oplus}\right) + \eth_{\partial X/Y}\end{aligned} \tag{36}$$

To tackle the large time limit, we can apply Proposition 4.23 and the commutation trick discussed afterwards for the correction term, up to changing the density dy' for $dy'dz'$. This results in (note this still works because of the order $b+1$ at ϕf_0):

$$\lim_{t\to\infty} {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) = {}^R\mathrm{Str}\left(\Pi_{\ker \eth}\right) + \int_{\partial X}{}^R\!\!\int_0^\infty \mathrm{str}_p\left(N_{\phi f_0, b+1}\left(e^{-t\eth^2}\right)\right)\frac{dT}{T^2}dy'dz'$$

i.e there is a contribution from zf, which corresponds to a renormalized index, and one from ϕf_0, similar to the contribution from bf_0 in the scattering case. We say more about this last term in §4.5.
This was the last ingredient in:

Theorem 4.28. Similarly to 4.25, the calculations above for the Dirac operator $\eth$ associated to $E \to X$, where X is a compact manifold whose boundary is the total space of a fibre bundle

$$Z \longrightarrow \partial X \xrightarrow{\phi} Y$$

and has an underlying metric on $\mathring{\mathcal{U}}$ differing only by polyhomogeneous higher order terms from the exact metric of the form

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z$$

so that

$$(\partial X, \phi^* g_Y + g_Z) \xrightarrow{\phi} (Y, g_Y)$$

is a Riemannian submersion, yield the local index formula:

$$\begin{aligned}{}^R\mathrm{ind}\left(\eth^+\right) &= {}^R\!\!\int_X \hat{\mathrm{A}}(X)Ch(E) \\ &+ \frac{1}{2}\int_{\partial X}\left[\mathrm{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b dy'dz' - \int_{\partial X}{}^R\!\!\int_0^\infty \mathrm{str}_p\left(N_{\phi f_0, b+1}\left(e^{-t\eth^2}\right)\right)\frac{dT}{T^2}dy'dz'\end{aligned}$$

under the assumption that the kernel of the vertical family $\left(\eth_{Z_y}\right)_{y\in Y}$ forms a bundle $\mathcal{K} \to Y$ over the base.

□

In the next section, we conjecture that the last terms correspond to:

$$-\frac{1}{2}\int_Y \hat{\mathrm{A}}(Y)\bar{\eta}(\partial X/Y) - \frac{1}{2}\eta\left(\eth_Y^{\mathcal{K}}\right)$$

Comparing it with the fully elliptic [LMP07, Theorem 1.3], the difference again lies in the large time limit: we have to renormalize the index and moreover obtain the eta-invariant of the induced base operator acting on sections of the kernel bundle of the vertical family. This resembles [Vai01, Theorem 5.29] up to finer comparison of their left-hand-sides.
Furthermore, this formula reduces to the asymptotically conical one (Theorem 4.25) when the fibres are trivial, since in this case $\hat{A}(Y)$ is the $\hat{A}$-genus of an odd-dimensional manifold, and $\eth_Y^{\mathcal{K}} = \eth_{\partial X}$ whenever $\partial X = Y$. On the contrary, if Y is a point, the integral of the vertical eta form over Y becomes the eta invariant of $\eth_{\partial X/Y} = \eth_{\partial X}$ and the last term vanishes, giving back the cusp case, which corrresponds to the b-index formula of [Mel93] (the APS formula). This bodes well with the fact that they are different compactifications of the very same cylindrical end (Example 4.1).
On a different note, we can improve the last term in the formula to its form in Theorem 1.5 by splitting the operator in fibrewise harmonic part and fully elliptic part, as discussed in the last segment of §4.5.

Remark 4.29. For the sake of completeness, even though we did not explicitly use it, we write here the lifted form of $\eth^2$ close to the boundary:

$$\begin{aligned}
\beta^*\eth^2 = &-(x'S+1)^4\partial_S^2 + (b-2)x'(x'S+1)^3\partial_S + b(x')^2(x'S+1)^2 + x'(x'S+1)^2 cl\left(\frac{dx}{x^2}\right) x'\beta^*\eth_Y \\
&- \frac{3}{4}(x')^2(x'S+1)^2\sum_{i=1}^{b} k\left(\tilde{U}_i\right) cl\left(\frac{dx}{x^2}\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) - (x')^3(x'S+1)^3 cl\left(\frac{dx}{x^2}\right) cl\left(T^\oplus\right) \\
&+ \frac{1}{4}(x')^2(x'S+1)^2\sum_{ijk} g_{\partial X/Y}\left([\tilde{U}_i, V_j], V_k\right) cl\left(\frac{dx}{x^2}\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl(V_j)cl(V_k) \\
&+ (x'S+1)^2\left(x'\beta^*\eth_Y\right)^2 + \eth_{\partial X/Y}^2 - \frac{5}{16}(x')^4(x'S+1)^4\left(cl\left(T^\oplus\right)\right)^2 \\
&+ \frac{1}{16}(x')^2(x'S+1)^2\left(\sum_{ijk} g_{\partial X/Y}\left([\tilde{U}_i, V_j], V_k\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl(V_j)cl(V_k)\right)^2
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
x'\beta^*\eth_Y = \sum_{i=1}^{b} cl\left(\frac{\tilde{U}_i^\flat}{x}\right)\Bigg(&\sum_{j=1}^{b} a_{ij}\partial_{u_j} + x'\sum_{j=1}^{f} b_{ij}\partial_{z_j} \\
&- \frac{(x')^2}{8}\sum_{jpq}^{b} g_Y\left(R_Y\left(\tilde{U}_i, \partial_{y_q}\right) u_q\tilde{U}_j, \tilde{U}_p\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\Bigg)
\end{aligned}$$

□

4.5 Towards a local index formula

This section is concerned with summarizing our current efforts to understand the last two terms in both index formulas above in terms of geometric invariants. This reflects the current state of the author's work on the problem and shows how the solution of the scattering case would essentially give the fibred boundary case as a corollary.

The scattering ϕf contribution

In [BF86, §II] (also discussed in [Mel93, §8.13]), the authors show the regularity of the integrand corresponding to the eta invariant in the Atiyah-Patodi-Singer setting, namely that

$$\eta(\eth_{\partial X}) = \int_0^\infty \mathrm{Tr}\left(\eth_{\partial X} e^{-t\eth_{\partial X}^2}\right)\frac{dt}{\sqrt{\pi t}}$$

converges, by proving that the trace term is $O(\sqrt{t})$ as $t \to 0$. To do this, they in turn relate this trace with the trace of the heat kernel of another operator, for which they prove a Lichnerowicz formula [BF86, (2.4)]. They introduce a Grassmann variable z (i.e. $z^2 = 0$) that facilitates finding the relation [BF86, (2.6)]:

$$\exp\left(-\frac{t\eth^2}{2} + z\sqrt{t}\eth\right) = \exp\left(-\frac{t\eth^2}{2}\right) + z\sqrt{t}\eth\exp\left(-\frac{t\eth^2}{2}\right)$$

Then, if z is thought of as a Clifford variable, the supertrace of the left hand side corresponds to the supertrace of the second term in the right hand side, since the first term in the right hand side has no z contribution, so cannot have full Clifford degree and thus the supertrace is oblivious to it. Recall that our problem consisted in dealing with the argument of the integral (26). As pointed out by Albin, this is of a form very similar to the one appearing in the eta invariant and the above expression, in particular if we think of $z \sim \varepsilon\left(\frac{dx}{x^2}\right)$, then

$$z\sqrt{t}\eth\exp\left(-\frac{t\eth^2}{2}\right) \sim \varepsilon\left(\frac{dx}{x^2}\right)\sqrt{t}\eth e^{-t\eth^2}$$

This suggests to follow suit with their approach and try to relate the argument of our supertrace with the heat kernel of another operator which we can compute. Then, on this new heat kernel we can perform a rescaling in the known fashion and relate its supertrace with the original term. The fact that $\varepsilon\left(\frac{dx}{x^2}\right)$ "plays the role of the Grassmann variable" indicates that we should strive for a rescaling also in this normal direction, i.e. as Vaillant performed in the face tff. Once these intuitions are clear, we can proceed with the computations to check their validity.

Recall that the term in (24) that does not allow rescaling in the normal direction is $x^2 cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}$, so we would like to get rid of it before rescaling in all directions. To get an expression similar to Bismut-Freed's, we would prefer to substract a multiple of $\eth$ instead of $\eth_{\partial X}$ from $\eth^2$. This is why we computed (25), which does the job. Moreover, the corresponding exponential decomposition looks like:

$$e^{-t\left(\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth\right)} = e^{-t\eth^2} + txcl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2} + O(x^2) \tag{38}$$

in the spirit of the Baker-Campbell-Hausdorff formula based on $[\eth^2, -xcl\left(\frac{dx}{x^2}\right)\eth] = O(x^2)$.

Relating this to our quantity of interest

$$\left[\mathrm{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b = \frac{1}{t}\left[\mathrm{str}_p\left(txcl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_{b+1}$$

(remember that $n = b + 1$). If after lifting to a neigborhood of $\phi\mathrm{f} \subset HX^2_{ac}$ we were able to show that taking supertraces and restricting to $(x')^b$ terms above kills all the terms in $O(x^2)$ and the term $e^{-t\eth^2}$, then we could write (compare with (26)):

$$\int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right)dt = \frac{1}{2}\int_0^\infty\int_{\partial X}\left[\mathrm{str}_p\left(e^{-t\left(\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth\right)}\right)\right]_{b+1} dy'\frac{dt}{t} \tag{39}$$

and it would be enough to compute the rescaling of this heat kernel at $\phi\mathrm{f}$ along all directions.

Computing the lift:

$$\beta^*\left(\not{\partial}^2 - xcl\left(\frac{dx}{x^2}\right)\not{\partial}\right) = -\left(x'S+1\right)^4\partial_S^2 + (b-1)\left(x'S+1\right)^3 x'\partial_S + \left(x'S+1\right)^2 (x')^2\frac{b}{2}$$
$$-\left(x'S+1\right)^2\sum_{i=1}^{b}\left(\partial_{u_i} - \frac{(x')^2}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right) cl\left(\frac{dy_j}{x}\right) cl\left(\frac{dy_p}{x}\right)\right)^2$$

Notice all Clifford terms are paired with an x' and we need to rescale in $b+1=n$ directions, thus we can think about the following rescaling: recall we have a fibration $\mathring{\phi}\mathrm{f} \to (0,\infty)\times\partial X$, in coordinates $(S,u,y',\tau)\to(\tau,y')$. The intersection of $\phi\mathrm{f}$ with a horizontal hyperplane of positive time $\{\tau_0\}\subset HX^2_{ac}$ then fibres over $\{\tau_0^2\}\times\mathrm{diag}_{\partial X}$. Here, we can use the identification $\mathrm{diag}_{\partial X}\cong\partial X\subset X\cong\mathrm{diag}_X$ to lift the filtration from X. First, we have the isomorphism $\mathrm{End}(E)\to X\cong\mathrm{diag}_X$, which we can restrict to the boundary to get $\mathrm{End}(E)\big|_{\partial X}\to\partial X\cong\mathrm{diag}_{\partial X}$. This comes with a filtration:

$$\mathrm{End}(E)\big|_{\partial X}\cong Cl\left({}^{ac}T^*X\big|_{\partial X}\right)\otimes\mathrm{End}_{Cl\left({}^{ac}T^*X\big|_{\partial X}\right)}(E)$$

Then, we can lift this filtration for each $\tau_0\in(0,\infty)$ along $\phi\mathrm{f}\cap\{\tau=\tau_0\}\to\{\tau_0^2\}\times\mathrm{diag}_{\partial X}\cong\partial X$ to produce:

$$\left(\beta^*_{xx'}\left(\mathrm{END}(E)\right)\big|_{\phi\mathrm{f}}\right)_{(S,u,y',\tau)}\cong\left(\mathrm{End}(E)\big|_{\partial X}\right)_{(0,y')}\cong\left(Cl\left({}^{ac}T^*X\big|_{\partial X}\right)\otimes\mathrm{End}_{Cl\left({}^{ac}T^*X\big|_{\partial X}\right)}(E)\right)_{(0,y')}$$

In the end, we arrive at the filtration (cf. [Vai01, Lemma 5.1(b)]):

$$\hat{\mathcal{F}}_k^{\phi\mathrm{f}}\cong Cl^k\left({}^{ac}T^*X\big|_{\partial X}\right)\otimes\mathrm{End}_{Cl\left({}^{ac}T^*X\big|_{\partial X}\right)}(E)$$

One could combine this with the tf rescaling, but we do not require it. We extend this by parallel transport along $\partial_{x'}$, rescale and obtain a bundle $\hat{\mathcal{G}}$ spanned by:

$$\rho_{\phi\mathrm{f}}\frac{dx}{x^2},\quad\rho_{\phi\mathrm{f}}\frac{dy_i}{x}$$

i.e. resembling the b-tangent bundle:

$$\hat{\mathcal{G}}\Big|_{\phi\mathrm{f}}\cong\bigoplus_{k=0}^{n}(N^*\phi\mathrm{f})^k\otimes\underbrace{\left(Cl^k\left({}^bT^*X\Big|_{\partial X}\right)\Big/Cl^{k-1}\left({}^bT^*X\Big|_{\partial X}\right)\right)}_{\Lambda^k\left({}^bT^*_{\partial X}X\right)}\otimes\mathrm{End}_{Cl\left({}^{ac}T^*X\big|_{\partial X}\right)}(E)$$

The calculus and related constructions follow similarly. For the leading order term after the rescaling (corresponding to the correct order contributing to the index formula) we consider the local rescaling $\delta_{x'}\omega=(x')^{-k}\omega$ for $\omega\in\Lambda^k\left({}^{ac}T^*X\right)$, from which we obtain:

$$\delta_{x'}\circ\beta^*\left(t\partial_t + t\left(\not{\partial}^2 - xcl\left(\frac{dx}{x^2}\right)\not{\partial}\right)\right)\circ\delta_{x'}^{-1} = \frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2$$
$$-\tau^2\sum_{i=1}^{b}\left(\partial_{u_i} - \frac{1}{8}\sum_{j,p,q=1}^{b} g_{\partial X}\left(R_{\partial X}(\partial_{y_i},\partial_{y_q})u_q\partial_{y_j},\partial_{y_p}\right)\varepsilon\left(\frac{dy_j}{x}\right)\varepsilon\left(\frac{dy_p}{x}\right)\right)^2 + O(x')$$

so the model problem reads:

$$\left(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2 H_{\partial X}\right)N^{\hat{\mathcal{G}}}_{\phi\mathrm{f},0}(\hat{h}) = 0$$

where $\hat{h}$ is the heat kernel of $\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth$. We know its solution already:

$$N^{\hat{\mathcal{G}}}_{\phi\mathrm{f},0}(\hat{h})dy' = \frac{1}{(4\pi\tau^2)^{\frac{n}{2}}} e^{-\frac{S^2}{4\tau^2}} \det{}^{1/2}\left(\frac{\tau^2 R_{\partial X}/2}{\sinh\left(\tau^2 R_{\partial X}/2\right)}\right) \exp\left(-\frac{1}{4\tau^2}\left\langle \frac{\tau^2 R_{\partial X}}{2} \coth\left(\frac{\tau^2 R_{\partial X}}{2}\right) u, u\right\rangle\right)$$

Applying the supertrace in particular restricts to $S = 0$ and $u = 0$, so we culminate with:

$$\int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) dt = \int_0^\infty \int_{\partial X} \mathrm{str}_p\left(\frac{1}{(4\pi\tau^2)^{\frac{n}{2}}} \det{}^{1/2}\left(\frac{\tau^2 R_{\partial X}/2}{\sinh\left(\tau^2 R_{\partial X}/2\right)}\right)\right)\frac{d\tau}{\tau}$$

$$= \frac{(-2i)^{\frac{n-1}{2}}}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \left[\tau^{-(n-1)} \det{}^{1/2}\left(\frac{\tau^2 R_{\partial X}/2}{\sinh\left(\tau^2 R_{\partial X}/2\right)}\right)\right]_{n-1} \frac{d\tau}{\tau^2} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \hat{\mathrm{A}}(\partial X)\frac{d\tau}{\tau^2}$$

but since the manifold X is even-dimensional, ∂X is odd-dimensional and thus its Â-genus vanishes. Hence, the face ϕf, after all, would not contribute! However, these calculations will prove useful in the fibred boundary case, where this term is not always trivial but rather related to the adiabatic degeneration of [BC89] (cf. Figure 31).

The fibred boundary ϕf contribution

The ϕ-case also has a ϕf face that belongs to the $t \in (0, \infty)$ regime, and whose McKean-Singer formula contribution is (cf. (26)):

$$\int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) dt = \frac{1}{2}\int_0^\infty \int_{\partial X} \left[\mathrm{str}_p\left(cl\left(\frac{dx}{x^2}\right)\eth e^{-t\eth^2}\right)\right]_b dy'dz'\frac{dt}{t}$$

With the same trick as (38) and (39) (and assuming we can argue why the supertrace makes the rest of the terms vanish), we could rewrite this into:

$$\int_0^\infty \partial_t\, {}^R\mathrm{Str}\left(e^{-t\eth^2}\right) dt = \frac{1}{2}\int_0^\infty \int_{\partial X} \left[\mathrm{str}_p\left(e^{-t\left(\eth^2 - xcl\left(\frac{dx}{x^2}\right)\eth\right)}\right)\right]_{b+1} dy'dz'\frac{dt}{t}$$

which asks for a rescaling in the normal and base directions. We already know how to construct this: we can start by considering the model conical base $Y^+ := (\mathbb{R}_+)_x \times Y$ with scattering bundle ${}^{ac}T^*Y^+$ spanned by $\{\frac{dx}{x^2}, \frac{dy_i}{x}\}$ and restrict this bundle to the boundary ${}^{ac}T^*Y^+\big|_Y \to Y$. The Clifford quantization of this bundle is denoted $Cl\left({}^{ac}T^*Y^+\big|_Y\right)$. This bundle can be lifted along $\phi\mathrm{f} \to \partial X \times_Y \partial X \to Y$ to produce $\beta^*\phi^*Cl\left({}^{ac}T^*Y^+\big|_Y\right)$. Thinking of this bundle fibrewise at each point of ϕf, we can write $\left(\beta^*\phi^*Cl\left({}^{ac}T^*Y^+\big|_Y\right)\right)_{(S,u,z,y',z',\tau)} \cong \left(Cl\left(\phi^*\left({}^{ac}T^*Y^+\big|_Y\right)\right)\right)_{y',z'}$, and as we did above, we drop the base point in the notation, keeping in mind they are bundles over different spaces.

At this point it is no surprise that we can use this bundle to decompose the endomorphism bundle at ϕf (see [Vai01, Lemma 5.1(b)]), obtaining:

$$\beta^*_{xx'}\left(\mathrm{END}(E)\right)\big|_{\phi\mathrm{f}} \cong Cl\left(\phi^*\left({}^{ac}T^*Y^+\big|_Y\right)\right) \otimes \mathrm{End}_{Cl\left(\phi^*\left({}^{ac}T^*Y^+\big|_Y\right)\right)}(E)$$

The rest of the rescaling is clear. Locally, we pair up each $cl\left(\frac{dx}{x^2}\right)$ and $cl\left(\frac{dy_i}{x}\right)$ with an x' through a map $\delta_{x'}$. Under this rescaling, $\beta^* \eth^2$ does not rescale due to the horizontal term:

$$\sum_{i=1}^{b} \left(\partial_{u_i} + \frac{x'}{2} cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl\left(\frac{dx}{x^2}\right) - \frac{(x')^2}{8} \sum_{jpq}^{b} g_Y\left(R_Y\left(\tilde{U}_i, \partial_{y_q}\right) u_q \tilde{U}_j, \tilde{U}_p\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\right)^2$$

$$= \sum_{i=1}^{b} \left(\partial_{u_i} - \frac{(x')^2}{8} \sum_{jpq}^{b} g_Y\left(R_Y\left(\tilde{U}_i, \partial_{y_q}\right) u_q \tilde{U}_j, \tilde{U}_p\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\right)^2$$

$$+ x' \sum_{i=1}^{b} cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl\left(\frac{dx}{x^2}\right) \left(\partial_{u_i} - \frac{(x')^2}{8} \sum_{jpq}^{b} g_Y\left(R_Y\left(\tilde{U}_i, \partial_{y_q}\right) u_q \tilde{U}_j, \tilde{U}_p\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\right)$$

The last summand resembles the lift of a Dirac operator on the base (up to the isomorphism $Cl(Y) \cong Cl\left(\frac{1}{x}T^*Y\right)$) times the normal Clifford direction, so we can define:

$$\eth_Y := \sum_{i=1}^{b} cl\left(\frac{\tilde{U}_i^\flat}{x}\right) \nabla^{\mathcal{S}_Y}_{\tilde{U}_i} = \sum_{i=1}^{b} cl\left(\frac{\tilde{U}_i^\flat}{x}\right) \left(\tilde{U}_i - \frac{1}{8} \sum_{jpq}^{b} g_Y\left(R_Y\left(\tilde{U}_i, \partial_{y_q}\right) y_q \tilde{U}_j, \tilde{U}_p\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)\right)$$

so the last summand above comes from the lift of a term $x^2 cl\left(\frac{dx}{x^2}\right) \eth_Y$ (much alike (24)) to ϕf by restricting to order 0 in x'. The other term is the one rescaling to H_Y. We can now compute:

$$\begin{aligned}
xcl\left(\frac{dx}{x^2}\right) \eth = {} & -x^3 \partial_x + x^2 cl\left(\frac{dx}{x^2}\right) \eth_Y + \frac{x^2 b}{2} \\
& + \frac{x^2}{4} cl\left(\frac{dx}{x^2}\right) \sum_{ijp} \left(\phi^* g_Y(S^\phi(V_j, V_p), \tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i, V_p], V_j)\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl(V_j) cl(V_p) \\
& + \frac{x^3}{4} cl\left(\frac{dx}{x^2}\right) \sum_{ijp} g_{\partial X/Y}(R^\phi(\tilde{U}_i, \tilde{U}_j), V_p) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl(V_k) \\
& + xcl\left(\frac{dx}{x^2}\right) \eth_{\partial X/Y} + \frac{x^2}{2} cl\left(\frac{dx}{x^2}\right) \sum_{ijp} \phi^* g_Y(S^\phi(V_i, V_j), \tilde{U}_p) cl(V_i) cl(V_j) cl\left(\frac{\tilde{U}_p^\flat}{x}\right) \\
& - \frac{x^3}{8} cl\left(\frac{dx}{x^2}\right) \sum_{ijp} g_{\partial X/Y}(R^\phi(\tilde{U}_j, \tilde{U}_p), V_i) cl(V_i) cl\left(\frac{\tilde{U}_j^\flat}{x}\right) cl\left(\frac{\tilde{U}_p^\flat}{x}\right)
\end{aligned}$$

and from it, the rescaled lifted difference:

$$\begin{aligned}
& \delta_{x'} \circ \left(\beta^* \left(\eth^2 - xcl\left(\frac{dx}{x^2}\right) \eth\right)\right) \circ \delta_{x'}^{-1} = -\partial_S^2 - H_Y + \mathbb{B}^2 \\
& + \frac{1}{4} \varepsilon\left(\frac{dx}{x^2}\right) \sum_{ijp} \left(\phi^* g_Y(S^\phi(V_j, V_p), \tilde{U}_i) - g_{\partial X/Y}([\tilde{U}_i, V_p], V_j)\right) \varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right) cl(V_j) cl(V_p) \\
& + \frac{1}{4} \varepsilon\left(\frac{dx}{x^2}\right) \sum_{ijp} g_{\partial X/Y}(R^\phi(\tilde{U}_i, \tilde{U}_j), V_p) \varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right) \varepsilon\left(\frac{\tilde{U}_j^\flat}{x}\right) cl(V_k) \\
& + \varepsilon\left(\frac{dx}{x^2}\right) \eth_{\partial X/Y} + \frac{1}{2} \varepsilon\left(\frac{dx}{x^2}\right) \sum_{ijp} \phi^* g_Y(S^\phi(V_i, V_j), \tilde{U}_p) cl(V_i) cl(V_j) \varepsilon\left(\frac{\tilde{U}_p^\flat}{x}\right)
\end{aligned}$$

$$- \frac{1}{8}\varepsilon\left(\frac{dx}{x^2}\right)\sum_{ijp} g_{\partial X/Y}(R^\phi(\tilde{U}_j,\tilde{U}_p),V_i)cl(V_i)\varepsilon\left(\frac{\tilde{U}_j^\flat}{x}\right)\varepsilon\left(\frac{\tilde{U}_p^\flat}{x}\right)+O(x')$$

$$= -\partial_S^2 - H_Y + \mathbb{B}^2 - \frac{3}{4}\varepsilon\left(\frac{dx}{x^2}\right)\sum_{i=1}^{b} k\left(\tilde{U}_i\right)\varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right)$$

$$- \frac{1}{4}\varepsilon\left(\frac{dx}{x^2}\right)\sum_{ijp} g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)\varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right)cl(V_j)cl(V_p) - \frac{1}{4}\varepsilon\left(\frac{dx}{x^2}\right)cl_0(T^\oplus)+\varepsilon\left(\frac{dx}{x^2}\right)\eth_{\partial X/Y}+O(x')$$

using that S^ϕ is symmetric [AR09, p. 9].
Assume, in analogy with the previous section, that we were able to show that the solution to the underlying model heat equation:

$$\left(\frac{1}{2}\tau\partial_\tau - \tau^2\partial_S^2 - \tau^2 H_Y - \tau^2\mathbb{B}^2 - \frac{3\tau^2}{4}\varepsilon\left(\frac{dx}{x^2}\right)\sum_{i=1}^{b} k\left(\tilde{U}_i\right)\varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right)\right.$$
$$\left. - \frac{\tau^2}{4}\varepsilon\left(\frac{dx}{x^2}\right)\sum_{ijp} g_{\partial X/Y}([\tilde{U}_i,V_p],V_j)\varepsilon\left(\frac{\tilde{U}_i^\flat}{x}\right)cl(V_j)cl(V_p)+\tau^2\varepsilon\left(\frac{dx}{x^2}\right)\left(\eth_{\partial X/Y}-\frac{cl_0(T^\oplus)}{4}\right)\right)N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)=0$$

(where $\mathcal{G}$ denotes the bundle obtained by just rescaling ϕf in the horizontal and normal directions) is of the form[115]:

$$\frac{1}{\sqrt{4\pi\tau^2}}e^{-\frac{S^2}{4\tau^2}}e^{-\tau^2 H_Y}e^{-\tau^2\mathbb{B}^2}\left(\tau^2\varepsilon\left(\frac{dx}{x^2}\right)\left(\eth_{\partial X/Y}+\frac{cl_0\left(T^\oplus\right)}{4}\right)\right)+A$$

where A is an endomorphism-valued section whose supertrace vanishes, then:

$$\frac{1}{2}\int_0^\infty\int_{\partial X}\left[\mathrm{str}_p\left(e^{-t\left(\eth^2-xcl\left(\frac{dx}{x^2}\right)\eth\right)}\right)\right]_{b+1}dy'dz'\frac{dt}{t}$$
$$=\int_0^\infty\int_{\partial X}\mathrm{str}_p\left(\frac{1}{\sqrt{4\pi}}\varepsilon\left(\frac{dx}{x^2}\right)\left(\eth_{\partial X/Y}+\frac{cl_0(T^\oplus)}{4}\right)e^{-\frac{S^2}{4\tau^2}}e^{-\tau^2 H_Y}e^{-\tau^2\mathbb{B}^2}\right)dy'dz'd\tau$$
$$=-\int_0^\infty\int_{\partial X}\left[\frac{(-2i)^{\frac{b}{2}}}{(4\pi\tau^2)^{\frac{b+1}{2}}}\det{}^{1/2}\left(\frac{\tau^2 R_Y/2}{\sinh\left(\tau^2 R_Y/2\right)}\right)\mathrm{str}_{\partial X/Y}\left(\left(\eth_{\partial X/Y}+\frac{cl_0(T^\oplus)}{4}\right)e^{-\tau^2\mathbb{B}^2}\right)\right]_b d\tau$$
$$=-\frac{1}{2}\int_{\partial X}\left[\det{}^{1/2}\left(\frac{\frac{R_Y}{4\pi i}}{\sinh\left(\frac{R_Y}{4\pi i}\right)}\right)\int_0^\infty\mathrm{str}_{\partial X/Y}\left(\left(\eth_{\partial X/Y}+\frac{cl_0(T^\oplus)}{4t}\right)e^{-t\mathbb{B}_t^2}\right)\right]_b\frac{dt}{2\sqrt{t}}$$
$$=-\frac{1}{2}\int_Y\hat{\mathrm{A}}(Y)\bar{\eta}(\partial X/Y)$$

which reproduces the Bismut-Cheeger result.

[115] This is at least the case if the fibre bundle ϕ is trivial, meaning $\partial X = Y\times Z$, since in that case, the Dirac operator has the form

$$\eth = cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x-\frac{xb}{2}\right)+x\eth_Y+\eth_{\partial X/Y}$$

(we could write $Z=\partial X/Y$) and all the mixed terms in horizontal and vertical directions vanish. In particular, $T^\oplus=0$, in other words, $\nabla^\oplus$ is torsion-free (it is the Levi-Civita connection). The model problem would be:

$$\left(\frac{1}{2}\tau\partial_\tau-\tau^2\partial_S^2-\tau^2 H_Y-\tau^2\mathbb{B}^2+\varepsilon\left(\frac{dx}{x^2}\right)\tau^2\eth_{\partial X/Y}\right)N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)=0$$

or, equivalently:

$$\left(\partial_t-\partial_S^2-H_Y-\mathbb{B}^2+\varepsilon\left(\frac{dx}{x^2}\right)\eth_{\partial X/Y}\right)N^{\mathcal{G}}_{\phi\mathrm{f},0}(h)=0$$

The scattering bf_0 contribution
We want to solve:

$$\begin{cases} \left(\partial_t + \eth^2\right) h(t,x,x',y,y') = 0, & t>0 \\ h(0,x,x',y,y') = \delta(x-x')\delta(y-y') \end{cases}$$

where (cf. (24))

$$\begin{aligned} \eth^2 &= -x^4\partial_x^2 + (b-2)x^3\partial_x + x^2\frac{b}{2}\left(1-\frac{b}{2}\right) + x^2\eth_{\partial X}^2 + x^2 cl\left(\frac{dx}{x^2}\right)\eth_{\partial X} \\ &= -\left(x^2\partial_x\right)^2 + bx^3\partial_x + x^2\left(\frac{b}{2}\left(1-\frac{b}{2}\right) + \eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}\right) \end{aligned}$$

is the square of the spin-Dirac operator at the infinite end of a cone. This means that away from the end $x \to 0$, we should obtain a solution similar to the cone case[116], which was already treated e.g. in [MV12, Prop. 3.2]. Inspired by this, we can change back to the conical radial coordinate $r = \frac{1}{x}$, so that:

$$x^2\partial_x \mathrel{\widehat{=}} -\partial_r, \quad \left(x^2\partial_x\right)^2 \mathrel{\widehat{=}} \partial_r^2$$

On top of that, recall that $\eth_{\partial X}$ is the Dirac operator with the Clifford structure induced on the boundary. This is a self-adjoint elliptic operator on a closed manifold and the same is true for $\overline{\eth_{\partial X}} := cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}$, and thus we can consider its complete orthonormal spectral decomposition in eigenvectors $\overline{\eth_{\partial X}}\phi_\lambda = \lambda\phi_\lambda$, with $\lambda \in \mathbb{R}$. Denote by E_λ the eigenspace for the eigenvector λ. Observe moreover that $\left(\overline{\eth_{\partial X}}\right)^2 = \eth_{\partial X}^2$. Then:

$$\eth^2\Big|_{E_\lambda} = -\partial_r^2 - \frac{b}{r}\partial_r + \frac{1}{r^2}\left(\frac{b}{2}\left(1-\frac{b}{2}\right) + \lambda^2 + \lambda\right)$$

We want to bring this back to the model "Bessel" heat equation:

$$\begin{cases} \left(\partial_t - \partial_r^2 + \frac{1}{r^2}\left(\nu^2 - \frac{1}{4}\right)\right) h'(t,r,r') = 0 & t>0 \\ h'(0,r,r') = \delta(r-r') \end{cases}$$

whose solution is standard:

$$h'(t,r,r') = \frac{\sqrt{rr'}}{2t} I_\nu\left(\frac{rr'}{2t}\right) e^{-\frac{r^2+(r')^2}{4t}}$$

[116] The best we could hope for is to find coordinates so that we can take a common factor x^2 and thus decouple the boundary directions from the normal direction in the equation (compare [KR22, Lemma 7.9]), which should lead to an eta invariant after mild computations. However, the coordinates in bf_0 do not seem to contribute to this goal, so we are stuck solving this Bessel-type equation. In particular, the most promising choice (with respect to the simplifying weight $\alpha = \frac{b}{2} - 1$ in terms of Footnote 117) is:

$$\tilde{T} = \frac{x^2}{\kappa^2} = x^2 t, \kappa = \frac{1}{\sqrt{t}}$$

(the rest are irrelevant) which yields

$$\beta^*\left(t\partial_t + t\eth^2\right) = \tilde{T}\partial_{\tilde{T}} - \frac{1}{2}\kappa\partial_\kappa - 4\tilde{T}^3\partial_{\tilde{T}}^2 + \tilde{T}\left(\eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}\right)$$

The leading order at the face bf_0 is $b+1$, so the normal problem takes the form:

$$\left(\tilde{T}\partial_{\tilde{T}} - \frac{b+1}{2} - 4\tilde{T}^3\partial_{\tilde{T}}^2 + \tilde{T}\left(\eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right)\eth_{\partial X}\right)\right) N_{\mathrm{bf}_0,b+1}(h) = 0$$

and due to the second term, we still cannot extract the common factor $\tilde{T}$.

Here, I_ν refers to the **modified Bessel function of the first kind**, which is closely related to the Bessel function of the first kind J_ν in that it is a complex version of it:

$$I_\nu(z) = i^{-\nu} J_\nu(iz) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}$$

To modify our heat equation to fit this form, we write $h_\lambda = r^\alpha h'_\lambda$ and choose the exponent α so as to get rid of the first order term on ∂_r[117]:

$$\begin{aligned}
\left(\partial_t - \eth^2\right) h_\lambda &= r^\alpha \partial_t h'_\lambda - \alpha(\alpha-1) r^{\alpha-2} h'_\lambda - 2\alpha r^{\alpha-1} \partial_r h'_\lambda - r^\alpha \partial_r^2 h'_\lambda \\
&\quad - \frac{b}{r}\left(\alpha r^{\alpha-1} h'_\lambda + r^\alpha \partial_r h'_\lambda\right) + \frac{1}{r^2}\left(\frac{b}{2}\left(1-\frac{b}{2}\right) + \lambda^2 + \lambda\right) r^\alpha h'_\lambda \\
&= r^\alpha \Bigg(\partial_t h'_\lambda - \partial_r^2 h'_\lambda + \frac{1}{r}\left(-2\alpha - b\right)\partial_r \\
&\quad + \frac{1}{r^2}\left(-\alpha(\alpha-1) - b\alpha + \frac{b}{2}\left(1-\frac{b}{2}\right) + \lambda^2 + \lambda\right)\Bigg) h'_\lambda
\end{aligned}$$

The factor $-2\alpha - b$ suggests we take $\alpha = -\frac{b}{2}$, which yields the equation:

$$\left(\partial_t - \partial_r^2 + \frac{1}{r^2}\left(\lambda^2 + \lambda\right)\right) h'_\lambda(t, r, r') = 0$$

For the initial condition, we make use of the behavior of the δ distribution under precomposition with a $\mathcal{C}^1$-function g:

$$\delta(g(z)) = \sum_{z:g(z)=0} \frac{\delta(z-z')}{|g'(z')|} \implies \delta(x - x') = \delta\left(\frac{1}{r} - \frac{1}{r'}\right) = \frac{\delta(r-r')}{|-(r')^{-2}|} = (r')^2 \delta(r - r')$$

The problem to solve then reads:

$$\begin{cases} \left(\partial_t - \partial_r^2 + \frac{1}{r^2}\left(\lambda^2+\lambda\right)\right) h'_\lambda(t,r,r') = 0 & t > 0 \\ h'_\lambda(0,r,r') = r^{-\alpha} h_\lambda(0,r,r') = r^{\frac{b}{2}} (r')^2 \delta(r-r') = (r')^{\frac{b+4}{2}} \delta(r - r') & \end{cases}$$

which up to the prefactor $(r')^{\frac{b+4}{2}}$ corresponds to the model Bessel heat problem with parameter:

$$\nu(\lambda) = \sqrt{\lambda^2 + \lambda + \frac{1}{4}} = \left|\lambda + \frac{1}{2}\right|$$

[117] A similar simplification could be performed already in the non-lifed situation by considering the operator $\eth$ acting on weighted Sobolev spaces with weight x^α, i.e.

$$\begin{aligned}
x^{-\alpha} \eth x^\alpha &= cl\left(\frac{dx}{x^2}\right)\left(x^2 \partial_x + \left(\alpha - \frac{b}{2}\right) x\right) + x \eth_{\partial X} \\
x^{-\alpha} \eth^2 x^\alpha &= -x^4 \partial_x^2 + (b - 2\alpha - 2) x^3 \partial_x + x^2 \left(\left(\frac{b}{2} - \alpha\right)\left(1 - \frac{b}{2} + \alpha\right) + \eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right) \eth_{\partial X}\right)
\end{aligned}$$

Treating the operator as acting on b-densities would correspond to $\alpha = \frac{b+1}{2}$ (cf. [Vai01, p. 26]), while the calculation above would be $\alpha = \frac{b}{2} - 1$:

$$\begin{aligned}
x^{-\frac{b+1}{2}} \eth^2 x^{\frac{b+1}{2}} &= -x^4 \partial_x^2 - 3x^3 \partial_x + x^2 \left(\eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right) \eth_{\partial X} - \frac{3}{4}\right) \\
x^{-\frac{b-2}{2}} \eth^2 x^{\frac{b-2}{2}} &= -x^4 \partial_x^2 + x^2 \left(\eth_{\partial X}^2 + cl\left(\frac{dx}{x^2}\right) \eth_{\partial X}\right)
\end{aligned}$$

The initial condition in the boundary directions follows from the orthogonality of the eigenfunctions when considering the whole spectrum at once, i.e.

$$\delta(y-y') = \sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} \phi_\lambda(y)\overline{\phi_\lambda(y')}$$

Hence we know the form of the solution:

$$h'(t,r,r',y,y') = \sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} \underbrace{(r')^{\frac{b+4}{2}} \frac{\sqrt{rr'}}{2t} I_{\nu(\lambda)}\left(\frac{rr'}{2t}\right) e^{-\frac{r^2+(r')^2}{4t}}}_{h'_\lambda(t,r,r')} \phi_\lambda(y)\overline{\phi_\lambda(y')}$$

To get back to the original kernel we use $h(t,r,r',y,y') = r^\alpha h'(t,r,r',y,y')$ and transform back to the scattering coordinates x, x':

$$h(t,x,x',y,y') = \sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} x^{\frac{b-1}{2}}(x')^{-\frac{b+5}{2}} \frac{1}{2t} I_{\nu(\lambda)}\left(\frac{1}{2txx'}\right) e^{-\frac{\frac{1}{x^2}+\frac{1}{(x')^2}}{4t}} \phi_\lambda(y)\overline{\phi_\lambda(y')}$$

We can lift this to the face bf_0, arriving at:

$$h(\kappa,\xi,T,\eta,y') = \sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} \frac{(T\xi+1)^{\frac{b-1}{2}}}{2T^3\kappa} I_{\nu(\lambda)}\left(\frac{1}{2T^2(T\xi+1)}\right) e^{-\frac{1+(T\xi+1)^2}{4T^2(T\xi+1)^2}} \phi_\lambda(T\eta+y')\overline{\phi_\lambda(y')}$$

Since we are interested in the supertrace, we can restrict to the diagonal $\xi,\eta\to 0$:

$$\mathrm{tr}(h)(\kappa,T,y') = \sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} \frac{1}{2T^3\kappa} I_{\nu(\lambda)}\left(\frac{1}{2T^2}\right) e^{-\frac{1}{2T^2}} |\phi_\lambda(y')|^2$$

We would like to relate this to the eta invariant integrand[118]

$$\begin{aligned}\mathrm{Tr}\left(\eth_{\partial X} e^{-t\eth^2_{\partial X}}\right)\frac{dt}{\sqrt{\pi t}} &= -\mathrm{Str}\left(cl\left(\frac{dx}{x^2}\right)\eth_{\partial X} e^{-t\eth^2_{\partial X}}\right)\frac{dt}{\sqrt{4\pi t}} \\ &\xrightarrow{\big|_{E_\lambda}} -\mathrm{Str}\left(\sum_{\lambda\in\mathrm{Spec}\left(\eth_{\partial X}\right)} \lambda e^{-t\lambda^2}|\phi_\lambda(y')|^2\right)\frac{dt}{\sqrt{4\pi t}}\end{aligned}$$

[118]We make use of the compatibility of the Clifford structure at X and ∂X, see e.g. [Mel93, §3.13] or [Liu25, (3.7), (9.60)].

We expect an eta invariant to appear here since local index formulas in conformal settings are heuristically related, which has to do with the fact that the geometrical invariants appearing in them (like the Â-genus) are conformal invariants. In particular, the asymptotically conical setting is conformal to the asymptotically cylindrical setting [Mel93], and the ϕ-setting, to the d-setting [Vai01]. See the argumentation in [LMP07] and [LM04].

The standard integral form of the eta invariant (e.g. [BF86, (2.22)] [Alb07, (6.6)] [Mel93, (In. 28)] [Vai01, p. 97]) for an operator P is:

$$\eta(P) = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{Tr}\left(Pe^{-tP^2}\right)\frac{dt}{\sqrt{t}}$$

but it is not clear how this connection should be made, even if we explicitly write the Bessel functions contributing[119]:

$$I_{\nu(\lambda)}\left(\frac{1}{2T^2}\right) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma\left(m+1+\left|\lambda+\frac{1}{2}\right|\right)}\left(\frac{1}{4T^2}\right)^{2m+\left|\lambda+\frac{1}{2}\right|}$$

The fibred boundary ϕf_0 contribution

Recall that the way we obtained the form of the compactified heat blow-up space in a neighborhood of $t=\infty$ was by relating the large time heat kernel with the low energy resolvent. The blow-up structure, i.e. which coordinate regimes have to be distinguished in the asymptotic, is thus inherited from that of the resolvent, which was studied in [GTV22]. In particular, since they treat the case of a non-fully elliptic differential operator, their approach to the problem consists in distinguishing between fibrewise harmonic sections[120] and its orthogonal complement. That work studies the Hodge Laplacian, but we can use this heuristic to treat the square of Dirac operators as generalized Laplacians (or just apply [KR22]).

A technical condition that is true for Hodge Laplace operators [GTV22, after Assumption 1.3] but in our more general setting we have to impose, is that the space of fibrewise harmonics forms a vector bundle over the base Y. This is analogous to the standard constant rank condition on the kernel (bundle) for the case of the families index theorem[121] [BGV04, p. 264] and the idea was already applied earlier in our context in the work of Vaillant [Vai01, p. 2, (39)]. We stick to Vaillant's notation and denote this bundle by $\mathcal{K} \coloneqq \left(\ker\left(\eth_{Z_y}\right)\right)_{y\in Y} \to Y$.

What we learn from the resolvent construction for the Hodge Laplacian is that the part acting on fibrewise harmonics behaves like a scattering operator (since the vertical part acts like 0), and the rest behaves like a fully elliptic operator [GTV22, §7]. For the fully elliptic part, as [LMP07] shows, the large time contribution is a true index, and there is no associated ϕf_0 contribution. At the level of blow-up spaces, this can be seen from the fact that this fully elliptic part is polyhomogeneous on a blow-up space where bf_0, ϕf_0, lb_0 and rb_0 are not needed [GTV22, Fig. 8 (right)]. On the other hand, the scattering part actually lives in the (compactified) scattering heat blow-up space (Figure 27, cf. [GTV22, Fig. 8 (left)]), where it contributes to the rescaled supertrace of the projection onto the kernel via zf and (presumably) with an eta invariant at bf_0. That is, the face ϕf_0 is just an artifice of trying to find a blow-up space which simultaneously harbors both regimes.

The heuristic resulting from these considerations is that the contribution from ϕf_0 should just reduce to that of the part of $\eth$ which acts on (sections of) ${}^{ac}T^*Y^+ \otimes \mathcal{K}$; if we denote the part of this restriction that commutes with $cl\left(\frac{dx}{x^2}\right)$ (equivalently, that does not have it as a prefactor)

[119] Chou is able to recover the eta invariant associated to the Dirac operator on the tip of the cone from his analysis of the zeta function with respect to a Hankel functional calculus, where the behaviour of Bessel functions is also key [Cho85, p. 35-36]. A similar approach here could be fruitful.

[120] I.e. elements in the kernel of the fibrewise/vertical Laplace operator Δ_F; from the perspective of Dirac opeartors, one would focus on the family of Dirac operators $\eth_{\partial X/Y}$, whose restriction over the fibre $Z_y \coloneqq \phi^{-1}(\phi(p))$ over a point $\phi(p) \in Y$ we denote $\eth_{Z_y}$.

[121] It makes sense that the families index theorem plays a role in our formula, as does in other instances of local index theorems in g_{klm} metrics. As we saw in the Atiyah-Patodi-Singer result, the presence of boundary gives rise to extra "boundary defect" terms which are associated to the induced Dirac operator on the boundary $\eth_{\partial X}$. In the case that the boundary is the total space of a fibre bundle, the induced Dirac operator on the boundary now can be seen as having a horizontal component, a vertical component and mixed terms (if the fibre bundle is not trivial). The vertical component restricts to each fibre Z_y to be a Dirac operator on the model manifold Z, so we have a family of Dirac operators acting on Z parametrized by the base Y, meaning part of the induced Dirac operator at the boundary is in fact (and can thus be treated like) a family of Dirac operators.

by $\eth_Y^{\mathcal{K}}$, then the heuristic of the asymptotically conical case tells us that it should produce a term $\frac{1}{2}\eta\left(\eth_Y^{\mathcal{K}}\right)$ (cf. [Vai01, p. 97]).

More precisely, consider first for clarity the case where $\partial X = Y \times Z$ (see Footnote 115), with Dirac operator:

$$\eth = cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x - \frac{xb}{2}\right) + x\eth_Y + \eth_{\partial X/Y} = \eth_{Y^+} + \eth_{\partial X/Y}$$

where $\eth_{Y^+}$ is the scattering Dirac operator on the asymptotically conical model base $Y^+ = \mathbb{R}_+ \times Y$ with metric $g_{Y^+} = x^{-2}dx^2 + x^{-1}g_Y$, which already appeared when describing the rescaling at ϕf with the Bismut-Freed trick (compare with (23)). In this case it is clear that $\eth|_{{}^{ac}T^*Y^+ \otimes \mathcal{K}} = \eth_{Y^+} \otimes id_{\mathcal{K}}$, and so the computations are exactly[122] as in the previous section (and allegedly yield $\frac{1}{2}\eta(\eth_Y)$).

We can treat the general case in an analogous manner by rewriting (36) as:

$$\eth = cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x - \frac{xb}{2}\right) + x\eth_Y^{\mathcal{K}} + \eth_{\partial X/Y}$$

with the help of (cf. D_Y in [Vai01, p. 44] or $\eth_{\phi\text{-hc}}^{H,\mathcal{K}}$ in [AR09, p. 13])

$$\eth_Y^{\mathcal{K}} := \eth_Y + \frac{3}{4}\sum_{i=1}^{b} k\left(\tilde{U}_i\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) - \frac{1}{4}\sum_{ijp} g_{\partial X/Y}\left([\tilde{U}_i, V_p], V_j\right) cl\left(\frac{\tilde{U}_i^\flat}{x}\right) cl\left(V_j\right) cl\left(V_p\right) - \frac{x}{2} cl\left(T^{\oplus}\right) \tag{40}$$

Consequently, the restriction to fibrewise harmonics reads:

$$\eth|_{{}^{ac}T^*Y^+ \otimes \mathcal{K}} = \left(cl\left(\frac{dx}{x^2}\right)\left(x^2\partial_x - \frac{xb}{2}\right) + x\eth_Y^{\mathcal{K}}\right) \otimes id_{\mathcal{K}}$$

Since this purely non-fully elliptic part is a scattering operator on the first factor and the identity on the second, its supertrace integral over $\phi\mathrm{f}_0$ corresponds to the integral over bf_0. The equation to solve is the lift to bf_0 of (cf. (24)):

$$\left(\partial_t + \eth^2\right) h = \left(\partial_t - x^4\partial_x^2 + (b-2)x^3\partial_x + x^2\frac{b}{2}\left(1 - \frac{b}{2}\right) + x^2\left(\eth_Y^{\mathcal{K}}\right)^2 + x^2 cl\left(\frac{dx}{x^2}\right)\eth_Y^{\mathcal{K}}\right) h = 0 \tag{41}$$

whose analysis and heat kernel construction should mirror that of the scattering case up to the replacement of $\eth_{\partial X}$ by $\eth_Y^{\mathcal{K}}$.

[122] Up to the fact that we have to integrate against dz' here, but this is taken care of by the fibrewise component $id_{\mathcal{K}}$, whose kernel is $\delta(z - z')$, so the corresponding fibre integral is 1.

5 Conclusion and open questions

> *On a more everyday level, it is common for people first starting to grapple with computers to make large-scale computations of things they might have done on a smaller scale by hand. They might print out a table of the first 10,000 primes, only to find that their printout isn't something they really wanted after all. They discover by this kind of experience that what they really want is usually not some collection of "answers"—what they want is understanding.*
>
> ON PROOF AND PROGRESS IN MATHEMATICS, *William P. Thurston*

We started this discussion by very briefly outlining the historical relevance and the development process of index theory in the last three quarters of a century, focusing on the heat kernel method to tackle the local index formula in singular settings. We would like to wrap things up with a word on some "applications" of index theory and an enumeration of a few possible near-future directions in the context of the methodology and classes of spaces we spent most of the time talking about.
For a "mathematical" application of index theory, we choose to talk about **rigidity of positive scalar curvature metrics**, which was for a long while a topic of interest in the community but has become "hot" in the last years, surely in part due to the very influential work [Gro23], which points out the relevance of index theory of Dirac operators as a tool to study positive scalar curvature (this is the reason why several of the results we have only concern spin manifolds). A particularly praised approach is that of Llarull's thesis [Lla98], which establishes a result previously conjectured by Gromov himself.
The link between spin structures and positive scalar curvature can be however traced back to the work of Lichnerowicz [Lic63], who applied Lichnerowicz's formula and the Atiyah-Singer index theorem to deduce that a manifold admitting a positive scalar curvature metric must have a vanishing Â-genus. In other words, he recognized this genus as the first known obstruction to the existence of positive curvature metrics (which applies to $4k$-dimensional spin manifolds)[123].
In the context of singular manifolds, an important article is [Cho85], which tackles the conical case and whose ideas have been later applied e.g. in [AGR16, §7] and [Liu25, §10]. Once we establish an index formula for a g_{klm} metric, we should be able to argue in a similar fashion. Indeed, by the Lichnerowicz formula, if the scalar curvature is non-negative:

$$\left\langle \eth\phi, \eth\phi \right\rangle = \left\langle \left(\nabla^{\mathcal{S}}\right)^* \nabla^{\mathcal{S}}\phi, \phi \right\rangle + \left\langle \frac{\text{scal}}{4}\phi, \phi \right\rangle \implies ||\eth\phi||^2_{L^2} \geq ||\eth\phi||^2_{L^2} - \left\langle \frac{\text{scal}}{4}\phi, \phi \right\rangle = ||\nabla^{\mathcal{S}}\phi||^2_{L^2} \geq 0$$

If $\phi \in \ker \eth$, then $\phi \in \ker \nabla^{\mathcal{S}}$ (i.e. it is parallel), meaning scal $\phi = 0$. Hence, if additionally the scalar curvature was positive somewhere, then ϕ would have to vanish at that point, thus at all points by $d\langle \phi, \phi \rangle = \langle \nabla^{\mathcal{S}}\phi, \phi \rangle + \langle \phi, \nabla^{\mathcal{S}}\phi \rangle = 0$. In other words, the kernel of the spinor Dirac operator is trivial and therefore ${}^R\text{ind}\left(\eth\right) = \text{ind}\left(\eth\right) = 0$, i.e. the operator has to be Fredholm. Moreover, the vertical family has trivial kernel (take a vertical section and extend it trivially in the base and normal directions to show that $\ker \eth_{\partial X/Y} \subset \ker \eth$).
Recall that the spinor-Dirac operator of a scattering metric was not fully elliptic and thus not Fredholm, so such a manifold does not admit a curvature non-negative everywhere and positive somewhere.
In the fibred boundary case, such a scalar curvature is possible, but it would mean the associated Dirac operator is fully elliptic and thus we are in the context of [LMP07, (1.4)]. Consequently, $\text{ind}\left(\eth\right) = 0$ forces $\int_X \hat{\text{A}}(X) = \frac{1}{2}\int_Y \hat{\text{A}}(Y)\overline{\eta}(\partial X/Y)$ to hold.

[123] This could also be seen as part of the "meaning" of the Â-genus (§2.2.3).

The resulting statement reads:

Proposition 5.1. A scattering metric cannot have non-negative scalar curvature everywhere and positive somewhere.
The spinor Dirac operator associated to a ϕ-metric of non-negative scalar curvature everywhere and positive somewhere has a vertical family $\eth_{\partial X/Y}$ of trivial kernel and satisfies $\mathrm{ind}\left(\eth^{+}\right) = 0$, provoking:

$$\int_X \hat{\mathrm{A}}(X) = \frac{1}{2}\int_Y \hat{\mathrm{A}}(Y)\bar{\eta}(\partial X/Y)$$

□
What could also be understood as an "application" of the index formula consists in using the corresponding analysis to investigate its form for concrete instances of Dirac operators, like the Gauß-Bonnet operator [Alb07, §7] or the signature operator [Vai01, §5.4] [Liu25, §12], producing the corresponding **Gauß-Bonnet** and **signature theorems** in singular settings. The Gauß-Bonnet theorem was already carried out in the ac-case in [Alb07, Corollary 7.3]. These are of course not the only interesting special cases (check e.g. [BGV04, Theorem 4.10]).
For a "physics" application (of the index theorem and the families index theorem), we would like to mention (local) **anomalies in quantum field theory**, for which [Fre21a, §8] is a nice review. These are obstructions to "renormalizing" the theory, i.e. making it feasible from a physical point of view, and can be related to the index of a Dirac operator. More surprisingly, in the realm of condensed matter physics, results concerning so-called topological insulators (like the robustness of the Hall conductance in the quantum Hall effect) can be tied up with ideas in index theory [KLWK16].
On the matter of **open problems** neighboring the above exposition, we present here a list with some ideas that we might consider tackling in the near future:

- Solve Conjecture 3.22 and develop a full understanding of the **asymptotics of heat kernels** of differential operators associated to iterated fibration structures; extend this to manifolds with corners and their corresponding iterated metric structures (here we are thinking about e.g. **manifolds with fibred corners** [ALMP12] [DLR15]).

- Also in that general class of manifolds, find the principle underlying the construction of heat triple spaces and describe the resulting **composition theorems**.

- **Characterize Fredholmness** of the differential operators adapted to the metric. When is it equivalent to ellipticity and invertibility of the corresponding normal operator? For weighted Sobolev spaces, when is Fredholmness dependent or independent of the weight? Are there also adapted notions of split calculi (in the sense of [GH14]), where Fredholmness behaves differently as in the usual Sobolev spaces?

- Fill in the gaps of the arguments in §4.5, thus obtaining a **local index formula** that interprets all of the terms in Theorem 4.25 and Theorem 4.28.

- Tackle the local index problem for **incomplete fibred cusp** metrics: this should be a mixture of the cone and cusp cases, as we tried to convey with Figure 24. The model problems should be similar to the incomplete cusp edge case [Liu25] but the fact that the base directions have a power of x in the metric produces more terms and might make the rescaling more complicated, as happens in our analysis of fibred boundary manifolds. Probably a geometric Witt condition is needed for unique self adjoint extension so that it specializes to cones when the fibre is trivial (compare [Cho85, §3]).

- Prove Gauß-Bonnet and signature theorems in these settings. The scattering Gauß-Bonnet theorem was already carried out in [Alb07, Corollary 7.3]. This can in particular provide more insight into the nature of the renormalized index.

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