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Solutions to non-linear Thiele BSDEs in the context of non-monotone information dynamics

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Zusammenfassung

(German)

In der Lebensversicherungsmathematik sind die zukünftigen kumulierten diskontierten Zahlungen eines Versicherungsvertrages von zentralem Interesse, diese sind jedoch zum aktuellen Zeitpunkt in der Regel nicht bekannt. Stattdessen werden bedingte Erwartungswerte bezüglich der aktuell verfügbaren Informationen betrachtet, welche dem Versicherer die Berechnung der sogenannten prospektiven Reserve ermöglichen.
In dieser Arbeit wird das gängige Mehrzustandsmodell der Lebensversicherung durch die Betrachtung von nicht-monotonen und eingeschränkten Informationsstrukturen erweitert. Gleichzeitig werden Zahlungen erlaubt, die nichtlinear von der prospektiven Reserve abhängen können und dadurch für eine Zirkularität in der Definition der Reserve sorgen. In der bestehenden Literatur, siehe zum Beispiel Christiansen und Djehiche [CD20], sind ein filtrierter Wahrscheinlichkeitsraum und die Anwendung der Martingaltheorie von zentraler Bedeutung um die Existenz und Eindeutigkeit von Lösungen für die entsprechende stochastische Rückwärts- Differentialgleichung (BSDE) der prospektiven Reserve zeigen zu können. Diese Martingal-Methoden erweisen sich jedoch unter der nicht-monotonen Informationsstruktur als nicht anwendbar, und stattdessen wird das infinitesimale Martingal-Konzept verwendet, welches in Christiansen [Chr21b] eingeführt wurde.
In dieser Arbeit zeigen wir die Existenz und Eindeutigkeit für den nicht-adaptierten Zahlungsprozess. Um dies zu erreichen verwenden wir das Fixpunkttheorem von Banach und nutzen den Zusammenhang zwischen Zahlungsprozess und der Reserve mit Hilfe der bedingten Erwartung aus. Diese Dualität ist ein charakteristisches Merkmal des Lebensversicherungsmodells und ermöglicht die Anwendung anderer Methoden als der in der BSDE-Theorie üblichen. Die Ergebnisse werden für zwei Modelle mit einem unterschiedlich starken Grad der Reserveabhängigkeit spezifiziert. Das erste Theorem kann für relativ allgemeine Informationsstrukturen verwendet werden, während das zweite Theorem nur für die in [Chr21b] vorgeschlagene Informationsstruktur gilt, dafür aber eine erweiterte Reserveabhängigkeit zulässt.
Die Resultate werden dann auf die Existenz und Eindeutigkeit der prospektiven Reserve ausgedehnt und eine BSDE-Formulierung analog zur Thiele BSDE wird präsentiert. Ein Theorem über die Berechnung von Nettoäquivalenzprämien als Startwertprobleme schließt die wichtigsten theoretischen Beiträge dieser Arbeit ab.
Zu den potenziellen Anwendungen der Theorie gehören gesetzliche Einschränkungen, wie sie im „Recht auf Löschung“ (Recht auf Vergessenwerden) der Allgemeinen Datenschutzverordnung 2016/679 festgelegt sind, sowie das Gleichbehandlungsgesetz, welches zu Unisex-Tarifen führt. Weitere Anwendungen sind Raucher-Tarife, bei denen die Information aufgrund der Datenverfügbarkeit eingeschränkt wird, und Modelle, bei denen die Markov-Annahme zur Modellvereinfachung verwendet wird, auch wenn die verfügbaren Daten diese Annahme nicht notwendigerweise unterstützen.

Abstract

(English)

In the field of life insurance mathematics, the discounted cumulative future payments of an insurance contract are of central interest, yet they are typically unknown at the present time. Given the available information, conditional expectations are considered instead, enabling the insurer to calculate the so-called prospective reserve.
This thesis builds upon existing multi-state life insurance models by incorporating non-monotone and restricted information structures, while simultaneously considering payments that may depend non-linearly on the prospective reserve, thus creating circularity in the definition of the reserve. In the existing literature, see for example Christiansen and Djehiche [CD20], a filtered probability space and the application of martingale theory are central to showing existence and uniqueness of solutions to the corresponding backward stochastic differential equation (BSDE) of the prospective reserve. However, these martingale methods are inapplicable in the context of non-monotone information structures, and to bypass this problem, the infinitesimal martingale concept, as introduced by Christiansen [Chr21b], is used instead.
In this thesis, we demonstrate the existence and uniqueness of the non-adapted payment process. To achieve this, we utilise the fixed-point theorem of Banach and exploit the interconnections between the payment process and reserve through the conditional expectation. This duality is a distinctive feature of the life insurance model, allowing the application of different methods than those typically used in BSDE theory. The results are specified for two main models, differing in their degree of reserve dependency. The first theorem can be applied for quite general information structures, whereas the second theorem is limited to the information structure proposed in [Chr21b], but allows for an extended reserve-dependency of the payments.
The results are extended to encompass the existence and uniqueness of the prospective reserve, and a BSDE formulation analogous to the Thiele BSDE is presented. An additional theorem about the calculation of net equivalent premiums as a starting value problem concludes the major theoretical contributions of this thesis.
Potential applications of the theory include legal restrictions as set out in the 'right to erasure' of the General Data Protection Regulation 2016/679, as well as the principle of equal treatment, resulting in unisex tariffs. Further applications include smoking tariffs, where information is restricted because of data availability, and models where the Markov assumption is used for model simplification, although the data might suggest otherwise.

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Chapter 1.

Introduction

Insurance is undoubtedly one of the largest industries in the world when measured in terms of revenue, and most people will encounter insurance contracts at some point in their lives. A multitude of professions are involved in the day-to-day operations of an insurance company, with the objective of establishing a clear and precise environment in which an insurance contract is viable. This leads us to the following general definition of an insurance policy:

> 'An insurance policy is a contract between a policyholder and an insurance company to exchange a fixed premium payment against uncertain benefit payments triggered by pre-described, but random events, that adversely affect the policyholder.' [1]

In this thesis, we are focusing on the mathematical, and especially stochastic, nature of the benefit payments as part of the insurance contracts and the methods that are used to examine them. Our main focus will be on multi-state models in life insurance and the theory of backward stochastic differential equations, applied to challenges related to reserving for and pricing of contracts in life insurance.

1.1. Life insurance

A life insurance policy is defined as an insurance contract where the underlying risk is connected to the human life. Life insurance contracts are used for two main reasons: firstly, to provide protection against the risks of death, illness or disability; and secondly, to facilitate retirement savings and wealth creation, compare for example Wilson [Wil15]. The following types of contract arise in the context of life insurance and are of interest. For a comprehensive overview of these contracts, see Olivieri and Pitacco [OP11].
Term life insurance is a contract, whereby a beneficiary will receive payments, upon the death of the policyholder during the period of the contract. Survival insurance is an insurance contract that provides a payment if the policyholder reaches a certain age. It can also be used for investment purposes and retirement savings. An endowment policy represents some combination of the aforementioned polices.
Insurance for disability, accidents and dread-diseases are contracts, where a monthly

[1] Own definition based on various legal, accounting and actuarial definitions, see for example [Far90]

benefit is paid. Such policies are commonly used for protection against the risk of being unable to generate income through work. Retirement contracts, which may be combined with other types of insurance, are utilized for private retirement savings. The policyholder is entitled to receive monthly benefit payments between the retirement age and the time of death.
Despite the differences between of the above contracts, they can be embedded in a common model, which will now be described.

For every contract type, the benefit payments are predetermined and pre-described within the contract conditions. This ensures that the circumstances under which a payment is triggered, and the nature of payment itself, are transparent. Subsequently, it is necessary to calculate the probability of these payments occurring. It is the responsibility of the insurance company to determine an appropriate premium for the contract and to maintain prospective reserves to cover the uncertain future expenses. Optimally, the expected value of premiums and benefits would be equal and the premium payment is said to be a net premium payment in that case, compare for example Olivieri and Pitacco [OP11]. In addition, the insurer calculates market prices for the contracts and manages their portfolio and the associated risks during the contract horizon.
A distinctive feature of life insurance contracts is that the time horizon is typically very long, and therefore an accurate basis for the pricing of a contract is crucial. For every contract, there is a maximal time, thereafter no further payments are happening. In most cases, the necessary reserves can be calculated by a backwards calculation with a final value of zero. In general, this would be presented as a backward stochastic differential equation, which also gets referred to as a Thiele BSDE. The Thiele BSDE is specific to the theory of life insurance and can usually be embedded into more general BSDE theory. Further details can be found at the beginning of the next chapter and in the research overview later in this chapter. The tasks of pricing and reserving are part of what we call the actuarial calculations in life insurance, and will be referenced in subsequent sections of this thesis, where a comprehensive mathematical introduction will be provided.

1.2. Motivation and problem formulation

Let us now motivate the extensions to the standard multi-state model in life insurance. The extension will occur in two specific ways in this thesis.

Firstly, we want to allow for general reserve-dependency of payments. In other words, the value of insurance (in particular, benefit) payments is not necessarily fixed, but may vary depending on the existing prospective reserve. This is for example needed for lapse payments, where it is fair, and also required of the insurance company by law, to repay the existing reserve, in the event of a contract being terminated. An additional possibility is the incorporation of capital management fees for, or investment returns on, the current reserve, if a general market model is used.
It is evident that this introduces complications to the reserving and pricing, since the reserve can now depend circularly on the reserves at a future time point. Allowing

for non-linearity in the dependency of payments introduces a further level of complexity.

Secondly, it is necessary to allow for a more general and potentially non-monotone information structure. Typically, the insurance company uses an information structure as the basis for all calculations, which is monotonously increasing in time. This has practical, as well as mathematical, reasons and leads to the usage of filtrations.
We now seek to eliminate the requirement for monotone information, and allow for a more general and potentially non-monotone information structure. There are several reasons, why this might be desirable. The General Data Protection Regulation of the EU (2016) [Eur16], Article 17, guarantees the insured a 'right to erasure', or a right to be forgotten. This data privacy regulation may challenge the above assumption of monotonic information structures.
Moreover, there are some anti-discrimination laws that have resulted in the emergence of unisex contracts, compare [Eur04] and the follow-up [Eur11]. In this case, the information structure is not non-monotone; however, not all available information can be used for the pricing of the contract. Furthermore, the unisex tariff may also lead to adverse selection bias, which must be considered by the insurance company and therefore comparisons with the model of full information are conducted.

Further reasons are based on the selection of the model. The insurance company has to decide on a model, which is used as a basis for the calculations. For simplicity of the numerical calculations, or data availability to generate good statistical estimators, the decision may be taken to exclude specific data. This approach, however, may result in model bias and sparseness of the model. In many cases computationally feasible results are preferred to perfect results. It should be noted that the data availability in a simpler model is typically superior, which may lead to better estimators, even if the model is biased.
If both models can be computed with reasonable precision, then the bias can be quantified. This would for example be the case with certain models, where the Markov assumption is applied on the probabilities in the multi-state model, but the actual data would suggest a different underlying model.

The combination of the two extensions results in a situation, where non-monotone information structures are used and a non-linear dependency on the reserve is present. It is yet to be seen how this should be modelled in detail, whether a BSDE formulation for the prospective reserve can still be achieved, and if existence and uniqueness of the solution can be shown. The solution would then allow for the calculation of reserves for a new class of insurance contracts.

1.3. Research status

The objective of this section is to provide an overview on the current state of research in the multi-state life insurance theory and on the two extensions, that were motivated in the previous section. Mathematical details will be given in the next chapters, and here we only provide a general overview of the literature.

The Thiele BSDE from the title of this thesis has its historical roots in the Danish actuary Thorvald N. Thiele, who is said to have been the first to describe (in an unpublished note) a recursion for calculating life insurance reserves in 1875. This is according to the timetable given in Milbrodt and Helbig [MH99], where major contributions to life insurance theory are listed. The theory has evolved since then, and for a proper mathematical formulation of more advanced recursions and BSDEs, we refer to Markov chain models, introduced in the paper of Hoem [Hoe69], and generalizations in Norberg [Nor91] and [Nor92], providing methods for the calculation of state-wise reserves. Møller [Møl93] then provides a general version of the Thiele BSDE, and Djehiche and Löfdahl [DL16], as well as Christiansen and Djehiche [CD20], formulate Thiele equations with reserve-dependency.

Another significant development was the introduction of Cantelli's theorem in 1914 ([Can14]), which has been subsequently generalised and modernised in Milbrodt and Helbig [MH99] and Christiansen, Denuit, and Dhaene [CDD13].
Cantelli's theorem provides sufficient conditions for the equality of reserves in different contracts, based on the Thiele equations. Applications includes models with and without lapse, where the reserves in both models are equal, if the lapse payment is set equal to the full reserve immediately before the time of lapse. This allows for the use of a reduced model, without needing to model reserve-dependency or the lapse behaviour. There are several reasons why the conditions for equality are not necessarily fulfilled in practice, and not the full reserve is paid out:
Reducing the lapse payment also reduces the incentive for the policyholder to terminate the contract. This is important for an insurance company, because it needs to manage its portfolio and maintain a certain number of contracts so that the application of averaging effects is still reasonable.
In addition, the insurance company has to consider the administrative costs associated with the insurance business. Further, the loss of planned interest income on the existing reserve will be reduced and could force the insurance company to consider short-term investments instead of more beneficial long-term investments. See, for example, the following report with notes on the lapse deduction in life insurance by the German Association of Actuaries in [Deu22b] [2]. The reasons they give explain why the reserve would not normally be paid out in full.
An additional problem is the estimation of lapse rates (or the modelling of lapse behaviour), as also personal and (macro)economic factors could play a role in the decision. The personal factor could involve a shift in guaranteed interest rate relative to what is available on the market through potentially different investment vehicles. This can lead to adverse selection in the portfolio, and increased surrender rates, if exercising the option is perceived as beneficial.
In summary, Cantelli's theorem needs to hold to be able to ignore lapse and avoid a reserve-dependent payment. Otherwise, lapse cannot be ignored and the existence and uniqueness of solutions for a contract with reserve-dependent payments must be investigated from scratch.

[2] `https://aktuar.de/de/wissen/fachinformationen/detail/stornoabzuege-in-der-lebensversicherung`, (accessed 12.2024)

An extension to reserve-dependent payments is adapted in Christiansen, Denuit, and Dhaene [CDD13], Djehiche and Löfdahl [DL16] and Christiansen and Djehiche [CD20], where the latter ones consider actual non-linear reserve-dependency. In the first reference, the reserve-dependency is mostly linear and embedded in the standard model by rearranging of summands. Examples are given as linear capital management fees and linear surrender payments, but the reserve-dependency is solved by using a different technical basis for the contract. The original interest rate is changed to compensate for the linear reserve-dependence. An example with non-linearity is only mentioned at the end, and reference is made to numerical methods.
Other references on reserve-dependence include the book by Asmussen and Steffensen [AS20] (Chapter VII.2), which also uses a linearisation approach to reserve-dependency and introduces an artificial interest rate to compensate for capital management fees. In Steffensen and Møller [SM07], the techniques are more so based on optimisation, but surrender and free-policy options are also considered.
Examples of reserve-dependent payments can also be found in Norberg [Nor91], where the retrospective reserve in a widow's annuity might be repaid if the beneficiary dies before the policyholder. An extension of the model is formulated, where the administrative costs depend partly on the reserve. Furthermore, Milbrodt and Helbig [MH99] present a version of Cantelli's theorem and therefore consider a reserve-dependent lapse payment as well.

The contributions by Gatzert and Schmeiser [GS06], and Gatzert [Gat09], take a more general approach to implicit options in life insurance and provide reasons and examples of contract design, but do not focus on the mathematical methods. These include situations where payments are modified after the contract start by the exercise of certain options, such as the 'paid-up' (or free policy) option, where the insured stops paying premiums and the insurance benefits change (decrease) accordingly.
It is also possible to consider contracts in which the payments are linked to investments, either through direct investment in a financial market (e.g. dividend payments), or through guaranteed minimum payments, in which the insurance is required to pay for potential losses. For further examples and the mathematical background of these types of contracts, see for example Steffensen and Møller [SM07] and Asmussen and Steffensen [AS20].
An alternative approach of reserve-dependency is presented in Gad, Juhl, and Steffensen [GJS15], where even the behaviour (the lapse/ surrender rate) of the insured is modelled as dependent on the reserve. Their reasoning is consistent with the behavioural rationale for lapse presented above. This approach will not be adopted in this thesis, as it creates dependency at a different part, i.e. in the intensities and not in the payment functions and would require vastly different techniques.

A relatively recent development is the paper by Christiansen and Djehiche [CD24], which considers reserve-dependent payments and investigates the existence and uniqueness of as-if-Markov reserves. This is motivated by the failure of Cantelli's theorem for as-if-Markov calculations, which in turn raises questions about the existence and uniqueness of the as-if-Markov reserves in models with reserve-dependent payments.

The theory of life insurance with non-linear reserve-dependency significantly relies on the methodologies derived from general BSDE theory. The Thiele equation can be considered

a Thiele BSDE in such circumstances and the BSDE is driven by jump processes, namely the transition counting processes.
Solving techniques for BSDEs based on general martingale theory are presented in a paper by Pardoux and Peng [PP90], who were the first to work on non-linear BSDEs in 1990. Advances were made by Karoui, Peng, and Quenez [KPQ97], Pham [Pha09], Delong [Del13] and a series of contributions by Cohen and Elliott, i.e. [CE08], [CE12], as well as the book [CE15]. In the context of BSDE theory, the existence and uniqueness of solutions are shown for a scenario where a terminal value and a non-linear dynamic are given. The solution is always adapted to the underlying filtration, and a predictable process in the martingale part is used to control the randomness, as demonstrated in Cohen and Elliott [CE15].
The BSDE theory has been developed to accommodate a variety of dynamics, including Brownian motion and general martingales, where a lot of foundational martingale theory is employed. The martingale theory is also significantly influenced by the concepts of a filtration, the martingale representation theorem, the inequality of Doob for martingales and the inequality of Burkholder-Davis-Gundy. These tools are applied to show the existence and uniqueness of solutions; however, they are only available for martingales and thus will not be used in this thesis.

To avoid further reliance on filtrations, it will first be necessary to establish a model for the non-monotone theory. In the paper by Christiansen [Chr21b], a general model is presented, which relies on the infinitesimal understanding of martingale properties. An explicit theory is subsequently developed for the case of marked point processes, which bears resemblance to the multi-state jump process modelling typically used in life insurance theory. Furthermore, Christiansen derives an equivalent of a Thiele equation in this context. In their paper, Christiansen and Furrer [CF21] extend the general Thiele equation to statewise reserves in the context of a Danish life insurance contract, thereby producing results that are comparable to those of standard life insurance theory.

At this time, the approach presented in Christiansen [Chr21b] is the only viable approach for modelling of a non-monotone theory. Consequently, the majority of the following chapter will introduce these concepts before we can consider the application of the model to life insurance contracts with reserve-dependent payments.
Note, that in the paper by Norberg [Nor99], a more general information setting is discussed, which does not require the use of a filtration. In one of their examples, calculations are conducted exclusively with respect to the current state of the insured, and the statewise surplus is investigated. This is not a non-monotone setting; however, this is arguably one of the closest to the idea presented in Christiansen [Chr21b].

Other approaches rely on non-monotonicity only at certain time points or enable the possibility of reverting to classical martingale representations through the imposition of independence assumptions. For further details, see [TW13] or [PP94]. Take note that these methods are not viable in our circumstances.
Another related concept is the use of information structures with shrinkage, as seen in the paper by Andersen and Lollike [AL23]. In their approach, lumping is introduced and averages with respect to smaller state space are calculated, reducing the computational

complexity. The reserve is then calculated for the reduced example and the conditional expectation is used for simplification.

It is evident from the two extensions presented in the preceding section that the classical theory of life insurance mathematics, and in particular the BSDE theory, cannot be used directly in this context. The primary concern is the martingale theory and its reliance on the underlying filtration, where the existing results cannot be translated to the novel circumstances.
Also recognise that we are solely working with a given first-order basis of the financial market and the transition rates in the life insurance model, as the research questions revolve around the pricing and reserving of contracts, with the main focus on the existence and uniqueness of a solution to the Thiele BSDEs. Consequently, our perspective is situated before the start of the contract, answering feasibility concerns instead of monitoring and managing a portfolio of existing contracts.

1.4. Outline of the thesis

Chapter 2 presents the theoretical background for the life insurance theory. Moreover, the non-monotone theory by [Chr21b] and the introduction of the infinitesimal martingale theory are discussed in detail. Subsequently, the marked point process construction is linked to the existing multi-state theory in life insurance and BSDE methodology is recapped.

Chapter 3 consists of a rigorous mathematical problem formulation, where the reserve-dependence of insurance payments is introduced and the first two main results about existence and uniqueness of the insurance payment process are presented. The detailed proofs of these results are provided. Two levels of reserve-dependency are distinguished, where the simpler variant works in a general setting, whereas the more complex dependency relies on the marked point process theory in the potentially non-monotone framework.

Chapter 4 presents further results and extensions to the theory introduced in Chapter 3. The results are extended to the corresponding prospective reserves of the payment processes and a BSDE formulation of the Thiele equation is presented in our model. Some existing results are adapted to the case of retrospective reserves.
The final main result is a theorem about the uniqueness of pricing for some of the insurance contracts from the preceding chapter. The proof uses a generalized Grönwall inequality.

Chapter 5 presents a number of potential applications and examples, while further research opportunities are discussed in Chapter 6. The last chapter also offers a short conclusion.

Chapter 2.

Theoretical background

We start with an introduction to the common multi-state model in life insurance and martingale theory. After that, we will introduce the infinitesimal martingale theory, that was developed by Christiansen [Chr21b]. Although the infinitesimal martingale concept will only be used in the special case of marked point process framework, the underlying ideas can also be formulated independently of these assumptions.
We finish this chapter with a recap of the standard life insurance theory and how the concepts connect. This includes the mathematical foundation to continue with the extension to non-linearity in the next chapter.

2.1. The mathematical basics in life insurance

This section builds on the general introduction to life insurance modelling in the introductory part and continues with some important definitions and results from the mathematics of multi-state modelling in life insurance.

Definition 2.1.1. *Insurance state process*
Let the finite set $\mathcal{S}$ be the state space of an insurance contract, where the insured can potentially be in each state $s \in \mathcal{S}$.
Then, let the $\mathcal{S}$-valued càdlàg pure jump process $S : [0, \infty) \to \mathcal{S}$ be the process, describing the evolution of an insurance policy, where at each time point a unique state is assigned to the insured.
For most applications it is assumed, that the insurance contract starts in a deterministic initial state $S_{0^-} := S_0 = s_0 \in \mathcal{S}$. This would usually be the state 'active'.

For the purpose of a more compact natation we also define the space of transitions between two states.

Definition 2.1.2. *Transition space*
Let $\mathcal{T} := \big\{(i, j) \in \mathcal{S}^2 \bigm| i \neq j\big\} \subset \mathcal{S}^2$ be the set of all transitions between two (different) states of the state space $\mathcal{S}$.
This definition does not mean, that a transition is actually possible and has a positive probability of occurrence. We also include transitions, which are almost surely not happening (for example a transition from 'dead' to 'active').

Definition 2.1.3. *Natural filtration*
Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of the insurance jump process S, i.e.

$$\mathcal{F}_t = \sigma\left(S_r,\, 0 \leq r \leq t\right) .$$

Then, we can also give a representation of the filtered probability space as

$$\left(\Omega,\, \mathcal{A},\, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0},\, \mathbb{P}\right) .$$

In order to properly model payments as part of an insurance contract, we need some further notation, to reflect when payments occur.

Definition 2.1.4. *State indication and transition counting processes*
The state indication processes

$$\mathbb{I}_t^i := \mathbb{1}_{\{S_t = i\}} ,$$

for $i \in \mathcal{S}$ indicate, whether the insured is in state i at time $t \geq 0$. These processes are used to model sojourn payments, which are payments that happen, while the insured is remaining in an insurance state.
The transition counting processes

$$N_{ij}(t) := \#\left\{r \in (0, t] \mid S_{r^-} = i,\, S_r = j\right\} ,$$

for $(i, j) \in \mathcal{T}$ indicate the total number of transition $i \to j$ from state i to j up to time t. These processes will be used to model transition payments, which will be triggered, when N jumps. The process N takes its values in $\mathbb{N}_0$ and is monotonously increasing.
Both the state indication process, as well as the transition counting process, inherit the càdlàg property from the process S.
We generally assume

$$\mathbb{E}\left[N_{ij}(t)\right] < \infty \quad \text{for } (i, j) \in \mathcal{T},\, t \geq 0 ,$$

to ensure that the state process S has at most finitely many jumps on compacts.

Assertion 2.1.5. *Equivalent representations of the current state*
The process S of the random pattern of states can be equivalently expressed by the initial state S_0 (or equivalently by $(\mathbb{I}_0^i)_{i \in \mathcal{S}}$) and the family of processes $(N_{ij})_{(i,j) \in \mathcal{T}}$, by using that

$$S_t = \sum_{i \in \mathcal{S}} i\, \mathbb{I}_t^i ,$$

where the counting processes can be updated from the initial value by

$$\mathbb{I}_t^i = \mathbb{I}_0^i + \sum_{j:\, j \neq i} \left(N_{ji}(t) - N_{ij}(t)\right) .$$

Compare to Djehiche and Löfdahl [DL16] for further details.

Let us formulate a small example for illustrating the above concepts.

Example 2.1.6. *An exemplary insurance state space*
A standard insurance space, that is sufficient to model various life insurance contracts, would be the space

$$\mathcal{S} = \{a,\, i,\, l,\, d\} \ ,$$

where a abbreviates 'active', i 'invalid', l 'lapse', and d abbreviates 'dead'. There are only five possible transitions, as can also be seen in the following figure.

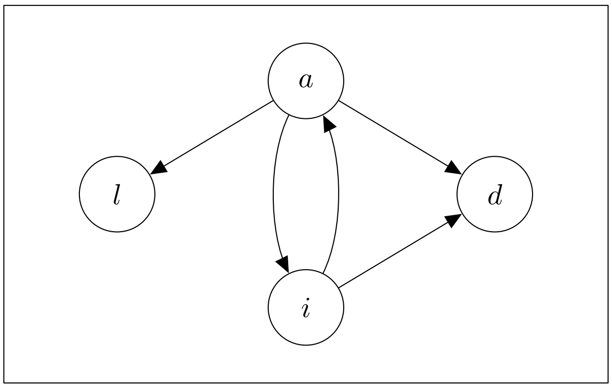

Figure 2.1.: An insurance state space with possible transitions

The states d and l are called absorbing states, since it is impossible to leave them, once they are entered. We allow for recovery from the state i to the state a. Only transitions with a non-zero probability of happening are denoted by an arrow in Figure 2.1.6.

Definition 2.1.7. *Contractual payments*
An insurance contract consist of several possible payments, where the details are agreed on in the contract upon signing.
The first category are payments, happening while the insured remains in a state, and are called sojourn payments, denoted as $b_i(t)$ for the payment in state $i \in \mathcal{S}$. This type of payment occurs, if the indication function $\mathbb{I}^i_{t^-} = 1$, i.e. the insured is in state i right before time t. At this time, we consider the sojourn payments to be absolute continuous and paid as a rate, while not allowing for lump sum payments just yet.
The second category are payments associated with transitions between two different states, and are called transition payments, denoted by $B_{ij}(t)$ for a payment happening upon a transition $i \to j$ for $(i,j) \in \mathcal{T}$ in t. It is always a lump sum payment, and will be triggered if $N_{ij}(t) - N_{ij}(t^-) = 1$, i.e. a new transition from i to j is recognized at time t.
We also differentiate between payments to the insurer (premiums) and payments to the insured (benefits). This will be marked by using a negative sign in the case of premiums, but all the payments will be grouped in a single summand.

At this point we will not specify any more details on the payment functions, in order to avoid repetitions later on.

Definition 2.1.8. *Accumulated cash flow*
The payments from above are merged as part of the accumulated cash flow during the time interval $[0, t]$, as

$$A(t) = \sum_{i \in \mathcal{S}} \int_{[0,t]} \mathbb{I}^i_{s-} \, b_i(s) \, \mathrm{d}s + \sum_{(i,j) \in \mathcal{T}} \int_{[0,t]} B_{ij}(s) \, \mathrm{d}N_{ij}(s)$$

in integral form, or equivalently in the differential notation with $A(0) = 0$ as

$$\mathrm{d}A(t) = \underbrace{\sum_{i \in \mathcal{S}} \mathbb{I}^i_{t-} \, b_i(t) \, \mathrm{d}t}_{\text{sojourn payments}} + \underbrace{\sum_{(i,j) \in \mathcal{T}} B_{ij}(t) \, \mathrm{d}N_{ij}(t)}_{\text{transition payments}}, \tag{2.1.1}$$

where the additive decomposition in sojourn and transition payments is marked.

This process A specifies the cumulative payments on the interval $[0, t]$. The insurance company is more interested in the accumulated future payments on (t, ∞) instead, which in our case, may be represented as $(t, T]$, since we assume to always have a maximum contract time T.

Definition 2.1.9. *Cumulative future payments*
The cumulative future payments of an insurance contract during the time interval $(t, T]$ are given as

$$X_t := \int_{(t,T]} \mathrm{d}A(s) \, .$$

When the representation from formula (2.1.1) is used, they are given as

$$X_t = \sum_{i \in \mathcal{S}} \int_{(t,T]} \mathbb{I}^i_{s-} \, b_i(s) \, \mathrm{d}s + \sum_{(i,j) \in \mathcal{T}} \int_{(t,T]} B_{ij}(s) \, \mathrm{d}N_{ij}(s) \, .$$

This process contains future payments, where the state of the insurer is usually unknown. The process X is therefore not adapted to the natural filtration $\mathbb{F}$ of the insurance state process, and the insurer has to use a different approach.

Definition 2.1.10. *Prospective reserve*
The so called prospective reserve $X^{\mathbb{F}}$ is the stochastic process, pointwise defined by

$$X^{\mathbb{F}}_t = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right]$$

as the optional projection of X onto $\mathbb{F}$.
This process is by definition adapted to the filtration $\mathbb{F}$ and is used by the insurer rather than the payment process X itself.

The following theorem guarantees properties of the stochastic process of the prospective reserve, which would otherwise only be a pointwise definition. There are a few possible preconditions, that guarantee a 'good' version and let us work with the paths of $X^{\mathbb{F}}$. The one presented here aligns with the usage in life insurance and is also the version used in Christiansen [Chr21b].

Theorem 2.1.11. *Existence of optional projection*
Let $X = (X_t)_{t \geq 0}$ be càdlàg process satisfying

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |X_s|\right] < \infty ,$$

for each $t \geq 0$.
Further, let $\mathbb{F}$ be a filtration fulfilling the usual conditions, compare Definition 2.2.1.
Then there exist the unique and càdlàg optional projection $X^{\mathbb{F}}$ of X with respect to $\mathbb{F}$, such that $X_t^{\mathbb{F}} = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right]$ almost surely for each $t \geq 0$.

Proof. See for example Bain and Crisan [BC09]. □

Let us further assume that the accompanying compensators of the counting processes $(N_{ij})_{i \neq j}$ have Lebesgue densities $\left(\mathbb{I}_{t-}^i \lambda_{ij}(t)\right)_{(i,j) \in \mathcal{T}}$, which are predictable processes and fulfil the integrability condition

$$\mathbb{E}\left[\int\limits_{(0,T]} \sum_{(i,j) \in \mathcal{T}} \mathbb{I}_{t-}^i \lambda_{ij}(t)\,\mathrm{d}t\right] < \infty .$$

With respect to the filtration $\mathbb{F}$, the compensated counting processes

$$M_{ij}(t) = N_{ij}(t) - \int\limits_{(0,t]} \mathbb{I}_{s-}^i \, \lambda_{ij}(s)\,\mathrm{d}s$$

are square-integrable martingales. Compare to the formulation in Christiansen and Djehiche [CD20] for further details.

We can then rearrange the representation of process A to be

$$\mathrm{d}A(t) = \sum_{i \in \mathcal{S}} \mathbb{I}_{t-}^i \, g_i(t)\,\mathrm{d}t + \sum_{(i,j) \in \mathcal{T}} B_{ij}(t)\,\mathrm{d}M_{ij}(t) ,$$

where the second group of summands joins all the $\mathbb{F}$-martingales, and where the functions g_i are now given as

$$g_i(t) = b_i(t) + \sum_{\substack{j \in \mathcal{S} \\ j \neq i}} \mathbb{I}_{t-}^i \, \lambda_{ij}(t) .$$

This formula for g_i also enables us to formulate an alternative representation for the prospective reserve. By application of the conditional expectation with respect to $\mathcal{F}_t$, the martingale parts vanish and it holds, that

$$X_t^{\mathbb{F}} = \mathbb{E}\left[\int\limits_{(t,T]} \mathrm{d}A(s) \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\int\limits_{(t,T]} \sum_{i \in \mathcal{S}} \mathbb{I}_{s-}^i \, g_i(s)\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] .$$

The following theorem then holds for the prospective reserve.

Theorem 2.1.12. *Thiele equations*
The prospective reserve $X^{\mathbb{F}}$ (associated with payment process A) satisfies a Backward Stochastic Differential Equation (BSDE) of the form

$$\begin{cases} \mathrm{d}X_t^{\mathbb{F}} = -f(t,\, X_t^{\mathbb{F}})\,\mathrm{d}t + Z(t)\,\mathrm{d}M(t) \\ X_T^{\mathbb{F}} = 0 \end{cases} \tag{2.1.2}$$

for a predictable process $Z = (Z_{ij})_{(i,j)\in\mathcal{T}}$, such that

$$\mathbb{I}_i(t^-)\, Z_{ij}(t) = \mathbb{I}_i(t^-)\left(\mathbb{E}\left[X_t^{\mathbb{F}}\,\middle|\,\mathcal{F}_{t^-}, S_t = j\right] - \mathbb{E}\left[X_t^{\mathbb{F}}\,\middle|\,\mathcal{F}_{t^-}, S_t = i\right]\right) \tag{2.1.3}$$

almost surely for each $t \geq 0$ and $i, j \in \mathcal{T}$.

Proof. The theorem follows as a special case of the BSDE in [CD20] or [DL16].
Explicit representations of function f and further details on the martingale representation theorem are presented in both sources. Moreover, a reformulation of A and the proof to a more general BSDE of the prospective reserve with discounting and lump sum sojourn payments can be found in the provided literature. □

Comment 2.1.13. *On the Markov-assumption*
If the Markov-assumption is used as an assumption on the state process S, then the above BSDE may be simplified and one can derive the representations:

$$\begin{aligned} X_t^{\mathbb{F}} &= \mathbb{E}\left[X_t \mid S_t\right] \;\; a.s. \\ Z_{ij}(t) &= \mathbb{E}\left[X_t^{\mathbb{F}}\,\middle|\,S_t = j\right] - \mathbb{E}\left[X_t^{\mathbb{F}}\,\middle|\,S_t = i\right] \;\; a.s. \end{aligned}$$

If further b_i and B_{ij} are deterministic functions, the BSDE will simplify to the case of Thiele backward equations, that can be solved through backward recursion by starting with the known final values.

The Markov-assumption is a powerful and standard assumption used for these problems, especially for the convenience of calculations. If it does not actually hold, than the original BSDE would be correct to use and a shrinking of information takes place when using the as-if-Markov reserves, which may result in an error in the calculation of the reserves. See also Christiansen [Chr21a] for further information.

One of the main reasons that we have developed the BSDE formulation (2.1.2) for the simple insurance example is, that it is a special case of the following more general BSDE

$$\begin{cases} \mathrm{d}Y_t = f(t, Y_{t^-}, Z_t)\,\mathrm{d}t - Z_t\,\mathrm{d}M_t \\ Y_T = \xi \end{cases} \tag{2.1.4}$$

where M is an $\mathbb{F}$-martingale, ξ is a final-value condition, and f is a generator function. The solution (Y, Z) consists of a $\mathbb{F}$-adapted process Y and a family of predictable processes Z.

Usually in the theory of BSDE solutions, see for example Pham [Pha09] or Cohen and Elliott [CE15], standard assumptions like Lipschitz- and integrability-conditions are assumed for f. As we have already mentioned in the introductory chapter, there are many advances in the BSDE literature, for example Pardoux and Peng [PP90] or El-Karaoui, Hamadene, and Matoussi [EHM08], where the BSDEs are considered in more general environments and the existence and uniqueness of solutions is investigated.
We will come back to this after the introduction of the non-monotone theory, when we are able to explain why existing methods fail and why a different way must be chosen in this thesis.

2.2. A general model for information dynamics

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space, and let

$$\mathcal{Z} = \{A \subseteq \Omega \mid \exists B : A \subseteq B \text{ with } \mathbb{P}(B) = 0\} \subset \mathcal{A}$$

the set of all subsets of $\mathbb{P}$-null sets. Further we consider a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, fulfilling the usual conditions, compare the following Definition 2.2.1.

Definition 2.2.1. *Filtration*
The family $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of σ-algebras is called a filtration, if it is increasing, i.e. if we have

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{A} \quad \text{for all } 0 \leq s \leq t\,.$$

(1) Define $\mathbb{F}^+ = \left(\mathcal{F}_t^+\right)_{t\geq 0}$ where $\mathcal{F}_t^+ := \bigcap\limits_{t<s} \mathcal{F}_s$. A filtration is right-continuous, if $\mathbb{F} = \mathbb{F}^+$.

(2) A filtration is complete, if $\mathcal{Z} \subseteq \mathcal{F}_t$ for every $t \geq 0$.

The filtration is said to be fulfilling the usual conditions, if it is right-continuous and complete.

We want to refer to $\mathcal{F}_t$ as the total observable information on the interval $[0, t]$. The concept of a filtration directly corresponds to a monotone perspective, since the relation $\mathcal{F}_s \subseteq \mathcal{F}_t$ holds for all $s \leq t$.
To model the non-monotone perspective, we assume that a subset of the available information may expire after a finite time. By subtracting the expired information from $\mathcal{F}_t$ we get the so called admissible information. The admissible information can also be represented by a family of complete sigma-algebras $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ and for every $t \geq 0$ the condition

$$\mathcal{G}_t \subseteq \mathcal{F}_t$$

holds, but in general we do not have the same monotonicity for $\mathcal{G}$ as we have for $\mathcal{F}$.

Since we have to define the monotone, as well as the non-monotone, perspective, we will keep up the duality when introducing concepts in the remainder of this section.

Definition 2.2.2. *Adapted process*
A stochastic process $X = (X_t)_{t\geq 0}$ is

(a) adapted to the filtration $\mathbb{F}$, if X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$.

(b) adapted to the (possibly non-monotone) family $\mathbb{G}$, if X_t is $\mathcal{G}_t$-measurable for every $t \geq 0$.

The second definition alone is not very useful because of the nature of $\mathbb{G}$, and we will also need the following concept with an incremental perspective.

Definition 2.2.3. *Incremental σ-algebra*
For any interval $(s,t] \subset [0,\infty)$ (with $0 \leq s \leq t < \infty$) we define

$$\mathcal{G}_{(s,t]} := \sigma\big(\mathcal{G}_u,\ u \in (s,t]\,\big)$$

as the sigma-algebra, that contains all the admissible information on the interval $(s,t]$.

This motivates the new perspective on the concept of adaptivity.

Definition 2.2.4. *Incrementally adapted process*
A process X is incrementally adapted to $\mathbb{G}$, if the increments $X_t - X_s$ are $\mathcal{G}_{(s,t]}$ -measurable for any interval $(s,t] \subset [0,\infty)$.

Comment 2.2.5. *Equivalence of the concepts in case of a filtration*
Incremental adaptivity and (usual) adaptivity are equivalent, if $\mathbb{G}$ is a filtration:
Let $\mathbb{G}$ be a filtration. Then for any interval $(s,t] \subset [0,\infty)$ it holds that $\mathcal{G}_{(s,t]} = \mathcal{G}_t$, since $\mathcal{G}_u \subseteq \mathcal{G}_t$ for every $u \in (s,t]$.
If X is incrementally adapted to $\mathbb{G}$, then $X_t - X_0$ is $\mathcal{G}_t$-measurable. Further, X_0 is always $\mathcal{G}_0$-measurable, therefore X_0 is also $\mathcal{G}_t$-measurable. By additivity X_t is $\mathcal{G}_t$-measurable.
If X is adapted to $\mathbb{G}$, then both X_t, as well as X_s are $\mathcal{G}_t$-measurable, and therefore $X_t - X_s$ is $\mathcal{G}_t$-measurable for any interval $(s,t] \subset [0,\infty)$.

We will now introduce the necessary concepts for the development of the infinitesimal perspective on martingales.

Definition 2.2.6. *Martingale*
Let $X = (X_t)_{t\geq 0}$ be a stochastic process and $\mathbb{F}$ a filtration.
X called an $\mathbb{F}$-martingale, if

(1) X is integrable, i.e. $\mathbb{E}\,[|X_t|] < \infty$ for all $t \geq 0$,

(2) X is $\mathbb{F}$-adapted,

(3) $\mathbb{E}\,[X_t \,|\, \mathcal{F}_s] = X_s$ almost surely for all $t \geq s \geq 0$.

For reference see Protter [Pro05].

Definition 2.2.7. *Sequence of partitions*
Let $[s,t] \subseteq [0,\infty)$ be an interval. We define the set of suitable sequences of partitions of the interval $[s,t]$, denoted $\mathfrak{T}([s,t])$, as the set that contains all sequences of the form

$$\left(\tau_n^{[s,t]}\right)_{n\in\mathbb{N}},$$

fulfilling the following conditions:

(a) For every $n \in \mathbb{N}$, $\tau_n^{[s,t]}$ is a partition of the interval $[s,t]$, which means that $\tau_n^{[s,t]} = \{t_0, t_1, t_2, \dots, t_n\}$ with $s = t_0 < t_1 < t_2 < \cdots < t_n = t$.

(b) The partitions are increasing in $n \in \mathbb{N}$: $\tau_n \subset \tau_{n+1}$.

(c) The maximum grid length converges to zero, i.e.

$$\left|\tau_n^{[s,t]}\right| := \max_{k\in\mathbb{N}_n} \{t_k - t_{k-1}\} \xrightarrow{n\to\infty} 0\,.$$

For the special case $s = 0$ the notation is relaxed to $\tau_n^t := \tau_n^{[0,t]}$.

The classical martingale property reads

$$\mathbb{E}\left[X_t - X_s \,|\, \mathcal{F}_s\right] = 0 \ a.s. \text{ for all } t \geq s \geq 0 \tag{2.2.1}$$

and can be considered on a partition from Definition 2.2.7, which then is the infinitesimal equivalent of (2.2.1). We then have

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,|\, \mathcal{F}_{t_k}\right] = 0 \ a.s.$$

for all $t \geq 0$, which holds as every individual summand is already zero.

The idea of Christiansen [Chr21b] is, to use this a similar property as the definition of infinitesimal martingales:

Definition 2.2.8. IF- *and* IB-*martingales*
A process X is called an

(1) infinitesimal forward martingale (IF-martingale) with respect to $\mathbb{G}$, if

(1.1) X is incrementally adapted to $\mathbb{G}$,

(1.2) for $(\tau_n^t)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$ we almost surely have

$$\lim_{n\to\infty} \sum_{\tau_n} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,|\, \mathcal{G}_{t_k}\right] = 0\,.$$

(2) infinitesimal backward martingale (IB-martingale) with respect to $\mathbb{G}$, if

(1.1) X is incrementally adapted to $\mathbb{G}$,

(1.2) for $(\tau_n)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$ we almost surely have

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,\middle|\, \mathcal{G}_{t_{k+1}}\right] = 0\,.$$

Definition 2.2.9. *Compensator*
We call the unique process C the compensator of an $\mathbb{F}$-adapted and integrable counting process X, if

(1) C is $\mathbb{F}$-predictable, i.e. C_t is $\mathcal{F}_{t-}$-measurable for $t \geq 0$,

(2) C is of finite variation,

(3) $C_0 = 0$,

(4) $X - C$ is a martingale with respect to $\mathbb{F}$.

Furthermore, C almost surely fulfils the equation

$$C_t - C_0 = \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,\middle|\, \mathcal{F}_{t_k}\right] \tag{2.2.2}$$

for each $t \geq 0$.

Assertion 2.2.10.
$X - C$ is an IF-martingale with respect to $\mathbb{F}$.

Proof. The two conditions from Definition 2.2.8 have to be checked.
First, $X - C$ has to be incrementally adapted to $\mathbb{F}$. This is clear, as this is just the normal adaptivity, since $\mathbb{F}$ is a filtration.

The $\mathbb{F}$-predictability of C also implies that for a sequence $\left(\tau_n^{[s,t]}\right)_{n\in\mathbb{N}}$

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[C_{t_{k+1}} - C_{t_k} \,\middle|\, \mathcal{F}_{t_k}\right] = C_t - C_0\,.$$

By linearity, and as a combination of the equation (2.2.2) and the above, we have

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - C_{t_{k+1}} - (X_{t_k} - C_{t_k}) \,\middle|\, \mathcal{F}_{t_k}\right] = C_t - C_0 - (C_t - C_0) = 0 \;a.s.$$

for a sequence $\left(\tau_n^{[0,t]}\right)_{n\in\mathbb{N}}$ for each $t \geq 0$. ☐

Once again, the above concepts are now extended to the infinitesimal theory, where some preliminary definitions are given first.

Definition 2.2.11. *Infinitesimal predictable process*
A process X is called

(1) infinitesimally forward predictable (IF-predictable) with respect to $\mathbb{G}$, if for any $(\tau_n^t)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$, we almost surely have

$$\lim_{n\to\infty} \sum_{\tau_n} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,|\, \mathcal{G}_{t_k}\right] = X_t - X_0$$

given that expectations and limits exist.

(2) infinitesimally backward predictable (IB-predictable) with respect to $\mathbb{G}$, if for any $(\tau_n^t)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$, we almost surely have

$$\lim_{n\to\infty} \sum_{\tau_n} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,|\, \mathcal{G}_{t_{k+1}}\right] = X_t - X_0$$

given that expectations and limits exist.

Definition 2.2.12. *Infinitesimal compensator*
A process C is called an

(1) infinitesimal forward compensator of X (IF-compensator) with respect to $\mathbb{G}$, if

(1.1) C is IF-predictable with respect to $\mathbb{G}$

(1.2) $X - C$ is an IF-martingale with respect to $\mathbb{G}$.

(2) infinitesimal backward compensator of X (IB-compensator) with respect to $\mathbb{G}$, if

(2.1) C is IB-predictable with respect to $\mathbb{G}$

(2.2) $X - C$ is an IB-martingale with respect to $\mathbb{G}$.

Similarly to the optional projection with respect to filtration $\mathbb{F}$, we can define the same for $\mathbb{G}$, even if we do know anything about the existence or properties of this definition.

Definition 2.2.13. *Optional projection*
Let the process X be integrable and càdlàg.
If there exists a unique càdlàg process $X^{\mathbb{G}}$, such that

$$X_t^{\mathbb{G}} = \mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right]$$

holds almost surely for every $t \geq 0$, then we call $X^{\mathbb{G}}$ the optional projection of X with respect to $\mathbb{G}$.

Definition 2.2.14. *Infinitesimal martingale representation for optional projections*
A decomposition of the form

$$X_t^{\mathbb{G}} - X_0^{\mathbb{G}} = C_t + M_t^{\mathrm{F}} + M_t^{\mathrm{B}}$$

is called an infinitesimal martingale representation if M^{F} is an IF-martingale, M^{B} is an IB-martingale and C is an IB- or IF-compensator with respect to $\mathbb{G}$.

2.3. Marked point processes

When we introduced the life insurance mathematics in the earlier section of this chapter, we had already seen the use of jump processes. The continuous process S is a jump process in the state space $\mathcal{S}$ and also every accompanying process, like $\mathbb{I}_t^i$ or N_{ij} has been a jump process.

Christiansen [Chr21b] uses a marked point process structure, where we will later embed the life insurance mathematics into.

Definition 2.3.1. *Special marked point process*
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space. Let $(E, \mathcal{E})$ be a measurable mark space, where E is a Polish space (a separable complete metrizable topological space) and $\mathcal{E} := \mathcal{B}(E)$ its Borel sigma algebra.

Then we call $(\tau_i, \zeta_i, \sigma_i)_{i \in \mathbb{N}}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ a special marked point process, with

(1) $\tau_i : (\Omega, \mathcal{A}) \to ([0,\infty], \mathcal{B}([0,\infty]))$ for $i \in \mathbb{N}$ as random times, indicating events,

(2) $\zeta_i : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ for $i \in \mathbb{N}$ as random variables, giving the marks,

(3) $\sigma_i : (\Omega, \mathcal{A}) \to ([0,\infty], \mathcal{B}([0,\infty]))$ for $i \in \mathbb{N}$ as another set of random times fulfilling $\tau_i \leq \sigma_i$.

We interpret each ζ_i as information that can be observed on the interval $[\tau_i, \sigma_i)$. That means, information ζ_i may be deleted after a finite holding time, and the deletion time is given by σ_i. The ordering $\tau_i \leq \sigma_i$ is a natural condition, since the deletions can only happen after the information introduction.

Comment 2.3.2. *On the revealing of additional information*
This notation does not assume any ordering of the random times. Therefore the number of events (especially the number of deletions) shall not be extractable in any form, and this can be realised by a reordering of the random times. Compare the Remark 3.1 in Christiansen [Chr21b].

We introduce a more pleasant notation, where the triplets are split into doublets, with subsets on the odd and even indices, where the consecutively odd and even number correspond to the introduction and (possibly) deletion of the same information.

Definition 2.3.3. *Doublet representation of the special marked point process*
Consider the equivalent sequence $\mathcal{I} := (T_i, Z_i)_{i \in \mathbb{N}}$, defined by

$$\begin{aligned} T_{2i-1} &:= \tau_i, & T_{2i} &:= \sigma_i, \\ Z_{2i-1} &:= \zeta_i, & Z_{2i} &:= Z_{2i-1} = \zeta_i, \end{aligned}$$

for every $i \in \mathbb{N}$, which is a more convenient way to represent the information structure, with a different representation for the odd and even indices.

Assumption 2.3.4. *Information availability*
Assume, that for all $i \in \mathbb{N}$ it holds

$$T_{2i-1}(\omega) < T_{2i}(\omega) \quad \text{for } \omega \in \{T_{2i} < \infty\} \,. \tag{2.3.1}$$

Formula 2.3.1 guarantees, that information is available for at least a short time, if it is available at all.

Assumption 2.3.5. *Finite jumps on compact intervals*
Assume, that

$$\mathbb{E}\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \le t\}}\right] < \infty, \quad \text{for } t \ge 0\,. \tag{2.3.2}$$

Formula 2.3.2 guarantees that there are at most finitely many random times on bounded intervals $[0,t]$. Especially, the existence of jump clusters is prohibited by this assumption.

In difference to the usual definition of marked point processes (see for example [Jac05]), a special structure is needed here, to be able to model the introduction and the deletion. With that in mind, each two consecutive odd and even indices have to be considered as one, where a slight difference to the marked point processes (MPP), in usual literature arises.
Based on the sequence $\mathcal{I} = (T_i, Z_i)_{i\in\mathbb{N}}$ we will now generate a class of random counting measures, but we have to introduce some additional notation first.

Definition 2.3.6. *Subsets of information indices*
In unity with the information sequence $\mathcal{I}$, let

$$\mathcal{N} := \{N \subset \mathbb{N} \ : \ |N| < \infty\}$$

be the set of all finite subsets of the natural numbers and let

$$\mathcal{M} := \{M \subset \mathbb{N}^{\mathrm{O}} \ : \ |M| < \infty\}$$

be the set of all finite subsets of the odd natural numbers $\mathbb{N}^{\mathrm{O}}$.

The differentiation between odd and even indices lets us address specific information by $\mathcal{M}$, which will later be used for the information state indication processes, where we only need to know, if the current time is in between the introduction and deletion time points, and we can therefore restrict the subset to only the odd natural numbers. The definition of $\mathcal{N}$ is later used for information state transitions and it therefore is not sufficient to know if the information is available, but we need to know, of it is introduced or deleted.

For any $I \in \mathcal{N}$ define the product space $E_I := E^{|I|}$ and $\mathcal{E}_I := \mathcal{B}(E_I)$ as the Borel sets of E_I. Further, introduce $Z_I := (Z_i)_{i\in I}$ as a short form notation of the vector of information in I.

Definition 2.3.7. *Counting measures*
Let $t \geq 0$ and $I \in \mathcal{N}$. Define

$$\mu_I\left([0,t] \times B\right) := \mathbb{1}_{\{T_i = T_j \leq t,\, i,j \in I\} \cap \{T_i \neq T_j,\, i \in I, j \notin I\}}\, \mathbb{1}_{\{Z_I \in B\}} \tag{2.3.3}$$

for $B \in \mathcal{E}_I$.

For each $I \in \mathcal{N}$, the measures $\{\mu_I(\,\cdot\,)(\omega) \mid \omega \in \Omega\}$, generated by their values on $[0,t] \times B$, form a random counting measure on $\big([0,\infty) \times E_I, \mathcal{B}([0,\infty) \times E_I)\big)$, i.e.

(1) For any fixed $A \in \mathcal{B}\big([0,\infty) \times E_I\big)$ the mapping

$$\omega \longmapsto \mu_I(A)(\omega)$$

is measurable from $(\Omega, \mathcal{A})$ to $(\mathbb{N}_0^+, \mathcal{B}(\mathbb{N}_0^+))$, where $\mathbb{N}_0^+ = \mathbb{N}_0 \cup \{\infty\}$.

(2) For almost each $\omega \in \Omega$ the mapping

$$A \longmapsto \mu_I(A)(\omega)$$

is a locally finite measure on $\big([0,\infty) \times E_I,\, \mathcal{B}([0,\infty) \times E_I)\big)$.

Comment 2.3.8. *On the distribution of time points T_i*
A special case occurs, if the different random times $(T_i)_{i \in \mathbb{N}}$ never coincide. Then, we only need to consider $\mu_{\{i\}}$ for $i \in \mathbb{N}$ and the index conditions in the definition of μ can be simplified. Since we always want to be able to allow for mass-deletion, we will not further focus on the simplified case.

We can now also express the observable and admissible information through the marked point process $\mathcal{I}$.

Definition 2.3.9. *Sigma algebras of observable and admissible information*
The observable information at time $t \geq 0$ is given by the complete filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, with

$$\mathcal{F}_t := \sigma\left(\{T_{2i-1} \leq s < T_{2i}\} \cap \{Z_{2i} \in B\} \,|\, s \in [0,t], B \in \mathcal{E}, i \in \mathbb{N}\right) \vee \mathcal{Z}\,,$$

letting the random times $T_i, i \in \mathbb{N}$ be stopping times.

The admissible information at time $t \geq 0$ is given by the family $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ of sub-sigma-algebras

$$\mathcal{G}_t := \sigma\left(\{T_{2i-1} \leq t < T_{2i}\} \cap \{Z_{2i} \in B\} \,|\, B \in \mathcal{E}, i \in \mathbb{N}\right) \vee \mathcal{Z}\,,$$

and the admissible information immediately before time $t > 0$ is given by the family $\mathbb{G}^- = (\mathcal{G}_t^-)_{t \geq 0}$

$$\mathcal{G}_t^- := \sigma\left(\{T_{2i-1} < t \leq T_{2i}\} \cap \{Z_{2i} \in B\} \,|\, B \in \mathcal{E}, i \in \mathbb{N}\right) \vee \mathcal{Z}\,,$$

where the operator „$\vee$“ denotes the sigma algebra, generated by unions of involved sets, see also A.2.6.

Proposition 2.3.10. *Alternative representation of the admissible information*
Suppose that $0 \notin E$. By defining the càdlàg process

$$\Gamma_t := \left(Z_{2i}\, \mathbb{1}_{\{T_{2i-1} \leq t < T_{2i}\}} \right)_{i \in \mathbb{N}}$$

the information $\mathcal{G}_t$ and $\mathcal{G}_t^-$ can be alternatively represented as

$$\begin{aligned} \mathcal{G}_t &= \sigma(\Gamma_t) \vee \mathcal{Z}, \quad t \geq 0 \\ \mathcal{G}_t^- &= \sigma(\Gamma_{t^-}) \vee \mathcal{Z}, \quad t > 0\,. \end{aligned}$$

Take note, that the process $(\Gamma_t)_t$ is infinite-dimensional and the left limit Γ_{t^-} is defined component-wise. Note that $\mathcal{G}_t^-$ usually differs from $\mathcal{G}_{t-}$ as the left side limit of $\mathcal{G}_t$. A counterexample can be found in Christiansen [Chr21b].

2.4. Optional projections

With the newly introduced information structures, one of the main concerns is the existence of optional projections, because of the important connection to the prospective reserves, as we have seen in the introduction.

Theorem 2.4.1. *Existence of optional projections*
Suppose that $X = (X_t)_{t \geq 0}$ is a càdlàg process and uniformly bounded, i.e. it holds

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |X_s| \right] < \infty \text{ , for } t \geq 0\,. \tag{2.4.1}$$

Then the optional projection $X_t^{\mathbb{G}}$ exists and we have

$$X_t^{\mathbb{G}} = \mathbb{E}\left[X_t \,|\, \mathcal{G}_t \right]$$

almost surely as well as

$$X_{t^-}^{\mathbb{G}} = \mathbb{E}\left[X_{t^-} \,\middle|\, \mathcal{G}_t^- \right]$$

almost surely for each $t > 0$.
If additionally X has integrable variation on compacts, then $X^{\mathbb{G}}$ has paths of finite variation on compacts.

Furthermore, there always exists a unique (here understood as uniqueness up to evanescence) càdlàg process $X^{\mathbb{F}}$, such that

$$X_t^{\mathbb{F}} = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t \right]$$

almost surely holds for each $t \geq 0$.

Proof. See Theorem 4.1 of [Chr21b] for the result with respect to $\mathbb{G}$. The second part (i.e. the case with filtration $\mathbb{F}$) is a repetition of the theorem in the introductory part, see Theorem 2.1.11. □

Let us continue by introducing some further notation.

Definition 2.4.2. *Jump times and jump information*
For $I \in \mathcal{N}$ we define the corresponding jump time to a set $I \in \mathcal{N}$

$$Q_I := \sup \{t \geq 0 \mid \mu_I([0,t] \times E_I) = 0\}$$

as the first time, where the random times T_i for $i \in I$ all coincide and equal Q_I and no other random times occur at the exact same moment, i.e. $T_j \neq Q_I$ for every $j \notin I$. The Q_I only depend on the random times $(T_i)_{i \in I}$, but not on the information (Z_I), since $Z_I \in E_I$ is always fulfilled.

For a more compact notation, we also define

$$R_I := (Q_I, Z_I) = (Q_I, (Z_i)_{i \in I}) \tag{2.4.2}$$

for $I \in \mathcal{N}$ as the tuple of jump time and the vector of information.

In Christiansen [Chr21b] it is now argued, that all the regular conditional probabilities of the form

$$\mathbb{P}(\,\cdot \mid Z_M)$$

as well as

$$\mathbb{P}(\,\cdot \mid Z_M,\, R_I)$$

exist on $(\Omega, \mathcal{A})$ for each $M \in \mathcal{M}$ and $I \in \mathcal{N}$ and that they are simultaneously unique up to a joint exception null set. We will not get into the details of the existence of the regular conditional expectations and refer to the literature, for example Klenke [Kle20], where the existence via stochastic kernels is argued for Polish spaces.

For a more compact notation, the conditional will from now on be written as an index, i.e.

$$\mathbb{P}_{M,R_I}(\,\cdot\,) := \mathbb{P}(\,\cdot \mid Z_M,\, R_I)$$

refers to an arbitrary but fixed regular version of the conditional probability. A similar construction is used for the conditional expectations, where for any integrable random variable X the expression

$$\mathbb{E}_{M,R_I}[X] := \int X \,\mathrm{d}\mathbb{P}_{M,R_I}$$

such that $\mathbb{E}_{M,R_I}[X]$ is the specific version of the conditional expectation $\mathbb{E}[Z \mid Z_M, R_I]$ that is obtained by integrating X with respect to the specific regular version that is picked for $\mathbb{P}(\cdot \mid Z_M,\, R_I)$.

The short forms $\mathbb{P}_M = \mathbb{P}_{M,R_\emptyset}$ and $\mathbb{E}_M = \mathbb{E}_{M,R_\emptyset}$ are used, if $I = \emptyset$ since $\mathbb{P}_{M,R_\emptyset}$ is a version of $\mathbb{P}(\,\cdot \mid Z_M)$. In addition to that, we need the set of all previous indices for indices in I, given as $I^- := \{i - 1 \mid i \in I\}$ and define the mappings

$$\mathbb{P}_{M,R_I=r}(\,\cdot\,) := \mathbb{P}(\,\cdot \mid Z_{M_I} = z,\ R_I = r)\Big|_{z=Z_{M_I}}$$

with $M_I := M \setminus (I \cup I^-)$, referring to arbitrary but fixed regular versions of the factorized conditional expectations on the right hand side. The definition guarantees, that no indices are arising twice in the condition, when M is reduced to M_I that everything already covered by R_I is left out in Z_M.
For any integrable random variable X we define

$$\mathbb{E}_{M,R_I=r}[X] := \int X \,\mathrm{d}\mathbb{P}_{M,R_I=r} \,.$$

The mapping $\mathbb{P}_{M,R_I=r}(\,\cdot\,)\big|_{r=R_I}$ then equals $\mathbb{P}_{M,R_I}(\,\cdot\,)$.

Definition 2.4.3. *Indication process for sets of admissible information*
For $M \in \mathcal{M}$ and $t \geq 0$, define $\mathcal{G}_t$-measurable sets

$$\begin{aligned} A_t^M &:= \bigcap_{i \in M} \{T_i \leq t < T_{i+1}\} \cap \bigcap_{i \in \mathcal{M} \setminus M} \{T_i \leq t < T_{i+1}\}^{\mathsf{c}} \\ &= \bigcap_{i \in M} \{T_i \leq t < T_{i+1}\} \cap \bigcap_{i \in \mathcal{M} \setminus M} (\Omega \setminus \{T_i \leq t < T_{i+1}\}) \end{aligned}$$

and the stochastic process $\mathbb{I}^M = (\mathbb{I}_t^M)_{t \geq 0}$ by setting

$$\mathbb{I}_t^M := \mathbb{1}_{A_t^M}, \quad t \geq 0$$

for every $t \geq 0$ as the indication function for the corresponding set.

The definition of the indicator can be understood as a indication of the information, that is available, and is therefore somewhat similar to the state indication process, that we have defined in the standard life insurance model. For every odd natural number, the condition guarantees, that Z_M is the only available information, if the indicator equals one. Every other piece of information has been already deleted, or has not been introduced until then.

Remark 2.4.4. *Properties of the indication process*
The indication process, as defined in Definition 2.4.3, has the following properties:

(1) The process $\mathbb{I}_M$ is $\mathbb{G}$-adapted, which is directly translated from the $\mathcal{G}_t$-measurability of the corresponding set A_t^M.

(2) The paths of $\mathbb{I}_t^M$ have finitely many jumps on compacts only, which is a direct consequence of Assumption 2.3.2, and guarantees the existence of left and right limits. More accurately, the processes can have a most 2 jump, from 0 to 1, once the set M is the active set, and back to 0, once another random time T_i is happening.

(3) The processes are càdlàg. They already are right-continuous by construction and the left limits can be represented as $\mathbb{I}_{t-}^M = \mathbb{1}_{A_{t-}^M}$ where

$$A_{t-}^M := \bigcap_{i \in M} \{T_i < t \leq T_{i+1}\} \cap \bigcap_{i \in \mathcal{M} \setminus M} \{T_i < t \leq T_{i+1}\}^{\mathsf{c}}$$

Let us also formulate a result on the interplay between some sets $I \in \mathcal{N}$ and $M \in \mathcal{M}$.

Definition 2.4.5. *Previous and subsequent indices*
For a set $A \subset \mathbb{N}$ define $A^- := \{i - 1 \mid i \in A\}$ as the set of previous indices and define $A^+ := \{i + 1 \mid i \in A\}$ as the set of subsequent indices.

Theorem 2.4.6. *Relationships between available information and transition indices*
The following two statement are true

(1) Let a set $M_1 \in \mathcal{M}$ be given. Let further a time point $s > 0$ and a set $I \in \mathcal{N}$ be given, such that the following conditions are fulfilled:

   (i) The time point s is the corresponding jump time for I, i.e. $Q_I = s$.

   (ii) The information from set M_1 was available pre jump, i.e. $\mathbb{I}_{s^-}^{M_1} = 1$.

   (iii) Only available information can be deleted, i.e. $(I^E)^- \subseteq M_1$.

   (iv) No information is introduced, if it is already available, i.e. $I^O \cap M_1 = \emptyset$.

   Then we can construct the set $M_2 \in \mathcal{M}$, that corresponds to the information available in s, as
$$M_2 = \left(M_1 \setminus (I^E)^-\right) \cup I^O$$
   and it holds that $I_s^{M_2} = I_{Q_I}^{M_2} = 1$.

(2) If $M_1 \neq M_2 \in \mathcal{M}$ are given and there exists an $s > 0$, such that $\mathbb{I}_{s^-}^{M_1} = 1$ and $\mathbb{I}_s^{M_2} = 1$, then the set $I \in \mathcal{N}$ with $Q_I = s$ can be constructed as
$$I = \left(M_2 \setminus M_1\right) \cup \left(M_1 \setminus M_2\right)^+$$

Proof. The two statements of the theorem are proven one after the other:

(1) Let $M_1 \in \mathcal{M}$ and $I \in \mathcal{N}$ be given with $Q_I = s$ and $\mathbb{I}_{Q_I^-}^{M_1} = 1$ according to the preconditions specified above.

   We look at the deletions and introductions of information separately:

   If these conditions are fulfilled, we know from the precondition, that $T_i = Q_I$ for every $i \in I^E$, and $T_j < Q_I \leq T_{j+1}$ for every $j \in M_1$. Therefore, $T_{j+1} = Q_I$ for every index in both sets. The corresponding information is deleted and remaining are all indices from $M_1 \setminus \left(I^E\right)^-$).

   For the new introduction we have that every $i \in I^O$ is an odd index and guarantees, that $T_i = Q_I$. Therefore also $T_i \leq Q_I < T_{i+1}$ (recall that there are no instant deletions) and the index i is part of the set M_2, where $\mathbb{I}_{Q_I}^{M_2} = 1$.

   Together, the new set M_2 is the union of the indices from these two cases and we get
$$M_2 = \left(M_1 \setminus \left(I^E\right)^-\right) \cup I^O .$$

(2) Let $\mathbb{I}_{s^-}^{M_1} = 1$ and $\mathbb{I}_s^{M_2} = 1$ for a $s > 0$. Then there are three possibilities for an index $j \in M_1 \cup M_2$:

(2.1) $j \in M_1$ and $j \in M_2$, i.e. $T_j < s < T_{j+1}$ by combination of the indicator processes, i.e the information was available before and is still available. The index j does not appear in I.

(2.2) $j \in M_1$ and $j \notin M_2$, i.e. $T_j < s \leq T_{j+1}$ and $s \geq T_{j+1}$ by the definition of the indicator processes.

This means then information Z_j is deleted, since by combining the inequalities, we have $T_{j+1} = s$. The index $j+1$ has to be added to the set I.

(2.3) $j \notin M_1$ and $j \in M_2$, i.e. $T_j \leq s < T_{j+1}$ and $s \leq T_j$ by the definition of the indicator processes.

This does mean, that information Z_j is introduced in time s and the index j has to be added to the set I.

As a combination of the above cases, we need to construct I as the indices from $M_2 \setminus M_1$, to consider those that are introduced, but not already there (cases (1) and (3)) and also the indices $(M_1 \setminus M_2)^+$, to consider those that are deleted (case (2)). We use the subsequent indices for the deletions.

In total we get

$$I = (M_2 \setminus M_1) \cup (M_1 \setminus M_2)^+ \,,$$

as stated in the assertion.

□

Comment 2.4.7. *Differences between state-indicators and information-indicators*
We are focussing on the information and the jump processes $(\mu_I)_I$ do note take into account what information has been available right before. This is different to the transition counters $(N_{ij})_{(i,j)\in\mathcal{T}}$, where the original and the future state are relevant. The above theorem gives a closer connection to this standard practice, if indicators are considered together with μ_I, since the original information is then given and enabling us to calculate the future information through part (1) of the theorem.

The following proposition provides a decomposition of the abstract conditional expectations.

Proposition 2.4.8. *Reformulation of conditional expectations I*
For any integrable random variable ξ and any sets $M \in \mathcal{M}$ and $I \in \mathcal{N}$ we almost surely have

$$\begin{aligned} \mathbb{I}_t^M \, \mathbb{E}\left[\xi \,|\, \mathcal{G}_t \vee \sigma(R_I)\right] &= \mathbb{I}_t^M \, \frac{\mathbb{E}_{M,R_I}\left[\mathbb{I}_t^M \, \xi\right]}{\mathbb{E}_{M,R_I}\left[\mathbb{I}_t^M\right]} \\ \mathbb{I}_{t^-}^M \, \mathbb{E}\left[\xi \,\middle|\, \mathcal{G}_t^- \vee \sigma(R_I)\right] &= \mathbb{I}_{t^-}^M \, \frac{\mathbb{E}_{M,R_I}\left[\mathbb{I}_{t^-}^M \, \xi\right]}{\mathbb{E}_{M,R_I}\left[\mathbb{I}_{t^-}^M\right]} \end{aligned} \tag{2.4.3}$$

under the convention that $\frac{0}{0} := 0$. The right hand sides are then indeed well defined, since whenever the denominator is zero, the numerator is zero as well.
Further, note that $\sigma(R_I)$ equals the trivial sigma-algebra if $I = \emptyset$.

Proof. See Proposition 4.2 in [Chr21b]. □

We also need a slightly different version of this proposition, which does not take into account the information about a change of the marked point process, but uses, that no stopping event is happening. We therefore first define the following indicator process.

Definition 2.4.9. *Stopping event indicator*
Define the process $\mathcal{J} = \left(\mathcal{J}_t\right)_{t\geq 0}$ as the collection of random variables

$$\mathcal{J}_t := \sum_{I\in\mathcal{N}} \mu_I\left(\{t\} \times E_I\right) \tag{2.4.4}$$

for $t \geq 0$.
For every t, this is an indication function for any stopping event happening at time t.

Proposition 2.4.10. *Reformulation of conditional expectations II*
For any integrable random variable ξ and any sets $M \in \mathcal{M}$ we almost surely have

$$\begin{aligned}\mathbb{I}_t^M\,\mathbb{E}\left[\xi\,|\,\mathcal{G}_t,\,\mathcal{J}_t = 0\right] &= \mathbb{I}_t^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\right]}\\ \mathbb{I}_{t-}^M\,\mathbb{E}\left[\xi\,\middle|\,\mathcal{G}_t^-,\,\mathcal{J}_t = 0\right] &= \mathbb{I}_{t-}^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\right]}\end{aligned} \tag{2.4.5}$$

under the convention that $\frac{0}{0} := 0$, which again guarantees for the right hand sides to be well defined.

Proof. Since this version is not explicitly stated in [Chr21b], we perform the proof, but only for the first equation and only with the non-completed versions of the sigma-algebras on the left hand side.

As a first step, note that

$$\mathbb{I}_{t-}^M\,\mathbb{I}_t^M = \mathbb{I}_t^M\,\mathbb{1}_{\{\mathcal{J}_t=0\}}$$

since no jump can have happened in t, if the left site equals 1.

Then, start with the right hand site of the equation, that is to show. By application of (2.4.3) in both nominator and denominator and because of $\mathbb{I}_t^M = (\mathbb{I}_t^M)^2$, we get

$$\mathbb{I}_t^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_{t-}^M\,\mathbb{I}_t^M\right]} = \mathbb{I}_t^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_t^M\,\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\,\mathbb{1}_{\{\mathcal{J}_t=0\}}\right]} = \frac{\mathbb{I}_t^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_t^M\,\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]}}{\mathbb{I}_t^M\,\frac{\mathbb{E}_M\left[\mathbb{I}_t^M\,\mathbb{1}_{\{\mathcal{J}_t=0\}}\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]}}$$

$$\overset{2.4.8}{=} \frac{\mathbb{I}_t^M \,\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi \,\middle|\, \mathcal{G}_t\right]}{\mathbb{I}_t^M \,\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]} = \mathbb{I}_t^M \frac{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]}$$

almost surely. Whenever we have $\mathbb{E}_M[\mathbb{I}_t^M] = 0$, we also have $\mathbb{E}_M[\mathbb{I}_t^M \,\mathbb{1}_{\{\mathcal{J}_t=0\}}] = 0$ and $\mathbb{E}_M[\mathbb{I}_t^M \,\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi] = 0$, so the expansion of the fraction can be done with the convention $\frac{0}{0} := 0$. Now, we need that

$$\mathcal{G}_t \cap \{\mathcal{J}_t = k\} = (\mathcal{G}_t \vee \sigma(\mathcal{J}_t)) \cap \{\mathcal{J}_t = k\} \subseteq (\mathcal{G}_t \vee \sigma(\mathcal{J}_t))$$

for every $t \geq 0$ and $k \in \{0,\, 1\}$.

This implies, that for any $G \in \mathcal{G}_t$ we get

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}} \frac{\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}} \frac{\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]} \,\middle|\, \mathcal{G}_t\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_G \frac{\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]} \,\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_G \,\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}}\,\xi \,\middle|\, \mathcal{G}_t\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}}\,\xi\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}}\,\xi \,\middle|\, \mathcal{G}_t \vee \sigma(\mathcal{J}_t)\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_G \,\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\mathbb{E}\left[\xi \,|\, \mathcal{G}_t \vee \sigma(\mathcal{J}_t)\right]\right]
\end{aligned}$$

almost surely. Because of the theorem of Radon-Nikodym and the definition of the conditional expectation, we can follow that

$$\mathbb{1}_{\{\mathcal{J}_t=k\}} \frac{\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=k\}} \,\middle|\, \mathcal{G}_t\right]} = \mathbb{1}_{\{\mathcal{J}_t=k\}} \,\mathbb{E}\left[\xi \,|\, \mathcal{G}_t \vee \sigma(\mathcal{J}_t)\right]$$

almost surely, which means that

$$\frac{\mathbb{E}\left[\xi\,\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]}$$

is a version of

$$\mathbb{E}\left[\xi \,|\, \mathcal{G}_t,\, \mathcal{J}_t = 0\right] .$$

A combination of both equations gives the assertion, since we then have

$$\mathbb{I}_t^M \frac{\mathbb{E}_M\left[\mathbb{I}_{t-}^M \,\mathbb{I}_t^M\,\xi\right]}{\mathbb{E}_M\left[\mathbb{I}_{t-}^M \,\mathbb{I}_t^M\right]} = \mathbb{I}_t^M \frac{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}}\,\xi \,\middle|\, \mathcal{G}_t\right]}{\mathbb{E}\left[\mathbb{1}_{\{\mathcal{J}_t=0\}} \,\middle|\, \mathcal{G}_t\right]} = \mathbb{I}_t^M \,\mathbb{E}\left[\xi \,|\, \mathcal{G}_t,\, \mathcal{J}_t = 0\right]$$

almost surely for every $M \in \mathcal{M}$ and $t \geq 0$. □

Lemma 2.4.11. *Properties of the conditional expectation*
Let $M \in \mathcal{M}$, $I \in \mathcal{N}$ and $r \in [0,\infty) \times E_I$.
For each càdlàg process X that satisfies condition (2.3.2), the stochastic processes, defined by

$$t \mapsto \mathbb{E}_{M,R_I}\left[\mathbb{I}_t^M X_t\right]$$
$$t \mapsto \mathbb{E}_{M,R_I=r}\left[\mathbb{I}_t^M X_t\right]$$

have càdlàg paths.
Moreover, their left limits can be obtained by replacing $\mathbb{I}_t^M X_t$ by $\mathbb{I}_{t-}^M X_{t^-}$.

Proof. See Lemma 4.3 in [Chr21b]. The proof is not stated explicitly and only the idea is given.

□

Proposition 2.4.12. *Finite supremum of the standardized indicator*
Under the conventions $\frac{0}{0} := 0$ and $\frac{1}{0} := +\infty$, for each $M \in \mathcal{M}$ we almost surely have

$$\sup_{t\in[0,\infty)} \frac{\mathbb{I}_t^M}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]} < \infty\,.$$

Proof. See Proposition 4.4 in [Chr21b].

□

Theorem 2.4.1, about the existence of the projection with respect to $\mathbb{G}$ and its properties, can now be proven.

Proof. The proof is given in [Chr21b], but since it is constructive, the candidate for the optional projection is important and repeated here, but we will dispense with the proof.

Define

$$Y_t := \sum_{M\in\mathcal{M}} \mathbb{I}_t^M \frac{\mathbb{E}_M\left[\mathbb{I}_t^M X_t\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]}\,, \qquad t \geq 0$$

where at most a countable number of conditional expectations are involved, and the corresponding regular versions are simultaneously unique up to evanescence.

The process $X_t^{\mathbb{G}}$ is then indeed almost surely equal to the process Y_t. It holds

$$Y_t = \sum_{M\in\mathcal{M}} \mathbb{I}_t^M \frac{\mathbb{E}_M\left[\mathbb{I}_t^M X_t\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]} = \sum_{M\in\mathcal{M}} \mathbb{I}_t^M\, \mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right] = \mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right] \tag{2.4.6}$$

almost surely, using equation (2.4.3).

For the whole proof, and especially the finite variation on compacts, see the proof of Theorem 4.1 in [Chr21b]. □

Comment 2.4.13. *On the decomposition of the optional projection*
The fractions

$$\frac{\mathbb{E}_M\left[\mathbb{I}_t^M X_t\right]}{\mathbb{E}_M\left[\mathbb{I}_t^M\right]}$$

are types of state-wise reserves and have been named non-classic state-wise reserves in the PhD dissertation [Fur20], where the paper [CF21] is a part of.
They are considered next to classical state-wise reserves and an application in Danish retirement (with observation of health status upon retirement) is discussed.

2.5. Infinitesimal theory for marked point processes

In this section we recap the results of [Chr21b] about the existence and explicit representations of infinitesimal compensators for a large class of incrementally adapted jump processes, including the counting process of the form

$$t \longmapsto \mu_I([0,t] \times B)$$

for $I \in \mathcal{N}$ and $B \in \mathcal{E}_I$.

Definition 2.5.1. *Infinitesimal compensator*
Continuing to use the convention $\frac{0}{0} := 0$, define

$$\nu_I\left([0,t] \times B\right) := \sum_{M \in \mathcal{M}} \int_{(0,t] \times B} \mathbb{I}_{u^-}^M \frac{\mathbb{P}_{M,R_I=(u,e)}\left(A_{u^-}^M\right)}{\mathbb{P}_M\left(A_{u^-}^M\right)} \mathbb{P}_M^{R_I}\left(\mathrm{d}(u,e)\right)$$

$$\rho_I\left([0,t] \times B\right) := \sum_{M \in \mathcal{M}} \int_{(0,t] \times B} \mathbb{I}_{u}^M \frac{\mathbb{P}_{M,R_I=(u,e)}\left(A_{u}^M\right)}{\mathbb{P}_M\left(A_{u}^M\right)} \mathbb{P}_M^{R_I}\left(\mathrm{d}(u,e)\right)$$

for $t \geq 0$, $B \in \mathcal{E}_I$ and $I \in \mathcal{N}$.

Proposition 2.5.2. *Extension to random measures*
For each $I \in \mathcal{N}$, the mappings ν_I and ρ_I can be uniquely extended to random measures on $([0,\infty) \times E_I, \mathcal{B}([0,\infty) \times E_I))$.

Proof. See the proof of Proposition 5.1 in [Chr21b]. □

Definition 2.5.3. *Short form notation*
Introduce the following short notation

$$F \bullet \mu((0,t] \times B) := \int_{(0,t] \times B} F(u,e)\, \mu(\mathrm{d}(u,e))$$

for any random measure μ and an integrable random function F.

Let us now state the main theorem of this section, where the compensators in context of the special marked point processes are given:

Theorem 2.5.4. IF- *and* IB-*compensator for the special MPPs*
For every $I \in \mathcal{N}$, let the mapping

$$(t, e, \omega) \longmapsto F_I(t, e)(\omega)$$

be jointly measurable and let

$$\mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(0,t]\times E_I} |F_I(u, e)| \, \mu_I(\mathrm{d}(u, e))\right] < \infty \,. \tag{2.5.1}$$

(1) If $F_I(t, e)$ is $\mathcal{G}_t^-$-measurable for each (t, e), then for each $B \in \mathcal{E}_I$ the jump process

$$t \longmapsto F_I \bullet \mu_I((0, t] \times B)$$

has the IF-compensator

$$t \longmapsto F_I \bullet \nu_I((0, t] \times B)\,.$$

(2) If $F_I(t, e)$ is $\mathcal{G}_t$-measurable for each (t, e), then for each $B \in \mathcal{E}_I$ the jump process

$$t \longmapsto F_I \bullet \mu_I((0, t] \times B)$$

has the IB-compensator

$$t \longmapsto F_I \bullet \rho_I((0, t] \times B)\,.$$

By choosing $F_I = 1$ the above statement yields that in particular ν_I is the IF-compensator of μ_I and that ρ_I is the IB-compensator of μ_I.

The proof of this theorem now follows in several steps. We start with an auxiliary lemma, where the existence of the integrals in the representation of the compensators is guaranteed, by only focussing on the integrator.

Lemma 2.5.5. *Finiteness of the compensator integrals*
For every $M \in \mathcal{M}$ and $t \geq 0$ we almost surely have

$$\sum_{I\in\mathcal{N}} \int_{[0,t]\times E_I} \mathbb{P}_M^{R_I}\,(\mathrm{d}(u, e)) < \infty\,. \tag{2.5.2}$$

Proof. We have that

$$\begin{aligned}\sum_{I\in\mathcal{N}} \int_{[0,t]\times E_I} \mathbb{P}_M^{R_I}\,(\mathrm{d}(u, e)) &= \sum_{I\in\mathcal{N}} \int_{[0,t]\times E_I} \mathbb{P}\left((Q_I, Z_I) \in \mathrm{d}(u, e) \,|\, Z_M\right) \\ &= \sum_{I\in\mathcal{N}} \mathbb{P}\left(Q_I \in [0, t],\, Z_I \in E_I \,|\, Z_M\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\mathbb{1}_{\{Q_I\in[0,t]\}}\,\mathbb{1}_{\{Z_I\in E_I\}}\,\middle|\,Z_M\right] \\
&\overset{(i)}{=} \sum_{I\in\mathcal{N}} \mathbb{E}\left[\mu_I([0,t]\times E_I)\,|\,Z_M\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}_M[\mu_I([0,t]\times E_I)] \\
&\overset{(ii)}{=} \mathbb{E}_M\left[\sum_{I\in\mathcal{N}} \mu_I([0,t]\times E_I)\right] \\
&\overset{(iii)}{\leq} \mathbb{E}_M\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{T_i\leq t\}}\right] \\
&\overset{(iv)}{<} \infty
\end{aligned}
$$

almost surely holds for each $M\in\mathcal{M}$ and $t\geq 0$.

We are using the definition of $R_I=(Q_I,Z_I)$, and rewrite the conditional distribution according to its definition. In step (i), the definition of Q_I is used and is lead back to the counting process μ_I. In step (ii) the monotone convergence theorem is applied. In step (iii), it is used that the definition of μ_I needs $(T_i)_{i\in I}$ to coincide and fulfil $T_i\leq t$. The total number of summands can be increased by not having the restriction about coincidence. In case of pairwise non-coinciding $T_i\leq t$ (for $i\in I$) , the number of summands is the same. The last step (iv) is a direct implication of Assumption 2.3.2. □

A few other steps are necessary to finally do the proof to Theorem 2.5.4. We will also recap the intermediate steps, since they are useful later on.

Proposition 2.5.6. *Explicit formulas for IF- and IB-compensators*
Let the preconditions on measurability and integrability of Theorem 2.5.4 be true.
Then for each $t>0$ and $B\in\mathcal{E}_I$ we almost surely have

$$
\lim_{n\to\infty}\sum_{\tau_n^t}\mathbb{E}\left[F_I\bullet\mu_I((t_k,t_{k+1}]\times B)\,|\,\mathcal{G}_{t_k}\right]=G_I\bullet\nu_I\left((0,t]\times B\right) \tag{2.5.3}
$$

for $(\tau_n)_{n\in\mathbb{N}}\in\mathfrak{T}([0,t])$, with

$$
G_I(u,e):=\sum_{M\in\mathcal{M}}\mathbb{I}^M_{u^-}\,\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}^M_{u^-}\,F_I(u,e)\right]}{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}^M_{u^-}\right]}
$$

and similarly we almost surely have

$$
\lim_{n\to\infty}\sum_{\tau_n^t}\mathbb{E}\left[F_I\bullet\mu_I\left((t_k,t_{k+1}]\times B\right)|\,\mathcal{G}_{t_{k+1}}\right]=B_I\bullet\rho_I\left((0,t]\times B\right) \tag{2.5.4}
$$

for $(\tau_n)_{n\in\mathbb{N}}\in\mathfrak{T}([0,t])$, with

$$
B_I(u,e):=\sum_{M\in\mathcal{M}}\mathbb{I}^M_{u}\,\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}^M_{u}\,F_I(u,e)\right]}{\mathbb{E}_{M,R_I=(u,e)}[\mathbb{I}^M_{u}]}\,.
$$

Proof. See Proposition 5.4 in [Chr21b].

□

Proposition 2.5.7. *Infinitesimal predictability of* IF- *and* IB-*compensators*
Given the preconditions of Theorem 2.5.4, for each $t > 0$ and $B \in \mathcal{E}_I$ we almost surely have

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[G_I \bullet \nu_I((t_k, t_{k+1}] \times B \,|\, \mathcal{G}_{t_k}\right] = G_I \bullet \nu_I((0,t] \times B) \tag{2.5.5}$$

as well as

$$\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[B_I \bullet \rho_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_{k+1}}\right] = B_I \bullet \rho_I((0,t] \times B) \tag{2.5.6}$$

for any suitable increasing sequence of partitions $(\tau_n)_{n\in\mathbb{N}} \in \mathfrak{T}([0,T])$.

Proof. See Proposition 5.5 in [Chr21b].

□

Although the proposition has the purpose to help to show Theorem 2.5.4 and therefore uses the special structure of the G_I, it is not necessary to assume it. The argument is rather, that the special G_I is $\mathbb{G}^-$ adapted, which could also be a direct precondition. We will make use of this and formulate an additional corollary.

Corollary 2.5.8. *Zero expectation of* IF- *and* IB-*martingales*
Let the preconditions of Theorem 2.5.4 be fulfilled.

(1) If $F_I(t,e)$ is $\mathcal{G}_t^-$ -measurable for each (t,e), then for each $B \in \mathcal{E}_I$ and each $t \geq 0$ we have

$$\mathbb{E}\left[F_I \bullet \mu_I\left((0,t] \times B\right)\right] = \mathbb{E}\left[F_I \bullet \nu_I\left((0,t] \times B\right)\right] .$$

(2) If $F_I(t,e)$ is $\mathcal{G}_t$ -measurable for each (t,e), then for each $B \in \mathcal{E}_I$ and for each $t \geq 0$ we have

$$\mathbb{E}\left[F_I \bullet \mu_I\left((0,t] \times B\right)\right] = \mathbb{E}\left[F_I \bullet \rho_I\left((0,t] \times B\right)\right] .$$

Proof. The proof will only be performed for (1), as the other case is analogous.

Start with the following auxiliary result. By similar arguments, as in the proof of equation (2.4.3), and by using that $F_I(t,e)$ is $\mathcal{G}_t^-$ -measurable for each (t,e), we get

$$\begin{aligned} G_I(t,e) &= \sum_{M\in\mathcal{M}} \mathbb{I}_{t-}^M \frac{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}_{t-}^M F_I(t,e)\right]}{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}_{t-}^M\right]} \\ &= \sum_{M\in\mathcal{M}} \mathbb{I}_{t-}^M F_I(t,e) \frac{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}_{t-}^M\right]}{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}_{t-}^M\right]} \\ &= \sum_{M\in\mathcal{M}} \mathbb{I}_{t-}^M F_I(t,e) \end{aligned}$$

$$= F_I(t,e)\ a.s.\,,$$

i.e. we have that $G_I(t,e) = F_I(t,e)$ almost surely.

By application of Proposition 2.5.6, and by directly using the above property, we get that

$$\begin{aligned} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I\left((t_k, t_{k+1}] \times B\right) \,|\, \mathcal{G}_{t_k}\right] &= G_I \bullet \nu_I((0,t] \times B) \\ &= F_I \bullet \nu_I((0,t] \times B) \end{aligned} \tag{2.5.7}$$

almost surely for any $\left(\tau_n^t\right)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$.

Further, without loss of generality, we suppose that F in a non-negative mapping (which can always be achieved by considering the decomposition of F in a positive and a negative part as $F = F^+ - F^-$).

If we apply the expectation on both sides of (2.5.7), we get:

$$\begin{aligned} \mathbb{E}\left[F_I \bullet \nu_I((0,t] \times B)\right] &\overset{(2.5.7)}{=} \mathbb{E}\left[\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right]\right] \\ &\overset{(i)}{=} \lim_{n\to\infty} \mathbb{E}\left[\sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right]\right] \\ &= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mathbb{E}\left[F_I \bullet \mu_I\left((t_k, t_{k+1}] \times B\right) \,|\, \mathcal{G}_{t_k}\right]\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\sum_{\tau_n^t} F_I \bullet \mu_I\left((t_k, t_{k+1}] \times B\right)\right] \\ &= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I\left((t_k, t_{k+1}] \times B\right)\right] \\ &\overset{(i)}{=} \mathbb{E}\left[\lim_{n\to\infty} \sum_{\tau_n^t} F_I \bullet \mu_I\left((t_k, t_{k+1}] \times B\right)\right] \\ &= \mathbb{E}\left[F_I \bullet \mu_I\left((0,t] \times B\right)\right]\ , \end{aligned}$$

where we applied the dominated convergence theorem in (i) to be able to exchange the limit and the expectation.

For each compact interval $[0,t]$ and almost each $\omega \in \Omega$, we define the set

$$\mathcal{M}_t(\omega) := \left\{ M \in \mathcal{M} \,\middle|\, \mathbb{I}_u^M(\omega) = 1 \quad \text{for at least one } u \in [0,t] \right\} \tag{2.5.8}$$

which is finite because of (2.3.2). For the first application of the dominated convergence theorem we have to add some details and we therefore use the rewriting

$$\tilde{X}_n := \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right]$$

$$
\begin{aligned}
&= \sum_{\tau_n^t} \sum_{M \in \mathcal{M}_t} \mathbb{I}_{t_k}^M \, \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right] \\
&= \sum_{\tau_n^t} \sum_{M \in \mathcal{M}_t} \frac{\mathbb{I}_{t_k}^M}{\mathbb{E}_M\left[\mathbb{I}_{t_k}^M\right]} \, \mathbb{E}_M\left[\mathbb{I}_{t_k}^M \, F_I \bullet \mu_I((t_k, t_{k+1}] \times B)\right] ,
\end{aligned}
$$

where the Proposition 2.4.12 and the almost sure finiteness of $\mathcal{M}_t$, together with

$$
0 \le \sum_{\tau_n^t} \mathbb{E}_M\left[\mathbb{I}_{t_k}^M \, F_I \bullet \mu_I((t_k, t_{k+1}] \times B)\right] \le \mathbb{E}_M[F_I \bullet \mu_I((0,t] \times B)]
$$

gives us the integrable majorant, as it is the precondition of Theorem 2.5.4 and we can therefore apply the dominated convergence theorem. □

The proof of Theorem 2.5.4 is then essentially a combination of the two previous propositions.

Proof. See Theorem 5.2 in [Chr21b].

We will still recap some major steps of the proof, but we are restricting the explicit formulation on the IF-parts of the statements.

The IF-predictability of the compensators follows from the second proposition, since we almost surely have

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[G_I \bullet \nu_I((0, t_{k+1}] \times B) - G_I \bullet \nu_I((0, t_k] \times B) \,|\, \mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[G_I \bullet \nu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right] \\
&\overset{2.5.7}{=} G_I \bullet \nu_I((0,t] \times B) \\
&= G_I \bullet \nu_I((0,t] \times B) - G_I \bullet \nu_I((0,0] \times B) .
\end{aligned}
$$

To show the IF-martingale property, we look at the following differences, given by the combination of the two propositions (used in the special case of $G = F \ a.s.$):

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) - G_I \bullet \nu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right] - \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[G_I \bullet \nu_I((t_k, t_{k+1}] \times B) \,|\, \mathcal{G}_{t_k}\right] \\
&= G_I \bullet \nu_I((0,t] \times B) - G_I \bullet \nu_I((0,t] \times B) \\
&= 0
\end{aligned}
$$

□

We will now apply these concepts to random variables and then stochastic processes. It is our goal to keep a duality between the standard case, where a filtered probability space is the foundation, and the new non-monotone case. Then, we will be able to compare the derived representations with each other and use both of them in the insurance theory.

Let us begin with the standard case and the existence of filtration $\mathbb{F}$:

Definition 2.5.9. *Classical compensator with respect to* $\mathbb{F}$
For every $I \in \mathcal{N}$, let λ_I be the classical compensator of μ_I with respect to $\mathbb{F}$ (from now on also called $\mathbb{F}$-compensator).
The compensator of a marked point process with respect to a filtration exists under the conditions we specified. See for example Jacod [Jac75], Karr [Kar91] or Jacobsen [Jac05] for general results, and Crépey [Cré13] for a similar construction for a compensator with density.

As a direct consequence of the martingale representation theorem, we get the following result:

Theorem 2.5.10. *Martingale representation theorem – Version 1*
Let ξ be an integrable random variable. The process

$$X_t^{\mathbb{F}} := \mathbb{E}\left[\xi \,|\, \mathcal{F}_t\right]$$

is a martingale with respect to the filtration $\mathbb{F}$ and can be represented as

$$X_t^{\mathbb{F}} = X_0^{\mathbb{F}} + \sum_{I \in \mathcal{N}} \int\limits_{(0,t] \times E_I} F_I(u,e)\ (\mu_I - \lambda_I)\left(\mathrm{d}(u,e)\right),$$

where the mapping

$$(u,e,\omega) \longmapsto F_I(u,e)(\omega)$$

is jointly measurable in (u,e,ω) and the mapping

$$\omega \longmapsto F_I(u,e)(\omega)$$

is $\mathcal{F}_{u^-}$-measurable (i.e. predictable) for each (u,e).

Proof. For reference see Karr [Kar91] or Elliott [Ell76].

□

This result is now extended to the non-monotone case:

Theorem 2.5.11. *Infinitesimal martingale representation theorem – Version 1*
Let ξ be an integrable random variable. Then, for the process $X_t^{\mathbb{G}} = \mathbb{E}\left[\xi \,|\, \mathcal{G}_t\right]$, the equation

$$\begin{aligned} X_t^{\mathbb{G}} = X_0^{\mathbb{G}} &+ \sum_{I \in \mathcal{N}} \int\limits_{(0,t] \times E_I} G_I(u^-, u, e)\,(\mu_I - \nu_I)(\mathrm{d}(u,e)) \\ &+ \sum_{I \in \mathcal{N}} \int\limits_{(0,t] \times E_I} G_I(u\ \ , u, e)\,(\rho_I - \mu_I)(\mathrm{d}(u,e)) \end{aligned} \tag{2.5.9}$$

almost surely holds for each $t \geq 0$ with

$$G_I(s,u,e) := \sum_{M\in\mathcal{M}} \mathbb{I}_s^M \left(\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}_s^M \xi\right]}{\mathbb{E}_{M,R_I=(u,e)}[\mathbb{I}_s^M]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M \xi\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M\right]} \right). \tag{2.5.10}$$

For each $I \in \mathcal{N}$ and $e \in E_I$, the process $u \mapsto G(u^-, u, e)$ is $\mathbb{G}^-$-adapted and the process $u \mapsto G(u, u, e)$ is $\mathbb{G}$-adapted.

If the mappings $F_I(u,e) = G_I(u^-, u, e)$ and $F_I(u,e) = G_I(u,u,e)$ satisfy the integrability condition in Theorem 2.5.4, then the representation (2.5.9) is the sum of IF-martingales and IB-martingales with respect to $\mathbb{G}$.

Proof. See proof of Theorem 6.1 in Christiansen [Chr21b].

□

It is now the goal to extend the results from the previous section to optional projections and we again start with the case of a filtration $\mathbb{F}$.

Theorem 2.5.12. *Martingale representation theorem – Version 2*
Suppose that X is a càdlàg process that satisfies condition (2.4.1).
Let furthermore $X_t - X_0$ be $\mathcal{F}_t$ -measurable for each $t \geq 0$. Then the optional projection of X with respect to $\mathbb{F}$, given as

$$X_t^{\mathbb{F}} = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right]$$

can be represented as

$$\mathrm{d}X_t^{\mathbb{F}} = \mathrm{d}X_t + \sum_{I\in\mathcal{N}} \int_{E_I} F_I(t,e)\,(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e)$$

for random mappings $F_I(t,e)$ that are $\mathcal{F}_{t^-}$-measurable processes for each (t,e).

Proof. Since this is a straightforward consequence of the previous Theorem 2.5.10, we give a sketch of the proof.
Under usage of the $\mathcal{F}_t$-measurability of $X_t - X_0$, we can express

$$\begin{aligned} & \mathbb{E}\left[X_0 \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{F}_0\right] \\ & = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{F}_0\right] - \mathbb{E}\left[X_t - X_0 \,|\, \mathcal{F}_t\right] \\ & = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{F}_0\right] - (X_t - X_0) \\ \Leftrightarrow \quad & X_t^{\mathbb{F}} - X_0^{\mathbb{F}} = (X_t - X_0) + \mathbb{E}\left[X_0 \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{F}_0\right] \end{aligned}$$

and then use the martingale representation Theorem 2.5.10 with random variable $\xi = X_0$.

We arrive at the following equation, where rearranging of the summands yields the desired representation (here in integral representation):

$$\mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{F}_0\right] = X_t - X_0 + \sum_{I\in\mathcal{N}} \int_{(0,t]\times E_I} F_I(u,e)(\mu_I - \lambda_I)(\mathrm{d}(u,e))$$

□

We will formulate an extension to the previous result in the non-monotone case, where two different variants are derivable, depending on the use of a forward- or backward-compensator:

Theorem 2.5.13. *Infinitesimal martingale representation theorem – Version 2*
Let X be a càdlàg process that satisfies condition (2.4.1)

(a) If X has an IB-compensator X^{IB} with respect to $\mathbb{G}$, then

$$\begin{aligned} X_t^{\mathbb{G}} - X_0^{\mathbb{G}} = X_t^{\mathrm{IB}} &+ \sum_{I \in \mathcal{N}} \int_{(0,t] \times E_I} G_I(u^-, u, e)\,(\mu_I - \nu_I)(\mathrm{d}(u,e)) \\ &+ \sum_{I \in \mathcal{N}} \int_{(0,t] \times E_I} G_I(u\ \ , u, e)\,(\rho_I - \mu_I)(\mathrm{d}(u,e)) \end{aligned}$$

almost surely holds with

$$G_I(s,u,e) := \sum_{M \in \mathcal{M}} \mathbb{I}_s^M \left(\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}_s^M X_{u^-}\right]}{\mathbb{E}_{M,R_I=(u,e)}[\mathbb{I}_s^M]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M X_{u^-}\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M\right]} \right). \tag{2.5.11}$$

(b) If X has an IF-compensator X^{IF} with respect to $\mathbb{G}$, then

$$\begin{aligned} X_t^{\mathbb{G}} - X_0^{\mathbb{G}} = X_t^{\mathrm{IF}} &+ \sum_{I \in \mathcal{N}} \int_{(0,t] \times E_I} G_I(u^-, u, e)\,(\mu_I - \nu_I)(\mathrm{d}(u,e)) \\ &+ \sum_{I \in \mathcal{N}} \int_{(0,t] \times E_I} G_I(u\ \ , u, e)\,(\rho_I - \mu_I)(\mathrm{d}(u,e)) \end{aligned}$$

almost surely holds with

$$G_I(s,u,e) := \sum_{M \in \mathcal{M}} \mathbb{I}_s^M \left(\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}_s^M X_u\right]}{\mathbb{E}_{M,R_I=(u,e)}[\mathbb{I}_s^M]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M X_u\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M\right]} \right). \tag{2.5.12}$$

Further, the mappings $G_I(u^-, u, e)$ are $\mathcal{G}_u^-$-measurable and the mappings $G_I(u\ \ , u, e)$ are $\mathcal{G}_u$-measurable.

Therefore, both integrals actually describe IF-martingales and IB-martingales with respect to $\mathbb{G}$, if $F_I(u,e) = G_I(u^-, u, e)$ and $F_I(u,e) = G_I(u,u,e)$ satisfy the integrability condition (2.5.1) in Theorem 2.5.4.

Corollary 2.5.14. *Simplifications in the case* $\mathbb{G} = \mathbb{F}$
In case of $\mathbb{G} = \mathbb{F}$, we have $\nu_I = \lambda_I$, $\rho_I = \mu_I$ and $X = X^{\mathrm{IB}}$ and the Theorem 2.5.13 can again be seen as a generalization of Theorem 2.5.12 in the filtration case.

Take note, that even in the case $\mathbb{G} \subsetneq \mathbb{F}$ we may still have that $X = X^{\mathrm{IB}}$ or $X = X^{\mathrm{IF}}$. We will later see this in the context of life-insurance mathematics, where the $\mathbb{G}^-$-measurability of the payments will guarantee this in case of the IF-compensator.

The situation $\mathbb{G} = \mathbb{F}$ means, that $\mathbb{G}$ is now also the sigma-algebra of full available information. It can generally be constructed by setting the deletion times to ∞. For a special construction in the case of a state process from classical life insurance theory, see Furrer [Fur20] (Section 2.2.2).

Proof. We use the existing results of Proposition 2.5.6 and Proposition 2.5.7. For $t \geq 0$, $I \in \mathcal{N}$ and $B \in \mathcal{E}_I$ we almost surely have

$$
\begin{aligned}
F_I \bullet \rho_I \left((0,t] \times B\right) &\overset{(2.5.5)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \rho_I \left((t_k, t_{k+1}] \times B\right) \,\middle|\, \mathcal{G}_{t_{k+1}}\right] \\
&\overset{(2.5.3)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,\middle|\, \mathcal{G}_{t_{k+1}}\right] \\
&\overset{\mathbb{F}=\mathbb{G}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \\
&= F_I \bullet \mu_I((0,t] \times B)
\end{aligned}
$$

for $F_I(u,e) = G_I(u,u,e))$ and $(\tau_n)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$.

Further, we almost surely have

$$
\begin{aligned}
F_I \bullet \nu_I \left((0,t] \times B\right) &\overset{(2.5.6)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \nu_I((t_k, t_{k+1}] \times B) \,\middle|\, \mathcal{G}_{t_k}\right] \\
&\overset{(2.5.4)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,\middle|\, \mathcal{G}_{t_k}\right] \\
&\overset{\mathbb{F}=\mathbb{G}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[F_I \bullet \mu_I((t_k, t_{k+1}] \times B) \,\middle|\, \mathcal{F}_{t_k}\right] \\
&= F_I \bullet \lambda_I((0,t] \times B)
\end{aligned}
$$

for $F_I(u,e) = G_I(u^-,u,e)$ and $(\tau_n)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$, since the definition of the $\mathbb{F}$-compensator λ_I also implies that

$$
\lambda_I((0,t] \times B) = \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mu_I((t_k, t_{k+1}] \times B) \,\middle|\, \mathcal{F}_{t_k}\right] .
$$

For the IB-compensator, we use the definition and $\mathbb{F} = \mathbb{G}$ to get

$$
X_t^{\mathrm{IB}} = \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,\middle|\, \mathcal{F}_{t_{k+1}}\right] = \lim_{n\to\infty} \sum_{\tau_n^t} \left(X_{t_{k+1}} - X_{t_k}\right) = X_t - X_0 \,.
$$

The IF- and IB-martingale versions have different representations of $(G_I)_I$, and to get the same simplified formula, a part from the G_I has to be added to the IF-compensator, such that X itself also appears in this representation.

Use the representation of G_I, and introduce X^-, then $\Delta X = X - X^-$ arise in an additional summand. Together with the definition of the IF-compensator we get

$$
\lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,\middle|\, \mathcal{F}_{t_k}\right]
$$

$$
\begin{aligned}
&+ \sum_{I\in\mathcal{N}} \int_{(0,t]\times E_I} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-}\left(\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}^M_{u^-}\Delta X_u\right]}{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}^M_{u^-}\right]} - \frac{\mathbb{E}_M\left[\mathbb{I}^M_{u^-}\mathbb{I}^M_u\Delta X_u\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{u^-}\mathbb{I}^M_u\right]}\right)(\mu_I-\lambda_I)(\mathrm{d}(u,e))\\
&= \lim_{n\to\infty}\sum_{\tau^t_n}\Bigg(\mathbb{E}\left[X_{t_{k+1}}-X_{t_k}\,|\,\mathcal{F}_{t_k}\right]\\
&\quad + \sum_{I\in\mathcal{N}} \int_{(t_k,t_{k+1}]\times E_I} \left(\mathbb{E}\left[\Delta X_u\,|\,\mathcal{F}_{u^-},\, R_I=(u,e)\right] - \mathbb{E}\left[\Delta X_u\,|\,\mathcal{F}_{u^-},\, \mathcal{J}_t=0\right]\right)(\mu_I-\lambda_I)(\mathrm{d}(u,e))\Bigg)\\
&\quad = \lim_{n\to\infty}\sum_{\tau^t_n}\left(X_{t_{k+1}}-X_{t_k}\right)\\
&\quad = X_t - X_0\,,
\end{aligned}
$$

by using, that the conditional expectations are compensated by exchanging the prediction with the actual changes, and thus arriving at the actual changes all along. The details are technical and will not be formulated.

Is can be seen, that we do not actually have $X^{\mathrm{IF}} = X$, what would be equivalent to the other case. In total, we achieve the same representation in both cases and the theorem is indeed a generalization of generalization of Theorem 2.5.12.

Note, that assertion follows more directly when using $F_I(u,e) = 1$, and then we have $\rho = \mu$ and $\nu = \lambda$. The Theorem 2.5.13 can therefore be seen as a more general results in comparison to Theorem 2.5.12, since we have now seen, how the parts of the formula simplify. □

We will now recap the idea of the proof to the theorem:

Proof. For details, see Theorem 7.1 in [Chr21b].

Use the following additive decomposition

$$
\begin{aligned}
\mathbb{E}\left[X_t\,|\,\mathcal{G}_t\right] - \mathbb{E}\left[X_0\,|\,\mathcal{G}_0\right] &= \lim_{n\to\infty}\sum_{\tau^t_n}\Big(\mathbb{E}\left[X_{t_{k+1}}\,|\,\mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_k}\right]\Big)\\
&= \lim_{n\to\infty}\sum_{\tau^t_n}\mathbb{E}\left[X_{t_{k+1}}-X_{t_k}\,|\,\mathcal{G}_{t_{k+1}}\right]\\
&\qquad + \lim_{n\to\infty}\sum_{\tau^t_n}\Big(\mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_k}\right]\Big)\\
&= X^{\mathrm{IB}}_t + \lim_{n\to\infty}\sum_{\tau^t_n}\Big(\mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_k}\right]\Big)\,.
\end{aligned}
$$

We can now use the Theorem 2.5.11 of the previous section for each summand of the form

$$
\mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_k}\,|\,\mathcal{G}_{t_k}\right]
$$

with $\xi = X_{t_k}$ and on each interval $(t_k, t_{k+1}]$.

Because of the càdlàg property of X, by applying the dominated convergence theorem path-wise for almost each $\omega \in \Omega$, we end up with the assertion. To give a little more details on the piecewise construction, we have

$$\begin{aligned}
&\mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{G}_0\right] \\
&= X_t^{\mathrm{IB}} + \lim_{n\to\infty} \sum_{\tau_n^t} \left(\sum_{I\in\mathcal{N}} \int\limits_{(t_k,t_{k+1}]\times E_I} G_I(u^-,u,e)\,(\mu_I - \nu_I)(\mathrm{d}(u,e)) \right. \\
&\qquad\qquad \left. + \sum_{I\in\mathcal{N}} \int\limits_{(t_k,t_{k+1}]\times E_I} G_I(u,u,e)\,(\rho_I - \mu_I)(\mathrm{d}(u,e)) \right)
\end{aligned}$$

where the G_I differ for each summand, but it holds

$$G_I(s,u,e) = \sum_{M\in\mathcal{M}} \mathbb{I}_s^M \left(\frac{\mathbb{E}_{M,R_I=(u,e)}\left[\mathbb{I}_s^M X_{t_k}\right]}{\mathbb{E}_{M,R_I=(u,e)}[\mathbb{I}_s^M]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M X_{t_k}\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M \mathbb{I}_u^M\right]} \right)$$

for $s \in (t_k, t_{k+1}]$.
The dominated convergence theorem is then applied, to end up with the desired representation, with the representation of G_I, then with X_{s^-} instead of X_{t_k}.

For part (b), we can use the following alternative decomposition

$$\begin{aligned}
\mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right] - \mathbb{E}\left[X_0 \,|\, \mathcal{G}_0\right] &= \lim_{n\to\infty} \sum_{\tau_n^t} \Big(\mathbb{E}\left[X_{t_{k+1}} \,|\, \mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_k} \,|\, \mathcal{G}_{t_k}\right] \Big) \\
&= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[X_{t_{k+1}} - X_{t_k} \,|\, \mathcal{G}_{t_k}\right] \\
&\quad + \lim_{n\to\infty} \sum_{\tau_n^t} \Big(\mathbb{E}\left[X_{t_{k+1}} \,|\, \mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_{k+1}} \,|\, \mathcal{G}_{t_k}\right] \Big) \\
&= X_t^{\mathrm{IF}} + \lim_{n\to\infty} \sum_{\tau_n^t} \Big(\mathbb{E}\left[X_{t_{k+1}} \,|\, \mathcal{G}_{t_{k+1}}\right] - \mathbb{E}\left[X_{t_{k+1}} \,|\, \mathcal{G}_{t_k}\right] \Big)
\end{aligned}$$

where we can proceed similarly with $\xi = X_{t_{k+1}}$, and the limit application while using the dominated convergence theorem results in the difference representation of G_I, where X_{u^-} is replaced by X_u, since now the right end of each interval is used.

The measurability conditions for $G_I(u^-,u,e)$ and $G_I(u,u,e)$ can once again be shown by using the equation (2.4.3) and will not be repeated again. □

2.6. Notational remarks for classic insurance contract

In the beginning of this chapter, we introduced a standard notation for life insurance modelling, where the focus lied on insurance states and the transitions between these states. Now that we have introduced the marked point process theory of Christiansen [Chr21b], we have learned of a different modelling, with a focus on information states and information introduction and deletion.
It remains to be seen, how a general contract can be cast in the necessary structure to fulfil the conditions needed for the marked point process theory, although it is intuitive, that a time point of changing insurance state also is a piece of information. This is the main target of this section and will also provide the basics for some of the examples in a later chapter, where the construction will not be repeated and is assumed to be done analogously to here. Similar constructions are done in Christiansen and Furrer [CF21], where the non-monotone theory is applied to an disability example with retirement, and in Milbrodt and Helbig [MH99] for a general connection between jump processes and marked point processes with a focus on the underlying mathematical details.

Remember that we are using a state space $\mathcal{S}$ and the development of the policy as the current state of the insured as the process $S : [0, \infty) \to \mathcal{S}$ from Definition 2.1.1 as a foundation for the reformulation.
In addition to the general model, we will use the following example with state space $\mathcal{S} = \{a, i, l, d\}$, where lapse ($l$) and death ($d$) are absorbing states, and recovery/reactivation from state invalid (i) is possible, compare also Figure 2.1.6.

The following Figure 2.2 provides an exemplary path of the state process for an insured person.

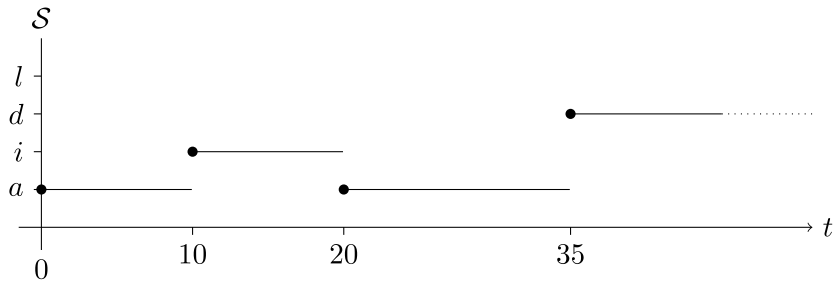

Figure 2.2.: An exemplary path of the insurance state process

The pattern of states in Figure 2.2 is given as a, i, a, d with a total of 3 transitions, a start in a, and a remaining in the absorbing state d.

We will now set the sequence $(T_i, Z_i)_{i \in \mathbb{N}^{\mathrm{O}}}$, where the focus is on the subsets of the odd indices. Set $T_1 = 0$ and use the following recursion for the rest. Let

$$T_{2i+1} := \inf \left\{ t \in (T_{2i-1}, \infty) \mid S_t \neq S_{T_{2i-1}} \right\}, \quad \text{for } i \in \mathbb{N},$$

where we set $\inf \emptyset = \infty$, if no further jump happens. The recursion continues with $T_{2j+1} = \infty$ for all $j \in \mathbb{N}_{\geq 2i+1}$ if $T_{2i-1} = \infty$ for an $i \in \mathbb{N}^{\mathrm{O}}$.

The remaining definitions of the marks now depends on what the underlying model is supposed to be:
In the Markov model, i.e. when S fulfils the Markov property, only the current state is of importance, and therefore the previous jumps, as well as the times of the jumps, are not encoded into the marks Z.
In the Semi-Markov model, the current state and the time since the last state change are of importance. Therefore, the information about previous jumps should be deleted, but the time of the current jump should be part of the information Z.
In a full information setting everything is getting saved, i.e. we then have $\mathcal{F}_t = \sigma(S_s : s \leq t)$.

Depending on three models, the marked point process has a different structure and we want to investigate this a little further for the example.

In the last model we do not delete information. Therefore, we set $T_{2i} = \infty$, for $i \in \mathbb{N}$. By joining together the time and the state of each jump, i.e. we will have and $Z_{2i-1} = Z_{2i} = (T_{2i-1}, S_{T_{2i-1}})$ for $i \in \mathbb{N}$, it is then also motivated to set $E = \mathbb{R}_{\geq 0} \times \mathcal{S}$ to include the above tuples. The structure of the filtration $\mathbb{F}$ corresponds to the natural filtration of the state process.

For our exemplary sequence, we get:

i	T_i	Z_i	Information
1	0	$(0, a)$	a at the beginning in 0
2	∞	$(0, a)$	no deletion
3	10	$(10, i)$	i entered at time 10
4	∞	$(10, i)$	no deletion
5	20	$(20, a)$	a entered at time 20
6	∞	$(20, a)$	no deletion
7	35	$(35, d)$	d entered at time 35
8	∞	$(35, d)$	no deletion

Table 2.1.: Exemplary sequence of a life insurance state – Full information

In this current setup, the insurance states are not unique, i.e. the re-entering of state a is allowed. This could be changed, by adding a second component to the setup, that counts the number of entries in each state. Then, the new state space would actually be $\tilde{\mathcal{S}} = \mathcal{S} \times \mathbb{N}$. Then, the combination of states would be unique.

By not considering the deletion of information, the indicator processes $\mathbb{I}_t^M$ will only be able to take value 1, if the set M is extended with each new state entry. In this example, we would have, that $\mathbb{I}_t^{\{1\}} = 1$ for $t \in [0, 10)$, $\mathbb{I}_t^{\{1,3\}} = 1$ for $t \in [10, 20)$ and so on but the indicator for just $M = \{3\}$ is always zero. Therefore, every relevant set M has to have the form $M = \mathbb{N}_{\leq k}^{\mathrm{O}}$ for $k \in \mathbb{N}^{\mathrm{O}}$. Similarly, the set $I \in \mathcal{N}$ can only have the representation

$I = \{l\}$ for $l \in \mathbb{N}^{\mathrm{O}}$. By Theorem 2.4.6 we have that for $M = \mathbb{N}^{\mathrm{O}}_{\leq k}$ and $I = \{k+2\}$, the next set would be given as $\tilde{M} = \mathbb{N}^{\mathrm{O}}_{\leq k+2}$.
This also explains, why the re-entering of state a is different, and how the information states are different from the insurance states. The indicator function also contains the information about the previous jumps, i.e. the whole history of the process.

In the Markov model, we assume that the process S fulfils the Markov assumption. We can set $T_{2i} = T_{2i+1}$, for $i \in \mathbb{N}$, as the information about the previous state is deleted, when a new state is entered. Further we have $Z_{2i-1} = Z_{2i} = S_{T_{2i-1}}$ for $i \in \mathbb{N}$ and we set $E = \mathcal{S}$. The information structures are then given by $\mathcal{F}_t = \sigma(S_r,\, 0 \leq r \leq t)$ and $\mathcal{G}_t = \sigma(S_t)$. Note, that since S actually fulfils the Markov property, the conditional expectations of the future payments with respect to this σ-algebras are equal.

For our exemplary sequence, we then have:

i	T_i	Z_i	Information
1	0	a	a at the beginning in 0
2	10	a	deletion when entering i
3	10	i	i entered at time 10
4	20	i	deletion when entering a
5	20	a	a entered at time 20
6	35	a	deletion when entering d
7	35	d	d entered at time 35
8	∞	d	no deletion

Table 2.2.: Exemplary sequence of a life insurance state – Markov information

We refrain from formulating the semi-Markov case as well. The time and state of the jumps would be included in the marks, and deletions of the previous information have to take place, which is why the semi-Markov case follows as a combination from the specified models.

The as-if Markov model, introduced by Christiansen [Chr21a], which calculates reserves with respect to Markov information, even if the process does not fulfil the Markov assumption, can not be included in this theory. A consistent definition of the marks is not possible, as the full information includes both time and state of the insured, compare the first example, and the Markov information only includes the current state, compare the second example. This would have to be modelled in a way, where the family of σ-algebras $\mathbb{G}$ does not use the representation in Christiansen [Chr21b].

Further take note, that it is possible to extend the model by adding other types of information, not originating from the state space. In that case, the definition of the T_i and Z_i would have to be adapted.

2.7. Solving technique for non-linear BSDEs

In this section, the solving techniques for BSDEs with non-linear structure are reviewed and the problems in the infinitesimal martingale theory are discussed.

Let us remember the BSDE 2.1.4 from the introduction is given as

$$\begin{cases} \mathrm{d}Y_t = f(t, Y_{t^-}, Z_t)\,\mathrm{d}t - Z_t\,\mathrm{d}M_t \\ Y_T = \xi \end{cases}$$

where M is an $\mathbb{F}$-martingale, ξ is a final-value condition, and f is a generator function and the family Z is predictable.

In the classical BSDE literature it is the target to show existence and uniqueness of a solution pair (Y, Z) in suitable spaces. It mostly is the approach to understand the BSDE (or rather the associated integral representation) as a fixed point equation, where a solution is a fixed point and vice versa. For the fixed point operator Φ, a fixed point would be a pair $({}^{*}Y, {}^{*}Z)$, satisfying

$$({}^{*}Y, {}^{*}Z) = \Phi\left(({}^{*}Y, {}^{*}Z)\right).$$

Major contribution in BSDE theory are for example the following papers by Pardoux and Peng [PP90], Pham [Pha09], and a series of papers by Cohen and Elliott [CE08], [CE12]. To summarize the approaches in these literature contributions, the main idea of the proof is to show the contraction property of the fixed point operator Φ, i.e. there exists a constant $C < 1$, such that for $n \in \mathbb{N}_0$

$$\left\| \left({}^{(n+1)}Y, {}^{(n+1)}Z\right) - \left({}^{(n)}Y, {}^{(n)}Z\right) \right\| \leq C \cdot \left\| \left({}^{(n)}Y, {}^{(n)}Z\right) - \left({}^{(n-1)}Y, {}^{(n-1)}Z\right) \right\|.$$

Existence and uniqueness of solutions can then be followed by application of the fixed point theorem of Banach, which also gives the opportunity to iterate the solution, by application of Φ for $n \in \mathbb{N}_0$ through

$$\left({}^{(n+1)}Y, {}^{(n+1)}Z\right) = \Phi\left(\left({}^{(n)}Y, {}^{(n)}Z\right)\right).$$

Since we want to have a solution pair (Y, Z), it is natural to consider separate norms for Y and Z. In most cases, the final norm is then a combination of both individual norms, for which the contraction property is shown and the fixed point theorem is applied to, see for example Pham [Pha09].
The contraction property is then shown by using a weighted norm, where the weight is specified in the proof and properties of the combined norm follow from properties of each individual norm. The weight is normally given as an exponential, therefore never being zero and not influencing the positive definiteness of the norm. The weighted norm can be shown to be equivalent to the standard norm.

For general BSDE methods, the adaptivity of the solution Y is a key property and one works directly with Y. In the proof, it is heavily relied on methods and tools, that are only available in the presence of classic martingales. These include various properties of martingales themselves, the martingale representation theorem, Doob's inequality and the

Burkholder-Davis-Gundy inequality.
These tools are mostly not available for the infinitesimal martingale theory, which is the main reason why we cannot rely on existing results and have difficulties adapting existing proofs to the new situations.

We are now making use of the following workaround, which is unique for the situation of the Thiele BSDE in multi-state life insurance theory. We can differentiate between the prospective reserve $X^{\mathbb{G}}$ and the payment process X, and the following relation

$$Y_t \triangleq X_t^{\mathbb{G}} = \mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right]$$

is exploited. It is the goal to make use of this 'additional layer' of the life insurance structure, and also to use the special additive structure of process X.

Together with our equivalent representation of the sum at risk $(Z_{ij}) \triangleq (G_I)$, we make use of a workaround, as sketched in the Figure 2.7, where the details of each step are formulated in the following chapters.

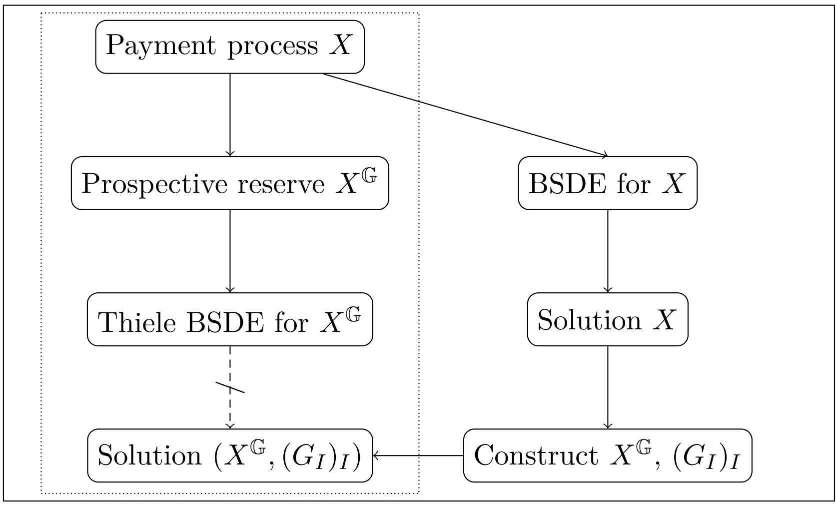

Figure 2.3.: Sketch of the main idea – The usual way is visualized in the dotted box, where the last (crossed out) step does not directly work for the infinitesimal martingale theory, and the workaround on the right side is used instead.

Comment 2.7.1. *On the modelling of dependency*
It is a common generalization in the BSDE literature, to directly model the dependency on Y and Z, even if dependency on Y would be enough for the application. This is motivated by the fact, that it is simpler to use a joint norm, compare for example to Djehiche and Löfdahl [DL16]. Leaving out the dependency on Z does not actually simplify the calculations.
In our case, we will not use a combined norm. The dependency will be extended in two steps, enabling us to use different methods and focus on the new challenges in each of the two steps. The dependency on Z (or rather G) will be more complex and require different tools and more restrict preconditions.

Chapter 3.

Life insurance with reserve-dependent payments

Instead of using a general BSDE formulation, we will now focus on the special structure of payments in life insurance, to be able to use the duality of payment process and reserve, as well as the given representation for the generator function f, in this context. We start by expanding the insurance model to the necessary structure, where reserve-dependent payments are possible, and the notation of Christiansen [Chr21b] is used.

3.1. The reserve-dependent payment process

We already introduced the sojourn and transition payments in the introductory part, but we are now doing it properly. The foundation for the non-linear dependence on the prospective reserve is set and this section is kept as general as possible, while further restrictions are introduced on the way.
Remember, that the insurer in general cannot calculate the future payments X_t on interval (t, ∞) itself, because the occurrence is unknown at time t and works with prospective reserves (optional projections), given by the conditional expectations

$$X_t^{\mathbb{F}} = \mathbb{E}\left[X_t \,|\, \mathcal{F}_t\right], \quad t \geq 0$$
$$X_t^{\mathbb{G}} = \mathbb{E}\left[X_t \,|\, \mathcal{G}_t\right], \quad t \geq 0$$

instead.

For simplicity, we now focus on the non-monotone case with information structure $\mathbb{G}$ to keep the notation simple, but the corresponding definitions with respect to $\mathbb{F}$ are also possible.

Definition 3.1.1. *Maximum contract time*
A time point $0 < T < \infty$ is called a possible maximum time of a contract, if the following condition holds

$$A(\mathrm{d}t) = 0, \qquad \text{for } t > T$$

almost surely holds for the contractual cash flow A of an individual insurance contract, which is yet to be specified.

We assume without loss of generality, that there always exists a $T < \infty$ with these properties. The time T is usually also agreed upon, when signing the contract. Special cases are retirement plans or insurances upon death, but even then a maximum time can be found, when the cumulative probability of death is reasonably close to one.

This definition has the advantage, that we only need to consider payments on the compact time interval $[0, T]$ instead of an infinite contract horizon.

When the Thiele BSDE was first formulated, we saw that the generator function could depend on Y^- and Z, where the Z_{ij} where given as the so called sums at risk, a difference between the statewise reserves Y_j and Y_i, and Y^- was the left limit of the reserve.

If a payment is happening in t and it is reserve dependent, then the existing reserve right before this time point has to be used for this payment. Therefore $X^{\mathbb{G}}_{t-}$ needs to be used for the reserve-dependency. In some cases, like lapse or death, the reserve in t would already be zero (since no further payments might be happening) and $X^{\mathbb{G}}_t$ could not be sensibly used for the reserve-dependency.

When decomposing the general prospective reserve in time t^- into statewise reserves, we have

$$X^{\mathbb{G}}_{t^-} = \mathbb{E}\left[X_{t^-} \,\middle|\, \mathcal{G}^-_t\right] = \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-} \cdot \mathbb{E}\left[X_{t^-} \,\middle|\, \mathcal{G}^-_t\right] = \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-} \cdot \frac{\mathbb{E}_M\left[\mathbb{I}^M_{t^-} X_{t^-}\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{t^-}\right]} \ a.s.,$$

which is a splitting into non-classical statewise reserves with respect to the information M (as defined in Furrer [Fur20]), but the classical state-wise reserves arising from the decomposition

$$X^{\mathbb{F}}_t = \sum_{i \in \mathcal{S}} \mathbb{I}^i_t \cdot X^{\mathbb{F},i}_t ,$$

where $X^{\mathbb{F},i}_t := \mathbb{E}\left[X_t \,|\, \mathcal{F}_{t^-},\, S(t) = i\right]$ (assuming them to be well-defined), can not be used. In standard life insurance theory, reserve-dependency on $X^{\mathbb{F},i}_t$ is feasible, since conditioning on $S(t) = i$ guarantees that the state-wise reserve is not affected by the transition happening in t. We then do not have to rely on the reserve in t^- right before the jump. In a case, where these state-wise reserve exist and are well-defined, it is also easier to solve a system of differential equations.

We can now continue to define the reserve-dependent payment functions.

Definition 3.1.2. *Sojourn payments*
Consider a function of the form

$$\begin{aligned} b : \mathcal{M} \times [0, \infty) \times \Omega &\longrightarrow \mathbb{R} \\ (M, t, \omega) &\longmapsto b(M, t, X^{\mathbb{G}}_{t^-})(\omega) \end{aligned}$$

as the rate of a sojourn payment, that will be paid at time t, if $\mathbb{I}^M_{t^-} = 1$ for all $M \in \mathcal{M}$ respectively. For simplicity, let us introduce the following notation, that is similar to the usual state-wise payment functions, we already know from standard life insurance theory. For every $M \in \mathcal{M}$, let

$$b_M(t, X_{t^-}^{\mathbb{G}})(\omega) := b(M, t, X_{t^-}^{\mathbb{G}})(\omega)$$

be a function from $[0, \infty) \times \Omega \to \mathbb{R}$.

We further need the following assumptions:

(1) The function $b(M, t, X_{t^-}^{\mathbb{G}})(\omega) : \mathcal{M} \times [0, \infty) \times \Omega \longrightarrow \mathbb{R}$ is jointly measurable in (M, t, ω).

(2) For every $M \in \mathcal{M}$ let b_M be bounded on every compact time interval, i.e. for $t \geq 0$ it holds

$$\left| b_M(s, X_{s^-}^{\mathbb{G}}) \right| \leq J_{\mathcal{M}}(s) \tag{3.1.1}$$

for an integrable majorant $J_{\mathcal{M}}$ and all $s \in [0, t]$.

(3) The functions $b_M(t, X_{t^-}^{\mathbb{G}})$ are $\mathbb{G}^-$-adapted for every $M \in \mathcal{M}$.

In addition to the payment functions themselves, we need the following integrator, to work in conjunction with them, in able to model both continuous and discrete sojourn payments.

Definition 3.1.3. *Integrator for continuous and discrete sojourn payments*
Let γ be the sum of a Lebesgue measure λ and a countable number of Dirac-measures $(\delta_{t_i})_{i \in \mathbb{N}}$, given as

$$\gamma(B) = \lambda(B) + \sum_{i=1}^{\infty} \delta_{t_i}(B) =: \lambda(B) + \delta(B)\,, \qquad B \in \mathcal{B}([0, \infty))$$

for deterministic time points $0 < t_1 < t_2 < \cdots$, increasing to infinity. Therefore, there are at most a finite number of discrete payments happening on every compact interval.

It now allows the sojourn payments to include discrete payments at the deterministic time points. By allowing the function b_M to be defined differently to the payment rate at the set of points $\{t_0, t_1, \dots\}$, this can be modelled, and the additional model flexibility does not influence the Lebesgue- integrability. Note, that it would be possible to have discrete payments as a separate summand, but that would complicate the notation.

The measure γ is monotone when used in the Lebesgue-Stieltjes integration. The Dirac measures correspond to step functions with a jump of 1 at the time points $\{t_0, t_1, \dots\}$ (also known as Heaviside-functions). We therefore do not have to focus on properties of the Dirac measures itself, since we can rewrite the integration as a Stieltjes integral with the corresponding step functions, which are also càdlàg. The measure γ, defined as the sum, is therefore both right-continuous and monotonously increasing. For details, also see Klenke [Kle20] (Example 1.58), where a similar cases is considered and the measure is shown to be a Lebesgue-Stieltjes measure, if the sequence of time points does not have a limit, which is also the case in our model. An explicit definition of the corresponding monotonously increasing and right-continuous (distribution) function F is also given as

$$F^{\delta}(x) = \#\{n \in \mathbb{N} \mid t_n \in [0, x]\}$$

for $x \geq 0$ such that the condition $\delta = \lambda_{F^{\delta}}$ is fulfilled for the corresponding Lebesgue-Stieltjes measure to function F^{δ}.

Comment 3.1.4. *On the sojourn payments*
The $\mathbb{G}^{-}$-adaptivity for the rates b_M is consistent with the dependence on $X_{t-}^{\mathbb{G}}$ and will guarantee a favourable structure of the IF-compensator later on. Sojourn payments will be paid with the given rate, if $\mathbb{I}_{t-}^{M} = 1$, i.e. the situation right before the current time point is decisive for the sojourn payments to be paid.

Furthermore, remember that a deletion is working in the same way like pretending, that the information never existed in the first place. This results from working with $\mathbb{I}_{s-}^{M}$, where the cases $s \leq T_i$ (before introduction) and $s > T_{i+1}$ (after deletion) for an index $i \in \mathbb{N}^{\mathrm{O}} \setminus M$ are not differentiated by the indicator process. This also means, that these cases are the same and this guarantees, that the deletion effect works like it is supposed to work.

Also note, that even when considering lump sum payments, the functions b_M can still assumed to be continuous almost everywhere, with an exception null set.

Let us continue with the transition payments.

Definition 3.1.5. *Transition payments*
For every $I \in \mathcal{N}$, consider a function of the form

$$\begin{aligned} B_I : [0, \infty) \times E_I \times \Omega &\longrightarrow \mathbb{R} \\ (t, e, \omega) &\longmapsto B_I(t, e, X_{t-}^{\mathbb{G}})(\omega) \end{aligned}$$

as a lump sum payment upon a transition $I \in \mathcal{N}$.
We need the following assumptions to hold:

(1) The functions $B_I(t, e, X_{t-}^{\mathbb{G}})(\omega) : [0, \infty) \times E_I \times \Omega \longrightarrow \mathbb{R}$ are jointly measurable in (t, e, ω) for every $I \in \mathcal{N}$.

(2) For every $I \in \mathcal{N}$ the B_I are bounded on compact time intervals, i.e. for every $t \geq 0$ it holds
$$\left| B_I(s, e, X_{s-}^{\mathbb{G}}) \right| \leq J_{\mathcal{N}}(s) \tag{3.1.2}$$
for an integrable majorant $J_{\mathcal{N}}$ and all $s \in [0, t]$.

(3) The functions $B_I(t, e, X_{t-}^{\mathbb{G}})(\omega)$ are $\mathbb{G}^{-}$-adapted for every $I \in \mathcal{N}$ and $e \in E_I$.

Let us also formulate a technical assumption about the reserve-dependency especially for time $t = 0$. We will demand, that the payments in 0 do not depend on the prospective reserve in 0^{-}, although our general model would allow this. For simplicity, the notation will be kept the same.

Assumption 3.1.6. *No reserve-dependency for payments at the start of a contract*
Let us assume, that the following conditions hold:

(a) $b_M(0, X_{0-}^{\mathbb{G}}) = b_M(0)$ *a.s.* for all $M \in \mathcal{M}$.

(b) $B_I(0, e, X_{0-}^{\mathbb{G}}) = B_I(0, e)$ *a.s.* for all $I \in \mathcal{N}$ with $e \in E_I$.

(c) $e_M(0, X_{0-}^{\mathbb{G}}) = e_M(0)$ *a.s.* for all $M \in \mathcal{M}$.

Therefore, all payments in 0 do not depend on the reserve $X_{0-}^{\mathbb{G}}$ in 0^-.

This is a technical condition, as well as a sensible assumption, since we are usually interested in construction of insurance contracts, where $X_{0-}^{\mathbb{G}} = 0$ *a.s.* anyway. This condition will later be investigated and we will use the condition $X_{0-}^{\mathbb{G}} = 0$ *a.s.* to call it a net equivalent premium.

Comment 3.1.7. *On transition payments*
A jump or transition payment is paid out in a case, where the process μ_I recognizes a jump. Take note, that in difference to the standard insurance theory, this does not depend on the current state or current information, but only on the changing information $e \in E_I$. A differentiation could be made as part of the definition of a payment B_I, where the family of indicator processes $(\mathbb{I}^M)_{M \in \mathcal{M}}$ might be used to differentiate certain cases, and also the information $e \in E_J$ in B_I matters.

The conditions introduced in definitions 3.1.2 and 3.1.5 are standing assumptions for the rest of the thesis, if it is not explicitly described to be different.

3.1.1. Assumptions and technical details of the discounting

An important concept for real life applications has not yet been introduced. We now also want to allow for discounting of the payments. In this section, we focus on the general idea of deterministic discounting.

Definition 3.1.8. *Discounting factor*
Consider the deterministic and piecewise continuous function

$$\varphi : [0, \infty) \to \mathbb{R}$$

as an interest short rate.

Let $t \geq s \geq 0$. The continuous discounting of a payment in the amount of 1 in t, over the interval $[s, t]$, will then be abbreviated by defining the function

$$\kappa(s, t) := \exp\left(- \int_s^t \varphi(u) \, \mathrm{d}u \right) = e^{- \int_s^t \varphi(u) \, \mathrm{d}u} .$$

Let us further introduce the following short notation for the case $s = 0$ and define

$$\kappa(t) := \kappa(0, t)$$

for the discounting of payments in t down to time 0.

The following properties of the discounting function will be helpful in the course of the thesis and are therefore stated for future referencing.

Assertion 3.1.9. *Properties of the discounting*
The discounting function κ has the following properties:

(1) It exists an upper bound D_κ, such that for every $s, t \in [0, T]$ with $s \leq t$ it holds that

$$\kappa(s,t) \leq D_\kappa\,, \tag{3.1.3}$$

i.e. we can always bound the discounting factor by D_κ.

(2) For every $t \in (0, T]$ it holds that

$$\kappa(0,t) = \kappa(0,t^-)\,, \tag{3.1.4}$$

i.e. the discounting function is left-continuous.

(3) The two parts of Definition 3.1.8 also imply the following representation of the original discounting for an interval $[s, t]$ as

$$\kappa(s,t) = \frac{\kappa(t)}{\kappa(s)}\,. \tag{3.1.5}$$

Further, the following general multiplication formula

$$\kappa(r,t) = \kappa(r,s) \cdot \kappa(s,t) \tag{3.1.6}$$

holds for $r \leq s \leq t$. The assumed order of r, s and t is not be necessary in general. Also, this generalizes the previous equation (3.1.5).

Proof. Only sketches of the proofs are provided.

(1) If φ is non-negative, then the complete argument of the exponential function is negative. In the general case, the interest rate is integrable, as the function is piecewise continuous. Therefore, the integral always exist and is a real number, which also implies that the complete discounting has to be a real number, for every pair of time points.

(2) Both the exponential function, as well as the integral are continuous. This especially implies the continuity of the discounting function.

(3) For the proof of the second formula, power laws for the exponential function and the additivity of the integral are used. The first equation follows as a special case, by setting $r = 0$ and using the short hand notation.

□

Comment 3.1.10. *Interpretation of the upper bound to the discounting factor*
The rational behind the upper bound to the discounting factor needs to be further explained. In case of a non-negative interest rate, the discount factor would be naturally bounded by 1, as the argument of the exponential function is non-positive.

Even if the interest rate is negative, the upper bound exists. If it would not exist, then long term pricing could not be done and therefore this is ruled out by the preconditions on the interest rate.

3.1.2. Construction of the payment process

We can now continue to construct the reserve-dependent payment process with discounting, by using the contractual payments, the corresponding integrators and the discounting factor.

Definition 3.1.11. *Cumulative cash flow*
The cumulative or aggregated cash flow $A(t)$ contains all contractual payments of the insurance contract on the interval $[0,t]$ and is given as the càdlàg process $(A_t)_{t\geq 0}$ with

$$A(t) = \sum_{M\in\mathcal{M}} \int\limits_{[0,t]} \mathbb{I}^M_{s^-}\, b_M(s, X^{\mathbb{G}}_{s^-})\, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{[0,t]\times E_I} B_I(s,e,X^{\mathbb{G}}_{s^-})\, \mu_I(\mathrm{d}(s,e)) \tag{3.1.7}$$

or in differential form as

$$A(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-}\, b_M(t, X^{\mathbb{G}}_{t^-})\, \gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I(t,e,X^{\mathbb{G}}_{t^-})\, \mu_I(\mathrm{d}t\times\mathrm{d}e)$$

with $A(0^-) = 0$ *a.s.* as a starting value.

Recall the condition $A(\mathrm{d}t) = 0$ for $t > T$ (with $T < \infty$), which allows us to only consider the cash flow on $[0,T]$, rather than $[0,\infty)$. This technical condition has the implication, that A is a càdlàg process with paths of finite variation on compacts.

The insurer then considers the process $X = (X_t)_{t\geq 0}$ of the aggregated discounted future payments for an insurance contract, given by

$$\begin{aligned} X_t &:= \int\limits_{(t,T]} e^{-\int_t^s \varphi(u)\,\mathrm{d}u}\, A(\mathrm{d}s) = \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, A(\mathrm{d}s) \\ &= \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \mathbb{I}^M_{s^-}\, b_M(s, X^{\mathbb{G}}_{s^-})\, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I(s,e,X^{\mathbb{G}}_{s^-})\, \mu_I(\mathrm{d}(s,e)) \end{aligned}$$

where the integrals involved are to be understand as path-wise Lebesgue-Stieltjes integrals, which here and later on, will always be denoted by writing the integrator as $\gamma(\mathrm{d}s)$ or $A(\mathrm{d}s)$, instead of $\mathrm{d}\gamma(s)$ or $\mathrm{d}A(s)$. Take note, that this can be done because the integrator is right-continuous (even càdlàg) and has paths of finite variation on compacts, which then allows for the path-wise version of the Lebesgue-Stieltjes integral.

The process X is potentially non-linearly depending on $X^{\mathbb{G}}$ and the additive structure of the sojourn and jump payments can be used. Further, also X has integrable variation on compacts, since this directly translates from the property of A.

The variable t now appears in the integration interval, as well as the discounting. We want to decouple the integrand, i.e. the discounting from t, such that t is only part of the integration interval, but not of the integrand as well. The multiplicative structure of v from 3.1.6 may be used for this purpose and we can now also define a second payment process, where the discounting is done on $[0, s]$ instead of $[t, s]$, decoupling the integrand from t, by

$$Y_t := \kappa(t) \cdot X_t = \int_{(t,T]} \kappa(s)\, A(\mathrm{d}s) \tag{3.1.8}$$

which in integral representation is given as

$$Y_t = \sum_{M \in \mathcal{M}} \int_{(t,T]} \kappa(s)\, \mathbb{I}^M_{s-}\, b_M\big(s, X^{\mathbb{G}}_{s-}\big)\, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int_{(t,T] \times E_I} \kappa(s)\, B_I\big(s, e, X^{\mathbb{G}}_{s-}\big)\, \mu_I(\mathrm{d}(s,e))\,.$$

We have indeed managed to decouple the integrand from the variable t, and by multiplying (3.1.8) with the deterministic factor $\kappa(t)^{-1}$, we can make X_t reappear. The case without discounting can be restored by setting $\varphi \equiv 0$, which leads to the exponential part being 1, and therefore $X = Y$.

Given the interchangeability, it is perfectly fine to guarantee the existence of the process Y rather than X. We will now show the existence and uniqueness of the (discounted to zero) payment process $\big(Y_t\big)_{t \geq 0}$.

Definition 3.1.12. *Information structures*
Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration of full information in the marked point process setting, according to Christiansen [Chr21b].
Let further $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be the, possibly non-monotone, family of sigma-algebras. In case of non-monotonicity, the notation would be based on [Chr21b].

Also take note, that independently of the setting, the condition

$$\mathcal{G}_t \subseteq \mathcal{F}_t, \quad \text{for all } t \geq 0$$

always holds.

3.2. Existence and uniqueness results for the payment process

We now want to prove the existence and uniqueness of the payment process, when allowing for reserve-dependent payments. This will indeed be done in two consecutive steps, where we first one only allows payments in time t to be dependent on the prospective reserve $X^{\mathbb{G}}_{t-}$.

3.2.1. Automorphism and recursion

Similar to the general BSDE methods, we have to create a fixed point equation together with an automorphism as a mapping. To do this, we break down the construction of our payment process into a three step iterative approach. It is important, that the reserve-dependent modelling of the payments has to be done by using the normal discounted cash-flow, while for technical reasons, we have to work with the process Y.

Let the iteration index $n \in \mathbb{N}$ be fixed, and let the process $\left({}^{(n)}Y_t\right)_{t\geq 0}$ be given as the current iteration, used as a predecessor for the new reserve. The iterative process consists of the following steps, that have to be performed in the specified order:

(1) Calculation of
$$
{}^{(n)}X_t = \frac{1}{\kappa(t)} \cdot {}^{(n)}Y_t = \exp\left(\int_0^t \varphi(r)\,\mathrm{d}r \right) \cdot {}^{(n)}Y_t
$$
as the usual version of the payment process by reversing the additional discounting of all payments on 0 to t.

(2) Application of Theorem 2.4.1, which guarantees the existence of the optional projection as a càdlàg process ${}^{(n)}X^{\mathbb{G}}$, with
$$
\begin{aligned}
{}^{(n)}X^{\mathbb{G}}_s &= \mathbb{E}\left[{}^{(n)}X_s \;\middle|\; \mathcal{G}_s \right] \\
{}^{(n)}X^{\mathbb{G}}_{s-} &= \mathbb{E}\left[{}^{(n)}X_{s-} \;\middle|\; \mathcal{G}_s^- \right]
\end{aligned}
$$
almost surely, where especially the second representation is needed as part of the payments.

(3) Construction of ${}^{(n+1)}Y$ by insertion of the results from (2) into the payments
$$
\begin{aligned}
{}^{(n+1)}Y_t = &\sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \kappa(s)\, \mathbb{I}^M_{s-}\, b_M(s, {}^{(n)}X^{\mathbb{G}}_{s-})\, \gamma(\mathrm{d}s) \\
&+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, B_I(s, e, {}^{(n-1)}X^{\mathbb{G}}_{s-})\, \mu_I(\mathrm{d}(s,e))\,.
\end{aligned}
$$

(4) Starting over with the newly constructed ${}^{(n+1)}Y$, which in total completes the iteration function Φ as
$$
{}^{(n+1)}Y = \Phi\left({}^{(n)}Y \right)
$$

We are now especially interested in expressing the difference of two consecutive iterations to be able to achieve the necessary contraction. Let ${}^{(n+1)}Y$ and ${}^{(n)}Y$ be constructed this way, then their difference can be expressed as
$$
{}^{(n+1)}Y_t - {}^{(n)}Y_t = \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \kappa(s)\, \mathbb{I}^M_{s-} \left(b_M(s, {}^{(n)}X^{\mathbb{G}}_{s-}) - b_M(s, {}^{(n-1)}X^{\mathbb{G}}_{s-}) \right) \gamma(\mathrm{d}s)
$$

$$+\sum_{I\in\mathcal{N}}\int\limits_{(t,T]\times E_I}\kappa(s)\ \left(B_I(s,e,{}^{(n)}X^{\mathbb{G}}_{s^-})-B_I(s,e,{}^{(n-1)}X^{\mathbb{G}}_{s^-})\right)\ \mu_I(\mathrm{d}(s,e))$$

and by grouping terms, the overall difference is expressed as a difference of the payment functions b_M and B_I, which makes it the natural next step to introduce Lipschitz conditions for the families $(b_M)_M$ and $(B_I)_I$. B_I may also depend on $e\in E_I$, but the $e\in E_I$ is the same for both functions, as it describes the information.

Comment 3.2.1. *On the evaluation of the payment functions*
To create the (possibly) non-linear dependency, we might not be able to multiply with the discounting factor as a scalar outside of the payment, i.e.

$$b_M\left(s,X^{\mathbb{G}}_{s^-}\right)=b_M\left(s,\frac{1}{\kappa(s)}Y^{\mathbb{G}}_{s^-}\right)\neq\frac{1}{\kappa(s)}\,b_M\left(s,Y^{\mathbb{G}}_{s^-}\right)$$

are (in general) not the same. Therefore, prior evaluation of X, i.e. discounting of Y, can not be disregarded and has to be carefully managed in the proof.

Assumption 3.2.2. *Lipschitz conditions*
Assume, that there exists a Lipschitz constant $L_{\mathcal{M}}>0$, independent of $M\in\mathcal{M}$, such that for all $X^{\mathbb{G}}_{s^-}(\omega)$, $\tilde{X}^{\mathbb{G}}_{s^-}(\omega)$ we $\mathrm{d}\mathbb{P}\times\mathrm{d}\gamma$ *a.e.* have

$$\left|b_M(s,X^{\mathbb{G}}_{s^-})(\omega)-b_M(s,\tilde{X}^{\mathbb{G}}_{s^-})(\omega)\right|\leq L_{\mathcal{M}}\cdot\left|X^{\mathbb{G}}_{s^-}(\omega)-\tilde{X}^{\mathbb{G}}_{s^-}(\omega)\right|$$

for all $M\in\mathcal{M}$. Under usage of the uniqueness of the current information state

$$\sum_{M\in\mathcal{M}}\mathbb{I}^M_{s^-}=1\tag{3.2.1}$$

for all $s\in[0,T]$, this directly implies, that $\mathrm{d}\mathbb{P}\times\mathrm{d}\gamma$ *a.e.* we have

$$\sum_{M\in\mathcal{M}}\mathbb{I}^M_{s^-}\left|b_M(s,X^{\mathbb{G}}_{s^-})(\omega)-b_M(s,\tilde{X}^{\mathbb{G}}_{s^-})(\omega)\right|\leq L_{\mathcal{M}}\cdot\left|X^{\mathbb{G}}_{s^-}(\omega)-\tilde{X}^{\mathbb{G}}_{s^-}(\omega)\right|\,.$$

Further assume that there exists a second Lipschitz constant $L_{\mathcal{N}}$, independent of $I\in\mathcal{N}$, such that for all $X^{\mathbb{G}}_{s^-}(\omega)$, $\tilde{X}^{\mathbb{G}}_{s^-}(\omega)$ and all $e\in E_I$ we $\mathrm{d}\mathbb{P}\times l_I(s,E_I)\mathrm{d}s$ *a.e.* have

$$\left|B_I(s,e,X^{\mathbb{G}}_{s^-})(\omega)-B_I(s,e,\tilde{X}^{\mathbb{G}}_{s^-})(\omega)\right|\leq L_{\mathcal{N}}\cdot\left|X^{\mathbb{G}}_{s^-}(\omega)-\tilde{X}^{\mathbb{G}}_{s^-}(\omega)\right|\,.$$

Let us further assume, that for all $s\in\{t_0,\,t_1,\dots\}$, we have the following stronger Lipschitz condition for the deterministic time points or reserve-dependent singular payments. Assume there exists a second Lipschitz constant $J<1$, independent of $I\in\mathcal{N}$ and $M\in\mathcal{M}$, such that for all $X^{\mathbb{G}}_{s^-}(\omega)$, $\tilde{X}^{\mathbb{G}}_{s^-}(\omega)$ and all $e\in E_I$ we $\mathrm{d}\mathbb{P}\times\mathrm{d}\lambda$ *a.e.* have

$$\Bigg|\sum_{M\in\mathcal{M}}\mathbb{I}^M_{s^-}(\omega)\left(b_M(s,X^{\mathbb{G}}_{s^-})(\omega)-b_M(s,\tilde{X}^{\mathbb{G}}_{s^-})(\omega)\right)$$

$$+ \sum_{I\in\mathcal{N}} \int_{E_I} \left(B_I(s,e,X^{\mathbb{G}}_{s-})(\omega) - B_I(s,e,\tilde{X}^{\mathbb{G}}_{s-})(\omega)\right) \mu_I(\{s\}\times \mathrm{d}e)(\omega)\Bigg| \\ \leq J \cdot \left|X^{\mathbb{G}}_{s-}(\omega) - \tilde{X}^{\mathbb{G}}_{s-}(\omega)\right|$$

for all time points $s \in \{t_0,\, t_1, \dots\}$.

Assumption 3.2.3. *Intensity of the compensator*
Suppose that each λ_I has a non-negative and càdlàg Lebesgue intensity l_I of the form

$$\lambda_I\left((0,t]\times A\right) = \int_{(0,t]} l_I(s,A)\,\mathrm{d}s$$

and further assume that each ν_I has a non-negative and càdlàg Lebesgue intensity n_I of the form

$$\nu_I\left((0,t]\times A\right) = \int_{(0,t]} n_I(s,A)\,\mathrm{d}s$$

for $A \in \mathcal{E}_I$ and each ρ_I has a non-negative and càdlàg Lebesgue intensity r_I of the form

$$\rho_I\left((0,t]\times A\right) = \int_{(0,t]} r_I(s,A)\,\mathrm{d}s$$

for $A \in \mathcal{E}_I$. All families of function are assumed to be non-negative and càdlàg.

Let us further introduce an joint majorant for the intensities. Let there be a constant $D < \infty$, such that

$$\sum_{I\in\mathcal{N}} |l_I(t,E_I)| \leq D$$

almost surely for all $t \in [0,T]$.

It is beneficial to assume this condition only for the intensities l_I of the $\mathbb{F}$-compensator λ_I and allows us to only focus on the classical $\mathbb{F}$-intensities, enabled by using the following Theorem about their connection. In the literature, similar results are referenced as the so called innovation theorem, but its usually only considered in case of two filtrations, or other specific cases that can not be used here. For reference see also [Aal78] and [Jac05] (Corollary 4.8.5 together with Proposition 4.8.4). It should be noted, that the density of ρ_I does only make sense in the model of Christiansen [Chr21b], when the IB-compensator is actually needed, as it would be $\rho_I = \mu_I$ if $\mathbb{F} = \mathbb{G}$.

Theorem 3.2.4. *Innovation theorem – Connections between $\mathbb{F}$- and $\mathbb{G}$- intensities*
Let $I \in \mathcal{N}$ be fixed. For every $A \in \mathcal{E}_I$

$$\int_{(0,t]} n_I(s,A)\,\mathrm{d}s = \int_{(0,t]} \mathbb{E}\left[l_I(s,A)\,\middle|\,\mathcal{G}^-_s\right]\,\mathrm{d}s$$

almost surely holds for every $t \geq 0$.

A special application of the result is the following: If the $\mathbb{F}$-intensities are equal to zero, then we directly know, that the same for the intensities of the IF-compensator has to hold.

Proof. Using the tower property and the respective properties of the compensators with respect to $\mathbb{F}$ and the IF-compensator with respect to $\mathbb{G}$, for every $t \geq 0$ and every $A \in \mathcal{E}_I$ we point-wise almost surely have

$$
\begin{aligned}
\int\limits_{(0,t]} n_I(s,A)\,\mathrm{d}s &= \nu_I((0,t]\times A) \\
&\overset{\text{Prop. 2.5.6}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mu_I\left((0,t_{k+1}]\times A\right) - \mu_I\left((0,t_k]\times A\right) \,|\, \mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mu_I\left((t_k,t_{k+1}]\times A\right) \,|\, \mathcal{G}_{t_k}\right] \\
&\overset{(i)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mathbb{E}\left[\mu_I\left((t_k,t_{k+1}]\times A\right) \,|\, \mathcal{F}_{t_k}\right] \,|\, \mathcal{G}_{t_k}\right] \\
&\overset{\mathbb{F}\text{-martingale}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\mathbb{E}\left[\lambda_I\left((t_k,t_{k+1}]\times A\right) \,|\, \mathcal{F}_{t_k}\right] \,|\, \mathcal{G}_{t_k}\right] \\
&\overset{(i)}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\lambda_I\left((t_k,t_{k+1}]\times A\right) \,|\, \mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau_n^t} \mathbb{E}\left[\int\limits_{(t_k,t_{k+1}]} l_I(s,A)\,\mathrm{d}s \,\middle|\, \mathcal{G}_{t_k}\right] \\
&\overset{\text{Fubini}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \int\limits_{(t_k,t_{k+1}]} \mathbb{E}\left[l_I(s,A) \,|\, \mathcal{G}_{t_k}\right] \mathrm{d}s \\
&\overset{\text{Form. (2.4.3)}}{=} \lim_{n\to\infty} \sum_{\tau_n^t} \int\limits_{(t_k,t_{k+1}]} \sum_{M\in\mathcal{M}} \mathbb{I}_{t_k}^M\, \mathbb{E}\left[l_I(s,A) \,|\, \mathcal{G}_{t_k}\right] \mathrm{d}s \\
&\overset{(ii)}{=} \sum_{M\in\mathcal{M}} \lim_{n\to\infty} \sum_{\tau_n^t} \int\limits_{(t_k,t_{k+1}]} \mathbb{I}_{t_k}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{t_k}^M\, l_I(s,A)\right]}{\mathbb{E}_M\left[\mathbb{I}_{t_k}^M\right]}\,\mathrm{d}s \\
&\overset{\text{Lemma 2.4.11}}{=} \sum_{M\in\mathcal{M}} \int\limits_{(0,t]} \mathbb{I}_{s-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{s-}^M\, l_I(s,A)\right]}{\mathbb{E}_M\left[\mathbb{I}_{s-}^M\right]}\,\mathrm{d}s \\
&\overset{\text{Form. (2.4.3)}}{=} \int\limits_{(0,t]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s-}^M\, \mathbb{E}\left[l_I(s,A) \,|\, \mathcal{G}_s^-\right] \mathrm{d}s \\
&= \int\limits_{(0,t]} \mathbb{E}\left[l_I(s,A) \,|\, \mathcal{G}_s^-\right] \mathrm{d}s
\end{aligned}
$$

for a sequence of partitions $(\tau_n^t)_{n\in\mathbb{N}}$ of the interval $[0,t]$.

In (i), the tower property was used, since $\mathcal{G}_{t_k} \subseteq \mathcal{F}_{t_k}$ for every t_k, and in (ii) the dominated convergence theorem was applied. The application of Lemma 2.4.11 yields the representation of the integral in s^-, as the t_k from the lower integral bound are part of the integrated function.

□

The following proposition states, that it is sufficient to only demand the assumption about an upper bound to only hold for the intensities of the classical $\mathbb{F}$-compensator, which is beneficial for application and also more in alignment with standard theory.

Proposition 3.2.5. *Inheritance of the upper bound for the intensities*
If there exists a constant $D < \infty$, such that

$$\sum_{I \in \mathcal{N}} |l_I(t, E_I)| \leq D$$

almost surely for all $t \in [0, T]$, then with the same constant it also holds, that

$$\sum_{I \in \mathcal{N}} |n_I(t, E_I)| \leq D$$

almost surely.

Proof. Through an application of the previous Theorem 3.2.4, in the special setting with $A = E_I$, we arrive at the first equality. An application of the inequality of Jensen A.2.12 and the conditional monotone convergence theorem in (i) then yields, that

$$\begin{aligned}
\sum_{I \in \mathcal{N}} |n_I(t, E_I)| \,\mathrm{d}t &= \sum_{I \in \mathcal{N}} |\mathbb{E}\left[l_I(t, E_I) \,|\, \mathcal{G}_{t-}\right]| \,\mathrm{d}t \\
&\overset{A.2.12}{\leq} \sum_{I \in \mathcal{N}} \mathbb{E}\left[\, |l_I(t, E_I)| \,|\, \mathcal{G}_{t-}\right] \mathrm{d}t \\
&\overset{\text{(i)}}{=} \mathbb{E}\left[\sum_{I \in \mathcal{N}} |l_I(t, E_I)| \,\middle|\, \mathcal{G}_{t-} \right] \mathrm{d}t \\
&\overset{\text{Precond.}}{\leq} \mathbb{E}\left[D \,|\, \mathcal{G}_{t-}\right] \mathrm{d}t = D \,\mathrm{d}t
\end{aligned}$$

what was to show. Take note, that the intensities are non-negative, which is why the first equality holds and the absolute value can be used without changing anything. □

Comment 3.2.6. *Comparison of the Lipschitz condition to classical BSDE methods*
If the Lipschitz condition is assumed for a generator function f, i.e there exists $L > 0$ such that for all $X^{\mathbb{G}}_{s^-}(\omega)$, $\tilde{X}^{\mathbb{G}}_{s^-}(\omega)$ we $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ *a.e.* we have

$$\left| f(\omega, t, X^{\mathbb{G}}_{t^-}) - f(\omega, t, \tilde{X}^{\mathbb{G}}_{t^-}) \right| \leq L \cdot \left| X^{\mathbb{G}}_{t^-}(\omega) - \tilde{X}^{\mathbb{G}}_{t^-}(\omega) \right|$$

then by using the special representation of f in the insurance example

$$f\left(\omega, t, X_{t^-}^{\mathbb{G}}\right) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^{M}\, b_M\left(t, X_{t^-}^{\mathbb{G}}\right) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I\left(t, e, X_{t^-}^{\mathbb{G}}\right) l_I(t, \mathrm{d}e)$$

the condition is similar, but the details are hidden. The conditions in our case are especially sufficient for the general case, when using the triangle inequality and defining L via $L_{\mathcal{M}}, L_{\mathcal{N}}$ and D.

3.2.2. Theorem – Existence and uniqueness

Let us now proceed by deciding on a norm, that allows us to show that

$$\begin{aligned}\left\| {}^{(n+1)}Y - {}^{(n)}Y \right\| &= \left\| \Phi\left({}^{(n)}Y\right) - \Phi\left({}^{(n-1)}Y\right) \right\| \\ &\leq C \cdot \left\| {}^{(n)}Y - {}^{(n-1)}Y \right\| ,\end{aligned}$$

where $C < 1$ would be the contraction constant.

In that case, our iteration function Φ would indeed be a contraction mapping and a fixed point theorem could be applied. Without specifying or checking the preconditions, as a consequence of the application of the Theorem of Banach A.3.1, a unique fixed point ${}^{*}Y$ would exist, fulfilling

$$\begin{aligned}{}^{*}Y_t &= \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}_{s^-}^{M}\, \kappa(s)\, b_M\left(s, \frac{1}{\kappa(s)}\, {}^{*}Y_{s^-}^{\mathbb{G}}\right) \gamma(\mathrm{d}s) \\ &\quad + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, B_I\left(s, e, \frac{1}{\kappa(s)}\, {}^{*}Y_{s^-}^{\mathbb{G}}\right) \mu_I(\mathrm{d}(s,e)) \\ &=: \Phi({}^{*}Y)_t \end{aligned} \tag{3.2.2}$$

where ${}^{*}Y_{t^-}^{\mathbb{G}} = \mathbb{E}\left[{}^{*}Y_{t^-} \,\middle|\, \mathcal{G}_t^{-}\right]$ *a.s.* as usual.

For an application of the fixed point Theorem of Banach A.3.1, we need to specify a non-empty complete metric space. Instead of choosing a metric, we specify a norm and use the induced metric corresponding to the norm.

Since we consider pathwise Lebesgue-Stieltjes integration, where the integrator, as a non-decreasing and right-continuous (distribution) function G, corresponds to a locally finite signed measure μ_G, that is then used to define the integral, compare Kallenberg [Kal21] for example.

The Banach space will now be set up for càdlàg functions on $[0, T]$, and we use a norm that is based on the Hahn-Banach decomposition of a function and is therefore isometrically isomorphic to the Banach space of signed Borel measures with support in $[0, T]$. This construction is done based on a similar approach for state-wise reserves in Christiansen [Chr10].

Definition 3.2.7. *Banach space of functions with bounded variation*
As a space we want to consider the processes with bounded variation on interval $[0,T]$. The function space

$$BV_{[0,T]} := \left\{ f : [0,T] \to \mathbb{R} \;\middle|\; f \text{ càdlàg},\ f(T) = 0,\ \|f\|_{V[0,T]} < \infty \right\} \tag{3.2.3}$$

is then actually a Banach space. We now use the correspondence of Jordan-Hahn decomposition of a function f as $f = f^+ - f^-$ and define the variation of f as $|f| := f^+ + f^-$ and the norm is given as

$$\|f\|_{V[0,T]} := \int\limits_{[0,T]} \mathrm{d}\,|f|_t$$

as the total variation norm.

Comment 3.2.8. *On the variation norm*
The above definition does only define a norm in the space (3.2.3), since we use $f(T) = 0$, which would otherwise be present as an additional summand. It is fine to leave it out in our application, since every payment process X fulfils this condition by definition and the additional summand $|f(T)|$ of the norm would always be zero.
The reason for the additional part is the definiteness of the norm. Every constant function has a variation of zero, but only the zero function has a norm of zero. This is shown in detail in [ABD13] (Proposition 1.10), where the absolute value $|f(a)|$ for a as the left bound of the interval $[a,b]$ is used, instead of the right value. The proof can be adapted to this case, where a backward perspective on the interval is taken.

We are now going to extend the concepts to stochastic processes, where the variation is defined as a path-wise property and which align with our situation in the life insurance theory with path-wise Lebesgue-Stieltjes integrals.

Definition 3.2.9. *Finite variation process*
Let X be a stochastic process on $\mathbb{R}_{\geq 0}$. Let $\tau = \{0 = t_0,\, t_1, \ldots, t_k = T\}$ be a partition of $[0,T]$ with increasing time points and of index k. Define the total variation of X as

$$V_{[0,T]}(X_.(\omega)) := \sup_{\tau} \sum_{i=1}^{k} \left| X_{t_k}(\omega) - X_{t_{k-1}}(\omega) \right| = \int\limits_{[0,T]} |\mathrm{d}X_.(\omega)|_s \, .$$

A process X is then said to be of finite variation, if almost every path is of finite variation, i.e. $V_{[0,t]}(X_.(\omega)) < \infty$ for every $t \geq 0$ and almost every $\omega \in \Omega$. See also Meintrup and Schäffler [MS05] for a similar construction and Protter [Pro05] for the representation via the integral.

A slightly different, but equivalent, norm is used for the solution space. A weighting factor for has to be included, because it is not possible to show the contraction property without the weight, which, as part of the norm, will be expressed as an exponential. Therefore, no properties of the norm are compromised and equivalence of the norms, with and without the weighting factor, can be shown.

Definition 3.2.10. *Solution space*
The space of càdlàg stochastic processes X on $[0,T]$ with final value $X(T)=0$ *a.s.* and with integrable variation is given as

$$BV^X_{[0,T]} := \left\{ X = (X_t)_{t\in[0,T]} : \Omega \times [0,T] \to \mathbb{R} \,\middle|\, X \text{ càdlàg},\, X_T = 0 \text{ } a.s.,\, \|X\|_{V[0,T]} < \infty \right\} \tag{3.2.4}$$

and the equivalent norm is defined as the expectation of the weighted variation norm as

$$\|X\|_{V[0,T]} := \mathbb{E}\left[\|X\|_{V[0,T],K,\zeta}\right] = \mathbb{E}\left[\int\limits_{[0,T]} e^{-K(\zeta(T)-\zeta(t))}\,\mathrm{d}\,|X|_t\right] \tag{3.2.5}$$

where $K>0$ is a constant, that will be chosen to guarantee the contraction property and the measure ζ is deterministic and has to be

$$\zeta(\mathrm{d}t) = \gamma(\mathrm{d}t) + D\,\mathrm{d}t$$

for the first theorem.

The definition of the weighting factor is also based on the construction in Christiansen [Chr10]. Take note, that as a major difference, we include the expectation as an outer operator, as the path-wise Lebesgue integration is not enough to identify zero as a random variable, as almost every path has the be zero.

The representation of ζ and K will result from the additive decomposition of the payments in the sojourn parts, where $\gamma(\mathrm{d}t)$ arises, and the jump part, where the second part arises as an upper bound of the compensators.

Comment 3.2.11. *On the chosen norm*
The above defined norm, together with the solution space, may be seen as a natural choice, since we are in a situation, where the payment process is a finite variation process. Therefore, the existence and uniqueness of solutions with respect to this Banach space and norm is not really a restriction, but arises naturally.

Theorem 3.2.12. *Existence and uniqueness of the payment process Y*
Under the conditions on Y and its parts, namely Assumptions 3.2.2, 3.3.8 and 3.2.3 about Lipschitz-conditions and the bound for the compensator density, and Definitions 3.1.1, 3.1.2 and 3.1.5 about the payment functions, the payment process Y exists and is unique up to indistinguishability as the solution of the integral equation

$$\begin{cases} Y_t = \displaystyle\sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s^-}\, b_M\left(s, \frac{1}{\kappa(s)} Y^{\mathbb{G}}_{s^-}\right) \gamma(\mathrm{d}s) \\ \qquad + \displaystyle\sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} B_I\left(s, e, \frac{1}{\kappa(s)} Y^{\mathbb{G}}_{s^-}\right) \mu_I(\mathrm{d}(s,e)) \\ Y_T = 0 \end{cases} \tag{3.2.6}$$

in the space of càdlàg processes with finite integrable variation, given in (3.2.4), equipped with the (equivalent) weighted norm

$$\|X\|_{V[0,T]} := \mathbb{E}\left[\|X\|_{V[0,T],K,\zeta}\right] = \mathbb{E}\left[\int\limits_{[0,T]} e^{-K(\zeta(T)-\zeta(t))}\,\mathrm{d}\,|X|_t\right]$$

where the constant

$$K := 2\cdot\frac{L}{1-J}$$

is the weighting factor with $L = \max\{L_{\mathcal{M}}, L_{\mathcal{N}}\}$, and ζ is defined via

$$\zeta(\mathrm{d}t) := \gamma(\mathrm{d}t) \,+\, D\,\mathrm{d}t\,.$$

Proof. The proof is performed in three steps.

Automorphism

We begin by showing that the mapping Φ, as specified by (3.2.2) is an automorphism on the solution space of processes with integrable variation. Therefore, let ${}^{(n)}Y \in BV^X_{[0,T]}$ be a process of integrable variation. We have to show that ${}^{(n+1)}Y$ is a process of integrable variation as well.

For every $0 \leq s < t \leq T$ and with $(\tau_m)_{m\in\mathbb{N}}$ as a sequence of partitions of the interval $[s,t]$ with $\{t_0 = s, \ldots, t_m = t\}$ and $\lim_{m\to\infty}|\tau_m| = 0$, we get

$$\begin{aligned}
&\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\mathrm{d}\left|{}^{(n+1)}Y\right|_u \\
&= \sup_{\tau_m}\sum_{j=1}^{m} e^{-K(\zeta(T)-\zeta(t_{j-1}))}\cdot\left|{}^{(n+1)}Y_{t_j} - {}^{(n+1)}Y_{t_{j-1}}\right| \\
&\leq \sup_{\tau_m}\sum_{j=1}^{m} e^{-K(\zeta(T)-\zeta(t_{j-1}))}\cdot\left|\int\limits_{(t_{j-1},t_j]}\sum_{M\in\mathcal{M}}\mathbb{I}^M_{u^-}\,\kappa(u)\,b_M\big(u,{}^{(n)}X^{\mathbb{G}}_{u^-}\big)\gamma(\mathrm{d}u)\right. \\
&\qquad\qquad\left. + \sum_{I\in\mathcal{N}}\int\limits_{(t_{j-1},t_j]\times E_I}\kappa(u)B_I\big(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}\big)\,\mu_I(\mathrm{d}(u,e))\right| \\
&\overset{(*)}{\leq} \sup_{\tau_m}\sum_{j=1}^{m}\left|\int\limits_{(t_{j-1},t_j]} e^{-K(\zeta(T)-\zeta(u))}\sum_{M\in\mathcal{M}}\mathbb{I}^M_{u^-}\,\kappa(u)\,b_M\big(u,{}^{(n)}X^{\mathbb{G}}_{u^-}\big)\,\gamma(\mathrm{d}u)\right. \\
&\qquad\left. + \sum_{I\in\mathcal{N}}\int\limits_{(t_{j-1},t_j]\times E_I} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\,B_I\big(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}\big)\,\mu_I(\mathrm{d}(u,e))\right| \\
&\overset{\triangle-\text{ineq.}}{\leq} \int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\sum_{M\in\mathcal{M}}\mathbb{I}^M_{u^-}\,\kappa(u)\,\left|b_M\big(u,{}^{(n)}X^{\mathbb{G}}_{u^-}\big)\right|\gamma(\mathrm{d}u)
\end{aligned}$$

$$+ \sum_{I \in \mathcal{N}} \int\limits_{(s,t] \times E_I} e^{-K(\zeta(T)-\zeta(u))} \kappa(u) \left| B_I(u, e, {}^{(n)}X^{\mathbb{G}}_{u-}) \right| \mu_I(\mathrm{d}(u,e))$$

where we also used in $(*)$, that ζ is monotonously increasing, to be able to pull the exponential part into the integral and the triangle inequality in the last step.

Then, by using the upper bound from above, for the norm it holds that

$$\begin{aligned}
&\left\| \Phi\left({}^{(n)}Y\right) \right\|_{V[0,T]} \\
&= \left\| {}^{(n+1)}Y \right\|_{V[0,T]} = \mathbb{E}\left[\left\| {}^{(n+1)}Y \right\|_{V[0,T],K,\zeta} \right] \\
&= \mathbb{E}\left[\int\limits_{[0,T]} \underbrace{e^{-K(\zeta(T)-\zeta(s))}}_{\leq 1} \mathrm{d} \left| {}^{(n+1)}Y \right|_s \right] \\
&\leq \mathbb{E}\left[\int\limits_{(0,T]} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{s-} \kappa(s) \underbrace{\left| b_M(s, {}^{(n)}X^{\mathbb{G}}_{s-}) \right|}_{\leq J_{\mathcal{M}}(s)} \gamma(\mathrm{d}s) \right] \\
&\quad + \mathbb{E}\left[\sum_{I \in \mathcal{N}} \int\limits_{(0,T] \times E_I} \kappa(s) \underbrace{\left| B_I(s, e, {}^{(n)}X^{\mathbb{G}}_{s-}) \right|}_{\leq J_{\mathcal{N}}(s)} \mu_I(\mathrm{d}(s,e)) \right] \\
&\leq \mathbb{E}\left[\int\limits_{(0,T]} J_{\mathcal{M}}(s) \cdot \underbrace{\kappa(s)}_{\leq D_\kappa} \cdot \underbrace{\sum_{M \in \mathcal{M}} \mathbb{I}^M_{s-}}_{=1} \gamma(\mathrm{d}s) \right] \\
&\quad + \mathbb{E}\left[\sum_{I \in \mathcal{N}} \int\limits_{(0,T] \times E_I} \underbrace{\kappa(s)}_{\leq D_\kappa} \cdot J_{\mathcal{N}}(s) \, \mu_I(\mathrm{d}(s,e) \right] \\
&\leq D_\kappa \cdot \mathbb{E}\left[\int\limits_{(0,T]} J_{\mathcal{M}}(s) \, \gamma(\mathrm{d}s) \right] + D_\kappa \cdot \mathbb{E}\left[\sum_{I \in \mathcal{N}} \int\limits_{(0,T] \times E_I} J_{\mathcal{N}}(s) \, \mu_I(\mathrm{d}(s,e)) \right] \\
&< \infty
\end{aligned}$$

where both parts are finite, respectively.

The discounting factor can be bounded by D_κ from above, compare (3.1.3), and for the functions we used the integrable majorant $J_{\mathcal{M}}$ and $J_{\mathcal{N}}$ that were introduced in Definitions 3.1.2 and 3.1.5 and are independent of the indices M and I. The second summand is finite, since it is the precondition for Theorem 2.5.4 and needed to guarantee the existence of that integral. Therefore, in total we have ${}^{(n+1)}Y \in BV^X_{[0,T]}$.

Contraction property

With similar arguments as before, we now prepare to look at the difference of two consecutive iterations. We again start by deriving an upper bound for the norm.

Start by disregarding the outer expectation. For every $0 \leq s < t \leq T$ we get

$$\begin{aligned}
&\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))} \,\mathrm{d} \left| {}^{(n+1)}Y - {}^{(n)}Y \right|_u \\
&= \sup_{\tau_m} \sum_{j=1}^{m} e^{-K(\zeta(T)-\zeta(t_{j-1}))} \left| \left({}^{(n+1)}Y_{t_j} - {}^{(n)}Y_{t_j} \right) - \left({}^{(n+1)}Y_{t_{j-1}} - {}^{(n)}Y_{t_{j-1}} \right) \right| \\
&\leq \sup_{\tau_m} \sum_{j=1}^{m} e^{-K(\zeta(T)-\zeta(t_{j-1}))}. \\
&\qquad \left| - \int\limits_{(t_{j-1},t_j]} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{u^-}\, \kappa(u) \left(b_M(u, {}^{(n)}X^{\mathbb{G}}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right) \gamma(\mathrm{d}u) \right. \\
&\qquad \left. - \sum_{I \in \mathcal{N}} \int\limits_{(t_{j-1},t_j] \times E_I} \kappa(u) \left(B_I(u,e, {}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right) \mu_I(\mathrm{d}(u,e)) \right| \\
&\overset{(*)}{\leq} \sup_{\tau_m} \sum_{j=1}^{m} \\
&\quad \left| - \int\limits_{(t_{j-1},t_j]} e^{-K(\zeta(T)-\zeta(u))} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{u^-}\, \kappa(u) \left(b_M(u, {}^{(n)}X^{\mathbb{G}}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right) \gamma(\mathrm{d}u) \right. \\
&\quad \left. - \sum_{I \in \mathcal{N}} \int\limits_{(t_{j-1},t_j] \times E_I} e^{-K(\zeta(T)-\zeta(u))} \kappa(u) \left(B_I(u,e, {}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right) \mu_I(\mathrm{d}(u,e)) \right| \\
&\overset{\triangle-\text{ineq.}}{\leq} \int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{u^-}\, \kappa(u) \left| b_M(u, {}^{(n)}X^{\mathbb{G}}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right| \gamma(\mathrm{d}u) \\
&\qquad + \sum_{I \in \mathcal{N}} \int\limits_{(s,t] \times E_I} e^{-K(\zeta(T)-\zeta(u))} \kappa(u) \left| B_I(u,e, {}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e, {}^{(n-1)}X^{\mathbb{G}}_{u^-}) \right| \mu_I(\mathrm{d}(u,e)),
\end{aligned}$$

where $(\tau_m)_{m \in \mathbb{N}}$ is again a sequence of partitions of $[s,t]$ with $\{t_0 = s, \ldots, t_m = t\}$ and $\lim_{m\to\infty} |\tau_m| = 0$ and otherwise similar arguments as in the situation above.

Include the outer expectation and as a consequence, for every $0 \leq s < t \leq T$, or equivalently for every subinterval $(s,t] \subseteq (0,T]$, we achieve the upper bound

$$\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))} \,\mathrm{d} \left| {}^{(n+1)}Y - {}^{(n)}Y \right|_u \right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-} \left|b_M(u, {}^{(n)}X^{\mathbb{G}}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G}}_{u^-})\right| \gamma(\mathrm{d}u)\right]$$
$$+\,\mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\left|B_I(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-})\right| \mu_I(\mathrm{d}(u,e))\right]$$
$$=: (\mathrm{I}) + (\mathrm{II})\,,$$

yielding an additive structure, where we can investigate both summands separately.

For the first part, we use the Lipschitz condition for b_M for every $M \in \mathcal{M}$ and get the following upper bound

$$(\mathrm{I}) = \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-} \underbrace{\left|b_M(u, {}^{(n)}X^{\mathbb{G}}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G}}_{u^-})\right|}_{\leq L_{\mathcal{M}}\cdot\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|} \gamma(\mathrm{d}u)\right]$$
$$\leq \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \underbrace{\sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-}}_{=1} \cdot L_{\mathcal{M}} \cdot \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| \gamma(\mathrm{d}u)\right]$$
$$\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{M}} \cdot e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| \gamma(\mathrm{d}u)\right]\,,$$

where we additionally used formula (3.2.1), for every $u \in (s,t] \subseteq (0,T]$ once there is no more dependency of function b_M on M.

For the second summand, we make use of the classical compensator λ with respect to $\mathbb{F}$ and its density l, to get

$$\begin{aligned}
&(\mathrm{II})\\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(T)-\zeta(u))}\kappa(u)\left|B_I(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-})\right| \mu_I(\mathrm{d}(u,e))\right]\\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} \kappa(u)\,G_I(u,e)\,\mu_I(\mathrm{d}(u,e))\right]\\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_I} \kappa(u)\,G_I(u,e)\,\mu_I(\mathrm{d}(u,e))\right]\\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[(\kappa G_I) \bullet \mu_I((s,t]\times E_I)\right]
\end{aligned}$$

$$
\begin{aligned}
&\overset{(i)}{=} \sum_{I\in\mathcal{N}} \mathbb{E}\left[(\kappa G_I) \bullet \lambda_I((s,t]\times E_I)\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \lambda_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \lambda_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(T)-\zeta(u))}\kappa(u) \left|B_I(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-})\right| \lambda_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\kappa(u) \sum_{I\in\mathcal{N}} \int\limits_{E_I} \underbrace{\left|B_I(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-})\right|}_{\leq L_{\mathcal{N}} \left|{}^{(n)}X_{u^-}{}^{\mathbb{G}} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|} l_I(t,\mathrm{d}e)\,\mathrm{d}u\right] \\
&\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\cdot e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \sum_{I\in\mathcal{N}} \int\limits_{E_I} \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| l_I(u,\mathrm{d}e)\,\mathrm{d}u\right] \\
&= \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\cdot e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| \sum_{I\in\mathcal{N}} l_I(u,E_I)\,\mathrm{d}u\right] \\
&\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\cdot e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| D\,\mathrm{d}u\right].
\end{aligned}
$$

In (i), we need that for every summand

$$
G_I(u,e) = e^{-K(\zeta(T)-\zeta(u)}\,\kappa(u)\cdot\left|B_I(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-})\right|
$$

is $\mathcal{F}^-_u$-measurable for every (u,e). This is guaranteed, since $\mathcal{G}_u \subseteq \mathcal{F}_u$ and we assumed the involved properties to be $\mathcal{G}^-_u$-measurable when formulating the necessary conditions. This holds both for the case, where $\mathbb{G}$ is a sub-filtration of $\mathbb{F}$, as well as for the case, where $\mathbb{G}$ has the special structure of [Chr21b].
It is beneficial to work with $\mathbb{F}$ as the outer structure here, since we do not need that strict conditions for the application of the classical martingale representation Theorem 2.5.12 in comparison to the analogue Theorem 2.5.13 in [Chr21b].
We then also used the existence of the density l_I for the compensator λ_I, which results in a representation as a du-integral. By design, when applying the Lipschitz-condition, the integrand does not further depend on e, and we can use the same majorant for the density process of the compensator.

Let us continue by defining $L := \max\{L_{\mathcal{M}}, L_{\mathcal{N}}\}$ as a joint Lipschitz constant, to further simplify the notation. In both summands, we arrive at a similar structure and we rejoin them to get the upper bound

$$
\begin{aligned}
&\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\mathrm{d}\left|{}^{(n+1)}Y - {}^{(n)}Y\right|_u\right] \\
&\leq \mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\,\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|\,\gamma(\mathrm{d}u)\right] \\
&\qquad + \mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\,\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|\,D\,\mathrm{d}u\right] \\
&= \mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\kappa(u)\,\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|\,\underbrace{\Big(\gamma(\mathrm{d}u) + D\,\mathrm{d}u\Big)}_{=\zeta(\mathrm{d}u)}\right] \\
&= \int\limits_{(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|\right]\,\zeta(\mathrm{d}u) \\
&\overset{(ii)}{\leq} \int\limits_{(s,t]} L\,e^{-K\,(\zeta(T)-\zeta(u))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\right]\,\zeta(\mathrm{d}u)\,,
\end{aligned}
$$

where we used the Theorem of Fubini-Tonelli to exchange the order of integration, which means that, the expectation can be evaluated first, where only the non-deterministic part has to be considered. Remember, that the discounting is deterministic.

In (ii), we applied the inequality of Jensen A.2.12 for conditional expectations to get

$$
\begin{aligned}
&\mathbb{E}\left[\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right|\right] \\
&= \mathbb{E}\left[\left|\mathbb{E}\left[{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\,\middle|\,\mathcal{G}^-_u\right]\right|\right] \\
&\overset{A.2.12}{\leq} \mathbb{E}\left[\mathbb{E}\left[\left|\,{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\,\middle|\,\mathcal{G}^-_u\right]\right] \\
&= \mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\right]\,,
\end{aligned}
\tag{3.2.7}
$$

which allows us to get rid of the optional projections. We can see one of the major advantages here, since we only need a sigma-algebra to perform this step, but the exact structure of the family $\mathbb{G}$ is not used and therefore not necessary to fix. In total, this step enables us to introduce the needed difference of predecessors for our iteration. We have

$$
\begin{aligned}
&\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\mathrm{d}\left|{}^{(n+1)}Y - {}^{(n)}Y\right|_u\right] \\
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\right]\,\zeta(\mathrm{d}u)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\mathbb{E}\left[\left|\kappa(u^-)\right|\,\left|{}^{(n)}X_{u^-}-{}^{(n-1)}X_{u^-}\right|\right]\,\zeta(\mathrm{d}u)\\
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(T)-\zeta(u))}\,\mathbb{E}\left[\left|{}^{(n)}Y_{u^-}-{}^{(n-1)}Y_{u^-}\right|\right]\,\zeta(\mathrm{d}u)\\
&\overset{(iii)}{\leq} \int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(T)-\zeta(u))}\,\mathbb{E}\left[\left|{}^{(n)}Y_{u}-{}^{(n-1)}Y_{u}\right|\right]\,\zeta(\mathrm{d}u)\\
&= \mathbb{E}\left[\int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(T)-\zeta(u))}\,\left|{}^{(n)}Y_{u}-{}^{(n-1)}Y_{u}\right|\,\zeta(\mathrm{d}u)\right]\\
&\overset{(iv)}{\leq} \mathbb{E}\left[\int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(T)-\zeta(u))}\left(\int\limits_{r\in[u,T]}\mathrm{d}\left|{}^{(n)}Y-{}^{(n-1)}Y\right|_r\right)\zeta(\mathrm{d}u)\right],
\end{aligned}
$$

where we used Formula (3.1.4), i.e. that $\kappa(t^-)=\kappa(t)$. Further it holds, that $\kappa(t)=|\kappa(t)|$, because of the non-negativity of the exponential function. Therefore the absolute difference can be scaled with the factor $\kappa(t)$, enabling us to change back from X to Y. Afterwards, the Theorem of Fubini-Tonelli is used in opposite direction to before.

Let us give some additional details about step (iii). Remember our definition of ζ as

$$
\zeta(\mathrm{d}t)=\gamma(\mathrm{d}t)+D\,\mathrm{d}t
$$

of a Lebesgue-part, multiplied with a constant, and the deterministic jumps, originating from the definition of γ.

A special consideration has to be given to the deterministic time point t_0, t_1, …, where the γ-part of ζ might introduce additional reserve-dependent payments. We now give details to the upper bound, that has been used in step (iii) of the proof. For the deterministic jump points $t\in\{t_0, t_1, \dots\}$, by splitting the possible payments on $[t,T]$ in time point t and the interval $(t,T]$, it holds

$$
\begin{aligned}
&\left|{}^{(n)}Y_{t^-}-{}^{(n-1)}Y_{t^-}\right|\\
&\leq \Bigg|\,{}^{(n)}Y_{t}-{}^{(n-1)}Y_{t}\\
&\qquad+\sum_{M\in\mathcal{M}}\mathbb{I}^M_{t^-}\,\kappa(t)\left(b_M(t,{}^{(n)}X^{\mathbb{G}}_{t^-})-b_M(t,{}^{(n-1)}X^{\mathbb{G}}_{t^-})\right)\mathbb{1}_{t\in\{t_0,\dots,t_n\}}\\
&\qquad+\sum_{I\in\mathcal{N}}\int_{E_I}\kappa(t)\left(B_I(t,e,{}^{(n)}X^{\mathbb{G}}_{t^-})-B_I(t,e,{}^{(n-1)}X^{\mathbb{G}}_{t^-})\right)\mu_I(\{t\}\times\mathrm{d}e)\Bigg|\\
&\leq \left|{}^{(n)}Y_{t}-{}^{(n-1)}Y_{t}\right|\\
&\qquad+\kappa(t)\left|\sum_{M\in\mathcal{M}}\mathbb{I}^M_{t^-}\left(b_M(t,{}^{(n)}X^{\mathbb{G}}_{t^-})-b_M(t,{}^{(n-1)}X^{\mathbb{G}}_{t^-})\right)\right.
\end{aligned}
$$

$$+\sum_{I\in\mathcal{N}}\int_{E_I}\Big(B_I\big(t,e,{}^{(n)}X^{\mathbb{G}}_{t^-}\big)-B_I\big(t,e,{}^{(n-1)}X^{\mathbb{G}}_{t^-}\big)\Big)\,\mu_I(\{t\}\times \mathrm{d}e)\Bigg|$$
$$\leq \left|{}^{(n)}Y_t-{}^{(n-1)}Y_t\right|+J\cdot\left|{}^{(n)}Y_{t^-}-{}^{(n-1)}Y_{t^-}\right|$$

by using the special Lipschitz-condition for simultaneous reserve-dependent payments. Therefore, by applying the expectation, we have

$$\begin{aligned}&\mathbb{E}\left[\left|{}^{(n)}Y_{t^-}-{}^{(n-1)}Y_{t^-}\right|\right]\\ &\leq \mathbb{E}\left[\left|{}^{(n)}Y_t-{}^{(n-1)}Y_t\right|\right]+J\cdot\mathbb{E}\left[\left|{}^{(n)}Y^{\mathbb{G}}_{t^-}-{}^{(n-1)}Y^{\mathbb{G}}_{t^-}\right|\right]\\ &\overset{(3.2.7)}{\leq} \mathbb{E}\left[\left|{}^{(n)}Y_t-{}^{(n-1)}Y_t\right|\right]+J\cdot\mathbb{E}\left[\left|{}^{(n)}Y_{t^-}-{}^{(n-1)}Y_{t^-}\right|\right]\end{aligned}$$

with similar steps as before, when the conditional inequality of Jensen A.2.12 was first used. By rearranging of terms, while using $J<1$, this also implies

$$\mathbb{E}\left[\left|{}^{(n)}Y_{t^-}-{}^{(n-1)}Y_{t^-}\right|\right]\leq\frac{1}{1-J}\cdot\mathbb{E}\left[\left|{}^{(n)}Y_t-{}^{(n-1)}Y_t\right|\right]\ ,$$

especially for all time points $t\in\{t_0,\,t_1,\,\dots\}$, but also for general t, if a deterministic jump is not even possible.

Further, in step (iv) we used, that the variation on the interval $[u,T]$ can be used as an upper bound for the absolute difference of the reserves in time u as

$$\left|{}^{(n)}Y_u-{}^{(n-1)}Y_u\right|\leq\int\limits_{r\in[u,T]}\mathrm{d}\left|{}^{(n)}Y-{}^{(n-1)}Y\right|_r \tag{3.2.8}$$

which is intuitive for the variation, and where equality would be given in a situation, when a function is monotone. We already discussed this in the preliminary discussion to the proof and we once again use the difference of the final values in T

$${}^{(n)}Y_T-{}^{(n-1)}Y_T=0$$

as the additional summand of the variation. This is possible, as the choice is arbitrary and the value in T is always included in the interval, since the payment process Y_u contains the payments over the interval $(u,T]$.

This step is only necessary, if the discrete sojourn payments at deterministic time points are used. Otherwise, the integral would only be a Lebesgue-integral, and the function could be exchanged without the additional factor, since both integrals would be the same dt-almost surely. One might use the simplification of $J=0$ in this case, simplifying the Lipschitz constant to just L.

We continue the proof by using the previous inequality for the complete interval $(0,T]$ and we arrive at the following upper bound, that now holds for the norm as

$$\left\|{}^{(n+1)}Y-{}^{(n)}Y\right\|_{V[0,T]}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\int\limits_{t\in(0,T]} \frac{L}{1-J}\cdot e^{-K(\zeta(T)-\zeta(t))}\left|{}^{(n)}Y_t-{}^{(n-1)}Y_t\right|\zeta(\mathrm{d}t)\right]\\
&\overset{(iv)}{\leq} \mathbb{E}\left[\int\limits_{t\in(0,T]} \frac{L}{1-J}\,e^{-K(\zeta(T)-\zeta(t))}\left(\int\limits_{r\in[t,T]}\mathrm{d}\left|{}^{(n)}Y-{}^{(n-1)}Y\right|_r\right)\zeta(\mathrm{d}t)\right]\\
&\overset{(v)}{\leq} \frac{L}{1-J}\cdot\mathbb{E}\left[\int\limits_{r\in[0,T]}\left(\int\limits_{t\in(0,r]} e^{-K(\zeta(T)-\zeta(t))}\,\zeta(\mathrm{d}t)\right)\mathrm{d}\left|{}^{(n)}Y-{}^{(n-1)}Y\right|_r\right]\\
&\leq \frac{L}{1-J}\cdot\mathbb{E}\left[\int\limits_{[0,T]}\left(\int\limits_{t\in(0,r]} e^{-K(\zeta(T)-\zeta(t))}\,\zeta(\mathrm{d}t)\right)\mathrm{d}\left|{}^{(n)}Y-{}^{(n-1)}Y\right|_r\right].
\end{aligned}
$$

In step (v), the order $0\leq t\leq r\leq T$ of the integration area and integration variables can be understood as a condition for t depending on r (i.e.$t\in[0,r]$) as well as on r depending on t ($r\in[t,T]$), which enables us to exchange the order of integration and additionally makes it possible to include the lower bound $r=0$, where the inner integral would be zero anyway.

We now have to use a general transformation formula to explicitly compute the inner integral and the details will be explained in detail. The calculations, as already mentioned when introducing the weighting factor, are inspired by Christiansen [Chr10].

If ζ is increasing, then the following equivalence

$$t\in(0,r]\;\Leftrightarrow\;\zeta(t)\in\big(\zeta(0),\zeta(r)\big]$$

together with the (inverse) quantile function

$$\zeta^{-1}(t)=\inf\{x\in\mathbb{R}\mid\zeta(x)\geq t\}$$

leads to

$$\zeta(r)-\zeta(0)=\lambda\big(\zeta\big((0,r]\big)\big)=\lambda\big(\zeta^{-1}\in(0,r]\big)=\mathscr{L}\big(\zeta^{-1}\mid\lambda\big)\big((0,r]\big)$$

and therefore the integration can be replaced by $\mathscr{L}(\zeta^{-1}\mid\lambda)(\mathrm{d}t)$, i.e. $\mathbb{P}(A)=\lambda(\zeta(A))$.

We rewrite the inner integral in the last line as

$$
\begin{aligned}
&\int\limits_{t\in(0,r]} e^{-K(\zeta(T)-\zeta(t))}\,\zeta(\mathrm{d}t)\\
&=\int\limits_{\mathbb{R}}\mathbb{1}_{\{t\in(0,r]\}}\cdot e^{-K\,(\zeta(T)-\zeta(t))}\,\mathscr{L}(\zeta^{-1}\mid\lambda)(\mathrm{d}t)\\
&=\int\limits_{\mathbb{R}}\underbrace{\mathbb{1}_{\{\zeta(t)\in(\zeta(0),\zeta(r)]\}}\,e^{-K\,(\zeta(T)-\zeta(t))}}_{=h(\zeta(t))}\,\underbrace{\mathscr{L}(\zeta^{-1}\mid\lambda)(\mathrm{d}t)}_{=\mathbb{P}^{\zeta}(\mathrm{d}t)}\\
&=\int\limits_{\mathbb{R}}\mathbb{1}_{\{t\in(\zeta(0),\zeta(r)]\}}\,e^{-K(\zeta(T)-t)}\,\mathbb{P}^{\zeta}(\mathrm{d}t)
\end{aligned}
$$

$$\overset{(vi)}{\leq} \int\limits_{(\zeta(0),\zeta(r)]} e^{-K(\zeta(T)-t)}\,\lambda(\mathrm{d}t) = \int\limits_{(\zeta(0),\zeta(r)]} e^{-K(\zeta(T)-t)}\,\mathrm{d}t\,,$$

where in (vi) we used, that

$$\mathbb{P}^{\zeta}(A) = \mathbb{P}\left(\zeta^{-1}(A)\right) = \lambda\left(\zeta(\zeta^{-1}(A))\right) \leq \lambda(A)$$

since

$$\zeta(\zeta^{-1}(A)) = \zeta\big(\{t \in \mathbb{R} \mid \zeta(t) \in A\}\big) \subseteq A\,.$$

We can now calculate the inner dt-integral and arrive at the following inequality

$$\begin{aligned}
&\left\| {}^{(n+1)}Y - {}^{(n)}Y \right\|_{V[0,T]} \\
&\leq \frac{L}{1-J} \cdot \mathbb{E}\left[\int\limits_{[0,T]} \int\limits_{(0,r]} e^{-K(\zeta(T)-\zeta(t))}\,\zeta(\mathrm{d}t)\,\mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r \right] \\
&\leq \frac{L}{1-J} \cdot \mathbb{E}\left[\int\limits_{[0,T]} \int\limits_{(\zeta(0),\zeta(r)]} e^{-K(\zeta(T)-t)}\,\mathrm{d}t\,\mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r \right] \\
&= \frac{L}{1-J} \cdot \mathbb{E}\left[\int\limits_{[0,T]} \frac{1}{K} \left(e^{-K(\zeta(T)-\zeta(r))} - e^{-K(\zeta(T)-\zeta(0))} \right) \mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r \right] \\
&\leq \frac{L}{(1-J)\cdot K} \cdot \mathbb{E}\left[\int\limits_{[0,T]} e^{-K(\zeta(T)-\zeta(r))}\,\mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r \right] \\
&= \frac{L}{(1-J)\cdot K} \cdot \left\| {}^{(n)}Y - {}^{(n-1)}Y \right\|_{V[0,T]}\,.
\end{aligned}$$

When choosing $K = 2 \cdot \dfrac{L}{1-J}$ and defining our contraction constant

$$C := \frac{L}{(1-J)\cdot K} = \frac{1}{2} < 1$$

we have indeed managed to show the contraction property with ζ as previously specified.

Application of the fixed point Theorem of Banach

Let us from now on assume, that $K = \frac{2 \cdot L}{1-J}$. Then we have a contraction and application of the fixed-point theorem of Banach guarantees existence and uniqueness of a process $Y = (Y_t)_{t \geq 0}$ fulfilling

$$Y_t = \sum_{M \in \mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s^-}\,\kappa(s)\,b_M\big(s, X^{\mathbb{G}}_{s^-}\big)\,\gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\,B_I\big(s, e, X^{\mathbb{G}}_{s^-}\big)\,\mu_I(\mathrm{d}(s,e))$$

in the space of càdlàg processes with integrable variation on $[0, t]$, where additionally

$$X_{s^-}^{\mathbb{G}} = \frac{1}{\kappa(s)} \mathbb{E}\left[Y_{s^-} \mid \mathcal{G}_s^-\right]$$

to express the dependency in the correct way and to emphasize the fact, that this is indeed a fixed point equation for process Y. $\square$

Comment 3.2.13. *On the benefits of this approach in comparison to standard techniques* This is a different result than usual, since we have shown the uniqueness and existence of the accumulated future payments, but not of the prospective reserves with respect to either $\mathbb{F}$ or $\mathbb{G}$. The recalculation of X from Y is simpler and since we assumed the discounting to be deterministic, this property also works for the optional projection (prospective reserve), where the expectation is also present. We further do not need to use the integration by parts formula in combination with the differential. This has usually be done, see for example [CD20] Proposition 3.5, where a theorem of [CE12] is used.

On the other hand, it remains unclear, what happens with the reserve. We have an iteration which might be used to calculate the reserve, in addition to the payment process. The details will be investigated in a later chapter.

The approach would allow us to use more general final values, if the proposed norm is completed with another summand, and the proof would stay the same, as the difference of the final values between two iterations would be zero. We will refrain from a formulation, as it is not needed in the context of life insurance contract, where our current final condition arises naturally.

3.3. Extension of the dependency

3.3.1. Motivation for the extended dependency

We have allowed the payments to only depend on the general reserve $X^{\mathbb{G}}_{t-}$ of the insurance contract, but we have not allowed the dependency on state-wise reserves or sums at risk. To motivate the next step in the dependency, we remember the BSDE representation of the reserve with respect to $\mathbb{G}$. Theorem 2.5.13 in [Chr21b], applied to the process X, gives us

$$\begin{cases} \mathrm{d}X^{\mathbb{G}}_t = \mathrm{d}X^{\mathrm{IF}}_t & + \sum_{I\in\mathcal{N}} \int_{E_I} G_I(t^-, t, e)(\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\ & + \sum_{I\in\mathcal{N}} \int_{E_I} G_I(t\ \ , t, e)(\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \\ X^{\mathbb{G}}_T = 0 & \end{cases}$$

which almost surely holds with

$$G_I(t^-, t, e) = \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t-} \left(\frac{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}^M_{t-} X_t\right]}{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}^M_{t-}\right]} - \frac{\mathbb{E}_M\left[\mathbb{I}^M_{t-}\mathbb{I}^M_t X_t\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{t-}\mathbb{I}^M_t\right]} \right),$$

$$G_I(t\ \ , t, e) = \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t} \left(\frac{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}^M_{t} X_t\right]}{\mathbb{E}_{M,R_I=(t,e)}\left[\mathbb{I}^M_{t}\right]} - \frac{\mathbb{E}_M\left[\mathbb{I}^M_{t-}\mathbb{I}^M_t X_t\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{t-}\mathbb{I}^M_t\right]} \right)$$

as an representation of the family $(G_I)_I$.

We will now also give a more compact notation for the integrands G_I, where the formulas (2.4.3) and (2.4.5) are used to rewrite the summands. Compare Christiansen [Chr21b] for details.

Assertion 3.3.1. *Rewriting of the sums at risk*
Without loss of generality suppose, that $0 \notin E$.
For any $t > 0$ and any integrable random variable ξ we define a short hand notation by using factorized conditional expectations as

$$\begin{aligned} \mathbb{E}\left[\xi \,\middle|\, \mathcal{G}_t\ \ , R_I = (t,e)\right] &:= \mathbb{E}\left[\xi \,|\, (\Gamma_t, R_I) = \cdot\ \right](\Gamma_t, (t,e)), \\ \mathbb{E}\left[\xi \,\middle|\, \mathcal{G}^-_t, R_I = (t,e)\right] &:= \mathbb{E}\left[\xi \,|\, (\Gamma_{t-}, R_I) = \cdot\ \right](\Gamma_{t-}, (t,e)), \qquad \text{for } e \in E_I, \\ \mathbb{E}\left[\xi \,\middle|\, \mathcal{G}_t\ \ , \mathcal{J}_t = 0\right] &:= \mathbb{E}\left[\xi \,\middle|\, (\Gamma_t\ \ , \mathcal{J}_t) = \cdot\ \right](\Gamma_t, 0), \\ \mathbb{E}\left[\xi \,\middle|\, \mathcal{G}^-_t, \mathcal{J}_t = 0\right] &:= \mathbb{E}\left[\xi \,|\, (\Gamma_{t-}, \mathcal{J}_t) = \cdot\ \right](\Gamma_{t-}, 0). \end{aligned}$$

Then, the integrands have the following alternative representations

$$G_I(t^-, t, e) = \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}^-_t, R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}^-_t, \mathcal{J}_t = 0\right], \tag{3.3.1}$$

$$G_I(t\ \ , t, e) = \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t\ \ , R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t\ \ , \mathcal{J}_t = 0\right] \tag{3.3.2}$$

almost surely for any $t > 0$, $I \in \mathcal{N}$ and $e \in E_I$, and where the random variable

$$\mathcal{J}_t = \sum_{I \in \mathcal{N}} \mu_I(\{t\} \times E_I)$$

indicates whether there is any stopping event at time t.

We give intuitive interpretations to these differences:
The first line (3.3.1) describes the difference in expectation between a change scenario an a remain scenario if we are currently at time t^- and are looking forward in time. The second line describes the difference in expectation between a change scenario and a remain scenario if we are currently at time t and are looking backwards in time. These differences are integrated with respect to the compensated forward and backward scenario dynamics in the BSDE representation. If we interpret the difference in formula (3.3.1), then this is similar to a general sum at risk, which also contains a remain scenario since the sum at risk a transition $i \to j$ contains the reserve in i with a negative sign.

The equations can also be formally shown, when we use similar arguments like we have in equations (2.4.3) and (2.4.5). Only the first line is performed and the second is analogous.

Proof. For any $I \in \mathcal{N}$, $e \in E_I$ and $t > 0$ we have that

$$\begin{aligned}
G_I(t^-, t, e) &= \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \left(\frac{\mathbb{E}_{M, R_I=(t,e)}\left[\mathbb{I}_{t^-}^M X_t\right]}{\mathbb{E}_{M, R_I=(t,e)}\left[\mathbb{I}_{t^-}^M\right]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{t^-}^M \, \mathbb{I}_t^M X_t\right]}{\mathbb{E}_M\left[\mathbb{I}_{t^-}^M \, \mathbb{I}_t^M\right]} \right) \\
&= \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \frac{\mathbb{E}_{M, R_I=(t,e)}\left[\mathbb{I}_{t^-}^M X_t\right]}{\mathbb{E}_{M, R_I=(t,e)}\left[\mathbb{I}_{t^-}^M\right]} - \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{t^-}^M \, \mathbb{I}_t^M X_t\right]}{\mathbb{E}_M\left[\mathbb{I}_{t^-}^M \, \mathbb{I}_t^M\right]} \\
&= \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \, \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, R_I = (t,e)\right] - \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \, \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, \mathcal{J}_t = 0\right] \\
&= \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, \mathcal{J}_t = 0\right]
\end{aligned}$$

holds almost surely, where of course the appropriate factorized conditional expectations have to be used. $\square$

3.3.2. Modelling of the extended dependency structure

The mathematical foundations to additionally allow for dependency on G_I have now to be formulated.

Similar to the standard life insurance theory, where Z_{ij} would be used, it is possible to use the G_I for every $I \in \mathcal{N}$ and $e \in E_I$. It is also imminent from the representation of the G_I, that the $M \in \mathcal{M}$ does actually has an important role as well and we remember that a transition in the information interpretation recognized both the M as well as the I, and non-zero summands would only arise from a viable combination of I, M and $e \in E_I$.

If we now think about the possibilities to allowing a dependence of our payments $(b_M)_M$ and $(B_I)_I$ on a representation of $(G_I)_I$ as well, then it makes sense to use this representation $G_J(t^-, t, e)$, similar to the fact that we use the general reserve prior to time t (i.e. $X^{\mathbb{G}}_{t^-}$) as well. The measurability condition of our functions $(b_M)_M$ and $(B_I)_I$ is then still fulfilled. Another advantage is the inclusion of the standard model, if we do not want to consider a situation with non-monotonicity. Then, the second integral vanishes and only the first representation is still part of the sum.

Let us introduce the following shorthand notation, to keep the representation of the payment functions reasonably short.

Definition 3.3.2. *Abbreviation for sum at risk dependency*
Define the notation

$$(G_J(t))_J := \big(G_J(t^-, t, f)_{f \in E_J}\big)_{J \in \mathcal{N}}$$

to model the two levels of dependency, where the $J \in \mathcal{N}$ is chosen in the first step and then the $f \in E_J$ is chosen in a second step.
Also note, that we use J (and $f \in E_J$) at this point, since this notation will be used as part of the family $(b_M)_M$, as well as the family $(B_I)_I$ and we therefore need to be careful about the re-usage of indexes to not confuse it with the I from the outer sum.

Let us now revisit the modelling of our contractual payments and extend it to the current needs of the extended model.

Definition 3.3.3. *Dependent sojourn payments*
Consider functions of the form

$$\begin{aligned} b : \mathcal{M} \times [0,\infty) \times \Omega &\longrightarrow \mathbb{R} \\ (M, t, \omega) &\longmapsto b\big(M, t, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)(\omega) \end{aligned}$$

as the rate of a sojourn payment, that will be paid at time t, if $\mathbb{I}^M_{t^-} = 1$, for all $M \in \mathcal{M}$ respectively, and let for $M \in \mathcal{M}$

$$b_M\big(t, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)(\omega) := b\big(M, t, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)(\omega)$$

be a function from $[0,\infty) \times \Omega \to \mathbb{R}$.

We further need the following assumptions:

(1) The function $b\big(M, t, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)$ is jointly measurable in (M, t, ω).

(2) For every $M \in \mathcal{M}$ let b_M be bounded on every compact time interval, i.e. for every $t \geq 0$ we have

$$\Big| b_M\big(s, X^{\mathbb{G}}_{s^-}, (G_I(s))_I\big) \Big| \leq J_{\mathcal{M}}(s) \tag{3.3.3}$$

for an integrable majorant $J_{\mathcal{M}}$ and for every $s \in [0,t]$.

(3) The functions $b_M\big(t, X^{\mathbb{G}}_{t^-}, (G_I(t))_I\big)$ are $\mathbb{G}^-$-adapted for every $M \in \mathcal{M}$.

Definition 3.3.4. *Dependent transition payments*
Consider functions of the form

$$\begin{aligned} B_I : [0,\infty) \times E_I \times \Omega &\longrightarrow \mathbb{R} \\ (t, e, \omega) &\longmapsto B_I\big(t, e, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)(\omega) \end{aligned}$$

for every $I \in \mathcal{N}$ as transition payments, happening upon a transition recognized by μ_I. We need the following assumptions to hold:

(1) The functions $B_I\big(t, e, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)(\omega)$ are jointly measurable in (t, e, ω) for every I.

(2) For each $I \in \mathcal{N}$ let B_I be bounded on every compact time interval, i.e. for every $t \geq 0$ we have

$$\Big| B_I\big(s, e, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big) \Big| \leq J_{\mathcal{N}}(s) \tag{3.3.4}$$

for an integrable majorant $J_{\mathcal{N}}$ and for every $s \in [0, t]$.

(3) The functions $B_I\big(t, e, X^{\mathbb{G}}_{t^-}, (G_J(t))_J\big)$ are $\mathbb{G}^-$-adapted for each $I \in \mathcal{N}$.

In addition to the functions b_M and B_I that slightly change, we can now only allow for continuous reserve-dependent for the sojourn payments and use $\mathrm{d}s$ instead of $\gamma(\mathrm{d}s)$. Take note, that the previous results are still valid and lump sum payments without reserve-dependency can still be included with the help of an additional summand.

3.3.3. Construction of the payment process

The payment process is now given as

$$\begin{aligned} X_t = &\sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \mathbb{I}^M_{s^-}\, b_M\big(s, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big)\,\mathrm{d}s \\ &+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I\big(s, e, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big)\,\mu_I(\mathrm{d}(s,e)), \end{aligned} \tag{3.3.5}$$

but we must work with $Y_t = \kappa(t) \cdot X_t$ instead of X_t again. The iteration is similar to before, but an additional step is necessary to evaluate the $(G_J)_J$. This works by using the construction property for the family $(G_J)_J$.

Let the iteration index $n \in \mathbb{N}$ be fixed and $({}^{(n)}Y)$ be given as the predecessor. Then the following steps have to be iterated:

(1) Calculation of

$$^{(n)}X_t = \frac{1}{\kappa(t)}\ {}^{(n)}Y_t$$

as the usual payment process by reversing the discounting of all payments to 0 back to t.

(2) Apply Theorem 2.4.1, which guarantees the existence of the optional projection (as a process), and it hold

$$\begin{aligned} {}^{(n)}X_s^{\mathbb{G}} &= \mathbb{E}\left[{}^{(n-1)}X_s \,\middle|\, \mathcal{G}_s \right] , \\ {}^{(n)}X_{s^-}^{\mathbb{G}} &= \mathbb{E}\left[{}^{(n-1)}X_{s^-} \,\middle|\, \mathcal{G}_s^- \right] \end{aligned}$$

almost surely, where the second representation is needed in the payments.

(3) Evaluate

$${}^{(n)}G_J(s^-, s, f) = \sum_{M \in \mathcal{M}} \mathbb{I}_{s^-}^M \left(\frac{\mathbb{E}_{M, R_J=(s,f)}\left[\mathbb{I}_{s^-}^M \, {}^{(n)}X_s \right]}{\mathbb{E}_{M, R_J=(s,f)}\left[\mathbb{I}_{s^-}^M \right]} - \frac{\mathbb{E}_M\left[\mathbb{I}_{s^-}^M \mathbb{I}_s^M \, {}^{(n)}X_s \right]}{\mathbb{E}_M\left[\mathbb{I}_{s^-}^M \mathbb{I}_s^M \right]} \right)$$

for every $J \in \mathcal{N}$ and $f \in E_J$.

(4) Construct ${}^{(n+1)}Y$ by inserting the results from (2) and (3) into the payments

$$\begin{aligned} {}^{(n+1)}Y_t = & \sum_{M \in \mathcal{M}} \int\limits_{(t,T]} \mathbb{I}_{s^-}^M \, \kappa(s) \, b_M \left(s, {}^{(n)}X_{s^-}^{\mathbb{G}}, \left({}^{(n)}G_J(s) \right)_J \right) \mathrm{d}s \\ & + \sum_{I \in \mathcal{N}} \int\limits_{(t,T] \times E_I} \kappa(s) \, H_I \left(s, e, {}^{(n)}X_{s^-}^{\mathbb{G}}, \left({}^{(n)}G_J(s) \right)_J \right) \mu_I(\mathrm{d}(s,e)) \, . \end{aligned}$$

(4) Start over with the newly constructed ${}^{(n+1)}Y$, which completes or iteration function Φ as

$${}^{(n+1)}Y = \Phi\left({}^{(n)}Y \right) \, .$$

Because of the changed payments, the Lipschitz-conditions also need to be adjusted. First, an appropriate norm has to be chosen for the $G = (G_J)_J$, to be able to state the Lipschitz conditions. The condition for the intensities of the compensators will be kept as a standing assumption.

Definition 3.3.5. *Semi-norm for the family of processes G*
For the family of processes $G = (G_J)_{J \in \mathcal{N}}$ in the setting with existing Lebesgue densities $n = (n_J)_{J \in \mathcal{N}}$ for the IF-compensator, define

$$\|G(s)\|_n := \sum_{J \in \mathcal{N}} \int\limits_{E_J} \left| G_J(s^-, s, f) \right| \, n_J(s, \mathrm{d}f) \, .$$

Assertion 3.3.6. *Semi-norm property*
The definition $\|G(s)\|_n$ constructs a semi-norm for the family $(G_J(s^-, s, f)_{f \in E_J})_{J \in \mathcal{N}}$.

Proof. The absolute homogeneity and the sub-additivity, and non-negativity are all direct consequences from the corresponding properties of the absolute value.

Let $G_J(s^-, s, f) = 0$ for all $f \in E_J$ and every $J \in \mathcal{N}$. Then every summand is zero and $\|G(s)\|_n = 0$ as well. From the non-negativity of both the absolute value and the intensity of the compensator, it follows that $\|G(s)\|_n$ is non-negative.

It only defines a semi-norm, as in such a case the norm would be zero, but it might be that n_J is zero (when μ_J is zero), and G would not necessarily also have to be zero. □

We also want to comment on why this choice of a semi-norm is sensible and in alignment with the standard theory in literature.

Comment 3.3.7. *Comparison to usual life insurance theory*
Having a semi-norm is sufficient for our case, as we only use it for the Lipschitz condition. This is similar to the literature, where an analogous construction takes place, see for example Djehiche and Löfdahl [DL16] (with a slightly different notation)

$$\|Z(s)\|_\mu^2 := \sum_{(i,j)\in\mathcal{T}} Z_{ij}^2(s)\,\mathbb{I}_i(s^-)\,\lambda_{ij}(s^-)$$

$$\|G(s)\|_n := \sum_{J\in\mathcal{N}} \int_{E_J} |G_J(s^-, s, f)|\; n_J(s, \mathrm{d}f)$$

and where the product $(\mathbb{I}_i\,\lambda_{ij})_{(i,j)\in\mathcal{T}}$ would be the intensities of the standard $\mathbb{F}$-compensators of the counting processes $(N_{ij})_{(i,j)\in\mathcal{T}}$, and we would be using the intensities of the IF-compensator.

Assumption 3.3.8. *Lipschitz conditions*
Assume, that there exists a Lipschitz constant $L_{\mathcal{M}} > 0$, independent of $M \in \mathcal{M}$, such that for all $X_{s^-}^{\mathbb{G}}(\omega), \tilde{X}_{s^-}^{\mathbb{G}}(\omega)$ and $(G_J(s))_J(\omega), (\tilde{G}_J(s))_J(\omega)$ we $\mathrm{d}\mathbb{P}\times\mathrm{d}s$ *a.e.* have

$$\begin{aligned}\Big|b_M(s, X_{s^-}^{\mathbb{G}}, (G_J(s))_J)(\omega) - b_M(s, \tilde{X}_{s^-}^{\mathbb{G}}, (\tilde{G}_J(s))_J)(\omega)\Big| \\ \leq L_{\mathcal{M}}\left(\Big|X_{s^-}^{\mathbb{G}}(\omega) - \tilde{X}_{s^-}^{\mathbb{G}}(\omega)\Big| + \Big\|G(s) - \tilde{G}(s)\Big\|_n(\omega)\right)\end{aligned}$$

for all $M \in \mathcal{M}$.

Further assume that there exists a second Lipschitz constant $L_{\mathcal{N}}$, independent of $I \in \mathcal{N}$, such that for all $X_{s^-}^{\mathbb{G}}(\omega), \tilde{X}_{s^-}^{\mathbb{G}}(\omega)$ and $(G_J(s))_J(\omega), (\tilde{G}_J(s))_J(\omega)$ and all $e \in E_I$ we $\mathrm{d}\mathbb{P}\times n_I(s, E_I)\mathrm{d}s$ *a.e.* have

$$\begin{aligned}\Big|B_I(s, e, X_{s^-}^{\mathbb{G}}, (G_J(s))_J)(\omega) - B_I(s, e, \tilde{X}_{s^-}^{\mathbb{G}}, (\tilde{G}_J(s))_J)(\omega)\Big| \\ \leq L_{\mathcal{N}}\left(\Big|X_{s^-}^{\mathbb{G}}(\omega) - \tilde{X}_{s^-}^{\mathbb{G}}(\omega)\Big| + \Big\|G(s) - \tilde{G}(s)\Big\|_n(\omega)\right).\end{aligned}$$

Define $L := \max\{L_{\mathcal{M}}, L_{\mathcal{N}}\}$ for simplicity.

3.3.4. Theorem – Existence and uniqueness II

We now proceed with the main theorem of this section.

Theorem 3.3.9. *Existence and uniqueness of the payment process Y*
Under the conditions on Y (from X in formula 3.3.5) and its parts, namely Assumptions 3.2.2 and 3.2.3 about the Lipschitz conditions and the upper bound for the compensator, Definitions 3.1.1, 3.3.3 and 3.3.4 of the payments, the existence and uniqueness of the payment process $Y \in BV^X([0,T])$ is guaranteed in the space of càdlàg processes with integrable variation from Definition 3.2.4, equipped with the (equivalent) weighted norm

$$\|X\|_{V[0,T]} := \mathbb{E}\left[\|X\|_{V[0,T],K,\zeta}\right] = \mathbb{E}\left[\int\limits_{[0,T]} e^{-K(\zeta(T)-\zeta(t))}\,\mathrm{d}\,|X|_t\right]$$

where $K = 2\,L\,(D+1)\,(2D+1)$ is the weighting factor and ζ is the identity

$$\zeta(\mathrm{d}t) = \mathrm{d}t\,.$$

Proof. The proof of the theorem is performed in three steps.

As a consequence of the application of the Theorem of Banach A.3.1, we want to show, that a unique fixed point ${}^*Y$ exists, fulfilling

$$\begin{aligned}
{}^*Y_t = &\sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s^-}\,\kappa(s)\,b_M\left(s, \frac{1}{\kappa(s)}\,{}^*Y^{\mathbb{G}}_{s^-}, \frac{1}{\kappa(s)}\left({}^*G_J(s)\right)_J\right)\gamma(\mathrm{d}s)\\
&+\sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\,B_I\left(s, e, \frac{1}{\kappa(s)}\,{}^*Y^{\mathbb{G}}_{s^-}, \frac{1}{\kappa(s)}\left({}^*G_J(s)\right)_J\right)\mu_I(\mathrm{d}(s,e))\\
=:&\ \Phi({}^*Y)_t
\end{aligned} \tag{3.3.6}$$

where ${}^*Y^{\mathbb{G}}_{t^-} = \mathbb{E}\left[{}^*Y_{t^-}\,\middle|\,\mathcal{G}^-_t\right]$ *a.s.* as usual and the respective representation holds for the $({}^*G_J)_{J\in\mathcal{N}}$ without explicit formulation.

Automorphism

We begin by showing that the mapping Φ is an automorphism on the space of processes with bounded variation.
Let ${}^{(n)}Y \in BV^X([0,T])$ be a process of integrable variation and show that ${}^{(n+1)}Y$ is a process of integrable variation as well. To ensure better readability, we do not repeat the parts that are identical for the first theorem.

It holds that

$$\left\|\Phi\left({}^{(n)}Y\right)\right\|_{V[0,T]} = \left\|{}^{(n+1)}Y\right\|_{V[0,T]} = \mathbb{E}\left[\left\|{}^{(n+1)}Y\right\|_{V[0,T],K,\zeta}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int\limits_{[0,T]} \underbrace{e^{-K(\zeta(T)-\zeta(t))}}_{\leq 1} \,\mathrm{d}\left|{}^{(n)}Y\right|_t\right] \\
&\leq \mathbb{E}\left[\int\limits_{(0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\,\kappa(s)\, \underbrace{\left|b_M\big(t, {}^{(n)}X^{\mathbb{G}}_{t-}, \big({}^{(n)}(G_J(s))\big)_J\big)\right|}_{\leq J_{\mathcal{M}}(t)} \,\mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(0,T]\times E_J} \kappa(s)\, \underbrace{\left|B_I\big(t, e, {}^{(n)}X^{\mathbb{G}}_{t-}, \big({}^{(n)}(G_J(s))\big)_J\big)\right|}_{\leq J_{\mathcal{N}}(t)}\, \mu_I(\mathrm{d}(t,e))\right] \\
&\leq \mathbb{E}\left[\int\limits_{(0,T]} D_\kappa\, J_{\mathcal{M}}(t)\,\mathrm{d}t\right] + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(0,T]\times E_I} D_\kappa\, J_{\mathcal{N}}(t)\, \mu_I(\mathrm{d}\,(t,e))\right] \\
&< \infty\,,
\end{aligned}
$$

where both parts are finite, respectively. In total we have ${}^{(n+1)}Y \in BV^X([0,T])$.

Contraction property

Note, that we shorten some steps of the proof, if the are essentially the same as in the previous proof to Theorem 3.2.12.

For every $0 \leq s < t \leq T$, or equivalently for every interval $(s,t] \subseteq (0,T]$ we get the following upward estimation for the variation of the difference of two iterated solutions by

$$
\begin{aligned}
&\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\mathrm{d}\left|{}^{(n+1)}Y - {}^{(n)}Y\right|_u\right] \\
&\leq \mathbb{E}\Bigg[\int\limits_{(s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u) \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u-}\cdot \\
&\qquad \left|b_M\left(u, {}^{(n)}X^{\mathbb{G}}_{u-}, \left({}^{(n)}G_J(u)\right)_J\right) - b_M\left(u, {}^{(n-1)}X^{\mathbb{G}}_{u-}, \left({}^{(n-1)}G_J(u)\right)_J\right)\right| \mathrm{d}u\Bigg] \\
&\quad + \mathbb{E}\Bigg[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\cdot \Big|B_I\left(u, e, {}^{(n)}X^{\mathbb{G}}_{u-}, \left({}^{(n)}G_J(u)\right)_J\right) \\
&\qquad - B_I\left(u, e, {}^{(n-1)}X^{\mathbb{G}}_{u-}, \left({}^{(n-1)}G_J(u)\right)_J\right)\Big|\, \mu_I(\mathrm{d}(u,e))\Bigg] \\
&=: (\mathrm{I}) + (\mathrm{II})\,,
\end{aligned}
$$

yielding an additive structure, where we again look at both summands separately.

For part (I), we use the Lipschitz condition for b_M for every $M \in \mathcal{M}$ together with formula (3.2.1) for every $u \in (s,t] \subseteq (0,T]$, once the dependency on M is not explicit any more, and get the following upper bound

$$(\mathrm{I}) \leq \mathbb{E}\left[\int_{(s,t]} L_{\mathcal{M}}\, e^{-K(\zeta(T)-\zeta(u))}\, \kappa(u)\Big(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\Big)\, \mathrm{d}u\right].$$

We need a few more steps for the second summand (II). For better readability, we introduce the following auxiliary short hand notation

$$\tilde{h}_t := e^{-K(\zeta(T)-\zeta(t))}\, \kappa(t)$$

which then allows us to further use the following short form

$$F_I(u,e) := \tilde{h}_u \cdot \left|B_I\left(u, e, {}^{(n)}X^{\mathbb{G}}_{u^-}, \left({}^{(n)}G_J(u)\right)_J\right) - B_I\left(u, e, {}^{(n-1)}X^{\mathbb{G}}_{u^-}, \left({}^{(n-1)}G_J(u)\right)_J\right)\right|$$

for the function, that is the integrand in the second summand (II).

They are non-negative and $\mathcal{G}_u^-$-measurable for every (u,e). This is assumed for B_I as a precondition and the deterministic exponential part and the discounting factor do not influence this. By application of the monotone convergence theorem, we can exchange the sum and the integral, which then allows us to exchange the each μ_I by ν_I summand-wise.

This is done in equality (i) and in this set-up relies on the special marked point processes structure of [Chr21b], since we now introduce the IF-compensator in each summand. We then also use the existence of the density n_I for the compensator ν_I, which does now result in a representation with a du integral.

$$\begin{aligned}
(\mathrm{II}) &= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(s,t]\times E_I} F_I(u,e)\ \mu_I(\mathrm{d}(u,e))\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int_{(s,t]\times E_I} F_I(u,e)\ \mu_I(\mathrm{d}(u,e))\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[F_I \bullet \mu_I((s,t]\times E_I)\right] \\
&\overset{(i)}{=} \sum_{I\in\mathcal{N}} \mathbb{E}\left[F_I \bullet \nu_I((s,t]\times E_I)\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int_{(s,t]\times E_I} F_I(u,e)\ \nu_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(s,t]\times E_I} F_I(u,e)\ \nu_I(\mathrm{d}(u,e))\right]
\end{aligned}$$

$$= \mathbb{E}\left[\int\limits_{(s,t]} \sum_{I\in\mathcal{N}} \int\limits_{E_I} F_I(u,e)\; n_I(u,\mathrm{d}e)\,\mathrm{d}u \right] .$$

After exchanging the counting process with the densities of the compensators, we can use the Lipschitz condition

$$\begin{aligned} &\left| B_I\left(u,e,{}^{(n)}X^{\mathbb{G}}_{u^-},\left({}^{(n)}G_J(u)\right)_J\right)(\omega) - B_I\left(u,e,{}^{(n-1)}X^{\mathbb{G}}_{u^-},\left({}^{(n-1)}G_J(u)\right)_J\right)(\omega)\right| \\ &\le L_{\mathcal{N}}\cdot\left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-}(\omega) - {}^{(n)}\tilde{X}^{\mathbb{G}}_{u^-}(\omega)\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n(\omega)\right) \end{aligned}$$

and arrive at an upper bound of

(II)

$$\begin{aligned} &\le \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\, e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\cdot \right. \\ &\qquad \left. \sum_{I\in\mathcal{N}} \int\limits_{E_I} \left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n)}\tilde{X}^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\right) n_I(u,\mathrm{d}e)\,\mathrm{d}u \right] \\ &= \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\, e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\cdot \right. \\ &\qquad \left. \left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n)}\tilde{X}^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\right) \sum_{I\in\mathcal{N}} n_I(u,E_I)\,\mathrm{d}u \right] \\ &\le \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\, e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n)}\tilde{X}^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\right) D\,\mathrm{d}u \right] . \end{aligned}$$

By design, when applying the Lipschitz-condition, the integrand does not further depend on e, and we can use the majorant for the density process of the compensator, which results in a representation with a du integral.

Furthermore, we use $L = \max\{L_{\mathcal{N}},\, L_{\mathcal{M}}\}$ as the previously defined joint Lipschitz constant to be able to join both parts in the next steps, making use of the similar additive structure we achieved.

We arrive at the upper bound

$$\begin{aligned} &\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(T)-\zeta(u))}\,\mathrm{d}\left|{}^{(n+1)}Y - {}^{(n)}Y\right|_u \right] \\ &\le \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{M}}\, e^{-K(\zeta(T)-\zeta(u))}\,\kappa(u)\left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\right) \mathrm{d}u \right] \end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}}\, e^{-K(\zeta(T)-\zeta(u))}\, \kappa(u) \left(\left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n)}\tilde{X}^{\mathbb{G}}_{u^-}\right| + \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n\right) D \,\mathrm{d}u\right] \\
\leq\ & \mathbb{E}\left[\int\limits_{(s,t]} L\, e^{-K(\zeta(T)-\zeta(u))}\, \kappa(u) \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| (\mathrm{d}u + D\,\mathrm{d}u)\right] \\
&+ \mathbb{E}\left[\int\limits_{(s,t]} L\, e^{-K(\zeta(T)-\zeta(u))}\, \kappa(u) \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n (\mathrm{d}u + D\,\mathrm{d}u)\right] \\
=\ & \mathbb{E}\left[\int\limits_{(s,t]} h_u \left|{}^{(n)}X^{\mathbb{G}}_{u^-} - {}^{(n-1)}X^{\mathbb{G}}_{u^-}\right| \mathrm{d}u\right] \\
&+ \mathbb{E}\left[\int\limits_{(s,t]} h_u \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n \mathrm{d}u\right] \\
=:\ & (\mathrm{III}) + (\mathrm{IV})\,,
\end{aligned}
$$

which is now a resorting of the previous additive composition. For reasons of readability, we update the short hand notation to

$$
h_u := (1+D)\, L\, e^{-K(\zeta(T)-\zeta(u))}\, \kappa(u) = (1+D)\, L \cdot \tilde{h}_u\,,
$$

starting in the last line, for the deterministic and non-negative part of the integral.

Part (III) is known from before and we will not repeat the steps from the first theorem at this point. Instead, we will focus on the unknown part (IV), where the difference of G is now present.

Let us start by plugging in the representation of G and regrouping expressions with the same condition in the expectation to introduce the differences of ${}^{(n)}X_t$ and ${}^{(n-1)}X_t$ early on.
Then, the triangle inequality and the conditional inequality of Jensen A.2.12 are used to arrive at a preferable representation with two non-negative summands. The monotone convergence theorem can be used to exchange the sum with the expectation, since we only have positive integrals (and integrands).

We arrive at

$$
\begin{aligned}
(\mathrm{IV}) &= \mathbb{E}\left[\int\limits_{(s,t]} h_u \left\|{}^{(n)}G(u) - {}^{(n-1)}G(u)\right\|_n \mathrm{d}u\right] \\
&= \mathbb{E}\left[\int\limits_{(s,t]} h_u \sum_{J\in\mathcal{N}} \int\limits_{E_J} \left|{}^{(n)}G_J(u^-,u,f) - {}^{(n-1)}G_J(u^-,u,f)\right| n_J(u,\mathrm{d}f)\,\mathrm{d}u\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\int\limits_{(s,t]} h_u \sum_{J\in\mathcal{N}} \int\limits_{E_J} \left|\mathbb{E}\left[{}^{(n)}X_u - {}^{(n-1)}X_u \,\middle|\, \mathcal{G}_u^-,\, R_J = (u,f)\right]\right| n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&\quad + \mathbb{E}\left[\int\limits_{(s,t]} h_u \sum_{J\in\mathcal{N}} \int\limits_{E_J} \left|\mathbb{E}\left[{}^{(n)}X_u - {}^{(n-1)}X_u \,\middle|\, \mathcal{G}_u^-,\, \mathcal{J}_u = 0\right]\right| n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&\leq \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \sum_{M\in\mathcal{M}} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_{M,R_J=(u,f)}\left[\mathbb{I}_{u^-}^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_{M,R_J=(u,f)}\left[\mathbb{I}_{u^-}^M\right]} \nu_J(\mathrm{d}(u,f))\right] \\
&\quad + \mathbb{E}\left[\int\limits_{(s,t]} \sum_{J\in\mathcal{N}} \int\limits_{E_J} \sum_{M\in\mathcal{M}} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\right]} n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \sum_{M\in\mathcal{M}} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_{M,R_J}\left[\mathbb{I}_{u^-}^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}_{u^-}^M\right]} \mu_J(\mathrm{d}(u,f))\right] \\
&\quad + \sum_{J\in\mathcal{N}} \sum_{M\in\mathcal{M}} \mathbb{E}\left[\int\limits_{(s,t]} \int\limits_{E_J} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\right]} n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&=: (\mathrm{V}) + (\mathrm{VI})\,,
\end{aligned}
$$

where we again achieve an additive structure, with new expressions (V) and (VI), that we now have to further investigate.

Start with the expression (VI), where we first want to exchange the $\mathcal{J}_u$ and continue with the original formulation. We get

$$
\begin{aligned}
(\mathrm{VI}) &= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]} \int\limits_{E_J} \sum_{M\in\mathcal{M}} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\right]} n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int\limits_{(s,t]} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\right]} \int\limits_{E_J} n_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int\limits_{(s,t]} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, \mathbb{I}_u^M\right]} n_J(u,E_J)\,\mathrm{d}u\right] \\
&\overset{(i)}{=} \sum_{J\in\mathcal{N}} \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int\limits_{(s,t]} \mathbb{I}_{u^-}^M \frac{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\, h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_M\left[\mathbb{I}_{u^-}^M\right]} n_J(u,E_J)\,\mathrm{d}u\right]\,,
\end{aligned}
$$

where the rewritten integrand does not depend on the index J. This enables us to do separate the integration by application of Theorem of Fubini, such that we remain only with the integration with respect to u.

In this situation, we now have a Lebesgue integral, where additionally the continuous density of the compensator is present. In the last step (i), we can exchange the integrand with a more preferable option, if both functions coincide everywhere with exception of an null set.

By case comparison, we have for the indication functions, that

$$\mathbb{I}^M_{u^-}(\omega) \cdot \mathbb{I}^M_u(\omega) = \mathbb{I}^M_{u^-}(\omega) \text{ for all } \omega \in \left\{ \mathbb{I}^M_{u^-}(\omega) = 1 \text{ and } \mathbb{I}^M_u(\omega) = 0 \right\}^{\mathsf{c}} .$$

The indicator processes have at most two jumps, and the only difference in the indicator functions is given for the jump from 1 back down to 0.

Especially, for a fixed $\omega \in \Omega$, we have that these processes are equal almost everywhere, i.e.

$$\mathbb{I}^M_{u^-}(\omega) \cdot \mathbb{I}^M_u(\omega) = \mathbb{I}^M_{u^-}(\omega) \ a.e.$$

which implies, that also the conditional expectation with respect to Z_M fulfil

$$\mathbb{E}_M \left[\mathbb{I}^M_{u^-} \, \mathbb{I}^M_u \right] (\omega) = \mathbb{E}_M \left[\mathbb{I}^M_{u^-} \right] (\omega) \ a.e.$$

and the same goes for the numerator. We therefore also have

$$\mathbb{E}_M \left[\mathbb{I}^M_{u^-} \, \mathbb{I}^M_u \, h_u \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \right] (\omega) = \mathbb{E}_M \left[\mathbb{I}^M_{u^-} \, h_u \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \right] (\omega) \ a.e..$$

We are now able to exchange $\mathbb{I}^M_{u^-}$ by $\mathbb{I}^M_{u^-} \cdot \mathbb{I}^M_u$ everywhere in the integrand. The functions are equal almost everywhere, which means that for the omega-wise Lebesgue integral, we can change the integrand, without changing the value of the integral.

Using this, and simplifying, we arrive at the following upper bound

$$\begin{aligned}
(\mathrm{VI}) &= \sum_{J \in \mathcal{N}} \mathbb{E} \left[\sum_{M \in \mathcal{M}} \int\limits_{(s,t]} \mathbb{I}^M_{u^-} \frac{\mathbb{E}_M \left[\mathbb{I}^M_{u^-} \, h_u \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \right]}{\mathbb{E}_M \left[\mathbb{I}^M_{u^-} \right]} n_J(u, E_J) \, \mathrm{d}u \right] \\
&= \sum_{J \in \mathcal{N}} \mathbb{E} \left[\int\limits_{(s,t]} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{u^-} \, \mathbb{E} \left[h_u \cdot \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \middle| \mathcal{G}^-_u \right] n_J(u, E_J) \, \mathrm{d}u \right] \\
&\overset{(3.2.1)}{=} \mathbb{E} \left[\int\limits_{(s,t]} h_u \cdot \mathbb{E} \left[\left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \middle| \mathcal{G}^-_u \right] \sum_{J \in \mathcal{N}} n_J(u, E_J) \, \mathrm{d}u \right] \\
&\leq \mathbb{E} \left[\int\limits_{(s,t]} h_u \cdot \mathbb{E} \left[\left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \middle| \mathcal{G}^-_u \right] \underbrace{\sum_{J \in \mathcal{N}} \left| n_J(u, E_J) \right|}_{\leq D} \, \mathrm{d}u \right] \\
&\leq D \cdot \mathbb{E} \left[\int\limits_{(s,t]} h_u \cdot \mathbb{E} \left[\left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \middle| \mathcal{G}^-_u \right] \mathrm{d}u \right] \\
&= D \cdot \int\limits_{(s,t]} h_u \, \mathbb{E} \left[\mathbb{E} \left[\left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \middle| \mathcal{G}^-_u \right] \right] \mathrm{d}u
\end{aligned}$$

$$= D \cdot \int\limits_{(s,t]} h_u \, \mathbb{E}\left[\left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right] \, \mathrm{d}u$$

$$= D \cdot \mathbb{E}\left[\int\limits_{(s,t]} h_u \, \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right| \, \mathrm{d}u\right]$$

where once again the law of total expectation has been applied to get rid of the condition in the conditional expectation. To be able to use similar arguments as in the first proof, we need to arrive at a representation where the conditional expectation is taken with respect to a sigma algebra. Here, we achieve to get $\mathcal{G}_u^-$, after reformulation from the condition $\mathcal{J}_u = 0$.

Let us continue with the summand (V) of the original equation. In this part, we also need to get rid of the conditional expectation and some additional results have to be formulated first. To rewrite the integrator, we want to use a formula, that is similar to the formula 2.4.5, but allowing for a jump time as the time point instead. We directly replicate the proof with similar reasoning, but adapted to the needs of this situation.

The σ-algebra $\sigma(\mathbb{I}^M_{Q_J^-})$ has the representation

$$\sigma(\mathbb{I}^M_{Q_J^-}) = \left\{\emptyset,\ \Omega,\ A^M_{Q_J^-},\ \left(A^M_{Q_J^-}\right)^{\mathsf{c}}\right\} = \sigma(A^M_{Q_J^-})$$

with generator $A^M_{Q_J^-}$. Let $\{\emptyset, \Omega\}$ be the trivial σ-algebra.

For any $H \in \sigma(\mathbb{I}^M_{Q_J^-})$ exists a $\tilde{H} \in \{\emptyset, \Omega\}$ and for any $\tilde{H} \in \{\emptyset, \Omega\}$ exists an $H \in \sigma(\mathbb{I}^M_{Q_J^-})$ such that

$$H \cap A^M_{Q_J^-} = \tilde{H} \cap A^M_{Q_J^-} \ .$$

As a consequence, it follows that for any $H \in \big(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\big)$ there exists a $\tilde{H} \in \big(\sigma(Z_M) \vee \sigma(R_J)\big)$ and for any $\tilde{H} \in \big(\sigma(Z_M) \vee \sigma(R_J)\big)$ there exists an $H \in \big(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\big)$ such that

$$H \cap A^M_{Q_J^-} = \tilde{H} \cap A^M_{Q_J^-}$$

or equivalently, in notation of the indicator functions

$$\mathbb{1}_H \, \mathbb{I}^M_{Q_J^-} = \mathbb{1}_{\tilde{H}} \, \mathbb{I}^M_{Q_J^-} \tag{3.3.7}$$

holds respectively. Further, the relation

$$\begin{aligned}\big(\sigma(Z_M) \vee \sigma(R_J)\big) \cap A^M_{Q_J^-} &= \big(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\big) \cap A^M_{Q_J^-} &(3.3.8)\\ &\subseteq \big(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\big) &(3.3.9)\end{aligned}$$

holds, implying that the random variable

$$\mathbb{I}^M_{Q_J^-} \, \frac{\mathbb{E}_{M,R_J}\left[\xi \, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]} \tag{3.3.10}$$

is $\left(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right)$-measurable.

For any $H \in \left(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right)$, by the law of total expectation, we achieve

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right] \\
&= \mathbb{E}\left[\mathbb{E}_{M,R_J}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right]\right] \\
&\overset{(3.3.7)}{=} \mathbb{E}\left[\mathbb{E}_{M,R_J}\left[\mathbb{1}_{\tilde{H}}\, \mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\tilde{H}}\, \mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\tilde{H}}\, \mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]\right] = \mathbb{E}\left[\mathbb{1}_{\tilde{H}}\, \mathbb{I}^M_{Q_J^-}\, \xi\right] \\
&\overset{(3.3.7)}{=} \mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \xi\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \xi \,\middle|\, \sigma(Z_M) \vee \sigma(R_I) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right]\right] \\
&\overset{(3.3.8)}{=} \mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \mathbb{E}\left[\xi \,\middle|\, \sigma(Z_M) \vee \sigma(R_I) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right]\right] ,
\end{aligned}
$$

where we used both equations (3.3.7) and (3.3.8) to guarantee the needed measurability in the respective steps of the equation, which enables us to exchange between conditional expectations with different conditions.

That means, that for any $H \in \left(\sigma(Z_M) \vee \sigma(R_J) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right)$ it holds

$$
\mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right] = \mathbb{E}\left[\mathbb{1}_H\, \mathbb{I}^M_{Q_J^-}\, \mathbb{E}\left[\xi \,\middle|\, \sigma(Z_M) \vee \sigma(R_I) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right]\right] .
$$

Therefore, the following equation almost surely holds

$$
\mathbb{I}^M_{Q_J^-}\, \frac{\mathbb{E}_{M,R_J}\left[\xi\, \mathbb{I}^M_{Q_J^-}\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]} = \mathbb{I}^M_{Q_J^-}\, \mathbb{E}\left[\xi \,\middle|\, \sigma(Z_M) \vee \sigma(R_I) \vee \sigma(\mathbb{I}^M_{Q_J^-})\right] \ a.s. \ .
$$

Now continue to rewrite the equation (V) and use, that the counting process μ_J can at most have one jump in the interval $(0, T]$. Remembering the definition, this jump time is given as the random variable Q_J, and this enables us to rewrite the integral with the counting process as a jump with zero or one summands.

To track if the jump actually happens during $(s,t]$, we use the indication function $\mathbb{1}_{\{Q_J\in(s,t]\}}$ for $(s,t]\subseteq(0,T]$. We then arrive at

$$
\begin{aligned}
&\text{(V)}\\
&= \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \mathbb{I}^M_{u^-}\,\frac{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{u^-}\,h_u\,\left|{}^{(n)}X_u - {}^{(n-1)}X_u\right|\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{u^-}\right]}\,\mu_J(\mathrm{d}(u,f))\right]\\
&= \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\mathbb{I}^M_{Q_J^-}\,\frac{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\cdot\mu_J\big((s,t]\times E_J\big)\right]\\
&= \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\,\mathbb{I}^M_{Q_J^-}\,\frac{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\right]}{\mathbb{E}_{M,R_J}\left[\mathbb{I}^M_{Q_J^-}\right]}\right]\\
&= \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\,\mathbb{I}^M_{Q_J^-}\,\mathbb{E}\left[h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\,\middle|\,\sigma(Z_M)\vee\sigma(R_J)\vee\sigma\big(\mathbb{I}^M_{Q_J^-}\big)\right]\right]\\
&\overset{(i)}{=} \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\,\mathbb{I}^M_{Q_J^-}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\,\middle|\,\sigma(Z_M)\vee\sigma(R_J)\vee\sigma\big(\mathbb{I}^M_{Q_J^-}\big)\right]\right]\\
&= \sum_{J\in\mathcal{N}}\sum_{M\in\mathcal{M}} \mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\,\mathbb{I}^M_{Q_J^-}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\right]\\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\sum_{M\in\mathcal{M}}\mathbb{I}^M_{Q_J^-}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\right]\\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\mathbb{1}_{\{Q_J\in(s,t]\}}\,h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\right]\\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[h_{Q_J}\,\left|{}^{(n)}X_{Q_J} - {}^{(n-1)}X_{Q_J}\right|\cdot\mu_J\big((0,T]\times E_J\big)\right]\\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} h_t\,\left|{}^{(n)}X_t - {}^{(n-1)}X_t\right|\,\mu_J(\mathrm{d}(t,f))\right],
\end{aligned}
$$

where the condition of the conditional expectation enables us to pull both

$$\mathbb{I}^M_{Q_J^-},\ \text{as well as}\ \mathbb{1}_{\{Q_J\in(s,t]\}}$$

into the expectation in step (i), since both indicators are measurable with respect to the joint σ-algebra by construction, since especially $R_J=(Q_J,Z_J)$ contains Q_J.

The integral in the last line can now be written as the limit of the grid-sum and the càdlàg property of the process X_u (and h_u) is used to determine the value of the integrand. This workaround is used, to not have to work with the stopping time sigma-algebra $\mathcal{F}_{Q_J^-}$.

Rewriting of the integral leads to

$$\begin{aligned}
(\mathrm{V}) &= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right| \mu_J(\mathrm{d}(u,f))\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]} h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right| \mu_J(\mathrm{d}u, E_J)\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\lim_{n\to\infty} \sum_{\tau_n^{[s,t]}} h_{t_{k+1}} \left|{}^{(n)}X_{t_{k+1}} - {}^{(n-1)}X_{t_{k+1}}\right| \cdot \mu_J((t_k,\, t_{k+1}], E_J)\right] \\
&\overset{(i)}{=} \sum_{J\in\mathcal{N}} \lim_{n\to\infty} \sum_{\tau_n^{[s,t]}} \mathbb{E}\left[h_{t_{k+1}} \left|{}^{(n)}X_{t_{k+1}} - {}^{(n-1)}X_{t_{k+1}}\right| \cdot \mu_J((t_k,\, t_{k+1}], E_J)\right] \\
&\overset{(ii)}{=} \sum_{J\in\mathcal{N}} \lim_{n\to\infty} \sum_{\tau_n^{[s,t]}} \mathbb{E}\left[\mathbb{E}\left[h_{t_{k+1}} \left|{}^{(n)}X_{t_{k+1}} - {}^{(n-1)}X_{t_{k+1}}\right| \cdot \mu_J((t_k,\, t_{k+1}], E_J) \,\middle|\, \mathcal{F}_{t_{k+1}}\right]\right] \\
&\overset{(iii)}{=} \sum_{J\in\mathcal{N}} \lim_{n\to\infty} \sum_{\tau_n^{[s,t]}} \mathbb{E}\left[\mathbb{E}\left[h_{t_{k+1}} \left|{}^{(n)}X_{t_{k+1}} - {}^{(n-1)}X_{t_{k+1}}\right| \,\middle|\, \mathcal{F}_{t_{k+1}}\right] \cdot \mu_J((t_k,\, t_{k+1}], E_J)\right] \\
&\overset{(iv)}{=} \sum_{J\in\mathcal{N}} \mathbb{E}\left[\lim_{n\to\infty} \sum_{\tau_n^{[s,t]}} \mathbb{E}\left[h_{t_{k+1}} \left|{}^{(n)}X_{t_{k+1}} - {}^{(n-1)}X_{t_{k+1}}\right| \,\middle|\, \mathcal{F}_{t_{k+1}}\right] \mu_J((t_k,\, t_{k+1}], E_J)\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]} \mathbb{E}\left[h_u \left|{}^{(n)}X_u - {}^{(n-1)}X_u\right| \,\middle|\, \mathcal{F}_u\right] \mu_J(\mathrm{d}u, E_J)\right] \\
&\overset{(v)}{=} \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \mu_J(\mathrm{d}u, E_J)\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \mu_J(\mathrm{d}(u,f))\right],
\end{aligned}$$

for any partition $\left(\tau_n^{[s,t]}\right)_{n\in\mathbb{N}} \in \mathfrak{T}([s,t])$ and for every subinterval $(s,t] \subseteq (0,T]$.

Additional reasoning for the marked steps in the last equation is provided now:

- (i) We interchange expectation and limit by using the monotone convergence theorem, which can be applied, since all summand are non-negative and we already know, that the limit exists. The sum then only consists of one summand at most, and we therefore also use the additivity of the expectation.
- (ii) The total expectation theorem is used once again, but here with full information sigma-algebra $\mathcal{F}_{t_{k+1}}$ for each summand individually.
- (iii) $\mu_J((t_k,\, t_{k+1}], E_J)$ is $\mathcal{F}_{t_{k+1}}$-measurable.

(*iv*) Step (*i*) in return. We also know, that the limit exists, since we have guaranteed the existence of the process of the optional projection with respect to $\mathbb{F}$, and we also took the corresponding version, that is càdlàg.

(*v*) Theorem III.20 from [Pro05] (Compare A.3.9) is applied, since

$$\mathbb{E}\left[h_u \cdot \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \,\middle|\, \mathcal{F}_u\right]$$

is a martingale by construction. It is bounded (all payments are bounded and only finitely many jumps may happen) and the integrator is of integrable variation, since it is a jump process with at most one jump.

By similar reasoning as before, we can now proceed by exchanging the process with its $\mathbb{F}$-compensator, because the $\mathbb{F}^-$-adaptivity of the integrand can now be used.

We achieve the upper bound

$$\begin{aligned}
(\mathrm{V}) &= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \mu_J(\mathrm{d}(u,f))\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_J} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \lambda_J(\mathrm{d}(u,f))\right] \\
&= \sum_{J\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \int\limits_{E_J} l_J(u,\mathrm{d}f)\,\mathrm{d}u\right] \\
&= \mathbb{E}\left[\int\limits_{(s,t]} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] \sum_{J\in\mathcal{N}} l_J(u,E_J)\,\mathrm{d}u\right] \\
&\le \mathbb{E}\left[\int\limits_{(s,t]} \mathbb{E}\left[h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right] D\,\mathrm{d}u\right] \\
&= D \int\limits_{(s,t]} h_u\, \mathbb{E}\left[\mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \,\middle|\, \mathcal{F}_{u^-}\right]\right] \mathrm{d}u \\
&= D \int\limits_{(s,t]} h_u\, \mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\right] \mathrm{d}u \\
&= D\,\mathbb{E}\left[\int\limits_{(s,t]} h_u \left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right| \mathrm{d}u\right]
\end{aligned}$$

by using similar steps as in previous proofs.

This time, we applied the expectation to the conditional expectations in both parts but with different conditions and once again we needed these steps to be able to introduce the difference in X instead of $X^{\mathbb{G}}$ or $X^{\mathbb{F}}$. By that, we have introduced the predecessor of our iteration in both parts.

As an intermediate step, we have now shown the upper bound for expression (IV) as

$$
\begin{aligned}
&\mathbb{E}\left[\int\limits_{(s,t]} h_u \left\| {}^{(n)}G(u) - {}^{(n-1)}G(u) \right\|_n \mathrm{d}u\right] \\
&\leq D\,\mathbb{E}\left[\int\limits_{(s,t]} h_u \left| {}^{(n)}X_u - {}^{(n-1)}X_u \right| \mathrm{d}u\right] + D\,\mathbb{E}\left[\int\limits_{(s,t]} h_u \left| {}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-} \right| \mathrm{d}u\right]
\end{aligned}
$$

for every $(s,t] \subseteq (0,T]$.

Therefore, by combining the previous results and by doing some regrouping, the complete upper bound can be given as

$$
\begin{aligned}
&\left\| {}^{(n+1)}Y - {}^{(n)}Y \right\|_{V[0,T]} \\
&\leq (1+D)\,\mathbb{E}\left[\int\limits_{(0,T]} h_t \left| {}^{(n)}X_{t^-} - {}^{(n-1)}X_{t^-} \right| \mathrm{d}t\right] + D\,\mathbb{E}\left[\int\limits_{(0,T]} h_t \left| {}^{(n)}X_t - {}^{(n-1)}X_t \right| \mathrm{d}t\right] \\
&= (1+2D)\,\mathbb{E}\left[\int\limits_{(0,T]} L\,(1+D)\, e^{-K(\zeta(T)-\zeta(t))}\,\kappa(t) \left| {}^{(n)}X_t - {}^{(n-1)}X_t \right| \mathrm{d}t\right] \\
&= L\,(1+D)\,(1+2D)\,\mathbb{E}\left[\int\limits_{(0,T]} e^{-K(\zeta(T)-\zeta(t))} \left| {}^{(n)}Y_t - {}^{(n-1)}Y_t \right| \mathrm{d}t\right] \\
&\leq L\,(1+D)\,(1+2D)\,\mathbb{E}\left[\int\limits_{t\in[0,T]} e^{-K(\zeta(T)-\zeta(t))} \int\limits_{r\in[t,T]} \mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r \mathrm{d}t\right] \\
&= L\,(1+D)\,(1+2D)\,\cdot\mathbb{E}\left[\int\limits_{r\in[0,T]} \int\limits_{t\in(0,r]} e^{-K(\zeta(T)-\zeta(t))}\,\mathrm{d}t\,\mathrm{d}\left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r\right] \\
&\leq \frac{L\,(1+D)\,(1+2D)}{K} \cdot \left\| {}^{(n)}Y - {}^{(n-1)}Y \right\|_{V[0,T]}
\end{aligned}
$$

as the steps in the proof hold for every subinterval $(s,t] \subseteq (0,T]$.

The last step is very similar to the original proof of the first Theorem 3.2.12, but with the more complex constant

$$K = 2\,L\,(1+D)\,(1+2D)$$

and we have shortened it. Note, that the solving of the inner integral is actually simpler in this case, since our integrator is given by the identity as $\zeta(t) = t$.

Application of the fixed point theorem of Banach

Let us from now on assume, that $K = 2\,L\,(1+D)\,(1+2D)$. Then we have a contraction and application of the fixed-point theorem of Banach guarantees existence and uniqueness of a process $Y = (Y_t)_{t\geq 0}$ fulfilling

$$Y_t = \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s^-}\,\kappa(s)\,b_M\big(s, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big)\,\mathrm{d}s \\ + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\,B_I\big(s, e, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big)\,\mu_I(\mathrm{d}(s,e))$$

in the space of càdlàg processes with integrable variation on $[0,t]$, where additionally

$$X^{\mathbb{G}}_{s^-} = \frac{1}{\kappa(s)}\,\mathbb{E}\left[Y_{s^-}\,\middle|\,\mathcal{G}^-_s\right]\ \ a.s.$$

and similarly

$$G_J(s) = \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\left(\frac{\mathbb{E}_{M,R_J=(s,e)}\left[\mathbb{I}^M_{s^-}X_s\right]}{\mathbb{E}_{M,R_J=(s,e)}\left[\mathbb{I}^M_{s^-}\right]} - \frac{\mathbb{E}_M\left[\mathbb{I}^M_{s^-}\mathbb{I}^M_s X_s\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{s^-}\mathbb{I}^M_s\right]}\right)\ \ a.s.$$

for $X_s = \frac{1}{\kappa(s)}\,Y_s$ and $J\in\mathcal{N}$, which then concludes the proof. □

Comment 3.3.10. *Combination of the two dependency structures*
The dependency structures I and II can be combined to a third and most general case, by using two separate summands for the sojourn payments, where the preconditions and reserve-dependency have to be matched to align with the existing situations.

A proof in that situation would mostly be a additive combination of the formulated proofs, where we already used the linearity of the summands. Summarizing, we would need the function γ to allow for jumps and would arrive at a more complicated constant K. A formulation of this case is not done, as it would need a lot of notation and preconditions with a lot of repetitions, while providing minimal to none new aspects in the formulation or the proof.

Chapter 4.

Actuarial calculations in life insurance

The examination of the payment process X instead of the prospective reserve $X^{\mathbb{G}}$ is convenient for our purposes, because it enables us to do the proof of the existence and uniqueness for the payment process, but it is by no means the preferred option for the insurance company to consider.

For pricing and reserving, the insurance company has to work with the prospective reserve instead of the unobservable future payments and is therefore more interested in existence and uniqueness of the prospective reserve as a process.

4.1. The prospective reserve

We will now first develop a BSDE for the reserve and then adapt the existence and uniqueness results to the prospective reserve. We have used two different representations of the payment process, where we have shown existence and uniqueness in both cases. The main takeaway for this section is, that by using a similar measurability condition, most parts do not really depend on the exact structure and yield similar results. For simplicity, we will formulate it once nevertheless, and in the notation of dependency structure I, where the γ is there and only dependence on the general reserve in t^- is considered.

Before we develop the BSDE, let us now first calculate the corresponding IF-compensator of the involved cumulative payments.

Definition 4.1.1. *Abbreviating notation for payment functions*
The càdlàg process $b_{\mathcal{M}}$ of the accumulated discounted sojourn payments is given as

$$b_{\mathcal{M}}(t) := \sum_{M\in\mathcal{M}} \int\limits_{[0,t]} \kappa(s)\, \mathbb{I}^M_{s^-}\, b_M\big(s, X^{\mathbb{G}}_{s^-}\big)\, \gamma(\mathrm{d}s)$$

for $t \geq 0$ in the notation of Theorem 3.2.12 and is given as

$$b_{\mathcal{M}}(t) := \sum_{M\in\mathcal{M}} \int\limits_{[0,t]} \kappa(s)\, \mathbb{I}^M_{s^-}\, b_M\big(s, X^{\mathbb{G}}_{s^-}, (G_J(s))_J\big)\, \mathrm{d}s$$

for $t \geq 0$ in the notation of the full dependency model in Theorem 3.3.9.

Further, the càdlàg process $B_{\mathcal{N}}$ of the accumulated discounted transition payments is given as

$$B_{\mathcal{N}}(t) := \sum_{I\in\mathcal{N}} \int\limits_{(0,t]\times E_I} \kappa(s)\, B_I\big(s,e,X^{\mathbb{G}}_{s^-}\big)\, \mu_I(\mathrm{d}(s,e))$$

for $t \geq 0$ in the notation of Theorem 3.2.12 and is given as

$$B_{\mathcal{N}}(t) := \sum_{I\in\mathcal{N}} \int\limits_{(0,t]\times E_I} \kappa(s)\, B_I\big(s,e,X^{\mathbb{G}}_{s^-},(G_J(s))_J\big)\, \mu_I(\mathrm{d}(s,e))$$

for $t \geq 0$ in the notation of the full dependency model in Theorem 3.3.9.

Assertion 4.1.2. IF-*compensator of the sojourn payments*
The process $b_{\mathcal{M}}$ (in both representations) is IF-predictable with respect to $\mathbb{G}$, i.e.

$$b^{\mathrm{IF}}_{\mathcal{M}}(t) = b_{\mathcal{M}}(t)\,.$$

Therefore the IF-compensator of $b_{\mathcal{M}}$ is $b_{\mathcal{M}}$ itself.

Proof. The proof is performed only in the notation of the first representation of $b_{\mathcal{M}}$, since the arguments do not depend on the explicit representation, but on the underlying properties like the assumed measurability condition.

By using some results from the introductory chapter, the dominated convergence theorem and the finiteness of the set $\mathcal{M}_t$, we get:

$$\begin{aligned}
b^{\mathrm{IF}}_{\mathcal{M}}(t) &= \lim_{n\to\infty} \sum_{\tau^t_n} \mathbb{E}\left[b_{\mathcal{M}}(t_{k+1}) - b_{\mathcal{M}}(t_k)\,\middle|\,\mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau^t_n} \mathbb{E}\left[\sum_{\tilde{M}\in\mathcal{M}} \int\limits_{(t_k,t_{k+1}]} \kappa(s)\,\mathbb{I}^{\tilde{M}}_{s^-}\, h(\tilde{M},s,X^{\mathbb{G}}_{s^-})\,\gamma(\mathrm{d}s)\,\middle|\,\mathcal{G}_{t_k}\right] \\
&\overset{\text{Form.3.2.1}}{=} \lim_{n\to\infty} \sum_{\tau^t_n} \sum_{M\in\mathcal{M}_t} \mathbb{I}^M_{t_k}\,\mathbb{E}\left[\int\limits_{(t_k,t_{k+1}]} \sum_{\tilde{M}\in\mathcal{M}} \kappa(s)\,\mathbb{I}^{\tilde{M}}_{s^-}\, h(\tilde{M},s,X^{\mathbb{G}}_{s^-})\,\gamma(\mathrm{d}s)\,\middle|\,\mathcal{G}_{t_k}\right] \\
&= \lim_{n\to\infty} \sum_{\tau^t_n} \sum_{M\in\mathcal{M}_t} \int\limits_{(t_k,t_{k+1}]} \mathbb{I}^M_{t_k}\,\mathbb{E}\left[\sum_{\tilde{M}\in\mathcal{M}} \kappa(s)\,\mathbb{I}^{\tilde{M}}_{s^-} h(\tilde{M},s,X^{\mathbb{G}}_{s^-})\,\middle|\,\mathcal{G}_{t_k}\right]\gamma(\mathrm{d}s) \\
&\overset{(i)}{=} \sum_{M\in\mathcal{M}_t} \lim_{n\to\infty} \sum_{\tau^t_n} \int\limits_{(t_k,t_{k+1}]} \mathbb{I}^M_{t_k}\,\mathbb{E}\left[\sum_{\tilde{M}\in\mathcal{M}} \kappa(s)\,\mathbb{I}^{\tilde{M}}_{s^-} h(\tilde{M},s,X^{\mathbb{G}}_{s^-})\,\middle|\,\mathcal{G}_{t_k}\right]\gamma(\mathrm{d}s) \\
&\overset{\text{Form. 2.4.3}}{=} \sum_{M\in\mathcal{M}_t} \lim_{n\to\infty} \sum_{\tau^t_n} \int\limits_{(t_k,t_{k+1}]} \mathbb{I}^M_{t_k} \frac{\mathbb{E}_M\left[\sum\limits_{\tilde{M}\in\mathcal{M}} \mathbb{I}^M_{t_k}\,\kappa(s)\,\mathbb{I}^{\tilde{M}}_{s^-} h(\tilde{M},s,X^{\mathbb{G}}_{s^-})\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{t_k}\right]}\gamma(\mathrm{d}s)
\end{aligned}$$

$$\overset{\text{Lemma 2.4.11}}{=} \sum_{M\in\mathcal{M}_t} \int_{(0,t]} \kappa(s)\,\mathbb{I}^M_{s-} \frac{\mathbb{E}_M\left[\mathbb{I}^M_{s-}\, b_M(s, X^{\mathbb{G}}_{s-})\right]}{\mathbb{E}_M\left[\mathbb{I}^M_{s-}\right]}\,\gamma(\mathrm{d}s)$$
$$\overset{\text{Form. 2.4.3}}{=} \sum_{M\in\mathcal{M}} \int_{(0,t]} \kappa(s)\,\mathbb{I}^M_{s-}\,\mathbb{E}\left[b_M(s, X^{\mathbb{G}}_{s-}) \,\middle|\, \mathcal{G}^-_s\right]\,\gamma(\mathrm{d}s)$$
$$\overset{(ii)}{=} \sum_{M\in\mathcal{M}} \int_{(0,t]} \kappa(s)\,\mathbb{I}^M_{s-}\, b_M(s, X^{\mathbb{G}}_{s-})\,\gamma(\mathrm{d}s)$$
$$= b_{\mathcal{M}}(t) - b_{\mathcal{M}}(0)$$

for a sequence $(\tau^t_n)_{n\in\mathbb{N}} \in \mathfrak{T}([0,t])$.

In (i), the dominated convergence theorem is applied to exchange the sum and the limit. In step (ii), we used that b_M is $\mathbb{G}^-$-adapted (which is the case for both representations of b_M) and always the case for the discounting, leading to the possibility of pulling it out of the conditional expectation.
Even if the $\mathbb{G}^-$-measurability would not have been an assumption, the so-called $\mathbb{G}^-$-average

$$\mathbb{E}\left[b_M(s, X^{\mathbb{G}}_{s-}) \,\middle|\, \mathcal{G}_{s-}\right]$$

would be left in the third to last line and similar results could be achieved by continuing with the $\mathbb{G}^-$-averaged payment.

The result of the calculation especially means, that $b_{\mathcal{M}}$ is IF-predictable with respect to $\mathbb{G}$, and that was to show. □

Assertion 4.1.3. IF-*compensator of the transition payments*
The IF-predictor of the process $B_{\mathcal{N}}$ (in both representations) with respect to $\mathbb{G}$ is given by

$$B^{\mathrm{IF}}_{\mathcal{N}}(t) = \sum_{I\in\mathcal{N}} (\kappa B_I) \bullet \nu_I([0,t] \times E_I)\,.$$

Proof. The proof is once again only performed for the first representation of $B_{\mathcal{N}}(t)$.

By rewriting V with the help of the abbreviating notation from the previous chapter (applied to every summand individually), we get

$$B_{\mathcal{N}}(t) = \sum_{I\in\mathcal{N}} (\kappa B_I) \bullet \mu_I((0,t] \times E_I)\,.$$

Since we assumed, that B_I is bounded and $\mathbb{G}^-$-adapted (in both representations), and the discounting factor does not compromise that, the preconditions in Theorem 2.5.4 are fulfilled (note, that vB_I is the F_I in our current application). The theorem states, that for every $I \in \mathcal{N}$

$$(\kappa B_I) \bullet \mu_I((0,t] \times E_I)$$

has the IF-compensator

$$(\kappa B_I) \bullet \nu_I((0,t] \times E_I)\,.$$

The application summand by summand can be performed, because Assumption 2.3.2 still holds. Therefore, almost surely only finitely many summands have non-zero contributions and the application summand by summand is possible. In total, we have that

$$B_{\mathcal{N}}^{\mathrm{IF}}(t) = \sum_{I\in\mathcal{N}} (\kappa B_I) \bullet \nu_I((0,t] \times E_I)\,.$$

$\square$

Assertion 4.1.4. IF-*compensator – Cumulative payments*
The process A of cumulated payments

$$A(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M\, b_M(t, X_{t^-}^{\mathbb{G}})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I(s, e, X_{t^-}^{\mathbb{G}})\,\mu_I(\mathrm{d}t \times \mathrm{d}e)$$

has the IF-compensator

$$A^{\mathrm{IF}}(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M\, b_M(t, X_{t^-}^{\mathbb{G}})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I(s, e, X_{t^-}^{\mathbb{G}})\,\nu_I(\mathrm{d}t \times \mathrm{d}e)\,.$$

Proof. The additive decomposition of A in sojourn and transition summands is used. Then, the two parts are similar to $b_{\mathcal{M}}$ and $B_{\mathcal{N}}$ (without discounting), and consequently a similar structure of the IF-compensator arises. Leaving out the discounting simplifies the proofs. $\square$

4.2. Thieles BSDE

We will begin to state the Thiele BSDE with respect to $\mathbb{F}$ for the projection $X_t^{\mathbb{F}}$. This is done to be able to compare the structure in both cases.
Take note, that the representation of the payment functions is different for dependency structures I and II (as well as the integrator γ), and we will only formulate the results for the simpler dependency structure I to avoid repetitions. The results hold for both cases, as the underlying preconditions are similar.

Theorem 4.2.1. *Thiele BSDE with respect to* $\mathbb{F}$
The prospective reserve with respect to $\mathbb{F}$ fulfils the following backward stochastic differential equation

$$\mathrm{d}X_t^{\mathbb{F}} = \varphi(t)\, X_{t^-}^{\mathbb{F}}\,\mathrm{d}t - A(\mathrm{d}t) - \sum_{I\in\mathcal{N}} \int\limits_{E_I} F_I(t,e)\,(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e) \tag{4.2.1}$$

with terminal condition $X_T^{\mathbb{F}} = 0$.

The integrand $F_I(t,e)$ is also known as the 'sum at risk' in actuarial practice and may be expressed as

$$F_I(t,e) = \mathbb{E}\left[X_{t^-} \,\middle|\, \mathcal{F}_t^-,\, R_I = (t,e)\right] - \mathbb{E}\left[X_{t^-} \,\middle|\, \mathcal{F}_t^-,\, \mathcal{J}_t = 0\right]\,.$$

Equation (4.2.1), together with final value condition $X^{\mathbb{F}}_T = 0$, then is a BSDE with solution pair $\left(X^{\mathbb{F}}, (F_I)_I\right)$.

Corollary 4.2.2. *Reformulation of the Thiele BSDE with respect to* $\mathbb{F}$
The above BSDE (4.2.1) may be expressed in the following equivalent form

$$\mathrm{d}X^{\mathbb{F}}_t = f(\omega, t, X^{\mathbb{F}}_{t^-})\,\mathrm{d}t + g(\omega, t, X^{\mathbb{F}}_{t^-})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} Z_I(t,e)\,(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e)$$

with natural final value condition $X^{\mathbb{F}}_T = 0$, and where the generator functions are given as

$$\begin{aligned} f(\omega, t, X^{\mathbb{F}}_{t^-}) &= \varphi(t)\, X^{\mathbb{F}}_{t^-} \; - \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I(t, e, X^{\mathbb{F}}_{t^-})\, l(t, \mathrm{d}e)\,, \\ g(\omega, t, X^{\mathbb{F}}_{t^-}) &= - \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-}\, b_M(t, X^{\mathbb{F}}_{t^-}) \end{aligned}$$

and

$$Z_I(s,e) = F_I(s,e) - B_I(s, e, X^{\mathbb{F}}_{s^-})\,.$$

Further reformulation and examples for Markovian models can be found in [DL16] and examples for non-Markovian models can be found in [CD20]. These cases will not be pursued in this thesis.

Comment 4.2.3. *On the case of a sub-filtration* $\mathbb{G}$ *of* $\mathbb{F}$
Until now, we have been focused on the development of BSDEs for the two main situations. There might also be a situation, where the information not the complete $\mathbb{F}$ (information shrinkage), but still monotone. If we do not assume the model of Christiansen [Chr21b], but instead use a sub-filtration $\mathbb{G}$ of $\mathbb{F}$, then the prospective reserve with respect to $\mathbb{G}$ does still exist and has a version that is càdlàg, as the Theorem can be applied to this situation as well. Further, we can still apply the classical martingale representation theorem, as long as our measurability conditions for the payments are formulated with respect to $\mathbb{G}$.

The needed $\mathbb{G}$-compensator can be calculated from the classical $\mathbb{F}$-compensator by using the innovation theorem (compare for example Theorem 3.4 in [Aal78]) or Proposition 4.8.4 (together with Corollary 4.8.5) in [Jac05], or the comments in Chapter 1) in the classical situation of two filtrations. The representation of $F_I(t,e)$ only holds in this case, as these rely on the notation of Christiansen [Chr21b], but a general application of the martingale representation theorem is still possible. In this case, the Thiele BSDE with respect to $\mathbb{G}$ would be similar to the one presented in 4.2.1, with the above mentioned adaptations, and we will not formulate it in detail, especially since the details can also depend on the actual representation of $\mathbb{G}$.

Proof. Let us consider the càdlàg process

$$Y_t := \kappa(t)\cdot X_t = \int\limits_{(t,T]} \kappa(s)\, A(\mathrm{d}s) \tag{4.2.2}$$

where all payments are discounted up to time zero, instead of just up to time t. The discounting is assumed to be deterministic and therefore the optional projection also fulfils

$$Y_t^{\mathbb{F}} = \kappa(t) \cdot X_t^{\mathbb{F}} \ a.s.$$

By using 4.2.2 we can compute the following two differentials for the payment process

$$\mathrm{d}Y_t = -\kappa(t)\, A(\mathrm{d}t)$$

and, by using the product rule, for the prospective reserve

$$\mathrm{d}Y_t^{\mathbb{F}} = -\varphi(t)\,\kappa(t)\, X_{t-}^{\mathbb{F}}\,\mathrm{d}t + \kappa(t)\,\mathrm{d}X_t^{\mathbb{F}}\,.$$

Then, the difference

$$Y_t - Y_0 = -\int\limits_{(0,t]} \kappa(s)\, A(\mathrm{d}s)$$

is $\mathcal{F}_t$ -measurable, which is the precondition of the martingale representation Theorem 2.5.12, and application of the theorem to $Y_t^{\mathbb{F}}$ yields

$$\begin{aligned}
&\kappa(t)\,\mathrm{d}X_t^{\mathbb{F}} - \varphi(t)\,\kappa(t)\, X_{t-}^{\mathbb{F}}\,\mathrm{d}t \\
&= \mathrm{d}Y_t^{\mathbb{F}} \overset{2.5.12}{=} \mathrm{d}Y_t + \sum_{I\in\mathcal{N}} \int\limits_{E_I} \tilde{F}_I(t,e)(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e) \\
&= -\kappa(t)\, A(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} \tilde{F}_I(t,e)(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e)\,.
\end{aligned}$$

By rearranging terms, we get

$$\kappa(t)\,\mathrm{d}X_t^{\mathbb{F}} = \kappa(t) \cdot \left(-A(\mathrm{d}t) + \varphi(t)\, X_{t-}^{\mathbb{F}}\,\mathrm{d}t + \sum_{I\in\mathcal{N}} \int\limits_{E_I} F_I(t,e)\,(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e) \right)$$

where $F_I(t,e) = \frac{1}{\kappa(t)}\,\tilde{F}_I(t,e)$. The reformulation to F was necessary, since the $\tilde{F}$ originally corresponds to $Y_t^{\mathbb{F}}$ and the specific structure implies, that the F now actually corresponds to $X_t^{\mathbb{F}}$, since the deterministic factor can also be pulled into the expectations. Respectively as we almost surely have

$$\begin{aligned}
F_I(t,e) &= \mathbb{E}\left[X_{t-} \,\middle|\, \mathcal{F}_t^-,\, R_I = (t,e)\right] - \mathbb{E}\left[X_{t-} \,\middle|\, \mathcal{F}_t^-,\, \mathcal{J}_t = 0\right] \\
&= \frac{1}{\kappa(t)} \left(\mathbb{E}\left[Y_{t-} \,\middle|\, \mathcal{F}_t^-,\, R_I = (t,e)\right] - \mathbb{E}\left[Y_{t-} \,\middle|\, \mathcal{F}_t^-,\, \mathcal{J}_t = 0\right] \right) \\
&= \frac{1}{\kappa(t)}\,\tilde{F}_I(t,e)\,.
\end{aligned}$$

An application of the Radon-Nikodym Theorem leads to the assertion. □

Proof. (Of the reformulation in Corollary 4.2.2.)

The representation of $A(\mathrm{d}t)$ as

$$A(\mathrm{d}t) = \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M \, b_M(t, X_{t^-}^{\mathbb{F}}) \, \gamma(\mathrm{d}t) + \sum_{I \in \mathcal{N}} \int_{E_I} B_I(t, e, X_{t^-}^{\mathbb{F}}) \, \mu_I(\mathrm{d}t \times \mathrm{d}e)$$

is used, which directly defines g, by matching the $\gamma(\mathrm{d}t)$-parts.

The second summand contains a μ_I integral and is shifted into the integral part of the BSDE, and the new $Z_I(t,e) := F_I(t,e) - B_I(t,e)$ is integrated. Since the integration is with respect to the $\mathbb{F}$-compensated measure $\mu_I - \lambda_I$, the newly arising λ_I part has to be compensated as well. Finally, the Lebesgue intensity l_I is used, to include this integral in the $\mathrm{d}t$ part of the BSDE, and therefore, as part of the function f. □

Similar results can be shown in the case with $\mathbb{G}$.

Theorem 4.2.4. *Thiele BSDE with respect to* $\mathbb{G}$
The following differential equation holds

$$\begin{aligned} \mathrm{d}X_t^{\mathbb{G}} = -A^{\mathrm{IF}}(\mathrm{d}t) + \varphi(t) X_{t^-}^{\mathbb{G}} \, \mathrm{d}t &+ \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t^-, t, e) \; (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\ &+ \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t , t, e) \; (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \end{aligned} \tag{4.2.3}$$

with terminal condition $X_T^{\mathbb{G}} = 0$ and where the representation of the integrands G_I is given by Theorem 2.5.13, and they can almost surely be written as

$$\begin{aligned} G_I(t^-, t, e) &= \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, \, R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t^-, \, \mathcal{J}_t = 0\right], \\ G_I(t , t, e) &= \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t , \, R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t , \, \mathcal{J}_t = 0\right] \end{aligned}$$

using the reformulation in 3.3.1.

This equation, combined with final value condition $X_T^{\mathbb{G}} = 0$, is a BSDE with solution pair $(X^{\mathbb{G}}, (G_I)_I)$. In difference to the classical Thiele equation, an IF-martingale as well as an IB-martingale appear.
First, the IF-martingale quantifies the impact of new information on the optional projection $X^{\mathbb{G}}$. The integrand $G_I(t^-, t, e)$ corresponds to the 'sum at risk'. Second, the IB-martingale quantifies the effect of information deletions on $X^{\mathbb{G}}$. As a deletion reduces the individual risk characteristic, this part may be interpreted as a risk transfer.
The integrability assumption and measurability (with respect to $\mathbb{G}^-$ and $\mathbb{G}$ respectively) is indeed satisfied for $G_I(t^-, t, e)$ and $G_I(t , t, e)$, and the projection $X^{\mathbb{G}}$ is by design a $\mathbb{G}$-adapted process.

We also want to derive a similar reformulation to the one in the standard filtration case. Such a complete reformulation is only achievable, if one is willing to use different representation of $G(t^-, t, e)$ and $Gt \quad , t, e)$, since only the IF-compensator is used as part of A^{IF} and it would lead to asymmetry (Compare for example Theorem 2.4.3 in [Fur20]).

Corollary 4.2.5. *Reformulation of the Thiele BSDE with respect to* $\mathbb{G}$
The above BSDE (4.2.3) may be expressed in the following equivalent form

$$
\begin{aligned}
\mathrm{d}X_t^{\mathbb{G}} = f(\omega, t, X_{t^-}^{\mathbb{G}})\,\mathrm{d}t + g(\omega, t, X_{t^-}^{\mathbb{G}})\,\gamma(\mathrm{d}t) &+ \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t^-, t, e)\ (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&+ \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t \quad , t, e)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)
\end{aligned}
$$

with final value $X_T^{\mathbb{G}} = 0$ and generator functions, defined as

$$
\begin{aligned}
f(\omega, t, X_{t^-}^{\mathbb{G}}) &= \varphi(t) X_{t^-}^{\mathbb{G}} \ - \sum_{I \in \mathcal{N}} \int_{E_I} B_I(t, e, X_{t^-}^{\mathbb{G}})\ n_I(t, \mathrm{d}e)\,, \\
g(\omega, t, X_{t^-}^{\mathbb{G}}) &= - \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^{M}\, b_M(t, X_{t^-}^{\mathbb{G}})
\end{aligned}
$$

and with $(G_I)_I$ remaining as before.

Proof. Let us again consider the càdlàg process

$$
Y_t = \kappa(t) \cdot X_t = \int_{(t,T]} \kappa(s)\, A(\mathrm{d}s)
$$

where all payments are discounted up to time zero. This is now not needed to fulfil the preconditions of the infinitesimal martingale representation theorem, but has to be done, to be able to compute the IF-compensator and that has also been the reason, why we precomputed the compensator of $b_{\mathcal{M}}$ and $B_{\mathcal{N}}$, with discounting to zero already included. The discounting is assumed to be deterministic and therefore the optional projection also almost surely fulfils

$$
Y_t^{\mathbb{G}} = \kappa(t) \cdot X_t^{\mathbb{G}}
$$

and we again use the differential

$$
\mathrm{d}Y_t^{\mathbb{G}} = \kappa(t)\,\mathrm{d}X_t^{\mathbb{G}} - \varphi(t)\,\kappa(t)\,X_{t^-}^{\mathbb{G}}\,\mathrm{d}t\,.
$$

We then apply Theorem 2.5.13 to Y and note that the preconditions are indeed fulfilled, as we need the càdlàg property of Y and the integrability condition specified in 2.4.1, to arrive at

$$
\mathrm{d}Y_t^{\mathbb{G}} = \mathrm{d}Y_t^{\mathrm{IF}} + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t^-, t, e)\,(\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e)
$$

$$+ \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t \quad , t, e) \, (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \, ,$$

where (in the reformulated form)

$$\begin{aligned} \tilde{G}_I(t^-, t, e) &= \mathbb{E}\left[Y_t \,\middle|\, \mathcal{G}_t^-, \, R_I = (t,e) \right] - \mathbb{E}\left[Y_t \,\middle|\, \mathcal{G}_t^-, \, \mathcal{J}_t = 0 \right] \;\; a.s., \\ \tilde{G}_I(t \quad , t, e) &= \mathbb{E}\left[Y_t \,\middle|\, \mathcal{G}_t \quad , \, R_I = (t,e) \right] - \mathbb{E}\left[Y_t \,\middle|\, \mathcal{G}_t \quad , \, \mathcal{J}_t = 0 \right] \;\; a.s.. \end{aligned}$$

To be able to work with the equation, we need to calculate the IF-compensator of Y. It holds, that by swapping in the representation for $A(\mathrm{d}t)$, we get from the definition of the compensator

$$\begin{aligned} Y_t^{\mathrm{IF}} &= \lim_{n \to \infty} \sum_{\tau_n} \mathbb{E}\left[Y_{t_{k+1}} - Y_{t_k} \,\middle|\, \mathcal{G}_{t_k} \right] \\ &= \lim_{n \to \infty} \sum_{\tau_n} \mathbb{E}\left[\int_{(t_{k+1}, T]} \kappa(s) \, A(\mathrm{d}s) - \int_{(t_k, T]} \kappa(s) \, A(\mathrm{d}s) \,\middle|\, \mathcal{G}_{t_k} \right] \\ &= \lim_{n \to \infty} \sum_{\tau_n} \mathbb{E}\left[- \int_{(t_k, t_{k+1}]} \kappa(s) \, A(\mathrm{d}s) \,\middle|\, \mathcal{G}_{t_k} \right] \\ &= \lim_{n \to \infty} \sum_{\tau_n} \mathbb{E}\left[- \int_{(t_k, t_{k+1}]} \sum_{M \in \mathcal{M}} \mathbb{I}_{t-}^M \, \kappa(s) \, b_M(s, X_{s-}^{\mathbb{G}}) \, \gamma(\mathrm{d}s) \,\middle|\, \mathcal{G}_{t_k} \right] \\ &\quad + \lim_{n \to \infty} \sum_{\tau_n} \mathbb{E}\left[- \int_{(t_k, t_{k+1}] \times E_I} \sum_{I \in \mathcal{N}} \kappa(s) \; B_I(s, e, X_{s-}^{\mathbb{G}}) \, \mu_I(\mathrm{d}(s,e)) \,\middle|\, \mathcal{G}_{t_k} \right] \\ &= -b_{\mathcal{M}}^{\mathrm{IF}}(t) - B_{\mathcal{N}}^{\mathrm{IF}}(t) \\ &= -b_{\mathcal{M}}(t) - B_{\mathcal{N}}^{\mathrm{IF}}(t) \end{aligned}$$

and in full form, we have

$$\mathrm{d}Y_t^{\mathrm{IF}} = - \sum_{M \in \mathcal{M}} \kappa(t) \, \mathbb{I}_{t-}^M \, b_M(t, X_{t-}^{\mathbb{G}}) \, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} \kappa(t) \, B_I(t, e, X_{t-}^{\mathbb{G}}) \, \nu_I(\mathrm{d}t \times \mathrm{d}e) \, .$$

All together, when using the formulas for the differential and the compensator, we get

$$\begin{aligned} &\kappa(t) \, \mathrm{d}X_t^{\mathbb{G}} - \varphi(t) \, \kappa(t) \, X_{t-}^{\mathbb{G}} \, \mathrm{d}t \\ &= \mathrm{d}Y_t^{\mathbb{G}} \\ &\overset{2.5.13}{=} \mathrm{d}Y_t^{\mathrm{IF}} + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I\,(t^-, t, e) \;\; (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\ &\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I\,(t \quad , t, e) \;\; (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \end{aligned}$$

$$
\begin{aligned}
&= - \sum_{M \in \mathcal{M}} \kappa(t)\, \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G}}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} \kappa(t)\, B_I\big(t, e, X^{\mathbb{G}}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\quad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t^-, t, e)\, (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\quad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t\quad, t, e)\, (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)\,,
\end{aligned}
$$

which by rearranging reads

$$
\begin{aligned}
&\kappa(t)\, \mathrm{d}X^{\mathbb{G}}_t \\
&= \kappa(t) \Bigg(\varphi(t)\, X^{\mathbb{G}}_{t^-}\, \mathrm{d}t - \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G}}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} B_I\big(t, e, X^{\mathbb{G}}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I\left(t^-, t, e\right)\, (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I\left(t\quad, t, e\right)\, (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \Bigg)\,,
\end{aligned}
$$

and where we substituted

$$
G_I(t^-, t, e) = \frac{1}{\kappa(t)}\, \tilde{G}_I(t^-, t, e)
$$

and similarly for the other representation.

An application of the Radon-Nikodym Theorem leads to

$$
\begin{aligned}
\mathrm{d}X^{\mathbb{G}}_t &= \varphi(t)\, X^{\mathbb{G}}_{t^-}\, \mathrm{d}t - \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G}}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} B_I\big(t, e, X^{\mathbb{G}}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t^-, t, e)\, (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t\quad, t, e)\, (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&= \varphi(t)\, X^{\mathbb{G}}_{t^-}\, \mathrm{d}t - A^{\mathrm{IF}}(\mathrm{d}t) + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t^-, t, e)\, (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t\quad, t, e)\, (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)\,.
\end{aligned}
$$

Take note, that the reformulation from $\tilde{G}$ to G is possible since

$$
\frac{1}{\kappa(t)}\, \tilde{G}_I\left(t^-, t, e\right) = \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}^-_t,\, R_I = (t, e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}^-_t,\, \mathcal{J}_t = 0\right] = G_I(t^-, t, e)\,,
$$

$$\frac{1}{\kappa(t)}\,\tilde{G}_I\,(t\;\;,t,e) = \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t\;,\; R_I = (t,e)\right] - \mathbb{E}\left[X_t \,\middle|\, \mathcal{G}_t\;,\; \mathcal{J}_t = 0\right] = G_I(t\;\;,t,e)$$

almost surely hold, and we therefore arrive at the representation from the assertion.

Let us also formulate the integral representation for both $X^{\mathbb{G}}$ and $Y^{\mathbb{G}}$ for later use. Together, with their natural final value condition as $X_T^{\mathbb{G}} = Y_T^{\mathbb{G}} = 0$, we have

$$\begin{aligned}
\kappa(t)\,X_t^{\mathbb{G}} = Y_t^{\mathbb{G}} = Y_T^{\mathbb{G}} \;&+ \sum_{M\in\mathcal{M}} \int_{(t,T]} \kappa(s)\,\mathbb{I}_{s^-}^M\, b_M\big(s, X_{s^-}^{\mathbb{G}}\big)\,\gamma(\mathrm{d}s) \\
&+ \sum_{I\in\mathcal{N}} \int_{(t,T]} \int_{E_I} \kappa(s)\,B_I\big(s,e,X_{s^-}^{\mathbb{G}}\big)\,n_I(s,\mathrm{d}e)\,\mathrm{d}s \\
&- \sum_{I\in\mathcal{N}} \int_{(t,T]\times E_I} \kappa(s)\,G_I(s^-,s,e)\,(\mu_I-\nu_I)(\mathrm{d}(s,e)) \\
&- \sum_{I\in\mathcal{N}} \int_{(t,T]\times E_I} \kappa(s)\,G_I(s\;\;,s,e)\,(\rho_I-\mu_I)(\mathrm{d}(s,e))
\end{aligned} \tag{4.2.4}$$

for the discounted reserve as well as

$$\begin{aligned}
X_t^{\mathbb{G}} = X_T^{\mathbb{G}} + \int_{(t,T]} \varphi(s)\,X_{s^-}^{\mathbb{G}}\,\mathrm{d}s \;&+ \sum_{M\in\mathcal{M}} \int_{(t,T]} \mathbb{I}_{s^-}^M\, b_M\big(s, X_{s^-}^{\mathbb{G}}\big)\,\gamma(\mathrm{d}s) \\
+ \sum_{I\in\mathcal{N}} \int_{(t,T]} \int_{E_I} &B_I\big(s,e,X_{s^-}^{\mathbb{G}}\big)\,n_I(s,\mathrm{d}e)\,\mathrm{d}s \\
- \sum_{I\in\mathcal{N}} \int_{(t,T]\times E_I} &G_I(s^-,s,e)\,(\mu_I-\nu_I)(\mathrm{d}(s,e)) \\
- \sum_{I\in\mathcal{N}} \int_{(t,T]\times E_I} &G_I(s\;\;,s,e)\,(\rho_I-\mu_I)(\mathrm{d}(s,e))
\end{aligned} \tag{4.2.5}$$

for the original prospective reserve. □

Proof. (Of the reformulation in Corollary 4.2.5)

We plug in the previously developed representation for $A^{\mathrm{IF}}(\mathrm{d}t)$ and arrive at

$$\begin{aligned}
\mathrm{d}X_t^{\mathbb{G}} = \varphi(t)\,X_{t^-}^{\mathbb{G}}\,\mathrm{d}t \;&- \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M\, b_M\big(t, X_{t^-}^{\mathbb{G}}\big)\,\gamma(\mathrm{d}t) - \sum_{I\in\mathcal{N}} \int_{E_I} B_I\big(t,e,X_{t^-}^{\mathbb{G}}\big)\,\nu_I(\mathrm{d}t\times\mathrm{d}e) \\
&+ \sum_{I\in\mathcal{N}} \int_{E_I} G_I\left(t^-,t,e\right)\;(\mu_I-\nu_I)(\mathrm{d}t\times\mathrm{d}e) \\
&+ \sum_{I\in\mathcal{N}} \int_{E_I} G_I\left(t\;\;,t,e\right)\;(\rho_I-\mu_I)(\mathrm{d}t\times\mathrm{d}e)
\end{aligned}$$

where we additionally insert the intensity of the compensator ν_I, such that we are able to introduce the function f for the combined dt -part, and g for the remaining $\gamma(\mathrm{d}t)$-part, to get to the representation from the assertion as

$$\begin{aligned}\mathrm{d}X_t^{\mathbb{G}} = f(\omega, t, X_{t^-}^{\mathbb{G}})\,\mathrm{d}t + g(\omega, t, X_{t^-}^{\mathbb{G}})\,\gamma(\mathrm{d}t) &+ \sum_{I\in\mathcal{N}} \int\limits_{E_I} G_I\,(t^-, t, e)\ (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\ &+ \sum_{I\in\mathcal{N}} \int\limits_{E_I} G_I\,(t\ \ , t, e)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e).\end{aligned}$$

□

4.2.1. Existence and uniqueness for the prospective reserve

As the last result of this section, we now want to discuss how solutions to the BSDE of the payment process and solutions to the BSDE of the prospective reserve are connected. The following theorem will both be summarizing the latest equations and also giving additional details on the equivalence.

We will not have an equivalence in the classical sense, i.e. one solution exists exact when the exists and vice versa, since we have already proven, that the solution to the payment process always exists, and can be constructed.
Instead, it rather is the question, if existence and uniqueness of the solution to the Thiele BSDE of the prospective reserve are guaranteed, and if the payment process can be constructed by using this solution of the Thiele BSDE. Additionally, it also is of interest, if the solutions can be constructed from each other and how the insurance company could calculate the reserves.

Theorem 4.2.6. *Existence and uniqueness of solution to the Thiele BSDE*
Let all assumptions of Theorem 3.2.12 (or respectively Theorem 3.3.9) be fulfilled.

Let $X \in BV^X([0,T])$ be the existing and unique solution to the BSDE of the payment process, i.e. the corresponding integral equation, or respectively (3.2.6) holds.

Then $X^{\mathbb{G}}$, defined by (2.1.11), together with $(G_I)_{I\in\mathcal{N}}$, defined by 2.5.12, fulfil the BSDE 4.2.3 of the prospective reserve and the solution is unique in the following sense:

(1) $X^{\mathbb{G}} \in BV^{X,\mathbb{G}}([0,T])$, defined as

$$\begin{aligned}BV^{X,\mathbb{G}}([0,T]) := \Big\{X = (X_t)_{t\in[0,T]} : \Omega \times [0,T] \to \mathbb{R}\ \Big|\ &X \text{ càdlàg}, X_t \text{ is } \mathcal{G}_t\text{-measurable}, \\ &X_{t^-} \text{ is } \mathcal{G}_t^-\text{-measurable}, X(T) = 0\ a.s., \|X\|_{V[0,T]} < \infty\Big\}\end{aligned}$$

(the subset of $BV^X([0,T])$ with additional measurability conditions with respect to the families $\mathbb{G}$ and $\mathbb{G}^-$), is unique up to indistinguishability.

(2) The family $(G_I)_{I\in\mathcal{N}} \in \mathcal{H}_n$, defined as

$$\mathcal{H}_n := \Bigg\{ G = (G_J)_{J\in\mathcal{N}} \,\Bigg|\, G_J : \Omega \times (0,T] \times E_J \to \mathbb{R},\ G_J(s^-,s,e) \text{ is } \mathcal{G}_s^- \text{-measurable},$$

$$G_J(s,s,e) \text{ is } \mathcal{G}_s\text{-measurable and}$$

$$\mathbb{E}\left[\int\limits_{(0,T]} \sum_{J\in\mathcal{N}} \int\limits_{E_J} |G_J(s^-,s,e)|\, n_J(s,\mathrm{d}e) + \int\limits_{E_J} |G_J(s,s,e)|\, r_J(s,\mathrm{d}e)\,\mathrm{d}s \right] < \infty \Bigg\}, \tag{4.2.6}$$

is unique up to equality $\mathrm{d}\mathbb{P} \times n_J(s,E_J)\,\mathrm{d}s$ and $\mathrm{d}\mathbb{P} \times r_J(s,E_J)\,\mathrm{d}s$ almost everywhere. Two processes $G = (G_J)_{J\in\mathcal{N}},\ \tilde{G} = (\tilde{G}_J)_{J\in\mathcal{N}} \in \mathcal{H}_n$ are considered equivalent, if

$$\mathbb{E}\left[\int\limits_{(0,T]} \sum_{J\in\mathcal{N}} \int\limits_{E_J} \Big| G_J(s^-,s,e) - \tilde{G}_J(s^-,s,e) \Big|\, n_J(s,\mathrm{d}e) \right.$$

$$\left. + \int\limits_{E_J} \Big| G_J(s,s,e) - \tilde{G}_J(s,s,e) \Big|\, r_J(s,\mathrm{d}e)\,\mathrm{d}s \right] = 0\,.$$

Further, the solution X can be reconstructed by using $X^{\mathbb{G}}$ (and $(G_I)_I$) as part of the reserve dependent payments as

$$X_t := \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \mathbb{I}^M_{s^-}\, b_M(s, X^{\mathbb{G}}_{s^-})\, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I(s,e,X^{\mathbb{G}}_{s^-})\, \mu_I(\mathrm{d}(s,e)) \tag{4.2.7}$$

depending on the version of reserve-dependency.

Proof. Let $X = (X_t)_{t\geq 0}$ be the existing and unique solution to the BSDE of the payment process, i.e. formula (3.2.6) holds.

We have to show existence and uniqueness of a solution to the Thiele BSDE of the prospective reserve.

Existence: By application of Theorem 2.1.11, we know that also the optional projection $X^{\mathbb{G}} = (X^{\mathbb{G}}_t)_{t\geq 0}$ of X exists and has a unique càdlàg version. It holds, that $X^{\mathbb{G}}_t = \mathbb{E}[X_t \,|\, \mathcal{G}_t]$ *a.s* for $t \geq 0$, and the Theorem 2.1.11 also guarantees the integrable variation. Further, we know, that the $(G_I)_{I\in\mathcal{N}}$ can be constructed by using conditional expectations of X, where the details for the existence can be found in Chapter 2 and especially Propositions 2.4.8 and 2.4.10. Therefore, the existence is guaranteed, and both $X^{\mathbb{G}}$ and $(G_I)_{I\in\mathcal{N}}$ fulfil the necessary conditions, as specified in the precondition. It can be seen that $(G_I)_{I\in\mathcal{N}} \in \mathcal{H}_n$, as this is the existence condition for Theorem 2.5.4, where the measure μ_I is changed with the corresponding IF- and IB-compensator intensities under the expectation.

Then the pair $\big(X^{\mathbb{G}}, (G_I)_{I\in\mathcal{N}}\big)$, fulfils the Thiele BSDE from Theorem 4.2.4. The existence of a solution is thereby guaranteed. The details to this have already been shown in the preparations leading up to this theorem, where we derived the representation of the Thiele BSDE and will not be repeated here.

Uniqueness: To show the uniqueness of a solution, we are directly going to focus on the case, where the $(G_I)_{I\in\mathcal{N}}$ is constructed from the already existing solution X of the payment process and therefore is the same for both solution pairs. This is done, because we want to keep the interpretable structure of $(G_I)_{I\in\mathcal{N}}$ as the sums at risk, respectively for the infinitesimal forward and backward view. The uniqueness condition in the precondition of the theorem is only to exclude cases, where the integrand in the IF- or IB-martingales is zero, and the choice for G_I would be arbitrary.

Let $\big(Y,(G_I)_{I\in\mathcal{N}}\big)$ be another solution to the Thiele BSDE from Theorem 4.2.4, such that

$$Y \neq X^{\mathbb{G}} \tag{4.2.8}$$

where inequality of the solution is now understood as not unique up to evanescence. Y is càdlàg, Y_t is $\mathcal{G}_t$-measurable and Y_{t^-} is $\mathcal{G}_t^-$-measurable. We have to show the uniqueness of the reserve itself.

Both discounted solutions $\kappa(t)\cdot Y_t$ and $\kappa(t)\cdot X_t^{\mathbb{G}}$ fulfil the BSDE 4.2.4 of the prospective reserve, i.e. we have

$$\begin{aligned}
\kappa(t)\cdot X_t^{\mathbb{G}} = \kappa(T)\cdot X_T^{\mathbb{G}} &+ \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \kappa(s)\,\mathbb{I}_{s^-}^M\, b_M\big(s, X_{s^-}^{\mathbb{G}}\big)\,\gamma(\mathrm{d}s)\\
&+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]} \int\limits_{E_I} \kappa(s)\, B_I\big(s,e,X_{s^-}^{\mathbb{G}}\big)\, n_I(s,\mathrm{d}e)\,\mathrm{d}s\\
&- \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, G_I(s^-,s,e)\,(\mu_I-\nu_I)(\mathrm{d}(s,e))\\
&- \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, G_I(s\quad,s,e)\,(\rho_I-\mu_I)(\mathrm{d}(s,e))
\end{aligned}$$

almost surely, as well as

$$\begin{aligned}
\kappa(t)\cdot Y_t = \kappa(T)\cdot Y_T &+ \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \kappa(s)\,\mathbb{I}_{s^-}^M\, b_M\big(s, Y_{s^-}\big)\,\gamma(\mathrm{d}s)\\
&+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]} \int\limits_{E_I} \kappa(s)\, B_I\big(s,e,Y_{s^-}\big)\, n_I(s,\mathrm{d}e)\,\mathrm{d}s\\
&- \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, G_I(s^-,s,e)\,(\mu_I-\nu_I)(\mathrm{d}(s,e))\\
&- \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\, G_I(s\quad,s,e)\,(\rho_I-\mu_I)(\mathrm{d}(s,e))
\end{aligned}$$

almost surely. and both BSDE representation have the same IF- and IB-martingale parts. Therefore, we can express the difference almost surely as

$$\kappa(t)\, X_t^{\mathbb{G}} - \kappa(t)\, Y_t = \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \kappa(s)\, \mathbb{I}_{s-}^M \left(b_M(s, X_{s-}^{\mathbb{G}}) - b_M(s, Y_{s-})\right) \gamma(\mathrm{d}s)$$
$$+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]} \int\limits_{E_I} \kappa(s) \left(B_I(s,e,X_{s-}^{\mathbb{G}}) - B_I(s,e,Y_{s-})\right) n_I(s,\mathrm{d}e)\, \mathrm{d}s\,.$$

With similar tools as in the proof of Theorem 3.2.12 we can then show that

$$\left\|\kappa\, X^{\mathbb{G}} - \kappa Y\right\|_{K,\zeta} \leq C \cdot \left\|\kappa\, X^{\mathbb{G}} - \kappa Y\right\|_{K,\zeta}$$

for a $C < 1$, which guarantees that $X^{\mathbb{G}} = Y$ in $BV^{X,\mathbb{G}}([0,T])$, and therefore uniqueness up to evanescence.

Construct

$$\tilde{X}_t := \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \mathbb{I}_{s-}^M\, b_M(s, Y_{s-})\, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I(s,e,Y_{s-})\, \mu_I(\mathrm{d}(s,e))\,,$$

for $t \in [0,T]$. Then we have $Y_t = \tilde{X}_t^{\mathbb{G}} = Y_t$, i.e. the Y can also be constructed from a payment process, and we directly get $X = \tilde{X}$ as the solution to the payment process BSDE is unique. This completes the proof, as both solution to the Thiele BSDE are equal up to indistinguishability and the payment process can be reconstructed from the prospective reserve.

The proof for the second part of the theorem with payment process X and extended dependency structure II is very similar. Since we keep the $(G_I)_I$ the same for both solutions, the payments also depend on the same $(G_I)_I$ and the Lipschitz condition for the payments b_M and B_I will only have a differences in the reserves, but not in the sums at risk. The steps of the proof are therefore identical, as additional steps are necessary, and we refrain from a formulation to avoid repetitions. □

Comment 4.2.7. *Computation of the reserve via the iteration*
Using the connection between the prospective reserve and the payment process, the same iteration that arose during the proof for the payment process, with the fixed point theorem of Banach, can also be used to compute iterations of the reserve.

It is a rather theoretical result and does not really work great in practice, as optional projections are in general not easy to calculate. Nevertheless it could be used as part of a numerical evaluation, for example in a Monte-Carlo estimation of the reserve. It is not the case, that a simple backward recursion arises, especially not, since we do not impose many assumptions on the model.

4.3. Extension to retrospective reserves

The research has focussed on prospective reserves for now and that was justified by the obvious applications in life insurance. Because of the symmetry in the definition, we can extend some results to retrospective reserves.

The general definition of retrospective reserves is quite similar to the one of the prospective reserves. Instead of calculating the conditional expected value of future payments, the conditional expected value of past (and present) payments is calculated. The two reserves have different advantages, and different reasons to consideration them. The retrospective reserve can for example be used, when the past performance of an individual contract is evaluated or balancing of the portfolio takes place. In the first case, it might be used for surplus and loss allocation. Past performance of a portfolio can also lead to management actions and adaptions to the future insurance business. A perspective like this is especially important in life insurance theory, because of the long time horizon that contracts tend to have.

The retrospective reserve is less commonly used in literature, in comparison to the prospective reserve, and the definition can vary, depending on the source. See for example the definition by Norberg [Nor91] and by Christiansen, Denuit, and Dhaene [CDD13] or the overview paper by Olivieri [Oli97] for some different definitions of retrospective reserves.

If the complete information about the past of an insured person is known, then the retrospective reserve would just be the actual premiums minus benefits of the contract up to that time, and we would call this the individual retrospective reserve. In general, it might be more valuable to calculate the retrospective reserve without individual information, but as a mean value of a portfolio to assess the past performance. Alternatively, the individual reserves would be considered with restricted information, for example only about the current state, but not with the complete history.

The assumptions on the multi-state model matter a lot in the case of retrospective reserves. For example, the Markov assumption, as an assumption about independency from the past, does not really align with the backward perspective, especially not, if we want to take an individual perspective. This has to be taken into account for the definition of state-wise retrospective reserves, where interpretation of a reserve might not really be possible, and even the past individual performance of the contract would not be accurately represented. When we consider information restriction, or non-monotone information structures, then the retrospective reserve can be of interest, as the past performance of a contract might be different when information is deleted or restricted and could especially be interesting for the calculation of lapse or surrender values, as well as an indication for individual past performance.

The retrospective reserve is now defined in a way, that we maintain symmetry to the prospective reserve, and can develop similar results. This is a reasonable approach in our model, as we impose no assumptions on the state process. A similar construction with a symmetric definition takes place in the paper of Christiansen [Chr21a], where the usage of forward- and backward-equations allows for a calculation of retrospective reserves in a symmetrical way compared to the prospective reserves.

4.3.1. Definition of the retrospective reserve

Let us begin by introducing a proper notation, that guarantees the definition of the payment process, the prospective, and the retrospective reserve.

The payment processes, now with all discounted future and past payments are now defined as

$$
\begin{aligned}
X_t^+ &= \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)} \, \mathrm{d}A(s) \quad \text{and} \\
X_t^- &= \int\limits_{[0,t]} \frac{\kappa(s)}{\kappa(t)} \, \mathrm{d}A(s)
\end{aligned}
$$

for $t \geq 0$ respectively, where the difference is in the integration area, and the prospective and retrospective reserves with respect to $\mathbb{G}$ are the unique càdlàg processes $X^{\mathbb{G},+}$ and $X^{\mathbb{G},-}$ with

$$
\begin{aligned}
X_t^{\mathbb{G},+} &= \mathbb{E}\left[X_t^+ \,\middle|\, \mathcal{G}_t\right] \;\; a.s. \quad \text{and} \\
X_t^{\mathbb{G},-} &= \mathbb{E}\left[X_t^- \,\middle|\, \mathcal{G}_t\right] \;\; a.s.
\end{aligned}
$$

for $t \geq 0$ respectively, where the right hand sides are otherwise just pointwise definitions. In the general setting, without reserve-dependency, we have that

$$
X_t^- = X_{0^-}^+ - X_t^+ .
$$

These rewritings are not possible any more, once a dependency structure is included into A. It would also be difficult to allow for reserve-dependency with respect to the opposing reserve, so the retrospective reserve can only depend on other retrospective reserves.

Let us come back to our setting and the payment process of the retrospective reserve will only depend on the retrospective reserves. We have

$$
X_t^- = \sum_{M \in \mathcal{M}} \int\limits_{[0,t]} \frac{\kappa(s)}{\kappa(t)} \, \mathbb{I}_{s^-}^M \, b_M(s, X_{s^-}^{\mathbb{G},-}) \, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int\limits_{[0,t] \times E_I} \frac{\kappa(s)}{\kappa(t)} \, B_I(s, e, X_{s^-}^{\mathbb{G},-}) \, \mu_I(\mathrm{d}(s,e)),
$$

where we have to use the retrospective reserve

$$
X_{s^-}^{\mathbb{G},-} = \mathbb{E}\left[X_{s^-}^- \,\middle|\, \mathcal{G}_s^-\right] \;\; a.s. \tag{4.3.1}
$$

as part of the payments.

Special consideration must also be given to the starting value. It has to hold that $X_{0^-}^- = 0$ *a.s.*, since the integration area is empty, but this is not directly symmetric to the final value for the prospective reserve. Here, it does not necessarily have to be the case, that $X_0^- = 0$, as potential payments in time point 0 are part of the payment process X_0^-. For convenience we also define $\mathbb{I}_{0^-}^M := \mathbb{I}_0^M$, and take note, that the reserve-dependent payments in 0 then do not really depend on the reserve.

4.3.2. Existence and uniqueness results for the payment process

A lot of notation has to be repeated, where the changes are sometimes only very minor. This section is kept as short as possible and for clarifying details consult the original formulation.

Definition 4.3.1. *Sojourn payments*
Consider a function of the form

$$\begin{aligned} b : \mathcal{M} \times [0,\infty) \times \Omega &\longrightarrow \mathbb{R} \\ (M,t,\omega) &\longmapsto b\big(M,t,X_{t^-}^{\mathbb{G},-}\big)(\omega) \end{aligned}$$

as the rate of a sojourn payment, that will be paid at time t, if $\mathbb{I}_{t^-}^M = 1$ for all $M \in \mathcal{M}$ respectively. For every $M \in \mathcal{M}$, let

$$b_M\big(t, X_t^{\mathbb{G},-}\big)(\omega) := b\big(M,t,X_{t^-}^{\mathbb{G},-}\big)(\omega)$$

be a function from $[0,\infty) \times \Omega \to \mathbb{R}$.

We further need the following assumptions:

(1) The function $b(t, X_{t^-}^{\mathbb{G},-})(\omega) : \mathcal{M} \times [0,\infty) \times \Omega \longrightarrow \mathbb{R}$ is measurable in (M,t,ω).

(2) For every $M \in \mathcal{M}$ let b_M be bounded on every compact time interval, i.e. for $t \geq 0$ it holds
$$\left| b_M\big(s, X_{s^-}^{\mathbb{G},-}\big) \right| \leq J_{\mathcal{M}}(s)$$
for an integrable majorant $J_{\mathcal{M}}$ and all $s \in [0,t]$.

(3) The functions $b_M\big(t, X_{t^-}^{\mathbb{G},-}\big)(\omega)$ are $\mathbb{G}^-$-adapted for every $M \in \mathcal{M}$.

Let us continue with the transition payments:

Definition 4.3.2. *Transition payments*
For every $I \in \mathcal{N}$, consider a function of the form

$$\begin{aligned} B_I : [0,\infty) \times E_I \times \Omega &\longrightarrow \mathbb{R} \\ (t,e,\omega) &\longmapsto B_I\big(t,e,X_{t^-}^{\mathbb{G},-}\big)(\omega) \end{aligned}$$

as a lump sum payment upon a transition $I \in \mathcal{N}$.
We need the following assumptions to hold:

(1) The functions $B_I(t,e,X_{t^-}^{\mathbb{G},-})(\omega) : \mathcal{N} \times [0,\infty) \times E_I \times \Omega \longrightarrow \mathbb{R}$ are jointly measurable in (t,e,ω) for every I.

(2) For every $I \in \mathcal{N}$ the B_I are bounded on compact time intervals, i.e. for every $t \geq 0$ it holds
$$\left| B_I(s, e, X_{s^-}^{\mathbb{G},-}) \right| \leq J_{\mathcal{N}}(s)$$
for an integrable majorant $J_{\mathcal{N}}$ and all $s \in [0, t]$.

(3) The functions $B_I\big(t, e, X_{t^-}^{\mathbb{G},-}\big)(\omega)$ are $\mathbb{G}^-$-adapted for every $I \in \mathcal{N}$ and $e \in E_I$.

Construction of the payment process

We can now continue to construct the retrospective reserve-dependent payment process with discounting, by using the contractual payments, the corresponding integrators and the discounting factor.

Definition 4.3.3. *Cumulative cash flow*
The cumulative or aggregated cash flow $A(t)$ contains all contractual payments of the insurance contract on the interval $[0, t]$ and is given as the càdlàg process $(A_t)_{t\geq 0}$ with

$$A(t) = \sum_{M \in \mathcal{M}} \int_{[0,t]} \mathbb{I}_{s^-}^M \, b_M(s, X_{s^-}^{\mathbb{G},-}) \, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int_{[0,t] \times E_I} B_I(s, e, X_{s^-}^{\mathbb{G},-}) \, \mu_I(\mathrm{d}(s, e)) \quad (4.3.2)$$

and $A(0^-) = 0$ *a.s.* as a starting value. Recall, that also $\mathrm{d}A(t) = 0$ for $t > T$ (with $T < \infty$).

The insurer then considers the process $X^- = (X_t^-)_{t\geq 0}$ of the aggregated discounted future payments for an insurance contract, given by

$$X_t^- := \int_{[0,t]} e^{-\int_t^s \varphi(u)\,\mathrm{d}u} \, A(\mathrm{d}s) = \int_{[0,t]} \frac{\kappa(s)}{\kappa(t)} \, A(\mathrm{d}s)$$

where the integrals involved are to be understand as path-wise Lebesgue-Stieltjes integrals.

This then leads to the following representation of the cash flow as

$$X_t^- = \sum_{M \in \mathcal{M}} \int_{[0,t]} \frac{\kappa(s)}{\kappa(t)} \, \mathbb{I}_{s^-}^M \, b_M(s, X_{s^-}^{\mathbb{G},-}) \, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int_{[0,t] \times E_I} \frac{\kappa(s)}{\kappa(t)} \, B_I(s, e, X_{s^-}^{\mathbb{G},-}) \, \mu_I(\mathrm{d}(s, e)).$$

We again decouple the decoupling the integrand from t, by

$$Y_t^- := \kappa(t) \cdot X_t^- = \int_{[0,t]} \kappa(s) \, A(\mathrm{d}s) \, . \quad (4.3.3)$$

We now want to prove again the existence and uniqueness of the payment process, when allowing for reserve-dependent payments. This will indeed be done in two consecutive steps, where we first one only allows payments in time t to be dependent on the retrospective reserve $X_t^{\mathbb{G}^-}$.

Automorphism and recursion

Let the iteration index $n \in \mathbb{N}$ be fixed, and let the process $\left({}^{(n)}Y^{-}{}_{t}\right)_{t\geq 0}$ be given as the current iteration, used as a predecessor for the new reserve. The iterative process consists of the following steps, that have to be performed in the specified order:

(1) Calculation of
$$ {}^{(n)}X_t^{-} = \frac{1}{\kappa(t)} \cdot {}^{(n)}Y_t^{-} $$
as the usual payment process by reversing the additional discounting of all payments on 0 to t.

(2) Application of Theorem 2.4.1, which guarantees the existence of the optional projection as a càdlàg process ${}^{(n)}X^{\mathbb{G},-}$, and it holds
$$ {}^{(n)}X_{s^-}^{\mathbb{G},-} = \mathbb{E}\left[{}^{(n)}X_{s^-}^{-} \,\middle|\, \mathcal{G}_s^{-}\right] $$
almost surely, and is needed as part of the payments.

(3) Construction of ${}^{(n+1)}Y^{-}$ by insertion of the results from (2) into the payments
$$ \begin{aligned} {}^{(n+1)}Y_t^{-} = & \sum_{M\in\mathcal{M}} \int\limits_{[0,t]} \kappa(s)\, \mathbb{I}_{s^-}^{M}\, b_M\big(s, {}^{(n)}X_{s^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}s) \\ & + \sum_{I\in\mathcal{N}} \int\limits_{[0,t]\times E_I} \kappa(s)\, B_I\big(s, e, {}^{(n-1)}X_{s^-}^{\mathbb{G},-}\big)\, \mu_I(\mathrm{d}(s,e))\,. \end{aligned} $$

(4) Starting over with the newly constructed ${}^{(n+1)}Y^{-}$, which in total completes the iteration function Φ as
$$ {}^{(n+1)}Y^{-} = \Phi\left({}^{(n)}Y^{-}\right)\,. $$

We now need the following updated Lipschitz-assumptions.

Assumption 4.3.4. *Lipschitz conditions*
Assume, that there exists a Lipschitz constant $L_{\mathcal{M}} > 0$, independent of $M \in \mathcal{M}$, such that for all $X_{s^-}^{\mathbb{G},-}(\omega)$, $\tilde{X}_{s^-}^{\mathbb{G},-}(\omega)$ we $\mathrm{d}\mathbb{P}\times\mathrm{d}\gamma$ *a.e.* have
$$ \left|b_M(s, X_{s^-}^{\mathbb{G},-})(\omega) - b_M(s, \tilde{X}_{s^-}^{\mathbb{G},-})(\omega)\right| \leq L_{\mathcal{M}} \cdot \left|X_{s^-}^{\mathbb{G},-}(\omega) - \tilde{X}_{s^-}^{\mathbb{G},-}(\omega)\right| $$
for all $M \in \mathcal{M}$.

Further assume that there exists a second Lipschitz constant $L_{\mathcal{N}}$, independent of $I \in \mathcal{N}$, such that for all $X_{s^-}^{\mathbb{G},-}(\omega)$, $\tilde{X}_{s^-}^{\mathbb{G},-}(\omega)$ and all $e \in E_I$ we $\mathrm{d}\mathbb{P}\times l_I(t, E_I)\mathrm{d}s$ *a.e.* have
$$ \left|B_I(s, e, X_{s^-}^{\mathbb{G},-})(\omega) - B_I(s, e, \tilde{X}_{s^-}^{\mathbb{G},-})(\omega)\right| \leq L_{\mathcal{N}} \cdot \left|X_{s^-}^{\mathbb{G},-}(\omega) - \tilde{X}_{s^-}^{\mathbb{G},-}(\omega)\right| $$

Let us further assume, that for all $t \in \{t_0, t_1, \dots\}$, we have the following stronger Lipschitz condition for the deterministic time points or reserve-dependent singular payments. Assume there exists a second Lipschitz constant $J < 1$, independent of $I \in \mathcal{N}$ and $M \in \mathcal{M}$, such that for all $X^{\mathbb{G},-}_{s^-}(\omega)$, $\tilde{X}^{\mathbb{G},-}_{s^-}(\omega)$ and $e \in E_I$ we $\mathrm{d}\mathbb{P} \times \mathrm{d}\lambda$ *a.e.* have

$$\begin{aligned}\Bigg| \sum_{M \in \mathcal{M}} \mathbb{I}^M_{s^-}(\omega) \Big(b_M(s, X^{\mathbb{G},-}_{s^-})(\omega) - b_M(s, \tilde{X}^{\mathbb{G},-}_{s^-})(\omega) \Big) \\ + \sum_{I \in \mathcal{N}} \int_{E_I} \Big(B_I(s, e, X^{\mathbb{G},-}_{s^-})(\omega) - B_I(s, e, \tilde{X}^{\mathbb{G},-}_{s^-})(\omega) \Big) \mu_I(\{s\} \times \mathrm{d}e)(\omega) \Bigg| \\ \le J \cdot \Big| X^{\mathbb{G},-}_{s^-}(\omega) - \tilde{X}^{\mathbb{G},-}_{s^-}(\omega) \Big|\end{aligned}$$

for all time points $s \in \{t_0, t_1, \dots\}$.

Theorem – Existence and uniqueness

We need to show

$$\begin{aligned}\Big\| {}^{(n+1)}Y^- - {}^{(n)}Y^- \Big\| &= \Big\| \Phi\left({}^{(n)}Y^-\right) - \Phi\left({}^{(n-1)}Y^-\right) \Big\| \\ &\le C \cdot \Big\| {}^{(n)}Y^- - {}^{(n-1)}Y^- \Big\|\end{aligned}$$

where $C < 1$ would be the contraction constant.

In that case, our iteration function Φ would indeed be a contraction mapping and a fixed point theorem could be applied. Without specifying or checking the preconditions, as a consequence of the application of the Theorem of Banach A.3.1, a unique fixed point ${}^*Y$ would exist, fulfilling

$$\begin{aligned}{}^*Y^-_t &= \sum_{M \in \mathcal{M}} \int_{[0,t]} \mathbb{I}^M_{s^-}\, \kappa(s)\, b_M\left(s, \frac{1}{\kappa(s)}\, {}^*Y^{\mathbb{G},-}_{s^-}\right) \gamma(\mathrm{d}s) \\ &\quad + \sum_{I \in \mathcal{N}} \int_{[0,t] \times E_I} \kappa(s)\, B_I\left(s, e, \frac{1}{\kappa(s)}\, {}^*Y^{\mathbb{G},-}_{s^-}\right) \mu_I(\mathrm{d}(s,e)) \\ &=: \Phi({}^*Y^-)_t \end{aligned} \tag{4.3.4}$$

where ${}^*Y^{\mathbb{G},-}_{t^-} = \mathbb{E}\left[{}^*Y^-_{t^-} \,\Big|\, \mathcal{G}^-_t \right]$ *a.s.* as usual.

Definition 4.3.5. *Solution space*
The space of càdlàg stochastic processes on $[0, T]$ with integrable variation is given as

$$BV^X_{[0,T]} := \Big\{ X = (X_t)_{t \in [0,T]} : \Omega \times [0,T] \to \mathbb{R} \,\Big|\, X \text{ càdlàg},\ X_{0^-} = 0 \ a.s.,\ \|X\|_{V[0,T]} < \infty \Big\} \tag{4.3.5}$$

and the norm equivalent norm is defined as a weighted expected variation norm as

$$\|X\|_{V[0,T]} := \mathbb{E}\left[\|X\|_{V[0,T],K,\zeta}\right] = \mathbb{E}\left[|X_0| + \int\limits_{[0,T]} e^{-K(\zeta(t)-\zeta(0))}\,\mathrm{d}\,|X|_t\right] \tag{4.3.6}$$

where $K > 0$ is a constant, that will be chosen to guarantee the contraction property and the measure ζ is deterministic and has to be

$$\zeta(\mathrm{d}t) = \gamma(\mathrm{d}t) + D\,\mathrm{d}t$$

and the additional summand in the norm is for the starting value of X_0, which might be different from 0.

Theorem 4.3.6. *Existence and uniqueness of the payment process Y*
Under the conditions on Y and its parts, namely Assumptions 4.3.4, 3.3.8 and 3.2.3, Definitions 3.1.1, 4.3.1 and 4.3.2, the payment process Y^- exists and is unique as the solution of the integral equation

$$\begin{cases} Y_t^- = \sum\limits_{M\in\mathcal{M}} \int\limits_{[0,t]} \mathbb{I}_{s^-}^M\, b_M\left(s, \dfrac{1}{\kappa(s)} Y_{s^-}^{\mathbb{G},-}\right)\gamma(\mathrm{d}s) \\ \qquad + \sum\limits_{I\in\mathcal{N}} \int\limits_{[0,t]\times E_I} B_I\left(s, e, \dfrac{1}{\kappa(s)} Y_{s^-}^{\mathbb{G},-}\right)\mu_I(\mathrm{d}(s,e)) \\ Y_{0^-}^- = 0 \end{cases} \tag{4.3.7}$$

in the space of càdlàg processes with finite integrable variation, given in (4.3.5), equipped with the (equivalent) weighted norm

$$\|X\|_{V[0,T]} = \mathbb{E}\left[|X_0| + \int\limits_{[0,T]} e^{-K(\zeta(t)-\zeta(0))}\,\mathrm{d}\,|X|_t\right],$$

where the constant

$$K := 2\cdot\frac{L}{1-J}$$

is the weighting factor with $L = \max\{L_{\mathcal{M}}, L_{\mathcal{N}}\}$, and ζ is defined via

$$\zeta(\mathrm{d}t) := \gamma(\mathrm{d}t) + D\,\mathrm{d}t\,.$$

Proof. The proof is again performed in three steps.

Automorphism

We begin by showing that the mapping Φ, as specified by (4.3.4) is an automorphism on the solution space of processes with bounded variation. Therefore, let ${}^{(n)}Y^- \in BV_{[0,T]}^X$ be a process of integrable variation. We have to show that ${}^{(n+1)}Y^-$ is a process of integrable variation as well. The definition of the payment process is different and the proof is not completely identical to the original proof, which is why the proof is performed again.

For every $0 \leq s < t \leq T$ and with $(\tau_m)_{m\in\mathbb{N}}$ as a sequence of partitions of the interval $[s,t]$ with $\{t_0 = s, \dots, t_m = t\}$ and $\lim_{m\to\infty} |\tau_m| = 0$, we get

$$
\begin{aligned}
&\int\limits_{[s,t]} e^{-K(\zeta(u)-\zeta(0))}\,\mathrm{d}\left|{}^{(n+1)}Y^-\right|_u \\
&= \sup_{\tau_m} \sum_{j=1}^{m} e^{-K(\zeta(t_j)-\zeta(0))} \cdot \left|{}^{(n+1)}Y^-_{t_j} - {}^{(n+1)}Y^-_{t_{j-1}}\right| \\
&\leq \sup_{\tau_m} \sum_{j=1}^{m} e^{-K(\zeta(t_j)-\zeta(0))} \cdot \Bigg| \int\limits_{(t_{j-1},t_j]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-}\,\kappa(u)\,b_M\big(u, {}^{(n}X^{\mathbb{G},-}_{u^-}\big)\gamma(\mathrm{d}u) \\
&\qquad\qquad + \sum_{I\in\mathcal{N}} \int\limits_{(t_{j-1},t_j]\times E_I} \kappa(u) B_I\big(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}\big)\,\mu_I(\mathrm{d}(u,e))\Bigg| \\
&\leq \int\limits_{(s,t]} e^{-K(\zeta(u)-\zeta(0))} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-}\,\kappa(u)\,\left|b_M\big(u,{}^{(n)}X^{\mathbb{G},-}_{u^-}\big)\right|\gamma(\mathrm{d}u) \\
&\quad + \sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u)\,\left|B_I\big(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}\big)\right|\,\mu_I(\mathrm{d}(u,e))
\end{aligned}
$$

Then, by using the upper bound from above for the norm, the following holds

$$
\begin{aligned}
&\left\|\Phi\left({}^{(n)}Y^-\right)\right\|_{V[0,T]} \\
&= \left\|{}^{(n+1)}Y^-\right\|_{V[0,T]} = \mathbb{E}\left[\left\|{}^{(n+1)}Y^-\right\|_{V[0,T],K,\zeta}\right] \\
&= \mathbb{E}\left[|X_0| + \int\limits_{[0,T]} \underbrace{e^{-K(\zeta(s)-\zeta(0))}}_{\leq 1}\,\mathrm{d}\left|{}^{(n+1)}Y^-\right|_s\right] \\
&\leq \underbrace{\mathbb{E}\left[|X_0|\right]}_{<\infty} + \mathbb{E}\left[\int\limits_{(0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\,\kappa(s)\,\underbrace{\left|b_M\big(s,{}^{(n)}X^{\mathbb{G},-}_{s^-}\big)\right|}_{\leq J_{\mathcal{M}}(s)}\gamma(\mathrm{d}s)\right] \\
&\qquad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(0,T]\times E_I} \kappa(s)\,\underbrace{\left|B_I\big(s,e,{}^{(n)}X^{\mathbb{G},-}_{s^-}\big)\right|}_{\leq J_{\mathcal{N}}(s)}\mu_I(\mathrm{d}(s,e))\right] \\
&\leq \mathbb{E}\left[|X_0|\right] + \mathbb{E}\left[\int\limits_{(0,T]} J_{\mathcal{M}}(s)\cdot\underbrace{\kappa(s)}_{\leq D_\kappa}\cdot\underbrace{\sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}}_{=1}\,\gamma(\mathrm{d}s)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(0,T]\times E_I} \underbrace{\kappa(s)}_{\le D_\kappa} \cdot J_{\mathcal{N}}(s)\, \mu_I(\mathrm{d}(s,e) \right] \\
&\le \mathbb{E}\left[|X_0|\right] + D_\kappa \cdot \mathbb{E}\left[\int_{(0,T]} J_{\mathcal{M}}(s)\, \gamma(\mathrm{d}s) \right] + D_\kappa \cdot \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(0,T]\times E_I} J_{\mathcal{N}}(s)\, \mu_I(\mathrm{d}(s,e)) \right] \\
&< \infty\,,
\end{aligned}
$$

where both parts are finite, respectively. Therefore, in total we have ${}^{(n+1)}Y^- \in BV^X_{[0,T]}$.

Contraction property

With similar arguments as before, we now prepare to look at the difference of two consecutive iterations. We again start by deriving an upper bound for the norm.

We start by disregarding the outer expectation. For every $0 \le s < t \le T$ we get

$$
\begin{aligned}
&\int_{[s,t]} e^{-K(\zeta(u)-\zeta(0))}\, \mathrm{d}\left| {}^{(n+1)}Y^- - {}^{(n)}Y^- \right|_u \\
&= \sup_{\tau_m} \sum_{j=1}^{m} e^{-K(\zeta(t_j)-\zeta(0))} \left| \left({}^{(n+1)}Y^-_{t_j} - {}^{(n)}Y^-_{t_j} \right) - \left({}^{(n+1)}Y^-_{t_{j-1}} - {}^{(n)}Y^-_{t_{j-1}} \right) \right| \\
&\le \int_{(s,t]} e^{-K(\zeta(u)-\zeta(0))} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-}\, \kappa(u) \left| b_M(u, {}^{(n)}X^{\mathbb{G},-}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G},-}_{u^-}) \right| \gamma(\mathrm{d}u) \\
&\quad + \sum_{I\in\mathcal{N}} \int_{(s,t]\times E_I} e^{-K(\zeta(u)-\zeta(0))}\, \kappa(u) \left| B_I(u,e, {}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e, {}^{(n-1)}X^{\mathbb{G},-}_{u^-}) \right| \mu_I(\mathrm{d}(u,e))
\end{aligned}
$$

where $(\tau_m)_{m\in\mathbb{N}}$ is again a sequence of partitions of $[s,t]$ with $\{t_0 = s, \dots, t_m = t\}$ and $\lim_{m\to\infty} |\tau_m| = 0$ and otherwise similar arguments as in the situation above.

As a consequence, for every $0 \le s < t \le T$, or equivalently for every subinterval $(s,t] \subseteq (0,T]$ we achieve the upper bound

$$
\begin{aligned}
&\mathbb{E}\left[\int_{[s,t]} e^{-K(\zeta(u)-\zeta(0))}\, \mathrm{d}\left| {}^{(n+1)}Y^- - {}^{(n)}Y^- \right|_u \right] \\
&\le \mathbb{E}\left[\int_{(s,t]} e^{-K(\zeta(u)-\zeta(0))} \kappa(u) \sum_{M\in\mathcal{M}} \mathbb{I}^M_{u^-} \left| b_M(u, {}^{(n)}X^{\mathbb{G},-}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G},-}_{u^-}) \right| \gamma(\mathrm{d}u) \right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(s,t]\times E_I} e^{-K(\zeta(u)-\zeta(0))} \kappa(u) \left| B_I(u,e, {}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e, {}^{(n-1)}X^{\mathbb{G},-}_{u^-}) \right| \mu_I(\mathrm{d}(u,e)) \right] \\
&=: \mathrm{(I)} + \mathrm{(II)}\,,
\end{aligned}
$$

yielding an additive structure, where we look at both summands separately.

For the first part, we use the Lipschitz condition for b_M for every $M \in \mathcal{M}$ and get the following upper bound

$$(\mathrm{I}) = \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-} \underbrace{\left|b_M(u, {}^{(n)}X^{\mathbb{G},-}_{u^-}) - b_M(u, {}^{(n-1)}X^{\mathbb{G},-}_{u^-})\right|}_{\leq L_{\mathcal{M}}\cdot\left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right|} \gamma(\mathrm{d}u)\right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \underbrace{\sum_{M\in\mathcal{M}} \mathbb{I}^M_{t^-}}_{=1} \cdot L_{\mathcal{M}} \cdot \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| \gamma(\mathrm{d}u)\right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{M}} \cdot e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u} - {}^{(n-1)}X^{\mathbb{G},-}_{u}\right| \gamma(\mathrm{d}u)\right],$$

where we additionally used formula (3.2.1), for every $u \in (s,t] \subseteq (0,T]$.

For the second summand, we make use of the classical compensator λ_I with respect to $\mathbb{F}$ and its density l_I, to get

$$\begin{aligned}
&(\mathrm{II}) \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(u)-\zeta(0))}\kappa(u) \left|B_I(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G},-}_{u^-})\right| \mu_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \mu_I(\mathrm{d}(u,e))\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \mu_I(\mathrm{d}(u,e))\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[(\kappa G_I) \bullet \mu_I((s,t]\times E_I)\right] \\
&\overset{(i)}{=} \sum_{I\in\mathcal{N}} \mathbb{E}\left[(\kappa G_I) \bullet \lambda_I((s,t]\times E_I)\right] \\
&= \sum_{I\in\mathcal{N}} \mathbb{E}\left[\int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \lambda_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} \kappa(u)\, G_I(u,e)\, \lambda_I(\mathrm{d}(u,e))\right] \\
&= \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{(s,t]\times E_I} e^{-K(\zeta(u)-\zeta(0))}\kappa(u) \left|B_I(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G},-}_{u^-})\right| \lambda_I(\mathrm{d}(u,e))\right]
\end{aligned}$$

$$= \mathbb{E}\left[\int\limits_{(s,t]} e^{-K(\zeta(u)-\zeta(0))}\kappa(u) \sum_{I\in\mathcal{N}} \int\limits_{E_I} \underbrace{\left|B_I(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G},-}_{u^-})\right|}_{\leq L_{\mathcal{N}} \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right|} l_I(t,\mathrm{d}e)\,\mathrm{d}u \right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}} \cdot e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \sum_{I\in\mathcal{N}} \int\limits_{E_I} \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| l_I(u,\mathrm{d}e)\,\mathrm{d}u \right]$$

$$= \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}} \cdot e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| \sum_{I\in\mathcal{N}} l_I(u,E_I)\,\mathrm{d}u \right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} L_{\mathcal{N}} \cdot e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| D\,\mathrm{d}u \right].$$

We again use Assumption 3.2.3 for the compensator λ, and for an application of the theorem we need, in (i) that for every summand

$$G_I(u,e) = e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \cdot \left|B_I(u,e,{}^{(n)}X^{\mathbb{G},-}_{u^-}) - B_I(u,e,{}^{(n-1)}X^{\mathbb{G},-}_{u^-})\right|$$

is $\mathcal{G}^-_u$-measurable for every (u,e).

Let us continue by defining $L := \max\{L_{\mathcal{M}}, L_{\mathcal{N}}\}$ as a joint Lipschitz constant, to further simplify the notation. In both summands, we arrive at a similar structure and we rejoin them to get the upper bound

$$\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(u)-\zeta(0))}\,\mathrm{d}\left|{}^{(n+1)}Y^- - {}^{(n)}Y^-\right|_u \right]$$

$$\leq \mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| \gamma(\mathrm{d}u) \right]$$

$$+\,\mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| D\,\mathrm{d}u \right]$$

$$= \mathbb{E}\left[\int\limits_{(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\kappa(u) \left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right| \underbrace{\Big(\gamma(\mathrm{d}u) + D\,\mathrm{d}u\Big)}_{=\zeta(\mathrm{d}u)} \right]$$

$$= \int\limits_{(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X^{\mathbb{G},-}_{u} - {}^{(n-1)}X^{\mathbb{G},-}_{u}\right|\right] \zeta(\mathrm{d}u)$$

$$\overset{(ii)}{\leq} \int\limits_{(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X_{u^-} - {}^{(n-1)}X_{u^-}\right|\right] \zeta(\mathrm{d}u)\,,$$

where we used the theorem of Fubini-Tonelli to exchange the order of integration. The expectation can be evaluated first and only the non-deterministic part has to be considered.

In (ii), we applied the inequality of Jensen A.2.12 for conditional expectations to get

$$\begin{aligned}
&\mathbb{E}\left[\left|{}^{(n)}X^{\mathbb{G},-}_{u^-} - {}^{(n-1)}X^{\mathbb{G},-}_{u^-}\right|\right] \\
&= \mathbb{E}\left[\left|\mathbb{E}\left[{}^{(n)}X^-_{u^-} - {}^{(n-1)}X^-_{u^-} \,\middle|\, \mathcal{G}^-_u\right]\right|\right] \\
&\overset{A.2.12}{\leq} \mathbb{E}\left[\mathbb{E}\left[\left|{}^{(n)}X^-_{u^-} - {}^{(n-1)}X^-_{u^-}\right| \,\middle|\, \mathcal{G}^-_u\right]\right] \\
&= \mathbb{E}\left[\left|{}^{(n)}X^-_{-u} - {}^{(n-1)}X^-_{u^-}\right|\right],
\end{aligned} \tag{4.3.8}$$

which allows us to get rid of the optional projections. This step enables us to introduce the needed difference of predecessors for our iteration and we get

$$\begin{aligned}
&\mathbb{E}\left[\int\limits_{[s,t]} e^{-K(\zeta(u)-\zeta(0))}\,\mathrm{d}\left|{}^{(n+1)}Y^- - {}^{(n)}Y^-\right|_u\right] \\
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\kappa(u)\,\mathbb{E}\left[\left|{}^{(n)}X^-_{u^-} - {}^{(n-1)}X^-_{u^-}\right|\right]\,\zeta(\mathrm{d}u) \\
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\mathbb{E}\left[\left|\kappa(u^-)\right|\,\left|{}^{(n)}X^-_{u^-} - {}^{(n-1)}X^-_{u^-}\right|\right]\,\zeta(\mathrm{d}u) \\
&\leq \int\limits_{u\in(s,t]} L\,e^{-K(\zeta(u)-\zeta(0))}\,\mathbb{E}\left[\left|{}^{(n)}Y^-_{u^-} - {}^{(n-1)}Y^-_{u^-}\right|\right]\,\zeta(\mathrm{d}u) \\
&\overset{(iii)}{\leq} \int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(u)-\zeta(0))}\,\mathbb{E}\left[\left|{}^{(n)}Y^-_{u} - {}^{(n-1)}Y^-_{u}\right|\right]\,\zeta(\mathrm{d}u) \\
&= \mathbb{E}\left[\int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(u)-\zeta(0))}\,\left|{}^{(n)}Y^-_{u} - {}^{(n-1)}Y^-_{u}\right|\,\zeta(\mathrm{d}u)\right] \\
&\overset{(iv)}{\leq} \mathbb{E}\left[\int\limits_{u\in(s,t]} \frac{L}{1-J}\cdot e^{-K(\zeta(u)-\zeta(0))}\cdot\right. \\
&\qquad\qquad \left.\left(\left|{}^{(n)}Y^-_0 - {}^{(n-1)}Y^-_0\right| + \int\limits_{r\in[0,u]} \mathrm{d}\left|{}^{(n)}Y^- - {}^{(n-1)}Y^-\right|_r\right)\zeta(\mathrm{d}u)\right],
\end{aligned}$$

where we used Formula (3.1.4) and further that $\kappa(t) = |\kappa(t)|$, enabling us to change back from X to Y. Afterwards, the Theorem of Fubini-Tonelli is used in opposite direction to before.

Let us give some additional details about step (iii). Remember our definition of ζ as

$$\zeta(\mathrm{d}t) = \gamma(\mathrm{d}t) + D\,\mathrm{d}t$$

of a Lebesgue-part, multiplied with a constant, and the deterministic jumps, originating from the definition of γ.

A special consideration has to be given to the deterministic time point $t_0, t_1, \dots$ again, where the γ-part of ζ might introduce additional reserve-dependent payments. We provide details to the upper bound in step (iii) of the proof. For the deterministic jump points $t \in \{t_0, t_1, \dots\}$, by splitting the possible payments on $[0, t]$ in time point t and keeping the interval $[0, t)$, it holds

$$
\begin{aligned}
&\left| {}^{(n)}Y^{-}_{t^-} - {}^{(n-1)}Y^{-}_{t^-} \right| \\
&\leq \left| {}^{(n)}Y^{-}_{t} - {}^{(n-1)}Y^{-}_{t} \right| \\
&\quad + \kappa(t) \Bigg| \sum_{M \in \mathcal{M}} \mathbb{I}^M_t \left(b_M(t, {}^{(n)}X^{\mathbb{G},-}_{t^-}) - b_M(t, {}^{(n-1)}X^{\mathbb{G},-}_{t^-}) \right) \\
&\qquad + \sum_{I \in \mathcal{N}} \int\limits_{E_I} \left(B_I(t, e, {}^{(n)}X^{\mathbb{G},-}_{t^-}) - B_I(t, e, {}^{(n-1)}X^{\mathbb{G},-}_{t^-}) \right) \mu_I(\{t\} \times \mathrm{d}e) \Bigg| \\
&\leq \left| {}^{(n)}Y^{-}_{t} - {}^{(n-1)}Y^{-}_{t} \right| + J \cdot \left| {}^{(n)}Y^{\mathbb{G},-}_{t^-} - {}^{(n-1)}Y^{\mathbb{G},-}_{t^-} \right|
\end{aligned}
$$

by using the special Lipschitz condition and therefore by application of the expectation also

$$
\begin{aligned}
&\mathbb{E}\left[\left| {}^{(n)}Y_{t^-} - {}^{(n-1)}Y_{t^-} \right| \right] \\
&\leq \mathbb{E}\left[\left| {}^{(n)}Y_{t} - {}^{(n-1)}Y_{t} \right| \right] + J \cdot \mathbb{E}\left[\left| {}^{(n)}Y^{\mathbb{G},-}_{t^-} - {}^{(n-1)}Y^{\mathbb{G},-}_{t^-} \right| \right] \\
&\overset{(3.2.7)}{\leq} \mathbb{E}\left[\left| {}^{(n)}Y^{-}_{t} - {}^{(n-1)}Y^{-}_{t} \right| \right] + J \cdot \mathbb{E}\left[\left| {}^{(n)}Y^{-}_{t^-} - {}^{(n-1)}Y^{-}_{t^-} \right| \right]
\end{aligned}
$$

with similar steps as before. By rearranging, this also implies

$$
\mathbb{E}\left[\left| {}^{(n)}Y^{-}_{t^-} - {}^{(n-1)}Y^{-}_{t^-} \right| \right] \leq \frac{1}{1-J} \cdot \mathbb{E}\left[\left| {}^{(n)}Y^{-}_{t} - {}^{(n-1)}Y^{-}_{t} \right| \right]
$$

for all time points $t \in \{t_0, t_1, \dots\}$, but also for general t, if a deterministic jump is not even possible.

Further, in step (iv) we used, that the variation on the interval $[0, u]$ can be used as an upper bound for the absolute difference of the reserves in time u as

$$
\left| {}^{(n)}Y^{-}_{u} - {}^{(n-1)}Y^{-}_{u} \right| \leq \left| {}^{(n)}Y^{-}_{0} - {}^{(n-1)}Y^{-}_{0} \right| + \int\limits_{r \in [0,u]} \mathrm{d} \left| {}^{(n)}Y - {}^{(n-1)}Y \right|_r , \tag{4.3.9}
$$

where we use the difference in time 0

$$
{}^{(n)}Y^{-}_{0} - {}^{(n-1)}Y^{-}_{0}
$$

as the additional summand of the variation, since Y^{-}_{u} contains all payments over the interval $[0, u]$. We are considering the difference between two consecutive iterations, where no reserve-dependency takes place and the additional summand therefore equals zero almost surely, and will therefore be left out in the expectation from now on.

This step is only necessary, if discrete sojourn payments at deterministic time points are possible. Otherwise, the integral would be dt-almost surely the same. Use the simplification $J = 0$ then, simplifying the Lipschitz constant to L.

We arrive at the following upper bound, that holds for the norm by using the above inequality for the complete interval $(0, T]$

$$
\begin{aligned}
&\left\| {}^{(n+1)}Y^- - {}^{(n)}Y^- \right\|_{V[0,T]} \\
&\leq \mathbb{E}\left[\left| {}^{(n)}Y_0^- - {}^{(n-1)}Y_0^- \right| + \int_{t\in(0,T]} \frac{L}{1-J} \cdot e^{-K(\zeta(t)-\zeta(0))} \left| {}^{(n)}Y_t^- - {}^{(n-1)}Y_t^- \right| \zeta(\mathrm{d}t) \right] \\
&\overset{(iv)}{\leq} \mathbb{E}\left[\int_{t\in(0,T]} \frac{L}{1-J} e^{-K(\zeta(t)-\zeta(0))} \left(\int_{r\in[0,t]} \mathrm{d}\left| {}^{(n)}Y^- - {}^{(n-1)}Y^- \right|_r \right) \zeta(\mathrm{d}t) \right] \\
&\overset{(v)}{\leq} \frac{L}{1-J} \cdot \mathbb{E}\left[\int_{r\in[0,T]} \left(\int_{t\in(r,T]} e^{-K(\zeta(t)-\zeta(0))} \zeta(\mathrm{d}t) \right) \mathrm{d}\left| {}^{(n)}Y^- - {}^{(n-1)}Y^- \right|_r \right] \\
&\leq \frac{L}{1-J} \cdot \mathbb{E}\left[\int_{[0,T]} \left(\int_{t\in(r,T]} e^{-K(\zeta(t)-\zeta(0))} \zeta(\mathrm{d}t) \right) \mathrm{d}\left| {}^{(n)}Y^- - {}^{(n-1)}Y^- \right|_r \right].
\end{aligned}
$$

In step (v), the order $0 \leq r \leq t \leq T$ of the integration area and integration variables can be understood as a condition for t depending on r (i.e. $t \in [r, T]$) as well as on r depending on t ($r \in [0, t]$), which enables us to exchange the order of integration and additionally makes it possible to include the lower bound $r = 0$, where the inner integral would be zero anyway.

We now have to use a general transformation formula to explicitly compute the inner integral and the details will be explained in detail.

Since ζ is increasing, the equivalence

$$
t \in (r, T] \;\Leftrightarrow\; \zeta(t) \in \big(\zeta(r), \zeta(T)\big]
$$

together with the quantile function leads to

$$
\zeta(T) - \zeta(r) = \lambda\Big(\zeta\big((r, T]\big)\Big) = \lambda\Big(\zeta^{-1} \in (r, T]\Big) = \mathscr{L}\big(\zeta^{-1} \mid \lambda\big)\big((r, T]\big)
$$

and therefore the integration can be replaced by $\mathscr{L}(\zeta^{-1} \mid \lambda)(\mathrm{d}t)$, i.e. $\mathbb{P}(A) = \lambda(\zeta(A))$.

We rewrite the inner integral in the last line as

$$
\int_{t\in(r,T]} e^{-K(\zeta(t)-\zeta(0))} \zeta(\mathrm{d}t)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \mathbb{1}_{\{t\in(r,T]\}} \cdot e^{-K(\zeta(t)-\zeta(0))}\, \mathscr{L}(\zeta^{-1} \mid \lambda)(\mathrm{d}t) \\
&= \int_{\mathbb{R}} \underbrace{\mathbb{1}_{\{\zeta(t)\in(\zeta(r),\zeta(T)]\}}\, e^{-K(\zeta(t)-\zeta(0))}}_{=h(\zeta(t))}\, \underbrace{\mathscr{L}(\zeta^{-1} \mid \lambda)(\mathrm{d}t)}_{=\mathbb{P}^{\zeta}(\mathrm{d}t)} \\
&= \int_{\mathbb{R}} \mathbb{1}_{\{t\in(\zeta(r),\zeta(T)]\}}\, e^{-K(t-\zeta(0))}\, \mathbb{P}^{\zeta}(\mathrm{d}t) \\
&\overset{(vi)}{\leq} \int_{(\zeta(r),\zeta(T)]} e^{-K(t-\zeta(0))}\, \lambda(\mathrm{d}t) = \int_{(\zeta(0),\zeta(r)]} e^{-K(t-\zeta(0))}\, \mathrm{d}t
\end{aligned}
$$

where in (vi) we used, that

$$
\mathbb{P}^{\zeta}(A) = \mathbb{P}\left(\zeta^{-1}(A)\right) = \lambda\left(\zeta(\zeta^{-1}(A))\right) \leq \lambda(A)\,.
$$

We can now calculate the inner $\mathrm{d}t$-integral and arrive at the following inequality:

$$
\begin{aligned}
&\left\|{}^{(n+1)}Y^{-} - {}^{(n)}Y^{-}\right\|_{V[0,T]} \\
&\leq \frac{L}{1-J} \cdot \mathbb{E}\left[\int_{[0,T]} \int_{(r,T]} e^{-K(\zeta(t)-\zeta(0))}\, \zeta(\mathrm{d}t)\, \mathrm{d}\left|{}^{(n)}Y^{-} - {}^{(n-1)}Y^{-}\right|_r\right] \\
&\leq \frac{L}{1-J} \cdot \mathbb{E}\left[\int_{[0,T]} \int_{(\zeta(r),\zeta(T)]} e^{-K(t-\zeta(0))}\, \mathrm{d}t\, \mathrm{d}\left|{}^{(n)}Y^{-} - {}^{(n-1)}Y^{-}\right|_r\right] \\
&= \frac{L}{1-J} \cdot \mathbb{E}\left[\int_{[0,T]} \frac{1}{K}\left(e^{-K(\zeta(r)-\zeta(0))} - e^{-K(\zeta(T)-\zeta(0))}\right) \mathrm{d}\left|{}^{(n)}Y^{-} - {}^{(n-1)}Y^{-}\right|_r\right] \\
&\leq \frac{L}{(1-J)\,K} \cdot \mathbb{E}\left[\int_{[0,T]} e^{-K(\zeta(r)-\zeta(0))}\, \mathrm{d}\left|{}^{(n)}Y^{-} - {}^{(n-1)}Y^{-}\right|_r\right] \\
&= \frac{L}{(1-J)\,K} \cdot \left\|{}^{(n)}Y^{-} - {}^{(n-1)}Y^{-}\right\|_{V[0,T]}\,,
\end{aligned}
$$

where the additional summand of the norm is zero. When choosing $K = 2 \cdot \dfrac{L}{1-J}$ and defining our contraction constant

$$
C := \frac{L}{(1-J)\,K} = \frac{1}{2} < 1
$$

we have indeed managed to show the contraction property with ζ as previously specified.

Application of the fixed point Theorem of Banach

Let us from now on assume, that $K = \frac{2 \cdot L}{1-J}$. Then we have a contraction and application of the fixed-point theorem of Banach guarantees existence and uniqueness of a process

$Y^- = (Y_t^-)_{t\geq 0} \in BV^X([0,T])$ fulfilling

$$Y_t^- = \sum_{M\in\mathcal{M}} \int\limits_{[0,t]} \mathbb{I}_{s^-}^M \, \kappa(s)\, b_M\big(s, X_{s^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{[0,t]\times E_I} \kappa(s)\, B_I\big(s, e, X_{s^-}^{\mathbb{G},-}\big)\, \mu_I(\mathrm{d}(s,e))$$

in the space of càdlàg processes with integrable variation on $[0,T]$, where additionally

$$X_{s^-}^{\mathbb{G},-} = \frac{1}{\kappa(s)}\, \mathbb{E}\left[Y_{s^-}^- \,\middle|\, \mathcal{G}_s^-\right]$$

almost surely to express the dependency in the correct way and to emphasize the fact, that this is indeed a fixed point equation for process Y^-. □

4.4. Thieles SDE

Let us now also develop the corresponding Thiele SDE for this case, where we will need the IF-compensators again. Let still $b_{\mathcal{M}}$ and $B_{\mathcal{N}}$ be defined as abbreviating notation for the respective payment functions and for computational details, we refer to the calculations prior to the Thiele BSDE.

Assertion 4.4.1. IF-*compensator – Cumulative payments*
The process A of cumulated payments

$$A(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M \, b_M\big(t, X_{t^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I\big(s, e, X_{t^-}^{\mathbb{G},-}\big)\, \mu_I(\mathrm{d}t \times \mathrm{d}e)$$

has the following IB-compensator

$$A^{\mathrm{IF}}(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M \, b_M\big(t, X_{t^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int\limits_{E_I} B_I\big(s, e, X_{t^-}^{\mathbb{G},-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e)\,.$$

Proof. The additive decomposition of A in sojourn and transition summands is used. Then, the two parts are similar to $b_{\mathcal{M}}$ and $B_{\mathcal{N}}$ (with a short rate set to 0), and consequently a similar structure of the IF-compensator arises. By leaving out the discounting, the argumentation is the same, but even simpler. □

The Thiele SDE with respect to $\mathbb{F}$ and $\mathbb{G}$ are now formulated.

Theorem 4.4.2. *Thiele SDE with respect to* $\mathbb{F}$
The prospective reserve with respect to $\mathbb{F}$ fulfils the following stochastic differential equation

$$\mathrm{d}X_t^{\mathbb{F},-} = \varphi(t)\, X_{t^-}^{\mathbb{F},-}\, \mathrm{d}t - A(\mathrm{d}t) - \sum_{I\in\mathcal{N}} \int\limits_{E_I} F_I(t,e)\, (\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e) \tag{4.4.1}$$

with initial condition $X_{0^-}^{\mathbb{F},-} = 0$.

The integrand $F_I(t,e)$ may be expressed as

$$F_I(t,e) = \mathbb{E}\left[X_{t^-}^{-} \,\middle|\, \mathcal{F}_t^{-},\, R_I = (t,e)\right] - \mathbb{E}\left[X_{t^-}^{-} \,\middle|\, \mathcal{F}_t^{-},\, \mathcal{J}_t = 0\right] .$$

Equation (4.2.1), together with initial value condition $X_{0^-}^{\mathbb{F},-} = 0$, then is a SDE with solution pair $(X^{\mathbb{F}}, (F_I)_I)$.

Corollary 4.4.3. *Reformulation of the Thiele SDE with respect to* $\mathbb{F}$
The above SDE (4.4.1) may be expressed in the following equivalent form

$$\mathrm{d}X_t^{\mathbb{F},-} = f(\omega, t, X_{t^-}^{\mathbb{F},-})\,\mathrm{d}t + g(\omega, t, X_{t^-}^{\mathbb{F},-})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int_{E_I} Z_I(t,e)\,(\mu_I - \lambda_I)(\mathrm{d}t \times \mathrm{d}e)$$

with natural initial value condition $X_{0^-}^{\mathbb{F},-} = 0$, and where the generator functions are given as

$$\begin{aligned} f(\omega, t, X_{t^-}^{\mathbb{F},-}) &= \varphi(t)\, X_{t^-}^{\mathbb{F},-} - \sum_{I\in\mathcal{N}} \int_{E_I} B_I(t, e, X_{t^-}^{\mathbb{F},-})\, l(t, \mathrm{d}e) \\ g(\omega, t, X_{t^-}^{\mathbb{F},-}) &= -\sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^{M}\, b_M(t, X_{t^-}^{\mathbb{F},-}) \end{aligned}$$

and

$$Z_I(s,e) = F_I(s,e) - B_I(s, e, X_{s^-}^{\mathbb{F},-}) .$$

Proof. See the proof to the Thiele BSDE with respect to $\mathbb{F}$. ☐

Proof. (Of the reformulation in Corollary 4.4.3.)

The representation of $A(\mathrm{d}t)$ as

$$A(\mathrm{d}t) = \sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^{M}\, b_M(t, X_{t^-}^{\mathbb{F},-})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}} \int_{E_I} B_I(t, e, X_{t^-}^{\mathbb{F},-})\,\mu_I(\mathrm{d}t \times \mathrm{d}e)$$

is used, which defines g by matching the $\gamma(\mathrm{d}t)$-parts.

The second summand contains a μ_I integral and is shifted into the integral part of the SDE, and the new $Z_I(t,e) := F_I(t,e) - B_I(t,e)$ is integrated. Since the integration is with respect to the $\mathbb{F}$-compensated measure $\mu_I - \lambda_I$, the newly arising λ_I part has to be compensated as well. Finally, the Lebesgue intensity l_I is used, to include this integral in the dt part of the BSDE, and therefore, as part of the function f. ☐

A similar result can once again be developed with respect to $\mathbb{G}$.

Theorem 4.4.4. *Thiele SDE with respect to* $\mathbb{G}$
The following differential equation holds

$$\begin{aligned}\mathrm{d}X_t^{\mathbb{G},-} = -A^{\mathrm{IF}}(\mathrm{d}t) + \varphi(t)X_{t^-}^{\mathbb{G},-}\,\mathrm{d}t + \sum_{I\in\mathcal{N}}\int_{E_I} G_I\left(t^-,t,e\right)\,(\mu_I-\nu_I)(\mathrm{d}t\times\mathrm{d}e)\\ + \sum_{I\in\mathcal{N}}\int_{E_I} G_I\left(t\ \ ,t,e\right)\,(\rho_I-\mu_I)(\mathrm{d}t\times\mathrm{d}e)\end{aligned} \tag{4.4.2}$$

with initial condition $X_{0^-}^{\mathbb{G},-} = 0$ *a.s.* and where the representation of the integrands G_I is given by Theorem 2.5.13, but they can almost surely also be written as

$$\begin{aligned}G_I(t^-,t,e) &= \mathbb{E}\left[X_t^-\,\middle|\,\mathcal{G}_t^-,\,R_I=(t,e)\right] - \mathbb{E}\left[X_t^-\,\middle|\,\mathcal{G}_t^-,\,\mathcal{J}_t=0\right]\,,\\ G_I(t\ \ ,t,e) &= \mathbb{E}\left[X_t^-\,\middle|\,\mathcal{G}_t\ ,\,R_I=(t,e)\right] - \mathbb{E}\left[X_t^-\,\middle|\,\mathcal{G}_t\ ,\,\mathcal{J}_t=0\right]\end{aligned}$$

with usage of the reformulation in 3.3.1.
Note, that the integrability assumption and measurability (with respect to $\mathbb{G}^-$ and $\mathbb{G}$ respectively) is indeed satisfied for $G_I(t^-,t,e)$ and $G_I(t\ \ ,t,e)$, and the projection $X^{\mathbb{G}}$ is by design a $\mathbb{G}$-adapted process.

Corollary 4.4.5. *Reformulation of the Thiele Equations with respect to* $\mathbb{G}$
The above BSDE (4.2.3) may be expressed in the following equivalent form

$$\begin{aligned}\mathrm{d}X_t^{\mathbb{G},-} = f(\omega,t,X_t^{\mathbb{G},-})\,\mathrm{d}t + g(\omega,t,X_t^{\mathbb{G},-})\,\gamma(\mathrm{d}t) + \sum_{I\in\mathcal{N}}\int_{E_I} G_I\left(t^-,t,e\right)\,(\mu_I-\nu_I)(\mathrm{d}t\times\mathrm{d}e)\\ + \sum_{I\in\mathcal{N}}\int_{E_I} G_I\left(t\ \ ,t,e\right)\,(\rho_I-\mu_I)(\mathrm{d}t\times\mathrm{d}e)\end{aligned}$$

with generator functions, defined as

$$\begin{aligned}f(\omega,t,X_{t^-}^{\mathbb{G},-}) &= \varphi(t)X_{t^-}^{\mathbb{G},-} - \sum_{I\in\mathcal{N}}\int_{E_I} B_I(t,e,X_{t^-}^{\mathbb{G},-})\,n_I(t,\mathrm{d}e)\\ g(\omega,t,X_{t^-}^{\mathbb{G},-}) &= -\sum_{M\in\mathcal{M}} \mathbb{I}_{t^-}^M\,b_M(t,X_{t^-}^{\mathbb{G},-})\end{aligned}$$

and with $(G_I)_I$ remaining as before.

Proof. Let us again consider the càdlàg process

$$Y_t^- = \kappa(t)\cdot X_t^- = \int_{[0,t]} \kappa(s)\,A(\mathrm{d}s)$$

where all payments are discounted up to time zero. This is now not needed to fulfil the preconditions of the infinitesimal martingale representation theorem, but has to be done, to be able to compute the IB-compensator and that has also been the reason, why we precomputed the compensator of $b_{\mathcal{M}}$ and $B_{\mathcal{N}}$, with discounting to zero already included.

The discounting is assumed to be deterministic and therefore the optional projection also almost surely fulfils

$$Y_t^{\mathbb{G},-} = \kappa(t) \cdot X_t^{\mathbb{G},-}$$

and we again use the differential

$$\mathrm{d}Y_t^{\mathbb{G},-} = \kappa(t)\,\mathrm{d}X_t^{\mathbb{G},-} - \varphi(t)\,\kappa(t)\,X_{t^-}^{\mathbb{G},-}\,\mathrm{d}t\,.$$

We then apply Theorem 2.5.13 to Y^- and note that the preconditions are indeed fulfilled, as we need the càdlàg property of Y^- and the integrability condition specified in 2.4.1, we arrive at

$$\begin{aligned}\mathrm{d}Y_t^{\mathbb{G},-} = \mathrm{d}Y_t^{\mathrm{IF},-} &+ \sum_{I\in\mathcal{N}} \int_{E_I} \tilde{G}_I(t^-,t,e)\,(\mu_I - \nu_I)(\mathrm{d}t\times\mathrm{d}e)\\ &+ \sum_{I\in\mathcal{N}} \int_{E_I} \tilde{G}_I(t\;\;,t,e)\,(\rho_I - \mu_I)(\mathrm{d}t\times\mathrm{d}e)\,,\end{aligned}$$

where (in the reformulated version) we have

$$\begin{aligned}\tilde{G}_I(t^-,t,e) &= \mathbb{E}\left[Y_t^- \,\middle|\, \mathcal{G}_t^-\,,\, R_I = (t,e)\right] - \mathbb{E}\left[Y_t^- \,\middle|\, \mathcal{G}_t^-\,,\, \mathcal{J}_t = 0\right] \;\; a.s.,\\ \tilde{G}_I(t\;\;,t,e) &= \mathbb{E}\left[Y_t^- \,\middle|\, \mathcal{G}_t\;\,,\, R_I = (t,e)\right] - \mathbb{E}\left[Y_t^- \,\middle|\, \mathcal{G}_t\;\,,\, \mathcal{J}_t = 0\right] \;\; a.s..\end{aligned}$$

To be able to work with the equation, we need to calculate the IB-compensator of Y^-. It holds, that by swapping in the representation for $A(\mathrm{d}t)$, we get from the definition of the compensator

$$\begin{aligned}Y_t^{\mathrm{IF},-} &= \lim_{n\to\infty} \sum_{\tau_n} \mathbb{E}\left[Y_{t_{k+1}}^- - Y_{t_k}^- \,\middle|\, \mathcal{G}_{t_k}\right]\\ &= \lim_{n\to\infty} \sum_{\tau_n} \mathbb{E}\left[-\int_{(t_k,t_{k+1}]} \kappa(s)\,A(\mathrm{d}s) \,\middle|\, \mathcal{G}_{t_k}\right]\\ &= -b_{\mathcal{M}}^{\mathrm{IF}}(t) - B_{\mathcal{N}}^{\mathrm{IF}}(t)\\ &= -b_{\mathcal{M}}(t) - B_{\mathcal{N}}^{\mathrm{IF}}(t)\end{aligned}$$

and in full form, we have

$$\mathrm{d}Y_t^{\mathrm{IF},-} = -\sum_{M\in\mathcal{M}} \kappa(t)\,\mathbb{I}_{s^-}^M\, b_M\big(t, X_{t^-}^{\mathbb{G},-}\big)\,\gamma(\mathrm{d}t) - \sum_{I\in\mathcal{N}} \int_{E_I} \kappa(t)\,B_I\big(t,e,X_{t^-}^{\mathbb{G},-}\big)\,\nu_I(\mathrm{d}t\times\mathrm{d}e)\,.$$

All together, when using the formulas for the differential and the compensator, we get

$$\begin{aligned}&\kappa(t)\,\mathrm{d}X_t^{\mathbb{G},-} - \varphi(t)\,\kappa(t)\,X_{t^-}^{\mathbb{G},-}\,\mathrm{d}t\\ &= \mathrm{d}Y_t^{\mathbb{G},-}\\ &\overset{2.5.13}{=} \mathrm{d}Y_t^{\mathrm{IF},-} + \sum_{I\in\mathcal{N}} \int_{E_I} \tilde{G}_I\left(t^-,t,e\right)\,\left(\mu_I - \nu_I\right)(\mathrm{d}t\times\mathrm{d}e)\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I\,(t\ \ , t, e)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&= - \sum_{M \in \mathcal{M}} \kappa(t)\, \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G},-}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} \kappa(t)\, B_I\big(t, e, X^{\mathbb{G},-}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\quad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t^-, t, e)\,(\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\quad + \sum_{I \in \mathcal{N}} \int_{E_I} \tilde{G}_I(t\ \ , t, e)\,(\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)\,,
\end{aligned}
$$

which by rearranging reads

$$
\begin{aligned}
&\kappa(t)\,\mathrm{d}X^{\mathbb{G},-}_t \\
&= \kappa(t) \Bigg(\varphi(t)\, X^{\mathbb{G},-}_{t^-} \mathrm{d}t - \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G},-}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} B_I\big(t, e, X^{\mathbb{G},-}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t^-, t, e)\ (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I\,(t\ \ , t, e)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \Bigg)\,,
\end{aligned}
$$

where we substituted

$$
G_I(t^-, t, e) = \frac{1}{\kappa(t)}\, \tilde{G}_I(t^-, t, e)
$$

and similarly for the other representation.

An application of the Radon-Nikodym Theorem leads to

$$
\begin{aligned}
\mathrm{d}X^{\mathbb{G},-}_t &= \varphi(t)\, X^{\mathbb{G},-}_{t^-} \mathrm{d}t - \sum_{M \in \mathcal{M}} \mathbb{I}^M_{t^-}\, b_M\big(t, X^{\mathbb{G},-}_{t^-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int_{E_I} B_I\big(t, e, X^{\mathbb{G},-}_{t^-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t^-, t, e)\,(\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t\ \ , t, e)\,(\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&= \varphi(t)\, X^{\mathbb{G},-}_{t^-} \mathrm{d}t - A^{\mathrm{IF}}(\mathrm{d}t) + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t^-, t, e)\,(\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&\qquad + \sum_{I \in \mathcal{N}} \int_{E_I} G_I(t\ \ , t, e)\,(\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)\,.
\end{aligned}
$$

The integral representation for both $X^{\mathbb{G},-}$ and $Y^{\mathbb{G},-}$ together with their natural starting value condition as $X_0^{\mathbb{G},-} = Y_0^{\mathbb{G},-}$, are given as

$$
\begin{aligned}
\kappa(t)\, X_t^{\mathbb{G},-} = Y_t^{\mathbb{G},-} = Y_0^{\mathbb{G},-} &- \sum_{M \in \mathcal{M}} \int\limits_{[0,t]} \kappa(s)\, \mathbb{I}_{s^-}^M\, b_M\big(s, X_{s^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}s) \\
&- \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} \kappa(s)\, B_I\big(s, e, X_{s^-}^{\mathbb{G},-}\big)\, n_I(s, \mathrm{d}e)\ \mathrm{d}s \\
&+ \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} \kappa(s)\, G_I(s^-, s, e)\, (\mu_I - \nu_I)(\mathrm{d}(s,e)) \\
&+ \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} \kappa(s)\, G_I(s\quad, s, e)\, (\rho_I - \mu_I)(\mathrm{d}(s,e))
\end{aligned}
\tag{4.4.3}
$$

for the discounted reserve as well as

$$
\begin{aligned}
X_t^{\mathbb{G},-} = X_0^{\mathbb{G},-} + \int\limits_{[0,t]} \varphi(s)\, X_{s^-}^{\mathbb{G}^-}\ \mathrm{d}s &- \sum_{M \in \mathcal{M}} \int\limits_{[0,t]} \mathbb{I}_{s^-}^M\, b_M\big(s, X_{s^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}s) \\
- \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} & B_I\big(s, e, X_{s^-}^{\mathbb{G},-}\big)\, n_I(s, \mathrm{d}e)\ \mathrm{d}s \\
+ \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} & G_I(s^-, s, e)\, (\mu_I - \nu_I)(\mathrm{d}(s,e)) \\
+ \sum_{I \in \mathcal{N}} \int\limits_{[0,t]\times E_I} & G_I(s\quad, s, e)\, (\rho_I - \mu_I)(\mathrm{d}(s,e))
\end{aligned}
\tag{4.4.4}
$$

for the original prospective reserve. □

Proof. (Of the reformulation 4.4.5)
We plug in the previously developed representation for $A^{\mathrm{IF}}(\mathrm{d}t)$ and arrive at

$$
\begin{aligned}
\mathrm{d}X_t^{\mathbb{G},-} = \varphi(t)\, X_{t^-}^{\mathbb{G},-}\mathrm{d}t\ &- \sum_{M \in \mathcal{M}} \mathbb{I}_{t^-}^M\, b_M\big(t, X_{t^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}t) - \sum_{I \in \mathcal{N}} \int\limits_{E_I} B_I\big(t, e, X_{t^-}^{\mathbb{G},-}\big)\, \nu_I(\mathrm{d}t \times \mathrm{d}e) \\
&+ \sum_{I \in \mathcal{N}} \int\limits_{E_I} G_I\left(t^-, t, e\right)\ (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&+ \sum_{I \in \mathcal{N}} \int\limits_{E_I} G_I\left(t\quad, t, e\right)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)
\end{aligned}
$$

where we insert the intensity of the compensator ν_I, to be able to introduce the function f for the dt-part, and g for the $\gamma(\mathrm{d}t)$-part. We arrive at

$$
\begin{aligned}
\mathrm{d}X_t^{\mathbb{G},-} = f\big(\omega, t, X_{t^-}^{\mathbb{G},-}\big)\, \mathrm{d}t + g\big(\omega, t, X_{t^-}^{\mathbb{G},-}\big)\, \gamma(\mathrm{d}t) &+ \sum_{I \in \mathcal{N}} \int\limits_{E_I} G_I\left(t^-, t, e\right)\ (\mu_I - \nu_I)(\mathrm{d}t \times \mathrm{d}e) \\
&+ \sum_{I \in \mathcal{N}} \int\limits_{E_I} G_I\left(t\quad, t, e\right)\ (\rho_I - \mu_I)(\mathrm{d}t \times \mathrm{d}e)\,.
\end{aligned}
$$

□

4.5. Calculation of premiums

The calculation of prospective reserves has been discussed in the previous section and we are now continuing with the pricing of contracts. Until now, we assumed that all payments are given and fulfil some necessary conditions. That does not necessarily take into account the nature of premiums. Some payments are of course pre-specified, but others, for example net equivalent premiums are calculated as a result from the remaining benefit payments, and can not be specified as a deterministic amount from the beginning.

In this section we discuss two ways to price a contract, where the second one allows for a general premium payment scheme.
In the simple case, a one-off premium is used at the beginning of the contract and no further premium payments happen until T. This is methodically easier, because we can view the insurance contract as benefit-only and compensate the necessary prospective reserve at the beginning of the contract by collecting a one-off premium. After that, we will allow for a more complicated premium payments scheme.

Let us start by defining what a net equivalent premium is supposed to be in our context, and for reference on premiums in general, and the following definitions in particular, refer to Olivieri and Pitacco [OP11].

Definition 4.5.1. *Net equivalent premium*
A premium of an insurance contract is called a net equivalent premium (also abbreviated as 'NEP'), if the following condition holds

$$X_{0^-}^{\mathbb{G}} = 0 \quad \mathbb{P} - a.s.. \tag{4.5.1}$$

The main interpretation is, that under usage of a net premium, the contract is actuarially fairly valued at the beginning of the contract.

4.5.1. A one-off premium payment at contract start

Definition 4.5.2. *One-off premium*
A premium is called a one-off premium, if it is paid as a lump sum payment at the beginning of the contract and no further premium payments are happening until the end of the contract. This type of premium payment is sometimes also called a set-up fee, or an instatement payment.

The mathematical definition for the one-off premium is given as

$$\pi := \mathbb{1}_{\{t=0\}} \cdot X_{0^-}^{\mathbb{G}} \quad \mathbb{P} - a.s.$$

as a premium in the state active and where $X_{0^-}^{\mathbb{G}}$ is the reserve, that is calculated for the contract with benefits only.
This type of pricing is therefore directly related to the calculation of the prospective reserve and may be done with already existing tools. This is not a reserve-dependent premium payment, as two different contracts are used respectively. The reserves of the complete contract and the reserve for the benefit-only contract are generally not the same.

Also note, that this premium payment is certain to be paid, since we assume the insured to start the contract in state active. It is not necessary to verify, that such a premium is indeed an equivalent net premium, since it is by definition.

4.5.2. General premium payments

The possibilities for premiums payments are now extended to more general premium payments, where a normed premium payment scheme is set and a premium level π, as a multiplicative constant for the whole premium scheme, has to be found.

Definition 4.5.3. *Premium scheme*
Assume that premium payments are only part of the sojourn payments, and are paid continuously. They can also depend on the prospective reserve (i.e. we have to assume dependency the structure from the Theorem 3.2.12).

Start by introducing a normed premium payment function for each $M \in \mathcal{M}$. Denote this type of payment by

$$e_M(s, X^{\mathbb{G}}_{s-})\,.$$

The premium payment scheme is then completed by multiplying each e_M by a premium level $\pi \in \mathbb{R}$. We may now write the complete sojourn payments as

$$\tilde{b}_M(s, X^{\mathbb{G}}_{s-}) := b_M(s, X^{\mathbb{G}}_{s-}) - \pi \cdot e_M(s, X^{\mathbb{G}}_{s-}) \tag{4.5.2}$$

for each $M \in \mathcal{M}$, where the $b_M(s, X^{\mathbb{G}}_{s-})$ contains benefits only and is therefore a slight change in notation to before.
The nature of the payment is now specified by the sign in front of the payment, therefore e_M and b_M for $M \in \mathcal{M}$ are assumed to be non-negative.

If we are considering a situation where the sojourn payments follow the structure (4.5.2), then we will introduce the following new notation for the corresponding payment process

$$X^{\pi} = (X^{\pi}_t)_{t \geq 0} \tag{4.5.3}$$

and the corresponding prospective reserve

$$X^{\mathbb{G},\pi} = (X^{\mathbb{G},\pi}_t)_{t \geq 0} \tag{4.5.4}$$

to emphasize the fact, that both are depending on the premium level π and to be able to differentiate between reserves, that are corresponding to different premium levels.

Until now we have considered BSDEs for the prospective reserve, where a final value is given. In addition to these final values, we now also need to restrain the BSDE with a second condition and make it a stochastic boundary value problem (sometimes abbreviated as SBVP). Therefore, we have introduced another variable π and we hope to be able to calculate a unique premium level, that results in an equivalent premium for a given contract.
If we do not have to work with reserve-dependent payments, then it would be possible to calculate the value of the benefits and premiums separately, and calculate the factor π as a quotient of these two properties, refer again to Olivieri and Pitacco [OP11] for some examples.

4.5.3. Existence and uniqueness of the premium level

The objective of this subsection is to show existence and uniqueness of a solution, albeit in a manner that differs from the approach taken previously. In addition to showing the existence of a unique payment process X we must now solve for a solution pair, consisting of the X^π and the premium level π. The premium level itself has to be shown to exist and be unique.

We start by showing, that for each premium level $\pi \in \mathbb{R}$, the payment process does exist and is unique. By using the results from the section about the Thiele BSDE, this then also implies the same for the prospective reserve.

Theorem 4.5.4. *Existence and uniqueness of X for a given premium level π*
Let the sojourn payments be given as in Definition 4.5.3 and also let the $(b_M)_M$ fulfil the necessary conditions of the previous section, i.e. boundedness and Lipschitz-condition with constant $L^b_{\mathcal{M}}$. Further, let the $(e_M)_M$ be bounded and also fulfil the Lipschitz condition with constant $L^e_{\mathcal{M}}$.

Then for every $\pi \in \mathbb{R}$, there exists a unique payment process $X^\pi = (X^\pi_t)_{t\geq 0} \in BV^X([0,T])$ fulfilling

$$
\begin{aligned}
X^\pi_t = &\sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s-} \, \Big(b_M(s, X^{\mathbb{G},\pi}_{s-}) - \pi \, e_M(s, X^{\mathbb{G},\pi}_{s-}) \Big) \, \mathrm{d}s \\
&+ \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} B_I(s, e, X^{\mathbb{G},\pi}_{s-}) \, \mu_I(\mathrm{d}(s,e)) \, .
\end{aligned}
$$

Proof. We verify, that the newly constructed sojourn payments

$$
\tilde{h}_M(s, X^{\mathbb{G},\pi}_{s-}) \, ,
$$

now depending on π, fulfil the Lipschitz conditions of our main Theorem. For a fixed π and $X^{\mathbb{G},\pi}_{s-}(\omega)$, $\hat{X}^{\mathbb{G},\pi}_{s-}(\omega)$ we get by triangle inequality that we $\mathrm{d}\mathbb{P} \times \mathrm{d}s$ *a.e.* have

$$
\begin{aligned}
&\left| \tilde{h}_M(s, X^{\mathbb{G},\pi}_{s-})(\omega) - \tilde{h}_M(s, \hat{X}^{\mathbb{G},\pi}_{s-})(\omega) \right| \\
&\qquad \leq \left| b_M(s, X^{\mathbb{G},\pi}_{s-})(\omega) - b_M(s, \hat{X}^{\mathbb{G},\pi}_{s-})(\omega) \right| + |\pi| \cdot \left| e_M(s, X^{\mathbb{G},\pi}_{s-})(\omega) - e_M(s, \hat{X}^{\mathbb{G},\pi}_{s-})(\omega) \right| \\
&\qquad \leq L^h_{\mathcal{M}} \left| X^{\mathbb{G},\pi}_{s-}(\omega) - \hat{X}^{\mathbb{G},\pi}_{s-}(\omega) \right| + |\pi| \; L^e_{\mathcal{M}} \left| X^{\mathbb{G},\pi}_{s-}(\omega) - \hat{X}^{\mathbb{G},\pi}_{s-}(\omega) \right| \\
&\qquad \leq \Big(L^h_{\mathcal{M}} + |\pi| \cdot L^e_{\mathcal{M}} \Big) \cdot \left| X^{\mathbb{G},\pi}_{s-}(\omega) - \hat{X}^{\mathbb{G},\pi}_{s-}(\omega) \right| \, ,
\end{aligned}
$$

by using the Lipschitz-properties for both b_M and e_M respectively and arriving at the new and combined Lipschitz constant $L_{\mathcal{M}} := L^h_{\mathcal{M}} + |\pi| \cdot L^e_{\mathcal{M}}$. This enables us to embed it into the theory of the dependency structure of the first type (even without γ) and to apply Theorem 3.2.12 to this situation, which guarantees the existence and uniqueness of the payment process or X^π, with a representation as specified in the assertion. $\square$

The theorem does not guarantee the premium to be a net equivalent premium just yet. To be able to show, that there is a net equivalent premium level, and that it is unique, we further need an assumption about the monotonicity of payments.

Assumption 4.5.5. *Monotonicity of payments*
If for an $\omega \in \Omega$ and a time point $t \in [0, T]$ two reserves $X_{t^-}^{\mathbb{G}}$ and $\tilde{X}_{t^-}^{\mathbb{G}}$ have the following order

$$X_{t^-}^{\mathbb{G}}(\omega) \leq \tilde{X}_{t^-}^{\mathbb{G}}(\omega)$$

then the corresponding reserve-dependent benefit payments fulfil the relations

$$b_M(t, X_{t^-}^{\mathbb{G}})(\omega) \leq b_M(t, \tilde{X}_{t^-}^{\mathbb{G}})(\omega)$$

for all $M \in \mathcal{M}$ and

$$B_I(t, e, X_{t^-}^{\mathbb{G}})(\omega) \leq B_I(t, e, \tilde{X}_{t^-}^{\mathbb{G}})(\omega)$$

for all $I \in \mathcal{N}$ with $e \in E_I$.

Further the reserve-dependent premium payments fulfil the relation

$$e_M(t, X_{t^-}^{\mathbb{G}})(\omega) \geq e_M(t, \tilde{X}_{t^-}^{\mathbb{G}})(\omega)$$

for all $M \in \mathcal{M}$.

These assumptions are quite natural, and have intuitive interpretations. In the case of a reserve-dependent lapse or death payment, a higher current reserve should lead to a higher payout. Vice versa, if the premium payment is reserve-dependent and the prospective reserve is bigger, then the premium payment should take the opposite relation.
Take note, that by allowing for payments containing a maximum or minimum, we at most have monotonicity in a non-strict way. We need these conditions, as they will enable us to use the Lipschitz-properties without the absolute value in certain cases, by using that we know the order of the payments, if the order of the reserves is known, as we are still using path-wise Lebesgue-Stieltjes integrals.

We will now formulate a second theorem to guarantee the existence and uniqueness of the net equivalent premium, in case that these additional assumptions hold.

Theorem 4.5.6. *Existence and uniqueness of π as a net equivalent premium*
Suppose that the assumptions of the previous theorem are fulfilled and Assumptions 3.1.6 (no reserve-dependency for payment in time 0) and 4.5.5 (monotonicity of payments) hold. Let also the Lipschitz conditions be given, such that they fulfil $L_M > 0$ and $L_N \leq 1$.

Further, assume that for every X the following condition

$$\mathbb{E}\left[\int_{[0,T]} \sum_{M \in \mathcal{M}} \mathbb{I}_{s^-}^M \, \kappa(s) \, e_M(s, X) \, \mathrm{d}s\right] \geq c \tag{4.5.5}$$

holds for a constant $c > 0$.

Then the proposed SBVP has a solution and both the payments process $X^\pi \in BV^X([0, T])$, as well as the premium level $\pi \in \mathbb{R}$ exist, and both are also unique.

Comment 4.5.7. *On the necessity of the precondition* (4.5.5)
Precondition (4.5.5) is a sensible assumption and it is essential to guarantee the strict monotonicity. It is crucial for certain arguments of the proof, as we will later see, but it is also motivated by the general construction of an insurance contract.
Without this condition, it would be possible to consider a contract, where all benefits are zero. Then also all premium payments can be set to zero, and this type of contract would always be equipped with an equivalent premium, naturally. In this case, the choice of the premium level is arbitrary and uniqueness would not be possible to achieve. This is constructed and exaggerated, but it gives an intuitive counterexample and therefore understates the importance of the precondition.

Proof. As an extension to the previous theorem, we have to show the existence and uniqueness of a premium level π, such that additionally the net equivalent premium condition $X_{0^-}^{\mathbb{G},\pi} = 0$ is fulfilled. This condition can also be rewritten in terms of the payments process, since $\mathcal{G}_0^-$ is the trivial sigma-algebra and the conditional expectation is indeed just a standard expected value. We then have

$$X_{0^-}^{\mathbb{G},\pi} = \mathbb{E}\left[X_{0^-}^{\pi} \,\middle|\, \mathcal{G}_0^-\right] = \mathbb{E}\left[X_{0^-}^{\pi}\right] =: F(\pi)$$

which we define as the function F, depending on $\pi \in \mathbb{R}$.

We will now proceed to show that there exists a $\pi \in \mathbb{R}$ such that $F(\pi) = 0$, and under further conditions, that the π is also unique. The existence will be shown by using the intermediate value theorem for F and the uniqueness will be shown by proving strict monotonicity of F.

We have to show the following three steps:

(1) Mapping F is continuous on $\mathbb{R}$, i.e. for every $\hat{\pi} \in \mathbb{R}$ it holds

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall \pi \in \mathbb{R} : |\hat{\pi} - \pi| < \delta \;\Rightarrow\; |F(\hat{\pi}) - F(\pi)| < \varepsilon.$$

(2) From the perspective of the insurance company, the insurance contract can be both underpriced and overpriced for different premium levels π, i.e.:

  (i) It exists a premium level $\pi_1 \in \mathbb{R}$, such that $F(\pi_1) < 0$.

  (ii) It exists a premium level $\pi_2 \in \mathbb{R}$, such that $F(\pi_2) > 0$.

(3) Under the additional preconditions, the mapping F is strictly monotonously decreasing, i.e. for all π_1, $\pi_2 \in \mathbb{R}$ with $\pi_1 < \pi_2$, it holds $F(\pi_1) > F(\pi_2)$.

Existence of the premium level

The statements (1) and (2) will guarantee the existence. We start by showing (1).

For two different premium levels π_1, $\pi_2 \in \mathbb{R}$, we get by application of the previous theorem, that the payment processes exists for both π_1 and π_2 and fulfils the respective fixed-point equation in both cases. Let us again use payment process, that is discounted down to 0, i.e. we use $Y_t = \kappa(t)\, X_t$ instead, but keep in mind, that this does not change our condition

in $t = 0$, as $Y_0 = X_0$ still holds.
The difference between the two discounted payment processes can be represented as

$$\begin{aligned} Y_t^{\pi_1} - Y_t^{\pi_2} = & \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}_{s^-}^{M}\,\kappa(s)\left(b_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - b_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right)\,\mathrm{d}s \\ & - \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \mathbb{I}_{s^-}^{M}\,\kappa(s)\left(\pi_1\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right)\,\mathrm{d}s \\ & + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \kappa(s)\left(B_I(s, e, X_{s^-}^{\mathbb{G},\pi_1}) - B_I(s, e, X_{s^-}^{\mathbb{G},\pi_2})\right)\mu_I(\mathrm{d}(s, e))\,. \end{aligned}$$

If we now apply our norm, and use similar arguments as in the proof of Theorem 3.2.12, combined with the extensions that were used in the section, where the discounting had been introduced, then we get the following inequality

$$\begin{aligned} & \|Y^{\pi_1} - Y^{\pi_2}\|_{V[0,T]} \\ = & \,\mathbb{E}\left[\int\limits_{[0,T]} e^{-K\,(\zeta(T)-\zeta(t))}\,\mathrm{d}\left|Y^{\pi_1} - Y^{\pi_2}\right|_t\right] \\ \le & \,\mathbb{E}\left[\int\limits_{(0,T]} e^{-K\,(\zeta(T)-\zeta(t))} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^{M}\,\kappa(s)\left|b_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - b_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right|\,\mathrm{d}s\right] \\ & + \mathbb{E}\left[\int\limits_{(0,T]\times E_I} e^{-K\,(\zeta(T)-\zeta(t))} \sum_{I\in\mathcal{N}} \kappa(s)\left|B_I(s, e, X_{s^-}^{\mathbb{G},\pi_1}) - B_I(s, e, X_{s^-}^{\mathbb{G},\pi_2})\right|\,\mu_I(\mathrm{d}(s, e))\right] \\ & + \mathbb{E}\left[\int\limits_{(0,T]} e^{-K\,(\zeta(T)-\zeta(t))} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^{M}\,\kappa(s)\left|\pi_1\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right|\,\mathrm{d}s\right]\,, \end{aligned}$$

where the first two lines are already known and one can proceed as before, but the new part has to be handled differently. With the triangle inequality and application of the Lipschitz condition of $(e_M)_M$ we get

$$\begin{aligned} & \left|\pi_1\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right| \\ = & \left|\pi_1\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) + \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right| \\ \le & \left|\pi_1 - \pi_2\right|\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) + |\pi_2| \left|e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right| \\ \le & \left|\pi_1 - \pi_2\right|\, e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) + |\pi_2|\, L_M^e \left|X_{s^-}^{\mathbb{G},\pi_1} - X_{s^-}^{\mathbb{G},\pi_2}\right|\,. \end{aligned}$$

This enables us to use the first part separately, as only the second part has a representation similar to the Lipschitz-condition for b_M and B_I. Note, that the new constant K is now depending on $|\pi_2|$. It is however not a problem, since the π_2 is arbitrary, but fixed, and can therefore be seen as constant in this application.

We then arrive at the following upper bound

$$\begin{aligned}\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} &\leq \frac{1}{2}\,\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} \\ &\quad + \mathbb{E}\left[\,\int_{(0,T]} e^{-K\,(\zeta(T)-\zeta(t))} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\,\kappa(s)\,|\pi_1-\pi_2|\,e_M(s,X^{\mathbb{G},\pi}_{s-})\,\mathrm{d}s\right] \\ &\leq \frac{1}{2}\,\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} \\ &\quad + |\pi_1-\pi_2|\cdot\mathbb{E}\left[\,\int_{(0,T]} e^{-K\,(\zeta(T)-\zeta(t))} \sum_{M\in\mathcal{M}} \kappa(s)\,\mathbb{I}^M_{s-}\,e_M(s,X^{\mathbb{G},\pi_1}_{s-})\,\mathrm{d}s\right] \\ &\leq \frac{1}{2}\,\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} + |\pi_1-\pi_2|\cdot C\end{aligned}$$

for a constant $C\in\mathbb{R}_{>0}$. Both the exponential part and the discounting are bounded by 1, and the expectation of the premium payment functions $(e_M)_M$ are bounded.

The same norm is present on both sides and rearranging of the above inequality leads to

$$\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} \leq 2\cdot C\cdot|\pi_1-\pi_2|\,. \tag{4.5.6}$$

Therefore, for every two π_1, $\pi_2\in\mathbb{R}$, we get by using the inequality of Jensen in (i) and by using that the exponential term is deterministic and positive, that

$$\begin{aligned}&e^{-K(\zeta(T)-\zeta(0))}\,|F(\pi_1)-F(\pi_2)| = e^{-K(\zeta(T)-\zeta(0))}\left|\mathbb{E}\left[X^{\pi_1}_{0^-}-X^{\pi_2}_{0^-}\right]\right| \\ &= e^{-K(\zeta(T)-\zeta(0))}\left|\mathbb{E}\left[Y^{\pi_1}_{0^-}-Y^{\pi_2}_{0^-}\right]\right| \overset{(i)}{\leq} e^{-K(\zeta(T)-\zeta(0))}\,\mathbb{E}\left[\left|Y^{\pi_1}_{0^-}-Y^{\pi_2}_{0^-}\right|\right] \\ &= \mathbb{E}\left[e^{-K(\zeta(T)-\zeta(0))}\left|Y^{\pi_1}_{0^-}-Y^{\pi_2}_{0^-}\right|\right] \overset{(*)}{\leq} \mathbb{E}\left[e^{-K(\zeta(T)-\zeta(0))}\int_{[0,T]}\mathrm{d}\left|Y^{\pi_1}-Y^{\pi_2}\right|_t\right] \\ &\leq \mathbb{E}\left[\,\int_{[0,T]} e^{-K(\zeta(T)-\zeta(t))}\,\mathrm{d}\left|Y^{\pi_1}-Y^{\pi_2}\right|_t\right] = \|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]}\end{aligned}$$

where it has additionally been used, that since $\zeta(0)\leq\zeta(t)$ for all $t\in[0,T]$ we also have

$$e^{-K(\zeta(T)-\zeta(0))} \leq e^{-K(\zeta(T)-\zeta(t))}\,.$$

Take note, that the upper bound in $(*)$ only holds by also considering the payments that happen in 0 (i.e. $Y^{\pi_1}_{0^-}-Y^{\pi_2}_{0^-}-Y^{\pi_1}_0+Y^{\pi_2}_0$), but the premiums are assumed to be continuous, and the transition payments are the same, as no reserve-dependency is allowed in 0, so the difference is zero.

It finally leads to the inequality

$$|F(\pi_1)-F(\pi_2)| \leq e^{K(\zeta(T)-\zeta(0))}\cdot\|Y^{\pi_1}-Y^{\pi_2}\|_{V[0,T]} \overset{(4.5.6)}{\leq} e^{K(\zeta(T)-\zeta(0))}\cdot 2\cdot C\cdot|\pi_1-\pi_2|\,.$$

Now we have all necessary pre-computations to show the continuity of F in π as a classic application of the ε-δ-definition.
Let there be given a fixed $\hat{\pi} \in \mathbb{R}$ and an $\varepsilon > 0$. Set

$$\delta := \frac{\varepsilon\, e^{-K\,(\zeta(T)-\zeta(0))}}{2\,C} > 0\,,$$

where we have to keep in mind, that the K is be depending on π_2, since we are now wanting to use the pre-computation with $\pi_1 = \pi$ and $\pi_2 = \hat{\pi}$.

Then, for every $\pi \in \mathbb{R}$ with $|\hat{\pi} - \pi| < \delta$, we get

$$\begin{aligned}|F(\hat{\pi}) - F(\pi)| &\leq e^{K(\zeta(T)-\zeta(0))} \left\| Y^{\pi} - Y^{\hat{\pi}} \right\|_{V[0,T]} \leq e^{K(\zeta(T)-\zeta(0))}\, 2\,C\,|\pi - \hat{\pi}| \\ &< e^{K(\zeta(T)-\zeta(0))}\, 2\,C\,\delta = \varepsilon\end{aligned}$$

which means, that the mapping F is indeed continuous in π and therefore step (1) of the proof is completed.

We continue with step (2), where the limit behaviour in the argument of F is investigated. Let $\pi \in \mathbb{R}$ be given. We may write (take note, that the second scaling factor is now 0)

$$\begin{aligned}
&F(\pi) - F(0) \\
&= \mathbb{E}\left[X^{\pi}_{0-} - X^{0}_{0-}\right] \overset{3.1.9}{=} \mathbb{E}\left[Y^{\pi}_{0-} - Y^{0}_{0-}\right] \\
&= \mathbb{E}\left[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\, \kappa(s) \left(b_M(s, X^{\mathbb{G},\pi}_{s-}) - b_M(s, X^{\mathbb{G},0}_{s-}) - \pi\, e_M(s, X^{\mathbb{G},\pi}_{s-})\right) \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{[0,T]\times E_I} \kappa(s) \left(B_I(s,e,X^{\mathbb{G},\pi}_{s-}) - B_I(s,e,X^{\mathbb{G},0}_{s-})\right) \mu_I(\mathrm{d}(s,e))\right] \\
&= \mathbb{E}\left[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\, \kappa(s) \left(b_M(s, X^{\mathbb{G},\pi}_{s-}) - b_M(s, X^{\mathbb{G},0}_{s-})\right) \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int\limits_{[0,T]\times E_I} \kappa(s) \left(B_I(s,e,X^{\mathbb{G},\pi}_{s-}) - B_I(s,e,X^{\mathbb{G},0}_{s-})\right) \mu_I(\mathrm{d}(s,e))\right] \\
&\quad - \pi\,\mathbb{E}\left[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi}_{s-})\, \mathrm{d}s\right]
\end{aligned}$$

which, by defining the remainder as $B(\pi)$, leads to

$$F(\pi) - F(0) = B(\pi) - \pi\,\mathbb{E}\left[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi}_{s-})\, \mathrm{d}s\right]\,,$$

where the B is bounded from above and from below, since we assumed the boundedness for all payment functions, i.e. there exists an $\tilde{C} > 0$ such that $|B(\pi)| \leq \tilde{C}$, as well as the discounting factor is bounded from above and below.
The second part is bounded away from 0 by precondition (4.5.5) of the theorem and this will result in an upper or lower bound depending on the sign of π. From the last representation it can be seen, that function F takes positive and negative values, by calculating the limits for $\pi \to \pm\infty$.

In more detail, we have:

(i) For $\pi \to \infty$ we get (without loss of generality assume $\pi > 0$ for simplicity):

$$\begin{aligned}
&\lim_{\pi \to \infty} F(\pi) \\
&= F(0) + \underbrace{\lim_{\pi \to \infty} B(\pi)}_{\leq \tilde{C}} - \lim_{\pi \to \infty} \pi \cdot \underbrace{\mathbb{E}\left[\int\limits_{[0,T]} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{s^-} \, \kappa(s) \, e_M(s, X^{\mathbb{G},\pi}_{s^-}) \, \mathrm{d}s \right]}_{\geq c > 0} \\
&= \lim_{\pi \to \infty} -\pi \cdot c = -\infty
\end{aligned}$$

Therefore, it exists a $\pi_1 \in \mathbb{R}$, such that $F(\pi_1) < 0$.

(ii) For $\pi \to -\infty$ we get (without loss of generality assume $\pi < 0$ for simplicity):

$$\begin{aligned}
&\lim_{\pi \to -\infty} F(\pi) \\
&= F(0) + \underbrace{\lim_{\pi \to -\infty} B(\pi)}_{\geq -\tilde{C}} - \lim_{\pi \to -\infty} \pi \cdot \underbrace{\mathbb{E}\left[\int\limits_{[0,T]} \sum_{M \in \mathcal{M}} \mathbb{I}^M_{s^-} \, \kappa(s) \, e_M(s, X^{\mathbb{G},\pi}_{s^-}) \, \mathrm{d}s \right]}_{\geq c > 0} \\
&= \lim_{\pi \to -\infty} -\pi \cdot c = \infty
\end{aligned}$$

Therefore, it exists a $\pi_2 \in \mathbb{R}$, such that $F(\pi_2) > 0$.

Since (1) and (2) now hold, the intermediate value Theorem A.3.2 can now be used and it guarantees, that there also exists a $\pi \in \mathbb{R}$ such that $F(\pi) = 0$, i.e. F has at least one zero value since it has values both over and below zero, and by continuity, it also has to take every value in between.

Uniqueness of the premium level

We continue with step (3) and additionally show uniqueness of the premium level, i.e. that exactly one π fulfils the equation $F(\pi) = 0$. To do that, we prove that F is strictly monotonously decreasing, which then guarantees, that at most one zero point can exist.

Let π_1, $\pi_2 \in \mathbb{R}$ with $\pi_1 < \pi_2$ be given. We have to show, that

$$F(\pi_1) = \mathbb{E}\left[X^{\pi_1}_{0^-}\right] > \mathbb{E}\left[X^{\pi_2}_{0^-}\right] = F(\pi_2)$$

holds.

By using formula (4.2.4) for the representation of the prospective reserve, and with Corollary 2.5.8, the following simplified equation for the expectation of the discounted prospective reserve holds. Take note, that a unique càdlàg version of the reserve exists. We are splitting the sojourn payments into two parts, as the difference of benefit and premium payments, and the martingale parts vanish under the expectation.

For every $t \in [0, T]$, we have

$$\begin{aligned} \mathbb{E}\left[\kappa(t)\, X_t^{\mathbb{G},\pi}\right] = \mathbb{E}\left[\kappa(T)\, X_T^{\mathbb{G},\pi}\right] &+ \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(t,T]} \kappa(s)\, \mathbb{I}_{s^-}^M\, b_M(s, X_{s^-}^{\mathbb{G},\pi})\,\mathrm{d}s\right] \\ &- \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(t,T]} \kappa(s)\, \mathbb{I}_{s^-}^M\, \pi\, e_M(s, X_{s^-}^{\mathbb{G},\pi})\,\mathrm{d}s\right] \qquad , \\ &+ \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(t,T]} \int_{E_I} \kappa(s)\, B_I(s, e, X_{s^-}^{\mathbb{G},\pi})\, n_I(s, \mathrm{d}e)\,\mathrm{d}s\right] \end{aligned} \tag{4.5.7}$$

which will later also be used as a formula for the differences.

Special considerations will later be placed on the summand that contains the premium payments. This is the only part, where the multiplication with π_i, $i = 1, 2$ arises, and we can not use the Lipschitz conditions in the usual way. Therefore, we split it into a non-positive and a weighted Lipschitz-part as follows

$$\begin{aligned} &\pi_1 \cdot e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2 \cdot e_M(s, X_{s^-}^{\mathbb{G},\pi_2}) \\ &= (\pi_1 - \pi_2)\cdot e_M(s, X_{s^-}^{\mathbb{G},\pi_2}) + \pi_1 \cdot \left(e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) - e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\right) \end{aligned}$$

which is similar to step (1) of the proof, but without using the absolute value. Instead, we keep the non-positive part, as $\pi_2 > \pi_1$, to get a negative upper bound in the later part of the proof.

The monotonicity condition for the premium payments is of great importance in this case, as we may have

$$e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) \geq e_M(s, X_{s^-}^{\mathbb{G},\pi_2}),$$

but then not generally

$$\pi_1 \cdot e_M(s, X_{s^-}^{\mathbb{G},\pi_1}) \geq \pi_2 \cdot e_M(s, X_{s^-}^{\mathbb{G},\pi_2}),$$

for $\pi_1 < \pi_2$. Also, we have to account for the negative sign, which is used for the summand of the premium payments.

Let us first develop an auxiliary result.

Theorem 4.5.8. *Temporal distribution of premium payments*
Let there be an interval $[\tau, \sigma] \subseteq [0, T]$, such that for every $u \in [\tau, \sigma]$ we have

$$X_u^{\mathbb{G},\pi_1} = X_u^{\mathbb{G},\pi_2} \ a.s.$$

i.e. both reserves are actually versions of each other on the interval $[\tau, \sigma]$.

Then, the following condition

$$\mathbb{E}\left[\int_{(\tau,\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}_{u^-}^M \, \kappa(u)\, e_M(u, X_{u^-}^{\mathbb{G},\pi_i})\,\mathrm{d}u\right] = 0$$

holds for both $i = 1$ and $i = 2$.

Proof. From the precondition, we have that

$$X_u^{\mathbb{G},\pi_1} = X_u^{\mathbb{G},\pi_2} \ a.s.$$

which then implies by law of total expectation

$$\mathbb{E}\left[X_u^{\pi_1}\right] = \mathbb{E}\left[X_u^{\pi_2}\right] \qquad (i)$$

for all $u \in [\tau, \sigma]$ as well as

$$X_{u^-}^{\mathbb{G},\pi_1} = X_{u^-}^{\mathbb{G},\pi_2} \ a.s. \qquad (ii)$$

for all $u \in (\tau, \sigma]$ as the left limit.

Let us now use the payment processes X^{π_1} and X^{π_2} for both premium levels π_1 and π_2 as unique solutions to their respective BSDEs. By applying the expectation, we are able to match the summands with each other and arrive at the following equivalent condition

$$\begin{aligned}
&\mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}_{u^-}^M \, \kappa(u)\, b_M(u, X_{u^-}^{\mathbb{G},\pi_1})\,\mathrm{d}u\right] - \pi_1\,\mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}_{u^-}^M \, \kappa(u)\, e_M(u, X_{u^-}^{\mathbb{G},\pi_1})\,\mathrm{d}u\right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(s,\sigma]\times E_I} \kappa(u)\, B_I(u, e, X_{u^-}^{\mathbb{G},\pi_1})\,\mu_I(\mathrm{d}(u,e))\right] + \mathbb{E}\left[\kappa(\sigma)\, X_\sigma^{\pi_1}\right] \\
&= \mathbb{E}\left[\kappa(s)\, X_s^{\pi_1}\right] \\
&\overset{(i)}{=} \mathbb{E}\left[\kappa(s)\, X_s^{\pi_2}\right] \\
&= \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}_{u^-}^M \, \kappa(u)\, b_M(u, X_{u^-}^{\mathbb{G},\pi_2})\,\mathrm{d}u\right] - \pi_2\,\mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}_{u^-}^M \, \kappa(u)\, e_M(u, X_{u^-}^{\mathbb{G},\pi_2})\,\mathrm{d}u\right] \\
&\quad + \mathbb{E}\left[\sum_{I\in\mathcal{N}} \int_{(s,\sigma]\times E_I} \kappa(u)\, B_I(u, e, X_{u^-}^{\mathbb{G},\pi_2})\,\mu_I(\mathrm{d}(u,e))\right] + \mathbb{E}\left[\kappa(\sigma)\, X_\sigma^{\pi_2}\right]
\end{aligned}$$

$$\overset{(ii)}{\Leftrightarrow} \pi_1 \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}^M_{u^-}\kappa(u)e_M(u, X^{\mathbb{G},\pi_1}_{u^-})\,\mathrm{d}u\right] = \pi_2 \mathbb{E}\left[\sum_{M\in\mathcal{M}} \int_{(s,\sigma]} \mathbb{I}^M_{u^-}\kappa(u)e_M(u, X^{\mathbb{G},\pi_2}_{u^-})\,\mathrm{d}u\right]$$

which holds for every $s \in [\tau, \sigma]$.

All summands with reserve-dependent payments have the same expectation, because equal reserves are plugged in into both summands, and the differences are therefore zero. For this step, it is essential for the premium level to be pulled outside of the expectation.
The last line can only be true, if both expectations are zero. Since we have $\pi_1 < \pi_2$, both sides of the equation only differ by these scaling parameters. Therefore the result implies, that the precondition of the proof may be slightly changed in certain situations. We do not have to consider the complete interval $[0, T]$, but can restrict the condition to the intervals outside of $(\tau, \sigma]$, if we already know, that the reserves are almost surely equal on such an interval. $\square$

The theorem guarantees, that premium payments can not be paid on the interval $(\tau, \sigma]$, if the preconditions are fulfilled. The result further implies, that we can not have $\tau = 0$ and $\sigma = T$, as this would contradict our precondition of the outer proof, see (4.5.5). Nevertheless, the case $\sigma = T$ would be desirable, as it creates the opportunity to focus on the interval $[0, \tau]$, for potential premium payments to arise and it therefore provides us with the possibility to use a smaller artificial final value, instead of T.

Non-strict monotonicity

Let us now continue with the main part of the proof. Define the random time

$$\sigma := \inf\left\{t \in [0, T] \;\middle|\; X^{\mathbb{G},\pi_2}_s \le X^{\mathbb{G},\pi_1}_s \text{ for every } s \in [t, T]\right\}$$

as a point, where the two reserves, in a backward view from T, first change their order.

The definition of σ is sketched in the following figure, to provide a visual aid of the situation and motivate the definition of σ as an overtaking condition in a backward perspective.

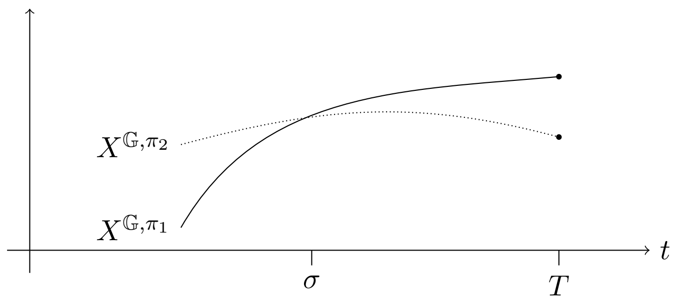

Figure 4.1.: Sketch of the overtaking condition

The definition of σ has the following implications, and the existence of a càdlàg version for the prospective reserves also allows for an evaluation at random times:

(I) It always is $0 \leq \sigma \leq T$ by definition. We do not need to define an exception for σ, as the condition is at least always fulfilled in the final value T, where even equality holds as $X_T^{\mathbb{G},\pi_2} = X_T^{\mathbb{G},\pi_1} = 0$.

(II) We have $X_\sigma^{\mathbb{G},\pi_2} \leq X_\sigma^{\mathbb{G},\pi_1}$. This means, that the order in the interval $[\sigma, T]$ remains the same until σ. The reserves are right-continuous and the definition of σ allows for equality, which guarantees that the order is kept until σ.

This property will later enable us to use the reserve in σ as a new final value.

(III) The order of the two reserves in σ^- depends on the behaviour of the reserves in σ. There are two possible scenarios on how the order of the reserves might change:

(1) The order changes continuously. It holds $X_{\sigma^-}^{\mathbb{G},\pi_2} \geq X_{\sigma^-}^{\mathbb{G},\pi_1}$, and the continuity implies the left-continuity. Then we have equality of the two reserves both in σ, as well as in σ^-.

(2) The order changes with a jump, because of a transition payment. In this case, the order has to be reversed and we have $X_{\sigma^-}^{\mathbb{G},\pi_2} > X_{\sigma^-}^{\mathbb{G},\pi_1}$. The difference even has to be positive, to be in alignment with the definition of the infimum.

Let us continue with a proper investigation in the situation in σ. We are especially interested in the case (III)(2) and want to develop further insights into the height of the jump and if this case actually appears with our assumptions.

By evaluation of the difference of the discounted reserves in σ^- and σ, we are able to isolate the summands, that are influenced by a possible jump. All the other parts are integrals with respect to a continuous integrator, and the difference is zero. The Lipschitz condition with constant $C = L_{\mathcal{N}} \leq 1$ is used. Also remember the left-continuity of the discounting process, i.e. $\kappa(0, \sigma^-) = \kappa(0, \sigma)$.

The cases in property (III) from above can be represented with the help of the following set

$$A := \left\{ X_{\sigma^-}^{\mathbb{G},\pi_2} > X_{\sigma^-}^{\mathbb{G},\pi_1} \right\}$$

and the corresponding indication functions

$$\mathbb{1}_A = \mathbb{1}_{\left\{ X_{\sigma^-}^{\mathbb{G},\pi_2} > X_{\sigma^-}^{\mathbb{G},\pi_1} \right\}}$$

if a jump happens from σ^- to σ, and

$$\mathbb{1}_{A^c} = \mathbb{1}_{\left\{ X_{\sigma^-}^{\mathbb{G},\pi_2} = X_{\sigma^-}^{\mathbb{G},\pi_1} \right\}}$$

in case no jump occurs in σ and where the two cases are covering every possible option, which means that we may use

$$\mathbb{1}_{\left\{ X_{\sigma^-}^{\mathbb{G},\pi_2} = X_{\sigma^-}^{\mathbb{G},\pi_1} \right\}} + \mathbb{1}_{\left\{ X_{\sigma^-}^{\mathbb{G},\pi_2} > X_{\sigma^-}^{\mathbb{G},\pi_1} \right\}} = 1 \tag{4.5.8}$$

and we get that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_2}_{\sigma^-}-X^{\mathbb{G},\pi_1}_{\sigma^-}\Big|\Big]-\mathbb{E}\Big[\ \kappa(0,\sigma)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma}-X^{\mathbb{G},\pi_1}_{\sigma}\Big)\Big]\Big|\\
&=\Big|\mathbb{E}\Big[(\mathbb{1}_A+\mathbb{1}_{A^{\mathsf{c}}})\cdot\Big(\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_2}_{\sigma^-}-X^{\mathbb{G},\pi_1}_{\sigma^-}\Big|-\kappa(0,\sigma)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma}-X^{\mathbb{G},\pi_1}_{\sigma}\Big)\Big)\Big]\Big|\\
&=\Big|\mathbb{E}\Big[\ \mathbb{1}_A\cdot\Big(\kappa(0,\sigma^-)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma^-}-X^{\mathbb{G},\pi_1}_{\sigma^-}\Big)-\kappa(0,\sigma)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma}-X^{\mathbb{G},\pi_1}_{\sigma}\Big)\Big)\Big]\Big|\\
&=\mathbb{E}\left[\ \mathbb{1}_A\cdot\sum_{I\in\mathcal{N}}\ \int\limits_{[\sigma,\sigma]\times E_I}\kappa(s)\Big(B_I(s,e,X^{\mathbb{G},\pi_1}_{s^-})-B_I(s,e,X^{\mathbb{G},\pi_2}_{s^-})\Big)\ \mu_I(\mathrm{d}(s,e))\right]\\
&\le\mathbb{E}\left[\ \mathbb{1}_A\cdot\sum_{I\in\mathcal{N}}\ \int\limits_{[\sigma,\sigma]\times E_I}\kappa(s)\underbrace{\Big|B_I(s,e,X^{\mathbb{G},\pi_1}_{s^-})-B_I(s,e,X^{\mathbb{G},\pi_2}_{s^-})\Big|}_{\le L_{\mathcal{N}}\cdot\left|X^{\mathbb{G},\pi_1}_{\sigma^-}-X^{\mathbb{G},\pi_2}_{\sigma^-}\right|}\ \mu_I(\mathrm{d}(s,e))\right]\\
&\le L_{\mathcal{N}}\cdot\mathbb{E}\left[\ \mathbb{1}_A\cdot\kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_1}_{\sigma^-}-X^{\mathbb{G},\pi_2}_{\sigma^-}\Big|\cdot\sum_{I\in\mathcal{N}}\int\limits_{E_I}\mu_I(\sigma,\mathrm{d}e)\right]\\
&\le L_{\mathcal{N}}\cdot\mathbb{E}\Big[\ \mathbb{1}_A\cdot\kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_1}_{\sigma^-}-X^{\mathbb{G},\pi_2}_{\sigma^-}\Big|\Big]\\
&\le L_{\mathcal{N}}\cdot\Big|\mathbb{E}\Big[\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_1}_{\sigma^-}-X^{\mathbb{G},\pi_2}_{\sigma^-}\Big|\Big]\Big|
\end{aligned}$$

where the differences in the reserves are zero for A^{c}, as discussed earlier.

The absolute value is only introduced for the last step, as it does not make a difference for an already non-negative number. Take note, that if $X^{\mathbb{G},\pi_2}_{\sigma^-}=X^{\mathbb{G},\pi_1}_{\sigma^-}$, then we also have $X^{\mathbb{G},\pi_2}_{\sigma}=X^{\mathbb{G},\pi_1}_{\sigma}$, since the equal reserves are used for the payment, and no other changes can occur at time point σ.

We further used the monotonicity property of the expectation, by using that

$$\sum_{I\in\mathcal{N}}\int\limits_{E_I}\mu_I(\sigma,\mathrm{d}e)\le 1\ a.s.\,,$$

since at most one jump can happen in σ. We see, that the structure of the arising inequality

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_2}_{\sigma^-}-X^{\mathbb{G},\pi_1}_{\sigma^-}\Big|\Big]-\mathbb{E}\Big[\ \kappa(0,\sigma)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma}-X^{\mathbb{G},\pi_1}_{\sigma}\Big)\Big]\Big|\\
&\qquad\le L_{\mathcal{N}}\cdot\Big|\mathbb{E}\Big[\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_1}_{\sigma^-}-X^{\mathbb{G},\pi_2}_{\sigma^-}\Big|\Big]\Big|
\end{aligned}$$

has a resemblance to the following inequality $|b-a|\le C\cdot|b|$ with

$$\begin{aligned}
a&=\mathbb{E}\Big[\ \kappa(0,\sigma)\cdot\Big(X^{\mathbb{G},\pi_2}_{\sigma}-X^{\mathbb{G},\pi_1}_{\sigma}\Big)\Big]\le 0\\
b&=\mathbb{E}\Big[\ \kappa(0,\sigma^-)\cdot\Big|X^{\mathbb{G},\pi_2}_{\sigma^-}-X^{\mathbb{G},\pi_1}_{\sigma^-}\Big|\Big]\ge 0
\end{aligned}$$

and $C=L_{\mathcal{N}}$, where conditions for solutions to this equation are as follows. Only in two cases there is potential for a for solution:

The first case $a = 0$ corresponds to the situation of no jump in σ. For $C = 1$, this gives us a solution for every $b > 0$, and if also $b = 0$, then for every $C > 0$, it is fulfilled.
The second case needs some additional details. If $a < 0$, solutions only exist if also $b < 0$, or if $C > 1$. These situation are not arising, since b is constructed with an absolute value and we further have a restriction for $CL_{\mathcal{N}} \leq 1$. The reserves therefore can not change their order in σ through a jump, where the intuitive idea is, that the previous difference can only be balanced, but a change of the order is not possible by a jump that is a multiple of the previous difference.

The proof is now continued with a contradiction argument.
Assume, that

$$\mathbb{P}(\sigma > 0) > 0 \tag{4.5.9}$$

holds, i.e. there exists a subset $N \subseteq \Omega$ with $\mathbb{P}(N) > 0$, such that $\sigma(\omega) > 0$ for every $\omega \in N$.

Let us now construct a second time τ.
For $\omega \in N$ we can always find and define a τ, as a time that fulfils $0 \leq \tau(\omega) < \sigma(\omega)$, and such that for every $s \in [\tau(\omega), \sigma(\omega))$ we have

$$X_s^{\mathbb{G},\pi_2}(\omega) > X_s^{\mathbb{G},\pi_1}(\omega)$$

as a implication of the right-continuity of the reserves. We are choosing the definition of τ based on the existence and we are not using an explicit formula (i.e. a supremum or infimum, compare the definition of σ) for its definition.
For the remaining $\omega \in N^{\mathsf{c}} = \Omega \setminus N$, we define $\tau(\omega) := \sigma(\omega)$, to complete the definition of τ.

As a consequence of the assumption we have $\mathbb{P}(N) > 0$, and therefore by construction of τ, we also have $\mathbb{P}(\sigma > \tau) > 0$, as

$$N = \{\sigma > 0\} = \{\sigma > \tau\}$$

holds. Additionally, the assumption implies, that

$$\mathbb{P}\left(X_\tau^{\mathbb{G},\pi_2} > X_\tau^{\mathbb{G},\pi_1}\right) > 0$$

and $X_\tau^{\mathbb{G},\pi_2} > X_\tau^{\mathbb{G},\pi_1}$ on N, i.e.

$$X_\tau^{\mathbb{G},\pi_2}(\omega) > X_\tau^{\mathbb{G},\pi_1}(\omega) \tag{4.5.10}$$

for every $\omega \in N$.

The definition of σ and τ has the following implications, in addition to the ones specified above:

(IV) For every $\omega \in N$ and for $s \in [\tau(\omega), \sigma(\omega))$ we have

$$X_s^{\mathbb{G},\pi_2}(\omega) > X_s^{\mathbb{G},\pi_1}(\omega)$$

and equality in time $\sigma(\omega)$. Therefore we also have

$$X_{s-}^{\mathbb{G},\pi_2}(\omega) > X_{s-}^{\mathbb{G},\pi_1}(\omega)$$

for every $s \in (\tau(\omega), \sigma(\omega))$ and $\omega \in N$.

We will now use similar arguments as in step (1) of the proof, with the difference that we do not have to show the contraction property and we therefore do not use the weighted norm. Instead, it is the goal to achieve an upper bound via application of the Lemma of Grönwall.

For $\omega \in N^{\mathsf{c}}$ we have that $\sigma(\omega) = 0$ (also $\tau(\omega) = 0$) and $X_s^{\mathbb{G},\pi_2}(\omega) \leq X_s^{\mathbb{G},\pi_1}(\omega)$ for all $s \in [0, T]$.

Let $t \in [0, T]$ be arbitrary, but fixed. As a combination of the interval, we get

$$(t, T] \cap (\tau, \sigma] = (t \vee \tau, T \wedge \sigma] .$$

By additivity of the integral and by using formula (4.5.7) with a new final value in $T \wedge \sigma = \sigma$ instead of T, and by including an indication function for N, we can follow that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0, (t \vee \tau) \wedge \sigma) \cdot \left| X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma} - X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma} \right|\right] \\
&= \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0, (t \vee \tau) \wedge \sigma) \cdot \left(X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma} - X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma} \right)\right] \\
&\overset{(4.5.7)}{=} \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0, \sigma) \cdot \left(X^{\mathbb{G},\pi_2}_{\sigma} - X^{\mathbb{G},\pi_1}_{\sigma} \right)\right] \\
&\quad + \mathbb{E}\Bigg[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s) \left(b_M(s, X^{\mathbb{G},\pi_2}_{s^-}) - b_M(s, X^{\mathbb{G},\pi_1}_{s^-}) \right) \mathrm{d}s \\
&\qquad - \mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s) \left(\pi_2\, e_M(s, X^{\mathbb{G},\pi_2}_{s^-}) - \pi_1\, e_M(s, X^{\mathbb{G},\pi_1}_{s^-}) \right) \mathrm{d}s \\
&\qquad + \mathbb{1}_N \cdot \sum_{I\in\mathcal{N}} \int\limits_{(t\vee\tau, T\wedge\sigma]\times E_I} \kappa(s) \left(B_I(s, e, X^{\mathbb{G},\pi_2}_{s^-}) - B_I(s, e, X^{\mathbb{G},\pi_1}_{s^-}) \right) \mu_I(\mathrm{d}(s, e)) \Bigg] \\
&= \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0, \sigma) \left(X^{\mathbb{G},\pi_2}_{\sigma} - X^{\mathbb{G},\pi_1}_{\sigma} \right)\right] \\
&\quad \mathbb{E}\Bigg[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s) \left(b_M(s, X^{\mathbb{G},\pi_2}_{s^-}) - b_M(s, X^{\mathbb{G},\pi_1}_{s^-}) \right) \mathrm{d}s \\
&\qquad + \mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, \pi_1 \left(e_M(s, X^{\mathbb{G},\pi_1}_{s^-}) - e_M(s, X^{\mathbb{G},\pi_2}_{s^-}) \right) \mathrm{d}s \\
&\qquad + (\pi_1 - \pi_2)\, \mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi_2}_{s^-})\, \gamma(\mathrm{d}s) \\
&\qquad + \mathbb{1}_N \cdot \sum_{I\in\mathcal{N}} \int\limits_{(t\vee\tau, T\wedge\sigma]\times E_I} \kappa(s) \left(B_I(s, e, X^{\mathbb{G},\pi_2}_{s^-}) - B_I(s, e, X^{\mathbb{G},\pi_1}_{s^-}) \right) \mu_I(\mathrm{d}(s, e)) \Bigg] \\
&\overset{(i)}{\leq} \mathbb{E}\Bigg[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s) \left| b_M(s, X^{\mathbb{G},\pi_2}_{s^-}) - b_M(s, X^{\mathbb{G},\pi_1}_{s^-}) \right| \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, \pi_1 \left| e_M(s, X^{\mathbb{G},\pi_1}_{s^-}) - e_M(s, X^{\mathbb{G},\pi_2}_{s^-})\right| \mathrm{d}s \\
&+ \mathbb{1}_N \cdot \sum_{I\in\mathcal{N}} \int\limits_{(t\vee\tau, T\wedge\sigma]\times E_I} \kappa(s) \left| B_I(s, e, X^{\mathbb{G},\pi_2}_{s^-}) - B_I(s, e, X^{\mathbb{G},\pi_1}_{s^-})\right| \mu_I(\mathrm{d}(s,e))\Bigg] \\
&+ (\pi_1 - \pi_2)\, \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, T\wedge\sigma]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi_2}_{s^-})\, \mathrm{d}s \right] \\
&+ \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma) \left(X^{\mathbb{G},\pi_2}_{\sigma} - X^{\mathbb{G},\pi_1}_{\sigma}\right)\right]
\end{aligned}
$$

where in (i) we introduced the absolute value in the first three summands. The difference under the integrals is positive, but in the premium summand, the multiplication with $(\pi_1 - \pi_2)$ creates a non-positivity. This is by design, as we intend to keep the summand without an absolute value and intend to use it as a upper bound in the end.

Let us proceed with similar steps as in previous proofs by using the Lipschitz condition in every one of the summands 1, 2 and 3 (in the three-lined expected value) for functions $(b_M)_M$, $(e_M)_M$ and $(B_I)_I$ respectively, and group these three summands by using a joint constant K, which will not explicitly be stated, but we keep in mind, that it originates from 3 summands and will also depend on $\pi_1 \in \mathbb{R}$.

In a slight change of notation, we include the constant K, that arises from the Lipschitz-constants and the upper bound of the compensator, into ζ and use this as the integrator. Take note, that ζ, defined as $\zeta(\mathrm{d}s) = K\mathrm{d}s$, is still monotonously increasing and has the same properties as before, but to be able to use the backward version A.3.8 of the Grönwall equality, we need a more general formulation of the integrator function.

With a minor reformulation of the integral bound and by using that $X_{s^-} = X_s$ ds *a.s.*, which is possible with our current integrator, we arrive at the inequality

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0, (t\vee\tau)\wedge\sigma) \left| X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma} - X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma}\right|\right] \\
&\leq \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, \sigma\wedge T]} \kappa(s) \left| X^{\mathbb{G},\pi_2}_{s^-} - X^{\mathbb{G},\pi_1}_{s^-}\right| \zeta(\mathrm{d}s)\right] \\
&\qquad + (\pi_1 - \pi_2)\, \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, \sigma\wedge T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi_2}_{s^-})\, \mathrm{d}s \right] \\
&\qquad + \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma) \left(X^{\mathbb{G},\pi_2}_{\sigma} - X^{\mathbb{G},\pi_1}_{\sigma}\right)\right] \\
&= \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, \sigma\wedge T]} \kappa(s) \left| X^{\mathbb{G},\pi_2}_{s} - X^{\mathbb{G},\pi_1}_{s}\right| \zeta(\mathrm{d}s)\right] \\
&\qquad + (\pi_1 - \pi_2)\, \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau, \sigma\wedge T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\, \kappa(s)\, e_M(s, X^{\mathbb{G},\pi_2}_{s^-})\, \mathrm{d}s \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma)\left(X_\sigma^{\mathbb{G},\pi_2} - X_\sigma^{\mathbb{G},\pi_1}\right)\right] \\
\overset{(i)}{\leq}\ & \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t,T]} \mathbb{1}_{\{\tau\leq s\leq\sigma\}}\, \kappa(s)\left|X_s^{\mathbb{G},\pi_2} - X_s^{\mathbb{G},\pi_1}\right| \zeta(\mathrm{d}s)\right] \\
&+ (\pi_1-\pi_2)\ \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau,\sigma\wedge T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\, \kappa(0,s)\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\,\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma)\left(X_\sigma^{\mathbb{G},\pi_2} - X_\sigma^{\mathbb{G},\pi_1}\right)\right] \\
\overset{(ii)}{\leq}\ & \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t,T]} \kappa(0,(s\vee\tau)\wedge\sigma)\left|X_{(s\vee\tau)\wedge\sigma}^{\mathbb{G},\pi_2} - X_{(s\vee\tau)\vee\sigma}^{\mathbb{G},\pi_1}\right| \zeta(\mathrm{d}s)\right] \\
&+ (\pi_1-\pi_2)\ \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\, \kappa(s)\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\,\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma)\left(X_\sigma^{\mathbb{G},\pi_2} - X_\sigma^{\mathbb{G},\pi_1}\right)\right] \\
\overset{(iii)}{=}\ & \int\limits_{(t,T]} \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,(s\vee\tau)\wedge\sigma)\left|X_{(s\vee\tau)\wedge\sigma}^{\mathbb{G},\pi_2} - X_{(s\vee\tau)\wedge\sigma}^{\mathbb{G},\pi_1}\right|\right] \zeta(\mathrm{d}s) \\
&+ (\pi_1-\pi_2)\ \mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau,\sigma\wedge T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\, \kappa(s)\, e_M(s, X_{s^-}^{\mathbb{G},\pi_2})\,\mathrm{d}s\right] \\
&+ \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,\sigma)\left(X_\sigma^{\mathbb{G},\pi_2} - X_\sigma^{\mathbb{G},\pi_1}\right)\right]
\end{aligned}
$$

which still holds for every $t \in [0,T]$.
In step (i), we first include an indicator function, which maps the same interval as the integration area. In a second step, the integration area is increased, which is possible with the non-negative integrator and while making use of $s \leq s\vee\tau$ and $\sigma\wedge T \leq T$.
In step (ii), further reformulations are taking place. We are using that

$$
\mathbb{1}_{\{\tau\leq s\leq\sigma\}}\, \kappa(s)\cdot\left|X_s^{\mathbb{G},\pi_2} - X_s^{\mathbb{G},\pi_1}\right| \leq \kappa(0,(s\vee\tau)\wedge\sigma)\cdot\left|X_{(s\vee\tau)\wedge\sigma}^{\mathbb{G},\pi_2} - X_{(s\vee\tau)\wedge\sigma}^{\mathbb{G},\pi_1}\right|.
$$

In the two cases $t \leq s < \tau$ and $\sigma < s \leq T$, the left hand side is zero and the inequality is trivially fulfilled, as the right hand side would be non-negative. For $\tau \leq s \leq \sigma$, equality holds on both sides, as we are simply in the case, where $(s\vee\tau)\wedge\sigma = s$. Take note, that in a stand alone view without the indication function, this would not hold, as we have $\kappa(t) \leq \kappa(r)$ for $t \geq r$ as a natural property of the discounting.
The last reformulation in step (iii) is a consequence of the Theorem of Fubini-Tonelli. Once we have gotten rid of the random integral bounds by working with the indication function, the order of integration can be exchanged and we arrive at the same representation as on the left side of the equation.

The above inequality holds for every $t \in [0, T]$ and in a simplified form then reads

$$f(t) \leq \alpha(t) + \int\limits_{(t,T]} f(s)\,\zeta(\mathrm{d}s)$$

with the following functions

$$\begin{aligned} f(t) &= \mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,(t\vee\tau)\wedge\sigma)\left|X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma} - X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma}\right|\right] \\ \alpha(t) &= \underbrace{(\pi_1-\pi_2)}_{<0}\cdot \underbrace{\mathbb{E}\left[\mathbb{1}_N \cdot \int\limits_{(t\vee\tau,\sigma\wedge T]} \sum_{M\in\mathcal{M}} \mathbb{I}^M_{s^-}\,\kappa(s)\,e_M(s,X^{\mathbb{G},\pi_2}_{s^-})\,\mathrm{d}s\right]}_{\geq 0} \\ &\quad + \underbrace{\mathbb{E}\left[\mathbb{1}_N\cdot\kappa(0,\sigma)\left(X^{\mathbb{G},\pi_2}_{\sigma} - X^{\mathbb{G},\pi_1}_{\sigma}\right)\right]}_{\leq 0} \end{aligned}$$

where α in total is a non-positive function.
Now, we can apply the backward version of the inequality of Grönwall from Corollary A.3.8 in a simplified form since our integration is done with respect to $\mathrm{d}s$, as $\zeta(\mathrm{d}s) = K\,\mathrm{d}s$. Then the inequality

$$\begin{aligned} f(t) &\leq \alpha(t) + e^{-Kt}\int\limits_{(t,T]} e^{Ks}\,\alpha(s)\,\zeta(\mathrm{d}s) \\ &= \alpha(t) + K\,e^{-Kt}\int\limits_{(t,T]} e^{Ks}\,\alpha(s)\,\mathrm{d}s \\ &\leq 0 \end{aligned}$$

holds for every $t \in [0, T]$.
From the definition of f, we conclude that for every $t \in [0, T]$ we have

$$\mathbb{E}\left[\mathbb{1}_N \cdot \kappa(0,(t\vee\tau)\wedge\sigma)\cdot\left|X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma} - X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma}\right|\right] = 0$$

and therefore, by the non-degeneracy of the expectation, we also have

$$X^{\mathbb{G},\pi_1}_{(t\vee\tau)\wedge\sigma}(\omega) = X^{\mathbb{G},\pi_2}_{(t\vee\tau)\wedge\sigma}(\omega)$$

for every $t \in [0, T]$ and $\omega \in N$.

By evaluating of the last equation only for $\omega \in N$ and for $t \in [\tau(\omega), \sigma(\omega)]$, where $(t \vee \tau(\omega)) \wedge \sigma(\omega) = t$, we get that

$$X^{\mathbb{G},\pi_1}_t = X^{\mathbb{G},\pi_2}_t$$

on N. As an implication, we have that especially $X^{\mathbb{G},\pi_2}_\tau = X^{\mathbb{G},\pi_2}_\tau$ on N, which can only be case, if N is a null set, as this was a direct consequence of our assumption and the construction of τ, i.e. formula (4.5.10). A contradiction has been found.

This now implies $\mathbb{P}(\sigma > 0) = 0$, as our assumption has to have been false. Therefore we have $\sigma = 0$ *a.s.* and by definition of σ, we know that it holds

$$X_s^{\mathbb{G},\pi_2} \le X_s^{\mathbb{G},\pi_1} \ a.s.$$

for every $s \in [0,T]$. The (non-strict) monotonicity has now been shown, and completes the first part of this proof.

Strict monotonicity

Let us now continue to show the strict monotonicity. As a the first result, we have already shown, that for every $s \in [0,T]$ it holds

$$X_s^{\mathbb{G},\pi_2} \le X_s^{\mathbb{G},\pi_1} \ a.s.$$

which is a direct consequence of the definition of σ.
Therefore, we also have

$$X_{s^-}^{\mathbb{G},\pi_2} \le X_{s^-}^{\mathbb{G},\pi_1} \ a.s.$$

for every $s \in (0,T]$, which can now be used together with the monotonicity condition of the payments.

Then, by using a similar structure to before, and by using the Assumption 4.5.5 on the monotonicity of payments , we get that

$$\begin{aligned}
&\mathbb{E}\left[X_{0^-}^{\mathbb{G},\pi_2} - X_{0^-}^{\mathbb{G},\pi_1}\right] \\
&\overset{(4.5.7)}{=} \mathbb{E}\left[X_T^{\mathbb{G},\pi_2} - X_T^{\mathbb{G},\pi_1}\right] \\
&\qquad + \mathbb{E}\Bigg[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\,\kappa(s)\left(b_M(s,X_{s^-}^{\mathbb{G},\pi_2}) - b_M(s,X_{s^-}^{\mathbb{G},\pi_1})\right)\mathrm{d}s \\
&\qquad\qquad + \int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\,\kappa(s)\left(\pi_1\,e_M(s,X_{s^-}^{\mathbb{G},\pi_1}) - \pi_2\,e_M(s,X_{s^-}^{\mathbb{G},\pi_2})\right)\mathrm{d}s \\
&\qquad\qquad + \sum_{I\in\mathcal{N}} \int\limits_{[0,T]\times E_I} \kappa(s)\left(B_I(s,e,X_{s^-}^{\mathbb{G},\pi_2}) - B_I(s,e,X_{s^-}^{\mathbb{G},\pi_1})\right)\mu_I(\mathrm{d}(s,e))\Bigg] \\
&\overset{(4.5.5)}{\le} \mathbb{E}\Bigg[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\,\kappa(s)\,\pi_1\left(e_M(s,X_{s^-}^{\mathbb{G},\pi_1}) - e_M(s,X_{s^-}^{\mathbb{G},\pi_2})\right)\mathrm{d}s \\
&\qquad\qquad + \int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\,\kappa(s)\,(\pi_1-\pi_2)\;e_M(s,X_{s^-}^{\mathbb{G},\pi_2})\,\mathrm{d}s\Bigg] \\
&\le (\pi_1-\pi_2)\;\mathbb{E}\Bigg[\int\limits_{[0,T]} \sum_{M\in\mathcal{M}} \mathbb{I}_{s^-}^M\,\kappa(s)\,e_M(s,X_{s^-}^{\mathbb{G},\pi_2})\,\mathrm{d}s\Bigg]
\end{aligned}$$

$$\overset{(4.5.5)}{\leq} (\pi_1 - \pi_2)\, c$$
$$< 0$$

where the last and second to last step is an application of condition 4.5.5 on the interval $[0, T]$, and where we used again, that the reserve-dependency does not happen in 0.

These calculations imply that

$$\mathbb{E}\left[X_{0^-}^{\mathbb{G},\pi_2}\right] < \mathbb{E}\left[X_{0^-}^{\mathbb{G},\pi_1}\right]$$

and we can now conclude that

$$F(\pi_2) < F(\pi_1)$$

which means, that our function F is strictly monotonously decreasing, and we therefore know that only a unique $\pi \in \mathbb{R}$ can fulfil $F(\pi) = 0$. This was the last part, that remained to be shown and now finishes the proof. □

This now means, that we do not only have existence and uniqueness of the payment process and the prospective reserve, but we can only price a contract, with a unique premium level, that exists as a scaling factor to an existing premium payment scheme.
We have to restrict ourselves to continuous premium payments (and sojourn payments in general), but this is only because of the reserve-dependency. A one-off premium payment is still possible, as it also does not depend on the reserve of the full payment process, but the one with only benefits.

Comment 4.5.9. *On the calculation of a premium level given a payment scheme*
The existence and uniqueness of the pricing factor π does not directly imply that a calculation can be done in a simple and inexpensive way.
The methodology of the proof, however, enables us to use numerical methods for the function $F(\pi)$, and make use of the already shown monotonicity and the continuity. The bisection method or Regula-falsi (false position) method can be used in an iteration to calculate the factor π, by starting with two start values, where F has an opposite sign. Such values should not be too hard to find and the monotonicity also guarantees linear convergence to the solution. The numerical details will not be formulated, but remember that the evaluation of F for a given π in each step of the iteration is not easily done by itself, since the iteration for the payment process and the reserve has to be used. It is therefore beneficial if the initial values of π are chosen, such that $F(\pi)$ is already close to zero.
Pre-calculation and easier evaluation may be possible, when the reserve-dependency is rather simple and parts of the payments do not even depend on the reserve at all, but we are not going to formulate these cases.

Chapter 5.

Examples and applications

In this chapter, we want to provide some examples, to which the theoretical results in this thesis can be applied to. For the development of examples, we will reformulate some of the examples from Djehiche and Löfdahl [DL16] and rewrite them in the notation of our model. At the end of this chapter, we will provide an overview of other applications, where we refrain from an explicit mathematical formulation.

The examples will give a little more context to the abstract life insurance structure from before, since we are now specifying the details of the contracts and define exemplary payments. We especially want to provide examples, where our main theorems can be applied, to provide a justification for the existence of our theory, but note that the examples are rather based on information restriction than non-monotone information.

5.1. Life policy with a guaranteed minimum death payment

Consider a life police, where the payment in case of the death of the policy holder is linked to the prospective reserve of the contract. A minimal payment is set for the case, that the existing reserve is lower than this threshold. This creates a simple non-linear dependence on the existing prospective reserve and we now look at the mathematical details of this example.

Example 5.1.1. *Life policy with guaranteed minimum payment*
Consider the finite state space $\mathcal{S} = \{a, d\}$, where a abbreviates 'active' and d abbreviates 'dead'. Let $S(t) : [0, \infty) \to \mathcal{S}$ be the insurance state of an insured person. $S(t)$ is a pure jump process and we assume it to be right continuous.
The full information setting is given as the natural filtration of S, with sigma-algebras

$$\mathcal{F}_t := \sigma\left(S(s), 0 \leq s \leq t\right)$$

and we do not consider any other information structure is this example, i.e. we set $\mathbb{F} = \mathbb{G}$.
Define

$$\eta = \inf\left\{u \geq 0 \mid S(u) = d\right\}$$

as a stopping time for the event of death (where $\inf \emptyset := \infty$). We want to keep as close as possible to the notation of Christiansen [Chr21b], although we only need $\mathbb{F}$ in this example.

Use $E = [0,\infty) \times \mathcal{S}$ as the information space of the marked point process and define the following times as part of the marked point process $(T_i, Z_i)_{i\in\{0,\ldots,4\}}$. Let T be the deterministic maximum contract time.

The marked point process is then given by

i	T_i	Z_i	Information
1	0	$(0,a)$	start in 0, active
2	∞	$(0,a)$	no deletion
3	η	(η,d)	time of transition, dead
4	∞	(η,d)	no deletion

Table 5.1.: Example 5.1.1 – Details on the marked point process

and as contractual payments, we define a continuous premium payment in state 'active', which is going to be paid up to time η and death benefit payment for the transition from 'active' to 'dead', if the transition happens prior to the final contract time T. The death benefit payment will be reserve dependent in a non-linear way. The details are now specified in the following two tables for the sojourn payment

$M \in \mathcal{M}$	Information	$b_M(s, X^{\mathbb{G}}_{s^-})$	$\mathbb{I}^M_{s^-} = 1$	Interpretation
$\{1\}$	'active'	$-\pi(s)\,\mathbb{1}_{\{s\leq T\}}$	$0 < s \leq \eta$	continuous premium

Table 5.2.: Example 5.1.1 – Overview of the sojourn payments

and the benefit payment (with $\mathbb{I}^{\{1\}}_{\eta^-} = 1$)

$I \in \mathcal{N}$	$e \in E_I$	Information	$B_I(s, e, X^{\mathbb{G}}_{s^-})$	Interpretation
$\{3\}$	(η_d, d)	'active' $\to$ 'dead' in η	$g_{\alpha_f,\alpha_p}(s, X^{\mathbb{G}}_{s^-})$	death payment

Table 5.3.: Example 5.1.1 – Overview of the transition payments

where we use the following payment

$$g_{\alpha_f,\alpha_p}(s,x) := \max\{\alpha_f(s),\, \alpha_p(s)\, x\} \cdot \mathbb{1}_{\{s\leq T\}}$$

in case of death, which is non-linearly reserve-depending on the prospective reserve with respect to $\mathbb{F}$.
Interpret

$$\alpha_f(s) : [0,\infty) \to \mathbb{R}_{>0}$$

as a guaranteed minimum payout and

$$\alpha_p(s) : [0,\infty) \to (0,1)$$

as a proportional reduction of the paid-out reserve. Assume that $\alpha_p(s) \in (0,1)$ for all s, since we would not pay out more than the reserve, but we do not specify the functions any further.

The accumulated future payments are given as the process $X = (X_t)_{t\geq 0}$, in its integral representation

$$\begin{aligned} X_t &= \sum_{M\in\mathcal{M}} \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)} \mathbb{I}^M_{s-} \, b_M(s, X^{\mathbb{G}}_{s-}) \, \gamma(\mathrm{d}s) + \sum_{I\in\mathcal{N}} \int\limits_{(t,T]\times E_I} \frac{\kappa(s)}{\kappa(t)} \, B_I(s, e, X^{\mathbb{G}}_{s-}) \, \mu_I(\mathrm{d}(s,e)) \\ &= -\int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)} \, \mathbb{I}_{\{s\leq\eta_d\}} \, \pi(s) \, \mathrm{d}s \; + \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)} \, g_{\alpha_f(s),\alpha_p(s)}(s, X^{\mathbb{G}}_{s-}) \, \mu_{\{3\}}(\mathrm{d}(s,(\eta,d))) \, . \end{aligned}$$

with a final condition $X_T = 0$ *a.s.*, and our Theorem 3.2.12 guarantees the existence of the payment process in the case of $\mathbb{F} = \mathbb{G}$.
We have to verify, that all involved functions $(b_M)_M$ and $(B_I)_I$ fulfil the necessary conditions. Measurability is given by definition. We only have to check the Lipschitz condition of $g_{\alpha_f,\alpha_p}(s,x)$, which can be done by case differentiation.

Comment 5.1.2. *On the extension to multiple absorbent states*
The model can be considered with multiple absorbent states, for example a dread-disease insurance contract. In that case, we extend the state space to $\mathcal{S} = \{a,\, d_1, \ldots, d_m\}$, but it is restricted to absorbent states, where the prospective reserve vanishes (apart from state a).

Comment 5.1.3. *On the differences in the formulation to standard life insurance theory*
Let us quickly comment on some major differences to the common multi-state life insurance theory and especially to the examples developed in Djehiche and Löfdahl [DL16], who define

$$\alpha_{ij}(t) = \max\left\{\alpha^0_{ij}(t), \alpha^1_{ij}(t)\,(V_j(t) - V_i(t))\right\}$$

as transition benefits for $i \neq j \in \mathcal{S}$, with state-wise reserves, but without defining $\mathcal{S}$, in comparison to our model, where the state space $\mathcal{S}$ is explicitely defined.

Every transition payment has the same structure and the differences of the current and future state-wise prospective reserve are used as part of this payment. This is then translated into a dependency on the family $(Z_{ij})_{i\neq j}$, where $Z_{ij}(t) = V_j(t) - V_i(t)$, while we do not consider state-wise reserves at all and restrict ourselves to payments that happen in case of a transition $a \to d$.
Also note, that by using the following representation of the general reserve

$$V_{S(t)}(t) = \sum_{i\in\mathcal{S}} \mathbb{I}_{\{S(t)=i\}}(t)\, V_i(t)$$

and by simplifying it with the condition that some reserves may be zero, it holds for $\mathcal{S} = \{a,\, d\}$ (with $V_d(t) \equiv 0$) as

$$V_{S(t)}(t) = \mathbb{I}_a(t)\, V_a(t) = \mathbb{I}_a(t)\; (V_a(t) - V_d(t))$$

which enables us to use the general reserve $X^{\mathbb{G}}_{t-}$, which is also just the reserve in active for $t \leq \eta_d$, instead of working with an equivalent of the sums at risk $(Z_{ij})_{i\neq j}$.

5.2. Guaranteed life endowment with a withdrawal option

Consider a slightly more complicated contract next, where the payment in case of death is still reserve-dependent, with a minimal payout set. We also want to allow for the insured to lapse the contract (withdraw from the contract) and get paid back the reserve.

Example 5.2.1. *Guaranteed life endowment with withdrawal option*
Consider the finite state spaces $\mathcal{S} = \{a, l, d\}$, where a abbreviates 'active', l abbreviates 'lapsed', and d abbreviates 'dead'.
Let $S(t) : [0, \infty) \to \mathcal{S}$ be the insurance state of an insured person. $S(t)$ is a pure jump process and we assume it to be right continuous. The setting of full information is given by the natural filtration of S, i.e. for $t \geq 0$ we define

$$\mathcal{F}_t := \sigma\left(S(s), 0 \leq s \leq t\right)$$

No other information structure is needed, so we set $\mathbb{G} = \mathbb{F}$. By defining

$$\eta_d = \inf\{u \geq 0 \mid S(u) = d\}, \qquad \eta_l = \inf\{u \geq 0 \mid S(u) = l\}$$

we have stopping times for the events of death and lapse (with $\inf \emptyset = \infty$) and use $\eta = \min\{\eta_l, \eta_d\}$.
Use $E = [0, \infty) \times \mathcal{S}$ as the information space of the marked point process and define the following times as part of the marked point process $(T_i, Z_i)_{i \in \{0,\dots,4\}}$. Again, let the deterministic time T be the maximum contract time.

The marked point process is then given by

i	T_i	Z_i	Information
1	0	$(0, a)$	start in 0, active
2	∞	$(0, a)$	no deletion
3	η	(η, d) or (η, l)	time of transition, lapse or dead
4	∞	(η, d) or (η, l)	no deletion

Table 5.4.: Example 5.2.1 – Details on the marked point process

Specify a continuous premium payments in state 'active', up to η, as a rate $\pi(s)$. The details are given in the following table for the sojourn payments

$M \in \mathcal{M}$	Information	$b_M\left(s, X^{\mathbb{G}}_{s-}\right)$	$\mathbb{I}^M_{s-} = 1$	Interpretation
$\{1\}$	'active'	$-\pi(s) \cdot \mathbb{1}_{\{s \leq T\}}$	$0 < s \leq \eta$	continuous premium

Table 5.5.: Example 5.2.1 – Overview of the sojourn payments

$I \in \mathcal{N}$	$e \in E_I$	Information	$B_I(s, e, X^{\mathbb{G}}_{s-})$	Interpretation
$\{3\}$	(η, d)	'active' $\to$ 'dead' in η_d	$\alpha_{ad}(s, X^{\mathbb{G}}_{s-})$	death payment
$\{3\}$	(η, l)	'active' $\to$ 'lapse' in η_l	$\alpha_{al}(s, X^{\mathbb{G}}_{s-})$	lapse payment

Table 5.6.: Example 5.2.1 – Overview of the transition payments

and as transition payments, we will have reserve-dependent death benefit and lapse payments, if the respective transition happens prior to T. These payment will be reserve dependent in a non-linear way and the details are given in the table (for $\mathbb{I}^{\{1\}}_{\eta^-} = 1$) and where we use the death payment

$$\alpha_{ad}(s, x) := \max\left\{\alpha^1_{ad}(s),\, \alpha^2_{ad}(s)\, x\right\} \mathbb{1}_{\{s \le T\}}\,,$$

which is non-linearly depending on the prospective reserve with respect to $\mathbb{G}$. Interpret $\alpha^1_{ad}(s) : [0, \infty) \to \mathbb{R}_{>0}$ as a guaranteed minimum, and $\alpha^2_{ad}(s) : [0, \infty) \to [0, 1)$ as a proportional reduction of the reserve. Further we use

$$\alpha_{al}(s, x) := \left(\alpha^0_{al}(s) + \max\left\{\alpha^1_{al}(s),\, \alpha^2_{al}(s)\, x\right\}\right) \cdot \mathbb{1}_{\{s \le T\}}$$

as the lapse payment, which is also non-linearly depending on the prospective reserve with respect to $\mathbb{G}$. Interpret $\alpha^0_{al}(s) : [0, \infty) \to \mathbb{R}_{>0}$ as a fixed payout and $\alpha^1_{al}(s) : [0, \infty) \to \mathbb{R}_{>0}$ together with $\alpha^2_{al}(s) : [0, \infty) \to [0, 1]$ as a proportional reduction of the paid-out reserve with a guaranteed minimum.

The accumulated discounted future payments are given by the process $X = (X_t)_{t \ge 0}$, in its integral representation

$$\begin{aligned}
X_t &= \sum_{M \in \mathcal{M}} \int\limits_{(t,T]} \mathbb{I}^M_{s^-} \frac{\kappa(s)}{\kappa(t)}\, b_M(s, X^{\mathbb{F}}_{s-})\, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int\limits_{(t,T] \times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I(s, e, X^{\mathbb{F}}_{s-})\, \mu_I(\mathrm{d}(s, e)) \\
&= - \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \mathbb{I}_{\{s \le \eta\}}\, \pi(s)\, \mathrm{d}s \\
&\quad + \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \alpha_{ad}(s, X^{\mathbb{F}}_{s-})\, \mu_{\{3\}}(\mathrm{d}(s, (\eta, d))) + \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, \alpha_{al}(s, X^{\mathbb{F}}_{s-})\, \mu_{\{3\}}(\mathrm{d}(s, (\eta, l))).
\end{aligned}$$

with final condition $X_T = 0$ *a.s.*, where Theorem 3.2.12 guarantees the existence and uniqueness of the reserve-dependent payment process. We have to verify, that all involved processes $(b_M)_M$ and $(B_I)_I$ fulfil the necessary conditions, which can once again be done by case differentiation.

5.3. A unisex tariff in life insurance with lapse

We now want to investigate unisex (or gender neutral) tariffs in a life insurance setting. This is combined with a non-linear dependency of the lapse payment on a prospective reserve and with a restricted information setting, as information about the gender is not considered in the calculations of premiums and other payments, since it forbidden by law.

In many cases, it is still valuable for the insurer to differentiate between male and female customers. For instance, data availability of morbidity (with significant differences) and lapsing behaviour (also because of adverse selection problems of non-fairly priced contracts) is usually differentiated. Despite these incentives to internally differentiate these contracts, the European Union has passed a law, that does not allow to sell differentiated contracts. For details on the council directive about 'the equal treatment of men and women in the access to and supply of goods and services', see [Eur04][1], which is the original EU ruling for gender equality in 2004. Additional clarifications about the application to insurance contracts and the deletion of the exception paragraph can be found in the statement [Eur11][2], after a ruling of the European court of justice on the application of gender equality for insurance contracts.

The information about gender of an insured can not be used to calculate premiums or payments, resulting in the term 'unisex-tariffs'. Internally, the insurer might want to keep the gender for the calculations, if data is available and significant differences are present. Otherwise, the insurer would increase its own exposition to risk.

Example 5.3.1. *Unisex tariff with lapse*
Consider the two-dimensional state process

$$S(t) = (S_1(t), S_2(t)) : [0, \infty) \to \mathcal{S} = \{a,\, l,\, d\} \times G\,,$$

that includes the insurance state and the gender of an insured person.
In addition to the classical insurance state spaces $\{a,\, l,\, d\}$ with absorbing state l and d, we consider the finite gender space $G = \{m,\, f\}$, where m abbreviates male and f female. A two-dimensional state space is created, where for example the entry $(a, m) \in \mathcal{S}$ means that an insured person is both active and male. The gender selection is considered to be random at the start of the contract.
The process $S = (S(t))_{t\geq 0}$ is assumed to be a right-continuous pure jump process, where the second component stays constant as the gender is considered non-changing. It is important to note, that jumps may only happen in the first component of the process S. The full information setting is given by the natural filtration of S, where the sigma-algebras are defined by

$$\mathcal{F}_t := \sigma\left(S(s),\, s \leq t\right) = \sigma\left(\left(S_1(s), S_2(s)\right),\, s \leq t\right) = \sigma\left(\left(S_1(s), S_2(t)\right),\, s \leq t\right)\,,$$

[1] `http://data.europa.eu/eli/dir/2004/113/oj`, (accessed 12.2024)
[2] `https://ec.europa.eu/commission/presscorner/detail/en/memo_11_123`, (accessed 12.2024)

and if we do not want to consider the gender for our calculations, we can leave out the second component and work with the reduced information

$$\mathcal{G}_t := \sigma\left(S_1(s),\, s \leq t\right),$$

which is still a filtration, i.e. monotonously increasing and the information about the gender is not deleted.
Although it is still monotone, it clearly contains less information than $\mathbb{F} = (\mathcal{F}_t)_t$ by design and it definitely holds $\mathcal{G}_t \subset \mathcal{F}_t$ for every $t \geq 0$, but note that $\mathbb{G}$ does now not correspond to the definition in Christiansen [Chr21b].

Define

$$\eta_l = \inf\left\{u \geq 0 \mid S_1(u) = l\right\}, \qquad \eta_d = \inf\left\{u \geq 0 \mid S_1(u) = d\right\}$$

as stopping times for the events of death and lapse (with $\inf \emptyset := \infty$), which are stopping times with respect to both sigma-algebras, since only the first component of S is used. For simplicity, also define $\eta = \min\{\eta_d,\, \eta_l\}$ and use T as the deterministic maximum contract time.

It still is the target to keep a structure, such that we can formulate $\mathbb{F}$ in the notation of [Chr21b]. We therefore use $E = [0, \infty) \times \mathcal{S}$ as the information space of the marked point process and define the following times as part of the marked point process $\left(T_i, Z_i\right)_{i \in \{0,\ldots,4\}}$.

The marked point process can be represented by:

i	T_i	Z_i	Information
1	0	$(0, a, m/f)$	start in 0, active, gender
2	∞	$(0, a, m/f)$	no deletion
3	η	$(\eta, d, m/f)$ or $(\eta, l, m/f)$	time of transition, lapse or dead, gender
4	∞	$(\eta, d, m/f)$ or $(\eta, l, m/f)$	no deletion

Table 5.7.: Example 5.3.1 – Representation of the marked point process

The payments have to be the same for both genders. This is naturally the case, if the payment is deterministic, but does not have to be the case for reserve-dependent payments.

A continuous premium payment rate and the deterministic survival benefit are presented in the following table about the sojourn payments

$M \in \mathcal{M}$	Information	$b_M(s, X^{\mathbb{G}}_{s-})$	$\mathbb{I}^M_{s-} = 1$	Interpretation
$\{1\}$	'active'	$-\pi \cdot \mathbb{1}_{\{s \leq T\}}$	$0 < s \leq \eta$	continuous premium
$\{1\}$	'active'	$S \cdot \mathbb{1}_{\{s = T\}}$	$0 < s \leq \eta$	survival benefit (in $t_0 := T$)

Table 5.8.: Example 5.3.1 – Overview of the sojourn payments

where we need to set $t_0 = T$ as a deterministic time point of the Dirac-measure in the definition of γ, to allow for the lump sum payments, but no reserve-dependency in this survival benefit takes place.

A deterministic death benefit D for the transition from 'active' to 'dead' and a reserve-dependent lapse payments for the transition from 'active' to 'lapse' are specified in the following table of transition payments (for $\mathbb{I}_{\eta^-}^{\{1\}} = 1$)

$I \in \mathcal{N}$	$e \in E_I$	Information	$B_I(s, e, X_{s^-}^{\mathbb{G}})$	Interpretation
$\{3\}$	(η, d, m)	a $\to$ d in η, male	$D\,\mathbb{1}_{\{s \leq T\}}$	death payment
$\{3\}$	(η, d, f)	a $\to$ d in η, female	$D\,\mathbb{1}_{\{s \leq T\}}$	death payment
$\{3\}$	(η, l, m)	a $\to$ l in η, male	$f_{\alpha,\beta}(s, X_{s^-}^{\mathbb{G}})$	lapse payment
$\{3\}$	(η, l, f)	a $\to$ l in η, female	$f_{\alpha,\beta}(s, X_{s^-}^{\mathbb{G}})$	lapse payment

Table 5.9.: Example 5.3.1 – Overview of the transition payments

where we use the lapse payment

$$f_{\alpha,\beta}(s, x) := ((1 - \beta) \cdot x - \alpha)^+ \,\mathbb{1}_{\{s \leq T\}} = \max\{0,\, (1 - \beta) \cdot x - \alpha\}\,\mathbb{1}_{\{s \leq T\}}\,,$$

which is non-linearly depending on the prospective reserve with respect to $\mathbb{G}$. Interpret $\alpha \in \mathbb{R}_{\geq 0}$ as a general lapse fee, used for the administrative effort, and $\beta \in [0, 1)$ as a proportional lapse fee, where both fees are constant in time.

The accumulated discounted future payments are given as the process $X = (X_t)_{t \geq 0}$ in its integral representation

$$\begin{aligned}
X_t &= \sum_{M \in \mathcal{M}} \int\limits_{(t,T]} \mathbb{I}_{s^-}^M \frac{\kappa(s)}{\kappa(t)}\, b_M(s, X_{s^-}^{\mathbb{G}})\, \gamma(\mathrm{d}s) + \sum_{I \in \mathcal{N}} \int\limits_{(t,T] \times E_I} \frac{\kappa(s)}{\kappa(t)}\, B_I(s, e, X_{s^-}^{\mathbb{G}})\, \mu_I(\mathrm{d}(s, e)) \\
&= -\int\limits_{(t,T]} \mathbb{I}_{\{s \leq \eta\}} \frac{\kappa(s)}{\kappa(t)} \pi \mathrm{d}s + \int\limits_{(t,T]} \mathbb{I}_{\{s \leq \eta\}} \frac{\kappa(s)}{\kappa(t)} S \cdot \mathbb{1}_{\{s = T\}} \gamma(\mathrm{d}s) \\
&\qquad + \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, D \left(\mu_{\{3\}}(\mathrm{d}(s, (\eta, d, m))) + \mu_{\{3\}}(\mathrm{d}(s, (\eta, d, f)))\right) \\
&\qquad + \int\limits_{(t,T]} \frac{\kappa(s)}{\kappa(t)}\, f_{\alpha,\beta}(s, X_{s^-}^{\mathbb{G}}) \left(\mu_{\{3\}}(\mathrm{d}(s, (\eta, l, m))) + \mu_{\{3\}}(\mathrm{d}(s, (\eta, l, f)))\right)
\end{aligned}$$

with final value $X_T = 0$ *a.s.*, and where $t_0 = T$ is the only time point for discrete sojourn payments and therefore a part of γ.

Theorem 3.2.12 can be applied to this situation and it guarantees that the reserve-dependent payment process exists and is unique. We can verify, that all involved payments fulfil the necessary conditions. Measurability is given by definition and we only have to check the Lipschitz condition of $f_{\alpha,\beta}(s, x)$. The reserve might actually be smaller than $\frac{\alpha}{1-\beta}$ (then $f_{\alpha,\beta} = 0$). This does have the implication, that the Lipschitz condition has to be checked for different cases (otherwise $L = 1 - \beta \in (0, 1]$ works, as is is the gradient of the linear function) and we could just use the upper bound $L = 1$ as a general Lipschitz constant for all payments.

The lapse payment is now the same for a male and a female insured, because it depends on the prospective reserve without gender information. This is exactly what we wanted to construct. As it can be seen in the equation for X_t, the possible differences in the integrators are considered, when the prospective reserve with respect to $\mathbb{F}$ is calculated. Another interesting thing is that lapse payment with the reserve $X^{\mathbb{G}}$ usually lies in-between the correct gender-based reserves and therefore the actual internal reserves are higher and lower respectively, than the reserve from the surrender payment, which explains why the insurer wants to keep the full information setting for the internal calculations.

The process X_t is neither adapted to $\mathbb{G}$ nor $\mathbb{F}$ and we have to use the optional projections instead. For both filtrations $\mathbb{G}$ and $\mathbb{F}$, the optional projections do exist, and the Thiele BSDE with respect to the standard formulation for filtrations can be formulated. We will not do this, as the sums at risk can not be represented in the notation of Christiansen [Chr21b]. Only a formulation of the Thiele BSDE with respect to $\mathbb{F}$ could be done, but we do not do it here, as it provides hardly any benefit compared to the representation of the payment process itself. As a final condition, to be able to solve the equation for the prospective reserve numerically, we would have $X_T^{\mathbb{G}} = 0 = X_T^{\mathbb{F}}$.

Theorem 4.5.4 can also be applied to guarantee the existence of a unique net equivalent premium, which we implicitly already used with the given continuous premium π. That a net equivalent premium exists would otherwise not have been guaranteed, as an application of Cantelli's theorem is not possible with the reduced lapse payment and without specifying the model any further. Note that the Theorem guarantees the existence of a premium level π as a multiplicative factor for the premium payment scheme, which in this case would be the constant rate $\mathbb{1}_{s\leq T}$ while in state active. Therefore, the precondition in formula (4.5.5) is fulfilled, if $\mathbb{P}(\eta > 0) > 0$, which would be the case for a reasonable model.

Comment 5.3.2. *Possible embedding into the non-monotone theory*
This model does not use the non-monotone information structure and it would be difficult to formulate an example with a proper notation, that actually utilizes a non-monotone structure.

Nevertheless, there is a possibility to embed this example into the theory of non-monotone information, but this approach is unintuitive and will not be continued in this thesis.
To keep the necessary notation in Christiansen [Chr21b], and have both $\mathbb{G}$ and $\mathbb{F}$ align with the definition, a different approach has to be followed for the gender information. The information about the gender could be introduced in $T_1 = 0$ and deleted in $T_2 = 1$, but the actual contract only takes place on the interval $[2, T + 2]$, and payments are defined accordingly. Then, the definition of $\mathbb{F}$ and $\mathbb{G}$ can be aligned with the needed structure, as $\mathcal{G}_t$ for $t \geq 2$ would never have the information about the gender.
In such a situation, the Theorem 3.3.9 could also be used, as the possibility for an inclusion of the sum at risk into the lapse payment would then be possible, instead of using the general reserve.

5.4. Other examples

In this section, a few short additional examples are provided. We are only going to specify the general idea, and argue the potential applications and limitations for the theory in this thesis, but we are not going to give a full mathematical formulation of the considered models.

5.4.1. German private health insurance

The private health insurance system in Germany is modelled in a unique way and would usually not be considered a part of (multi-state) life insurance theory. Nevertheless, the general life insurance model can also be represented as a multi-state model.

In the standard model, the private health insurance relies on the so called 'Kopfschaden' per age, which would be something like an average of yearly health cost per age. This is similar to a conditional expectation, like we need in our life insurance model.

The private health insurance uses a deterministic forecast for the actual medical expenses for every year of the contract, and the situation allows for a later adaptations of the premiums, if the forecast differs in a significant way. It can be considered a model with information reduction, where the actual true cash flow is stochastic and not observable and the insurance companies calculate the individual claims based on the industry values, but not their own company values. These industry values are provided by the KVAV (supervisory unit for the private health insurance, which provides the regulations), in exchange for participation of the insurance company in the collection of all data.

The original model is set up in a way, that the only randomness lies in the potential occurrence of death or lapse, but al the payments are given as deterministic, and we could use a state space $\mathcal{S} = \{a,\, l,\, d\}$. If we want to model it differently and stochastic, then the following would be a potential application of restricted information structures.

We can model a two-dimensional right-continuous jump process $S = (S_t^1, S_t^2)_{t\geq 0}$ by specifying

$$S = (S^1, S^2) : [0,\infty) \longrightarrow \mathcal{S} = \{a,\, l,\, d\} \times K\,,$$

where the first component is specifying the state of an insured person, as a standard state from $\mathcal{S} = \{a,\, d,\, l\}$ (active, dead, lapse), and the second component S^2 is a health related state, where medial costs of B_k are arising.

The filtration of the complete information $\mathbb{F} = \left(\mathcal{F}_t\right)_{t\geq 0}$ is given by

$$\mathcal{F}_t := \sigma\left(S_r,\, r \leq t\right) = \sigma\left(\left(S_r^1, S_r^2\right),\, 0 \leq r \leq t\right)$$

and could be used for the individual performance evaluation of a contract, and the reduced information $\mathbb{G} = \left(\mathcal{G}_t\right)_{t\geq 0}$ is given by

$$\mathcal{G}_t := \sigma\left(S_r^1,\, 0 \leq r \leq t\right)$$

which is simply a restriction to the first component of S.

Note, that $\mathcal{G}_t \subset \mathcal{F}_t$ holds for every $t \geq 0$, i.e. the information structure $\mathbb{G}$ has less information than $\mathbb{F}$ at every time point, but $\mathbb{G}$ is nonetheless still a filtration.

The payment process could then be given as

$$\begin{aligned} A(\mathrm{d}t) = -\mathbb{1}_{\{S^1(t^-)=a\}} \cdot \pi(t)\,\mathrm{d}t + \sum_{k\in K} \mathbb{1}_{\{S^1(t^-)=a\}} \cdot \mathbb{1}_{\{S^2(t^-)=k\}} \cdot B_k\,\mathrm{d}t \\ + b_{al}\big(t, X^{\mathbb{G}}_{t^-}\big)\,\mathrm{d}\mu^{S^1}_{al}(t) \end{aligned}$$

where π would be a general premium payment and the B_k are the health costs for an actively insured person with health level $k \in K$. The lapse payment would be a reserve-dependent value with respect to $\mathbb{G}$. This lapse value, or transfer value, is better known as the 'Übertragswert' and it is the value, that the insured will receive, when the contract is terminated. The name is motivated by the fact, that this value is supposed to transfer to a different insurance company without reduction, which allows for healthy competition between insurance companies.

The 'Kopfschaden', or rather the difference of the 'Kopfschaden' and the premium, from the classical private health insurance model would then be something similar to a reserve with respect to $\mathbb{G}$, and where the time-development of $X^{\mathbb{G}}$ would include the changing value for the 'Kopfschaden'. The premium $\pi(t)$ would be independent of the $k \in K$, and could be calculated to be constant in time, to meet the net equivalent condition as an initial value for the reserve with respect to $\mathbb{G}$.

A proper formulation will not be done at this point, and is especially difficult because the payments in the German health insurance are usually lump sum payments, and the age for the age-based 'Kopfschaden' is also difficult to represent.

5.4.2. Life insurance with health data

In general, life insurance contracts could be used together with health data collection via wristband devices, where deletions might happen because of data privacy controls or because the information is considered outdated. This is an application where non-standard data is used in combination with a common life insurance multi-state model. A brief introduction to such a model is also given in Christiansen [Chr21b].
In other countries, mostly outside of the European Union, life insurance contracts with health tracking do already exist. The insurance companies offering these types of contracts, do not have to work with nearly as strict data protection laws, and the population has a higher acceptance for these contracts.

Healthy behaviour on the part of the insured could be rewarded, if data suggests, that the expected health costs are to be lower. This would particularly be true for cardiovascular disease, where the link to exercise, steps, resting heart rate and other metrics could be used with wristband devices.

However, we will not further elaborate on this example as it would be difficult to specify the model without knowing the impact on transition probabilities and health costs. Data availability (and non-simplicity) will always be one of the main reasons, why certain models will not go into production. For non-standard ideas, the insurer would have to collect its own data, so the model can only be developed when many clients are already using it and are providing the baseline data, to setup a model. Alternatively, without a proper model in the background, policyholders could receive fixed benefits (a bonus program) for participating in such a tariff, but a systematic and model-based reduction in premium would be difficult to implement.

An example of such a contract can be found in the 'Vitality Plus'[3] life insurance policy offered by John Hancock Life Insurance Company for the US American insurance market, which appears to be the first mention of such a product, but other companies are now offering similar tariffs. Take note that the mathematical details of this insurance policy are of course not publicly available, but the collection of health data via wristband devices is definitely a part of it and the benefits seem to be more like a bonus program than through an underlying model. Some additional information can also be found in a web article by the American Hospital Association[4], and a 30% reduction in hospitalization costs for participants in these types of insurance programmes is mentioned.

[3] Compare `https://www.johnhancock.com/life-insurance/vitality-program.html` (accessed 12.2024)

[4] `https://www.aha.org/aha-center-health-innovation-market-scan/2018-10-01-now-you-can-share-fitbit-data-save-life` (accessed 12.2024)

5.4.3. Life policies with smoking behaviour

In the German life insurance market, it is quite common to have tariffs, where a differentiation between smoking and non-smoking takes place. The German actuarial association provides a death table with differentiated risks, and also provides a note on the methodology of this data. The death table was introduced in 2008 and details can be found in [Deu08][5]. A verification of the continuous applicability can be found in [Deu22a] [6]. Take note, that application of this specific life table is only intended for insurance upon death, and where a health check is part of the underwriting process.

The collection of data for the development of life tables with smoking behaviour based on data from German insurance companies is described as difficult. The DAV research does not find significant differences between smokers and non-smokers from their local insurance portfolios. There are a variety of reasons for that. Most of the German database with a differentiation in smoking behaviour has been developed recently. Further, the portfolio distribution of smokers and non-smokers, the smoking behaviour (how often), the actual classification can influence the findings. Especially the definition of non-smoking usually only takes into account the past 12 or 24 months and therefore does not really recognize long term effects correctly, while also suffering from the problem, that most of the data can not be verified. Additionally there might be methodical differences between companies and the health care system can also influence the realised deaths. Premature deaths are happening less often, if proper health care is provided.
In comparison to the German data, other countries like the UK, USA and Canada provide age-related factors for the excess of deaths in smokers. In the end, the German death table is based on the data from foreign markets, while verifying the applicability through other parameters of the data. Details can be found in the linked note, but to summarise, the German insurance market was still not able to generate a death table based on their own data. For some cases, they also suggest the use of safety margins, and also emphasize importance of the total percentage of smokers in the total portfolio, which is one of the main risk factors.

It seems, that the underlying statistical model is inaccurate, especially about the differences of smoking and non-smoking. If the death table is used despite that, it might introduce some errors for smokers who stop smoking and are then considered non-smokers. Historically, former smokers might also be currently part of the death table statistics for non-smokers, which is part of the problem, why no significant differences are found.
The current model introduces an error, which is generally accepted, as more complex and better statistical model are not easily found anyway. It might be better to use a model, where a proper differentiation between former smokers and smokers can take place and this could be done in our model. The information about the history of a smoker can be deleted, but the effects would be accounted for through the IF-and IB-martingale dynamics. The statistical model is not clear in our model either, but the questionable assumptions about the history of smoking are not needed.

[5] `https://aktuar.de/de/wissen/fachinformationen/detail/herleitung-der-sterbetafel-dav-2008-t-fuer-lebensversicherungen-mit-todesfallcharakter` (accessed 12.2024)

[6] `https://aktuar.de/de/wissen/fachinformationen/detail/raucher-und-nichtrauchersterbetafeln-fuer-lebensversicherungen-mit-todesfallcharakter/` (accessed 12.2024)

5.4.4. Partial lapse and contract modifications

There are insurance contracts, where the conditions of the contract can be changed through the exercising of options during the contract horizon. These options include partial lapse, like for example the free policy option, where premium payment cease and the benefits need to be adjusted to reflect level of premiums paid up to the exercising time point.

Under certain conditions, such a contract modification can maintain the actuarial equivalence, and the sum at risk for the contract modification would be zero. Then, an application of Cantelli's theorem would allow to ignore the option. In such a case, the difference in reserves between the old and new contract needs to be taken into account and the old reserve could be considered a single premium for the new contract with reduced benefits. The true cost of the contract change is then reflected as the difference between the existing reserve and the future reserve.

If Cantelli's theorem cannot be applied, which is generally the case for non-Markovian models or if not the complete reserve is paid out, then the partial lapse can not be ignored in the model. It could also be problematic for the insurers asset and liability management and fees could be introduced to respond to the new reserving requirements. In general, the life insurance cash flow becomes reserve-dependent upon the contract modification and the dependency would be with respect to the exercising time point. Note, that there are also some papers that work with scaled payments to accommodate these contract modifications, see for example Furrer [Fur22] and Christiansen and Furrer [CF22]. Christiansen and Djehiche [CD20] consider multiple contract modifications, develop Cantelli's theorem for non-Markovian models and investigate conditions that maintain actuarial equivalence.

In general, the needed reserve-dependency for partial lapse is a different kind of dependency than our model allows for. In our situation, the partial lapse could be modelled by lapsing the existing and entering a new contract, where the the reduced reserve of the old contract is used as a lapse-payment and is paid back to the insured. This payment can then be considered a one-off premium payment for a new contract.

Chapter 6.

Conclusion and further research opportunities

In this thesis, the existence and uniqueness of the payment process and the prospective reserve have been proven for a rather general class of insurance products with reserve-dependent payments. The main focus has been on the incorporation of restricted and non-monotone information structures, with the latter being based on the infinitesimal martingale theory of Christiansen [Chr21b]. The existence and uniqueness of solutions for these cases was previously unclear, as they could not be included into the existing theoretical framework. Conversely, the new theorems presented in this thesis still include existing results, by simplification, as special cases.

In general, the study has led to the emergence of a new type of non-linear BSDE, similar to the Thiele BSDE. In the proof of existence and uniqueness for the solution, we have presented a novel way of working with the Thiele BSDE, focusing on the relationship between the payment process and the prospective reserve. This approach was initially introduced as a workaround to avoid relying on the properties of martingales, as adapting similar results turned out to be difficult. Furthermore, we were able to develop conditions under which the pricing of such an insurance contract is possible and the nature of the proof also provided us with a pricing method for these contracts. Finally, we considered some examples to illustrate the potential applications of our theory, but kept the formulation at a theoretical level.

Further research opportunities are available in the field of multi-state life insurance theory. This is due to the fact that already existing extensions can be combined in various ways, leading to new models with a priori unclear properties. This section will present a few potential extensions starting from the research presented in this thesis.
We only allow for a deterministic discounting rate, which is a part where further stochasticity can be introduced and adaptions to the methodology of the proofs would be required. Moreover, we have imposed quite restrictive conditions on the payment functions, which is standard practice in the existing literature on BSDEs, but it may be possible to relax these conditions. The exercising of options, such as the free-policy options, would also necessitate the introduction of scaled payments, with the scaling factor depending on the time point of exercise. This introduces a further dependency into the payments that the model does currently not allow for.
Furthermore, we impose the additional restriction that the intensities of the counting processes must be absolutely continuous, contributing to favourable conditions for the proofs. These are some potential adjustments, where even more general models can be incorporated in the theory.

The development of new techniques and models is driven by a number of reasons. These might be legal and regulatory requirements, as well as advances in statistical and numerical methods, and the increasing computational power that allows for more complex models than were previously feasible.

The primary objective of this thesis was to demonstrate the existence and uniqueness of solutions, which does not necessarily imply a straightforward and desirable possibility to calculate these solutions. The main theorem in each chapter provides the reader with a recursion formula derived from the fixed point theorem of Banach or the regula falsi method. However, the calculations remain challenging. This is in contrast to standard Markov models, where the solution can be obtained through backward recursion.

Moreover, we always just assume to have a the first order basis for the intensities of the counting processes without providing any information on the statistical problems that come with this. The calculation of the IF- and IB-compensators is difficult, since particularly models of high complexity require a large amount of data in order to ensure the robustness and correctness of the estimations for the involved intensities. This is especially difficult, if many different and uncommon information states are to be considered. If correct and robust calculation would not be possible, it might even be preferable to use a model with higher model error, but with better estimation properties, especially if in addition to the increased accuracy, the calculation would also be more feasible.
A trade-off must be made between systematic model risk and unsystematic estimation risk. In general, estimation risk would be vanishing for large sample sizes; however, if only a small number of samples is available, the higher model risk may be preferable. It is also important to recognize that, in at least some examples (see the smoking example in the previous chapter), the statistical foundations of these models are not sound, or even knowingly inaccurate, but the simple model is still used for convenience. This may be attributed to a lack of data availability or to the fact that the optimal model is unknown.

Note, that new approaches have been developed and they improve the estimation situation. For example, the as-if-Markov theory presented in [Chr21a] provides the necessary concepts to do estimation without the need for a parametric model, by relying on methods related to the landmark Nelson-Aalen estimators and applied to the specific nature of the multi-state insurance model. It has to be mentioned that the as-if-Markov model can not be included in the non-monotone framework, as a consistent definition of the information structure is not possible, but these new approaches are drivers of statistical advancements and can still be used in the more general case. Further advances have been made by Bladt and Furrer [BF23], providing statistical analysis for the so-called conditional Aalen-Johansen estimator. It enables the non-parametric estimation of state occupation probabilities for a wide range of finite state jump processes and by allowing for the inclusion of covariates, and especially for continuous conditioning.

In the non-monotone framework, additional fundamental results may be proven and more complex reserve-dependencies may be realised. The main results presented in Christiansen [Chr21b] have been proven within the setting of marked point processes, but the concepts of IF- and IB-martingales are also defined more generally in the introduction of the aforementioned paper. Consequently, an extension of this theory may be possible and could lead to a more general version of the IF- and IB-martingale representation theorems. Even in the marked point process setting, not all properties are investigated and there are some reverse relations, that are of interest. The question arises, whether an infinitesimal martingale representation necessarily implies, that the underlying process is an infinitesimal martingale. This would be particularly significant in developing an analogous result to Cantelli's theorem, whereby multiple models (for example with and without the possibility of lapsing the contract) are compared and must fulfil the same Thiele BSDE, with the lapse payment defined as the existing reserve. It is preferable to have Cantelli's theorem, because less calculations are involved and the sparsity of statistical data is even more limited for states like lapse and the problems with reserve-dependency could be avoided all along.

Summarizing, there are a variety of fundamental results, special properties in life insurance theory and in the statistical estimation theory, where the investigation is beyond the scope of this thesis and where further research opportunities arise.

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Appendix A.

Mathematical tools

To increase the readability of this thesis, some definitions, theorems, and other mathematical statements are collected in this chapter of the appendix instead of the main part.

References are provided, and the majority of proofs are not formulated in this thesis.

A.1. Stochastic analysis

Definition A.1.1. *Stochastic process*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $X = (X_t)_{t\geq 0}$ is called a stochastic process on $\mathbb{R}_{\geq 0}$, if every $X_t : \Omega \to \mathbb{R}$ is a random variable, i.e. measurable.

Let $X = (X_t)_{t\geq 0}$ be a stochastic process. The mapping

$$X_{\cdot}(\omega) : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad t \longmapsto X_t(\omega)$$

is called a path of X for every $\omega \in \Omega$.

(a) X is called (right-) continuous, if almost every path is (right-) continuous.

(b) X is called càdlàg, if it almost surely has sample paths that are right-continuous with existing left limits

$$X_{t-} = X_{t^-} := \lim_{s \nearrow t} X_s = \lim_{\substack{s \to t \\ s < t}} X_s$$

where both notations may be used, if there is no danger of ambiguity.

(c) For a càdlàg process we also define the accompanying process ΔX via $\Delta X_t := X_t - X_{t-}$.

See Protter [Pro05] or Klenke [Kle20] for comparison.

Definition A.1.2. *Indistinguishable (equivalent) processes*
Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two stochastic processes on the same probability space.

We call them indistinguishable, if for almost all $\omega \in \Omega$ it holds $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$, i.e. almost all paths of X and Y are equal.

Equivalently, we may write:

$$\mathbb{P}\left(X_t = Y_t \text{ for all } t \geq 0\right) = 1$$

This concept is sometimes also called equivalence up to evanescence.

See also Meintrup and Schäffler [MS05] for details.

Definition A.1.3. *Version (modification) of a process*
Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two stochastic processes on the same probability space.

We call Y a version or modification of X, if

$$\mathbb{P}\left(X_t = Y_t\right) = 1$$

for all $t \geq 0$.

Theorem A.1.4. *Implications for Definitions A.1.3 and A.1.2*
If the two stochastic processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are indistinguishable, they are versions of each other.

If additionally X and Y are (right-) continuous and versions of each other, then they are also indistinguishable.

See also Meintrup and Schäffler [MS05] or Klenke [Kle20] for details.

A.2. Sigma-algebras

Definition A.2.1. *Generated σ-algebra*
For every $\mathcal{E} \subseteq \mathcal{P}(\Omega)$, there exists a smallest σ-algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subseteq \sigma(\mathcal{E})$ and it has the representation

$$\sigma(\mathcal{E}) := \bigcap_{\substack{\mathcal{A}\subseteq\mathcal{P}(\Omega) \text{ is } \sigma\text{-Algebra} \\ \mathcal{E}\subseteq\mathcal{A}}} \mathcal{A}\,.$$

$\sigma(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$.

Further, let $\mathcal{A}$ be a σ-algebra. If it holds, that $\mathcal{A} = \sigma(\mathcal{E})$ for any $\mathcal{E} \subseteq \mathcal{P}(\Omega)$, then we call $\mathcal{E}$ a generator of $\mathcal{A}$.

See Meintrup and Schäffler [MS05] or Klenke [Kle20] for details.

Definition A.2.2. *Trace σ-algebra*
Let there be a $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ and a $A \in \mathcal{P}(\Omega) \setminus \{\emptyset\}$. Define

$$\mathcal{A} \cap A := \{A \cap B \mid B \in \mathcal{A}\} \subseteq \mathcal{P}(A)\,.$$

If $\mathcal{A}$ is a σ-algebra on Ω, then $\mathcal{A} \cap A$ is a σ-algebra on A and it is called the trace-σ-algebra.

See also Meintrup and Schäffler [MS05] or Klenke [Kle20] for details.

Corollary A.2.3. *Trace generated σ-algebra*
Let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ and a non-empty $A \subseteq \Omega$ be given. Then it holds

$$\sigma\left(\mathcal{E} \cap A\right) = \sigma\left(\mathcal{E}\right) \cap A\,.$$

Especially for $\mathcal{A} = \sigma(\mathcal{E})$, we have

$$\sigma\left(\mathcal{E} \cap A\right) = \sigma\left(\mathcal{E}\right) \cap A = \mathcal{A} \cap A$$

which means that $\mathcal{E} \cap A$ is a generator of the trace σ-algebra.

See Klenke [Kle20] for reference.

Definition A.2.4. *Mapping-generated σ-algebra*
Let $(\Omega', \mathcal{A}')$ be a measurable space and Ω a non-empty space. Let $X : \Omega \to \Omega'$ be a mapping. The pre-image

$$X^{-1}(\mathcal{A}') := \left\{ X^{-1}(A') \,\middle|\, A' \in \mathcal{A}' \right\}$$

is the smallest σ-algebra with respect to which X is measurable. We say that $\sigma(X) := X^{-1}(\mathcal{A}')$ is the σ-algebra on Ω that is generated by mapping X.

See Klenke [Kle20] for reference.

Definition A.2.5. *Union σ-algebra – Arbitrary mappings*
Let Ω be a non-empty set. Let I be an arbitrary index set. For any $i \in I$ let $(\Omega_i, \mathcal{A}_i)$ be a measurable space and let $X_i : \Omega \to \Omega_i$ be an arbitrary mapping. Then

$$\sigma\left(X_i,\, i \in I\right) := \sigma\left(\bigcup_{i \in I} \sigma(X_i)\right) = \sigma\left(\bigcup_{i \in I} X_i^{-1}(\mathcal{A}_i)\right)$$

is called the σ-algebra on Ω that is generated by the family $(X_i)_{i \in I}$. This is the smallest σ-algebra with respect to which all X_i are measurable.

See also Klenke [Kle20] for details.

Corollary A.2.6. *Union σ-algebra*
Let Ω be a non-empty set. Let I be an arbitrary index set. For any $i \in I$ let $\mathcal{A}_i$ be a σ-algebra on Ω. Then

$$\bigvee_{i \in I} \mathcal{A}_i = \sigma\left(\mathcal{A}_i,\, i \in I\right) := \sigma\left(\bigcup_{i \in I} \mathcal{A}_i\right)$$

is called the σ-algebra on Ω that is generated by the $(\mathcal{A}_i)_{i \in I}$. If I is finite, we will use the notation

$$\bigvee_{i \in I} \mathcal{A}_i = \bigvee_{i=1}^{n} \mathcal{A}_i = \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n\,.$$

See Klenke [Kle20]. Use the identity mapping in the previous Definition A.2.5 to see this. Also note, that the union itself is not generally a σ-algebra, but it an be used as a generator.

Definition A.2.7. *Conditional expectation*
Let X be an integrable random variable on an probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a σ-algebra.

A random variable Y is called a version of the conditional expectation of X under $\mathcal{G}$, if

(1) Y is $\mathcal{G}$-measurable,

(2) $\mathbb{E}[\mathbb{1}_G X] = \mathbb{E}[\mathbb{1}_G Y]$ for every $G \in \mathcal{G}$. This is also denoted as $\mathbb{E}[X \mid \mathcal{G}]$.

The conditional expectation is almost surely unique and different versions may exist. For a random variable Z we define $\mathbb{E}[X \mid Z] = \mathbb{E}[X \mid \sigma(Z)]$.

See also Meintrup and Schäffler [MS05] or Klenke [Kle20] for details.

Theorem A.2.8. *Semimartingales*
An adapted process with càdlàg paths of finite variation on compacts is a semimartingale, and an adapted process with càdlàg paths of finite variation is a total semimartingale.

See Protter [Pro05] Theorem II.7 for reference.

Theorem A.2.9. *Stochastic integral and pathwise Lebesgue-Stieltjes integral*
If X is a semimartingale that has paths of finite variation on compacts, then the stochastic integral $H.X$ is indistinguishable from the Lebesgue-Stieltjes Integral, that is computed path-by-path or pathwise.

A proof is given in Protter [Pro05] (Theorem II.17). We further glance over the definition of both semimartingales as well as stochastic integral and refer to the given, or any other, established source.

Theorem A.2.10. *Integration by parts*
Let X and Y be two semimartingales. Then XY is also a semimartingale and the following formula holds:

$$XY = \int X_- \,\mathrm{d}Y + \int Y_- \,\mathrm{d}X + [X, Y]$$

where $[X, Y]$ denotes the bracket process, or the quadratic covariation, of X and Y.

A proof is given in Protter [Pro05].

Theorem A.2.11. *Bracket process for a pure jump semimartingale*
Let X be a quadratic pure jump semimartingale, i.e. we have

$$[X, X]_t = \sum_{0 \le s \le t} (\Delta X_s)^2$$

and let Y be any semimartingale. Then we have

$$[X, Y]_t = X_0 \, Y_0 + \sum_{0 \le s \le t} \Delta X_s \, \Delta Y_s \,.$$

A proof is given in Protter [Pro05].

Theorem A.2.12. *Conditional inequality of Jensen (Probabilistic setting)*
Let $\phi : I \to \mathbb{R}$ be a convex function on the interval I and let $X : I \to \mathbb{R}$ be an integrable random variable. Further let $\mathcal{G}$ be a σ-algebra. If $\phi(X)$ is an integrable random variable as well, we have that

$$\phi\left(\mathbb{E}\left[X \mid \mathcal{G}\right]\right) \leq \mathbb{E}\left[\phi\left(X\right) \mid \mathcal{G}\right]$$

For reference see or Klenke [Kle20], where a proof is given both for the normal and for the conditional version, or see Protter [Pro05].

A.3. Analysis

Theorem A.3.1. *Fixed point theorem of Banach*
Let (X, d) be a non-empty complete metric space with a mapping $\Phi : X \to X$. Let Φ be a strict contraction, i.e. there exists an $L < 1$, such that for all $x, y \in X$ it holds

$$d\left(\Phi(x),\, \Phi(y)\right) \leq L \cdot d\left(x,\, y\right) \ .$$

Then Φ has a unique fixed-point $x^* \in X$ and it holds $x^* = \Phi\left(x^*\right)$. Further, x^* can be obtained by iterated application of Φ via

$$x_{n+1} := \Phi\left(x_n\right), \quad n \geq 0$$

for a given starting value x_0 and the following error-estimations

$$\begin{aligned} d\left(x_n,\, x^*\right) &\leq \frac{L^n}{1-L} \cdot d\left(x_1,\, x_0\right) && \text{A PRIORI} \\ d\left(x_{n+1},\, x^*\right) &\leq \frac{L}{1-L} \cdot d\left(x_{n+1},\, x_n\right) && \text{A POSTERIORI} \end{aligned}$$

hold. Note that the order of convergence is 1, i.e. the convergence is given in a linear manner.

For reference see Werner [Wer18].

Theorem A.3.2. *Intermediate value theorem of the real analysis*
Let $a, b \in \mathbb{R}$ with $a < b$. Further, consider a continuous function $f : [a, b] \to \mathbb{R}$. Then for every $s \in [f(a), f(b)]$ (or $s \in [f(b), f(a)]$) there exists an $t \in [a, b]$ such that $f(t) = s$.

Especially if $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there exists an $t \in [a, b]$ such that $f(t) = 0$.

For reference, see Forster [For16] (This particular version is given as a corollary of Theorem 11.1) or Theorem 3.3 in Protter and Morrey [PM91].

Definition A.3.3. *Stieltjes exponential*
For any càdlàg function of finite variation $f : [0, \infty) \to \mathbb{R}$ we define

$$\begin{aligned}\mathfrak{E}(f(t)) &:= \mathfrak{E}(f(\,\cdot\,),\, 0 \le s \le t) \\ &:= e^{f(t)} \prod_{0 \le s \le t} (1 + \Delta f(s))\, e^{-\Delta f(s)} = e^{f^{c}(t)} \prod_{0 \le s \le t} (1 + \Delta f(s))\end{aligned}$$

as the so-called Stieltjes exponential of f, where f^{c} is the continuous part of f.

Take note, that the definition takes into account the whole path of f from 0 to t, and the notation abbreviates this.

See Definition 4.1 in Cohen and Elliott [CE12].

Lemma A.3.4. *Properties of Stieltjes exponential*
In the situation of Definition A.3.3, we have the following properties for all $t \ge 0$:

(1) $\mathfrak{E}(f(t))$ is a càdlàg function.

(2) $\mathfrak{E}(f(t)) > 0$, if $\Delta f(s) > -1$ for every $s \in [0, t]$, and the inverse $\mathfrak{E}(f(t))^{-1}$ is well-defined in that case.

(3) It holds, that $\mathfrak{E}(f(t)) = \mathfrak{E}(f(t^-))\,(1 + \Delta f(t))$ and therefore also

$$\Delta\, \mathfrak{E}(f(t)) = \mathfrak{E}\left(f(t^-)\right)\, \Delta f(t)$$

and

$$\mathrm{d}\, \mathfrak{E}(f(t)) = \mathfrak{E}\left(f(t^-)\right)\, \mathrm{d}f(t)\,.$$

(4) If $f : [0, \infty] \to \mathbb{R}$ is a pure jump function

$$f(t) = \sum_{0 \le s \le t} \Delta f(s) \tag{A.1}$$

with $f(0) = 0$, then we have

$$\mathfrak{E}(f(t)) = \prod_{0 \le s \le t} (1 + \Delta f(s))$$

Proof. Let $f : [0, \infty) \to \mathbb{R}$ be a càdlàg function of finite variation.

(1) The càdlàg property is directly translated from the càdlàg property of f.

(2) See the proof of Lemma 4.2 in Cohen and Elliott [CE12].

(3) It holds that

$$\begin{aligned}\mathfrak{E}(f(t)) &= e^{f(t)} \prod_{0 \le s \le t} (1 + \Delta f(s))\, e^{-\Delta f(s)} \\ &= e^{\Delta f(t)}\, (1 + \Delta f(t))\, e^{-\Delta f(t)}\, e^{f(t^-)} \prod_{0 \le s < t} (1 + \Delta f(s))\, e^{-\Delta f(s)} \\ &= (1 + \Delta f(t))\, \mathfrak{E}\left(f(t^-)\right)\end{aligned}$$

which directly implies the representation of the jumps as

$$\begin{aligned}\Delta\, \mathfrak{E}\left(f(t)\right) &= \mathfrak{E}\left(f(t)\right) - \mathfrak{E}\left(f(t^-)\right) \\ &= \left(1+\Delta f(t)\right)\, \mathfrak{E}\left(f(t^-)\right) - \mathfrak{E}\left(f(t^-)\right) = \mathfrak{E}\left(f(t^-)\right)\, \Delta f(t)\end{aligned}$$

and the differential follows similarly with the help of the first two equations as

$$\mathrm{d}\, \mathfrak{E}\left(f(t)\right) = \mathfrak{E}\left(f(t^-)\right)\, \mathrm{d}f(t)$$

which completes the proof.

(4) Clear.

□

Definition A.3.5. *Jump inversions*
Let $f : [0,\infty) \to \mathbb{R}$ be a càdlàg function of finite variation.

(a) If $\Delta f(s) > -1$ for all s, then we define the so called left-jump inversion of f by

$$\overline{f}(t) := f(t) - \sum_{0\le s\le t} \frac{(\Delta f(s))^2}{1+\Delta f(s)}\,.$$

(b) If $\Delta f(s) < 1$ for all s, then we define the so called right-jump inversion of f by

$$\tilde{f}(t) := f(t) + \sum_{0\le s\le t} \frac{(\Delta f(s))^2}{1-\Delta f(s)}\,.$$

In the case of a pure jump function f, sing the representation A.1, the left- and right inversions also simplify to

$$\begin{aligned}\overline{f}(t) &= f(t) - \sum_{0\le s\le t} \frac{(\Delta f(s))^2}{1+\Delta f(s)} = \sum_{0\le s\le t} \left(\Delta f(s) - \frac{(\Delta f(s))^2}{1+\Delta f(s)}\right) = \sum_{0\le s\le t} \frac{\Delta f(s)}{1+\Delta f(s)} \\ \tilde{f}(t) &= f(t) + \sum_{0\le s\le t} \frac{(\Delta f(s))^2}{1-\Delta f(s)} = \sum_{0\le s\le t} \left(\Delta f(s) + \frac{(\Delta f(s))^2}{1-\Delta f(s)}\right) = \sum_{0\le s\le t} \frac{\Delta f(s)}{1-\Delta f(s)}\,.\end{aligned}$$

Compare Definition 4.2 in Cohen and Elliott [CE12].

Lemma A.3.6. *Inversions of Stieltjes exponentials*
For a function f, that fulfils the necessary conditions from Definition A.3.5, the left and right jump inversions are finite and satisfy

$$\begin{aligned}\mathfrak{E}(f(t))^{-1} &:= \mathfrak{E}(-\overline{f}(t)) \\ \mathfrak{E}(-f(t)) &:= \mathfrak{E}(\tilde{f}(t))^{-1}\end{aligned}$$

A proof is given in Cohen and Elliott [CE12] (Lemma 4.4).

Theorem A.3.7. *Backward inequality of Grönwall for – Stieltjes version (Cohen, Elliott)* Let u be a process, such that for a non-negative Stieltjes measure f with $\Delta f(t) < 1$, and α^- is an $\tilde{f}$-integrable process and u^- is f-integrable and fulfils

$$u(t) \leq \alpha(t) + \int_{(t,T]} u(s^-)\,\mathrm{d}f(s)$$

then also

$$u(t) \leq \alpha(t) + \mathfrak{E}(-f(t)) \int_{(t,T]} \mathfrak{E}(\tilde{f}(s^-))\,\alpha(s^-)\,\mathrm{d}\tilde{f}(s)\,.$$

In case of a constant function $\alpha(t) = \alpha$, the above equation simplifies to

$$u(t) \leq \alpha\,\mathfrak{E}(-f(t))\,\mathfrak{E}(-f(T))^{-1}\,.$$

See Cohen and Elliott [CE12] for a similar formulation of the backward equation.

A proof is performed by Cohen and Elliott for a slightly different version and with some typographical mistakes, which is why we will give a complete proof for the theorem as well.

Proof. We have

$$\begin{aligned}\mathrm{d}\tilde{f}(t) &= \mathrm{d}f(t) + \frac{\Delta f(t)}{1-\Delta f(t)}\,\mathrm{d}f(t) = \frac{1}{1-\Delta f(t)}\,\mathrm{d}f(t) = \left(1 + \frac{\Delta f(t)}{1-\Delta f(t)}\right)\mathrm{d}f(t) \\ &= (1+\Delta\tilde{f}(t))\,\mathrm{d}f(t)\,.\end{aligned} \tag{A.2}$$

Define the auxiliary function

$$w(t) := \mathfrak{E}(\tilde{f}(t)) \int_{(t,T]} u(s^-)\,\mathrm{d}f(s)$$

such that the precondition reads

$$u(t) \leq \alpha(t) + \mathfrak{E}(\tilde{f}(t))^{-1}\,w(t) = \alpha(t) + \mathfrak{E}(-f(t))\,w(t)$$

and whose differential we calculate by using the product rule for stochastic integrals (as a consequence of A.2.10 and A.3.4 property (3)), to get

$$\begin{aligned}\mathrm{d}w(t) &= \mathrm{d}\left(\mathfrak{E}(\tilde{f}(t)) \cdot \int_{(t,T]} u(s^-)\,\mathrm{d}f(s)\right) \\ &= \mathfrak{E}(\tilde{f}(t))\,\mathrm{d}\left(\int_{(t,T]} u(s^-)\,\mathrm{d}f(s)\right) + \left(\int_{(t^-,T]} u(s^-)\,\mathrm{d}f(s)\right)\mathrm{d}\mathfrak{E}(\tilde{f}(t)) \\ &\overset{A.3.4}{=} \mathfrak{E}(\tilde{f}(t^-))\,\left(1+\Delta\tilde{f}(t)\right)\,(-u(t^-)\,\mathrm{d}f(t)) + \left(\int_{[t,T]} u(s^-)\,\mathrm{d}f(s)\right)\mathfrak{E}(\tilde{f}(t^-))\,\mathrm{d}\tilde{f}(t)\end{aligned}$$

$$
\begin{aligned}
&= -u(t^-)\,\mathfrak{E}(\tilde f(t^-))\,\left(1+\Delta\tilde f(t)\right)\,\mathrm{d}f(t) + \left(\int\limits_{[t,T]} u(s^-)\,\mathrm{d}f(s)\right)\mathfrak{E}(\tilde f(t^-))\,\mathrm{d}\tilde f(t) \\
&\overset{A.2}{=} \left(-u(t^-) + \int\limits_{[t,T]} u(s^-)\,\mathrm{d}f(s)\right)\mathfrak{E}(\tilde f(t^-))\,\mathrm{d}\tilde f(t) \\
&\overset{\text{Precond.}}{\geq} -\alpha(t^-)\,\mathfrak{E}(\tilde f(t^-))\,\mathrm{d}\tilde f(t)\,.
\end{aligned}
$$

As the precondition holds on the whole interval, we can also evaluate the inequality for the left limit to get

$$
-\alpha(t^-) \leq -u(t^-) + \int\limits_{[t,T]} u(s^-)\,\mathrm{d}f(s)
$$

and which then implies, as $\mathfrak{E}(\tilde f(t^-))$ is non-negative, and $\mathrm{d}\tilde f(t)$ as well, that by integration with $w(T) = 0$ we have

$$
w(t) \leq \int\limits_{(t,T]} \mathfrak{E}(\tilde f(s^-))\,\alpha(s^-)\,\mathrm{d}\tilde f(s)\,.
$$

By using the connection from u to w, we have

$$
u(t) \leq \alpha(t) + \mathfrak{E}(\tilde f(t))\,w(t) \leq \alpha(t) + \mathfrak{E}(-f(t)) \int\limits_{(t,T]} \mathfrak{E}(\tilde f(s^-))\,\alpha(s^-)\,\mathrm{d}\tilde f(s)
$$

yielding the assertion.

Take note, that $\mathfrak{E}(f(t))$ is positive, if $\Delta f(s) > -1$ for every s (see also the Lemma A.3.4), and $\mathfrak{E}(-f(s^-))$ is always positive, since it holds

$$
\Delta(-f(s)) > -1 \;\Leftrightarrow\; \Delta f(s) < 1
$$

which means, that the necessary jump condition is always fulfilled, if the precondition of the theorem is fulfilled.

In case of a constant $\alpha(t) = \alpha$, we get

$$
\begin{aligned}
u(t) &\leq \alpha\left(1 + \mathfrak{E}(-f(t)) \int\limits_{(t,T]} \mathfrak{E}(\tilde f(s^-))\,\mathrm{d}\overline{f}(s)\right) = \alpha\left(1 + \mathfrak{E}(-f(t)) \int\limits_{(t,T]} \mathrm{d}\mathfrak{E}(\tilde f(s))\right) \\
&= \alpha\left(1 + \mathfrak{E}(-f(t))\left(\mathfrak{E}(\tilde f(T)) - \mathfrak{E}(\tilde f(t))\right)\right) = \alpha\left(1 - 1 + \mathfrak{E}(-f(t))\,\mathfrak{E}(\tilde f(T))^{-1}\right) \\
&= \alpha\,\mathfrak{E}(-f(t))\,\mathfrak{E}(\tilde f(T)) = \alpha \cdot \mathfrak{E}(-f(t))\,\mathfrak{E}(-f(T))^{-1}
\end{aligned}
$$

where we used property (3) of Lemma A.3.4 and $\mathfrak{E}(-f(t)) = \mathfrak{E}(\tilde f(t))^{-1}$ to rewrite the integral. □

Corollary A.3.8. *Backward inequality of Grönwall – Continuous Version*
Let u be a process, such that with $f(t) = Kt$ with $K \in \mathbb{R}_{>0}$, and an $\tilde{f}$-integrable process α, u is (f-)integrable and fulfils

$$u(t) \leq \alpha(t) + K \int\limits_{(t,T]} u(s)\,\mathrm{d}s$$

then also

$$u(t) \leq \alpha(t) + e^{-Kt}\, K \int\limits_{(t,T]} e^{Ks}\,\alpha(s)\,\mathrm{d}s\,.$$

The proof will be formulated, but similar results can be found in the Analysis literature.

Proof. The integrator f, defined as $f(t) = Kt$, does not have any jumps and therefore always fulfils the precondition of the previous Theorem A.3.7. We are now going to give the explicit representations of the involved parts for the special structure of v and formulate the result thereafter. By using the special structure of f, we directly see, that the precondition aligns with the structure of Theorem A.3.7, as

$$u(t) \leq \alpha(t) + \int\limits_{(t,T]} u(s)\,\mathrm{d}f(s) = \alpha(t) + K \int\limits_{(t,T]} u(s)\,\mathrm{d}s$$

and therefore the assertion of the Theorem also holds.

The integrator f does not have any jumps, and by using Definitions A.3.3 and A.3.5 as well as Lemma A.3.6, we achieve the following simplifications for the involved expression as

$$\mathfrak{E}(f(t)) = e^{f(t)} \cdot 1 = e^{Kt}\,, \qquad \mathfrak{E}(f(t))^{-1} = \mathfrak{E}(-f(t)) = e^{-Kt}$$
$$\overline{f}(t) = f(t) - 0 = f(t) = Kt\,, \qquad \tilde{f}(t) = f(t) + 0 = f(t) = Kt\,.$$

Therefore, by plugging in these formulas, we arrive at

$$u(t) \leq \alpha(t) + \mathfrak{E}(-f(t)) \int\limits_{(t,T]} \mathfrak{E}(\tilde{f}(s^-))\,\alpha(s)\,\mathrm{d}\tilde{f}(s) = \alpha(t) + e^{-Kt} \int\limits_{(t,T]} e^{Ks}\,\alpha(s)\,K\,\mathrm{d}s$$

what was to show. □

Theorem A.3.9. *Compensator for the integral of a bounded martingale (Protter)*
Let A be an increasing process of integrable variation and let M be a bounded martingale. Then for X_t and C_t defined as follows

$$X_t := \int\limits_0^t M_s\,\mathrm{d}A_s$$
$$C_t := \int\limits_0^t M_{s^-}\,\mathrm{d}A_s$$

it holds, that C_t is the compensator of X_t.

For reference see Protter [Pro05], Theorem III.20.

Appendix B.

Academic education / Affidavit

Curriculum Vitae

13.10.1997	born in Wilhelmshaven, Germany
07.2006 - 06.2014	High school student at Europaschule Gymnasium Westerstede graduated with Abitur (High school diploma)
10.2014 - 11.2017	Undergraduate studies (Bachelor) in mathematics at Carl von Ossietzky Universität Oldenburg graduated with B.Sc. Mathematics
10.2016 - 09.2019	Student assistant in teaching at Carl von Ossietzky Universität Oldenburg Institute of Mathematics
10.2017 - 10.2019	Graduate studies (Master) in mathematics at Carl von Ossietzky Universität Oldenburg graduated with M.Sc. Mathematics
10.2020 - 10.2021	Mentor / Teaching assistant at the Center für lebenslanges Lernen (C3L), Carl von Ossietzky Universität Oldenburg
11.2019 - 12.2022	Research and teaching assistant at Carl von Ossietzky Universität Oldenburg Institute of Mathematics
11.2019 - 03.2025	Doctoral studies at Carl von Ossietzky Universität Oldenburg

Table B.1.: Curriculum Vitae

Eidesstattliche Erklärung

Hiermit versichere ich, Jannes Tjark Rastedt, dass ich die Arbeit mit dem Titel „Solutions to non-linear Thiele BSDEs in the context of non-monotone information dynamics“ selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Außerdem versichere ich, dass ich die allgemeinen Prinzipien wissenschaftlicher Arbeit und Veröffentlichung, wie sie in den Leitlinien guter wissenschaftlicher Praxis festgelegt sind, befolgt habe.
Ich versichere weiterhin, dass weder diese Arbeit noch Teile davon an einer anderen Universität eingereicht wurden.

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Westerstede, den 03.03.2025