# Towards a classification of simple non-isolated Cohen-Macaulay codimension 2 singularities

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Fakultät für Mathematik und Naturwissenschaften
der
Carl von Ossietzky Universität Oldenburg

zur Erlangung des Grades und Titels
Doctor rerum naturalium (Dr. rer. nat.)

angenommene Dissertation von
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Disputationstermin: 16.10.2024

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# Zusammenfassung

In dieser Arbeit wird die Klassifikation von einfachen *isolierten* Cohen-Macaulay Kodimension 2 Singularitäten zu einer nicht-vollständigen Klassifikation von einfachen *nicht-isolierten* Cohen-Macaulay Kodimension 2 Singularitäten verallgemeinert. Cohen-Macaulay Kodimension 2 Singularitäten zeichnen sich dadurch aus, dass sie, anders als determinantielle Singularitäten im Allgemeinen, in 1:1 Korrespondenz zu Klassen von Matrizen von der Form $n \times (n+1)$ stehen, $n \in \mathbb{N}$, dessen Einträge Nicht-Einheiten sind. Deformationen der Raumkeime korrespondieren zu Störungen einer definierenden Matrix und machen den Begriff der Einfachheit greifbar. Durch diese Korrespondenz erhalten wir eine gut handhabbare Klasse von Singularitäten, welche über den bekannten von vollständigen Durchschnitten hinausgehen und dadurch interessante weiterführende Beispiele liefern. Lässt man von der Isoliertheitsanforderung ab und betrachtet nicht-isolierte Singularitäten, steigert sich die mögliche Matrixgröße und die Dimension des Umgebungsraumes für immer noch einfache Singularitäten. Auf vorhandene Klassifikationen, wie die der einfachen Hyperflächensingularitäten und einfachen isolierten Cohen-Macaulay Kodimension 2 Singularitäten wird zurückgegriffen, um einige neue einfache, nicht-isolierte determinantielle Singularitäten in Normalform angeben zu können.

# Abstract

In this work, the classification of simple *isolated* Cohen-Macaulay codimension 2 singularities is generalized to a non-complete classification of simple *non-isolated* Cohen-Macaulay codimension 2 singularities. Cohen-Macaulay codimension 2 singularities are characterized by the fact that, unlike determinantal singularities in general, they correspond one-to-one with classes of matrices of the form $n \times (n+1)$, where $n \in \mathbb{N}$, and the entries are non-units. Deformations of the space germs correspond to perturbations of a defining matrix, making the concept of simplicity accessible. Through this correspondence, we obtain a well-manageable class of singularities that go beyond the known complete intersections, thereby providing interesting further examples. By dropping the requirement of isolatedness and considering non-isolated singularities, the possible matrix size and the dimension of the ambient space for simple singularities increase. Existing classifications, such as those of simple hypersurface singularities and simple isolated Cohen-Macaulay codimension 2 singularities, are used to present some new simple non-isolated determinantal singularities in normal form.

# Chapter 1

# Introduction

Different sets of polynomial equations might have solution sets with identical geometric properties. An equivalence relation between these sets of polynomials can collect those who share identical geometric properties. In a classification, we are usually concerned with a particular type of geometric object and try to find nice representatives for equivalence classes of these objects under a given equivalence relation. When studying geometric objects, their singular points are of particular interest which leads to a local examination of these objects in a neighbourhood of a singular point. In this work, we start a classification of a particular type of singularities of complex varieties;the so-called simple non-isolated Cohen-Macaulay codimension 2 singularities.

Hypersurface singularities were the first to be studied, having been a focus of research for over a century. They represent the easiest case of varieties, as they are defined by a single equation. They can be generalized by complete intersection singularities, which are defined by several equations, subjects to the condition that the number of equations equals the codimension of the singularity. In this work, we consider determinantal singularities, again, a generalization of complete intersection singularities. The defining equations are given by minors of a matrix and the codimension of the resulting variety has to conincide with the expected dimension due to the dependencies given by the matrix structure.

There are many fascinating aspects to study, one of which is deformation theory. What happens if we perturb the defining equations of a variety? For hypersurface and complete intersection singularities, any perturbation of the defining equations gives rise to a well-behaved family,a so-called deformation. For determinantal singularities in contrast, this is no longer the case. Unsuitable perturbations may result in families containing singularities with empty solution sets or undesirable dimension drops. Flatness is the technical tool that ensures well-behaved families of perturbed equations for any singularities. In the case of Cohen-Macaulay codimension 2 singularities, the flatness condition yields that the admissible perturbations of the equations are precisely those induced by arbitrary pertur-

bations of a defining matrix.
Depending on the complexity of the singularitiy, the outcome of deformations can result in infinitely many non-equivalent types of singularities. We are interested in singularities which deform only into finitely many non-equivalent types. We call these singularities simple.

In the early 70's, V. Arnold [2] classified simple hypersurface singularities and discovered the famous ADE-singularities. About a decade later, M. Giusti [19] extended this work to a classification of simple complete intersection singularities. Shortly after that, Wall [39] classified unimodal complete intersection singularities, generalizing simple complete intersection singularities to singularities which deform into 1-parameter families of singularities.
Among hypersurface and complete intersection singularities, there is no hope of finding simple non-isolated ones. In these cases, non-isolatedness is equivalent to the fact that the Tjurina module, a vector space which classifies all non-trivial first order deformations, has infinite dimension and therefore, the singularity cannot be simple. This is different for determinantal singularities, as non-isolated determinantal singularities can still possess a finite dimensional Tjurina module. Some rigid singularities, for example, are a trivial case of simple non-isolated singuarities. Hence, the classification of simple non-isolated singularities is a meaningful question I want to contribute to in this work.

For simple determinantal singularities, some classifications have already been completed. In the language of map germs, the history of classifying simple singularities continued with a partial classification of skew-symmetric matrices by G. Haslinger [25] in 2001, a complete classification of skew symmetic matrices by W. Bruce [5] in 2003 and a joined work of W. Bruce and F. Tari [7] on the classification of simple mappings of square matrices in 2004. [1] In 2022, W. Bruce, V. Goryunov and G. Haslinger classified singularities of complex skew-symmetric matrix families of even size which are simple under a natural equivalence relation [6]. This work provides a unified approach to all three previously mentioned classifications.
Working with complex space germs, A. Frühbis-Krüger [15] classified simple space curve singularities in 1999, followed by the classification of simple isolated Cohen-Macaulay codimension 2 singularities in a joint work with A. Neumer [16] in 2010.

In this work, we will extend the classification of simple isolated Cohen-Macaulay codimension 2 singularities to include simple non-isolated ones. In Chapter 2, we introduce (determinantal) singularities, their properties, and deformation theory. In Chapter 3, we provide the motivation for the classification and introduce initial methods such as the

[1] Through an alternative approach of Goryunov [23] who studied vanishing cycles of matrix singularities in 2021, a part of the results was rearranged and reformulated.

counting argument. In Chapter 4-7, we reduce the problem to a classfication of smaller ambient space dimension, recognize the cases which have to be studied and for one of the cases we reduce the classification to smaller matrix size. In this case, we classify all simple Cohen-Macaulay codimension 2 singularities within this range of size and dimension by finding candidates for simple singularities and check the adjacencies afterwards. The final results can be found in Theorem 7.1. In Chapter 8, we outline the remaining tasks needed to complete the classification.

# Chapter 2

# Basics

We introduce the following notations: Let $N \in \mathbb{N}$. Then the power series ring $\mathbb{C}\{x_1, \ldots, x_N\}$ shall be abbreviated by $\mathbb{C}\{\underline{x}\}$. Any ring $R$ shall be a commutative ring with 1.

## 2.1 Commutative Algebra and Algebraic Geometry

In this section, we give a foundation of basic definitions from commutative algebra and algebraic geometry in order to define Cohen-Macaulay modules and to introduce singularities in the next section.
We quickly recall the definitions of sheaves, sections, stalks and germs. For further details, see [29, Chapter 2.2].

**Definition 2.1.** [29, Chapter 2.2][Sheaf, Stalk, Germ]

1. Let $X$ be a topological space, a *presheaf* $\mathcal{F}$ of Abelian groups (resp. rings) on $X$ consists of the following set of data

   - Abelian groups (resp. rings) $\mathcal{F}(U)$ for any open subset $U$ of $X$
   - a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every pair of open subsets $V \subseteq U$

   satifying the following conditions:

   (a) $\mathcal{F}(\emptyset) = 0$

   (b) $\rho_{UU} = \mathrm{id}$

   (c) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ holds for any three open subsets $W \subseteq V \subseteq U$

   An element $s \in \mathcal{F}(U)$ is called *section of $\mathcal{F}$ over $U$*.

2. A *sheaf* $\mathcal{F}$ is a presheaf satisfying the following two additional properties:

   (a) Uniqueness: If $U \subseteq X$ is an open subset with an open covering $\{U_i\}_{i \in I}$, $I$ some index set, $s \in \mathcal{F}(U)$ and $s_{|U_i} = 0$ for all $i$, then $s = 0$.

(b) Glueing local sections: Using the notations above, for all sections $s_i \in \mathcal{F}(U_i)$ satisfying $s_{i|U_i \cup U_j} = s_{j|U_i \cup U_j}$ there exists a $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$.

3. Let $\mathcal{F}$ be a presheaf on $X$. A *stalk of $\mathcal{F}$ at $x$* is the group

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the direct limit is taken over the open neighbourhoods $U$ of $x$.
For a section $s \in \mathcal{F}(U)$ and $x \in U$, the *germ $s_x$ of $s$ at $x$* is given by the image of $s$ in $\mathcal{F}_x$.

4. A *ringed space* $(X, \mathcal{O}_X)$ is a tuple of a topological space $X$ with a sheaf of rings $\mathcal{O}_X$.

To build a foundation of commutative algebra we collect definitions, examples and theorems from [12, 20, 22, 29, 40].

**Definition 2.2.** [22, Definition 7.3.1][Flatness]
Let $R$ be a ring. An $R$–module $M$ is called *flat* iff, for every injective homomorphism $N \to L$, the induced map $N \otimes_R M \to L \otimes_R M$ is again injective.

To be able to talk about the dimension of singularities we need a precise definition of „dimension“ of rings and modules, which will be the objects that describe the singularities algebraically.

**Definition 2.3.** [22, Definition 3.3.1.][Height and Dimension]
Let $R$ be a noetherian ring, $P \subset R$ a prime ideal, $I \subseteq R$ an arbitrary ideal and $M$ a finitely generated $R$-module.
The *height* of these ideals is defined as

$$\begin{aligned} \operatorname{height}(P) &:= \sup\{r \in \mathbb{N} \mid P \supsetneq Q_r \supsetneq \cdots \supsetneq Q_0 \text{ chain of prime ideals}\}, \\ \operatorname{height}(I) &:= \inf\{\operatorname{height}(P) \mid I \subseteq P \text{ prime ideal in } R\ \}. \end{aligned}$$

The *Krull dimension of $R$* is defined as

$$\dim R := \sup\{\operatorname{height}(\mathfrak{m}) \mid \mathfrak{m} \text{ maximal ideal in } R\ \}$$

and the *codimension of $R/I$* is defined as

$$\operatorname{codim}(R/I) := \dim R - \dim R/I.$$

The *dimension of the module $M$* is defined as

$$\dim(M) := \dim(R/\operatorname{Ann}_R(M)),$$

where $\mathrm{Ann}_R(M)$ denotes the annihilator of $M$.

**Example 2.4.**

- Let $R = \mathbb{Z}$, $p \in \mathbb{Z}$ a prime number. The prime ideals $p\mathbb{Z}$ have height

  $$\mathrm{height}(p\mathbb{Z}) = 1,$$

  which yields $\dim \mathbb{Z} = 1$.

- Let $K$ be a field, then $\{0\}$ is the only maximal ideal, hence, $\dim K = 0$.

- Let $R$ be an arbitrary noetherian ring and $A = R[x_1, \ldots, x_n]$, $n \in \mathbb{N}$, $x_1, \ldots, x_n$ variables. Then

  $$\dim A = \dim R + n.$$

  In particular, for a field $K$ we have $\dim K[x_1, \ldots, x_n] = n$ (see [29, Lemma 5.16.]). A chain of prime ideals of maximal length can be, for example, the following:

  $$\{0\} \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, \ldots, x_n \rangle.$$

**Definition 2.5.** [12, Definition 6.5.1.][Regular Sequences and Depth]
Let $R$ be a ring, $M$ an $R$-module and $I \subset R$ an ideal.

1. An ordered sequence of elements $x_1, \ldots, x_n \in R$ is called *$M$-regular sequence* iff
   (a) $\langle x_1, \ldots, x_n \rangle M \neq M$
   (b) $x_i$ is a non-zerodivisor of $M/\langle x_1, \ldots, x_{i-1} \rangle M$ for each $i = 1, \ldots, n$
2. If $IM \neq M$, the maximal length of an $M$-regular sequence contained in $I$ is called the *$I$-depth of $M$* and denoted by $\mathrm{depth}(I, M)$. If $IM = M$ define $\mathrm{depth}(I, M) := \infty$. If $R$ is local with maximal ideal $\mathfrak{m}$, then $\mathrm{depth}(\mathfrak{m}, M)$ is called the *depth of $M$* and denoted by $\mathrm{depth}(M)$.

To get an intuitive idea of regular sequences we introduce relations.

**Definition 2.6.** [22, Definition 2.5.1.]

1. A *relation* or *syzygy* between $n$ elements $f_1, \ldots, f_n$ of an $R$-module $M$ is a $n$-tuple $(r_1, \ldots, r_n) \in R^n$ satisfying
   $$\sum_{i=1}^{n} r_i f_i = 0.$$

2. The set of all relations between $f_1, \ldots, f_n$ is a submodule of $R^n$, it coincides with the kernel of the ring homomorphism

$$\varphi : \bigoplus_{i=1}^{n} R\varepsilon_i \longrightarrow M, \ \varepsilon_i \mapsto f_i,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is the canonical basis of $R^n$. The *module of syzygies* w.r.t. $f_1, \ldots, f_n$ is defined as

$$syz(f_1, \ldots, f_n) := \ker(\varphi).$$

**Example 2.7.**

- Let $M = K[x_1, x_2, x_3, x_4]$, $K$ an arbitrary field. Consider the sequence

$$x_3x_4, \ x_1x_4 - x_2x_3, \ x_4^2.$$

This is not a regular sequence, as $x_4^2$ is a zerodivisor in $K[x_1, x_2, x_3, x_4]/\langle x_3x_4, x_1x_4 - x_2x_3\rangle$ :

$$x_4^2 \cdot x_1 = x_1x_4 \cdot x_4 \equiv x_2x_3 \cdot x_4 = x_2 \cdot x_3x_4 \equiv x_2 \cdot 0 = 0.$$

- In case of a ring $R$ (as a module of rank 1 over itself) we get an easy reformulation for regular sequences: If the only relations amongst the elements of the sequence are the trivial relations, i.e., $x_ix_j - x_jx_i = 0$, the sequence is regular: If $x_i$ is a zerodivisor in $R/\langle x_1, \ldots, x_{i-1}\rangle$, there is $a \in R$ and $a_1, \ldots, a_{i-1} \in R$ such that $ax_i = \sum\limits_{j=1}^{i-1} a_jx_j$, i.e., $0 = \sum\limits_{j=1}^{i-1} a_jx_j - mx_i$. We get a non-trivial relation $(a_1, \ldots, a_{i-1}, a)$.

- Let $R = \mathbb{C}[x, y, z]$, then $(xy, xz, yz)$ is not an $R$-regular sequence, as $xz \cdot y = xy \cdot z$, hence, $xz$ is a zerodivisor in $\mathbb{C}[x, y, z]/\langle xy\rangle$.

Dimension, i.e., height, and depth are different ways to describe the „size“ of a ring or module. In general, there is the inequality

$$\text{depth}(M) \leq \dim(M).$$

In other words, the longest regular sequence cannot be longer than the longest chain of prime ideals. If the depth and the dimension of a module coincide, we speak of „Cohen-Macaulay“ rings and modules:

**Definition 2.8.** [12, Definition 6.5.1.][Cohen-Macaulay]
Let $R$ be a local Noetherian ring and $M$ a finitely generated $R$-module.

1. $M$ is *Cohen-Macaulay*, iff $\text{depth}(M) = \dim(M)$.

2. $R$ is *Cohen-Macaulay*, iff $R$ is Cohen-Macaulay as $R$-module.

Later, we use the abbreviation „CM“ for Cohen-Macaulay rings and modules.

The following examples can be found in [12, Example 6.5.6.].

**Example 2.9.**

1. Smooth spaces are Cohen-Macaulay.
2. Complete intersection singularities are Cohen-Macaulay.
3. Reduced space curve singularities are Cohen-Macaulay (as their depth is positive and bounded by the dimension which is one).
4. $\mathbb{C}[x,y,z]/\langle xy, yz\rangle$ is not Cohen-Macaulay:

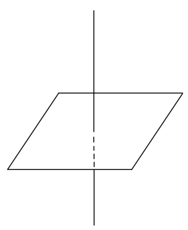


Figure 2.1: $V(xy, yz)$

Consider $x-z$, which is a non-zerodivisor but the local ring $\mathbb{C}[x,y,z]/\langle xy, yz, x-z\rangle \cong \mathbb{C}[x,y]/\langle x^2, xy\rangle$ has an embedded point, i.e., an embedded prime in the prime factorization of $\langle x^2, xy\rangle$. Therefore, the depth of the germ $(V(xy, yz), 0)$ in the origin is one but the dimension is two. The variety $V(xy, yz)$ is illustrated in figure 2.1.

5. $\mathbb{C}[x,y,z]/\langle xy, yz, xz\rangle$, the union of the three coordinate axes, is Cohen-Macaulay.
6. $\mathbb{C}[x,y,z,v]/\langle xz, xv, yz, yv\rangle$, the union of the two planes $x = y = 0$ and $z = v = 0$ intersecting only in the origin, is not Cohen-Macaulay: Consider $y-v$, which is a non-zerodivisor but the local ring $\mathbb{C}[x,y,z,v]/\langle xz, xv, yz, yv\rangle \cong \mathbb{C}[x,y,z]/\langle xy, xz, yz, y^2\rangle$ has an embedded point, given by the embedded prime ideal $(y^2, x, z)$ in the prime factorization
$$(xz, xy, yz, y^2) = (x, y) \cap (y, z) \cap (y^2, x, z).$$
Therefore, the depth of the germ $(V(xz, xv, yz, yv), 0)$ in the origin is 1 but the dimension is 2.

## 2.2 Singularities

We introduce singularities as complex space germs and dimension thereof, as well as deformation theory. All definitions and results can be found in [1, 12, 17, 20, 22, 31, 32].

### 2.2.1 Space germs

First, we introduce the objects, we call „singularities“.

**Definition 2.10.** [20, I-Definition 4.14.][Singularities]

1. Let $D \subseteq \mathbb{C}^N$ be an open subset and $A \subseteq D$.

    (a) $A$ is called *analytic at* $p \in D$, iff there is a neighbourhood $U \subseteq D$ containing $p$ and a set of holomorphic functions $f_1 \dots, f_k \in \mathcal{O}(U)$ such that

    $$A \cap U = V(f_1, \dots, f_k) := \{a \in U \mid f_1(a) = \dots = f_k(a) = 0\}.$$

    (b) $A$ is called a *locally analytic subset of* $D$, iff $A$ is analytic at every $p \in A$.

2. A *complex space* is a tuple $(X, \mathcal{O}_X)$, where $(X, \mathcal{O}_X)$ is a ringed space, $X$ is a hausdorff topological space and $\mathcal{O}_X$ is a structure sheaf satisfying the following property:
   For all $p \in X$ there is a neighbourhood $U \subset X$ containing $p$ and a locally closed analytic set $A$, $\mathcal{O}_A$ the sheaf of holomorphic functions on $A$ such that

   $$(U, \mathcal{O}_{X|U}) \cong (A, \mathcal{O}_A).$$

   In other words, $(X, \mathcal{O}_X)$ is locally isomorphic to a locally closed analytic set.

3. A *pointed complex space* $(X, x)$ is a complex space $(X, \mathcal{O}_X)$ with a distinguished point $x \in X$.

4. A *complex space germ* $(X, x)$ (also called *singularity*) is the set germ of the pointed complex space $X$ at $x$, locally equipped with the stalk $\mathcal{O}_{X,x}$.

When we restrict to locally analytic sets and holomorphic function we usually consider the euclidean topology. In the following, when we think about neighbourhoods and other topological concepts, we use the euclidean topology.

So, locally, i.e., in an arbitrarily small neighbourhood of a distinguished point, we have a set of holomorphic functions defining a singularity. The vanishing locus of these functions is a complex variety, restricting to set germs, we get complex space germs that describe the complex variety locally. To change from the analytic to the algebraic point of view, we can translate the holomorphic functions to complex (convergent) power series. These power series define an ideal in the ring $\mathbb{C}\{\underline{x}\}$ of convergent power series. The ring defined by the quotient of the power series ring by this ideal defines the local ring of the complex space germ. The local ring is the stalk of the structure sheaf of the ringed space defining the complex variety globally. In special cases, this local ring is just a quotient of the polynomial ring. So, on the analytic/geometric side, we see the set-theoretic object and the holomorphic functions describing it, on the algebraic side we see the polynomials resp. power series. Some algebraic methods will be important to examine singularities through their structure sheaves.

A natural question that arises is about the geometric meaning of the appearing numbers, for example in the ambient space but also the number of generators, the number of tangent directions or the length of the longest chains inside of the defining stalks. Therefore, we introduce some concepts of „dimension“ of a complex space in a point. Using these concepts, we can define „singular“ and „regular“ points.

**Definition 2.11.** [20, I-Definition 1.39.][Singular and Regular Points]
Let $X$ be a complex space, $p \in X$ and $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$. We define

1. the *dimension of $X$ at $p$* as $\dim_p X :=$ Krull dimension of $\mathcal{O}_{X,p}$,
2. the *embedding dimension of $X$ at $p$* as $\operatorname{edim}_p X := \dim_{\mathbb{C}} \mathfrak{m}_p/\mathfrak{m}_p^2$,
3. a point $p \in X$ to be *regular*, iff $\dim_p X = \operatorname{edim}_p X$ and
4. a point $p \in X$ to be *singular*, iff $p$ is not regular.

As usual, the *codimension* $\operatorname{codim}_p X$ of $X$ at $p$ is given by

$$\operatorname{codim}_p X = N - \dim_p X,$$

where $N$ is the dimension of the ambient space. The distinguished point of complex space germs is usually singular as those are the interesting points we want to study. For this reason, we often call complex space germs „singularities“.

If the distinguished point is some point outside of the origin, we get a complex space germ at the origin by an easy coordinate change. In the origin, we can calculate with the ordinary complex power series ring. As these calculations are much easier, it is more convenient to calculate in the origin and transform back to the point outside of the origin afterwards. Therefore, it is common to choose the origin as the distinguished point of the complex space germ. In the setting we will introduce later, the difference of these complex space germs (outside of the origin or translated into the origin) is irrelevant, it will be caught by coordinate changes.

Some first examples give an idea of the objects we call singularities:

**Example 2.12.**

1. $(\mathbb{C}^N, 0)$ is the germ of the smooth variety $\mathbb{C}^N$ locally at the origin ($\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^N,0} = \mathbb{C}\{x_1, \ldots, x_N\}$).

2. $(V(f), 0) \subset (\mathbb{C}^2, 0)$, $f = x^2 - y^2$ is the germ of the variety $V(f)$ (illustrated in figure 2.2) locally at the origin ($\mathcal{O}_{V(f),0} = \mathbb{C}\{x, y\}/\langle f \rangle$). This is a real $A_1$ singularity.

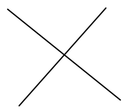


Figure 2.2: $V(x^2 - y^2)$

3. $(V(x), 0) \subset (\mathbb{C}^2, 0)$ is the germ of the smooth variety $V(x)$ (illustrated in figure 2.3) locally at the origin ($\mathcal{O}_{V(x),0} \cong \mathbb{C}\{y\}$).



Figure 2.3: $V(x)$

4. $(V(f, g), 0) \subset (\mathbb{C}^3, 0)$, $f = x^2, g = y^2 - z^3$ is the germ of the variety $V(f, g)$ (illustrated in figure 2.4) locally at the origin ($\mathcal{O}_{V(f,g),0} = \mathbb{C}\{x, y, z\}/\langle f, g \rangle$). We recognize the cusp, embedded in the 2-dimensional plane.

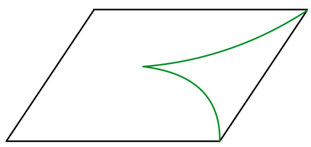


Figure 2.4: $V(x^2, y^2 - z^3)$

5. $(V(xy, xz, yz), 0) \subset (\mathbb{C}^3, 0)$ is the germ of the variety $V(xy, xz, yz)$ (illustrated in figure 2.5) locally at the origin ($\mathcal{O}_{V(xy,xz,yz),0} = \mathbb{C}\{x, y, z\}/\langle xy, xz, yz \rangle$). This is the union of the three coordinate axes.

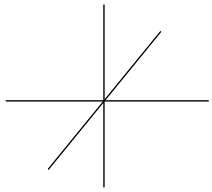


Figure 2.5: $V(xy, yz, xz)$

Singularities, which are locally defined by one equation are well-studied and the easiest situation to deal with. Of course, it is interesting to examine more general objects but it is helpful to have certain control and structure while studying different properties. To generalize the problem, we want the codimension of a singularity to coincide with the minimal number of generators of the variety.

**Definition 2.13** (Hypersurface and Complete Intersection Singularities)**.**
Let $(X, 0)$ be a singularity with $\mathcal{O}_{X,0} = \mathbb{C}\{\underline{x}\}/\langle f_1, \ldots, f_k\rangle$, where $f_1, \ldots, f_k$ are chosen to be a minimal system of generators (there is no system of less holomorphic functions defining the same singularity).

1. $(X, 0)$ is a *hypersurface singularity* iff $k = 1$
2. $(X, 0)$ is a *complete intersection singularity* iff $k = \operatorname{codim}\langle f_1, \ldots, f_k\rangle$

Example 2.12.2 and 2.12.3 are easy examples for hypersurface singularities, whereas example 2.12.4 is a complete intersection singularity (the codimension is 2 and the minimal number of generators is also 2). Example 2.12.5 is not a complete intersection singularity, as the codimension is 2 but at least 3 generators are necessary to define this singularity. This is an example for determinantal singularities and more precisely, CMC2 singularities.

**Definition 2.14.** A singularity $(X, x)$ is called *isolated*, if there is a neighbourhood $U \subset X$ of $x$, such that no point $y \in U \setminus \{x\}$ is singular.

### 2.2.2 Deformation theory

The goal of this section is the definition of simplicity which is necessary in order to classify the simple non-isolated CMC2 singularities. Simplicity, or in general, modality, measures the „degrees of freedom“ in a deformation. Naively, a deformation is a family of singularities obtained by perturbing the equations of the original singularity in a suitable way. Formally, a deformation is defined using the information stored in a Cartesian diagram.

**Definition 2.15.** [20, I-Definition 1.46.][Fibre Product and Cartesian Diagram]
Let $X, Y, T$ be complex spaces and $f : X \to T, g : Y \to T$ morphisms of complex spaces. The *(analytic) fibre product of* $X$ *and* $Y$ is a triple $(X \times_T Y, \pi_X, \pi_Y)$ consisting of a complex space $X \times_T Y$ and morphisms $\pi_X : X \times_T Y \to X, \pi_Y : X \times_T Y \to Y$ such that

$$f \circ \pi_X = g \circ \pi_Y$$

and the following universal property holds:
For any complex space $Z$ and morphism $h : Z \to X, h' : Z \to Y$ such that

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "h'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & T \end{tikzcd}$$

commutes there is a unique morphism $\varphi : Z \to X \times_T Y$ such that

$$\begin{tikzcd} Z \arrow[dr, "\varphi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h'"'] & & \\ & X \times_T Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & T \end{tikzcd}$$

commutes. If the fibre product exists it is unique up to a unique isomorphism. For an arbitrary choice of a complex space $W$ with morphisms $\tilde{g} : W \to X$ and $\tilde{f} : W \to Y$ we use the notation

$$\begin{tikzcd} W \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & T \end{tikzcd}$$

for a commutative diagram providing the universal property of the fibre product and call it *Cartesian diagram.*

The intuition about the fibre product is a nice interaction between $X$ and $Y$ in the sense that there are proper projections of the product of $X$ and $Y$ over $T$ to its components and mapping from an arbitrary complex space to $X$ or $Y$ factorizes into a product of a unique morphism to the product and the projection onto the respective component. In the definition of deformation, one of the complex spaces, say $Y$, is only one point. In that case the factorization of a morphism onto $X$ through the product gives a morphism onto a fiber of $X$. These morphisms are the tools to study the fibres of $X$, additional properties of the Cartesian diagram provide for example further properties about the fibres.

**Definition 2.16.** [20, II-Definition 1.1.][Deformations]

Let $(X, x)$ and $(S, s)$ be complex space germs and consider the cartesian diagram

$$\begin{tikzcd} (X,x) \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathcal{X},x) \arrow[d, "\varphi"] \\ \{\mathrm{pt}\} \arrow[r, hook] & (S,s) \end{tikzcd}$$

where $\{\mathrm{pt}\}$ denotes the reduced point considered as a complex space germ, $\varphi : (\mathcal{X}, x) \longrightarrow (S, s)$ is a flat morphism with $(X, x) \cong (\mathcal{X}_s, x) := (\varphi^{-1}(s), x)$. A *deformation of* $(X, x)$ *over* $(S, s)$ is a family of singularities given by the fibres of $\varphi$ over the base space. $(\mathcal{X}, x)$ is called the total space, $(S, s)$ the base space, and $(\mathcal{X}_s, x) \cong (X, x)$ the special fibre of the deformation.

Shortly, we write

$$(X, x) \overset{\iota}{\hookrightarrow} (\mathcal{X}, x) \overset{\varphi}{\to} (S, s)$$

for a deformation.

The base space can be seen as the space of parameters. It is crucial that $\varphi$ is a flat morphism, that is, $\mathcal{O}_{(S,s)}$ is a flat $\mathcal{O}_{(\mathcal{X},x)}$-module via the induced morphism $\varphi^{\#} : \mathcal{O}_{(S,s)} \to \mathcal{O}_{(\mathcal{X},x)}$. Recall, this means for any monomorphism $M \to N$ of $\mathcal{O}_{(\mathcal{X},x)}$-modules the induced map $\mathcal{O}_{(S,s)} \otimes_{(\mathcal{X},x)} M \to \mathcal{O}_{(S,s)} \otimes_{(\mathcal{X},x)} N$ is a monomorphism too. This injectivity ensures a close relation between the nearby fibres $\varphi^{-1}(t)$ and the special fibre $\varphi^{-1}(s)$. For instance, if the representatives of the complex spaces are pure dimensional, flatness implies equal dimension of the fibres (see [20, Chapter II-1.1]). The tensor product can be interpreted as a realization of perturbing the equations defining the complex space locally. A tensor product of the stalk of the complex space germ with some algebra encoding the parameter space is the algebraic way to describe a perturbation of the equations. Checking flatness might be hard, therefore, the following proposition supplies a tool to check flatness algorthmically in our setting. The flatness of the morphism mapping from total space to base space can be replaced by an equivalent property, the *lifting property*. This property is a condition on the relations amongst the generators of the special fibre, they shall be liftable to relations amongst the generators of the total space.

**Proposition 2.17.** [31, Proposition 7.2.2.]

Let $I = \langle f_1, \ldots, f_k \rangle \subset \mathcal{O}_{\mathbb{C}^N,0}$ be an ideal, $(S, s)$ a complex space germ and $\tilde{I} = \langle F_1, \ldots, F_k \rangle \subset \mathcal{O}_{\mathbb{C}^N \times S,(0),s}$ a lifting of $I$, i.e., $F_i$ is a preimage of $f_i$ under the surjection

$$\mathcal{O}_{\mathbb{C}^N \times S,(0,s)} \twoheadrightarrow \mathcal{O}_{\mathbb{C}^N,0}.$$

Then the following are equivalent:

1. $\mathcal{O}_{\mathbb{C}^N \times S,(0,s)}/\tilde{I}$ is $\mathcal{O}_{(S,s)}$-flat.

2. Any relation $(r_1, \dots, r_k)$ among $f_1, \dots, f_k$ lifts to a relation $(R_1, \dots, R_k)$ among $F_1, \dots, F_k$. That is, for each $(r_1, \dots, r_k)$ satisfying

$$\sum_{i=1}^{k} r_i f_i = 0$$

there exists $(R_1, \dots, R_k)$ such that

$$\sum_{i=1}^{k} R_i F_i = 0, \text{ with } R_i \in \mathcal{O}_{\mathbb{C}^N \times S,(0,s)}$$

and the image of $R_i$ in $\mathcal{O}_{\mathbb{C}^N,0}$ is $r_i$.

To calculate relations in a computer algebra system (e.g. Singular, OSCAR, see [10,11]) we can calculate the so-called *syzygy-module* (i.e., the module of relations). It is easy to check whether the relations lift by checking the liftability for the generators of the syzygy-module. Now, we know an algebraic tool to work with deformations and we have the geometric idea of perturbing the defining equations. The diagram defining deformations allows us to define morphisms between deformations and with that, induced deformations. In some cases there is a deformation that induces any other deformation. Such a deformation is useful to get an idea of the singularities arising in arbitrary deformations.

**Definition 2.18.** [20, II-Definition 1.2.][Morphisms of Deformations]
Given two deformations

$$(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathcal{X}, x) \overset{\phi}{\to} (S, s), \quad (i', \phi') : (X, x) \overset{i}{\hookrightarrow} (\mathcal{X}', x') \overset{\phi}{\to} (S', s')$$

of $(X, x)$ over $(S, s)$ and $(S', s')$ respectively. A *morphism of deformations* from $(i, \phi)$ to $(i', \phi')$ consists of two morphisms $(\psi, \varphi)$ such that the following diagram commutes:

$$\begin{tikzcd} & (X,x) \arrow[dl, hook', "i'"'] \arrow[dr, hook, "i"] & \\ (\mathcal{X}',x') \arrow[rr, "\psi"] \arrow[d, "\phi'"'] & & (\mathcal{X},x) \arrow[d, "\phi"] \\ (S',s') \arrow[rr, "\varphi"] & & (S,s) \end{tikzcd} .$$

Two deformations over the same base space $(S, s)$ are isomorphic if there exists a morphism $(\psi, \mathrm{id})$ such that $\psi$ is an isomorphism.

**Definition 2.19.** [20, II-Definition 1.3.][Induced Morphisms]
Let $(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathcal{X}, x) \overset{\phi}{\to} (S, s)$ be a deformation of the complex space germ $(X, x)$, $(T, t)$ another complex space germ and $\varphi : (T, t) \to (S, s)$ a morphism of germs. The fibre product of $\phi$ and $\varphi$ is given by the following commutative diagram of germs

$$\begin{tikzcd} & (X,x) \arrow[dl, hook', "\varphi^* i"'] \arrow[dr, hook, "i"] & \\ (\mathcal{X},x) \times_{(S,s)} (T,t) \arrow[rr, "\tilde{\varphi}"] \arrow[d, "\varphi^*\phi"'] & & (\mathcal{X},x) \arrow[d, "\phi"] \\ (T,t) \arrow[rr, "\varphi"] & & (S,s) \end{tikzcd},$$

where $\tilde{\varphi}$ is induced by the projection on the first component, $\varphi^*\phi$ is induced by the projection on the second component and $\varphi^* i = (\tilde{\varphi}^{-1}_{|(\varphi^*\phi)^{-1}(t)}) \circ i$. We call $(\varphi^* i, \varphi^*\phi)$ the *deformation induced by* $\varphi$ *from* $(i, \phi)$ and $\varphi$ the *base change map.*

A „versal“ deformation is a deformation which contains in some sense all information about any possible deformation. We can think of it as a basis for all deformations.

**Definition 2.20.** [31, Definition 7.2.13.][Versality]
A deformation $(X, x) \overset{i}{\hookrightarrow} (\mathcal{X}, x) \overset{\phi}{\to} (S, s)$ is called *versal* iff, for any given deformation $(j, \psi) : (X, x) \overset{j}{\hookrightarrow} (\mathcal{Y}, y) \overset{\psi}{\to} (T, t)$ the following holds:
For any closed embedding

$$k : (T', t) \hookrightarrow (T, t)$$

of complex space germs and any morphism

$$\varphi' : (T', t) \hookrightarrow (S, s)$$

such that $(\varphi'^* i, \varphi'^* \phi)$ is isomorphic to $(k^* j, k^* \psi)$ there exists a morphism $\varphi : (T, t) \to (S, s)$ satisfying

1. $\varphi \circ k = \varphi'$ and

2. $(j, \psi) = (\varphi^* i, \varphi^* \phi)$.

That is, there exists a commutative diagram with Cartesian squares

$$
\begin{tikzcd}
 & (X,x) \arrow[dl, hook', "k^*i"'] \arrow[d, hook, "j"] \arrow[dr, hook, "i"] & \\
(\mathcal{X},x)\times_{(S,s)}(T,t) \arrow[r, hook] \arrow[d, "k^*\psi"'] \arrow[dr, phantom, "\square"] & (\mathcal{Y},y) \arrow[r, dashed, "\exists"] \arrow[d, "\psi"] \arrow[dr, phantom, "\square"] & (\mathcal{X},x) \arrow[d, "\phi"] \\
(T',t') \arrow[r, hook, "k"'] \arrow[rr, bend right, "\varphi'"'] & (T,t) \arrow[r, dashed, "\exists\varphi"'] & (S,s)
\end{tikzcd}
\quad .
$$

Loosely speaking, one can say that a deformation is versal, if every deformation can be induced from it by some base change.

Naturally, the question arises, which singularities appear in a versal family. In the following sections, we will define suitable equivalence relations for the appearing objects. Depending on these equivalence relations, we call the equivalence classes a „type“ of singularity. If after deforming a singularity, only finitely many types of singularities appear, we call a singularity simple.

**Definition 2.21.** [17, Chapter 4][Simplicity]
A singularity is called *simple* if only a finite number of non-equivalent singularities appear in its versal family (w.r.t. a given equivalence relation for a specific class of singularities).

A useful method to study the versal family of a complex space germs are infinitesimal deformations, which are deformations over a fat point:

**Definition 2.22.** [20, II-Definition 1.19.][Infinitesimal Deformations]

1. The complex space germ $T_\varepsilon$ consists of one point with local ring $\mathbb{C}[\varepsilon] := \mathbb{C}[t]/\langle t^2\rangle$, where $t$ is an indeterminate.

2. For any complex space germ $(X,0)$ define

$$T^1_{(X,0)} := \underline{\mathrm{Def}}_{(X,0)}(T_\varepsilon),$$

   where $\underline{\mathrm{Def}}_{(X,0)}(T_\varepsilon)$ is the set of isomorphism classes of deformations of $(X,0)$ over $T_\varepsilon$. A deformation of $(X,0)$ over $T_\varepsilon$ is called *infinitesimal deformation of* $(X,0)$ or *first order deformation of* $(X,0)$.

These infinitesimal deformations give an idea of the behavior of deformations in an arbitrarily small neighbourhood of the origin. In good cases, for example for hypersurface singularities, complete intersection singularities and CMC2 singularities, infinitesimal deformations can be lifted to deformations but in general the existence of obstructions does not permit to lift all infinitesimal deformations. The $T^1$ is often referred to as *Tjurina module*.

Another perspective on the versal family of a complex space germ can be reached by examining the equivalence classes or orbits of the defining object of the complex space germ w.r.t. a suitable equivalence relation or a suitable group action on the defining objects, respectively.

**Remark 2.23.** If the term „neighbourhood“ is defined, this allows an alternative definition for simplicity:
If there is a neighbourhood of the defining object meeting only finitely many equivalence classes (or intersecting only finitely many orbits of the group action, respectively), the singularity is simple.

We will examine these ideas in concrete situtaions in the following sections.

## 2.3 Hypersurface Singularities

The easiest singularities are those defined by only one equation. The definitions and results in this section can be found in [16, 20].

**Definition 2.24** (Hypersurface singularities)**.**
A singularity $(X, 0) \subseteq (\mathbb{C}^N, 0)$ is a *hypersurface singularity*, if $\mathcal{O}_{X,x} \cong \mathbb{C}\{\underline{x}\}/\langle f\rangle$ for some $f \in \mathbb{C}\{\underline{x}\}$, i.e., $(X, 0) = (V(f), 0) = (f^{-1}(0), 0)$.

**Remark 2.25.** To define a neighbourhood of a power series $f \in \mathbb{C}\{\underline{x}\}$ we define the distance of $f$ and some $g \in \mathbb{C}\{\underline{x}\}$ for example by

$$\|f - g\| := \frac{1}{\operatorname{ord}(f - g)}.$$

Now, an example for a neighbourhood can be a ball with radius $\varepsilon > 0$ in the topological space $(\mathbb{C}\{\underline{x}\}, \|\cdot\|)$.

To study a hypersurface singularity as a complex space germ it is helpful to study the defining power series as an algebraic object. There are two common equivalence relations which gather power series to be identified up to coordinate changes or coordinate changes combined with multiplication with a unit, respectively, in other words, isomorphic map germs (right equivalence) or isomorphic complex space germs (contact equivalence). Using these equivalence classes, we can examine properties of the complex space germs which are invariant under the equivalence relation and therefore, focus on the interesting information. For hypersurface singularities we can define „right-“ and „contact-equivalence“.

**Definition 2.26.** [20, I-Definition 2.9.][Right- and Contact-Equivalence]
Let $f, g \in \mathbb{C}\{\underline{x}\}$. Then

- $f$ is called *right-equivalent* to $g$, $f \sim_r g$, if there exists an automorphism $\varphi$ of $\mathbb{C}\{\underline{x}\}$ such that $f = \varphi(g)$.
- $f$ is called *contact-equivalent* to $g$, $f \sim_c g$, if there exists an automorphism $\varphi$ of $\mathbb{C}\{\underline{x}\}$ and a unit $u \in \mathbb{C}\{\underline{x}\}^\star$ such that $f = u \cdot \varphi(g)$.

**Remark 2.27.** [20, I-Remark 2.9.1.]
Let $f, g \in \mathbb{C}\{\underline{x}\}$ define map germs $f, g : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$.

1. Right-equivalence implies contact-equivalence.
2. Any automorphism $\varphi$ of $\mathbb{C}\{\underline{x}\}$ determines a biholomorphic local coordinate change. On the level of map germs, the group of local coordinate changes acts from the right. Hence, $f \sim_r g$ iff the diagram

$$\begin{tikzcd} (\mathbb{C}^N, 0) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & (\mathbb{C}^N, 0) \arrow[dl, "g"] \\ & (\mathbb{C}, 0) & \end{tikzcd}$$

   commutes. As map germs, $f$ and $g$ are isomorphic.
3. $f \sim_c g \iff (V(f), 0) \cong (V(g), 0) \iff \mathbb{C}\{\underline{x}\}/\langle f \rangle \cong \mathbb{C}\{\underline{x}\}/\langle g \rangle$, i.e., the analytic algebras defined by $f$ and $g$ respectively are isomorphic.

Now, singularities can be collected into equivalence classes. In some sense, we can choose the „nicest“ representative. The splitting lemma (see 2.28) is a tool to find such a representative. Recall, that for $f \in \{\underline{x}\}$ the *Hessian matrix* is defined as

$$H(f) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

By the splitting lemma, $f$ can be splitted (w.r.t. the equivalence relation) into a sum of squares and a „residual part“ of order 3, containing only those variables which did not appear in degree 2.

**Lemma 2.28.** [20, I-Theorem 2.47.][Splitting lemma]
Let $f \in \mathfrak{m}^2 \subseteq \mathbb{C}\{\underline{x}\}$. If $\operatorname{rk} H(f)(\underline{0}) = k$, then

$$f \sim_r x_1^2 + \cdots + x_k^2 + g(x_{k+1}, \ldots, x_n).$$

$g$ is called the *residual part of* $f$. It is uniquely determined up to right-equivalence.

**Remark 2.29.** In particular

$$f \sim_c x_1^2 + \cdots + x_k^2 + g(x_{k+1}, \ldots, x_n)$$

with $g \in \mathfrak{m}^3$ and the residual part is uniquely determined up to contact equivalence.

In this work, we are interested in the geometric objects defined by the germ of a complex space, not in the power series. Therefore, in the case of hypersurface singularities we consider only contact-equivalence.

In general, power series are given by an infinite set of data. It would be more convenient to work with polynomials or even better, polynomials of bounded degree. In fact, we can get this convenience in many cases, when we use finitely determined power series w.r.t. contact-equivalence. To be in control of the monomials in a power series, we can truncate the power series after some fixed degree $k \in \mathbb{N}$. In some situations, we get valuable information about the original power series out of that.

**Definition 2.30.** [20, I-Definition 2.20.][Jets]
Let $f \in \mathbb{C}\{\underline{x}\}$. Then

$$j_k f := \text{image of } f \text{ in } \mathbb{C}\{\underline{x}\}/\mathfrak{m}^{k+1}$$

denotes the $k$-jet of $f$ and

$$J_k := \mathbb{C}\{\underline{x}\}/\mathfrak{m}^{k+1}$$

is the vector space of all $k$-jets.

Whether two power series belong to the same equivalence class might be visible amongst the monomials up to some degree $k \in \mathbb{N}$. In this case, there is a representative of the equivalence class which is a polynomial of degree less or equal to $k$. We can choose this truncated polynomial representative and work in an easier setting with bounded degree instead of studying an infinite set of data.
This phenomenom is called „determinacy“ and it gives a bound for the degree of relevant monomials of a power series which can change the equivalence class. In other words, above this bound, monomials do not change the equivalence class anymore.

**Definition 2.31.** [20, I-Definition 2.21.][Finite Determinacy]

1. $f \in \mathbb{C}\{\underline{x}\}$ is called *contact-k-determined*, or shortly *k-determined*, if for each $g \in \mathbb{C}\{\underline{x}\}$ with $j_k f = j_k g$ we have $f \sim_c g$.

2. The minimal such $k \in \mathbb{N}$ is called *contact-determinacy degree* of $f$.

3. $f$ is called *finitely contact-determined*, shortly *finitely determined*, if $f$ is $k$-determined for some $k \in \mathbb{N}$.

**Example 2.32.**

1. $xy \in \mathbb{C}\{x, y, z\}$ is not finitely determined as hypersurface singularity, because any pure power of $z$ changes the contact-equivalence class of the hypersurface singularity.

2. $xy + z^k \in \mathbb{C}\{x, y, z\}$ is $k$-determined as all monomials (of degree higher than $k$) divisible by $x$ or $y$ can be cancelled by $xy$ via a coordinate change in $x$ or $y$, while all monomials divisible by $z^k$ can be cancelled by $z^k$ via a suitbale coordinate change in $z$, for instance,

$$xy + z^k - x^3 + z^{k+2} \mapsto \tilde{x}\tilde{y} + \tilde{z}^k$$

under the coordinate change

$$\tilde{x} := x, \quad \tilde{y} := y - x^2, \quad \tilde{z} := z \cdot \sqrt[k]{1 + z^2}.$$

The following criterion is a convenient method to check finite determinacy:

**Theorem 2.33.** [20, I-Theorem 2.23.] [Finite Determinacy Theorem]
$f$ is contact $k$-determined if

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^2 \cdot \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \rangle + \mathfrak{m} \cdot \langle f \rangle.$$

Finite determinacy makes the calculations finite. Yet, the remaining calculations can still be too involved to be solved. If we still see too many monomials, even if we consider only those of lower degree, it is helpful to prioritize some variables in the consideration of the degree and obtain refined determinacy bounds. Therefore, we can endow the variables with a weight and define a weighted degree. The slices of fixed weighted degree contain less monomials, depending on the choice of weights for the variables. This gives the option to focus the attention on specific parts of the power series and get the most information out of only a few monomials. Formally, we define:

**Definition 2.34.** [20, I-Definition 2.11.] [Quasihomogeneous Polynomial]
A polynomial $f = \sum_{\alpha \in \mathbb{N}^N} a_\alpha \underline{x}^\alpha \in \mathbb{C}\{\underline{x}\}$ is called *weighted homogeneous* or *quasihomogeneous of type* $(\omega; d) = (\omega_1, \ldots, \omega_N, d)$ if $\omega_i, d$ are positive integers satisfying

$$d = \omega \cdot \alpha := \omega_1 \alpha_1 + \cdots + \omega_N \alpha_N$$

for each $\alpha \in \mathbb{N}^N$ with $a_\alpha \neq 0$. The numbers $\omega_i$ are called *weights* and $d$ the *weighted degree* of $f$. We define the *weighted order* of an arbitrary power series $g = \sum_{\alpha \in \mathbb{N}^N} c_\alpha \underline{x}^\alpha$ as

$$\nu_\omega(g) := \min\{d \mid \exists \alpha \in \mathbb{N}^N : d = \omega \cdot \alpha, c_\alpha \neq 0\}.$$

**Remark 2.35.** [20, I-Remark 2.11.1.]

1. Quasihomogeneous polynomials of type $(\omega, d)$ remain quasihomogeneous of type $(\omega, d)$ under coordinate changes which respect the weights.

2. A quasihomogeneous polynomial $f$ of type $(\omega, d)$ satisfies the *Euler relation*

$$d \cdot f = \sum_{i=2}^{N} \omega_i x_i \frac{\partial f}{\partial x_i}.$$

One version to classify the simple hypersurface singularities, a less technical and rather intuitive version, uses the „quasihomogeneous initial jet“ of a power series and „weighted finite determinacy“. Given a weighting for the variables, the monomials of weighted degree higher than some bound $k$ can be truncated as before. The remaining possibilities for an arbitrary quasihomogeneous polynomial up to degree $k$ can be easily classified.
For the classification of simple CMC2 singularities, we will generalize the idea of the „quasihomogeneous initial jet“, so we define:

**Definition 2.36.** Given a weight $\omega \in \mathbb{N}^N$ and $f \in \mathbb{C}\{\underline{x}\}$, the *quasihomogeneous initial jet of* $f$ is given by

$$j_\omega f := j_{\nu_\omega(f)} f,$$

i.e., the polynomial given by the monomials of lowest weighted degree.

When it comes to deformations of a singularity, we want to understand the meaning of the flatness property that recognizes perturbations of the defining equation(s) from genuine deformations. Hypersurface singularities are defined by only one power series which can easily be perturbed by adding random monomials (adding constants translates a singularity out of the origin, that is the only inconvenient kind of perturbation). This leads to the definition of „unfoldings“ of hypersurface singularities.

**Definition 2.37.** [20, I-Definition 2.5.] [Unfolding]

Let $f \in \mathbb{C}\{\underline{x}\}$. An *unfolding of* $f$ is a power series $F \in \mathbb{C}\{\underline{x}, \underline{t}\} := \mathbb{C}\{x_1, \ldots, x_N, t_1, \ldots, t_\tau\}$ such that $F(\underline{x}, 0) = f(\underline{x})$.

**Remark 2.38.** An unfolding $F \in \mathbb{C}\{\underline{x}, \underline{t}\}$ of a power series $f \in \mathbb{C}\{\underline{x}\}$ induces a deformation

$$\begin{tikzcd} (V(f),0) \arrow[r, hook] \arrow[d] & (V(F),0) \arrow[d, "pr_2"] \\ \{\mathrm{pt}\} \arrow[r, hook] & (\mathbb{C}^\tau, 0) \end{tikzcd}$$

as the projection $pr_2$ to the second component is a flat morphism in the case of a hypersurface singularity $(V(F), 0)$.
Conversely, a deformation of a hypersurface singularity is an unfolding of the defining power series.

We see that deformations are easier to grasp in the case of hypersurface singularities. This simplification is also visible if we consider the $T^1$, i.e., the set of equivalence classes of infinitesimal deformations (or first order deformations).

**Remark 2.39.** [20, I-Corollary 1.17.]

Let $(X, 0)$ be a hypersurface singularity defined by $f \in \mathbb{C}\{\underline{x}\}$. Then

$$T^1_{(X,0)} = \mathbb{C}\{\underline{x}\}/\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\rangle =: T(f)$$

is a $\mathbb{C}\{\underline{x}\}$-algebra which is referred to as the *Tjurina algebra*.

The *Tjurina number* $\tau := \dim_{\mathbb{C}} T(f)$ of a power series $f \in \mathbb{C}\{\underline{x}\}$ is an important analytical invariant for hypersurface singularities. A generalization of Tjurina algebra and Tjurina number usually describes the respective $T^1$ and the dimension thereof. One of the most interesting properties of the Tjurina algebra of a power series is the connection to deformations of the corresponding hypersurface singularity.

**Proposition 2.40.** [20, I-Corollary 1.17.]

Let $f \in \mathbb{C}\{\underline{x}\}$ and $g_1, \ldots, g_\tau \in \mathbb{C}\{\underline{x}\}$ a $\mathbb{C}$-basis of $T(f)$. Define $(\mathcal{X}, 0) := (V(F), 0) \subset (C^N \times \mathbb{C}^\tau, 0)$ by

$$F(\underline{x}, \underline{t}) := f(\underline{x}) + \sum_{j=1}^{\tau} t_j g_j(\underline{x}).$$

Then $(X, 0) \hookrightarrow (\mathcal{X}, 0) \xrightarrow{pr_2} (\mathbb{C}^\tau, 0)$, is a versal deformation of $(X, 0)$, where $pr_2$ is the canonical projection on the second component.

Hence, any deformation can be induced by the versal deformation given by a basis of $T(f)$. We are now familiar with basic concepts for hypersurface singularities, some of these concepts can be generalized to the singularities we introduce next, i.e., complete intersection singularities, CMC2 singularities and EIDS.

In order to formulate simplicity in terms of orbits of a group acting on the set of power series, we reformulate the equivalence relation on a power series using the following group:

**Definition 2.41.** [20, I-Definition 2.30.] [Right- and Contact-Group]
The group $\mathcal{R} := \mathrm{Aut}(\mathbb{C}\{\underline{x}\})$ of automorphisms of the analytic algebra $\mathbb{C}\{\underline{x}\}$ is called the *right group*. The group $\mathcal{K} := \mathbb{C}\{\underline{x}\}^\star \ltimes \mathrm{Aut}(\mathbb{C}\{\underline{x}\})$ [1] is called the *contact group*, where the product in $\mathcal{K}$ is given by

$$(\tilde{u}, \tilde{\varphi})(u, \varphi) = (\tilde{u}\tilde{\varphi}(u), \tilde{\varphi}(\varphi)).$$

Remark, that $\mathcal{R}$ and $\mathcal{K}$ are not finite dimensional (as $\mathbb{C}\{\underline{x}\}$ is not finite dimensional). Therefore, they do not define algebraic groups. By truncating the power series, we can enforce finite dimension and an algebraic group action.
The contact group acts on $\mathbb{C}\{\underline{x}\}$ by

$$\begin{aligned} \mathcal{K} \times \mathbb{C}\{\underline{x}\} &\longrightarrow \mathbb{C}\{\underline{x}\}, \\ ((u, \varphi), f) &\mapsto u \cdot \varphi(f). \end{aligned}$$

This group action can be used to reformulate contact equivalence, as

$$f \sim_c g \iff f \in \mathcal{K} \cdot g,$$

where $\mathcal{K} \cdot g$ describes the orbit of $g$ under $\mathcal{K}$. Instead of an equivalence class, we can now talk about orbits of a power series. Simplicity of a hypersurface singularity can now be expressed as follows:
A hypersurface singularity $(X, 0)$ defined by a power series $f \in \mathbb{C}\{\underline{x}\}$ is simple iff there is a neighbourhood of $f$ in $\mathbb{C}\{\underline{x}\}$ which intersects only finitely many $\mathcal{K}$-orbits.

The simple hypersurface singularities were classified by Arnold in [2, §4-6], these are the so called ADE-singularities:

[1] The *semidirect product* $G \ltimes H$ of two groups $G$ and $H$ via a homomorphism $\theta : H \to \mathrm{Aut}(G)$ is defined as the cartesian product $G \times H$ with the group operation $(g, h)(g', h') = (g \cdot_G \theta(h)(g'), hh')$, see [27, Chapter II.6, Exercise 1].

**Example 2.42.**

Table 2.1: ADE-singularities

| $A_k$, $k \geq 1$ | $D_k$, $k \geq 4$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|
| $x^{k+1}$ | $x^2y + y^{k-1}$ | $x^3 + y^4$ | $x^3 + xy^3$ | $x^3 + y^5$ |

For the classification of simple hypersurface singularities, it is useful to introduce the „corank“ of a given power series $f \in \mathbb{C}\{\underline{x}\}$.

**Definition 2.43.** [20, I-Definition 2.45.]
Let $f \in \mathbb{C}\{x_1, \dots, x_N\}$. The *corank of $f$* is defined as

$$\operatorname{crk}(f) := N - \operatorname{rk}(H(f)(0)),$$

where $H(f)$ is the Hessian matrix of $f$.

**Theorem 2.44.** [20, I-Theorem 2.55.]

1. The ADE-singularities are (contact-)simple.
2. Let $f \in \mathfrak{m}^2 \subseteq \mathbb{C}\{\underline{x}\}$ be not contact-equivalent to one of the ADE classes, then either
   (a) $\operatorname{crk}(f) \geq 3$, or
   (b) $\operatorname{crk}(f) = 2$, i.e., $f \sim_c g(x_1, x_2) + x_3^2 + \dots + x_N^2$ with
      i. $g \in \mathfrak{m}^4$, or
      ii. $g \in \langle x_1, x_2^2 \rangle^3$.

   In any of these cases, $f$ is not (contact-)simple.

To classify the simple singularities in the previous theorem it is important to know boundary singularities which are not simple. Exceeding a specific (weighted) degree, all singularities deform into a non-simple one. In the proof of the previous theorem, the following non-simple boundary singularities (in the sense that any other singularity despite the ADE-singularities deforms into one of the following non-simple ones) are necessary:

**Proposition 2.45.** Consider

1. $X_9$, defined by $x^4 + y^4 + ax^2y^2 \in \mathbb{C}\{x, y\}$ with $a \in \mathbb{C} \setminus \{2, -2\}$,
2. $J_{10}$, defined by $x^3 + y^6 + axy^4 \in \mathbb{C}\{x, y\}$ with $a \in \mathbb{C}$,
3. $P_8$, defined by $x^3 + y^3 + z^3 + axyz \in \mathbb{C}\{x, y, z\}$ with $a \in \mathbb{C}$.

All of these singularities are non-simple.

*Proof.* For the classification of the simple non-isolated Cohen-Macaulay codimension 2 singularities, we will introduce the counting argument as a method to exclude non-simple singularities (see 3.4). Using the counting argument with the following set of weights for $x, y$ and weight matrix $D$ (which is just one entry with the weighted degree of a generic power series in $x, y$) proves that the given singularities are not simple:

1. $\omega := (1, 1), D := (4)$
2. $\omega = (2, 1), D := (4)$
3. $\omega = (1, 1, 1), D := (3)$

$\square$

Another interesting and important result is the classification of simple hypersurface singularities „with section". Therefore, we introduce deformations and $T^1$ „with section". Even if the following concepts are introduced for curves in [20], they do not restrict on curves only. We can use the proposition for hypersurface singularities, too.

**Definition 2.46.** [20, II-Definition 2.1.][Deformation with Section]
Let $(X, 0) \subseteq (\mathbb{C}^N, 0)$ define a hypersurface singularity. A *deformation with section* of $(X, 0)$ over a complex germ $(S, s)$ consists of a deformation

$$(X, 0) \overset{\iota}{\hookrightarrow} (\mathcal{X}, x_0) \overset{\varphi}{\to} (S, s)$$

and a *section* of $\varphi$, that is, a morphism $\sigma : (S, s) \to (\mathcal{X}, x_0)$ satisfying

$$\varphi \circ \sigma = \mathrm{id}_{(S,s)} .$$

**Definition & Remark 2.47.** [20, Chapter II.2.1]

1. The ideal
$$I_\sigma := \ker(\sigma^\sharp : \mathcal{O}_{\mathbb{C}^2 \times S,(0,s)} \to \mathcal{O}_{S,s}).$$
determines the section $\sigma$.

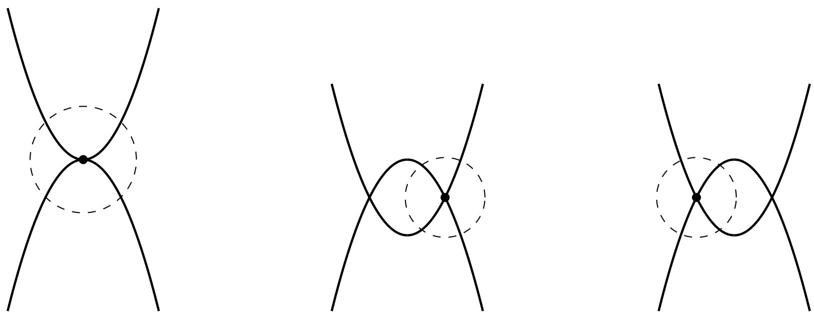
2. The *trivial section* is the section $\sigma$ which satisfies
$$\sigma(S, s) = (\{0\} \times S, s), \quad \text{i.e., } I_\sigma = \langle x, y \rangle.$$

**Example 2.48.** Consider $f = y^2 - x^4 \in \mathbb{C}\{x, y\}$ defining a hypersurface singularity in $(\mathbb{C}^2, 0)$. Let $(\mathbb{C}, 0)$ be the base space of the following two deformations defined by $f \rightsquigarrow f + t, t \in \mathbb{C}$ with section:

$$\sigma_1 : (\mathbb{C}, 0) \to (\mathcal{X}, 0),\ t \mapsto (\sqrt{t}, t)$$
$$\sigma_2 : (\mathbb{C}, 0) \to (\mathcal{X}, 0),\ t \mapsto (-\sqrt{t}, t)$$

Depending on the section, the distinguished point of the germs differs. In the following picture, we see the original fibre of the deformations and for each of $\sigma_1, \sigma_2$ a singularity of the family of deformations with section. We can picture the deformation with section as a deformation along the line between the original fibre and the respective distinguished point.

Figure 2.6: Deformation with Section



**Proposition 2.49.** [20, II-Corollary 2.3.] [$T^1$ with section]
Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a hypersurface singularity defined by $f$. Then

$$T^{1,sec}_{(X,0)} := \underline{\mathrm{Def}}^{sec}_{(X,0)}(T_\varepsilon) = \mathfrak{m}\Big/\langle f\rangle + \mathfrak{m}\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\rangle$$

where $\underline{\mathrm{Def}}^{sec}_{(X,0)}(T_\varepsilon)$ is the set of isomorphism classes of deformations of $(X, 0)$ over $T_\varepsilon$ with section $\sigma$ and $\mathfrak{m} \subseteq \mathbb{C}\{\underline{x}\}$ denotes the maximal ideal.

The simple hypersurface singularities with trivial section are exactly the ADE-singularites (see the proof of theorem 3.5 in [16]):

**Proposition 2.50** (Simple Hypersurface Singularities with Trivial Section)**.** If $\sigma$ is the trivial section, then the simple hypersurface singularites with section $\sigma$ are given by the ADE-singularities.

## 2.4 Complete Intersection Singularities

The following definitions and results in this section can be found in [12, 20–22]. Complete intersection singularities are a generalization of hypersurface singularities, which can be generalized again by defining determinantal singularities. The local ring of a complete intersection singularity can be defined by an ideal which is generated by more than one element as long as these generators have only trivial relations. This fact is captured by the codimension of the singularity which has to coincide with the minimal number of generators for the defining ideal.

**Definition 2.51.** [20, p. 88, footnote][Complete Intersection Singularity]
Let $(X,0)$ be a complex space germ with local ring $\mathcal{O}_{X,0} \cong \mathbb{C}\{\underline{x}\}/I$ and $\operatorname{codim} \mathcal{O}_{(X,0)} = k$. $(X,0)$ is called *complete intersection singularity* if $I$ can be generated by $k$ elements.

This definition expresses a condition on the relations of the generators of a complete intersection singularity.

**Remark 2.52.** A minimal generating set of the ideal is a regular sequence. Hence, the only relations between these generators are the trivial relations.

Our goal is to identify complex space germs with isomorphic local rings via a suitable equivalence relation, i.e.,

$$(X,0) \sim (Y,0) \iff \mathcal{O}_{X,0} \cong \mathcal{O}_{Y,0}.$$

Therefore, we define the following group action inducing such equivalence classes of singularities.

**Definition 2.53.** [21, Definition 2.1.1.][Contact-Group]
The *contact group* $\mathscr{K}$ is the set of pairs of germs of holomorphic functions $(h, H)$, where $h : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, $H : (\mathbb{C}^N \times \mathbb{C}^p, 0) \to (\mathbb{C}^N \times \mathbb{C}^p, 0)$ such that $pr_1 \circ H = h$, $(pr_2 \circ H)(x, 0) = 0$, where $pr_1$ and $pr_2$ are the canonical projections into $\mathbb{C}^N$ and $\mathbb{C}^p$, respectively.

As before, we can define „jets“ and „finite determinacy“.

**Definition 2.54.**

1. Let $\underline{f} = (f_1, \ldots, f_s) \in \mathbb{C}\{\underline{x}\}^s$. Then the *k-jet of* $\underline{f}$ is defined by

$$j_k\underline{f} := (j_k f_1, \ldots, j_k f_s).$$

2. $\underline{f} \in \mathbb{C}\{\underline{x}\}^s$ is called *finitely-$\mathcal{K}$-determined*, if $\underline{f}$ is *$k$-$\mathcal{K}$-determined* for some $k \in \mathbb{N}$, i.e., for all $\underline{g} \in \mathbb{C}\{\underline{x}\}^s$ with $j_k\underline{f} = j_k\underline{g}$ the map germs defined by $\underline{f}$ and $\underline{g}$ are $\mathcal{K}$-equivalent.

In analogy to the hypersurface case, the $T^1$-module can be simplified and it is called the „Tjurina module“ of a complete intersection singularity.

**Remark 2.55.** [20, II-Theorem 1.16]

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a complete intersection singularity defined by a minimal set $f_1, \dots, f_k \in \mathbb{C}\{\underline{x}\}$ of generators. Then

$$T^1_{(X,0)} = \mathbb{C}\{\underline{x}\}^k \Big/ \left(Df \cdot \mathbb{C}\{\underline{x}\}^N + \langle f_1, \dots, f_k\rangle \cdot \mathbb{C}\{\underline{x}\}^k\right),$$

where

$$Df := \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq i\leq k\\ 1\leq j\leq N}} : \mathbb{C}\{\underline{x}\}^N \to \mathbb{C}\{\underline{x}\}^k$$

denotes the Jacobian matrix of $f := (f_1, \dots, f_k)$.

If $T^1_{(X,0)}$ is a finite dimensional $\mathbb{C}$-vector space, let $g_1, \dots, g_\tau \in \mathbb{C}\{\underline{x}\}^k$ represent a basis thereof, where $g_i = (g_{i,j})_j$. Set

$$F_i(\underline{x}, \underline{t}) := f_i + \sum_{j=1}^{\tau} t_j g_{i,j}(\underline{x})$$

and define $(\mathcal{X}, 0) := (V(F_1, \dots, F_k), 0) \subset (\mathbb{C}^N \times \mathbb{C}^\tau, 0)$. Then

$$(X, 0) \overset{\iota}{\hookrightarrow} (\mathcal{X}, 0) \overset{\phi}{\to} (\mathbb{C}^N \times \mathbb{C}^\tau, 0)$$

is a versal deformation of $(X, 0)$, where $\iota$ is induced by the inclusion $(\mathbb{C}^N, 0) \subset (\mathbb{C}^N \times \mathbb{C}^\tau, 0)$, $\phi$ is induced by the projection $(\mathbb{C}^N \times \mathbb{C}^\tau, 0) \to (\mathbb{C}^\tau, 0)$.

The following two propositions provide equivalences between isolatedness, the dimension of the $T^1$, finite determinacy and later, as a conclusion, simplicity. It will motivate the consideration of simple non-isolated singularities amongst determinantal singularities, but we will see that there are no simple non-isolated complete intersection singularities. First, we observe that isolated complete intersection singularities have a finite dimensional $T^1$.

**Proposition 2.56.** [12, Theorem 10.2.15.]

Let $(X, 0)$ be a germ of a complex space with an isolated singularity. Then $T^1_{(X,0)}$ is a finite-dimensional $\mathbb{C}$-vector space.

Further, we extract the equivalence between finite dimension and finite determinacy from [21].

**Proposition 2.57.** [21, Theorem 3.0.1.]
Let $\underline{f} \in \mathfrak{m}^k_{\mathbb{C}\{\underline{x}\}}$. The following are equivalent:

1. $\underline{f}$ is finitely-$\mathcal{K}$-determined
2. $\dim_{\mathbb{C}} \left(\mathbb{C}\{\underline{x}\}^k \Big/ \left(Df \cdot \mathbb{C}\{\underline{x}\}^N + \langle f_1, \ldots, f_k\rangle \cdot \mathbb{C}\{\underline{x}\}^k\right)\right) < \infty$

In the end, complete intersection singularities are the first step to generalize hypersurface singularities:

**Remark 2.58.** Every hypersurface singularity is a complete intersection singularity, defined by a single generator $f \in \mathbb{C}\{\underline{x}\}$. Krull's principle ideal theorem [22, Theorem 5.6.8] ensures that

$$\operatorname{codim}(\mathbb{C}\{\underline{x}\}/\langle f\rangle) = 1.$$

To generalize complete intersection in the next step, we consider determinantal singularities.

## 2.5 Determinantal Singularities

Determinantal ideals generalize complete intersection singularities in the sense that they satisfy a condition on the dimension, namely, the codimension of the singularity coincides with the dimension which is expected by the structure of a defining matrix. To define the singularitiy via a matrix, consider the vanishing locus of the minors of fixed size. For these objects, many ideas can be explored, some concepts can be generalized but often, we cannot find unique correspondances. In this section, we will give an introduction and point out delicate subtleties. The definitions and results can be found in [13–15, 17, 32, 33, 35].

### 2.5.1 Presentation

The definition of determinantal ideals and determinantal singularities is chosen as in [17] (cf. Definition 1.1.and Definition 1.13.).

**Definition 2.59** (Determinantal Ideals and Singularities)**.**

1. An ideal $I \subseteq \mathbb{C}\{\underline{x}\}$ is called *determinantal ideal of type* $(m, n, t)$ iff there is a matrix $M \in \operatorname{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ and $1 \leq t \leq \min\{m, n\}$ such that $I$ can be generated by the $t$-minors of $M$. We write $I = \operatorname{minor}(M, t)$.

2. A *determinantal singularity of type* $(m, n, t)$ is a singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ defined by a determinantal ideal $I$ of type $(m, n, t)$ such that $\operatorname{codim} I = (m-t+1)(n-t+1)$. Translating to represetatives of the germ, this means: For every $p \in X$ there is a neighbourhood $U$ of $p$ in $\mathbb{C}^N$ (a representative of the germ) and a determinantal ideal $I$ of type $(m, n, t)$ such that

$$X \cap U = V(I) = V(\text{minor}(M, t)), \quad \mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^N}/I.$$

The codimension property ensures the choice of a suitable matrix. For instance, the 2-minors of the matrix

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & y^2 \end{pmatrix}$$

define the ideal $\langle 0, xy, xy^2 \rangle = \langle xy \rangle$ which defines a hypersurface singularity (a singularity of codimension 1). The expected codimension due to the matrix structure would be $(m-t+1)(n-t+1)) = (2-2+1)(3-2+1) = 2$ and does not coincide with the codimension of the singularity. In this example, we see that substracting the $y$-th multiple of the second column to the third column gives rise to an empty column. Pathological examples like this can be avoided by forcing the codimension of the singularity to be the expected one.

In some classes of matrices, there is additional structure amongst the minors which has an impact on the expected codimension of the vanishing locus. Hence, the codimension property has to be adapted to matrix size and matrix structure, it depends on the degrees of freedom in a matrix. Famous classes are for example „symmetric determinantal singularities“ or „skew symmetric determinantal“ singularities.

**Definition 2.60.** [17, Definition 1.13.] [(Skew) Symmetric Determinantal Singularities]

1. A *symmetric determinantal singularity (of type* $(m, m, t)$*)* is a singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ defined as a germ of the vanishing locus of the minors of a symmetric square matrix $M \in \operatorname{Mat}(m, m, \mathbb{C}\{\underline{x}\})$ such that

$$\operatorname{codim}_0 X = \frac{1}{2}(m-t+1)(m-t+2).$$

2. A *skew symmetric determinantal singularity (of type* $(m, m, t)$*)* is a singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ defined as a germ of the vanishing locus of the minors of a skew symmetric square matrix $M \in \operatorname{Mat}(m, m, \mathbb{C}\{\underline{x}\})$ such that

$$\operatorname{codim}_0 X = \frac{1}{2}(m-2t+1)(m-2t+2).$$

**Example 2.61.**

1. Hypersurface singularities are determinantal singularities of type (1,1,1). Let $f \in \mathbb{C}\{\underline{x}\}$ and consider $M = \begin{pmatrix} f \end{pmatrix}$, $t = 1$, then $I = \langle f \rangle$ is a determinantal ideal of type $(1,1,1)$ and $(X,0)$ defines the hypersurface singularity given by $f$.

2. Complete intersection singularities are determinantal singularities of type $(1,k,1)$. Let $f_1,\dots,f_k$ be a set of generators defining a complete intersection singularity. Consider $M = \begin{pmatrix} f_1,\dots,f_k \end{pmatrix}$ and $t = 1$, then $I = \langle f_1,\dots,f_k \rangle$ is a determinantal ideal of type $(1,k,1)$ and $\operatorname{codim} I = (1-1+1)(k-1+1) = k$. Hence, $(V(I),0)$ is a complete intersection singularity.

3. Let $M = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix}$ and $t = 2$, then $I = \langle xy, xz, yz \rangle$ is a determinantal ideal of type (2,3,2) and $(V(I),0)$ is a determinantal singularity. The matrix $\begin{pmatrix} x & x+z & z \\ 0 & y+z & z \end{pmatrix}$ defines the same determinantal ideal and singularity. The intersection of the three planes given by $V(xy,xz,yz)$, which is the union of the three coordinate axes (see Example 2.12.2.5).

4. The next example can be found in [17]:
The 2-minors of $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}$ and the 2 minors of $\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$ define the same Ideal
$$I = \langle x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_0x_4 - x_1x_3, x_1x_3 - x_2^2,\ x_1x_4 - x_2x_3,\ x_2x_4 - x_3^2 \rangle.$$
Consider the singularity $(X,0) \subseteq (\mathbb{C}^5,0)$ defined by $X = V(I)$. Calculations show (see [17]) that
$$\dim X = \dim \mathbb{C}\{\underline{x}\}/I = 2.$$
We can realize this singularity as a determinantal singularity of type $(2,4,2)$ using the first matrix. The expected codimension due to the matrix structure is
$$(2-2+1)(4-2+1) = 3$$
and coincides with the codimension of the singularity.
At the same time, we can realize this singularity as a symmetric determinantal singularity of type $(3,3,2)$ as the expected codimension due to the matrix structure is
$$\frac{1}{2}(3-2+1)(3-2+2) = \frac{2\cdot 3}{2} = 3$$
and coincides with $\operatorname{codim} \mathbb{C}\{\underline{x}\}/I$.
Hence, there may be different approaches we can use to study the singularity in

the context of a specific kind of matrices and their minors. Each approach provides different information, as we will see when we study their deformations.
Remark: There are different perspectives on a singularity, each of these perspectives provides methods to study it and gives new information, but there is no 1:1 correspondence between a singularity and a matrix defining it. The information we get only captures a specific part of the behaviour of the singularity. It strongly depends on the chosen algebraic description.

The realization of singularities as determinantal singularities is the one we shall focus on. Still, there are many different matrices defining the same determinantal ideal or at least the same „type“ of determinantal ideal (up to some equivalence relation).
Firstly, some operations we perform on a matrix do not change the determinantal ideal defined by the minors of a matrix, others do not change the topological type of the singularity defined by the minors. Therefore, we can work with the equivalence class of matrices erasing this irrelevant information.

**Definition 2.62.** [32, Definition 2.1.1.]
$\mathcal{G} := \big(\operatorname{GL}(m, \mathbb{C}\{\underline{x}\})\big) \times \big(\operatorname{GL}(n, \mathbb{C}\{\underline{x}\})\big) \ltimes \mathcal{R}$ is called *geometric group* and it's a subgroup of the contact group $\mathcal{K}$ .

**Remark 2.63.**

1. By Gauß, row and column operations of a matrix $M \in \operatorname{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ do not change the determinantal ideal $I$ of type $(m, n, t)$ defined by the $t$-minors of $M$.
2. As in [15], let $A, B \in \operatorname{Mat}(m, n, \mathbb{C}\{\underline{x}\})$. We define an equivalence relation by

$$A \sim B :\iff \exists (P, Q, \phi) \in \mathcal{G} : A = P \cdot \phi^\star(B) \cdot Q,$$

where $\phi^\star$ is a map applying $\phi$ to every entry of $B$. We say that $A$ and $B$ are GL-equivalent (in other literature often $\mathcal{G}$-equivalent).

Secondly, for every determinantal singularity we can choose a defining matrix with non-units in its entries. In analogy to minimizing a free resolution, the entries of the defining matrix can be chosen in the maximal ideal of the power series ring.

**Remark 2.64.** If a determinantal singularity is defined by a matrix which has units in some entries, then instead, this singularity can be defined by a matrix of smaller size without units in its entries, i.e., with entries in the maximal ideal of $\mathbb{C}\{\underline{x}\}$:
Let $M \in \operatorname{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ be a matrix with a unit $u \in \mathbb{C}\{\underline{x}\}^\star$ in one of its entries. Using

suitable row and column operations we can find the unit in the lower right corner of $M$. We write

$$M = \begin{pmatrix} \star & v \\ w & u \end{pmatrix},$$

where $v \in \mathrm{Mat}(1, n-1, \mathbb{C}\{\underline{x}\}), w \in \mathrm{Mat}(m-1, 1, \mathbb{C}\{\underline{x}\})$. Divide every entry in the last column by the unit $u$. Subtracting a suitable multiple of the last row from the other rows, we get

$$\begin{pmatrix} \star & v \\ w & u \end{pmatrix} \sim \begin{pmatrix} \star & vu^{-1} \\ w & 1 \end{pmatrix} \sim \begin{pmatrix} \star & 0 \\ w & 1 \end{pmatrix}.$$

Now, using the last column, we can erase $w$ by subtracting a suitable multiple of the last column from the other columns

$$\begin{pmatrix} \star & 0 \\ w & 1 \end{pmatrix} \sim \begin{pmatrix} \star & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\tilde{M} \in \mathrm{Mat}(m-1, n-1, \mathbb{C}\{\underline{x}\})$. The maximal minors of $M$ define the same ideal as the maximal minors of the latter matrix, which are equal to the maximal minors of the matrix $\tilde{M}$ in the upper left corner (determinant of block matrices). If $\tilde{M}$ still contains a unit, we iterate these GL-operations until we get a matrix $\hat{M} \in \mathrm{Mat}(m-k, n-k, \mathfrak{m}_{\mathbb{C}\{\underline{x}\}})$ with $k < \min\{m, n\}$ such that

$$M \sim \begin{pmatrix} \hat{M} & 0 \\ 0 & (1)_k \end{pmatrix}.$$

The ideal generated by the maximal minors of $M$ is equal to the one generated by the maximal minors of $\hat{M}$ and none of the entries in $\hat{M}$ is a unit.
Without loss of generality, we choose $\hat{M}$ as a defining matrix for the determinantal singularity defined by $M$. In the power series ring this translates to the fact that no constant terms appear in the matrix $\hat{M}$.

Analoglously to the case of hypersurface singularities, we can get a lot of information out of the monomials up to a fixed degree. By truncating the power series we created an algebraic group acting on them, now, we can also truncate the terms whose degree exceeds some $k \in \mathbb{N}$ in each entry of a matrix defining a determinantal singularity.

**Definition 2.65.** [17, Chapter 2.1]
The *k-jet* of a matrix $M \in \mathrm{Mat}(m, n, \mathbb{C}\{x_1, \dots, x_N\})$ is the matrix of $k$-jets of the entries:

$$j_k M = (j_k(M_{ij}))_{ij}.$$

As before, „determinacy“ reduces the problem of studying matrices containing power series in its entries to a problem of studying matrices containing polynomials up to a fixed degree. „Finite determinacy“ allows to choose a truncated representative of a GL-equivalence class of matrices.

**Definition 2.66.** [17, Definition 2.6.]
A matrix $M \in \mathrm{Mat}(m, n, \mathbb{C}\{x_1, \ldots, x_N\})$ is *k-finitely determined* if

$$j_k M = j_k N \implies M \sim N.$$

Similarly to the grading of the polynomial ring w.r.t. slices of quasihomogeneous polynomials, we want to find a grading of matrices into quasihomogeneous matrices. In order to reach this goal, we need a weight matrix, additionally to the weights for the variables. The weighted slices w.r.t. a suitable weight will be smaller, which means we have to treat fewer cases and a vector space dimension argument will deliver a tool for the classification of simple singularities.

**Definition 2.67.** [15, Definition 2.4.]
A matrix $M \in \mathrm{Mat}(m, n, \mathbb{C}\{x_1, \ldots, x_N\})$ is called *quasihomogeneous of type* $(D; \omega) \in \mathrm{Mat}(n, m, \mathbb{N}_0) \times \mathbb{N}^N$, if

1. every entry $M_{ij}$ is quasihomogeneous of weighted degree $D_{ij}$ w.r.t. $\omega$ and
2. there are relative row and column weights, i.e., $D_{ij} - D_{ik} = D_{lj} - D_{ik}$ for all $1 \leq i, l \leq n$ and $1 \leq j, k \leq m$.

For an arbitrary matrix $L \in \mathrm{Mat}(m, n, \mathbb{C}\{x_1, \ldots, x_N\})$ the relative matrix weight is defined as

$$\nu_{(D;\omega)}(L) := \inf_{i,j}\{\nu_\omega(L_{ij}) - D_{ij}\}.$$

Using jet-spaces, we can consider the truncated objects w.r.t. some weight or without weights (that are different from $\omega = (1, \ldots, 1)$). For finite determinacy, Miriam Pereira proved the following criterion for finite determinacy (see [32]):

**Theorem 2.68.** [32, Theorem 2.3.1.] [Infinitesimal Criterion of Finite Determinacy]
Let $M \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ and $\mathfrak{m}$ the maximal ideal of $\mathbb{C}\{\underline{x}\}$. Let $k \in \mathbb{N}$ be a natural number such that

$$\mathfrak{m}^{k+1} \cdot \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\}) \subseteq \mathfrak{m}^2 \cdot \langle \frac{\partial M}{\partial x_1}, \ldots, \frac{\partial M}{\partial x_N} \rangle + \mathfrak{m} \cdot \mathrm{im}(g),$$

where

$$g : \mathrm{Mat}(m, m, \mathbb{C}\{\underline{x}\}) \times \mathrm{Mat}(n, n, \mathbb{C}\{\underline{x}\}) \to \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\}), \quad (A, B) \mapsto AM + MB.$$

Then, $M$ is $k$-finitely determined.

This criterion can be adapted to quasihomogeneous matrices w.r.t. to a weight and weight matrix, as in the hypersurface case, which yields a weighted determinacy criterion. For finite determinacy of determinantal singularities there are some powerful theorems. A theorem from Maria Aparecida Soares Ruas and Imran Ahmed (see [1]) states that finite determinacy holds for sufficiently general matrices, which means it is an expectable property.

The following theorem by Jim Damon (cf. [9], Theorem 9.3 and Theorem 10.2) gives further important equivalences regarding the action of a geometric subgroup [2] $\mathcal{N}$ of the contact group $\mathcal{K}$:

**Theorem 2.69.** Let $M \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ and $\mathcal{N}$ be a geometric subgroup of $\mathcal{K}$. Then the following statements are equivalent:

1. $\dim_{\mathbb{C}} \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})/T\mathcal{N}_e(M) < \infty$.
2. $M$ is $\mathcal{N}$-finitely determined.
3. A $\mathcal{N}$-versal unfolding exists.

### 2.5.2 Generic determinantal singularity and EIDS

The „generic determinanatal singularity“ is first of all an easy example for determinantal singularities and at the same time represents a whole class of determinantal singularities. We define the generic determinantal variety in terms of [13, Chapter 1]:

**Definition 2.70.** We define

$$M_{m,n} := \mathrm{Mat}(m, n, \mathbb{C}) \cong \mathbb{C}^{m \cdot n}$$

and the subvariety

$$M^t_{m,n} := \{M \in M_{m,n} \mid \mathrm{rk}(M) < t\}$$

of codimension $(m-t+1)(n-t+1)$ in $M_{m,n}$. We call $M^t_{m,n}$ *generic determinantal variety of type* $(m, n, t)$.

The singular locus of $M^t_{m,n}$ is given by $M^{t-1}_{m,n}$ (cf. [13]). The union of sets $M^i_{m,n} \setminus M^{i-1}_{m,n}$ for $i \in \{1, \ldots, t\}$ yields a partition of $M^t_{m,n}$. To identify this partition as a „Whitney stratification“, we introduce the definition thereof.

[2] By [9], a geometric subgroup is „defined geometrically“ in that the germs of diffeomorphisms are defined by their behavior with respect to projections, subspaces, groups acting on the spaces and other such geometric objects. Such „geometric“ subgroups inherit certain natural properties from $\mathcal{K}$.

**Definition 2.71.** [31, Definition 4.1.1.] [Stratification]
A *stratification* of an algebraic variety $X$ is a finite partition

$$X = \bigcup_{i=1}^{s} X_i$$

of $X$ into locally closed connected smooth subvarieties $X_1, \cdots, X_s$, called the *strata* of the stratification, such that the closure of each stratum is a union of strata.

**Definition 2.72.** [31, Definition 4.3.2.] [Whitney Stratification]
A *Whitney stratification* of an algebraic variety $X$ is a stratification $(X_\alpha)_{\alpha \in A}$ of $X$ which satisfies the following *Whitney condition*[3]:

1) A pair of strata $(X_\alpha, X_\beta)$ is (b)-regular at $y \in X_\beta$ iff for all sequences $\{x_i\} \in X_\alpha$ and $\{y_i\} \in X_\beta$ with $y = \lim_{i \to \infty} y_i$ such that (in a chart) $\{T_{x_i} X\}$ tends to $\tau$ and the lines $\overline{x_i y_i}$ tend to $\lambda$, we have $\lambda \in \tau$. A stratification is (b)-regular iff every pair of strata is (b)-regular at each point.

and the frontier condition:

2) For any $(\alpha, \beta) \in A \times A$ such that $X_\alpha \cap \overline{X_\beta} \neq \emptyset$ we have $X_\alpha \subseteq \overline{X_\beta}$.

**Remark 2.73.** [13, Chapter 1]
$(M_{m,n}^{i+1} \setminus M_{m,n}^{i})_{0 \leq i \leq t-1}$ defines a Whitney stratification of $M_{m,n}^t$.

From different point of views, it makes sense to introduce determinantal singularities in a different way. For instance, they might be introduced using the generic determinantal variety.

**Remark 2.74.** [13, Chapter 1] An alternative definition of a determinantal singularity can be given by considering $M \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ as a map

$$M : \mathbb{C}^N \longrightarrow M_{m,n}$$

and the *determinantal variety*

$$X_M^t = M^{-1}(M_{m,n}^t).$$

The germ $(X_M^t, 0)$ is a determinantal singularity of type $(m, n, t)$ defined by $M$, iff

$$\operatorname{codim} X_M^t = (m - t + 1)(n - t + 1).$$

[3] There are two Whitney conditions, (a)-regularity and (b)-regularity. As (b)-regularity implies (a)-regularity (see [31]), (b)-regularity suffices for the definition of a Whitney stratification.

$M_{m,n}^t$ is the most convenient example for determinantal singularities. Even if the singular locus gives non-isolated singularities, there is only one singular point (w.l.o.g. the origin) which shows interesting behaviour, we can control the other singular points in some sense (they are rigid). These singularities are called „essentially isolated determinantal singularities (EIDS)“.

**Definition 2.75.** [13, Chapter 1] Let $M : \mathbb{C}^N \to M_{m,n}$ be a holomorphic map germ defining a determinantal singularity by the vanishing locus of the ideal generated by the $t$-minors of $M$.
A point $x \in M^{-1}(M_{m,n}^t)$ is called *essentially non-singular*, iff the map $M$ is transverse to the stratum $M_{m,n}^i \setminus M_{m,n}^{i-1}$ containing $M(x)$ (i.e., $i = \mathrm{rk}(M(x)) + 1$).
A determinantal singularity is called an *Essentially Isolated Determinantal Singularity (EIDS)* iff $M$ has only essentially non-singular points in a punctured neighbourhood of the origin.

A theorem from Miriam Pereira (see [32]) gives an equivalence between finite determinacy and the property to be an EIDS:

**Theorem 2.76.** [32, Theorem 2.4.1.]
Let $M \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ define a determinantal singularity. Then the following statements are equivalent:

1. $M$ is GL-finitely determined.
2. $M$ defines an EIDS.

### 2.5.3 Unfoldings and Deformations

Deformations contain a wealth of information about the behaviour of perturbations of the equations defining a singularity. Unfoldings, on the other hand, provide a rather coarse method for perturbing these defining equations. The main idea behind unfoldings is to add additional monomials, each with extra parameters, to the equations. Deformations are more involved as they contain extra conditions to exclude pathological behaviour. In favorable circumstances, there is a correspondence between unfoldings and deformations, such as in the case of hypersurface singularities and complete intersection singularities. For matrices, it is known that a family of perturbations is flat, as shown in [28]. Consequently, unfoldings of matrices induce deformations of the singularity defined by those matrices. However, the converse is not true — not every deformation can be realized by an unfolding of a matrix. Even when studying unfoldings, certain difficulties require careful consideration.

**Definition 2.77.** [17, Definition 1.16.]

Let $M \in \mathrm{Mat}(m.n, \mathbb{C}\{\underline{x}\})$ and let $(X_M, 0)$ be a determinantal singularity of type $(m, n, t)$ defined by a matrix $M$.

1. An *unfolding of $M$ on $k$ parameters* is a map

$$(\mathbb{C}^N, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^{m\times n}, 0) \times (\mathbb{C}^k, 0), (x, t) \mapsto (\mathbf{M}(x, t), t).$$

2. A *determinantal deformation of* $(X_M, 0)$ is a deformation of $(X_M, 0)$ induced by an unfolding of $M$.

3. A given deformation $(X, 0) \hookrightarrow (\mathcal{X}, 0) \xrightarrow{\pi} (S, 0)$ of an arbitrary singularity $(X, 0) \subset (\mathbb{C}^N, 0)$ is called determinantal, iff there exists a matrix $A \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ and an integer $t$ such that $(X, 0) \cong (X_A^t, 0)$ is determinantal of type $(m, n, t)$ and if there exists an unfolding

$$\mathbf{A} : (\mathbb{C}^N, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^{m\times n}, 0)$$

of $A$ together with a commutative diagram

$$\begin{tikzcd}
(X,0) \arrow[r, hook] \arrow[d] & (\mathcal{X},0) \arrow[r] \arrow[d, "\pi"] & (\mathcal{X}_A,0) \arrow[d] \\
\{0\} \arrow[r] & (S,0) \arrow[r, "\Psi"] & (\mathbb{C}^k,0)
\end{tikzcd},$$

where $(\mathcal{X}_A, 0) \to (\mathbb{C}^k, 0)$ is the family induced from $\mathbf{A}$.

**Proposition 2.78.** [17, Lemma 1.15.]

Let $M \in \mathrm{Mat}(m.n, \mathbb{C}\{\underline{x}\})$ and let $(X_M, 0)$ be a determinantal singularity of type $(m, n, t)$ defined by a matrix $M$. Every unfolding of $M$ induces a deformation

$$\begin{tikzcd}
(X_M,0) \arrow[r, hook] \arrow[d] & (\mathcal{X}_M,0) \arrow[r, hook] \arrow[d, "\pi"] & (\mathbb{C}^N,0)\times(\mathbb{C}^k,0) \arrow[dl] \\
\{0\} \arrow[r, hook] & (\mathbb{C}^k,0) &
\end{tikzcd}$$

of the germ $(X_M, 0)$.

As any unfolding induces a deformation it is easy to believe that there are simply more unfoldings, but all the information encoded in the unfoldings can be embedded into deformation theory. In fact, we see pathological cases in any direction. Some information encoded in unfoldings gets lost when inducing a deformation while not all deformations can be induced by unfoldings.

**Example 2.79.** [17, Example 1.3.13.]

1. Let $M = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$ and $(X_M^2, 0)$ the determinantal singularity defined by the 2-minors of $M$.
   A versal unfolding of $M$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, induced deformations perturb the complex space germ, which is defined by $x^2 - yz$ by the term $-t^2$, $t \in \mathbb{C}$. A versal deformation is given by the perturbation of $x^2 - yz$ by the term $u \in \mathbb{C}$. Hence, unfoldings give a 2:1 cover of all deformations.

2. Let $M = \begin{pmatrix} x_1 & 0 & x_3 & \alpha x_4 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}$, where $\alpha \in \mathbb{C} \setminus \{0, 1\}$ and $(X_M^2, 0)$ the determinantal singularity defined by the 2-minors of $M$.
   Geometrically, this is the union of the four coordinate axes in $(\mathbb{C}^4, 0)$. A versal unfolding of $M$ is given by $(\mathbb{C}^5, 0)$, generated by

$$\begin{pmatrix} 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   A versal deformation is given by the 5-dimensional cone of the Segre embedding. Hence, $\begin{pmatrix} 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is non-trivial unfolding but induces a trivial deformation.

The choice of a presentation matrix for a singularity leads to different theoretical frameworks, affecting deformation theory. Consequently, generalizing known concepts and finding statements about the complex space germ independent of the presentation can be challenging.

### 2.5.4 CMC2 singularities

In the context of Cohen-Macaulay codimension 2 singularities, we can avoid the difficulties of the presentation of the singularity. A very important theorem in this thesis is the Hilbert-Burch Theorem. Thanks to the Hilbert-Burch Theorem, the theory of determinantal singularities is significantly simplified in this case.

**Theorem 2.80.** [14, Theorem 20.15.] [Hilbert-Burch Theorem]
Let $R$ be a local ring, $I \subset R$ an ideal of $R$ and $n \in \mathbb{N}$.

1. Let $\mathcal{F} : 0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$ be an exact sequence and $F_1 \cong R^n$. Then

   (a) $F_2 \cong R^{n-1}$,

(b) there is a non-zero-divisor $a \in R$ such that $I = a \cdot \text{minor}(\varphi_2, n-1)(\varphi_2)$, where $\text{minor}(\varphi_2, n-1)$ is the ideal generated by the $(n-1)$-minors of the representation matrix of $\varphi_2$,

(c) the $i$-th entry of the representation matrix of $\varphi_1$ is $(-1) \cdot a$ times the minor obtained from $\varphi_2$ by leaving out the $i$-th row and

(d) $\text{depth}(\text{minor}(\varphi_2, n-1)) = 2$.

2. If $\varphi_2$ is an arbitrary $(n-1) \times n$ matrix with $\text{depth}(\text{minor}(\varphi_2, n-1)) \geq 2$, $a \in R$ a non-zerodivisor and $\varphi_1$ defined as $(-1) \cdot a$ times the minor obtained from $\varphi_2$ by leaving out the $i$-th row. Then

$$0 \longrightarrow R^{n-1} \xrightarrow{\varphi_2} R^n \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

is a free resolution of $R/I$ and $I = a \cdot \text{minor}(\varphi_2, n-1)$.

This leads to a very important application of Cohen-Macaulay singularities of codimension 2 (short: CMC2).

**Theorem 2.81.** [15, Proposition 2.1.][Application of Hilbert-Burch] Let $M \in \text{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ and $\Delta$ the ideal given by the $n$-minors of $M$. Then

a) $\mathbb{C}\{\underline{x}\}/\Delta$ is Cohen Macaulay and $\text{codim}(\Delta) = 2$

b) If $X \subset \mathbb{C}^N$ is Cohen-Macaualay, $\text{codim}(X) = 2$ and $X = V(I)$ for some ideal $I \subseteq \mathbb{C}\{\underline{x}\}$, then $I = u\Delta$, $u \in \mathbb{C}\{\underline{x}\}$ a unit.

c) Any perturbation of $M$ gives rise to a deformation of $X$.

d) Any deformation of $X$ can be generated by a perturbation of the matrix $M$.

There are two main observations of this theorem:

1. The CMC2 singularities correspond to matrices of size $n \times (n+1)$, where all entries are non-units. The size of this matrix is fixed, the matrix is unique up to GL-equivalence.

2. Every deformation of a CMC2 singularity corresponds to a determinantal deformation of the matrix defining the singularity, i.e., for simplicity it suffices to check the determinantal deformations of the defining matrix.

The first step towards a classification of the simple non-isolated CMC2 singularities leads to the infinitesimal deformations. These deformations in an „infinitesimally small neighbourhood of the origin“ are encoded in the $T^1$ often referred to as *Tjurina module*. For CMC2 singularities the $T^1$ has a nice presentation:

**Theorem 2.82.** [15, Lemma 2.6.]
The $T^1$ of a matrix $M$ defining a CMC2 singularity of type $(n, n+1, n)$ is given by

$$T^1(M) = \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})/\langle \frac{\partial M}{\partial x_i}, \mathrm{im}\, g \rangle,$$

where $i = 1, \ldots, N$ and

$$\begin{aligned} g : \mathrm{Mat}(n, n, \mathbb{C}\{\underline{x}\}) \oplus \mathrm{Mat}(n+1, n+1, \mathbb{C}\{\underline{x}\}) &\longrightarrow \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\}) \\ (A, B) &\mapsto AM + MB. \end{aligned}$$

The proof of this theorem in [15] does not force the singularities to be isolated, the statement works for non-isolated CMC2 singularities as well.
Using the $T^1$ we find methods for precise calculations excluding non-simple singularities.

# Chapter 3

# Determinantal Singularities - Properties and Methods

## 3.1 Simple non-isolated complete intersection singularities do not exist

Let $(X, 0)$ be a complete intersection singularity defined by $\underline{f} := (f_1, \ldots, f_k) \in \mathbb{C}\{x_1, \ldots, x_N\}^k$, i.e., $X = V(f_1, \ldots, f_k)$. In [21, p. 19], we find the definition of the *extended tangent space* $T\mathcal{K}_e\underline{f}$ which we immediately identify as

$$\langle f_1, \ldots, f_k \rangle \cdot \mathbb{C}\{\underline{x}\}^k + D\underline{f} \cdot \mathbb{C}\{\underline{x}\}^N.$$

By Proposition 2.56, we know about a non-isolated singularity $(X, 0)$ that

$$\dim T^1_{(X,0)} = \infty.$$

Referring to Proposition 2.57, we conclude that $\underline{f} = (f_1, \ldots, f_k)$ is not finitely-$\mathcal{K}$-determined. A singularity which is not finitely determined, cannot be simple, as monomials of arbitrary degree appear in $T^1_{(X,0)}$, hence, in a versal deformation. So, a non-isolated complete intersection singularity cannot be simple.

## 3.2 Examples of simple non-isolated determinantal singularities

Let

$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix}$$

define a determinantal singularity of type $(3, 4, 3)$, i.e., the singularity locally defined by the vanishing of the four 3-minors. As this singularity can be expressed as the preimage of the generic determinantal variety (see Definition 2.70), the singular locus of this singularity is given by the determinantal singularity defined by the 2-minors of $M$ (see [1, p. 1]). This is an example for a CMC2 singularity, hence, the deformations correspond to the perturbations of the matrix $M$. Consider any possible perturbation of $M$:

- Perturbations by constant terms create a unit in at least one entry. Via GL-operations, we get one of the following matrices

  $$\begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ x_{21} & x_{22} & x_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  The first two matrices define cylinders over the generic determinantal variety of type $(2, 3, 2)$ and $(1, 2, 1)$, respectively. The last matrix defines a cylinder over a point. All of these singularities are still generic determinantal varieties, but before deforming, we found them within strata outside of the origin. As we perturbed by a constant term, the singularity moves outside of the origin but it remains the same singularity. In the origin, the singular point becomes smoother.

- Linear perturbations terms can be caught by a linear coordinate transformation, renaming all variables. We get $M$ again.

- Any perturbation terms of degree at least 2 can be cancelled by a coordinate change of the form

  $$x_{ij} \mapsto x_{ij} - \mathrm{hot}(\underline{x}),$$

  where $\mathrm{hot}(\underline{x})$ is polynomial collecting all monomials in any variable of $\underline{x}$ of degree at least 2.

Therefore, the only deformation is the trivial one (which is not changing the singularity type). Such a singularity is called *rigid* and a rigid singularity is particularly simple. Hence, we get an example for a simple non-isolated singularity.

## 3.3 Simple (non-isolated) determinantal singularities are EIDS

**Proposition 3.1.** Let $A \in \mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ define a simple determinantal singularity $(X_A, 0)$. Then $A$ defines an EIDS.

*Proof.* By Theorem 2.76, every finitely determined matrix defines an EIDS. Therefore, we want to show, that a simple determinantal singularity is finitely determined. For this, we prove that their $T^1$ has finite dimension.
To define a neighbourhood in $\mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$ we define

$$\|A - B\| := \min\{\mathrm{ord}(a_{i,j} - b_{i,j}) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\},$$

where $A = (a_{i,j})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ and $B = (b_{i,j})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$. In a small neighbourhood of $A$ we find matrices $B$ such that $\|A - B\|$ is big, more precisely, for arbitrary $k \in \mathbb{N}$ there is a $B$ close enough to $A$ such that $\|A - B\| > k$. Now, we prove that $\dim T^1(A) < \infty$:
Assume that $\dim T^1(A) = \infty$. As matrices with finite dimensional $T^1$ are dense in the module of matrices $\mathrm{Mat}(m, n, \mathbb{C}\{\underline{x}\})$, for every $A$ and every $k$ we find a matrix $A_k$ such that $\|A - A_k\| \geq k$ and $\dim T^1(A_k) < \infty$. As $A$ and $A_k$ have different dimensional $T^1$, they are not in the same GL-orbit, so $A_k$ is not $k$-finitely determined (otherwise the same $k$-jet would force $A$ and $A_k$ to be in the same GL-orbit). As $\dim T^1(A_k) < \infty$, $A_k$ is finitely determined with some determinacy bound $k' > k$. Again, we find some $A_{k'}$ close to $A$ such that $\|A - A_{k'}\| \geq k'$ and $\dim T^1(A_{k'}) < \infty$. The dimensions of $T^1(A)$, $T^1(A_k)$ and $T^1(A_{k'})$ are pairwise different, so the three matrices have disjoint orbits. Repeatedly, we can increase the determinacy bound, decrease the size of the neighbourhood of $A$ and find infinitely many orbits of matrices in an arbitrarily small neighbourhood of $A$. This is a contradiction to the simplicity of the singularity defined by $A$. Hence, $\dim T^1(A) < \infty$. As all monomials of degree higher than $\dim T^1(A)$ can be expressed by monomials of lower degree, $A$ is finitely determined. Hence, $A$ defines an EIDS. □

## 3.4 Counting argument

The counting argument is a criterion, which excludes matrices that do not define a simple singularity. More precisely, for sufficiently general matrices (we call them generic matrices), the counting argument shows non-simplicity, while all non-generic matrices can deform into a generic matrix and therefore, do not define simple singularities either. We will now examine the idea of the counting argument. The main results can be found in [15].
In the first step, the variables $x_1, \ldots, x_N$ get weights $\omega_1, \ldots, \omega_N \in \mathbb{N}$, which leads to relative matrix weights for an arbitrary matrix $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ and therefore, $T^1(M)$ becomes a graded module over the graded ring $\mathbb{C}\{\underline{x}\}$ (cf. [15, p. 3998]):

$$T^1(M) = \bigoplus_{\nu \in \mathbb{Z}} T^1_\nu(M).$$

The dimension of the 0-th slice of this module is of particular interest for the counting argument. We want to see that the 0-th slice of the $T^1$ contains the starting jet of a matrix.

**Definition 3.2.** Given a weight $\omega \in \mathbb{N}^N$ and a matrix $M \in \mathrm{Mat}(n, m, \mathbb{C}\{\underline{x}\})$ the *starting jet* is defined as

$$j_\omega M := (j_{\nu_\omega(M_{ij})} M_{ij}).$$

In other words, the starting jet is the matrix that, in each entry, contains only monomials of lowest degree (which is the order of the entry).

If the starting jet w.r.t. a weight $\omega$ is a quasihomogeneous matrix of type $(D, \omega)$ for a suitable $D \in \mathrm{Mat}(n, m, \mathbb{C}\{\underline{x}\})$, then:

1. Every monomial in $M$ which does not appear in the starting jet has positive weight, i.e.,
$$T^1(M) = \bigoplus_{\nu \in \mathbb{N}_0} T^1_\nu(M).$$
2. The 0-th slice $T^1_0(M)$ contains only monomials of weighted degree 0, therefore
$$T^1_0(M) = T^1_0(j_\omega M).$$

To examine $T^1_0(M)$ we define the following sets (cf. [15, p.4000]):
For a weighted degree $d$ let $m_{d,1}, \dots, m_{d,r(\omega,d)}$ be the monomials of weighted degree $d$ w.r.t. weight $\omega$. Set

$$S_1 := \{m_{\omega_j,i} \frac{\partial M}{\partial x_j} \mid 1 \le i \le r(\omega, \omega_j), 1 \le j \le m\}$$
$$S_2 := \{m_{D_{l1} - D_{j1},i} R_{\ell,j} \mid 1 \le i \le r(\omega, D_{\ell 1} - D_{j1}), 1 \le \ell, j \le n\}$$
$$S_3 := \{m_{D_{1l} - D_{1j},i} C_{\ell,j} \mid 1 \le i \le r(\omega, D_{1\ell} - D_{1j}), 1 \le \ell, j \le n+1\}$$

Define $Q_{(D,\omega)}$ as the set of quasihomogeneous matrices of type $(D, \omega)$. Now, we know

$$T^1_0(M) = T^1_0(j_\omega M) = Q_{(D,\omega)} / (S_1 \cup S_2 \cup S_3).$$

There are at least two relations between the elements of $S_1, S_2, S_3$ (cf. [15, p.4000]):

- $\sum_{i=1}^{n} R_{i,i} = \sum_{j=1}^{n+1} C_{j,j}$
- $\sum_{i=1}^{m} \omega_i x_i \frac{\partial M}{\partial x_i} = \sum_{i=1}^{n+1} D_{1i} C_{i,i} + \sum_{i=1}^{n} (D_{i1} - D_{11}) R_{i,i}$

Hence, the dimension of an orbit in $T_0^1(M)$ is bounded by $\#S_1 + \#S_2 + \#S_3 - 2$, where $\#S_i$ denotes the cardinality of the sets $S_i$, $i \in \{1, 2, 3\}$. If this number is smaller than $\dim Q_{(D,\omega)}$, a neighbourhood in $Q_{(D,\omega)}$ cannot be covered by finitely many orbits (w.r.t. degree 0 operations, i.e., GL-operations of the matrix using monomials and coordinate changes of weighted degree 0). This translates to the following: There is a quasihomogeneous matrix $N$ of type $(D, \omega)$ such that for infinitely many $t_1, t_2 \in \mathbb{C}$ the elements $j_\omega M + t_1 N$ and $j_\omega M + t_2 N$ are in different equivalence classes in $T_0^1(M)$. We can conclude, that $M + t_1 N$ and $M + t_2 N$ must be in different equivalence classes in $T^1(M)$ as the 0-th slice of these matrices is already in different equivalence classes. Therefore, $M$ deforms into infinitely many equivalence classes and does not define a simple singularity. This argumentation gives a criterion to exclude non-simple matrices.

**Proposition 3.3** (Counting argument)**.** If $\#S_1 + \#S_2 + \#S_3 - 2 < \dim Q_{(D,\omega)}$, the matrix $M$ cannot define a simple singularity.

# Chapter 4

# First steps of the classification

In this chapter, we want to reduce the classification of CMC2 singularities to a classification of matrixes with bounded ambient space dimension and bounded number of variables in the 1-jet. We will recall the results of previous classifications of CMC2 singularities before proceeding to find a reduction to these known results.

## 4.1 Reduction of matrix structure

We will use the counting argument to start with a sufficiently small number of candidates for simple singularities. As we have seen, without loss of generality, we can consider matrices with vanishing 0-jet (non-units in each entry). Next, we want to get information about the 1-jet of simple matrices. Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$. Consider the 1-jet $j_1 M$ of $M$.

Each of the $n^2 + n$ entries of $j_1 M$ is a linear form $\ell_i$ of the $N$ variables of the power series ring over $\mathbb{C}$. We write

$$\ell_i = \sum_{j=1}^{N} a_j^{(i)} x_j,$$

with suitable coefficients $a_j^i \in \mathbb{C}$ for each $1 \leq i \leq n^2 + n$. Now, we define the matrix $C := (a_j^{(i)})_{i,j} \in \mathrm{Mat}(n^2 + n, N, \mathbb{C})$ containing the coefficients of these linear forms. Each row describes the vector of coefficients of a linear form. We have $\mathrm{rk}(C)$ linearly independent linear forms, say $\ell_1, \ldots, \ell_{\mathrm{rk}(C)}$. Therefore,

$$x_i := \begin{cases} \ell_i & 1 \leq i \leq \mathrm{rk}(C) \\ x_i & \mathrm{rk}(C) < i \leq N \end{cases}$$

is a valid coordinate change. Clearly, $\mathrm{rk}(C) \leq \min\{n^2 + n, N\}$, so we will now consider 2 different cases:

1. $\mathrm{rk}(C) = n^2 + n$ (necessarily $n^2 + n \leq N$):

We get

$$j_1M \sim \begin{pmatrix} x_1 & \dots & x_{n+1} \\ \dots & \dots & \dots \\ x_{n^2} & \dots & x_{n^2+n} \end{pmatrix}.$$

This is the generic matrix that contains the monomials $x_1, \dots, x_{n^2+n}$ in its $n^2+n$ entries, filled from left to right. Now via these linear monomials all monomials of higher degree which might appear in $M$ might be cancelled by a suitable coordinate change, so

$$M \sim \begin{pmatrix} x_1 & \dots & x_{n+1} \\ \dots & \dots & \dots \\ x_{n^2} & \dots & x_{n^2+n} \end{pmatrix}.$$

This is the rigid singularity of size $n \times (n+1)$.

2. $\mathrm{rk}(C) < n^2 + n$:
   Before we perform the coordinate change, $l_1, \dots, l_{\mathrm{rk}(C)}$ are linearly independent. As soon as any linear form $l_i$ with $i > \mathrm{rk}(C)$ is added to this family, the family becomes linearly dependent. Hence, every $l_i$ with $i > \mathrm{rk}(C)$ can be expressed as a linear form in $l_1, \dots, l_{\mathrm{rk}(C)}$. After the coordinate change

$$x_i := \begin{cases} l_i & 1 \leq i \leq \mathrm{rk}(C) \\ x_i & \mathrm{rk}(C) < i \leq N \end{cases},$$

   the first $\mathrm{rk}(C)$ entries of $j_1M$ are $x_1, \dots, x_{\mathrm{rk}(C)}$ and the last $n^2+n-\mathrm{rk}(C)$ entries contain linear forms in $x_1, \dots, x_{\mathrm{rk}(C)}$.

To find conditions of simple matrices about the numbers $\mathrm{rk}(C)$ and $N$, we will consider two different pairs of weight matrix and vector $(D, \omega)$:

1. We will consider $(D, \omega)$, where $\omega = (1, \dots, 1)$ and $D = (D_{ij})$ with $D_{ij} = 1$ for all $i, j$. In this case, the 1-jet of every matrix is a quasihomogeneous matrix of type $(D, \omega)$, independent of the number of variables appearing in the 1-jet. The counting argument will give us a condition on the number $N$ of variables which is the dimension of the ambient space.

2. We will consider $(D, \omega)$, where $\omega = (\underbrace{2, \dots, 2}_{s}, \underbrace{1, \dots, 1}_{N-s})$ and $D = (D_{ij})$ with $D_{ij} = 2$ for an arbitrary number $1 \leq s \leq N$. We will see, that the counting argument gives a condition on $s$. If we consider the 1-jet of a matrix and set $s$ to be $\mathrm{rk}(C)$ then $x_1, \dots, x_s$ are the only variables appearing in the 1-jet. Again, the 1-jet is quasihomogeneous of type $(D, \omega)$ but now we get a condition on the number of variables appearing sufficiently general in the 1-jet of a simple matrix.

## 4.2 Dimension of ambient space

Consider $(D,\omega) = ((1)_{\substack{1\le i\le n \\ 1\le j\le n+1}}, \underbrace{(1,\dots,1)}_{N})$ and a quasihomogeneous matrix of type $(D,\omega)$. We obtain the following values for the 0-th slice of $T^1(j_1M)$ in this grading:

$$\begin{aligned}
\dim Q_{(D,\omega)} &= N\cdot n\cdot(n+1) = N(n^2+n)\\
\#S_1 &= N^2\\
\#S_2 &= n^2\\
\#S_3 &= (n+1)^2
\end{aligned}$$

Hence, if $M$ is simple, then

$$\begin{aligned}
N(n^2+n) &\le N^2+n^2+(n+1)^2-2 = N^2+n^2+n^2+2n+1-2\\
&= N^2+2n^2+2n-1 = N^2+2(n^2+n)-1,
\end{aligned}$$

which implies $N(n^2+n)-2(n^2+n)\le N^2-1$ and therefore

$$(N-2)(n^2+n) < N^2.$$

If $n=2$, this inequality holds for every $N$ i.e., for all $N$ we have $6(N-2)<N^2$.
If $n>2$, then $n^2+n\ge 12$ and we get

| $N$ | $N^2$ | $(N-2)(n^2+n)$ | |
|---|---|---|---|
| 1 | 1 | $(-1)(n^2+n)$ | |
| 2 | 4 | 0 | |
| 3 | 9 | $(n^2+n)$ | $\ge 12$ |
| 4 | 16 | $2(n^2+n)$ | $\ge 24$ |
| 5 | 25 | $3(n^2+n)$ | $\ge 36$ |
| 6 | 36 | $4(n^2+n)$ | $\ge 48$ |
| 7 | 49 | $5(n^2+n)$ | $\ge 60$ |
| 8 | 64 | $6(n^2+n)$ | $\ge 72$ |
| 9 | 81 | $7(n^2+n)$ | $\ge 84$ |
| 10 | 100 | $8(n^2+)$n | $\ge 96$ |
| 11 | 121 | $9(n^2+n)$ | $\ge 108$ |

So, if $M$ is simple and $n>2$, we need $N>10$. Using this fact, we get the condition

$$\begin{aligned}
(N+3) = (N+2)+1 &> (N+2)+\frac{4}{8}\\
&\ge (N+2)+\frac{4}{N-2} = \frac{(N-2)(N+2)+4}{N-2}\\
&= \frac{N^2}{N-2} > \frac{(N-2)(n^2+n)}{N-2} = n^2+n
\end{aligned}$$

Hence, $N+3>n^2+n$ and therefore,

$$N \geq n^2+n-2.$$

By looking at this set of weight and weight matrix, we get the information, that, in case of $n>2$, simple matrices can only appear for at least $n^2+n-2$ variables, i.e., in an ambient space of dimension at least $n^2+n-2$.

## 4.3 Variables in the 1-jet

Let $1 \leq s \leq N$ be an arbitrary number. Consider $(D,\omega) = ((2)_{\substack{1\leq i\leq n\\ 1\leq j\leq n+1}},)(\underbrace{2,\ldots,2}_{s},\underbrace{1,\ldots,1}_{N-s})$.
In this situation, there are $(N-s)$ monomials of weighted degree 1 and $s+\binom{N-s+1}{2}$ variables of weighted degree 2. We obtain the following values for the 0-th slice of $T^1(j_\omega M)$ in this grading:

$$\begin{aligned}
\dim Q_{(D,\omega)} &= (n^2+n)\left(s+\binom{N-s+1}{2}\right)\\
\#S_1 &= s\left(s+\binom{N-s+1}{2}\right)+(N-s)^2\\
\#S_2 &= n^2\\
\#S_3 &= (n+1)^2
\end{aligned}$$

Hence, if $M$ is simple, then

$$(n^2+n)\left(s+\binom{N-s+1}{2}\right) \leq s\left(s+\binom{N-s+1}{2}\right)+(N-s)^2+n^2+(n+1)^2-2$$

respectively

$$\begin{aligned}
0 &\leq (s-n^2-n)\left(s+\binom{N-s+1}{2}\right)+(N-s)^2+n^2+(n+1)^2-2\\
&= (s-n^2-n)\left(s+\frac{(N-s)(N-s+1)}{2}\right)+(N-s)^2+n^2+n^2+2n+1-2\\
&= (s-n^2-n)\left(s+\frac{(N-s)^2+(N-s)}{2}\right)+(N-s)^2+2n^2+2n-1 \quad (*)
\end{aligned}$$

1. If $s \geq n^2+n$, then $(s-n^2+n) \geq 0$, $\binom{N-s+1}{2}=0$ and as $n,N\in\mathbb{N}$, $s\leq N$, the whole sum in $(*)$ is positive.

2. If $s = n^2 + n - 1$, then $s - n^2 - n = -1$ and

$$(*) = n^2 + n + \underbrace{\frac{(N-s)^2}{2} - \frac{(N-s)}{2}}_{\geq 0} > 0$$

So, simple singularities can occur.

3. If $s = n^2 + n - 2$, then $s - n^2 - n = -2$ and

$$\begin{aligned}(*) &= -2s - (N-s)^2 - (N-s) + (N-s)^2 + 2n^2 + 2n - 1 \\ &= n^2 + n - N \geq 0\end{aligned}$$

is only possible if $N \leq n^2 + n$. So, simple singularities can occur, but only with up to $n^2 + n$ variables in total.

4. If $1 \leq s \leq n^2 + n - 4$, then $s - n^2 - n \leq -4$ and by using that simple singularities can only occur for $N \geq n^2 + n - 2$ we get

$$\begin{aligned}(*) &\leq -4s - (N-s)^2 - 2N + 2s + (N-s)^2 + 2n^2 + 2n - 1 \\ &\leq -2s - (N-s)^2 + 3 \leq 0\end{aligned}$$

if $s \geq 2$ or $(s = 1, N \geq 2)$. So, the only simple singularities can occur if $s = N = 1 \geq n^2 + n - 2$ which implies $n = 1$. These simple hypersurface singularities in $(\mathbb{C}, 0)$ are already completely classified (only $A_k$).

5. If $s = n^2 + n - 3$, then $s - n^2 - n = -3$ and

$$\begin{aligned}(*) &= -3s - \frac{3}{2}(N-s)^2 - \frac{3}{2}(N-s) + (N-s)^2 + 2n^2 + 2n - 1 \\ &= -\frac{1}{2}n^4 - n^3 + Nn^2 + 3n^2 + Nn - \frac{1}{2}N^2 + \frac{7}{2}n - \frac{9}{2}N - 1 \quad =: g(n, N)\end{aligned}$$

Calculations (see Appendix) show that

   (a) for $n \geq 3$: $g(n, N) < 0$
   (b) for $n = 2$: $g(n, N) < 0 \iff N > 4$, in the case $N \leq 3$ the only simple singularities are the simple space curve singularities in $(\mathbb{C}^3, 0)$ and fat points in $(\mathbb{C}^2, 0)$, which are isolated
   (c) for $n = 1$: the only simple singularities are simple complete intersection singularities, which are isolated

So, we see that the number $s$ must be at least $n^2 + n - 2$ (respectively at least $n^2 + n - 3$ if $n = 2$) to be able to find a simple singularity. This means that at least $n^2 + n - 2$ variables have to appear in the 1-jet of a simple matrix (respectively at least $n^2 + n - 3$ variables if $n = 2$). We conclude the following proposition:

**Proposition 4.1.** Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ and $C$ its matrix of coefficients. Then $M$ can only define a simple singularity, if

1. $N \geq n^2 + n - 2$,
2. if $n = 2$: $\mathrm{rk}(C) \geq n^2 + n - 3$, i.e., the 1-jet $j_1 M$ contains at least $n^2 + n - 3$ variables which appear sufficiently general,
3. if $n > 2$: $\mathrm{rk}(C) \geq n^2 + n - 2$, i.e., the 1-jet $j_1 M$ contains at least $n^2 + n - 2$ variables which appear sufficiently general.

**Remark 4.2.** There are four cases to find candidates for matrices which define a simple singularity: Among the matrices with

1. $n^2 + n$ variables appearing in the 1-jet,
2. $n^2 + n - 1$ variables appearing in the 1-jet,
3. $n^2 + n - 2$ variables appearing in the 1-jet and
4. 3 variables appearing in the 1-jet of a $(2 \times 3)$-matrix.

**Observation 4.3.** Consider $m$ linear forms $\ell_1, \ldots, \ell_m$ in $\mathbb{C}\{x_1, \ldots, x_N\}$, where at most $k$ of them are linearly independent. In the case of $N > k$ the map

$$\mathbb{C}\{\underline{x}\} \longrightarrow \mathbb{C}\{\underline{x}\}, x_i := \ell_i \quad \text{for} \quad 1 \leq i \leq k$$

is an automorphism. Therefore, any other linear form must depend only on $x_1, \ldots, x_k$ (otherwise this linear form $\ell$ together with $\ell_1, \ldots, \ell_k$ are linearly independent).

**Observation 4.4.** In the case of $n^2 + n$ variables in the 1-jet, the coordinate change described in the previous observation (with $k = n$) gives the following 1-jet:

$$j_1 M \sim \begin{pmatrix} x_1 & \ldots & x_{n+1} \\ \ldots & \ldots & \ldots \\ x_{n^2} & \ldots & x_{n^2+n} \end{pmatrix}.$$

The matrix $M$ might have additional monomials of higher degree in its entries, for example, consider the 2-jet

$$j_2 M \sim \begin{pmatrix} x_1 + f_1 & \ldots & x_{n+1} + f_{n+1} \\ \ldots & \ldots & \ldots \\ x_{n^2} + f_{n^2} & \ldots & x_{n^2+n} + f_{n^2+n} \end{pmatrix},$$

$f_1, \dots, f_{n^2+n} \in \mathfrak{m}^2_{\mathbb{C}\{\underline{x}\}} \setminus \mathfrak{m}_{\mathbb{C}\{\underline{x}\}}$. The coordinate change

$$x_i := x_i + f_i, \quad 1 \le i \le n^2 + n$$

erases all monomials of degree 2. In this manner, we can iteratively change coordinates to clean all monomials with increasing degree. In the power series ring, all monomials can be erased this way and we get

$$M \sim \begin{pmatrix} x_1 & \dots & x_{n+1} \\ \dots & \dots & \dots \\ x_{n^2} & \dots & x_{n^2+n} \end{pmatrix}.$$

This matrix defines the so called rigid singularity, as all perturbations of the matrix can be removed by coordinate changes, so the singularity did not change at all. Therefore, it defines a simple singularity, as it deforms only into itself.

**Observation 4.5.** If the 1-jet of $M$ contains $k$ linearly independent linear forms $\ell_1, \dots, \ell_k$, $k \in \{1, \dots n^2 + n\}$, the coordinate change

$$x_i := \ell_i$$

leads to a matrix with $x_1, \dots, x_k$ in $k$ of its entries. The other entries are linear forms depending only on $x_1, \dots, x_k$. Hence, $j_1 M$ is equivalent to a matrix containing only $x_1, \dots, x_k$ and $k$ of its entries are each given by one of $x_1, \dots, x_k$.

**Remark 4.6.** As the case of simple isolated singularities with 3 variables in the 1-jet of a $(2 \times 3)$-matrix is treated in [16], this work will focus on the cases of $n^2 + n - 1$ and $n^2 + n - 2$ variables appearing in the 1-jet of a $(n \times n + 1)$-matrix.

## 4.4 List of simple isolated CMC2 singularities

The simple isolated CMC2 singularities have been completely classified by Anne Frühbis-Krüger and Alexander Neumer (see [16]). The list is divided by the number of variables appearing in the 1-jet, which are 4, 5 or 6. The 1-jet is mentioned in the first column.

**Theorem 4.7.** [16, Theorem 3.6]
The simple isolated Cohen-Macaulay codimension 2 singularities in $(\mathbb{C}^6, 0)$ are listed in the following table:

Table 4.1: List of simple isolated CMC2 singularities in $(\mathbb{C}^6, 0)$

| Jet-Type | Type | Presentation Matrix | |
|---|---|---|---|
| $J^{(6,1)}$ | $\Omega_1$ | $\begin{pmatrix} x & y & z \\ v & w & a \end{pmatrix}$ | |
| $J^{(5,1)}$ | $\Omega_k$ | $\begin{pmatrix} x & y & z \\ v & w & x+a^{k+1} \end{pmatrix}$ | $k \geq 2$ |
| $J^{(5,2)}$ | $A_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k+1}+y^2+a^2 \end{pmatrix}$ | $k \geq 1$ |
| | $D_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k-1}+xy^2+a^2 \end{pmatrix}$ | $k \geq 4$ |
| | $E_6^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^4+a^2 \end{pmatrix}$ | |
| | $E_7^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+xy^3+a^2 \end{pmatrix}$ | |
| | $E_8^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^5+a^2 \end{pmatrix}$ | |
| | | $\begin{pmatrix} x & y & z \\ v & w & ax+y^k+a^l \end{pmatrix}$ | $k \geq 2, l \geq 3$ |
| | | $\begin{pmatrix} x & y & z \\ v & w & x^2+y^2+a^3 \end{pmatrix}$ | |
| $J^{(4,1)}$ | $F_{q,r}^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+wa & y+w^r+a^r \end{pmatrix}$ | $q, r \geq 2$ |
| | $G_5^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+w^2 & y+a^3 \end{pmatrix}$ | |
| | $G_7^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+w^2 & y+a^4 \end{pmatrix}$ | |
| | $H_{q+3}^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+w^2+a^q & y+wa^2 \end{pmatrix}$ | $q \geq 3$ |
| | $I_{2q-1}^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+w^2+a^3 & y+a^q \end{pmatrix}$ | $q \geq 4$ |
| | $I_{2r+2}^{\#}$ | $\begin{pmatrix} x & y & z \\ v & x+w^2+a^3 & y+wa^r \end{pmatrix}$ | $r \geq 3$ |
| $J^{(4,2)}$ | | $\begin{pmatrix} x & y & z \\ v & x+w^q+a^r & y^k+aw \end{pmatrix}$ | $q, r, k \geq 2$ |
| | | $\begin{pmatrix} x & y & z \\ v & x+w^2 & a^2+yw \end{pmatrix}$ | |
| | | $\begin{pmatrix} x & y & z \\ v & x+aw & a^2+yw+w^k \end{pmatrix}$ | $k \geq 3$ |
| | | $\begin{pmatrix} x & y & z \\ v & x+w^k & a^2+yw+w^3 \end{pmatrix}$ | $k \geq 3$ |
| | | $\begin{pmatrix} x & y & z \\ v & x+aw^k & a^2+yw+w^3 \end{pmatrix}$ | $k \geq 2$ |
| | | $\begin{pmatrix} x & y & z \\ v & x+w^3 & a^2+yw \end{pmatrix}$ | |
| | | $\begin{pmatrix} x & y & z \\ v & x+w^k & a^2+y^2+w^3 \end{pmatrix}$ | $k \geq 3$ |
| | | $\begin{pmatrix} x & y & z \\ v & x+aw^k & a^2+y^2+w^3 \end{pmatrix}$ | $k \geq 2$ |

# Chapter 5

# Candidates of type $(n, n+1, n)$ with $n^2 + n - 1$ variables in the 1-jet

In this chapter, we find a normal form for matrices containing $n^2 + n - 1$ variables in the 1-jet. For these objects, we will study the $T^1$ and find the connection to matrices of smaller size. In this way, we can reduce the problem to matrices of size $2 \times 3$. The following proposition summarizes the results in this chapter:

**Proposition 5.1.** Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{x_{1,1}, \ldots, x_{n,n+1}, a_1, \ldots, a_r\})$. Then for some $t \in \{0, \ldots, n-1\}$ and $P \in \mathfrak{m}^2$, which depends only on $\{x_{i,j} \mid t < i \leq n-1,\ t < j \leq n\}$ and variables $a_1, \ldots, a_r$ which did not appear in the 1-jet, we have

$$M \sim \begin{pmatrix} x_{1,1} & \cdots & \cdots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \cdots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \cdots & \cdots & x_{n,n} & \sum\limits_{i=1}^{t} x_{i,i} + P \end{pmatrix}.$$

$M$ cannot define a simple singularity, if $t < n-2$ and the simple candidates of size $n \times n+1$ are determined by the simple candidates of size $2 \times 3$.

## 5.1 Reduction of 1-jets

For reasons of convenience, we rename $x_1, \ldots, x_{n^2+n-1}$ to

$$x_{1,1}, \ldots, x_{n-1,n+1}, x_{n,1}, \ldots, x_{n,n}.$$

Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}, \underline{a}\})$, where $\underline{x} = (x_{1,1}, \ldots, x_{n-1,n+1}, x_{n,1}, \ldots, x_{n,n})$ are the variables appearing in the 1-jet, and $\underline{a} = (a_1, \ldots, a_r)$ are the variables not appearing in the 1-jet (i.e., $N = n^2 + n - 1 + r$). Consider the 1-jet of $M$, i.e.,

$$j_1 M = \begin{pmatrix} \ell_{1,1}(\underline{x}) & \dots & \ell_{1,n+1}(\underline{x}) \\ \vdots & & \vdots \\ \ell_{n,1}(\underline{x}) & \dots & \ell_{n,n+1}(\underline{x}) \end{pmatrix},$$

which is a matrix consisting of $n^2 + n$ linear forms $\ell_{i,j}$ in $\underline{x}$, but only $n^2 + n - 1$ of these linear forms are linearly independent. We place the linearly dependent linear form $\ell_{n,n+1}$ in the lower right entry. By a coordinate change, we can simplify the linear forms to new variables $x_{1,1}, \dots, x_{n-1,n+1}, x_{n,1}, \dots, x_{n,n}$ resulting in

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & \ell(\underline{x}) \end{pmatrix},$$

where $\ell(\underline{x}) = \sum_{1 \le i,j \le n} \lambda_{i,j} x_{i,j}$.

Next, we erase all terms in the last entry containing variables from the last column using row operations. By abuse of notation, the entries of the last row can be cleaned by the coordinate changes

$$x_{n,j} := x_{n,j} + \sum_{i=1}^{n-1} \lambda_{i,n+1} x_{i,j}, \quad 1 \le j \le n.$$

We obtain

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & \sum\limits_{1 \le i,j \le n} \tilde{\lambda}_{i,j} x_{i,j} \end{pmatrix},$$

where $\tilde{\lambda}_{i,j} = \lambda_{i,j} + \lambda_{i,n+1}\lambda_{n,j}$. Analogously, we continue by cleaning all terms in the last entry containing variables from the last row to arrive at

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & \sum\limits_{\substack{1 \le i \le n-1 \\ 1 \le j \le n}} \hat{\lambda}_{i,j} x_{i,j} \end{pmatrix},$$

where $\hat{\lambda}_{i,j} = \tilde{\lambda}_{i,j} + \tilde{\lambda}_{n,j}\tilde{\lambda}_{i,n+1}$.

Depending on the number of non-vanishing coefficients in the last entry we can encounter different possible 1-jets of the matrices representing simple CMC2 singularities.
If all coefficients in $\sum \hat{\lambda}_{i,j} x_{i,j}$ vanish, then the first possibility for a normal form of the 1-jet of a matrix with $n^2+n-1$ variables in the 1-jet is given by

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & 0 \end{pmatrix} =: J_{(n^2+n-1,0)}.$$

If at least one of the coefficients does not vanish, i.e., there is $\hat{\lambda}_{s,t} \neq 0$, then we arrive at

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & x_{1,1} + \sum\limits_{\substack{2 \leq i \leq n-1 \\ 2 \leq j \leq n}} \mu_{i,j} x_{i,j} \end{pmatrix},$$

where $(\mu_{i,j})_{i,j} \in \mathbb{C}$ are some new coefficients, in the following manner:
To switch the positions of $x_{1,1}$ and $x_{s,t}$, we switch the first and $s$-th row as well as the first and $t$-th column. The coordinate change, which defines the variable in the $i$-th row and $j$-th column as $x_{i,j}$, corrects the enumeration of the variables. Now, by abuse of notation, the last entry of the matrix has a non-vanishing coefficient $\mu_{1,1}$ of $x_{1,1}$, with the coordinate change

$$x_{1,1} := \mu_{1,1} x_{1,1}$$

this coefficient vanishes. We define the last entry as

$$h = x_{1,1} + \sum_{\substack{2 \leq i \leq n-1 \\ 2 \leq j \leq n}} \mu_{i,j} x_{i,j}.$$

Using the term $x_{1,1}$ in $h$, we can erase all variables of the sum which appear in the first row or the first column of the matrix. We use the coordinate change

$$x_{1,1} := x_{1,1} - \sum_{j=2}^{n} \mu_{1,j} x_{1,j},$$

add the $\mu_{1,j}$-th multiple of the $j$-th column to the first column and finish with the coordinate change

$$x_{i,1} := x_{i,1} - \sum_{j=2}^{n} \mu_{1,j} x_{i,j}, \quad 2 \leq i \leq n.$$

Analogously, we erase variables from the first column in the last entry of the matrix. The remaining polynomial $\sum_{\substack{2\le i\le n-1\\ 2\le j\le n}} \mu_{i,j}x_{i,j}$ in the last entry either vanishes, then

$$j_1M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & x_{1,1} \end{pmatrix} =: J_{(n^2+n-1,1)}$$

or another coefficient in $\sum_{\substack{2\le i\le n-1\\ 2\le j\le n}} \mu_{i,j}x_{i,j}$ doesn't vanish. We pick an arbitrary variable with non-vanishing coefficient and define it to be $x_{2,2}$, the coefficient of $x_{2,2}$ is $\mu_{2,2}$. We perform, by abuse of notation, the coordinate change

$$x_{2,2} := \mu_{2,2}x_{2,2}.$$

We can multiply the second row with $\mu_{2,2}$ and redefine all variables in the second row via

$$x_{2,j} := \mu_{2,2}x_{2,j} \text{ for } 1 \le j \le n+1$$

to be monic again. Now, we have

$$j_1M \sim \begin{pmatrix} x_{1,1} & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n} & x_{1,1} + x_{2,2} + \sum_{\substack{3\le i\le n-1\\ 3\le j\le n}} \mu_{i,j}x_{i,j} \end{pmatrix}$$

and we iterate the previous GL-operations on

$$\begin{pmatrix} x_{2,2} & \dots & x_{2,n} & x_{2,n+1} \\ \vdots & & & \vdots \\ x_{n-1,2} & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,2} & \dots & x_{n,n} & x_{2,2} + \sum_{\substack{3\le i\le n-1\\ 3\le j\le n}} \mu_{i,j}x_{i,j} \end{pmatrix}$$

replacing $x_{1,1}$ by $x_{2,2}$ in each step. Either, there are no further terms in the last entry of the 1-jet or we iterate this procedure with $x_{3,3}, \dots, x_{n-1,n-1}$. All possible 1-jets in normal form are given by

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \cdots & \cdots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \cdots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \cdots & \cdots & x_{n,n} & \sum\limits_{i=1}^{t} x_{i,i} \end{pmatrix} =: J_{(n^2+n-1,t)}$$

for $t \in \{0, \ldots, n-1\}$, where we set $\sum\limits_{i=1}^{0} x_{i,i} := 0$.

For detailed instructions of the neccessary GL-operations, we introduce the following elementary matrices which describe the operation on the matrix by a multipliciation with the corresponding elementary matrix from the left (resp. right):

- Let $P_{i,j}$ be the matrix which switches the $i$-th and $j$-th row (resp. column).
- Let $S_i(\mu)$ be the matrix which multiplies the $i$-th row (resp. column) with $\mu$.
- Let $Q_{i,j}(\mu)$ be the matrix which adds the $\mu$-th multiple of the $j$-th row to the $i$-th row resp. adds the $\mu$-th multiple of the $i$-th column to the $j$-th column.

The following table gives an overview of GL-operations on the matrix $M$ to get the normal form:

Table 5.1: GL-operations on $M$ to archieve a normal form in the 1-jet

| matrix multiplication from | matrix | coordinate change |
|---|---|---|
| | | $x_{i,j} := l_{i,j}, \quad l(\underline{x}) := l_{n,n+1}(\underline{x})$ |
| left | $\prod_{i=1}^{n-1} Q_{i,n}(-\lambda_{i,n+1})$ | |
| | | $x_{n,j} := x_{n,j} + \sum_{i=1}^{n-1} \lambda_{i,n+1} x_{i,j}, \quad 1 \leq j \leq n$ |
| right | $\prod_{j=1}^{n} Q_{j,n+1}(-\tilde{\lambda}_{n+1,j})$ | |
| | | $x_{i,n+1} := x_{i,n+1} + \sum_{j=1}^{n} \tilde{\lambda}_{n,j} x_{i,j}, \quad 1 \leq i \leq n-1$ |
| left | $P_{1,s}$ | |
| | | $x_{s,j} := x_{1,j}, x_{1,j} := x_{s,j}, \quad 1 \leq j \leq n+1$ |
| right | $P_{1,t}$ | |
| | | $x_{i,t} := x_{i,1}, x_{i,1} := x_{i,t}, \quad 1 \leq i \leq n$ |
| | | $x_{1,1} := \mu_{1,1}^{-1} x_{1,1}$ |
| | | $x_{i,1} := \mu_{1,1} x_{i,1}, \quad 2 \leq i \leq n$ |
| | | $x_{1,1} := x_{1,1} - \tilde{\mu}_{1,j} x_{1,j}, \quad 2 \leq j \leq n$ |
| right | $\prod_{j=2}^{n} Q_{j,1}(\tilde{\mu}_{1,j})$ | |
| | | $x_{i,1} := x_{i,1} - \sum_{j=2}^{n} \tilde{\mu}_{1,j} x_{i,j}, \quad 2 \leq i \leq n$ |
| | | $x_{1,1} := x_{1,1} - \tilde{\mu}_{i,1} x_{i,1}, 2 \leq i \leq n-1$ |
| left | $\prod_{i=2}^{n-1} Q_{i,1}(\tilde{\mu}_{i,1})$ | |

## 5.2 Normal forms of matrices with $n^2+n-1$ variables in the 1-jet

Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}, \underline{a}\})$ be a matrix with $n^2+n-1$ variables in the 1-jet. We perform the required GL-operations to get, for some $t \in \{0, \dots, n-1\}$, the 1-jet

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & \dots & x_{n,n} & \sum\limits_{i=1}^{t} x_{i,i} \end{pmatrix}.$$

After these GL-operations, the matrix $M$ can still contain polynomials in the entries of order higher than one. Say

$$M \sim \begin{pmatrix} x_{1,1}+\rho_{1,1} & \dots & \dots & x_{1,n}+\rho_{1,n} & x_{1,n+1}+\rho_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1}+\rho_{n-1,1} & & \dots & x_{n-1,n}+\rho_{n-1,n} & x_{n-1,n+1}+\rho_{n-1,n+1} \\ x_{n,1}+\rho_{n,1} & \dots & \dots & x_{n,n}+\rho_{n,n} & \sum\limits_{i=1}^{t} x_{i,i}+\rho \end{pmatrix},$$

where $\rho_{i,j}, \rho \in \mathfrak{m}^2$, $1 \le i \le n, 1 \le j \le n+1$. With the coordinate change

$$\mathbb{C}\{\underline{x}, \underline{a}\} \longrightarrow \mathbb{C}\{\underline{x}, \underline{a}\},\ x_{i,j} := x_{i,j} - \rho_{i,j}$$

we get

$$M \sim \begin{pmatrix} x_{1,1} & \dots & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & \dots & x_{n,n} & \sum\limits_{i=1}^{t} x_{i,i}+P \end{pmatrix}$$

with some polynomial $P \in \mathfrak{m}^2$. Again, all variables in the last row or column and all variables in the first $t$ rows or columns can be erased from $P$ in a similar way as for the 1-jet. Iteratively, starting with the variables in the last row and column, continuing with the variables in the first row and column, continuing with the variables in the second row and column and proceeding to the $t$-th row and column, for every variable in these positions, we can erase all monomials in $P$ which are divisible by this variable. In this procedure, we only change the coefficients of variables, we have not erased from $P$ already. In details:
We denote by $\mathrm{mult}(m)$ the sum of all monomials that are divisible by a monomial $m$.

Table 5.2: GL-operations on $M$ to archieve a normal form

| matrix multiplication from | matrix | coordinate change |
|---|---|---|
| left | $Q_{n,s}(-\frac{\mathrm{mult}(x_{i,n+1})}{x_{i,n+1}})$ | |
| | | $x_{n,j} := x_{n,j} + \frac{\mathrm{mult}(x_{i,n+1})}{x_{i,n+1}} x_{i,j}$ |
| right | $Q_{j,n+1}(-\frac{\mathrm{mult}(x_{n,j})}{x_{n,j}})$ | |
| | | $x_{i,n+1} := x_{i,n+1} + \frac{\mathrm{mult}(x_{n,j})}{x_{n,j}} x_{i,j}$ |
| | | $x_{s,s} := x_{s,s} - \mathrm{mult}(x_{s,j})$ |
| right | $Q_{j,s}(\frac{\mathrm{mult}(x_{s,j})}{x_{s,j}})$ | |
| | | $x_{i,s} := x_{i,s} - \frac{\mathrm{mult}(x_{s,j})}{x_{s,j}} x_{i,j}$ |
| | | $x_{s,s} := x_{s,s} - \mathrm{mult}(x_{i,s})$ |
| left | $Q_{s,i}(\frac{\mathrm{mult}(x_{i,s})}{x_{i,s}})$ | |
| | | $x_{s,j} := x_{s,j} - \frac{\mathrm{mult}(x_{i,s})}{x_{i,s}} x_{i,j}$ |

To explain the operations from the table, first, we clean $P$ from the variables from the last row and column (i.e., for $1 \leq i \leq n-1$ and $1 \leq j \leq n$), then we continue with the variables from the $s$-th row and column, $1 \leq s \leq t$, apart from the variables on the diagonal (i.e., for $2 \leq i \leq n$ and $2 \leq j \leq n+1$).

The remaining polynomial in the last entry does not depend on the variables of the first $t$ rows, the first $t$ columns or the last row or last column anymore. We find a polynomial $\tilde{P} \in \mathfrak{m}^2$, which depends on $\{x_{i,j} \mid t < i \leq n-1,\ t < j \leq n\}$ and the variables $a_1, \ldots, a_r$ which did not appear in the 1-jet such that

$$M \sim \begin{pmatrix} x_{1,1} & \cdots & \cdots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \cdots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \cdots & \cdots & x_{n,n} & \sum\limits_{i=1}^{t} x_{i,i} + \tilde{P} \end{pmatrix}.$$

## 5.3 $T^1$

We will show inductively that

$$T^1(M) \cong T^1\left(\begin{pmatrix} x_{t+1,t+1} & \cdots & x_{t+1,n} & x_{t+1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,t+1} & \cdots & x_{n,n} & \tilde{P} \end{pmatrix}\right).$$

We want to show that, for every position $(i,j)$ in the first $t$ rows and first $t$ columns (i.e., $1 \leq i \leq t$ or $1 \leq j \leq t$), there is a relation cancelling any entry in position $(i,j)$ and there is a relation cancelling the variable $x_{i,j}$ in all other entries as well. Let

$$E_{s,v} := (a_{i,j})_{i,j}, \text{ with } a_{i,j} := \begin{cases} 1 & s\text{-th row, } v\text{-th column} \\ 0 & \text{else} \end{cases}.$$

Let $1 \leq s \leq n$ and $1 \leq v \leq n+1$ where ($s \leq t$ or $v \leq t$). We can erase every information in all entries except from the last one:

1. If $x_{s,s}$ is one of the first $t$ diagonal entries, i.e., $s \leq t$:
   The relation $\frac{\partial M}{\partial x_{s,s}} = E_{s,s} + E_{n,n+1} = 0$ translates to

   $$E_{s,s} = -E_{n,n+1}$$

   and shifts every information from position $(s,s)$ to position $(n, n+1)$.

2. If $x_{s,v}$ is in the first $t$ rows or columns, not on the diagonal, i.e., $s \neq v$ and ($s \leq t$ or $v \leq t$):
   The relation

   $$\frac{\partial M}{\partial x_{s,v}} = E_{s,v} = 0$$

   erases every information in the position $(s,v)$.

3. If $x_{s,v}$ is elsewhere, i.e., $t < s, v$ and $(s,v) \neq (n, n+1)$ :
   There is some $f \in \mathfrak{m}$ such that

   $$\frac{\partial M}{\partial x_{s,v}} = E_{s,v} + f \cdot E_{n,n+1} = 0,$$

   which translates to

   $$E_{s,v} = -f \cdot E_{n,n+1} = 0.$$

Therefore, in all entries of the matrix $M$ except for the last entry all variables can be erased.
To calculate $T^1(M)$ we also consider the relations given by row and column operations. Therefore, we consider

$$R_{i,k} := E_{i,k} \cdot M,$$

which places the $k$-th row of $M$ in the $i$-th row and

$$C_{j,\ell} := M \cdot E_{\ell,j},$$

which places the $\ell$-th column of $M$ in the $j$-th column. With these matrices we find further relations cleaning the last entry. We get the following relations to erase monomials from the last entry of $M$:

1. No diagonal entries in last entry:

$$x_{s,s}E_{n,n+1} = -R_{s,s} + \sum_{\substack{j=1\\j\neq s}}^{n+1} x_{s,j}E_{s,j} + x_{s,s}(E_{s,s} + E_{n,n+1}) = 0$$

2. No monomials $x_{s,v}$ with $s < v$ in the last entry:

$$x_{s,v}E_{n,n+1} = -C_{s,v} + \sum_{\substack{i=1\\i\neq s}}^{n} x_{i,v}E_{i,s} + x_{s,v}(E_{s,s} + E_{n,n+1}) = 0$$

3. No monomials $x_{s,v}$ with $s > v$ in the last entry:

$$x_{s,v}E_{n,n+1} = -R_{v,s} + \sum_{\substack{j=1\\j\neq v}}^{n} x_{s,j}E_{v,j} + x_{s,v}(E_{v,v} + E_{n,n+1}) = 0$$

With these relations, there is an isomorphism

$$T^1(M) \cong T^1\left(\begin{pmatrix} x_{t+1,t+1} & \cdots & x_{t+1,n} & x_{t+1,n+1} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & x_{n-1,n+1} \\ x_{n,t+1} & \cdots & x_{n,n} & \tilde{P} \end{pmatrix}\right).$$

## 5.4 Reduction to type $(2, 3, 2)$

We were able to find normal forms for the case of $n^2 + n - 1$ variables in the 1-jet. In this case, it is very convenient to find a connection to classifications that were already studied. Now, we show a reduction of the classification to smaller matrix size.

### 5.4.1 Reduction of matrix size

In this section, we want to show that the interesting information to find simple singularities depends on a submatrix of a defining matrix of a CMC2 singularity, where the variables on the diagonal do not appear linearly in the last entry.

**Proposition 5.2.** Let

$$M \sim \begin{pmatrix} x_{1,1} & \cdots & \cdots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \cdots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \cdots & \cdots & x_{n,n} & \sum_{i=1}^{t} x_{i,i} + \tilde{P} \end{pmatrix},$$

where $\tilde{P} \in \mathfrak{m}^2$ is a polynomial which depends only on $\{x_{i,j} \mid t < i \leq n-1,\ t < j \leq n\}$ and $a_1, \ldots, a_r$. If $t \leq n-3$, then $M$ does not define a simple singularity.

**Corollary 5.3.** Let

$$M \sim \begin{pmatrix} x_{1,1} & \ldots & \ldots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \ldots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \ldots & \ldots & x_{n,n} & \sum_{i=1}^{t} x_{i,i} + \tilde{P} \end{pmatrix}$$

be a matrix defining a simple CMC2 singularity, $\tilde{P}$ as above in 5.2. Then

$$T^1(M) \cong T^1\left(\begin{pmatrix} x & y & z \\ v & w & \tilde{P} \end{pmatrix}\right),$$

$\tilde{P} \in \mathbb{C}\{x, y, z, v, w, \underline{a}\}$ of order 2.

This result has a strong effect: In the case of CMC2 singularities, the deformations of the determinantal singularity defined by $M$ correspond to the unfoldings of $M$, which are in fact arbitrary perturbations of the matrix. Furthermore, a basis of $T^1(M)$ yields a versal unfolding of $M$, therefore, finding all the types of singularities appearing in the versal family of deformations reduces to the problem of finding all types of singularities appearing in the versal family of deformations of a $(2\times 3)$-matrix. We reduced the problem of handling deformations of singularities defined by matrices of arbitrary size $n \times (n+1)$ to a problem concerning only matrices of fixed size $2 \times 3$.

To proof this result, we split the proposition in two parts. We proof for $t = n-3$ and for $t < n-3$, that $M$ does not define a simple singularity.

### 5.4.2 Case $t = n-3$

First, we check that there are no simple matrices in the case that $t = n-3$.

**Lemma 5.4.** Let

$$M \sim \begin{pmatrix} x_{1,1} & \ldots & \ldots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \ldots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \ldots & \ldots & x_{n,n} & \sum_{i=1}^{t} x_{i,i} + \tilde{P} \end{pmatrix}$$

and $\tilde{P}$ as above in 5.2. If $t = n-3$ then $M$ does not define a simple singularity.

*Proof.* In this case, we have

$$T^1(M) \cong T^1\left(\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & P \end{pmatrix}\right),$$

where $P \in \mathbb{C}\{\underline{x}\}$. We split the proof in two parts, first proof non-simplicity for 11 variables in total, then we proof non-simplicity in higher dimensional ambient spaces.

**Ambient space of dimension 11:**

The smallest ambient space of this determinantal singularity we can choose is the 11-dimensional space

$$\mathbb{C}\{x_{i,j},\ 1 \leq i \leq 3,\ 1 \leq j \leq 4, (i,j) \neq (3,4)\}.$$

In this case, we use the weight matrix

$$D = \begin{pmatrix} 2 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 4 \end{pmatrix}$$

with the weight vector

$$\omega = (2,2,2,3,2,2,2,3,3,3,3).$$

For the counting argument (see section 3.4) we count the number of positions $\operatorname{Pos}(d)$ in $D$ of weight $d$, variables $\operatorname{Var}(d)$ with weight $d$ and the number of monomials $\operatorname{Mon}(d)$ with weighted each weighted degree $d$:

| $d$ | $\operatorname{Pos}(d)$ | $\operatorname{Var}(d)$ | $\operatorname{Mon}(d)$ |
|---|---|---|---|
| 2 | 6 | 6 | 6 |
| 3 | 5 | 5 | 5 |
| 4 | 1 | 0 | $\binom{6+2-1}{2} = 21$ |

Now,

$$\dim Q_{(D,\omega)} = \operatorname{Pos}(2)\cdot\operatorname{Mon}(2) + \operatorname{Pos}(3)\cdot\operatorname{Mon}(3) + \operatorname{Pos}(4)\cdot\operatorname{Mon}(4),$$

and

$$\#S_1 = \operatorname{Var}(2)\cdot\operatorname{Mon}(2) + \operatorname{Var}(3)\cdot\operatorname{Mon}(3) + \operatorname{Var}(4)\cdot\operatorname{Mon}(4), \quad \#S_2 = 5, \quad \#S_3 = 10.$$

The counting argument yields: A determinantal singularity with this quasihomogeneous starting jet cannot be simple if

$$\dim Q_{(D,\omega)} > \#S_1 + \#S_2 + \#S_3 - 2,$$

but we know

$$\begin{aligned} &\dim Q_{(D,\omega)} > \#S_1 + \#S_2 + \#S_3 - 2 \\ \iff &\dim Q_{(D,\omega)} - \#S_1 > \#S_2 + \#S_3 - 2 \\ \iff &\mathrm{Mon}(4) > 13 \\ \iff &21 > 13. \end{aligned}$$

So, a matrix $M$ with 1-jet

$$j_1 M \sim \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & 0 \end{pmatrix}$$

always defines a non-simple singularity in $(\mathbb{C}^{11}, 0)$.

**Ambient space of dimension higher than 11:**

This argument can be generalized to a higher dimensional ambient space. Consider a matrix $M \in \mathrm{Mat}(3, 4, \mathbb{C}\{\underline{x}, a_1, \ldots, a_r\})$ defining a singularity in $(\mathbb{C}^{11+r}, 0)$ with 11 variables in the 1-jet. The additional variables can only appear in the last entry.
Consider the matrix

$$M \sim \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & P + \sum\limits_{s=1}^{r} a_s^2 \end{pmatrix}.$$

All mixed terms $\lambda a_s x_{i,j}$ of degree 2 can be cancelled with a coordinate change of the form

$$a_s \mapsto a_s - \frac{\lambda}{2} x_{i,j}.$$

This originates terms in $x_{i,j}$. We can split the polynomial in the last entry in a (new) polynomial $P$ which depends only on $x_{i,j}$ and $\sum\limits_{s=1}^{r} a_s^2$. $T^1(M)$ contains $\frac{\partial M}{\partial a_s}$ and we know

$$\frac{\partial M}{\partial a_s} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a_s \end{pmatrix},$$

hence, $T^1(M)$ as factor module in $\mathrm{Mat}(3,4,\mathbb{C}\{\underline{x},\underline{a}\})$ is isomorphic to

$$T^1\left(\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & P \end{pmatrix}\right)$$

as factor module in $\mathrm{Mat}(3,4,\mathbb{C}\{\underline{x}\})$. So, every non-vanishing monomial in $\mathrm{Mat}(3,4,\mathbb{C}\{\underline{x}\})$ gives a non-vanishing monomial in $\mathrm{Mat}(3,4,\mathbb{C}\{\underline{x},\underline{a}\})$.
Hence, considering the singularity in $(\mathbb{C}^{11},0)$, defined by

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & P \end{pmatrix},$$

and an infinite family of deformations thereof (which arise as the $\mathbb{C}$-span of a non-vanishing monomial of the $T^1$) gives an infinite family of deformations of the singularity in $(\mathbb{C}^{11+r},0)$ defined by $M$. Therefore, a singularity defined by $M$ with deform into finitely many types as well and cannot be simple.
Now, every singularity defined by some $M \in \mathrm{Mat}(3,4,\mathbb{C}\{\underline{x},\underline{a}\})$ with

$$j_1M \sim \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & 0 \end{pmatrix}$$

deforms into the non-simple singularity defined by

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & P+\sum\limits_{s=1}^{r} a_s^2 \end{pmatrix},$$

where $P \in \mathbb{C}\{\underline{x}\} \cap \mathfrak{m}^2$ (add the perturbation terms $\sum\limits_{s=1}^{r} \lambda_s a_s^2$ in the last entry and perform a suitable coordinate change to erase the parameters $\lambda_s$). The parameters ensure, that we consider this singularity in an arbitrarily small neighbourhood of the origin.
Therefore, no matrix in $\mathrm{Mat}(3,4,\mathbb{C}\{\underline{x},\underline{a}\})$ can define a simple singularity.

□

### 5.4.3 Case $t < n-3$

We can exclude all matrices with $t < n-3$, they never define a simple singularity.

**Lemma 5.5.** Let

$$M \sim \begin{pmatrix} x_{1,1} & \dots & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & \dots & x_{n,n} & \sum_{i=1}^{t} x_{i,i} + \tilde{P} \end{pmatrix}$$

and $\tilde{P}$ as in 5.2. If $t < n-3$ then $M$ does not define a simple singularity.

*Proof.* Consider a matrix $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ with

$$j_1 M \sim \begin{pmatrix} x_{1,1} & \dots & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & \dots & x_{n,n} & \sum_{i=1}^{t} x_{i,i} \end{pmatrix},$$

$t < n-3$. Perturb the matrix by adding the perturbation terms $\sum_{i=t+1}^{n-3} \lambda_i x_{i,i}$, $\lambda_i \in \mathbb{C}$ and performing a suitable coordinate change to get a matrix with 1-jet

$$\begin{pmatrix} x_{1,1} & \dots & \dots & x_{1,n} & x_{1,n+1} \\ \vdots & & & \vdots & \vdots \\ x_{n-1,1} & & \dots & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & \dots & x_{n,n} & \sum_{i=1}^{n-3} x_{i,i} \end{pmatrix}.$$

This matrix cannot define a simple singularity (see lemma 5.4), hence $M$, which deformes into a matrix defining a non simple singularity, cannot define a simple singularity either.

$\square$

# Chapter 6

# Candidates of type $(2, 3, 2)$ with 5 variables in the 1-jet

To classify the simple CMC2 singularities of arbitrary size, we only need to classify the simple CMC2 singularities of type $(2, 3, 2)$. First, we find candidates for such singularities, later we check simplicity by calculating the adjacencies.

## 6.1 Full list of candidates

In this chapter, we achieve the following list of candidates with their Tjurina numbers $\tau$ in the last column. The candidates will be acquired in detailed considerations about their 1-jets, their $T^1$ and an examination of the matrices itselves throughout the sections 6.2 to 6.5, the calculation for the Tjurina numbers can be found in section 6.5.3 with the calculation of the Tjurina algebra.

**Proposition 6.1.** Candidates for simple non-isolated CMC2 singularities are given by the following list:

Table 6.1: Candidates for simple non-isolated CMC2 singularities

| Type | Presentation matrix | | $\tau$ |
|---|---|---|---|
| $\hat{A}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k+1} + \sum_{i=2}^{r} a_i^2 \end{pmatrix}$ | $k \geq 1$ | $k$ |
| $\hat{D}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k-1} + a_1 a_2^2 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k$ |
| $\hat{E}_6$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^4 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 6 |
| $\hat{E}_7$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_1 a_2^3 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 7 |
| $\hat{E}_8$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^5 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 8 |
| $A_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k+1} + y^2 + \sum_{i=1}^{r} a_i^2 \end{pmatrix}$ | $k \geq 1$ | $k+2$ |
| $D_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k-1} + xy^2 + \sum_{i=1}^{r} a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k+2$ |
| $E_6^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + y^4 + \sum_{i=1}^{r} a_i^2 \end{pmatrix}$ | | 8 |
| $E_7^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + xy^3 + \sum_{i=1}^{r} a_i^2 \end{pmatrix}$ | | 9 |
| $E_8^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + y^5 + \sum_{i=1}^{r} a_i^2 \end{pmatrix}$ | | 10 |
| $S_{k,\ell}$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + y^k + a_1^\ell + \sum_{i=2}^{r} a_i^2 \end{pmatrix}$ | $k \geq 2, \ell \geq 3$ | $k+\ell-1$ |
| $Q$ | $\begin{pmatrix} x & y & z \\ v & w & x^2 + y^2 + a_1^3 + \sum_{i=2}^{r} a_i^2 \end{pmatrix}$ | | 6 |
| $D_k^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^{k-1} + a_1 a_2^2 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k+2$ |
| $E_6^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_2^4 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 8 |
| $E_7^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_1 a_2^3 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 9 |
| $E_8^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_2^5 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | | 10 |

## 6.2 Reduction of 1-jets

Applying the reduction of the 1-jets to $2 \times 3$ matrices the possible 1-jets for matrices with 5 variables in the 1-jet are

$$\underbrace{\begin{pmatrix} x & y & z \\ u & v & x \end{pmatrix}}_{J^{(5,1)}}, \underbrace{\begin{pmatrix} x & y & z \\ u & v & 0 \end{pmatrix}}_{J^{(5,2)}}.$$

## 6.3 $T^1$

The $T^1$ is one of the most important tools in this work. It provides information about the orbits of the singularities with respect to GL-equivalence and information about the behaviour of deformations.

### 6.3.1 Calculation

Let $M \in \operatorname{Mat}(2, 3, \mathbb{C}\{x, y, z, v, w, a_1, \ldots, a_r\})$ be a matrix with 5 variables in the 1-jet. Remember, as $x, y, z, v, w$ appear as linear terms, the first 5 entries of $M$ can be transformed into $x, y, z, v, w$ by coordinate transformations. By row and colum operations we can delete all terms in the last entry containing $z, v, w$. Hence, we get the normal form

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x, y, a_1, \ldots, a_r) \end{pmatrix}, \ p \in \mathfrak{m}.$$

For further examination, we consider $T^1(M)$. Therefore, we calculate a standard basis of the module $I$ factored out in $T^1(M)$. We know:

$$\begin{aligned} I = \Bigg\langle & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{\partial P}{\partial y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial a_i} \end{pmatrix}, \\ & \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ x & y & z \end{pmatrix}, \begin{pmatrix} v & w & P \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ v & w & P \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & 0 \\ 0 & v & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & v \end{pmatrix}, \begin{pmatrix} y & 0 & 0 \\ w & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y & 0 \\ 0 & w & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ P & 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & z & 0 \\ 0 & P & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & P \end{pmatrix} \Bigg\rangle \end{aligned}$$

After the valid ideal operations, the generators can be simplified to

$$I = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{\partial P}{\partial y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial a_i} \end{pmatrix}, \right.$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} v & w & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix},$$
$$\left. \begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix} \right\rangle$$

The suitable ordering for a standard basis calculation is the one giving priority to the component, i.e., $> = (c, >)$ which is given by

$$\underline{x}^{\alpha}\, e_i > \underline{x}^{\beta}\, e_j :\Longleftrightarrow\ i < j \text{ or } \underline{x}^{\alpha} > \underline{x}^{\beta},$$

where $e_i$ is the matrix with 1 in the $i$-th entry (counting from left to right and row by row) and 0 in all other entries (see [22], p. 136). The $s$-polynomial is 0 if the leading monomials are in different entries and as usual if the leading monomials are in the same entry. We define $S$ to be the set of generators of $I$ (the second set of generators). Following the algorithm in [22, p.240], we do the standard basis calculation starting with generators having the leading monomial in the first component, these are

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial x} \end{pmatrix}}_{=:A_1}, \underbrace{\begin{pmatrix} v & w & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:A_2}, \underbrace{\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:A_3}, \underbrace{\begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:A_4}.$$

Now, we calculate normal forms of the $s$-polynomials of these matrices and add them to $S$ if they do not vanish.

$$\begin{aligned}
\mathrm{NF}\left(\mathrm{spoly}(A_1,A_2)\right) =& v\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \frac{\partial P}{\partial x}\end{pmatrix} - \begin{pmatrix}v & w & 0\\ 0 & 0 & 0\end{pmatrix} + w\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \frac{\partial P}{\partial y}\end{pmatrix}\\
& -\frac{\partial P}{\partial x}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & v\end{pmatrix} - \frac{\partial P}{\partial y}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & w\end{pmatrix} = 0\\
\mathrm{NF}\left(\mathrm{spoly}(A_1,A_3)\right) =& x\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \frac{\partial P}{\partial x}\end{pmatrix} - \begin{pmatrix}x & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & x\frac{\partial P}{\partial x}\end{pmatrix}\\
\mathrm{NF}\left(\mathrm{spoly}(A_1,A_4)\right) =& x\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \frac{\partial P}{\partial x}\end{pmatrix} - \begin{pmatrix}y & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & y\frac{\partial P}{\partial x}\end{pmatrix}\\
\mathrm{NF}\left(\mathrm{spoly}(A_2,A_3)\right) =& x\begin{pmatrix}v & w & 0\\ 0 & 0 & 0\end{pmatrix} - v\begin{pmatrix}x & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - w\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\end{pmatrix} = 0\\
\mathrm{NF}\left(\mathrm{spoly}(A_2,A_4)\right) =& y\begin{pmatrix}v & w & 0\\ 0 & 0 & 0\end{pmatrix} - v\begin{pmatrix}y & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - w\begin{pmatrix}0 & y & 0\\ 0 & 0 & 0\end{pmatrix} = 0\\
\mathrm{NF}\left(\mathrm{spoly}(A_3,A_4)\right) =& y\begin{pmatrix}x & 0 & 0\\ 0 & 0 & 0\end{pmatrix} - x\begin{pmatrix}y & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = 0.
\end{aligned}$$

Add $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & x\frac{\partial P}{\partial x}\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & y\frac{\partial P}{\partial x}\end{pmatrix}$ to $S$.

The elements of $S$ with leading monomial in the second component are

$$\underbrace{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \frac{\partial P}{\partial y}\end{pmatrix}}_{=:A_1}, \underbrace{\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\end{pmatrix}}_{=:A_2}, \underbrace{\begin{pmatrix}0 & y & 0\\ 0 & 0 & 0\end{pmatrix}}_{=:A_3}, \underbrace{\begin{pmatrix}0 & z & 0\\ 0 & 0 & 0\end{pmatrix}}_{=:A_4}.$$

$$\begin{aligned}
\mathrm{NF}(\mathrm{spoly}(A_1,A_2)) =& x\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \frac{\partial P}{\partial y}\end{pmatrix} - \begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & x\frac{\partial P}{\partial y}\end{pmatrix}\\
\mathrm{NF}(\mathrm{spoly}(A_1,A_3)) =& y\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \frac{\partial P}{\partial y}\end{pmatrix} - \begin{pmatrix}0 & y & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & y\frac{\partial P}{\partial y}\end{pmatrix}\\
\mathrm{NF}(\mathrm{spoly}(A_1,A_4)) =& z\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \frac{\partial P}{\partial y}\end{pmatrix} - \begin{pmatrix}0 & z & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{\partial P}{\partial y}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & z\end{pmatrix} = 0\\
\mathrm{NF}(\mathrm{spoly}(A_2,A_3)) =& \mathrm{NF}(\mathrm{spoly}(A_2,A_4)) = \mathrm{NF}(\mathrm{spoly}(A_3,A_4)) = 0.
\end{aligned}$$

Add $\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & x\frac{\partial P}{\partial y}\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & y\frac{\partial P}{\partial y}\end{pmatrix}$ to $S$.

There is exactly one element in $S$ having its leading monomial in the 3rd component, exactly one having its leading monomials it the 4th component and exactly one elments having its leading monomial in the 5th component. The will remain in $S$.

The elements with leading monomial in the 6. component are:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial P}{\partial a_i} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x\frac{\partial P}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y\frac{\partial P}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x\frac{\partial P}{\partial y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y\frac{\partial P}{\partial y} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y\frac{\partial P}{\partial a_i} \end{pmatrix}$$

We see, that we only need to find a standard basis $\tilde{S}$ for

$$\left\langle x\frac{\partial P}{\partial x}, y\frac{\partial P}{\partial x}, x\frac{\partial P}{\partial y}, y\frac{\partial P}{\partial y}, \frac{\partial P}{\partial a_i}, P \right\rangle$$

in $\mathbb{C}\{x, y, a_i\}$ and then all standard basis elements with leading monomial in the 6. component will be given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix}, s \in \tilde{S}.$$

A module which is factorized by an ideal is isomorphic to the same module factorized by the leading ideal (see [22], p. 228, Remark 3.3.12), so we get

$$\begin{aligned}
T^1(M) &\cong \mathrm{Mat}(2, 3, \mathbb{C}\{x, y, z, v, w, \underline{a}\})/\langle S \rangle \\
&\cong \mathrm{Mat}(2, 3, \mathbb{C}\{x, y, z, v, w, \underline{a}\})/\langle L(S) \rangle \\
&\cong \mathbb{C}\{x, y, z, v, w, \underline{a}\}/\langle L(\tilde{S}), z, v, w \rangle \\
&\cong \mathbb{C}\{x, y, \underline{a}\}/\langle L(\tilde{S}) \rangle \\
&\cong \mathbb{C}\{x, y, \underline{a}\}/\Big( \langle x, y \rangle \langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \rangle + \langle P, \frac{\partial P}{\partial a_1}, \ldots, \frac{\partial p}{\partial a_r} \rangle \Big).
\end{aligned}$$

### 6.3.2 Interpretation

Every monomial that vanishes in the $T^1$ can be interpreted as a perturbation term (inducing a deformation) that will be catched by a linear coordinate transformation (this would not change the singularity type) or row and column operations of the matrix (this does not change the singularity at all). Hence, only non-vanishing monomials in the $T^1$, considered as a perturbation term inducing a deformation of the singularity as complex space germ, have the ability to give rise to a new singularity type. Hence, the matrices perturbed by the elements of the $T^1$ cover all singularty types appearing in the versal family of deformations.

## 6.4 1-jet $J^{(5,1)}$

The candidates for simple singularities with 1-jet $J^{(5,1)}$ are given by the following proposition.

**Proposition 6.2.** The candidates for simple matrices with 1-jet $J^{(5,1)}$ are given by the following list:

Table 6.2: Candidates for simple non-isolated CMC2 singularities with 1-jet $J^{(5,1)}$

| Type | Presentation Matrix | |
|---|---|---|
| $\hat{A}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k+1} + \sum_{i=2}^{r} a_i^2 \end{pmatrix}$ | $k \geq 1$ |
| $\hat{D}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k-1} + a_1 a_2^2 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | $k \geq 4$ |
| $\hat{E}_6$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^4 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | |
| $\hat{E}_7$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_1 a_2^3 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | |
| $\hat{E}_8$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^5 + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$ | |

*Proof.* We consider a matrix with 1-jet

$$\begin{pmatrix} x & y & z \\ u & v & x \end{pmatrix}.$$

The original matrix is of the form

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x, y, a_1, \ldots, a_r) \end{pmatrix}.$$

In this case, we have the linear term $x$ in the polynomial $P$. By a coordinate transformation of $x$ we can erase all terms in $p$ divisible by $x$ or $y$. Hence, $P = x + h(\underline{a})$, $h \in \mathfrak{m}^2$. We get

$$T^1(M) \cong \mathbb{C}\{x, y, \underline{a}\}/\Big(\langle x, y\rangle + \langle h, \{\frac{\partial h}{\partial a_i}\}_i\rangle\Big) \cong \mathbb{C}\{\underline{a}\}/\langle h, \{\frac{\partial h}{\partial a_i}\}_i\rangle \cong T(h).$$

Hence, the $T^1$ of this matrix is equivalent to the Tjurina algebra of the polynomial in the last entry as hypersurface singularity in $(\mathbb{C}^r, 0)$ (remember: $\underline{a} = (a_1, \ldots, a_r)$).

Now, all deformations are given by at most the deformations of $h$ as hypersurface singularity in $(\mathbb{C}^r, 0)$. We know that the simple hypersurface singularties are the so called ADE-singularities. So, the candidates for simple singularities in this case are:

$$\begin{pmatrix} x & y & z \\ v & w & x + h(a_1, \dots, a_r) \end{pmatrix},$$

where $h$ is an ADE-singularity.

□

## 6.5 1-jet $J^{(5,2)}$

To find the candidates in this case, first, we consider matrices in an ambient space of dimension 7, i.e., additionally to the 5 variables $x, y, z, v, w$ which appear in the 1-jet of a matrix $M \in \mathrm{Mat}(2, 3, \mathbb{C}\{x, y, z, v, w, a, b\})$ there are two variables $a, b$ which do not appear in the 1-jet of the matrix $M$. In lower dimension, simple singularities are isolated, these are already classified in [16]. In dimension 7, we will distinguish all possible constellations with monomials in the 2-jet. Later we can use this knowledge to give all candidates in ambient spaces of arbitrary dimension.

### 6.5.1 Ambient space of dimension 7

As we cannot get more information out of the 1-jet, we will increase the visible monomials of $M$ by considering the 2-jet instead of the 1-jet. Now, we will see further monomials of degree 2. By distinguishing all possible constellations, we will find further candidates for simple matrices and exclude the ones that are definitely non-simple. This examination leads to the following results:

**Proposition 6.3.** The candidates for simple matrices with 1-jet $J^{(5,2)}$ are given by the following two lists:

Table 6.3: Candidates for simple non-isolated CMC2 singularities with 1-jet $J^{(5,2)}$

| Type | Presentation Matrix | | Type | Presentation Matrix | |
|---|---|---|---|---|---|
| $A_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k+1}+y^2+a^2+b^2 \end{pmatrix}$ | $k \geq 1$ | $A_k^{\star}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^{k+1}+b^2 \end{pmatrix}$ | $k \geq 1$ |
| $D_k^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k-1}+xy^2+a^2+b^2 \end{pmatrix}$ | $k \geq 4$ | $D_k^{\star}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^{k-1}+ab^2 \end{pmatrix}$ | $k \geq 4$ |
| $E_6^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^4+a^2+b^2 \end{pmatrix}$ | | $E_6^{\star}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+b^4 \end{pmatrix}$ | |
| $E_7^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+xy^3+a^2+b^2 \end{pmatrix}$ | | $E_7^{\star}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+ab^3 \end{pmatrix}$ | |
| $E_8^{\#}$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^5+a^2+b^2 \end{pmatrix}$ | | $E_8^{\star}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+b^5 \end{pmatrix}$ | |
| $S_{k,\ell}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+y^k+a^2+b^l \end{pmatrix}$ | $k \geq 2$ $\ell \geq 3$ | | | |
| $Q$ | $\begin{pmatrix} x & y & z \\ v & w & x^2+y^2+a^2+b^3 \end{pmatrix}$ | | | | |

*Proof.* To get the candidates we distinguish the cases depending on the terms appearing in the 2-jet:

1. One of $a$ and $b$ appears as a square (with some non-vanishing coefficient) in $j_2M$: For the original matrix $M$ this means that, without loss of generality, some term $u \cdot a^2$ appears in the last entry, where $u \in \mathbb{C}\{\underline{x}\}^{\star}$. We use the coordinate change

$$a \mapsto \frac{1}{\sqrt{u}}a,$$

such that $a^2$ becomes a monic term in the last entry of $M$ and we repeat coordinate changes

$$a \mapsto a - \frac{m}{a}$$

for every monomial $m$ divisible by $a$ (except from $a^2$), therefore, we start with monomials of lowest degree (due to some monomial ordering) and continue with rising degree. At some point, we exceed the bound for finite determinacy and we see a representative for the equivalence class of $M$. We get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & a^2+P(x,y,b) \end{pmatrix},$$

with some $p \in \mathfrak{m}^2$. In this case,

$$T^1(M) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \end{pmatrix} \,\middle|\, h \in \mathbb{C}\{x,y,a,b\} \middle/ \left\langle P, a, \frac{\partial P}{\partial b}, x\frac{\partial P}{\partial x}, y\frac{\partial P}{\partial y}, y\frac{\partial P}{\partial x}, x\frac{\partial P}{\partial y} \right\rangle \right\}$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \end{pmatrix} \,\middle|\, h \in \mathbb{C}\{x,y,b\} \middle/ \left\langle P, \frac{\partial P}{\partial b}, x\frac{\partial P}{\partial x}, y\frac{\partial P}{\partial y}, y\frac{\partial P}{\partial x}, x\frac{\partial P}{\partial y} \right\rangle \right\}.$$

The only perturbation terms that can arise are those we have seen in the classification of the simple isolated singularities in $(\mathbb{C}^6, 0)$ with 1-jet $\begin{pmatrix} x & y & z \\ v & w & 0 \end{pmatrix}$. We add the following singularities to the list of candidates for simple singularities:

| | | | | | |
|---|---|---|---|---|---|
| $A_k^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k+1}+y^2+a^2+b^2 \end{pmatrix}$ | $k \geq 1$ | $E_8^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^5+a^2+b^2 \end{pmatrix}$ | |
| $D_k^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k-1}+xy^2+a^2+b^2 \end{pmatrix}$ | $k \geq 4$ | $S_{k,\ell}$ | $\begin{pmatrix} x & y & z \\ v & w & ax+y^k+a^2+b^\ell \end{pmatrix}$ | $k\geq 2$, $\ell \geq 3$ |
| $E_6^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+y^4+a^2+b^2 \end{pmatrix}$ | | $Q$ | $\begin{pmatrix} x & y & z \\ v & w & x^2+y^2+a^2+b^3 \end{pmatrix}$ | |
| $E_7^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3+xy^3+a^2+b^2 \end{pmatrix}$ | | | | |

2. None of $a, b$ appears as a square in $j_2M$ but $ab$ appears in $j_2M$:
   We use a coodinate change to get the monic term $ab$ in the last entry of $M$, followed by the coordinate change $b \mapsto a+b$. Now, the monomial $a^2$ appears in the last entry of $M$ and we get the first case.

3. In $j_2M$ the variables $a$ and $b$ appear only mixed with $x$ or $y$ (sufficiently general):
   The possible monomials of degree 2 divisible by $a$ or $b$ are $ax, ay, bx, by$. Consider

$$P(x,y,a,b) = \lambda_1 ax + \lambda_2 abx + \mu_1 ay + \mu_2 by + h_1(x,y) + h_2(x,y,a,b)$$

with sufficiently general coefficients $\lambda_1, \lambda_2, \mu_1, \mu_2$ and $h_1 \in \mathfrak{m}^2 \backslash \mathfrak{m}^3$, $h_2 \in \mathfrak{m}^3$. Without loss of generality, let $\lambda_1$ be non-zero (one of these coefficients has to be non-zero in order to be sufficiently general). We try to find a better defining matrix of the GL-equivalence class of

$$M \sim \begin{pmatrix} x & y & z \\ v & w & \lambda_1 ax + \lambda_2 abx + \mu_1 ay + \mu_2 by + h_1(x,y) + h_2(x,y,a,b) \end{pmatrix}.$$

Let $x$ cancel the non-vanishing factor $\lambda_1$ by $x \mapsto (\lambda_1)^{-1}x$. Redefine $h_1$ and $h_2$ and clean the first column of $M$ with a column operation and a coordinate change in $v$. As $a, b$ did not appear in the other entries of the matrix, we can perform arbitrary coordinate changes in $a$ and $b$. By the coordinate change $a \mapsto a - \lambda_2 b$ we get rid of the monomial $bx$ in the last entry. Now, after redefining $h_2$, we have

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + \mu_1 ay + \mu_2 by + h_1(x,y) + h_2(x,y,a,b) \end{pmatrix}.$$

Again, let $y$ cancel $\mu_2 \neq 0$, redefine $h_1, h_2$ and clean the second column with a column operation and a coordinate change in $w$. Then, the last entry is given by

$$ax + \mu_1 ay + by + h_1(x,y) + h_2(x,y,a,b).$$

Use the coordinate change $b \mapsto b - \mu_1 a$ and redefine $h_1, h_2$ to get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + by + h_1(x,y) + h_2(x,y,a,b) \end{pmatrix}.$$

Every monomial $m$ divisible by $x$ can be erased by $a \mapsto a - \frac{m}{x}$ (except from $m = ax$ as $a - a = 0$). Using this coordinate change iteratively all monomials divisible by $x$ (except from $ax$) disappear. Analogously, monomials $m$ which are divisible by $y$ can be erased using the coordinate change $b \mapsto b - \tilde{m}$. Now, $h_1$ is erased, therefore, after redefining $h_2$ to a new polynomial $h \in \mathfrak{m}^3$, we get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + by + h(a,b) \end{pmatrix}.$$

We plug in $P = ax + by + h$ into the calculation of $T^1(M)$:

$$\begin{aligned} T^1(M) &\cong \mathbb{C}\{x,y,a,b\}/\Big(\langle x,y\rangle\langle\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\rangle + \langle P, \frac{\partial P}{\partial a}, \frac{\partial P}{\partial b}\rangle\Big) \\ &\cong \mathbb{C}\{x,y,a,b\}/\Big(\langle P, ax, ay, bx, by, x + \frac{\partial h}{\partial a}, y + \frac{\partial h}{\partial b}\rangle\Big) \end{aligned}$$

A standard basis of the latter ideal

$$\Big\langle P, ax, ay, bx, by, x + \frac{\partial h}{\partial a}, y + \frac{\partial h}{\partial b}\Big\rangle$$

is given by

$$\tilde{S} = \{x + \frac{\partial h}{\partial a}, y + \frac{\partial h}{\partial b}, ax, ay, bx, by, a\frac{\partial h}{\partial a}, b\frac{\partial h}{\partial a}, a\frac{\partial h}{\partial ab}, b\frac{\partial h}{\partial b}, h\}$$

as the following short calculation shows:

- $s$-polynomials of monomials are always 0
- $s$-polynomials of $x + \frac{\partial h}{\partial a}$ and $y + \frac{\partial h}{\partial b}$ with other monomials give the new elements in the standard basis

- $\text{spoly}(x+\frac{\partial h}{\partial a}, y+\frac{\partial h}{\partial b}) = y\frac{\partial h}{\partial a} - x\frac{\partial h}{\partial b}$ reduces to 0 as $h \in \mathfrak{m}^3$ and each of $y\frac{\partial h}{\partial a}, x\frac{\partial h}{\partial b}$ is either divisible by $ay$ or $by$ resp. $ax$, $bx$
- $\text{spoly}(h, ax) = \begin{cases} \text{tail}(hax) & a \nmid \text{LM}(h) \\ \text{tail}(hx) & a \mid \text{LM}(h) \end{cases}$ reduces to 0 as each monomial (in both cases) is either divisible by $ax$ or $bx$ (each monomial of $h$ is divisible by $a$ or $b$),
- $\text{spoly}(h, bx)$ reduces to 0 (analogously)

Hence,

$$T^1(M) \cong \mathbb{C}\{x,y,a,b\}/\langle L(\tilde{S})\rangle \cong \mathbb{C}\{a,b\}/\langle a\frac{\partial h}{\partial a}, b\frac{\partial h}{\partial a}, a\frac{\partial h}{\partial b}, b\frac{\partial h}{\partial b}, h\rangle$$

This is the $T^1$ of a hypersurface singularity in $(\mathbb{C}^2, 0)$ with section $(a=0, b=0)$. In this case, we get the candidates for simple singularities by looking at [16, p.19]. We see that $h$ has to be an ADE-singularity:

| | | |
|---|---|---|
| $A_k^\star$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^{k+1}+b^2 \end{pmatrix}$ | $k \geq 1$ |
| $D_k^\star$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^{k-1}+ab^2 \end{pmatrix}$ | $k \geq 4$ |
| $E_6^\star$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+b^4 \end{pmatrix}$ | |
| $E_7^\star$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+ab^3 \end{pmatrix}$ | |
| $E_8^\star$ | $\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+b^5 \end{pmatrix}$ | |

The singularities defined by $A_k^\star$ are already defined by $S_{2,k+1}$ (we already treated the case that one of $a$,$b$ appears in the 2-jet) and do not need to be mentioned in the final list of candidates (to check the calculation start with $A_k^\star$ and perform $b \mapsto b - \frac{y}{2}$, continue by cancelling the coefficient of $y^2$ and finish by cleaning the first two columns).

4. In $j_2M$ the variables $a$ and $b$ appear only mixed with $x$ or $y$ (not sufficiently general but in degree 2):
Again, consider the original matrix which is of the following shape

$$M \sim \begin{pmatrix} x & y & z \\ v & w & \lambda_1 ax + \lambda_2 bx + \mu_1 ay + \mu_2 by + h_1(x,y) + h_2(x,y,a,b) \end{pmatrix},$$

where $h_1$ is at least of order 2 and $h_2$ is at least of order 3. At least one of $\lambda_1, \lambda_2, \mu_1, \mu_2$ does not vanish, otherwise $a$ and $b$ would not appear. Without loss of generality,

$\lambda_1 \neq 0$. We can repeat the first coordinate changes and get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + \mu_1 ay + \mu_2 by + h_1(x,y) + h_2(x,y,a,b) \end{pmatrix}.$$

If one of $\mu_1, \mu_2$ does not vanish we get the previous (suffieciently general) case, hence, $\mu_1 = \mu_2 = 0$. As before, we can iteratively erase all monomials $m$ divisible by $x$ (except from $ax$) with a suitable coordinate change of the form $a \mapsto a - \frac{m}{x}$. We get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + h_1(y) + h_2(y,a,b) \end{pmatrix}.$$

The only possible monomial in $h_1$ which we cannot see in $h_2$ is $y^2$.

(a) If $h_1(y) = \lambda y^2$, then let $y$ cancel $\lambda$, clean the second column and redefine $h_2$, i.e.,

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + y^2 + h_2(y,a,b) \end{pmatrix}.$$

With respect to weight matrix and weight vector

$$(D,\omega) = \left( \begin{pmatrix} 4 & 3 & 7 \\ 3 & 2 & 6 \end{pmatrix}, (4,3,7,3,2,2,2) \right)$$

we can apply the counting argument to the quasihomogeneous starting jet

$$j_\omega M \sim \begin{pmatrix} x & y & z \\ v & w & ax + y^2 + j_3 h_2(a,b) \end{pmatrix}$$

and get

$$\dim Q_{(D,\omega)} - \#S_1 - \#S_2 - \#S_3 + 2 = 1 > 0.$$

Hence, $M$ cannot define a simple singularity.

(b) If $h_1(y) = 0$, then

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + h_2(y,a,b) \end{pmatrix}$$

has the quasihomogeneous starting jet

$$j_\omega M \sim \begin{pmatrix} x & y & z \\ v & w & ax + j_3 h_2(y,a,b) \end{pmatrix} \text{ w.r.t.}$$

$$(D,\omega) = \left( \begin{pmatrix} 4 & 2 & 5 \\ 5 & 3 & 6 \end{pmatrix}, (4,2,5,5,3,2,2) \right).$$

By the counting argument, we have $\dim Q_{(D,\omega)} - \#S_1 - \#S_2 - \#S_3 + 2 = 19 > 0$, so $M$ cannot be simple.

5. The last entry of $j_2M$ (hence any entry of $j_2M$) does not contain either of $a, b$: Consider the original matrix

$$M \sim \begin{pmatrix} x & y & z \\ v & w & h_1(x,y) + h_2(x,y,a,b) \end{pmatrix},$$

where $h_1 \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$, $h_2 \in \mathfrak{m}^3$, i.e., for some coefficients $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ we have

$$M \sim \begin{pmatrix} x & y & z \\ v & w & \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 + h_2(x,y,a,b) \end{pmatrix}.$$

(a) If $\lambda_1 = \lambda_2 = \lambda_3 = 0$ then

$$M \sim \begin{pmatrix} x & y & z \\ v & w & h_2(x,y,a,b) \end{pmatrix}.$$

By perturbing with $ax$ we get the matrix of the previous case (4b), which does not define a simple singularity. Hence, $M$ deforms into a non-simple singularity, therefore, $M$ does not define a simple singularity either.

(b) If one of $\lambda_1, \lambda_3$ does not vanish, w.l.o.g. $\lambda_1 \neq 0$, perform the coordinate changes

$$x \mapsto \frac{x}{\sqrt{\lambda_1}}, \quad x \mapsto \frac{\lambda_2 y}{2},$$

to get

$$j_2M \sim \begin{pmatrix} x & y & z \\ v & w & x^2 + \lambda y^2 \end{pmatrix},$$

with some new $\lambda \in \mathbb{C}$. Perturb with $ay$, perform a suitable coordinate change ($a \mapsto a - \lambda y$) to erase $\lambda y^2$, then we get a matrix

$$\begin{pmatrix} x & y & z \\ v & w & ay + x^2 + h(x,a,b) \end{pmatrix},$$

with some $h \in \mathfrak{m}^3$ (remember: monomials which are divisible by $y$ can be erased by coordinate changes in $a$). By the argument in 4a), this matrix does not define a simple singularity and as $M$ deforms into this matrix, $M$ cannot define a simple singularity either.

(c) If $\lambda_1 = \lambda_3 = 0$ but $\lambda_2 \neq 0$ then by performing $x \mapsto x - \frac{\lambda_2}{2} y$ we get rid of the monomial $xy$ and remain with a matrix of the form

$$\begin{pmatrix} x & y & z \\ v & w & \frac{\lambda_2^2}{4} y^2 + h_2(x,y,a,b) \end{pmatrix}.$$

Using a coordinate change in $y$ to cancel the coefficient and redefining $h_2$ we get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & y^2 + h_2(x,y,a,b) \end{pmatrix}.$$

Perturbing with $ax$ yields a matrix

$$\begin{pmatrix} x & y & z \\ v & w & ax + y^2 + j_3h_2(x,y) \end{pmatrix},$$

which does not define a simple singularity, hence, $M$ cannot define a simple singularity either.

□

### 6.5.2 Ambient space of dimension higher than 7

The main result in this section is the reduction of the classification to a smaller ambient space dimension which contains proper information about the behaviour of deformations. This proposition is a variant of the splitting lemma we know for hypersurface singularities:

**Proposition 6.4.** Let

$$M \begin{pmatrix} x & y & z \\ v & w & P(x,y,a_1,\dots,a_r) \end{pmatrix} \in \mathrm{Mat}(2,3,\mathbb{C}\{x,y,z,v,w,a_1,\dots,a_r\})$$

define a simple CMC2 singularity. Then

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x,y,a_1,a_2) + \sum_{i=3}^{r} a_i^2 \end{pmatrix},$$

where $P$ is a polynomial of the previous classification in ambient space dimension 7.

*Proof.* In order to proof the statement, we first check what happens in ambient space dimension 8, i.e., if we have a matrix $M \in \mathrm{Mat}(2,3,\mathbb{C}\{x,y,z,v,w,a,b,c\})$ with 1-jet of the form

$$\begin{pmatrix} x & y & z \\ v & w & 0 \end{pmatrix}$$

and three further variables $a,b,c$. Therefore, we focus on $j_2M$ and examine the different cases:

1. One of $a,b,c$ appears as a square in the last entry of $j_2M$:
   Without loss of generality, $c^2$ appears in the last entry, so after a coordinate change to erase the coefficient we repeat the GL-operations we performed in 7-dimensional

ambient space to get

$$M \sim \begin{pmatrix} x & y & z \\ v & w & c^2 + P(x,y,a,b) \end{pmatrix}.$$

In this case, we have

$$\begin{aligned} T^1(M) &\cong \mathbb{C}\{x,y,a,b,c\}/\langle P, \frac{\partial P}{\partial x}, \ldots, \frac{\partial P}{\partial b}, c\rangle \\ &\cong \mathbb{C}\{x,y,a,b\}/\Big(\langle x,y\rangle\langle P, \frac{\partial P}{\partial x}, \ldots, \frac{\partial P}{\partial b}\rangle\Big) \\ &\cong T^1\left(\begin{pmatrix} x & y & z \\ v & w & P(x,y,a,b) \end{pmatrix}\right). \end{aligned}$$

The candidates for simple matrices in 7-dimensional ambient space give all candidates for this case (with additional $c^2$).

2. None of the variables appear as a square, but one of $ab, ac, bc$ appears, without loss of generality, $ab$ appears:
Let $\lambda ab$ be the term appearing in the last entry. Perform $a \mapsto a - \frac{\lambda}{2}b$. Then $b^2$ appears with some coefficient. The matrix is GL-equivalent to the matrices of the first case and we do not see further candidates.

3. $a$,$b$,$c$ appear only mixed with $x, y$:
Let the last entry be $P = \lambda_1 ax + \lambda_2 bx + \lambda_3 cx + \mu_1 ay + \mu_2 by + \mu_3 cy$.

   (a) all coefficients vanish:
   The matrix cannot define a simple singularity using the weight matrix $\begin{pmatrix} 2 & 2 & 3 \\ 5 & 5 & 6 \end{pmatrix}$ and weight vector $\omega = (2,2,3,5,5,2,2,2)$.

   (b) not all coefficients vanish:
   Without loss of generality, let $\lambda_1 \neq 0$. Use $a \mapsto \frac{a}{\lambda_1}$ to erase the coefficient of $ax$, afterwards use $a \mapsto a - \lambda_2 b - \lambda_3 c$. Then, with suitable $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3 \in \mathbb{C}$ we have
   $$M \sim \begin{pmatrix} x & y & z \\ v & w & a_1 x + \tilde{\mu}_1 ay + \tilde{\mu}_2 by + \tilde{\mu}_3 cy \end{pmatrix}.$$
   The coordinate change $x \mapsto x - \tilde{\mu}_1 y$ and a column operation (adding the $\tilde{\mu}_1$ multiple of the second column to the first column) yields
   $$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + \tilde{\mu}_2 by + \tilde{\mu}_3 cy \end{pmatrix}.$$

      i. If $\tilde{\mu}_2 = \tilde{\mu}_3 = 0$ then we can use the same weight vector and weight matrix as in the next case to show that this matrix cannot define a simple singularity.

ii. If, without loss of generality, $\tilde{\mu}_2 \neq 0$, then $b \mapsto \frac{b}{\tilde{\mu}_2}$ and $b \mapsto b - \tilde{\mu}_3 c$ yields

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax + by \end{pmatrix}.$$

Using $D = \begin{pmatrix} 4 & 4 & 5 \\ 5 & 5 & 6 \end{pmatrix}$ and $\omega = (4,4,5,5,5,6)$ a matrix with this starting jet cannot define a simple singularity.

Hence, we do not find additional candidates for simple matrices.

Iteratively, we increase the dimension of the ambient space (i.e., the number of variables in the power series ring $\mathbb{C}\{\underline{x}\}$). In every step, the only new candidates for simple matrices are those whose $T^1$ is isomorphic to the $T^1$ of one the previous simple matrices with less variables. This is the case iff we can see the new variable as a square in the last entry after some GL-operations.

$\square$

### 6.5.3 Tjurina algebras and Tjurina numbers of the candidates

1. Let $M$ define an $\hat{A}_k$, $\hat{D}_k$, or one of $\hat{E}_6, \hat{E}_7, \hat{E}_8$. Let $f \in \mathbb{C}\{\underline{a}\}$ be the respective ADE-singularity in the last entry (added to $x$). Then

$$T^1(M) = \mathbb{C}\{\underline{a}\}/\langle f, \frac{\partial f}{\partial a_1}, \dots, \frac{\partial f}{\partial a_r}\rangle \cong T(f)$$

and

$$\tau(M) = \dim_{\mathbb{C}} T^1 \left( \mathbb{C}\{\underline{a}\}/\langle f, \frac{\partial f}{\partial a_1}, \dots, \frac{\partial f}{\partial a_r}\rangle \right) = \dim_{\mathbb{C}} T(f).$$

The Tjurina numbers $\tau$ in the table of simple candidates are the Tjurina numbers of the ADE-singularities.

2. Let $M$ define an $A_k^\sharp$, $D_k^\sharp$, or one of $E_6^\sharp, E_7^\sharp, E_8^\sharp$. Then

$$\tau(M) = \tau(\tilde{M}),$$

where $\tilde{M}$ is the matrix given by the intersection of $M$ with $Mat(2,3,\mathbb{C}\{x,y,z,v,w,a_1\})$, i.e., the part of $M$ containing only the variables in $x, y, z, v, w, a_1$. The Tjurina numbers $\tau$ can be copied from [16].

3. If

$$M \sim \begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + f(a_1, a_2) + \sum_{i=3}^{r} a_i^2 \end{pmatrix},$$

we calculated

$$T^1(M) \cong \mathbb{C}\{a_1, a_2\}/\langle a_1 \frac{\partial f}{\partial a_1}, a_2 \frac{\partial f}{\partial a_1}, a_1 \frac{\partial f}{\partial a_2}, a_2 \frac{\partial f}{\partial a_2}, f\rangle.$$

For reasons of readability we change the notation to

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+f(a,b) \end{pmatrix}, \quad T^1(M) \cong \mathbb{C}\{a,b\}/\langle a\frac{\partial f}{\partial a}, b\frac{\partial f}{\partial a}, a\frac{\partial f}{\partial b}, b\frac{\partial f}{\partial b}, f\rangle.$$

(a) Let $f = a^{k+1} + b^2$ be an $A_k$, then the quotient is factored by

$$\langle a^{k+1},\ b^2,\ ab\rangle,$$

so a basis of the $T^1$ is given by $1, a, a^2, \ldots, a^k, b$. Then

$$\tau(M) = k + 2.$$

(b) Let $f = a^2 b + b^{k-1}$ be an $D_k$ $(k \geq 4)$, then

$$\frac{\partial f}{\partial a} = 2ab, \quad \frac{\partial f}{\partial b} = a^2 + (k-1)b^{k-2},$$

so the quotient $T^1(M)$ is factored by

$$\begin{aligned} &\langle a^2b,\ ab^2,\ a^3 + (k-1)ab^{k-2},\ a^2b + (k-1)b^{k-1}\rangle \\ =&\langle a^2b,\ ab^2,\ a^3,\ b^{k-1}\rangle. \end{aligned}$$

A basis of the $T^1$ is given by $1, a, a^2, b, \ldots, b^{k-2}, ab$. Then

$$\tau(M) = k + 2.$$

(c) Let $f = a^3 + b^4$ be an $E_6$-singularity, then the quotient is factored by

$$\langle a^3,\ ab^3,\ a^2b,\ b^4\rangle,$$

so a basis of the $T^1$ is given by $1, a, a^2, ab, ab^2, b, b^2, b^3$. Then

$$\tau(M) = 8.$$

(d) Let $f = a^3 + ab^3$ be an $E_7$-singularity, then

$$\frac{\partial f}{\partial a} = 3a^2 + b^3, \quad \frac{\partial f}{\partial b} = 3ab^2,$$

so the quotient $T^1(M)$ is factored by

$$\begin{aligned} &\langle 3a^3 + ab^3,\ 3a^2b + b^4,\ a^2b^2,\ ab^3\rangle \\ =&\langle a^3,\ 3a^2b + b^4,\ a^2b^2,\ ab^3\rangle. \end{aligned}$$

We calculate a standard basis (see [22], p.54) w.r.t. negative degree reverse lexicographical ordering (see [22], p.14):

- s-polynomials of monomials are 0,
- $\mathrm{spoly}(3a^2b + b^4, a^3) = 3a^3b + ab^4 - 3a^3b = ab^4$ reduces to 0,
- $\mathrm{spoly}(3a^2b + b^4, a^2b^2) = 3a^2b^2 + b^5 - 3a^2b^2 = b^5$,
- $\mathrm{spoly}(3a^2b + b^4, ab^3) = 3a^2b^3 + b^6 - 3a^2b^3 = b^6$ reduces to 0,
- $\mathrm{spoly}(3a^2b + b^4, b^5) = 3a^2b^5 + b^7 - 3a^2b^5 = b^7$ reduces to 0.

Hence, the quotient $T^1(M)$ is factored by

$$\langle a^3,\ 3a^2b + b^4,\ a^2b^2,\ ab^3, b^5\rangle.$$

With a standard basis we can factor by the leading ideal (see [22], p. 228, Remark 3.3.12), which is

$$\langle a^3,\ a^2b,\ a^2b^2,\ ab^3, b^5\rangle,$$

and which yields a basis $1, a, a^2, b, b^2, b^3, b^4, ab, ab^2$ of $T^1(M)$. Then

$$\tau(M) = 9.$$

(e) Let $f = a^3 + b^5$ be an $E_8$-singularity, then the quotient is factored by

$$\langle a^3,\ a^2b,\ ab^4,\ b^5\rangle,$$

so a basis of the $T^1$ is given by $1, a, a^2, b, b^2, b^3, b^4, ab, ab^2, ab^3$. Then

$$\tau(M) = 10.$$

# Chapter 7

# Complete classification of simple non-isolated CMC2 singularities of type $(n, n+1, n)$ with $n^2+n-1$ variables in the 1-jet

To finish the classification of simple non-isolated CMC2 singularities, we need to calculate all adjacencies to check if there are indeed only finitely many. In this chapter, we finish the classification of simple non-isolated CMC2 singularities of type $(n, n+1, n)$ with $n^2+n-1$ variables in the 1-jet.

## 7.1 Full list of simple CMC2 singularities of type $(n, n+1, n)$ with at least $n^2+n-1$ variables in the 1-jet

The following theorem summarizes all results of this work. The proof will be split into several steps treated in the following sections.

**Theorem 7.1.** The simple non-isolated CMC2 singularities of type $(n, n+1, n)$ with at least $n^2+n-1$ variables in the 1-jet are given by the following list:

Table 7.1: Simple non-isolated CMC2 singularities of type $(n, n+1, n)$ with at least $n^2+n-1$ variables in the 1-jet

| Type | Presentation matrix | | $\tau$ |
|---|---|---|---|
| $\hat{A}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k+1} + \sum_{i=2}^r a_i^2 \end{pmatrix}$ | $k \geq 1$ | $k$ |
| $\hat{D}_k$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^{k-1} + a_1 a_2^2 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k$ |
| $\hat{E}_6$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^4 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 6 |
| $\hat{E}_7$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_1 a_2^3 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 7 |
| $\hat{E}_8$ | $\begin{pmatrix} x & y & z \\ v & w & x + a_1^3 + a_2^5 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 8 |
| $A_k^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k+1} + y^2 + \sum_{i=1}^r a_i^2 \end{pmatrix}$ | $k \geq 1$ | $k+2$ |
| $D_k^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^{k-1} + xy^2 + \sum_{i=1}^r a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k+2$ |
| $E_6^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + y^4 + \sum_{i=1}^r a_i^2 \end{pmatrix}$ | | 8 |
| $E_7^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + xy^3 + \sum_{i=1}^r a_i^2 \end{pmatrix}$ | | 9 |
| $E_8^\#$ | $\begin{pmatrix} x & y & z \\ v & w & x^3 + y^5 + \sum_{i=1}^r a_i^2 \end{pmatrix}$ | | 10 |
| $S_{k,\ell}$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + y^k + a_1^\ell + \sum_{i=2}^r a_i^2 \end{pmatrix}$ | $k \geq 2, \ell \geq 3$ | $k+\ell-1$ |
| $Q$ | $\begin{pmatrix} x & y & z \\ v & w & x^2 + y^2 + a_1^3 + \sum_{i=2}^r a_i^2 \end{pmatrix}$ | | 6 |
| $D_k^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^{k-1} + a_1 a_2^2 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | $k \geq 4$ | $k+2$ |
| $E_6^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_2^4 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 8 |
| $E_7^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_1 a_2^3 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 9 |
| $E_8^\star$ | $\begin{pmatrix} x & y & z \\ v & w & a_1 x + a_2 y + a_1^3 + a_2^5 + \sum_{i=3}^r a_i^2 \end{pmatrix}$ | | 10 |

## 7.2 Reduction of the adjacencies to type $(2, 3, 2)$

**Proposition 7.2.** Let $M \in \text{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ be a matrix with $n^2+n-1$ variables in the 1-jet in normal form. The adjacencies of the singularity defined by the $n$-minors of $M$ is determined by the adjacencies of the singularity defined by the lower right $2 \times 3$ submatrix. More precisely: The adjacencies of a $n \times (n+1)$ matrix are given by the $n \times (n+1)$ matrix with the adjacencies of the lower right $2 \times 3$ submatrix in the lower right corner.

*Proof.* Let $M \in \text{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ be a matrix with $n^2+n-1$ variables in the 1-jet. Perform GL-operations to get $M$ in normal form (see section 5.2). We know by the calculation of the $T^1$ that a versal family of any simple candidate may be expressed using only perturbation terms in the lower right $2 \times 3$ submatrix. Hence, the adjacencies of the original matrix are given by the matrices arising by perturbing the matrix with these perturbation terms in the lower right $2 \times 3$ submatrix. $\square$

## 7.3 Adjacencies of the candidates of type $(2, 3, 2)$

To find all adjacencies of the CMC2-singularities of type (2,3,2) we first treat the cases with enough information in the 1-jet. Later we check the cases where the last entry only contains monomials of higher order.

### 7.3.1 1-jet adjacencies

An easy observation is the behaviour in the 1-jet. The folowing lemma treats the deformations into the rigid singularity.

**Lemma 7.3.** A perturbation of

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x,y,\underline{a}) \end{pmatrix} \in \text{Mat}(2, 3, \mathfrak{m}_{\mathbb{C}\{x,y,z,v,w,\underline{a}\}})$$

with a matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_i \end{pmatrix}$$

yields the rigid singularity for any $1 \leq i \leq r$.

*Proof.* Without loss of generality, we call the 6-th variable in the last entry of the pertubation matrix $a_1$. With a coordinate change in $a_1$, precisely $a_1 := a_1 + P(x, y, \underline{a})$, any other term appearing in the last entry vanishes. We get

$$\begin{pmatrix} x & y & z \\ v & w & a_1 \end{pmatrix}$$

which is the rigid singularity (and particularly simple).

□

**Observation 7.4.** The adjacencies of 1-jets of CMC2-singularities of type (2,3,2) in $\mathbb{C}\{x, y, z, v, w, \underline{a}\}$ are the following:

$$\begin{pmatrix} x & y & z \\ v & w & a_1 \end{pmatrix}$$
$$\uparrow$$
$$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$$
$$\uparrow$$
$$\begin{pmatrix} x & y & z \\ v & w & 0 \end{pmatrix}$$

**Proposition 7.5.** Let $M \in \mathrm{Mat}(2, 3, \mathfrak{m}_{\mathbb{C}\{\underline{x},\underline{a}\}})$ and $f \in \mathbb{C}\{a_1, a_2\}$ of order at least 2. If

$$M \sim \begin{pmatrix} x & y & z \\ v & w & x + f(a_1, a_2) + \sum_{i=3}^{r} a_i^2 \end{pmatrix}$$

then the adjacencies of $M$ are determined by the adjacencies of $f(a_1, a_2)$.

*Proof.* The only adjacencies of these matrices are those with 1-jet $\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ v & w & a_1 \end{pmatrix}$ or those with a unit (constant term) in the last entry.

Perturbation by a unit: A constant perturbation term leads to a unit either in the first or in the last entry. In any case, we can clean the entries to get a cylinder over a point, i.e., $\langle x, y \rangle \times \mathbb{C}^{3+r}$. This is a simple singularity as the deformations are defined by the deformations of the smooth point, which is rigid in $\mathbb{C}^2$.

Perturbation by $a_1$: If we perturb the last entry w.l.o.g. by $a$, then we find the rigid singularity

$$\begin{pmatrix} x & y & z \\ v & w & a_1 \end{pmatrix}.$$

Perturbation by monomials of higher degree: Perturbation terms in $x, y, z, v, w$ are irrelevant, they may be cancelled by row and column operations respectively they may be cancelled by $x$. Only perturbation terms in the variables $a_1, \dots, a_r$ can change the equivalence class. However, the resulting singularity will still be of the same form with some $\tilde{f}(a_1, a_2)$ in the last entry, again corresponding to a plane curve singularity (as GL-equivalence of the matrix and contact equivalence of the hypersurface singularity in its last entry coincide). In particular, there is a generalized splitting lemma and the corank of $f$ cannot be smaller than the one of $\tilde{f}$. Therefore, the adjacencies of the matrix are determined by the adjacencies of $f$.

$\square$

Attention: Constant perturbation terms may lead to a splitting of the singularity into several singularities whose Tjurina numbers sum up to at most the Tjurina number of the original singularity.

### 7.3.2 2-jet-adjacencies

**Proposition 7.6.** Let

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x,y,\underline{a}) \end{pmatrix},$$

where $P(x,y,\underline{a}) \in \mathbb{C}\{x,y,\underline{a}\}$ of order at least 2 and

$$H_{\underline{a}}(P) = \left(\frac{\partial^2 P}{\partial a_i \partial a_j}\right)_{1 \le i,j \le r},$$

$\rho := \operatorname{crk} H_{\underline{a}(P)(0)}$. Then the adjacencies of $M$ are determined by the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & P(x,y,a_1,\dots,a_\rho) \end{pmatrix}$$

in $(\mathbb{C}^{5+\rho}, 0)$ where the last entry is extended by

$$\sum_{i=\rho+1}^{r} a_i^2.$$

*Proof.* As we can perform arbitrary coordinate changes in $a_1, \dots, a_r$, by Morse lemma, we can split the last entry into a polynomial in $x, y, a_1, \dots, a_\rho$ and a sum $\sum_{i=\rho+1}^{r} a_i^2$. Any monomials divisible by $a_{\rho+1}, \dots, a_r$ can be cancelled by a coordinate change in $a_{\rho+1}, \dots, a_r$.

$\square$

As we have seen, no matrices with $\operatorname{crk} H_{\underline{a}(P)(0)} \ge 3$ can be simple. Therefore, we check the adjacencies of the candidates with $\operatorname{crk} H_{\underline{a}(P)(0)} \le 2$.

Let

$$M \sim \begin{pmatrix} x & y & z \\ v & w & P(x,y,a,b) \end{pmatrix},$$

where $P(x,y,a,b) \in \mathbb{C}\{x,y,a,b\}$ is a polynomial of order 2. Let

$$H_{a,b}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 a} & \frac{\partial^2 P}{\partial a \partial b} \\ \frac{\partial^2 P}{\partial b \partial a} & \frac{\partial^2 P}{\partial^2 b} \end{pmatrix}$$

be the Hessian matrix of $P$ but only in the variables $a, b$. The evaluation in 0 shows the coefficients of the monomials in the 2-jet of $P$. Depending on the rank of $H_{a,b}(P)$ in 0, we can either reduce the work of finding the adjacencies to well-known cases or start a new calculation.

**Proposition 7.7.** Let $M$ be as described.

1. If $\operatorname{rk} H_{a,b}(P)(0) = 2$, the adjacencies of $M$ are determined by the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & P(x,y,0,0) \end{pmatrix}$$

as a CMC2 singularity in $(\mathbb{C}^5, 0)$.

2. If $\operatorname{rk} H_{a,b}(P)(0) = 1$, the adjacencies of $M$ are determined by the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & P(x,y,a,0) \end{pmatrix}$$

as a CMC2 singularity in $(\mathbb{C}^6, 0)$.

*Proof.*

1. $\operatorname{rk} H_{a,b}(P)(0) = 2$:
By Morse lemma, $P$ contains $a^2 + b^2$ up to legitimate coordinate changes. Therefore, any monomials divisible either by $a$ or $b$ in degree at least 2 can be cancelled by $a^2$ or $b^2$ via coordinate change. Assume $P$ to be w.l.o.g. the polynomial after the coordinate change giving rise to $a^2 + b^2$ and without any mixed terms of $a, b$ with $x, y$ (those can be cancelled by $a$ and $b$ respectively changing the coefficients of $x, y$ which can be cancelled by $x$ and $y$ respectively). The adjacencies of $M$ are determined by the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & P(x,y,0,0) \end{pmatrix}$$

as a CMC2 singularity in $(\mathbb{C}^5, 0)$.

2. $\operatorname{rk} H_{a,b}(P)(0) = 1$:
   As $H_{a,b}(P)$ is a symmetric matrix, after a suitable coordinate change one of $a$ and $b$ has to appear as a square. Assume $P$ to be w.l.o.g. the polynomial that arises after a coordinate change giving rise to $b^2$ and without any other monomial divisible by $b$. Then, the adjacencies of $M$ are determined by the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & P(x,y,a,0) \end{pmatrix}$$

   as a CMC2 singularity in $(\mathbb{C}^6, 0)$.

$\square$

There is one more case to consider, the case $\operatorname{rk} H_{a,b}(P)(0) = 0$. In this case, none of $a, b$ appears as a square, $ab$ does not appear either. For further examination we consider the full Hessian matrix

$$H_{x,y,a,b}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} & \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \\ \frac{\partial^2 P}{\partial a \partial x} & \frac{\partial^2 P}{\partial a \partial y} & 0 & 0 \\ \frac{\partial^2 P}{\partial b \partial x} & \frac{\partial^2 P}{\partial b \partial y} & 0 & 0 \end{pmatrix}.$$

In this case, we cannot reduce the problem to known matrices and their adjacencies. Therefore, we will approach the classification by using the rank of $H_{x,y,a,b}(P)$. We identify all normal forms for the 2-jets of matrices, determine the 2-jet and 3-jet adjacencies to complete the calculation of the adjacencies.
To find normal forms for $P$ or at least $j_2 P$, we must avoid coordinate changes that mix $x, y$ with $a, b$, as these changes alter the singularity type due to the matrix structure of $M$. Our goal to find all adjacencies of those matrices with $\operatorname{rk} H_{a,b}(P)(0) = 0$. To determine the adjacencies within the 2-jet, we need all normal forms that appear in the 2-jet of the last entry.

**Proposition 7.8.** Let $\operatorname{rk} H_{a,b}(P)(0) = 0$. Then all normal forms of $j_2 P$ (up to coordinate changes not mixing $x, y$ with $a, b$) are given by:

Table 7.2: normal form(s) of $j_2 P$

| $\operatorname{rk}(H_{x,y,a,b}(P)(0))$ | normal form(s) of $j_2 P$ |
|---|---|
| 4 | $ax + by$ |
| 3 | $y^2 + ax$ |
| 2 | $x^2 + y^2$, $ax$ |
| 1 | $x^2$ |
| 0 | 0 |

*Proof.* Let $\operatorname{rk} H_{a,b}(P)(0) = 0$. Consider $H_{x,y,a,b}(P)(0)$. After coordinate changes in $a, b$ the lower right corner becomes 0. We have

$$H_{x,y,a,b}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} & \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \\ \frac{\partial^2 P}{\partial a \partial x} & \frac{\partial^2 P}{\partial a \partial y} & 0 & 0 \\ \frac{\partial^2 P}{\partial b \partial x} & \frac{\partial^2 P}{\partial b \partial y} & 0 & 0 \end{pmatrix}.$$

As this matrix is symmetric, we get some information about the derivatives in the mixed terms of $x, y$ with $a, b$. We distinguish:

1. $\operatorname{rk} H_{x,y,a,b}(P)(0) = 4$:
   As the lower right block has rank 0, the lower left block must have rank 2 and by symmetry of the Hessian matrix, the upper right block must have rank 2 as well. Precisely, this means that the submatrix

   $$\begin{pmatrix} \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \end{pmatrix}$$

   must have rank 2, we say that the coefficients of the monomials $ax, ay, bx, by$ are „sufficiently general“. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ be the coefficients of $ax, ay, bx, by$. Define $a_{new} := \alpha a + \gamma b$ and $b_{new} := \beta a + \delta b$, then

   $$\alpha ax + \beta ay + \gamma bx + \delta by = a_{new}x + b_{new}y.$$

   As the determinant of

   $$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

   does not vanish, this defines a coordinate change (i.e., an automorphism on $\mathbb{C}\{x, y, a, b\}$). Now, with coefficients $\varepsilon, \zeta, \eta \in \mathbb{C}$, further terms $\varepsilon x^2, \zeta xy, \eta y^2$ in $x, y$ might appear in $P$. Define $a_{new} := a + \varepsilon x + \zeta y$ and $b_{new} := b + \eta y$ to get

   $$ax + by + \varepsilon x^2 + \zeta xy + \eta y^2 = (a + \varepsilon x + \zeta y)x + (b + \eta y)y = a_{new}x + b_{new}y.$$

   Hence,

   $$j_2 P \sim ax + by.$$

2. $\operatorname{rk} H_{x,y,a,b}(P)(0) = 3$:
   For reasons of symmetry, the block

   $$\begin{pmatrix} \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \end{pmatrix}$$

must have rank 1. Consider the terms $\alpha ax, \beta ay, \gamma bx, \delta by$ with coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. As the rank of

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is 1, there is $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\beta = \lambda\alpha$, $\delta = \lambda\gamma$, w.l.o.g. $\alpha \neq 0$. Via the coordinate change $x \mapsto x - \lambda y$ we get

$$\alpha ax + \beta ay + \gamma bx + \delta by \mapsto \alpha ax - \lambda\alpha ay + \lambda\alpha ay + \gamma bx - \lambda\gamma by + \lambda\gamma by = \alpha ax + \gamma bx.$$

Now, use $a \mapsto \frac{1}{\alpha}(a - \gamma b)$ to get

$$\alpha ax + \gamma bx \mapsto ax.$$

To reach $\operatorname{rk} H_{x,y,a,b}(P)(0) = 3$, we need to have

$$H_{x,y,a,b}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} & \alpha & \gamma \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} & \beta & \delta \\ \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \star & \star & \alpha & \gamma \\ \star & \star & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \star & \star & 1 & 0 \\ \star & \star & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

In order to find a matrix of rank 3, the coefficient of $y^2$ does not vanish. We check, that we can find the normal form $j_2 P \sim y^2 + ax$ with legitimate coordinate changes: Let $\lambda, \mu, \rho \in \mathbb{C}$, $\rho \neq 0$ and consider

$$\lambda x^2 + \mu xy + \rho y^2 + ax.$$

A coordinate change in $x, y$ yields

$$\lambda x^2 + \mu xy + \rho y^2 + ax \;\sim\; \lambda x^2 + \rho y^2 + ax \;\sim\; \lambda x^2 + y^2 + ax.$$

Now, perform the coordinate change $a \mapsto a - \lambda x$, then

$$\lambda x^2 + y^2 + ax \;\sim\; y^2 + ax.$$

Hence,

$$j_2 P \sim y^2 + ax.$$

3. $\operatorname{rk} H_{x,y,a,b}(P)(0) = 2$:
   There are two subcases to consider.
   (a) $\operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} \end{pmatrix}\right) = 2$, $\operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \end{pmatrix}\right) = 0$ : Then, $j_2 P \sim x^2 + y^2$.

(b) $\operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} \end{pmatrix}\right) = 0, \quad \operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \end{pmatrix}\right) = 1$ : Then, $j_2 P \sim ax$.

4. $\operatorname{rk} H_{x,y,a,b}(P)(0) = 1$:
   We know that

$$\operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} \end{pmatrix}\right) = 1, \quad \operatorname{rk}\left(\begin{pmatrix} \frac{\partial^2 P}{\partial x \partial a} & \frac{\partial^2 P}{\partial x \partial b} \\ \frac{\partial^2 P}{\partial y \partial a} & \frac{\partial^2 P}{\partial y \partial b} \end{pmatrix}\right) = 0$$

   Therefore,

$$j_2 P \sim x^2.$$

5. $\operatorname{rk} H_{x,y,a,b}(P)(0) = 0$: In this case,

$$j_2 P \sim 0.$$

$\square$

The adjacencies for $\operatorname{rk} H_{a,b}(P)(0) \in \{0,1\}$, we can cite the adjacencies from [16] but we can also extract the normal forms for the 2-jet of the last entry from the previous proof. Set

$$H_{x,y}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} \end{pmatrix}, \quad H_{x,y,a}(P) = \begin{pmatrix} \frac{\partial^2 P}{\partial^2 x} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial a} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial^2 y} & \frac{\partial^2 P}{\partial y \partial a} \\ \frac{\partial^2 P}{\partial a \partial x} & \frac{\partial^2 P}{\partial a \partial y} & \frac{\partial^2 P}{\partial^2 a} \end{pmatrix}.$$

**Corollary 7.9.**

1. If $\operatorname{rk} H_{a,b}(P)(0) = 2$, then $j_2 P \sim \begin{cases} x^2 + y^2 + a^2 + b^2 & , \operatorname{rk} H_{x,y}(P)(0) = 2 \\ y^2 + a^2 + b^2 & , \operatorname{rk} H_{x,y}(P)(0) = 1 \\ a^2 + b^2 & , \operatorname{rk} H_{x,y}(P)(0) = 0 \end{cases}$.

2. If $\operatorname{rk} H_{a,b}(P)(0) = 1$, then $j_2 P \sim \begin{cases} y^2 + ax + b^2 & , \operatorname{rk} H_{x,y,a}(P)(0) = 3 \\ x^2 + y^2 + b^2 & , \operatorname{rk} H_{x,y,a}(P)(0) = 2, \operatorname{rk} H_{x,y}(P)(0) = 2 \\ ax + b^2 & , \operatorname{rk} H_{x,y,a}(P)(0) = 2, \operatorname{rk} H_{x,y}(P)(0) = 1 \\ x^2 + b^2 & , \operatorname{rk} H_{x,y,a}(P)(0) = 1, \operatorname{rk} H_{x,y}(P)(0) = 0 \end{cases}$.

With the 1-jet adjacencies, the adjacencies of matrices containing $a^2, b^2$ in its last entry and the other normal forms of the 2-jet of $M$, we find the following (possibly incomplete) adjacency diagram:

**Proposition 7.10.** The 2-jet adjacencies of matrices with 5 variables in the 1-jet and 7 variables in total are demonstrated in the following graph:

$$
\begin{tikzcd}
 & & \begin{pmatrix} x & y & z \\ v & w & x+a^2+b^2 \end{pmatrix} & & & \\
 & \begin{pmatrix} x & y & z \\ v & w & x+a^2 \end{pmatrix} \arrow[ur] & & \begin{pmatrix} x & y & z \\ v & w & x^2+y^2+a^2+b^2 \end{pmatrix} \arrow[ul] & & \\
\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix} \arrow[ur] & & & \begin{pmatrix} x & y & z \\ v & w & ax+y^2+b^2 \end{pmatrix} \arrow[ull] \arrow[u] & & \begin{pmatrix} x & y & z \\ v & w & y^2+a^2+b^2 \end{pmatrix} \arrow[ull] \\
 & & \begin{pmatrix} x & y & z \\ v & w & ax+b^2 \end{pmatrix} \arrow[ur] & & \begin{pmatrix} x & y & z \\ v & w & x^2+y^2+b^2 \end{pmatrix} \arrow[ul] & \begin{pmatrix} x & y & z \\ v & w & a^2+b^2 \end{pmatrix} \arrow[u] \\
 & & \begin{pmatrix} x & y & z \\ v & w & ax+by \end{pmatrix} \arrow[u] \arrow[uull] & & \begin{pmatrix} x & y & z \\ v & w & x^2+b^2 \end{pmatrix} \arrow[u] & \\
 & & \begin{pmatrix} x & y & z \\ v & w & ax+y^2 \end{pmatrix} \arrow[u] & & & \\
\begin{pmatrix} x & y & z \\ v & w & x^2+y^2 \end{pmatrix} \arrow[urr] \arrow[uuuu] & & \begin{pmatrix} x & y & z \\ v & w & ax \end{pmatrix} \arrow[u] \arrow[uurr] & & & \\
 & & \begin{pmatrix} x & y & z \\ v & w & x^2 \end{pmatrix} \arrow[ull] \arrow[u] & & & \\
 & & \begin{pmatrix} x & y & z \\ v & w & 0 \end{pmatrix} \arrow[u] \arrow[uuuuurrr] & & &
\end{tikzcd}
$$

We know, that

$$\begin{pmatrix} x & y & z \\ v & w & x^2+b^2 \end{pmatrix}, \begin{pmatrix} x & y & z \\ v & w & ax+y^2 \end{pmatrix}$$

define non-simple matrices, hence, those are boundary singularities.

So, for all candidates for simple singularities except from $M$ with

$$j_2M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by \end{pmatrix},$$

we know the adjacencies and prooved simplicity. Now, we treat the final candidate. As a last step for the proof of simplicity and analogous to [20, I-Theorem 2.51], we look at the 3-jet:

**Observation 7.11.** In the 3-jet of the last entry of $M$, we can see further terms in $a, b$. Let $P = ax + by + f(a, b)$ be the last entry of $M$, $\operatorname{ord} f \geq 3$. There are three options:

- $j_3f$ factors into three different factors, i.e., $j_3f \sim ab(a+b)$.
- $j_3f$ factors into two different factors, i.e., $j_3f \sim ab^2$.
- $j_3f$ has one unique linear factor (of multiplicity 3), i.e., $j_3f \sim a^3$.

The calculations are the same as for hypersurface singularities, linear coordinate changes in $a, b$ can be caught by coordinate changes amongst $x, y$ (and matrix operations). The 3-jet adjacencies of $j_3P$ with stable 2-jet are the following:

$$ax + by \to ax + by + a^3 \to ax + by + ab^2 \to ax + by + ab(a+b).$$

**Proposition 7.12.** Let $M \in \operatorname{Mat}(2, 3, \mathbb{C}\{\underline{x}, \underline{a}\})$ be a candidate for a simple singularity of the form

$$M \sim \begin{pmatrix} x & y & z \\ v & w & a_1x + a_2y + f(a_1, a_2) + \sum\limits_{i=3}^{n} a_i^2 \end{pmatrix},$$

$f \in \mathbb{C}\{a_1, a_2\}$ as in the list of candidates. Then the adjacencies of $M$ are determined by the adjacencies of $f(a_1, a_2)$ as hypersurface singularity with section.

*Proof.* Let $M$ be one of the candidates for simple matrices with

$$j_2M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by \end{pmatrix},$$

i.e.,

$$M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+f(a,b) \end{pmatrix},$$

where $f \in \mathbb{C}\{a,b\}$ defines a $D_k$-singularity or $E_6, E_7, E_8$.

If $M$ deforms into a matrix sucht that $j_2M$ changes, we see by observation 7.10 that we deform only into one of the matrices

$$\tilde{M} = \begin{pmatrix} x & y & z \\ v & w & G(x,y,a,b) \end{pmatrix}$$

such that

1. $\operatorname{rk} H_{a,b}(G)(0) \geq 1$, then w.l.o.g. there are only the adjacencies of

$$\begin{pmatrix} x & y & z \\ v & w & G(x,y,a,0) \end{pmatrix}$$

(with additional $b^2$ in the last entry) with $\tau(\tilde{M}) \leq \tau(M)$.

2. $G = x + g(a,b)$, $g \in \mathbb{C}\{a,b\}$ of order 3, then the adjacencies are given by hypersurface adjacencies with $\tau(\tilde{M}) = \tau(g) \leq \tau(f)$.

Now, we check the adjacencies with stable $j_2M$. Therefore, we check the adjacencies for $D_k^\star$, $E_6^\star$, $E_7^\star$, $E_8^\star$ separately.

- $D_k^\star$:
  If $f$ defines a $D_k$-singularity, then

$$T^1(M) \cong \mathbb{C}\{a,b\}/\langle a^2b,\ ab^2,\ a^3,\ b^{k-1}\rangle,$$

  so any perturbation can be expressed using the monomials $1, a, a^2, b, \ldots, b^{k-2}, ab$. With a stable 2-jet, we are not interested in perturbations using monomials of degree lower than 3.
  If $k = 4$, then, by a linear coordinate change in $a, b$, the 3-jet is

$$j_3M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+a^3+ab^2 \end{pmatrix} \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+ab(a+b) \end{pmatrix}.$$

  This is already the most general 3-jet, so $M$ can only deform with stable 3-jet but there a no monomials which do not change the 3-jet.
  If $k > 4$, then

$$j_3M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+ab^2 \end{pmatrix},$$

which can either deform into the 3-jets

$$\begin{pmatrix} x & y & z \\ v & w & ax+by+ab^2 \end{pmatrix} \text{ or } \begin{pmatrix} x & y & z \\ v & w & ax+by+ab(a+b) \end{pmatrix}.$$

In the first case, i.e., if the 3-jet is stable, perturbation terms $b^l$, $3 \leq l \leq k-2$ lead to $D_l^\star$-singularities. In the second case, we only see the $D_4^\star$-singularity.

- $E_6^\star$:
  If $f$ defines a $E_6$-singularity, then

$$T^1(M) \cong \mathbb{C}\{a,b\}/\langle a^3,\ ab^3,\ a^2b,\ b^4\rangle,$$

so any perturbation can be expressed using the monomials $1, a, a^2, ab, ab^2, b, b^2, b^3$. All of them change the 3-jet, so there are no adjacencies with 3-jet

$$\begin{pmatrix} x & y & z \\ v & w & ax+by+a^3 \end{pmatrix}$$

except from the deformation of $M$ into itself. Using the perturbation term $ab^2$ in the last entry we get to the adjacencies with 3-jet

$$\begin{pmatrix} x & y & z \\ v & w & ax+by+ab^2 \end{pmatrix}.$$

With the same argument as for $D_k^\star$ and the upper semicontinuity of the Tjurina number, we see only $D_l$ singularities with $\tau = l+2 \leq 8 = \tau(E_6^\star)$, i.e., $l \leq 6$ (and their adjacencies).

- $E_7^\star$:
  If $f$ defines a $E_7$-singularity, then

$$T^1(M) \cong \mathbb{C}\{a,b\}/\langle a^3,\ a^2b,\ a^2b^2,\ ab^3, b^5\rangle,$$

so any perturbation can be expressed using the monomials $1, a, a^2, b, b^2, b^3, b^4, ab, ab^2$. Perturbations containing $b^4$ might lead to adjacencies with the same 3-jet

$$j_3M \sim \begin{pmatrix} x & y & z \\ v & w & ax+by+a^3 \end{pmatrix}.$$

By finite determinacy, after perturbing with a $b^4$, we find at most an $E_6^\star$. The other adjacencies have 3-jet

$$\begin{pmatrix} x & y & z \\ v & w & ax + by + ab(a+b) \end{pmatrix}.$$

The only singularities with that 3-jet such that an adjacency is possible are the $D_l^\star$-singularities with $l \leq 7$ (as semicontinuity yields $\tau = l + 2 \leq 9 = \tau(E_7^\star)$) and their adjacencies.

- $E_8^\star$:
  If $f$ defines a $E_8$-singularity, then

$$T^1(M) \cong \mathbb{C}\{a, b\}/\langle a^3,\ a^2b,\ ab^4,\ b^5\rangle.$$

  Any perturbation can be expressed using the monomials $1, a, a^2, b, b^2, b^3, b^4, ab, ab^2, ab^3$. With stable 3-jet we find adjacencies to $E_6^\star$ and $E_7^\star$ (these are the types we can get if the terms $b^4$ or $ab^3$ are involved). Other adjacencies change the 3-jet, again, we find $D_l^\star$ singularities with $l \leq 8 = \tau(M) - 2$ and their adjacencies.

So, for all candidates, we find exactly the adjacencies of $f$ as hypersurface singularity, (which are finitely many).

□

**We conclude:**
All singularities in the list of candidates define indeed simple singularities, as there are only finitely many adjacencies for each candidate.

# Chapter 8

# Outlook: CMC2 singularities of type $(n, n+1, n)$ with $n^2+n-2$ variables in the 1-jet

To complete the classification of simple non-isolated CMC2 singularities, three points remain to be studied:

- Classification of normal forms for 1-jets containing $n^2+n-2$ variables: We were able to encounter 3 possibilities for the positions of the two 1-forms in the 2 non-generic entries. For each of these possibilities, normal forms for 1-jets have to be classified.
- Classification of $2\times 3$ matrices with four variables in the 1-jet: One of the remaining challenges is to develop a classification for $2 \times 3$ matrices containing four variables in the 1-jet. Achieving this will expand our understanding of higher-dimensional singularities and their classifications.
- Reduction from $n \times (n+1)$ matrices to the lower right $2 \times 3$ submatrix: Another critical task is to establish a reduction method from $n \times (n+1)$ matrices to the lower right $2 \times 3$ submatrix. Successfully achieving this reduction will close the classification for this case and provide an approach to dealing with more complex matrix structures.

These open tasks highlight the efforts required to complete the classification. Future research focusing on these areas will not only validate the methods developed in this thesis but also lay the foundation for further advancements in the field of singularity theory. By addressing these challenges, we can achieve a more comprehensive and nuanced understanding of the behavior of singularities across various dimensions and configurations.

We sketch the first approach towards the classification in the case of $n^2+n-2$ variables:

**Proposition 8.1.** Let $M \in \mathrm{Mat}(n, n+1, \mathbb{C}\{\underline{x}\})$ be a matrix with $n^2+n-2$ variables appearing in the 1-jet. Then $j_1 M$ is equivalent to one of the following matrices

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & x_{n-2,n+1} \\ \vdots & & x_{n-1,n-1} & x_{n-1,n} & \beta \\ x_{n,1} & \dots & x_{n,n-1} & \alpha & x_{n,n+1} \end{pmatrix}, \quad \begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & x_{n-2,n+1} \\ \vdots & & x_{n-1,n-1} & x_{n-1,n} & x_{n-1,n+1} \\ x_{n,1} & \dots & x_{n,n-1} & \alpha & \beta \end{pmatrix},$$

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} & x_{1,n} & x_{1,n+1} \\ \vdots & & \vdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & x_{n-2,n+1} \\ \vdots & & x_{n-1,n-1} & x_{n-1,n} & \alpha \\ x_{n,1} & \dots & x_{n,n-1} & x_{n,n} & \beta \end{pmatrix},$$

with 1-forms $\alpha, \beta \in \mathfrak{m}_{\mathbb{C}\{\underline{x}\}}$.

*Proof.* We start with $n^2+n-2$ variables appearing as general as possible in the 1-jet. a coordinate change replaces all but two entries by a single variable. Changing rows and columns, renumbering the variables, the two entries that are left can be placed in the lower right corner. Now, we have three possibilities left.

□

**Remark 8.2.** There are results on the classification of simple isolated singularities defined by $(2 \times 3)$-matrices with 4 variables appearing in the 1-jet in [16]. There is hope to extend the classification to simple non-isolated CMC2 singularities and to find a similar reduction of matrices of arbitrary size to submatrices of size $2 \times 3$.

**Open question:**

- Is it possible to reduce matrices of size $n \times (n+1)$ defining simple singularities to matrices of size $2 \times 3$?
- Is it possible to reduce the number of variables in a $(2 \times 3)$-matrix carrying relevant information?

# Appendix A

# Calculations

## A.1 Calculations for the reduction of the 1-jet:

Consider

$$g(n,N) = -\frac{1}{2}n^4 - n^3 + Nn^2 + 3n^2 + Nn - \frac{1}{2}N^2 + \frac{7}{2}n - \frac{9}{2}N - 1.$$

1. Let $n \geq 3$. In this case, the polynomial $g(n,N) \in \mathbb{N}[n,N] \subseteq \mathbb{R}[n,N] \subseteq \mathbb{C}[n,N]$ has a negative value for example in $n = 3, N = 1$:

   $$g(3,1) = -\frac{1}{2} \cdot 81 - 27 + 9 + 27 + 3 - \frac{1}{2} + \frac{7}{2} \cdot 3 - \frac{9}{2} - 1 = -24.$$

   As a polynomial in $N$ we write $g(n,N)$ as

   $$g(n,N) = -\frac{1}{2}N^2 + N(n^2 + n - \frac{9}{2}) - \frac{1}{2}n^4 - n^3 + 3n^2 + \frac{7}{2}n - 1 \in \underbrace{\mathbb{N}[n,N]}_{\subseteq \mathbb{R}[n,N] \subseteq \mathbb{C}[n,N]}.$$

   We find the complex roots by the following calculation:

   $$\begin{aligned}
   & 0 = -\frac{1}{2}N^2 + N(n^2 + n - \frac{9}{2}) - \frac{1}{2}n^4 - n^3 + 3n^2 + \frac{7}{2}n - 1 \\
   \iff & 0 = N^2 - 2N(n^2 + n - \frac{9}{2}) + n^4 + 2n^3 - 6n^2 - 7n + 2 = 0 \\
   \iff & N = n^2 + n - \frac{9}{2} \pm \sqrt{\Big(n^2 + n - \frac{9}{2}\Big)^2 - \big(n^4 + 2n^3 - 6n^2 - 7n + 2\big)} \\
   \iff & N = n^2 + n - \frac{9}{2} \pm \sqrt{n^4 + 2n^3 - 8n^2 - 9n + \frac{81}{4} - n^4 - 2n^3 + 6n^2 + 7n - 2} \\
   \iff & N = n^2 + n - \frac{9}{2} \pm \sqrt{-2n^2 - 2n + \frac{89}{4}}.
   \end{aligned}$$

So, the complex roots of the polynomial are given by

$$N = n^2 + n - \frac{9}{2} \pm \sqrt{-2n^2 - 2n + \frac{89}{4}}.$$

For some $n$, the roots are always non-real:
$-2n^2 - 2n + \frac{89}{4} < 0$ holds iff $11 = \frac{88}{8} < \frac{89}{8} < n^2 + n$, which means for all $n \geq 3$ there are no real roots. But, considering $g(n, N)$ as a real polynomial without any real roots and with a known negative value $g(3, 1) < 0$, there is no positive value $g(n, N)$ for any $n \geq 3$ and $N$ arbitrary.

2. Let $n = 2$.

$$\begin{aligned} g(2, N) &= -\frac{1}{2} \cdot 16 - 8 + N \cdot 4 + 3 \cdot 4 + N \cdot 2 - \frac{1}{2} \cdot N^2 + \frac{7}{2} \cdot 2 - \frac{9}{2} \cdot N - 1 \\ &= -\frac{1}{2} \cdot N^2 + \frac{3}{2} \cdot N + 2 = -\frac{1}{2}(N^2 - 3N - 4) = -\frac{1}{2}(N+1)(N-4) < 0 \end{aligned}$$

iff $N > 4$ (or $N < -1$, but as $N \in \mathbb{N}$ this does not happen). In this case, we consider at most space curves in $(\mathbb{C}^3, 0)$ and we know there are no simple non-isolated space curves.

3. Let $n = 1$. We consider complete intersection singularities and we know there are no simple non-isolated complete intersection singularities.

## A.2 Calculations for the counting argument:

1. Weights:

$$\left( \begin{pmatrix} 4 & 3 & 7 \\ 3 & 2 & 6 \end{pmatrix}, (4, 3, 7, 3, 2, 2, 2) \right)$$

Values:

| $i$ | Pos($i$) | Var($i$) | Mon($i$) |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 1 | 3 | 3 |
| 3 | 2 | 2 | 2 |
| 4 | 1 | 1 | 7 |
| 5 | 1 | 1 | 7 |
| 6 | 1 | 0 | 19 |
| 7 | 1 | 1 | ? |

Calculation:

$$\dim Q_{(D,\omega)} - \#S_1 = (\mathrm{Pos}(2) - \mathring{Var}(2))\mathrm{Mon}(2) + (\mathrm{Pos}(6) - \mathring{Var}(6))\mathrm{Mon}(6) = 13, \;\; \#S_2 = 2, \#S_3 = 12$$

$$\Longrightarrow \dim Q_{(D,\omega)} - \#S_1 - \#S_2 - \#S_3 + 2 = 1 > 0$$

2. Weights:

$$(D,\omega) = \left( \begin{pmatrix} 4 & 2 & 5 \\ 5 & 3 & 6 \end{pmatrix}, (4,2,5,5,3,2,2) \right)$$

Values:

| $i$ | Pos($i$) | Var($i$) | Mon($i$) |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 1 | 3 | 3 |
| 3 | 1 | 1 | 1 |
| 4 | 1 | 1 | 7 |
| 5 | 2 | 2 | 5 |
| 6 | 1 | 0 | 32 |

Calculation:

$$\dim Q_{(D,\omega)} - \#S_1 = (\mathrm{Pos}(2) - \mathrm{Var}(2))\mathrm{Mon}(2) + (\mathrm{Pos}(6) - \mathrm{Var}(6))\mathrm{Mon}(6) = 26, \;\; \#S_2 = 2, \#S_3 = 7$$

$$\Longrightarrow \dim Q_{(D,\omega)} - \#S_1 - \#S_2 - \#S_3 + 2 = 19 > 0$$

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Oldenburg, 30. Oktober 2024.

Aline Bartel