Geometric Resolution of Generalized Semi-Classical Operators

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Zusammenfassung

In dieser Dissertation werden verallgemeinerte, semi-klassische Operatoren geometrisch durch Blow-ups aufgelöst und anschließend Quasimoden auf den aufgelösten Räumen konstruiert. Das Ziel ist dabei die bekannten Methoden zur Konstruktion von WKB-Approximation $u = e^{i\varphi/h}A$ von Lösungen der Schrödinger-Gleichung $(-h^2\partial_x^2 + V)u = 0$, für $V \in C^\infty(\mathbb{R})$, auf eine größere Klasse von Operatoren $P = P(x, h, \partial_x)$ zu verallgemeinern. Das zentrale Hilfsmittel ist dabei das neu eingeführte *Newton Polygon* $\mathcal{P}(\Lambda(P))$ eines semi-klassischen Operators P, welches genutzt wird um qualitative und quantitative Aussagen über die Existenz von Quasimoden vorherzusagen. Darüber hinaus betrachten wir Operatoren, dessen Lösungen der induzierten Eikonalgleichungen Multiplizitätssprünge aufweisen. Diese werden algorithmisch durch die Anwendung verketteter, quasihomogener Blow-ups aufgelöst, in dem Sinne, dass wir hinreichend reguläre Quasimoden von WKB-Art für den Operator β^*P auf dem aufgelösten Raum $\beta\colon M \to \mathbb{R}\times\mathbb{R}_+$ konstruieren können.

Short Summary

In this thesis we resolve generalized semi-classical operators geometrically by means of blow-up and construct quasimodes on the resolved spaces. The goal is to generalize the well-known methods for constructing WKB approximations $u = e^{i\varphi/h}A$ for solutions of the Schrödinger equation $(-h^2\partial_x^2 + V)u = 0$, for $V \in C^\infty(\mathbb{R})$, to a wider class of operators $P = P(x, h, \partial_x)$. The central tool is the newly introduced *Newton polygon* $\mathcal{P}(\Lambda(P))$ of a semi-classical operator P, which is used to predict qualitative and quantitative statements about the existence of quasimodes. Moreover, we consider operators whose solutions of their induced eikonal equation have jumps of multiplicity. These are algorithmically resolved by chaining quasihomogeneous blow-ups, in the sense that we can construct sufficiently regular WKB-type quasimodes for the operator β^*P on the resolved space $\beta\colon M \to \mathbb{R}\times\mathbb{R}_+$.

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Contents

1 Introduction

Motivation

Semi-classical operators were first introduced in the early 20th century and have been a central object of research in mathematics and physics ever since. In general, these are differential operators $P = P(x, h, \partial_x)$ whose coefficients may depend on a small, non-negative parameter $h \geq 0$. Especially for the most famous object, the (time independent) *Schrödinger operator* on $\mathbb{R}$ with negative potential $V \in C^\infty(\mathbb{R})$

$$-h^2\partial_x^2 + V$$

acting on smooth functions, there is great coverage about its properties in the literature. The singular leading coefficient $-h^2$ complicates the computation of families of solutions v_h of its corresponding equation $(-h^2\partial_x^2 + V)v_h = 0$ as $h \to 0$. Thus, a natural first step to address this problem is the construction of approximate solutions u_h, so called *quasimodes*. These satisfy

$$\left(-h^2\partial_x^2 + V\right) u_h = O(h^\infty)\,,$$

where the right hand side means that $\left(-h^2\partial_x^2 + V\right) u_h = f_h$ is not necessarily zero for $h > 0$, but vanishes faster than any polynomial in h in a suitable norm on any compact subset $K \subset \mathbb{R}$, as $h \to 0$.

Such a quasimode can be obtained by stating an *ansatz* and validating that particular functions of this form can satisfy the remainder condition afterwards. The most important ansatz, i.e. an initial guess for u_h, is given by the so-called *WKB*-approximation for $x \in \mathbb{R}$,

$$u_h(x) = e^{i\varphi(x)/h} A_h(x)\,,$$

stating that potential quasimodes are given by a wave function, with an oscillatory behavior of $\varphi(x)/h$ for all $x \in \mathbb{R}$ as $h \to 0$ and an amplitude in form of a power series $A_h(x) = \sum_k a_k(x)h^k$ in h. It is named after *Gregor Wentzel*, *Hendrik Anthony Kramers* and *Léon Brillouin*, who have proposed this ansatz in the early 20th century independently, see [Bri26],[Kra26],[Wen26]. It can be derived as an educated guess after simplifying the potential V to a point where one is able to solve the Schrödinger equation explicitly. Plugging this ansatz into the equation and expanding the remainder of this WKB-ansatz in powers of h

$$\left(-h^2\partial_x^2 + V\right) e^{i\varphi/h} A_h = e^{i\varphi/h}\left[\left((\varphi')^2 + V\right) a_0 + O(h)\right]$$

yields a leading order term $((\varphi')^2 + V)a_0$ that vanishes if either $a_0 \equiv 0$ or $\varphi' = \sqrt{-V}$ holds. This immediately yields that the *phase* φ is given by

$$\varphi(x) = \int_{x_0}^{x} \sqrt{-V(t)}dt$$

for any choice of $x_0 \in \mathbb{R}$, which is smooth if V is non-zero. Further computations show that all functions a_k in the expansion of A_h have to solve a recurrent first order equation

$$(-2\varphi'\partial_x - \varphi'')\, a_k = a''_{k-1} \tag{1.1}$$

for $k \in \mathbb{N}_0$, where $a_{-1} := 0$, to improve the remainder estimate of $\left(-h^2\partial_x^2 + V\right) e^{i\varphi/h}A_h = f_h$.

One can immediately see that depending on the choice of potential $V \in C^\infty(\mathbb{R})$ this ansatz can cause problems, resulting in non-smooth solutions φ and a_k. In particular, if V vanishes at the point $x = 0$ then $\varphi = \int \sqrt{-V}$ is not smooth at $x = 0$. Even worse, the leading coefficient $-2\varphi'$ of the recursive equation is non-smooth and vanishes at $x = 0$, resulting in a non-elliptic differential equation. The homogeneous equation for a_0 can be rephrased to

$$\left(x\partial_x - \frac{x\varphi''(x)}{2\varphi'(x)}\right) a_0 = 0$$

whose solution a_0 is non-smooth at $x = 0$ in general. The structure of the inhomogeneous equation is the same as in (1.1) with the second derivative of the previous solution as inhomogeneity. Thus, this recurrent equation will lead to an increasingly singular behavior of a_k at $x = 0$ as $k \to \infty$.

In this case, there are two different kinds of singularities of the Schrödinger operator simultaneously: the vanishing leading coefficient of $-h^2\partial_x^2$ as $h \to 0$ and the vanishing potential V in the point $x = 0$. Viewing (x, h) as combined coordinates on the half space $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$, these singularities with their very different impacts overlap at $(x, h) = (0, 0)$, resulting in WKB-type approximations with non-uniform remainder estimates at $x = 0$. The most important way to resolve these overlapping singularities in this thesis is given by the *quasihomogeneous blow-up* of $(0, 0)$ in the half space $\beta\colon [\mathbb{H}, (0, 0)]_r \to \mathbb{H}$ *with parameter* $r \in \mathbb{Q}_+$. The blow-up removes the origin $(0, 0)$ and adds the end point of each curve $\gamma(c) := \{h^r = cx\}$ with respect to $c \in \overline{\mathbb{R}}$ to the space, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. For the Schrödinger operator this corresponds to the introduction of singular coordinates (τ, h) with $\tau = x/h^{2/3}$, i.e. $r = 2/3$. For $V(x) = cx$ the pullback of the Schrödinger operator in these coordinates yields the *Airy* operator

$$\beta^*\left(-h^2\partial_x^2 + V\right) = -h^2(\beta^*\partial_x)^2 + (\beta^*V)(\tau h^{2/3}) = -h^{2/3}(-\partial_\tau^2 + c\tau)$$

on the front face $\{(\tau, 0)\colon\ \tau \in \mathbb{R}\}$. This is a significant improvement since it is a product of the vanishing coefficient h and an elliptic operator $-\partial_\tau^2 + c\tau$ for all $\tau \in \mathbb{R}$, eliminating both singularities. Simultaneously, we can construct a WKB-approximation on both $x > 0$ and $x < 0$. Matching these WKB-approximations with the Airy function on the front face can be done using a separate set of coordinates induced by the blow-up. This is performed explicitly in the popular *method of matched asymptotic expansion*. Another more geometric method of equal value is described in [Gri17] and [Sob18] and will be the foundation of this thesis. It uses a family of principal symbols along the boundary faces of the blown-up space to construct global polyhomogeneous functions whose leading terms are solutions of the recursive differential equations. This systematic approach also works for more complicated blown-up spaces.

As one can easily verify, these methods are not limited to the Schrödinger operator. Analyzing generalizations of this operator with coefficients h^α for $\alpha > 0$ and differentiation order $k \in \mathbb{N}$,

$$-h^\alpha \partial_x^k + V\,,$$

one is able to construct quasimodes with minimal changes to the standard WKB-ansatz. For a vanishing potential V, one can find a suitable quasihomogeneous blow-up, allowing for the same procedure as with the Schrödinger operator with linear potential and the Airy functions. Thus, the central question of this thesis is whether one can give a uniform approach, including blow-ups, to determine a full set of independent quasimodes for a general class of operators.

Results

The main result of this thesis, Theorem 5.2.18, states that we can eliminate all relevant singularities of generalized semi-classical operators by a finite number of blow-ups of the underlying space. Thereafter, Theorem 5.3.22 constructs quasimodes for generalized semi-classical operators on these blown-up spaces. Thus, Theorem 5.2.18 resolves a generalized semi-classical operator in the sense that we are able to construct an associated basis of quasimodes on the corresponding space.

The class of generalized semi-classical operators we are interested in contains operators

$$\sum_{k=0}^{m} A_k(x,h)\partial_x^k = \sum_{(k,\alpha)} a_{(k,\alpha)}(x)h^\alpha \partial_x^k$$

on intervals $I \subset \mathbb{R}$ whose coefficients A_k are smooth in x and polyhomogeneous in h. More precisely, we consider operators whose coefficients A_k are asymptotically equal to formal power series in h with arbitrary, real powers. In a simplified version without blow-ups, omitting technicalities, the results of this thesis with their constructive proofs are of the form as follows.

Theorem. *Let $P = \sum_{k=0}^m A_k(x,h)\partial_x^k$ be a generalized semi-classical operator on $I \subset \mathbb{R}$. Then there are phase functions φ_j, numbers $\delta_j \geq 0$ and amplitudes B_j, for $j = 1, \dots, m$, such that the functions*

$$u_j = e^{\varphi_j / h^{\delta_j}} B_j$$

are independent quasimodes of P, i.e. for each $j = 1, \dots, m$ we have

$$\left(e^{-\varphi_j/h^{\delta_j}} P e^{\varphi_j/h^{\delta_j}}\right) B_j = O(h^\infty)\,.$$

The functions u_j are called *generalized WKB-quasimodes*, due to the power δ_j of h in the denominator of the phase. The quasimodes we construct should at least be polyhomogeneous or if they are a product of an exponential term and an amplitude, then both the phase function and the amplitude should be polyhomogeneous on I. Thus, there are three central questions which we discuss in the main part of this thesis:

(i) What are sufficient conditions on the coefficients of the operator so that a generalized WKB-ansatz yields quasimodes with *smooth* phase functions and amplitudes? (Chapter 3)

(ii) What are sufficient conditions on the coefficients of the operator so that a generalized WKB-ansatz yields quasimodes with *polyhomogeneous* phase functions and amplitudes? (Chapter 4)

(iii) Can the remaining generalized semi-classical operators be regularized by a finite chain of blow-ups, in the sense that they allow for exponential-polyhomogeneous quasimodes? (Chapter 5)

Each of these questions will be addressed in a separate chapter in the given order and can be answered completely in Theorems 3.3.9, 4.3.3 and 5.3.22. The first two theorems are simple versions of Theorem 5.3.22 due to the higher levels of regularity. All theorems yield clear conditions for the types of regularity and a blow-up algorithm to desingularize the rest. In particular, this covers parameter-dependent families of ordinary differential equations such as the *Bessel* equation.

Apart from answering these questions, the methods presented in the following chapters have more positive features. Based on the combinatorial data given a priori by the operator, these algorithms can predict the correct type of generalized WKB-ansatz and the complete chain of blow-ups required to regularize it. This data only includes the powers of h and ∂_x in the asymptotic expansion of the semi-classical operator for the prediction of the adequate WKB-type. To determine the chain of blow-ups, it only requires the vanishing orders of the coefficients $a_{(k,\alpha)}$ in the relevant points as additional data. The algorithmic approach generates an explicit and iterative blow-up scheme, which will be encoded in an oriented blow-up graph.

Coming from the general theory of irregular singular ordinary differential equations, this thesis is a generalization towards constructing quasimodes for perturbed linear ordinary differential equations. The evaluation of combinatorial data and the geometric resolution algorithm results in a reduction of the complexity of the initial problem. The construction of a set of *independent* solutions can thus be reduced to a family of first order b-elliptic differential equations and in rare cases to irregular singular ordinary differential equations.

We want to emphasize that it is beyond the scope of this thesis to determine explicit solutions of the equations induced by the semi-classical operators or to show that the constructed quasimodes are close to these. Although explicit solutions arise in individual cases for very special operators, these are rather generic coincidences. Thus, this thesis is located in the early level of asymptotic solvability of generalized semi-classical equations analogous to the works of Wentzel, Kramers and Brillouin (see [Wen26],[Kra26],[Bri26]), which themselves were starting points for the development of the theory of semi-classical operators.

Regular Operators

In Chapter 3 we will investigate generalized semi-classical operators with sufficient regularity properties. The goal is to answer the first question (i) about the existence of suitable adjustments to the WKB-ansatz in a more general setting based on data given by the operator itself. In the end we will have a systematic approach to construct quasimodes for any generalized semi-classical operator matching the classical approach for the Schrödinger operator. Additionally, we are able to determine the relevant types of WKB-ansatz and the regularity of the quasimode for each operator based on combinatorial data.

The essential object to reduce an operator to its combinatorial data is the so called *set of exponents* $\Lambda \subset \mathbb{N}\times\mathbb{R}$. It consists of pairs of exponents $(k, \alpha) \in \Lambda$ for which the corresponding coefficient $a_{(k,\alpha)}$ in the asymptotic expansion of the operator

$$\sum_{(k,\alpha)\in\Lambda} a_{(k,\alpha)} h^\alpha \partial_x^k$$

does not vanish everywhere. The chapter starts with an exploration of different examples for which we can compute some quasimodes directly. These examples include a generalized WKB-ansatz

$$e^{\varphi/h^\delta} A_h$$

with arbitrary powers $\delta > 0$ of the denominator of general and complex valued phases to construct corresponding quasimodes. One of the early, important discoveries is that multiple WKB-type functions with different values of $\delta > 0$ are needed to construct sufficiently many quasimodes.

This generalized type of WKB-ansatz can be formalized to a family of symbol maps Σ_δ, depending on the power $\delta > 0$. It can be obtained directly from the classical symbol p of any generalized semi-classical operator $P = p(x, h, \partial_x)$. Applying this operator to a generalized WKB-ansatz yields a conjugated operator acting on the amplitude A_h

$$Pe^{\varphi/h^\delta} A_h = e^{\varphi/h^\delta} \cdot p\left(x, h, \partial_x + \frac{\varphi'}{h^\delta}\right) A_h \, .$$

Introducing two independent, non-commutating variables ξ and ζ, the *δ-symbol* $\Sigma_\delta(P)$ of an operator $P = p(x, h, \partial_x)$,

$$\Sigma_\delta(P) = p\left(x, h, \xi + \frac{\zeta}{h^\delta}\right) ,$$

is an element in the quotient algebra $C^\infty(\mathbb{R})\langle\xi, \zeta\rangle/\mathcal{I}$, where $\mathcal{I} := [[\xi, \zeta], \zeta]$. It is designed to mirror the conjugation of P with a generalized WKB-ansatz with e^{φ/h^δ}. The lack of commutativity of ξ and ζ reflects the lack of commutativity of the vector field ∂_x with non-constant functions

$$\left[\partial_x, e^{\varphi/h^\delta}\right] = e^{\varphi/h^\delta} \cdot \frac{\varphi'}{h^\delta} .$$

Since ζ appears in the δ-symbol with an additional factor $h^{-\delta}$ for $\delta > 0$, the leading part $E_\delta(P)$ as $h \to 0$ of the δ-symbol

$$\Sigma_\delta(P)(x, h, \xi, \zeta) \sim h^{l_\delta} E_\delta(P)(x, \zeta) + o(h^{l_\delta})$$

is independent of ξ and is called *δ-principal symbol*. Thus by construction, the δ-principal symbol $E_\delta(P)$ is a polynomial in ζ with smooth coefficients in x. It replaces the semi-classical principal symbol for the Schrödinger operator and can be used to state families of eikonal equations

$$E_\delta(P)(\cdot, \varphi') = 0$$

for generalized semi-classical operators.

The δ-principal symbol will be trivial for most values of $\delta > 0$, resulting in a monomial in ζ,

$$E_\delta(P)(\cdot, \zeta) = a \cdot \zeta^k ,$$

for some $k \in \mathbb{N}$ and $a \in C^\infty(\mathbb{R})$. This implies that for most values of $\delta > 0$ the solutions of their corresponding eikonal equation are given by $\varphi \equiv c$, for some $c \in \mathbb{C}$. They are trivial in the sense that their exponential terms commute with the operator

$$Pe^{c/h^\delta} A_h = e^{c/h^\delta} \cdot p\left(x, h, \partial_x + \frac{c'}{h^\delta}\right) A_h = e^{c/h^\delta} P A_h ,$$

and thus have no impact in the construction of quasimodes. To understand for which $\delta > 0$ this does not happen, note that for each summand in $E_\delta(P)$ there has to be a corresponding pair of exponents $\lambda = (k, \alpha) \in \Lambda$. More specifically, these pairs of exponents have to be on a line with slope δ and there must not be points underneath that line. Thus, the search for non-trivial values $\delta > 0$ for the principal symbol can be reduced to the existence of edges $\mathcal{L}$ with slope δ in the lower boundary of the Newton polygon $\mathcal{P}(\Lambda)$ of the set of exponents Λ. Deviating from the literature, the *Newton polygon* $\mathcal{P}(\Lambda)$ is defined to be the convex hull of

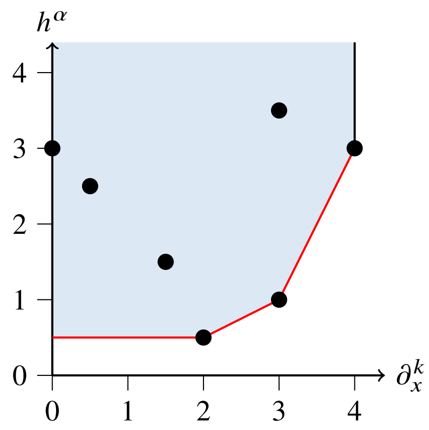

Figure 1.1: Newton polygon $\mathcal{P}(\Lambda)$ of a two dimensional set Λ. The red boundary is the lower boundary $\partial_- \mathcal{P}(\Lambda)$.

the union of all second quadrant quarter spaces attached to each point $\lambda \in \Lambda$ (see Figure 1.1). In particular, the δ-principal symbol with respect to the slope $\delta(\mathcal{L}) > 0$ of any edge can be phrased as

$$E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = \sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda \\ \lambda = (k, \alpha)}} a_\lambda \zeta^k ,$$

allowing us to reduce the analysis of the eikonal equations to a computation of simple, combinatorial objects. If we require the coefficients of $E_{\delta(\mathcal{L})}(P)$ corresponding to the end points of $\mathcal{L}$ to not vanish anywhere, we obtain non-zero solutions $\zeta(x)$ of $E_{\delta(\mathcal{L})}(P)(x, \zeta(x))$, for $x \in I$. Their number, counting multiplicities, corresponds to the horizontal width of the edge.

This notion of Newton polygon also generates a horizontal edge of non-negative width. Expanding the semi-classical operator $P = h^{l_0} T_0 + o\left(h^{l_0}\right)$, its leading operator T_0 can be associated to the points in the horizontal edge $\mathcal{L}_0 \subset \partial \mathcal{P}(\Lambda)$, yielding

$$T_0 = \sum_{\substack{\lambda \in \mathcal{L}_0 \cap \Lambda \\ \lambda = (k, \alpha)}} a_\lambda \partial_x^k .$$

One can use this operator T_0 in the same way one can use a more general operator T_δ in the case of $\delta > 0$ as we will describe briefly.

By further expanding $p(x, h, \partial_x + \varphi'/h^\delta)$ to the power of $h^{l_\delta + \delta}$ there is always a first order differential operator

$$T_\delta = \sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda \\ \lambda = (k, \alpha)}} a_\lambda k (\varphi')^{k-1} \left(\partial_x + \frac{k-1}{2} \frac{\varphi''}{\varphi'} \right) ,$$

which will be referred to as *induced transport operator*. It is the lowest power of h allowing for the presence of a differential operator of degree higher than 0. The induced transport operator coincides with $(-2\varphi'\partial_x - \varphi'')$ when applied to the Schrödinger operator and induces the recurrent first order differential equation

$$T_\delta a_k = f_k \,,$$

required to determine the functions a_k in the expansion of the amplitude A_h in the generalized WKB-ansatz. Due to the correspondence of the eikonal polynomial and the induced transport operator to edges $\mathcal{L}$ in the Newton polygon and their associated coefficient functions a_λ, with $\lambda \in \mathcal{L} \cap \Lambda$, we are able to state requirements for the existence of quasimodes with smooth amplitudes in x based on these particular coefficients. The leading coefficient of the induced transport operator

$$(\partial_\xi E_\delta(P))(\cdot, \varphi')$$

vanishes at a point $x_0 \in \mathbb{R}$ if $\varphi'(x_0)$ is not a simple solution of $E_\delta(P)$ in x_0. Simplicity is sufficient in many cases to construct a quasimode with smooth amplitude. To construct a maximal set of independent quasimodes, whose number matches the order of the operator P, one additionally has to require that both coefficients associated to the endpoints of $\mathcal{L}$ are nowhere vanishing. This property is called *δ-regularity* and yields non-vanishing solutions φ' in the desirable amount. It is necessary for the regularity of phase functions of adjacent edges and hence amplitudes, which we will discuss in detail in Section 3.3. The absence of δ-regularity yields families of phase functions that are either unbounded or vanishing in the non-δ-regular point x_0 and will be the focus of Chapters 4 and 5.

In a small share of cases, the second lowest term in the expansion of $p(x, h, \partial_x + \varphi'/h^\delta)$ is not given by T_δ but by some multiplication operator. This happens if there are points in $\Lambda \setminus \mathcal{L}$ which are geometrically not separated far enough from $\mathcal{L}$ itself. These artifacts can be handled easily by an extension of the phase function to a *full phase function*

$$\Phi = \frac{\varphi}{h^\delta} + \sum_{l=1}^{N} \frac{\psi_l}{h^{\gamma_l}} \,,$$

for $0 < \gamma_l < \delta$ and some $\psi_l \in C^\infty(\mathbb{R})$, which can be computed explicitly. Conjugating the semi-classical operator P with e^Φ then leads to a leading operator

$$T = T_\delta + V \,,$$

for some multiplication operator V. The ellipticity of T is determined by the ellipticity of T_δ. It results in the first theorem of this thesis, Theorem 3.3.9.

Theorem. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^\Lambda(I)$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Let $\varphi \in C^\infty(I)$ be a simple solution of $E_\delta(P)(\cdot, \varphi') = 0$ on I with $\varphi' \not\equiv 0$. Let $T = T_\delta + V$ be as above.*

Then there is a full phase function $\Phi = \varphi/h^{\delta} + o\left(h^{-\delta}\right)$ corresponding to φ', such that

$$e^{-\Phi} P e^{\Phi} = h^{l_{\delta}+\delta} T + o\left(h^{l_{\delta}+\delta}\right) .$$

Moreover, there is a quasimode $u = e^{\Phi(h)} A$, with $A := \sum_{k=0}^{\infty} a_k h^{\beta_k}$, where $a_k \in C^{\infty}(I)$ and $a_0 \in \ker T$ with $a_0 \not\equiv 0$, such that

$$\left(e^{-\Phi} P e^{\Phi}\right) A = O(h^{\infty}) .$$

If P is δ-separated, then $\Phi(h) = \varphi/h^{\delta}$. Additionally, if P is strongly δ-separated, then $T = T_{\delta,\varphi}$.

In particular, this theorem says that if a solution φ' of an eikonal equation $E_{\delta}(P)(\cdot, \varphi') = 0$ is simple, then we can extend it to a finite sum of phase functions Φ and obtain a smooth amplitude A, such that $e^{\Phi} A$ is a quasimode associated to φ'. In addition, if all non-trivial solutions of all eikonal polynomials $E_{\delta}(P)(x, \zeta)$ in a single point $x \in \mathbb{R}$ are simple and non-zero, then we can construct a maximal set of independent quasimodes in a neighborhood of x. Their number matches the order of P.

In a final discussion in Subsection 3.3.3 we are briefly extending the scope of this chapter by allowing operators with non-simple solutions φ of one of their corresponding eikonal equations. Allowing for solutions with constant multiplicities higher than one results in $T_{\delta} \equiv 0$. However, for every degree of multiplicity $\rho \in \mathbb{N}$ there is a natural, elliptic transport operator $T_{\delta,\varphi',\rho}$ with $\deg T_{\delta,\varphi',\rho} = \rho$. Using this induced transport operator, we can reproduce the results about regularity of the quasimode for these generalized semi-classical operator with some technical adjustments.

Resolved Operators

In Chapter 4 our focus shifts towards generalized semi-classical operators lacking δ-regularity. These are operators having an edge $\mathcal{L}$ with positive slope $\delta = \delta(\mathcal{L})$ in their associated Newton polygon $\mathcal{P}(\Lambda)$ whose coefficients a_{λ} and a_{μ} corresponding to the endpoints of the edge are vanishing at a point $0 \in \mathbb{R}$. It turns out that certain non-δ-regular operators still admit WKB-type quasimodes with polyhomogeneous phase functions and amplitudes on the half line $\mathbb{R}_{+}$, assuming all zeros are in $x = 0$. In this chapter we analyze when this is the case.

Relaxing regularity on generalized semi-classical operators can lead to a broad variety of phenomena. The most famous example of such an operator is the Schrödinger operator with linear potential

$$-h^2 \partial_x^2 + x .$$

The behavior of the differential of the phase function $\varphi'(x) \sim x^{1/2}$ as $x \to 0$ will result in a non-polyhomogeneous amplitude. Another variation of the Schrödinger operator

$$-h^2 x^4 \partial_x^2 + 1$$

is more singular at first glance. The associated eikonal equation $-(x^2\varphi')^2 + 1 = 0$ is solved by the functions $\varphi_\pm(x) = \pm 1/x + c$, for any $c \in \mathbb{C}$, leading to a recurrent equation

$$\left(-2x^2\partial_x + x\right) a_k = x^4 a''_{k-1}\,,$$

whose solutions a_k, $k \in \mathbb{N}$, improve their behavior towards $x = 0$ in every iteration. In particular, the quasimode $e^{\varphi_\pm/h} \sum a_k h^{\beta_k}$ is polyhomogeneous on $\mathbb{R}^2_+$, whereas the Schrödinger operator with linear potential requires a blow-up for the polyhomogeneity of its quasimodes.

The essential difference between these two and all other examples in Section 4.1 is the distribution of *weight* in x, where $x^l\partial_x^k$ has weight $l - k$. The central criteria to classify whether a non-regular operator admits an exponential-polyhomogeneous quasimode on $\mathbb{R}^2_+$ is the ordering and minimality in growth weight along the lower boundary of the Newton polygon. This will lead to the notion of *essential points* $\lambda \in \Lambda$ in Chapter 4, which can be contained in the interior of the Newton polygon. Whenever an edge $\mathcal{L}$ is spanned by two essential points it will be called $\mathcal{L}$-resolved in $x_0 = 0$, which leads to the construction of polyhomogeneous phase functions and amplitudes.

This extension of combinatorial data including weights can be captured by means of a set of exponents in the following way. For $0 \in \mathbb{R}_+$ we associate the *localized* set of exponents Λ_0. It expands points $\lambda = (k, \alpha) \in \Lambda$ to triples

$$(k, \alpha, \omega) \in \Lambda_0$$

with weights $\omega = l - k$ arising from powers l of x in the Taylor series of a_λ with non-vanishing coefficients. We introduce the new notion of a Newton *polyhedron* $\mathcal{P}(\Lambda_0)$ which we use to determine whether certain WKB-type quasimodes are of exponential-polyhomogeneous type. Projecting this localized set of exponents to its first and third entries $(k, \omega) \in \pi_{k,\omega}(\Lambda_0)$, the essential points of Λ are exactly those corresponding to the points on the lower boundary of $\mathcal{P}(\pi_{k,\omega}(\Lambda_0))$. In particular, an operator is $\mathcal{L}$-resolved, i.e. $\mathcal{L}$ is spanned by two essential points, if and only if there is an edge in $\partial\mathcal{P}(\Lambda_0)$ whose projection onto $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\pi_{k,\omega}(\Lambda_0))$ is an edge in their respective lower boundary. Thus, one can characterize non-regular, but resolved operators in terms of of an object $\mathcal{L}_0 \subset \partial_-\mathcal{P}(\Lambda_0)$ in the combinatorial geometry of the data given by the semi-classical operator.

We also need to extend these results to the following question. Given a semi-classical operator

$$\sum_{k=0}^{m} A_k(x, y) V^k$$

with vector field $V = x\partial_x - y\partial_y$, we want to specify sufficient conditions for A_k such that the phase functions and amplitudes of a WKB-type quasimode are polyhomogeneous on the quarter space $\mathbb{R}^2_+$ up to the corner $\{x = y = 0\}$. These types of operators arise naturally after blowing up the point $(0,0) \in \mathbb{R}^2_+$, since the pullback of a vector field with vanishing coefficient

$$\beta^*(x\partial_x) = x\partial_x - y\partial_y$$

is of that form.

In particular, we need to check under which condition quasimodes constructed locally at both boundary faces admit a joint exponential-polyhomogeneous extension to the whole manifold. This means that we need to find pairs of solutions of eikonal equations at adjacent boundary faces, such that the fractions φ_1/h^{δ_1} and φ_2/h^{δ_2} are polyhomogeneous at the corner and match.

Analogously to the first case where $V = x\partial_x$, we can compute the three-dimensional set of exponents at $\{x = y = 0\}$ with respect to the powers of x, y and V in the expansion of $\sum_{k=0}^m A_k(x,y)V^k$. Interestingly, the same condition as in the horizontal case is sufficient. The existence of an edge in the lower boundary of the Newton polyhedron of the corner point $x_0 = 0$ yields exponential-polyhomogeneous quasimodes. Having such a three-dimensional edge lets us relate the eikonal equation of one projected edge to the eikonal equation of the other projected edge with coefficients coinciding asymptotically at the corner. This asymptotic matching at the corner is an essential part of the resolution process in Chapter 5 and is proven in Proposition 4.3.4.

Resolving General Operators

In Chapter 5 we are investigating general non-δ-regular operators, which are not $\mathcal{L}$-resolved in the sense explained above. In general, the core problem of these non-regular operators is the existence of multiple solutions φ'_j of the eikonal equation

$$E_{\delta(\mathcal{L})}(P)(\cdot,\varphi') = 0$$

that vanish at a point $x_0 \in I$. This leads to a jump of multiplicity of $\varphi'_j(x_0)$ in the eikonal polynomial, resulting in a non-elliptic transport operator in x_0. Since the semi-classical operator is non-regular, the coefficients a_λ corresponding to the relevant edge $\mathcal{L}$ vanish at the same point x_0. While jumps in multiplicity of solutions φ'_j of δ-regular operators can be resolved by a single blow-up, non-regular operators require multiple blow-ups to resolve them.

In the beginning of Chapter 4 we have already shown how to resolve some examples of unresolved operators by the successive use of quasihomogeneous blow-ups $\beta_t\colon M \to \mathbb{H}$. To do this algorithmically, we have to address this problem with a systematic approach. Suppose we have done some blow-ups, resulting in a *manifold with corners* M and total blow-down map $\beta\colon M \to \mathbb{H}$. Since $\dim M = 2$ it has only 0- and 1-dimensional faces. The former are called corners, the latter are simply called *arcs* of M. We also use coordinates near each face

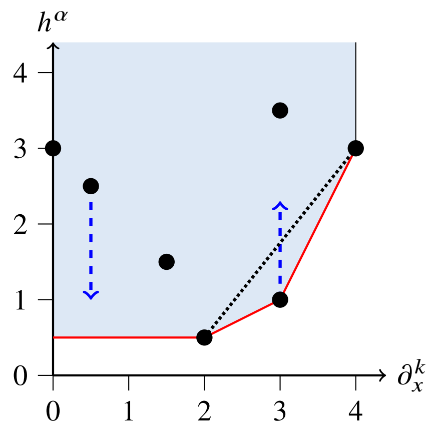

Figure 1.2: Set of exponents with induced action by the blow-up indicated by the dashed blue arrows. A collision is triggered when the second left point hits the lower boundary $\partial_{-}\mathcal{P}(\Lambda)$ or the second right point hits the dotted line.

pulled back from the coordinates (x, h) on $\mathbb{H}$ in a certain manner. Thus, we can associate two dimensional sets of exponents Λ_H to each face H and three dimensional, localized sets of exponents Λ_p to each point of ∂M. In particular, this allows us to compute the vanishing orders of the coefficients at each point p for each face H. We can determine whether the operator $\beta^* P$ is δ-regular in Λ_H or resolved in a non-regular point $p \in H$.

The central observation in the beginning of Chapter 5 is the effect of a blow-up on the powers of the coefficients

$$\beta_t^* \left(h^\alpha x^\omega (x\partial_x)^k \right) = h^{\alpha + t\omega} \tau^\omega (\tau \partial_\tau)^k$$

with corresponding quasi-projective coordinates $\tau = x/h^t$ and h. This shows how the Newton polygons and polyhedra on the front face of the blown up space are related to the original ones. Thus, we can analyze the effects of blow-ups on a purely combinatorial level again. A very important early result is that if an operator is $\mathcal{L}$-resolved in a point on a boundary face, its pullback will be $\beta_t^* \mathcal{L}$-resolved for all blow-ups of this point, where $\beta_t^* \mathcal{L}$ is the edge spanned by the transformed endpoints. On the other hand, edges in the lower boundary for which the operator is non-regular and not resolved in a certain point will behave significantly different. These edges will be altered drastically by pairs of exponents outside of the edge with relatively lower associated vanishing order (see Figure 1.2). We think of t as time, which we increase continuously. For $t > 0$ big enough such a pair of exponents will either:

(i) break the edge into two edges if the pair is centered between the end points of the edge,

(ii) absorb the edge into a bigger edge if the pair is outside of the vertical strip above the edge,

(iii) or it will collide with one of the boundary points.

The values t where these phenomena occur are called *collision times* and will be very important for the resolution algorithm. Until that time, including the collision time itself, the edge and its successor are in the lower boundary of their respective set of exponents, thus there is a three-dimensional edge in the boundary of the Newton polyhedron associated with the corner of the adjacent boundary faces. Only up to the collision times can we apply the result of the previous chapter about the existence of matching solutions of eikonal equations. The general idea for a resolution algorithm is then given by choosing an edge, blowing up unresolved points in boundary faces with respect to the collision time associated to the edge and repeat that process on the new boundary face with respect to the successors of the edge if there are still unresolved points remaining.

Since all processes can be described and relevant quantities can be measured combinatorially, we are able to state a resolution algorithm solely based on combinatorial data of semi-classical operators. Given an edge of the Newton polygon, a solution of its associated eikonal equation and an unresolved point, it should provide a blow-up diagram, in which all unresolved points are blown up successively. In order to do so, we have to determine these points, which are mainly roots of polynomials and compute the collision times with respect to the successor in each of these points to obtain a blow-up as intermediate step of the iteration. This diagram will be displayed in the form of a graph $\mathcal{G}$, more precisely a directed tree, encoding the complete blow-up scheme. The vertices of this tree are tuples

$$(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{G}$$

consisting of the partially blown-up space Y at this state, a boundary hypersurface $H \subset \partial Y$, and a successor $(\mathcal{L}_H, \zeta_H)$, where $\mathcal{L}_H$ is the edge succeeding the initially chosen edge and ζ_H is a solution of the eikonal equation induced by $\mathcal{L}_H$ that matches the solution ζ of the previous vertex.

The main theorem of this thesis, Theorem 5.2.18, states that the resulting *resolution tree* is finite for each semi-classical operator. In particular, this means that the resolution algorithm terminates.

Theorem. *Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $P = \sum_{k=0}^{m} A_k(x,h)\partial_x^k$, such that the leading coefficient A_m analytic in x. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda(P))$, $I \subset \mathbb{R}$ be sufficiently small and ζ be a solution of $E_\delta(P)(\cdot,\zeta) = 0$ on I.*

Then the resolution tree $\mathcal{G}$ with respect to $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ is finite.

The proof of this statement is completely combinatorial and combines elementary methods. Firstly, one shows that splitting and merging of edges can only happen finitely many times. Secondly, collisions of pairs of exponents from the interior with endpoints of the relevant edge result in the scattering of zeros along the new blown up boundary face. Since the degree of these polynomial coefficients associated to the pairs $\lambda \in \Lambda_H$ on a boundary face H cannot increase after the application of blow-ups, the total degree of all zeros is bounded and will be constant along the branches of $\mathcal{G}$ after finitely many blow-ups. Each of the singularities in these branches can be resolved directly.

We want to emphasize that this algorithm does not only provide the complete scheme of blow-ups required. The resolution graph $\mathcal{G}$ stores vertices $(Y, H, \mathcal{L}_H, \zeta_H)$ containing possibly multiple edges and solutions of eikonal equations. Therefore, it determines a family of phase functions at each boundary face which is required to construct a quasimode based on initial data $\mathcal{L}$ and ζ on $\mathbb{H}$. As an additional step to the algorithm, it will also resolve crossings of solutions of the eikonal equation, which are undesirable and can be resolved by a single blow-up.

In the end, all that remains to be done is the construction of amplitudes locally at each boundary hypersurface, which admit a joint extension to the interior of the resolved space. This is a bookkeeping problem, since most transport operators at each boundary hypersurface are first order b-differential operators and thus already provide polyhomogeneous solutions in general. This process can be iterated, allowing for a successive improvement on the vanishing order of the remainder at each boundary face, respectively. Theorem 5.3.22 summarizes this process, stating that for each semi-classical operator and pair of initial data $(\mathcal{L}, \zeta)$ there is an exponential-polyhomogeneous quasimode on the resolution space. In particular, this theorem proves that the resolution process described in Algorithm 1 resolves the generalized semi-classical operator. As a direct consequence, we can show that for a semi classical operator P with $\deg P = m$ there are up to m possibly trivial resolution spaces, resulting in a set of m independent quasimodes for P.

Related Literature

Literature in semi-classical theory often starts with examples related to *Schrödinger operators* (see Example 3.1.3) and generalizes certain aspects of these operators, aiming for different results. [BM72], [Was85], [Was87] and [Fed93] look into differential operators with holomorphic coefficients in both x and h, but also analyze turning points, the connection problem, perturbed systems and non-linear equations. Some take a more detailed look into special generalizations: the Schrödinger operator with magnetic fields in [Hel80] in multiple dimensions or the analysis of asymptotic behavior of solutions of certain fourth order ordinary differential equations with similarities to the Schrödinger operator in [LR60].

Other functional analytic treatments work with semi-classical Fourier integral or pseudo differential operators, related to compactly supported or Schwartz kernels depending on $h > 0$, with a much wider scope in [Zwo12] or [GS90]. The book of Kato [Kat95] is also related to this thesis, but uses a functional analytic approach to analyze eigenvalues and eigenvectors of operators on Hilbert spaces with analytic perturbations.

Newer developments related to asymptotic solutions of semi-classical operators introduce the concept of *exact WKB solutions*, using the so called *Borel-Laplace transformation* to obtain solutions for a class of semi-classical operators that are analytic in x and of Gevrey type in h. These can be found in [Nik23] and [Vor99]. Other work analyzing operators of Gevrey type in h are [MS00] and [Sib00].

A different direction can be found in the generalization of Schrödinger operators to fiber bundles in [Lam14], [LT17], [Teu03] or [LR13]. They construct and analyze quasimodes also in terms of eigenvalue crossings in an operator theoretic way. Crossings of these eigenvalue sections lead to singular behavior of the quasimodes and can be resolved with the methods described in Chapter 5.

The works of [Gri17], [KS22] and [Sob18] are very strongly related to this thesis. They attempt to construct quasimodes for the Schrödinger operator, including a geometric resolution in form of a blow-up for a minimum of the potential V in [Gri17] and turning points in the other two works. In addition, [KS22] can show closeness of WKB quasimodes to explicit solutions, using a dynamical systems approach. This thesis is an extension of [Sob18] which was based on the approach in [Gri17] to construct quasimodes by performing blow-ups and using model operators at each boundary hypersurface. The extension of local solutions at boundary hypersurfaces to a single function on the blown-up space is closely related to the methods of matched asymptotic expansion and multiple scales described in [Hol13] and [BO99].

2 Preliminaries

2.1 Singular Geometry

Singular spaces occur naturally in the setting of perturbed objects. A family of functions $(f_h)_h$ on $\mathbb{R}$ depending on a parameter $h \geq 0$ can also be rephrased as a function f depending on both x and h, i.e. $f(x,h) \coloneqq f_h(x)$. Thus, f is a function defined on $\mathbb{R} \times \mathbb{R}_+$, which is not a regular manifold. The definitions and statements presented in this section, but also further topics, can be found in [Mel96]. The notion of *manifold with corners* in this section is less general than the one presented in [Joy12].

2.1.1 Manifolds with Corners

Singular manifolds are topological spaces which can, analogously to manifolds, be represented locally over so called *model spaces*. These are intersections of finitely many half spaces with boundary.

Definition 2.1.1 (Model space)**.** Let $n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$. We define

$$\mathbb{R}^n_k \coloneqq \{x \in \mathbb{R}^n : \ x_i \geq 0, \ 1 \leq i \leq k\}$$

to be the *model space* with corners of codimension k. This set is naturally equipped with the relative topology induced by the standard topology of $\mathbb{R}^n$. For each $l \in \mathbb{N}_0$ we can define a space of *l-differentiable functions* on $\mathbb{R}^n_k$ by taking all differentiable functions in the interior of $\mathbb{R}^n_k$ whose differentials are uniformly bounded on each compact set of $\mathbb{R}^n_k$, i.e.

$$C^l\left(\mathbb{R}^n_k\right) \coloneqq \left\{ f \in C^l\left(\mathbb{R}^{n\circ}_k\right) : \sup_{\substack{|\alpha| \leq l \\ x \in K \cap \mathbb{R}^{n\circ}_k}} | D^\alpha f(x) | < \infty, \ \text{for all } K \Subset \mathbb{R}^n_k \right\}. \tag{2.1}$$

Functions are called *smooth*, if they are l-differentiable for all $l \in \mathbb{N}$. We denote the codimension

$$\operatorname{codim}(x) \coloneqq | \{i \leq k : \ x_i = 0\} | \tag{2.2}$$

of $x \in \mathbb{R}^n_k$ in the model space. Let $k_1, k_2 \in \mathbb{N}_0$, $\Omega_1, \ \Omega_2 \subset \mathbb{R}^n_{k_i}$ be open sets. A map

$$F\colon \Omega_1 \longrightarrow \Omega_2$$

is called *diffeomorphism*, if F is a homeomorphism and if the components of F and F^{-1} are smooth functions (in the sense of (2.1)).

Remark 2.1.2. We also allow $k = 0$ in Definition 2.1.1. Since there is no $i \in \mathbb{N}$ such that $1 \leq i \leq 0$ it follows that $\mathbb{R}^n_0 = \mathbb{R}^n$.

Having a notion of smoothness one can define smooth coordinates from topological to model spaces (see Figure 2.1).

Definition 2.1.3 (Coordinates)**.** Let M be a Hausdorff topolocigal space, let I be a countable set and $i \in I$, let $\Omega, \Omega_i \subset M$ and $\Omega', \Omega'_i \subset \mathbb{R}^n_k$ and let $\varphi\colon \Omega \to \Omega'$ and $\varphi_i\colon \Omega_i \to \Omega'_i$ be maps.

(i) The components of φ are called *coordinates at a corner* if φ is a homeomorphism. We then call (Ω, φ) a *coordinate system*.

(ii) Assume $(\Omega_i, \varphi_i), (\Omega_i, \varphi_j)$ are coordinate systems for $i, j \in I$. They are called (smoothly) *compatible* if $\varphi_i \circ \varphi_j^{-1}\colon \varphi_j(\Omega_i \cap \Omega_j) \to \varphi_i(\Omega_i \cap \Omega_j)$ is a diffeomorphism .

(iii) A set $A = \{(\Omega_i, \varphi_i)\}_{i \in I}$ of smooth compatible coordinate systems is called a *smooth atlas* if the corresponding Ω_i cover M.

One can naturally lift the notion of manifolds to the non-regular case with this notion of coordinates. Note that manifolds with corners will have strong properties concerning their boundary faces.

Definition 2.1.4 (t-Manifold)**.** Let M be a paracompact, topological Hausdorffspace, let $A = \{(\Omega_i, \varphi_i)\}_{i \in I}$ be a smooth atlas with coordinates $\varphi_j\colon \Omega_j \to \mathbb{R}^n_{k_j}$, $\Omega_j \subset M$ and let $k := \max\{k_j\}_j$.

The pair (M, A) is called a *t-manifold* of dimension n and codimension k. For any $p \in M$ we set

$$\operatorname{codim}(p) := \operatorname{codim}(\varphi_j(p))\,. \tag{2.3}$$

Remark 2.1.5. We will omit the atlas and will only refer to the topological space M in the context of t-manifolds and *manifolds with corners* after Definition 2.1.11 throughout the thesis.

Definition 2.1.6 (Boundary hypersurface)**.** Let M be a t-manifold. For any $m \in \mathbb{N}_0$ we denote

$$\partial_m M := \operatorname{codim}^{-1}(m)$$

and call the closure of an arbitrary connected component of $\partial_1 M$ *boundary hypersurface*.

Definition 2.1.7. Let M be a 2-dimensional manifold with corners and $H \subset \partial_1(M)$ be a boundary hypersurface. Then we call H an *arc* of M.

Remark 2.1.8. Calling hypersurfaces of 2-dimensional manifolds with corners arcs will unravel some possible misunderstandings when analyzing the boundary of polygons associated to boundary hypersurfaces in the following chapters.

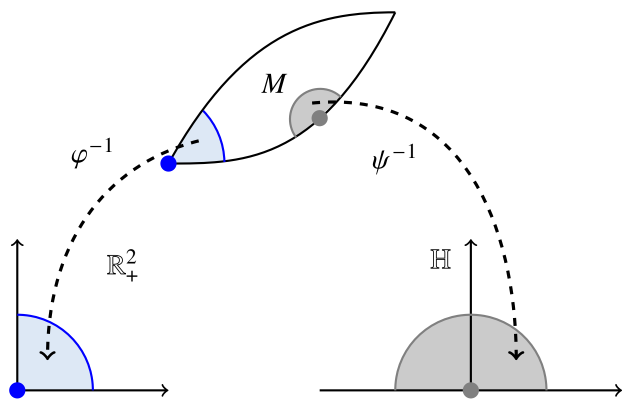

Figure 2.1: A manifold with corners M with local coordinate patches at the boundary.

Remark 2.1.9. Note that $\partial_m M \subset \partial M$ is a subset of the boundary of M and not a collection of boundary hypersurfaces.

We are interested in a type of t-manifold for which the boundary hypersurfaces behave well. Namely, these are submanifolds that are everywhere locally of product type and will be called *p-submanifolds*.

Definition 2.1.10 (p-Submanifolds)**.** Let (M, A) be a t-manifold of dimension n and codimension k.

We call $S \subset M$ an *embedded submanifold* of dimension n', if the image of S is a invertible linear transformation of a model space, i.e.

$$\text{for all } p \in S \text{ there are } (\Omega, \varphi) \in A,\ k' \in \mathbb{N}_0,\ G \in GL(n, \mathbb{R}) :$$
$$\varphi\colon S \cap \Omega \to G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}\right) \cap \Omega' ,$$

in a neighborhood $\Omega' \subset \mathbb{R}^n$ of 0.

Additionally, we call S *p-submanifold* if there is an $l \in \mathbb{N}$ and a set $L = \{x \in \mathbb{R}^n_k : x_{l+1} = \ldots = x_k = \ldots = x_n = 0\}$, such that

$$\varphi(S \cap \Omega) = L \cap \varphi(\Omega) \tag{2.4}$$

everywhere locally.

We can directly define *manifolds with corners*.

Definition 2.1.11 (Manifold with corners)**.** A *manifold with corners* is a t-manifold M such that each boundary hypersurface $H \subset \partial_1 M$ is a p-submanifold.

Requiring boundary hypersurfaces to be p-submanifolds ensures that there are global tubular neighborhoods of each boundary hypersurface. This is the case when there exist boundary defining functions, as shown in the following Lemma 2.1.12.

Lemma 2.1.12 (Boundary defining function)**.** *Let M be a manifold with corners and $H \subset \partial_1 M$ be a boundary hypersurface.*

Then there exists a boundary defining function $\rho \in C^\infty(M)$ for H, i.e.

$$\rho \geq 0, \quad H = \{\rho = 0\}\,.$$

Remark 2.1.13. Every coordinate system at $p \in H$ can be chosen such that ρ is the first coordinate of the corresponding map.

2.1.2 Blow-Ups

A tool we will use frequently to resolve functions on a manifold with corners which are smooth in the interior but not polyhomogeneous on the whole manifold is the *blow-up* of submanifolds. Using it we can replace the critical submanifold by its inward pointing normal sphere, unraveling ambiguities of these functions and thus effectively extending the spaces of smooth and polyhomogeneous functions.

Example 2.1.14. Let $x \in \mathbb{R}_+$ and $f_h(x) := h/(x+h)$. Evaluated at $x = 0$ this is a constant family of numbers $(f_h(0))_h = (1)_h$, for $h > 0$, whereas $f_0(x) = 0$, for all $x \geq 0$. Thus, there is an ambiguity of the value of $f(x,h) := f_h(x)$ at $(0,0) \in \mathbb{R}^2_+$, where the function is not naturally defined.

Approaching $0 \in \mathbb{R}^2_+$ on a fixed ray other than $\{x = 0\}$ or $\{h = 0\}$ yields a similar result. For $c > 0$ the value of f along the ray $R_c := \{x = ch\}$ is constant and given by $f(ch,h) = h/((c+1)h) = 1/(c+1)$. Excluding $0 \in \mathbb{R}^2_+$ and adding a separate endpoint for each ray then yields a manifold with corners, which resolves the ambiguity of f and results in smoothness of the function on the manifold.

We will only introduce the notions of blow-ups and quasihomogeneous blow-ups in the context of two-dimensional manifolds in this thesis.

Definition 2.1.15 (Blow-up)**.** The *blow-up* of $0 \in \mathbb{R}^2_+$ is defined by the *blown-up space*

$$\left[\mathbb{R}^2_+, 0\right] := \mathbb{R}_+ \times \mathbb{S}_+$$

and the *blow-down* map β

$$\begin{aligned} \beta\colon\ \left[\mathbb{R}^2_+, 0\right] &\to \mathbb{R}^2_+ \\ (r,\omega) &\mapsto r \cdot \omega, \end{aligned} \tag{2.5}$$

where $\mathbb{S}_+ := \mathbb{S} \cap \mathbb{R}^2_+$ is the quarter of the circle $\mathbb{S} := \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 = 1\}$ in the first quadrant. The set

$$\mathrm{ff} := \beta^{-1}(0)\,,$$

is called *front face* of the blown-up space $\left[\mathbb{R}^2_+, 0\right]$.

Remark 2.1.16. The set $\left[\mathbb{R}^2_+, 0\right]$ generated by the blow-up $\beta\colon \left[\mathbb{R}^2_+, 0\right] \to \mathbb{R}^2_+$ is a manifold with corners. This will be shown in more generality in Lemma 2.1.20

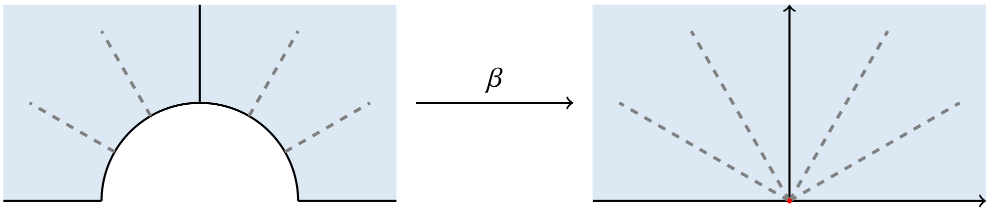

Figure 2.2: The blow-up of 0 in $\mathbb{H}$. The rays indicate how linear spaces through 0 get lifted.

Remark 2.1.17. Analogously, one can blow up the half space $\mathbb{H} := \mathbb{R}\times\mathbb{R}_+$ (see Figure 2.2). The blown-up space is given by $[\mathbb{H}, 0] := \mathbb{R}_+ \times \mathbb{S}_+$, where $\mathbb{S}_+ := \mathbb{S} \cap \mathbb{H}$ is the upper half circle. This procedure is not limited to the half space; interior points can be considered, too, generating a blown-up space $[\mathbb{R}^2, 0] := \mathbb{R}_+ \times \mathbb{S}$. The blow-down map is same as in Definition 2.1.15 in both cases. We will only consider blow-ups of boundary points throughout this thesis.

Remark 2.1.18. Blowing up 0 in a model space $\mathbb{R}^2_+$ results in a new boundary hypersurface $\mathrm{ff} = \beta^{-1}(0)$, which is equipped with more structure than general hypersurfaces of manifolds with corners. Let $T_0^+\mathbb{R}^2_+$ be the inward pointing cone of the tangent space at $0 \in \mathbb{R}^2_+$. By construction the front face

$$\mathrm{ff} = {T_0^+\mathbb{R}^2_+ \setminus \{0\}}\big/{\mathbb{R}_{>0}}\,,$$

is a projective space. Thus, for any choice of coordinates (x, y) on $\mathbb{R}^2_+$ around 0 there are induced projective coordinates

$$\eta := \frac{y}{x} \quad \text{and} \quad \xi := \frac{x}{y}\,,$$

forming coordinate systems (x, η) and (ξ, y) in neighborhoods of each boundary point of ff in $[\mathbb{R}^2_+, 0]$ (see Figure 2.3). Choosing other coordinates $(\widetilde{x}, \widetilde{y}) = (x \cdot a(x, y), y \cdot b(x, y))$, $a(0, 0), b(0, 0) \neq 0$ then induce similar projective coordinates along ff given by

$$\widetilde{\eta} := \frac{\widetilde{y}}{\widetilde{x}} = \eta \cdot \frac{b(x, x\eta)}{a(x, x\eta)}_{|x=0} = \eta \cdot \frac{b(0, 0)}{a(0, 0)} \quad \text{and} \quad \widetilde{\xi} = \xi \cdot \frac{a(0, 0)}{b(0, 0)}\,.$$

Thus, there is an invariant notion of polynomials on arcs since the definition is independent of the choice of coordinates with the transformation of coefficients described above.

These induced projective coordinates also determine a choice of trivialization of (certain) open neighborhoods of arcs in $\beta\colon Y \to X$ yielding a non-coordinate invariant but accurate description of *minimal* index sets of polyhomogeneous functions in these neighborhoods.

One often needs multiple blow-ups to resolve a function. In certain cases this can be reduced to a single, *quasihomogeneous* blow-up, where we replace a single point with, for instance, the endpoints of a family of parabolas (see Figure 2.4). Having this refined notion of blow-ups will be very useful in the resolution of singular operators in Chapter 5.

Definition 2.1.19 (Quasihomogeneous blow-up, [Beh21])**.** Let $\kappa_h, \kappa_x \in \mathbb{N}$, let $r\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ be defined by $r(x, h) := (x^{\kappa_h} + h^{\kappa_x})^{1/(\kappa_h\kappa_x)}$ and denote the unit sphere deformed by r with

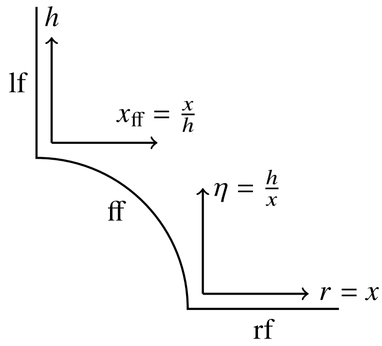

Figure 2.3: Local induced projective coordinates on $[\mathbb{R}^2_+, 0]$ at each corner of the front face ff.

$\mathbb{S}^1_r := \{(\omega, \eta) \in \mathbb{R}^2_+ : \ r(\omega, \eta) = 1\}$. The *quasihomogeneous blow-up* of $(0,0) \in \mathbb{R}^2_+$ with respect to $t := \kappa_x/\kappa_h$ is defined by the set

$$\left[\mathbb{R}^2_+, 0\right]_{\kappa_x, \kappa_h} := \mathbb{R}_+ \times \mathbb{S}^1_r$$

and the blow-down map β

$$\begin{aligned} \beta \colon \mathbb{R}_+ \times \mathbb{S}^1_r &\longrightarrow \mathbb{R}^2_+ \\ (R, (\omega, \eta)) &\mapsto (R^{\kappa_x}\omega, R^{\kappa_h}\eta) \ . \end{aligned}$$

As in the homogeneous case of Definition 2.1.15 there are induced quasi-projective coordinates on $\left[\mathbb{R}^2_+, 0\right]_{\kappa_x, \kappa_h}$ based on the choice of coordinates on $\mathbb{R}^2_+$.

Lemma 2.1.20 (Properties of the quasihomogeneous blow-up)**.** *Let* $\beta \colon \left[\mathbb{R}^2_+, 0\right]_{\kappa_x, \kappa_h} \to \mathbb{R}^2_+$ *be a quasihomogeneous blow-up. Let* $t := \kappa_x/\kappa_h$*.*

Then the following holds

(i) $\left[\mathbb{R}^2_+, 0\right]_{\kappa_x, \kappa_h}$ *is a manifold with corners.*

(ii) The maps $\beta^* \left(x/h^t, h^{1/\kappa_h}\right)$ *and* $\beta^* \left(x^{1/\kappa_x}, h/x^{(t^{-1})}\right)$ *form compatible coordinate systems.*

Proof. (i) Since we can describe $\mathbb{S}^1_r \subset \mathbb{R}^2_1$ at $(0,1)$ as a graph, i.e.

$$\mathbb{S}^1_r \cap U = \{(g(h), h) : \ h \in [0, \varepsilon)\} \, ,$$

it is a *p-submanifold* of $\mathbb{R}^2_1$. Therefore $\left[\mathbb{R}^2_1, 0\right]_{\kappa_x, \kappa_h}$ is a product of manifolds with corners.

(ii) By definition of β we have $\beta^* x = r^{\kappa_x}\omega$ and therefore

$$\beta^* x^{1/\kappa_x} = r \cdot \omega^{1/\kappa_x} = r \cdot (1 - \eta^{\kappa_x})^{1/(\kappa_x \kappa_h)} \, .$$

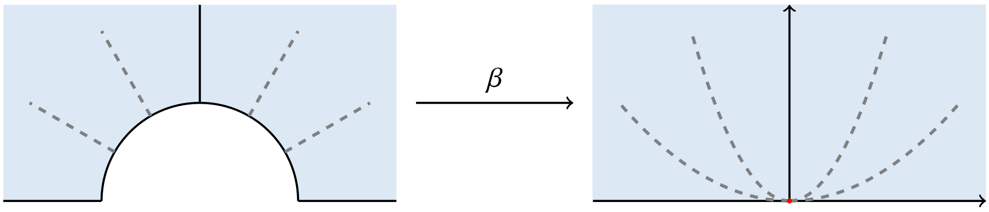

Figure 2.4: Quasihomogeneous blow-up of 0 in $\mathbb{H}$ with corresponding curves being resolved on $[\mathbb{H}, 0]_{\kappa_x,\kappa_h}$.

So $\beta^* x^{1/\kappa_x}$ is smooth in terms of (r, η) for $\eta < 1$ and vanishes to first order at $\mathcal{A}$. Thus, it is a defining function at $\mathcal{A}$. Analogously, the pullback

$$\beta^*(h/x^{(t^{-1})}) = r^{\kappa_h}\eta/(r^{\kappa_h}\omega^t) = \eta/(1-\eta^{\kappa_x})^{1/(\kappa_x\kappa_h)}$$

is smooth and boundary defining for O and $\eta < 1$. Moving on, we have $\beta^* h^{1/\kappa_h} = r\eta^{1/\kappa_h} = r(1-\omega^{\kappa_h})^{1/(\kappa_h\kappa_x)}$ and $\beta^*(x/h^t) = \omega/\eta^{(t^{-1})} = \omega/((1-\omega^{\kappa_h})^{1/\kappa_h})$.

□

Remark 2.1.21. One can also blow-up points and certain submanifolds in manifolds (with corners), which we will not discuss in detail since we do all necessary computations in local coordinates. To emphasize that certain blow-ups are iterated, we will call these concatenations *chains of blow-ups*. We refer to [Beh21], [Gri01], [KM15] and [Mel96] for the introduction of blow-ups on manifolds.

In Chapter 5 we will perform blow-ups of points $(p_k)_{k\in\mathbb{N}}$ in the boundary which are nested sequentially on each newly generated front face, i.e. $p_{k+1} \in \beta_{(k)}^{-1}(p_k)$. Since there can be multiple of these nested sequences in parallel we need to show that blowing up points in different sequences is independent of the order of blow-ups. The following theorem shows a standard result of commutativity of quasihomogeneous blow-ups in the case of transversally intersecting p-manifolds.

Theorem 2.1.22 (Commutativity of Blow-Ups, [Beh21])**.** *Let X be a manifold with corners and $Y, Z \subset X$ be two p-submanifolds that intersect transversally, i.e. for each $p \in Y \cap Z$ we have $T_pY + T_pZ = T_pX$.*

Then any quasihomogeneous structures Π_Y, Π_Z of Y and Z, respectively, intersect each other cleanly and their iterated blow-ups in any order are diffeomorphic to each other

$$\big[[X, Z]_{\Pi_Z}, Y\big]_{\Pi_Y} \cong \big[[X, Y]_{\Pi_Y}, Z\big]_{\Pi_Z}\ .$$

A direct consequence of Theorem 2.1.22 is given by the following corollary about the commutativity of nested blow-ups.

Corollary 2.1.23 (Nested Sequences)**.** *Let X be a manifold with boundary, $p_j \in \partial X$, $j = 1, 2$, $p_1 \neq p_2$, and let $\beta_{p_j} : [X, \{p_j\}] \to X$ be their associated blow-ups. Let $q \in \beta^{-1}(p_1)$ and $\beta_q : [[X, \{p_1\}], q] \to [X, \{p_1\}]$.*

$$\begin{tikzcd}
& {[X,p_1]} \arrow[dl, "\beta_{p_1}"'] & & {[[X,p_1],q]} \arrow[ll, "\beta_q"'] & \\
X & & Y \arrow[ul, "\beta_{Y_1}"'] \arrow[dl, "\beta_{Y_2}"] & & Z \arrow[ul, "\beta_{Z,q}"'] \arrow[ll, "\beta_Z"'] \arrow[dlll, bend left, "\beta_{Z,p_2}"] \\
& {[X,p_2]} \arrow[ul, "\beta_{p_2}"] & & &
\end{tikzcd}$$

Figure 2.5: Blow-up diagram of two nested sequences with intermediate space Y.

Then there are spaces Y, Z, there are blow-down maps $\beta_{Z_2}\colon Z \to [X, \{p_2\}]$, $\beta_Y\colon Z \to Y$, $\beta_q\colon Z \to [[X, \{p_1\}], q]$ and $\beta_{Y_j}\colon Y \to [X, \{p_j\}]$, $j = 1, 2$, such that the corresponding diagram in Figure 2.5 commutes.

Proof. By Theorem 2.1.22 $\beta_{p_j}\colon [X, \{p_j\}] \to X$ commute in the sense that $[[X, p_1], p_2] \cong [[X, p_2], p_1]$. We denote $Y := [[X, p_1], p_2]$ and $\beta_{Y_j}\colon Y \to [X, \{p_j\}]$. Since $q, \beta^* p_2$ are transversally intersecting their respective blow-ups $\beta_q\colon [[X, \{p_1\}], q] \to [X, \{p_1\}]$ and $\beta_{Y_2}\colon Y \to [X, \{p_2\}]$ commute. We call their common blown-up space $Z := [Y, q]$ with corresponding blow-down maps $\beta_Y\colon Z \to Y$ and $\beta_q\colon Z \to [[X, \{p_1\}], q]$. Finally, the consecutive blow-down map $\beta_{Z_2} := \beta_{Y_2} \circ \beta_Y\colon Z \to [X, p_2]$ completes the diagram. □

Remark 2.1.24. The statement of Corollary 2.1.23 remains valid if one performs quasihomogeneous instead of homogeneous blow-ups.

Remark 2.1.25. Let M be a manifold with corners and $p \in \partial M$. Due to construction in Definition 2.1.19 the blown-up spaces $[M, p]_{p,q}$ and $[M, p]_{kp,kq}$ have different smooth structures, for all $k > 1$. We will only be interested in polyhomogeneity and less in the index sets themselves, since they are depending on the choice of smooth structure on $[M, p]_{p,q}$. Thus, we will refer to quasihomogeneous blow-ups only with respect to the quotient of weights $t \in \mathbb{Q}_{>0}$, i.e.

$$\beta\colon\ [M, p]_t \to M\,,$$

by choosing the unique reduced fraction.

Remark 2.1.26. There is a more general notion of blow-ups than the ones presented in Section 2.1. Kottke and Melrose introduce the concept of *Generalized Blow-Up* in [KM15], where they construct blown-up spaces by gluing the interiors of different model spaces along one boundary face of each in opposing orientation. Since this approach directly corresponds to the introduction of quasi-projective coordinates, one can get a quick access to understand quasihomogeneous blow-ups of model spaces.

2.2 Singular Analysis

Singular analysis is a broad term that is used by many different fields in various contexts. Here we are primarily interested in asymptotic analysis and its correspondence to singular geometry in order to resolve these. Therefore we will cover asymptotic series and the notion of polyhomogeneous and oscillating functions on manifolds with corners as well as some basic results, in particular asymptotic solutions of ordinary differential equations.

We will frequently use the O-notation for asymptotic statements, usually as $x \to 0$ or $h \to 0$. Recall that for $\alpha \in \mathbb{R}$ and a function $f\colon (0,\infty) \to \mathbb{C}$ we say that $f = O(h^\alpha)$, if

$$\limsup_{h\to 0} |h^{-\alpha} f(h)| < \infty\,,$$

and that $f = o\,(h^\alpha)$, (little o), if

$$\lim_{h\to 0} |h^{-\alpha} f(h)| = 0\,.$$

2.2.1 Asymptotic Summation

We start this section with the notion of *asymptotic power series*. Asymptotic solutions of semi-classical equations in Chapter 3 will be of this type.

Definition 2.2.1 (Asymptotic Expansion)**.** Let $(V, \|\cdot\|)$ be a normed vector space over a field $\mathbb{K}$ and $f\colon (0,\infty) \to V$. Let $(\alpha_k)_k \subset \mathbb{R}$ be increasing and unbounded. For each $k \in \mathbb{N}$ let $v_k \in V$.

(i) We say that f is *asymptotically equal* to the *asymptotic series* $\sum_{k=0}^{\infty} v_k h^{\alpha_k}$ as $h \to 0$, if for all $m \in \mathbb{N}_0$ we have that

$$\left\| f(h) - \sum_{k=0}^{m-1} v_k h^{\alpha_k} \right\| = O(h^{\alpha_m})\,,$$

as $h \to 0$. In this case we write

$$f(h) \sim \sum_{k=0}^{\infty} a_k h^{\alpha_k} \quad \text{as } h \to 0\,.$$

(ii) If $f(h) \sim \sum_{k=0}^{\infty} v_k h^{\alpha_k}$ as $h \to 0$, then we call v_0 the *leading term* of $f(h)$.

(iii) We denote the *space of asymptotic series* by V_h.

Remark 2.2.2. One can define an equivalence relation on functions $f, g\colon (0,\infty) \to V$ by saying that $f(h) \sim g(h)$ if and only if $f(h) - g(h) \sim 0$.

Multiplication with scalars $\lambda \in \mathbb{K}$ is defined by multiplying each summand of the series. Two series can be added by expanding each series so that they both include all appearing powers of h and adding every pair of summands with mutual power of h afterwards. This way,

V_h is a $\mathbb{K}$-vector space, if V is a $\mathbb{K}$-vector space. The most common space will be given by the asymptotic space of smooth functions over intervals $I \subset \mathbb{R}$

$$C_h^\infty(I) = \left\{ \sum_{k=0}^{\infty} u_k h^{\alpha_k} \ : \ u_k \in C^\infty(I) \text{ and } \alpha_k \nearrow \infty \right\} .$$

In the case of smooth functions we will need a relaxed version of asymptotic equality.

Definition 2.2.3. Let $I \subset \mathbb{R}$ be an interval, $u\colon (0,\infty) \to C^\infty(I)$ be a family of smooth functions and $\sum_{j=0}^\infty u_j h^{\alpha_j} \in C_h^\infty(I)$. We say that u is asymptotically equal to $\sum_{j=0}^\infty u_j h^{\alpha_j}$ *locally everywhere*, if for all compact sets $K \subset I$ we have that

$$u(h)_{|K} \sim \sum_{j=0}^{\infty} u_{j_{|K}} h^{\alpha_j} .$$

The notion of asymptotic expansion also directly applies to functions $f \in C^\infty(\mathbb{R})$, which can have all sorts of asymptotic behavior as $x \longrightarrow \pm\infty$. A special class asymptotic power series associated to smooth functions on an interval I are the so called *Puiseux* series.

Definition 2.2.4 (Puiseux series)**.** Let $c_k \in \mathbb{C}$, let $n \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$. The asymptotic series

$$\sum_{k=k_0}^{\infty} c_k x^{k/n} ,$$

is called a *Puiseux* series at $x = 0$.

One can define Puiseux series to measure asymptotic behavior of smooth functions f in any point x_0 by a simple shift to $(x - x_0)^{k/n}$. To describe the asymptotic behavior at $x_0 = \pm\infty$ one needs to either allow asymptotic series of the form $\sum_{k=-\infty}^{k_0} c_k x^{k/n}$ or, equivalently, introduce a inverted coordinate $r := x^{-1}$ and express it as a Puiseux series in terms of r.

We can use the notion of asymptotic series to define a notion of approximate solutions of asymptotic linear maps.

Definition 2.2.5 (Quasimodes)**.** Let $(V, \|\cdot\|)$ be a normed space and $h > 0$. Let $L(h)$ be a family of endomorphisms of V and let $v \in V_h$ be an asymptotic series. Let $s \in \mathbb{R}$.

We say that v is a *quasimode* or approximate solution of order s for the equation $L(h)u = 0$, if

$$\|L(h)v\| = O(h^s) .$$

We say that v is a quasimode for the equation $L(h)u = 0$, if it is a quasimode of order r for all $r \in \mathbb{R}$.

In the context of differential operators one cannot directly extend the notion of quasimodes from asymptotic series (or more general polyhomogeneous functions) to functions with exponential behavior depending on $h > 0$. Let $\varphi \in C^\infty(\mathbb{R})$ be positive, then for a family of differential operators $P(h)$ we have that

$$P(h)e^{\pm\varphi/h} = O\left(h^{\pm\infty}\right) .$$

This asymptotic behavior on the right hand side is determined completely by the behavior exponential function reproducing itself after differentiation. Having a product of such an exponential function $e^{\varphi/h}$ and an asymptotic series A, one can measure the behavior of the resulting asymptotic series after applying $P(h)$. Thus, one can say that $e^{\varphi/h}A$ is a quasimode for $P(h)$ if

$$\left(e^{-\varphi/h}P(h)e^{\varphi/h}\right)A = O\left(h^{\infty}\right) ,$$

i.e. the amplitude A is a quasimode of the conjugated operator. This concept of quasimodes naturally extends to polyhomogeneous and asymptotically exponential functions.

2.2.2 Polyhomogeneous and Exponential Functions

Next we introduce the notion of polyhomogeneous functions, which allows for a thorough bookkeeping of the powers in its asymptotic expansion. We will require these asymptotic expansions to be stable under the application of b-operators as in (2.6). As a byproduct of polyhomogeneous functions we can present a notion of functions with asymptotic exponential behavior.

This subsection starts with the introduction of *index sets*, accounting for the powers of non vanishing coefficients in the asymptotic expansion of polyhomogeneous functions. These will play a central role in the construction of quasimodes in Chapters 3 and 4 and the resolution of operators in Chapter 5.

Definition 2.2.6 (Index Set)**.** A set $I \subset \mathbb{C} \times \mathbb{N}_0$ is called *index set*, if it satisfies the following properties:

(i) For all $s \in \mathbb{R}$ the set $I_{\leq s} := \{(z,k) \in I : \operatorname{Re}(z) \leq s\}$ is finite.

(ii) If $(z,k) \in I,\ 0 \leq l \leq k$, then $(z,l) \in I$.

Additionally, if we have

(iii) if $(z,k) \in I$, then $(z+1,k) \in I$,

then I is called a C^∞−index set.

Usually we will encounter *log-free* index sets with real exponents, i.e. $I \subset \mathbb{R} \times \{0\}$. Since these are totally ordered by $\leq$, we introduce a notion to indicate the absence of the *lowest entries* of I.

Definition 2.2.7. Let $I \subset \mathbb{R} \times \{0\}$ be an index set, $n \in \mathbb{N}_0$ and $\alpha_k \in \mathbb{N}$ for each $k \in \mathbb{N}_0$, such that $I = \{(\alpha_k, 0) : k \in \mathbb{N}_0\}$. Then we denote

$$I_n := \{(\alpha_k, 0) : n \leq k\} .$$

Alternatively, we will refer to I_0 also as I_+. Since I is discrete, we have that $\bigcap_n I_n = \emptyset$.

Definition 2.2.8 (Index-Family)**.** Let M be a manifold with corners. An *index family* is an assignment $\mathcal{I}$ from the set of boundary hypersurfaces to the C^∞-index sets.

The notion of Definition 2.2.7 lifts directly to index families $\mathcal{I} = (I(H))_{H \subset \partial M}$, by setting $\mathcal{I}_n := (I(H)_n)_{H \subset \partial M}$. With the notion of index sets we can refine the concept of asymptotic series and define *polyhomogeneous functions* on the half space.

Definition 2.2.9 (Polyhomogeneous Functions on $\mathbb{R}^n_1$)**.** Let $u \colon \operatorname{int}\left(\mathbb{R}^n_1\right) \to \mathbb{C}$ be a smooth function and I be an index set. The function u is called *polyhomogeneous* with respect to I at $\partial\mathbb{R}^n_1$, if there are $a_{z,k} \in C^\infty\left(\mathbb{R}^{n-1}\right)$ for each $(z,k) \in I$, such that for all $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^{n-1}$ and $s \in \mathbb{R}$

$$(x\partial_x)^j \left(\partial_y\right)^\alpha \left(u(x,y) - \sum_{(z,k)\in I} a_{z,k}(y) x^z \log^k x \right) = O\left(x^s\right). \tag{2.6}$$

holds. Having this, we write

$$u(x,y) \sim \sum_{(z,k)\in I} a_{z,k}(y) x^z \log^k x \tag{2.7}$$

and denote the space of polohomogeneous functions as

$$\mathcal{A}^I\left(\mathbb{R}^n_1\right) := \left\{ u \in C^\infty\left(\operatorname{int}\left(\mathbb{R}^n_1\right)\right) \text{ is polyhomogeneous on } \mathbb{R}^n_1 \text{ with respect to } I \right\}. \tag{2.8}$$

The notion of *b-operators* appears throughout Chapter 4 in order to characterize b-ellipticity of transport operators. Apart from that, we can also use b-operators to define polyhomogeneous functions on model spaces with codimension 2, where one needs to be able to partially develop the asymptotic series along each boundary hypersurface.

Definition 2.2.10 (b-Differential Operator)**.** Let $n \geq 2$. We denote coordinates on the model space $\mathbb{R}^n_2 = \mathbb{R}^2_+ \times \mathbb{R}^{n-2}$ by (x,y). Let $\alpha \in \mathbb{N}_0^2, \beta \in \mathbb{N}_0^{n-2}$ and $b_{\alpha,\beta} \in C^\infty\left(\mathbb{R}^n_2\right)$. A differential operator $P \in \operatorname{Diff}\left(\mathbb{R}^n_2\right)$ (acting on smooth functions) is called *b-differential operator* if it is a finite sum of the form

$$P = \sum_{\alpha \in \mathbb{N}_0^2, \beta \in \mathbb{N}_0^{n-2}} b_{\alpha,\beta}(x,y) \, (x\partial_x)^\alpha \, \partial_y^\beta,$$

using multi-index notation. We denote the space of b-differential operators as

$$\operatorname{Diff}_b^*\left(\mathbb{R}^n_2\right) := \left\{ P \in \operatorname{Diff}\left(\mathbb{R}^n_2\right) : \text{P is a b-differential operator} \right\}.$$

Given $s,t \in \mathbb{R}$ we define the *conormal spaces*

$$\mathcal{A}^{(s,t)}\left(\mathbb{R}^n_2\right) := \left\{ u \in C^\infty\left(\operatorname{int}\left(\mathbb{R}^n_2\right)\right) : Pu = O\left(x_1^s x_2^t\right) \text{ for all } P \in \operatorname{Diff}_b^*\left(\mathbb{R}^n_2\right) \right\}.$$

Definition 2.2.11 (Polyhomogeneous Functions on $\mathbb{R}^n_2$)**.** Let $u\colon \operatorname{int}\left(\mathbb{R}^n_2\right) \to \mathbb{C}$ be a smooth function and (I_1, I_2) be an index family assigned to the boundary hypersurfaces of $\mathbb{R}^n_k$. u is called *polyhomogeneous* on $\mathbb{R}^n_k$ with respect to (I_1, I_2) if there are $N \in \mathbb{R}$ and functions

$$a^{(1)}_{\omega,l} \in \mathcal{A}^{I_1}\left(\mathbb{R}^{n-1}_1\right), \quad a^{(2)}_{\omega,l} \in \mathcal{A}^{I_2}\left(\mathbb{R}^{n-1}_1\right), \quad (\omega, l) \in I_i \,,$$

such that for all $s \in \mathbb{R}$ we have

$$\begin{aligned} u(x,y) &= \sum_{(z,k)\in I_1} a^{(1)}_{z,k}(x_2, y) x_1^z \log^k x_1 + r^{(1)}_s \\ u(x,y) &= \sum_{(z,k)\in I_2} a^{(2)}_{z,k}(x_1, y) x_2^z \log^k x_2 + r^{(2)}_s \,, \end{aligned} \tag{2.9}$$

where $r^{(1)}_s \in \mathcal{A}^{(s,-N)}\left(\mathbb{R}^n_2\right)$ and $r^{(2)}_s \in \mathcal{A}^{(-N,s)}\left(\mathbb{R}^n_2\right)$.

Using local charts one can lift the notion of polyhomogeneity to general manifolds with corners.

Definition 2.2.12 (Polyhomogeneous Functions on Manifolds with Corners)**.** Let M be a manifold with corners (of codimension 2 or less), $H \subset \partial_1(M)$ be a boundary hypersurface and $\mathcal{I}$ a C^∞ index family. A smooth function $u\colon \operatorname{int}(M) \to \mathbb{C}$ is called *polyhomogeneous* on M with respect to $\mathcal{I}$, if u is polyhomogeneous with respect to any boundary chart, i.e.

$$\phi\colon U \to V\colon\; \phi^* u \in \mathcal{A}^{\mathcal{I}'}(U)\,, \tag{2.10}$$

where we define $\mathcal{I}'\left(\phi^{-1}(V \cap H)\right) := \mathcal{I}(H)$ and

$$\mathcal{A}^{\mathcal{I}'}(U) := \left\{u \in C^\infty\left(U \cap \operatorname{int}\left(\mathbb{R}^n_2\right)\right) :\; pu \in \mathcal{A}^{\mathcal{I}'}(\mathbb{R}^n_2) \text{ for all } p \in C^\infty_0(U)\right\}\,.$$

We will denote the space of polyhomogeneous functions with respect to arbitrary index families on M by $\mathcal{A}(M)$.

As a byproduct we can define oscillating functions using the notion of polyhomogeneous functions.

Definition 2.2.13 (Exponential-Polyhomogeneous Functions)**.** Let M be a manifold with corners and $H_j \subset \partial M$ be the boundary hypersurfaces of M, for $j \leq r$, $r \in \mathbb{N}$. Let $N \in \mathbb{N}$, $n \leq N$ and $I(H_j, n) \subset \mathbb{C}\times\mathbb{N}_0$ be a family of index sets. Let $h\colon M \to \mathbb{R}_+$ be a global boundary defining function. Let $\Gamma \subset \bigcup_j T^*_{\mathbb{C}} H^\circ_j$ be a union of projectible, non-intersecting submanifolds over each H_j, $j = 1, \ldots, r$. Let $c\colon \Gamma \to \mathbb{N}_0$ be locally constant.

The set $\mathcal{E}\mathcal{A}^I(M;\Gamma)$ is called space of *exponential-polyhomogeneous functions* on M, Γ and I. It consists of elements

$$u = \sum_{n=1}^{N} e^{\Phi_n} A_n \in C^\infty(M^\circ)\,,$$

where $A_n \in \mathcal{A}^{\mathcal{I}(n)}(M)$ and $\Phi_n \in \mathcal{A}(M)$, such that for all Φ_n and $H \in \mathcal{M}_1(M)$ there is a $\delta \in \mathbb{R}_{>0}$ and a function $\varphi_{n,H,\delta}$ with

$$\Phi_n = \frac{\varphi_{n,H,\delta}}{h^\delta} + c(\Gamma_{n,H,\delta})\frac{\log(h)}{h^\delta} + O(h^{-\delta}) \quad \text{at } H^\circ ,$$

where $\Gamma_{n,H,\delta} \subset \Gamma \cap T^*_{\mathbb{C}}H^\circ_j$, for $\Gamma_{n,H,\delta} := \operatorname{graph}(d\varphi_{n,H,\delta})$.

Remark 2.2.14. We will omit the function c in the notion of exponential-polyhomogeneous functions and refer to it as $\mathcal{E}\mathcal{A}^{\mathcal{I}}(M;\Gamma)$ throughout this thesis, since it can be derived from Γ itself.

A small but important statement is the existence of primitives of one dimensional, oscillating functions having the same type of oscillation.

Lemma 2.2.15. *Let $f \in C^\infty((0,\infty))$, p be a Puiseux series and $A \in \mathcal{A}(\mathbb{R}_+)$ such that $f(x) \sim e^{p(x)}A(x)$ as $x \to +\infty$.*

Then there is a $B \in \mathcal{A}(\mathbb{R}_+)$ and a primitive F of f such that

$$F(x) \sim e^{p(x)}B(x) ,$$

as $x \to +\infty$.

Proof. There are two cases depending on the lowest power N/r of $p(x) = \sum_{k=N}^\infty a_k x^{k/r}$, $r \in \mathbb{N}$, $N \in \mathbb{Z}$.

In the case of $N > 0$, applying partial integration yields

$$\int e^p A = \int p' e^p \frac{A}{p'} = e^p \frac{A}{p'} - \int e^p \underbrace{\left(\frac{A}{p'}\right)'}_{=O(x^{-N/r}\cdot A)} .$$

On the other hand, in the case of $N < 0$ we have

$$\int e^p A = e^p A - \int e^p \underbrace{\left(p' \cdot \widetilde{A}\right)'}_{=O(x^{-N/r}\cdot A)} ,$$

where $\widetilde{A}' = A$. Iterating this argument and using de L'Hôpital we conclude that

$$e^{-p} \int e^p A \sim A/p' + (A/p')'/p' + \ldots + O(x^{-\infty})$$

in the first case and $e^{-p} \int e^p A \sim A + (\widetilde{A}p')' + \ldots + O(x^{-\infty})$ in the second case. $\square$

2.2.3 Borel lemma

A standard result called *Borel lemma* says that for every asymptotic power series in t with smooth coefficients on $K \subset \mathbb{R}^n$ there is a small neighborhood I of $t = 0$ and a smooth function f on $K \times I$ being asymptotically equal to that power series.

Lemma 2.2.16 (Classical Borel lemma, [Hör03]). *For $j = 0, 1, \ldots$ let $f_j \in C_0^\infty(K)$ where K is a compact subset of $\mathbb{R}^n$, and let I be a compact neighborhood of 0 in $\mathbb{R}$. Then one can find $f \in C_0^\infty(K \times I)$ such that*

$$\frac{\partial^j f}{\partial t^j}(x, 0) = f_j(x), \ j = 0, 1, \ldots .$$

Remark 2.2.17. Note that this statement is phrased without using the Taylor series directly but by evaluating its coefficients at $t = 0$. It remains valid for asymptotic power series with non-integer powers, yielding polyhomogeneous functions f as a result.

There is an immediate extension to the case of manifolds with corners, which we will illustrate for our case in two dimensions.

Lemma 2.2.18 (Borel lemma). *Let $\mathbb{R}^2_+$, let $c \in \mathbb{C}$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $f, g \in \mathcal{A}(\mathbb{R}_+)$ with index sets $E_f, E_g \subset \mathbb{R} \times \{0\}$, respectively, such that $f(x) \overset{x\to 0}{\sim} c \cdot x^{\gamma_1} + o(y^{\gamma_2})$ and $g(y) \overset{y\to 0}{\sim} c \cdot y^{\gamma_2} + o(y^{\gamma_2})$. Then there is a function $h \in \mathcal{A}(\mathbb{R}^2_+)$ with index family $\mathcal{E} = (E_g, E_f)$, such that $h(\cdot, y) \overset{y\to 0}{\sim} f \cdot y^{\gamma_2} + o(y^{-\gamma_2})$ and $h(x, \cdot) \overset{x\to 0}{\sim} g \cdot x^{\gamma_1} + o(x^{-\gamma_1})$.*

Proof. Let $\widetilde{f}$ be an extension of $f \cdot \sum_{\gamma \in E_g} y^\gamma$ to $\mathbb{R}^2_+$ by the classical Borel lemma, $\widetilde{f_y}$ be defined by $\widetilde{f}(x, \cdot) \overset{x\to 0}{\sim} \widetilde{f_y} \cdot x^{\gamma_1} + o(x^{\gamma_1})$ and $\widetilde{g}$ be an extension of $(g - \widetilde{f_y}) \cdot \sum_{\gamma \in E_f} x^\gamma$. Define $h := \widetilde{f} + \widetilde{g}$. Then the leading part of h at $\{y = 0\}$

$$h(\cdot, y) \overset{y\to 0}{\sim} f \cdot (y^{\gamma_2} + o(y^{\gamma_2})) + (c \cdot y^{\gamma_2} - c \cdot y^{\gamma_2} + o\left(y^{\gamma_2}\right)) \cdot \widetilde{g}_x = f \cdot y^{\gamma_2} + o(y^{\gamma_2}) \, ,$$

coincides with f, while the leading part of h at $\{x = 0\}$

$$h(x, \cdot) \overset{x\to 0}{\sim} \widetilde{f_y} \cdot x^{\gamma_1} + o(x^{\gamma_1}) + \left(g - \widetilde{f_y}\right) x^{\gamma_1} + o(x^{\gamma_1}) = g \cdot x^{\gamma_1} + o(x^{\gamma_1}) \, ,$$

coincides with g. □

2.2.4 Singular Ordinary Differential Equations

The following theorem plays a central role in the construction of quasimodes in Chapters 4 and 5. It gives a relation between asymptotic and exact solutions of an ordinary differential equation which we will use to match quasimodes at the corners of the blown-up half space.

Theorem 2.2.19 (Existence of Solutions, [Was87]). *Let $m \in \mathbb{N}$, $a_k \in C^\infty(\mathbb{R}_+)$, $k \leq m$, be nowhere vanishing on $(0, \infty)$ and analytic in 0.*

Then for each asymptotic solution $u(x) \sim e^{p(x)} A(x)$, as $x \to 0$, where $p(x)$ is a Puiseux series and A is polyhomogeneous, of the ordinary differential equation

$$\sum_{k=0}^{m} a_k y^{(k)} = 0 \, , \tag{2.11}$$

there is a solution y with $y(x) \sim u(x)$, as $x \to 0$. Moreover if $\{u_1, \dots, u_n\}$ is a set of independent, asymptotic solutions then the set of corresponding solutions $\{y_1, \dots, y_n\}$ is linear independent.

Remark 2.2.20. Proposition 2.3.6 shows how to construct asymptotic solutions of Equation 2.11 by the use of a tool called *Newton polygon*, which will be introduced in Subsection 2.3.2

As a direct consequence of Theorem 2.2.19 we can show the existence of an inhomogeneous solution of (2.11), whose asymptotic behavior matches that of the imhomogeneity.

Corollary 2.2.21. *Let $m \in \mathbb{N}$, $a_k \in C^\infty(\mathbb{R}_+)$, $k \leq m$, be analytic in 0. Let q be a Puiseux series, $B \in \mathcal{A}(\mathbb{R}_+)$ and $f \in C^\infty((0,\infty))$ with $f(x) \sim e^{q(x)}B(x)$, as $x \to 0$.*

Then there is a solution u of

$$\sum_{k=0}^{m} a_k y^{(k)} = f\,,$$

with $u(x) \sim e^{q(x)}\widetilde{B}(x)$, as $x \to 0$, for some $\widetilde{B} \in \mathcal{A}(\mathbb{R}_+)$.

Proof. By Theorem 2.2.19 there are solutions $u_1, \dots, u_m$ of the equation $\sum_{k=0}^m a_k y^{(k)} = 0$, with $u_j \sim e^{p_k} A_k$, for some Puiseux series p_k and $A_k \in \mathcal{A}(\mathbb{R}_+)$. Using Cramer's rule it holds that $y_p := \sum_{k=1}^m c_k \cdot u_k$ is a solution of $\sum_{k=0}^m a_k y^{(k)} = f$, where

$$c_k := \int \frac{W_k(u_1, \dots, u_m)}{W(u_1, \dots, u_m)}\,,$$

W is the Wronskian of its entries and W_k is the Wronskian with the k-th column replaced by $F := (0, \dots, 0, f)^T$. Thus, we have

$$\frac{W_k(u_1, \dots, u_m)}{W(u_1, \dots, u_m)} \sim e^{q - p_k} B_k\,,$$

for all $k \leq m$ and for some $B_k \in \mathcal{A}(\mathbb{R}_+)$. By Lemma 2.2.15 we have $c_k \sim e^{q-p_k}\widetilde{B}_k$ for some $B \in \mathcal{A}(\mathbb{R}_+)$. Thus, we can conclude that

$$y_p = \sum_{k=1}^m c_k \cdot u_k \sim \sum_{k=1}^m e^{q-p_k}\widetilde{B}_k e^{p_k} A_k = e^q \sum_{k=1}^m A_k \widetilde{B}_k\,,$$

completing the proof. □

2.3 Elementary Perturbation Theory

This section contains two elementary objects in the field of perturbation theory which will play an essential role in the construction of quasimodes in Chapters 3 - 5. We extract the computation of commutators with respect to concatenations involving the exponential function and n-th powers of vector fields from Chapter 3. These are needed to determine the relevant parts of semi-classical operators when constructing quasimodes. Related to these computations throughout this thesis is the concept of Newton polygons, which we will introduce directly after. It is used in different contexts to compute asymptotic solutions of the corresponding equation. Extending this concept to semi-classically perturbed differential operators is the basis of this thesis.

2.3.1 Faà di Bruno's formula

We begin with some basic computations involving Faà di Bruno's formula. For this we will introduce the faculty of $m \in \mathbb{N}_0^n$, $m = (m_1, \ldots, m_n)$, given by $m! := m_1! \cdot \cdots \cdot m_n!$.

Lemma 2.3.1 (Faà di Bruno's formula)**.** *Let $n \in \mathbb{N}$ and $f, g \in C^\infty(\mathbb{R})$. Then the n-th derivative of $f \circ g$ is given by*

$$\partial_x^n (f \circ g) = \sum_{m \in M_n} \frac{n!}{m!} \left(f^{(m_1 + \ldots + m_n)} \circ g \right) \prod_{k=1}^{n} \left(\frac{g^{(k)}}{k!} \right)^{m_k} , \tag{2.12}$$

where the summation on the right side is over the set

$$M_n := \left\{ (m_1, \ldots, m_n) \in \mathbb{N}_0^n : \sum_{k=0}^{n} k \cdot m_k = n \right\} . \tag{2.13}$$

Faà di Bruno's formula (2.12) plays an important role in the construction of quasimodes with exponential behavior. Applying it to $f = \exp$ and $g = \varphi/h^\delta$ yields

$$\partial_x^n e^{\varphi/h^\delta} = \sum_{m \in M_n} \frac{n!}{m!} e^{\varphi/h^\delta} \prod_{k=1}^{n} \left(\frac{\varphi^{(k)}}{h^\delta k!} \right)^{m_k} ,$$

which can be ordered in powers of h. Its lowest order term

$$\partial_x^n e^{\varphi/h^\delta} = h^{-n\delta} e^{\varphi/h^\delta} \frac{n!}{(n, 0, \ldots, 0)!} (\varphi')^n = h^{-n\delta} e^{\varphi/h^\delta} \cdot (\varphi')^n + O\left(h^{-(n-1)\delta} \right) \tag{2.14}$$

is the single term corresponding to $(n, 0, \ldots, 0) \in M_n$. Conjugating the n-th power of ∂_x by $\exp(\varphi/h^\delta)$ and subtracting $h^{-n\delta}(\varphi')^n$ then yields an expansion

$$h^{-(n-1)\delta} \left[n(\varphi')^{n-1} \left(\partial_x + \frac{n-1}{2} \frac{\varphi''}{\varphi'} \right) \right] + O\left(h^{-(n-2)\delta} \right) , \tag{2.15}$$

which is a direct consequence of (2.12) due to $(n-2, 1, 0, \ldots, 0) \in M_n$ and the Leibniz rule. Essentially, the summation condition of the set M_n in (2.13) and in particular its weights associated to the entries of the tuple $(m_1, \ldots, m_n)$ reduces the amount of terms of $e^{-\varphi/h^\delta} \partial_x^n e^{\varphi/h^\delta}$ with low powers of h. Another important observation is given by the fact, that the conjugated operator

$$e^{-\varphi/h^\delta} \partial_x^n e^{\varphi/h^\delta} = \sum_{k=0}^{n} h^{-(n-k)\delta} P_k$$

only admits for terms with powers $(n-k)\delta$, $k \in \{0, \ldots, n\}$, with associated differential operators P_k of complementary degree $\deg P_k = k$, of which we have determined the first two in (2.14) and (2.15).

2.3.2 Newton Polygon

The Newton polygon is a tool first described by Isaac Newton (see [New60]) to approximate curves γ in $\mathbb{R}^2$ parametrized over x that asymptotically solve the equation $f(x, \gamma(x)) = 0$ at $x = 0$ for polynomials $f \in \mathbb{R}[X, Y]$. By taking pairs of the powers $(k, l) \in \mathbb{R}^2$ of X and Y in each summand of f, its associated Newton polygon is the convex hull of the union over all first quadrant quarter spaces attached to any of these points (k, l). Its edges of finite length help detect leading orders in x of asymptotic solutions γ. As one can easily show, these leading orders are given by the negative slope of the edge and the multiplicity of the first order solution by the horizontal width of the edge.

Obtaining a first order approximation is very valuable since improvement of these can often be achieved by a general scheme applying to any given approximation. Proposition 3.1.7 in the beginning of Chapter 3 is one of these schemes that will be used frequently in this thesis. Arriving at that scheme in order to construct quasimodes is the central task of the first chapter, in particular Section 3.2.

We present our notion of Newton polygon (compare Figure 2.6) which we will use extensively. In the upcoming definition the set $\operatorname{conv}(A)$ is the convex hull of $A \subset \mathbb{N}_0 \times \mathbb{R} \subset \mathbb{R}^2$ in the Euclidean space.

Definition 2.3.2 (Newton Polygon)**.** Let $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be discrete and bounded from below in its second argument.

(i) We call

$$\mathcal{P}(\Lambda) := \operatorname{conv}(\{(x, y) \in \mathbb{R}^2 : \text{ there exists } (k, \alpha) \in \Lambda \text{ with } x \leq k \ \wedge \ y \geq \alpha\})$$

the *Newton polygon* with respect to Λ.

(ii) Let $\lambda = (k_\lambda, \alpha_\lambda), \mu = (k_\mu, \alpha_\mu) \in \partial\mathcal{P}(\Lambda)$. A line $\mathcal{L} := \overline{\lambda\mu}$ is called *lower edge* of $\mathcal{P}(\Lambda)$ if $k_\lambda \neq k_\mu$. The union of all lower edges is denoted by

$$\partial_{\text{-}}\mathcal{P}(\Lambda)$$

and is called *lower boundary* of $\mathcal{P}(\Lambda)$.

(iii) Let $\mathcal{L} := \overline{\lambda\mu} \subset \partial_{\text{-}}\mathcal{P}(\Lambda)$ be a lower edge, for $\lambda = (k_\lambda, \alpha_\lambda), \mu = (k_\mu, \alpha_\mu) \in \Lambda$. We denote the *horizontal width* of $\mathcal{L}$ by

$$|\mathcal{L}| := |k_\lambda - k_\mu|,$$

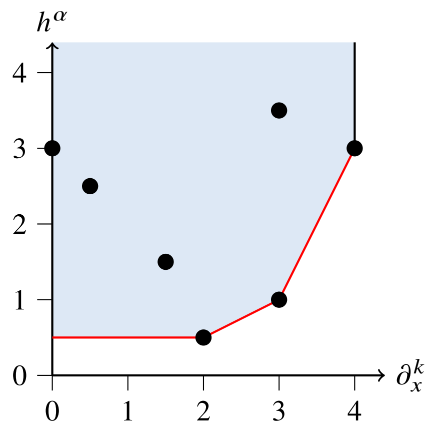

Figure 2.6: Newton polygon $\mathcal{P}(\Lambda)$ of a two dimensional set Λ. The red boundary is the lower boundary $\partial_{\!-}\mathcal{P}(\Lambda)$.

and its *slope* by

$$\delta(\mathcal{L}) := \frac{|\alpha_\lambda - \alpha_\mu|}{|k_\lambda - k_\mu|}.$$

Remark 2.3.3. Definition 2.3.2 is slightly different than the usual definition of Newton polygons as in [Kol07] or [Zie95]. More commonly, one says that, given a finite set of points Λ, the Newton polygon associated to Λ is its convex hull, i.e. $\mathcal{P}(\Lambda) := \operatorname{conv}(\Lambda)$. Thus, the classical Newton polygon is a subset of the object we defined as Newton polygon. However, all edges with positive slopes coincide for both types of definitions. In our notion of Newton polygon all edges with non-positive slope in the lower boundary will be shifted to the horizontal edge from the y-axis to the base point of the edge with the lowest positive point. This will make statements regarding non-positive edges throughout Chapters 3-5 easier and more natural in this context.

Remark 2.3.4. Note that we explicitly allow the existence of multiple edges with identical slope. Whenever there is an edge $\mathcal{L} := \overline{\lambda\mu} \subset \partial_{\!-}\mathcal{P}(\Lambda)$ with a point $\tau \in \mathcal{L} \cap \Lambda$, such that $\tau \neq \lambda, \mu$, the segments $\overline{\lambda\tau}$ and $\overline{\tau\mu}$ are edges in $\partial_{\!-}\mathcal{P}(\Lambda)$, too. However, we exclude lower edges with zero width.

Remark 2.3.5. Geometrically, the Newton polygon is the convex hull of the union of shifted second-quadrant quarter spaces attached to any point in the set Λ. Note that one classically takes the first quadrant in the construction.

Using the Newton polygon we can prove Proposition 2.3.6, which lets us compute and quantify types of asymptotic solutions of ordinary differential equations. Theorem 2.2.19 then guarantees the existence of explicit solutions having the same asymptotic expansions as these constructed asymptotic solutions.

Proposition 2.3.6. *Let $m \in \mathbb{N}$, $a_k \in C^\omega(\mathbb{R}_+)$, $k = 1, \dots, m$, and $\sum_{k=0}^m a_k(x)(x\partial_x)^k y = 0$ be the corresponding ordinary b-differential equation with analytic coefficients. Let $\Lambda := \{(k, l) : l = \operatorname{ord}_0(a_k)\}$ be the set of zero orders of a_k at $x = 0$ and $\mathcal{P}(\Lambda)$ be the associated*

Newton polygon. Then for every edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ there are asymptotic solutions $u_{k,\mathcal{L}}$ of $\sum_{j=0}^m a_j(x)(x\partial_x)^j y = 0$ at $x = 0$, for $k = 1, \dots, |\mathcal{L}|$, with

$$u_{k,\mathcal{L}}(x) \sim e^{\varphi_{k,\mathcal{L}}(x)} A_{k,\mathcal{L}}(x)\,,$$

for some polyhomogeneous functions $A_{k,\mathcal{L}}$, where $\varphi_{k,\mathcal{L}}(x) = c_k \log(x)$, if $\delta = 0$, and where $\varphi_{k,\mathcal{L}}(x) = c_k/x^\delta$, if $\delta \neq 0$, where δ is the slope of $\mathcal{L}$, $c_k \in \mathbb{C}$ are the solutions of

$$\sum_{(j,l)\in\Lambda\cap\mathcal{L}} \alpha_j c^j = 0\,,$$

where $a_j(x) = \alpha_j x^{\mathrm{ord}_0(a_j)} + o(x^{\mathrm{ord}_0(a_j)})$.

Proof. We only prove the special case where $\mathrm{ord}_0(a_k) = k\delta + l_\delta$, for some $\delta > 0$ and $l_\delta \in \mathbb{R}$, and where all solutions c_k of $\sum_{(j,l)\in\Lambda\cap\mathcal{L}} \alpha_j c^j = 0$ are simple. In that case we can write

$$P = \sum_{j=0}^m a_j(x)(x\partial_x)^j = \sum_{j=0}^m b_j(x) x^{j\delta + l_\delta}(x\partial_x)^j\,,$$

with $\alpha_j = b_j(0) \neq 0$. Choosing the ansatz $u = e^\varphi A$ for an asymptotic solution, the expansion of Pu yields

$$e^\varphi x^{l_\delta} \sum_{j=0}^m \left(b_j \cdot x^{j\delta}(x\varphi')^j + \left[b_j \cdot x^{j\delta} j(x\varphi')^{j-1}((x\partial_x) + \frac{j-1}{2}\frac{x\varphi''}{\varphi'}) \right] + \dots \right) A\,,$$

where the lowest order terms with respect to x, as $x \to 0$, are amongst $x^{j\delta}(x\varphi')^j$, if $\varphi(x) \sim x^{-\gamma}$ with $\gamma > 0$, and b_0, if $\gamma \leq 0$. Only the first has a chance to eliminate the lowest order remainder term, if $\gamma = \delta$, where we have

$$\underbrace{b_j \cdot x^{j\delta}(x\varphi')^j}_{\sim x^0} + \underbrace{\left[b_j \cdot x^{j\delta} j(x\varphi')^{j-1}((x\partial_x) + \frac{j-1}{2}\frac{x\varphi''}{\varphi'}) \right]}_{\sim x^\delta} + O(x^{2\delta})\,,$$

for each $j = 0, \dots, m$. Thus, φ has to solve

$$\sum_{j=0}^m b_j \cdot (x^{\delta+1}\varphi')^j = 0\,,$$

or equivalently $x^{\delta+1}\varphi' = -\zeta_k$, for a solution ζ_k of $\sum_{j=0}^m b_j \cdot \zeta^j = 0$. Since $b_j(0) \neq 0$, we have that $\zeta_k(0) = c_k \neq 0$, for all $k = 1, \dots, m$, and hence

$$\varphi(x) \sim -\frac{c_k}{\delta} x^{-\delta}\,.$$

Thus, Pu reduces asymptotically to

$$e^{\varphi} x^{l_\delta} \left(x^\delta \sum_{j=0}^{m} \left[b_j(0) k c_j^{j-1} \right] \cdot \left(x\partial_x - \frac{j-1}{2}\delta \right) + O(x^{2\delta}) \right) A(x) \,,$$

which allows for an asymptotic solution of A as $x \to 0$, since

$$T := \sum_{j=0}^{m} b_j(0) \cdot ((x\partial_x) - \delta(j-1)/2)$$

is b-elliptic by assuming that c_k is a simple root. □

Remark 2.3.7. The proof of the special case includes all analytical details that appear in the proof of the general case. The additional complication there lies within the combinatorial data Λ. The full proof is similar to the proof of Theorem 3.3.11, where we show how to deal with multiple edges $\mathcal{L} \subset \underline{\partial}\mathcal{P}(\Lambda)$ and non-separated lower boundaries, resulting in higher order terms of the phase function. These are not fully determined by $\mathcal{L}$ itself.

Note that for $\delta = 0$ this is an elliptic b-operator. Thus, $e^{\varphi(x)} \sim x^{c_k}$ as $x \to 0$, which is absorbed by the polyhomogeneous amplitude A.

2.4 Semi-Classical Analysis

In this last section of Chapter 2 we are going to present semi-classical perturbations of differential operators. This will include the *Schrödinger* operator and standard methods of constructing asymptotic solutions, i.e. quasimodes, in the semi-classical limit as $h \to 0$. Most prominent throughout this thesis will be the method known as *WKB*-method, presented in Subsection 2.4.1. The section ends with an outline about *turning points* in Subsection 2.4.2, where some of the problems of singular operators in Chapter 5 are presented briefly using the WKB-method.

2.4.1 WKB-Method

A standard method of constructing quasimodes of the time independent Schrödinger operator

$$P := -h^2\partial_x^2 + V \,, \tag{2.16}$$

for some potential $V \in C^\infty(I)$, $I \subset \mathbb{R}$, is the so called *WKB*-method. It is named after *Gregor Wentzel*, *Hendrik Anthony Kramers* and *Léon Brillouin*, who discovered this method of approximation simultaneously but independently in the early twentieth century (see [Wen26],[Kra26],[Bri26]). It is a standard subject in undergrad physics courses (see [Nol03]), which also comes to use in the construction of semi-classical Fourier integral operators (see [Zwo12]). The WKB-method is an ansatz based procedure that can be easily motivated by the special case of constant potentials $V \equiv c^2$, $c \in \mathbb{C} \setminus \{0\}$. In that particular case, the solutions of

$$-h^2 y'' + c^2 y = 0\,,$$

are given by $y_\pm(x) := e^{\pm cx/h} a$, for any $a \in \mathbb{C}$, i.e. simple, exponential functions with constant phases and amplitudes. Increasing the absolute value of c leads to a faster oscillation or stronger (real) exponential growth or decay.

The ansatz

This gives reason to assume that in the case of non-constant, nowhere vanishing potentials V a solution y might be of similar shape

$$y_\pm(x,h) = e^{\varphi_\pm(x)/h} A_\pm(x,h)\,,$$

with non-constant phase functions $\varphi_\pm$ and amplitudes $A_\pm$, which is the first part of the ansatz of the WKB-method. In the context of this thesis phase functions are complex valued in general. We will refer to them as phase functions even if they are real valued. Applying this to (2.16) then yields

$$\left(-h^2\partial_x^2 + V\right) e^{\varphi/h} A = -h^2 e^{\varphi/h} \left(\left(\frac{\varphi'}{h}\right)^2 A + \frac{\varphi''}{h} A + 2\frac{\varphi'}{h} A' + A'' \right) + e^{\varphi/h} V A\,,$$

which we will not be able to solve in general. Thus, as a second part of the ansatz we reduce the scope of finding solutions of (2.16) to finding quasimodes of the Schrödinger equation. These quasimodes $u = e^{\varphi(x)/h} A(x,h)$ then should satisfy

$$\left(e^{-\varphi(x)/h}(-h^2\partial_x^2 + V) e^{\varphi(x)/h}\right) A(x,h) = O(h^\infty)\,,$$

i.e. the remainder $f(x,h) := e^{-\varphi(x)/h}\left(-h^2\partial_x^2 + V\right) u(x,h)$ vanishes faster than any polynomial in h as $h \to 0$, uniformly on any compact interval $K \subset \mathbb{R}$. By reducing the scope to quasimodes the WKB method reduces the ansatz to finding an asymptotic power series in powers of h

$$A(x,h) \sim \sum_{k=0}^{\infty} a_k(x) h^k\,.$$

Applying this asymptotic ansatz to (2.16) then yields

$$\left(-h^2\partial_x^2 + V\right) e^{\varphi/h} A \sim e^{\varphi/h} \sum_{k=0}^{\infty} \left(-h^{k+2} \left(\left(\frac{\varphi'}{h}\right)^2 a_k + \frac{\varphi''}{h} a_k + 2\frac{\varphi'}{h} a_k' + a_k'' \right) + V a_k h^k \right)\,.$$

The goal is to choose φ, a_k successively, such that the remainder

$$f(x,h) \sim e^{\varphi/h} \sum_{k=0}^{\infty} \left(-h^{k+2} \left(\left(\frac{\varphi'}{h}\right)^2 a_k + \frac{\varphi''}{h} a_k + 2\frac{\varphi'}{h} a_k' + a_k'' \right) + V a_k h^k \right)$$

vanishes faster than h^k, as $k \to \infty$. Thus, we sort the asymptotic expansion of f by powers of h and check whether the implicit equations of its coefficients admit solutions. For the lowest two powers the expansion

$$f(x,h) \sim h^0 \left(-(\varphi')^2 + V\right) a_0 + h^1 \left(-(2\varphi'\partial_x + \varphi'')a_0 + (-(\varphi')^2 + V)a_1\right) + O(h^2)\,, \quad (2.17)$$

yields two first order differential equations.

Eikonal Equation

In order to achieve $f(x,h) = O(h)$ its lowest order coefficient in the expansion of (2.17) must vanish. We can directly exclude setting $a_0 \equiv 0$ since this would result in the same problem recurring for $(-(\varphi')^2 + V)a_1 = 0$ in the coefficient of h^1. Thus, the only option is to choose φ solving the so called *eikonal equation*

$$-(\varphi')^2 + V = 0\,. \quad (2.18)$$

In the case of the Schrödinger operator its solutions are of the form

$$\varphi_{\pm,x_0}(x) \coloneqq \pm \int_{x_0}^{x} \sqrt{V(t)}dt\,, \quad (2.19)$$

for any given choice of base point $x_0 \in \mathbb{R}$. Note that its solution φ is smooth since the potential V is vanishing nowhere. Choosing any such solution we can proceed with the cancellation of the higher order terms of the remainder.

Transport Equations

It is important to notice that φ itself does not appear in the asymptotic expansion of $f(x,h)$ but φ'. Thus, it is independent of the choice of base point $x_0 \in \mathbb{R}$ in (2.19) and does only depend on the choice of root of the algebraic equation in (2.18). Choosing the solution $\varphi \coloneqq \varphi_{+,0}$ for the rest of this subsection, the lowest order term of the remainder of $f(x,h)$

$$f(x,h) \sim h^1 \left(-(2\varphi'\partial_x + \varphi'')a_0\right) + O(h^2)\,,$$

vanishes if and only if a_0 is the solution of the first order linear differential equation

$$-\left(2\sqrt{V}\partial_x + \sqrt{V}'\right) a_0 = 0\,.$$

This equation is called homogeneous *transport equation* and its solutions are given by $c \cdot \sqrt[4]{V}$, for any $c \in \mathbb{C}$. Interestingly, all higher order terms in the expansion of the remainder, after choosing $a_0(x) \coloneqq \sqrt[4]{V(x)}$,

$$f(x,h) \sim \sum_{k=2}^{\infty} h^k \left(-(2\varphi'\partial_x + \varphi'')a_{k-1} - a''_{k-2}\right)\,,$$

are of the same shape. Thus, all higher order equations are iterations of the transport equation

$$-\left(2\sqrt{V}\partial_x + \sqrt{V}'\right) a_k = a''_{k-1}\,,$$

with inhomogeneities depending only on the previously determined solution of the iteration. Denote $T_{\varphi'} := -(2\varphi'\partial_x + \varphi'')$ and $R := -\partial_x^2$. Then the recurrent system of inhomogeneous equations

$$T_{\varphi'} a_k = -R a_{k-1}$$

is called *transport system*. Since $\varphi' = \sqrt{V}$ vanishes nowhere, $T_{\varphi'}$ is an elliptic first order differential operator and thus all solutions a_k of the transport system are smooth on $\mathbb{R}$. Applying the classical Borel lemma 2.2.16 to the asymptotic series $\sum_{k_0}^{\infty} a_k \cdot h^k$ we obtain a function $A \in C^\infty(\mathbb{R} \times \mathbb{R}_{>0})$ with $A(\cdot, h) \sim \sum_{k_0}^{\infty} a_k \cdot h^k$, such that

$$\left(e^{-\varphi/h}(-h^2\partial_x^2 + V)e^{\varphi/h}\right) A = O(h^\infty)\,,$$

completing the WKB-method with a generated quasimode $u = e^{\varphi/h}A$.

Remark 2.4.1. Allowing for higher order perturbations of V with respect to h can be implemented easily using this approach and would only result in multiple remainder operators $R_1, \ldots, R_m$ recurring alternately.

2.4.2 Turning Points & Regimes

After demonstrating how the WKB-method can be used to construct quasimodes an immediate question is whether this method also works if the potential V vanishes at certain isolated points.

Airy Equation

The simplest example of this kind is the Schrödinger operator with linear potential

$$-h^2 y'' + xy = 0\,. \tag{2.20}$$

We can explicitly solve this equation by substituting $\xi := xh^{-2/3}$, which then yields

$$h^{2/3}\,(-\widetilde{y}'' + \xi\widetilde{y}) = 0\,,$$

the Airy equation in the coordinate ξ independently of h. A system of fundamental solutions is given by the first and second Airy function $y_1(x,h) := \mathrm{Ai}(x/h^{2/3}) = \mathrm{Ai}(\xi)$ and $y_2(x,h) := \mathrm{Bi}(x/h^{2/3}) = \mathrm{Bi}(\xi)$, which oscillate as $\xi \to -\infty$ and have real exponential behavior as $\xi \to \infty$ with Ai decaying exponentially (see [Olv97]). These different types of exponential behavior in (x,h) are challenging at $x = 0$. For any positive h we observe that $u_j(x,h)$ is oscillating, as $x \to -\infty$. Conversely, the same holds for any fixed $x < 0$ as $h \to 0$. But for $x = 0$ both $u_j(0,h)$, $j = 1, 2$, are constant as $h \to 0$.

Linear Vanishing Potentials

This problem is recurrent for Schrödinger operators with more general, vanishing potentials

$$P := -h^2\partial_x^2 + xV\,,$$

for $V \in C^\infty(\mathbb{R})$, where $V > 0$ and $c := V(0)$. Since we cannot compute the solutions of the corresponding homogeneous equation $Py = 0$ in general the WKB-method can be an approach to construct quasimodes of this equation. For $I_+ := \{x > 0\}$ and $I_- := \{x < 0\}$ this was already covered in Subsection 2.4.1. For $I := I_-$ the phase functions

$$\varphi_{\pm,x_0}(x) := \pm\int_{x_0}^{x} \sqrt{tV(t)}dt\,,$$

are differentiable but not smooth at $x = 0$. In particular, its differential

$$\varphi_+'(x) = \sqrt{xV(x)}\,,$$

is only continuous in $x = 0$ and vanishes $\varphi_+'(x) \sim c\sqrt{x}$ as $x \to 0$. Thus, the associated transport operator of the differential of $\varphi := \varphi_{+,0}$

$$T_{\varphi'} := -\left(2\sqrt{xV(x)}\partial_x + \sqrt{xV(x)}'\right) = -2\sqrt{V(x)}x^{-1/2}\left(x\partial_x - \frac{1}{4}(V(x))^{-1}\right)\,,$$

is a b-differential operator. Its kernel is spanned by $a_0 \in \mathcal{A}(I)$ with $a_0(x) \sim x^{1/(4c)}$, since $c = V(0) > 0$. Thus, the recurrent inhomogeneous transport equation

$$T_{\varphi'}a_1 = -\partial_x^2 a_0\,,$$

results in a chain of deteriorating behavior of $a_k(x) \sim x^{c_k}$ with $c_k := 1/(4c) - 3k/2$, if $c_k \neq 0$, and $a_k(x) \sim \log(x)$, if $c_k = 0$, as $x \to 0$.

Assuming $c_k = 1/(4c) - 3k/2 \neq 0$ for all $k \in \mathbb{N}_0$, the asymptotic series

$$\sum_{k=0}^{\infty} a_k(x)h^k \sim x^{1/(4c)}\sum_{k=0}^{\infty}\alpha_k x^{-3/2k}h^k\,, \tag{2.21}$$

for some $\alpha_k \in \mathbb{R}$, does not allow for an extension to $x = 0$. In particular, the amplitude is not polyhomogeneous, if restricted to the quarter space $\mathbb{R}_+^2$.

On the other hand, changing coordinates again to $\xi := h^{-2/3}x$, results in

$$P = h^{2/3}\left(-\partial_\xi^2 + V(0)\xi\right) + O(h^{4/3})\,,$$

a regular perturbation of the Airy operator in $h > 0$ (see [Kat95]). Thus, in a small area around $\{x = 0\}$ the solution behaves essentially as the Airy functions in regards to the leading order in h.

Matched Asymptotic Expansions

Constructing WKB-approximation for both *outer regimes* $\{x > 0\}$ and $\{x < 0\}$ on one hand and solving the Airy equation "at" $\{x = 0\}$ leaves us with three *regimes* with significantly different and independent approximative solutions. A popular method to relate these is the so called method of *matched asymptotic expansion*. Briefly, it says that $\xi = x/h^{2/3}$ is a rescaled variable on an infinitisimally small regime $\{x = 0\}$, the so called *transition* or *boundary layer*. Its limit points $\xi \to \pm\infty$ correspond to the (two) limit points $x \to 0_\pm$ of either side, allowing us to relate both outer regimes. By solving the Airy equation at the transition layer one obtains information about the asymptotic behavior at the boundary this layer. Writing these asymptotic expansions in classical coordinates x (and h), it suggests a choice of lowest power of h in (2.21) by the vanishing order of Ai as $\xi \to \pm\infty$ and a choice of α_0 by its coefficient.

Matched asymptotic expansions is a general method that can be applied for a wide class of perturbed differential equations where the approximations of the homogeneous solutions are not uniform. Detailed descriptions of this and other related methods can be found in [Hol13], [KC96] and [Nol03].

Blow-ups and Model Operators

A relatively new approach to construct uniform approximations of homogeneous solutions of the Schrödinger operator with linear potentials as in (2.20) uses techniques from singular geometry such as blow-ups to distinguish between different *regimes*. As described in [Gri17] and [Sob18] including the semi-classical parameter $h \geq 0$ into the *geometry* of the problem has some significant conceptual advantages. The family of solutions $u_h(x) = Ai(x/h^{2/3})$ of (2.20) is a function on the joint space $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$,

$$u(x, h) \coloneqq u_h(x, h)\,,$$

where it is constant along the curves $\left\{x = ch^{2/3}\right\}$. The introduction of the rescaled variable $\xi = x/h^{2/3}$ corresponds to a change of coordinates from (x, h) to (ξ, h), outside of $\{h = 0\}$. Thus, the behavior in the *transition layer*, i.e. in $\{x = 0\}$, as $h \to 0$, corresponds to the behavior of $\lim_{h\to 0} u(\xi, h) = u(\xi, r)$, for any $r > 0$, since u is constant along the curves $\{\xi = c\}$, $c \in \mathbb{R}$. The ambiguity of $(\xi, 0)$ corresponding to the origin in $\mathbb{H}$ for all $\xi \in \mathbb{R}$ can be resolved by the introduction of a quasihomogeneous blow-up

$$\beta\colon\ [\mathbb{H}, 0]_{2/3} \to \mathbb{H}\,,$$

where $(\xi, 0)$ correspond to different points on the front face $\beta^{-1}(0)$ for different $\xi \in \mathbb{R}$. The transformed operator $h^{2/3}(-\partial_\xi^2 + \xi) + O(h)$ is nothing but the asymptotic expansion of the pullback of $P = -h^2\partial_x^2 + x$ via β at the front face. The WKB-approximations on the other hand are asymptotic solutions at either the left and right face, which are the lift of the outer regimes to the blown-up half space. Interestingly, the asymptotic power series

$$x^{1/(4c)} \sum_{k=0}^{\infty} \alpha_k x^{-3/2k} h^k = r^{1/(2c)} \sum_{k=0}^{\infty} \alpha_k \eta^k \,,$$

is polyhomogeneous at $\{r = 0\}$ with respect to the induced coordinates $\eta = h/x^{3/2}$ and $r = \sqrt{x}$ at the corner of front and right face. The matching in the method of matched asymptotic expansion then corresponds to the choice of solutions at either face for which there exists an extension from the boundary faces to the interior of $[\mathbb{H}, 0]_{2/3}$ by the Borel lemma.

3 Regular Operators

In this chapter we lay out the basic principles in the analysis of semi-classical operators. After giving a definition of *generalized* semi-classical operators on an interval I, which is the main object of this thesis, our focus in the beginning is on operators with non-vanishing leading coefficients. As it will turn out, these operators can be reduced to regular transport systems, i.e. families of recurrent differential equations with corresponding elliptic transport operators, by the choice of an appropriate *WKB*-ansatz.

The summands in the expansion of a generalized semi-classical operator $P = P(h)$ can be ordered both in powers of h, for $h > 0$, and order of differentiation. This allows for a Newton polygon analysis of the operator in these two variables in Section 3.2. Similar to the classical application of Newton polygons for algebraic curves in the plane, one is able to relate the edges of the polygon $\mathcal{P}(\Lambda)$ to polynomials $E_\delta(P)(x,\zeta)$ with smooth coefficients, which is the key finding of this chapter. For $x \in I$ these polynomials can be used to express appropriate eikonal equations

$$E_\delta(P)(x, \varphi'(x)) = 0\,,$$

linking combinatorial data to a suitable, generalized *WKB*-ansatz $u = e^{\varphi/h^\delta} A$ for each edge of the polygon with slope δ.

The chapter ends with a discussion about the range of applicability of this Newton polygon analysis. In particular, it will display sufficient requirements for the solvability of eikonal equations and their induced transport system with smooth solutions. Eventually, we are able to prove a regular version of the main theorem, Theorem 3.3.9 in Section 3.3.1. Additionally assuming δ-separation of P as in Definition 3.3.3, we can state the theorem in a simpler version.

Theorem. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(I)$. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Let $\varphi \in C^\infty(I)$ be a simple, non-trivial solution of $E_\delta(P)(\cdot, \varphi') = 0$ on I. Assume that P is δ-separated. Let T be the leading operator in the expansion of $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$ as $h \to 0$.*

Then there is a quasimode $u = e^{\varphi/h^\delta} A$, with $A := \sum_{k=0}^{\infty} a_k h^{\beta_k}$, where $a_k \in C^\infty(I)$ and $a_0 \in \ker T$ with $a_0 \not\equiv 0$, such that

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} A = \mathcal{O}(h^\infty)\,.$$

With minor adjustments in Theorem 3.3.11 one can show that there is a basis of independent quasimodes for semi-classical operators, if it satisfies a condition called δ-regularity.

3.1 Generalized Semi-Classical Operators

In this section we introduce the central objects of this thesis, the so called *generalized semi-classical operators* and their associated *sets of exponents*. The latter one will allow us to reduce the analysis of perturbed differential operators to combinatorial problems in parts. This section will be exploratory and driven by examples, emphasizing some of the basic problems of quasimode construction. Noteworthily, in Subsection 3.1.2 we will present Proposition 3.1.7 about the existence of quasimodes of so called *regular* perturbations. It takes an essential role in the construction of quasimodes in Theorem 3.3.9. Finally, we will introduce a generalized *WKB*-ansatz for the ad-hoc construction of quasimodes with exponential behavior of some generalized semi-classical operators.

3.1.1 Generalized Semi-Classical Operators

Definition & Remarks

Definition 3.1.1 (Generalized Semi-Classical Operators)**.** Let $I \subset \mathbb{R}$ be an interval and $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a discrete set, which is bounded in its first argument and bounded from below in its second argument. For each $\lambda = (k, \alpha) \in \Lambda$ let $a_\lambda \in C^\infty(I)$ be not identically zero on I and have only finite order zeros.

The formal operator sum defined by

$$P := \sum_{\substack{\lambda\in\Lambda \\ \lambda=(k,\alpha)}} a_\lambda h^\alpha \partial_x^k\,, \tag{3.1}$$

is called a *generalized semi-classical operator* with respect to Λ. The set $\Lambda = \Lambda(P)$ is called *set of exponents* of P. The *space of generalized semi-classical operators* with respect to Λ on I is defined by

$$\mathrm{Diff}^{\Lambda}(I) := \left\{ \sum_{\substack{\lambda=(k,\alpha) \\ \lambda\in\Lambda'}} a_\lambda h^\alpha \partial_x^k \;:\; \Lambda' \subset \Lambda,\ \text{and } a_\lambda \in C^\infty(I),\ \text{for each } \lambda \in \Lambda' \right\}. \tag{3.2}$$

Remark 3.1.2. Since we require coefficient functions of $P = \sum_\lambda a_\lambda h^\alpha \partial_x^k$ to be non-identically zero, the sum of two generalized semi-classical operators P, Q with $\Lambda(P) = \Lambda(Q) = \Lambda$ can possibly have smaller sets of exponents $\Lambda(P+Q) \subset \Lambda$. In particular, if $Q = -P$ then $\Lambda(P+Q) = \emptyset$. Thus, $\mathrm{Diff}^{\Lambda}(I)$ in (3.2) is the smallest vector space containing all operators P with $\Lambda(P) = \Lambda$.

Note that this class of operators with smooth coefficients having finite order zeros is larger than the classes of semi-classical operators often considered in the literature.

The quasimodes we will construct in this section are not functions of (x, h) on the combined space $I \times \mathbb{R}_+$ but rather asymptotic series with respect to the parameter h. Thus, the most natural class of operators is given by asymptotic series of classical operators with respect to h itself. One can think of these as asymptotic expansions of $\sum_{k=0}^n A_k(x,h)\partial_x^k$, $n \in \mathbb{N}_0$, as

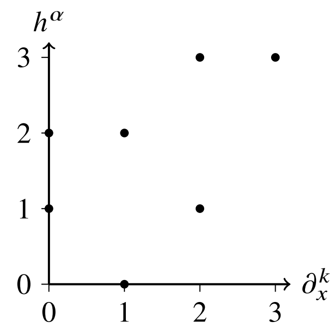

Figure 3.1: The set of exponents $\Lambda = \Lambda(P) = \{(0,1),(0,2),(1,0),(1,2),(2,1),(2,3),(3,3)\}$ of the operator P in Example 3.1.4.

$h \to 0$, where each A_k is smooth in its first and polyhomogeneous in its second argument. Thus, these are naturally contained in this setting. In particular, standard semi-classical operators are generalized semi-classical operators with sets of exponents bounded from below by $\Delta_{\mathbb{R}^2_+} = \{(x,x) : x \geq 0\}$.

Examples

Example 3.1.3. The very first example of a generalized semi-classical operator on an interval $I \subset \mathbb{R}$ is given by the *Schrödinger* operator

$$P := -h^2\partial_x^2 + V,$$

with a potential $V \in C^\infty(I)$ as in Subsection 2.4.1. We can directly compute its associated set of exponents $\Lambda = \Lambda(P) = \{(0,0),(2,2)\}$. Since its set of exponents $\Lambda \subset \Delta_{\mathbb{R}^2_+}$ is contained in the diagonal of $\mathbb{R}^2_+$, P is a semi-classical operator.

We have already analyzed the Schrödinger operator in Section 2.4, whose quasimodes' behavior is heavily dependent on the sign of the potential V. The classical *WKB*-ansatz $u = e^{i\varphi/h}A$ of quasimodes with oscillatory-polyhomogeneous behavior, where $\varphi \in C^\infty(I)$ and $A \in \mathcal{A}(I \times \mathbb{R}_+)$, relies on the fact that $\Lambda \subset \Delta_{\mathbb{R}^2_+}$. As a consequence, the lowest order term with respect to h of

$$Pe^{i\varphi/h}A = e^{i\varphi/h}\left((\varphi')^2 + V + O(h)\right)A \tag{3.3}$$

vanish if and only if the eikonal equation

$$(\varphi')^2 + V = 0$$

is satisfied. Allowing for more points $\lambda \in \Lambda$ contained in or even outside of the diagonal $\Delta_{\mathbb{R}^2_+}$ increases the difficulty of constructing quasimodes significantly.

Example 3.1.4. Let $I \subset \mathbb{R}$ be an interval and define

$$P := h^3x^3\partial_x^3 + (hx + h^3x^2)\partial_x^2 + (1 + h^2)\partial_x + h(1 + hx)\,.$$

The operator P is a generalized semi-classical operator and its associated set of exponents is given by $\Lambda = \Lambda(P) = \{(0,1),(0,2),(1,0),(1,2),(2,1),(2,3),(3,3)\}$ (see Figure 3.1). In particular, P is not a semi-classical operator since $(1,2),(2,1) \in \Lambda$ are not contained in $\Delta_{\mathbb{R}_+^2}$.

It is not obvious how or even if one is able to construct quasimodes for P due to multiple reasons at once. Since P lacks parity between the powers of h and the order of differentiation, it is unclear if one can reestablish a promising *WKB*-type ansatz for the construction of quasimodes. Also many of the coefficients of P vanish at $\{x = 0\}$ which might impose additional complications. Having the Schrödinger operator with linear potential in mind, the question arises whether one is able to find a quasimode on I itself or whether one has to resolve $I \times \mathbb{R}_+$ in a suitable sense by the introduction of blow-ups.

3.1.2 Polyhomogeneous Quasimodes

The easiest case of generalized semi-classical operators is given by a regular perturbation. These are formal sums of differential operators $P = h^\alpha T + o(h^\alpha)$, $\alpha \in \mathbb{R}$, whose lowest order term $T \in \mathrm{Diff}(I)$ with respect to h is elliptic. Taking any element $u_0 \in \ker T$ yields a *first order* quasimode, i.e. $Pu_0 = Tu_0 + fh^\beta + o(h^\beta) = fh^\beta + o(h^\beta)$. Since T is elliptic, one can find a function u_1 such that Tu_1 eliminates the new, highest order remainder f of $Pu_0 = fh^\beta + o(h^\beta)$ by extending the quasimode to $u_0 + u_1h^{\beta-\alpha}$. Applying P to this yields an expansion

$$P(u_0 + u_1h^{\beta-\alpha}) = h^\alpha T(u_1h^{\beta-\alpha}) + fh^\beta + o(h^\beta) = h^\beta\,(Tu_1 + f) + o(h^\beta)\,.$$

Thus, u_1 has to satisfy $Tu_1 = -f$. The function $u = u_0 + u_1h^{\beta-\alpha}$ is a second order quasimode of P. The recurrent equations of the form $Tu_k = Ru_j$, $j < k$, are called inhomogeneous *transport equations*.

This subsection's goal is to prove the essential result that one can construct arbitrarily good quasimodes u for any generalized semi-classical operator $P = h^\alpha T + o(h^\alpha)$, $\alpha \in \mathbb{R}$, with surjective leading term T by iteratively solving transport equations. Since $h \geq 0$ is a parameter commuting with any generalized semi-classical operator $P = P(h)$, multiplying any quasimode u of P with a polynomial $p \in \mathbb{C}[h]$ yields a new quasimode $p \cdot u$ of P. Thus, we need to specify a new notion of *independent* quasimodes.

Definition

Definition 3.1.5 (Independent Quasimodes)**.** Let $\Gamma \subset T^*_{\mathbb{C}}I$ be a union of projectible submanifolds, $\mathcal{I}$ be a family of index sets corresponding to the leaves of Γ and $\mathcal{F} \subset \mathcal{EA}^{\mathcal{I}}(I \times \mathbb{R}_+;\Gamma)$ be a subset of the space of exponential-polyhomogeneous functions.

$\mathcal{F}$ is called *independent*, if the following holds:

(i) For all $\delta > 0$ and $\varphi' \in C^\infty(I)$ the set

$$\left\{a_0 \in C^\infty(I) : \; e^{\varphi/h^\delta} a \in \mathcal{F} \text{ with } a = a_0 h^\alpha + o(h^\alpha)\right\}$$

is linearly independent over $C^\infty(I)$.

(ii) $\{a_0 \in C^\infty(I) : \; a \in \mathcal{F} \text{ with } a = a_0 h^\alpha + o(h^\alpha)\}$ is linearly independent over $C^\infty(I)$.

It is important to note that Definition 3.1.5 implicitly says that any two phase functions φ_1, φ_2 are equivalent, in the sense that $\{e^{\varphi_1/h^\delta} a, e^{\varphi_2/h^\delta} a\}$ is linearly dependent for every $\delta > 0$, if $(\varphi_1 - \varphi_2)' = 0$. We will refer to these constant phase functions as *trivial*.

Remark 3.1.6. The second case contains exponential behavior of the form e^{φ/h^δ}, $\delta \leq 0$, which is not singular as $h \to 0$. Let $\delta \leq 0$, $\varphi \in C^\infty(I)$ and $a = a_0 h^\alpha + o(h^\alpha)$ be polyhomogeneous. Then $e^{\varphi/h^\delta} a$ itself is polyhomogeneous with leading part a_0, since $e^{\varphi/h^\delta} = 1 + o(1)$.

Statements & Remarks

The following proposition is the central tool in the construction of quasimodes of generalized semi-classical operators. In Section 3.3 we will show that one can reduce the construction of exponential-polyhomogeneous quasimodes to this proposition with the techniques developed in Section 3.2.

Proposition 3.1.7. *Let V, W be vector spaces, $I \subset \mathbb{N}$ be a family of indices and for $k \in I$ let $R_k : V \to W$ be linear and $P := T + \sum_{k \in I} h^{\alpha_k} R_k$, where $\{\alpha_k : k \in I\} \subset \mathbb{R}_{>0}$ is discrete. Suppose that T is surjective.*

Then for each $u_0 \in \ker T$ there is a formal sum $u = \sum_{k \in J} u_k h^{\beta_k}$ and $J \subset \mathbb{N}$, with $u_k \in V$ for each $k \in J$ and $\{\beta_k : k \in J\} \subset \mathbb{R}_{>0}$ discrete, such that

$$Pu = O(h^\infty) .$$

Proof. Without loss of generality assume that $\ker T \neq \{0\}$ and $\alpha_k < \alpha_{k+1}$ for all $k \in I$. Denote $R(h) := P - T$, let $u_0 \in \ker T \setminus \{0\}$ and $\beta_0 := 0$. Then $Pu_0 = Tu_0 + O(h^{\alpha_1}) = O(h^{\alpha_1})$ since $R(h) = O(h^{\alpha_1})$. Then $R(h)u_0 \in \bigoplus_k h^{\alpha_k} W$. Since T is surjective there is a $u_1 \in C^\infty(\mathbb{R})$, $u_1 \neq 0$, s.t.

$$Tu_1 = -R_1 u_0 ,$$

with $\beta_1 := \alpha_1$. Note that $R - h^{\alpha_1} R_1 = o(h^{\beta_1})$. Thus, for $u := u_0 + h^{\beta_1} u_1$, it follows that

$$Pu = Tu_0 + R(h)u_0 + T\left(u_1 h^{\beta_1}\right) + R(h)\left(u_1 h^{\beta_1}\right) = 0 + (R(h) - R_1)u_0 + h^{\beta_1} R u_1 = o(h^{\beta_1}) .$$

Proceeding this way, we can inductively construct a quasimode $u_{(N)} = \sum_{k=0}^N u_k h^{\beta_k}$ with $u_k \in W$, such that

$$Pu_{(N)} = O\left(h^{\beta_N}\right) .$$

What remains to be shown is that $(\beta_n)_{n\in\mathbb{N}}$ itself is discrete. To prove that, we will use a different order of solving the inhomogeneous transport equations, possibly risking redundancies. Let $N \in \mathbb{N}$ be arbitrary. We claim that there is an $M \in \mathbb{N}$ such that $\beta_M \geq N$.

Since $(\alpha_n)_{n\in I}$ is discrete there is a $m_0 \in \mathbb{N}$ such that $\alpha_{m_0} \leq N < \alpha_{m_0+1}$, where we define $\alpha_n := +\infty$, for $n \notin I$. Thus, we can split the formal series of operators

$$P = T + \sum_{k=1}^{m_0} h^{\alpha_k} R_k + O(h^N)$$

into three parts: the transport operator T, all terms with powers of h greater than N and a finite sum in between. Thus, applying P to u_0 yields at most finitely many remainder terms $f_{0,j} := R_j u_0$ on the right hand side,

$$Pu_0 = \sum_{k=1}^{m_0} f_{0,j} h^{\alpha_j} + O(h^N)\,,$$

with powers of h less that N. On the other hand, we have $\sum_{k=1}^{m_0} f_{0,j} h^{\alpha_j} = O(h^{\alpha_1})$, since $\alpha_j \geq \alpha_1 > 0$. We simultaneously solve all inhomogeneous transport equations corresponding to $f_{0,j}$, i.e.

$$Tu_{1,j} = f_{0,j}\,,$$

and sort the collection of remainder terms $f_{1,j} := R_j u_{1,j}$ in the expansion

$$P\left(u_0 + \sum_{j=1}^{m_0} u_{1,j} h^{\alpha_j}\right) = R(h)\left(\sum_{j=1}^{m_0} u_{1,j} h^{\alpha_j}\right) = \sum_{j=1}^{m_1} f_{1,j} h^{\alpha_{1,j}} + O(h^N)\,,$$

where $m_1 := m_0^2$. Most importantly, $\alpha_{1,j} \geq 2\alpha_1$, since $R(h) = O(h^{\alpha_1})$. Thus, repeating this process $n := \lfloor N/\alpha_1 \rfloor + 1$ times, it follows that

$$P\left(u_0 + \sum_{k=1}^{n}\sum_{j=1}^{m_0} u_{k,j} h^{\alpha_{k,j}}\right) = \sum_{j=1}^{m_{n+1}} f_{m_{n+1},j} h^{\alpha_{n+1,j}} + O(h^N) = O(h^N)\,,$$

since $\alpha_{n+1,j} \geq (n+1)\alpha_1 \geq N$, for all $j = 1, \ldots, m_{n+1}$. Ordering $u_{k,j}$ in by their exponents with respect to h and relabeling $\alpha_{k,j}$ accordingly in successive order by β_k yields that after solving at most $M := 1 + \sum_{k=1}^{n} m_k = \sum_{k=0}^{n} m_0^k$ inhomogeneous equations in the indicated order, we have that

$$Pu_M = O\left(h^N\right)\,.$$

In particular, all further additions to u_M will be of order $o(h^{\beta_M})$ and $\{\beta_k : k \in J\} \subset \mathbb{R}_{>0}$ is discrete. □

Remark 3.1.8. Due to the alternative order in which we have solved the inhomogeneous transport equations in the proof of Proposition 3.1.7 the exponents $\alpha_{k,j}$ are in ascending order in j for each fixed k, but are unordered along k for each fixed j. In general it is possible that $\alpha_{k,j} = \alpha_{l,i}$ for $(k, j) \neq (l, i)$, which does not violate the proof. This method of counting will occur repeatedly throughout this thesis.

Corollary 3.1.9. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}_+$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(I)$, where $P = T + \sum_{k=0}^{\infty} h^{\alpha_k} R_k$. Assume that T is elliptic and denote $d := \deg T$.*

Then there are independent quasimodes $u_j = \sum_k u_{j,k} h^{\beta_{j,k}}$, $j = 1, \dots, d$, with $u_{j,0} \in \ker T$ and $u_{j,k} \in C^\infty(I)$.

Proof. Let T be elliptic and $d := \deg T$. Then by standard ODE theory it is surjective as an operator from $C^\infty(I) \to C^\infty(I)$ and $\dim \ker T = d$. Let $u_{j,0}$, $j = 1, \dots, d$ be a basis of $\ker T$. Then we can apply Proposition 3.1.7 for each $u_{j,0}$. □

Definition 3.1.10 (Transport Operator)**.** For a semi-classical operator

$$P = T + \sum_{k=0}^{\infty} h^{\alpha_k} R_k \,, \tag{3.4}$$

with $\alpha_k > 0$, we call its leading term T with respect to h the *transport operator* associated to P. Any equation of the form $Tu = 0$ or $Tu = -R_k f$, for any given $f \in C^\infty(I)$ and $k \in \mathbb{N}$, is called *(inhomogeneous) transport equation*.

Remark 3.1.11. Note that Λ being a set of exponents already implies that $(\alpha_n)_{n\in\mathbb{N}}$ is strictly increasing and unbounded. This is due to the discreteness of Λ.

Examples

Example 3.1.12. Revisiting Example 3.1.3 with $P = -h^2\partial_x^2 + V$, let $\varphi_\pm$ be the solutions of $(\varphi')^2 + V = 0$. Thus, expanding (3.3) completely yields

$$e^{-i\varphi_\pm/h} P e^{i\varphi_\pm/h} = h\left(-2i\varphi'_\pm \partial_x - i\varphi''_\pm\right) + h^2\left(-\partial_x^2\right) .$$

If $V(x) \neq 0$ for all $x \in I$ then $\varphi'_\pm(x) = \sqrt{-V(x)} \neq 0$ for all $x \in I$. In particular, the transport operator T of the conjugated operator $\exp(-i\varphi/h)P\exp(i\varphi/h)$, given by

$$T := -2i\varphi'_\pm \partial_x - i\varphi''_\pm$$

is an elliptic first order differential operator. Thus, we can apply Corollary 3.1.9 and obtain a polyhomogeneous quasimode $u_\pm$ of $h\left(-2i\varphi'_\pm\partial_x - i\varphi_\pm\text{''}\right) + h^2\left(-\partial_x^2\right)$ for each phase function $\varphi_\pm$. Thus,

$$\{e^{\varphi_\pm} u_\pm\}$$

is a set of independent quasimodes, since $\varphi_+ \neq \varphi_-$. Note that the amount of independent quasimodes matches the degree of P.

Remark 3.1.13. The application of Corollary 3.1.9 coincides with the procedure presented in Subsection 2.4.1, where we displayed how to construct a quasimode by solving the inhomogeneous transport equation iteratively.

Example 3.1.14. Let $P := -h^2\partial_x^2 + (\partial_x - V)$ on $I \subset \mathbb{R}$ and $V \in C^\infty(I)$. The set of exponents is given by $\Lambda = \{(0,0),(1,0),(2,2)\}$ and thus P is not a standard but a generalized semi-classical operator. Even though P is similar to the Schrödinger operator, the leading term $T := \partial_x - 1$ of P with respect to h is elliptic and has degree $\deg T = 1$. Thus, we can apply Corollary 3.1.9 and obtain a quasimode u with leading term $u_0 = \exp$.

This result is not satisfying in its current state. While we are able to create some polyhomogeneous quasimodes if $\deg T \neq 0$, the number of quasimodes $\deg T$ obtained this way is lower than the degree of P in general. Determining other potential quasimodes with exponential-polyhomogeneous behavior as in the standard semi-classical case will be the central task of the upcoming subsection.

3.1.3 Exponential Behavior

To obtain the missing quasimode in Example 3.1.14 corresponding to the mismatch of $\deg P$ and $\deg T$ we have to approach generalized semi-classical operators in an adjusted way compared to Section 2.4. In the case of the Schrödinger operator with strictly negative potential $V \in C^\infty(I)$, $V < 0$,

$$P := -h^2\partial_x^2 + V\,,$$

we applied the *WKB*-ansatz in Examples 3.1.3 and 3.1.12 to obtain two quasimodes. The most obvious justification for this ansatz is shown in the case where V is constant. Then the solutions of $Pu = 0$ can be computed directly and are given by $u_\pm = \exp(\pm\sqrt{V}x/h)$. The WKB-ansatz extrapolates this in the anticipation that for general potentials $V \in C^\infty(I)$ solutions u of $Pu = 0$ are asymptotically of the form $u = e^{\varphi/h}A$ for some phase function $\varphi \in C^\infty(I)$ and amplitude $A \in C_h^\infty(I)$. The *generalized WKB*-ansatz takes this ansatz and generalizes it, anticipating that any potential quasimode u of P can have exponential behavior $u = e^{\varphi/h^\delta}A$, where $A \in C_h^\infty(I)$, $\varphi \in C^\infty(I)$ and arbitrary $\delta \in \mathbb{R}$.

Since the exponential factor is restored after differentiation we can conjugate $P = -h^2\partial_x^2 + V$ with e^{φ/h^δ} and obtain

$$e^{-\varphi/h^\delta}Pe^{\varphi/h^\delta} = -h^2\left(\left(\frac{\partial_x\varphi}{h^\delta}\right)^2 + 2\left(\frac{\partial_x\varphi}{h^\delta}\right)\partial_x + \left(\frac{\partial_x^2\varphi}{h^\delta}\right) + \partial_x^2\right) + V\,. \tag{3.5}$$

If $\delta < 0$ then all newly generated summands are relatively small compared to $-h^2\partial_x^2$ and will not impact any initial solution. The case of $\delta = 0$ only has impact on ∂_x^2, transforming it into a more general second order differential operator $\partial_x^2 + 2\varphi'\partial_x + (\varphi')^2 + \varphi''$. Choosing $\delta > 0$, on the other hand, has a significant impact on the structure of (3.5). Due to the higher multiplicity of φ', the two lowest terms of the conjugated operator are given by

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} = -h^{2-2\delta}(\varphi')^2 + V + O\left(h^{2-\delta}\right) ,$$

with parity of exponents 0 and $2-2\delta$ of h of the first two terms gained at $\delta = 1$. The lowest order term $\widetilde{T}$ of $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$ with respect to $\delta \in \mathbb{R}$ is given by

$$\widetilde{T} = \begin{cases} V\,, & \text{if } \delta < 1\,, \\ -(\varphi')^2 + V\,, & \text{if } \delta = 1\,, \\ -(\varphi')^2\,, & \text{if } \delta > 1\,, \end{cases}$$

which is a multiplication operator in all of these cases. Hence, we have $\deg T = 0$, which means that the only chance of obtaining quasimodes is by erasing the multiplication operator as whole. This is only possible in the cases of $\delta = 1$ and $\delta > 1$ by an appropriate choice of φ satisfying

$$\begin{cases} (\varphi')^2 = V\,, & \text{if } \delta = 1\,, \\ (\varphi')^2 = 0\,, & \text{if } \delta > 1\,. \end{cases}$$

Since V was chosen to be strictly negative in the beginning of Subsection 3.1.3, any solution $\varphi = \pm \int i\sqrt{-V}$ in the case of $\delta = 1$ takes only imaginary values and thus $e^\Phi = e^{\varphi/h^\delta}$ is oscillating as $h \to 0$. In the case of $\delta > 1$, all solutions φ of $\varphi' = 0$ are trivial. In particular, $e^{\varphi/h}A$ is not independent from A and $e^{-\varphi/h}Pe^{\varphi/h} = P$. When $\delta = 1$ and $\varphi = \int i\sqrt{-V}$, the lowest order term $T = (-2\varphi'\partial_x - \varphi'')$ of $e^{-\varphi/h}Pe^{\varphi/h}$ was already computed in Example 3.1.12.

Allowing for arbitrary powers $\delta > 0$ in the ansatz will turn out to be extremely fruitful, especially when it is analyzed in the context of sets of exponents. Before we continue following the approach described in Section 2.4, we want to illustrate the applicability of the *generalized WKB-ansatz* with some slightly more complicated examples of generalized semi-classical operators.

Examples

The easiest operator where we can apply the generalized *WKB*-ansatz is the direct extension of the Schrödinger operator, where we allow for any positive power $\gamma > 0$ of h in front of the derivative.

Example 3.1.15. Let $I \subset \mathbb{R}$ and define $P := -h^{2\gamma}\partial_x^2 + V$, for some $V \in C^\infty(I)$. Its associated set of exponents $\Lambda = \Lambda(P) = \{(0,0),(2,2\gamma)\}$ is contained in the diagonal if and only if $\gamma = 1$ and thus it is a generalized semi-classical operator. By applying a generalized *WKB*-ansatz with e^{φ/h^γ}, one can directly and analogously repeat the computations in Examples 3.1.3 and 3.1.12. On the other hand, by defining $\widetilde{h} := h^\gamma$, this is exactly the same setting as in the mentioned examples. Hence its quasimodes are of the form

$$u_{\pm} = e^{\pm i \int \sqrt{-V}/\widetilde{h}} A = e^{\pm i \int \sqrt{-V}/h^{\gamma}} A \, .$$

We revisit Example 3.1.14, where we have computed a quasimode for $P = -h^2\partial_x^2 + (\partial_x - 1)$, and try to compute the missing quasimode using the generalized *WKB*-ansatz.

Example 3.1.16. Let $I \subset \mathbb{R}$ and define $P := -h^2\partial_x^2 + (\partial_x - 1)$. Since its highest order derivative also vanishes as $h \to 0$ compared to every other term of P, we expect there to be a quasimode obtainable by the use of the generalized *WKB*-method. Applying it, the lowest order term of

$$e^{-\varphi/h^{\delta}} P e^{\varphi/h^{\delta}} = -h^{2-2\delta}(\varphi')^2 - h^{2-\delta}(2\varphi'\partial_x + \varphi'') - h^2\partial_x^2 + h^{-\delta}\varphi' + \partial_x - 1$$

is given by

$$\widetilde{T} = \begin{cases} \varphi' \, , & \text{if } 0 < \delta < 2 \, , \\ -(\varphi')^2 + \varphi' \, , & \text{if } \delta = 2 \, , \\ -(\varphi')^2 \, , & \text{if } \delta > 2 \, . \end{cases}$$

Thus, for the value $\delta = 2$ there is a WKB-ansatz whose leading operator vanishes with a non-trivial phase $\varphi(x) = x$. Again, following Equation 3.5 with $\delta = 2$ and $\varphi(x) = x$ the leading operator in the expansion of $e^{-\varphi/h^2} P e^{\varphi/h^2}$ is given by

$$T = (1 - 2\varphi')\partial_x - (1 + \varphi'') = -\left(\partial_x + 1\right) \, .$$

Since this is an elliptic operator we can apply Proposition 3.1.7 and obtain a quasimode $u = e^{x/h^2} A(x, h)$ with $A \in C_h^{\infty}(I)$.

3.2 Combinatorial Geometry I: Newton Polygons

The observations made in Subsection 3.1.3 allowing for the construction of exponential-polyhomogeneous quasimodes are vague at this stage. The aim of this section is to work through these observations systematically to establish a framework in which we can see a priori in what form, if any, quasimodes exist for any operator. In the core of this section, we will present the notion of *δ-symbols* $\Sigma_\delta(P)$ in Definition 3.2.3, a family of symbols generalizing the standard semi-classical symbol, whose leading term $E_\delta(P)$ can be used to express eikonal equations. These symbols can be used efficiently with the introduction of *Newton polygons* for semi-classical operators in Definition 2.3.2. Newton polygons allow an analysis of generalized semi-classical operators based on their combinatorial data in form of the set of exponents. This results in Proposition 3.2.15 using the Newton polygon to determine the relevant values δ for $\Sigma_\delta(P)$. Eventually, these values of δ coincide with the relevant values for the generalized WKB-ansatz in Section 3.1.

3.2.1 Semi-Classical δ-Principal Symbol

Motivation

Revisiting Example 3.1.15 with its operator $P = -h^{2\gamma}\partial_x^2 + V$, the successful application of a generalized *WKB*-ansatz of the form e^{φ/h^δ} with $\delta = \gamma$ was not far-fetched. The commutator of the vector field ∂_x and the exponential term

$$\left[\partial_x, e^{\varphi/h^\delta}\right] = h^{-\delta} \cdot \varphi' e^{\varphi/h^\delta} \tag{3.6}$$

has a decreased homogeneity with respect to h if and only if $\delta > 0$. Since P is a second order differential operator it appears quadratically exactly once. Thus, one only needs to choose $\delta \in \mathbb{R}$ such that the powers associated to the squared commutator and the potential V match, i.e.

$$2\gamma - 2\delta = 0\,.$$

The operator $P = -h^2\partial_x^2 + (\partial_x - V)$ in Example 3.1.16 with the value $\delta = 2$ has proven to be successful for the generalized WKB-ansatz e^{φ/h^δ}. The main concern in this example was the differential term ∂_x in the middle, which will generate an additional commutator whose associated power $-\delta < 0$ is negative, in particular when choosing $\delta = 1$ such that $(\varphi')^2$ and V have the same homogeneity 0 in h. Thus, the only possibility of leveling different leading powers of h can occur between the terms which arise from $h^2\partial_x^2$ and ∂_x,

$$e^{\varphi/h^\delta}\left(-h^{2-2\delta}(\varphi')^2 + h^{-\delta}\varphi'\right) \tag{3.7}$$

which happens for $\delta = 2$. In general any summand of order n of a generalized operator $P \in \mathrm{Diff}^{\Lambda}(I)$ will generate $(n+1)$ summands after conjugation with e^{φ/h^δ}. A direct computation in (2.15) shows that

$$e^{-\varphi/h^\delta}\partial_x^n e^{\varphi/h^\delta} = h^{-n\delta}(\varphi')^n + h^{-(n-1)\delta}\left[n(\varphi')^{n-1}\left(\partial_x + \frac{n-1}{2}\frac{\varphi''}{\varphi'}\right)\right] + O\left(h^{-(n-2)\delta}\right)\,.$$

the lowest of these terms with respect to h is the n-th power of the phase function's first derivative φ'. Equation 3.6 shows that the conjugation of $P = p(x, h, \partial_x)$ can be expressed again using its symbol

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} = p\left(x, h, \partial_x + \frac{\varphi'}{h^\delta}\right)\,.$$

In particular, Equation 2.15 shows that its leading term

$$p\left(x, h, \partial_x + \frac{\varphi'}{h^\delta}\right) = h^{l_\delta} E_\delta(P)(x, \varphi') + o(h^{l_\delta})\,, \tag{3.8}$$

is always given by an element $E_\delta(P) \in C_h^\infty(I)[\zeta]$, for some value $l_\delta \in \mathbb{R}$. This is important since all higher terms in the expansion of $p(x, h, \partial_x + \varphi'/h^\delta) = \widetilde{p}(x, h, \partial_x, \varphi')$ are depending on two non-commutating variables ζ and ξ, evaluated at $\zeta = \varphi'$ and $\xi = \partial_x$. We want to keep track for both the power of φ' and the order of differentiation of each term in $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$ independently. An important commutator relation between vector fields and functions was already given in (3.6), saying that $[\partial_x, f] = f'$ for any function $f \in C^\infty(I)$. In particular, the commutator of ∂_x and f is an element of $C^\infty(I)$ itself, hence the iterated commutator

$$[[\partial_x, f], g] = 0\,, \tag{3.9}$$

vanishes identically, for any $g \in C^\infty(I)$. This is reflected in our definition of symbol algebras.

Definitions & Properties

Definition 3.2.1 (Semi-Classical Symbol Algebra)**.** Let $I \subset \mathbb{R}$ be an interval and $\mathcal{A}_h(I) \coloneqq C_h^\infty(I)\langle \xi, \zeta \rangle$ be the non-commutative, associative, unital algebra over $C_h^\infty(I)$ generated by ξ and ζ. Let $\mathcal{I} \leq \mathcal{A}$ be the two-sided ideal generated by $[[\xi, \zeta], \zeta]$.

Then we call

$$\mathcal{S}_h(I) \coloneqq {}^{\mathcal{A}_h(I)}\!/_{\mathcal{I}}$$

the *semi-classical symbol algebra* over I.

We need to check if the (re-)substitution of $\xi = \partial_x$ and $\zeta = \varphi'$ is well defined on the quotient space $\mathcal{S}_h(I)$.

Lemma 3.2.2. *Let $I \subset \mathbb{R}$ be an interval, $f \in C^\infty(I)$ and $\mathfrak{v} \in \mathfrak{X}(I)$. Then the associated evaluation homomorphism*

$$\iota_{f,\mathfrak{v}} \colon \mathcal{A}_h(I) \to C_h^\infty(I) \otimes \mathit{Diff}(I)\,,$$

defined by the basis element mappings

$$1 \mapsto 1\,, \quad \xi \mapsto \mathfrak{v}\,, \quad \zeta \mapsto f\,,$$

descends to a map

$$\iota_{f,\mathfrak{v}} \colon \mathcal{S}_h(I) \to C_h^\infty(I) \otimes \mathit{Diff}(I)\,.$$

Proof. This is an immediate consequence of $\iota_{f,\mathfrak{v}|_{\mathcal{I}}} \equiv 0$ since

$$\iota_{f,\mathfrak{v}}([[\xi, \zeta], \zeta]) = [[\mathfrak{v}, f], f] = 0$$

by (3.9). □

For convenience, we denote the equivalence class of $a \in \mathcal{A}_h(I)$ in $\mathcal{S}_h(I)$ by a again and for any $a \in \mathcal{S}_h(I)$ we use the short notation $a(h, x, \mathfrak{v}, f) \coloneqq (\iota_{f,\mathfrak{v}})(h, x)$.

The idea of the upcoming definition of semi-classical δ-symbol is to map generalized semi-classical operators $P = p(x, h, \partial_x)$ to the non-commutative polynomial $p(x, h, \xi+\zeta/h^\delta)$. This way the full δ-symbol Σ_δ distinguishes between differential terms ∂_x and contributions of the commutator in (3.6) of the conjugated operator $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$.

Definition 3.2.3 (Semi-Classical δ-Symbol)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents, $\delta > 0$ and $P \in \mathrm{Diff}^\Lambda(I)$ be a generalized semi-classical operator, $P = \sum_{\lambda\in\Lambda} a_\lambda h^\alpha \partial_x^k$.

The *full δ-symbol* Σ_δ is the map

$$\begin{aligned} \Sigma_\delta \colon \mathrm{Diff}^\Lambda(I) &\to \mathcal{S}_h(I) \\ P &\mapsto \sum_{\substack{\lambda\in\Lambda \\ \lambda=(k,\alpha)}} a_\lambda h^\alpha \left(\xi + \frac{\zeta}{h^\delta}\right)^k . \end{aligned}$$

The *δ-principal symbol* of P is the map

$$E_\delta \colon \mathrm{Diff}^\Lambda(I) \to C^\infty(I)[\zeta]$$

defined by the leading part of $\Sigma_\delta(P)$ with respect to h with associated power $l_\delta \in \mathbb{R}$, i.e.

$$\Sigma_\delta(P)(h, x, \xi, \zeta) \sim h^{l_\delta} E_\delta(P)(x, \zeta) + o(h^{l_\delta}), \quad h \to 0 . \tag{3.10}$$

Lemma 3.2.4. *Let $I \subset \mathbb{R}$ and $P \in \mathit{Diff}^\Lambda(I)$. Then we have*

$$l_\delta = \min\{\alpha - k\delta : (k, \alpha) \in \Lambda\} .$$

Proof. This is a direct consequence of $E_\delta(P)$ being the leading part of $\Sigma_\delta(P)$ and Equation 2.14. ◻

Remark 3.2.5. By definition of the full δ-symbol, we have

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} = \Sigma_\delta(P)(h, x, \partial_x, \varphi') = h^{l_\delta} E_\delta(P)(x, \varphi') + o(h^{l_\delta}) . \tag{3.11}$$

In particular, we have that the δ-symbol maps vector fields

$$h^\gamma \partial_x \mapsto h^\gamma \left(\xi + \frac{\zeta}{h^\delta}\right)$$

to its conjugated symbolic counterpart. It is noteworthy that for trivial phase functions φ the full δ-symbol

$$\Sigma_\delta(P)(h, x, \xi, 0) = \sigma(P)(h, x, \xi)$$

coincides with the classical full symbol of operators P.

Recall that, by construction, the δ-principal symbol is independent of ζ. Since it is the lowest order term with respect to h in the expansion of $\Sigma_\delta(P)(x,h,\xi,\zeta)$, we can express the eikonal equation in terms of $E_\delta(P)$ namely

$$E_\delta(P)(x,\varphi') = 0\,,$$

of the associated conjugation of P by e^{φ/h^δ} as in (3.11). The δ-principal symbol E_δ coincides with the standard semi-classical symbol for $\delta = 1$ when applied to any standard semi-classical operator $Q = q(x,h,h\partial_x)$. In particular, we have $l_1 = 0$ if $q(x,0,\xi) \not\equiv 0$. Note that in the case of generalized semi-classical operators l_δ can take arbitrary values in $\mathbb{R}$, even if $\delta = 1$.

Definition 3.2.6 (Eikonal Equation)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}\times\mathbb{R}$ be a set of exponents, $\delta > 0$ and $P \in \mathrm{Diff}^\Lambda(I)$ be a generalized semi-classical operator. Let E_δ be the δ-principal symbol.

We call $E_\delta(P) \in C^\infty(I)[\zeta]$ the *eikonal polynomial* of P with respect to δ. The first order differential equation

$$E_\delta(P)(\cdot,\varphi') = 0\,,$$

is called *eikonal equation* with respect to P and δ. Its solutions φ are called *simple*, if for each $x \in I$ the solution $\varphi'(x)$ of $E_\delta(P)(x,\varphi'(x)) = 0$ is simple. Further, a solution φ of an eikonal equation is call *trivial*, if $\varphi' \equiv 0$.

Additionally, we call $E_\delta(P)$ *trivial*, if for any solution φ' of $E_\delta(P)(\cdot,\varphi') = 0$ it follows that $\varphi' \equiv 0$.

For a generalized semi-classical operator P and almost all values of $\delta > 0$ their associated eikonal polynomials $E_\delta(P)$ are monomials in ζ, i.e.

$$E_\delta(P)(x,\zeta) = a(x)\zeta^k\,,$$

with $a \in C^\infty(I)$. In these cases the solutions $\varphi \in C^\infty(I)$ of the induced eikonal equations

$$E_\delta(P)(x,\varphi') = a(x)(\varphi'(x))^k = 0\,,$$

are trivial.

For those values of $\delta > 0$ where $E_\delta(P)$ is not trivial, the non-trivial solutions φ' of the associated eikonal equation are smooth under certain conditions. In particular, φ is smooth, too.

Proposition 3.2.7. *Let $I \subset \mathbb{R}$, $d \in \mathbb{N}$, $a_k \in C^\infty(I)$, $1 \le k \le d$, and $E \in C^\infty(I)[\zeta]$, be a polynomial of degree $d \ge 1$, $E(x,\zeta) = \sum_{k=0}^d a_k(x)\zeta^k$, with $a_d(0) \neq 0$. Assume that for $x = 0$ all roots $\zeta_{1,0},\dots,\zeta_{d,0}$ of $E(0,\cdot)$ are simple.*

Then there is a neighborhood $U \subset I$ of 0 and smooth maps $\zeta_k\colon U \to \mathbb{C}$, such that $E(\cdot,\zeta_k)_{|U} = 0$ and $\zeta_k(0) = \zeta_{k,0}$, for all $1 \le k \le d$.

Proof. Let $d \in \mathbb{N}$, $d \geq 1$, let $a_k \in C^\infty(\mathbb{R})$, $a_d(0) \neq 0$, and define $E := \sum a_k \zeta^k \in C^\infty(U)[\zeta]$. Since all solutions $\zeta_{1,0}, \ldots, \zeta_{d,0}$ of $E(0, \cdot) = 0$, the differential $(\partial_\zeta E)(0, \zeta_{k,0})$ is invertible for all $1 \leq k \leq d$. Hence we can apply the implicit function theorem which yields functions ζ_k on a maximal interval $U \subset I$ such that for all $1 \leq k \leq d$

$$E(\cdot, \zeta_k)_{|U} = 0 \,.$$

In particular, we have that $\zeta_k(0) = \zeta_{k,0}$. □

Examples

We compute the images of the full and principal δ-symbol of some known operators to show that it coincides with the standard semi-classical symbol and yields a way of approximating the missing quasimode of Example 3.1.16.

Example 3.2.8. Let $P := -h^2\partial_x^2 + V$ for a potential $V \in C^\infty(I)$ and an interval $I \subset \mathbb{R}$. The eikonal polynomial $E_\delta(P)$ of P is given by

$$E_\delta(P)(x,\zeta) = \begin{cases} V(x)\,, & \text{if } 0 < \delta < 1\,, \\ -\zeta^2 + V(x)\,, & \text{if } \delta = 1\,, \\ -\zeta^2\,, & \text{if } \delta > 1\,. \end{cases}$$

For $\delta = 1$ the full δ-symbol is given by

$$\Sigma_1(P)(x,h,\xi,\zeta) = h^0(-\zeta^2 + V(x)) + h\,(-2\zeta\xi - [\xi,\zeta]) + h^2(-\xi^2)\,,$$

where we used that $-(\zeta\xi + \xi\zeta) = -2\zeta\xi - [\xi,\zeta]$.

Example 3.2.9. Let $P := -h^2\partial_x^2 + \partial_x - V$ for a potential $V \in C^\infty(I)$ and an interval $I \subset \mathbb{R}$. Then the eikonal polynomial $E_\delta(P)$ of P is given by

$$E_\delta(P)(x,\zeta) = \begin{cases} \zeta\,, & \text{if } 0 < \delta < 2\,, \\ -\zeta^2 + \zeta\,, & \text{if } \delta = 2\,, \\ -\zeta^2\,, & \text{if } \delta > 2\,. \end{cases}$$

For $\delta = 2$ the full δ-symbol is given by

$$\Sigma_2(P)(x,h,\xi,\zeta) = h^{-2}(-\zeta^2 + \zeta) + h^0\,((1 - 2\zeta)\xi + [\xi,\zeta] - V(x)) + h^2(-\xi^2)\,.$$

It is still unclear how to systematically obtain all relevant values of $\delta > 0$ yielding non-trivial phase functions as solutions of the eikonal equation associated to δ. The tool allowing us to detect these values geometrically based on the set of exponents $\Lambda(P)$ is the so called *Newton polygon*. We will introduce it for semi-classical operators in the upcoming subsection.

3.2.2 Newton Polygon

The central problem concerning the δ-principal symbol is that we do not have a method to compute the relevant values of $\delta > 0$ for which the eikonal polynomial $E_\delta(P)$ is not trivial. This subsection aims to introduce a Newton polygon approach based on the set of exponents Λ to determine $E_\delta(P)$. The relevant values of $\delta > 0$ for the eikonal polynomial $E_\delta(P)$ are given by the slopes of the edges $\mathcal{L}$ in the lower boundary of the Newton polygon. The summands of $E_\delta(P)$ correspond one-to-one with the pairs of exponents $\lambda \in \mathcal{L} \cap \Lambda$.

Remark 3.2.10. There have been Newton polygon approaches in the literature to construct Puiseux series approximations of non-linear ordinary differential equations. They date back to the 19th century (see [BB56] and [Fin89]) and after that have been used in a few modern works (see [Can05], [DJ97] or [GS91] for instance). These approaches are limited to polynomial differential equations corresponding to $F \in \mathbb{C}[x, y_1, \dots, y_n]$, given by

$$F(x, y, y', \dots, y^{(n)}) = 0\,,$$

whose approximate solutions are Puiseux series in certain cases. In this context, the Newton polygon is the convex hull of tuples (k, α) for powers $x^k y^\alpha$, where $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, of non-vanishing coefficients $A_{k,\alpha} \in \mathbb{C}$ of F. The use of a Newton polygon approach to construct quasimodes semi-classical operators appears to be new.

Motivation

Returning to the *Schrödinger*-operator we can see some important invariants concerning the δ-principal symbol. Only for the value of $\delta = 1$ the application of the δ-principal symbol to the Schrödinger operator

$$E_1(P)(x, \zeta) = -\zeta^2 + V(x)\,,$$

imposes an eikonal equation $E_1(P)(\cdot, \varphi') = 0$ with non-trivial solutions $\varphi \in C^\infty(I)$ for general potentials $V \in C^\infty(I)$. Increasing the order of differentiation or the power of h on both summands of P equally, i.e. $P_1 := -h^2\partial_x^{k+2} + V\partial_x^k = (-h^2\partial_x^2 + V)\partial_x^k$ and $P_2 := -h^{2+\alpha}\partial_x^2 + h^\alpha V = h^\alpha(-h^2\partial_x^2 + V)$, has no effect on the choice of $\delta = 1$ nor the outcome

$$E_1(P)(x, \zeta) = E_1(P_1)(x, \zeta) = E_1(P_2)(x, \zeta) = -\zeta^2 + V\,.$$

The only difference is the value of l_1, i.e. the power $l_\delta(P)$ of the coefficient $h^{l_\delta(P)}$ associated to $E_1(P_1)$ in the asymptotic expansion of $\Sigma_1(P_1)$ and $\Sigma_1(P_2)$. In particular, $l_1(P) = 0$, $l_1(P_1) = -k$ and $l_1(P_2) = \alpha$.

As a consequence, the relevant values of of δ with respect to P are related to the relative position of the points $(0,0), (2,2) \in \Lambda(P)$ or $(k,0), (k+2,2) \in \Lambda(P_1)$ and $(0,\alpha), (2, 2+\alpha) \in \Lambda(P_2)$, respectively. Another important point is that additional terms $h^\alpha\partial_x^k$ only impact $E_1(P + h^\alpha\partial_x^k)$ if their relative position is *low enough* regarding $(0,0)$ and $(2,2)$. By linearity

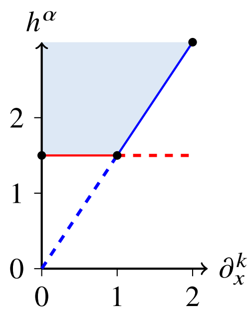

Figure 3.2: The set of exponents $\Lambda := \{(0,3/2),(1,3/2),(2,3)\}$ with the two half spaces defined by $\overline{(0,3/2),(1,3/2)}$ and $\overline{(1,3/2),(2,3)}$ and their corresponding intersection in light blue. The point omitted in each case is in the upper half space respectively.

of Σ_δ we can compute the δ-principal symbol of $h^\alpha \partial_x^k$ independently, yielding

$$\Sigma_\delta\left(P + h^\alpha \partial_x^k\right) = h^0 E_\delta(P) + h^{\alpha - k\delta}\zeta^k + \text{h.o.t.}\,.$$

There are three different cases based on the choices of α and k. If $\alpha > k\delta$, then we have

$$E_\delta\left(P + h^\alpha \partial_x^k\right) = E_1(P)\,.$$

On the other hand, if $\alpha < k\delta$, then

$$E_\delta\left(P + h^\alpha \partial_x^k\right) = \zeta^k\,.$$

In the intermediate case $\alpha = k\delta$ both terms are part of the δ-principal symbol

$$E_\delta\left(P + h^\alpha \partial_x^k\right) = E_1(P) + \zeta^k\,.$$

These distinctions are also relative to the positions of $(0,0)$ and $(2,2)$ on one hand and (k,α) on the other hand, in particular the relation between α and $k\delta$ reflects the position of (k,α) relative to half spaces defined by the straight line with slope $\delta = 1$ going through $(0,0)$ and $(2,2)$ (see Figure 3.2).

If (k,α) is in the upper half space defined by $\mathcal{L} := \overline{(0,0),(2,2)}$, then the additional summand $h^\alpha \partial_x^k$ has no impact on the δ-principal symbol. Only if (k,α) is in the lower half space with respect to $\mathcal{L}$ the summand $h^\alpha \partial_x^k$ affects or even dominates the δ-principal symbol $E_1(P + h^\alpha \partial_x^k)$.

This point of view by means of half spaces naturally confirms the invariance of E_δ with respect to increases of the differentiation order or the power of h since these only shift the associated set of exponents $\Lambda(P + h^\alpha \partial_x^k)$ horizontally or vertically as whole.

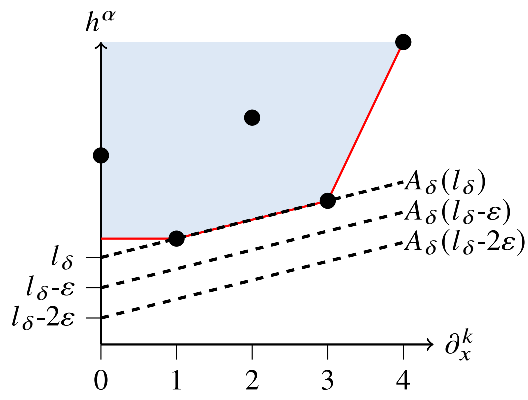

Figure 3.3: Relation between $A_\delta(t)$ and $\partial\mathcal{P}(\Lambda)$ for some $\varepsilon > 0$. l_δ is the unique value where both sets intersect for the first time.

Recall that by (2.14) the lowest order summands with respect to h of $P = \sum_{\lambda\in\Lambda} a_\lambda h^\alpha \partial_x^k$ after conjugation with e^{φ/h^δ} for $\delta > 0$ are amongst the summands

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda=(k,\alpha)}} a_\lambda h^{\alpha-k\delta}(\varphi')^k\,. \tag{3.12}$$

We can geometrically determine the lowest order summand(s) corresponding to h^{l_δ}. Recall that sets of exponents are bounded from below. Let

$$A_\delta(t) := \{(x,y)\in\mathbb{R}^2:\ y-\delta x = t\}\,, \tag{3.13}$$

$t\in\mathbb{R}$, be a family of affine spaces partitioning $\mathbb{R}^2$. Then the power l_δ of the coefficient of $E_\delta(P)$ in (3.10) is given by the value $t\in\mathbb{R}$ for which $A_\delta(t)$ intersects with Λ for the first time (see Figure 3.3), i.e.

$$l_\delta = \min\{t\in\mathbb{R}:\ A_\delta(t)\cap\Lambda\neq\emptyset\}\,. \tag{3.14}$$

Even more importantly, this geometric point of view shows that $E_\delta(P)$ is a monomial in ζ if and only if $|A_\delta(l_\delta)\cap\Lambda| = 1$. Note that for all $\delta > 0$ we have that $|A_\delta(l_\delta)\cap\Lambda|\geq 1$. In particular, if we choose $\delta > 0$ such that $A_\delta(t)$ is parallel to the one of the *lower* edges of the convex hull of Λ, $E_\delta(P)$ is not a monomial. Hence the values of δ where $E_\delta(P)$ is not a monomial are given by the positive slopes of edges of $\operatorname{conv}(\Lambda)$.

Example 3.2.11. Let $I = \mathbb{R}$ and $P := -h^2\partial_x^2 + h^{1/2}\partial_x + V$. Its associated set of exponents is given by $\Lambda(P) = \{(0,0),(1,1/2),(2,2)\}$. We can use the slopes of the edges spanned by $(0,0),(1,1/2)$ and $(1,1/2),(2,2)$ to compute the relevant values of δ. These are given by $\delta_1 = 1/2$ and $\delta_2 = 3/2$. For $\delta = 1/2$ we have

$$\Sigma_{1/2}(P)(x,h,\xi,\zeta) = -h^2\left(\xi+\frac{\zeta}{h^{1/2}}\right)^2 + h^{1/2}\left(\xi+\frac{\zeta}{h^{1/2}}\right) + V(x) = \zeta + V(x) + o(1)\,,$$

and hence

$$E_{1/2}(P)(x,\zeta) = \zeta + V(x)\,.$$

On the other hand for $\delta = 3/2$ we have

$$\Sigma_{3/2}(P)(x,h,\xi,\zeta) = -h^2\left(\xi + \frac{\zeta}{h^{3/2}}\right)^2 + h^{1/2}\left(\xi + \frac{\zeta}{h^{3/2}}\right) + V(x) = h^{-1}\left(-\zeta^2+\zeta\right) + o\left(h^{-1}\right)\,,$$

and thus its δ-principal symbol is given by

$$E_{3/2}(P)(x,\zeta) = -(\zeta-1)\zeta\,.$$

These values $\delta_1 = 1/2$ and $\delta_2 = 3/2$ provide single, non-trivial solutions of their corrsponding eikonal equations $\varphi_1' + V = 0$ and $-(\varphi_2' - 1)\varphi_2' = 0$ each.

All information required to determine the relevant values of $\delta > 0$ were given by the slopes of the lines separating the space. Thus, we focus our investigation on edges $\mathcal{L}$ between points in Λ having the property that Λ is contained in a single half space defined by $\mathcal{L}$. The central object containing these edges based on the set of exponents is the *Newton polygon*.

Definition & Proposition

Definition 3.2.12 (Newton Polygon of a Semi-Classical Operator)**.** Let P be a generalized semi-classical operator and $\Lambda = \Lambda(P)$ its set of exponents. The *Newton polygon* of P is the Newton polygon $\mathcal{P}(\Lambda)$ of Λ as in Definition 2.3.2.

The lower boundary $\partial_{-}\mathcal{P}(\Lambda)$ is the same as in Definition 2.3.2 (compare Figure 3.4). Thus, we have a notion of edges in $\mathcal{P}(\Lambda)$ associated to P. Recall that for $\mathcal{L} := \overline{(k_1,\alpha_1),(k_2,\alpha_2)} \subset \partial_{-}\mathcal{P}(\Lambda)$ its width is given by $|\mathcal{L}| = |k_2 - k_1|$ and its slope is given by $\delta(\mathcal{L}) = |\alpha_2 - \alpha_1|/|\mathcal{L}|$.

Definition 3.2.13 (Minimal Point)**.** Let P be a generalized semi-classical operator and Λ its set of exponents. The unique point $\lambda_{\min} = (k^*,\alpha^*) \in \Lambda$ with $\alpha^* := \min\{\alpha : (k,\alpha) \in \Lambda\}$ and $k^* := \max\{k : (k,\alpha^*) \in \Lambda\}$ is called *minimum* of Λ.

Remark 3.2.14. The minimal point $\lambda_{\min} \in \Lambda$ of $\mathcal{P}(\Lambda)$ corresponds one-to-one with the maximal, horizontal edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$, if $|\mathcal{L}| > 0$. Note that $|\mathcal{L}| = 0$ is not allowed, since there is no well-defined notion of slope for edges with no width.

The Newton polygon Λ of a generalized semi-classical operator P is divided into halves by its minimal point $\lambda_{\min} \in \Lambda$. The set of points $\lambda \in \Lambda$ contained in the horizontal affine space that includes $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ impact the shape of the transport operator T of $P = h^{l_0}T + o(h^{l_0})$, where $\deg T = |\mathcal{L}|$. On its right side the lower boundary $\partial_{-}\mathcal{P}(\Lambda)$ consists of edges with positive slopes. Summing up their widths coincides with the difference $\deg P - \deg T$.

The following proposition relates the Newton polygon of a generalized semi-classical operator to its associated family of eikonal polynomials. In particular, it links all relevant values

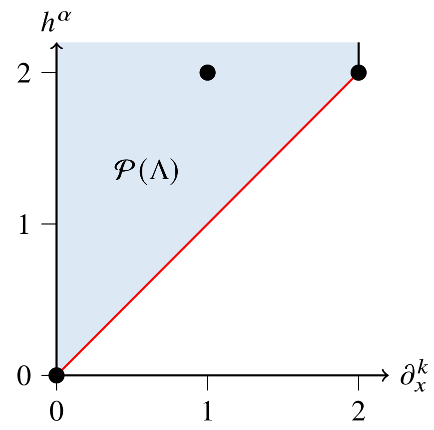

Figure 3.4: The Newton polygon $\mathcal{P}(\Lambda)$ of the set of exponents $\Lambda := \{(0,2),(1,4),(2,4)\}$. The red line is the lower boundary $\partial_{\_}\mathcal{P}(\Lambda)$.

$\delta \in \mathbb{R}_{>0}$ where $E_\delta(P)$ is not trivial to the geometry of $\mathcal{P}(\Lambda)$. Note that there are always two points in the boundary of an edge $\partial\mathcal{L}$.

Proposition 3.2.15. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents, $a_\lambda \in C^\infty(I)$ and $P = \sum_{\lambda\in\Lambda} a_\lambda h^\alpha \partial_x^k$ be a generalized semi-classical operator. Then the following holds:*

(i) For each $\delta > 0$ it holds that $E_\delta(P) \not\equiv 0$.

(ii) The eikonal polynomial $E_\delta(P)$ is non-trivial if and only if there is an edge $\mathcal{L} \subset \partial_{\_}\mathcal{P}(\Lambda)$ with slope $\delta > 0$.

(iii) Let $\mathcal{L} \subset \partial_{\_}\mathcal{P}(\Lambda)$ be an edge with slope $\delta > 0$. Then the eikonal polynomial is given by

$$E_\delta(P)(x,\zeta) = \sum_{\substack{\lambda\in\mathcal{L}\cap\Lambda \\ \lambda=(k,\alpha)}} a_\lambda(x)\zeta^k\,.$$

(iv) Let $\mathcal{L} \subset \partial_{\_}\mathcal{P}(\Lambda)$ be an edge with slope $\delta > 0$, $x_0 \in I$ and $a_\lambda(x_0) \neq 0$ for all $\lambda \in \partial\mathcal{L}$. If all non-zero roots $\zeta_j(x_0)$ of $E_\delta(P)(x_0,\zeta) = 0$ are simple, then there is an interval $U \subset I$ and simple, non-trivial solutions φ_j of $E_\delta(P)(\cdot,\varphi_j') = 0$ with $\varphi_j'(x_0) = \zeta_j(x_0)$ on U.

Proof. (i) This is true by construction, since $E_\delta(P)$ is the leading term of the expansion of $\Sigma_\delta(P)$ with respect to h.

(ii) This was already discussed in (3.12) and thereafter.

(iii) Let $\mathcal{L} \subset \partial_{\_}\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Since $\mathcal{L} \subset A_\delta(l_\delta)$, this is a direct consequence of $A_\delta(l_\delta) \cap \Lambda = \mathcal{L} \cap \Lambda$.

(iv) Let $\mathcal{L} := \overline{(k_1,\alpha_1),(k_2,\alpha_2)} \subset \partial_{\_}\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Then $|\mathcal{L}| \geq 1$ and we have $|\mathcal{L} \cap \Lambda| \geq 2$. Thus, the induced eikonal polynomial

$$E_\delta(P)(\cdot,\zeta) = \sum_{\substack{\lambda\in\mathcal{L}\cap\Lambda \\ \lambda=(k,\alpha)}} a_\lambda \cdot \zeta^k$$

is not a monomial. It is a polynomial of degree $k_2 \geq 1$, which can be factorized into

$$E_\delta(P)(\cdot,\zeta) = \zeta^{k_1} \sum_{\substack{\lambda\in\mathcal{L}\cap\Lambda \\ \lambda=(k,\alpha)}} a_\lambda \cdot \zeta^{k-k_1} \,.$$

Its latter factor is a polynomial of degree $k_2 - k_1 = |\mathcal{L}|$ whose highest and lowest order coefficient does not vanish at $x = x_0$, since $a_\lambda(x_0) \neq 0$ for $\lambda \in \partial\mathcal{L}$. Thus, there are non-zero roots $\zeta_1, \dots, \zeta_{|\mathcal{L}|}$, which are all simple by assumption. Applying Proposition 3.2.7, there is an interval $U \subset I$ and there are functions $\zeta_j \in C^\infty(U)$, $j = 1, \dots, |\mathcal{L}|$, such that

$$E_\delta(P)(\cdot,\zeta_j)_{|U} = 0\,.$$

For $x \in U$, integration yields the solutions

$$\varphi_j(x) = \int_{x_0}^{x} \zeta_j(t)dt\,,$$

of the eikonal equation $E_\delta(P)(\cdot,\varphi') = 0$ with $\varphi_j'(x_0) = \zeta_j(x_0)$ for each $j = 1, \dots, |\mathcal{L}|$. □

What remains to be investigated in order to construct quasimodes is the existence and regularity of transport operators of a generalized semi-classical operator P after solving any eikonal equation. The eikonal polynomials $E_\delta(P)$ are in a direct one-to-one relation with the positively sloped edges $\mathcal{L} \subset \partial_{\text{-}}\mathcal{P}(\Lambda(P))$. Thus, their appearance changes drastically, depending on the selected edge. For any solution φ of the eikonal equation $E_\delta(P)(\cdot,\varphi') = 0$, we expect the transport operator of the conjugated operator

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$$

to be in a direct relation with the edge $\mathcal{L} \subset \partial_{\text{-}}\mathcal{P}(\Lambda)$ of slope δ, in general.

3.2.3 Semi-Classical δ-Regularity

Motivation

Before moving on we want to take a step back and evaluate what we have done so far. The initial problem was that arbitrary, generalized semi-classical operators P cannot be analyzed in general by standard semi-classical symbols $\sigma_{\text{s-cl}}$ mapping $h\partial_x \mapsto \xi$. In most cases the principal symbol part of the full symbol $\sigma_{\text{s-cl}}(P)(x,h,\xi)$ admits less than $\deg P$ non-trivial solutions of its imposed eikonal equation $\sigma_{\text{s-cl}}(P)(\cdot,0,\varphi') = 0$, if it exists at all. This happens due to $\sigma_{\text{s-cl}}(P)(\cdot,0,\xi)$ having degree less than P itself. There is another undesirable case of $\sigma_{\text{s-cl}}(P)(\cdot,0,\xi)$ having full degree but $\sigma_{\text{s-cl}}(P)(\cdot,0,\xi) = \xi^k Q(\xi)$, where $\deg Q = m - k$. In that case, k of the m solutions of $\sigma_{\text{s-cl}}(P)(\cdot,0,\varphi') = 0$ are trivial and thus irrelevant in the construction of independent quasimodes.

The first deficiency is a severe failure of standard *semi-classical ellipticity*, i.e. the leading coefficient $a_m \in C^\infty(I)$ of the highest polynomial power ξ^m of $\sigma_{\text{s-cl}}(P)(\cdot,h,\xi)$ vanishes not only at a single point $x \in I$ but vanishes identically as $h \to 0$. The other case of $\sigma_{\text{s-cl}}(P)(\cdot, 0, \xi) = \xi^k Q(\xi)$ usually does not fall under P not being semi-classically elliptic, but can also pose severe problems in creating initial solutions. This can be seen for the operator $-h^2\partial_x^2 + hV$, where $V \in C^\infty(I)$ with $V > 0$. Its semi-classical principal symbol

$$\sigma_{\text{s-cl}}(P)(\cdot, 0, \xi) = \xi^2\,,$$

is semi-classically elliptic. But none of the solutions $\varphi =$ const. of the corresponding eikonal equation have an impact on the construction of a quasimode since $[P, e^{\varphi/h}] = 0$. The leading term of P remains to be hV, which is a multiplication operator and thus is not a useful choice as a transport operator in Proposition 3.1.7.

The semi-classical δ-symbol can compensate for these deficiencies of $\sigma_{\text{s-cl}}$. A family of admissible values of $\delta > 0$ can be determined by the use of the Newton polygon $\mathcal{P}(\Lambda(P))$ of P. The corresponding family of semi-classical δ-principal symbols $E_\delta(P)$ each admit non-trivial solutions φ of their associated eikonal equations $E_\delta(P)(\cdot, \varphi') = 0$. Their introduction reduces the non-regularity of generalized semi-classical operators from the severe failure to the non-ellipticity at points $x \in I$ where the multiplicity of solutions φ' of the eikonal equations jump, as we will show.

Definitions & Proposition

We start with the definition of δ-regular operators. Despite its name, an operator is not required to be δ-regular to admit quasimodes with smooth phase functions and amplitudes. However, it will be required in the existence of a basis of independent quasimodes in general, as stated in Remark 3.3.12.

Definition 3.2.16 (δ-Regularity)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents, $a_\lambda \in C^\infty(I)$ and $P = \sum_{\lambda \in \Lambda} a_\lambda h^\alpha \partial_x^k$ be a generalized semi-classical operator. Let $\delta > 0$ and $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ be the maximal edge of the lower boundary with slope $\delta > 0$.

The operator P is called *δ-regular* in $x \in I$, if $a_\lambda(x) \neq 0$, for all $\lambda \in \partial\mathcal{L}$. It is called *$\delta$-regular*, if it is δ-regular in every point $x \in I$.

If P is not δ-regular (in $x \in I$) it is called *δ-singular* (in x).

Remark 3.2.17. As proven in Proposition 3.2.15, δ-regularity in $x \in I$ guarantees that $|\mathcal{L}|$ solutions $\varphi'(x)$ of $E_\delta(P)(x, \varphi'(x)) = 0$ are non-zero. Thus, δ-regularity can and will be used to count the number of non-trivial solutions. However, we will show the central property regarding the ellipticity of the transport operator is the simplicity, or more general the constant multiplicity, of solutions $\varphi' \in C^\infty(I)$ of the eikonal equations. The notion of δ-regularity affects this indirectly. Let $\mathcal{L} = \overline{\lambda\mu}$ with slope $\delta > 0$. If P is δ-singular in $x_0 \in I$, then either the leading coefficient a_μ or the lowest coefficient a_λ of the eikonal polynomial $E_\delta(P)$ vanishes at x_0.

In the first case this results in the existence of unbounded solutions φ' of the eikonal equation. This will lead to the question whether these solutions admit exponential-polyhomogeneous quasimodes, which will be answered in Chapter 4. In the second case this results in solutions of the eikonal equation vanishing in x_0. The latter case leads to a crossing of solutions in x_0 in most cases, since $|\mathcal{L} \cap \Lambda| \leq |\mathcal{L}|$. The presence of more than two points on an edge is generic! Since the crossing point coincides with the vanishing point of the coefficient a_λ, the resolution of P becomes much harder than in the case of δ-regular operators having a solution φ' of an eikonal equation that jumps in multiplicity. In the δ-regular case the non-constant multiplicity can be resolved by a single blow-up, as we will show in Algorithm 1. However, in the δ-singular case, where the crossing point coincides with the vanishing point of a_λ, the behavior of P under the pullback of a blow-up is much different and requires a thorough analysis, presented in Chapter 5.

The central object of this subsection is the so called *induced δ-transport operator*. Associated to each summand $a_\lambda \zeta^k$ of $E_\delta(P)$ with $\lambda = (k, \alpha) \in \mathcal{L} \cap \Lambda$, there is a natural partial sum in the expansion of $\Sigma_\delta(P)$

$$a_\lambda \cdot \left(\xi\zeta^{k-1} + \zeta\xi\zeta^{k-2} + \ldots + \zeta^{k-1}\xi\right) = a_\lambda \cdot k\zeta^{k-1}\left(\xi + \frac{k-1}{2}[\xi, \zeta]\right). \tag{3.15}$$

This expression is the second lowest term with respect to h in the expansion of $\partial_x^k \circ \exp(\varphi/h^\delta)$ as shown in (2.15). Summing over all of these sums for each $\lambda \in \mathcal{L} \cap \Lambda$ and evaluating it at $\xi = \partial_x$ and $\zeta = \varphi'$ yields a first order differential operator, the *induced δ-transport operator*.

Definition 3.2.18 (Induced Transport Operator)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents, $a_\lambda \in C^\infty(I)$ and $P = \sum_{\lambda \in \Lambda} a_\lambda h^\alpha \partial_x^k$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial_{\_}\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$. Let φ be a solution of $E_\delta(P)(\cdot, \varphi') = 0$.

Then we call

$$T_{\delta,\varphi'}(P) := \sum_{\substack{\lambda=(k,\alpha)\\ \lambda \in \mathcal{L}\cap\Lambda}} a_\lambda \cdot k(\varphi')^{k-1}\left(\partial_x + \frac{k-1}{2}\frac{\varphi''}{\varphi'}\right) \tag{3.16}$$

the *induced δ-transport operator* of P with respect to φ.

Remark 3.2.19. Returning to Proposition 3.2.7.(ii) and (3.15) we have

$$E_\delta(P)(\cdot, \varphi') = \iota_{\varphi',\partial_x}\left[\sum_{\lambda \in \mathcal{L}\cap\Lambda} a_\lambda \cdot \zeta^k\right]$$

and parallel to that

$$T_{\delta,\varphi'}(P) = \iota_{\varphi',\partial_x}\left[\sum_{\lambda \in \mathcal{L}\cap\Lambda} a_\lambda \cdot \left(\xi\zeta^{k-1} + \zeta\xi\zeta^{k-2} + \ldots + \zeta^{k-1}\xi\right)\right]. \tag{3.17}$$

Recall that $\xi\zeta \neq \zeta\xi$. Whenever it is clear from context, we will refer to the induced δ-transport operator as T_δ.

The upcoming proposition shows the importance of simplicity of solutions of eikonal equations for their associated induced transport operators.

Proposition 3.2.20. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(I)$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial_{\text{-}}\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$ and let φ be a solution of $E_\delta(P)(\cdot, \varphi') = 0$ on I.*

If φ' is a simple solution, then $T_{\delta,\varphi'}$ is elliptic.

Proof. By definition, $T_{\delta,\varphi}(P)$ is elliptic, if and only if the coefficient $\sum_{\lambda \in \mathcal{L}\cap\Lambda} a_\lambda \cdot k(\varphi')^{k-1}$ of $T_{\delta,\varphi}(P)$ in (3.16) has no zeros $x \in I$. This coefficient can be rephrased in terms of the eikonal polynomial

$$\sum_{\lambda \in \mathcal{L}\cap\Lambda} a_\lambda \cdot k(\varphi')^{k-1} = (\partial_\zeta E_\delta(P))(\cdot, \varphi') \,.$$

Since φ' is a simple solution of $E_\delta(P)(\cdot, \varphi') = 0$, it holds that $(\partial_\zeta E_\delta(P))(x, \varphi'(x)) \neq 0$, for all $x \in I$. □

Examples

Vanishing phase functions are not a new phenomenon of generalized semi-classical operators and can be seen in the following example.

Example 3.2.21. Let $I = (0, \infty)$ and $P := h^2\partial_x^2 + h\partial_x - x$. Its associated set of exponents is given by $\Lambda = \{(0,0), (1,1), (2,2)\}$ and its Newton polygon $\mathcal{P}(\Lambda)$ consists of a single edge $\mathcal{L} = \overline{(0,0),(2,2)} = \mathcal{P}(\Lambda(P))$ with slope $\delta = 1$. Its induced eikonal equation

$$E_1(P)(x, \varphi') = (\varphi')^2 + \varphi' - x = 0\,,$$

has solutions $\varphi'_\pm(x) := -1/2 \pm \sqrt{1/4 + x}$. While φ_- is bounded from above by -1, the other solution

$$\varphi'_+(x) \sim x + O(x^2)\,, \quad \text{as } x \to 0$$

vanishes to first order. Its induced transport operator with respect to φ'_+ is given by

$$T_{\delta,\varphi'_+} = \left(2\varphi'_+(x) + 1\right)\partial_x + \frac{1}{2}\varphi''_+(x) = \left(2\sqrt{1/4 + x}\right)\partial_x + \frac{1}{4}\frac{1}{\sqrt{1/4 + x}}\,.$$

This example shows us that there can be both vanishing and non-vanishing phase functions φ solving the eikonal equation with respect to a single edge. In both cases $T_{\delta,\varphi'}$ is elliptic, since φ'_+ and φ'_- are simple for $x \neq -1/4$.

Example 3.2.22 (Schrödinger Operator with Linear Potential)**.** Let $P := -h^2\partial_x^2 + x$ on the interval $I := (0, \infty)$. Its associated set of exponents is given by $\Lambda = \{(0,0), (2,2)\}$ and the Newton polygon $\mathcal{P}(\Lambda)$ consists of a single edge $\mathcal{L} = \overline{(0,0),(2,2)} = \mathcal{P}(\Lambda(P))$ with slope

$\delta = 1$. Its induced eikonal equation

$$E_\delta(P)(x, \varphi') = -(\varphi'(x))^2 + x = 0\,,$$

has solutions $\varphi_\pm(x) \coloneqq \pm 2/3x^{3/2}$. Choosing $\varphi \coloneqq \varphi_+$, the corresponding conjugated operator is given by

$$e^{-\varphi/h} P e^{\varphi/h} = h\left(-2\sqrt{x}\partial_x + \frac{1}{2\sqrt{x}}\right) + h^2\partial_x^2\,. \tag{3.18}$$

The induced transport operator is given by $T_{\delta,\varphi'_+} = -2x^{-1/2}\,(x\partial_x - 1/4)$.

There are two new features presented by this example. The first one is that solutions φ of the induced eikonal equation $E_\delta(P)(\cdot, \varphi') = 0$ are not smooth in general, but polyhomogeneous at the boundary of I. The second new phenomenon is given by the fact that the induced transport operator $T_{\delta,\varphi'}$ is a b-operator with an additional vanishing order of $x^{-1/2}$. The non-ellipticity of the induced transport operator $T_{\delta,\varphi'}$ at $x = 0$ is caused by the jump in multiplicity of the solutions $\varphi'_\pm(x) = \sqrt{x}$ at $x = 0$.

It is important to notice that we have not shown yet that for any solution φ of $E_\delta(P)(\cdot, \varphi') = 0$ of P the induced transport operator $T_{\delta,\varphi'}$ is the transport operator, i.e. the leading term of $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$, in general. Despite being called induced δ-transport operator, there might be lower terms with respect to h in the expansion of the conjugated operator. But we will show in Subsection 3.3.2 that it can be turned into one eventually. We move on to the final step, the construction of quasimodes.

3.3 Construction of Quasimodes I: Regular Operators

In the final section of this chapter we prove the existence of quasimodes of generalized semi-classical operators P solutions φ' with constant multiplicity. The proof is constructive and, in addition, is capable of generating sufficiently many independent quasimodes matching the length of all corresponding edges $\mathcal{L} \subset \partial_{-}\mathcal{P}(P)$ with the number of generated quasimodes.

What remains to be investigated after Section 3.2 is the existence of sufficient conditions such that $T_{\delta,\varphi'}(P)$ coincides with the transport operator of P after conjugation in the sense of Equation 3.4 for any solution φ' of $E_\delta(P)(\cdot, \varphi') = 0$. We will show in the beginning of this section that simplicity of the eikonal solution φ' is not sufficient. In Subsection 3.3.3 we will briefly discuss the approach of constructing quasimodes when the solutions φ' of the eikonal polynomial are not simple but of constant higher multiplicity.

3.3.1 δ-Separation

In the first part of Section 3.3 we present a new phenomenon of semi-classical operators. Whenever their corresponding set of exponents Λ is too dense, the induced and explicit transport operator do not coincide. The aim of this subsection is to characterize this phenomenon geometrically on Λ and introduce the concept of *full phase functions* to resolve this defect.

Motivation & Examples

Let $P \in \mathrm{Diff}^{\Lambda}(I)$ be a generalized semi-classical operator and let φ be a simple solution of the associated eikonal equation $E_\delta(P)(\cdot, \varphi') = 0$. The lowest order term in the expansion of $e^{-\varphi/h^\delta} P e^{\varphi/h^\delta}$ of differential order greater than 0 is given by the induced δ-transport operator T_δ. This is a direct consequence of T_δ being the lowest order symbolic term containing ξ in (3.17). However, there can still be a multiplication operator $V \in C^\infty(I)$, such that

$$h^{-l_\delta} \cdot e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} = h^\varepsilon V + h^\delta T_\delta + o\left(h^\delta\right), \tag{3.19}$$

for some $0 < \varepsilon \leq \delta$, due to $[V, e^{\varphi/h^\delta}] = 0$.

Example 3.3.1. Let $I = \mathbb{R}$ and $P := -h^2\partial_x^2 + (1 + h^{1/2})$. The lower boundary of its associated Newton polygon $\partial_{-}\mathcal{P}(\Lambda(P)) = \mathcal{L}$ consists of a single edge $\mathcal{L} := \overline{(0,0),(2,2)}$. The solutions of the eikonal equation $E_1(P)(\cdot, \varphi') = 0$ are given by $\varphi'_\pm(x) := \pm 1$ and hence are simple. Conjugating P by the exponential term induced by $\varphi = \varphi_+$ yields

$$e^{-x/h} P e^{x/h} = h^{1/2} + h\,(-2\partial_x) + h^2\left(-\partial_x^2\right), \tag{3.20}$$

which does not admit polyhomogeneous quasimodes due to its leading part $h^{1/2} \cdot 1$ being elliptic and of order 0.

The term $-2\partial_x$ in (3.20) in Example 3.3.1 is the induced δ-transport operator $T_{\delta,\varphi'}(P)$ of $P = -h^2\partial_x^2 + (1 + h^{1/2})$ for $\delta = 1$ and $\varphi(x) = x$. The presence of $h^{1/2}$ geometrically implies the presence of points $\mu \in \Lambda(P)$ in the strip between $A_\delta(l_\delta)$ and $A_\delta(l_\delta + \delta)$.

However, the following example shows that one is still able to construct a quasimode in this setting with a few adjustments.

Example 3.3.2. Let $I \subset \mathbb{R}$ be an interval and $a, V \in C^\infty(I)$, where a is positive. Let $P := h^{3/2}\partial_x^2 + h^{1/2} a \partial_x + V$. Its set of exponents is given by $\Lambda = \{(0,0), (1, 1/2), (2, 3/2)\}$. Thus, the associated Newton polygon has two lower edges corresponding to the slopes $\delta_1 = 1/2$ and $\delta_2 = 1$. Choosing $\delta = \delta_2 = 1$ the conjugated operator is given by

$$e^{-\varphi/h} P e^{\varphi/h} = h^{-1/2}(\varphi')\,(a + \varphi') + V + h^{1/2}\,((2\varphi' + a)\partial_x + \varphi'') + h^{3/2}\partial_x^2\,,$$

where $\varphi \in C^\infty(I)$. The corresponding eikonal equation $E_1(P)(\cdot, \varphi') = 0$ is solved by $\varphi = -\int a$. Then the conjugated operator

$$e^{-\varphi/h} P e^{\varphi/h} = V + h^{1/2}\,(-a\partial_x - a') + h^{3/2}\partial_x^2\,,$$

is similar to P and in particular it does not satisfy the requirements of Proposition 3.1.7. The only obvious option remaining is the conjugation with respect to $\varepsilon = 1/2$ and an arbitrary $\psi \in C^\infty(I)$. Denote $\Phi := \varphi/h + \psi/h^{1/2}$. By doing so we obtain an operator

$$e^{-\Phi} P e^{\Phi} = (V - a(\psi')) + h^{1/2}\left(-a\partial_x - a' + (\psi')^2\right) + h\,(2\psi'\partial_x + \psi'') + h^{3/2}\partial_x^2\,,$$

whose corresponding eikonal equation

$$E_{1/2}\left(e^{-\varphi/h}Pe^{\varphi/h}\right)(\cdot,\psi') = V - a\psi' = 0\,,$$

can be solved by $\psi = +\int \frac{V}{a}$. Thus, the double conjugated operator

$$e^{-\left(\varphi/h+\psi/h^{1/2}\right)}Pe^{\left(\varphi/h+\psi/h^{1/2}\right)} = h^{1/2}\left(-a\partial_x - a' + (\frac{V}{a})^2\right) + h\left(2\frac{V}{a}\partial_x + (\frac{V}{a})'\right) + h^{3/2}\partial_x^2\,,$$

has an elliptic leading term $T = -a\partial_x + (a' + (V/a)^2)$. Applying Proposition 3.1.7 then yields a quasimode of the form

$$u = e^{i\left(\varphi/h+\psi/h^{1/2}\right)}\sum_{k=0}^{\infty} u_k h^{k/2}\,.$$

The essence of these examples can be extracted easily. Let $P \in \mathrm{Diff}^{\Lambda}(I)$, $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$, $\delta := \delta(\mathcal{L}) > 0$ and φ' be a solution of $E_\delta(P)(\cdot,\varphi') = 0$. The existence of points

$$\lambda \in \Lambda \cap \mathrm{conv}\,(A_\delta(l_\delta) \cup A_\delta(l_\delta + \delta))^\circ$$

lead to the defect of $T_{\delta,\varphi}$ not being the leading term of $e^{-\varphi/h^\delta}Pe^{\varphi/h^\delta}$ with respect to h (see Figure 3.5). This is a direct consequence of (2.15). If $\Lambda \cap \mathrm{conv}\,(A_\delta(l_\delta) \cup A_\delta(l_\delta + \delta))^\circ$ is empty and

$$\Lambda \cap A_\delta\,(l_\delta + \delta) \neq \emptyset\,,$$

we have that in the expansion of the conjugated operator

$$e^{-\varphi/h^\delta}Pe^{\varphi/h^\delta} = h^{l_\delta+\delta}\left(T_{\delta,\varphi} + \sum_{k=0}^{m} a_{(k,l_\delta+k\delta)}\cdot(\varphi')^k\right) + o\left(h^{l_\delta+\delta}\right)$$

the transport operator T of P with respect to δ and φ

$$T = T_{\delta,\varphi} + \sum_{k=0}^{m} a_{(k,l_\delta+k\delta)}\cdot(\varphi')^k\,, \tag{3.21}$$

is a perturbation of the induced transport operator. Note that $a_{(k,l_\delta+k\delta)} \equiv 0$ if and only if $(k, l_\delta + k\delta) \notin \Lambda$.

Definitions & Remarks

One can a priori tell whether the phenomenon in Example 3.3.2 appears for a semi-classical operator P based on the geometry of $\Lambda(P)$.

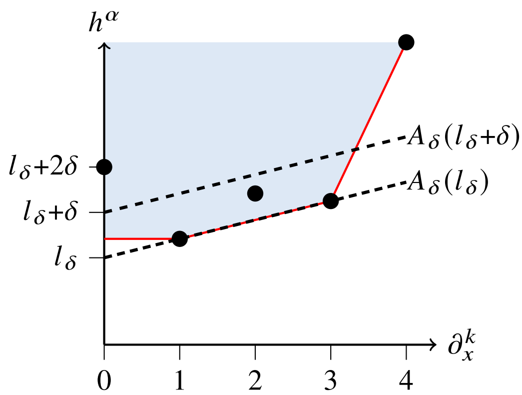

Figure 3.5: A non-δ-separated set Λ.

Definition 3.3.3 (δ-Separation)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(I)$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$.

Then we call P *δ-separated*, if

$$\Lambda \cap \mathrm{conv}\,(A_\delta(l_\delta) \cup A_\delta(l_\delta + \delta))^\circ = \emptyset\,, \tag{3.22}$$

with l_δ given by (3.14) and A_δ defined in (3.13). If additionally

$$\Lambda \cap A_\delta\,(l_\delta + \delta) = \emptyset\,,$$

then we call P *strictly* δ-separated.

Remark 3.3.4. Note that P can be δ-separated with respect to multiple values of $\delta > 0$.

3.3.2 Existence of Full Phase Functions

A key finding in Example 3.3.2 is that introducing a second, minor phase function with a lower power of h resolved the defect caused by non-separateness. Analyzing the behavior of a semi classical operator under the inclusion of these minor phase functions is the focus of this subsection.

In the upcoming definition we introduce a short notation for iteratively conjugated operators and classify partial sums of phase functions.

Definition 3.3.5 (Full Phase Function)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(I)$ with $\mathrm{ord}(P) = m$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with slope $\delta > 0$ and $\varphi \in C^\infty(I)$ be a simple solution of $E_\delta(\cdot, \varphi') = 0$. Let $n \in \mathbb{N}_0$, $(\varepsilon, \psi) \in (0, \delta)^n \times C^\infty(I)^n$ and let $\Phi := \varphi/h^\delta + \sum_{l=1}^m \psi_l/h^{\varepsilon_l}$ be the sum of phase functions.

If the leading term T in the expansion of the conjugated operator

$$P_\Phi := \exp\,(-\Phi) \circ P \circ \exp\,(\Phi)\;,$$

is given by (3.21), we call Φ a *full phase function* of P with respect to δ.

Often one has to add multiple smaller phase functions to obtain a full phase function. It will be useful to emphasize iteration steps in the process of constructing full phase functions. Thus, we will introduce the notation

$$P_{\delta,\varphi} := \exp(-\varphi/h^{\delta})P\exp(\varphi/h^{\delta})$$

and $P_{(\varepsilon,\psi)}$ accordingly, for $(\varepsilon,\psi) \in (0,\delta)^n \times C^{\infty}(I)^n$.

Example 3.3.6. Example 3.3.2 already showed us an operator with a small, correcting phase and corresponding full phase function.

If P is δ-separated for some $\delta > 0$, then any phase function φ solving the eikonal equation $E_{\delta}(P)(\cdot,\varphi') = 0$ is a full phase function.

Example 3.3.7. Revisiting Example 3.3.1 with $P := -h^2\partial_x^2 + (1+h^{1/2})$ we can resolve the non-separateness as in Example 3.3.2 by introducing an additional minor phase function $\psi_1/h^{1/2}$, with $\psi_1 = 1/2x$.

Motivation

The remaining question is when are we able to construct full phase function for non-δ-separated operators. Let $\mathcal{L} \subset \mathcal{P}(\Lambda(P))$ be an edge with slope $\delta > 0$ and let $\varphi \in C^{\infty}(I)$ be a simple solution of $E_{\delta}(P)(\cdot,\varphi') = 0$. Since $P_{\delta,\varphi}$ is non-δ-separated in general, the conjugated operator is given by

$$P_{\delta,\varphi} = 0 + \sum_{j=1}^{r} F_j h^{\gamma_j} + h^{\delta}T + o(h^{\delta}), \tag{3.23}$$

for $\gamma_j \in (0,\delta)$ ordered and $F_j \in C^{\infty}(I)$, with T given by (3.21), where we assumed without loss of generality that $l_{\delta} = 0$. Notice that for $\varepsilon := \delta - \gamma_1$ and any $\psi_1 \in C^{\infty}(I)$ we have

$$\left(P_{\delta,\varphi}\right)_{\varepsilon,\psi_1} = h^{\gamma_1}\left(F_1 + [T,\psi_1]\right) + \sum_{j=1}^{r_1} F_{1,j}h^{\gamma_{1,j}} + h^{\delta}T + o(h^{\delta}) \tag{3.24}$$

since $T_{\varepsilon,\psi_1} = h^{\gamma_1-\delta}[T_{\delta,\varphi},\psi_1] + T$. The additional terms in the sum arise from the conjugation of the remaining differential operators in $P_{\delta,\varphi} - h^{\delta}T$. We can solve the *sub-eikonal* equation for the *sub-oscillation*

$$E_{\delta-\gamma_1}(T)(\cdot,\psi') = F_1 + [T,\psi] = 0 \tag{3.25}$$

since φ' is a simple solutions and hence $T = T_{\delta} + \sum_{k=0}^{m} a_{(k,l_{\delta}+k\delta)} \cdot (\varphi')^k$ is elliptic and of order $\operatorname{ord}(T) = 1$. It is not clear yet if we have resolved the problem partially, since the number of multiplication operators increased. This problem is very similar to the infinite asymptotic expansion of $R(h)$ in the proof of Proposition 3.1.7 and is a matter of efficient counting as the upcoming subsection will show.

Central Statement

A central tool of this subsection is the observation that simple phase functions φ always admit full phase functions, as stated in the next proposition.

Proposition 3.3.8. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and let $P \in \mathit{Diff}^{\Lambda}(I)$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Let $\varphi \in C^\infty(I)$ be a simple, non-trivial solution of $E_\delta(\cdot, \varphi') = 0$.*

Then there is a full phase function $\Phi \in C_h^\infty(I)$ with respect to P and δ with $\Phi = \varphi/h^\delta + o(h^\delta)$.

Proof. Since φ' is simple on I we have that T_δ is elliptic by Proposition 3.2.20. Without loss of generality we assume that $l_\delta = 0$. With T given by (3.21) and for $n, r \in \mathbb{N}_0$, $F_j, G_k, f_l \in C^\infty(I)$ and $\beta_k, \gamma_l \in (0, \delta)$ ordered, we can write

$$P_{\delta,\varphi} = \sum_{j=1}^{r} F_j h^{\gamma_j} + h^\delta T + \sum_{k=1}^{n} G_k h^{\delta+\beta_k} + \sum_{l=1}^{r} f_l h^{\delta+\gamma_l} \partial_x + \, O(h^{2\delta}) \,.$$

Letting $\varepsilon_1 := \delta - \gamma_1$ and $\psi_1 \in C^\infty(I)$ then yields the following leading terms in the expansion of $(P_{\delta,\varphi})_{\varepsilon_1,\psi_1}$

$$(F_1 + [T_\delta, \psi])h^{\gamma_1} + \sum_{j=2}^{r} F_j h^{\gamma_j} + h^\delta T + \sum_{k=1}^{n} G_k h^{\delta+\beta_k} + \sum_{l=1}^{r} f_l \cdot (\psi_1') h^{\gamma_1+\gamma_l} + \, O(h^{2\gamma_1}) \,,$$

since for all terms Q of differential order k we have $Q = O(h^{k\delta})$ and thus $Q_{\varepsilon_1,\psi_1} = O(h^{k\gamma_1})$. By solving the sub-eikonal equation $E_{\delta-\gamma_1}(T_\delta)(\cdot, \psi') = F_1 + [T_\delta, \psi] = 0$ we are able to increase the vanishing order from $P_{\delta,\varphi} = O(h^{\gamma_1})$ to $(P_{\delta,\varphi})_{\varepsilon_1,\psi_1} = O(h^{\min\{\gamma_2, 2\gamma_1\}})$ at the cost of lowering the remainder estimate to $O(h^{2\gamma_1})$. Repeating this step for all remaining F_l, $l = 2, \dots, m$, simultaneously and solving all corresponding sub-eikonal equations then yields

$$(P_{\delta,\varphi})_{\widehat{\varepsilon},\widehat{\psi}} = h^\delta T + \sum_{k=1}^{n} G_k h^{\delta+\beta_k} + \, O(h^{2\gamma_1}) \,,$$

for $\widehat{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_l := \delta - \gamma_l$ and $\widehat{\psi} := (\psi_1, \dots, \psi_m)$, where ψ_j are the corresponding solutions of the sub-eikonal equations. If $2\gamma_1 \geq \delta$ then the operator $(P_{\delta,\varphi})_{\widehat{\varepsilon},\widehat{\psi}}$ satisfies the requirements of Proposition 3.1.7 and thus $\Phi := \varphi/h^\delta + \sum_{l=1}^{m} \psi_l/h^{\varepsilon_l}$ is a full phase function. Otherwise we can repeat this construction starting with $(P_{\delta,\varphi})_{\widehat{\varepsilon},\widehat{\psi}}$ and increase the remainder $O(h^{2\gamma_1})$ by h^{γ_1} for each iteration until $N\gamma_1 \geq \delta$ for some $N \in \mathbb{N}$. □

We are able to prove the main theorem of this chapter in two versions. Theorem 3.3.9 shows the existence of quasimodes in one-to-one relation to solutions ζ of an eikonal polynomial $E_\delta(P)(\cdot, \zeta) = 0$. As a consequence, Theorem 3.3.11 shows that under sufficient conditions there is a full set of independent quasimodes of a semi-classical operator, i.e. their number matches the degree of the operator.

Theorem 3.3.9. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(I)$. Let $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Let $\varphi \in C^\infty(I)$ be a simple, non-trivial solution of $E_\delta(P)(\cdot, \varphi') = 0$ on I. Let T be given by* (3.21).

Then there is a full phase function Φ corresponding to φ', such that

$$P_\Phi = h^{l_\delta + \delta} T + o\left(h^{l_\delta + \delta}\right) .$$

Moreover, there is a quasimode $u = e^{\Phi(h)} A$, with $A := \sum_{k=0}^\infty a_k h^{\beta_k}$, where $a_k \in C^\infty(I)$ and $a_0 \in \ker T$ with $a_0 \not\equiv 0$, such that

$$P_\Phi A = O(h^\infty) .$$

If P is δ-separated, then $\Phi(h) = \varphi / h^\delta$. Additionally, if P is strongly δ-separated, then $T = T_{\delta,\varphi}$.

Proof. Let $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda)$ be an edge with slope $\delta > 0$ and let φ be a simple solution of $E_\delta(\cdot, \varphi') = 0$. Since φ is a simple, non-trivial solution, Proposition 3.2.20 yields that T_δ is an elliptic first order differential operator and in particular T given in (3.21) is elliptic, too. Since T is elliptic, applying Proposition 3.3.8 yields a $\Phi \in C_h^\infty(I)$, with $\Phi = \varphi/h^\delta + o(h^\delta)$, such that

$$P_\Phi = 0 + h^{\delta + l_\delta} T + o\left(h^{\delta + l_\delta}\right) ,$$

for $l_\delta \in \mathbb{R}$ given by (3.14). Hence T is the leading operator of P_Φ. Proposition 3.1.7 then yields a polyhomogeneous quasimode $\sum_k a_k h^{\beta_k}$ for P_Φ with $a_k \not\equiv 0$ and further $u = e^\Phi \sum_k a_k h^{\beta_k}$ is a quasimode of P. □

Example 3.3.10. Let $n \in \mathbb{N}$, let $0 \leq k < n$ and $A_k \in C^\infty(I)[h]$ be polynomial in h, where we assume that $A_0(0,0) \neq 0$. Let

$$P := h^n \partial_x^n + \sum_{k=0}^{n-1} A_k(x,h) h^k \partial_x^k$$

be a semi-classical operator. Its corresponding set of exponents $\Lambda(P)$ is contained in the cone spanned by the h-axis and the diagonal $\Delta_{\mathbb{R}_+} := \{(x,x) : x \in \mathbb{R}_+\}$ and its lower boundary coincides with the edge $\mathcal{L} = \overline{(0,0),(n,n)}$. These operators correspond to a class of operators considered in Chapter 5 of [Fed93], for which the author sketches a way of obtaining the phase functions and leading term of the amplitude, under the assumptions that all solutions φ_j', for $j = 1, \ldots, n$, of the eikonal equation

$$E_1(P)(x, \zeta) = 0 ,$$

are simple, for all $x \in I$. Applying Theorem 3.3.9 yields the full asymptotic expansion of the amplitudes in that setting for each φ_j', with corresponding leading terms $a_{j,0} \in C^\infty(I)$ of the amplitudes, up to a constant determined by

$$T_{1,\varphi_j'} a_{j,0} = \left(\sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap \Lambda}} a_\lambda \cdot k(\varphi')^{k-1} \left(\partial_x + \frac{k-1}{2} \frac{\varphi''}{\varphi'} \right) \right) a_{j,0} = 0 \,.$$

Requiring that all solutions $\varphi' \in C^\infty(I)$ eikonal equations $E_\delta(P)(\cdot, \zeta) = 0$ for all slopes $\delta > 0$ of the Newton polygon $\partial \mathcal{P}(\Lambda)$ are simple, simultaneously, implies that almost all solutions φ' are bounded away from 0 on I. This will be discussed briefly in Remark 3.3.12. Thus, the following theorem about the existence of a *basis* of independent quasimodes, whose number coincides with the order of the operator requires instead the existence of a point $x_0 \in I$ where all eikonal polynomials have only non-zero roots.

Theorem 3.3.11. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in$ Diff$^\Lambda(I)$ with* $\operatorname{ord} P = m$. *Let $\Delta \subset \mathbb{R}_{>0}$ be the set of slopes corresponding to maximal edges $\mathcal{L}_\delta \subset \partial \mathcal{P}(\Lambda)$ for $\delta \in \Delta$. Denote their length by $L_\delta := |\mathcal{L}_\delta|$. Assume that there is an $x_0 \in I$ such that for each positive $\delta \in \Delta$ the eikonal polynomial $E_\delta(P)(x_0, \zeta) = 0$ has simple roots $\zeta_{\delta,j} \neq 0$ for each $j = 1, \ldots, L_\delta$.*

Then there is an interval $U \subset I$ and there are independent (exponential)-polyhomogeneous quasimodes $u_{\delta,j} = e^{\Phi_{\delta,j}} A_{\delta,j}$ of P with $A_{\delta,j} \not\equiv 0$, for each $\delta \in \Delta$ and $j = 1, \ldots, L_\delta$, such that

$$P_{\Phi_{\delta,j}} A_{\delta,j} = O(h^\infty) \,.$$

For each positive $\delta \in \Delta$ and $j = 1, \ldots, L_\delta$ the phase functions satisfy $\Phi_{\delta,j} = \varphi_{\delta,j}/h^\delta + o(h^{-\delta})$ for $\varphi_{\delta,j} \in C^\infty(U)$ with $\varphi'_{\delta,j}(x_0) = \zeta_{\delta,j}$ and $E_\delta(P)(\cdot, \varphi'_{\delta,j}) = 0$. For $\delta = 0$ we have that $\Phi_{0,j} \equiv 0$ for all $j = 1, \ldots, L_0$.

In particular, the number of independent quasimodes $u_{\delta,j}$ is equal to $m = \sum_{\delta \in \Delta} L_\delta$.

Remark 3.3.12. Note that in the statement of Theorem 3.3.11 we do not require simplicity of solutions of eikonal equations and instead reduce it to the existence of local non-zero roots $\zeta_{\delta,j}$ of the eikonal polynomials $E_\delta(P)(x_0, \zeta)$ at $x_0 \in I$. This is equivalent to the existence of simple solutions and δ-regularity in a neighborhood of $x_0 \in I$, as we will show. Reducing this to simple roots $\zeta_{\delta,j} \in \mathbb{C}$ by allowing roots $\zeta_{\delta,j} = 0$, excluding the trivial roots, for one $j = 1, \ldots, L_\delta$, has negative impacts on solutions of eikonal equations on adjacent edges.

Let $\mathcal{L}_1, \mathcal{L}_2 \subset \partial \mathcal{P}(\Lambda)$ be two maximal, adjacent edges to the left with slopes $\delta_k := \delta(\mathcal{L}_k)$, $k = 1, 2$, and $\delta_1 < \delta_2$. Allowing for a vanishing solution $\varphi'_2(x_0) = 0$ of

$$E_{\delta_2}(P)(x_0, \zeta) = \zeta^l \cdot \prod_{j=1}^{L_2} (\zeta - \zeta_{\delta_2,j}) = \zeta^l \cdot \sum_{j=0}^{L_2} a_{2,j}(x_0) \zeta^j$$

implies $a_{2,0}(x_0) = 0$. On the other hand, $a_{2,0}$ is the leading coefficient of $E_{\delta_1}(P)(\cdot, \zeta)$, resulting in the existence of an unbounded solutions φ'_1 at x_0. Since we aim to find an interval $U \subset I$ on which we have the maximal amount of independent quasimodes, the point $x_0 \in I$ has to be excluded from U.

This can be relaxed to requiring that all but one solution are non-zero, with the exception of $\zeta_{\delta_1,1}$, where the slope of the zero root

$$\delta_1 := \min\{\delta : \text{ there is a } \mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda), \text{with } \delta = \delta(\mathcal{L})\}\,,$$

has to be minimal, following the discussion above.

Proof. Without loss of generality assume that all edges are ordered from left to right with corresponding slopes δ_k and widths $L_k := |\mathcal{L}_{\delta_k}|$. Since $\zeta_{\delta_2,j} \neq 0$ for all $j = 1, \ldots, L_2$, the associated eikonal polynomial at $x_0 \in I$ is of the form

$$E_{\delta_2}(P)(x_0, \zeta) = \zeta^{L_1} \cdot \sum_{j=0}^{L_2} a_j(x_0)\zeta^j\,,$$

with $a_0(x_0), a_{L_2}(x_0) \neq 0$. Thus, there is a maximal neighborhood $U_2 \subset I$ of x_0, such that $a_0(x), a_{L_2}(x) \neq 0$ for all $x \in U_2$. This is equivalent to P being δ_2-regular. Thus, we can apply Proposition 3.2.15 and obtain smooth, simple solutions $\varphi_{\delta_2,j}$ of $E_{\delta_2}(P)(\cdot, \varphi'_{\delta_2,j}) = 0$ on $\widetilde{U}_2 \subset U_2$ with $\varphi'_{\delta_2,j}(x_0) = \zeta_{\delta_2,j}$ for $j = 1, \ldots, L_2$. Applying Theorem 3.3.9 then yields full phase functions $\Phi_{\delta_2,j} = \varphi_{\delta_2,j}/h^{\delta_2} + o\left(h^{\delta_2}\right)$ and amplitudes $A_{\delta_2,j}$, $j = 1, \ldots, L_2$, such that

$$P_{\Phi_{\delta_2,j}} A_{\delta_2,j} = O\left(h^\infty\right)\,,$$

on $\widetilde{U}_2$. Repeating the same argument for all remaining positive edge $\mathcal{L}_k$, $k > 1$, then yields a set of independent quasimodes

$$u_{\delta_k,j} = e^{\Phi_{\delta_k,j}} A_{\delta_k,j}\,,$$

of P on $U := \bigcap_k \widetilde{U}_k$. Moreover, for the horizontal edge $\mathcal{L}_1$ we have that $P = h^{l_0} T_0 + o\left(h^{l_0}\right)$ with

$$T_0 = a_0(x)\partial_x^{L_1} + \sum_{k=0}^{L_1-1} b_k(x)\partial_x^k\,,$$

for some $b_k \in C^\infty(I)$ and the coefficient a_0 of $E_{\delta_2}(P)(x, \zeta)$. Since $a_0(x) \neq 0$ for all $x \in U_2$ it holds that T_0 is elliptic. Thus, applying Proposition 3.1.7 yields L_1 polyhomogeneous quasimodes $A_{0,j}$ for P on U_2. In total, this yields

$$m = \sum_{\delta_k \in \Delta} L_k$$

independent quasimodes on U, of which L_0 are polyhomogeneous. $\square$

3.3.3 Multiple Roots

In the end of this chapter we want to briefly discuss the phenomenon of operators P with edges $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$ whose solutions ζ of $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ have multiplicities $r \in \mathbb{N}$, $r \geq 1$, globally on I. Due to the structure of T_δ in (3.16), it can be displayed as

$$T_{\delta,\zeta} = \partial_\zeta(E_\delta(P))(\cdot,\zeta)\cdot\partial_x + \sum_{\substack{\lambda=(k,\alpha)\\ \lambda\in\mathcal{L}\cap\Lambda}} a_\lambda \frac{k(k-1)}{2}\zeta^{k-2}[\partial_x,\zeta]\,,$$

where $\delta = \delta(\mathcal{L})$. Thus, the leading coefficient $\partial_\zeta(E_\delta(P))(\cdot,\zeta)$ of $T_{\delta,\zeta}$ vanishes, since for solutions ζ of multiplicity $r \in \mathbb{N}$ it holds that $\partial_\zeta^\rho(E_\delta(P))(\cdot,\zeta) = 0$, for all $\rho < r$. The canonical replacement for the induced transport operator in these cases is the operator induced by the r-th lowest summand in the expansion of (2.15).

Definitions & Proposition

Definition 3.3.13 (r-δ-Transport Operator)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(I)$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$ and let $r \in \mathbb{N}$. For any $k \in \mathbb{N}_0$ and $\sigma \in S_k$ let $(\xi^r\zeta^{k-r})^\sigma$ be the permutation of factors of the product $\xi^r\zeta^{k-r}$.

Then we call

$$T_{\delta,\varphi',r}(P) := \iota_{\varphi',\partial_x}\left[\sum_{\substack{\lambda=(k,\alpha)\\ \lambda\in\mathcal{L}\cap\Lambda}}\sum_{\sigma\in S_k} a_\lambda\cdot(\xi^r\zeta^{k-r})^\sigma\right],$$

the r-th order *induced transport operator* with respect to δ.

Note that $[\zeta,\xi] \neq 0$. Thus, $\xi^r\zeta^{k-r} \neq (\xi^r\zeta^{k-r})^\sigma$ for many $\sigma \in S_k$. The simplicity of a solution φ' of $E_\delta(P)(\cdot,\varphi') = 0$ to show the ellipticity of $T_{\delta,\varphi'}$ can be replaced by requiring φ' to have maximal, constant multiplicity for $T_{\delta,\varphi',r}$.

Proposition 3.3.14. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^\Lambda(I)$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$ and let $r \in \mathbb{N}$. Let φ' be a solution of $E_\delta(P)(\cdot,\varphi') = 0$ with constant multiplicity r.*

Then the r-δ-induced transport operator $T_{\delta,\varphi',r}$ of P has order r and is elliptic.

Proof. Let φ' be a solution of $E_\delta(P)(\cdot,\varphi') = 0$ with constant multiplicity $r \in \mathbb{N}$. The r-δ-induced transport operator $T_{\delta,\varphi',r}$ is a differential operator of the form

$$T_{\delta,\varphi',r} = \left(\sum_{\lambda\in\Lambda\cap\mathcal{L}} a_\lambda\cdot\frac{k!}{(k-r)!}\cdot(\varphi')^{k-r}\right)\partial_x^r + \sum_{j=0}^{r-1} A_j\partial_x^j\,,$$

with a leading coefficient that can be rephrased using (2.12) to

$$(\partial_\zeta^r E_\delta(P))(\cdot,\varphi') = \sum_{\lambda\in\Lambda\cap\mathcal{L}} a_\lambda\cdot\frac{k!}{(k-r)!}\cdot(\varphi')^{k-r}\,.$$

The multiplicity r of φ' with respect to $E_\delta(P)(\cdot,\zeta) = 0$ is constant. Thus, the leading coefficient of $T_{\delta,\varphi',r}$ does not vanish, i.e. $(\partial_\zeta^r E_\delta(P))(x,\varphi'(x)) \neq 0$ for all $x \in I$. In particular, the r-δ-induced transport operator is elliptic on I and $\operatorname{ord} T_{\delta,\varphi',\rho} = r$. □

Due to the higher order coefficient $h^{l_\delta + r\delta}$ of $T_{\delta,\varphi',r}$ in the expansion of $P_{\delta,\varphi'}$ we need to extend the margins of separation for them to be useful.

Definition 3.3.15 (r-δ-Separation)**.** Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \operatorname{Diff}^\Lambda(I)$ be a generalized semi-classical operator. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ be an edge of the lower boundary with slope $\delta > 0$ and let $r \in \mathbb{R}$.

Then we call P *r-δ-separated*, if

$$\Lambda \cap \operatorname{conv}\left(A_\delta(l_\delta) \cup A_\delta(l_\delta + r\delta)\right)^\circ = \emptyset\,,$$

with l_δ given by (3.14) and A_δ defined in (3.13). Additionally, if

$$\Lambda \cap A_\delta\left(l_\delta + r\delta\right) = \emptyset\,,$$

then we call P *strictly* r-δ-separated. P is called δ-separated, if there is an $r \in \mathbb{N}$ such that P is r-δ-separated.

Remark 3.3.16. As in (3.21) the transport operator T for r-δ-separated operators P can be computed combinatorially and is given by

$$T = T_{\delta,\varphi',r} + \sum_{k=0}^{m} a_{(k,l_\delta+(r+k)\delta)} \cdot (\varphi')^k\,, \tag{3.26}$$

where $a_{(k,l_\delta+(r+k)\delta)} \equiv 0$, if and only if $(k, l_\delta + (r+k)\delta) \notin \Lambda$.

Remark 3.3.17. We will not cover the question of how to resolve general non-separated operators with multiplicities higher than one. Other than in Proposition 3.3.8, higher order multiplicities $r > 1$ of the solution φ' of $E_\delta(P)(\cdot,\varphi') = 0$ impose significantly worse starting points to resolve $P_{\delta,\varphi'}$. This is due to the existence of points $(k,\alpha) \in \Lambda(P_{\delta,\varphi})$ with $0 < k < r$ and $\alpha < l_\delta + r\delta$.

In general, $\Lambda(P_{\delta,\varphi})$ is contained in a upward facing, closed cone with base point $(0, l_\delta)$ with a vertical side and a side with slope δ (see Figure 3.6). Non-separateness corresponds to the existence of points λ in the interior of the triangle $\Delta((0,l_\delta),(0,l_\delta + r\delta),(r,l_\delta + r\delta))$ for all $r > 0$, see Figure 3.6. One can eliminate these points by repeating the general process of constructing quasimodes for $P_{\delta,\varphi}$ with respect to edges $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P_{\delta,\varphi}))$ that are contained in that triangle, i.e.

$$\mathcal{L} \subset \Delta((0,l_\delta),(0,l_\delta + r\delta),(r,l_\delta + r\delta))\,.$$

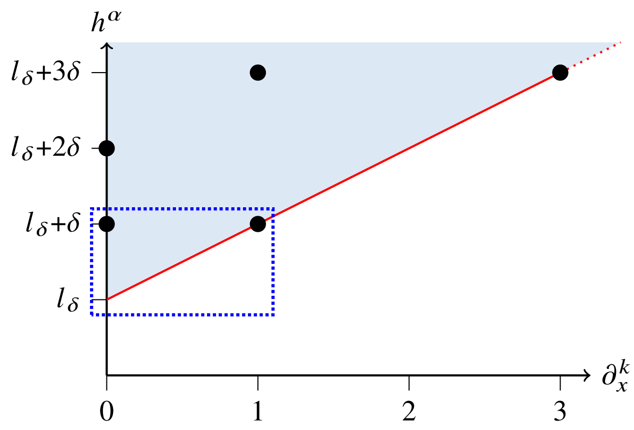

Figure 3.6: The set of exponents $\Lambda(e^{-\varphi/h^\delta}Pe^{\varphi/h^\delta})$ of a semi-classical operator P after conjugation. The red dashed lines show the extension of the cone based at $(0, l_\delta)$. The blue dotted box highlights the triangle in the cone underneath $\{\alpha = l_\delta + \delta\}$, which is empty if and only if P is 1-δ-separated.

Central Statement

We will prove the last version of the existence of quasimodes in the regular case, Theorem 3.3.18, including solutions of their corresponding eikonal equations with higher order multiplicities.

Theorem 3.3.18. *Let $I \subset \mathbb{R}$ be an interval, $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(I)$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with slope $\delta > 0$. Let $\varphi \in C^\infty(I)$ be a solution of $E_\delta(P)(\cdot, \varphi') = 0$ with constant multiplicity $r \in \mathbb{N}$ and assume that P is r-δ-separated. Let T be given by* (3.26).

Then there are quasimodes $u_j = e^{\varphi/h^\delta}A_j$, where $A_j := \sum_{k=0}^\infty a_{j,k}h^{\beta_k}$ with $a_{j,k} \in C^\infty(I)$, and $a_{j,0} \in \ker T$ linearly independent, for $j = 1, \dots, r$, with $a_{j,0} \not\equiv 0$, such that

$$P_{\delta,\varphi}A_j = O(h^\infty)\,.$$

Additionally, if P is strongly r-δ-separated, then $T = T_{\delta,\varphi',r}$.

Proof. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ be an edge of $P \in \mathrm{Diff}^\Lambda(I)$ with slope δ and let φ be a solution of $E_\delta(\cdot, \varphi') = 0$ with multiplicity $r \in \mathbb{N}$. Since P is r-δ-separated we have

$$e^{-\varphi}Pe^{\varphi} = 0 + h^{r\delta + l_\delta}T + o(h^{r\delta + l_\delta})\,,$$

for $l_\delta \in \mathbb{R}$ given by (3.14) and T defined in (3.26). By Remark 3.3.16 we have $T = T_{\delta,\varphi',r}$, if and only if $\Lambda \cap A_\delta(l_\delta + r\delta) = \emptyset$. Since φ' has constant multiplicity r, Proposition 3.3.14 yields that $\varphi'(x) \neq 0$, for all $x \in I$. Hence $T_{\delta,\varphi,r}$ is an elliptic differential operator of order r, and in particular T is elliptic, too. Thus, we can apply Proposition 3.1.7 to the leading term $T = T_{\delta,\varphi,r}$ of $P_{\varphi,\delta} = h^{l_\delta + r\delta}T + o(h^{l_\delta + r\delta})$, yielding polyhomogeneous quasimodes $A_j := \sum_k a_{j,k}h^{\beta_k}$. Hence $u_j = e^{\varphi/h^\delta}\sum_k a_{j,k}h^{\beta_k}$ is a quasimode of P, for any $j = 1, \dots, r$. $\square$

The case where the multiplicity of $\varphi'(x)$ in $E_{\delta(\mathcal{L})}(P)(x,\varphi'(x)) = 0$ is not continuous and thus has points with higher order zeros is the essential problem of the following chapters and will cause many new phenomena.

4 Resolved Operators

In this chapter we are interested in the construction of quasimodes for δ-singular operators. We will present the notion of *resolved* operators, a subclass of singular operators, which admit a construction of exponential-polyhomogeneous quasimodes with the methods developed in Chapter 3. We will see that the lack of regularity results in phase functions and amplitudes not being smooth in general. Thus, it is natural to consider operators on the quarter space $\mathbb{R}^2_+$ instead of the half space $\mathbb{H}$.

The singularities of these operators arise when coefficients corresponding to pairs of exponents λ, μ spanning an edge $\mathcal{L}$ vanish at $0 \in \mathbb{R}_+$. These singularities result in pointwise jumps in multiplicity, i.e. crossing points, or in unbounded behavior of solutions of the eikonal equation in $x = 0$. Crossing points lead to a vanishing leading coefficient of the induced transport operator

$$T_{\delta,\varphi'}(P) = \sum_{\substack{\lambda=(k,\alpha)\\ \lambda\in\mathcal{L}\cap\Lambda}} a_\lambda \cdot k(\varphi')^{k-1}\left(\partial_x + \frac{k-1}{2}\frac{\varphi''}{\varphi'}\right),$$

whose zero $x = 0$ coincides with the zero of a_λ. Thus, one needs to have an overview of the vanishing orders of all coefficients a_λ in $x = 0$ in the expansion of P. Unbounded phase functions lead to a b-transport operator in the same way crossing solutions of the eikonal equations do. The focus of this chapter is to determine when these singular operators admit polyhomogeneous amplitudes.

The new quantity we have to take into account to determine a priori whether an operator admits the construction of a quasimode is the vanishing order of the coefficients. As we will show in Section 4.2, this is possible if the increase of homogeneity in x of the coefficients along a lower edge $\mathcal{L}$ is minimal compared to all other points in Λ. The minimality in increase directly corresponds to the considerations of the Newton polygon and thus leads to the introduction of the three-dimensional *Newton polyhedron*, additionally accounting for these homogeneities. Using these polyhedrons and an appropriate notion of lower boundary, we are able to characterize resolved operators geometrically, which can easily be transferred to Chapter 5 and the resolution of general singular operators.

In contrast to Chapter 3, it is convenient to treat generalized semi-classical operators in terms of b-differential operators, although it is not necessary to do so. Due to the excessive use of blow-ups in Chapter 5, there are two different types of b-vector fields V that have to be considered when treating singular operators of the form $P = \sum_k A_k(x,h)V^k$: horizontal b-vector fields $V = x\partial_x$ and hyperbolic b-vector fields $V = x\partial_x - y\partial_y$. In the end of this chapter we will be able to prove the existence of quasimodes constructively for both cases, leading to

Theorems 4.3.3 and 4.3.10. The chapter starts with a sequence of examples in Section 4.1 exploring the phenomena occurring when dropping δ-regularity. These examples motivate the definitions presented in Section 4.2 but can be skipped entirely.

4.1 Examples of Singular Operators

This section presents many examples of δ-singular operators and approaches to construct quasimodes for most of them. Recall Definition 3.2.16 that P is called δ-regular on $I \subset \mathbb{R}$, if the maximal edge $\mathcal{L} = \overline{\lambda\mu} \subset \partial\mathcal{P}(\Lambda(P))$ with slope $\delta > 0$ has width $|\mathcal{L}| > 0$ and if for all $x \in I$ we have that $a_\lambda(x) \neq 0$ and $a_\mu(x) \neq 0$. Its absence can lead to unbounded or crossing solutions of its associated eikonal equation

$$E_\delta(P)(\cdot, \varphi') = 0\,.$$

We will outline some classes of singular operators and their properties to show what sort of undesirable side effects occur with these. In particular, we will split this section into two parts: Subsection 4.1.1 shows examples of so called *resolved* operators, which will be introduced in Section 4.2. These are operators that admit exponential-polyhomogeneous quasimodes corresponding to an edge $\mathcal{L}$, despite being $\delta(\mathcal{L})$-singular. We aim to emphasize important observations in their construction. In its center is the distribution of *weights* along $\Lambda(P)$. The weight $\omega(\lambda)$ of a point $\lambda = (k, \alpha)$ is the homogeneity in x of its associated summand in the expansion of P and coincides with the difference $\mathrm{ord}_0(a_\lambda) - k$.

Subsection 4.1.2 exceeds the scope of Chapter 4 by presenting so called *unresolved* operators, i.e. operators that do not admit exponential-polyhomogeneous quasimodes. It briefly sketches possible ways to resolve these operators by the successive use of quasihomogeneous blow-ups in $0 \in \mathbb{R}_+$. The goal is to show how the analysis of the distribution of weights $\omega(\lambda)$ can be used to determine the appropriate parameters for the required blow-ups and to define a final state that does not require further resolution.

4.1.1 Examples I: Resolved Operators

This subsection contains examples of operators which will later be referred to as *resolved*. We will construct quasimodes following the methods developed in Chapter 3 and show that these have polyhomogeneous phases and amplitudes on $\mathbb{R}^2_+$.

Singular Unperturbed Operators

A part of the class of perturbed, singular operators are unperturbed operators. In particular, we will refer to Theorem 2.2.19, which says that for every asymptotic solution of a singular ordinary differential equation $\sum_k a_k y^{(k)} = 0$ there is a solution of this equation matching the approximative solution asymptotically. Unperturbed operators can and will occur repeatedly in the resolution of singularities in Chapter 5.

Example 4.1.1. Let $P := x^3\partial_x^2 + x\partial_x + 1$ and $I := \mathbb{R}_+$. We use the Newton polygon of the orders of differentiation and powers of x to compute asymptotic solutions of $Pu = 0$ at the singular point $x = 0$ of P according to Proposition 2.3.6. Its set of exponents (in powers of ∂_x^k and x^ω) is given by $\Lambda = \{(0,0), (1,1), (2,3)\}$. Choosing the edge $\mathcal{L} := \overline{(0,0),(1,1)} \subset \partial\mathcal{P}(\Lambda)$, its corresponding initial solution is given by $u_0(x) := x^{-1}$. This yields

$$Pu_0 = (x\partial_x + 1)x^{-1} + (x^3\partial_x^2)x^{-1} = 0 + 2\,.$$

A first order correction term u_1 would need to solve $Pu_1 = -2$, which is satisfied by $u_1 := -2$. In particular,

$$u_0 + u_1 = x^{-1} - 2$$

is a solution of $Pu = 0$. Following Proposition 2.3.6, another initial solution of $Pu = 0$ is given by $v_0 := e^{1/x}$, yielding

$$Pv_0 = (x^3\partial_x^2 + x\partial_x)e^{1/x} + e^{1/x} = e^{1/x}\,.$$

This can be rephrased to

$$e^{-1/x}Pe^{1/x} = -(x\partial_x - 2) + x\left((x\partial_x)^2 - x\partial_x\right)\,,$$

which is a system of b-transport equations with transport operators $T := x\partial_x - 2$ and remainder operator $R := x((x\partial_x)^2 - x\partial_x)$. An initial solution $a_0 \in \ker T$ is given by $a_0(x) := x^2$ and the first inhomogeneous term a_1 is the solution of $Ta_1 = Ra_0$, i.e. $Ta_1 = 6x^3$. Having determined these exponential-polyhomogeneous, asymptotic solutions of $Pu = 0$, we can apply Theorem 2.2.19 and obtain a second solution v which has $v_0 \cdot \sum_{k=0}^\infty a_k$ as asymptotic expansion as $x \to 0$.

Remark 4.1.2. All sets of exponents discussed after this example are with respect to h^α and ∂_x^k. However, powers of x^ω will be included in a 3-dimensional extension of sets of exponents in Section 4.2. These will be called *localized sets of exponents* and lead to the introduction of *Newton polyhedra* in Definitions 4.2.23 and 4.2.24.

Partial Regularity

We want to emphasize that δ-regularity was only required in Theorem 3.3.11 to obtain a basis of independent quasimodes for operators with multiple edges. In particular, if $\partial\mathcal{P}(\Lambda)$ consists of a single edge $\mathcal{L}$, simplicity of solutions φ' of the eikonal equation is sufficient for the existence of exponential-polyhomogeneous quasimodes, wherever these exists. Example 4.1.3 shows that there can be two simple solutions of a δ-singular operator of which one solution even extends smoothly to the δ-singular point $x = 0$. Thus, there is one exponential-polyhomogeneous quasimode whose amplitude is smooth in x on $\mathbb{R}_+$. The existence of an

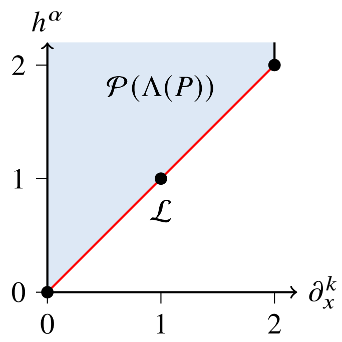

Figure 4.1: The set of exponents $\Lambda(P)$ associated to $P := x^2h^2\partial_x^2 + h\partial_x - 1$ and the lower edge $\mathcal{L}$ of the associated polygon $\mathcal{P}(\Lambda(P))$.

exponential-polyhomogeneous quasimode for the other, unbounded solution of the eikonal equation will be analyzed in greater generality Section 4.2.

Example 4.1.3 (Partial Regularity)**.** Let $P := x^2h^2\partial_x^2 + h\partial_x - 1$ and $I := \mathbb{R}_+$. The associated set of exponents is given by $\Lambda(P) = \{(0,0),(1,1),(2,2)\}$ (see Figure 4.1). Thus, P is not δ-regular with respect to its only edge $\mathcal{L} := \mathcal{P}(P) = \overline{(0,0),(2,2)}$ with slope $\delta(\mathcal{L}) = 1$, since $a_{(2,2)}(x) = x^2$ vanishes at $\{x = 0\}$. We analyze the behavior of the phase functions on I° as $x \to 0$. The δ-principal symbol of P for $\delta = 1$ is given by

$$E_\delta(P)(x,\zeta) = x^2\zeta^2 + \zeta - 1\,,$$

and the corresponding eikonal equation $E_\delta(P)(\cdot,\zeta) = 0$ has solutions

$$\zeta_\pm(x) = -\frac{1}{2x^2} \pm \sqrt{\frac{1}{4x^4} + \frac{1}{x^2}} = -\frac{1}{2x^2}\left(1 \mp \sqrt{1+4x^2}\right)\,, \tag{4.1}$$

which are simple for all $x > 0$. In particular, ζ_+ is defined and smooth in $x = 0$, since $\zeta_+(x) \sim -1/(2x^2)\left(1 - 1 + O(x^2)\right)$ as $x \to 0$. The asymptotic expansion of the conjugation of P with $\varphi'_+ := \zeta_+$ is a sum of its induced transport operator

$$T_{\delta,\varphi'_+} = (2x^2\varphi'_+ + 1)\partial_x + x^2\varphi''_+\,,$$

with coefficient h^1 and a single remainder operator $R_\delta = x^2\partial_x^2$ with coefficient h^2. Since both solutions $\zeta_\pm$ are simple, we can apply Theorem 3.3.9. However, since only ζ_+ is defined in $x = 0$, the theorem yields a quasimode u_+ corresponding to ζ_+ on $\mathbb{R}_+$ with smooth phase and amplitude. The other quasimode u_- corresponding to ζ_- is restricted to $(0,\infty)$. With our methods we cannot decide yet whether u_- exponential-polyhomogeneous on $\mathbb{R}_+$.

Remark 4.1.4. One can link the existence of the smooth solution φ'_+ of the eikonal equation in Example 4.1.3 to the existence of a regular *subedge* $\overline{(0,0),(1,1)} \subset \mathcal{L}$, i.e. the coefficients $a_{(0,0)}$ and $a_{(1,1)}$ do not vanish in $x = 0$.

Singular Minimum

Let P be a semi-classical operator with set of exponents $\Lambda \subset \mathbb{N}_0 \times \mathbb{R}$. A regular minimal point $\lambda_{\min} = (k_{\min}, \alpha_{\min}) \in \Lambda$, i.e. $a_{\lambda_{\min}}(0) \neq 0$, with its associated horizontal edge $\mathcal{L}_{\text{hor}} \subset \partial\mathcal{P}(\Lambda)$ and elliptic transport operator

$$T_0 = \sum_{\substack{\lambda \in \mathcal{L}_{\text{hor}} \cap \Lambda \\ \lambda = (k,\alpha)}} a_\lambda \partial_x^k$$

allows for a direct construction of polyhomogeneous quasimodes which are smooth in x via Proposition 3.1.7. However, $a_{\lambda_{\min}}(0) \neq 0$ is not required for the existence of quasimodes with polyhomogeneous behavior on $\mathbb{R}^2_+$. In some cases, we can allow vanishing coefficients $a_{\lambda_{\min}}$ of minimal points in $x = 0$.

The point $\lambda_{\min} = (k_{\min}, \alpha_{\min})$ corresponds to a summand

$$a_{\lambda_{\min}} h^{\alpha_{\min}} \partial_x^{k_{\min}} ,$$

in the asymptotic expansion of P and is the leading term of the transport operator T_0. In particular any transport equation will be of the form

$$T_0 u = R_k v ,$$

where R_k is a summand in $P = T + \sum_{k=0}^{\infty} h^{\beta_k} R_k$, v is known and we want to compute u. If $a_{\lambda_{\min}}$ vanishes in $x = 0$, these recurrent equations can produce a family of regular solutions if and only if the homogeneity in x of $a_{\lambda_{\min}} h^{\alpha_{\min}} \partial_x^{k_{\min}}$ is minimal amongst all summands in the expansion of P. More specific, denote the zero order of $a_\lambda = (k, \alpha) \in \Lambda$ at $x = 0$ by $l(\lambda)$ and the corresponding homogeneity in $x = 0$ by $\omega(\lambda) := l(\lambda) - k$. Then the minimality condition can be phrased as

$$\forall\, \lambda \in \Lambda\colon\ \omega(\lambda_{\min}) \leq \omega(\lambda) .$$

The number $\omega(\lambda) \in \mathbb{Z}$ will be called *weight* of λ in $x = 0$ in Definition 4.2.3.

Example 4.1.5 (Singular Minimum)**.** Let $P := x\partial_x + h$ on $I := \mathbb{R}_+$. Although the coefficient $a_{(1,0)}(x) = x$ of the highest order differential term vanishes at $x = 0$, the function $u_0 \equiv 1$ is a first order quasimode, i.e. $Pu_0 = O(h)$. Its first inhomogeneous transport equation

$$x\partial_x u_1 = -1$$

has a solution $u_1 = -\log$, $u = u_0 + hu_1$, and all succeeding solutions of $x\partial_x u_k = -u_{k-1}$ will have increasing powers of log, i.e. $u_k = (-1)^k \log^k/(k!)$. This suggests that we should make an ansatz

$$u = e^{\varphi/h^\delta} A ,$$

where φ and A are polyhomogeneous functions and $\delta = -1$. Then φ has to solve the equation

$$x\varphi' + 1 = 0\,.$$

Choosing the solution $\varphi := -\log$ for the ansatz and conjugating $P = x\partial_x + h$ yields a b-operator

$$e^{\log(x)\cdot h}\,(x\partial_x + h)\,e^{-\log(x)\cdot h} = x\partial_x\,.$$

Then any constant amplitude $A \in \mathbb{C}\setminus\{0\}$ yields not only a quasimode, but a solution $u(x,h) = e^{-\log(x)h}A$ of

$$Pu = 0\,.$$

Semi-Classical b-Operators

Another important example are semi-classical b-operators. Their invariance under the pullback of a blow-up, in the sense that they remain to be b-operators, will become relevant in the resolution of singular operators in Subsection 5.2.2. The essential difference to quasimodes of δ-regular operators is given by the possible presence of logarithms in the phase functions and a b-operator as an associated induced transport operator. The presence of logarithms is directly linked to the zero *weight* $\omega(\lambda)$ of each point $\lambda \in \Lambda$ for a b-operator.

Example 4.1.6 (Semi-Classical b-Operator)**.** Let $P := x^2h^2\partial_x^2 + 1$ and $I := \mathbb{R}_+$. The associated set of exponents is given by $\Lambda(P) = \{(0,0),(2,2)\}$. Thus, P is δ-singular with respect to its only edge $\mathcal{L} = \overline{(0,0),(2,2)} \subset \partial_{\!-}\mathcal{P}(\Lambda(P))$ with slope $\delta(\mathcal{L}) = 1$. In particular, it is a semi-classical b-operator, since

$$P = x^2h^2\partial_x^2 + 1 = h^2(x\partial_x)^2 - h^2(x\partial_x) + 1\,.$$

We construct its phases and amplitudes to analyze their behavior as $x \to 0$. According to the slope $\delta = 1$, its phase functions $\varphi_\pm$ are solutions of the eikonal equation

$$(x\varphi')^2 + 1 = 0$$

and are given by $\varphi_\pm := \pm i\log$ on $(0,\infty)$. The induced transport operator corresponding to $\varphi := \varphi_+$

$$T_{\delta,\varphi'} = 2x^2(\varphi')\partial_x + x^2\varphi'' = 2ix\partial_x - i$$

is a b-differential operator on $\mathbb{R}_+$. Instead of applying Theorem 3.3.9 for φ' on $(0,\infty)$, we compute the amplitude explicitly to analyze its behavior as $x \to 0$.

In combination with the only remainder operator $R_\delta = x^2\partial_x^2$ in the expansion of $P_{\delta,\varphi}$, the recurrent system of inhomogeneous transport equations admit polyhomogeneous solutions with resonances

$$i\,(2x\partial_x - 1)\,a_{k+1} = c_k\sqrt{x}\cdot\left(\log^k + o(\log^k)\right)$$

in the asymptotic series $a = \sum_{k=0}^{\infty} a_k h^k$ of the quasimode. Normalizing $T_{\delta,\varphi'}$ via conjugation with $\sqrt{x}$ yields

$$\widetilde{P} := \sqrt{x}^{-1} P_{\delta,\varphi}\sqrt{x} = h\cdot(2i(x\partial_x)) + h^2\cdot\left((x\partial_x)^2 - \frac{1}{4}\right),$$

showing that the origin of the increasing powers of log lies within the non-zero term $-1/4$ of $\sqrt{x}^{-1}R_\delta\sqrt{x}$. This can be addressed uniformly by the same means as in the case of sub-oscillations. Computing the difference in powers of h of the summands hT_δ and $h^2(-1/4)$, i.e. $2-1=1$, we can extend the phase function by a term with homogeneity of the difference in the powers of h to erase this term. Thus, we introduce an additional phase function $\phi_1 = -i/8\cdot\log$, yielding

$$\widetilde{P}_{\phi_1,-1} = h\cdot(2i(x\partial_x)) + h^2\cdot(x\partial_x)^2 + h^3\cdot\left(-\frac{2}{8}i(x\partial_x) + \frac{1}{8}i\right) + h^4\cdot\left(-\frac{1}{64}\right).$$

Although there is a new constant summand $h^4\cdot(-2i/8)$ present in the asymptotic expansion of $\widetilde{P}_{\phi_1,-1}$, its corresponding power in h is higher than the power of its predecessor $h^2(-1/4)$. Thus the first two summands a_0, a_1 of $u(x,h) = e^{(i/h+1/2-i/8h)\log(x)}A(x,h)$ are free of resonances and consist of single powered log terms, where the term $1/2$ emerged from the normalization. Continuing this procedure, we end up with an asymptotic phase function

$$\Phi = \left(i\frac{1}{h} + \sum_{k=0}^{\infty} c_k h^k\right)\cdot\log\,, \tag{4.2}$$

which yields a significantly simplified conjugated operator

$$P_\Phi = e^{-\Phi}Pe^{\Phi} = \widetilde{P}\circ\partial_x + O(h^\infty)\,,$$

where $\widetilde{P}$ is a b-operator. Hence $A\equiv 1$ is a quasimode solving $(e^{-\Phi}Pe^{\Phi})A = O(h^\infty)$ and consequently $u = e^{\Phi}A$ is a quasimode for P.

Double Exponential Behavior

A completely new type of behavior can be observed in the construction of quasimodes of semi-classical operators P with an edge $\mathcal{L} = \overline{\lambda\mu}$ with increasing weights from left to right endpoint, i.e. $\omega(\lambda) < \omega(\mu)$. Then there are solutions of the corresponding eikonal equation $E_\delta(P)(\cdot,\varphi') = 0$ which are unbounded as $x\to 0$ and hence the quasimode oscillates at both $\{h=0\}$ and $\{x=0\}$. In Example 4.1.7 we will construct quasimodes for a semi-classical operator of this type. For general operators one has to measure the *increase in weight* from λ to μ and compare it to the rest of the set of exponents to check whether this operator admits an exponential-polyhomogeneous quasimode. This leads to the notion of $\mathcal{L}$-resolved operators in Definition 4.2.18.

Example 4.1.7 (Double Exponential Behavior)**.** Let $P := h^2x^4\partial_x^2 + 1$ and $I := \mathbb{R}_+$. Its associated set of exponents is $\Lambda = \{(0,0),(2,2)\}$ whose Newton polygon $\mathcal{P}(\Lambda)$ consists of a single edge $\mathcal{L} := \overline{(0,0),(2,2)}$ with slope $\delta(\mathcal{L}) = 1$. The eikonal polynomial for $\delta = 1$ is given by

$$E_1(P)(x,\zeta) = x^4\zeta^2 + 1$$

and hence $\zeta_\pm := \pm ix^{-2}$ are the two simple solutions of its associated eikonal equation $E_1(P)(\cdot,\zeta) = 0$ on $(0,\infty)$.

Conjugating P with $\exp(-ir/(xh))$ then yields

$$e^{i/(xh)}Pe^{-i/(xh)} = h \cdot 2ix\,(x\partial_x - 1) + h^2 \cdot x^4\partial_x^2\,.$$

The recurrent transport equation $2ix(x\partial_x - 1)u_k = -x^4u_{k-1}''$ will successively increase the vanishing order of its solutions u_k by one. However, in this particular example we have $\ker\,(x\partial_x - 1) \subset \ker\partial_x^2$. Hence there is a polyhomogeneous amplitude $u_0(x) := x$, such that

$$P(e^{-i/(xh)}x) = 0\,.$$

In particular, $u(x) := e^{-i/(xh)}x$ is a solution of the equation $Pu = 0$. It is exponential-polyhomogeneous on $\mathbb{R}_+^2$.

Another important case of double exponential behavior that will occur frequently in Chapter 5 is operators $P = P(x,y,V)$, where $V = x\partial_x - y\partial_y$. Whenever we blow up $0 \in \mathbb{R}\times\mathbb{R}_+$ homogeneously, the pullback of the initial b-vector field $x\partial_x$ turns into a b-vector field $\beta^*(x\partial_x) = x\partial_x - y\partial_y$, at the right corner, where $y := h/x$. Thus the question arises of how potential quasimodes behave at the corner and how one is able to construct any in a neighborhood of the whole boundary $\partial[I\times\mathbb{R}_+,0]$.

Example 4.1.8. Let $V := x\partial_x - y\partial_y$ on $\mathbb{R}_+^2$, $F \in C^\infty(\mathbb{R}_+^2)$ with $F(p) \neq 0$ for all $p \in \partial\mathbb{R}_+^2$ and $P := (xy)^2V^2 + F$. One approach to construct quasimodes on $\mathbb{R}_+^2$ is to construct quasimodes at both boundary hypersurfaces, $H_1 := \{y = 0\}$ and $H_2 := \{x = 0\}$, and try to extend these to an exponential-polyhomogeneous function on $\mathbb{R}_+^2$. Let $\varphi_j \in C^\infty(H_j)$ for $j = 1,2$.

Conjugating P with $\varphi_1(x)/y$ and $\varphi_2(y)/x$ yield leading terms

$$x^2\left(x\varphi_1' + \varphi_1\right)^2 + F_{|H_1}$$

at H_1 as $y \to 0$ and

$$y^2\left(-y\varphi_2' - \varphi_2\right)^2 + F_{|H_2}$$

at H_2 as $x \to 0$. Thus, either solution of the respective eikonal equation

$$\begin{aligned}(x\varphi_1' + \varphi_1) &= \frac{\sqrt{-F_{|H_1}}}{x} \\ (y\varphi_2' + \varphi_2) &= \frac{\sqrt{-F_{|H_2}}}{y},\end{aligned}$$

is given by

$$\varphi_1(x) := \frac{1}{x}\int_{x_0}^{x} \frac{\sqrt{-F|_{H_1}(t)}}{t} dt \qquad \text{and} \qquad \varphi_2(y) := \frac{1}{y}\int_{y_0}^{y} \frac{\sqrt{-F|_{H_2}(t)}}{t} dt,$$

which behave as

$$\varphi(x) \sim \sqrt{-F(0,0)}\log(x)/x + o(x^{-1}) \text{ at } H_2 \text{ as } x \to 0$$

and

$$\varphi(y) \sim \sqrt{-F(0,0)}\log(y)/y + o(y^{-1}) \text{ at } H_2 \text{ as } y \to 0,$$

since $F(0,0) \neq 0$. Thus, the pair $\varphi_1(x)/y$ and $\varphi_2(y)/x$ admit a polyhomogeneous extension Φ to $\mathbb{R}^2_+$. Conjugating P with Φ then yields

$$P_\Phi = (xy) \cdot \Big((xy)(V\Phi)V + (xy)(V^2\Phi)\Big) + (xy)^2 \cdot V^2,$$

where $(V\Phi)(x,y) \sim \sqrt{-F(0,0)}\log(xy)/(xy) + o((xy)^{-1})$. Note that $(xy)V\Phi(x,y) \sim O(1)$. Thus, the sum

$$T := (xy)(V\Phi)V + (xy)(V^2\Phi),$$

is a first order, elliptic b-operator and has homogeneity 0 in (xy). Constructing amplitudes can be done either directly for the transport system $(xy)T + (xy)^2V^2$ or consecutively for every boundary hypersurface as we will show in Section 4.3.2

4.1.2 Examples II: Unresolved Operators

The second subsection we will discuss a selection of singular operators that do not allow for an immediate construction of exponential-polyhomogeneous quasimodes with the theory developed in Chapter 3 or tools similar to this approach. These will later be referred to as *unresolved operators*. They are not resolved in the sense that their distribution of *weights* $l - k$ of summands $x^l h^\alpha \partial_x^k$ in the expansion of an operator P is not ordered as required in Definition 4.2.18. We will present and develop ad hoc methods to resolve the operators using quasihomogeneous blow-ups and relate the relevant parameters $t > 0$ of these blow-ups to the weights of the points in the set of exponents.

Singular Minimum

The first example of this section is a variation of Example 4.1.5. By increasing the *weight* of the minimal point, it does not have minimal weight anymore. As mentioned before, this leads to a decrease in regularity of the functions a_k at $\{x = 0\}$ in the asymptotic expansion of the polyhomogeneous quasimode A as $k \to \infty$.

Example 4.1.9 (Singular Minimum II)**.** Let $P := x^4\partial_x^2 + h$ and $I := \mathbb{R}_+$. The set of exponents is given by $\Lambda = \{(0,1),(2,0)\}$ and its Newton polygon $\mathcal{P}(\Lambda)$ contains only one lower edge $\mathcal{L} = \overline{(0,0),(2,0)}$. Hence the operator $T := x^4\partial_x^2$ is the transport operator of P and $R := h$ is the only remainder operator.

A first order quasimode of $Pu = 0$ is given by $u_0(x) := 1$. Following the construction shown in Proposition 3.1.7, any subsequent term u_k has to be a solution of

$$x^4 u_k'' = -u_{k-1}\,,$$

for $k \geq 1$. Hence, for $k = 1$ it is given by $u_1(x) = -1/(6x^{-2})$. Proceeding in this way all, subsequent terms u_k will have more and more negative powers in x, namely $u_k(x) \sim x^{-2k}$. In particular, the asymptotic series $u = \sum u_k h^k$ does not allow a polyhomogeneous extension to $\mathbb{R}_+^2$.

Schrödinger Operator with Vanishing Potential

Probably the most famous example in semi-classical analysis is given by the Schrödinger operator

$$P := -h^2\partial_x^2 + V$$

for some potential $V \in C^\infty(I)$ on an interval $I := \mathbb{R}_+$. This operator allows for an exponential-polyhomogenous quasimode if $V > 0$ on I. However, when $V(0) = 0$ the Schrödinger operator is not δ-regular and its minimal point $(0,0)$ with coefficient $a_{(0,0)} = V$ does not have minimal weight. Example 4.1.10 continues the previous discussion about Schrödinger operators with linear vanishing potentials in Subsection 2.4.2 and shows their resolution by the use of blow-ups.

Quasimodes of the equation $Pu = 0$ and their corresponding resolution in terms of blow-ups for this operator are well known for linear and quadratic potentials. In either case, a quasihomogeneous blow-up of 0 in the combined space $(x,h) \in \mathbb{R}_+^2$ is included in the construction of a quasimode to make its asymptotic expansions uniform along the boundary hypersurfaces. The reason a blow-up is able to resolve the operator lies within the lower weight -2 of the summand $-h^2\partial_x^2$ compared to the weight 0 of V at $x = 0$. As we will show, pulling back a summand $x^l h^\alpha \partial_x^k$ by a homogeneous blow-up $\beta\colon [\mathbb{R}_+^2, 0] \to \mathbb{R}_+^2$ will lead to a shift in powers of h on the front face

$$\beta^*\left(x^l h^\alpha \partial_x^k\right) = x_{\mathrm{ff}}^l h^{\alpha-\omega}\partial_{x_{\mathrm{ff}}}^k\,,$$

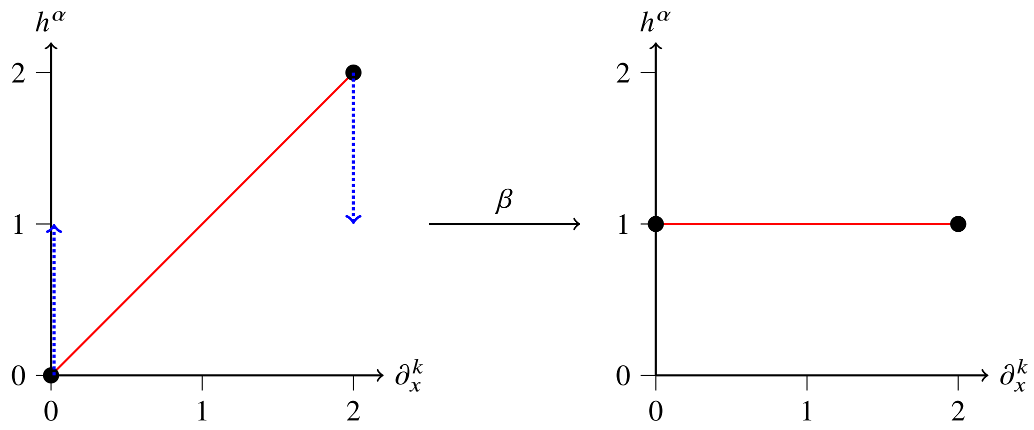

Figure 4.2: The set of exponents $\Lambda(P)$ associated to $P = -h^2\partial_x^2 - x$ in Example 4.1.10 and its transformation induced by the quasihomogeneous blow-up of 0 corresponding to the quasihomogeneous projective coordinate $x_{\mathrm{ff}} := x/h^\gamma$ with $\gamma = 2/3$. The dotted, blue arrows indicate the direction in which the points $\lambda \in \Lambda(P)$ are shifting. The second diagram represents the set of exponents $\Lambda(\beta^*P)$ of β^*P at the front face of $[\mathbb{R}^2_+, \{0\}]_\gamma$.

where $\omega = l - k$ and $x_{\mathrm{ff}} := x/h$ is the induced coordinate along the front face. This shift of powers also results in a transformation of the set of exponents at the front face, which will be the core of the resolution algorithm in Chapter 5. We will use this idea with quasihomogeneous blow-ups to achieve that the point of the Schrödinger operator $(2, 2) \in \Lambda$ with lowest weight becomes the minimal point on the front face.

Example 4.1.10 (Singular Potential)**.** Let $P := -h^2\partial_x^2 - x$ and $I := \mathbb{R}_+$. Its set of exponents is given by $\Lambda(P) = \{(0, 0), (2, 2)\}$ and P is δ-singular for $\delta = 1$. In particular, the solutions of its eikonal equation

$$(\varphi')^2 + x = 0\,,$$

given by $\varphi'_\pm(x) = \pm i\sqrt{x}$, are not simple in $x = 0$.

By application of the chain rule one can see that any solution u of $Pu = 0$ has non-trivial behavior in terms of the quasi-projective coordinate $x_{\mathrm{ff}} = x/h^\gamma$, since

$$\left(h^2\partial_x^2 + x\right) u\left(\frac{x}{h^\gamma}\right) = h^{2-2\gamma}u''\left(\frac{x}{h^\gamma}\right) + h^\gamma\left(\frac{x}{h^\gamma}\right)\cdot u\left(\frac{x}{h^\gamma}\right) = 0$$

is a non-trivial differential equation if $\gamma = 2/3$. Introducing the associated quasihomogeneous blow-up of $0 \in \mathbb{R}^2_+$ with weights 2 and 3 and with front face $\mathrm{ff} := \beta^{-1}(0)$ yields

$$(\beta^*P)_{|\mathrm{ff}} = h^{1/3}\left(-\partial_{x_{\mathrm{ff}}}^2 - x_{\mathrm{ff}}\right)\,.$$

It is important to notice that the set of exponents associated to $(\beta^*P)_{|\mathrm{ff}}$

$$\Lambda(\beta^*P_{|\mathrm{ff}}) = \{(0, 1/3), (2, 1/3)\}\,,$$

consists of two leveled points and thus its associated Newton polygon contains only a horizontal lower edge $\mathcal{L}$, compare Figure 4.2. Moreover, $(\beta^* P)_{|\mathrm{ff}}$ is an elliptic differential operator on $\mathrm{ff} \setminus \{x_{\mathrm{ff}} = +\infty\} \cong \mathbb{R}_+$ and thus we can apply Proposition 3.1.7 along $\mathrm{ff} \setminus \{x_{\mathrm{ff}} = +\infty\}$. To determine the behavior of $\beta^* P$ at the right face rf we compute the pullback of ∂_x in the quasi-projective coordinates $\eta := h/x^{3/2}$ and $r := x^{1/2}$ according to Definition 2.1.19,

$$\beta^* \partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \eta}{\partial x}\partial_\eta = \frac{1}{2}r^{-1}\partial_r - \frac{3}{2}r^{-2}\eta\partial_\eta = r^{-2}\left(\frac{1}{2}r\partial_r - \frac{3}{2}\eta\partial_\eta\right).$$

Thus, $\beta^* \partial_x$ is the product of a b-vector field $V := \frac{1}{2}r\partial_r - \frac{3}{2}\eta\partial_\eta$ with an additional factor r^{-2}. It holds that $(\beta^* \partial_x)^2 = r^{-4}(V^2 - 2V)$. The pullback of P is then given by

$$(\beta^* P)_{|\mathrm{rf}} = (r^6\eta^2)r^{-4}\widetilde{P} + r^2 = r^2\left(\eta^2(V^2 - 2V) + 1\right). \tag{4.3}$$

Conjugating the pullback with $e^{\varphi(r)/\eta}$ yields an eikonal equation

$$\left(\frac{1}{2}r\varphi' + \frac{3}{2}\varphi\right)^2 + 1 = 0,$$

which can be transformed to $[(r^3\varphi)']^2 = -4r^4$. In particular, there are two smooth, non-vanishing solutions $\varphi_\pm(r) := \pm 2i/3$ on the right face.

It remains to be investigated that the oscillation of the solutions of $\left(\partial_{x_{\mathrm{ff}}}^2 + x_{\mathrm{ff}}\right)u = 0$ at its singular point $x_{\mathrm{ff}} = +\infty$ matches the oscillation of φ at rf and if this is generally true. This will be done in Section 5.3.

Regular Remainder Terms (Splitting)

The transformation of the set of exponents seen in Example 4.1.10 by introducing a rescaled variable $x_{\mathrm{ff}} = x/h^\gamma$ gives an ad hoc idea of how to resolve some singular operators. In Example 4.1.11 we increase the complexity of the δ-singular operator by allowing for three summands, where one corresponding point $\lambda \in \Lambda$ will be contained in the interior of the Newton polygon and will have the lowest weight. Due to its lower weight, the interior point λ will decrease its distance to the subspace spanned by edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ after the pullback via the quasihomogeneous blow-up β_t, as t grows. This inevitably leads to the *collision* of the point λ and the $\mathcal{L}$ for sufficiently large values of t on the front face (see Figure 4.3). Its biggest impact is the change of the eikonal equation on the front face, now being defined by the coefficients of all three points.

Geometrically, the edge $\mathcal{L}$ of the Newton polygon will break into two edge, if the interior point contained in the interior of the vertical strip above the $\mathcal{L}$. Determining the maximal parameter t for which the edge is stable leads to the notion of *collision time* in Definition 5.2.1. We interpret the parameter t as time in which the points $\lambda \in \Lambda$ move in the plane. Collision times are an essential part of the resolution in Algorithm 1.

Example 4.1.11 (Splitting Edge)**.** Let $P := x^2h^3\partial_x^2 + h^3\partial_x - x$ and $\mathbb{R}_+$. The operator's set of exponents is given by $\Lambda(P) = \{(0,0), (1,3), (2,3)\}$ and its associated polygon is bounded

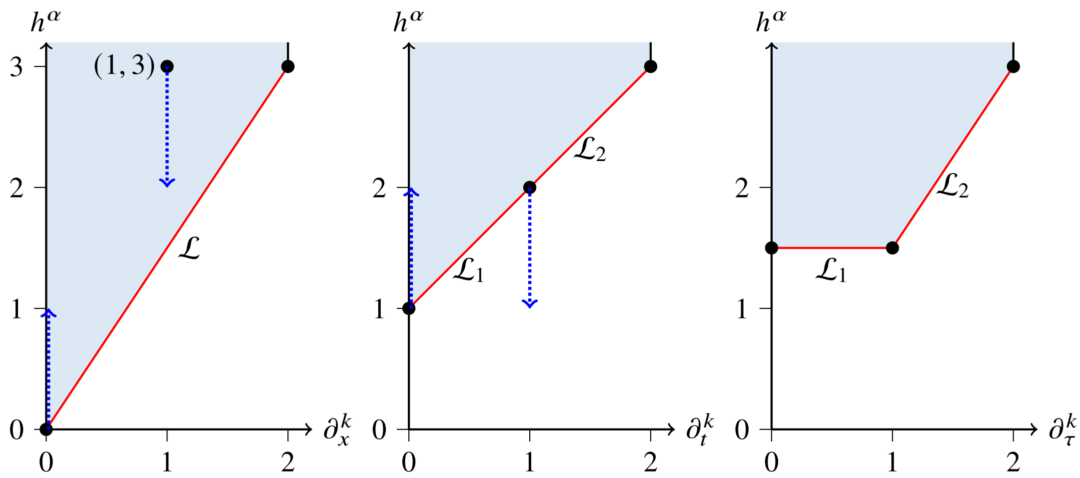

Figure 4.3: The picture on the left shows the set of exponents $\Lambda(P)$ and its corresponding Newton polygon $\mathcal{P}(\Lambda(P))$ associated to $P := x^2h^3\partial_x^2 + h^3\partial_x - x$ in Example 4.1.11. Due to the relatively low weight of the middle summand in the asymptotic expansion of P, the central point $(1,3)$ shortens its distance towards $\mathcal{L}$ and eventually breaks through, splitting $\partial\mathcal{P}(\Lambda(P))$ into two edges. The picture in the middle and on the right show the sets of exponents after the application of a quasihomogeneous blow-up with parameters $s = 1$ and $s = 3/2$. The splitting of $\mathcal{L} \subset \partial\mathcal{P}(\Lambda(P))$ happens once $s > 1$.

from below by a single edge $\mathcal{L} := \overline{(0,0),(2,3)} = \partial\mathcal{P}(P)$. Thus, P is δ-singular for $\delta = 3/2$ on $\mathbb{R}_+$, since $a_{(0,0)}(0) = a_{(2,3)}(0) = 0$. We expect both quasimodes to have similar asymptotic behavior as $x \longrightarrow 0$. The associated eikonal equation

$$(x\varphi')^2 - x = 0$$

has two solutions $\varphi_\pm(x) := \pm 2\sqrt{x}$, which are simple on $(0,\infty)$. The transport operator of P_{δ,φ'_+},

$$T = T_{\delta,\varphi'_+} + \varphi'_+ = 2x^{-1/2}\left(x^2\partial_x - \frac{1}{4}x + 1\right),$$

is a scattering operator with additional factor $x^{-1/2}$, where T_{δ,φ'_+} was defined in (3.16). Moreover, the remainder operator is given by $R := \partial_x + x^2\partial_x^2$ with ∂_x having the lowest homogeneity as $x \longrightarrow 0$. Thus, one has to solve the recurrent transport equations of the form

$$2x^{-1/2}\left(x^2\partial_x - \frac{1}{4}x + 1\right)a_k = \left(\partial_x + x^2\partial_x^2\right)a_{k-1},$$

effectively resulting in a decrease of regularity of a_k in x in steps of $3/2$ at $x = 0$ as $k \longrightarrow \infty$. This defect is due to the relatively low homogeneity $0 - 1 = -1$ of $x^0\partial_x$ compared to $2 - 2 = 0$ of $x^2\partial_x^2$.

We assume that there is a small scale $t \in \mathbb{R}_{>0}$ and an associated blow-up $\beta_t \colon [\mathbb{R}^2_+, 0]_t \to \mathbb{R}^2_+$ at least partially resolving P on its front face $\mathrm{ff} := \beta_t^{-1}(0)$. However, we have no method of determining any such scale.

For $t = 1$, the quasihomogeneous blow-up $\beta_t \colon [\mathbb{R}^2_+, 0]_t \to \mathbb{R}^2_+$ defined by $x_{\mathrm{ff},t} := x/h^t$ changes the geometry of $\mathcal{P}(\Lambda(\beta_t^* P_{|\mathrm{ff}}))$ significantly, compared to $\mathcal{P}(\Lambda)$ (see Figure 4.3). It is the smallest value t such that the shifted interior point starting at $\lambda = (2,3)$ is contained in the lower boundary. For each $t > 1$ it holds that the lower boundary of the polygon associated to $\beta_t^* P_{|\mathrm{ff}}$ splits into two non-collinear edges

$$\partial_{\!-}\mathcal{P}(\Lambda(\beta_t^* P_{|\mathrm{ff}})) = \mathcal{L}_1 \cup \mathcal{L}_2 \,,$$

with $\mathcal{L}_1 := \overline{(0,t),(1,3-t)}$ and $\mathcal{L}_2 := \overline{(1,3-t),(2,3)}$ (see Figure 4.3). At $t = 1$ these two lines are contained in $\mathcal{L}$. The corresponding operator on the front face of the homogeneous blow-up $\beta = \beta_1$ is given by

$$(\beta^* P)_{|\mathrm{ff}} = h^3 x_{\mathrm{ff}}^2 \partial_{x_{\mathrm{ff}}}^2 + h^2 \partial_{x_{\mathrm{ff}}} - h x_{\mathrm{ff}} \,,$$

with projective coordinates $x_{\mathrm{ff}} := x/h$ and h. It is δ-singular for $\delta = 1$ since the coefficients of both boundary points vanish at $x_{\mathrm{ff}} = 0$, i.e. $a_{(2,3)}(0) = a_{(0,1)}(0) = 0$.

The observation that $\mathcal{L}$ splits into two edges at ff if $t > 1$ suggests that we should treat the parts of $\beta^* P$ associated to either $\mathcal{L}_1$ and $\mathcal{L}_2$ differently. The operator

$$P_{\mathcal{L}_1} := \sum_{\lambda \in \mathcal{L}_1} a_\lambda \partial_{x_{\mathrm{ff}}}^k = h^2 \partial_{x_{\mathrm{ff}}} - h x_{\mathrm{ff}} \,,$$

corresponding to the first subedge $\mathcal{L}_1$ has a singular potential, resembling the Schrödinger operator covered in Example 4.1.10. Reproducing the strategy to resolve the singularity of the latter we introduce yet another quasihomogeneous blow-up $\beta_{1/2}$ of the left corner $\{0\} \subset \mathrm{ff}$ with respect to $t = 1/2$, yielding

$$(\beta_t^* \beta^* P)_{|\mathrm{lff}} = h^3 \tau^2 \partial_\tau^2 + h^{3/2} \left(\partial_\tau - \tau \right) \,,$$

where $\tau := x_{\mathrm{ff}}/h^{1/2}$ is the induced coordinate along $\mathrm{lff} := \beta_t^{-1}(0)$. Its lowest order summand $T := \partial_\tau - \tau$ is elliptic and thus admits polyhomogeneous quasimodes on the most recent front face lff when we apply Corollary 3.1.9. There are two caveats which we will have to check in Chapter 5, precisely Lemma 5.3.16 and Proposition 5.1.16:

(i) The compatibility of elements $u_0 \in \ker T$ with the oscillation u_{ff} at ff with respect to $\mathcal{L}_1 \subset \partial_{\!-}\mathcal{P}(\beta^* P)$ at the central corner $\mathrm{lff} \cap \mathrm{ff}$, as in Example 4.1.10.

(ii) The compatibility of local solutions at the bottom corner $\mathrm{ff} \cap \mathrm{rf}$, where we have oscillatory quasimodes at both ff and rf.

Returning to the (first) front face ff, the partial polynomial of $E_{\delta(\mathcal{L})}(\beta^* P)(x_{\mathrm{ff}}, \zeta) = x_{\mathrm{ff}}^2\zeta^2 + \zeta - x_{\mathrm{ff}}$ corresponding to $\mathcal{L}_2 = \overline{(1,2),(2,3)}$ given by

$$E_{\mathcal{L}_2}(\beta^* P) = (x_{\mathrm{ff}}^2 \cdot \zeta + 1)\zeta$$

essentially resembles the situation in Example 4.1.6. Its solution $\widetilde{\zeta}_2(x_{\mathrm{ff}}) := -1/x_{\mathrm{ff}}^2$ is a first order solution of the full eikonal equation $E_{\delta(\mathcal{L})}(\beta^* P)(x_{\mathrm{ff}}, \zeta) = 0$, since

$$E_{\delta(\mathcal{L})}(\beta^* P)(x_{\mathrm{ff}}, \widetilde{\zeta}_2(x_{\mathrm{ff}})) = x_{\mathrm{ff}}^2(-x_{\mathrm{ff}}^{-2})^2 - x_{\mathrm{ff}}^{-2} - x_{\mathrm{ff}} = 0 - x_{\mathrm{ff}} = O(x_{\mathrm{ff}})\,.$$

Thus, $\widetilde{\zeta}_2$ determines a solution $\zeta_\pm$ of the full eikonal equation $E_{\delta(\mathcal{L})}(\beta^* P)(x_{\mathrm{ff}}, \zeta) = 0$,

$$\zeta_\pm(x_{\mathrm{ff}}) := -\frac{1}{2x_{\mathrm{ff}}^2} \pm \sqrt{\frac{1}{4x_{\mathrm{ff}}^4} + \frac{1}{x_{\mathrm{ff}}}} = -\frac{1}{2x_{\mathrm{ff}}^2}\left(1 \mp \sqrt{1 + 4x_{\mathrm{ff}}^3}\right), \tag{4.4}$$

which satisfies $\zeta_-(x_{\mathrm{ff}}) \sim \widetilde{\zeta}(x_{\mathrm{ff}})$ as $x_{\mathrm{ff}} \to 0$. Thus, $\exp(\varphi/h)$ is a first order solution of $\beta^* P_{|\mathrm{ff}}$ corresponding to the subedge $\mathcal{L}_2$, where $\varphi_2' := \zeta_-$. In particular, the equations

$$(\beta^* P)_{1,\varphi_2} = h^2\left((2x_{\mathrm{ff}}^2\varphi_2' + 1)\partial_{x_{\mathrm{ff}}} + x_{\mathrm{ff}}^2\varphi_2''\right) + h^3 x_{\mathrm{ff}}^2\partial_{x_{\mathrm{ff}}}^2 = \frac{h^2}{x_{\mathrm{ff}}}\left(a \cdot x_{\mathrm{ff}}\partial_{x_{\mathrm{ff}}} - 2b(x_{\mathrm{ff}})\right) + h^3 x_{\mathrm{ff}}^2\partial_{x_{\mathrm{ff}}}^2$$

holds, where $a(0) = b(0) = 1$. Thus, the transport operator $T_\delta = 1/x_{\mathrm{ff}} \cdot (a \cdot x_{\mathrm{ff}}\partial_{x_{\mathrm{ff}}} - 2b)$ is a b-operator whose homogeneous solutions $a_0 \in \ker T_\delta$ behave asymptotically as $a_0(x_{\mathrm{ff}}) \sim x_{\mathrm{ff}}^2$ as $x_{\mathrm{ff}} \to 0$. Due to the additional factor $1/x_{\mathrm{ff}}$ in front of the b-operator T_δ, the 0-homogeneous remainder operator $R_\delta = x_{\mathrm{ff}}^2\partial_{x_{\mathrm{ff}}}^2$ does not produce resonances, and, thus we can apply Proposition 3.1.7 to construct an asymptotic amplitude $A = \sum_{k=0}^\infty a_k h^k$ for the quasimode corresponding to $\mathcal{L}_2$ at the front face.

Remark 4.1.12. If the interior point with lowest weight is contained outside of the vertical strip over the edge the same principle holds, essentially. But instead of splitting the edge itself, the point will hit the horizontal part of the Newton polygon, split it into two parts and merge the right part with the original edge.

Covered Points

Recall the discussion before Example 4.1.11. Let P be a semi-classical operator with set of exponents $\Lambda = \{\lambda, \mu, \tau\}$ and $\partial_- \mathcal{P}(\Lambda) = \overline{\mu\tau}$. One case we have not discussed was the presence of an interior point $\lambda \in \Lambda \cap \mathcal{P}(\Lambda)^\circ$ with minimal weight, which is placed on the boundary of the vertical strip above the lower edge $\mathcal{L}$. As before, the point λ will collide with $\mathcal{L}$ for sufficiently large $t > 0$ after being pulled back by a quasihomogeneous blow-up $\beta_t \colon [\mathbb{R}_+^2, 0]_t \to \mathbb{R}_+^2$. It will have the same impact on the eikonal equation as in the case of splitting in Example 4.1.11. However, the exact collision of the interior point λ with a boundary point results in the scattering of zeros of the leading coefficients of the edge along the front face.

Assume that $\lambda = (2,2)$ with $a_\lambda(x) = -1$, $\mu = (0,0)$ with $a_\mu(x) = 1$ and $\tau = (2,1)$ with $a_\tau(x) = x^2$. In particular, the corresponding semi-classical operator is given by $P = (x^2 h - h^2)\partial_x^2 + 1$. Then $(\beta_t^* a_\tau)(x_{\mathrm{ff}}) = x_{\mathrm{ff}}^2$, for all $t > 0$. But for the collision time $t = 2$ we have that the pulled back operator

$$(\beta_2^* P) = h \cdot \left(x_{\mathrm{ff}}^2 - 1\right) \partial_{x_{\mathrm{ff}}}^2 + 1$$

has a corresponding set of exponents consisting of two points. In particular, its leading coefficient vanishes at $x_{\mathrm{ff}} = \pm 1$ instead of $x_{\mathrm{ff}} = 0$.

This scattering of zeros along the front face will complicate the resolution of general semi-classical operators in Chapter 5. In the following example, we will show how one can obtain a quasimode by the use of consecutive blow-ups.

Example 4.1.13 (Covered Point)**.** Let $P := (hx - h^2(1+x))\partial_x + 1$ on $\mathbb{R}_+$. Its associated set of exponents is $\Lambda(P) = \{(0,0),(1,1),(1,2)\}$ and its Newton polygon $\mathcal{P}(\Lambda(P))$ is bounded from below by a single edge $\mathcal{L} = \overline{(0,0),(1,1)} \subset \partial_- \mathcal{P}$. Since the operator associated to $\mathcal{L}$

$$P_{\mathcal{L}} = hx\partial_x + 1$$

is a semi-classical b-differential operator of degree 1, one might expect a solution as in Example 4.1.6. The solution φ of the eikonal equation for $\delta = 1$ induced by the eikonal polynomial

$$E_\delta(P)(x,\zeta) = x\zeta + 1\,,$$

is given by $\varphi := -\log$ on $(0,\infty)$. However, the transport operator of $P_{\delta,\varphi'}$

$$T = x\partial_x + \left(\frac{1}{x} + 1\right)$$

is not a b-differential operator with smooth coefficients and neither is the remainder operator

$$R := (1+x)\partial_x\,.$$

Normalizing $P_{\delta,\varphi'}$ by conjugation with $u_0(x) := e^{1/x} \cdot x^{-1}$, where $u_0 \in \ker T$, yields a new remainder

$$R_0 = (1+x)\partial_x - \frac{(1+x)^2}{x^2}$$

whose associated inhomogeneous transport equation

$$Tu_k = R_0 u_{k-1}$$

does not admit uniformly polyhomogeneous solutions at $x = 0$. This is due to the non-vanishing coefficient $(1+x)$ of R, which, unlike the case of Example 4.1.6, cannot be erased

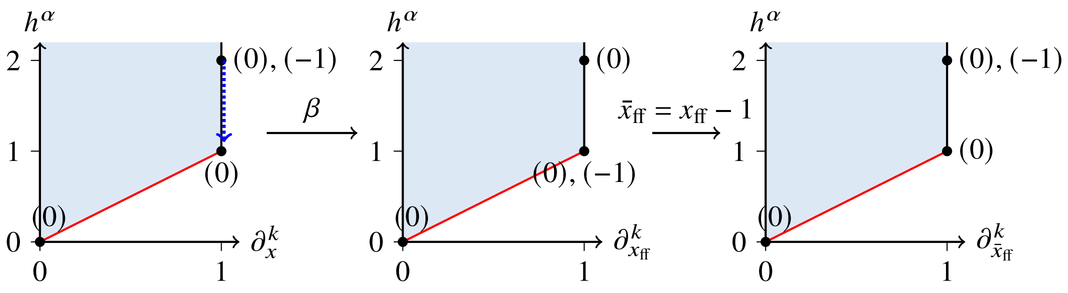

Figure 4.4: The transformations of the set of exponents $\Lambda(P)$ of $P := (hx - h^2(1+x))\partial_x + 1$ in Example 4.1.13 due to the homogeneous blow-up of $0 \in \mathbb{R}^2_+$ and the shift of the coordinate $\bar{x}_{\mathrm{ff}} := x_{\mathrm{ff}} - 1$ along the front face. The weights (in the brackets) of the points involved lead to a full recreation of the singularity at the beginning of the transformations on the front face at $\bar{x}_{\mathrm{ff}} = 0$.

by slower, logarithmic extension of the phase function, since these additions will only cancel out constant terms of the remainder operators. Due to the different weight in space of the points $(1,1)$ and $(1,2)$, we apply the same approach as in Example 4.1.11. This suggests a homogeneous blow-up of 0 in $\mathbb{R}^2_+$, yielding an operator

$$(\beta^* P)_{|\mathrm{ff}} = (h(x_{\mathrm{ff}} - 1) + h^2 x_{\mathrm{ff}})\partial_{x_{\mathrm{ff}}} + 1\,,$$

at the front face ff, where $x_{\mathrm{ff}} := x/h$ is the induced projective coordinate. As in the previous example, there is a new singularity at $x_{\mathrm{ff}} = 1$, where the leading term

$$h(x_{\mathrm{ff}} - 1)$$

of the operator $(\beta^* P)_{|\mathrm{ff}}$ has a singularity in $x_{\mathrm{ff}} = 1$. Introducing a coordinate shift $\bar{x}_{\mathrm{ff}} := x_{\mathrm{ff}} - 1$ then yields

$$(\beta^* P)_{|\mathrm{ff}} = (h\bar{x}_{\mathrm{ff}} + h^2(\bar{x}_{\mathrm{ff}} - 1))\partial_{\bar{x}_{\mathrm{ff}}} + 1\,,$$

which brings us back to the starting point with a singular point in $\bar{x}_{\mathrm{ff}} = 0$ (see Figure 4.4). Thus, it is not possible to resolve this operator immediately. Analyzing the coefficient $A_1(x,h) = hx - h^2(1+x)$ of $P = A_1\partial_x + A_0$ itself might be required. The zero level set of A_1

$$N := \left\{(x,h) : \frac{h(1+x)}{x} = 1\right\}$$

is not a p-submanifold on $\mathbb{R}^2_+$ (see Figure 4.5). However, being pulled back via blow-up $\beta_{\mathrm{hom}} \colon [\mathbb{R}^2_+; \{0\}] \longrightarrow \mathbb{R}^2_+$ to the blown-up space, the zero level set $\beta^*_{\mathrm{hom}} N = \{(x_{\mathrm{ff}} - 1)/x_{\mathrm{ff}} = h\}$ of $\beta^*_{\mathrm{hom}} A_1$ is a p-submanifold of $[\mathbb{R}^2_+; \{0\}]$. The introduction of a new coordinate $\widetilde{x}_{\mathrm{ff}} := (x_{\mathrm{ff}} - 1)/x_{\mathrm{ff}}$ shows that $\beta^*_{\mathrm{hom}} N = \{h - \widetilde{x}_{\mathrm{ff}} = 0\}$ is the zero level set of a homogeneous polynomial. Blowing up $h = \widetilde{x}_{\mathrm{ff}} = 0$ homogeneously and finally blowing up the pullback

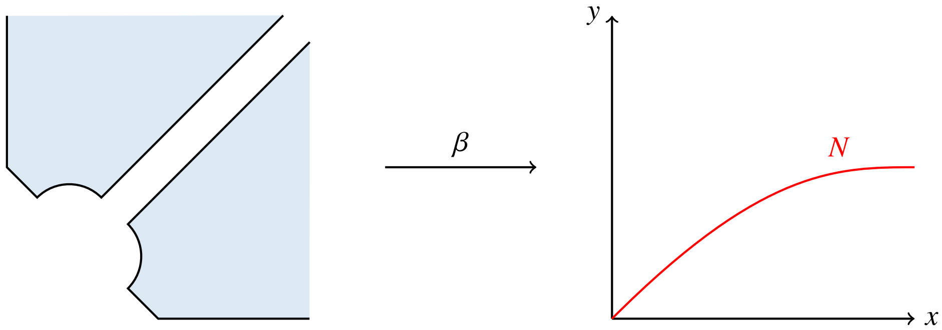

Figure 4.5: The resolution of $A_1(x,h) := hx - h^2(1+x)$ along its zero set $N \subset \mathbb{R}^2_+$. The process of regularizing A_1 erases the problem in Example 4.1.13, where the term $h^2(1+x)$ would create a loop in the direct attempt of resolving $P = A_1\partial_x + A_0$.

$\beta^*\beta^* N = \{1 - s = 0\}$, $s = \widetilde{x}_{\mathrm{ff}}/h$, yields

$$(\beta^*\beta^*_{\mathrm{hom}} A_1)(s,h) \sim h^4 s^2(1+s)\,, \quad \text{as } s \to 0\,.$$

In particular, $h^4s^2(1+s)$ is a regular, polyhomogeneous leading coefficient of $\beta^*\beta^*_{\mathrm{hom}}P$ at the new boundary $\beta^*\beta^*_{\mathrm{hom}}N = \{1 - s = 0\}$.

4.2 Combinatorial Geometry II: Newton Polyhedra

4.2.1 Semi-Classical b-Operators

In this subsection, we introduce the notion of *resolved* operators. These are a subclass of singular operators that admit exponential-polyhomogeneous quasimodes on $\mathbb{R}^2_+$. The relevant characteristic distinguishing the examples of Subsections 4.1.1 and 4.1.2 is their corresponding distribution of weights $l-k$ of the summands $x^l h^\alpha \partial_x^k$ of P. Controlling these combined weights $l-k$ of every summand in the expansion of P as $h \to 0$ is essential in the definition of resolved operators and is crucial in Subsection 5.2.2 as termination condition for the algorithm resolving general singular operators.

Definitions

An efficient way to approach resolved operators is to shift to the b-operator notation with the corresponding change in the sets of exponents.

Definition 4.2.1 (b-Set of Exponents)**.** Let $\Lambda \subset \mathbb{N}\times\mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$. Then the *b-set of exponents*, ${}^b\Lambda \subset \mathbb{N}\times\mathbb{R}$, is defined as the unique, minimal set, such that

$$P = \sum_{\lambda\in\Lambda} a_\lambda(x) h^\alpha \partial_x^k = \sum_{\lambda\in{}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k\,. \tag{4.5}$$

Remark 4.2.2. Note that by construction of the Newton polygon we have $\partial\mathcal{P}({}^b\Lambda) = \partial\mathcal{P}(\Lambda)$.

The essential tool to determine whether P is resolved is the aforementioned *weight* or homogeneity in x of a summand in the expansion of P. The weights $\omega(\lambda)$ correspond to the vanishing order of $\widetilde{a}_\lambda$ in $0 \in \mathbb{R}_+$ when switching to semi-classical b-operators.

Comparing weights of points along an edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ contains the relevant information to determine the asymptotic behavior of solutions of the eikonal equation $E_{\delta(\mathcal{L})}(P) = 0$. Their distribution affects the regularity of the solutions of the eikonal equations, as we saw in Examples 4.1.7 and 4.1.10.

Definition 4.2.3 (Weight of a b-Coefficient)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$ with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$. Then we call

$$ {}^b\omega(\lambda) := \mathrm{ord}_0(\widetilde{a}_\lambda) $$

the *weight* of $\lambda \in {}^b\Lambda$. The *relative increase of weights* along $\mathcal{L} := \overline{\lambda\mu}$, where $\lambda = (k_\lambda, \alpha_\lambda)$ and $\mu = (k_\mu, \alpha_\mu) \in {}^b\Lambda$, is defined by

$$ \gamma(\mathcal{L}) := \frac{{}^b\omega(\mu) - {}^b\omega(\lambda)}{k_\mu - k_\lambda} . $$

Remark 4.2.4. The conversion of a_λ to $\widetilde{a}_\lambda$ in (4.5) reduces the order of zeros for $k \geq 1$. However, P and $x^m P$ have the same quasimodes. Thus, without loss of generality, we can assume that ${}^b\omega(\lambda) \geq 0$.

Example 4.2.5. Let $P := h^2 x^2 \partial_x^2 - 1 = h^2 (x\partial_x)^2 - h^2 (x\partial_x) - 1$. Then the relative increase of weight along $\mathcal{L} := \overline{(0,0),(2,2)} \subset {}^b\Lambda(P)$ is given by $\gamma(\mathcal{L}) = 0$.

Remark 4.2.6. The increase of weights $\gamma(\mathcal{L})$ of an edge $\mathcal{L} \subset \partial_{-}\mathcal{P}({}^b\Lambda)$ in Definition 4.2.3 is the direct analogue to the slope

$$ \delta(\mathcal{L}) = \frac{\alpha(\mu) - \alpha(\lambda)}{k_\mu - k_\lambda} , $$

of an edge. This is a very important point of view and will be further developed in Subsection 4.2.3.

We want to transfer our notion of δ-symbol to operators generated by the b-vector field $x\partial_x$.

Definition 4.2.7 (b-δ-Symbol)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$ with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$ and let $\delta > 0$.

The b-δ*-symbol*, ${}^b\Sigma_\delta$, is the map

$$ \begin{aligned} {}^b\Sigma_\delta \colon \mathrm{Diff}^\Lambda(\mathbb{R}_+) &\to \mathcal{S}_h(\mathbb{R}_+) \\ \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k &\mapsto \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha \left(\xi + \frac{\zeta}{h^\delta}\right)^k . \end{aligned} $$

Its leading part ${}^bE_\delta$ is called b-δ*-principal symbol*.

Using the b-δ-symbol ${}^{\mathrm{b}}\Sigma_\delta$ for $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$, the associated eikonal equation ${}^{\mathrm{b}}E_\delta(P)(x, x\varphi') = 0$ can be reduced to solving the equation

$$(x\partial_x)\,\varphi = \zeta_j\,, \tag{4.6}$$

for any solution ζ_j of ${}^{\mathrm{b}}E_\delta(P)(x,\zeta) = 0$.

We can compute the asymptotic behavior of $\varphi(x)$ as $x \to 0$ for solutions φ of the eikonal equation, which corresponds to the increase of weight of the associated edge.

Proposition 4.2.8. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^\Lambda(\mathbb{R}_+)$ be an operator with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$. Let $\mathcal{L} := \overline{\lambda_1\lambda_n} \subset \partial_{\text{-}}\mathcal{P}({}^b\Lambda)$ be an edge with slope $\delta > 0$ such that $\Lambda \cap \mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$. Assume that $\gamma(\mathcal{L}) \leq \gamma(\overline{\lambda_i\lambda_j})$, for all $1 \leq i,j \leq n$. Suppose that P is δ-singular in $x = 0$ according to Definition 3.2.16. Let φ be a solution of the eikonal equation*

$${}^bE_\delta(P)(\cdot, x\varphi') = 0\,,$$

and $\gamma := \gamma(\mathcal{L})$ be the increase of weight of $\mathcal{L}$. Then for some $c \in \mathbb{C}$ we have

$$\varphi(x) \sim \begin{cases} c\log(x) & \text{if } \gamma = 0 \\ cx^{-\gamma} & \text{else} \end{cases}\,,$$

as $x \to 0$.

Proof. Let $\lambda_j = (k_j, k_j\delta + l_\delta)$ and $\mathcal{L} := \overline{\lambda_1\lambda_n} \subset \partial_{\text{-}}\mathcal{P}({}^b\Lambda)$ with $\Lambda \cap \mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$. Thus, the b-eikonal polynomial

$${}^bE_\delta(P)(\cdot,\zeta) = \left(\sum_{l=1}^{n} a_{\lambda_l} \cdot \zeta^{k_l - k_1}\right)\zeta^{k_1}$$

has solutions ζ_l, $l = 1, \ldots, k_n - k_1$, of its associated eikonal equation ${}^bE_\delta(P)(\cdot,\zeta) = 0$. Since $\gamma(\mathcal{L}) \leq \gamma(\overline{\lambda_i\lambda_j})$, the Newton polygon associated to $E_\delta(P)$ with respect to the powers of x and ζ has only one edge in its lower boundary with slope $\gamma = \gamma(\mathcal{L})$. Thus, for some $c_j \in \mathbb{C}$ it holds that

$$\zeta_j(x) \sim c_j \cdot x^{-\gamma}\,,$$

as $x \to 0$. Since φ_j is the solution of the b-differential equation $(x\partial_x)\varphi_j = \zeta_j$, the statement holds. $\square$

Remark 4.2.9. In the simple and usual case where $\mathcal{L} \cap \Lambda = \{\lambda, \mu\}$, we can compute the asymptotic behavior of ζ_l directly. We have that $E_\delta(P)(\cdot,\zeta) = (a_\mu \cdot \zeta^{k_\mu - k_\lambda} + a_\lambda)\zeta^{k_\lambda}$ and, hence,

$$\zeta_l = (-1)^{j/(k_2-k_1)} \cdot \left|\frac{a_\lambda}{a_\mu}\right|^{1/(k_2-k_1)} .$$

In particular, these solutions of the eikonal equation then behave as

$$\zeta_l(x) \sim c_l \cdot x^{({}^b\omega(\lambda) - {}^b\omega(\mu))/(k_2-k_1)} = c_l \cdot x^{-\gamma(\mathcal{L})} ,$$

as $x \to 0$, where c_l is the leading coefficient of $(-1)^{l/(k_2-k_1)} |a_\lambda/a_\mu|^{1/(k_2-k_1)}$.

Due to the symbolic nature of Definition 4.2.7 we can make use of Remark 3.2.19 to define b-transport operators. Recall the evaluation homomorphism $\iota_{f,V}$ associated to a vector field V and a function f, mapping symbols $a(\zeta,\xi) \in \mathcal{S}_h(\mathbb{R}_+)$ to operators $\iota_{f,V} a = a(f, V)$.

Definition 4.2.10 (Induced b-Transport Operator)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$ with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$. Let $\delta > 0$ and let $\mathcal{L} \subset \partial\mathcal{P}({}^b\Lambda)$ be an edge with slope δ. Let φ be a solution of ${}^{\mathrm{b}}E_\delta(P)(x, x\varphi') = 0$.

Then we call

$${}^bT_{\delta,\varphi'}(P) := \iota_{x\varphi', x\partial_x} \left[\sum_{\lambda \in \mathcal{L} \cap {}^b\Lambda} \widetilde{a}_\lambda \cdot \left(\xi\zeta^{k-1} + \zeta\xi\zeta^{k-2} + \ldots + \zeta^{k-1}\xi \right) \right] ,$$

the *induced* b-δ*-transport operator* associated to δ and φ'.

Remark 4.2.11. Assume that ζ_1 is a simple root of ${}^{\mathrm{b}}E_\delta(P)(x,\zeta) = 0$ with $\zeta_1(x) \sim cx^{-\gamma}$ for some $\gamma \geq 0$ and that φ is a solution of the corresponding eikonal equation (4.6). Note that this can only be the case if for some $l_\gamma \in \mathbb{R}$ and $\widehat{a}_\lambda(x) := x^{-(\gamma k + l_\gamma)} \widetilde{a}_\lambda(x)$ bounded for all $\lambda \in \mathcal{L} \cap {}^b\Lambda$, we have that

$${}^{\mathrm{b}}E_\delta(P)(x,\zeta) = \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widetilde{a}_\lambda(x)\zeta^k = \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widehat{a}_\lambda(x) x^{\gamma k + l_\gamma} \cdot \zeta^k = x^{l_\gamma} \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widehat{a}_\lambda(x)(x^\gamma \zeta)^k .$$

Then $\varphi(x) = x^{-\gamma} f(x)$ for some function f with $f(0) \neq 0$ and, for some function V, the induced b-transport operator

$$\begin{aligned} {}^bT_{\delta,\varphi'}(P) &= \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widetilde{a}_\lambda(x) \left(x\partial_x \circ (x\varphi')^{k-1} + \ldots + (x\varphi')^{k-1} x\partial_x \right) \\ &= \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widehat{a}_\lambda(x) x^{\gamma k - \gamma(k-1) + l_\gamma} \left(x\partial_x + V(x)\right) = x^{\gamma + l_\gamma} \sum_{\substack{\lambda=(k,\alpha) \\ \lambda \in \mathcal{L} \cap {}^b\Lambda}} \widehat{a}_\lambda(x) \left(x\partial_x + V(x)\right) \end{aligned}$$

is a first order b-operator with additional vanishing factor $x^{\gamma + l_\gamma}$. Note that $V(0) \neq 0$ since $\varphi(x) = x^{-\gamma} f(x)$.

4.2.2 Resolved Operators

Restoring a notion of regularity for semi-classical b-operators is more difficult than in the regular case in Chapter 3. We need to compare weights ${}^b\omega(\lambda)$ of points on the edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with weights of points in the interior of the Newton polygon $\Lambda \cap \mathcal{P}(\Lambda)^\circ$. The necessity was shown in Example 4.1.11.

The upcoming definition of *essential points* plays the central role in determining whether an operator can be considered resolved. To make it more accessible, we revisit some of the examples in Section 4.1 to analyze why we were not able to construct exponential-polyhomogeneous quasimodes regarding the distribution of weights.

Examples 4.1.7 and 4.1.10 (and possible variations of them) show that the minimal point according to Definition 3.2.13 has minimal weight, necessarily (see Figure 4.6). Example 4.1.6 shows that there can be multiple points with minimal weight. Example 4.1.7 shows that relative increases of weight along an edge $\mathcal{L} \subset \partial\mathcal{P}({}^b\Lambda)$ should be allowed. However, these should be minimal, as the singular behavior in Example 4.1.11 shows.

Note that we will only treat simple solutions ζ of eikonal equations $E_\delta(P)(\cdot, \zeta) = 0$ to exclude jumps in multiplicity of the roots at isolated points. This phenomenon will be addressed in Section 5.2. For an edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$, recall the definitions of its slope $\delta(\mathcal{L})$ and relative increase of weight $\gamma(\mathcal{L})$ in Definitions 2.3.2 and 4.2.3. We present a recursive argument to select points in ${}^b\Lambda$ with pairwise minimal increase in weight.

Lemma 4.2.12. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$ be an operator with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$. Denote the differential order by $m := \operatorname{ord}(P)$.*

(i) Let $\mu_0 = (k_0, \alpha_0) \in {}^b\Lambda$ be the minimal point of ${}^b\omega^{-1}(0) \subset {}^b\Lambda$.

(ii) If $k_{n-1} < m$ for $n \geq 1$, let $\mu_n = (k_n, \alpha_n) \in {}^b\Lambda$ be the unique point with $k_{n-1} < k_n$, such that

(a) For all $\lambda = (k, \alpha) \in {}^b\Lambda$ with $k_{n-1} < k$, we have $\gamma(\overline{\mu_{n-1}\mu_n}) \leq \gamma(\overline{\mu_{n-1}\lambda})$,

(b) For all $\lambda = (k, \alpha) \in {}^b\Lambda$ satisfying (a). we have $\delta(\overline{\mu_{n-1}\mu_n}) \leq \delta(\overline{\mu_{n-1}\lambda})$,

(c) k_n is maximal among all points satisfying (a) and (b).

Then this iteration terminates after $N \in \mathbb{N}_0$ steps, where $N \leq m$.

Proof. This is a direct consequence of $0 \leq k_{n-1} < k_n$ for $\mu_n = (k_n, \alpha_n) \in {}^b\Lambda$ and $k \leq m$, for all $(k, \alpha) \in {}^b\Lambda$. □

Definition 4.2.13 (Essential Points)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$ with $P = \sum_{\lambda \in {}^b\Lambda} \widetilde{a}_\lambda(x) h^\alpha (x\partial_x)^k$.

The points $\mu_0, \ldots, \mu_N \in {}^b\Lambda$, with $N \in \mathbb{N}_0$, determined in 4.2.12 for P, are called *essential points* of P in 0. The point μ_0 is called *essential minimum.*

Remark 4.2.14. There are some additions in Definition 4.2.13 to the discussion prior. Whenever there are pairs of exponents with the same increase of weight stacked over the same value

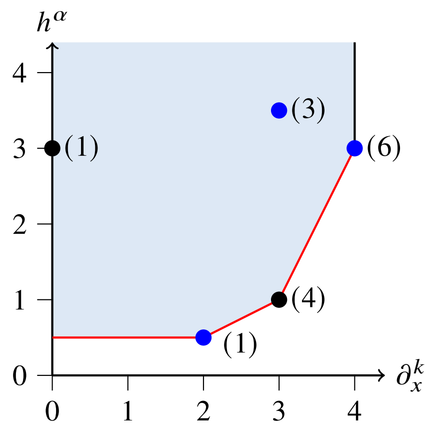

Figure 4.6: Illustration of the set of exponents $\Lambda(P) = \{(0,3), (2,0.5), (3,1), (3,3.5), (4,3)\}$ with respective weights of an operator P in the brackets at each point and its Newton polygon $\mathcal{P}(\Lambda(P))$. The blue points are the essential points of P in $\Lambda(P)$. The essential minimum is given by $\mu_0 = (2,0.5)$. Of the three points $(3,1), (3,3.5), (4,3)$ on the right side of μ_0, the lowest increase of weight is given by $\mu_1 = (3,3.5)$, since $(3-1)/(3-2) = 2 < 5/2 = (6-1)/(4-2)$. There is only one remaining point on the right side of μ_1. Thus $\mu_2 = (4,3)$ is the last essential point. In particular, P is not resolved, since $\mu_1 \notin \partial_{-}\mathcal{P}(\Lambda(P))$.

of k, we consider only the pair $\lambda = (k,\alpha)$ with the lowest value of α. Also, if there are multiple collinear points with identical increase of weight, then only the rightmost is another essential point.

Remark 4.2.15. The iteration in Lemma 4.2.12 can also be phrased in the following way. Let ${}^b\Lambda_0 := {}^b\omega^{-1}(0)$ and μ_0 be its minimal point. Let $\mu_n = (k_n, \alpha_n) \in {}^b\Lambda$, where

$$\mu_{n+1} := \operatorname*{arg\,max}_{\mu=(k,\alpha)} \left\{ k > k_n \,:\, \delta(\overline{\mu_n\mu}) = \min_{k_*>k_n} \{\delta(\overline{\mu_n\mu_*}) \,:\, \gamma(\overline{\mu_n\mu_*}) = \min_{\widehat{k}>k_n} \gamma(\overline{\mu_n\widehat{\mu}})\} \right\}, \quad (4.7)$$

if the set is not empty.

Example 4.2.16. Let $P := h^2x^4\partial_x^2 + 1$ as in Example 4.1.7. Then $\mu_0 = (0,0)$ is the essential minimum and $\mu_1 = (2,2)$ is another essential point with weight ${}^b\omega((2,2)) = 2$. In particular, $N = 1$.

Example 4.2.17. Let $P := h^3x^2\partial_x^2 + h^3\partial_x - x$ as in Example 4.1.11. Then $\mu_0 = (1,3)$ is the essential minimum and $\mu_1 = (2,3)$ is the only other essential point, since ${}^b\omega((1,3)) = -1$ and ${}^b\omega((0,0)) = 1$. Note that $\mu_0 \notin \partial_{-}\mathcal{P}({}^b\Lambda)$.

Regular operators as in Definition 3.2.16 cannot be characterized well in terms of essential points. However, it is not necessary to do so, since essential points have been introduced to describe a class of singular operators whose quasimodes have polyhomogeneous amplitudes on the model space $\mathbb{R}^2_+$.

As we observed in Section 4.1, we are not able to construct exponential-polyhomogeneous quasimodes of singular operators whenever essential points are contained in the interior of

$\partial_{-}\mathcal{P}(\Lambda)$. Moreover, if the minimal point does not coincide with the essential minimum, there are no polyhomogeneous quasimodes either.

Definition 4.2.18 (Resolved Operators)**.** Let $\Lambda \subset \mathbb{N}\times\mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R}_+)$ with $P = \sum_{\lambda\in{}^b\Lambda} \widetilde{a}_\lambda(x)h^\alpha(x\partial_x)^k$. Let $\mu_0, \dots, \mu_N$ be the essential points of P and $\mathcal{L} := \overline{\lambda\mu} \subset \partial_{-}\mathcal{P}({}^b\Lambda)$ be an edge with slope δ. Suppose that P is δ-singular in 0.

Then we call P *$\mathcal{L}$-resolved* in 0, if either

(i) $\mu = \mu_0$ is the minimal point of ${}^b\Lambda$, or

(ii) $\lambda = \mu_k$ and $\mu = \mu_{k+1}$, for some $k \geq 0$.

If the semi-classical operator P is $\mathcal{L}$-resolved for all $\mathcal{L} \subset \partial_{-}\mathcal{P}({}^b\Lambda)$, we call P *resolved* in 0.

Example 4.2.19. The operator $P := h^2x^4\partial_x^2 + 1$ as in Example 4.2.16 is resolved, since its only edge $\mathcal{L} = \overline{(0,0),(2,2)}$ is spanned by essential points $\mu_0 = (0,0)$ and $\mu_1 = (2,2)$.

Example 4.2.20. The operator $P := h^3x^2\partial_x^2 + h^3\partial_x - x$ as in Example 4.2.17 is unresolved with respect to its only edge $\mathcal{L} = \overline{(0,0),(2,2)}$. The only essential point $\mu_0 = (1,3)$ is contained in the interior of the Newton polygon.

Example 4.2.21. The operator $P := x^3h^2\partial_x^2 + h\partial_x - 1$ as in Example 4.1.3 is partially resolved. Multiplying P with an additional factor x, the essential points are given by $\mu_0 = (1,1)$ and $\mu_1 = (2,2)$ with weights ${}^b\omega(\mu_0) = 0$ and ${}^b\omega(\mu_1) = 1$. Its only edge in the lower boundary $\mathcal{L} = \overline{(0,0),(2,2)}$ consists of two segments. Since one of them is spanned by μ_0 and μ_1, it is partially resolved with respect to $\delta = 1$.

Remark 4.2.22. We want to emphasize again that δ-regularity of $P = \sum_{\lambda\in\Lambda} a_\lambda h^\alpha \partial_x^k$ is a property which can be verified by only knowing the coefficients a_λ for $\lambda \in \Lambda \cap \partial_{-}\mathcal{P}(\Lambda)$. In particular, knowing the shape of $\partial_{-}\mathcal{P}(\Lambda)$, we can determine the exponential behavior, regularity and the amount of independent amplitudes based only on the coefficients associated to each maximal edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ and its associated eikonal polynomial.

In contrast to that, it is not possible to determine whether a singular operator is resolved with respect to any edge $\mathcal{L} \subset \partial_{-}\mathcal{P}({}^b\Lambda)$ in general, even if we know all weights along the lower boundary. The relatively lower weights of essential points μ_k in the interior of the Newton polygon let them dominate on the front face of a quasihomogeneous blow-up $\beta_t\colon [\mathbb{R}^2_+, 0]_t \to \mathbb{R}^2_+$ for some $t > 0$. Scraping these to the lower boundary of $\partial_{-}\mathcal{P}({}^b\Lambda)$ is the main task of resolving these operators in Chapter 5 and involves successive quasihomogeneous blow-ups.

4.2.3 Newton Polyhedra

The goal of this subsection is to describe resolved operators using the language of Newton polygons as in Chapter 3. By construction, essential points are pairs of exponents (k, α) whose increase of weight, as in Definition 4.2.3, from the previous essential point is minimal compared to all other points. The minimality of the increase of weights is the direct analogue of the discussion about slopes of edges $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ in Section 3.2. Thus, we are interested in an extension of the notion of Newton polygons to three dimensional sets, including not only the order of differentiation k and power α of h^α, but also the *weight* $\omega = l - k$ of $x^l \partial_x^k$. This results in the definition of *Newton polyhedra* of an operator; a three-dimensional convex set in $\mathbb{N}_0 \times \mathbb{R} \times \mathbb{N}_0$, whose projection onto the first two entries $\mathbb{N}_0 \times \mathbb{R}$ coincides with the Newton polygon of the operator.

We start this subsection by introducing the set of triples (k, α, ω) associated to an operator at the point $x = 0$. Since these weights depend on the choice of a point $p \in \{h = 0\} \subset \mathbb{R}^2_+$ we will refer to the sets of triples as localized set of exponents.

Definition 4.2.23 (Localized Set of Exponents)**.** Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$, let $(k, \alpha, \omega) \in \mathbb{N}_0 \times \mathbb{R} \times \mathbb{N}_0$ and $a_{k,\alpha,\omega} \in \mathbb{C} \setminus \{0\}$, such that $P = \sum_{k,\alpha,\omega} a_{k,\alpha,\omega} \cdot x^\omega h^\alpha (x\partial_x)^k$ at $(0, 0)$. Then we call

$$\Lambda_0 := \left\{(k, \alpha, \omega) :\ a_{k,\alpha,\omega} \neq 0\right\} ,$$

the *localized set of exponents of P at* $0 \in \mathbb{R}^2_+$.

The upcoming definition of Newton polyhedra is a direct extension from Newton polygons in Definition 2.3.2. Its lower boundary refers to the lower boundaries of its projections. Recall that the set $\mathrm{conv}(A)$ is the convex hull of $A \subset \mathbb{N}_0 \times \mathbb{R} \times \mathbb{N}_0 \subset \mathbb{R}^3$ in the Euclidean space.

Definition 4.2.24 (Newton Polyhedron)**.** Let $\Lambda_0 \subset \mathbb{N} \times \mathbb{R}^2$ be discrete and bounded from above in the first and from below in the latter two arguments.

Then we call

$$\mathcal{P}(\Lambda_0) := \mathrm{conv}\left(\{(x, y, z) :\ \exists\, (k, \alpha, \omega) \in \Lambda_0 \text{ with } x \leq k \text{ and} y \geq \alpha \text{ and} z \geq \omega\}\right)$$

its *Newton polyhedron* and

$$\partial_{-}\mathcal{P}(\Lambda_0) := \left\{\lambda \in \partial\mathcal{P}(\Lambda_0) :\ \pi_{(k,\alpha_j)}(\lambda) \in \partial_{-}\mathcal{P}(\pi_{(k,\alpha_j)}(\Lambda_0)) \text{ for } j = 2, 3\right\}$$

the *lower boundary* of the Newton polyhedron $\mathcal{P}(\Lambda_0)$, with projections $\pi_{(k,\alpha_j)}$, for $j = 2, 3$, defined by

$$\begin{aligned} \pi_{(k,\alpha_j)} : \mathbb{N} \times \mathbb{R}^2 &\to \mathbb{N} \times \mathbb{R} \\ (k, \alpha_2, \alpha_3) &\mapsto (k, \alpha_j) . \end{aligned}$$

Remark 4.2.25. We want to stress the fact that requiring a general point in the boundary $\lambda \in \partial\mathcal{P}(\Lambda_0)$ to project to both lower boundaries via $\pi_{(k,\alpha_j)}$, $j = 2, 3$, usually leads to $\partial_{-}\mathcal{P}(\Lambda_0)$ being “incomplete” and often being empty. The lower boundary can only contain points and edges with a combined horizontal length of at most $m = \max\{k \in \mathbb{N} : (k, \alpha, \omega) \in \Lambda_0\}$.

We keep the indexing of $\pi_{(k,\alpha_j)}$ via (k, α_j) for $j = 2, 3$ to indicate whether the second or third entry will be present in the image of the projection.

This notion of lower boundary of a Newton polyhedron gives us a geometric interpretation of essential points of $\Lambda(P)$.

Proposition 4.2.26. *Let $\Lambda \subset \mathbb{N}\times\mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda)$ be an edge and assume that P is $\mathcal{L}$-singular.*

Then the following is true:

(i) If $\mu = (k,\alpha) \in \partial_{-}\mathcal{P}(\Lambda)$ is essential, then $(k,\alpha,\omega(\mu)) \in \partial_{-}\mathcal{P}(\Lambda_0)$.

(ii) P is $\mathcal{L}$-resolved if there exists $\mathcal{L}' \subset \partial_{-}\mathcal{P}(\Lambda_0)$ with $\pi_{(k,\alpha_3)}(\mathcal{L}') = \mathcal{L}$.

Proof. (i). Proof by Induction. Assume that $\mu = (k,\alpha) \in \partial_{-}\mathcal{P}(\Lambda)$ is essential. If $\mu = \mu_0$ is the essential minimum then $(k,\alpha,\ {}^{b}\omega(\mu)) \in \partial_{-}\mathcal{P}(\Lambda_0)$, since ${}^{b}\omega(\mu) \leq {}^{b}\omega(\lambda)$, for all $\lambda = (l,\beta) \in \Lambda$ and $l \leq k$ if their weights ${}^{b}\omega(\lambda)$ and ${}^{b}\omega(\mu_0)$ are equal. Assume that $\mu_0,\ldots,\mu_{N-1} \in \partial_{-}\mathcal{P}(\Lambda)$ are essential in ascending order, for $N \geq 1$, $\mu_j = (k_j,\alpha_j)$ and $\omega_j := {}^{b}\omega(\mu_j)$, with $(k_j,\alpha_j,\omega_j) \in \partial_{-}\mathcal{P}(\Lambda_0)$. Let μ_N be the next essential point. By Definition 4.2.13, we have that $\gamma(\overline{\mu_{N-1}\mu})$ is minimal if $\mu = \mu_N$ for all sets of exponents (k,α) with $k \geq k_{N-1}$. Hence $(k_{N-1},\omega_{N-1}) \in \partial_{-}\mathcal{P}(\pi_{k,\omega}(\Lambda_0))$ implies that $(k_N,\omega_N) \in \partial_{-}\mathcal{P}(\pi_{k,\omega}(\Lambda_0))$ and thus $(k_N,\alpha_N,\omega_N) \in \partial_{-}\mathcal{P}(\Lambda_0)$.

(ii). This is a direct consequence from (1). □

Note that Proposition 4.2.26 deals only with essential points that are already contained in the lower boundary $\partial_{-}\mathcal{P}(\Lambda)$. Identifying the remaining essential points in Λ_0 will become important in Chapter 5 when we show that for every operator P there is a quasihomogeneous blow-up β_t such that all essential points are contained in $\partial_{-}\mathcal{P}(\Lambda(\beta_t^{*}P))$.

Remark 4.2.27. The converse of the first statement in Proposition 4.2.26 is not true in general. Whenever there are more than two points on an edge $\mathcal{L}' \subset \partial_{-}\mathcal{P}(\Lambda_0)$, all (relatively) interior points $\mu \in \mathcal{L}' \setminus \partial\mathcal{L}'$ are contained in the lower boundary of $\mathcal{P}(\Lambda_0)$ but are not essential.

4.3 Construction of Quasimodes II: Resolved Operators

In this section, we show that there is an algorithm to construct exponential-polyhomogeneous quasimodes of singular, but resolved operators $P = \sum_{k=0}^{m} A_k V^k$, where V is a b-vector field on $\mathbb{R}^2_+$. In contrast to regular operators in Chapter 3, neither the phase functions nor the amplitudes will be smooth. They will however be polyhomogeneous in general. There are two different types of b-vector fields on the quarter space: those whose image under projection onto the boundary ${}^{b}\pi_{H_j,*}(V)$ vanishes on one boundary hypersurface H_j, $j = 1,2$, i.e. $V = x\partial_x$, and those whose image does not vanish, i.e. $V = x\partial_x - y\partial_y$. The latter case will become important in the resolution process of Chapter 5, where we are frequently facing vector fields of this type whenever we pull back a b-vector field $x\partial_x$ to a blown-up space of $\{x = h = 0\}$.

4.3.1 Horizontal b-Vector Fields

b-Transport Equation

A recurring phenomenon in Section 4.1 in that the transport operator T of the operators discussed in the examples is a b-operator. Moreover, all remainder operators R_k of

$\exp(-\varphi/h^{\delta})P\exp(\varphi/h^{\delta}) = T + \sum_k R_k h^{\alpha_k}$ were b-operators with various weights. It is important to note that these weights follow a strict pattern indicated by $\varphi(x) \sim cx^{-\gamma}$, as $x \to 0$, for some $\gamma > 0$. As we have computed in Remark 4.2.11, the induced transport operator

$$ {}^{b}T_{\delta,\varphi'} = x^{\gamma+l_\gamma} \sum_{\lambda\in\mathcal{L}\cap{}^{b}\Lambda} \widehat{a}_\lambda(x)\,(x\partial_x + V(x))\ , $$

has an additional factor of $x^{\gamma+l_\gamma}$. This is the direct analogue of the power $\delta + l_\delta$ of h for the induced transport operator

$$ P_{\delta,\varphi'} = h^{l_\delta+\delta}T_{\delta,\varphi'} + o\left(h^{l_\delta+\delta}\right)\ , $$

in its counterpart $\pi_{k,\omega}(\Lambda_0)$. For any pair of exponents $\lambda = (k,\alpha) \in {}^{b}\Lambda$, with weight $\omega = {}^{b}\omega(\lambda)$, conjugating its associated differential operator

$$ \exp(-\varphi/h^{\delta})\left(a_\lambda h^{\alpha}(x\partial_x)^k\right)\exp(\varphi/h^{\delta}) = h^{l_\delta}(x\varphi')^k a_\lambda + \text{h.o.t.} $$

results in a lowest order summand with weight $-\gamma k + {}^{b}\omega(\lambda)$ and all higher order summands with weights of the form $-\gamma\cdot(k-j) + {}^{b}\omega(\lambda)$. There are two important things to notice. Firstly, all weights increase in steps with size γ. Secondly, if P is resolved, i.e. φ is the solution of an eikonal equation related to an edge spanned by essential points μ_k and μ_{k+1} with increase of weight γ, the weight of λ satisfies ${}^{b}\omega(\lambda) \geq \gamma k + l_\gamma$. Hence, the weight of the lowest order summand after conjugation is bounded from below,

$$ -\gamma k + {}^{b}\omega(\lambda) \geq -\gamma k + \gamma k + l_\gamma = l_\gamma\ . $$

Thus, the weights of the lowest order summands of the expansion of the conjugation of $\left(a_\lambda h^{\alpha}\partial_x^k\right)$ are not separated in $\pi_{k,\omega}(\Lambda_0)$ by default, if P is resolved. However, the only point $\lambda \in \pi_{k,\omega}(\Lambda_0)$ below $\gamma + l_\gamma$ has height l_γ. This can be resolved analogously to Proposition 3.3.8 and is the reason that the sum in Equation 4.8 starts at $l = -1$.

This motivates the following proposition which shows that we are able to construct polyhomogeneous quasimodes of transport systems induced by resolved operators.

Proposition 4.3.1 (b-Normal Form)**.** *Let $I \subset \mathbb{N}_0$. For $k \in I$, let R_k be b-differential operators with smooth coefficients, $\gamma \in \mathbb{R}$ and $T := x^{1+\gamma}\partial_x$. Let $P := T + \sum_{k\in I} h^{\alpha_k}R_k$, where $\{\alpha_k\ :\ k \in I\}$ is discrete and positive. Assume that $R_m = x^{\mathrm{ord}(R_m)\gamma}\widetilde{R}_m$, for some smooth b-operator $\widetilde{R}_m$ for each $m \in I$.*

Then there is a phase function Φ and an asymptotic sum of b-operators $\widetilde{P}$ such that

$$ \exp(-\Phi)P\exp(\Phi) = \widetilde{P}V + O(h^{\infty})\ , $$

where

$$\Phi(x) = \left(\sum_{j=0}^{\infty} b_{j,0}(x) h^{\beta_{j,0}}\right) \cdot \log(x) \;+\; \sum_{\substack{l=-1\\ l\neq 0}}^{\infty} \left(\sum_{j=0}^{\infty} b_{j,l}(x) h^{\beta_{j,l}}\right) \cdot x^{l\gamma}\,, \tag{4.8}$$

for some $b_{j,l} \in C^\infty(\mathbb{R}_+)$ and $\beta_{j,l} \in \mathbb{R}_{>0}$.

Proof. Denote $R := \sum_{k\in I} h^{\alpha_k} R_k$, then $P = T + R$. Note that for any m with differential order $\operatorname{ord}(R_m) = 0$, i.e. $R_m = -f$ with $f = O(x^k)$, the solution b of the eikonal equation

$$(x^{1+\gamma}\partial_x) b = f\,,$$

is polyhomogeneous with $b = O(x^{k-\gamma})$, if $k \neq \gamma$ and $b = O(\log(x))$, if $k = \gamma$. Conjugating any b-operator R_n with $\exp(b)$ then yields

$$e^{-b} R_n e^b = \underbrace{(xb')^l \cdot x^{l\gamma}}_{=O(1)} + \text{h.o.t.}\,,$$

where we used that $x^{1+\gamma} b' = O(f(x))$ and $R_n = x^{l\gamma}\widetilde{R}_n$ for $\operatorname{ord}(R_n) = l$. In particular, the assumption that $\check{R}_m = x^{k\gamma}\widetilde{R}_m$ for some b-operator $\widetilde{R}_m$ whenever $\operatorname{ord}(\check{R}_m) = k$ remains true for all b-operators in the expansion of $\exp(-bh^\beta) P \exp(bh^\beta) = \sum_{l=0}^{\infty} h^{\check{\alpha}_l}\check{R}_l$.

Thus any non-differential term $h^{\alpha_m} P_m = -h^{\alpha}_m f_m$, $f_m \in C^\infty(\mathbb{R}_+)$ can be erased by the conjugation of $P = T + R$ with $\exp(\psi \cdot h^{\alpha_m})$, where ψ satisfies

$$x\psi' = x^{-\gamma} f_m\,.$$

Although this conjugation "enlarges" the asymptotic sum $\exp(-\psi \cdot h^{\alpha_m}) P \exp(\psi \cdot h^{\alpha_m})$ compared to P in general, we can erase all constant terms $h^{\alpha_m} R_m = -h^{\alpha}_m f_m$ with $\alpha_m \leq N$ for all $N \in \mathbb{N}$ eventually. This is analogue to the proof of Proposition 3.3.8.

Summing over all $\psi_k h^{\beta_k}$ and sorting the asymptotic expansion of the sum Φ by powers of $x^{l\varepsilon}$ we get

$$\Phi = \left(\sum_{j=0}^{\infty} b_{j,0} h^{\beta_{j,0}}\right) \cdot \log \;+\; \sum_{\substack{l=-1\\ l\neq 0}}^{\infty} \left(\sum_{j=0}^{\infty} b_{j,l} h^{\beta_{j,l}}\right) \cdot x^{l\varepsilon}\,,$$

for some $\beta_{j,l} \in \mathbb{R}$ and $b_{j,l} \in C^\infty(\mathbb{R}_+)$. $\square$

Remark 4.3.2. Note that for $\gamma = 0$ this is coincides with the statement of Corollary 3.1.9 for b-operators where the transport operator T is b-elliptic.

Construction

We are now able to combine all previous results from Chapter 4 and show that quasimodes constructed on $(0, \infty)$ are exponential-polyhomogeneous at $x = 0$ for resolved operators P. In particular, the number of independent quasimodes corresponds to the width of the edge $\mathcal{L}$ as in the statement of Theorem 3.3.9 for regular operators. We can use Proposition 3.2.15 for the existence of phase functions on the interior of $\mathbb{R}_+$, where phases are simple. Proposition 4.2.8 then quantifies the asymptotic behavior of these phase functions at the boundary $x = 0$ and, in the case of resolved operators, allows us to apply Proposition 4.3.1 to obtain polyhomogeneous amplitudes.

Theorem 4.3.3 (Construction of Quasimodes I: Horizontal b-Vector Fields)**.** *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(\mathbb{R}_+)$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ be an edge with $|\mathcal{L}| = L$ and slope $\delta > 0$. Assume that P is δ-singular in 0 and $\mathcal{L}$-resolved. Assume that all solutions φ_j' of $E_\delta(P)(\cdot, \zeta) = 0$ are simple on $(0, \infty)$, for $j = 1, \ldots, L$. Then the following holds:*

(i) If $\delta > 0$, then there are polyhomogeneous amplitudes A_j with $A_{j|\{h=0\}} \not\equiv 0$ and independent, exponential-polyhomogeneous quasimodes $u_j = e^{\Phi_j} A_j$, with $\Phi_j = \varphi_j / h^\delta + o(h^\delta)$, for $j = 1, \ldots, L$, such that

$$P_{\Phi_j} A_j = O(h^\infty) .$$

(ii) If $\delta = 0$, then there are polyhomogeneous quasimodes A_j with $A_{j|\{h=0\}} \not\equiv 0$, for $j = 1, \ldots, L$, such that

$$P A_j = O(h^\infty) .$$

Proof. Without loss of generality assume that ${}^b\omega(\lambda) \geq 0$ for all $\lambda \in {}^b\Lambda$ and that $\lambda_{\min} = (n, 0)$ for some $n \in \mathbb{N}$. Let $m := \operatorname{ord}(P)$ and $\gamma = \gamma(\mathcal{L})$.

1. Consider the case $\delta > 0$. Then there are essential points μ_k, μ_{k+1}, for some $0 \leq k \leq m$, such that $\mathcal{L} = \overline{\mu_k \mu_{k+1}}$. Since P is δ-singular in 0 and all solutions φ_j' of $E_\delta(P)(\cdot, \zeta) = 0$ are simple, we can apply Proposition 3.2.15 on $(0, \infty)$ and obtain solutions φ_j of the associated eikonal equation for $j = 1, \ldots, L$. Further, we can apply Proposition 4.2.8 for each φ_j since $\gamma(\mathcal{L}) \leq \gamma(\overline{\lambda\mu})$, for all $\lambda, \mu \in \mathcal{L}$, which yields that $\varphi_j(x) \sim x^{-\gamma}$ (or $\varphi_j(x) \sim \log(x)$, if $\gamma = 0$) as $x \to 0$. Applying Proposition 3.3.8 on $(0, \infty)$ for each φ_j' then yields $0 < \delta_{j,i} < \delta$, and full phases

$$\Phi_j = \frac{\varphi_j}{h^\delta} + \sum_{i=1}^{N} \frac{\psi_{j,i}}{h^{\delta_{j,i}}} .$$

By the same argument as in the beginning of Subsection 4.3.1 and Proposition 4.3.1 we have that $\psi_{j,i}(x) = o(1)$ or $\psi_{j,i}(x) \sim \log(x)$. Finally, we can apply Proposition 4.3.1 to P_{Φ_j} and obtain functions Ψ_j such that

$$\left(P_{\Phi_j}\right)_{\Psi_j} = \widetilde{P} \circ x\partial_x + O(h^\infty) ,$$

for an asymptotic sum of b-operators $\widetilde{P}$. In particular, $A_j \coloneqq \exp(\Psi_j)$ satisfies

$$P_{\Phi_j} A_j = O(h^\infty)\,.$$

2. Since P is $\mathcal{L}$-resolved and $\delta = 0$, we have that $\mathcal{L} = \overline{(0,0),(L,0)}$ and in particular $\lambda_{\min} = (L,0) \in {}^b\Lambda$. Without loss of generality we can assume that $a_{\lambda_{\min}}(0) \neq 0$. Thus,

$$P = T + \sum_{k=0}^{\infty} h^{\alpha_k} R_k\,,$$

is a sum of b-operators and that T is a L-th order b-elliptic operator. By Proposition 3.1.7, there are polyhomogeneous quasimodes A_j, where $A_j = a_{j,0} + o(1)$ with $a_{j,0}$ spanning $\ker T$, for each $j = 1, \dots, L$.

□

4.3.2 Hyperbolic b-Vector Fields

Another important case in Chapter 5 is operators of the form $P \coloneqq \sum_{k=0}^m A_k V^k$, where $V \coloneqq x\partial_x - y\partial_y$ is a b-vector field whose image under projection does not vanish at either face $\{y = 0\}$ and $\{x = 0\}$. These vector fields occur whenever we blow up singular points $p \in \partial\mathbb{H}$ of P with respect to h and x, resulting in the pullback of $x\partial_x$ of the form $\beta^*(x\partial_x) = x\partial_x - y\partial_y$.

Although we have not taken this case into consideration in Section 4.2 explicitly, it easily fits into the theory of resolved operators. Constructing quasimodes in this case reduces to the analysis of Newton polygons in powers of x and y at either boundary face. If these edges have a common, three dimensional pre-image $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_0)$, the corresponding eikonal polynomials *essentially* coincide at the corner $\{x = y = 0\}$, allowing for a polyhomogeneous extension Φ of pairs of solutions φ_j to the quarter space $\mathbb{R}^2_+$.

Eikonal Equations

Choosing a hypersurface and a corresponding edge is ambiguous for an operator P with a symmetrical vector field $V = x\partial_x - y\partial_y$. We will phrase all statements in terms of lower edges $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_0)$ of the Newton polyhedron at the corner $0 \in \mathbb{R}^2_+$. We will distinguish between eikonal polynomials at different faces of $\mathbb{R}^2_+$ by adding the relevant face in the index, i.e. $E_{H,\delta}(P)$.

Proposition 4.3.4. *Let $V \coloneqq x\partial_x - y\partial_y$ on $\mathbb{R}^2_+$, $k \in \mathbb{N}_0$, $A_k \in \mathcal{A}(\mathbb{R}^2_+)$ be log-free, $m \in \mathbb{N}$, $P \coloneqq \sum_{k=0}^m A_k V^k$ and $\Lambda_0 \subset \mathbb{N}_0 \times \mathbb{R} \times \mathbb{N}_0$ be the associated set of exponents at $\{x = y = 0\}$. Denote $H_1 \coloneqq \{y = 0\}$ and $H_2 \coloneqq \{x = 0\}$ and let Λ_{H_j}, $j = 1, 2$, be their respective sets of exponents. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_0)$ and denote $\delta_j \coloneqq \delta(\mathcal{L}_j)$, for $\mathcal{L}_j \coloneqq \pi_{(k,\alpha_j)}(\mathcal{L}) \subset \partial_{-}\mathcal{P}(\Lambda_{H_j})$. Assume that $\delta_1 \neq \delta_2$, where $\delta_j \geq 0$ and for $P \sim \sum_{\lambda \in \Lambda_0} \alpha_\lambda x^\omega y^\alpha V^k$ at $0 \in \mathbb{R}^2_+$ let $c \in \mathbb{C}$ be a* *solution of*

$$\sum_{\substack{\lambda\in\mathcal{L}\cap\Lambda_0\\ \lambda=(k,\alpha)}} \alpha_\lambda c^k = 0\,.$$

Assume that all non-trivial solutions of $E_{H_j,\delta_j}(P)(\cdot,\zeta) = 0$ *are simple on* H_j°, $j = 1, 2$.

(i) If $\delta_1, \delta_2 > 0$ *then there is a phase* $\Phi \in \mathcal{A}(\mathbb{R}^2_+)$ *with* $\Phi(x, y) \sim y^{-\delta_1}\varphi_1(x)$ *at* H_1 *and* $\Phi(x, y) \sim x^{-\delta_2}\varphi_2(y)$ *at* H_2 *such that for* $j = 1, 2$

$$E_{H_j,\delta_j}(P)(\cdot, V\varphi_j) = 0 \quad \text{at } H_j\,,$$

and

$$\Phi(x, y) \sim \frac{c}{\delta_1 - \delta_2}\frac{1}{x^{\delta_2}y^{\delta_1}} \;+\; h.o.t.\,,$$

at the corner $0 \in \mathbb{R}^2_+$.

(ii) If $\delta_2 = 0$ *then there is a solution* $\varphi_1 \in C^\infty(H_1)$ *of* $E_{H_1,\delta_1}(P)(\cdot, V\varphi_1) = 0$ *with* $\varphi_1(0) \neq c$ *and a solution u of the equation*

$$\sum_{\substack{\lambda\in\Lambda_{H_2}\cap\mathcal{L}_2\\ \lambda=(k_\alpha)}} a_\lambda y^{(k)} = 0$$

on H_2 *such that*

$$u(y) \sim e^{c/(\delta_2\cdot y^{\delta_1})} + o\left(y^{-\delta_1}\right)\,.$$

Proof. (i). Let $\delta_1, \delta_2 > 0$ and φ_j be simple solutions of $E_{\delta_j,H_j}(P)(\cdot, (V - \delta_l)\varphi_j) = 0$ at each face H_j° respectively, $l, j = 1, 2$, $l \neq j$. All we need to show is that $\varphi_1/y^{-\delta_1}$ and $\varphi_2/x^{-\delta_2}$ match pairwise at the corner $0 \in \mathbb{R}^2_+$, in the sense that they allow for a polyhomogeneous extension Φ to the interior of $\mathbb{R}^2_+$. We normalize these solutions φ_j by $\widetilde{\psi}_1 := x^{\delta_2}\cdot\varphi_1$ and $\widetilde{\psi}_2 := y^{\delta_1}\cdot\varphi_2$. It suffices to show that $\widetilde{\psi}_1, \widetilde{\psi}_2$ are continuous and $\widetilde{\psi}_1(0) = \widetilde{\psi}_2(0)$.

Since $\mathcal{L}$ is an edge in $\partial\mathcal{P}(\Lambda_0)$ there are $b_j \in \mathbb{R}$, $j = 1, 2$, such that

$$\mathcal{L} = \{(k,\, \delta_1 k + b_1,\, \delta_2 k + b_2) :\; (k, \delta_1 k + b_1) \in \mathcal{L}_1 \cap \Lambda_{H_1}\}\,.$$

We have to distinguish two different cases depending on the shape of $\mathcal{P}(\Lambda_0)$. If there is a surface F adjacent to $\mathcal{L}$ in $\mathcal{P}(\Lambda_0)$ that is perpendicular to either $\{\alpha = 0\}$ or $\{\omega = 0\}$ in $\mathbb{R}^3_+$, then denote $A_j := \pi_{(k,\alpha_j)}(F \setminus \mathcal{L})$. Otherwise set $A_j := \emptyset$.

Without loss of generality assume $A_2 \neq \emptyset$ and let $\widetilde{\mu} \in A_2$. Then there is $\varepsilon > 0$ such that $\widetilde{\mu} = (n, \delta_1 n + b_1 + \varepsilon, \delta_2 n + b_2)$ for some $n \in \mathbb{N}$.

Let $Q := V - (\delta_1 - \delta_2)$ and $u_j := e^{\widetilde{\psi}_j/y^{\delta_1}x^{\delta_2}}$. Then there are numbers $l_{\delta_j,H_j} \in \mathbb{R}$ and $a_\lambda \in \mathcal{A}(H_1)$, $b_\lambda, b_\mu \in \mathcal{A}(H_2)$, with $\lambda \in \mathcal{L} \cap \Lambda_0$ and $\mu \in A_2$ such that the conjugation of P

with u_j at $0 \in \mathbb{R}^2_+$ yields

$$u_1^{-1} P u_1 = y^{l_{\delta_1, H_1}} \cdot \left(\sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_0 \\ \lambda = (k, \delta_1 k + b_1, \delta_2 k + b_2)}} a_\lambda(x) \cdot x^{-k\delta_2} (Q\widetilde{\psi}_1)^k \; + o(y^{l_{\delta_1, H_1}}) \right)$$

$$u_2^{-1} P u_2 = x^{l_{\delta_2, H_2}} \cdot \left(\sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_0 \\ \lambda = (k, \delta_1 k + b_1, \delta_2 k + b_2)}} b_\lambda(y) \cdot y^{-k\delta_1} (Q\widetilde{\psi}_2)^k + \sum_{\mu \in A_2} b_\mu(y) \cdot y^{-k\delta_1} (Q\widetilde{\psi}_2)^k \right) + o(x^{l_{\delta_2, H_2}}) \,.$$

Since all pairs $\lambda \in \mathcal{L} \cap \Lambda_0$ are aligned along the edge $\mathcal{L}$, we can rewrite

$$\begin{aligned} \sum_{\lambda \in \mathcal{L} \cap \Lambda_0} a_\lambda(x) \cdot x^{-k\delta_2} (Q\widetilde{\psi}_1)^k &= \sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_0 \\ \lambda = (k, \delta_1 k + b_1, \delta_2 k + b_2)}} \widetilde{a}_\lambda(x) \cdot x^{k\delta_2 + b_2 - k\delta_2} (Q\widetilde{\psi}_1)^k \\ &= x^{b_2} \cdot \sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \widetilde{a}_\lambda(x) \cdot (Q\widetilde{\psi}_1)^k \,, \end{aligned}$$

with $\widetilde{a}_\lambda(0) \neq 0$ for all $\lambda \in \partial\mathcal{L}$, and

$$\begin{aligned} &\sum_{\lambda \in \mathcal{L} \cap \Lambda_0} b_\lambda(y) \cdot y^{-k\delta_1} (Q\widetilde{\psi}_2)^k + \sum_{\mu_l \in A_2} b_\mu(y) \cdot y^{-k\delta_1} (Q\widetilde{\psi}_2)^k \\ &= \sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \widetilde{b}_\lambda(y) \cdot y^{b_1} (Q\widetilde{\psi}_2)^k \quad + \sum_{\mu_l \in A_2} \widetilde{b}_\mu(y) \cdot y^{b_2 + \varepsilon_\mu} (Q\widetilde{\psi}_2)^k \end{aligned}$$

for finitely many $\varepsilon_\mu > 0$ and $\widetilde{b}_\lambda(0) \neq 0$, for all

$$\lambda \in \partial\mathcal{L} = \partial\{(k,\ \delta_1 k + b_1,\ \delta_2 k + b_2) :\ (k, \alpha) \in \mathcal{L}_1\} \,.$$

In particular, we have $\widetilde{a}_\lambda(0) = \widetilde{b}_\lambda(0)$ for all $\lambda \in \mathcal{L} \cap \Lambda_0$, since all coefficients of P are polyhomogeneous on $\mathbb{R}^2_+$ and $\mathcal{L} \in \partial\mathcal{P}(\Lambda_0)$ is minimal. Thus, all solutions $\zeta_{i,1}, \zeta_{i,2}$ for $i = 1, \ldots, |\mathcal{L}|$, of

$$\sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \widetilde{a}_\lambda(x) \cdot (\zeta_{i,1})^k = 0 \,,$$
$$\sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \widetilde{b}_\lambda(y) \cdot (\zeta_{i,2})^k = 0 \,,$$

coincide pairwise at p with $\zeta_{i,1}(0) = \zeta_{i,2}(0) = c$, where $0 \neq c \in \mathbb{C}$, is a root of

$$\sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \alpha_\lambda c^k = 0 \,,$$

with $\alpha_\lambda \in \mathbb{C}$ given by $P = \sum_{\lambda \in \Lambda_0} \alpha_\lambda x^\omega y^\alpha V^k$ at $0 \in \mathbb{R}^2_+$. Solving both $Q\widetilde{\psi}_1 = \zeta_1$ and $Q\widetilde{\psi}_2 = \zeta_2$ then yields $\widetilde{\psi}_1(0) = \widetilde{\psi}_2(0) = c/(\delta_1 - \delta_2)$, since $Q = V - (\delta_1 - \delta_2)$ is a first order b-differential operator with $\delta_1 \neq \delta_2$.

(ii). Let $\delta_2 = 0$ and $\delta_1 > 0$. We have $\widetilde{\psi}_1(0) = \psi_1(0) = c/\delta_1$, following the computations above. Applying Proposition 2.3.6 to the leading operator of $P \sim y^{l_{H_2,\delta_2}} P_0 + o(y^{l_{H_2,\delta_{H_2}}})$ at H_2 given by

$$P_0 = \sum_{\substack{\lambda \in \Lambda_{H_2} \cap \mathcal{L}_2 \\ \lambda = (k_\alpha)}} a_\lambda \partial_y^k$$

then yields asymptotic solutions $u = e^{\varphi(y)} A(y)$ of $P_0 u = 0$ at the corner $0 \in \mathbb{R}^2_+$ on H_2, with $\varphi(y) \sim c/\delta_2 y^{-\delta_2} + o(y^{-\delta_2})$. □

It is important to emphasize that the existence of an edge in the lower boundary of a Newton polyhedron $\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda_0)$ is only a sufficient condition for the existence of matching phase functions by Proposition 4.3.4 in the asymptotic construction. In the following example we will analyze an operator admitting an exponential-polyhomogeneous solution despite not having a lower edge in the Newton polyhedron at the corner $0 \in \mathbb{R}^2_+$.

Example 4.3.5. Let $V = x\partial_x - y\partial_y$, $P := yV + (x + y)$ on $\mathbb{R}^2_+$, denote $H_1 := \{y = 0\}$, $H_2 := \{x = 0\}$ and let $\Lambda_0 := \{(0,1,0), (0,0,1), (1,0,1)\}$ be the associated set of exponents localized at $0 \in \mathbb{R}^2_+$. At H_j, $j = 1, 2$, the sets of exponents at both faces are given by $\Lambda_{H_1} = \{(0,0), (0,1), (1,1)\}$ and $\Lambda_{H_2} = \{((0,0), (0,1), (1,0))\}$, where the difference in the last tuples corresponds to the coefficient y of V. Thus, there is no edge $\mathcal{L} \in \partial_- \mathcal{P}(\Lambda_0)$ with $\pi_{(k,\alpha_{j+1})}(\mathcal{L}) \subset \partial_- \mathcal{P}(\Lambda_{H_j})$ for $j = 1, 2$. Attempting to construct local approximations corresponding to Λ_{H_j} for $\mathcal{L}_1 := \overline{(0,0), (1,1)} \subset \partial_- \mathcal{P}(\Lambda_{H_1})$ and $\mathcal{L}_2 := \overline{(0,0), (1,0)} \subset \partial_- \mathcal{P}(\Lambda_{H_2})$ results in an exponential ansatz at H_1 since $\delta(\mathcal{L}_1) = 1$, and a polyhomogeneous ansatz at H_2 since $\delta(\mathcal{L}_2) = 0$. Thus, the eikonal polynomial at H_1

$$E_{\delta_1}(P)(x, \zeta) = \zeta + x$$

has a single solution $\zeta := -x$ and the corresponding eikonal equation

$$(x\partial_x + 1)\varphi = -x$$

is solved by $\varphi(x) := -x/2$. Conjugation of P with $\exp(\varphi(x)/y)$ then yields

$$P_{\varphi/y} = y \cdot (V + 1) \ .$$

Applied to functions, it has a kernel spanned by $A(x) := x^{-1}$. In particular, the function $u_1(x, y) := \exp(\varphi(x)/y)A(x) = e^{-x/(2y)}x^{-1}$ is a solution of $Pu = 0$. On the other hand, $\delta(\mathcal{L}_2) = 0$ yields a local solution at H_2, which is given by $u_{H_2,0}(y) := y$ and solves

$$y(-y\partial_y + 1)u_{H_2,0}(y) = 0 \,,$$

where $-y^2\partial_y + y$ is the leading part of $P = y(x\partial_x - y\partial_y) + (x+y)$ with respect to powers in x. For higher order corrections $u_{H_2,k}(y) \cdot x^k$ the family of inhomogeneous transport equations

$$y(-y\partial_y + (1+k))u_{H_2,k}(y) = -u_{H_2,k-1}(y)\,,$$

has solutions $u_{H_2,1}(y) = -1/2$ for $k = 1$ and $u_{H_2,k}(y) = (-1)^k/k!y^{-k+1}$ for $k \in \mathbb{N}$, $k > 1$. Thus, the asymptotic series of correction terms converges, namely

$$u_2(x,y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y \left(\frac{x}{y}\right)^k = y \exp(-x/y)$$

and coincides with the exponential behavior of $\varphi(x)/y = -x/(2y)$. In particular, we have $(xy)u_2(x,y) = u_1(x,y)$, which means that we have essentially constructed the same solution locally at each face, since $[P, xy] = 0$.

There is another important point of view for Example 4.3.5. b-Vector fields of the form $V = x\partial_x - y\partial_y$ usually appear at the corner of the front face after blowing up an interior point of a boundary hypersurface. Thus the blow-down of P results in a semi-classical, one dimensional, first order differential operator, which can be analyzed completely.

Example 4.3.6. Changing the coordinates in Example 4.3.5 corresponding to the integral curves yields a system of coordinates (t, h) in the interior of $\mathbb{R}^2_+$ defined by $t := x$ and $h := xy$. Thus the vector field transforms as

$$\begin{aligned} V = x\partial_x - y\partial_y &= x\left(\frac{\partial_t}{\partial_x}\partial_t + \frac{\partial_h}{\partial_x}\partial_h\right) - y\left(\frac{\partial_t}{\partial_y}\partial_t + \frac{\partial_h}{\partial_y}\partial_h\right) \\ &= t\left(\partial_t + \frac{h}{t}\partial_h\right) - \frac{h}{t}\left(0 + t\partial_h\right) = t\partial_t\,. \end{aligned}$$

In these coordinates the operator P is given by

$$P = h\partial_t + t + \frac{h}{t}\,.$$

Computing the set of exponents relative to $h \to 0$ yields $\Lambda = \{((0,0),(0,1),(1,1))\}$ with a single lower edge $\mathcal{L} := \overline{(0,0),(1,1)}$. It is important to note that P is $\mathcal{L}$-resolved in these coordinates, since there are two essential points $\mu_0 := (0,0)$ and $\mu_1 := (1,1)$, spanning $\mathcal{L}$. Solving the family of ODEs directly yields a family of solutions u_h, $h > 0$, given by

$$u_h(t) = \frac{1}{t} e^{-t^2/(2h)}\,.$$

Thus, we can compute the pullback of $u(t,h) := u_h(t)$ in terms of the coordinate change β from (x,y) to (t,h) and obtain

$$(\beta^* u)(x,y) = \frac{1}{x} e^{-x/(2y)}\,,$$

which is exponential-polyhomogeneous on $\mathbb{R}^2_+$, despite $\Lambda_0(P)$ not having a lower edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_0)$. In particular, $\beta^* u$ coincides with the solution computed in Example 4.3.5 up to a power of xy.

Transport Equations

Analogously to Subsection 4.3.1 the induced transport operator as in Definition 3.1.10 will be given by the b-vector field $V = x\partial_x - y\partial_y$ multiplied with some powers of x and y, i.e. $T = x^{l\delta_2+\delta_2} y^{l\delta_1+\delta_1} V$. Thus we need to make sure that solutions along both boundary hypersurfaces H_j match asymptotically at the corner, allowing for a polyhomogeneous extension to $\mathbb{R}^2_+$.

Lemma 4.3.7. *Let $V := x\partial_x - y\partial_y$ on $\mathbb{R}^2_+$. Let $G \in C^\infty(\mathbb{R}^2_+)$, $s := G(0,0)$ and $P := V - G$. Then the following holds:*

(i) There is an asymptotic solution $u \in \mathcal{A}(\mathbb{R}^2_+)$ of $Pu = 0$, i.e. $Pu = o(u)$ at $\partial\mathbb{R}^2_+$, with $u(x,y) \sim u_1(x) + o(1)$ as $y \to 0$ and $u(x,y) \sim u_2(y)x^s + o(x^s)$ as $x \to 0$.

(ii) For each $f \in \mathcal{A}(\mathbb{R}^2_+)$ log-free, there is a polyhomogeneous function $u = O(f)$, such that $Pu - f = o(f)$ at $\partial\mathbb{R}^2_+$.

Proof. (i) Let $g_1(x) := G(x,0)$, $g_2(y) := G(0,y)$ and $s := G(0,0)$. Let $u_1 \in \mathcal{A}(\mathbb{R}_+)$, be a solution of $(x\partial_x - g_1)u_1 = 0$ and $u_2 \in \mathcal{A}(\mathbb{R}_+)$ be a solution of $(-y\partial_y - (g_2 - s))u_2 = 0$. Then $u_1(x) \sim x^s + o(x^s)$ and $u_2(y) \sim 1 + o(1)$. By the Borel lemma 2.2.18 the pair $(u_1, u_2 \cdot x^s)$ admits a polyhomogeneous extension u to $\mathbb{R}^2_+$ and $Pu \sim (x\partial_x - g_1)u_1 = 0$ at $\{y = 0\}$ and

$$Pu \sim (V - (g_2 - s))u_2 x^s = -y\partial_x u_2 = 0$$

at $\{x = 0\}$. Hence $Pu = o(u)$ at $\partial\mathbb{R}^2_+$.

(ii) Let $f \in \mathcal{A}(\mathbb{R}^2_+)$ and denote the leading parts of f at $\partial\mathbb{R}^2_+$ by f_1 and f_2 respectively, i.e. $f(x,y) \sim f_1(x)y^\alpha +$ h.o.t. as $y \to 0$ and $f(x,y) \sim f_2(y)x^\beta +$ h.o.t. as $x \to 0$. If $\beta > s$, let $u_1 \in \mathcal{A}(\mathbb{R}_+)$ be the solution of $(x\partial_x - g_1)u_1 = f_1$ with $u_1(x) \sim x^\beta$, and if $\beta \le s$ let u_1 be any solution. If $\alpha > s$ let u_2 be the solution of $(y\partial_x + (g_2 - s))u_2 = -f_y$ with $u_2(y) \sim y^\alpha$, or any solution otherwise. By the Borel lemma the pair $(u_1 y^\beta, u_2 x^\alpha)$ admits a polyhomogeneous extension u to $\mathbb{R}^2_+$ and

$$Pu - f \sim (x\partial_x - g_1)u_1 - f_1 = 0$$

at $\{y = 0\}$ and

$$Pu - f \sim (y\partial_y + (g_2 - s))u_2 + f_y = 0$$

at $\{x = 0\}$. □

Remark 4.3.8. If u is an asymptotic solution of $(V - F)u = 0$ at $\partial\mathbb{R}^2_+$, then $(xy)^t u$ is an asymptotic solution for all $t \in \mathbb{R}$, since any b-vector field V commutes with xy as operators, i.e. $[V, xy] = 0$.

Remark 4.3.9. Although it is not necessary, one can do more than proposed in Lemma 4.3.7. Recall that $\mathbb{R}^2_+ \cong [\mathbb{R}^2_+, 0] \setminus \mathrm{lf}$, where $[\mathbb{R}^2_+, 0]$ is the space resulting from the homogeneous blow-up of 0 in $\mathbb{R}^2_+$, i.e. $\beta\colon [\mathbb{R}^2_+, 0] \to 0$. In particular, $x\partial_x - y\partial_y = \beta^*(x\partial_x)$ in the interior of either space and the differential equation $(V - G)u = f$ corresponds to the ordinary b-differential equation $(x\partial_x + \widetilde{G})u = \widetilde{f}$. Hence we can compute explicit solutions of these differential equations, which are given in terms of push-forwards for some $c \in \mathbb{C}$, i.e.

$$u(x, h) = e^{-(\int_0^x \widetilde{G}(t,h)\frac{dt}{t})} \left(c + \int_0^x e^{(\int_0^s \widetilde{G}(t,h)\frac{dt}{t})} \widetilde{f}(s, h) \frac{ds}{s} \right) .$$

These solutions are not unique in powers of xy, since $[V, xy] = 0$, as mentioned before. Their asymptotics can be computed explicitly based on the behavior of the integral kernel

$$K(x, s, h) = e^{(\int_x^s \widetilde{G}(t,h)dt/t)} \chi_{s \le x} ,$$

using the singular asymptotics lemma or the push-forward theorem. These can be found in [Mel96] and [Gri01].

Construction

Combining both Proposition 4.3.4 and Lemma 4.3.7, we are able to prove that local quasimodes at either adjacent boundary hypersurface H_j corresponding to an edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_0)$ can be extended as a joint, single quasimode to the interior of $\mathbb{R}^2_+$. The whole process is summarized in the following theorem.

Theorem 4.3.10 (Construction of Quasimodes II: Hyperbolic b-Vector Fields)**.** *Let $\mathcal{I}$ be an index family on $\mathbb{R}^2_+$, $n \in \mathbb{N}$, $A_k \in \mathcal{A}^{\mathcal{I}}(\mathbb{R}^2_+)$, $0 \le k \le n$, $V := x\partial_x - y\partial_y$ and $P := \sum A_k V^k$ with associated set of exponents Λ_0 at $0 \in \mathbb{R}^2_+$. Denote $H_1 := \{y = 0\}$, $H_2 := \{x = 0\}$ and let Λ_{H_j}, be the set of exponents generated by the asymptotic expansion of P at H_j, $j = 1, 2$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_0)$ and denote $\mathcal{L}_j := \pi_{(k,\alpha_{j+1})}(\mathcal{L})$, $j = 1, 2$, and $\delta_j := \delta(\mathcal{L}_j)$. Let all roots ζ_j, $j = 1, 2$, of $E_{\delta_j}(\cdot, \zeta_j) = 0$ be simple and let P be strictly δ_j-separated at H_j.*

Then for each root $c \in \mathbb{C}$ of $\sum_{\lambda \in \mathcal{L} \cap \Lambda_0} \alpha_\lambda c^k = 0$ as in Proposition 4.3.4 there are functions $\Phi, A \in \mathcal{A}(\mathbb{R}^2_+)$ and φ_j with $\varphi_j' = \zeta_j$, such that $\Phi(x, y) \sim \varphi_1(x) y^{\delta_1} +$ h.o.t. at H_1 and $\Phi(x, y) \sim \varphi_2(y) x^{\delta_2} +$ h.o.t. at H_2, with

$$\Phi(x, y) \sim \frac{c}{\delta_1 - \delta_2} \frac{1}{x^{\delta_2} y^{\delta_1}} + \textit{h.o.t.} ,$$

at $0 \in \mathbb{R}^2_+$. The function $u = e^{\Phi} A$ is a WKB-type quasimode of P on $\mathbb{R}^2_+$, i.e.

$$P_\Phi A = O((xy)^\infty) .$$

Proof. By Proposition 4.3.4 there is a phase Φ whose leading parts φ_j, $j = 1,2$ at H_j are solutions of $V\varphi_j = \zeta_j$, with ζ_j solutions of $E_{\delta_j}(\cdot, \zeta_j) = 0$ corresponding to $c \in \mathbb{C}$, i.e. $\varphi_1(x) \sim c/(\delta_1 - \delta_2)x^{-\delta_2}$ and $\varphi_2(y) \sim c/(\delta_1 - \delta_2)y^{-\delta_2}$. Since $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_0)$, conjugating P with e^Φ yields an operator of the form

$$Q := e^{-\Phi} P e^{\Phi} = Q_0 + \sum_{k=1}^{\infty} x^{\alpha_k} y^{\beta_k} Q_k ,$$

and without loss of generality we can assume that the transport operator is given by $Q_0 = V - G$ for some $G \in C^\infty(\mathbb{R}^2_+)$. Denote $s := G(0,0)$. Both sequences $(\alpha_k)_k$ and $(\beta_k)_k$ and strictly increasing, positive and have no accumulation point. This is a direct consequence of $\pi_{(k,\alpha_j)}(\mathcal{L}) \subset \partial\mathcal{P}(\Lambda_{H_j})$, analogous to the remarks in the beginning of Subsection 4.3.1. Applying Lemma 4.3.7 (1.) to Q_0 then yields a zeroth order quasimode u_0, satisfying

$$Qu_0 = Q_0 u_0 + (\sum_{k=1}^{\infty} x^{\alpha_k} y^{\beta_k} Q_k) u_0 = O(x^{\alpha_1} y^{\beta_1}) .$$

Its lowest remainder term is given by $f_{0,1} := x^{\alpha_1} y^{\beta_1} Q_1 u_0$. By Lemma 4.3.7, (ii) solving the inhomogeneous equation $Q_0 u_1 = f_{0,1}$ then yields a first order quasimode $u_0 + u_1$ with

$$Q(u_0 + u_1) = O(x^{\min\{2\alpha_1, \alpha_2\}} y^{\min\{2\beta_1, \beta_2\}}) .$$

Since $\{(\alpha_k, \beta_k) : k \in \mathbb{N}\}$ is discrete in the Euclidean topology, the same principle as in Theorem 3.3.11 holds and for any threshold $N \in \mathbb{N}$ there is a finite number $M \in \mathbb{N}$ of iterations, such that

$$Q\left(\sum_{k=1}^{M} u_k\right) = O((xy)^N) .$$

Thus $A := \sum_{k=1}^{\infty} u_k$ is an amplitude and $u := e^\Phi A$ is a quasimode of P. $\square$

5 Resolution of Operators

This chapter analyses unresolved operators and their resolutions by using successive blow-ups in the half space $\mathbb{H}$. Section 5.1 starts with the lift of the notion of sets of exponents Λ_H to arcs $H \subset \partial M$ of a blown-up half space $\beta\colon M \to \mathbb{H}$. The lifts can be computed explicitly and correspond to the parameters $t > 0$ of the blow-up maps β_t. Using a specific family of *induced coordinates* at each arc, we are able to specify a notion of δ-regular and $\mathcal{L}$-resolved operators along these arcs. Finally, we can transfer the results of Proposition 4.3.4 to blown-up spaces in order to extend local solutions of eikonal equations at adjacent arcs. Eventually, we are able to specify a list of conditions in Section 5.1 to check whether a chain of blow-ups has resolved an unresolved operator.

Section 5.2 advances in analyzing the effects of blow-ups $\beta_t\colon Y \to \mathbb{H}$ on zeros of the coefficients $\beta_t^* a_\lambda$ on the new front face. These are the relevant quantities for δ-regularity and $\mathcal{L}$-resolved operators introduced in Section 5.1. Increasing the parameter $t > 0$ deforms the Newton polygon and splits up single or merges multiple edges. When this phenomenon occurs, it is not clear if lower three-dimensional edges of the Newton polyhedron still exist at the corner of the front face. This leads to the introduction of *collision times* in Definition 5.2.1. These are maximal parameters $t(\mathcal{L})$ for a blow-up and are associated to an edge $\mathcal{L}$, such that a lower three-dimensional edge remains at the corner of the new front face. They are easy to compute and important for the resolution of operators. The successive use of blow-ups with collision times as parameters then allows for an algorithmic regularization of the operator's associated unresolved points. A new problem compared to the setting in Chapters 3 and 4 that arises is the necessity to construct quasimodes for multiple edges and solutions simultaneously at some arcs. This issue is already present in the construction of quasimodes for the Schrödinger operator with linear potential and only leads to some technical difficulties in the resolution process. We will display the full algorithm in a blow-up graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, accounting for all spaces, arcs, edges and solutions of eikonal equations

$$(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$$

that appear in any step of the resolution. This is specified in Algorithm 1, which terminates when the operator $\beta^* P$ is either regular or resolved on all arcs and when its solutions of eikonal equations have constant multiplicities. The finiteness of the algorithm is proven in Theorem 5.2.18. We apply the resolution algorithm to the Schrödinger operator with vanishing potential and the Bessel operator in Subsections 5.2.3 and 5.2.4.

In Section 5.3 the goal is to recover the systematic approach to construct and improve asymptotic amplitudes on blown-up spaces as in Proposition 3.1.7. The complexity arises from solving the transport equations at all arcs $H \subset \partial M$ generated in the resolution process of the previous section simultaneously. The problems that emerge are mostly technical challenges which can be addressed by ordering the edges $\mathcal{L}$ and solutions ζ of the associated eikonal polynomial at the arcs. Due to the variety of edges and phases at a single arc one has to solve multiple transport equations corresponding to different *induced transport operators* at each step of the iteration, since these are directly linked to the choice of $\mathcal{L}$ and ζ. Eventually, we are able to present suitable spaces of exponential-polyhomogeneous functions

$$\mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma(\mathcal{G}))$$

with a family of index sets $\mathcal{I}(\mathcal{G})$ and a collection of phase functions $\Gamma(\mathcal{G})$. Having these, we can solve a family of homogeneous transport equations on M and extend these local solutions to an exponential-polyhomogeneous quasimode u of β^*P with low order on M. The order of this initial quasimode can be improved by eliminating the leading parts of its image $F = (\beta^*P)u$, resulting in a chain of inhomogeneous transport equations at each arc. Theorem 5.3.22 summarizes this process, namely we are able to construct quasimodes to a given set of initial data $(\mathcal{L}_0, \zeta_0)$ on their corresponding resolution space $\beta\colon M \to \mathbb{H}$.

Afterwards we extend the scope and take a look at the vector valued Schrödinger operator in Subsection 5.3.5. We perform an ad-hoc resolution by means of a quasihomogeneous blow-up.

5.1 Combinatorial Geometry III: Resolution

In this section we lift the central tools developed in Chapters 3 and 4 to blown-up half spaces $\beta\colon M \to \mathbb{H}$. We start with the introduction of sets of exponents associated to arcs $H \subset \partial M$ and the effects of blow-ups towards these sets on the newly generated front face. In particular, we will present a family of induced coordinates along the arcs, allowing for a unique choice of coordinates. Having these we can reintroduce a notion of δ-regularity and $\mathcal{L}$-resolved operators on arcs and can discuss the behavior of essential points under blow-ups.

Eventually, we can lift a notion of eikonal polynomials $E_{H,\delta}(\beta^*P)$ to arcs $H \subset \partial M$. At the end of this section we prove in Proposition 5.1.16 that under the presence of lower three dimensional edges over corners of adjacent arcs H_1 and H_2, there is a pair (φ_1, φ_2) of solutions of eikonal equations at both adjacent arcs such that $(\varphi_1/h^{\delta_1}, \varphi_2/h^{\delta_2})$ has a polyhomogeneous extension to the interior of M. Although this statement is a direct transfer of the local statement of Proposition 4.3.4, it will be crucial in the resolution of operators in Section 5.2, systematically using the effects of blow-ups to sets of exponents.

5.1.1 Sets of Exponents and Blow-Ups

We start this section with the introduction and discussion of sets of exponents at arcs on a manifold M and the effects of blow-ups to sets of exponents.

Induced Action

To analyze how blow-ups affect sets of exponents Λ we have to investigate their effects on semi-classical operators itself when being pulled back. Let $\kappa_h, \kappa_x \in \mathbb{N}$, let $t := \kappa_x/\kappa_h$ and let $\beta_t : M \to \mathbb{H}$ be the corresponding quasihomgeneous blow-up with $M := [\mathbb{H}, 0]_t$. Then the pullback of coefficients of P on the front face $\beta_t^{-1}(0)$

$$\beta_t^* \left(x^l h^\alpha \partial_x^k\right) = x_{\text{ff}}^l \cdot \widehat{h}^{\kappa_x(l-k)+\kappa_h \alpha} \cdot \partial_{x_{\text{ff}}}^k \tag{5.1}$$

changes only the powers of h, with coordinates $x_{\text{ff}} := x/h^t$ and $\widehat{h} = h^{1/\kappa_h}$ as in Definition 2.1.19. Since $[\partial_{x_{\text{ff}}}, \widehat{h}] = 0$, we can rescale $\widehat{h}$ to h

$$h^{\alpha + t\cdot\omega} = \widehat{h}^{\kappa_x(l-k)+\kappa_h \alpha} \,,$$

with $\omega := l - k$, which relates the coordinate change only to the quotient $t = \kappa_x/\kappa_h$ and simplifies later computations along multiple arcs. On the other hand, at the corner we have

$$\beta_t^*(\partial_x) = r^{-\kappa_x} \left(\frac{1}{\kappa_x} r\partial_r - \frac{1}{t}\eta\partial_\eta \right) , \tag{5.2}$$

which is a b-vector field on M with negative weight at the front face, where $r := x^{1/\kappa_x}$ and $\eta := h/r^{\kappa_h}$, following Lemma 2.1.20. Since r and $\widehat{h}$ coincide at the corner away from the right face H, rescaling r as

$$r_H := r^{\kappa_h} = x^{1/t}$$

then leads to the same weight of both coordinates h and r_H at the front face. Thus, we can write

$$\beta_t^*(\partial_x) = \frac{1}{t} \cdot r_H^{-t} \left(r_H \partial_{r_H} - \eta\partial_\eta \right) = \frac{1}{t} r_H^{-t} V \,,$$

where $V := r_H \partial_{r_H} - \eta\partial_\eta$ is a regular b-vector field at the corner.

Denote $\omega = l - k$. In a more general way, coefficients of operators with vanishing amplitudes transform as

$$\beta_t^* \left(x^l h^\alpha \partial_x^k\right) = (\beta^* x)^l (\beta^* h)^\alpha (\beta^*(x^k \partial_x^k)) = r_H^{\alpha + t\omega} \eta^\alpha V^k + O(V^{k-1}) \tag{5.3}$$

to leading differential order V^k. Thus, by construction, the power of r_H at H matches that of h at the front face, after pulling back $x^l h^\alpha \partial_x^k$. Recursively, we can introduce coordinates of the same type at each arc arising from succeeding blow-ups. This leads to the definition of induced coordinates.

Definition 5.1.1 (Induced Coordinates)**.** Let (x, h) be coordinates on $\mathbb{H}$, let $t \in \mathbb{Q}_{>0}$ and $\beta : M \to \mathbb{H}$ be a sequence of quasihomogeneous blow-ups of 0. Denote $H_0 := \partial\mathbb{H}$ and let

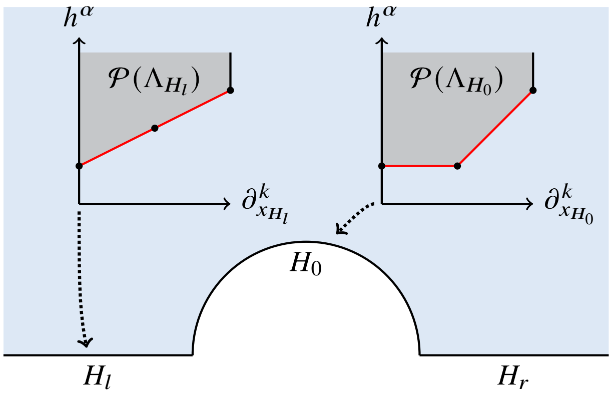

Figure 5.1: The Newton polygons $\mathcal{P}(\Lambda_{H_0})$ and $\mathcal{P}(\Lambda_{H_l})$ at front face H_0 and left face H_l on the blown-up space $\beta\colon [\mathbb{H}, 0] \to \mathbb{H}$.

$H \subset \partial M$ be a boundary face. Let $t > 0$, $p \in H^\circ$ and $\beta_t\colon [M, p]_t \to M$ be the corresponding blow-up.

Then the coordinates defined recursively from H to $\beta^{-1}(p)$ in the following scheme are called *induced coordinates* of $(x_{H_0}, h) := (x, h)$ on M.

(i) *Interior coordinates* (x_{ff}, h) at the front face $\mathrm{ff} := \beta_t^{-1}(p)$, where $x_{\mathrm{ff}} := (x_H - p)/h^t$, with induced coordinates (x_H, h) on H.

(ii) *Corner coordinates* (r_H, η_H) at the corner $\mathrm{ff} \cap \beta_t^* H$, where $\eta_H := h/(x_H - p)^{1/t}$ and $r_H := x_H^{1/t}$, with induced coordinates (x_H, h) on H.

With the introduction of *induced coordinates* we have a unique way to lift the notion of sets of exponents in Definition 4.2.23 to manifolds with corners.

Definition 5.1.2 (Set of Exponents on Manifolds)**.** Let $\beta\colon M \to \mathbb{H}$ be a chain of quasihomogeneous blow-ups of $0 \in \mathbb{H}$, let P be a semi-classical operator on $\mathbb{H}$ and $H \subset \partial M$ be an arc. For $\lambda = (k, \alpha) \in \mathbb{N}_0 \times \mathbb{R}$ let $a_\lambda \in C^\infty(H)$, such that $\beta^* P = \sum_\lambda a_\lambda h^\alpha \partial^k_{x_H}$ at H with induced coordinates (x_H, h) at H.

Then we call

$$\Lambda_H(\beta^* P) := \{\lambda = (k, \alpha) : \ a_\lambda \not\equiv 0\} \ ,$$

the *set of exponents of $\beta^* P$ at H*.

Expanding $a_\lambda \in C^\infty(H)$ at $p \in \partial H$ then yields a direct way to lift the notion of the localized version, corresponding to Definition 4.2.23.

Definition 5.1.3 (Localized Set of Exponents)**.** Let $\beta\colon M \to \mathbb{H}$ be a sequence of quasihomogeneous blow-ups of $0 \in \mathbb{H}$ and denote $V := \beta^* \partial_x$. Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R})$ and let $H \subset \partial M$ be an arc. Let $p \in \partial H$ and $\alpha_{k,\alpha,\omega} \in \mathbb{C}$, for $\lambda = (k, \alpha, \omega) \in \mathbb{N}_0 \times \mathbb{R} \times \mathbb{Z}$, be the coefficients of $\beta^* P = \sum_\lambda \alpha_{k,\alpha,\omega} r^\omega \eta^\alpha V^k$ at p.

Then we call

$$\Lambda_p(\beta^* P) := \big\{(k, \alpha, \omega) : \ \alpha_{k,\alpha,\omega} \neq 0\big\}$$

the *localized* set of exponents of $\beta^* P$ at $p \in \partial H$, where (r_H, η_H) are the induced coordinates at $p \in \partial H$.

One can easily return to the two-dimensional set of exponents by omitting the latter entry of each element in Λ_p. Projecting the convex hull of Λ_p on either the (k, α_2) or (k, α_3) plane as in Definition 4.2.24 yields a way to lift the notion of lower boundary from two to higher dimensions.

By Definitions 2.3.2 and 4.2.24 we have a notion of lower boundary and edges corresponding to the Newton polygons and Newton polyhedra of an operator $\beta^* P$ on $\beta \colon M \longrightarrow \mathbb{H}$ (see Figure 5.1). The following definition formalizes the computations (5.1)-(5.3).

Definition 5.1.4 (Transformation Maps)**.** Let $t \in \mathbb{Q}_+$ and $\beta \colon M \longrightarrow \mathbb{H}$ be a chain of quasi-homogeneous blow-ups of $0 \in \mathbb{H}$, let $H \subset \partial M$ be an arc and $p \in H^\circ$. Let $t > 0$ and $\beta_t \colon [M, p]_t \longrightarrow M$ be the associated blow-up of p. Let $\lambda, \mu \in \Lambda_H$ and $\mathcal{L} = \overline{\lambda\mu} \subset \partial_- \mathcal{P}(\Lambda_H)$ be an edge.

(i) The matrices mapping $(k, \alpha, \omega) \mapsto (k, \alpha + t\omega, \omega)$ and $(k, \alpha, \omega) \mapsto (k, \alpha, \alpha + t\omega)$,

$$L_+(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \qquad L_-(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}$$

are called *transformation maps* induced by a blow-up corresponding to t.

(ii) We call $\omega_p(\lambda) := \mathrm{ord}_p(a_\lambda) - k_\lambda$ the *weight* of $\lambda \in \Lambda_H$ at $p \in H$.

(iii) We call $\lambda(t) := \pi_{(k,\alpha)}\left(L_+(t)(k_\lambda, \alpha_\lambda, \omega(\lambda))\right) \in \Lambda_{\beta_t^{-1}(p)}$ and $\mathcal{L}(t) := \overline{\lambda(t)\mu(t)}$ the *transformation* of λ and $\mathcal{L}$ by β_t at H, respectively.

Remark 5.1.5. Note that Definitions 4.2.3 and 5.1.4 have a coinciding notion of weights, i.e. $\omega_0(\lambda) = {}^b\omega(\lambda)$. The latter notion $\omega_0 \colon \Lambda \longrightarrow \mathbb{Z}$ in 5.1.4 refers to the coefficients of P with respect to $V = \partial_{x_H}$, which is a non-vanishing vector field in $0 \in \mathbb{H}$.

Remark 5.1.6. Note that k_λ and α_λ are constant along H, but $\omega(\lambda) = \omega_p(\lambda) = \mathrm{ord}(a_\lambda) - k_\lambda$ depends heavily on the choice of the point $p \in H$. By design we have that

$$\omega(\lambda) = \min\{\omega : \ (k_\lambda, \alpha_\lambda, \omega) \in \Lambda_p\}.$$

5.1.2 Resolved Operators on Hypersurfaces

The main reason for the non-existence of exponential-polyhomogenous quasimodes in Chapter 4 is the presence of essential points $\mu_k \in \Lambda$ in the interior of the Newton polygon $\mathcal{P}(\Lambda)$. Recall that P is called $\mathcal{L}$-resolved for $\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda)$, if and only if the edge is spanned by essential points, i.e. $\mathcal{L} = \overline{\mu_k\mu_{k+1}}$. Thus, we study the impact of blow-ups for essential points of P at a point $p \in \partial M$ in the subsection.

Definition 5.1.7 (δ-Regularity on Blown-Up-Spaces)**.** Let P be a generalized semi-classical operator on $\mathbb{R}$, let $\beta\colon M \to \mathbb{H}$ be a chain of quasihomogeneous blow-ups, $H \subset \partial M$ be an arc. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ be a maximal edge with slope $\delta := \delta(\mathcal{L})$.

The operator β^*P is called *δ-regular* in $p \in H^\circ$, if $a_\lambda(p) \neq 0$, for all $\lambda \in \partial\mathcal{L}$. It is called δ-regular on H if it is *δ-regular* in all points $p \in H^\circ$.

If P is not δ-regular (in $p \in H$) it is called *δ-singular* (in $p \in H$).

Definition 5.1.8 (Essential Points on Blown-Up Spaces)**.** Let P be a generalized semi-classical operator on $\mathbb{R}$, let $\beta\colon M \to \mathbb{H}$ be a quasihomogeneous blow-up, $H \subset \partial M$ be an arc and (x_H, h) be its corresponding pair of induced coordinates. Let $p \in H^\circ$, assume that $\omega_p(\tau) \geq 0$ for all $\tau \in \Lambda_H$. Let $\lambda, \mu \in \Lambda_H$ and define

$$\gamma(\overline{\lambda\mu}) := (\omega_p(\mu) - \omega_p(\lambda))/(k_\mu - k_\lambda)\,.$$

The points $\mu_0, \ldots, \mu_N \in \Lambda_H$, with $N \in \mathbb{N}_0$, determined in 4.2.12 for β^*P, are called *essential points* of β^*P in p. The point μ_0 is called *essential minimum.*

Remark 5.1.9. As in Definition 4.2.13 there are at most $m = \operatorname{ord}(P)$ essential points associated to each point $p \in \partial H$. If $\mu = (n, \alpha_\mu)$ is an essential point with $n < m$ then there is always another essential point $\mu^* \in \Lambda$ since $\gamma(\overline{\mu\lambda}) \in \mathbb{Q}$ and $\delta(\overline{\mu\lambda}) \in \mathbb{R}_+$ are bounded from below for all $\lambda = (k, \alpha)$ with $k > n$.

We can immediately lift the definition of $\mathcal{L}$-resolved operators to blown-up half spaces $\beta\colon M \to \mathbb{H}$.

Definition 5.1.10 (Resolved Operators on Blown-Up Spaces)**.** Let P be a generalized semi-classical operator on $\mathbb{R}$, let $\beta\colon M \to \mathbb{H}$ be a chain of quasihomogeneous blow-ups, $H \subset \partial M$ be an arc, $p \in H^\circ$ and (x_H, h) be its corresponding pair of induced coordinates. Let $\mu_0, \ldots, \mu_N$ be the essential points of β^*P at $p \in H$ and $\mathcal{L} := \overline{\lambda\mu} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ be an edge with slope δ. Suppose that β^*P is δ-singular in p.

Then we call the operator β^*P *$\mathcal{L}$-resolved* in p, if either μ is the minimal point of Λ_H and $\mu = \mu_0$ or $\lambda = \mu_k$ and $\mu = \mu_{k+1}$, for some $k \geq 1$.

If β^*P is $\mathcal{L}$-resolved in p for all $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_H)$, we call it *resolved* in p.

Resolution of Essential Points

Having a notion of essential points on blown-up half-spaces $\beta_t\colon [\mathbb{H}, 0]_t \to \mathbb{H}$, we can analyze their transformation from $\mathbb{H}$ to $[\mathbb{H}, 0]_t$ under the application of the blow-up β_t. Note that there is a canonical successor of $0 \in \mathbb{H}$ in the front faces, given by $0_{\mathrm{ff}} \in \beta^*\{x = 0\} \cap \mathrm{ff}$, where $\mathrm{ff} := \beta^{-1}(0)$. These points are important since $\beta_t^* a_\lambda$ will vanish in 0_{ff} if $a_\lambda(0) = 0$.

In Proposition 5.1.11 we will show that essential points are invariant under the transformation induced by β_t at the succeeding zeros 0_{ff}. Moreover, we will show that the lower boundary will be spanned by essential points eventually, for t big enough. This can be done either with a single or a chain of quasihomogeneous blow-ups at their respective zero points.

Proposition 5.1.11. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents, $P \in \mathit{Diff}^{\Lambda}(\mathbb{R}_+)$ and $\mu_0, \ldots, \mu_N$ be the essential points of P at $0 \in \{h = 0\}$. Then the following is true:*

(i) $\mu_0(t), \ldots, \mu_N(t)$ *are the essential points of $\Lambda_{\beta_t^{-1}(0)}$ with respect to $\beta_t^* P$ at 0_{ff}, for all $t \geq 0$.*

(ii) There is a $T \geq 0$ such that for all $t \geq T$ and for each $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda_{\beta_t^{-1}(0)})$ with $\delta(\mathcal{L}) > 0$ there is a $k < N$ with $\mathcal{L} = \overline{\mu_k(t)\mu_{k+1}(t)}$.

Proof. (i.) Let $\mu_0, \ldots, \mu_N$ be the essential points of P, where $\mu_j \in \omega^{-1}(0)$, $j \leq N$. Since $\mathrm{ord}_{0_{\mathrm{ff}}}(\beta_t^* a_\lambda) = \mathrm{ord}_0(a_\lambda)$ we have that $\mu_0(t), \ldots, \mu_n(t)$ are essential points for some $n \leq N$. In particular since $\omega(\mu) = \omega(\mu(t))$, for all $t \in \mathbb{R}_+$, we have

$$\min \gamma(\overline{\mu_n(t)\lambda}) = \min \gamma(\overline{\mu_n \mu}) = \gamma(\overline{\mu_n \mu_{n+1}}) = \gamma(\overline{\mu_n(t)\mu_{n+1}(t)}) \, .$$

Since μ_{n+1} is the lowest point of all points μ satisfying (4.7), $\mu_{n+1}(t)$ is an essential point. Iterating this argument we find that $\mu_0(t), \ldots, \mu_N(t)$ are essential points.

(ii.) Let $\mu_j = (k_j, \alpha_j)$, $j \in \{m, m+1\}$, be essential points and assume that there is a point $\lambda = (k, \alpha) \in \Lambda \cap \partial_{\!-}\mathcal{P}({}^b\Lambda)$, such that $k_m \leq k \leq k_{m+1}$. Then there is an $a_m \in \mathbb{R}$ such that $\omega(\mu_j) = \gamma_m \cdot k_j + a_m$, for $j = m, m+1$ and $\gamma_m := \gamma(\overline{\mu_m \mu_{m+1}})$. Since λ is not essential, we have that $\gamma(\overline{\mu_m \lambda}) > \gamma(\overline{\mu_m \mu_{m+1}})$. In particular, $\omega(\lambda) > \omega(\mu_{m+1})$ and hence there is a $T \geq 0$ such that $\partial_{\!-}\mathcal{P}(\{\mu_m(t), \lambda(t), \mu_{m+1}(t)\}) = \partial_{\!-}\mathcal{P}(\{\mu_m(t), \mu_{m+1}(t)\})$, for all $t \geq T$. □

Remark 5.1.12. Recall that by Definition 5.1.10 the latter statement means that $\beta_t^* P$ is resolved for all edges and for all $t \geq T$.

Remark 5.1.13. It is important to emphasize that the lower boundary of a polygon outside of the blown-up point remains unchanged, i.e. $\partial_{\!-}\mathcal{P}(\Lambda_{\beta_t^* H}) = \partial_{\!-}\mathcal{P}(\Lambda_H)$. This is due to the choice of induced coordinates in Definition 5.1.1, for any blow-up of $p \in H \subset \partial M$. In particular, any edge $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda_H)$ with positive slope will be mapped to itself in $\Lambda_{\beta_t^* H}$. Only the weights $\omega(\lambda)$ of $\lambda \in \Lambda_H$ are affected by the pullback of Λ_H to $\beta_t^* H$.

Combinatorial Requisites of Matching

To analyze the behavior of potential quasimodes of $\beta^* P$ based on its associated edges $\mathcal{L} \subset \partial_{\!-}\mathcal{P}(\Lambda_H)$ over arcs $H \subset M$ we need to reintroduce a notion of eikonal polynomials. Induced coordinates in Definition 5.1.1 give us a unique way to display the operator at H and thus are an apparent choice.

Definition 5.1.14 (δ-Principal Symbol on Blown-up Spaces)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be an operator, let $\beta \colon M \to \mathbb{H}$ be a blow-up and $H \subset \partial M$ an arc. Let $\mathcal{L}_H \subset \partial_{\!-}\mathcal{P}(\Lambda_H)$ be an edge with $\delta := \delta(\mathcal{L}) > 0$ and $\beta^* P = \sum_{\lambda \in \Lambda_H} a_\lambda h^\alpha \partial_{x_H}^k$ at H with induced coordinates (x_H, h).

Then the *δ-principal symbol* of β^*P at H is the polynomial

$$E_{\delta,H}(\beta^*P)(\cdot,\zeta_H) := \sum_{\lambda\in\Lambda_H\cap\mathcal{L}_H} a_\lambda \zeta_H^k .$$

We generalize the height l_δ of $\Lambda(P)$ after conjugation in Definition 3.2.3 to the blown-up case.

Definition 5.1.15. Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R}_+)$, $\beta\colon M \to \mathbb{R}_+^2$ be a blow-up, $H \subset \beta^{-1}(\{h=0\})$ and $\delta \geq 0$. Then define

$$l_{\delta,H} := \min\{\alpha - k\delta :\ (k,\alpha) \in \Lambda(\beta^*P)_H\} .$$

Matching at the Corner

It was discussed in Chapters 3 and 4 when an eikonal equation has smooth solutions on $\mathbb{H}$ or $\mathbb{R}_+^2$. The same principles apply for an eikonal equation

$$E_{\delta,H}(\beta^*P)(\cdot,\partial_{x_H}\varphi_{H,\delta}) = 0$$

induced by β^*P at an arc $H \subset \partial M$ of a blown up space $\beta\colon M \to \mathbb{H}$. Thus, we need to compare solutions of eikonal equations at adjacent arcs H and H' to show when these admit a polyhomogeneous extension to a neighborhood of both arcs. In particular, these solutions $\varphi_{H,\delta}$ are asymptotic solutions of the eikonal equation associated to the asymptotic expansion of the operator

$$\beta^*P = \sum_{\substack{\lambda\in\Lambda_p\\ \lambda=(k,\alpha,\omega)}} \alpha_\lambda r_H^\omega \eta_H^\alpha V^k ,$$

in induced coordinates (r_H,η_H) at the intersection $p \in H \cap H'$ with $V = r_h\partial_{r_H} - \eta_H\partial_{\eta_H}$. This is the same situation locally as in Subsection 4.3.2, allowing us to solve *matching* pairs of eikonal equations in the presence of lower three dimensional edges in $\partial_-\mathcal{P}(\Lambda_p)$.

Proposition 5.1.16. *Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R})$, let $\beta\colon M \to \mathbb{H}$ and $H_j \subset \partial M$, $j=1,2$, be adjacent arcs with $p \in H_1 \cap H_2$ and denote $H := H_1$. Let $V := r\partial_r - \eta\partial_\eta$ be a b-vector field with induced coordinates (r,η) at p. Assume there is an edge $\mathcal{L} \subset \partial_-\mathcal{P}(\Lambda_p(\beta^*P))$ and denote $\delta_j := \delta(\mathcal{L}_j)$, for $\mathcal{L}_j := \pi_{(k,\alpha_j)}(\mathcal{L}) \subset \partial_-\mathcal{P}(\Lambda_{H_j})$. Assume that $\delta_1 \neq \delta_2$ and $\delta_j \geq 0$. Let $c \in \mathbb{C}$ be a solution of*

$$\sum_{\substack{\lambda\in\mathcal{L}\cap\Lambda_0\\ \lambda=(k,\alpha,\omega)}} \alpha_\lambda c^k = 0 ,$$

for $\beta^*P \sim \sum_{\lambda\in\Lambda_p} \alpha_\lambda r^\omega \eta^\alpha V^k$ at p. Assume that all non-trivial solutions of the eikonal equation $E_{H_j,\delta_j}(\beta^*P)(\cdot,\zeta) = 0$ are simple on H_j°, $j=1,2$.

(i) *If* $\delta_1, \delta_2 > 0$ *then there is a phase function* $\Phi \in \mathcal{A}(M)$ *with* $\Phi(r_H, \eta_H) \sim \eta_H^{-\delta_1}\widetilde{\varphi}_1(r_H)$ *at* H_1 *and* $\Phi(r_H, \eta_H) \sim r_H^{-\delta_2}\widetilde{\varphi}_2(\eta_H)$ *at* H_2*, such that*

$$E_{H_j,\delta_j}(\beta^* P)(\cdot, V\varphi_j) = 0\,,$$

where $\varphi_1(x_H) \coloneqq \widetilde{\varphi}_1(r_H)\cdot(h/\eta_H)^{\delta_1}$ *and* $\varphi_2(x_{H_2}) \coloneqq \widetilde{\varphi}_2(\eta_H)\cdot(h/r_H)^{\delta_2}$*, and*

$$\Phi(r_H, \eta_H) \sim \frac{c}{\delta_1 - \delta_2}\cdot\frac{1}{r_H^{\delta_2}\cdot\eta_H^{\delta_1}} \;+\; h.o.t.\,,$$

at the corner $p \in H_1 \cap H_2$.

(ii) *If* $\delta_2 = 0$ *then there is a solution* $\varphi_1 \in C^\infty(H_1)$ *of* $E_{H_1,\delta_1}(\beta^* P)(\cdot, V\varphi_1) = 0$ *with* $\varphi_1(0) \neq c$ *and a solution* u *of the equation*

$$\sum_{\substack{\lambda\in\Lambda_{H_2}\cap\mathcal{L}_2\\ \lambda=(k_\alpha)}} a_\lambda y^{(k)} = 0$$

on H_2 *such that*

$$u(y) \sim e^{c/(\delta_2\cdot y^{\delta_1})} + o\left(y^{-\delta_1}\right)\,.$$

Proof. This is a consequence of Proposition 4.3.4, with the slight difference that h is a global boundary defining function on M. Thus, we need to rescale $\widetilde{\varphi}_1(r_H)/\eta_H^{\delta_1} \coloneqq \varphi_1(x_H)/h^{\delta_1}$ and rescale analogously for $j = 2$. Then for each $c \in \mathbb{C}$ solving $\sum_{\lambda\in\mathcal{L}\cap\Lambda_0}\alpha_\lambda c^k = 0$ we can apply Proposition 4.3.4, yielding a phase function Φ with $\Phi(r_H, \eta_H) \sim \widetilde{\varphi}_1(r_H)/\eta_H^{\delta_1}$ and analogously for $j = 2$. □

5.2 Resolution of Operators

This section is the programmatic center of Chapter 5 with the derivation of a resolution algorithm for unresolved operators. The essential new tool in this section is the so called *collision time* $t(\mathcal{L})$ of an edge $\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda)$ in Definition 5.2.1, determining a priori a range of values $t \in (0, t(\mathcal{L})]$ for which quasihomogeneous blow-ups $\beta_t : Y \longrightarrow \mathbb{H}$ partially resolve operators. It is designed in such a way that it guarantees the existence of a lower three dimensional edge over the corner at the end of the front face, which is sufficient for the existence of extendable pairs of solutions of eikonal equations.

The resolution algorithm itself will generate a graph, accounting for all spaces, arcs, edges and solutions of the eikonal equation that appear during the stepwise resolution of operators. It terminates once the operator in question is either regular or resolved at all points and if all solutions of eikonal equations in question do not jump in multiplicity pointwise. At the end of the section we show that the algorithm terminates for all relevant types of operators and also discuss possible extensions. Due to the simplification of regularity to collision times and graphs, the proof of Theorem 5.2.18 will be mostly combinatorial.

5.2.1 Collisions

The key in the resolution of operators will be the so called *collision time*. It is the smallest parameter t such that there is another pair of exponents $\lambda \in \Lambda_{\mathrm{ff}}$ which is collinear with the transformed edge $\mathcal{L}(t)$ (see Figure 5.2). Afterwards we will show that the collision time is always positive, guaranteeing the short time existence of lower, three dimensional edges after blowing up single points.

Definition 5.2.1 (Collision Time)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$. Let $\beta\colon M \to \mathbb{H}$ be a blow-up, $H \subset \partial M$ an arc and Λ_H be the set of exponents of $\beta^* P$ at H. Let $\mathcal{L} = \overline{\lambda, \mu} \subset \partial\mathcal{P}(\Lambda_H)$ be an edge. Let $p \in H^\circ$ and $\beta_t\colon [M, p]_t \to M$ be the quasihomogeneous blow-up of p in M with respect to $t > 0$.

Then we call

$$t(\mathcal{L}) := \inf\{t > 0 : \text{ there exists } \nu \in \Lambda \setminus \mathcal{L} \text{ with } \nu(t) \in \operatorname{span}(\mathcal{L}(t))\}\ ,$$

the *collision time* of $\mathcal{L}$, where $\mathcal{L}(t) = \overline{\lambda(t)\mu(t)}$.

Proposition 5.2.2. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$. Let $\beta : M \to \mathbb{H}$ be a blow-up, $H \subset \partial M$ an arc and Λ_H be the set of exponents of $\beta^* P$ at H. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_H)$ be an edge. Then we have $t(\mathcal{L}) > 0$.*

An essential part of the proof will be the following basic lemma about collinearity of points scattering in the plane along rays.

Lemma 5.2.3 (Collision Lemma)**.** *Let $\Lambda \subset \mathbb{R}^2 \times \mathbb{R}^2$ be finite and assume that $p_1, p_2, p_3 \in \mathbb{R}^2$ are not collinear, for each $(p_j, v_j) \in \Lambda$, $j = 1, 2, 3$. Denote $\Lambda(t) := \{(p + tv, v) : (p, v) \in \Lambda\}$.*

Then $\tau(\Lambda) := \{t \in \mathbb{R} :$ three points in $\Lambda(t)$ are collinear$\}$ is finite.

Proof. Without loss of generality assume that $(p_1, v_1) = 0$. Then the points 0, p_2 and p_3 are collinear at time $t \in \mathbb{R}$, if and only if

$$\det(p_2 + tv_2, p_3 + tv_3) = 0\,.$$

Since $\det(p_2 + tv_2, p_3 + tv_3)$ is a second degree polynomial there are only two solutions. Repeating this process for each triple $(p_1, v_1), (p_2, v_2), (p_3, v_3)$ then yields

$$|\tau(\Lambda)| \leq 2|\Lambda|\binom{|\Lambda|-1}{2} = |\Lambda|(|\Lambda| - 1)(|\Lambda| - 2)\,.$$

□

Proof of Proposition 5.2.2. To show that the collision time of $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_H)$ is always positive it is sufficient to consider an ε-strip around $A(l_{\delta(\mathcal{L})})$ for arbitrary small $\varepsilon > 0$. Since the

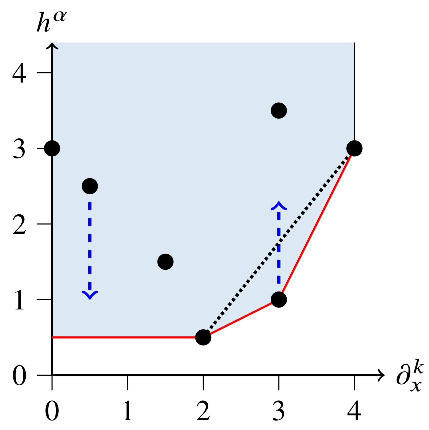

Figure 5.2: Set of exponents with induced action by the blow-up indicated by the dashed blue arrows. A collision is triggered when the second left point hits the lower boundary $\partial\mathcal{P}(\Lambda)$ or the second right point hits the dotted line.

transformation of $\lambda \in \Lambda_p$ is continuous with respect to t, pairs of exponents μ outside of that strip have a lower bound on the distance to $\operatorname{span}(\mathcal{L})$. Since $\mathcal{P}(\Lambda_H)$ is convex, the intersection of the ε-strip with Λ_H is finite. Thus, we can apply Lemma 5.2.3 and show that $t(\mathcal{L}) > 0$. □

Conveniently, the collision time also detects whether an operator is already $\mathcal{L}$-resolved as the following proposition shows.

Proposition 5.2.4. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^\Lambda(\mathbb{R})$. Let $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ with slope $\delta(\mathcal{L}) \geq 0$. Assume that P is $\delta(\mathcal{L})$-singular in $0 \in \mathbb{R}^2_+$.*

Then $t(\mathcal{L}) = \infty$ if and only if P is $\mathcal{L}$-resolved in 0.

Proof. For the 'if' direction, let $\mathcal{L} = \overline{\lambda\mu}$ for some $\lambda, \mu \in \Lambda$ and let $\mu_0, \ldots, \mu_k$ be all essential points on the left side of λ. Assume that $\mu_{k+1} \neq \lambda$, i.e. P is not $\mathcal{L}$-resolved in 0. Thus, $\gamma(\overline{\mu_k\mu_{k+1}}) < \gamma(\overline{\mu_k\lambda})$ and eventually $\mu_{k+1}(t) \in \operatorname{span}(\mathcal{L}(t))$ for some $t > 0$, which contradicts $t(\mathcal{L}) = \infty$.

For the other direction, let $\mathcal{L} = \overline{\lambda\mu}$ and assume that $t := t(\mathcal{L}) < \infty$. Thus, there is a set of exponents $\tau \in \Lambda$ such that $\tau(t) \in \operatorname{span}(\mathcal{L}(t))$. This can only be true in one of the following cases: Either $k_\tau \neq k_\lambda$ and $\gamma(\overline{\lambda\tau}) < \gamma(\mathcal{L})$, or $k_\tau = k_\lambda$ and $\omega(\tau) < \omega(\lambda)$, or the analogue for τ and μ. In any case it follows that either λ or μ cannot be essential. In particular, P is not $\mathcal{L}$-resolved in 0. □

Remark 5.2.5. The statement of Proposition 5.2.4 can be lifted directly to arcs on blown-up half spaces $\beta\colon M \to \mathbb{H}$.

The following proposition justifies the attempt to resolve singular operators P by the successive use of blow-ups at its singular points $p \in H$ in Subsection 5.2.2. Proposition 5.2.6 states that for short periods of time $0 < t \leq t(\mathcal{L})$ the transformed edge $\mathcal{L}(t) \subset \partial\mathcal{P}(\Lambda_{\mathrm{ff}})$ and $\beta^*_{t,\text{-}}\mathcal{L} \subset \partial\mathcal{P}(\Lambda_H)$ are images of projections of a three dimensional edge $\mathcal{L}_t \subset \partial\mathcal{P}(\Lambda_{p_t})$, where $p_t \in \beta^*_t H \cap \mathrm{ff}$. Thus, we can apply Proposition 5.1.16 and obtain a phase function Φ solving both eikonal equations corresponding to $\mathcal{L}(t)$ and $\beta^*_{t,\text{-}}\mathcal{L}$ for all $0 < t \leq t(\mathcal{L})$.

Proposition 5.2.6. *Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$. Let $\beta \colon M \to \mathbb{H}$ be a blow-up, $H \subset \partial M$ be an arc and Λ_H be the set of exponents of $\beta^* P$ at H. Let $\mathcal{L}_0 \coloneqq \overline{\lambda\mu} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ and $t_0 \coloneqq t(\mathcal{L}_0)$ be the collision time of $\mathcal{L}_0$ at $p \in H$. For each $t > 0$ let ff_t be the front face corresponding to $\beta_t \colon [M, p]_t \to M$ and p_t be the unique point in $\beta_t^* H \cap \mathit{ff}_t$.*

Then for all $0 < t \leq t_0$ there exists an edge $\mathcal{L}_t \subset \partial_{-}\mathcal{P}(\Lambda_{p_t})$, such that

$$\pi_{(k,\alpha_2)}(\mathcal{L}_t) = \mathcal{L}_0(t) \text{ and } \pi_{(k,\alpha_3)}(\mathcal{L}_t) = \mathcal{L}_0 \,.$$

Proof. This is a direct consequence of the continuity of L_+ with respect to $t \in (0, \infty)$. □

Since edges might split up or merge after blowing up the singular point at their collision times, we need to keep track of their succeeding edges. These can be multiple edges *succeeding* a single edge in general after blow-up.

Definition 5.2.7 (Successor)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$. Let $\beta \colon M \to \mathbb{H}$ be a blow-up, $H \subset \partial M$ an arc and Λ_H be the set of exponents of $\beta^* P$ at H. Let $\mathcal{L} \coloneqq \overline{\lambda\mu} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ and $t_0 \coloneqq t(\mathcal{L})$ be the collision time of $\mathcal{L}$ at $p \in H$. Let $0 < t \leq t_0$ and $\beta_t \colon [M, p]_t \to M$ be the corresponding blow-up. Denote $\mathrm{ff}_t \coloneqq \beta_t^{-1}(p)$.

Assume there is an edge $\mathcal{L}_0 \subset \partial_{-}\mathcal{P}(\Lambda_{\mathrm{ff}_t})$, such that $\mathcal{L}_0 \subset \mathcal{L}(t)$ or $\mathcal{L} \subset \mathcal{L}_0$. Let ζ and ζ_0 be solutions of $E_{H,\delta(\mathcal{L})}(\cdot, \zeta) = 0$ and $E_{\mathrm{ff}_t,\delta(\mathcal{L}_0)}(\cdot, \zeta_0) = 0$, respectively, such that they admit an extension of their phase functions to $[M, p]_t$ in the sense of Proposition 5.1.16.

Then we call the pair $(\mathcal{L}_0, \zeta_0)$ *successor* of $(\mathcal{L}, \zeta)$ at ff_t.

Remark 5.2.8. We want to emphasize that $t \in (0, t_0)$ can be chosen arbitrarily in Definition 5.2.7, since incomplete resolutions will only shift singular points to the next front face.

Remark 5.2.9. Another complication arises in the presence of horizontal edges during the resolution process. The associated homogeneous solutions at the corresponding arc can have different exponential behavior at both corners of the arc. This possibly requires multiple solutions of the relevant eikonal equation on the other side of the arc to match the behavior of the homogeneous solutions. However, these solutions of the eikonal equation will be added manually in the resolution algorithm, in Step 6, since these are additional data on the same adjacent arc whereas successors are pairs of edges and solutions on a *deeper level* of blow-ups.

5.2.2 Resolution Algorithm

Keeping track of the successors of an initial choice of an edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$ and a corresponding solution ζ of its induced eikonal equation $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ requires us to specify the regions where these solutions exist.

Definition 5.2.10 (Regular Boundary)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ and $\beta \colon M \to \mathbb{H}$ be a blow-up. Let $H \subset \partial M$ be an arc and $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ be a lower edge with slope $\delta > 0$.

Then we call

$$H_{\mathrm{sing}}(\mathcal{L}) \coloneqq \{p \in H : \; \beta^* P \text{ is } \delta\text{-singular at } p\}$$

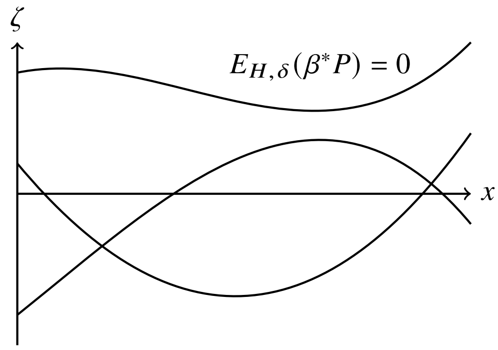

Figure 5.3: Zero set of an eikonal polynomial $E_{H,\delta}(\beta^* P)$ of $\beta^* P$ over an arc H in $T^*_{\mathbb{C}} H$. The multiple leaves may intersect at various points or diverge at the boundary of H.

singular boundary of H with respect to $\mathcal{L}$. Its complement $H_{\mathrm{reg}}(\mathcal{L}) \coloneqq H \setminus H_{\mathrm{sing}}(\mathcal{L})$ is called the *regular boundary*.

A new phenomenon we encounter are intersecting graphs of solutions ζ_j of

$$E_{H,\delta}(\beta^* P)(\cdot, \zeta_j) = 0\,.$$

The pointwise jumps in multiplicity of solutions of solutions of the eikonal equation in a point $p \in H$ lead to an increase in the vanishing order of the leading coefficient of the induced transport operator. However, these jumps can be resolved as we will show in Algorithm 1.

Definition 5.2.11 (Crossing Points and Simple Operators)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ and $\beta\colon M \to \mathbb{H}$ be a blow-up. Let $H \subset \partial M$ be an arc, $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda_H)$ be a lower edge with slope $\delta > 0$ and ζ be a solution of $E_{H,\delta}(\beta^* P)(\cdot, \zeta) = 0$ on $H_{\mathrm{reg}}(\mathcal{L})$. Let $p \in H$ and denote the multiplicity of $\zeta(p)$ of $E_{H,\delta}(P)(p, \zeta) = 0$ by $m_\zeta(\mathcal{L})(p)$. Define the global multiplicity of ζ along H by

$$m_\zeta(\mathcal{L}) \coloneqq \min\{m_\zeta(\mathcal{L})(p) : \ p \in H\}\,.$$

Then we call

$$H_{\mathrm{cross}}(\mathcal{L}, \zeta) \coloneqq \{p \in H^{\circ}_{\mathrm{reg}} : \ m_\zeta(\mathcal{L})(p) > m_\zeta(\mathcal{L})\}$$

crossing points in H with respect to $(\mathcal{L}, \zeta)$.

We call $\beta^* P$ *simple* along $H_{\mathrm{reg}}(\mathcal{L})$ with respect to ζ, if $H_{\mathrm{cross}}(\mathcal{L}, \zeta) = \emptyset$.

Remark 5.2.12. The term *crossing points* refers to the crossing of the graphs of solutions of $E_{H,\delta}(P)(p, \zeta) = 0$ and the resulting non-constant multiplicity (see Figure 5.3).

Example 5.2.13. Let $P \coloneqq x^2 h^2 \partial_x^2 + h\partial_x + 1$ on $\mathbb{R}_+$. Its associated Newton polygon is given by the single edge $\mathcal{L} = \overline{(0,0), (2,2)}$ and has a single singular point $0 \in \mathbb{R}_+$, i.e. $\mathbb{R}_{+,\mathrm{reg}}(\mathcal{L}) = (0, \infty)$. Its solutions of the eikonal equation $E_1(P)(\cdot, \zeta) = 0$ are given by

$$\zeta_\pm(x) = -\frac{1}{2x^2} \pm \sqrt{\frac{1}{4x^4} - \frac{1}{x^2}} = -\frac{1}{2x^2}\left(1 \mp \sqrt{1 - 4x^2}\right),$$

and coincide in $x = 1/2$ and thus have a crossing point. Thus, P is not simple along $\mathbb{R}_+$ since $m_{\zeta_\pm}(\mathcal{L}) = 1 < 2 = m_{\zeta_\pm}(\mathcal{L})(1/2)$.

Algorithm

We can now specify Algorithm 1 to resolve singular semi-classical operators using the notion of successors in Definition 5.2.7. The goal is to blow up $\mathbb{H}$ successively until $\beta^* P$ is resolved in all singular points according to Definition 5.1.10 and is simple along the interior of all arcs $H \subset \partial M$. We start on the half space $\mathbb{H}$ with a generalized semi-classical operator $P \in \mathrm{Diff}^\Lambda(\mathbb{R})$, an edge $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ and a solution ζ of $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ on an interval $I \subset \partial\mathbb{H}_{\mathrm{reg}}$. There are three different types of problems we need to address step-by-step in the algorithm to eliminate all singularities:

(i) The resolution of singular points at which P is non-resolved.

(ii) The elimination of crossing points at which other solutions of $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ intersect the initially chosen solution.

(iii) The division of the space at a singular point in the interior of a hypersurface at which P is already resolved.

In the first part *Resolution of Singular points* of Algorithm 1 we deal with singular points $p \in I$ at which P is unresolved regarding an edge $\mathcal{L}$. Introducing a quasihomogeneous blow-up $\beta_t \colon [\mathbb{H}, p]_t \to \mathbb{H}$ corresponding to its collision time $t = t(\mathcal{L})$ thus generates a successor $(\mathcal{L}_0, \zeta_0)$ at $\beta_t^{-1}(p)$ of $\mathcal{L}$ and ζ chosen initially. This successor $(\mathcal{L}_0, \zeta_0)$ of $(\mathcal{L}, \zeta)$ is unique. However, if $\mathcal{L}$ is horizontal, then the homogeneous solutions of the leading operator of $\beta_t^* P$ at H can have a different exponential asymptotic behavior at both corner of the front face. Thus, we need to consider additional pairs $(\mathcal{L}', \zeta')$ on the other side of the front face. This can be observed in Subsection 2.4.2 with the Airy function for the Schrödinger operator on the blown-up space.

By construction, all pairs $(\mathcal{L}, \zeta)$ on all arcs have to be taken into consideration to be able to construct phase functions on the blown-up space $[\mathbb{H}, p]_t$ matching the local solution of $\partial_x \varphi = \zeta$ at I. To account for this amount of simultaneous problems, we will introduce a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ capturing the resolution process step-by-step. Its vertices are 4-tupels $\mathfrak{y} := (Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ consisting of: the (partially) blown-up space $\beta_Y \colon Y \to \mathbb{H}$ at that step $\mathfrak{y}$ of the iteration, an arc H, an edge $\mathcal{L}_H \subset \partial\mathcal{P}(\Lambda_H)$ and a solution ζ_H of $E_{\delta(\mathcal{L}_H)}(\beta_Y^* P)(\cdot, \zeta_H) = 0$ on H_{reg}.

To minimize confusion, we will refer to edges of the (directed) graph as *arrows*. In particular, the tree with arrows connecting vertices can be interpreted as a blow-up diagram. Vertices $\mathfrak{x} := (X, H_0, \mathcal{L}, \zeta), \mathfrak{y} \in \mathcal{V}$ are connected by arrows, i.e. ordered pairs of vertices $(\mathfrak{y}, \mathfrak{x}) \in \mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, if there is a point $p \in H_0$ with finite collision time $t = t(\mathcal{L})$, such that the associated blow-up $\beta_t \colon [X, p]_t \to X$ and the successor $(\mathcal{L}_H, \zeta_H)$ of $(\mathcal{L}, \zeta)$ coincide with $\mathfrak{y}$. In particular, it holds that $Y = [X, p]$ and $H = \beta_t^{-1}(p)$. Thus, the graph $\mathcal{G}$ is a directed tree whose arrows correspond to partial blow-downs.

To order the resolution process, we sort the arcs from right to left. For any point $p \in H_0$ we can split it into a left side $H_{0,l}$ and right side $H_{0,r}$. Since there is no singular point in $H_{0,r}$, we can lift the right side with the corresponding vertex to the next level in the tree by adding the vertex $(Y, H_{0,r}, \mathcal{L}, \zeta)$ and the arrow connecting it with $\mathfrak{x}$. Putting these leaves to the end of their own branch will simplify the construction of solution spaces. These are spaces of exponential-polyhomogeneous functions in Definition 5.3.11 and thus refer to families of edges and solutions $(\mathcal{L}, \zeta)$ over each arc H.

If at some point a vertex $(Y, H, \mathcal{L}_H, \zeta_H)$ is added with $\delta(\mathcal{L}_H) = 0$, one might need to add additional vertices $(Y, \beta_t^* H_{0,l}, \mathcal{L}, \zeta_j)$ for multiple solutions ζ_j of $E_\delta(\mathcal{L})(P)(\cdot, \zeta) = 0$, $j = 1, \ldots, r$, $r \in \mathbb{N}$. This might be required to match the asymptotic expansion of the homogeneous solution of the transport operator of $\beta_t^* P$ at H. A prominent example for which this is necessary is the resolution of the Schrödinger equation with linear potential. Its local approximate solution at the front face, the Airy function, has real exponential behavior on one side, but oscillatory behavior on the other side. Thus, one needs to add another solution $\zeta_\pm$ of the eikonal equation on the oscillatory side of the front face.

In the second step *Resolution of Crossing Points*, we eliminate potential crossings in $(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$ of solutions ζ, ζ_i, $i = 1, \ldots, r$, $r \in \mathbb{N}$, of $E_{H,\delta}(\beta^* P)(\cdot, \zeta) = 0$ at H, with $\delta = \delta(\mathcal{L})$. If there is an r-fold intersection at a crossing point $p \in H_{\mathrm{reg}}$ of solutions ζ, ζ_i, $i = 1, \ldots, r$, we can write

$$E_\delta(\beta^* P)(x_H, \zeta_H) = \sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_H \\ \lambda = (k, \alpha)}} (a_\lambda(p) - \widetilde{a}_\lambda(x_H))\, \zeta_H^k \,,$$

where $\widetilde{a}_\lambda(x_H) = O(x_H - p)$. Since $a_\lambda(p) \neq 0$ for all $\lambda \in \partial\mathcal{L}$ by regularity, we have

$$\sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_H \\ \lambda = (k, \alpha)}} a_\lambda(p) \zeta_H^k = (\zeta_H - c)^{r+1} Q(\zeta_H) \,,$$

for some $c \in \mathbb{C}^*$ and $Q \in C^\infty(H)[\zeta_H]$ with $Q(\zeta(p)), Q(\zeta_j(p)) \neq 0$, for $j = 1, \ldots, r$. Blowing up p quasihomogeneously with parameter $t = \delta$ then yields a horizontal edge $\mathcal{L}_p$ as a single successor with corresponding symbol

$$E_{H_p,0}(x_{\mathrm{ff}}, \zeta_{\mathrm{ff}}) = (\zeta_{\mathrm{ff}} - c)^{r+1} Q(\zeta_{\mathrm{ff}})$$

along $\beta_\delta^{-1}(p)$. In particular, the leading operator at the front face H_p

$$\beta_\delta^* \beta^* P = h^{l_{H_p,0}} \sum_{\substack{\lambda \in \mathcal{L} \cap \Lambda_H \\ \lambda = (k, \alpha)}} a_\lambda(p) \partial_{x_{\mathrm{ff}}}^k + o(h^{l_{H_p,0}})$$

is an elliptic operator along H_p. These blow-ups add at most one vertex per crossing point to the resolution graph.

In the third and final step *Resolution of Rays* we resolve the space by trivially blowing up a submanifold of M transversally intersecting the boundary. This might be necessary in general due to the presence of singular points in the interior of arcs $H \subset \partial M$ where $\beta^* P$ is resolved. In general, the quasimodes' phases and amplitudes will have polyhomogeneous behavior at these submanifolds, which have to be blown-up for technical reasons. This blow-up will be accounted for by two additional vertices in the graph for both sides of the blown-up submanifold.

These considerations are formally written in Algorithm 1 whose output is the *resolution tree*.

Definition 5.2.14 (Resolution Tree)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$ with slope δ, let $I \subset \partial\mathbb{H}_{\mathrm{reg}}(\mathcal{L})$ and ζ be a solution of $E_\delta(P)(\cdot, \zeta) = 0$ on I.

The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ generated in Algorithm 1 with initial data $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ is called *resolution tree* of P and $(\mathcal{L}, \zeta)$ and I.

Based on the resolution tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ generated in Algorithm 1 we can construct the *resolution space*, including all blow-ups contained in $\mathcal{E}$. We say that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *finite* if $\mathcal{V}$ is finite.

Definition 5.2.15 (Resolution Space)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents and $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$. Let $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$, $I \subset \partial\mathbb{H}_{\mathrm{reg}}(\mathcal{L})$ and ζ be a solution of $E_\delta(P)(\cdot, \zeta) = 0$ on I. Let $\mathcal{G}$ be the resolution tree of Algorithm 1 for $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$. Assume that $\mathcal{G}$ is finite.

We call the space $\beta \colon M \longrightarrow \mathbb{H}$ generated by the blow-up graph $\mathcal{G}$ in the sense of Corollary 2.1.23 the *resolution space* of P and $(\mathcal{L}, \zeta)$.

Remark 5.2.16. Step 6 of the *Resolution of Singular Points* in Algorithm 1 will not add new leaves in most cases. However, if Step 5 of the algorithm adds solutions $\zeta \neq \zeta_H$ we need to complete the tree with edges and solutions matching $(\mathcal{L}_H, \zeta)$ successively on all arcs positioned left of the left face H_l corresponding to Step 6.

Remark 5.2.17. We will only present initial data $\mathfrak{a} = (\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ of Algorithm 1 for an operator P in form of an edge $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$ with *positive* slope and corresponding solution ζ of the eikonal equation $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$. While one can also initiate the algorithm with a horizontal edge and corresponding homogeneous solution of the transport equation, it does not require any changes in the algorithm but complicates the notation in the proof of Theorem 5.2.18 due to the different cases.

Algorithm 1: Construction of the Resolution Tree

Data: Semi-classical operator P and $\mathfrak{a} := (\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$, where $\mathcal{L} \subset \partial_{-}\mathcal{P}(\Lambda(P))$ and ζ solving $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ on $I \subset \mathbb{H}_{\text{reg}}$.

Result: Resolution tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Resolution of Singular Points:

while $\exists\, \mathfrak{v} = (X, H, \mathcal{L}_H, \zeta_H), \mathfrak{u} \in \mathcal{V}$ *with* $(\mathfrak{v}, \mathfrak{u}) \in \mathcal{E}$*, and a rightmost point* $p \in H^\circ$*, s.t.* $\beta_X^* P$ *is neither* $\delta(\mathcal{L}_H)$*-regular nor* $\mathcal{L}$*-resolved in* p **do**

1. Determine the collision time t_p of $\mathcal{L}_H$ in $\Lambda(\beta_X^* P)_H$.
2. Add $\mathfrak{w} := (Y, H_p, \mathcal{L}_{H_p}, \zeta_{H_p})$ to $\mathcal{V}$ and $(\mathfrak{w}, \mathfrak{v})$ to $\mathcal{E}$, where $Y := [X, p]_{t_p}$, $H_p := \beta_{t_p}^{-1}(p)$, for the successor $(\mathcal{L}_{H_p}, \zeta_{H_p})$ of $(\mathcal{L}_h, \zeta_H)$ in Λ_{H_p}.
3. Add $\mathfrak{y} := (Y, H_r, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ and $(\mathfrak{y}, \mathfrak{v}) \in \mathcal{E}$.
4. If $\delta(\mathcal{L}_{H_p}) > 0$, then add $\mathfrak{z} := (Y, H_l, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ and $(\mathfrak{z}, \mathfrak{v}) \in \mathcal{E}$.
5. If $\delta(\mathcal{L}_{H_p}) = 0$, let u be the solution of $(\beta_Y^* P)u = 0$ at H_p matching ζ_H at H_r. For all solutions ζ of $E_{H_l, \delta(\mathcal{L}_H)}(\beta_Y^* P)(\cdot, \zeta) = 0$ matching u at H_l add $\mathfrak{x} := (Y, H_l, \mathcal{L}_H, \zeta) \in \mathcal{V}$ and $(\mathfrak{x}, \mathfrak{v}) \in \mathcal{E}$.
6. If there is $(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$, an adjacent arc H' on the left side of H and a pair $(\mathcal{L}', \zeta')$, s.t. $(\mathcal{L}, \zeta)$ is a successor of $(\mathcal{L}', \zeta')$, then add $\mathfrak{y} = (Y, H', \mathcal{L}', \zeta') \in \mathcal{V}$ and $(\mathfrak{y}, \mathfrak{a}) \in \mathcal{E}$.

Resolution of Crossing Points:

while $\exists\, \mathfrak{v} = (X, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ *and* $p \in H^\circ$*, s.t.* $\beta_X^* P$ *is* $\delta(\mathcal{L}_H)$*-regular in* p *and* $m_{\zeta_H}(\mathcal{L}_H)(p) > m_{\zeta_H}(\mathcal{L}_H)$ **do**

Add $\mathfrak{w} := (Y, H_p, \mathcal{L}_{H_p}, \zeta_{H_p})$ to $\mathcal{V}$ and $(\mathfrak{w}, \mathfrak{v})$ to $\mathcal{E}$, where $Y := [X, p]_\delta$, $H_p := \beta_\delta^{-1}(p)$, with successor $(\mathcal{L}_{H_p}, \zeta_{H_p})$ of $(\mathcal{L}_H, \zeta_H)$ in Λ_{H_p}.

Resolution of Rays:

forall $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ *and for all* $p \in H^\circ$ **do**

if $\beta_Y^* P$ *is* $\delta(\mathcal{L}_H)$*-singular and resolved in* p **then**

Add $\mathfrak{w}_r := (Z, H_r, \mathcal{L}, \zeta)$ and $\mathfrak{w}_l := (Z, H_l, \mathcal{L}, \zeta)$ to $\mathcal{V}$ and $(\mathfrak{w}_l, \mathfrak{v})$, $(\mathfrak{w}_r, \mathfrak{v})$ to $\mathcal{E}$, where $Z := [Y, N_p]$ is the trivial blow-up along the ray $N_p := \{x_H - p = 0\} \subset Y$.

Before we prove that Algorithm 1 terminates in Theorem 5.2.18, we want to perform the complete analysis and algorithmic resolution to characteristic examples to demonstrate their application. For this we will revisit the standard example of *Schrödinger* operators in Subsection 5.2.3 since it will already include the inconvenient property of multiple successors. After that we will discuss the *Bessel* equation in Subsection 5.2.4 which is not given by a semi-classical operator by design but can be brought into the form easily. Although we can only construct quasimodes for this family of equations, we want to underline that this algorithm derives the resolution of the Bessel operator by a double blow-up of $x = \nu = \infty$ quickly. Simultaneously, it also predicts an Airy-type behavior at the center face of the second blow-up.

5.2.3 Schrödinger Operator

Let $I = \mathbb{R}$, $k \in \mathbb{N}$, let $V \in C^\infty(I)$ with $V(x) = x^k \cdot \widetilde{V}(x)$ and $\widetilde{V}(x) \neq 0$ for all $x \in \mathbb{R}$, and

$$P := -h^2\partial_x^2 + V\,,$$

be the associated *Schrödinger* operator. This is the general case of Example 4.1.10.

Resolution

Its associated set of exponents is given by $\Lambda(P) = \{(0,0),(2,2)\}$ with a single edge in the lower boundary $\mathcal{L} := \overline{(0,0),(2,2)} = \partial_{-}\mathcal{P}(\Lambda)$. Since the potential V is vanishing in $x = 0$, the operator is not δ-regular for its only slope $\delta = \delta(\mathcal{L}) = 1$ according to Definition 3.2.16. The weight $\omega((2,2)) = -2$ of μ_0 is smaller than then weight $\omega((0,0)) = k$ of $(0,0)$. Thus, following Definition 4.2.13, the only essential point μ_0 of $\Lambda(P)$ in $x = 0$ is given by $\mu_0 = (2,2)$. Since μ_0 is not the minimal point on the lower boundary of $\partial_{-}\mathcal{P}(\Lambda)$, P is not resolved in $x = 0$ with respect to $\mathcal{L}$.

In order to resolve P, the first collision time is given by $t(\mathcal{L}) = 2/(k+2)$, since both points $(0,0)$ and $(2,2)$ shift when applying a quasihomogeneous blow-up and have same powers of h if $2 - 2t = kt$. The associated blow-up is given by $\beta_{t(\mathcal{L})}\colon [\mathbb{H},0]_{t(\mathcal{L})} \to \mathbb{H}$, where $x_{\mathrm{ff}} := x/h^{t(\mathcal{L})}$ is the induced coordinate along ff. Denote $M := [\mathbb{H},0]_{t(\mathcal{L})}$. The leading term of $\beta^*_{t(\mathcal{L})}P$ at ff is given by

$$\beta^*_{t(\mathcal{L})}P = h^{2-4/(k+2)}\left(-\partial^2_{x_{\mathrm{ff}}} + x^k_{\mathrm{ff}}\cdot\widetilde{V}(0)\right) + O\left(h^{2-2/(k+2)}\right)\,.$$

It is an elliptic, ordinary differential operator of order two. In particular, $\beta^*_{t(\mathcal{L})}P$ has no singular points on ∂M, since $0 \in \mathbb{R}$ is the only singular point on the original space $\mathbb{H}$.

To check whether $\beta^*_{t(\mathcal{L})}P$ has any crossing points, i.e. jumps in multiplicities of solutions of the eikonal equation, we analyze

$$E_{\mathcal{L}}(P)(x,\zeta) = -\zeta^2 + x^k\widetilde{V}(x) = 0\,,$$

at $\beta^*_{t(\mathcal{L})}\{h = 0, x \neq 0\}$. Its solutions are given by $\zeta_\pm(x) := \pm x^{k/2}\sqrt{\widetilde{V}(x)}$, whose graphs intersect for $x \neq 0$ if and only if $\widetilde{V}(x) = 0$. Since this is not possible, there are no crossing points. Note that $V(x_0) = 0$ for $x_0 \neq 0$ would also imply that x_0 is a singular point. Thus, the resolution is almost complete.

As initial data for the resolution tree we choose $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-)$, where ζ_- is non-positive and real-valued on $\mathbb{R}_+$. Following Step 2 in Algorithm 1, we add $(M, \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_-)$ and attach

$$\begin{tikzcd}
 & (M, \partial\mathbb{H}_r, \mathcal{L}, \zeta_{\partial\mathbb{H}_r,-}) \arrow[ddl, dashed] \\
 & (M, \partial\mathbb{H}_l, \mathcal{L}, \zeta_{\partial\mathbb{H}_l,\pm}) \arrow[dl] \\
(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-) & (M, \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},-}) \arrow[l]
\end{tikzcd}$$

Figure 5.4: The resolution tree of $P = -h^2\partial_x^2 + V$ with respect to $\mathcal{L} = \overline{(0,0),(2,0)}$ and $\zeta_+(x) = x^{k/2}\widetilde{V}(x)^{1/2}$. The dashed arrow corresponds to the lift of $\partial\mathbb{H}$ to the blown-up space $[\mathbb{H}, 0]_{t(\mathcal{L})}$. The middle arrow corresponds to two vertices. The addition of ζ_+ to the tree at the lift of $\partial\mathbb{H}_l$ to match the Airy function's asymptotic behavior on the front face.

it to $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-)$, where $\zeta_{\mathrm{ff},-}$ is the unique solution of $-\zeta_{\mathrm{ff}}^2 + x_{\mathrm{ff}}^k\widetilde{V}(0) = 0$ matching ζ_- at the right corner of ff. Following Step 3, we lift the initial data to the blown-up space and add $(M, \partial\mathbb{H}_r, \mathcal{L}, \zeta_{\mathbb{H}_r,-})$ on the right side of the front face, attached to $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-)$. Since the Airy function, corresponding to $\zeta_{\mathrm{ff},-}$, has oscillatory behavior at the left corner of ff, we need to add vertices for both lifted solutions $\zeta_{\mathbb{H}_l,\pm}$, according to Step 5. Thus, we add two vertices $(M, \partial\mathbb{H}_l, \mathcal{L}, \zeta_\pm)$, attached to $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-)$. The vertex set is given by

$$\mathcal{V} = \{(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-), (M, \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},-}), (M, \partial\mathbb{H}_l, \mathcal{L}, \zeta_\pm), (M, \partial\mathbb{H}_r, \mathcal{L}, \zeta_-)\}$$

with edges from all vertices containing M pointing towards $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta_-)$ (see Figure 5.4).

Eikonal Equations

To construct quasimodes, we first have to determine the associated eikonal equations and solve them accordingly. Their associated varieties are given by

$$\Gamma_{\partial\mathbb{H}_r} = \{(x, \zeta) : \ \zeta^2 - V(x) = 0, \ x > 0\},$$

on $\beta^* I$, by $\Gamma_{\partial\mathbb{H}_l} = \{(x, \zeta) : \ \zeta^2 - V(x) = 0, \ x < 0\}$ and by $\Gamma = \emptyset$ on ff. The solutions of the eikonal equations are given by

$$\varphi_\pm(x) := \pm \int_{x_0}^{x} t^{k/2}\sqrt{\widetilde{V}(t)}dt,$$

for either $x, x_0 > 0$ or $x, x_0 < 0$ depending on the choice of face $\partial\mathbb{H}_l, \partial\mathbb{H}_r$.

Transport Equations

Depending on the choice of phases $\varphi_\pm$, the transport operators at $\partial\mathbb{H}_l, \partial\mathbb{H}_r$ are given by the b-operator

$$T_{\partial\mathbb{H}_{l/r},\delta,\pm} = \pm\left(-2x^{k/2}\sqrt{\widetilde{V}(x)}\partial_x - \left(x^{k/2}\sqrt{\widetilde{V}(x)}\right)'\right).$$

The transport operator on the front face is given by the leading part of $\beta^* P$ at ff

$$T_{\mathrm{ff}} = -\partial_{x_{\mathrm{ff}}}^2 + x_{\mathrm{ff}}^k \cdot \widetilde{V}(0) \,.$$

For $k = 1$ and $\widetilde{V}(0)$ this coincides with the Airy operator in Example 3.1.3. Depending on the choice of a homogeneous solution u_0 at ff, we can determine the phases $\varphi_\pm$ required to match at both corners of ff. Iteratively solving the transport equations as in Lemmas 5.3.20 and 5.3.21 then yields quasimodes $u \in \mathcal{E}\mathcal{A}^{(\mathcal{I})}(M; \Gamma)$ for $\beta_{t(\mathcal{L})}^* P$.

5.2.4 Bessel Equation

Let $I := \mathbb{R}_+$ and $\nu \in \mathbb{R}_{>0}$. The *Bessel* operator on $\mathbb{R}_+$ is given by

$$\widetilde{P} := x^2 \partial_x^2 + x\partial_x + (x^2 - \nu^2) \,.$$

It is a b-operator that does not fit into the class of semi-classical operators immediately. However, we are able to analyze the exponential-polyhomogenous behavior of its quasimodes as $x \to +\infty$, $\nu \to \infty$ by changing coordinates to $r := x^{-1}$ and $h := \nu^{-1}$. This way we obtain an associated semi-classical operator on $\mathbb{R}_+^2$, where

$$P := r^2 h^2 \widetilde{P} = h^2 r^4 \partial_r^2 + h^2 r^3 \partial_r + h^2 - r^2 \,,$$

which we can analyze. Note that multiplying $\widetilde{P}$ with monomial factors does not impact the behavior of the quasimodes and can be used to normalize semi-classical operators.

With the resolution of the operator P we derive the same resolution space as in [She22]. The asymptotic behavior of the asymptotic solutions at the boundary faces coincide with the results in that work. However, this work describes the asymptotic expansions of solutions of the Bessel equations, while we are only able to compute asymptotic solutions.

Resolution

Its associated set of exponents

$$\Lambda(P) = \{(0,0), (0,2), (1,2), (2,2)\} \,,$$

has only a single edge $\mathcal{L} := \overline{(0,0), (2,2)}$ in the lower boundary of its Newton polygon $\partial_- \mathcal{P}(\Lambda)$. Their corresponding coefficient functions $a_{(0,0)}(r) = r^2$ and $a_{(2,2)}(r) := r^4$ are both vanishing in $r = 0$ and non-vanishing everywhere else. Thus, P is not $\delta(\mathcal{L})$-regular on $\{h = 0\}$ with a single non-regular point in $r = 0$ according to Definition 3.2.16. The weights of all points in $r = 0$ are given by

$$\omega((0,0)) = 2 \,,\ \omega((0,2)) = 0 \,,\ \omega((1,2)) = 2 \text{ and } \omega((2,2)) = 2 \,.$$

Thus, the essential points are given by $\mu_0 := (0,2)$ and $\mu_1 := (2,2)$, since the increase of weight from μ_0 to $(2,2)$ is minimal among all points $\lambda \in \Lambda(P)$, $\lambda \neq (0,2)$. That means that there is a minimal point in $r = 0$ that is not on the lower boundary $\partial_- \mathcal{P}(\Lambda)$, implying that P is not resolved on $\{h = 0\}$ in the sense of Definition 4.2.18.

In order to resolve the singular point $r = 0$ in $\{h = 0\}$, we have to compute the collision time of $\mathcal{L}$. Since all points have weight $\omega(\lambda) = 2$ with the exception of $\omega((0,2)) = 0$, the only collision will be due to $(0,0)$ and $(0,2)$. The collision time is given by $t(\mathcal{L}) = 1$. The homogeneous blow-up $\beta_1 : [\mathbb{R}^2_+, 0] \to \mathbb{R}^2_+$ of $0 \in \mathbb{R}^2_+$ associated to it results in an operator

$$\beta_1^* P = h^4 t^4 \partial_t^2 + h^4 t^3 \partial_t + h^2 (1 - t^2)\,,$$

at the front face $\beta_1^{-1}(0)$ in its induced coordinates h and $t := r/h$. The associated set of exponents

$$\Lambda_{\mathrm{ff}}(\beta_1^* P) = \{(0,2), (1,4), (2,4)\}$$

remains to have a single edge $\mathcal{L}_{\mathrm{ff}}$ in the lower boundary of its Newton polygon $\partial_- \mathcal{P}(\Lambda_{\mathrm{ff}})$. What has changed is the localization and relevance of the non-regular points $p \in \partial\mathrm{ff}$ in comparison to $\{h = 0\} \subset \mathbb{R}^2_+$. The relevant coefficient functions $b_{(0,2)}(t) = t^4$ and $b_{(2,4)}(t) = (1-t)(1+t)$ corresponding to $\mathcal{L}_{\mathrm{ff}}$ vanish at $t = 0$ and $t = \pm 1$, respectively. In particular, for $t = 0$ we have that the corresponding weights of $\lambda \in \Lambda_{\mathrm{ff}}$ are

$$\omega_0((0,2)) = 0\,,\ \omega_0((1,4)) = 2 \text{ and } \omega_0((2,4)) = 2\,.$$

Thus, the essential points are given by $(0,2)$ and $(2,4)$, which span the lower boundary $\partial_- \mathcal{P}(\Lambda_{\mathrm{ff}})$, implying that $\beta_1^* P$ is resolved in $t = 0$.

For $t = 1$, on the other hand, the weights are given by

$$\omega_1((0,2)) = 1\,,\ \omega_1((1,4)) = 0 \text{ and } \omega_1((2,4)) = 0\,.$$

Thus, there is only one essential point in $t = 1$ given by $(2,4)$, showing that $\beta_1^* P$ is still singular in $t = 1$. Rewriting the pullback of the semi-classical operator with the coordinate shift $\bar{t} := t - 1$, we obtain

$$\beta_1^* P = h^4 \left(1 + 4\bar{t} + 6\bar{t}^2 + 4\bar{t}^3 + \bar{t}^4\right) \partial_{\bar{t}}^2 + h^4 \left(1 + 3\bar{t} + 3\bar{t}^2 + \bar{t}^3\right) \partial_{\bar{t}} + h^2 \bar{t}(2 + \bar{t})\,.$$

The collision time of $\mathcal{L}_{\mathrm{ff}}$ is given by $t(\mathcal{L}_{\mathrm{ff}}) = 2/3$, yielding a blow-up $\beta_{2/3} \colon M \to [\mathbb{R}^2_+, 0]$, where

$$M := [[\mathbb{R}^2_+, 0], \{h = \bar{t} = 0\}]\,.$$

We will refer to the newly generated front face $\mathrm{cf} := \beta_{2/3}^{-1}(\{\bar{t} = h = 0\})$ as *center face*, to $\mathrm{lf} := \beta_{2/3}^*(\mathrm{ff} \setminus \{\bar{t} \geq 0\})$ as *left face* and to $\mathrm{rf} := \beta_{2/3}^*(\mathrm{ff} \setminus \{\bar{t} \leq 0\})$ as *right face*. Pulling back

$\beta_1^* P$ to cf results in

$$\beta_{2/3}^*(\beta_1^* P) = h^{2+2/3}\left(\partial_s^2 + 2s\right) + O\left(h^{3+1/3}\right), \tag{5.4}$$

where h and $s := \bar{t}/h^{2/3}$ are the induced coordinates at the center face. Denote $\beta := \beta_1 \circ \beta_{2/3}$. The leading operator $\partial_s^2 + 2s$ of $\beta^* P$ is essentially the Airy operator. In particular, it is elliptic on the interior of cf. Thus, the chain of blow-ups

$$\beta\colon\ [[\mathbb{R}_+^2, 0], \{h = \bar{t} = 0\}] \to [\mathbb{R}_+^2, 0] \to \mathbb{R}_+^2$$

resolves all singular points of P in the relative interior of ∂M. We still need to check if $\beta^* P$ has crossing points on ∂M. This can only be done by computing the solutions of the eikonal equations associated to $\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda)$ and $\mathcal{L}_{\mathrm{ff}} \subset \partial_- \mathcal{P}(\Lambda_{\mathrm{ff}})$ with slope δ and δ_{ff}, respectively. The eikonal polynomials are given by

$$E_\delta(P)(r, \zeta) = r^4 \zeta^2 - r^2,$$

on $\beta^* \mathbb{R}_{>0}$ and

$$E_{\delta_{\mathrm{ff}}}(t, \zeta) = t^4 \zeta^2 + (1 - t^2),$$

on $\mathrm{lf} \cup \mathrm{rf}$, where we omitted $\beta^* P$ in the argument of E. In the first case, the solutions are given by

$$\zeta_\pm(r) := \pm 1/r,$$

for $r > 0$, and in the latter case they are given by

$$\zeta_{\mathrm{ff},\pm}(t) := \pm\sqrt{1 - t^2}/t^2,$$

for $t > 0$, $t \neq 1$. Neither ζ_+ nor $\zeta_{\mathrm{ff},+}$ have any zeros on their respective domains. Thus, both ζ_+, ζ_- and $\zeta_{\mathrm{ff},+}, \zeta_{\mathrm{ff},-}$ do not intersect themselves pairwise, resulting in a constant multiplicity of roots of $E_{\delta(\mathcal{L})}(r, \zeta)$ and $E_{\delta(\mathcal{L}_{\mathrm{ff}})}(t, \zeta)$ on their respective domains on M. Thus, P is completely resolved by the chain of blow-ups.

To summarize this resolution, we construct its associated resolution tree (see Figure 5.5). Its root is given by the vertex $(\mathbb{R}_+^2, \{h = 0\}, \mathcal{L}, \zeta_+)$. Pointing towards the root, there are two vertices. The first vertex $([\mathbb{R}_+^2, 0], \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$ is due to the first blow-up of $0 \in \mathbb{R}_+^2$, corresponding to Step 2 in Algorithm 1. The second vertex $([\mathbb{R}_+^2, 0], \{h = 0\}_r, \mathcal{L}, \zeta_+)$ is the lift of the root according to Step 3. Note that there is no left side of $\{h = 0\}$ in $[\mathbb{R}_+^2, 0]$ after blowing up $(0, 0) \in \mathbb{R}_+^2$.

Following this vertex there is a second level of depth given by $(M, \mathrm{cf}, \mathcal{L}_{\mathrm{cf}}, \zeta_{\mathrm{cf},+})$, where $(\mathcal{L}_{\mathrm{cf}}, \zeta_{\mathrm{cf},+})$ is the successor of $(\mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$ according to Step 2. Note that $\zeta_{\mathrm{cf},+}$ is a solution of $\sigma(\partial_s^2 + 2s)(s, \zeta) = 0$. Its associated homogeneous solution of the equation $(\partial_s^2 + 2s)u = 0$

$$\begin{tikzcd}
 & & (M, \mathrm{lf}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},\pm}) \arrow[dl] \\
(\mathbb{R}^2_+, \{h=0\}, \mathcal{L}, \zeta_+) & ([\mathbb{R}^2_+, 0], \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+}) \arrow[l] & (M, \mathrm{cf}, \mathcal{L}_{\mathrm{cf}}, \zeta_{\mathrm{cf},+}) \arrow[l] \\
 & ([\mathbb{R}^2_+, 0], \{h=0\}_r, \mathcal{L}, \zeta_+) \arrow[ul, dashed] & (M, \mathrm{rf}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+}) \arrow[ul, dashed]
\end{tikzcd}$$

Figure 5.5: The resolution tree of semi-classical version of the Bessel operator $P = h^2 r^4 \partial_r^2 + h^2 r^3 \partial_r + h^2 - r^2$. The dashed arrows correspond to lifted vertices according to Step 3 in Algorithm 1. The two vertices in the top right with signs $\pm$ are required to match the asymptotic behavior of the Airy function(s) at the center face cf, corresponding to $(M, \mathrm{cf}, \mathcal{L}_{\mathrm{cf}}, \zeta_{\mathrm{cf},+})$.

has two successors $(\mathcal{L}, \zeta_{\mathrm{ff},\pm})$ on $\mathrm{lf} = \mathrm{ff}_l$. Thus, we add two vertices $(M, \mathrm{lf}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},\pm})$ pointing to $([\mathbb{R}^2_+, 0], \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$, according to Step 5. Finally, we lift $([\mathbb{R}^2_+, 0], \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$ to $(M, \mathrm{rf}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$, which points to $([\mathbb{R}^2_+, 0], \mathrm{ff}, \mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$ according to Step 3, where $\mathrm{rf}{=}\mathrm{ff}_r$.

Note that due to the simplicity of P and the symmetry of the solutions of the eikonal equations related by a change of sign, all resolution trees starting on $\mathbb{R}^2_+$ lead to the same chain of resolving blow-ups $\beta\colon M \to \mathbb{R}^2_+$.

Eikonal Equations

It is worth noting that

$$\zeta_{\mathrm{ff},\pm}(t) = \pm\sqrt{1-t^2}/t^2 \sim \pm 1/t$$

as $t \to +\infty$. Equivalently, the corresponding solutions of the b-symbol ${}^{\mathrm{b}}E_{\mathcal{L}_{\mathrm{ff}}}(t, {}^{\mathrm{b}}\zeta) = 0$, given by ${}^{\mathrm{b}}\zeta_{\mathrm{ff},\pm}(t) = \pm\sqrt{1-t^2}/t$ are bounded and non-vanishing as $t \to +\infty$. Thus, there are non-zero log-coefficients in the phase functions at $\mathrm{rf} \cap \beta^*\{h=0\}$ for any quasimode generated in the construction process. One can immediately see the logarithmic behavior of the phases φ solving

$$\partial_r \varphi_\pm = \pm\zeta_\pm$$

as $r \to 0$ along $\{h=0\}$ due to the behavior of $\zeta_\pm(r) = \pm 1/r$. Yet the presence of logarithms did not show the behavior on the more refined, blown-up space $\beta\colon M \to \mathbb{R}^2_+$.

For simplicity, we start the construction of our quasimode at the center face, where we choose $u_{\mathrm{cf},0}(s) := \mathrm{Ai}(\sqrt{1/2}s)$, accounting for the additional 2 in front of the monomial s in $\partial_s^2 + 2s$. Due to the oscillatory behavior of $\mathrm{Ai}(s)$ as $s \to -\infty$, i.e. $p \to \mathrm{cf} \cap \mathrm{lf}$ along cf, there are two successors $(\mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},+})$ and $(\mathcal{L}_{\mathrm{ff}}, \zeta_{\mathrm{ff},-})$ required to match Ai at lf. Their respective solutions are given by the phases

$$\varphi_{\mathrm{lf},\pm}(t) := \pm \int_{t_0}^{t} \frac{\sqrt{1-\tau^2}}{\tau^2} d\tau ,$$

for some $t_0 \in (0,1)$, solving $\partial_t \varphi_{\mathrm{lf},\pm} = \zeta_{\mathrm{ff},\pm}$. On the right face rf we can define the single required phase $\varphi_{\mathrm{rf},+}$ by the same expression with the difference that $t, t_0 > 1$. Due to $t, t_0 > 1$, we have $1 - t^2 > 0$ and thus $\varphi_{\mathrm{rf},+}(t) > 0$ for all $t > 1$, whereas $\varphi_{\mathrm{lf},\pm}(t) \in i\mathbb{R}$ for each $t \in (0,1)$. On rr $:= \beta^*\{h = 0\}$ the solution of the eikonal equation solving $\partial_r \varphi = 1/r$ and matching $\varphi_{\mathrm{rf},+}$ is given by

$$\varphi_{\mathrm{rr}}(r) := \log(r) .$$

Transport Equations

We start on the very right face rr $= \beta^*\{h = 0\}$. Recall from the beginning of this example that $P = h^2 r^4 \partial_r^2 + h^2 r^3 \partial_r + h^2 - r^2$. Conjugating $\beta^* P$ with $\exp(\varphi_{\mathrm{rr}}/h)$ at rr yields

$$e^{-\varphi_{\mathrm{rr}}/h} (\beta^* P) e^{\varphi_{\mathrm{rr}}/h} = h^1 \cdot \left(2r^3 \partial_r - r^2\right) \; + \; h^2 \cdot \left(r^4 \partial_r^2 + r^3 \partial_r + 1\right) .$$

Thus, the transport operator is given by $T_{\mathrm{rr}} := r^2((1 + 2r)\partial_r + 1)$ and the only remainder operator is $R_{\mathrm{rr}} := r^4 \partial_r^2 + r^3 \partial_r + 1$. Repeating the same process simultaneously at lf $\cup$ rf yields

$$e^{-\varphi_\pm/h} (\beta^* P) e^{\varphi_\pm/h} = h^3 \cdot t\sqrt{1-t^2}\,(2t\partial_t - 1) \; + \; h^4 \cdot \left(t^4 \partial_t^2 + t^3 \partial_t\right) ,$$

with transport operators $T_\pm := t\sqrt{1-t^2}(2t\partial_t - 1)$ and remainder operator $R_\pm := t^3(t\partial_t + 1)\partial_t$. We have already displayed the transport operator on the central face $T_{\mathrm{cf}} := \partial_s^2 + 2s$ in (5.4). Due to the form of

$$\beta_1^* P = h^4 \left(1 + 4\bar{t} + 6\bar{t}^2 + 4\bar{t}^3 + \bar{t}^4\right) \partial_{\bar{t}}^2 + h^4 \left(1 + 3\bar{t} + 3\bar{t}^2 + \bar{t}^3\right) \partial_{\bar{t}} + h^2 \bar{t}(2 + \bar{t})$$

there is no reasonable way to display all remainder operators associated to $\beta^* P$ on cf. However, their corresponding powers of h increase in increments of $2/3$, starting with $2 + 2/3$ for T_{cf}. Since the increments are all of size $2/3$, the index set can be chosen such that $\mathcal{I}(\mathcal{G})(\mathrm{cf}) := 2/3\mathbb{N}$. Due to the increments of size 1 at any other face, all other index sets coincide and are given by $\mathbb{N}$.

5.2.5 Finiteness of Resolutions

The goal of this subsection is to prove the main result of Section 5.2, Theorem 5.2.18, showing that all resolution trees are finite. The proof is mostly combinatorial due to the language of graphs used in Algorithm 1. Since we are only considering smooth coefficients a_λ in the class of generalized semi-classical operators, all coefficients in the asymptotic expansion of $\beta^* P$ at the front faces will be polynomial after any blow-up. The branching of $\mathcal{G}$ is a result of the presence of multiple singular points, which is stated implicitly in the while condition of *Resolution of Singular Points* in Algorithm 1. It occurs when an interior point collides with a corner point on the lower boundary under blow-up, since their coefficient's leading terms

$$\beta_t^* \left(h^\alpha x^\omega + h^{\alpha+\varepsilon} x^{\omega-n}\right) = h^{\alpha+t\omega} \left(x_{\mathrm{ff}}^\omega + x_{\mathrm{ff}}^{\omega-n}\right)$$

have the same homogeneity on the front face. The corresponding new leading term $x_{\mathrm{ff}}^\omega + x_{\mathrm{ff}}^{\omega-n}$ has zeros outside of $\{x_{\mathrm{ff}} = 0\}$.

The zero set of the leading coefficient $A_m = A_m(x, h)$ will be of special interest and needs to be considered separately in the resolution of P as in Example 4.1.13. This is covered in the resolutions of zeros in the beginning of the proof of Theorem 5.2.18 below. In a special case, it results in the trivial blow-up of $\beta^*\{x = 0\}$ for the resolution space M in the *Resolution of Rays* of Algorithm 1. The chain of successive zeros is of special interest. If $f \in C^\infty(\mathbb{R})$ vanishes to l-th order in 0, then for every blow-up $\beta_t \colon [\mathbb{H}, 0]_t \to \mathbb{H}$ of 0 we have

$$(\beta_t^* f)(x_{\mathrm{ff}}, h) = h^{tl} x_{\mathrm{ff}}^l + o\left(h^{tl}\right) ,$$

i.e. the leading part of $\beta_t^* f$ vanishes in $x_{\mathrm{ff}} = 0$ on ff to l-th order as well.

Theorem 5.2.18. *Let $P \in \mathit{Diff}^\Lambda(\mathbb{R})$ be a semi-classical operator, $P = \sum_{k=0}^m A_k(x, h)\partial_x^k$, such that the leading coefficient A_m analytic in x. Let $\mathcal{L} \subset \partial_{\text{-}}\mathcal{P}(\Lambda(P))$, let $I \subset \partial\mathbb{H}_{reg}$ and ζ be a solution of $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ on I.*

Then the resolution tree $\mathcal{G}$ of $(\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ is finite.

In other words, Theorem 5.2.18 says that Algorithm 1 creates a resolution space $\beta \colon M \to \mathbb{H}$ for P. By construction, $\beta^* P$ is resolved and simple on ∂M.

Proof. We first prove a special case. Assume that the leading coefficient is $A_m(x, h) = x^{l_m} h^{\alpha_m}$ and thus

$$\{A_m(x, h) = 0\} \cap \{h > 0\} = \{x = 0\} \cap \{h > 0\} .$$

Under this assumption, we will prove the following for statements:

(i) The branch $\mathcal{B}_0 \subset \mathcal{G}$ resolving iterated zeros along each newly generated front face is finite.

(ii) For each path $\mathcal{B} \subset \mathcal{G}$ there are only finitely many collisions involving splitting or merging of edges.

(iii) There are at most finitely many paths in $\mathcal{G}$.

(iv) Each path $\mathcal{B} \neq \mathcal{B}_0$ is finite.

(i) We start by resolving the sequence of zeros along the arcs. These are given by consecutive points $0_{\mathrm{ff}} \in \beta^*\{x = 0\} \cap \mathrm{ff}$. By Proposition 5.1.11 there is a $T \geq 0$ such that $\beta_t^* P$ is resolved at 0 on the recent arc for all $t \geq T$. We show that after finitely many collisions at the sequence of zeros, the sum of collision times $\sum t_p$ exceeds T. Let μ_l be the essential points of Λ, for $l \in \{1, \dots, n\}$ and denote

$$\alpha_{\max} \coloneqq \max_{\mu_l \in \Lambda} \{\alpha_l : \ \mu_l = (k_l, \alpha_l)\} \ .$$

The value $\alpha_{\max}$ is finite and thus there are at most finitely many points in $\Lambda \cap \{\alpha \leq \alpha_{\max}\}$. Since μ_l is essential we have $\omega_0(\mu_l) \leq \omega_0(\lambda)$ for all $\lambda = (k_l, \alpha) \in \Lambda$. Hence there are no collisions regarding any points outside of $\Lambda \cap \{\alpha \leq \alpha_{\max}\}$ and therefore there are at most finitely many collisions in $[0, T]$. Thus, the sequence of blow-ups regularizing the chain of zeros is finite.

(ii) Assume for some vertex $v_0 \coloneqq (X, H, \mathcal{L}_H, \zeta_H)$ in the path $\mathcal{B}_0$ resolving 0 that there is a point $p \in H_{\text{sing}}(\mathcal{L}_H, \zeta_H)$. We distinguish between two types of collisions with $\mathcal{L}_H$:

1. There exists $\lambda \in \Lambda_H \setminus \mathcal{L}_H\,, \ \ t \in \mathbb{Q}_+ \colon \ \lambda(t) \in \partial\mathcal{L}_H(t)$.

2. There exists $\mu \in \Lambda_H \setminus \mathcal{L}_H\,, \ \ t \in \mathbb{Q}_+ \colon \ \mu(t) \in \operatorname{span}(\mathcal{L}_H(t)) \setminus \partial\mathcal{L}_H(t)$.

We will call them collisions of *first type* and of *second type*, respectively. Let $\mathcal{B} \neq \mathcal{B}_0$ be a path starting in v_0 at $p \neq 0$. For convenience we re-label the ordered vertices $\mathfrak{v}_j \in \mathcal{B}$ to $\mathfrak{v}_j \coloneqq (Y_j, H_j, \mathcal{L}_j, \zeta_j)$ with $\beta_j \colon Y_j \longrightarrow Y_{j-1}$ and $p_j \in H_j$, s.t. $\beta_{j+1}^{-1}(p_j) = H_{j-1}$. In particular $(Y_0, H_0, \mathcal{L}_0, \zeta_0) = (X, H, \mathcal{L}_H, \zeta_H)$. We show that there are at most finitely many collisions of the second type along $\mathcal{B}$.

The idea is to use that every collision of second type requires the existence of a point in the interior of $\mathcal{P}(\Lambda)$ with a weight that is lower than of the points spanning the edge $\mathcal{L}$. Since weights of $\lambda = (k, \alpha)$ are bounded from below by $-k$, there can be only finitely many collisions of this type.

Recall that $m = \operatorname{ord}(P)$. Let $\Omega_j \coloneqq (\omega_{j,1}, \dots, \omega_{j,m}) \in \mathbb{Z}^m$ be the list of lowest weights at each column of $\partial\mathcal{P}(\Lambda_{H_j})$ up to the j-th vertex v_j defined in the following way:

(a) $\omega_{0,i}$ is the weight of the point in $\{i\} \times \mathbb{R} \cap \partial\mathcal{P}(\Lambda_{H_0}) \cap \Lambda_{H_0}$ if this set is non-empty. Otherwise $\omega_{0,i}$ is the maximum of the weights of $\partial\mathcal{L}_0$, where $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda_{H_0})$ is the unique edge with $\mathcal{L}_0 \cap \{i\} \times \mathbb{R} \neq \emptyset$.

(b) For collisions of second type with respect to λ and $\mathcal{L}_j$ on the j-th leaf of the branch, let $\omega_{j,i} \coloneqq \omega_{p_{j+1}}(\lambda)$, if $\lambda = (i, \alpha_i)$. Otherwise define $\omega_{j,i} \coloneqq \omega_{j-1,i}$.

(c) If the j-th collision is of first type, we let $\omega_{j+1,i} \coloneqq \omega_{p_{j+1}}(\lambda)$.

Furthermore Thus, the sequence $(\Omega_j)_j$ is ordered, decreasing in $j \in \mathbb{N}$ and by construction bounded from below by $(0, -1, \dots, -m)$. Since collisions of second type of $\mathcal{L}_j \coloneqq \overline{\lambda_{j,1}\lambda_{j,2}}$ with λ require that $\omega_{p_j}(\lambda) < \omega_q(\lambda_{j,i})$, $i = 1, 2$, the sequence $(\Omega_j)_j$ is strictly decreasing following these collisions. Thus, after finitely many iterations all collisions are of first type.

(iii) Recall that there are multiple paths at a vertex $\mathfrak{v} = (Y, H, \mathcal{L}, \zeta)$ in $\mathcal{G}$ if there are multiple singular points corresponding to $\mathcal{L} \coloneqq \overline{\lambda_1\lambda_2}$. These originate in collisions of first type. Without loss of generality assume that λ collides with λ_1. In particular, this branching can only occur if $a_{\lambda(t_p)} := \beta_{t_p}^*(a_\lambda + a_{\lambda_1})$ has multiple roots along the arc $\mathrm{ff}_p := \beta_{t_p}^{-1}(p)$. Since $\deg(a_{\lambda(t_p)}) = l$, with $l := \mathrm{ord}_p(a_\lambda)$, there are at most l different roots along ff_p. More precisely, we have

$$\sum_{q\in \mathrm{ff}_p} \mathrm{ord}_q(a_{\lambda(t_p)}) \le \deg(a_{\lambda(t_p)}) \le \mathrm{ord}_p(a_\lambda)\,.$$

Branching implies that $\mathrm{ord}_q(a_{\lambda(t_p)}) < \mathrm{ord}_p(a_\lambda)$. Since $\mathrm{ord}_q(a_\lambda) \ge 0$ for all $\lambda \in \Lambda_H$, this can happen only finitely often.

(iv) Return to $\mathfrak{v}_1 = (Y_1, H_1, \mathcal{L}_1, \zeta_1)$ and recall that $H_1 = \beta_1^{-1}(p)$ with $p \ne 0$, where we continue to use the index notation of $\mathfrak{v}_j$, $\beta_j : Y_j \to Y_{j-1}$ and $p_j \in H_j$ for the chain of blow-ups. Thus, the leading term a_{μ_1} of $\beta_{Y_1}^* A_m$ with respect to h as $h \to 0$ is non-zero everywhere along the arc H_1, since $A_m(x,h) = x^{l_m} h^{\alpha_m}$. In particular, Λ_{H_1} has only one essential point $\mu_1 = (m, \alpha^*)$. Thus, $\beta_{Y_1}^* P$ can only be resolved if μ_1 is the minimal point of Λ_{H_1} or if any successor $\mathcal{L}_j$, $j \ge 1$, of $\mathcal{L}_1$ is regular (as in Definition 3.2.16). Assume that $\mathcal{B}$ is infinite and that $\mathcal{L}_j$ is singular for all $j \ge 1$.

Since a_{μ_1} is non-zero everywhere on H_1, all leading terms a_{μ_j} of $\beta_{Y_j}^* A_m$ are non-zero everywhere and $\mu_j = \mu_{j-1}(t_{p_{j-1}})$ remains to be the only essential point of Λ_{H_j}. Due to $A_m(x,h) = x^{l_m} h^{\alpha_m}$ we have $\omega_{p_1}(\mu_1) < \omega_{p_1}(\lambda)$ for all $\lambda \in \Lambda_{H_1}$, $\lambda \ne \mu_1$. In particular, there is a time T after which $\beta_T^* \mu_1$ is the minimal point of $\beta_T^* \Lambda$. Thus, all we need to show is that there is $M \in \mathbb{N}$ such that

$$\sum_{j=2}^{M} t_{p_j} \ge T\,. \tag{5.5}$$

Without loss of generality assume that all collisions in $\mathcal{B}$ are of first type and are with the left boundary point $\lambda_j := (k_1, \alpha_j) = \lambda_{j,1}$ of $\mathcal{L}_j$. Note that the order of zeros of a_{λ_j} at p_j is decreasing and converges to $L \ge 1$. Since $\mathrm{ord}_{p_j}(a_{\lambda_j}) \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\mathrm{ord}_{p_j}(a_{\lambda_j}) = L$ for all $n \ge N$. To prove that $\mathcal{B}$ is finite we show that the sequence of partial sums $(\sum_{j=2}^n t_{p_j})_n$ can be bounded from below similar to the harmonic series.

Let $\widetilde{\Lambda}_0 := \{(j,\alpha) \in \Lambda_{H_0} : \alpha - \alpha_0 < LT\}$ be the set of points in Λ_{H_0} which can potentially collide with $\mathcal{L}_0$ in time less than or equal to T. Also let

$$\widetilde{\Lambda}_j := \{(k,\alpha) \in \Lambda_{H_j} : \alpha - \alpha_j < L(T - \sum_{l=0}^{j-1} t_{p_l})\}$$

be the set of points in Λ_{H_k} that can potentially collide with $\mathcal{L}_k$.

A necessary condition for $\omega_{p_j}(\lambda_j) = L$, as $j \to \infty$, is that there is a point $\mu \in \widetilde{\Lambda}_{j-1}$, with $\omega_{p_{j-1}}(\widetilde{\mu}) = L - 1$, that collides with λ_{j-1} at time $t_{p_{j-1}}$. Otherwise the polynomial a_{λ_j} would be given by

$$a_{\lambda_j}(\xi) = c_L \xi^L + \sum_{l=0}^{L-2} c_l \xi^l\,,$$

for some $c_l \in \mathbb{R}$, $l = 0, \dots, L$, not all vanishing, which has no root of order L. Since points of order L or higher keep their vertical distance towards λ_j, the distance between λ_j and $B_j := \{\widetilde{\mu} \in \Lambda_{H_j} : \omega_{p_j}(\widetilde{\mu}) \geq L\}$ is bounded from below by

$$t^* := \min\{\alpha - \alpha_0 : (k_1, \alpha) \in \Lambda_{H_0}, \alpha > \alpha_j\},$$

independently of $j \in \mathbb{N}_0$. Thus, all $n_0 \in \mathbb{N}$ collisions in the time period $\sum_{k=0}^{n_0} t_k < t^*$ are only due to points within $\widetilde{\Lambda}_0 \cap (\mathbb{N}_0 \times [\alpha^*, \alpha^* + t^*))$. After these first n_0 collisions there are at most finitely many points in $\widetilde{\Lambda}_1 \cap (\mathbb{N}_0 \times [\alpha^*, \alpha^* + t^*))$, and again all collisions within $\sum_{k=n_0+1}^{n_1} t_k < t^*$ are only due to points in that region. Repeating this process yields that after finitely many iterations there is $M \in \mathbb{N}$ such that $T - \sum_{k=0}^{n_M} t_k < 0$, proving (5.5). Hence, the only essential point $\mu \in \Lambda_{H_{n_M}}$ is the minimal point.

The addition of successors in Step 6 of *Resolution of Singular Points* whose purpose is to account for horizontal edges leads only to a finite number of iterations of (ii)-(iv) in the proof of Algorithm 1 for these additional branches. Since both (ii) and (iii) are adding only finitely many blow-ups to each arc generated by (i), the tree $\mathcal{G}$ is finite. Hence we proved the theorem under the assumption that $A_m(x, h) = x^{l_m} h^{\alpha_m}$.

For the general case let A_m be analytic in x. Without loss of generality assume that $0 \in I$ and $A_m(0, 0) = 0$. Let $n \in \mathbb{N}$ and $N_j \subset \mathbb{H}$, $j = 1, \dots, n$, be the zero level sets of A_m, i.e.

$$\{A_m = 0\} = \bigcup_j N_j .$$

We fix N_1 and resolve A_m with respect to this zero set. Then there are at most $l = 1, \dots, n-1$ iterated, quasihomogeneous blow-ups β_l of $0 \in \mathbb{H}$ such that $\beta^* N_1 \cap \beta^* N_j = \emptyset$ for $j \geq 2$, with $\beta := \Pi_l \beta_l$. If Algorithm 1 has not terminated already, we add these blow-ups β_j to the resolution process, since they can be treated as additional collisions. Note that these do not compromise the ability of extending phase functions to the interior of M. We denote the complete chain of blow-ups also including β_j, $j = 1, \dots, n$, by $\widetilde{\beta}$. If $N_1 = \{x = 0\}$, then the leading term $\widetilde{\beta}^* A_m$ is simple, i.e. satisfies the same properties as in the special case after weaving in all blow-ups β_j, $j = 1, \dots, n-1$. On the other hand, if $N_1 \neq \{x = 0\}$, then we can apply the implicit function theorem for analytic functions (see [KP13]) to obtain a regular curve $c_1 : \mathbb{R}_+ \to \mathbb{H}$ with $c_1(\mathbb{R}_+) = N_1$. Introducing a new coordinate $\bar{h}(x_{\mathrm{ff}}, h) := h - (\widetilde{\beta}^* c_1)(x_{\mathrm{ff}})$ results in the leading term $\widetilde{\beta}^* A_m$ being simple in $(x_{\mathrm{ff}}, \bar{h})$. Thus, Algorithm 1 terminates eventually. □

5.3 Construction of Quasimodes III: Unresolved Operators

In this section we will lay out the tools and methods required for our approach to construct quasimodes on resolved spaces. After having shown that operators P can be resolved by finitely many quasihomogeneous blow-ups, the question that remains is how one is able to solve all appearing transport equations simultaneously. To do so, we will introduce a family of transport operators $T_{H,\mathcal{L},\zeta}$ linked to the vertices $(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$ of the resolution tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. This yields a local description of the transport equations at every arc H. After showing that one can solve these equations consecutively, we introduce the solution and remainder spaces, allowing us to account for the asymptotic behavior of the functions' amplitudes. With these spaces we will present a way to describe transport equations after blow-ups based on [Gri17] and with the addition of multiple boundary faces as in [Sob18]. Theorem 5.3.22 then states that the iteration of inhomogeneous transport equations eliminating leading orders of remainder terms results in an arbitrarily good quasimode.

We want to emphasize that we will only present initial data of the form $\mathfrak{a} = (\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ for Algorithm 1 in Section 5.3, following Remark 5.2.17. This does not affect the statements throughout the section. Of all these statements, only Theorem 5.3.22 references the initial date in the construction of the resolution space. Both edges with positives slope and horizontal edges will be addressed separately in the statement of Theorem 5.3.22.

5.3.1 Transport Operators

We begin this section by re-introducing the notion of *induced transport operators*. Since there are multiple arcs $H \subset \partial M$ in a resolved manifold $\beta\colon M \to \mathbb{H}$ this will be a family of operators, depending on the arc. Moreover, due to the branching of resolution trees $\mathcal{G}$ associated to its generators $(\mathcal{L}_0, \zeta_0)$, there will be multiple induced transport operators at each arc associated to each successor $(\mathcal{L}, \zeta)$. Thus, we will define multiple new objects whose purpose is to sort vertices of $\mathcal{G}$ to simplify the transport equations at every arc H eventually.

Eikonal Variety

In the upcoming definition we will specify the relevant solutions ζ of the respective eikonal polynomial at arcs $H \subset \partial M$. These can be derived from the resolution tree $\mathcal{G}$.

Definition 5.3.1 (Eikonal Variety)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot, \zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Then we call

$$\Gamma(\mathcal{G}) := \bigcup_{\substack{\mathrm{ord}((Y,H,\mathcal{L},\zeta))=1 \\ (Y,H,\mathcal{L},\zeta)\in\mathcal{V}}} \mathrm{graph}(\zeta) \times \{\delta(\mathcal{L})\} \tag{5.6}$$

the *eikonal variety*, where $\left\{E_{H,\delta(\mathcal{L})}(\beta^* P) = 0\right\} \subset T^*_{\mathbb{C}} H^\circ$, for each $H \subset \partial M$.

Remark 5.3.2. Only including vertices of order one in (5.6) in Definition 5.3.1 restricts the variety to end points of branches when considering relevant arcs, edges and solutions. The lift of right sides of blown-up arcs as described in Algorithm 1 was designed to create short branches with terminal vertices $(Y, H_r, \mathcal{L}, \zeta)$, such that $\beta^*_Y P$ is $\mathcal{L}$-resolved on H_r.

It is useful to sort the resolution tree $\mathcal{G}$, yielding a set of edges $\mathcal{L}$ and a set of solutions ζ corresponding to edges at arcs H.

Definition 5.3.3. Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial_- \mathcal{P}(\Lambda)$ be an edge, $I \subset \partial \mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot, \zeta_0) = 0$ on I and $\beta \colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Let $H \in \mathcal{M}_1(M)$.

Then we call

$$L(H; \mathcal{G}) \coloneqq \{\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda_H) : \text{ there exists } Y \text{ and } \zeta \text{ with } (Y, H, \mathcal{L}, \zeta) \in \mathcal{V}\}$$

the *edges associated* to H, and

$$Z(H, \mathcal{L}; \mathcal{G}) \coloneqq \{\zeta : \text{ there exists } Y \text{ with } (Y, H, \mathcal{L}, \zeta) \in \mathcal{V}\}$$

the *solutions* of $E_{H,\delta(\mathcal{L})}(\beta^* P)(\cdot, \zeta) = 0$, for $\mathcal{L} \subset \partial_- \mathcal{P}(\Lambda_H)$.

Remark 5.3.4. Note that we identify each connected component of $\beta^* H \subset \partial M$ with $H \subset \partial Y$, where $\beta \colon M \to Y$.

The following proposition illustrates how exponential-polyhomogeneous functions associated to an eikonal variety $\Gamma(\mathcal{G})$ look locally, including the logarithmic behavior at some arcs.

Proposition 5.3.5. *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, let $\mathcal{L}_0 \subset \partial_- \mathcal{P}(\Lambda)$ be an edge, $I \subset \partial \mathbb{H}_{reg}$ be a regular interval and ζ_0 a solution of $E_\delta(P)(\cdot, \zeta_0) = 0$ on I. Let $\beta \colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$ and $\Gamma = \Gamma(\mathcal{G})$ be the corresponding eikonal variety. Let $\mathcal{I}$ be a family of index sets on M corresponding to Γ.*

Then for each function $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}}(M; \Gamma)$ and arc $H \in \mathcal{M}_1(M)$ we have

$$u = \sum_{\mathcal{L} \in L(H; \mathcal{G})} \sum_{\zeta \in Z(H, \mathcal{L}; \mathcal{G})} e^{\varphi_{\mathcal{L},\zeta}/h^\delta \, + \, c \cdot \log(h)/h^\delta} A_{H,\mathcal{L},\zeta} \text{ at } H^\circ , \tag{5.7}$$

with $A_{H,\mathcal{L},\zeta}$ being polyhomogeneous in a neighborhood of H°, for some $c \colon \Gamma \to \mathbb{N}$ locally constant and where the phase functions $\varphi_{\mathcal{L},\zeta}$ are the solutions of

$$E_{H,\delta(\mathcal{L})}(\cdot, \partial_{x_H} \varphi_{\mathcal{L},\zeta}) = 0 ,$$

for each $\mathcal{L} \in L(H; \mathcal{G})$ and $\zeta \in Z(H, \mathcal{L}; \mathcal{G})$.

Proof. By construction, every exponential-polyhomogeneous function $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}}(M; \Gamma)$ is of the form $u = \sum_{k=1}^N e^{\Phi_j} A_j$ for some $\Phi_j, A_j \in \mathcal{A}$. In particular, at every arc $H \in \mathcal{M}_1(M)$ we have

$$\Phi_j = \varphi_{j,\mathcal{L},\zeta}/h^\delta + c \cdot \log(h)/h^\delta ,$$

where $\delta = \delta(\mathcal{L})$, $c = c(H, \mathcal{L}, \zeta)$, for some $\mathcal{L} \in L(H; \mathcal{G})$ and $\operatorname{graph}(d\varphi_{j,\mathcal{L},\zeta}) \times \{\delta\} \subset \Gamma$ at H°. Thus, there are at most $|Z(H, \mathcal{L}; \mathcal{G})|$ different sheets and hence different phases φ for each $\mathcal{L} \in L(H;\mathcal{G})$. Therefore, we can write

$$u = \sum_{k=1}^{N} e^{\Phi_j} A_j = \sum_{\mathcal{L} \in L(H;\mathcal{G})} \sum_{\zeta \in Z(H,\mathcal{L};\mathcal{G})} e^{\varphi_{\mathcal{L},\zeta}/h^\delta + c\cdot\log(h)/h^\delta} A_{H,\mathcal{L},\zeta} ,$$

with $\delta = \delta(\mathcal{L})$ and, as a consequence of graph$(d\varphi_j) \subset \{E_{H,\delta(\mathcal{L})}(\beta^* P) = 0\}$, it follows that $E_{H,\delta(\mathcal{L})}(\beta^* P)(\cdot, \partial_{x_H}\varphi_{\mathcal{L},\zeta}) = 0$. □

Transport Operator

The upcoming definition of induced transport operators along arcs is straightforward and coincides with Definition 3.2.18 for the boundary of the half space $\mathbb{H}$. These operators reflect the symbolic nature of the transport operators as shown in Remark 3.2.19.

Definition 5.3.6 (Transport Operator on Resolved Manifolds)**.** Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Let $H \in \mathcal{M}_1(M)$, $\mathcal{L} \in L(H;\mathcal{G})$ and $\zeta \in Z(H,\mathcal{L};\mathcal{G})$ be simple. Assume that $\beta^* P$ is $\delta(\mathcal{L})$-separated and let φ be a solution of $\partial_{x_H}\varphi = \zeta$.

If $\delta(\mathcal{L}) > 0$ we call

$$T_{H,\mathcal{L},\zeta} := \sum_{\substack{\lambda \in \mathcal{L}\cap\Lambda_H \\ \lambda=(k,\alpha)}} a_\lambda k(\varphi')^{k-1} \left(\partial_{x_H} + \frac{k-1}{2}\frac{\varphi''}{\varphi'} \right)$$

the *induced transport operator* of $\beta^* P$ at H with respect to h and $(\mathcal{L},\zeta)$.

If $\delta(\mathcal{L}) = 0$ we call the leading term

$$T_{H,0} := (\beta^* P)_H ,$$

of $(\beta^* P)$ at H° with respect to h the *transport operator*.

Immediately, it holds that $T_{H,\mathcal{L},\zeta}$ is elliptic on H° for $\mathcal{L}$ with $\delta(\mathcal{L}) > 0$, if φ' is a simple, non-trivial solution of $E_{\delta(\mathcal{L})}(\beta^* P)(\cdot,\zeta) = 0$.

Remark 5.3.7. We can define transport operators for higher order multiplicities as in Definition 3.3.13 at each arc $H \subset \partial M$ for any choice of $\mathcal{L}$ and ζ. Since these only occur in extreme marginal cases we will restrict the algorithmic construction in Subsection 5.3.3 to simple solutions ζ of eikonal polynomials $E_{H,\delta}(\beta^* P)$ with $\delta > 0$.

The following proposition shows that we are able to solve transport equations on resolved manifolds as in the model cases of Chapters 3 & 4. This is important to determine the jumps for the minimal exponents in the asymptotic expansion of the amplitudes between arcs.

Proposition 5.3.8 (Homogeneous Transport Solution)**.** *Let $P \in \mathit{Diff}^\Lambda(\mathbb{R})$, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$, let $I \subset \partial\mathbb{H}_{reg}$, ζ_0 a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space and $\Gamma = \Gamma(\mathcal{G})$ be the corresponding eikonal variety. Let $H_1, H_2 \in \mathcal{M}_1(M)$ be adjacent hypersurfaces, $p \in H_1 \cap H_2$, $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_p(\beta^* P))$ and $\mathcal{L}_j := \pi_{(k,\alpha_j)}(\mathcal{L}) \subset \partial\mathcal{P}(\Lambda_{H_j})$*

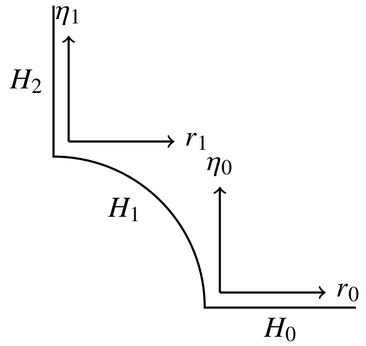

Figure 5.6: Illustration of adjacent arcs with their induced quasi-projective coordinate systems (r_j, η_j).

with slopes δ_j, for $j = 1, 2$. Let ζ_j be simple solutions of $E_{H_j,\delta_j}(\beta^*P)(\cdot, \zeta_j) = 0$ on H_j° matching in the sense of Proposition 5.1.16. Suppose that β^*P is strictly δ_j-separated at H_j, $j = 1, 2$.

Then for each solution u_1 of $T_{H_1,\delta_1,\zeta_j}u_1 = 0$ there is a solution u_2 of $T_{H_2,\delta_2,\zeta_2}u_2 = 0$ and there is $s \in \mathbb{R}$, such that the pair (u_1, u_2h^s) has a polyhomogeneous extension to M. In particular, s is given by $u_1(r) \sim r^s$ as $r \to 0$.

Proof. By Proposition 5.1.16 there is a function $\Phi \in \mathcal{A}(M)$ such that the leading terms φ_j of $h^{\delta_j}\Phi \sim \varphi_j + o(1)$ at H_j° satisfy $\partial_{x_{H_j}}\varphi_j = \zeta_j$. Since P is δ_j-separate at H_j, $j = 1, 2$, the operators T_{H_j,δ_j,ζ_j} from Definition 3.2.18 are the transport operators at both faces. They are elliptic, since ζ_j are simple solutions of $E_{H_j,\delta_j}(\beta^*P)(\cdot, \zeta_j) = 0$. These can be rephrased to

$$\sum_{\substack{\lambda \in \mathcal{L}_j \cap \Lambda_{H_j} \\ \lambda = (k,\alpha)}} a_\lambda k(\varphi_j')^{k-1}\left(\partial_{x_{H_j}} + \frac{k-1}{2}\frac{\varphi_j''}{\varphi_j'}\right) = (\partial_\zeta E_{H_j,\delta_j}(\beta^*P))\partial_{x_{H_j}} + \frac{\varphi_j''}{2}\partial_\zeta^2 E_{H_j,\delta_j}(\beta^*P)\,.$$

For $j = 1, 2$ and for

$$f_j := x_{H_j}\varphi_j''/2 \cdot (\partial_\zeta^2 E_{H_j,\delta_j})/(\partial_\zeta E_{H_j,\delta_j})(\beta^*P)$$

we have $s := f_1(0) = f_2(0) \neq 0$ by Proposition 4.3.4 since $\mathcal{L} \subset \partial\mathcal{P}(\Lambda_p)$. Thus, they admit an extension F to M and we can apply Lemma 4.3.7 at $p \in H_1 \cap H_2$ to $T := \beta^*(x\partial_x) + F$ and obtain an asymptotic solution u. In particular, for $u \sim u_1 + o(1)$ at H_1 and $u \sim u_2h^s + o(h^s)$ at H_2 we have $T_{H_j,\delta_j,\zeta_j}u_j = 0$, completing the proof. □

5.3.2 Solution- & Remainder Spaces

The goal of this subsection is to find a suitable space of exponential-polyhomogeneous functions $\mathcal{E}\mathcal{A}^{\mathcal{I}}(M;\Gamma)$ on the resolved space $\beta\colon M \to \mathbb{H}$, containing quasimodes u for the operator β^*P on M. We use the eikonal variety $\Gamma(\mathcal{G})$ associated to a resolution tree to specify the exponential behavior of potential quasimodes at each arc $H_0 \subset \partial M$. What remains to be determined are the index families $\mathcal{I}(\mathcal{G})$ associated to each phase corresponding to each sheet in $\Gamma(\mathcal{G})$. Due to the asymptotic behavior of homogeneous solutions of the transport equations at each arc, the corresponding index sets will have slightly shifted minima going from one arc H_0 to an adjacent arc H. Incorporating these shifts $\varepsilon_{H,\mathcal{L},\zeta}$ for each combination of arcs, edges and solutions is crucial in the definition of solution- and remainder spaces. These will become especially important in the definition of leading parts.

Index Shift

Proposition 5.3.8 has significant implications on the structure of potential solution spaces. Let $\mathfrak{v}_j = (Y_j, H_j, \mathcal{L}_j, \zeta_j) \in \mathcal{V}$, for $j = 1, 2$, be leaves of order one in the resolution tree of P. Suppose that these leaves are matching in the sense that the arcs are adjacent, i.e. there is $p \in H_1 \cap H_2$, there is an three-dimensional edge $\mathcal{L} \subset \Lambda(\beta^* P)_p$ with $\pi_{(k,\alpha_{j+1})} = \mathcal{L}_j$ and matching solution ζ_j, for $j = 1, 2$. Assume that H_2 is on the left side of H_1. Denote $x_H := x_{H_1}$.

Then the value

$$s(\mathfrak{v}_2) := \lim_{x_H \to 0} \frac{x_H \varphi_1''(x_H)}{2} \frac{(\partial_\zeta^2 E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P))(x_H, \varphi_1'(x_H))}{(\partial_\zeta E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P))(x_H, \varphi_1'(x_H))} \tag{5.8}$$

is the shift of the minimal value in the index set at H_2 from H_1 with respect to the leaves $\mathfrak{v}_j$, $j = 1, 2$. The solution space will be a space of exponential-polyhomogeneous functions with a family of index sets corresponding to these leaves. Note that $s(\mathfrak{v}_2)$ exists and is real valued.

Lemma 5.3.9. *In the setting above, let $s(\mathfrak{v}_2)$ be the value defined in* (5.8)*. Then $s(\mathfrak{v}_2) \in \mathbb{R}$.*

Proof. At $x_H = 0$ we can factorize the eikonal polynomial of $\beta^* P$ at H_1

$$E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)(\cdot, \zeta) = \prod_{l=1}^{L} (\zeta - \zeta_l) \ ,$$

where $L \geq 2$ is the length of the associated edge in $\partial\mathcal{P}(\Lambda_{H_1})$. By assumption, we have $(\zeta_1/\zeta_2)(0) = 1$. Differentiating $E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)$ in ζ then yields

$$\begin{aligned}
\left(\partial_\zeta E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)\right)(\cdot, \zeta) &= \sum_{k=1}^{L} \prod_{\substack{l=1 \\ l \neq k}}^{L} (\zeta - \zeta_j) \\
\left(\partial_\zeta^2 E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)\right)(\cdot, \zeta) &= \sum_{\substack{j,k=1 \\ j \neq k}}^{L} \prod_{\substack{l=1 \\ l \neq j,k}}^{L} (\zeta - \zeta_j) \, .
\end{aligned}$$

Sorting this and the second derivative by vanishing order at $x_H = 0$ and with respect to ζ_1 then yields

$$\left(\partial_\zeta E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)\right)(\cdot,\zeta) = (\zeta - \zeta_2)\prod_{j=2}^{L}(\zeta - \zeta_j) + O(\zeta - \zeta_1)$$

$$\left(\partial_\zeta^2 E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)\right)(\cdot,\zeta) = 2\cdot\prod_{j=2}^{L}(\zeta - \zeta_j) + O(\zeta - \zeta_2) + O(\zeta - \zeta_1)\,,$$

where the two Os in the second equation mean that some summands only vanish at $x_H = 0$ and most summands vanish globally on H_1 if $\zeta = \zeta_1$. Thus, as $x_H \to 0$ we have

$$\frac{x_H\zeta_1'(x_H)}{2}\,\frac{(\partial_\zeta^2 E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P))(x_H,\zeta_1(x_H))}{(\partial_\zeta E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P))(x_H,\zeta_1(x_H))} \sim \frac{x_H\zeta_1'(x_H)}{\zeta_1(x_H) - \zeta_2(x_H)} + o(1)\,.$$

Denoting $\zeta_j'(x_H) \sim c_j x^{\gamma_j}$, for $j = 1, 2$, and taking the limit, we have

$$s(\mathfrak{v}_2) = \frac{c_1}{c_1 - c_2} \neq 0\,,$$

since ζ_1, ζ_2 intersect cleanly in $x_H = 0$. Since ζ_1, ζ_2 are implicit functions of the real valued polynomial $E_{H_1,\delta(\mathcal{L}_1)}(\beta^* P)$ they coincide with the implicit differential evaluated at $x_H = 0$ and are real valued. In particular, we have $s(\mathfrak{v}_2) \in \mathbb{R}$. □

Solution Space

Without loss of generality we can assume that any quasimode we construct for initial data $\mathcal{L} \subset \partial_{\mathcal{P}}(\Lambda)$, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ and ζ solving $E_{\delta(\mathcal{L})}(P)(\cdot,\zeta) = 0$ is bounded at $\partial\mathbb{H}_r$ on the resolved space $\beta\colon M \to \mathbb{H}$, since $[P, h] = 0$. Thus, the index set $\mathcal{I}(\mathcal{G}; \partial\mathbb{H}_r, \mathcal{L}, \zeta) \subset \mathbb{R}_+ \times \{0\}$ at $\partial\mathbb{H}_r$ is log-free and does not contain negative polynomial powers. We will denote the minimal entry of $\mathcal{I}(\mathcal{G}; \partial\mathbb{H}_r, \mathcal{L}, \zeta)$ by $\varepsilon_{\partial\mathbb{H}_r,\mathcal{L},\zeta} := 0$. The number in (5.8) shifts the minimal value of the index sets going from one hypersurface to an adjacent hypersurface. Thus, we need to sum all values $s(\mathfrak{v})$ for a chain of leaves at the end of branches to determine the minimal value of $\mathcal{I}(\mathcal{G}; H, \mathcal{L}_H, \zeta_H)$ at the final leaf $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$.

Let $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ be a leaf at the end of a branch and $(\mathfrak{v}_n)_{n=0}^N$ with

$$\mathfrak{v}_n := (Y_n, H_n, \mathcal{L}_n, \zeta_n)$$

be the unique family of pairwise matching leaves at the end of branches starting from $\mathfrak{v}_0 := (\mathbb{H}, \partial\mathbb{H}, \mathcal{L}, \zeta)$ to $\mathfrak{v}_N := (Y, H, \mathcal{L}_H, \zeta_H)$, for some $N \in \mathbb{N}$. For $1 \le n \le N$ let $s(\mathfrak{v}_n)$ in (5.8) corresponding to the pair $(\mathfrak{v}_{n-1}, \mathfrak{v}_n)$. Then the lowest entry of $\mathcal{I}(\mathcal{G}; H_n, \mathcal{L}_n, \zeta_n)$ is given by

$$\varepsilon_{H_n,\mathcal{L}_n,\zeta_n} := \sum_{n=1}^{N} s(\mathfrak{v}_n)\,. \tag{5.9}$$

Recall that transport equations for determining the amplitude $u \sim \sum u_k h^{\gamma_k}$ are asymptotically of the form

$$Tu_k = -R_j u_{k-1}\,,$$

where T is the transport operator of the expansion of the conjugated operator

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} = h^{\alpha_0} T + \sum_{l=1}^{\infty} h^{\alpha_l} R_l\,.$$

The other operators R_l are differential operators in the expansion with a higher order in h. Thus, the non-negative parts of the pairwise differences $(\alpha_l - \alpha_0)_+$ and their linear combinations over $\mathbb{N}_0$ for all $l \in \mathbb{N}$ are the relevant exponents γ_k in the expansion of u. This motivates the upcoming definition of induced index sets.

Definition 5.3.10 (Induced Index Sets)**.** Let $\Lambda \subset \mathbb{N} \times \mathbb{R}$ be a set of exponents. Denote $\alpha_{\min} := \min\{\alpha :\ (k,\alpha) \in \Lambda\}$. Then we define the induced index set of Λ,

$$E(\Lambda) := \langle \{(\alpha - \alpha_{\min})_+ |\ (k,\alpha) \in \Lambda\} \rangle_{\mathbb{N}}\,,$$

to be the free semi group over $\mathbb{N}$ generated by the pairwise non-negative differences in the first entries of Λ.

Using this notion of induced index sets we are able to introduce the spaces of potential quasimodes of an operator P.

Definition 5.3.11 (Solution Space)**.** Let $P \in \mathrm{Diff}^\Lambda(\mathbb{R})$ be an operator, $\mathcal{L}_0 \subset \partial_- \mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y,H,\mathcal{L}_H,\zeta_H) \in \mathcal{V}$ of order one. Let Γ be the associated eikonal variety and for each leaf $(Y,H,\mathcal{L},\zeta) \in \mathcal{V}$ of order one let $\varepsilon_{H,\mathcal{L},\zeta}$ be the transport shift given by (5.9). Let

$$\mathcal{I}(\mathcal{G}; H, \mathcal{L}, \zeta) := E\left(\Lambda\left((\beta^* P_H)_{\delta(\mathcal{L}),\zeta}\right)\right) + \varepsilon_{H,\mathcal{L},\zeta}$$

be a family of index sets at H induced by all leaves in $\mathcal{G}$ of order one and denote

$$\mathcal{I}(\mathcal{G}) := (\mathcal{I}(\mathcal{G}; H, \mathcal{L}, \zeta))_{H,\mathcal{L},\zeta}\,.$$

Then the space

$$\mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$$

is called *solution space* of $\beta^* P$ and $\mathcal{G}$.

It will become clear at the end of this chapter that it is very important to keep an accurate record of the remainder functions $f = Pu$ during the iterative construction of quasimodes.

Their asymptotic behavior will be used to measure the quality of the quasimode after finite iterations. In the upcoming definition, we will introduce the notion of *remainder space*. This turns out to be the image of the solution space with respect to the operator $\beta^* P$ on the resolution space $\beta : M \to \mathbb{H}$.

Definition 5.3.12 (Remainder Space)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be an operator, $\mathcal{L}_0 \subset \partial_{-}\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot, \zeta_0) = 0$ on I and $\beta \colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ of order one. Let Γ be the associated eikonal variety. Denote $\alpha_{\min}(H, \mathcal{L}, \zeta) \coloneqq \min\{\alpha :\ (k, \alpha) \in \Lambda_H((\beta^* P)_{\delta(\mathcal{L}),\zeta})\}$ and let

$$\mathcal{J}(\mathcal{G}; H, \mathcal{L}, \zeta) \coloneqq \langle \mathcal{I}(\mathcal{G}; H, \mathcal{L}, \zeta) + \{\alpha :\ (k, \alpha) \in \Lambda_H((\beta^* P)_{\delta(\mathcal{L}),\zeta})\}\rangle_{\mathbb{N}}$$

be the semi group generated by $\mathcal{I}(\mathcal{G})$ for leaves in $\mathcal{G}$ of order one, α_j at H with respect to $\mathcal{L}$ and ζ. Then the space

$$\mathcal{E}\mathcal{A}^{\mathcal{J}(\mathcal{G})}(M; \Gamma)$$

is called *remainder space* of $\beta^* P$ with respect to $\mathcal{G}$.

The upcoming definition of leading parts is crucial in the construction of quasimodes. It simplifies the transport equations at any arc H and evaluating remainder terms.

Definition 5.3.13 (Space of Leading Parts and Leading Part Operator)**.** Let $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial_{-}\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{\mathrm{reg}}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot, \zeta_0) = 0$ on I and $\beta \colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ of order one. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety and let $k \in \mathbb{N}_0$.

For any $H \in \partial_1(M)$ let $\sum_{\mathcal{L}} \sum_{\zeta} e^{\Phi_{H,\mathcal{L},\zeta}} A_{H,\mathcal{L},\zeta}$ be the expansion of $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M; \Gamma)$ at H°. Let $a_{H,\mathcal{L},\zeta}$ be the leading part of $h^{-\alpha_{H,\mathcal{L},\zeta}} A_{H,\mathcal{L},\zeta} \sim a_{H,\mathcal{L},\zeta} + o(1)$ to the power $\alpha_{H,\mathcal{L},\zeta} \in \mathbb{R}$ at H°. The operator LP defined by

$$u \mapsto (a_{H,\mathcal{L},\zeta})_{H,\mathcal{L},\zeta}$$

is called *leading part operator*. Its restriction to the filtered subspace $\mathcal{E}\mathcal{A}^{\mathcal{I}_k(\mathcal{G})}(M; \Gamma)$ is denoted by

$$\mathrm{LP}_k \coloneqq \mathrm{LP}_{|\mathcal{E}\mathcal{A}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma)}\,.$$

Further we call

$$\mathcal{S}^k(\partial M; \Gamma) \coloneqq \mathrm{LP}_k\left(\mathcal{E}\mathcal{A}^{\mathcal{I}_k(\mathcal{G})}(M; \Gamma)\right)$$

the *space of leading parts* to k-th order.

Remark 5.3.14. We will also apply LP and its restrictions LP_k to elements $f \in \mathcal{EA}^{\mathcal{J}}(M;\Gamma)$ in the same sense without introducing any new terminology.

Remark 5.3.15. For $u \in \mathcal{EA}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ the function $\mathrm{LP}_0(u)_{H,\mathcal{L},\zeta}$ does not vanish on H if and only if $\alpha_{H,\mathcal{L},\zeta} = \varepsilon_{H,\mathcal{L},\zeta}$.

To extend families of asymptotic solutions at different arcs to the interior of M, we need a more sophisticated version of the Borel lemma 2.2.18, respecting the exponential behavior at corners and in particular the successors going from one arc to an adjacent arc.

Lemma 5.3.16 (Extended Borel lemma)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be an operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{reg}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y,H,\mathcal{L}_H,\zeta_H) \in \mathcal{V}$ of order one. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety and $\mathcal{I}(\mathcal{G})$ be the induced index family. Let $(a_{H,\mathcal{L},\zeta})_{H,\mathcal{L},\zeta}$ be a family of functions such that $a_{H,\mathcal{L},\zeta} \in \mathcal{A}(H)$ such that*

$$a_{H,\mathcal{L},\zeta}(r_{H'}) \sim r_{H'}^{\varepsilon_{H',\mathcal{L}',\zeta'}-\varepsilon_{H,\mathcal{L},\zeta}},$$

for all matching leaves $(Y,H,\mathcal{L},\zeta),(Y,H',\mathcal{L}',\zeta') \in \mathcal{V}$ of order one, adjacent hypersurface H' and successor $(\mathcal{L}',\zeta')$.

Then there is a function $u \in \mathcal{EA}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ such that

$$LP(u) = (a_{H,\mathcal{L},\zeta})_{H,\mathcal{L},\zeta}\,.$$

Proof. Since all solutions ζ for all $\mathcal{L}$ and H in $\mathcal{G}$ are simple, there are global phase functions $\varphi_{H,\mathcal{L},\zeta}$ on H° with

$$\partial_{x_H}\varphi_{H,\mathcal{L},\zeta} = \zeta_{H,\mathcal{L},\zeta}\,,$$

for all leaves $(Y,H,\mathcal{L},\zeta) \in \mathcal{V}$ at the end of branches. Since $(Y',H',\mathcal{L}',\zeta')$ and $(Y,H,\mathcal{L},\zeta)$ are matching leaves, the corresponding phase functions $\varphi_{H,\mathcal{L},\zeta}$ and $\varphi_{H',\mathcal{L}',\zeta'}$ can be extended polyhomogeneously to M.

Let $v_{H,\mathcal{L},\zeta} := e^{\varphi_{H,\mathcal{L},\zeta}} h^{\varepsilon_{H,\mathcal{L},\zeta}} a_{H,\mathcal{L},\zeta}$ be a family of local exponential-polyhomogeneous functions at H with respect to Γ. Since $\varepsilon_{H',\mathcal{L}',\zeta'} - \varepsilon_{H,\mathcal{L},\zeta} = s((Y',H',\mathcal{L}',\zeta'))$ and

$$a_{H,\mathcal{L},\zeta}(r_{H'}) \sim r_{H'}^{\varepsilon_{H',\mathcal{L}',\zeta'}-\varepsilon_{H,\mathcal{L},\zeta}},$$

the amplitudes $h^{\varepsilon_{H,\mathcal{L},\zeta}} a_{H,\mathcal{L},\zeta}$ and $a_{H',\mathcal{L}',\zeta'}$ can be extended to M. Thus, starting at the vertex $(\mathbb{H},\partial\mathbb{H}_r,\mathcal{L},\zeta)$ for initial values $(\mathcal{L},\zeta)$ on $\mathbb{H}$, Propositions 5.1.16 and 5.3.8 determine a function $u \in \mathcal{EA}^{\mathcal{I}}(M;\Gamma)$ with

$$LP(u) = (a_{H,\mathcal{L},\zeta})_{H,\mathcal{L},\zeta}\,.$$

$\square$

We have to correct the polyhomogeneous amplitudes of the exponential-polyhomogeneous functions $u \in \mathcal{EA}^{\mathcal{I}}(M;\Gamma(\mathcal{G}))$ in the iterative construction of quasimodes. Thus, we can make use of a short exact sequence of similar solution spaces.

Lemma 5.3.17 (Short Exact Sequence)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be an operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{reg}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ of order one. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety and let $k \in \mathbb{N}_0$.*

Then the sequence

$$0 \to \mathcal{EA}^{\mathcal{I}_{k+1}(\mathcal{G})}(M;\Gamma) \xrightarrow{\iota} \mathcal{EA}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma) \xrightarrow{LP_k} \mathcal{S}^k(\partial M;\Gamma) \to 0$$

is exact, where the left map is the canonical inclusion ι and the right map LP_k is the leading part operator.

Proof. The proof for $k > 0$ is the same as the proof for $k = 0$. By construction we have $\mathrm{LP}(\mathcal{EA}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)) = \mathcal{S}(\partial M;\mathcal{G})$. Since h is a global boundary defining function, it holds that $\mathrm{LP}(\mathcal{EA}^{\mathcal{I}_+(\mathcal{G})}(M;\Gamma)) = \emptyset$. Let $u \in \mathcal{EA}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ such that $\mathrm{LP}(u) = 0$. Thus, the leading part $a_{H,\mathcal{L},\zeta}$ of $A_{H,\mathcal{L},\zeta}$ vanishes globally at each H for each $\mathcal{L}$ and ζ, implying that $u \in \mathcal{EA}^{\mathcal{I}_+(\mathcal{G})}(M;\Gamma)$. □

Remark 5.3.18. If $\mathcal{I}(\mathcal{G})$ can be generated by a finite amount of elements for each $(H, \mathcal{L}, \zeta)$ then one can simplify the construction of quasimodes by using only the principal symbol of LP. In order to do so, one needs to expand $\mathcal{I}(\mathcal{G})$ so that it is generated by the greatest common divisor of all generating elements mentioned above and shifted afterwards by $\varepsilon_{H,\mathcal{L},\zeta}$.

5.3.3 Model Operators & Compatibility

In this subsection we derive a method to construct quasimodes on a resolved manifold M. This is done via three subsequent lemmas, showing that we can solve a family of transport equations at all arcs simultaneously. The following lemma demonstrates the core of the iteration in the construction of quasimodes. It shows how solution- and remainder spaces are related by the operator β^*P and that the leading part of β^*Pu is given by the image of the transport operator applied to the leading part of u at any arc H for any edge $\mathcal{L}$ and solution ζ. In particular, this lemma shows that the elimination of leading parts of remainders improves the quasimode and that this can be done by adding correction terms.

Lemma 5.3.19 (Leading Part and Model Operator Lemma)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{reg}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ of order one. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety. Let $H \in \mathcal{M}_1(M)$ be an arc, $k \in \mathbb{N}_0$, let $f \in \mathcal{EA}^{\mathcal{J}_k(\mathcal{G})}(M;\Gamma)$*

and let $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ *such that*

$$u = \sum_{\mathcal{L}\in L(H;\mathcal{G})} \sum_{\varphi'\in Z(H,\mathcal{L};\mathcal{G})} e^{\varphi_{\mathcal{L},\zeta}/h^{\delta}} a_{H,\mathcal{L},\varphi'} h^{\varepsilon_{H,\mathcal{L},\varphi'}} + \text{ h.o.t. at } H^\circ .$$

Then the following hold:

(i) $\beta^* P \colon \mathcal{E}\mathcal{A}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma) \to \mathcal{E}\mathcal{A}^{\mathcal{J}_k(\mathcal{G})}(M;\Gamma)$,

(ii) At H° *we have*

$$(\beta^* P)u = \sum_{\mathcal{L}\in L(H;\mathcal{G})} \sum_{\varphi'\in Z(H,\mathcal{L};\mathcal{G})} e^{\varphi_{\mathcal{L},\zeta}/h^{\delta}} h^{l_{H,\delta(\mathcal{L})}+\delta(\mathcal{L})+\varepsilon_{H,\mathcal{L},\varphi'}} T_{H,\mathcal{L},\varphi'} a_{H,\mathcal{L},\varphi'} + \text{h.o.t.},$$

(iii) $f \in \mathcal{E}\mathcal{A}^{\mathcal{J}_{k+1}(\mathcal{G})}(M;\Gamma)$ *if and only if* $LP_k(f) = 0$.

Proof. (i) This is true by construction. (ii) This is an immediate consequence of the construction of $L(H;\mathcal{G})$ and $Z(H,\mathcal{L};\mathcal{G})$ for $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$. (iii) The proof is analogue to the proof of Lemma 5.3.17, exchanging $\mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ with $\mathcal{E}\mathcal{A}^{\mathcal{J}(\mathcal{G})}(M;\Gamma)$. □

The method described above starts with the family of homogeneous transport equations at each arc H. One effect of the resolution algorithm is that crossing points turn into constant multiplicities of solutions. However, since the corresponding edge is horizontal, it is separated by default. Thus, the presence of constant multiplicities of solutions for non-horizontal edges is highly generic and will be excluded from Theorem 5.3.22 for simplicity. Recall the notion of truncated, real index sets I_n in Definition 2.2.7.

Lemma 5.3.20 (Initial Step Lemma)**.** *Let* $P \in \mathrm{Diff}^{\Lambda}(\mathbb{R})$ *be an operator,* $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ *be an edge,* $I \subset \partial\mathbb{H}_{reg}$ *be a regular interval,* ζ_0 *be a solution of* $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ *on* I *and* $\beta\colon M \to \mathbb{H}$ *be the associated resolution space with resolution tree* $\mathcal{G}$*. Assume* ζ_H *is simple on* H° *for all* $(Y,H,\mathcal{L}_H,\zeta_H) \in \mathcal{V}$ *of order one. Let* $\Gamma = \Gamma(\mathcal{G})$ *be the associated eikonal variety and assume that* $m_\zeta(\mathcal{L}) = 1$ *for all* $(Y,H,\mathcal{L},\zeta) \in \mathcal{V}$ *with* $\delta(\mathcal{L}) > 0$*. Assume that* β^*P *is strictly* $\delta(\mathcal{L})$*-separated for every* $(Y,H,\mathcal{L},\zeta) \in \mathcal{V}$*.*

Then there is $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$*, with* $LP(u) \neq 0$*, such that*

$$(\beta^* P)u \in \mathcal{E}\mathcal{A}^{\mathcal{J}_+(\mathcal{G})}(M;\Gamma). \tag{5.10}$$

Proof. Let $H_0 := \beta^* I$, assume that all boundary faces are on the right of H_0 and index them by order from left to right. For H_0, the only choice of edges and solutions is given by $\mathcal{L}_0$ and ζ_0, respectively. Let $a_0 \neq 0$ be a solution of $T_{H_0,\mathcal{L}_0,\zeta_0} a_0 = 0$. The asymptotic behavior of u_0 towards $H_0 \cap H_1$ yields boundary value problems for $T_{H_1,\mathcal{L}',\zeta'}(a_{H_1,\mathcal{L}',\zeta'} h^{\varepsilon_{H_1,\mathcal{L}',\zeta'}})$, for each leaf $\mathfrak{v}'$ with respect to H_1, $\mathcal{L}'$ and ζ'. Repeating this argument for all $\mathfrak{v} \in \mathcal{V}$ successively yields a family of homogeneous solutions $(a_{H,\mathcal{L},\zeta} h^{\varepsilon_{H,\mathcal{L},\zeta}})_{H,\mathcal{L},\zeta}$ matching successively at each corner. Thus, by Lemma 5.3.16, we have $(a_{H,\mathcal{L},\zeta}) \in \mathcal{S}(\partial M;\mathcal{G})$ and by Lemma 5.3.17 there is $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ such that $(\beta^* P)u \in \mathcal{E}\mathcal{A}^{\mathcal{J}_+(\mathcal{G})}(M;\Gamma)$. □

The leading terms of the remainders of the homogeneous solutions can now be eliminated by adding suitable correction terms $v \in \mathcal{EA}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma)$. This will lead to inhomogeneous transport equations in the upcoming lemma.

Lemma 5.3.21 (Iterated Improvement Lemma)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{reg}$ be a regular interval, ζ_0 be a solution of $E_{\delta(\mathcal{L}_0)}(P)(\cdot,\zeta_0) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Assume ζ_H is simple on H° for all $(Y,H,\mathcal{L}_H,\zeta_H) \in \mathcal{V}$ of order one. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety and assume that the multiplicity $m_\zeta(\mathcal{L}) = 1$ for all $(Y,H,\mathcal{L},\zeta) \in \mathcal{G}$ with $\delta(\mathcal{L}) > 0$. Assume that β^*P is strictly $\delta(\mathcal{L})$-separated for every $(Y,H,\mathcal{L},\zeta) \in \mathcal{V}$. Suppose that there is $u \in \mathcal{EA}^{\mathcal{I}(\mathcal{G})}(M;\Gamma)$ such that $(\beta^*P)u \in \mathcal{EA}^{\mathcal{J}_k(\mathcal{G})}(M;\Gamma)$.*

Then there is $v \in \mathcal{EA}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma)$ such that

$$(\beta^*P)\,(u+v) \in \mathcal{EA}^{\mathcal{J}_{(k+1)}(\mathcal{G})}(M;\Gamma)\,. \tag{5.11}$$

Proof. Let $v \in \mathcal{EA}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma)$. Denote $F := (\beta^*P)u$. Since $F \in \mathcal{EA}^{\mathcal{J}_k(\mathcal{G})}(M;\Gamma)$, we have

$$\mathrm{LP}[(\beta^*P)\,(u+v)] = \mathrm{LP}_k\,[F + (\beta^*P)v] \in \mathcal{S}^{\mathcal{J}_k}(\partial M;\Gamma)\,.$$

By Lemma 5.3.19 we have $[F + (\beta^*P)v] \in \mathcal{EA}^{\mathcal{J}_{k+1}(\mathcal{G})}(M;\Gamma)$ if and only if $\mathrm{LP}_k([F + (\beta^*P)v]) = 0$ which, again by Lemma 5.3.19, is equivalent to v solving the equation

$$T_{H,\mathcal{L},\zeta}\left(\mathrm{LP}_k(v)_{H,\mathcal{L},\zeta}\right) = -\mathrm{LP}_k(F)_{H,\mathcal{L},\zeta}\,, \tag{5.12}$$

for all H, $\mathcal{L}$ and ζ with $\delta = \delta(\mathcal{L})$. For each arc H the transport operator of the equation $T_{H,\mathcal{L},\zeta}b_{H,\mathcal{L},\zeta} = -f_{H,\mathcal{L},\zeta}$ is elliptic in the interior of H since M is resolved, ζ is a simple solution and β^*P is $\delta(\mathcal{L})$-separated at H. Thus, there are solutions $b_{H,\mathcal{L},\zeta}$ which are polyhomogeneous at the corner $H \cap H'$ for all adjacent hypersurfaces with respect to $\mathcal{J}_k(\mathcal{G};H',\mathcal{L}',\zeta')$ for their respective successors $\mathcal{L}'$ and ζ'. By Lemma 5.3.19 we have

$$(\beta^*P)u = \sum_{\mathcal{L}}\sum_{\zeta} e^{\varphi_{\mathcal{L},\zeta}/h^\delta}\,T_{H,\mathcal{L},\zeta}A_{H,\mathcal{L},\zeta} + \text{h.o.t.}$$

and $(\beta^*Pu) \in \mathcal{EA}^{\mathcal{J}_k(\mathcal{G})}(M;\Gamma)$. Applying Lemma 5.3.16 to the family $(b_{H,\mathcal{L},\zeta}h^{\gamma_{H,\mathcal{L},\zeta}})_{H,\mathcal{L},\zeta}$ yields a function $v \in \mathcal{EA}^{\mathcal{I}_k(\mathcal{G})}(M;\Gamma)$ satisfying (5.12) at every boundary hypersurface H, which is equivalent to $(\beta^*P)\,(u+v) \in \mathcal{EA}^{\mathcal{J}_{(k+1)}(\mathcal{G})}(M;\Gamma)$. □

5.3.4 Construction

Finally, we are able to prove the existence of quasimodes for any semi-classical operator on a resolved manifold M. The theorem is stated in a way that it shows the existence of a quasimode u on M coinciding with the regular solution constructed in Theorem 3.3.11 on the initial regular interval $I \subset \mathbb{H}_{\mathrm{reg}}$. Note that this is equivalent to an initial value problem which is unique up to a factor $A(h)$, since $[P, A(h)] = 0$.

The proof itself is relatively short since most of the steps involved in the construction of quasimodes have been sourced out into Theorem 5.2.18 and Lemmas 5.3.19-5.3.21.

Theorem 5.3.22 (Existence of Quasimodes)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L}_0 \subset \partial_{\text{-}}\mathcal{P}(\Lambda)$ be an edge and*

(i) A regular interval $I \subset \partial\mathbb{H}_{reg}$ and a simple solution ζ_0 of $E_{\delta(\mathcal{L}_0)}(P)(\cdot, \zeta_0) = 0$ on I, if $\delta(\mathcal{L}_0) > 0$, or

(ii) A function $a_0 \in \ker T_0$, if $\delta(\mathcal{L}_0) = 0$.

In either case, let $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $\Gamma = \Gamma(\mathcal{G})$ be the associated eikonal variety and assume that $m_{\zeta}(\mathcal{L}) = 1$ for all $(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$ of order one with $\delta(\mathcal{L}) > 0$. Assume that $\beta^* P$ is strictly $\delta(\mathcal{L})$-separated for every $(Y, H, \mathcal{L}, \zeta) \in \mathcal{V}$. Additionally, if $\delta(\mathcal{L}_0) > 0$, let φ_0 be a solution of $\varphi_0' = \zeta_0$ and $a_0 \in \ker T_{\partial\mathbb{H}, \mathcal{L}_0, \zeta_0}$ be a solution of the homogeneous transport equation on I.

Then there is a quasimode $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M; \Gamma)$ such that $(\beta^* P)u \in \mathcal{E}\mathcal{A}^{\emptyset}(M; \Gamma)$ and

(i) $u = e^{\varphi_0/h^{\delta(\mathcal{L}_0)}}(a_0 + o(1))$ at I, if $\delta(\mathcal{L}_0) > 0$, or

(ii) $u = a_0 + o(1)$ at I, if $\delta(\mathcal{L}_0) = 0$.

Proof. In either case let $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Applying Lemma 5.3.20 yields a unique, initial quasimode $u_0 \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M; \Gamma)$ with either

(i) $u_0 = e^{\varphi_0/h^{\delta(\mathcal{L}_0)}} a_0$ at I satisfying (5.10), in (i), or

(ii) $u_0 = a_0$, where $a_0 \in \ker T_0$, in (ii).

In particular, we have $(\beta^* P)u_0 \in \mathcal{E}\mathcal{A}^{\mathcal{J}_+(\mathcal{G})}(M; \Gamma)$ and can apply Lemma 5.3.21 iteratively. Thus, for every $k \in \mathbb{N}$ we to obtain improved quasimodes $u_k \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M; \Gamma)$ with $(\beta^* P)u_k \in \mathcal{E}\mathcal{A}^{\mathcal{J}_k(\mathcal{G})}(M; \Gamma)$ and

(i) $u_k = e^{\varphi_0/h^{\delta(\mathcal{L}_0)}} a_0 + o(1)$ at I, in (i), or

(ii) $u_k = a_0 + o(1)$ at I, in (ii).

Since $\mathcal{J}(\mathcal{G})$ is discrete for every $(H, \mathcal{L}, \zeta)$, there is a quasimode $u \in \mathcal{E}\mathcal{A}^{\mathcal{I}(\mathcal{G})}(M; \Gamma)$ such that $(\beta^* P)u \in \mathcal{E}\mathcal{A}^{\emptyset}(M; \Gamma)$ with leading part $e^{\varphi_0/h^{\delta}} a_0$ or a_0 in either case, respectively. □

Remark 5.3.23. The statement that the quasimode u satisfies $(\beta^* P)u \in \mathcal{E}\mathcal{A}^{\emptyset}(M; \Gamma)$ is stronger than requiring $(\beta^* P)u = O(h^{\infty})$ at all boundary hypersurfaces. The latter condition is already true for all functions $\beta^* v$, where $v \in \mathcal{A}^{\emptyset}(\mathbb{H})$, for instance $v(x, h) := e^{-1/h}x$. The statement thus requires the quasimode's remainder $(\beta^* P)u$ to vanish faster than u multiplied by any polyhomogeneous function $\beta^* f$ for any $f \in \mathcal{A}^J(\mathbb{H})$ with non-trivial index set J.

Remark 5.3.24. These quasimodes are highly non-unique. Since $[P,h]=0$, it holds that the product of a function $f=f(h)$ and any quasimode u is a quasimode as well.

Corollary 5.3.25. *Let $P\in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator and let $m\in\mathbb{N}$ be the order of P. Let $L_0\in\mathbb{N}_0$, let $\mathcal{L}_0\subset\partial_{-}\mathcal{P}(\Lambda)$ be the horizontal edge with $|\mathcal{L}_0|=L_0$ and let $a_{1,0},\dots,a_{L_0,0}$ be a basis of $\ker T_{\delta(\mathcal{L}_0)}$. For each unique tuple $(\mathcal{L}_0,a_{j,0})$, $j=1,\dots L_0$, and $(\mathcal{L}_j,\zeta_j)$, with $\delta(\mathcal{L}_j)>0$ and ζ_j solving $E_{\delta(\mathcal{L}_j)}(P)(\cdot,\zeta_j)=0$, for $j=L_0+1,\dots,m$, let $\mathcal{G}_{(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)}$ be the resolution tree and*

$$\beta_{(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)}\colon M_{(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)}\to\mathbb{H}$$

be the resolution space. Assume that for all pairs $(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)$ and for all leaves $(Y,H,\mathcal{L}_H,\zeta_H)\in\mathcal{V}_{(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)}$ of order one with $\delta(\mathcal{L}_H)>0$ the solution ζ_H of $E_{H,\delta(\mathcal{L}_H)}(P)(\cdot,\zeta_H)=0$ is simple and $\beta^*_{(\mathcal{L}_0,a_{j,0})/(\mathcal{L}_j,\zeta_j)}P$ is strictly $\delta(\mathcal{L})$-separated.

Then there are independent, exponential-polyhomogeneous quasimodes $u_1,\dots,u_m$, satisfying

$$\left(\beta^*_{(\mathcal{L}_0,a_{j,0})}P\right)u_j\in\mathcal{E}\mathcal{A}^{\emptyset}(M_{(\mathcal{L}_0,a_{j,0})},\Gamma_{(\mathcal{L}_0,a_{j,0})})\,,$$

with $u_j=a_{j,0}+o(1)$ at $\partial\mathbb{H}_r$, for $j=1,\dots,L_0$, and

$$\left(\beta^*_{(\mathcal{L}_j,\zeta_j)}P\right)u_j\in\mathcal{E}\mathcal{A}^{\emptyset}(M_{(\mathcal{L}_j,\zeta_j)},\Gamma_{(\mathcal{L}_j,\zeta_j)})\,,$$

with $u_j=e^{\varphi_j/h^{\delta(\mathcal{L}_j)}}A_j$ at $\partial\mathbb{H}_r$, where $\varphi_j'=\zeta_j$, for $j=L_0+1,\dots,m$.

Note that $\mathcal{L}_i=\mathcal{L}_j$ is not excluded for $0<i,j\le m$, $i\neq j$. However, if there are $i,j>0$ with $i\neq j$ and $\mathcal{L}_i=\mathcal{L}_j$, then $\zeta_i\neq\zeta_j$.

Proof. This is a direct application of Theorem 5.3.22. For any tuple

$$(\mathcal{L}_0,a_{1,0}),\dots,(\mathcal{L}_0,a_{L_0,0}),(\mathcal{L}_{L_0+1},\zeta_{L_0+1}),\dots,(\mathcal{L}_m,\zeta_m)$$

the application of this theorem yields a quasimode u_j with

(i) $u_j=a_{j,0}+o(1)$ at $\partial\mathbb{H}_r$, for $j=1,\dots,L_0$, and

(ii) $u_j=e^{\varphi_j/h^{\delta(\mathcal{L}_j)}}\left(a_j+o(1)\right)$ at $\partial\mathbb{H}_r$, for $j=L_0+1,\dots,m$, where $\varphi_j'=\zeta_j$.

Since $(a_{1,0},\dots,a_{L_0,0})$ is a basis of T_0 on $\partial\mathbb{H}_r$ and $(\mathcal{L}_j,\zeta_j)$ are unique for $j=L_0+1,\dots,m$, the collection $(u_1,\dots,u_m)$ is an independent basis of quasimodes. In addition, these quasimodes satisfy

$$\left(\beta^*_{(\mathcal{L}_0,a_{j,0})}P\right)u_j\in\mathcal{E}\mathcal{A}^{\emptyset}(M_{(\mathcal{L}_0,a_{j,0})},\Gamma_{(\mathcal{L}_0,a_{j,0})})\,,$$

at $\partial\mathbb{H}_r$, for $j=1,\dots,L_0$, and

$$\left(\beta^*_{(\mathcal{L}_j,\zeta_j)}P\right)u_j\in\mathcal{E}\mathcal{A}^{\emptyset}(M_{(\mathcal{L}_j,\zeta_j)},\Gamma_{(\mathcal{L}_j,\zeta_j)})\,,$$

at $\partial\mathbb{H}_r$, where $\varphi'_j = \zeta_j$, for $j = L_0 + 1, \ldots, m$. □

Separation

In Theorem 5.3.22 we have excluded operators P with resolution $\beta\colon M \to \mathbb{H}$ whose pullback $\beta^* P$ to M is not separated at every arc $H \subset \partial M$. To deal with this phenomenon, we can lift the result of Proposition 3.3.8 to $\beta^* P$ on the resolution space M for each arc H individually.

Proposition 5.3.26 (Resolution of Non-Separation)**.** *Let $P \in \mathit{Diff}^{\Lambda}(\mathbb{R})$ be a semi-classical operator, $\mathcal{L} \subset \partial\mathcal{P}(\Lambda)$ be an edge, $I \subset \partial\mathbb{H}_{reg}$ be a regular interval, ζ be a simple solution of $E_{\delta(\mathcal{L})}(P)(\cdot, \zeta) = 0$ on I and $\beta\colon M \to \mathbb{H}$ be the associated resolution space with resolution tree $\mathcal{G}$. Let $(Y, H, \mathcal{L}_H, \zeta_H) \in \mathcal{V}$ be a leaf of order one and $\varphi_{H,\mathcal{L}_H,\zeta_H}$ be a solution of $\partial_{x_H}\varphi = \zeta_H$.*

Then there is $N \in \mathbb{N}$, subphases $\psi_{H,\mathcal{L}_H,\zeta,j} \in C^\infty(H)$ and $0 < \varepsilon_j < \delta(\mathcal{L})$, for $j = 1, \ldots, N$, such that for the function

$$\Phi := \frac{\varphi_{H,\mathcal{L}_H,\zeta_H}}{h^{\delta(\mathcal{L})}} + \sum_{j=1}^{N} \frac{\psi_{H,\mathcal{L}_H,\zeta,j}}{h^{\varepsilon_j}} .$$

the conjugated operator $(\beta^* P)_\Phi$ is separated at H.

Proof. This is the same proof as in Proposition 3.3.8 in induced, local coordinates (x_H, h) at H. □

Remark 5.3.27. The reason why we have excluded non-separated operators from Theorem 5.3.22 is because we did not show that these full phase functions Φ in Proposition 5.3.26 match at the corner of two adjacent arcs H and H'. However, we expect that these phase functions match pairwise and that a proof involves a thorough analysis of the points in the localized set of exponents at $p \in H \cap H'$ and their correspondence to the subphases at both arcs. This can be very cumbersome, so another approach may be preferable.

5.3.5 Further Applications: Vector Bundles

A generalization where we want to apply the geometric resolution are semi-classical operators on a vector bundle. We analyze the case of intersecting eigenbands. Let $V \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ be self-adjoint for all $x \in \mathbb{R}$ and denote the *Schrödinger operator* on vector bundles by

$$P := -h^2 \partial_x^2 + V .$$

Let $\lambda_1, \lambda_2 \in C^\infty(\mathbb{R})$ be the eigenbands of V, i.e. for all $x \in \mathbb{R}$ and $j = 1, 2$ we have

$$\det\big(V(x) - \lambda_j(x) I\big) = 0 .$$

Assume that these eigenbands cross transversally in $x = 0$, i.e. $\lambda_1(0) = \lambda_2(0) = 0$ and that $\lambda'_1(0) < 0 < \lambda'_2(0)$. Let $I := \mathbb{R}_{>0}$ and $v_1, v_2 \in C^\infty(I, \mathbb{R}^2)$ be an orthonormal eigenframe with respect to V, i.e. for each $x > 0$ and $j = 1, 2$ we have $\|v_j(x)\|_2 = 1$,

$$\big(V(x) - \lambda_j(x)I\big)\, v_j(x) = 0$$

and $\langle v_1(x), v_2(x)\rangle = 0$. Note that $\langle v_j, v_j'\rangle = 0$, since (v_1, v_2) is an orthonormal frame. We want to construct WKB-type quasimodes of the form

$$u_j = e^{\varphi_j/h} \sum_{k=0}^{\infty} w_{k,j} h^k$$

for P and $j = 1, 2$, where $w_{k,j} \in C^\infty(I, \mathbb{R}^2)$. Expanding the remainder of Pu_1 then yields

$$h^0 \left(-(\varphi_1')^2 I + V\right) w_{0,1} + h \left((-2\varphi'\partial_x - \varphi'' I)\, w_{0,1} + \left(-(\varphi_1')^2 I + V\right) w_{1,1}\right) + O\left(h^2\right) .$$

Thus, the WKB function u_1 can only be a quasimode if the lowest order term in the expansion vanishes, i.e.

$$\left(-(\varphi_1')^2 I + V\right) w_{0,1} = 0 .$$

A necessary condition for the solvability of the equation is that the determinant of the operator

$$\det\left(-(\varphi_1')^2 I + V\right)$$

vanishes, for all $x \in I$. This determinant vanishes, if and only if $-(\varphi_1')^2$ coincides with one of the eigenbands. We choose

$$\varphi_1 := \int \sqrt{\lambda_1} dx .$$

Ultimately, the lowest order term in the expansion of Pu_1

$$(-\lambda_1 I + V)\, w_{0,1}$$

vanishes, if and only if $w_{0,1}(x) \in \ker\left(-\lambda_1(x) I + V(x)\right)$ for all $x \in I$, i.e.

$$w_{0,1} = a_0 \cdot v_1 ,$$

for any $a_0 \in C^\infty(I, \mathbb{R}^2)$. Hence we still need to determine a_0. The next order term with coefficient h in the expansion of Pu_1 then can be written as

$$\left(-2\sqrt{\lambda_1}\partial_x - \sqrt{\lambda_1}' I\right)(a_0 \cdot v_1) + (-\lambda_1 I + V)\, w_{1,1} .$$

Denote $T := -2\sqrt{\lambda_1}\partial_x - \sqrt{\lambda_1}'$. Requiring the next order term to vanish induces different equations on each subbundle

$$\begin{cases} Ta_0 & = 0 \\ (-\lambda_1 I + V)\, w_{1,1} & = -2\sqrt{\lambda_1} a_0 \langle v_1', v_2 \rangle v_2 \end{cases} .$$

Thus, the second set of equations fully determines the first term $w_{0,1}$ in the expansion of u_1 by the choice of $a_0 \in \ker T$. The latter equation does not determine $w_{1,1}$ entirely, since $\ker(\lambda_1 I - V)$ coincides with the eigenbundle spanned by v_1. Denote $w_{1,1} = a_1 v_1 + b_1 v_2$ for $a_1, b_1 \in C^\infty(I)$. Then the latter equation can be written as

$$(\lambda_2 - \lambda_1)\, b_1 v_2 = -2\sqrt{\lambda_1} a_0 \langle v_1', v_2 \rangle v_2 .$$

This is a trivial equation without any differentiation and the solution is given by

$$b_1 = \frac{-2\sqrt{\lambda_1} a_0 \langle v_1', v_2 \rangle}{\lambda_1 - \lambda_2} . \tag{5.13}$$

This solution justifies the reduction of the base of the vector bundle from $\mathbb{R}$ to $I = \mathbb{R}_{>0}$ in the case of crossing eigenbands λ_1, λ_2, leading to a singular behavior of u_1 at $x = 0$, if $v_1' \neq 0$. Denoting all terms in the expansion of u_1 by

$$w_{k,1} = a_k v_1 + b_k v_2 ,$$

the following set of equations is given by

$$\begin{cases} Ta_1 & = a_0'' + 2a_0 \langle v_1'', v_1 \rangle + 2\sqrt{\lambda_1} b_1 \langle v_2', v_1 \rangle \\ (\lambda_2 - \lambda_1)\, b_2 & = 2(\sqrt{\lambda_1} a_1 + a_0') \langle v_1', v_2 \rangle + a_0 \langle v_1'', v_2 \rangle + T b_1 \end{cases} .$$

Thus, the singular behavior of b_1 in (5.13) transfers from sections of the second eigenbundle, spanned by v_2, to the first eigenbundle if these are non-constant. Since T is a b-operator at $x = 0$, the dominant contribution for the asymptotic behavior of b_2 at $x = 0$ is given by $a_1/(\lambda_1 - \lambda_2)$. Thus, their asymptotic behavior as $x \to 0$ worsens as k increases by powers of $-3k/2$, indicating a relevant scale of $h/x^{3/2}$, as for the standard Schrödinger operator with linear potential.

Expanding $V(x) = \sum_{j=1}^\infty V_j x^j$ and $v_i(x) = \sum_{j=0}^\infty v_{i,j} x^j$ at $x = 0$ for $i = 1, 2$, the quasihomogeneous blow-up

$$\beta : [\mathbb{H}, 0]_{2/3} \to \mathbb{H}$$

then yields an asymptotic series

$$\beta^* \left(\left(-h^2 \partial_x^2 + V(x) \right) v_i(x) \right) = h^{2/3} \left(-\partial_\xi^2 + \xi V_1 \right) v_{i,0} + O\left(h^{4/3} \right) ,$$

where $V_1 v_{i,0} = \lambda_i'(0) v_{i,0}$ for $i = 1, 2$. Since $\lambda_1'(0) < 0 < \lambda_2'(0)$, we have two initial, independent quasimodes

$$\widetilde{w}_{0,1} = \mathrm{Ai}(\xi) v_{1,0} \quad \text{and} \quad \widetilde{w}_{0,2} = \mathrm{Ai}(-\xi) v_{2,0}$$

on the front face $\beta^{-1}(0)$. The different signs of ξ in the Airy function correspond to the dichotomy of signs of λ_1 and λ_2 on both sides of $x = 0$. Solving all higher order corrections, one obtains a quasimode at the front face.

Note that one needs both $\widetilde{w}_{0,1}$ and $\widetilde{w}_{0,2}$ with different powers of h to match the asymptotic behavior of u_j as $x \to 0$. This is due to the different behavior of $a_0(x) \sim x^{1/4}$ and $b_0(x) = 0$, which transfers to an offset in powers of $a_1(x) \sim x^{1/4-3/2}$ and $b_1(x) \sim x^{1/4-2}$.

Bibliography

[BB56] C. Briot and J. Bouquet. “Propriétés des fonctions définies par des équations différentielles”. In: *Journal de l’Ecole Polytechnique* 36 (1856), pp. 133–198.

[Beh21] Malte Behr. *Quasihomogeneous blow-ups and pseudodifferential calculus on SL(n,R)*. eng. PhD Thesis. Oldenburg, 2021.

[BM72] M. V. Berry and K. E. Mount. “Semiclassical approximations in wave mechanics”. In: *Rep. Prog. Phys.* 35 (1972), pp. 315–397.

[BO99] Carl M. Bender and Steven A. Orszag. *Asymptotic Methods and Perturbation Theory*. Advanced Mathematical Methods for Scientists and Engineers. Springer, 1999.

[Bri26] Léon Brillouin. “La mécanique ondulatoire de Schrödinger; une méthode générale de resolution par approximations successives”. In: *Compt. Rend. Hebd. Seances Acad. Sci.* 183.1 (1926), pp. 24–26.

[Can05] José Cano. “The Newton Polygon Method for Differential Equations”. In: *Computer Algebra and Geometric Algebra with Applications*. Ed. by P. Olver H. Li and G. Sommer. Vol. 3519. Lecture Notes in Computer Science. Springer, 2005, pp. 18–30.

[DJ97] J. Della Dora and F. Jung. “About the Newton Polygon Algorithm for Non Linear Ordinary Differential Equations”. In: *Proceedings of the 1997 ISSAC* (1997), pp. 298–304.

[Fed93] Mikhail V. Fedoryuk. *Asymptotic Analysis: Linear Ordinary Differential Equations*. eng. Springer, 1993.

[Fin89] H. Fine. “On the functions defined by differential equations, with an extension of the Puiseux Polygon construction to these equations”. In: *Amer. Jour. of Math* XI (1889), pp. 317–328.

[Gri01] Daniel Grieser. “Basics of the b-calculus”. In: *Approaches to Singular Analysis*. Ed. by J.B.Gil et al. Vol. 125. Operator Theory: Advances and Applications. Advances in Partial Differential Equations. Basel: Birkhäuser, 2001, pp. 30–84.

[Gri17] Daniel Grieser. “Scales, blow-up and quasimode constructions”. In: *Geometric and Computational Spectral Theory*. Ed. by A. Girouard et al. Vol. 700. Contemp. Math. AMS, 2017, pp. 207–266.

[GS90] Victor Guillemin and Shlomo Sternberg. *Geometric Asymptotics*. eng. Mathematical Surveys and Monographs 14. AMS, 1990.

[GS91] D. Y. Grigoriev and M. Singer. "Solving Ordinary Differential Equations in Terms of Series with Real Exponents". In: *Transactions of the AMS* 327 (1991), pp. 329–351.

[Hel80] Bernard Helffer. *Semi-Classical Analysis for the Schrödinger Operator and Applications*. Lecture Notes in Mathematics. New York, Berlin, Heidelberg: Springer-Verlag, 1980.

[Hol13] M. H. Holmes. *Introduction to Perturbation Methods*. Springer New York, 2013. DOI: 10.1007/978-1-4614-5477-9.

[Hör03] Lars Hörmander. *Distribution Theory and Fourier analysis*. eng. Reprint of the 2. ed. 1990. Classics in mathematics. Springer, 2003.

[Joy12] Dominic Joyce. "On manifolds with corners". In: *Advances in Geometric Analysis*. Ed. by P. Olver H. Li and G. Sommer. Vol. 21. Advanced Lectures in Mathematics. International Press, 2012, pp. 225–258.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators*. Springer Berlin Heidelberg, 1995. DOI: 10.1007/978-3-642-66282-9.

[KC96] J. Kevorkian and J. D. Cole. *Multiple Scale and Singular Perturbation Methods*. Springer New York, 1996. DOI: 10.1007/978-1-4612-3968-0.

[KM15] Chris Kottke and Richard B. Melrose. "Generalized Blow-Up of Corners and Fiber Products". In: *Transactions of the AMS* 367 (2015), pp. 651–705.

[Kol07] János Kollár. *Lectures on Resolution of Singularities*. Annals of Mathematics Studies 166. Princeton, Oxford: Princeton University Press, 2007.

[KP13] Steven G. Krantz and Harold R. Parks. *The Implicit Function Theorem: History, Theory, and Applications*. eng. Modern Birkhäuser Classics. Birkhäuser, 2013.

[Kra26] H. A. Kramers. "Wellenmechanik und halbzahlige Quantisierung". ger. In: *The European physical journal. A, Hadrons and nuclei* 39.10-11 (1926), pp. 828–840.

[KS22] K. Uldall Kristiansen and P. Szmolyan. *A dynamical systems approach to WKB-methods: The simple turning point*. 2022. arXiv: 2207.00252 [math.DS].

[Lam14] Jonas Lampart. *The adiabatic limit of Schrödinger operators on fibre bundles*. eng. PhD Thesis. Tübingen, 2014.

[LR13] Matthias Ludewig and Elke Rosenberger. "Asymptotic Eigenfunctions for Schrödinger Operators on a Vector Bundle". In: *Reviews in Mathematical Physics* 32 (Sept. 2013).

[LR60] C. C. Lin and A. L. Rabenstein. "On the Asymptotic Solutions of a Class of Ordinary Differential Equations of the Fourth Order: I. Existence of Regular Formal Solutions". In: *Transactions of the American Mathematical Society* 94.1 (1960), pp. 24–57.

[LT17] Jonas Lampart and Stefan Teufel. "The adiabatic limit of Schrödinger operators on fibre bundles". In: *Mathematische Annalen* 367 (2017), pp. 1647–1683.

[Mel96] R. B. Melrose. *Differential Analysis on Manifolds with Corners*. Sept. 11, 1996. URL: http://www-math.mit.edu/~rbm/book.html.

[MS00] R. Schäfke M. Canalis-Durand J.P. Ramis and Y. Sibuya. "Gevrey solutions of singularly perturbed differential equations". In: *Journal for die reine und angewandte Mathematik* 518 (2000), pp. 95–129.

[New60] Isaac Newton. *The correspondance of Isaac Newton*. Vol. 2. New York: Cambridge Univ. Press, 1960.

[Nik23] Nikita Nikolaev. "Existence and uniqueness of exact WKB solutions for second-order singularly perturbed linear ODEs". English. In: *Communications in Mathematical Physics* 400.1 (2023), pp. 463–517.

[Nol03] Wolfgang Nolting. *Grundkurs Theoretische Physik 5/2: Quantenmechanik - Methoden und Anwendungen*. ger. 8. Aufl. Springer-Lehrbuch. Berlin Heidelberg: Springer Spektrum, 2003.

[Olv97] Frank W. J. Olver. *Asymptotics and Special Functions*. AKP classics. Wellesley: A K Peters, 1997.

[She22] David A. Sher. *Joint asymptotic expansions for Bessel functions*. 2022. URL: https://arxiv.org/abs/2203.06329.

[Sib00] Yasutaka Sibuya. "The Gevrey Asymptotics in the Case of Singular Perturbations". In: *Journal of Differential Equations* 165.2 (2000), pp. 255–314.

[Sob18] Dennis Sobotta. *Quasimode Construction in the Presence of Turning Points via Geometric Resolution Analysis*. eng. Master's Thesis. Oldenburg, 2018.

[Teu03] Stefan Teufel. *Adiabatic perturbation theory in quantum mechanics*. Vol. 1821. Lecture Notes in Mathematics. Springer, 2003.

[Vor99] André Voros. "Airy function - exact WKB results for potentials of odd degree". In: *Journal of Physics A: Mathematical and General* 32.7 (1999), p. 1301.

[Was85] Wolgang Richard Wasow. *Linear Turning Point Theory*. Applied Mathematical Sciences 54. New York, Berlin, Heidelberg: Springer-Verlag, 1985.

[Was87] Wolgang Richard Wasow. *Asymptotic Expansions for Ordinary Differential Equations*. New York: Dover Publications, Inc., 1987.

[Wen26] Gregor Wentzel. "Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik". ger. In: *The European physical journal. A, Hadrons and nuclei* 38.6-7 (1926), pp. 518–529.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*. eng. [corr. and] updated 7. print. 2007. Vol. 152. Graduate texts in mathematics. Springer, 1995.

[Zwo12] Maciej Zworski. *Semiclassical Analysis*. eng. Graduate Studies in Mathematics Volume 138. AMS, 2012.

Index

Symbols

Declaration

Hereby, I declare that the presented dissertation with the title

Geometric Resolution of Generalized Semi-Classical Operators

is my own work and that I have not used any other sources except the sources stated in the text or in the bibliography. The dissertation has, neither as a whole, nor in part, been submitted for assessment in a doctoral procedure at another university.

I confirm that I am aware of the guidelines of good scientific practice of the Carl von Ossietzky University Oldenburg and that I have observed them. Furthermore, I declare that I have not availed myself of any commercial placement or consulting services in connection with my doctoral procedure.

Oldenburg, February 12, 2024

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