L^2-equivariant index theorem for the Atiyah-Singer-Dirac operator on globally hyperbolic spacetimes

Von der Fakultät für Mathematik und Naturwissenschaften

der Carl von Ossietzky Universität Oldenburg

zur Erlangung des Grades und Titels

Doktor der Naturwissenschaften (Dr. rer. nat)

angenommene Dissertation von

Herrn Orville Damaschke

geboren am 9.August 1992 in Delmenhorst

Oldenburg, 2023

Gutachter: Prof. Boris Vertman
Institut für Mathematik
Carl von Ossietzky Universität Oldenburg

Prof. Alexander Strohmaier
Department of pure Mathematics - School of Mathematics
University of Leeds

Tag der Disputation: 17. Juli 2023

Abstract

We consider the Lorentzian Atiyah-Singer-Dirac operator on a temporal compact, globally hyperbolic spin manifold of even dimension with non-compact Cauchy hypersurface which induces two boundary hypersurfaces due to the compactness of the time domain. We specify the non-compactness of the Cauchy hypersurface by considering it as a Galois covering with closed base manifold. The Dirac operator of interest can be viewed as lift of the same Dirac operator on the base manifold. We equip the lifted Dirac operator with (generalised) Atiyah-Patodi-Singer boundary conditions and show Fredholmness in this setting, or more precisely, Breuer-/L^2-Fredholmness of the lifted Dirac operator under the mentioned boundary conditions. The presented method of proof is based on the descriptions and results for compact hypersurfaces, provided by Christian Bär and Alexander Strohmaier for ordinary Atiyah-Patodi-Singer boundary conditions; we also consider generalised Atiyah-Patodi-Singer boundary conditions for which we rely on results from Christian Bär and Sebastian Hannes for the compact setting.

The starting point of the presented derivation is the well-posedness of the initial value problem for the Lorentzian Atiyah-Singer-Dirac operator on globally hyperbolic spacetimes with initial values in certain Sobolev spaces and inhomogeneities which are (locally) square-integrable in time and of (local) Sobolev degree in space. The well-posedness of the associated homogeneous problem implies the existence of an evolution operator which acts unitarily between L^2-spaces in the space domain. It turns out to be a Fourier integral operator. The L^2-spaces of spinors on the boundary hypersurfaces are decomposed into two eigenspaces of the corresponding hypersurface Dirac operator of the lifted Dirac operator such that they are appropriate with the boundary conditions. The spectral decomposition will be applied to the evolution operator which splits into four spectral evolution operators, mapping a spectral subspace from one boundary hypersurface to another spectral subspace of the other boundary hypersurface. The unitarity of the evolution operator and its regularity as Fourier integral operator show that some of the spectral evolution operators are Breuer-Fredholm. With further functional-analytic methods it follows from this observation that the Dirac operator of interest is Breuer-Fredholm as well. The algebraic index is expressed by the spectral flow of the hypersurface Dirac operators, varying in the time parameter. The geometric index follows by comparing with the index of the Riemannian Dirac-operator.

We prove everything in our setting in the language of von Neumann algebras with respect to Galois coverings where the L^2- and Sobolev spaces are (free) Hilbert modules with unitary left representations of a discrete group action. In passing, we take a closer look on Seeley's theorem and the spectral flow which are two main ingredients of the proof.

Zusammenfassung

Wir betrachten den Lorentzschen Atiyah-Singer-Dirac-Operator auf Zeit-kompakten global hyperbolischen Spin Mannigfaltigkeiten gerader Dimension mit nicht kompakter, raumartiger Cauchy-Hyperfläche, welche aufgrund der Kompaktheit der Zeit-Domäne zwei raumartige Randhyperflächen impliziert. Insbesondere betrachten wir die Cauchy-Hyperfläche als Galois-Überlagerung mit geschlossener Basismannigfaltigkeit. Der zu betrachtende Dirac Operator entspricht dem gleichen Dirac Operator auf der Basismannigfaltigkeit, welcher auf die Überlagerung geliftet wurde. Wir beweisen, dass dieser Lift unter (verallgemeinerten) Atiyah-Patodi-Singer Randbedingungen Breuer-Fredholm ist. Die hier vorgestellte Beweisstrategie basiert auf den Ausführungen und Ergebnissen für kompakte Hyperflächen von Christian Bär und Alexander Strohmaier für gewöhnliche Atiyah-Patodi-Singer Randbedingungen; der Fall verallgemeinerter Atiyah-Patodi-Singer Randbedingungen beruht auf Ergebnissen von Christian Bär und Sebastian Hannes für den Fall kompakter Cauchy-Hyperflächen.

Der Ausgangspunkt der hier vorgestellten Herleitung ist die Wohlgestelltheit des Anfangswertproblems für den Lorentzschen Atiyah-Singer-Dirac Operator auf global hyperbolischen Raumzeiten mit Anfangswerten in gewissen Sobolev-Räumen und Inhomogenitäten, die (lokal) quadrat-integrabel in der Zeit und von (lokaler) Sobolev-Ordnung in den Raumkoordinaten sind. Die Wohlgestelltheit des dazugehörigen homogenen Problems impliziert einen Evolutionsoperator, der unitär zwischen L^2-Räume in den Raumkoordinaten agiert. Es stellt sich heraus, dass dieser ein Fourier-Integraloperator ist. Die L^2-Räume bezüglich Spinore auf den Randhyperflächen werden passend zu den gewählten Randbedingungen in zwei Eigenräume des Rand-Dirac Operators zerlegt. Diese Zerlegung impliziert eine Zerlegung des Evolutionsoperators in spektrale Evolutionsoperatoren, welche zwischen den Eigenräume der Rand-Hyperflächen abbilden. Da der Evolutionsoperator unitär und als Fourier-Integraloperator aufgefasst werden kann, sind einige der spektralen Evolutionsoperatoren Breuer-Fredholm. Hieraus folgt durch funktional-analytische Methoden, dass der zu untersuchende Dirac-Operator ebenso Breuer-Fredholm ist. Der algebraische Index wird durch den spektralen Fluss der Hyperflächen-Dirac-Operatoren ausgedrückt, welche durch die Zeit parametrisiert sind. Der geometrische Index folgt durch einen Vergleich mit dem Index des Riemannschen Dirac-Operators.

In unserem Setting beweisen wir die Resultate in der Sprache von von-Neumann-Algebren bezüglich Galois-Überlagerungen, in der die L^2- und Sobolev-Räume als (freie) Hilbert-Module mit unitärer Darstellung der Linkswirkung einer diskreten Gruppe auftreten. Insbesondere betrachten wir das Theorem von Seeley und den spektralen Fluss unter diesen Modifikationen als zwei wichtige Schlüsselelemente in der Beweisführung.

Acknowledgements

I first want to thank Alexander Strohmaier and Christian Bär for providing their works as base for several conceivable generalisations of their result and in particular I thank Alexander Strohmaier for suggesting the here presented generalisation to me as my PhD-thesis and for some feedback during my research.

Secondly, I am very grateful to Boris Vertman who supervised me and my research with helpful feedback, ideas, and ulterior remarks as well as advices within and besides this thesis. I also want to thank Daniel Grieser and Jørgen Olsen Lye for helpful discussions, suggestions and clarifications. I also would like to thank all collegues of the IfM in Oldenburg, I spent my time with during my appointment, for discussions and necessary distractions within and besides my research.

Furthermore, I want to thank all my beloved friends and companians as well as my family for their support and indulgence, especially during the last months of finishing this thesis.

Contents

List of Figures

Part I.

Introduction

1. Background, structure and results of this thesis

Before we introduce the structure of the thesis and our main results, we will review the evolution of index theory and the results of Bär and Strohmaier and further so far known extensions on which this thesis is based on. Atiyah and Singer presented in [AS63] the first index theorem which connects the analytic index of an elliptic differential operator on an oriented closed manifold to the topological index of the operator which is defined by the Chern character of the (principal symbol of the) operator and the Todd class of the underlying manifold. The latter index is non-trivial if the dimension of the manifold is even. In a series of papers (see [AIMSII68] and [AS71]) Atiyah, Segal and Singer refined their results and proofs and introduced several extensions, such as the extension to elliptic pseudo-differential operators, to elliptic complexes, and a Lie group equivariant index theorem. All of their proofs rely on K-theoretic methods. In 1973 Atiyah, Bott and Patodi introduced an alternative and more analytic proof method of the index theorem for an elliptic differential operator which is based on the heat equation; see [ABP73] or the textbooks [BGV03] and [Roe99]. Getzler extended this method to elliptic pseudo-differential operators in [Get83]. We want to point out two extensions which become important in this thesis.

[*A*] *Manifolds with boundary*: Suppose E, F are two vector bundles over a compact manifold M with boundary ∂M and $D : C^\infty(M,E) \to C^\infty(M,F)$ a linear first order elliptic differential operator. Atiyah, Patodi and Singer have shown in [APS75a] that this operator is Fredholm under additional conditions next to the required ellipticity:

(A) the neighbourhood of the boundary has product structure, i.e. any neighbourhood of ∂M is of the form $[0,\epsilon) \times \partial M$ with $\epsilon > 0$.

(B) the operator D takes the form

$$D|_{[0,\epsilon)\times\partial M} = \beta\,(\partial_n + A)$$

in the neighbourhood of the boundary where β is a bundle isomorphism from $E|_{\partial M}$ to $F|_{\partial M}$, ∂_n the normal derivative with respect to inwards pointing normal coordinates and A is a self-adjoint elliptic differential operator on ∂M.

(C) in order to have an elliptic boundary problem, one needs to impose boundary conditions: let $P_{\geq 0}$ be the spectral projection of A which corresponds to non-negative eigenvalues. The Atiyah-Patodi-Singer boundary condition is the global boundary condition

$$P_{\geq 0}u|_{\partial M} = 0 \tag{1.1}$$

for all $u \in C^\infty(M,E)$.

For geometric operators like the Dirac operator, (B) can be easily achieved by imposing the equivalent condition that the Riemannian metric near the boundary has product structure:

$$g|_{[0,\epsilon)\times\partial M} = \mathrm{d}n^{\otimes 2} + g_{\partial M}$$

where $g_{\partial M}$ is the induced metric on the boundary and does not depend on the boundary-defining coordinate n. We define $C^\infty_{\mathrm{APS}}(M,E)$ as the space of smooth sections which satisfy the boundary condition (1.1). We write D_{APS} for the operator D with domain $C^\infty_{\mathrm{APS}}(M,E)$. The result can then be phrased as follows.

Theorem 1.0.1 (cf. Theorem 3.10 in [APS75a]). *Let M be a compact manifold with boundary ∂M, E, F vector bundles over M and $D : C^\infty(M,E) \to C^\infty(M,F)$ a linear elliptic first order differential operator such that the above requirements (A) and (B) are satisfied. Then the operator $D_{\mathrm{APS}} : C^\infty_{\mathrm{APS}}(M,E) \to C^\infty(M,F)$ is Fredholm with index*

$$\mathrm{ind}(D) = \int_M a(M) - \frac{\eta(A) + \dim \ker(A)}{2} \tag{1.2}$$

where $a(M)$ is a density-valued local term and $\eta(A)$ the eta-invariant of the boundary operator A.

The concrete definition of the eta-invariant will be given in subsection 8.2.1. Theorem 1.0.1 has been extended to pseudo-differential operators by Piazza in [Pia93]. The local term $a(M)$ coincides with the local term in the boundaryless case. If M is a Riemannian spin manifold, E and F spinor bundles, associated to the two half-spin representations, and D the (Atiyah-Singer-)Dirac operator, Theorem 1.0.1 reduces to the following result.

Corollary 1.0.2 (cf. Theorem 4.2 in [APS75a]). *The Dirac operator is Fredholm with index (1.2) with local term $a(M)$, described by the $\hat{A}$-genus $\hat{\mathbb{A}}$: $a(M) = \hat{\mathbb{A}}(M)$.*

If we twist the Dirac operator with a Hermitian bundle $\mathcal{E}$ over M, the local term becomes $a(M) = \hat{\mathbb{A}}(M) \wedge \mathrm{ch}(\mathcal{E})$ where $\mathrm{ch}(\mathcal{E})$ is the (total) Chern character of $\mathcal{E}$. In both cases, the hypersurface operator is itself a (twisted) Dirac operator which enters in the boundary contribution; see [APS75a, Eq.(4.3)]. The concrete definitions of the $\hat{A}$-genus and the Chern character will be given in subsection 10.3.3.

[*B*] *Extension to non-compact manifolds*: Two main difficulties arise if one considers non-compact manifolds M: first of all, one needs to impose practical growth conditions at infinity, i.e. outside any compact subset of M. In addition, the null-spaces of elliptic differential operators are in general infinite-dimensional. The first difficulty can be resolved by choosing square-integrability with respect to a natural inner product as growth condition which leads to Hilbert spaces. However, this does not resolve the second problem and thus the definition of the index as difference of dimensions becomes meaningless. But one is able to introduce a renormalised, but in general real-valued dimension in a von Neumann algebra if certain geometric as well as analytic conditions are given as follows: let Γ be a discrete and freely acting group on a non-compact manifold M such that the orbit space $M_\Gamma := M/\Gamma$ is compact. We furthermore need to assume that a chosen unitary left action representation operator of the group action commutes with the elliptic operator D. The non-compact manifold M can then be seen as Galois covering with compact base manifold M_Γ and the elliptic operator D, acting on sufficient regular sections of vector bundles over M, can be viewed as lift of an elliptic operator $\underline{D}$ which acts on sufficient regular section of vector bundles over M_Γ. We impose a positive density on M_Γ and a (positive definite) Hermitian bundle metric for the vector bundle E_Γ over M_Γ such that they lift to a Γ-invariant positive density on M_Γ and a Γ-invariant (positive definite) Hermitian bundle metric of a vector bundle bundle E over M. The Hilbert space of square-integrable sections of E with respect to these Γ-invariant lifted quantities $L^2_\Gamma(M,E)$ becomes a Γ-module with respect to the von Neumann algebra $\mathscr{B}_\Gamma(L^2_\Gamma(M,E))$ of bounded operators on $L^2_\Gamma(M,E)$ which commutes with the action of Γ. This algebra has a natural trace Tr_Γ which defines the mentioned renormalised dimension: let $\mathscr{H}$ be any Γ-submodule[1] of $L^2_\Gamma(M,E)$ and $P_{\mathscr{H}}$ the orthogonal projection onto $\mathscr{H}$ which is again an element of the von Neumann algebra $\mathscr{B}_\Gamma(L^2_\Gamma(M,E))$; the Γ-dimension of $\mathscr{H}$ is

$$\dim_\Gamma(\mathscr{H}) := \mathrm{Tr}_\Gamma\left(P_{\mathscr{H}}\right) \quad .$$

Let F be another vector bundle over the covering M with a Γ-invariant (positive definite) Hermitian bundle metric. The L^2-kernel of the operator $D : C^\infty(M,E) \to C^\infty(M,F)$

$$\mathsf{ker}_{L^2}(D) := \left\{u \in L^2_\Gamma(M,E) \,|\, Du = 0\right\}$$

and of its L^2-adjoint D^*

$$\mathsf{ker}_{L^2}(D^*) := \left\{v \in L^2_\Gamma(M,F) \,|\, D^*v = 0\right\}$$

then have finite Γ-dimensions and a meaningful Γ-index can be defined by

$$\mathrm{ind}_\Gamma(D) = \dim_\Gamma \mathsf{ker}_{L^2}(D) - \dim_\Gamma \mathsf{ker}_{L^2}(D^*) \quad .$$

The finiteness of this renormalised index is guaranteed if the operator D is Γ-Fredholm[2]. Atiyah has shown the following result.

Theorem 1.0.3 (cf. Introduction and Theorem 3.8 in [Ati76]). *In the introduced setting, the elliptic Γ-invariant differential operator D, which acts between (different) sections of vector bundles over the Galois covering M and is a lift of the elliptic operator $\underline{D}$, is Γ-Fredholm and its Γ-index satisfies*

$$\mathrm{ind}_\Gamma(D) = \mathrm{ind}(\underline{D}) \quad .$$

If Γ is a finite group with cardinality $|\Gamma|$, then M is already compact and one recovers the finite-covering index formula

$$\mathrm{ind}_\Gamma(D) = |\Gamma|\,\mathrm{ind}(\underline{D}) \quad .$$

The proof can be extended to elliptic classical pseudo-differential operators which is e.g. presented in [Shu, Thm.3.11.3]. The omitted details and concepts will be given in Chapter 5 of this thesis.

[1] We will clarify later that the Γ-modules needs to be free or even more projective.

[2] This concept of Breuer- or L^2-Fredholmness in a von Neumann setting will be explained in subsection 5.2.3 which we will introduce with the terminology Γ-Fredholmness.

A combination of Theorem 1.0.3 and Corollary 1.0.2 for Dirac-type operators on Riemannian manifolds has been worked out by Ramachandran.

Theorem 1.0.4 (cf. Theorem 1.1 in [Ram93]). *Let M be a Galois covering with Galois group Γ of a compact Riemannian manifold M_Γ with boundary and D a Dirac-type operator on M which acts on a graded Clifford/spinor bundle $\mathcal{S}(M)$ and is a lift of a Dirac-type operator $\underline{D}$ on M_Γ, acting on a graded Clifford/spinor bundle $\mathcal{S}(M_\Gamma)$. If M_Γ and $\underline{D}$ have product structure near the boundary, then the operator D_{APS} under Atiyah-Patodi-Singer boundary conditions is Γ-Fredholm with Γ-index*

$$\mathrm{ind}_\Gamma(D_{\mathrm{APS}}) = \int_{M_\Gamma} a(M_\Gamma) - \frac{\eta_\Gamma(A) + \dim_\Gamma \ker(A)}{2} \tag{1.3}$$

where A the lifted hypersurface Dirac operator, $\eta_\Gamma(A)$ its Γ-eta-invariant and $a(M_\Gamma)$ is formally the same density as in (1.2).

Theorem 6.6 of the same reference shows that the operator D_{APS} is Γ-Fredholm as map from $L^2_\Gamma(M, \mathcal{S}(M)) \to L^2_\Gamma(M, \mathcal{S}(M))$ with domain $H^1_{\Gamma,\mathrm{APS}}(M, \mathcal{S}(M))$ which are Sobolev sections of $\mathcal{S}(M)$ with respect to the inner product of $L^2_\Gamma(M, \mathcal{S}(M))$, satisfying the Atiyah-Patodi-Singer boundary condition. Further here omitted details and definitions will be given in Chapter 5 and subsection 8.2.4.

Another worth mentioning extension is the Callias index theorem for the Callias-Dirac operator on an odd-dimensional non-compact manifold which is equipped with a Callias potential, regulating the behaviour outside any compact subset of the manifold. We refer to [Cal78] and [BS78] for further remarks. An index theorem for elliptic Fourier integral operators for certain canonical relations is discussed in [EM98] ; further explainations and a local index formula for these Fourier integral operator are presented in [LNT01]. Another extension to open manifolds has been proven for manifolds with bounded geometry by Roe in [Roe88a] and [Roe88b].

On the other hand, an index theorem can be used for geometric as well as application-oriented questions in physics, especially in Quantum Field Theory. For the latter, the ellipticity condition is in general not satisfied for geometrically related operators such as the Dirac operator. In this situation a Riemannian structure is necessary, but not given in the context of physical theories which are constructed in the framework of Special or General Relativity. However, the Atiyah-Singer index theorem has been used in order to relate the topology of a underlying manifold M as space in a physical system to the zero modes of a physical differential operator (i.e. its kernel solutions) on M. The index of a Dirac operator in particular plays an important role in calculating the excess of dimensions of the spaces of zero modes with respect to one chirality, i.e. a particular choice of the grading of the spinor bundle, to the space of zero modes with respect to the opposite chirality. This is known as (gravitational) chiral anomaly. The a priori lack of ellipticity of the Dirac operator due to a Lorentzian manifold as physical spacetime is resolved by translating everything in the Euclidean or Riemannian geometric picture before applying the Atiyah-Singer index theorem for the Dirac operator. Afterwards, the Wick rotation[3] for translating into

[3]The time coordinate is treated as complex variable, constraint to the real axis. The Wick rotation rotates the real time axis to a imaginary time axis. In this way, the Lorentzian spacetime can be interpreted as Euclidean spacetime with an imaginary time coordinate.

the Euclidean setting is reperformed to the Lorentzian setting. Another prominent use of index theorems in physics is the ghost number anomaly in BRST symmetry theories where the Gauss-Bonnet theorem and the Riemann-Roch theorem as special cases of index theorems in two dimensions are used. For these two and other quantum anomalies we refer to [FS04] . The Callias index theorem is used in Topological Quantum Field Theory of odd-dimensional manifolds and in Chern-Simons (gauge) theory; see [NS] for a rigorous treatment with physical applications. It is also conceivable to use index theorems for sorting out (parts of a) gravitational action in some modifying geometric theories of gravity because they turn out to be invariant with respect to the field variation which leads to the field equations. Last, but not least, the proof of the Atiyah-Singer theorem with the help of the heat equation in [ABP73] can be interpreted in terms of supersymmetry; see [AG83].

Besides its great use in relativistic physics, it is though somehow unsatisfying to work with index theorems of geometric operators on Lorentzian spacetimes just by analogy or with the help of the Wick rotation if practicable at all. Furthermore, an index theorem for such in general non-elliptic operators is also of mathematical interest. A major step in this direction has been done by Bär and Strohmaier in 2015 for the Atiyah-Singer-Dirac operator $D : C^\infty(M, \mathcal{S}(M)) \to C^\infty(M, \mathcal{S}(M))$ for positive-chirality spinors on an even-dimensional globally hyperbolic, temporal compact[4] spin manifold with compact spacelike Cauchy hypersurface and two boundary hypersurfaces on which Atiyah-Patodi-Singer boundary conditions are enjoined. The boundary hypersurfaces are again spacelike Cauchy hypersurfaces. The spinor bundle $\mathcal{S}(M)$ decomposes into two subbundles $\mathcal{S}^\pm(M)$ which sections are spinors of positive respectively negative chirality. Hence D becomes a map from $C^\infty(M, \mathcal{S}^+(M))$ to $C^\infty(M, \mathcal{S}^-(M))$. They worked out an index theorem similar to Theorem 1.0.1 with Corollary 1.0.2. The domain $FE^0_{\mathrm{APS}}(M, \mathcal{T}, D)$ of smooth sections of $\mathcal{S}^+(M)$ with respect to the L^2-graph norm for the Dirac operator, subject to Atiyah-Patodi-Singer boundary conditions, turns out to be the correct setting for Fredholmness. Roughly speaking, spinors of this space are continuous in time and of Sobolev degree 0 and thus square-integrable with respect to space coordinates such that the action of D is in $L^2(M, \mathcal{S}^-(M))$.

Theorem 1.0.5 (cf. Main Theorem in [BS19])**.** *Let (M, g) be a temporal compact, time-oriented, even-dimensional globally hyperbolic Lorentzian spin manifold with boundary ∂M and compact smooth spacelike hypersurface Σ such that the boundary is $\partial M = \Sigma_2 \sqcup \Sigma_1$, with Σ_2 as the Cauchy boundary in the future of Σ_1. The Atiyah-Singer-Dirac operator*

$$D_{\mathrm{APS}} : FE^0_{\mathrm{APS}}(M, \mathcal{T}, D) \to L^2(M, \mathcal{S}^-(M))$$

is Fredholm with index

$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \hat{\mathrm{A}}(M) + \int_{\partial M} T\hat{\mathrm{A}}(g) - \frac{\dim\ker(A_1) + \dim\ker(A_2) + \eta(A_1) - \eta(A_2)}{2} \quad . \tag{1.4}$$

[4] *Temporal compactness* means that the time domain $\mathcal{T}(M)$ of the globally hyperbolic manifold M is a compact time interval, i.e. there exist $t_1, t_2 \in \mathbb{R}$ such that $\mathcal{T}(M) = [t_1, t_2]$. We will also use this terminology to express that we restrict the possibly non-compact time domain of M to any compact time interval $[t_1, t_2]$. Thus, any restriction $M|_{[t_1,t_2]}$ becomes temporal compact in the original sense.

$\hat{\mathbb{A}}(M)$ is the $\hat{A}$-genus, manufactured from the curvature of the Levi-Civita connection with respect to g, $T\hat{\mathbb{A}}(g)$ its corresponding transgression form and A_1, A_2 are the hypersurface Dirac operators on Σ_1 and respectively Σ_2. Comparing with Theorem 1.0.1 and Corollary 1.0.2 shows that it is the Lorentzian analogue of the Atiyah-Patodi-Singer index theorem for the spin-Dirac operator with two boundary hypersurfaces. The additional term is given by the transgression form which vanishes, if the boundary is totally geodesic, or equivalently the manifold and Dirac operator have product structure near the boundary. We note that the product structure assumption is a priori essential for (1.0.1), but is not required for Theorem 1.0.5. In the same paper, they worked out an index theorem for anti-Atiyah-Patodi-Singer boundary conditions which are the corresponding orthogonal boundary conditions. The Dirac operator D_{aAPS} in this case has the index

$$\mathrm{ind}(D_{\mathrm{aAPS}}) = -\mathrm{ind}(D_{\mathrm{APS}}) \quad .$$

Furthermore, they generalised their results to spin-Dirac operators with twisting bundles for spinors with values in another Hermitian vector bundle. Moreover, they allowed that M has Spin^c-structure such that M carries an associated Hermitian (determinant) line bundle. The square-root of this bundle is then twisted with an artificial global spin structure on M. Bär and Hannes extended the results to generalised Atiyah-Patodi-Singer boundary conditions in [BH18] where the spectral cuts in the spectrum of the self-adjoint boundary Dirac operators are allowed to be any other point than just zero. Altogether, the most general formulation of these theorems is stated as follows.

Theorem 1.0.6 (cf. Theorem 7.1 in [BS19] with section 4.2 in [BH18]). *Let (M, g) be a temporal compact, time-oriented, even-dimensional globally hyperbolic Lorentzian manifold with boundary ∂M and compact smooth spacelike hypersurface Σ such that the boundary is $\partial M = \Sigma_2 \sqcup \Sigma_1$, where Σ_2 is the Cauchy hypersurface in the future of Σ_1. Assume in addition that M is equipped with a Spin^c-structure by means of a Hermitian line bundle $L \to M$ with metric connection ∇^L, and carries a Hermitian vector bundle $E \to M$ with metric connection ∇^E. The twisted Spin^c-Dirac operator on sections of the graded twisted Spin^c-spinor bundle $\mathcal{S}_{L,E}(M) = \mathcal{S}^+_{L,E}(M) \oplus \mathcal{S}^-_{L,E}(M)$,*

$$D^{E_L}_{\mathrm{APS}(\mathrm{a}_1,\mathrm{a}_2)} : FE^0_{\mathrm{APS}(\mathrm{a}_1,\mathrm{a}_2)}(M, D^{E_L}) \to L^2(M, \mathcal{S}^-_{L,E}(M)) \quad ,$$

is Fredholm under generalised Atiyah-Patodi-Singer boundary conditions with spectral cuts $a_1, a_2 \in \mathbb{R}$ at Σ_1 and respectively Σ_2, and its index is given by

$$\begin{aligned}\mathrm{ind}\left(D^{E_L}_{\mathrm{APS}(a_1,a_2)}\right) = &\int_M \hat{\mathbb{A}}(M) \wedge \mathrm{e}^{\mathrm{c}_1(L)/2} \wedge \mathrm{ch}(E) + \int_{\partial M} \mathcal{TG}(g, E, L) \\ &- \frac{\dim \ker\left(A_1^{E_L}\right) + \dim \ker\left(A_2^{E_L}\right) + \eta(A_1^{E_L}) - \eta(A_2^{E_L})}{2} \\ &+ \chi_{\{a_1>0\}} \dim\left(L^2_{(0,a_1]}(\mathcal{S}_{L,E}(\Sigma_1))\right) - \chi_{\{a_1<0\}} \dim\left(L^2_{[a_1,0)}(\mathcal{S}_{L,E}(\Sigma_1))\right) \\ &+ \chi_{\{a_2<0\}} \dim\left(L^2_{(a_2,0]}(\mathcal{S}_{L,E}(\Sigma_2))\right) - \chi_{\{a_2>0\}} \dim\left(L^2_{(0,a_2]}(\mathcal{S}_{L,E}(\Sigma_2))\right) .\end{aligned} \tag{1.5}$$

In comparison to (1.4), there are additional contributions from the first Chern form $\mathrm{c}_1(L)$ of the curvature of ∇^L and the Chern character $\mathrm{ch}(E)$ of the curvature of ∇^E. $\mathcal{TG}(g, E, L)$ is the transgression form of the wedge product of the $\hat{A}$-genus with these other two characteristic classes. $A_1^{E_L}$ and $A_1^{E_L}$ are the twisted hypersurface Dirac operators on the boundary hypersurfaces Σ_1 and Σ_2. The additional boundary contributions due to the modification are the dimensions of the spaces L_I^2 which are ranges of L^2-spaces under a spectral projection onto an interval $I \subset \mathbb{R}$, describing the range of regarded eigenvalues of the boundary Dirac operators. The interval is determined by the spectral cut points a_1 or a_2, depending on the chosen hypersurface. The characteristic function controls which part of this extra term contributes, depending on the sign of the spectral cut points. (1.5) reduces to (1.4) if we set $a_2 = 0 = a_1$ and ignore any extra structure besides a global spin structure. It is mandatory that the twisting bundle do carry a positive definite bundle metric. The induced hypersurface Dirac operators are otherwise no longer self-adjoint such that the spectral contributions do have to be analysed with more care. This implies that natural bundles from TM needs to be excluded as twisting bundles. This problem has been resolved and is presented in [BS20]. The globally hyperbolic manifolds is assumed to have product structure near the boundaries, but any Dirac-type operator can be regarded which is not necessarily self-adjoint and therefore dropped the assumptions of a positive definite bundle metric on E. Based on their results, Bär and Strohmaier gave a rigorous derivation of the chiral anomaly on curved backgrounds in [BS16] where they used Theorem 1.0.6 with product structure near the boundary and $a_1 = 0 = a_2$. The index then occurs once as relative left-handed charge and again as relative right-handed charge which differs from the first by a sign. The difference of these charges is non-zero and describes the relative chiral current, manifesting the anomaly. The twisting bundle is interpreted as associated vector bundle of a compact gauge group in the physical picture. If we for example choose the compact gauge group to be the circle group $\mathsf{U}(1)$, the twisting spinor bundle can be viewed as Spin^c-spinor bundle which mimics electromagnetic potentials in the physical interpretation.

We briefly sketch the steps of the proofs of Theorem 1.0.5 and Theorem 1.0.6. We recall that temporal compactness and global hyperbolicity of M imply that at each point in time $t \in [t_1, t_2] =: \mathcal{T}(M)$ $(t_1, t_2 \in \mathbb{R})$ there is as slice $\Sigma_t = \{t\} \times \Sigma$. In particular, we have $\Sigma_1 := \Sigma_{t_1}$ and $\Sigma_2 := \Sigma_{t_2}$. Hence M can be viewed as foliation of spacelike compact Cauchy hypersurfaces $\{\Sigma_t\}_{t \in \mathcal{T}(M)}$. We denote the restriction of the spinor bundles $\mathcal{S}^{\pm}(M)$ to a hypersurface Σ_t with $\mathcal{S}^{\pm}(\Sigma_t)$.

(I) *Well-posedness of the Cauchy problem*: The starting point in showing Fredholmness and its index is the well-posedness of the initial value problem for D on M with initial values in Sobolev spaces $H^s(\mathcal{S}^+(\Sigma_2))$, $s \in \mathbb{R}$, and inhomogeneities in $L^2(\mathcal{T}(M), H^s(\mathcal{S}^+(\Sigma_\bullet)))$. The operational description of the well-posedness of the Cauchy problem is that the direct sum of the restriction operator res_t and D

$$\mathsf{res}_t \oplus D \,:\, FE^s(M, D) \quad\to\quad H^s(\mathcal{S}^+(\Sigma_t)) \oplus L^2(\mathcal{T}(M), H^s(\mathcal{S}^-(\Sigma_\bullet)))$$

is an isomorphism of Banach spaces for all $s \in \mathbb{R}$ and all $t \in \mathcal{T}(M)$. The space $FE^s(M, D)$ is defined as $FE^0_{\mathrm{APS}}(M, D)$ without imposing boundary conditions and for arbitrary Sobolev degree s. In analytic terms it means that the Cauchy problem with inital time $t = t_0 \in \mathcal{T}(M)$

$$Du \in L^2(\mathcal{T}(M), H^s(\mathcal{S}^-(\Sigma_\bullet))) \quad \text{with} \quad u|_{\Sigma_{t_0}} \in H^s(\mathcal{S}^+(\Sigma_{t_0}))$$

has a unique solution $u \in FE^s(M, D)$.

(II) *Dirac-wave evolution operator*: The homogenous problem from step (I) says that

$$\mathsf{res}_t \,:\, FE^s(M,D) \cap \ker(D) \quad \to \quad H^s(\mathcal{S}^+(\Sigma_t))$$

is an isomorphism of topological vector spaces for all $s \in \mathbb{R}$ and all $t \in \mathcal{T}(M)$. This induces the *(Dirac-)wave evolution operator*

$$Q(t,\tau) \,:\, H^s(\mathcal{S}^+(\Sigma_\tau)) \quad \to \quad H^s(\mathcal{S}^+(\Sigma_t))$$

for all $t, \tau \in \mathcal{T}(M)$ which is an isomorphism of Hilbert spaces for all Sobolev degrees s. In particular, it is a unitary map if $s = 0$. One further important property is that $Q(t,\tau)$ can be described as Fourier integral operator of order 0 with canonical relation which is associated to a canonical map, given by the flow of lightlike geodesics from Σ_τ to Σ_t for all t, τ. This is shown by constructing Q with the help of solution operators of initial value problems for normally hyperbolic operators of real principal type.

(III) *Boundary conditions*: The imposed Atiyah-Patodi-Singer boundary conditions on ∂M induces orthogonal splittings of the L^2-spaces

$$\begin{array}{rcl} L^2(\mathcal{S}^+(\Sigma_1)) & = & L^2_{[0,\infty)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{(-\infty,0)}(\mathcal{S}^+(\Sigma_1)) \\ L^2(\mathcal{S}^+(\Sigma_2)) & = & L^2_{(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \oplus L^2_{(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \end{array} .$$

These imply a splitting of $Q(t_2,t_1)$ by restricting its domain and range to one of the two spectral subspaces of $L^2(\mathcal{S}^+(\Sigma_1))$ and $L^2(\mathcal{S}^+(\Sigma_2))$. The wave evolution operator can then be represented as a (2×2)-matrix of the form

$$Q(t_2,t_1) = \begin{pmatrix} Q_{++}(t_2,t_1) & Q_{+-}(t_2,t_1) \\ Q_{-+}(t_2,t_1) & Q_{--}(t_2,t_1) \end{pmatrix} \tag{1.6}$$

where the entries are compositions of Q with the spectral projectors $P_{\gtrless 0}(t)$, $P_{\geq 0}(t)$ and $P_{\leq 0}(t)$ onto the eigenspaces of the hypersurface Dirac operator at time t for eigenvalues cutted at 0:

$$Q_{++}(t_2,t_1) := P_{>0}(t_2) \circ Q(t_2,t_1) \circ P_{\geq 0}(t_1)$$
$$Q_{--}(t_2,t_1) := P_{\leq 0}(t_2) \circ Q(t_2,t_1) \circ P_{<0}(t_1)$$
$$Q_{+-}(t_2,t_1) := P_{>0}(t_2) \circ Q(t_2,t_1) \circ P_{<0}(t_1)$$
$$Q_{-+}(t_2,t_1) := P_{\leq 0}(t_2) \circ Q(t_2,t_1) \circ P_{\geq 0}(t_1)\,.$$

These are the *spectral (Dirac-)wave evolution operators*. A regularity analysis of $Q_{\pm\pm}(t_2,t_1)$ and $Q_{\pm\mp}(t_2,t_1)$ shows that all entries in (1.6) are again Fourier integral operators with respect to the same canonical relation for Q, but the off-diagonal entries in (1.6) have in fact order (-1) due to the vanishing of their principal symbols of order 0. This implies that the off-diagonal entries are compact operators on L^2-spaces. The unitarity of Q implies $Q^*Q = \mathbb{1}$ and $QQ^* = \mathbb{1}$. The matrix representation (1.6) then indicates

$$\begin{array}{rcrcl}
(Q_{++}(t_2,t_1))^* Q_{++}(t_2,t_1) & + & (Q_{-+}(t_2,t_1))^* Q_{-+}(t_2,t_1) & = & \mathbb{1} \\
(Q_{+-}(t_2,t_1))^* Q_{+-}(t_2,t_1) & + & (Q_{--}(t_2,t_1))^* Q_{--}(t_2,t_1) & = & \mathbb{1} \\
Q_{++}(t_2,t_1)(Q_{++}(t_2,t_1))^* & + & Q_{+-}(t_2,t_1))(Q_{+-}(t_2,t_1))^* & = & \mathbb{1} \\
Q_{-+}(t_2,t_1)(Q_{-+}(t_2,t_1))^* & + & Q_{--}(t_2,t_1))(Q_{--}(t_2,t_1))^* & = & \mathbb{1}
\end{array}$$

and further equations which are not important for the further proceeding. As the $Q_{\pm\mp}(t_2,t_1)$ are compact, the adjoint and compositions among each other are compact as well. One observes from these four equations that the diagonal entries of (1.6) are in fact Fredholm with their adjoints as their parametrices.

(IV) *Index of $Q_{\pm\pm}(t_2,t_1)$*: $Q(t_2,t_1)$ is Fredholm with index 0 on L^2-spaces as it maps as unitary operator. Hence the difference

$$Q(t_2,t_1) - \begin{pmatrix} Q_{++}(t_2,t_1) & 0 \\ 0 & Q_{--}(t_2,t_1) \end{pmatrix} = \begin{pmatrix} 0 & Q_{+-}(t_2,t_1) \\ Q_{-+}(t_2,t_1) & 0 \end{pmatrix}$$

is a compact operator and thus $Q(t_2,t_1)$ and $Q_{++}(t_2,t_1) \oplus Q_{--}(t_2,t_1)$ have the same index. Since the index of the direct sum is the sum of the indices, one gains

$$\begin{aligned}
& \operatorname{ind}(Q(t_2,t_1)) = 0 = \operatorname{ind}(Q_{++}(t_2,t_1)) + \operatorname{ind}(Q_{--}(t_2,t_1)) \\
\Leftrightarrow\quad & \operatorname{ind}(Q_{--}(t_2,t_1)) = -\operatorname{ind}(Q_{++}(t_2,t_1)) \quad .
\end{aligned}$$

In particular, one can directly compute $\operatorname{ind}(Q_{++}(t_1,t_1))$ to be $\dim \mathsf{ker}\,(A_1)$. Telescoping the difference $\operatorname{ind}(Q_{--}(t_2,t_1)) - \operatorname{ind}(Q_{--}(t_1,t_1))$ with respect to a partition of $\mathcal{T}(M)$ allows to relate the index of $Q_{--}(t_2,t_1)$ with the spectral flow $\operatorname{sf}\{A_t\}_{t\in\mathcal{T}(M)}$ of the smooth family of self-adjoint and Fredholm hypersurface Dirac operators $\{A_t\}_{t\in\mathcal{T}(M)}$ ($A_{t_2} = A_2$ and $A_{t_1} = A_1$):

$$\operatorname{ind}(Q_{--}(t_2,t_1)) = \operatorname{sf}\{A_t\}_{t\in\mathcal{T}(M)} - \dim \mathsf{ker}\,(A_2) = -\operatorname{ind}(Q_{++}(t_2,t_1)) \quad .$$

(V) *Fredholmness and Indices of D_{APS} and D_{aAPS}*: Let $\mathbb{P}_{\mathrm{APS}}$ and $\mathbb{P}_{\mathrm{aAPS}}$ denote the operators which generate the (anti-)Atiyah-Patodi-Singer boundary conditions. Due to some functional-analytic lemmas the operators $\mathbb{P}_{\mathrm{APS}} \oplus D$ and $\mathbb{P}_{\mathrm{aAPS}} \oplus D$ are Fredholm if and only if $\mathbb{P}_{\mathrm{APS}}|_{\mathsf{ker}(D)}$ and respectively $\mathbb{P}_{\mathrm{aAPS}}|_{\mathsf{ker}(D)}$ are Fredholm. The ranges of the latter are closed and the kernels and cokernels are isomorphic to the kernels of $Q_{\pm\pm}(t_2,t_1)$ and $Q^*_{\pm\pm}(t_1,t_2)$. As the latter are Fredholm, $\mathbb{P}_{\mathrm{APS}} \oplus D$ and $\mathbb{P}_{\mathrm{aAPS}} \oplus D$ become Fredholm, too. One concludes with the same argument that $D|_{\mathsf{ker}(\mathbb{P}_{\mathrm{APS}})}$ and $D|_{\mathsf{ker}(\mathbb{P}_{\mathrm{aAPS}})}$ are Fredholm and, as they coincide with D_{APS} and respectively D_{aAPS}, the Dirac operators of interest become Fredholm with indices

$$\begin{aligned}
\operatorname{ind}(D_{\mathrm{APS}}) &= \operatorname{ind}(Q_{--}(t_2,t_1)) = \operatorname{sf}\{A_t\}_{t\in\mathcal{T}(M)} - \dim \mathsf{ker}\,(A_2) \\
&= -\operatorname{ind}(Q_{++}(t_2,t_1)) = -\operatorname{ind}(D_{\mathrm{aAPS}}) \quad .
\end{aligned} \tag{1.7}$$

(VI) *Geometric index of $D_{\mathrm{(a)APS}}$*: One considers the Dirac operator $\check{D}$ which differs from D by a sign flip in the temporal part of the metric g, making it a Riemannian metric. This operator is elliptic. If we consider in addition a product structure near the two boundary hypersurfaces, we can apply Corollary 1.0.2, showing

$$\mathrm{ind}(\check{D}_{\mathrm{APS}}) = \int_M \hat{\mathbb{A}}(\check{\nabla}) - \frac{\dim \mathsf{ker}\,(A_1) + \dim \mathsf{ker}\,(A_2) + \eta(A_1) - \eta(A_2)}{2}$$

where $\hat{\mathbb{A}}(\check{\nabla})$ is the $\hat{A}$-genus, computed from the curvature of the Levi-Civita connection $\check{\nabla}$ with respect to the auxiliary Riemannian metric. As two equal characteristic classes with different choices of the connection are in the same cohomology class, the genus $\hat{\mathbb{A}}(\check{\nabla})$ differs from $\hat{\mathbb{A}}(M)$ in an exact form, given by the transgression form $T\hat{\mathbb{A}}(g)$ which turns out to be independent by the particular choice of the auxiliary Riemannian metric as consequence of the product structure in the Riemannian case. Hence we can rewrite the integral part in Lorentzian geometric quantities:

$$\mathrm{ind}(\check{D}_{\mathrm{APS}}) = \int_M \hat{\mathbb{A}}(M) + \int_{\partial M} T\hat{\mathbb{A}}(g) - \frac{\dim \mathsf{ker}\,(A_1) + \dim \mathsf{ker}\,(A_2) + \eta(A_1) - \eta(A_2)}{2} .$$

As D and $\check{D}$ only differ in their temporal directions along any hypersurface Σ_t, they come with the same family of self-adjoint and Fredholm hypersurface Dirac operators $\{A_t\}_{t\in\mathcal{T}(M)}$. From the geometrical expression of $\mathrm{ind}(\check{D}_{\mathrm{APS}})$ and the constance of the index, one can express the spectral flow of this family in geometric terms:

$$\mathrm{sf}\,\{A_t\}_{t\in\mathcal{T}(M)} = \int_M \hat{\mathbb{A}}(M) + \int_{\partial M} T\hat{\mathbb{A}}(g) - \frac{\dim \mathsf{ker}\,(A_1) - \dim \mathsf{ker}\,(A_2) + \eta(A_1) - \eta(A_2)}{2} .$$

With (1.7), the index formula (1.4) follows which concludes the proof of Theorem 1.0.5.

The steps of the proofs transfer to the twisted Spin^c-case which only makes the computations of the principal symbol in step (IV) and the index formula in the last step more involved. If generalised Atiyah-Patodi-Singer boundary conditions are imposed, one can show that the matrix entries of Q with respect to the corresponding orthogonal splitting of L^2-spaces differ from compositions of $Q_{\pm\pm}(t_2,t_1)$ and $Q_{\pm\mp}(t_2,t_1)$ in finite-rank projections. Since these finite-rank projections are Fredholm, the diagonal matrix entries for generalised Atiyah-Patodi-Singer boundary conditions are Fredholm while the off-diagonal entries are compact. The indices of the finite-rank projections exactly give the additional spectral contribution for spectral cuts different than zero. Steps (I), (II) and (VI) are not influenced by this modification. The proof of the Fredholmness of the Dirac operators for these generalised boundary conditions works analogously. Hence formula (1.5) and finally Theorem 1.0.6 are shown. An alternative proof with the additional assumption of a product structure near the boundaries is based on Feynman parametrices of the Dirac operators. This other method of proof is also presented in [BS19] and has been used in [BS20] to extend Theorem 1.0.5 to any Dirac-type operator with a possibly twisting bundle, equipped with a possibly indefinite bundle metric.

In this thesis we want to present a Lorentzian pendent of Theorem 1.0.4, based on Theorem 1.0.5 or rather Theorem 1.0.6. We consider the Fredholm property and the indices of the Lorentzian (Atiyah-Singer-)Dirac operator $\mathcal{D}$ and its chiral decomposits $D_\pm$ on an even-dimensional, temporal compact, globally hyperbolic spin manifold $(M,\mathscr{g})$ such that the Cauchy hypersurface Σ is a Galois covering with compact base with respect to a Galois group Γ. Such manifolds M are referred on as spatial Γ-manifolds in this thesis as the covering is proactive on the spacelike Cauchy hypersurface. The compact base manifold M_Γ of the covering is therefore isomorphic to the product $[t_1,t_2]\times\Sigma_\Gamma$ with $\Sigma_\Gamma := \Sigma/\Gamma$ as compact base manifold of the covering Σ. All vector bundles over M_Γ can be lifted to vector bundles on the covering by lifting the time-independent spatial action of the group to each foliating hypersurface. We call such vector bundles Γ-vector bundles. We extend the domains $FE^0_{\mathrm{APS}(a_1,a_2)}$ and $FE^0_{\mathrm{aAPS}(a_1,a_2)}$ and in particular $FE^0_{\mathrm{APS}} = FE^0_{\mathrm{APS}(0,0)}$ and $FE^0_{\mathrm{aAPS}} = FE^0_{\mathrm{aAPS}(0,0)}$ to the Γ-setting by replacing the Sobolev spaces of degree s in the definition of $FE^s(M,D)$ with Γ-Sobolev spaces. The Γ-Fredholmness of D as well as its pendent $\tilde{D}$, acting on negative spinor fields, and thus $\mathcal{D}$ has been presented in [Dam21] for ordinary Atiyah-Patodi-Singer boundary conditions and their orthogonal boundary conditions in the untwisted case.

Main Theorem 1 (cf. Theorem 7.6, 7.9 and 1.2 in [Dam21], 2021)**.** *Let M be a temporal compact, even-dimensional globally hyperbolic spatial Γ-manifold, $\mathcal{S}^\pm(M)\to M$ Γ-spin bundles of positive and respectively negative chirality of the chiral decomposition $\mathcal{S}(M) = \mathcal{S}^+(M)\oplus\mathcal{S}^-(M)$; the Γ-invariant Dirac operators*

$$\begin{aligned}
D_{\mathrm{APS}}: &\quad FE^0_{\Gamma,\mathrm{APS}}(M,\mathcal{T},D) \to L^2_\Gamma(M,\mathcal{S}^-(M))\\
D_{\mathrm{aAPS}}: &\quad FE^0_{\Gamma,\mathrm{aAPS}}(M,\mathcal{T},D) \to L^2_\Gamma(M,\mathcal{S}^-(M))\\
\tilde{D}_{\mathrm{APS}}: &\quad FE^0_{\Gamma,\mathrm{APS}}(M,\mathcal{T},\tilde{D}) \to L^2_\Gamma(M,\mathcal{S}^+(M))\\
\tilde{D}_{\mathrm{aAPS}}: &\quad FE^0_{\Gamma,\mathrm{aAPS}}(M,\mathcal{T},\tilde{D}) \to L^2_\Gamma(M,\mathcal{S}^+(M))\\
\mathcal{D}_{\mathrm{APS}}: &\quad FE^0_{\Gamma,\mathrm{APS}}(M,\mathcal{T},\mathcal{D}) \to L^2_\Gamma(M,\mathcal{S}(M))\\
\mathcal{D}_{\mathrm{aAPS}}: &\quad FE^0_{\Gamma,\mathrm{aAPS}}(M,\mathcal{T},\mathcal{D}) \to L^2_\Gamma(M,\mathcal{S}(M))
\end{aligned}$$

as lifts of Dirac operators on the base manifold are Γ-Fredholm under Atiyah-Patodi-Singer and respectively anti-Atiyah-Patodi-Singer boundary conditions on the Cauchy boundary hypersurfaces Σ_1 and Σ_2.

The Γ-indices of $\mathcal{D}_{\mathrm{APS}}$ and $\mathcal{D}_{\mathrm{aAPS}}$ are related to those of D and $\tilde{D}$ with the corresponding boundary conditions:

$$\mathrm{ind}_\Gamma(\mathcal{D}_{\mathrm{APS}}) = \mathrm{ind}_\Gamma(D_{\mathrm{APS}}) + \mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{APS}}) = -\mathrm{ind}_\Gamma(\mathcal{D}_{\mathrm{aAPS}}) \quad . \tag{1.8}$$

The skew-adjointness of $\mathcal{D}$ on one hand implies that the Γ-indices of $\mathcal{D}_{(\mathrm{a})\mathrm{APS}}$ vanish such that

$$\mathrm{ind}_\Gamma(D_{\mathrm{APS}}) = -\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{APS}}) \quad \text{and} \quad \mathrm{ind}_\Gamma(D_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{aAPS}}) \quad .$$

On the other hand, we will show in this thesis that these equivalences are true without using skew-adjointness of $\mathcal{D}$ by calculating the Γ-indices directly. We prove Γ-Fredholmness in the more general case that the spinors are twisted with a Hermitian Γ-vector bundle $E\to M$ with positive definite bundle metric which is a lift of the vector bundle $E_\Gamma\to M_\Gamma$. The spin-structure is replaced with a Spin^c-structure which comes with a Hermitian line bundle $L \to M$ as lift of $L_\Gamma \to M_\Gamma$, and generalised Atiyah-Patodi-Singer boundary conditions.

The following statement sums up our results as Γ-version of Theorem 1.0.6.

Main Theorem 2. *Let $a_1, a_2 \in \mathbb{R}$, M a temporal compact, globally hyperbolic spatial Γ-manifold with compact base M_Γ, $\mathcal{S}^{\pm}_{L,E}(M) \to M$ Γ-spin bundles of positive and respectively negative chirality which is twisted with a Hermitian Γ-vector bundle $E \to M$ and twisted with the square-root of a Hermitian Γ-line bundle $L \to M$ for a Spin^c-structure; we set $E_\Gamma \to M_\Gamma$ and $L_\Gamma \to M_\Gamma$ for the vector bundles over the compact base which lift to $E \to M$ respectively $L \to M$. Under these assumptions, the Γ-invariant Dirac operators*

$$\begin{aligned}
D^{E_L}_{\mathrm{APS}(a_1,a_2)} &: \quad FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \to L^2_\Gamma(\mathcal{S}^-_{L,E}(M)) \\
D^{E_L}_{\mathrm{aAPS}(a_1,a_2)} &: \quad FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \to L^2_\Gamma(\mathcal{S}^-_{L,E}(M)) \\
\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)} &: \quad FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}^+_{L,E}(M)) \\
\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)} &: \quad FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}^+_{L,E}(M))
\end{aligned}$$

as lifts of Dirac operators on the base manifold are Γ-Fredholm under generalised Atiyah-Patodi-Singer and respectively generalised anti-Atiyah-Patodi-Singer boundary conditions on the Cauchy boundary hypersurfaces Σ_1 and Σ_2 for all choices of a_1, a_2; their Γ-indices are

$$\begin{aligned}
\mathrm{ind}_\Gamma\left(D^{E_L}_{\mathrm{APS}(a_1,a_2)}\right) = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{T}\mathcal{G}(g, E_\Gamma, L_\Gamma) \\
&- \frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) - \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2) \\
&+ \chi_{\{a_2<0\}} \dim_\Gamma L^2_{\Gamma,(a_2,0]}(\mathcal{S}_{L,E}(\Sigma_2)) - \chi_{\{a_2>0\}} \dim_\Gamma L^2_{\Gamma,(0,a_2]}(\mathcal{S}_{L,E}(\Sigma_2)) \\
&+ \chi_{\{a_1>0\}} \dim_\Gamma L^2_{\Gamma,[0,a_1)}(\mathcal{S}_{L,E}(\Sigma_1)) - \chi_{\{a_1<0\}} \dim_\Gamma L^2_{\Gamma,[a_1,0)}(\mathcal{S}_{L,E}(\Sigma_1)) \\
= &\,\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) = -\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) = -\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) \quad .
\end{aligned}$$

The Γ-eta invariant η_Γ and the Γ-dimension of the null-space of the hypersurface Dirac operator have been condensed to the Γ-xi invariant

$$\xi_\Gamma(A^{E_L}_t) = \frac{\eta_\Gamma(A^{E_L}_t) + \dim \mathsf{ker}\left(A^{E_L}_t\right)}{2}$$

and the odd-dimensionality of Σ implies that the spinor bundle has no grading, i.e. $\mathcal{S}(\Sigma_t) = \mathcal{S}^{\pm}(\Sigma_t)$ for all $t \in \mathcal{T}(M)$. For $a_1 = 0 = a_2$ the Γ-indices for Main Theorem 1 follow.

Corollary 1.0.7. *The Γ-indices for Main Theorem 1 are*

$$\begin{aligned}
\mathrm{ind}_\Gamma(D_{\mathrm{APS}}) = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma) + \int_{\partial M_\Gamma} T\hat{\mathbb{A}}(g) - \xi_\Gamma(A_1) - \xi_\Gamma(-A_2) \\
&- \frac{1}{2}\left(\rho_\Gamma(A_{t_2}, \underline{A}_{t_2}) - \rho_\Gamma(A_{t_1}, \underline{A}_{t_1})\right) \\
= &\,\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(D_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{APS}}) \quad .
\end{aligned}$$

One observes by comparing the Γ-indices from Corollary 1.0.7 and (1.4) that the difference of Cheeger-Gromov-rho invariants

$$\frac{1}{2}\left(\rho_\Gamma(A_{t_2}, \underline{A}_{t_2}) - \rho_\Gamma(A_{t_1}, \underline{A}_{t_1})\right)$$

of the hypersurface Dirac operators A_t on Σ and $\underline{A}_t$ on Σ_Γ occur as additional term which is in general a non-trivial contribution and vanishes if the group Γ is reduced to the identity element.

The structure of the thesis is oriented on the presented steps of the proof. The first five chapters in Part II introduce necessary notations, background informations and some preparatory material. Chapter 2 deals with general concepts from functional analysis and some notions from differential geometry. In Chapter 3 we concentrate on pseudo-Riemannian manifolds where we take a closer look on globally hyperbolic manifolds and Riemannian topological cylinder manifolds. In Chapter 4 we recall basic facts of function spaces and operators on manifolds where we take a closer look on Fourier integral operators as one of the main needed concepts in our proof. Chapter 5 is dedicated to Galois coverings, von Neumann algebras associated to a Galois group and functional calculus in the von Neumann setting, in particular function and operator spaces for those von Neumann algebras which are associated to Galois coverings. Chapter 6 gives an introduction of spin structures and Dirac operators on pseudo-Riemannian manifolds in general. We then specify to spin structures in the presence of hypersurfaces, in particular the presence of a spacelike Cauchy hypersurface in globally hyperbolic spacetime and a base hypersurface in a Riemannian topological cylinder manifold.

In Part III we prove in four chapters Main Theorem 1, Main Theorem 2 and Corollary 1.0.7. The first two steps are considered in Chapter 7. We prove well-posedness results of the Cauchy problems for the Dirac operators D and $\tilde{D}$ in the more general situation that the manifold is a globally hyperbolic spin manifold with non-compact, but complete Cauchy hypersurface. We extract from the homogeneous problem two Dirac-wave evolution operators for D and $\tilde{D}$ and show that they are unitary on compactly supported L^2-sections and in particular Fourier integral operators of order zero with the same canonical relation as in the main reference [BS19] if the globally hyperbolic manifold is temporal compact. We then specify the non-compactness to Galois coverings of our interest. The well-posedness result in the general case is used in this special situation and the Fourier integral operator character of the evolution operator carries over as well if M is temporal compact. The evolution operator becomes Γ-invariant in this special case. For this situation we will also observe that the domains of the Dirac operator in the well-posedness result are in fact Hilbert Γ-modules.

Important results about regularity of projectors and the spectral flow in the group von Neumann setting are prepared in Chapter 8. These concepts are widely known in the compact case. We present a version of Seeley's theorem of complex powers in the Galois covering case which we deduce from the known extensions for manifolds with bounded geometry. Spectral projectors onto eigenspaces of (essentially) self-adjoint operators on a Riemannian Γ-manifold with eigenvalues in any compact interval of the spectrum in $\mathbb{R}$ are smoothing in the sense that they map between any Γ-Sobolev spaces. Our version of

Seeley's theorem allows to consider such spectral projectors with eigenvalues in any subinterval of $\mathbb{R}$ as Γ-invariant operators which differ from a Γ-invariant properly supported pseudo-differential operator of order 0 in such a smoothing operator. Thus, we can introduce the ordinary and generalised boundary conditions of Atiyah, Patodi and Singer by means of spectral projectors with similar regularity properties as spectral projectors on compact manifolds. These boundary conditions induce orthogonal splittings of L^2_Γ-spaces as Hilbert Γ-modules. Afterwards, we introduce the concepts of Fredholm pairs and the spectral flow in the von Neumann setting of our interest which we introduce as Γ-Fredholm pairs and Γ-spectral flow. We define the latter one with functional-analytic concepts and show an analytic expression in terms of the Γ-eta invariant.

These two concepts then become important in Chapter 9 where we proceed as in the steps (III) and (IV). We first focus on the splittings of the evolution operators with respect to ordinary (anti-)Atiyah-Patodi-Singer boundary conditions. We show that each spectral wave evolution operator is again a Γ-invariant Fourier integral operator of order 0. A principal symbol analysis shows that the off-diagonal elements in (1.6) have vanishing principal symbol of order 0. We show Γ-Fredholmness of the diagonal elements in two ways: we clarify that the off-diagonal elements are Γ-compact such that the equations of the unitary properties imply the wanted properties. An alternative argumentation for this step is based on the observation that error terms in the unitarity equations only imply an initial parametrix. A Neumann series argument shows that this can be improved and one can construct full left- and right-parametrices after finitely many iterations. This procedure is known from the classical theory of pseudo-differential operators where unitarity replaces ellipticity for constructing an initial parametrix. This prove has been presented in [Dam21]. Γ-Fredholmness for generalised boundary conditions then follows from the ordinary case. The Γ-index of the diagonal elements in (1.6) can be expressed with the Γ-spectral flow and additional terms due to generalised boundary conditions.

We finally prove the main results as in the steps (V) and (VI) in Chapter 10. We first prove that certain functional-analytic facts for boundary-value problems carry over to Hilbert Γ-modules. We then prove Γ-Fredholmness of D with the help of this modified argument. We finally derive the Γ-index formula in Main Theorem 2 with the help of Theorem 1.0.4 and the deduced analytic expression of the Γ-spectral flow.

We end this part with some remarks about further conceivable generalisations and open problems which are based on our result or are otherwise interesting. The appendix chapters are stored in the Backmatter.

Parts of this thesis are taken literally or are paraphrased from [Dam21] and it is planned to publish these results in the near future.

Part II.

Background and preparations

2. Basics and notations

In this subsection we fix some notations and recall basic material from Differential Geometry and Functional Analysis which we assume is familiar for the reader of this thesis. In the first section we also give a more extended recapitulation about unbounded operators which occurs to be useful at some points.

2.1. Basics from Functional Analysis

For a vast description of these topics we suggest to consult the classical literature, e.g. [Tre06], [Yos95], and [Kat76].

2.1.1. Basic notations and concepts

Let V be a *topological* $\mathbb{C}$*-vector space (TVS)*, i.e. a topological space which is compatible with a $\mathbb{C}$-linear structure. This is the most general setting in functional analysis from which one can derive several known concepts, depending on the additional structure V is allowed to have:

Definition 2.1.1. A TVS V is

(a) a *locally convex set* if there is a basis of neighbourhoods $\{V_i\}_{i\in J}$ in V which consists of convex sets, i.e. the straight line between any two points $p, q \in V_i$ lies entirely in V_i for all $i \in J$.

(b) a *metrizable TVS* if it is Hausdorff and has a countable basis of neighbourhoods of $0 \in V$. Equivalently, the underlying topology of V is defined by a metric. We call V moreover *complete* if every Cauchy sequence in V converges in V with respect to this metric.

(c) a *Fréchet space* if V is metrizable, complete and locally convex.

(d) a *normable/normed space* if the topology on V is defined by a norm $\|\cdot\|_V$. A complete normed space is a *Banach space*.

(e) a *pre-Hilbert space* if a positive definite sesquilinear form $\langle \cdot \mid \cdot \rangle_V : V \times V \to \mathbb{C}$ is defined on V. We call it *Hilbert space* if it is complete with respect to the topology, defined by the induced norm $\|u\|_V = \sqrt{\langle u \mid u \rangle_V}$ for $u \in V$.

(f) an *LF-space* if V is a countable strict inductive limit of Fréchet spaces; in other words, V is a union of countable many Fréchet spaces $\{V_j\}$ such that the natural injection $V_j \subset V_{j+1}$ is an isomorphism for each j in a countable index set. The initial topology on each V_j coincides with the induced topology from V_{j+1}. The union is called *LB-space* or *LH-space* if we replace Fréchet spaces with Banach respectively Hilbert spaces in the definition.

An alternative definition of a locally convex set is based on the notion of a *seminorm* which is an assignment $p : V \ni v \mapsto p(v)$ with the following three properties:

(a) (p is subadditive) $p(v+u) \leq p(v) + p(u)$ for all $u, v \in V$.

(b) (p is positive homogeneous) $p(\lambda v) = |v|\, p(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$.

(c) $p(0) = 0$.

A locally convex set is then characterised by a family of seminorms $\{p_i\}_{i\in J}$ which determines the topology of the TVS. If on the other hand $p(v) = 0$ implies $v = 0$ in (c), the seminorm becomes a norm with $\|\cdot\|_V := p(\cdot)$. If the TVS is clear from the context we neglect the subscript and write $\|\cdot\|$. We also use the short term *inner product* for a positive definite sesquilinear form. A (Hermitian) sesquilinear form, which is (a priori) not positive definite, will be denoted with $\left(\cdot \mid \cdot \right)_V$ to distingush them from a positive definite sesquilinear form $\left\langle \cdot \mid \cdot \right\rangle_V$ on V. In order to have a well-defined Hausdorff topology on V one needs to consider those indefinite sesquilinear forms which are *non-degenerate*, i.e. the map

$$v \mapsto \left(u \mapsto \left(u \,\middle|\, v\right)_V\right) \tag{2.1}$$

is an isomorphism. This is automatically satisfied for inner products. From now on the designation $\mathcal{H}$ always stands for a Hilbert space with inner product. If V is an LF-space with respect to the countable collection $\{V_j\}$ and F any locally convex TVS one defines a map $u : V \to F$ to be continuous if and only if $u|_{V_j} : V_j \to F$ is a continuous map for all members in the collection. Any LF-space is complete. If V is finite dimensional with $\dim(V) = n$, then V is isomorphic to $\mathbb{C}^n$, complete, and any functional as well as any map from V onto any locally convex TVS are continuous. In the course of this thesis we are going to meet several examples of these spaces in a more or less direct way.

The *anti-dual* $\overline{V}^*$ of a TVS V is the space of maps $F : V \to \mathbb{C}$ which are anti-linear:

$$F(\lambda u + v) = \overline{\lambda} F(u) + F(v)$$

for $u, v \in V$ and $\lambda \in \mathbb{C}$. If the space of functionals is just linear, we denote the dual by V^*. The map (2.1) is a continuous linear map between V and $\overline{V}^*$ as normed spaces. In the Hilbert space case it becomes in addition an isometry due to non-degeneracy/positive definiteness of the sesquilinear form. In this way one can identify an element in $\overline{V}^*$ with an element in V in a unique way such that $V \cong \overline{V}^*$ (Fréchet-Riesz Theorem). Hence the elements of the (anti-)dual of any Hilbert space $\mathcal{H}$ can be described with elements in $\mathcal{H}$. A modification of Hilbert spaces with indefinite, but non-degenerate (Hermitian) sesquilinear form $\left(\cdot \mid \cdot \right)_V$ is a *Krein space* V which decomposes into a direct sum of two complete subspaces from which one is positive definite and one negative definite. A fundamental symmetry $\mathcal{J}$, which is a self-adjoint and unitary operator, can be used to distinguish between the positive and negative definite subspaces. This symmetry then allows to introduce an inner product via $\left\langle v \,\middle|\, u\right\rangle_V = \left(\mathcal{J} v \,\middle|\, u\right)_V$ for $u, v \in V$. We refer to [vdD18, Sec.5.1] and contained references for more informations about Krein spaces and fundamental symmetries. We note that the main theorems of functional analysis for Banach spaces have a generalisation to the level of TVSs; see [Tre06, Sec.17/18/33] for details. The closed range theorem is one exception which only holds for operators between Banach spaces.

In this thesis we mostly focus on Hilbert spaces and on some Banach spaces and bounded mappings between them. Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces; we write $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of bounded (or equivalently continuous) linear operators. This space is itself a Banach space with respect to the norm

$$\|A\|_{\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)} := \sup_{u\in\mathcal{H}/\{0\}} \left\{ \frac{\|Au\|_{\mathcal{H}_2}}{\|x\|_{\mathcal{H}_1}} \right\} \tag{2.2}$$

for an operator $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$. We write $\mathscr{B}(\mathcal{H}_1)$ for $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_1)$. The composition of two operators $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in \mathscr{B}(\mathcal{H}_2, \mathcal{H}_3)$ is again bounded: $B \circ A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_3)$; the norm of the composition is smaller or equal the product of the norms: $\|B \circ A\|_{\mathscr{B}(\mathcal{H}_1,\mathcal{H}_3)} \leq \|B\|_{\mathscr{B}(\mathcal{H}_2,\mathcal{H}_3)} \cdot \|A\|_{\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)}$. If the image of the open ball $\mathring{\mathbb{B}}_{\mathcal{H}_1}(0)$ in $\mathcal{H}_1$ under an operator $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is relatively compact, we call the operator compact. We designate the set of those operators with

$$\mathscr{K}(\mathcal{H}_1, \mathcal{H}_2) := \left\{ A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2) \,\middle|\, A\left(\mathring{\mathbb{B}}_{\mathcal{H}_1}(0)\right) \text{ is relatively compact} \right\}$$

which is a closed subset of $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ and consequently also a Banach space with respect to the same operator norm (2.2). We write $\mathscr{K}(\mathcal{H}_1)$ for $\mathscr{K}(\mathcal{H}_1, \mathcal{H}_1)$. Another subset of $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of finite-rank operators

$$\mathscr{R}(\mathcal{H}_1, \mathcal{H}_2) := \{A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2) \mid \dim \mathsf{ran}\,(A) < \infty\} \quad . \tag{2.3}$$

These operators can be represented in a canonical way: let $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$; a canonical a rank-1 operator is an operator of the form

$$\theta_{v,u}(w) := \langle u \,|\, w \rangle_{\mathcal{H}_1} v \quad , \tag{2.4}$$

acting on a $w \in \mathcal{H}_1$. Let $\{u_i\}$ and $\{v_j\}$ be orthonormal bases in $\mathcal{H}_1$ respectively $\mathcal{H}_2$, then we can write any operator $T \in \mathscr{R}(\mathcal{H}_1, \mathcal{H}_2)$ with $m = \dim \mathsf{ran}\,(T)$ like

$$Tw = \sum_{i=1}^{m} \theta_{v_i,u_i}(w) = \sum_{i=1}^{m} \alpha_i \langle u_i \,|\, w \rangle_{\mathcal{H}_1} v_i \tag{2.5}$$

where only finitely many basis elements from both Hilbert spaces contribute and the coefficients α_i are positive. These operators are clearly compact as their range is bounded and finite-dimensional and thus relatively compact. It is well-known that the norm-closure of $\mathscr{R}(\mathcal{H}_1, \mathcal{H}_2)$ in $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ coincides with set of compact operators. From spectral theory of compact operators one observes that (2.5) corresponds to the singular value decomposition of a compact operator for $m \to \infty$; the values α_i are then the singular values which accumulate to zero. Compact and finite-rank operators have the property that they are left-ideals for $\mathscr{B}(\mathcal{H}_2)$ and right-ideals for $\mathscr{B}(\mathcal{H}_1)$ and thus two-sided ideals if $\mathcal{H}_1 = \mathcal{H}_2$. Several further ideals can be introduced as follows: if the sequence of singular values of a compact operator is in $l^p(\mathbb{N})$ for $p \in [1, \infty)$, i.e.

$$\sum_{i\in\mathbb{N}} \alpha_i^p < \infty \quad ,$$

this operator is called *p-Schatten-class operator* and we define the set of these operators with

$$\mathscr{S}^p(\mathcal{H}_1,\mathcal{H}_2) := \{A \in \mathscr{K}(\mathcal{H}_1,\mathcal{H}_2) \,|\, \{\alpha_i\}_i \in l^p(\mathbb{N})\} \quad .$$

As finite-rank operators come with terminating sequences of singular values, they are contained in any $l^p(\mathbb{N})$ such that $\mathscr{R} \subset \mathscr{S}^p$ for all $p \in [1,\infty)$. In fact it is a dense subspace in $\mathscr{S}^1$. Two specific cases are *Hilbert-Schmidt operators* for $p = 2$ and *trace-class operators* for $p = 1$. If $1 \leq p < q < \infty$, then $\mathscr{S}^p \subset \mathscr{S}^q$. The composition of operators in different Schatten-classes are trace-class if their orders are a conjugated pair[5]. More precisely, if p and q are conjugated pairs and $A \in \mathscr{S}^p(\mathcal{H}_1,\mathcal{H}_2)$ and $B \in \mathscr{S}^p(\mathcal{H}_2,\mathcal{H}_3)$, then the composition satisfies $B \circ A \in \mathscr{S}^1(\mathcal{H}_1,\mathcal{H}_3)$. On the other hand one can express any trace-class operator as a finite sum of compositions of two operators in Schatten-classes with orders which are conjugated number pairs. The easiest and most known realisation is that any trace-class operator is a finite sum of compositions of Hilbert-Schmidt operators. For $A \in \mathscr{S}^1(\mathcal{H}_1)$ we define its *trace* to be the operation

$$\mathrm{Tr}\,(A) := \sum_{i\in\mathbb{N}} \langle Au_i \,|\, v_i \rangle_{\mathcal{H}_1} \tag{2.6}$$

for two orthonormal bases $\{u_i\}$ and $\{v_j\}$ in $\mathcal{H}_1$. If $A \in \mathscr{S}^1(\mathcal{H}_1,\mathcal{H}_2)$ for $\mathcal{H}_2 \neq \mathcal{H}_1$, we either need to consider A as an operator on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$, in order to apply the definition of the trace, or we take $|A|$ as operator on $\mathcal{H}_1$ from the polar decomposition of A. The latter way of computing the trace is motivated from the fact that an operator A is in $\mathscr{S}^1(\mathcal{H}_1,\mathcal{H}_2)$ if and only if $\mathrm{Tr}\,(|A|) < \infty$. In a nutshell, we have the following inclusions of ideals in $\mathscr{B}$:

$$\mathscr{R} \subset \mathscr{S}^p \subset \mathscr{K} \subset \mathscr{B} \tag{2.7}$$

for any $p \in [1,\infty)$. We note that the composition $B \circ A$ of two bounded operators is trace-class if and only if one of them is trace-class, both Hilbert-Schmidt operators or in any other Schatten-classes such that their orders are a conjugated pair. In this case, one has the additional property $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$ and this carries over to the rule that one can cyclically commute operators of a finite composition under the trace as long as the new compositions are well-defined and remain trace-class.

The inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ induces the notion of an adjoint operator A^* which is defined via

$$\langle Au \,|\, v \rangle_{\mathcal{H}_2} = \langle u \,|\, A^*v \rangle_{\mathcal{H}_1} \quad . \tag{2.8}$$

In fact, if $A \in \mathscr{A}(\mathcal{H}_1,\mathcal{H}_2)$, where $\mathscr{A}$ is in any operator space in (2.7), the adjoint satisfies $A^* \in \mathscr{A}(\mathcal{H}_2,\mathcal{H}_1)$. Adjoining an operator is an anti-linear map which satisfies $A^{**} = A$ and $\|A^*\|_{\mathscr{A}(\mathcal{H}_2,\mathcal{H}_1)} = \|A\|_{\mathscr{A}(\mathcal{H}_1,\mathcal{H}_2)}$ and thus $\|A^*A\|_{\mathscr{B}(\mathcal{H}_1)} = \|AA^*\|_{\mathscr{B}(\mathcal{H}_2)} = \|A\|^2_{\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)}$. One also observes from (2.8) that $(BA)^* = A^*B^*$ for $B \in \mathscr{B}(\mathcal{H}_2,\mathcal{H}_3)$. We can define certain further characterisations of operators with respect to adjoining.

Definition 2.1.2. Let $A \in \mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)$; we call A

(a) an *isometry* if $\langle Au \,|\, Av \rangle_{\mathcal{H}_2} = \langle u \,|\, v \rangle_{\mathcal{H}_1}$ for all $u, v \in \mathcal{H}_1$. This is equivalent to $A^*A = \mathbb{1}_{\mathcal{H}_1}$.

(b) a *coisometry* if $\langle A^*u \,|\, A^*v \rangle_{\mathcal{H}_1} = \langle u \,|\, v \rangle_{\mathcal{H}_2}$ for all $u, v \in \mathcal{H}_2$. This is equivalent to $AA^* = \mathbb{1}_{\mathcal{H}_2}$.

[5] For $p, q \in [1,\infty)$ we call (p, q) a *conjugated (number) pair* if $1 = 1/p + 1/q$.

(c) *unitary* if A is an isometry and a coisometry.

If moreover $\mathcal{H}_1 = \mathcal{H} = \mathcal{H}_2$, we say A is

(d) *self-adjoint* if $\langle Au \mid v \rangle_{\mathcal{H}} = \langle u \mid Av \rangle_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$.

(e) *skew-adjoint* if $\langle Au \mid v \rangle_{\mathcal{H}} = -\langle u \mid Av \rangle_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$.

(f) *normal* if $AA^* = A^*A$.

(g) an *orthogonal projection* if $A^2 = A = A^*$ and $\mathsf{ker}\,(A) \perp \mathsf{ran}\,(A)$.

Any projection is bounded with norm 1 and a positive operator. If $V \subset \mathcal{H}$, we write P_V for the projection from $\mathcal{H}$ onto V. An equivalent and more practical characterisation of unitary operators is as isometries with dense ranges in $\mathcal{H}_2$. The trace of the adjoint of a trace-class operator A is the complex conjugate of the trace of A: $\mathrm{Tr}\,(A^*) = \overline{\mathrm{Tr}\,(A)}$.

If V is a TVS, one can define the *dual pairing* as map $[\,\cdot \mid \cdot\,]_V : \overline{V}^* \times V \to \mathbb{C}$ such that

$$[w \mid v]_V = w(v)$$

for $v \in V$ and $w \in \overline{V}^*$. If A is a map on V, we write $A^\dagger$ for the *dual/adjoint operator* with respect to this dual pairing which satisfies

$$[w \mid Av]_V = [A^\dagger w \mid v]_V \quad .$$

2.1.2. A short recap about unbounded operators

As we are going to deal with differential operators, we need to recall some basic facts about unbounded operators as it becomes important to consider operators in one Hilbert space instead of two different Hilbert spaces. An *unbounded operator* between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is a linear operator $A : \mathsf{dom}_{\mathcal{H}_1}(A) \to \mathcal{H}_2$ such that the domain $\mathsf{dom}_{\mathcal{H}_1}(A)$ is a linear subspace in $\mathcal{H}_1$. We designate the set of these operators with $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$. Known operations between unbounded operators and certain subspaces have to be refined by taking the domain into account: let A, A' be unbounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and B an unbounded operator with domain in $\mathcal{H}_2$, then

(a) $\mathsf{ker}\,(A) := \{u \in \mathsf{dom}_{\mathcal{H}_1}(A) \mid Au = 0\}$;

(b) $\mathsf{ran}\,(A) := \{v \mid v = Au \text{ for a } u \in \mathsf{dom}_{\mathcal{H}_1}(A)\}$;

(c) The inverse operator A^{-1} of A exists if and only if $\mathsf{ker}\,(A) = \{0\}$ (i.e. A is injective), $\mathsf{dom}_{\mathcal{H}_2}(A^{-1}) = \mathsf{ran}\,(A)$, $\mathsf{ran}\,\left(A^{-1}\right) = \mathsf{dom}_{\mathcal{H}_1}(A)$, and $A^{-1}v = u$ if and only if $v = Au$;

(d) $\mathsf{dom}_{\mathcal{H}_1}(B \circ A) := \{u \mid u \in \mathsf{dom}_{\mathcal{H}_1}(A) : Au \in \mathsf{dom}_{\mathcal{H}_2}(B)\}$;

(e) $\mathsf{dom}_{\mathcal{H}_1}(\alpha A + \beta A') := \mathsf{dom}_{\mathcal{H}_1}(A) \cap \mathsf{dom}_{\mathcal{H}_1}(A')$ for all $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

We say that A is *densely defined* if $\mathsf{dom}_{\mathcal{H}_1}(A)$ is dense in $\mathcal{H}_1$ such that the adjoint operator becomes well-defined by $\langle Au \mid v \rangle_{\mathcal{H}_2} = \langle u \mid A^*v \rangle_{\mathcal{H}_1}$ and its domain is

$$\mathsf{dom}_{\mathcal{H}_2}(A^*) = \left\{ v \in \mathcal{H}_2 \,\middle|\, \mathsf{dom}_{\mathcal{H}_1}(A) \ni u \mapsto \langle Au \mid v \rangle_{\mathcal{H}_2} \text{ is a continuous linear funtional} \right\} \quad .$$

If for a densely defined operator A there exists a $C > 0$ such that $\|\mathcal{H}_2\|_{Au} \leq C\,\|u\|_{\mathcal{H}_1}$ for $u \in \mathsf{dom}_{\mathcal{H}_1}(A)$, then A can be uniquely extended to a bounded linear operator from the closure of $\mathsf{dom}_{\mathcal{H}_1}(A)$ to $\mathcal{H}_2$. In this way one observes that $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$. The *graph* of a linear operator A is

$$\mathrm{Graph}(A) := \left\{ (u, Au) \,\middle|\, u \in \mathsf{dom}_{\mathcal{H}_1}(A) \right\} \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \quad .$$

The linear operator is *closed* if Graph(A) is a closed subset in $\mathcal{H}_1 \oplus \mathcal{H}_2$. If this is not the case, but the closure of Graph(A) is a graph of a linear operator $\overline{A}$, A is called *closeable* and $\overline{A}$ its *closure*. Being closed is equivalent with the implication that if $u_n \to u$ and $Au_n \to v$, then $v = Au$. If already $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, then A is always closeable and closed if and only if $\mathsf{dom}_{\mathcal{H}_1}(A)$ equals or is closed in $\mathcal{H}_1$. If the operator is closed, then its graph is a closed subspace of a Hilbert space which induces a Hilbert space structure on the graph. Thus, the graph becomes itself a Hilbert space with the *graph norm*

$$\|u\|^2_{\mathrm{Graph}(A)} = \|u\|^2_{\mathcal{H}_1} + \|Au\|^2_{\mathcal{H}_2} \quad .$$

Equipping the domain $\mathsf{dom}_{\mathcal{H}_1}(A)$ of the closed operator A with this norm makes it a Hilbert space and $A : \mathsf{dom}_{\mathcal{H}_1}(A) \to \mathcal{H}_2$ becomes bounded.

Let $\mathcal{H}_1 = \mathcal{H} = \mathcal{H}_2$. A densely defined operator $A \in \mathscr{L}(\mathcal{H})$ is *symmetric/formally self-adjoint* if $\langle Au \,|\, v \rangle_{\mathcal{H}} = \langle u \,|\, Av \rangle_{\mathcal{H}}$ for $u, v \in \mathsf{dom}_{\mathcal{H}}(A)$ which is equivalent with $\mathsf{dom}_{\mathcal{H}}(A) \subset \mathsf{dom}_{\mathcal{H}}(A^*)$. If in addition $\mathsf{dom}_{\mathcal{H}}(A) \supset \mathsf{dom}_{\mathcal{H}}(A^*)$ and thus $\mathsf{dom}_{\mathcal{H}}(A) = \mathsf{dom}_{\mathcal{H}}(A^*)$, the operator is *self-adjoint* and $A = A^*$. If the closure of a linear operator is self-adjoint, we call the operator *essentially self-adjoint*. We have the following further useful characterisations of self-adjoint operators: let $A \in \mathscr{L}(\mathcal{H})$ be densely defined, then

(1) A is self-adjoint if and only if A is closed, symmetric and $\mathsf{ker}\,(A^* \pm \mathbb{i}) = \{0\}$;

(2) A is self-adjoint if and only if A is symmetric and $\mathsf{ran}\,(A \pm \mathbb{i}) = \mathcal{H}$;

(3) A is self-adjoint if and only if A is symmetric and the spectrum $\sigma(A)$ is real;

(4) if A is also closed, then A^*A is self-adjoint and $\mathbb{1}_{\mathcal{H}} + A^*A$ has bounded inverse.

(5) (Hellinger-Toeplitz) If A is symmetric and $\mathsf{dom}_{\mathcal{H}}(A) = \mathcal{H}$, then A is bounded;

(6) if A is moreover bounded, then it is a positive operator (i.e. $\langle Au \,|\, u \rangle_{\mathcal{H}} \geq 0$ for $u \in \mathsf{dom}_{\mathcal{H}}(A)$) if and only if A is self-adjoint with $\sigma(A) \subset [0, \infty)$;

(7) if A is moreover bounded, then it is a positive operator if and only if there exists a bounded (or self-adjoint) operator B on $\mathcal{H}$, such that $A = B^*B$ (or $A = B^2$).

We observe that an unbounded operator behaves somehow nicely if it is densely defined and closed; see for example [Kat76, Chap.3 §5] for a precise analysis. We introduce the notation $\mathscr{C}(\mathcal{H}_1, \mathcal{H}_2)$ for the space of closed and densily defined operators between Hilbert spaces. Because any bounded operator is closed if its domain is closed, we observe $\mathscr{B} \subset \mathscr{C}$, such that the inclusion (2.7) can be extended to

$$\mathscr{R} \subset \mathscr{S}^p \subset \mathscr{K} \subset \mathscr{B} \subset \mathscr{C} \subset \mathscr{L} \tag{2.9}$$

for any $p \in [1, \infty)$. Now we recall the definition of a Fredholm operator.

Definition 2.1.3. An operator $A \in \mathscr{C}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *Fredholm operator* if the dimensions of $\mathsf{ker}\,(A)$ and of $\mathsf{coker}\,(A)$ are finite. The *index* of a Fredholm operator is defined as

$$\mathrm{ind}(A) = \dim \mathsf{ker}\,(A) - \dim \mathsf{coker}\,(A) \quad .$$

It is not necessary that the operator is densely defined, but becomes helpful in practice. Fredholmness implies that $\mathsf{ran}\,(A)$ is closed in $\mathcal{H}_2$, such that $\mathsf{coker}\,(A) \cong (\mathsf{ran}\,(A))^{\perp} = \mathsf{ker}\,(A^*)$. Thus, one could have also defined Fredholmness of an operator to be equivalent with closed (densely defined) operators for which $\dim \mathsf{ker}\,(A) < \infty$, $\mathsf{ran}\,(A)$ is closed, and $\mathrm{codim}\,\mathsf{ran}\,(A) = \dim \mathsf{ker}\,(A^*) < \infty$ and the index can be calculated with

$$\mathrm{ind}(A) = \dim \mathsf{ker}\,(A) - \mathrm{codim}\,\mathsf{ran}\,(A) = \dim \mathsf{ker}\,(A) - \dim \mathsf{ker}\,(A^*) \quad .$$

This quantity is locally constant and $\mathbb{Z}$-valued. We will distinguish between bounded Fredholm operators and closed Fredholm operators with the designations

$$\begin{aligned} \mathscr{F}(\mathcal{H}_1, \mathcal{H}_2) := \quad & \mathscr{BF}(\mathcal{H}_1, \mathcal{H}_2) := \{A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2) \,|\, A \,\text{is Fredholm}\} \ , \\ & \mathscr{CF}(\mathcal{H}_1, \mathcal{H}_2) := \{A \in \mathscr{C}(\mathcal{H}_1, \mathcal{H}_2) \,|\, A \,\text{is Fredholm}\} \ . \end{aligned}$$

A more practical characterisation of (bounded) Fredholm operators is the existence of an inverse modulo some ideal.

Proposition 2.1.4. *Let $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$; $A \in \mathscr{F}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if there exist left- and right-parametrices $B_1 \in \mathscr{B}(\mathcal{H}_1), B_2 \in \mathscr{B}(\mathcal{H}_2)$ with*

$$B_2 A - \mathbb{1}_{\mathcal{H}_2} \in \mathscr{J}(\mathcal{H}_2) \quad \textit{and} \quad AB_1 - \mathbb{1}_{\mathcal{H}_1} \in \mathscr{J}(\mathcal{H}_1)$$

where $\mathscr{J}$ stand for any ideal in (2.7).

We will use several functional calculi in the forthcoming thesis. We refer to [Kad83] for a vast description of the holomorphic, continuous and Borel functional calculus. In addition, we refer to [Haa06, Chap.2] for the holomorphic calculus of *sectorial operators* for which we recall some basics.

Definition 2.1.5. An operator $A \in \mathscr{C}(\mathcal{H}_1)$ is *sectorial of angle* ω if its spectrum $\sigma(A)$ lies inside the closure of a sector[6]

$$S_\omega := \left\{ z \in \mathbb{C} \setminus \{0\} \;\middle|\; |\arg(z)| < \omega \right\}$$

for $\omega \in (0, \pi]$ and the resolvent $\mathsf{R}(\lambda, A)$ satisfies

$$\sup_{\lambda \in \mathbb{C} \setminus \overline{S}_\varpi} \left\{ \|\lambda \mathsf{R}(\lambda, A)\|_{\mathscr{B}(\mathcal{H}_1)} \right\} < \infty \quad \forall\, \varpi \in (\omega, \pi) \quad .$$

An important example of such an operator are multiplication operators for which

$$\|\mathsf{R}(\lambda, A)\|_{\mathscr{B}(\mathcal{H}_1)} \leq \frac{1}{\mathrm{dist}(\lambda, \overline{S}_\omega)}$$

[6] Some authors consider the sector to be contained in the resolvent set $\rho(A)$ which is just a change of the point of view.

holds for $\omega \in (0, \pi]$ and $\lambda \in \mathbb{C} \setminus \overline{S}_\omega$. The functional calculus of normal operators then implies that these operators are sectorial. Let A be sectorial of angle ω and $f(z)$ be a bounded holomorphic function on the sector S_ϕ with angle $\phi \in (\omega, \pi)$ such that the function decays regularly at $z = 0$ and $|z| \to \infty$. One can define the operator $f(A)$ with a Banach space-valued Cauchy integral formula:

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1(\varpi)} f(z) \mathsf{R}(z; A) \, \mathrm{d}z \tag{2.10}$$

with $\Gamma_1(\varpi)$ as boundary path of the sector S_ϖ for $\varpi \in (\omega, \phi)$, given by the parametrisation

$$\Gamma_1(\varpi) := \begin{cases} -\rho \mathrm{e}^{\mathrm{i}\varpi} & \rho \in (-\infty, 0] \\ \rho \mathrm{e}^{-\mathrm{i}\varpi} & \rho \in (0, \infty) \end{cases} \quad \text{for} \quad . \tag{2.11}$$

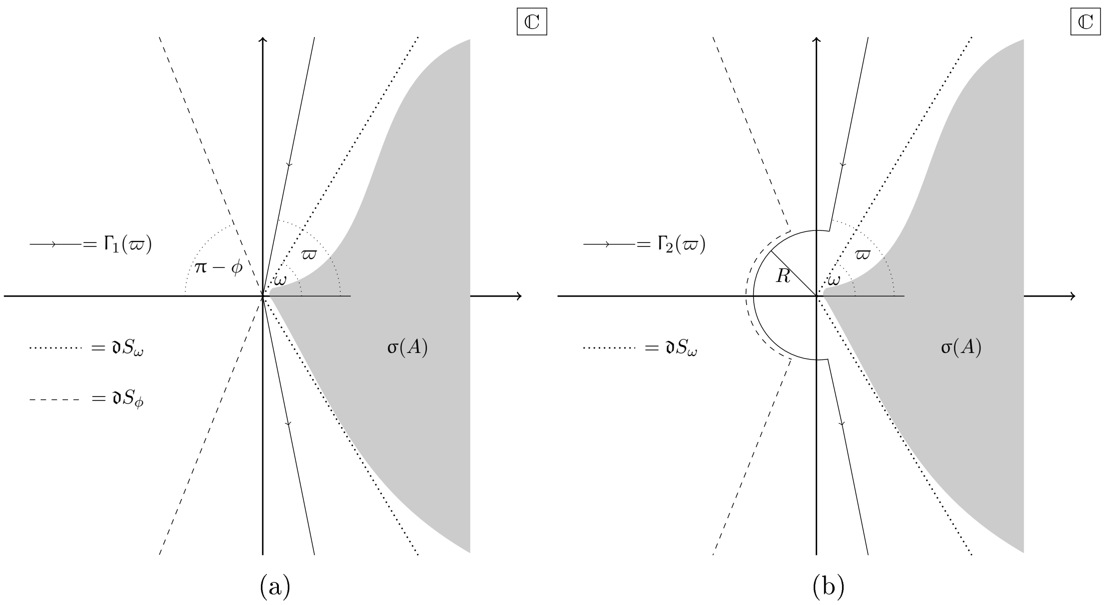

Figure 2.1.: Illustration of the contours (2.11) in (a) and (2.12) in (b).

The condition on f, being only holomorphic at zero, can be relaxed, but then we need to circle around the origin in the complex plane with some radius $R > 0$ and we have to replace $\Gamma_1(\varpi)$ with

$$\Gamma_2(\varpi) := \begin{cases} -\rho \mathrm{e}^{\mathrm{i}\varpi} & \rho \in (-\infty, -R] \\ R\mathrm{e}^{-\mathrm{i}\alpha} \quad \text{for} & \alpha \in (-\varpi, \varpi) \\ \rho \mathrm{e}^{-\mathrm{i}\varpi} & \rho \in [R, \infty) \end{cases} . \tag{2.12}$$

(2.10) then induces a holomorphic functional calculus which is investigated in [Haa06, Chap.2 & 3] in some detail.

2.2. Basics from Differential Geometry

From now on M will always stand for a smooth manifold which is assumed to be Hausdorff and satisfies the second axiom of countability, i.e. there is a countable basis for the topology of M. The dimension of the manifold is denoted with $\dim(M)$. The set of smooth functions on M is denoted with $C^\infty(M)$. A *submanifold* Σ of a smooth manifold is a subset $\Sigma \subset M$ which carries the structure of a smooth manifold. The *tangent* and the *cotangent bundle* of M are

$$TM := \bigsqcup_{p\in M} T_pM \quad \text{and} \quad T^*M := \bigsqcup_{p\in M} T_p^*M$$

where T_pM is the tangent space and $T_p^*M = (T_pM)^*$ the cotangent space, each at p. From these two bundles one can define the (r, s)-*tensor bundle*

$$\mathcal{T}_s{}^r M := \bigsqcup_{p\in M} (\mathcal{T}\, r_s)_p M$$

where $(\mathcal{T}_s{}^r)_p M := (T_p^*M)^{\otimes r} \otimes (T_pM)^{\otimes s}$ is a r-fold tensor product of the cotangent space with the s-fold tensor product of the tangent space. These are some examples of fibre bundles. In general, a space E is a *fibre bundle* over M with fibre F if there exists a continuous surjective function $\pi : E \to M$ and for all p a neighbourhood $U_p \subset M$ and a homeomorphism $\psi : \pi^{-1}(U_p) \to U_p \times F$ such that the following diagram commutes:

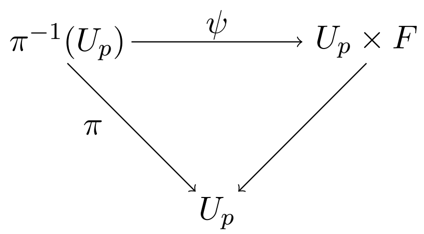

Figure 2.2.: Diagram for the definition of a fibre bundle.

The manifold M is called *base* of the fibre bundle. *Smooth sections* of a fibre bundle are smooth maps $\sigma : M \to E$ such that $(\pi \circ \sigma)(p) = p$ for all $p \in M$. The set of all smooth sections is denoted by $C^\infty(M, E)$. If the base is clear from the context or by the notation of the fibre bundle, we just write $C^\infty(E)$. We denote any fibre bundle with $E \to M$ and we just write E if the base is clear from the context or the notation of E. We recall some important examples of fibre bundles:

(a) *trivial bundle*: $E = M \times F$ and π is the projection on the first factor of the Cartesian product.

(b) *coverings*: the fibre F is a discrete set and π is a covering map. More informations about coverings are given in Section 5.1.

(c) *vector bundle of rank* k: the fibre F is a real/ complex vector space of dimension k.

(d) *principal* G-*bundle*: a fibre bundle P with projection $\pi : P \to M$ and a continuous right action on P with respect to a topological group G (structure group), such that the action is free and transitive, and $\pi(p \cdot g) = \pi(p)$ is satisfied for all $p \in P$ and $g \in \mathsf{G}$.

(e) *associated vector bundle*: if $P \to M$ is a principal G-bundle and $\rho : \mathsf{G} \to \mathsf{Aut}(V)$ a real or complex representation of the group with vector space V as representation space, the quotient space $P \times V/\mathsf{G}$ with equivalence relation $[p, v] \sim [p \cdot g, \rho(g^{-1})v]$ for $p \in P, v \in V$ and all $g \in \mathsf{G}$ becomes a vector bundle.

The tangent and cotangent bundle are vector bundles of rank n. An example of a principal bundle is the *frame bundle* of a vector bundle $E \to M$ which we denote with $P_{\mathsf{GL}}(E)$. It is a principal bundle with structure group $\mathsf{GL}_n(\mathbb{R})$ and group action $(A \cdot g)(v) = A(g \cdot v)$ where $A : \mathbb{R}^n \to E_p$ $(p \in M)$, $g \in \mathsf{G}$ and $v \in \mathbb{R}^n$. Sections of these bundles are *frames*, i.e. they assign to each fibre E_p at each point $p \in M$ a basis. In this sense, the vector bundle E can be viewed as an associated vector bundle $E \cong P_{\mathsf{GL}}(E) \times \mathbb{R}^n/\mathsf{GL}_n(\mathbb{R})$. The frame bundle of TM is denoted by $P_{\mathsf{GL}}(M)$. By restricting the structure group to $\mathsf{O}_n(\mathbb{R})$ (orthogonal transformation in $\mathbb{R}^n$) one can introduce orthogonal frames and moreover oriented and orthonormal frames if the structure group can be restricted to $\mathsf{SO}_n(\mathbb{R})$. Further examples of such bundles are going to occur at different points.

For two given (complex) vector bundles $E, F \to M$ one can construct further vector bundles which come with different notations. Denote with E_p and F_p the fibres of the vector bundles over $p \in M$, then one can introduce the

(a) *dual bundle* E^* for which the fibres over a point p are given by the dual spaces E_p^*.

(b) *conjugate bundle* $\overline{E}$ for which the fibres over a point p are given by $\overline{E}_p$, i.e. each element in E_p is complex conjugated.

(c) *Whitney/direct sum bundle* $E \oplus F$ for which the fibres over a point p are given by the vector space direct sum $E_p \oplus F_p$. A k-fold direct sum of one vector bundle is denoted by $E^{\oplus k} := E \oplus E \oplus ... \oplus E$ (k summands).

(d) *tensor bundle* $E \otimes F$ for which the fibres over a point p are given by the tensor vector space $E_p \otimes F_p$. The k-fold tensor product of a vector bundle is denoted by $E^{\otimes k} := E \otimes E \otimes ... \otimes E$ (k factors).

(e) *vector subbundle* $F \subset E$ which is defined to be the vector bundle $F \to M$ such that F_p is a vector subspace of E_p for each $p \in M$.

(f) *exterior power bundle* $E \wedge F$ for which the fibres over a point p are given by $E_p \wedge F_p$. It is a subbundle of the tensor bundle $E \otimes F$, consisting of totally antisymmetric combinations of tensor products. The k-fold exterior power of vector bundle is defined by $E^{\wedge k}$.

(g) *symmetric power bundle* $E \odot F$ for which the fibres over a point p are given by $E_p \odot F_p$. It is a subbundle of $E \otimes F$ which consists of totally symmetric combinations of tensor products. k-fold symmetric powers are denoted by $E^{\odot k}$.

(h) *homomorphism bundle* $\mathsf{Hom}(E, F) := F \otimes E$ for which the fibres are linear vector space maps from E_p to F_p. The *endomorphism bundle* is $\mathsf{End}(E) := \mathsf{Hom}(E, E)$.

One observes from (d) that the (r, s)-tensor bundle is in fact also a vector bundle over M. Smooth sections for some of these vector bundles get their own notation: $\mathfrak{X}(M) := C^\infty(TM)$ and $\Omega^k(M) := C^\infty((T^*M)^{\wedge k})$ for $k \in \{0, 1, ..., \dim(M)\}$. Another set of smooth sections are densities on a manifold of dimension n: for $r \in \mathbb{R}$ a map $\mu : \Omega^n(M) \to C^\infty(M)$ is a $r \in \mathbb{R}$ an *r-density* if it satisfies $\mu(\lambda u) = |\lambda|^r \mu(u)$ for all $u \in \Omega^n(M) \setminus \{0\}$ and $\lambda \in C^\infty(M, \mathbb{R}_{>0})$. The set of all r-densities on M is denoted by $|\Omega^n(M)|^r$. One can also combine vector bundles which have different bases. Let $f : M' \to M$ be a smooth map between manifolds M and M'. The *pullback bundle* of a vector bundle $E \to M$ is a vector bundle over M', defined by

$$f^*(E) := \{(p, e) \in M' \times E \,|\, f(p) = \pi(e)\} \tag{2.13}$$

where $\pi : E \to M$. Let E be as introduced and $F \to M'$. The Cartesian product $M' \times M$ comes with two projections $\pi_M : M' \times M \to M$ and $\pi_{M'} : M' \times M \to M'$. The pullback of the bundles E and F to bundles over the Cartesian product allows to take the tensor product which defines the *exterior tensor bundle*

$$F \boxtimes E := \pi_{M'}^*(F) \otimes \pi_M^*(E) \quad .$$

In the context of operators on manifolds in subsection 4.1.2, which maps sections of $E \to M$ to sections of $F \to M'$, the external tensor product is combined with half-densities on the Cartesian product $M \times M'$ which are taken into account to have an invariant notation of integration. We will write $\mathbf{Hom}(E, F)$ for $\left|\Omega^{n+n'}(M' \times M)\right|^{\frac{1}{2}} \otimes F \boxtimes E^*$ where $\left|\Omega^{n+n'}(M' \times M)\right|^{\frac{1}{2}} \cong \left|\Omega^{n'}(M')\right|^{\frac{1}{2}} \boxtimes |\Omega^n(M)|^{\frac{1}{2}}$ with $n' = \dim(M')$.

An *orientation of a manifold* can be described in different ways. After we have introduced principal bundles one can think about the orientation as a reduction of the structure group for the frame bundle $P_{\mathsf{GL}}(M)$ from $\mathsf{GL}_n(\mathbb{R})$ to

$$\mathsf{GL}_n^+(\mathbb{R}) := \{A \in \mathsf{GL}_n(\mathbb{R}) \,|\, \det(A) > 0\} \quad . \tag{2.14}$$

This is equivalent to the existence of an atlas such that the transition functions are orientation preserving linear maps in the tangent space. In this sense, the manifold becomes orientable. Another equivalent description of orientation is given through the existence of a nowhere vanishing section ω of $(T^*M)^{\wedge \dim(M)}$ which is called *volume form*.

A connection on fibre bundles can be introduced in different ways. The common axiomatic way for vector bundles is a Koszul connection: a *covariant derivative*[7] is a map $\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ such that ∇^E is $\mathbb{R}$-linear and obeys the Leibniz rule $\nabla^E(fs) = (\mathrm{d}f) \otimes s + f\nabla^E s$ for all $f \in C^\infty(M)$, $s \in C^\infty(M, E)$. The *covariant derivative in direction X* of a vector field $X \in \mathfrak{X}(M)$ is an assignment $\nabla^E_X : \mathfrak{X}(M) \otimes C^\infty(M, E) \to C^\infty(M, E)$ which satisfies for $s_1, s_2 \in C^\infty(M, E)$ and $c \in \mathbb{R}$

(a) $X \mapsto \nabla^E_X s$ is $C^\infty(M)$-linear: $\nabla^E_{fX} s = f\nabla^E_X s$ for all $f \in C^\infty(M)$;

(b) $s \mapsto \nabla^E_X s$ is $\mathbb{R}$-linear: $\nabla^E_X(cs_1 + s_2) = c\nabla^E_X s_1 + \nabla^E_X s_2$;

(c) Leibniz rule: $\nabla^E_X(fs) = (Xf)s + f\nabla^E_X s$ for all $f \in C^\infty(M)$.

[7] As we will only consider connections on a vector bundle, we will parallely use the term connection for covariant derivative.

A connection on one or different vector bundles induce a connection on several combinations:

(a) *dual connection* $\nabla_X^{E^*}$: suppose E^* is the dual bundle to E with respect to the (pointwise) dual pairing $\langle \cdot \mid \cdot \rangle_{E_p} : E_p^* \times E_p \to \mathbb{C}$; the dual connection is defined to be the covariant derivative $\nabla_X^{E^*}$ which satisfies

$$\langle \nabla_X^{E^*} s_1 \mid s_2 \rangle_{E_p} := X \langle s_1 \mid s_2 \rangle_{E_p} - \langle s_1 \mid \nabla_X^E s_2 \rangle_{E_p}$$

at each point $p \in M$ and $s_1 \in E_p^*, s_2 \in E_p$.

(b) *tensor product connection* of two connections ∇^E and ∇^F for vector bundles E, F over M:

$$\nabla_X^{E \otimes F}(s_1 \otimes s_2) := (\nabla_X^E s_1) \otimes s_2 + s_1 \otimes (\nabla_X^F s_2)$$

for $s_1 \in C^\infty(M, E)$ and $s_2 \in C^\infty(M, F)$. This carries over to total symmetric and total antisymmetric tensor products:

$$\nabla_X^{E \odot F}(s_1 \odot s_2) := (\nabla_X^E s_1) \odot s_2 + s_1 \odot (\nabla_X^F s_2) \quad ,$$
$$\nabla_X^{E \wedge F}(s_1 \wedge s_2) := (\nabla_X^E s_1) \wedge s_2 + s_1 \wedge (\nabla_X^F s_2) \quad .$$

(c) *direct sum connection*: $\nabla_X^{E \oplus F}(s_1 \oplus s_2) = (\nabla_X^E s_1) \oplus (\nabla_X^F s_2)$.

(d) *homomorphism/endomorphism connection*: let $V \in C^\infty(M, \mathsf{Hom}(E, F))$ and $s \in C^\infty(M, E)$, then the homomorphism covariant derivative is defined by

$$\left(\nabla_X^{\mathsf{Hom}(E,F)} V \right)(s) := \nabla_X^F (V(s)) - V(\nabla_X^E s) \quad .$$

The endomorphism connection is thus given by replacing F with E which we denote with $\nabla_X^{\mathsf{End}(E)}$.

(e) *pullback connection*: let $f : M' \to M$ be a smooth map and $E \to M$ a vector bundle with connection ∇^E; the connection on the pullback bundle $f^*(E)$ in direction of a vector field $X \in \mathfrak{X}(M')$ is defined to be

$$\nabla_X^{f^*E}(f^*s) = \left(f^* \nabla^E \right)_X (f^*s) := f^* \left(\nabla_{f_*X}^E s \right) \tag{2.15}$$

where $s \in C^\infty(M, E)$ and f_*X is the (pointwise) pushforward of the vector field.

Every connection gives rise to the notion of *bundle curvature*:

$$\mathcal{R}_{X,Y}^E(s) := \nabla_X^E \nabla_Y^E s - \nabla_Y^E \nabla_X^E s - \nabla_{[X,Y]}^E s \tag{2.16}$$

with $X, Y \in \mathfrak{X}(M)$ and $s \in C^\infty(M, E)$. Another term, which is induced by a connection on a vector bundle, is parallel transport. Let γ be a curve on the manifold M; the *parallel transport* is a collection of vector space homomorphisms

$$\mathcal{P}_t^\tau(\gamma) : E_{\gamma(t)} \to E_{\gamma(\tau)}$$

which satisfy

(a) $\mathcal{P}_t^t = \mathbb{1}_{E_\gamma(t)}$;

(b) $\mathcal{P}_{\top}^{\tau}(\gamma) \circ \mathcal{P}_t^{\top}(\gamma) = \mathcal{P}_t^{\tau}(\gamma)$;

(c) $\mathcal{P}_t^{\tau}(\gamma)$ depends smoothly on the curve γ and on the parameters t, τ.

If we want to stress the vector bundle E, we write $\mathcal{P}^E$. Property (a) and (b) imply that the parallel transport is in fact a vector space isomorphism with inverse $(\mathcal{P}(\gamma)_\tau^t)^{-1} = \mathcal{P}(\gamma^{-1})_t^\tau$ where γ^{-1} denotes the curve γ with reverse orientation. A connection induces a parallel transport operator by integrating the system of ODE's $\nabla^E_{\dot\gamma} Y(t) = 0$ with initial condition $Y(0) = Y_0$ such that formally $Y(t) = \mathcal{P}_0^t Y_0$. On the other hand, having a parallel transport allows to define a covariant derivative: let $s \in C^\infty(M, E)$ and $X \in T_pM$, then

$$\nabla_X^E s|_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_t^0(\gamma)s(\gamma(t))\right|_{t=0} = \lim_{h\to 0} \frac{\mathcal{P}_h^0(\gamma)s(\gamma(h)) - s(\gamma(0))}{h} \tag{2.17}$$

where the curve γ is an integral curve with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = X$.

3. Lorentzian and Riemannian geometry

We recall some basic concepts of pseudo-Riemannian geometry and globally hyperbolic manifolds. At the end we briefly introduce the notion of manifolds with bounded geometry. We refer to the main literatur in [O'N83] for a complete overview of topics in pseudo-Riemannian geometry. Some supporting material has been taken from [Bau81, Sec.0.15], [BGM05, Sec.4] and in particular from [Lee19].

3.1. Pseudo-Riemannian geometry

3.1.1. General aspects

Let M be a n-dimensional manifold. A smooth section g of $C^\infty((T^*M)^{\odot 2})$ is called *pseudo-Riemannian metric* if at each $p \in M$ the bilinear form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is non-degenerate. For an arbitrary, but fixed point $p \in M$ let $\{e_i\}_{i=1}^n$ a basis of the tangent space T_pM such that any $v \in T_pM$ can be written as linear combination $v = \sum_{i=1}^n a_i e_i$. We can choose the basis in such a way that it diagonalise the quadratic form $g_p(v,v)$:

$$g_p(v,v) = \sum_{i=1}^{r(p)} a_i^2 - \sum_{i=1}^{s(p)} a_{i+r(p)}^2 =: \big(v \,\big|\, v\big)_{\mathbb{R}^n} \tag{3.1}$$

with $r(p)+s(p) \leq n$. (3.1) is the indefinite bilinear form on $\mathbb{R}^n$ with respect to its standard basis. T_pM decomposes into $T_pM = T_p^+M \oplus T_p^-M \oplus T_p^0M$ where

$$\begin{aligned} T_p^\pm M &:= \left\{ v \in \dot{T}_pM \,\middle|\, g_p(v,v) \gtrless 0 \right\} \quad ; \\ T_p^0 M &:= \left\{ v \in T_pM \,|\, g_p(v,v) = 0 \right\} \quad . \end{aligned} \tag{3.2}$$

We can relate their dimensions to the numbers r and s: $r(p) = \dim(T_p^+M)$ and $s(p) = \dim(T_p^-M)$; the dimension $d(p) := \dim(T_p^0M)$ describes the defect $d(p) = n-(r(p)+s(p))$. The triple $(r(p), s(p), d(p))$ is the *signature* of the quadratic form at p. Sylvester's law of inertia says that these numbers do not depend on the chosen basis. Moreover, the assumed smoothness and non-degeneracy imply that these numbers are independent of the point and thus (r,s,d) is a global invariant and $d = 0$ in particular. Hence we write (r,s) for $(r,s,0)$ which is the only case of interest. We call the metric g *Riemannian* and the tupel (M,g) *Riemannian manifold* if $s = 0$ and g becomes positive definite[8]. If $s = 1$, we call the metric *Lorentzian*[9] and the pair (M,g) a Lorentzian manifold. The three subspaces of

[8] $(r,s) = (0,n)$ implies a negative definite metric which becomes Riemannian after rescaling with (-1); we only refer to $(r,s) = (n,0)$ as the Riemannian case as the other possibility differs in a global sign.

[9] This is the *east-coast convention* which we are going to use in this thesis. Another common way to define a Lorentzian metric is the choice $s = n-1$ which is the so-called *west-coast convention*.

T_pM induce the following causal characterisations: a vector $v \in T_pM$ is called

(a) *spacelike* if $v \in T_p^+M$, i.e. $g_p(v,v) > 0$;

(b) *lightlike/null-vector* if $v \in \dot{T}_p^0 M$, i.e. $g_p(v,v) = 0$ for $v \neq 0$;

(c) *timelike* if $v \in T_p^- M$, i.e. $g_p(v,v) < 0$.

These definitions[10] transfer to vector fields $X \in \mathfrak{X}(M)$ and curves $\gamma \in C^\infty(I, M)$, provided that $X_p \in T_pM$ and respectively $\dot{\gamma}(p)$ are spacelike, lightlike or timelike at each point $p \in M$.

A Riemannian metric can always be locally constructed and globally extended with a partition of unity. The situation for general pseudo-Riemannian manifolds is more involved as the gluing of local pseudo-Riemannian metrics does not need to be non-degenerate. In order to check that a smooth manifold admits a pseudo-Riemannian metric, one needs to consider reductions of structure groups. The *pseudo-orthogonal group* is

$$\mathsf{O}(r,s) := \left\{ A \in \mathsf{GL}(n) \,\middle|\, \big(Av \,\big|\, Au\big)_{\mathbb{R}^n} = \big(v \,\big|\, u\big)_{\mathbb{R}^n} \,\forall\, u, v \in \mathbb{R}^n \right\} \quad , \tag{3.3}$$

i.e. the group of all linear mappings which left (3.1) invariant. We write $\mathsf{SO(r,s)}$ for the *special pseudo-orthogonal group* to indicate those elements in $\mathsf{O}(r,s)$ for which their determinant is one. The existence of a Riemannian metric is equivalent to the reduction of the frame bundle $P_{\mathsf{GL}}(M)$ to the orthogonal frame bundle $P_{\mathsf{O}(n)}(M)$. This transfers to the pseudo-Riemannian case: a smooth manifold admits a pseudo-Riemannian metric if the frame bundle is reduceable to the *pseudo-orthonormal bundle* $P_{\mathsf{O}(r,s)}(M)$. The tangent bundle is then viewed as associated vector bundle

$$TM = P_{\mathsf{O}(r,s)}(M) \times \mathbb{R}^n / \mathsf{O}(r,s) \quad .$$

Sections of $P_{\mathsf{O}(r,s)}(M)$ are local pseudo-(Riemannian-)orthonormal tangent frames $\{e_i\}_{i=1}^n$ with the property that $e_1, ..., e_r$ are spacelike and $e_{r+1}, ..., e_{r+s}$ are timelike:

$$g(e_i, e_j) = \epsilon_i \delta_{ij} \quad \text{with} \quad \epsilon_i = g(e_i, e_i) = \begin{cases} 1 & \\ & \text{for} \\ -1 & \end{cases} \begin{matrix} i = 1, ..., r \\ \\ i = r+1, ..., r+s \end{matrix} \quad . \tag{3.4}$$

We call such a basis *Riemann*-orthonormal respectively *Lorentz-orthonormal* if the metric is Riemannian respectively Lorentzian. Due to non-degeneracy of the metric, there is no lightlike basis element. Thus, any lightlike vector is a combination of timelike and spacelike basis elements. Moreover, the pseudo-orthonormal frame bundle can be reduced to a principal bundle with structure group $\mathsf{O}(r) \times \mathsf{O}(s)$ which then decomposes into a direct sum of principal bundles with respect to each factor:

$$P_{\mathsf{O}(r,s)}(M) \;\rightarrow\; P_{\mathsf{O}(r)\times\mathsf{O}(s)}(M) = P_{\mathsf{O}(r)}(M) \oplus P_{\mathsf{O}(s)}(M) \quad .$$

This implies a splitting of the tangent bundle into two subbundles:

$$TM = \big(P_{\mathsf{O}(r)}(M) \times \mathbb{R}^n / \mathsf{O}(r)\big) \oplus \big(P_{\mathsf{O}(s)}(M) \times \mathbb{R}^n / \mathsf{O}(s)\big) =: T^+M \oplus T^-M \quad .$$

[10] In other contexts where we need to distinguish between spacelike and timelike properties, we also use the synonyms spatial respectively temporal.

These two subbundles $T^{\pm}M$ are in fact the disjoint unions over $T^{\pm}_pM$ from (3.2) over all $p \in M$:

$$\begin{aligned} T^+M &= P_{\mathsf{O}(r)}(M) \times \mathbb{R}^n/\mathsf{O}(r) = \bigsqcup_{p\in M} T^+_pM \quad , \\ T^-M &= P_{\mathsf{O}(s)}(M) \times \mathbb{R}^n/\mathsf{O}(s) = \bigsqcup_{p\in M} T^-_pM \quad . \end{aligned} \tag{3.5}$$

T^+M is the spacelike subbundle whereas T^-M is called timelike bundle. Both are orthogonal to each other. Certain topological characteristic classes like the Stiefel-Whitney classes are independent of the concrete choice of the orthogonal splitting. Orientability in the pseudo-Riemannian setting becomes more involved: a pseudo-Riemannian manifold is

(a) *space-oriented* if the first Stiefel-Whitney class of T^+M vanishes, i.e. T^+M has orientation;

(b) *time-oriented* if the first Stiefel-Whitney class of T^-M vanishes, i.e. T^-M has orientation;

(c) *time and space-oriented* if the first Stiefel-Whitney classs of $T^{\pm}M$ vanish, i.e. $T^{\pm}M$ have orientation;

(d) *(fully) oriented* if the first Stiefel-Whitney class of TM vanishes.

Note that space and time orientablility does not necessarily imply full orientability of M. Another characterisation of orientability of a pseudo-Riemannian manifold can be phrased by means of principal bundles: a pseudo-Riemannian manifold is oriented/orientable if the pseudo-orthonormal frame bundle in addition reduces to a frame bundle with structure group $\mathsf{SO}(r,s)$ which has two connected components. Time and space orientability can be rephrased with subgroups of the pseudo-orthogonal group, explaining the inequivalence of fully orientability and concurrent time and space orientability. We refer to [Bau81, Satz 0.51] and the corresponding section in [O'N83, Chap.9] for more details.

The non-degeneracy of the metric allows to relate elements in T_pM to those in T^*_pM and vice versa in a unique way. This is achieved by the two *musical isomorphisms*, known as *flat* and *sharp isomorphisms*:

$$\begin{aligned} \flat : T_pM &\to T^*_pM \quad : \quad X \mapsto X^\flat(\cdot) := g_p(X,\cdot) \\ \sharp : T^*_pM &\to T_pM \quad : \quad \xi \mapsto \xi^\sharp \end{aligned} \quad ; \tag{3.6}$$

the latter one is defined in such a way that $g_p(\xi^\sharp, X) = \xi(X)$ for any $X \in T_pM$.

3.1.2. Globally hyperbolic manifolds

A subset Σ of a $(n+1)$-dimensional time-oriented Lorentzian manifold M with Lorentzian metric g is called *Cauchy hypersurface* if Σ is a smooth, embedded hypersurface and every inextendable timelike curve in M meets Σ exactly once. If M admits several Cauchy hypersurfaces, then all of them are homeomorphic to each other. A time-oriented Lorentzian manifold is called *globally hyperbolic* if and only if it contains a Cauchy hypersurface; see [Ger70, Thm.11]. Geroch as well as later on Bernal and Sánchez proved several results in order to classify these kinds of Lorentzian spaces and their properties: if a Cauchy hypersurface Σ is fixed, one can find a function on M which level sets are foliating the spacetime with Σ being one of them:

Theorem 3.1.1 (Theorem 1.2 in [BS06]). *Suppose (M, g) is a globally hyperbolic manifold and Σ a spatial Cauchy hypersurface; there exists a function $\mathcal{T} \in C^\infty(M, \mathbb{R})$ such that*

(1) $\Sigma_{t_0} = \Sigma$ for a fixed chosen $t_0 \in \mathbb{R}$,

(2) $\Sigma_t := \mathcal{T}^{-1}(t)$ is a Cauchy hypersurface $\forall\, t \in \mathbb{R} \setminus \{t_0\}$ if non-empty.

Geroch's topological splitting theorem says that any globally hyperbolic manifold M is homeomorphic to $\mathbb{R} \times \Sigma$. The following result shows that the manifold is even isometrically related to this product manifold:

Theorem 3.1.2 (*Geroch's splitting theorem*, Theorem 1.1 in [BS05] & [Ger70]). *(M, g) is isometric to the product manifold $\mathbb{R} \times \Sigma$ with Lorentzian metric*

$$g = -N^2 \,\mathrm{d}\mathcal{T}^{\otimes 2} + g_{\mathcal{T}}$$

where $\mathcal{T}$ is a surjective smooth function on M, $N \in C^\infty(M, \mathbb{R}_{>0})$ and $g_{\mathcal{T}}$ is a smooth one-parameter family of smooth Riemannian metrics on Σ, satisfying

(1) $\mathrm{grad}(\mathcal{T})$ is a past-directed timelike gradient on M,

(2) each hypersurface Σ_t is a spacelike Cauchy hypersurface with Riemannian metric $g_{\mathcal{T}}$, where $\Sigma_{t_0} := \Sigma$, and

(3) span $\{\mathrm{grad}|_p(\mathcal{T})\}$ is orthogonal to $T_p\Sigma_t$ with respect to $g_{\mathcal{T}}|_p$ at each $p \in \mathbb{R} \times \Sigma$.

$\mathcal{T}$ is referred to as *Cauchy temporal function*. The time domain for a certain globally hyperbolic manifold is denoted with $\mathcal{T}(M)$. The existence of such a function ensures that each level set can be interpreted as a slice $\{t\} \times \Sigma$ which from now is meant by Σ_t. The result can be extended to non-spacelike, non-smooth or achronal[11], but at least non-acausal Cauchy hypersurfaces; see [BS06] for more details. Theorem 3.1.2 furthermore suggests that along $\mathrm{grad}(\mathcal{T})$ the spacetime is foliated by these level sets wherefore the function N is called *lapse function (of the foliation)*. If $\mathcal{T}(M)$ does not contain any critical points of $\mathcal{T}$, then each level set is regular and the regular level set theorem ensures that Σ_t is a closed embedded submanifold of codimension one. Each embedding $\iota_t : \Sigma_t \hookrightarrow M$ becomes a proper map. In the following we will use $\mathrm{d}t$ and ∂_t instead of $\mathrm{d}\mathcal{T}$ and $\mathrm{grad}(\mathcal{T})$ to stress the time differentials/derivatives as coordinate (co-)vector with respect to a hypersurface Σ_t for $t \in \mathcal{T}(M)$. Hence and henceworth we will rewrite the metric of a globally hyperbolic manifold as

$$g = -N^2 \,\mathrm{d}t^{\otimes 2} + g_t \quad . \tag{3.7}$$

An alternative defintion of global hyperbolic manifolds is given by means of causal sets: for any $p \in M$ define

$$\mathcal{J}^+(p) := \{q \in M \,|\, \exists \text{ causal future-directed curve } \gamma : p \rightsquigarrow q\} \quad \text{and}$$
$$\mathcal{J}^-(p) := \{q \in M \,|\, \exists \text{ causal past-directed curve } \gamma : p \rightsquigarrow q\} \quad .$$

[11] No two points in Σ can be connected with a timelike curve.

For any subset $A \subset M$ put $\mathcal{J}^{\pm}(A) := \bigcup_{p\in A} \mathcal{J}^{\pm}(p)$ as *future* respectively *past light cone* of A. The *causal domain*, *domain of influence* or *light cone* of A is the union $\mathcal{J}(A) := \mathcal{J}^{+}(A) \cup \mathcal{J}^{-}(A)$.

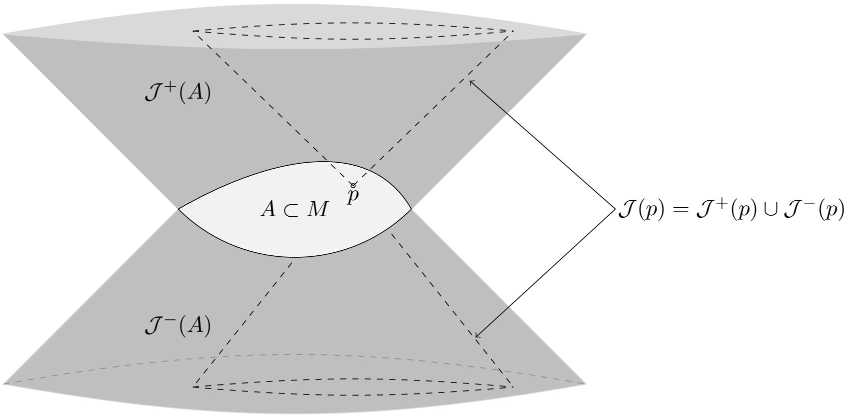

Figure 3.1.: Depiction of a future and past light cone of a subset $A \subset M$ and a point $p \in A$.

In comparison, the *domain of dependence*, *causal diamond* or *Cauchy developement* is defined as $\mathcal{D}(A) = \mathcal{D}^{+}(A) \cup \mathcal{D}^{-}(A)$ for $A \subset M$ such that A is achronal, where

$$\mathcal{D}^{+}(A) := \{p \in M \mid \text{every past inextendible causal curve through } p \text{ meets } A\} \quad \text{and}$$
$$\mathcal{D}^{-}(A) := \{p \in M \mid \text{every future inextendible causal curve through } p \text{ meets } A\}$$

are the *future* are respectively *past domain of dependence* of a subset A.

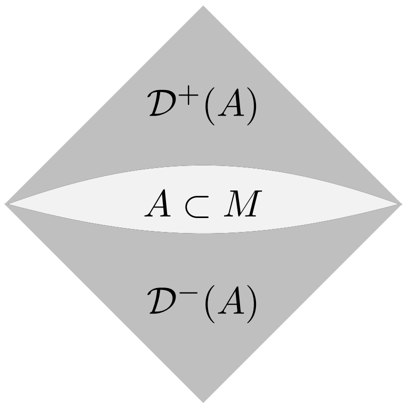

Figure 3.2.: Depiction of a future and past domain of dependence of a subset $A \subset M$.

In particular $A \subset \mathcal{D}^{\pm}(A) \subset \mathcal{J}^{\pm}(A)$. The concept of light cones can be used to rephrase global hyperbolicity in terms of causal sets:

Theorem 3.1.3 (Theorem 3.2 in [BS07]). *Given a time-oriented Lorentzian manifold (M, g); the following claims are equivalent to each other:*

(1) (M, g) is globally hyperbolic.

(2) (M, g) satisfies

(a) $\mathcal{J}^+(p) \cap \mathcal{J}^-(q)$ *is compact for all* $p, q \in M$,

(b) (M, g) *is causal, i.e. has no causal loops, and*

(c) (M, g) *is strongly causal, i.e. given a neighbourhood* $\mathcal{U}_p$ *for any* $p \in M$*, there exists a smaller neighbourhood* $\mathcal{V}_p \subset \mathcal{U}_p$*, containing* p*, such that any causal future-directed or past-directed curve on* M *with endpoints in* $\mathcal{V}_p$ *is entirely contained in* $\mathcal{U}_p$.

In terms of the domain of dependence, a globally hyperbolic manifold can be depicted as causal diamond of its Cauchy hypersurface Σ: $M = \mathcal{D}(\Sigma)$. A subset $A \subset M$ is called *spatially/spacelike compact* if A is a closed subset and there exists a compact subset $K \subset M$ such that $A \subset \mathcal{J}(K)$. The intersection of a spatially compact subset with any Cauchy hypersurface is compact. In contrast to this definition, one calls the whole manifold M spatially compact if every spacelike Cauchy hypersurface of M is compact.

A notion of timelike compactness can be introduced as well: a closed subset $A \subset M$ is *future/past compact* if $A \cap \mathcal{J}^\pm(K)$ is compact for every compact $K \subset M$; it is called *temporal/timelike compact* if A is both future and past compact. We call in contrast the whole manifold M temporal compact if $\mathcal{T}(M)$ is a closed interval. This is equivalent by saying that there exist $t_1, t_2 \in \mathbb{R}$ such that $\mathcal{T}(M) = [t_1, t_2]$. M is then viewed as the causal diamond $\mathcal{J}^+(\Sigma_1) \cap \mathcal{J}^-(\Sigma_2)$ for $\Sigma_1 = \Sigma_{t_1}$ and $\Sigma_2 = \Sigma_{t_2}$. The reader should recall footnote 4 on page 5.

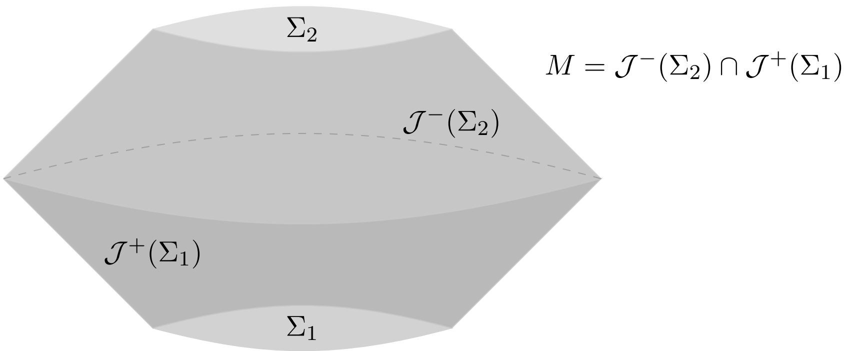

Figure 3.3.: Depiction of a temporal compact M.

3.1.3. Metric-affine connections

Let $\{e_i\}_{i=1}^n$ be a not necessarily pseudo-orthonormal tangent frame. The dual tangent or cotangent frame can be constructed by applying the flat-isomorphism from (3.6) to each basis element in the tangent frame; we denote the dual frame with $\{e^i\}_{i=1}^n$ which satisfies $e^i = (e_i)^\flat$ and $e^i(e_j) = \epsilon_i \delta^i_j$. The exterior derivative on each element of the dual frame defines the *anholonomity two-forms*

$$\Xi^i := -\,\mathrm{d}e^i = \frac{1}{2}\Xi_{jk}{}^i e^k \wedge e^j \tag{3.8}$$

with the structure coefficients $\Xi_{jk}{}^i$, defined by $[e_j, e_k] = \Xi_{jk}{}^i e_i$. A frame is called *holonomic* if all structure coefficients are zero and thus $[e_j, e_k] = 0$ or $\mathrm{d}e^i = 0$ for all $i, j, k \in \{1, ..., n\}$.

A connection on TM with respect to this local frame is given by the matrix coefficients $(\omega)^i_j$ of the connection one-form $\omega \in \Omega^1(M, TM)$:

$$\nabla^{TM}_{e_i} e_j := \sum_{k=1}^{n} \omega^k_j(e_i) e_k \quad . \tag{3.9}$$

The one-form coefficients can be expressed by means of the dual basis $\{e^i\}_{i=1}^n$ and coefficients $\Gamma^i_{kj} := \omega^i_j(e_k)$ such that

$$\omega^i_j = \sum_{k=1}^{n} \Gamma^i_{kj} e^k \quad . \tag{3.10}$$

In order to fix a connection on the tangent bundle, one needs to impose conditions, given by structure equations. For metric-affine connections one usually considers three geometric quantities which determine the connection. For this, we first introduce the notion of a *solder form* for any vector bundle E which is a vector-valued one-form $\theta \in \Omega^1(M, E)$, mapping as linear isomorphism from TM to E. For $E = TM$ such a form is given by the identity map. As we have a pseudo-Riemannian metric g, it induces the flat isomorphism from TM to T^*M in (3.6). Hence it gives rise to a solder form $\theta \in \Omega^1(M, TM)$ on T^*M as per $\theta(e_i) = (e_i)^\flat = e^i$. The exterior covariant derivative d^∇ of the solder form defines the vector-valued *torsion two-form* Θ of the connection:

$$\Theta^i := \mathrm{d}^\nabla \theta(e_i) = \mathrm{d}e^i + \sum_{k=1}^{n} \omega^i_k \wedge e^k = -\Xi^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} e^j \wedge e^k =: \frac{1}{2} T_{jk}{}^i e^j \wedge e^k \tag{3.11}$$

where $T_{jk}{}^i = \Gamma^i_{jk} - \Gamma^i_{kj} - \Xi_{jk}{}^i$ are the coefficients of the *torsion tensor* $T \in C^\infty(\mathcal{T}_1{}^2(M))$:

$$\mathfrak{X}(M)^{\times 2} \ni (X, Y) \quad \mapsto \quad T(X, Y) = \nabla^{TM}_X Y - \nabla^{TM}_Y X - [X, Y] \quad .$$

The *curvature two-form* $\Omega \in \Omega^2(M, TM)$ is defined by $\Omega = \mathrm{d}^\nabla \omega$ and locally given by the components $\mathcal{R}_{lkj}{}^i$:

$$\begin{aligned} \Omega^i_j &= \mathrm{d}\omega^i_j + \sum_{l=1}^{n} \omega^i_l \wedge \omega^l_j = \sum_{k,l=1}^{n} \left[e_l \Gamma^i_{kj} + \sum_{m=1}^{n} \left(\Gamma^i_{lm} \Gamma^m_{kj} - \frac{1}{2} \Gamma^i_{mj} \Xi_{lk}{}^m \right) \right] e^l \wedge e^k \\ &=: \frac{1}{2} \sum_{k,l=1}^{n} \mathcal{R}_{lkj}{}^i e^l \wedge e^k \end{aligned} \tag{3.12}$$

where

$$\mathcal{R}_{lkj}{}^i = e_l \Gamma^i_{kj} - e_k \Gamma^i_{lj} + \sum_{m=1}^{n} \left[\Gamma^i_{lm} \Gamma^m_{kj} - \Gamma^i_{km} \Gamma^m_{lj} - \Gamma^i_{mj} \Xi_{lk}{}^m \right]$$

are the coefficients of the *curvature tensor* $\mathcal{R} \in C^\infty(\mathcal{T}_3{}^1(M))$:

$$\mathfrak{X}(M)^{\times 3} \ni (X, Y, Z) \quad \mapsto \quad \mathcal{R}(X, Y)Z := \nabla^{TM}_X \nabla^{TM}_Y Z - \nabla^{TM}_Y \nabla^{TM}_X Z - \nabla^{TM}_{[X,Y]} Z \, . \tag{3.13}$$

The *non-metricity tensor* is a smooth section of $\mathcal{T}_3{}^0(M)$, defined by

$$\mathcal{Q}(Z, X, Y) := -(\nabla^{TM}_Z g)(X, Y) = g(\nabla^{TM}_Z X, Y) + g(X, \nabla^{TM}_Z Y) - Z g(X, Y)$$

for $X, Y, Z \in \mathfrak{X}(M)$. We abbreviate $g_{ij} := g(e_i, e_j)$ and $\Gamma_{ij,k} := g(\nabla^{TM}_{e_i} e_j, e_k)$ such that we can write its components as

$$\begin{aligned} \mathcal{Q}_{ijk} &= -(\nabla^{TM}_{e_i} g)(e_j, e_k) = g(\nabla^{TM}_{e_i} e_j, e_k) + g(e_j, \nabla^{TM}_{e_i} e_k) - e_i g(e_j, e_k) \\ &= \Gamma_{ij,k} + \Gamma_{ik,j} - e_i g_{jk} \quad . \end{aligned}$$

The *non-metricity one-form* is then defined by

$$\mathcal{Q}_{jk} = \sum_{i=1}^{n} \mathcal{Q}_{ijk} e^i \quad . \tag{3.14}$$

The equations (3.14), (3.11) and (3.12) are in this order the zeroth, first and second Maurer-Cartan structural equations which are used to determine the connection coefficients. The *Levi-Civita connection* is defined by setting torsion and non-metricity to zero, giving two determining equations from (3.11) and (3.14):

$$\mathcal{Q}_{jk} = 0 \quad \Leftrightarrow \quad \Gamma_{ij,k} + \Gamma_{ik,j} = e_i g_{jk} \tag{3.15}$$

$$\Theta^i = 0 \quad \Leftrightarrow \quad -\Xi^i = \mathrm{d}e^i = -\Gamma^i_{jk} e^j \wedge e^k \quad . \tag{3.16}$$

(3.16) can be used to derive the connection coefficients from the anholonomity one-form. The vanishing of the torsion implies

$$\Gamma^i_{jk} = \Gamma^i_{kj} + \Xi_{jk}{}^i \quad ; \tag{3.17}$$

it shows that the anholonomity perturbs the symmetric part of the connection coefficients. (3.15) describes how the anti-symmetric part of the connection coefficients are determined. The vanishing non-metricity implies the anti-symmetry of the connection one-forms: for a vector field Y we get after choosing a pseudo-orthonormal basis as in (3.4)

$$\begin{aligned} 0 &= Y g(e_i, e_j) = g(e_i, \nabla_Y e_j) + g(e_j, \nabla_Y e_i) = \sum_{k=0}^{n} \left[\omega^k_j(Y) g(e_i, e_k) + \omega^k_i(Y) g(e_j, e_k) \right] \\ &= \omega^j_i(Y) + \omega^i_j(Y) \quad . \end{aligned} \tag{3.18}$$

Combining (3.15) and (3.17) finally shows that the connection coefficients are given by

$$\Gamma_{ij,k} = \frac{1}{2} \left[e_i(g_{jk}) + e_j(g_{ik}) - e_k(g_{ij}) + \sum_{m=1}^{n} g_{mk} \Xi_{ji}{}^m + g_{im} \Xi_{kj}{}^m + g_{jm} \Xi_{ki}{}^m \right] \quad . \tag{3.19}$$

If we choose a holonomic frame, we get the *Christoffel symbols of first kind*:

$$\Gamma_{ij,k} = \frac{1}{2} \left[e_i(g_{jk}) + e_j(g_{ik}) - e_k(g_{ij}) \right] \quad .$$

The *Christoffel symbols of second kind* Γ^k_{ij} are then determined by $\Gamma_{ij,k} = \sum_{l=1}^{n} g_{lk} \Gamma^l_{ij}$. Thus, the Levi-Civita connection is with (3.19) locally given by

$$\nabla^{\mathrm{LC}}_{e_i} e_j := \sum_{k=1}^{n} \Gamma^k_{ij} e_k \quad . \tag{3.20}$$

An example of a holonomic basis is the coordinates frame $\{\partial_i\}_{i=1}^n$ for which each Christoffel symbols take the known form. The remaining equation (3.12) is left to determine the curvature tensor components. We suggest [McC92] for more informations about metric-affine connections.

It is a general fact that the Levi-Civita connection on a pseudo-Riemannian manifold is the unique connection which is torsion-free and compatible with the pseudo-Riemannian metric (vanishing non-metricity). It can be abstractly characterised by the *Koszul formula*

$$2g(\nabla_X^{LC} Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \quad . \tag{3.21}$$

Several further notions of curvature can be defined from (3.13) like the *Riemannian/metric curvature tensor*

$$\mathcal{R}(X,Y,Z,W) := g(\mathcal{R}(X,Y)Z,W)$$

for $X,Y,Z,W \in \mathfrak{X}(M)$ and from here the Ricci-endomorphism, the Ricci curvature, sectional curvatures and the scalar curvature $\mathcal{R} := \mathcal{R}_M$. For more about these curvature notions we refer to the sections in [O'N83, Chap.3] and [Lee19, Chap.7]. Further notions of curvature can be defined and become more involved after taking any metric-affine connection.

3.2. Pseudo-Riemannian submanifolds

We add some basic facts from pseudo-Riemannian submanifold theory and give some useful results for the coming analysis.

Let (M,g) be a pseudo-Riemannian manifold of dimension $(n+k)$ and $i : \Sigma \hookrightarrow M$ the embedding of a smooth n-dimensional submanifold Σ. If the pullback i^*g is a pseudo-Riemannian metric on Σ, we say that Σ is a pseudo-Riemannian submanifold with induced metric $g_\Sigma := i^*g$, commonly known as *first fundamental form*. The tangent spaces at each point $p \in \Sigma$ decompose into a direct sum of non-degenerate subspaces

$$T_pM = T_p\Sigma \oplus N_p\Sigma \tag{3.22}$$

where $T_p\Sigma$ is the tangent space and $N_p\Sigma$ the normal space of Σ. Depending on the dimension of the normal space which determines the codimension of Σ in M, the signature of the induced metric depends on the causal character of vectors in $N_p\Sigma$. We decompose the normal space into the two orthogonal subspaces

$$N_p^{\pm}\Sigma := \left\{ v \in N_p\Sigma \,|\, g|_{N_p\Sigma}(v,v) \gtrless 0 \right\} \quad .$$

The non-degeneracy of the subspace assures that the lightlike normal bundle N_Σ^0 has zero dimension. If the codimension of Σ is $k = \dim(N_p\Sigma)$ and similarly $k_\pm = \dim(N_p^\pm\Sigma)$ for all p, then $k = k_+ + k_-$ and the signature of the induced metric becomes $(r-k_+, s-k_-)$ if the signature of g is (r,s).

The decomposition of the tangent spaces along the submanifold (3.22) induce an orthogonal projection $\pi_p : T_pM \to T_p\Sigma$ for $p \in \Sigma$ such that $T_p\Sigma = \pi_p(T_pM)$ and $N_p\Sigma = \pi_p^\perp(T_pM)$. These projection then maps a local frame $\{e_i\}_{i=1}^{n+k}$ for T_pM into a local frame $\{e_i\}_{i=1}^{n}$ for $T_p\Sigma$ respectively into a local frame $\{e_{i+n}\}_{i=1}^{k}$ for $N_p\Sigma$. The orthogonal projection lifts to a smooth bundle homomorphism from TM to $T\Sigma$ respectively to the normal bundle $N\Sigma$, so we can apply it to sections of $T_\Sigma M := TM|_\Sigma$. Let ∇ be the Levi-Civita connection on the tangent bundle TM. In order to decompose this connection into a tangential and a normal part, we need to consider two vector fields X, Y on Σ which must be extended to vector fields on M or an open subset of M, containing Σ. Denoting them with the same letters, the Levi-Civita connection decomposes along Σ to

$$\nabla_X Y|_\Sigma = \pi\left(\nabla_X Y\right) + \pi^\perp\left(\nabla_X Y\right) \quad . \tag{3.23}$$

The normal component is the map $\mathrm{II} : \mathfrak{X}(M) \times \mathfrak{X}(M)$ to $C^\infty(N\Sigma)$, given by $\mathrm{II}(X, Y) := \pi^\perp\left(\nabla_X Y\right)$. This is a $C^\infty(\Sigma)$-bilinear and symmetric form, known as the *second fundamental form*. The value of this form at a point p on Σ depends only on the vectors X_p and Y_p; hence it is independent of the extension of the vector fields on Σ. The *induced Levi-Civita connection* is defined by $\nabla_X^{T\Sigma} Y := \pi\left(\nabla_X Y\right)$. With these two definitions (3.23) becomes the *Gauss formula*:

$$\nabla_X Y|_\Sigma = \nabla_X^{T\Sigma} Y + \mathrm{II}(X, Y) \quad . \tag{3.24}$$

If we replace the vector field Y with a normal vector field $n \in C^\infty(N\Sigma)$, one can express its extrinsic covariant derivative with the *Weingarten map/shape operator* $\mathcal{W}_p(n, \cdot) : T_p\Sigma \to T_p\Sigma$ in direction of n:

$$g(\mathcal{W}(n, X), Y) = g(n, \mathrm{II}(X, Y)) = g(n, \nabla_X Y) \quad . \tag{3.25}$$

It is a $C^\infty(\Sigma)$-linear map and due to the symmetry of g and the second fundamental form it is furthermore a self-adjoint map. The *Weingarten equation* then expresses the extrinsic covariant derivative of the normal vector along the submanifold:

$$\nabla_X^{T\Sigma} n = -\mathcal{W}(n, X) \quad . \tag{3.26}$$

If Σ has codimension 1 in M, we call it a *pseudo-Riemannian hypersurface*. If both M and Σ are oriented, we might pick a global unit normal vector field $\mathfrak{n}$ such that $\mathfrak{n}(p) \in N_p\Sigma$ for any $p \in \Sigma$. However, this is in general not true, but as long as the computations are local, we can assume that there is a small enough neighbourhood on which such a unit normal vector field exists. We explain in Appendix A how a transverse vector field induces a normal vector field with respect to the pseudo-Riemannian metric g for the case that the hypersurface is a boundary. The causal character of the unit normal vector defines a causal character of the hypersurface: we call a hypersurface *spacelike* if $\mathfrak{n}(p)$ is timelike at any point $p \in \Sigma$. In contrast we call a hypersurface *timelike* if the unit normal vector field is everywhere spacelike. (3.24) remains true and due to the fact that the normal bundle is trivial and spanned by $\mathfrak{n}$, one can project the second fundamental form onto the normal component which defines the *scalar second fundamental form*:

$$\mathrm{II}_g(X, Y) := g(\mathrm{II}(X, Y), \mathfrak{n}) = g(\nabla_X Y, \mathfrak{n}) \tag{3.27}$$

for $X, Y \in \mathfrak{X}(\Sigma)$. It is an element in $C^\infty((T^*M)^{\odot 2})$ with the same symmetry, inherited from the second fundamental form. The Gauss formula (3.24) in this setting then becomes

$$\nabla_X Y|_\Sigma = \nabla_X^{T\Sigma} Y + \mathrm{II}_g(X,Y)\mathfrak{n} \quad . \tag{3.28}$$

The Weingarten map is still defined like in (3.25), but we just write $\mathcal{W}$ for $\mathcal{W}(\mathfrak{n},\cdot)$ due to the unique choice of a normal vector field. The causal character of the unit normal vector fields now influences the Weingarten equation: pairing (3.25) with $g(\cdot,\mathfrak{n})$ gives

$$\begin{aligned} g(\nabla_X Y,\mathfrak{n}) &= \mathrm{II}_g(X,Y)g(\mathfrak{n},\mathfrak{n}) \overset{(3.27)}{=} g(\mathrm{II}(X,Y),\mathfrak{n})g(\mathfrak{n},\mathfrak{n}) \\ &\overset{(3.25)}{=} g(\mathcal{W}(X),Y)g(\mathfrak{n},\mathfrak{n}) \end{aligned}$$

and on the other hand we have

$$g(\nabla_X Y,\mathfrak{n}) = Xg(Y,\mathfrak{n}) - g(Y,\nabla_X\mathfrak{n}) = -g(\nabla_X\mathfrak{n},Y) \quad ;$$

combining these two equations yields

$$\mathcal{W}(X) = -g(\mathfrak{n},\mathfrak{n})\nabla_X\mathfrak{n} \tag{3.29}$$

due to non-degeneracy and $g(\mathfrak{n},\mathfrak{n}) = \pm 1$. (3.28) then takes the form

$$\nabla_X Y|_\Sigma = \nabla_X^{T\Sigma} Y - g(\nabla_X\mathfrak{n},Y)\mathfrak{n} = \nabla_X^{T\Sigma} Y + g(\mathfrak{n},\mathfrak{n})g(\mathcal{W}(X),Y)\mathfrak{n} \quad . \tag{3.30}$$

As $\mathcal{W}$ is still self-adjoint, we can apply the spectral theorem for self-adjoint maps in finite dimensions, saying that for any $p \in \Sigma$ the Weingarten map $\mathcal{W}_p : T_p\Sigma \to T_p\Sigma$ has a pseudo-orthonormal basis of eigenvectors and real eigenvalues $\kappa_1(p),...,\kappa_n(p)$. These eigenvalues are the *principal curvatures* and the eigenvectors are the principal curvature directions at p. The determinant and the trace of the Weingarten map at p define further curvature notions which are specific for hypersurfaces:

(a) *mean curvature:* $$H(p) = \frac{1}{n}\sum_{i=1}^n \kappa_i(p) = \frac{1}{n}\mathrm{tr}_{T_pM}(\mathcal{W}_p) =: \frac{1}{n}\mathrm{tr}_{g_\Sigma}(\mathcal{W}_p) \ , \tag{3.31}$$

(b) *Gauss-Kronecker curvature:* $$K(p) = \prod_{i=1}^n \kappa_i(p) = \det(\mathcal{W}_p) \quad . \tag{3.32}$$

For more informations about pseudo-Riemannain manifolds in this context we refer to chapter 7 in [Lee19] and chapter 4 in [O'N83]. Before we consider the two important cases for the forthcoming analysis, we recall how the Levi-Civita connection coefficients of $\nabla = \nabla^{TM}$ along Σ and $\nabla^{T\Sigma}$ are related to each other. Suppose $\{e_i\}_{i=1}^n$ is a pseudo-orthonormal local tangent frame with respect to the induced metric on Σ. As the embedding is a homeomorphism on its image, we can define a local pseudo-orthonormal tangent frame for TM with respect to g by pointwise pushforward and adding $\iota_*\mathfrak{n}$ to the basis. (3.28) implies on one hand that

$$\nabla_Y e_i|_\Sigma = \nabla_Y^{T\Sigma} e_i + g(\mathfrak{n},\mathfrak{n})\mathrm{II}_g(Y,e_i)\mathfrak{n} = \sum_{k=1}^n (\omega^{T\Sigma})_i^k e_k + g(\mathfrak{n},\mathfrak{n})\mathrm{II}_g(Y,e_i)\mathfrak{n} \quad . \tag{3.33}$$

On the other hand: let Y be a vector field on Σ which we lift to a vector field on M via pushforward[12]; with (3.9) and the pullback connection (item (e) on page 28) we get

$$\begin{aligned}\nabla_Y e_i|_\Sigma &= i^*\left(\nabla_{i_*Y} i_* e_i\right) = (i^*\nabla)_Y e_i \\ &= \sum_{k=0}^{n} (i^*\omega)_i^k(Y) e_k = (i^*\omega)_i^0(Y)\mathfrak{n} + \sum_{k=1}^{n}(i^*\omega)_i^k(Y)e_k \quad .\end{aligned} \tag{3.34}$$

Comparing (3.33) with (3.34) shows that

$$\begin{aligned}(i^*\omega)_i^k(Y) &= (\omega^{T\Sigma})_i^k \qquad &\text{for} \quad k \in \{1,..,n\} \quad , \\ (i^*\omega)_i^0(Y) &= g(\mathfrak{n},\mathfrak{n})\mathrm{II}_g(Y,e_i) \qquad &\text{for} \quad k = 0 \quad .\end{aligned} \tag{3.35}$$

The other coefficients are determined by (3.18).

We now focus on a Riemannian hypersurface in an ambient manifold; the latter one is either an oriented Riemannian manifold or a space and time-oriented Lorentzian manifold. We assume additionally that the ambient manifold is a *topological cylinder manifold*, i.e. it is isometric to $\mathbb{R}\times\Sigma$. The Lorentzian case then becomes a globally hyperbolic manifold. We want to investigate both cases together and consider the metric

$$g^{[\epsilon]} := \epsilon N^2 \,\mathrm{d}t^{\otimes 2} + g_t \tag{3.36}$$

for $\epsilon \in \{\pm 1\}$. A *geometric cylinder manifold* in comparison is a topological cylinder manifold with metric

$$g^{[\epsilon]} = \epsilon \,\mathrm{d}t^{\otimes 2} + g_\Sigma \tag{3.37}$$

where g_Σ denotes the t-independent Riemannian metric on the hypersurface Σ. As in (3.7), N is again a lapse function and $\{g_t\}_{t\in\mathbb{R}}$ with $g_t := g_{\Sigma_t}$ is a smooth one-parameter family of Riemannian metrics on Σ. **For $\epsilon = -1$ we are in the Lorentzian case with $g^{[-1]} = g$. For $\epsilon = +1$ we are in the Riemannian case which we will denote by $\breve{g} := g^{[+1]}$ for later use** and is referred to as the *flipped metric* of (3.7).

Let $\{e_i(t)\}_{i=1}^n$ be a local tangent frame on the slice Σ_t which is Riemann-orthonormal with respect to g_t. We can lift this basis to a local frame $\{e_0(t)\}\cup\{e_i(t)\}_{i=1}^n$ in M which is orthonormal with respect to $g^{[\epsilon]}$ for each $t\in\mathbb{R}$. This implies that $e_0(t)$ has to be perpendicular to each slice. We construct e_0 to be parallel to $(-\partial_t)$ and we conclude from the orthonormality assumption that

$$g^{[\epsilon]}(e_0(t), e_i(t)) = 0 \ \ \forall\, i \in \{1,...,n\} \quad \text{and} \quad \epsilon = g^{[\epsilon]}(e_0(t), e_0(t)) = \epsilon N^2$$

and thus $e_0(t) = -\frac{1}{N}\partial_t$. Because of its often appearance this vector gets from now on the designation ν and it comes with several important properties.

Lemma 3.2.1 (cf. Proposition 4.1 (14) in [BGM05])**.** *For $p\in\Sigma_t$ let $X,Y\in T_p\Sigma_t$; the following expressions for ν are fulfilled near the point p:*

(1) ν is autoparallel with respect to ∇, i.e. $\nabla_\nu \nu = 0$, and

(2) $g^{[\epsilon]}(\mathcal{W}(X),Y) = \frac{\epsilon}{2N}\partial_t g_t(X,Y)$.

[12] To be more precise, i_*Y is a section of the pullback bundle $i^*(TM)$ along Σ.

Proof.

(1) We lift and extend $X \in T_p\Sigma_\tau$ to a vector field on M near (τ, p) which we denote with the same symbol. The action of ν on $g^{[\epsilon]}(X,\nu) = 0$ gives

$$\begin{aligned} 0 &= \nu g^{[\epsilon]}(X,\nu) = g^{[\epsilon]}(\nabla_\nu X,\nu) + g^{[\epsilon]}(\nabla_\nu \nu, X) \overset{(3.29)}{=} -\epsilon g^{[\epsilon]}(\mathcal{W}(X),\nu) + g^{[\epsilon]}(\nabla_\nu\nu, X) \\ &\overset{(*)}{=} g^{[\epsilon]}(\nabla_\nu \nu, X) \end{aligned}$$

where we have used in $(*)$ that the Weingarten map acts as a bundle endomorphism by which $\mathcal{W}(X) \perp \nu$. The action of ν on $g^{[\epsilon]}(\nu,\nu) = \epsilon$ leads to

$$0 = \nu g^{[\epsilon]}(\nu,\nu) = 2g^{[\epsilon]}(\nabla_\nu\nu,\nu)$$

with vanishing non-metricity of the Levi-Civita connection. We observe that due to non-degeneracy of $g^{[\epsilon]}$ the tangent as well as the normal parts of $\nabla_\nu\nu$ are vanishing and therefore ν is a geodesic.

(2) Extend the vectors X, Y to vector fields on Σ_t and lift them to M. We have $g^{[\epsilon]}(X,Y) = g_t(X,Y)$. If we apply ν to this product, we get with the self-adjointness of the Weingarten map

$$\begin{aligned} \nu g_t(X,Y) &= \nu g^{[\epsilon]}(X,Y) = g^{[\epsilon]}(\nabla_\nu X, Y) + g^{[\epsilon]}(X, \nabla_\nu Y) \\ &\overset{(3.29)}{=} -\epsilon\left[g^{[\epsilon]}(\mathcal{W}(X),Y) + g^{[\epsilon]}(X,\mathcal{W}(Y))\right] = -2\epsilon g^{[\epsilon]}(\mathcal{W}(X),Y) \quad . \end{aligned}$$

Thus we have

$$g^{[\epsilon]}(\mathcal{W}(X),Y) = -\frac{\epsilon}{2}\nu g_t(X,Y) = \frac{\epsilon}{2N}\partial_t\left[g_t(X,Y)\right]$$

and the claim is proven. □

(3.30) takes the form

$$\nabla_X Y|_{\Sigma_t} = \nabla_X^{T\Sigma_t} Y + \epsilon g^{[\epsilon]}(\mathcal{W}(X),Y)\nu \quad . \tag{3.38}$$

With this we can calculate the Christoffel symbols $\Gamma^M_{ij,k}$ where latin indices will stand for tangential or spacelike coordinates, "0" for the normal or timelike direction $e_0 = \nu$,∇ for the Levi-Civita connection, and Σ short for Σ_t for each t with hypersurface metric g_Σ:

$$\begin{aligned} \Gamma^M_{jk,l} &= g^{[\epsilon]}(\nabla_{e_j}e_k, e_l) \overset{(3.38)}{=} g^{[\epsilon]}(\nabla^{T\Sigma}_{e_j}e_k, e_l) = g_\Sigma(\nabla^{T\Sigma}_{e_j}e_k, e_l) = \Gamma^\Sigma_{jk,l} \\ \Gamma^M_{jk,0} &= g^{[\epsilon]}(\nabla_{e_j}e_k,\nu) \overset{(3.38)}{=} \epsilon g^{[\epsilon]}(\mathcal{W}(e_j),e_k) g^{[\epsilon]}(\nu,\nu) = \epsilon^2 g^{[\epsilon]}(\mathcal{W}(e_j),e_k) \\ &= g^{[\epsilon]}(\mathcal{W}(e_j),e_k) \\ \Gamma^M_{j0,l} &= g^{[\epsilon]}(\nabla_{e_j}\nu, e_l) \overset{(*)}{=} e_j g^{[\epsilon]}(\nu, e_l) - g^{[\epsilon]}(\nu,\nabla_{e_j}e_l) \overset{(3.38)}{=} -\epsilon g^{[\epsilon]}(\nu,\nu) g^{[\epsilon]}(\mathcal{W}(e_j),e_l) \\ &= -\epsilon^2 g^{[\epsilon]}(\mathcal{W}(e_j),e_l) = -g^{[\epsilon]}(\mathcal{W}(e_j),e_l) = -\Gamma^M_{jl,0} \\ \Gamma^M_{j0,0} &= \Gamma^M_{00,0} = \Gamma^M_{00,l} = 0 \quad . \end{aligned} \tag{3.39}$$

In $(*)$ we used that the connection is compatible with the metric. The trivial symbols are determined by the fact that ν is an autoparallel due to Lemma 3.2.1 (1).

The volume form on M and its induced volume forms on each slice are given as follows: in local coordinates $\{x^i\}_{i=1}^n$ for Σ we define

$$\mathsf{dvol} := \sqrt{g_t}N\,\mathrm{d}t \wedge \bigwedge_{i=1}^{n} \mathrm{d}x^i \text{ and } \mathsf{dvol}_{\Sigma_t} := \iota_{-\nu}\,\mathsf{dvol} = \sqrt{g_t}\bigwedge_{i=1}^{n} \mathrm{d}x^i \quad \text{for } t \text{ fixed} \tag{3.40}$$

where $g_t = \det\left(g_t\right)$ and $\iota_{-\nu}$ is the interior product. Later, we need time-derivatives of expressions like

$$I_f(t) = \int_{\Sigma_t} f_t\,\mathsf{dvol}_{\Sigma_t} \tag{3.41}$$

where f_t is any integrable and t-differentiable function on Σ_t. In order to do so, one needs to introduce variations with respect to a reference hypersurface. For $\tau \in \mathbb{R}$ let Σ_τ be such a reference hypersurface with Riemannian metric g_τ; its embedding into M is denoted with i_τ. A *variation map of* i_τ is a map $e \in C^\infty(\mathcal{T}(M) \times \Sigma_\tau, M)$ such that

(1) $e_t := e(t,\cdot)$ is a Riemannian or spacelike immersion for all $t \in \mathcal{T}(M)$ such that $e_\tau = i_\tau$, and

(2) if the boundary of Σ_τ is non-empty: $e_t|_{\partial\Sigma_\tau} = i_\tau|_{\partial\Sigma_\tau}$ for all $t \in \mathcal{T}(M)$.

Hence any slice can be viewed as $\Sigma_t = e_t(\Sigma_\tau)$ with $\partial\Sigma_t = \partial\Sigma_\tau$ if the boundary is non-empty. A particular choice is given by the exponential map along the geodesics of a normal field:

$$e_t(p) = \exp_{i(p)}\left[(t-\tau)\phi(p)\nu(p)\right] \quad ;$$

the function $\phi \in C_c^\infty(\Sigma_0)$ is introduced to guarantee a variation under compact support in case of non-compact hypersurfaces. We then get the following result.

Lemma 3.2.2. *Let M be a topological cylinder manifold of dimension $(n+1)$, $\tau \in \mathbb{R}$ and $i_\tau : \Sigma_\tau \to M$ the embedding of a hypersurface. The variation under compact support of the integral (3.41) at $t = \tau$ for a time-differentiable function f_t on Σ_t is*

$$\frac{\mathrm{d}}{\mathrm{d}t}I_f(t)\bigg|_\tau = -\int_{\Sigma_\tau} \phi(p)\Big[\epsilon n H_\tau(p) + (\nu f_t)\,|_{\tau,p}\Big]\,\mathsf{dvol}_{\Sigma_\tau}(p)$$

where $\phi \in C_c^\infty(\Sigma_\tau)$ and H_τ is the mean curvature of Σ_τ.

Proof. With $\Sigma_t = e_t(\Sigma_\tau)$ and pullback properties we get

$$I_f(t) = \int_{e_t(\Sigma_\tau)} f_t(p)\,\mathsf{dvol}_{\Sigma_t}(p) = \int_{\Sigma_\tau} e_t^*(p)\left(f_t\,\mathsf{dvol}_{\Sigma_t}\right) = \int_{\Sigma_\tau} (f_\bullet \circ e_t)(p)(e_t^*\,\mathsf{dvol}_{\Sigma_t})|_p$$

where $\bullet$ stands for the (unfixed) time-dependence. The time derivative at $t = \tau$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}I_f(t)\bigg|_\tau = \int_{\Sigma_\tau} \frac{\mathrm{d}}{\mathrm{d}t}(f_\bullet \circ e_t)(p)\bigg|_\tau \mathsf{dvol}_{\Sigma_\tau} + (f_\bullet \circ i_0)(p)\,\frac{\mathrm{d}}{\mathrm{d}t}e_t^*\,\mathsf{dvol}_{\Sigma_t}\bigg|_{\tau,p} \quad . \tag{3.42}$$

The first term can be computed by the chain rule and the derivative of the exponential map, giving

$$\frac{\mathrm{d}}{\mathrm{d}t}(f_\bullet \circ e_t)(p)\Big|_\tau = \mathrm{d}f_\bullet|_{i_0(p)}\left(\phi(p)\nu(p)\right) = \phi(p)\left(\nu f_t\right)|_{\tau,p} \quad . \tag{3.43}$$

Let $\{x^i(t)\}_{i=1}^n$ be local coordinates at $e_t(p)$ on a level set Σ_t. The pullback by e_t of these local coordinates implies coordinates on Σ_τ, denoted by $\{x^i(\tau)\}_{i=1}^n$ such that the pullback of the volume form is

$$e_t^* \,\mathsf{dvol}_{\Sigma_t}|_p = (\sqrt{g_\bullet} \circ e_t)(p)) \bigwedge_{i=1}^n \mathrm{d}(e_t^* x^i(t))|_p = (\sqrt{g_\bullet} \circ e_t)(p) \bigwedge_{i=1}^n \mathrm{d}x^i(\tau) \quad .$$

Performing the derivative with respect to t yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e_t^* \,\mathsf{dvol}_{\Sigma_t}\Big|_{\tau,p} &= \frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{g_\bullet} \circ e_t)(p)\Big|_\tau \bigwedge_{i=1}^n \mathrm{d}x^i(\tau) = \phi(p)(\nu\sqrt{g_t})|_{\tau,p} \bigwedge_{i=1}^n \mathrm{d}x^i(\tau) \\
&= -\frac{\phi(p)}{2N(\tau,p)\sqrt{g_\tau(p)}}\partial_t[(\det(g_t)]|_{\tau,p} \bigwedge_{i=1}^n \mathrm{d}x^i(\tau) \qquad (3.44) \\
&\overset{(*)}{=} -\frac{\phi(p)}{2N(\tau,p)}\mathrm{tr}_{g_t}\left(\partial_t g_t\right)|_{\tau,p} \,\mathsf{dvol}_{\Sigma_\tau} \\
&\overset{(**)}{=} -\epsilon\phi(p)\mathrm{tr}\left(\mathcal{W}_p\right)|_\tau \,\mathsf{dvol}_{\Sigma_\tau} = -n\epsilon\phi(p)H_\tau(p)\,\mathsf{dvol}_{\Sigma_\tau} \qquad (3.45)
\end{aligned}$$

where one has used Jacobi's formula for the derivative of the determinant in $(*)$, and Lemma 3.2.1 (2) and (3.31) at $(**)$. Plugging (3.43) and (3.45) into (3.42) leads to the claimed expression. □

3.3. Manifolds of bounded geometry

We end this introductory chapter with a short description about manifolds of/with bounded geometry, based on material from [Shu92] and [Kor91]. Suppose Σ is a connected Riemannian manifold of dimension n. The Riemannian metric g induces a distance function $d : \Sigma \times \Sigma \to \mathbb{R}$ by taking the infimum of lengths of arcs, connecting two points and measured with respect to the Riemannian metric. The manifold becomes a metric space in this way. The exponential geodesic map $\exp_p : T_p\Sigma \to \Sigma$ is defined through $\exp_p(v) = \gamma(1)$ where γ is a geodesic, given in such a parametrisation that $\gamma(0) = p$ and $\dot\gamma(0) = v$. If we assume Σ to be complete, we can define the geodesic $\gamma(t)$ on all $\mathbb{R}$. If we take a ball $\mathring{\mathbb{B}}_r^n(0)$ of radius $r > 0$ around the zero vector in $T_p\Sigma$, the ball becomes a diffeomorphism onto its image $U_r(p) = \exp(\mathring{\mathbb{B}}_r^n(0))$ which is an open neighbourhood of $p \in \Sigma$. Let

$$r(p) := \sup\left\{ r > 0 \,\middle|\, \exp_p : \mathring{\mathbb{B}}_r^n(0) \to U_r(p) \text{ diffeomorphism}\right\} \quad .$$

The infimum over all points $p \in \Sigma$ then defines the *injectivity radius*

$$r_{\mathsf{inj}} := \inf_{p\in\Sigma}\{r(p)\} \quad . \tag{3.46}$$

For $r_{\mathsf{inj}} > 0$ and $r \in (0, r_{\mathsf{inj}})$ the exponential map $\exp_p : \mathring{\mathbb{B}}^n_r(0) \to U_r(p)$ is a diffeomorphism for all $p \in \Sigma$ which motivates the following definition.

Definition 3.3.1. Let Σ be a Riemannian manifold; we say that Σ is a *manifold of/with bounded geometry* if the following two conditions are satisfied:

(a) $r_{\mathsf{inj}} > 0$;

(b) the transition maps from $U_r(p) \cap U_r(q)$ to $\mathbb{R}^n$ are bounded for any fixed $r \in (0, r_{\mathsf{inj}})$ and any points p, q with open neighbourhoods $U_r(p), U_r(q)$ such that $U_r(p) \cap U_r(q) \neq \emptyset$.

(a) implies that Σ is complete and (b) is equivalent to the boundedness of all covariant derivatives of the Riemannian curvature tensor. Examples of such spaces are homogenous spaces with invariant metrics, covering manifolds of compact manifolds[13] as well as leaves of foliations on a compact manifold. We call a vector bundle E to be of bounded geometry if all derivatives of the vector bundle transition functions on $U_r(p) \cap U_r(q) \neq \emptyset$ are bounded for all $p, q \in \Sigma$. Examples of such vector bundles are $\Sigma \times \mathbb{C}$ and complexifications of $T\Sigma$, $T^*\Sigma$ as well as any other complexified natural bundle.

A manifold of bounded geometry can be covered by countably many balls. To be more precise, there exists a radius $R > 0$ such that for all $r \in (0, R)$ the manifold Σ can be covered by open balls $\mathring{\mathbb{B}}_r(p_i)$ around countably many points $p_i \in \Sigma$:

$$\Sigma = \bigcup_i \mathring{\mathbb{B}}_r(p_i) \quad ; \tag{3.47}$$

this covering has the property that if we double the radii, but fixing the centres of the open balls in Σ, this covering has a finite number of non-empty intersections in this covering. This implies the existence of a suitable partition of unity (see [Shu92, Lem.1.2] and [Shu92, Lem.1.4]).

Lemma 3.3.2. *For every $R > 0$ there exists a partition of unity $\{\phi_i\}$ on Σ such that*

(1) $\phi_i \in C^\infty_{\mathsf{c}}(\Sigma, \mathbb{R}_{\geq 0})$ with $\operatorname{supp}(\phi_i) \subset \mathring{\mathbb{B}}_{2r}(p_i)$ for all p_i from (3.47).

(2) all derivatives are uniformly bounded.

With this partition of unity one can define Sobolev spaces $H^s(\Sigma)$ on manifolds with bounded geometry for any $s \in \mathbb{R}$ as follows:

$$\|u\|^2_{H^s(\Sigma)} := \sum_i \|\phi u\|^2_{H^s(\mathring{\mathbb{B}}_{2r}(p_i))} \quad . \tag{3.48}$$

$H^s(\Sigma)$ is then defined to be the completion of $C^\infty_{\mathsf{c}}(\Sigma)$ with respect to (3.48). This can be easily extended to sections of a vector bundle of bounded geometry. (3.47) will occur in subsection 5.1.2 in the setting of Galois coverings with compact base manifolds.

[13] The case of such coverings is proven in [CG91].

4. Function spaces and operators on manifolds

In this chapter we recapitulate some notions and facts about function spaces and operators, mapping between smooth sections of vector bundles over manifolds. The first section deals with general notions. We also take a closer look on Sobolev spaces for compact and in particular non-compact manifolds. We recall in the second section the necessary definitions and facts about Fourier integral operators from the global point of view. The last section deals with special function spaces on globally hyperbolic manifolds which become important in Part II.

The main literature for this chapter are the books [Hö03], [Hö07] and [Hö09] with some supporting material from [SA11] and [IS20]. The last section is based on results and facts, taken from [BTW15] and [Bä14].

4.1. Function spaces and operators on manifolds

4.1.1. Function spaces on manifolds

Let $E \to M$ be any $\mathbb{C}-$(anti-)linear smooth vector bundle over a pseudo-Riemannian manifold M with metric g. We equip the tangent bundle TM and via (3.6) the cotangent bundle T^*M with the Levi-Civita connection, and the vector bundle comes with a related Koszul connection ∇^E. The l'th covariant derivative $\nabla^l := (\nabla^E)^l$ is induced by the connection on $(T^*M)^{\otimes l} \otimes E$ via l-fold application of ∇^E. Let $K \subset M$ be compact, $m \in \mathbb{N}_0$ and $\|\cdot\|_{(T^*M)^{\otimes l}\otimes E}$ a norm on the vector bundle $(T^*M)^{\otimes l} \otimes E$. The space $C^\infty(M,E)$ is equipped with the seminorm

$$\|f\|_{K,m} := \max_{l\in\{0,\ldots,m\}} \max_{x\in K} \left\{ \left\| (\nabla^l f)(x) \right\|_{(T^*M)^{\otimes l}\otimes E} \right\} \tag{4.1}$$

for $f \in C^\infty(M,E)$. Compactness of K implies that different norms and connections on the vector bundle lead to equivalent seminorms. Since M can be exhausted by compact subsets $\{K_i\}_{i\in\mathbb{N}}$, we can select a seminorm $\|\cdot\|_{K_i,m}$ for each compact subset. Moreover, $C^\infty(M,E)$ is a complete such that it becomes a Fréchet space according to Definition 2.1.1 (3). If the support is contained in a closed subset $A \subset M$, one defines

$$C^\infty_A(M,E) := \{u \in C^\infty(M,E) \,|\, \operatorname{supp}(u) \subset A\} \quad .$$

The union over all compact subsets of M defines $C^\infty_c(M,E)$ as *space of compactly supported sections* of E:

$$C^\infty_c(M,E) := \bigcup_{\substack{K\subset M\\ K \text{ compact}}} C^\infty_K(M,E) \quad . \tag{4.2}$$

This is an LF-space with continuous inclusion mapping $C^\infty_K \hookrightarrow C^\infty_\mathsf{c}$ for $K \Subset M$. Any linear map between this space and other locally convex topological vector spaces is continuous if any restriction $C^\infty_K(M,E)$ to any compact subspace K is continuous. We first assume that M is orientable such that a volume form dvol on M exists. If the vector bundle E comes with a bundle metric $(\cdot\,|\,\cdot)_E$ which is a pointwise sesquilinear form[14] $(\cdot\,|\,\cdot)_{E_p} : E_p \times E_p \to \mathbb{C}$, another sesquilinear form

$$C^\infty_\mathsf{c}(M,E) \times C^\infty_\mathsf{c}(M,E) \ni (u,v) \;\mapsto\; (u\,|\,v)_{C^\infty_\mathsf{c}(M,E)} := \int_M (u\,|\,v)_{E_p}\, \mathsf{dvol}\,(p) \tag{4.3}$$

can be introduced. If the vector spaces E_p for each $p \in M$ are equipped with an inner product $\langle\,\cdot\,|\,\cdot\,\rangle_E(p) : E \times E \to \mathbb{C}$, they become Hilbert spaces and (4.3) becomes positive definite if we replace $(\,\cdot\,|\,\cdot\,)_{E_p}$ with $\langle\,\cdot\,|\,\cdot\,\rangle_{E_p}$. We also write $(\,\cdot\,|\,\cdot\,)_E(p)$ for $(\,\cdot\,|\,\cdot\,)_{E_p}$ and similarly $\langle\,\cdot\,|\,\cdot\,\rangle_E(p)$ for $\langle\,\cdot\,|\,\cdot\,\rangle_{E_p}$. The bundle metric is called *Hermitian* if it is Hermitian at each point. If the vector bundle does not come with a further structure, we can consider the anti-dual vector bundle $\overline{E}^* \to M$ such that a pointwise dual pairing $[\,\cdot\,|\,\cdot\,]_E(p) = [\,\cdot\,|\,\cdot\,]_{E_p} : \overline{E}^*_p \times E_p \to \mathbb{C}$ can be introduced. Integrating over the manifold yields a distributional pairing for smooth and compactly supported sections of E:

$$C^\infty_\mathsf{c}(M,\overline{E}^*) \times C^\infty_\mathsf{c}(M,E) \ni (\psi,\phi) \;\mapsto\; [\psi\,|\,\phi]_{C^\infty_\mathsf{c}(M,E)} := \int_M [\psi\,|\,\phi]_E(p)\, \mathsf{dvol}\,(p) \quad . \tag{4.4}$$

If the bundle metric is Hermitian, (4.3) induces a L^2-inner product

$$\langle u\,|\,v\rangle_{L^2(M,E)} := \int_M \langle u\,|\,v\rangle_E(p)\, \mathsf{dvol}\,(p) \tag{4.5}$$

for u,v as in (4.3) and $\langle\,\cdot\,|\,\cdot\,\rangle_E(p)$ positive definite and Hermitian. This inner product induces an L^2-norm $\|u\|^2_{L^2(M,E)} := \langle u\,|\,u\rangle_{L^2(M,E)}$. The completion of $C^\infty_\mathsf{c}(M,E)$ with respect to this norm defines the Hilbert space $L^2(M,E)$ of *square-integrable sections* of E.

Furthermore, one can consider special spaces, related to $L^2(M,E)$, which come with different support properties. Let K be a compact subset of M; the space of L^2-sections with support in K is defined by

$$L^2_K(M,E) := \left\{u \in L^2(M,E)\,|\,\mathrm{supp}\,(u) \subset K\right\} \quad .$$

This is a Hilbert space with induced topology of $L^2(M,E)$. The space of L^2*-sections with compact support* is the LH-space

$$L^2_\mathsf{c}(M,E) := \bigcup_{\substack{K \subset M \\ K\,\text{compact}}} L^2_K(M,E) \quad .$$

These two spaces are subsets of $L^2(M,E)$. On the other hand one could consider functions which are locally in $L^2(M,E)$. This can be realised by multiplying with a cut-off function $\psi \in C^\infty_\mathsf{c}(M)$ such that the product is L^2 on the support of the cut-off function. This

[14] Here and in the following we choose the sesquilinearity in such a way that the first entry is anti-linear.

defines the space of *local L^2-sections* as another example of a Fréchet space:

$$L^2_{\mathsf{loc}}(M,E) := \left\{ u : M \to E \,|\, \psi u \in L^2_{\mathrm{supp}(\psi)}(M,E) \quad \forall\, \psi \in C^\infty_{\mathsf{c}}(M) \right\} \quad .$$

If the manifold is not orientable, we need to take densities instead of volume forms and for any kind of pairings half-densities. We abbreviate the space of half-densities $|\Omega^n(M)|^{\frac{1}{2}}$ with $|\Omega|^{\frac{1}{2}}_M$. The dual pairing (4.4) then takes the form

$$\left[\psi \,|\, \phi\right]_{C^\infty_{\mathsf{c}}(M,E)} := \int_M \left[\psi \,|\, \phi\right]_E(p) \tag{4.6}$$

for $(\psi,\phi) \in C^\infty_{\mathsf{c}}(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M) \times C^\infty_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ where the pointwise pairing $\left[\psi \,|\, \phi\right]_{E_p}$ is now density-valued. Similarly, one can neglect the volume form in (4.3) and (4.5) after replacing the sections with half-density-valued sections, such that the sesquilinear form or inner product becomes density-valued. As long as the intersection of supports of the two sections ϕ and ψ is compact, (4.4) or rather (4.6) still make sense. Fixing ϕ to be smooth and compactly supported, we call the section ψ a *regular distribution* if (4.4) or (4.6) are still meaningful.

With $C^{-\infty}(M,E)$ we designate distributional sections as dual space of smooth sections with compact support. A distribution on E is a $\mathbb{C}$-linear functional on test functions $\phi \in C^\infty_{\mathsf{c}}(M, \overline{E}^*)$ with respect to a dual or rather distributional pairing $\left[u \,|\, \phi\right]_{C^\infty_{\mathsf{c}}(M,\overline{E}^*)}$ which is a map

$$C^{-\infty}(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M) \times C^\infty_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M) \to \mathbb{C} \quad ,$$

coinciding with (4.6) for regular distributions. In the same way we define the space $C^{-\infty}_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ of compactly supported distributions as the dual space $(C^\infty(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M))^*$.

4.1.2. Operators between sections of manifolds

Let $E \to M$ be as in the former subsection and $F \to N$ another $\mathbb{C}$-(anti-)linear vector bundle over a manifold N; the ranks of the vector bundles are m_E for E and m_F for F. The dimension of the manifolds are denoted with m respectively n. Denote with P a linear operator which a priori maps from $C^\infty_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ to $C^{-\infty}(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$. The *formal dual operator* $P^\dagger$ with respect to (4.6) is a map from $C^\infty_{\mathsf{c}}(N, \overline{F}^* \otimes |\Omega|^{\frac{1}{2}}_N)$ to $C^{-\infty}(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M)$ with the defining property

$$\left[P^\dagger \psi \,|\, \phi\right]_{C^\infty_{\mathsf{c}}(M,E)} := \left[P\phi \,|\, \psi\right]_{C^\infty_{\mathsf{c}}(N,\overline{F}^*)}$$

for $\psi \in C^\infty_{\mathsf{c}}(N, \overline{F}^* \otimes |\Omega|^{\frac{1}{2}}_N)$ and $\phi \in C^\infty_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$. This can be used to extend the linear operator P to an operator on distributions: if $P^\dagger$ maps from $C^\infty_{\mathsf{c}}(N, \overline{F}^* \otimes |\Omega|^{\frac{1}{2}}_N)$ to $C^\infty_{\mathsf{c}}(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M)$, the operator P can be extended to a map from $C^{-\infty}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ to

$C^{-\infty}(N, F \otimes |\Omega|_N^{\frac{1}{2}})$ and the action on a distribution $u \in C^{-\infty}(M, E \otimes |\Omega|_M^{\frac{1}{2}})$ is defined by

$$[Pu \,|\, \chi]_{C_c^\infty(N,\overline{F}^*)} := [u \,|\, P^\dagger \chi]_{C_c^\infty(M,\overline{E}^*)}$$

for $\chi \in C_c^\infty(N, \overline{F}^* \otimes |\Omega|_N^{\frac{1}{2}})$. The *formal adjoint operator* P^* of an operator P, mapping from $C_c^\infty(M, E \otimes |\Omega|_M^{\frac{1}{2}})$ to $C_c^\infty(N, F \otimes |\Omega|_N^{\frac{1}{2}})$, is the dual operator with respect to the inner product (4.5) with the defining property

$$\langle P^*\psi \,|\, \phi \rangle_{L^2(N,F)} := \langle \psi \,|\, P\phi \rangle_{L^2(M,E)}$$

for ψ, ϕ as before and P^* maps from $C_c^\infty(N, F \otimes |\Omega|_N^{\frac{1}{2}})$ to $C_c^\infty(M, E \otimes |\Omega|_M^{\frac{1}{2}})$.

If the operator P is continuous, then there is a bijective correspondence between P and a *Schwartz kernel* K which is a half-density-valued distribution on the Cartesian product $N \times M$ such that the action of P on a function ϕ can be represented as

$$(P\phi)(x) = \int_M K(x, y) u(y)$$

in the distributional sense. This is the statement of the *Schwartz Kernel Theorem*.

Theorem 4.1.1 (cf. Theorem 5.2.1 in [Hö03], p.93 in [Hö07])**.** *Given two manifolds M, N as well as vector bundles $E \to M$ and $F \to N$ and suppose $K \in C^{-\infty}(N \times M, \mathbf{Hom}(E, F))$ defines the operator $P_K : C_c^\infty(M, E \otimes |\Omega|_M^{\frac{1}{2}}) \to C^{-\infty}(N, F \otimes |\Omega|_N^{\frac{1}{2}})$ through*

$$[P_K\psi \,|\, \phi]_{C_c^\infty(N,F)} := [K \,|\, \psi \otimes \phi]_{C_c^\infty(N\times M, F \boxtimes E^*)}$$

for all $\phi \in C_c^\infty(M, E \otimes |\Omega|_M^{\frac{1}{2}})$ and $\psi \in C_c^\infty(N, F \otimes |\Omega|_N^{\frac{1}{2}})$; this assignment $K \mapsto P_K$ is a bijection from $C^{-\infty}(N \times M, \mathbf{Hom}(E, F))$ to continuous elements in $\mathscr{L}(C_c^\infty(M, E \otimes |\Omega|_M^{\frac{1}{2}}), C^{-\infty}(N, F \otimes |\Omega|_N^{\frac{1}{2}}))$.

In this way we can characterise an operator by the properties of its kernel. The dual operator has a Schwartz kernel which is an element in $C^{-\infty}(M \times N, \mathbf{Hom}(\overline{F}^*, \overline{E}^*))$. An operator P with Schwartz kernel K is said to be *properly supported* if both projections $\pi_N : N \times M \to N$ and $\pi_M : N \times M \to M$ are proper maps on $\mathrm{supp}\,(K) \subset N \times M$, i.e. for compact supsets $K_N \subset N$ and $K_M \subset M$ the intersections with the preimages of these projections

$$\mathrm{supp}\,(K) \cap (\pi_M)^{-1}(K_M) \quad \text{and} \quad \mathrm{supp}\,(K) \cap (\pi_N)^{-1}(K_N)$$

are compact. A more handy characterisation is the following: the operator P is properly supported if and only if

$$\begin{aligned} &(1) \quad \forall\, K_M \Subset M \,\exists\, K'_N \Subset N : \mathrm{supp}\,(u) \subset K_M \Rightarrow \mathrm{supp}\,(Pu) \subset K'_N \quad , \\ &(2) \quad \forall\, K_N \Subset N \,\exists\, K'_M \Subset M \;: \mathrm{supp}\,(v) \subset K_N \Rightarrow \mathrm{supp}\left(P^\dagger u\right) \subset K'_M \quad . \end{aligned} \tag{4.7}$$

In a nutshell, an operator is properly supported if and only if it maps compactly supported sections to compactly supported sections. This has been proven in [SA11, Prop.3.4] for operators on one manifold. The proof carries over to the here presented general case. The composition $Q \circ P$ of two operators then becomes well-defined if at least P is properly supported. Moreover, the composition of properly supported operators is again properly supported.

We now turn our attention to differential and pseudo-differential operators for which we need to consider the case $M = N$. The standard, non-geometric introduction of differential operators is based on local coordinates $\{x^l\}_{l=1}^m, \{y^l\}_{l=1}^m$ on a neighbourhood $\mathcal{U} \subset M$ and two trivialisations

$$\phi_E : E \to \mathcal{U} \times \mathbb{C}^{m_E} \quad , \quad \phi_F : F \to \mathcal{U} \times \mathbb{C}^{m_F}$$

of the vector bundles. The action of a continuous map $P : C^\infty(M, E) \to C^\infty(M, F)$ on $u \in C^\infty_{\mathsf{c}}(\mathcal{U}, E)$ is locally given by a $(m_F \times m_E)$-matrix (P_{ij}):

$$[\phi_F(Pu)|_{\mathcal{U}}]_i = \sum_{j=1}^{m_E} P_{ij}(\phi_E u)_i$$

for $i \in \{1, ..., m_F\}$. We call P a *(partial-)differential operator of order k* if for all open neighbourhoods and trivialisations the matrix elements P_{ij} are linear differential operators of order k:

$$P_{ij} = \sum_{|\boldsymbol{\alpha}| \leq k} P_{ij\boldsymbol{\alpha}}(x) \partial_x^{\boldsymbol{\alpha}} \tag{4.8}$$

where $\boldsymbol{\alpha}$ is a multi-index in $\mathbb{N}_0^m$, $P_{ij\boldsymbol{\alpha}}(x)$ smooth functions on $\mathcal{U}$ and $\partial_x^{\boldsymbol{\alpha}}$ are combinations of partial derivatives. We denote the set of those linear partial-differential operators of order k with $\mathsf{Diff}^k(M, \mathsf{Hom}(E, F))$ and write $\mathsf{Diff}^k(E, F)$ if the manifold is clear from the context; in addition we set $\mathsf{Diff}^k(E)$ for $\mathsf{Diff}^k(M, \mathsf{End}(E))$[15]. The Schwartz kernel of each matrix entry P_{ij} is the distribution

$$K_{ij}(x, y) = \sum_{|\boldsymbol{\alpha}| \leq k} P_{ij\boldsymbol{\alpha}}(x) \partial_x^{\boldsymbol{\alpha}} \delta(x - y) \tag{4.9}$$

where $\delta(x - y)$ stands for the Dirac distribution along the diagonal in $M \times M$. A *pseudo-differential operator of order $r \in \mathbb{R}$* is a continuous map $P : C^\infty_{\mathsf{c}}(M, E) \to C^\infty(M, F)$ if for every open neighbourhood and any trivialisation of the vector bundles each entry of the matrix (P_{ij}) of local operators is given as an integral operator with Schwartz kernel

$$K_{ij}(x, y) = \int_{\mathbb{R}^m} \exp\left(\mathbb{i}\langle x - y \,|\, \xi \rangle_{\mathbb{R}^m}\right) a_{ij}(x, \xi) \frac{\mathrm{d}\xi}{(2\pi)^{m/2}} \tag{4.10}$$

where the a_{ij} are elements in $C^\infty(T^*M, \pi^*_{T^*M}(\mathsf{Hom}(E, F)))$ such that their pullbacks with the inverse map of the trivialisation $T^*M \to \mathcal{U} \times \mathbb{R}^m$ are symbols of order r on $\mathcal{U} \times \mathbb{R}^m$; π_{T^*M} is the bundle projection $T^*M \to M$. The concrete definition and properties of symbols are comprehensively described in the main references of this chapter; a condensed introduction can be found in [IS20, App.A]. We designate the set of pseudo-differential operators of order r with $\Psi^r(M, \mathsf{Hom}(E, F))$.

[15] We will use this convention for any other occuring space of operators, acting between sections on one and the same manifold.

We recall some basic facts about differential and pseudo-differential operators. Let E, F and M be as before and $G \to M$ another vector bundle, $k, l \in \mathbb{N}_0$, and $r, s \in \mathbb{R}$.

Lemma 4.1.2.

(1) if $P \in \mathsf{Diff}^k(M, \mathsf{Hom}(E,F))$, then $\operatorname{supp}(Pu) \subset \operatorname{supp}(u)$ for all k and $u \in C^\infty(M,E)$.

(2) $\mathsf{Diff}^k(M, \mathsf{Hom}(E,F)) \subset \Psi^r(M, \mathsf{Hom}(E,F))$ for all k with $k \leq r$.

(3) given two operators $Q \in \Psi^s(M, \mathsf{Hom}(F,G))$ and $P \in \Psi^r(M, \mathsf{Hom}(E,F))$; if one of them is properly supported, then $Q \circ P \in \Psi^{r+s}(M, \mathsf{Hom}(E,G))$; if in particular $Q \in \mathsf{Diff}^k(M, \mathsf{Hom}(F,G))$ and $P \in \mathsf{Diff}^l(M, \mathsf{Hom}(E,F))$, then the composition $Q \circ P$ is an operator in $\mathsf{Diff}^{k+l}(M, \mathsf{Hom}(E,G))$;

(4) if $P \in \Psi^r(M, \mathsf{Hom}(E,F))$, then $P^\dagger \in \Psi^r(M, \mathsf{Hom}(\overline{F}^*, \overline{E}^*))$; in particular: if $P \in \mathsf{Diff}^k(M, \mathsf{Hom}(E,F))$, then $P^\dagger \in \mathsf{Diff}^k(M, \mathsf{Hom}(\overline{F}^*, \overline{E}^*))$;

(5) if $P \in \Psi^r(M, \mathsf{Hom}(E,F))$ is properly supported, then it continuously maps

$$\begin{aligned} C^\infty_{\mathsf{c}}(M,E) &\to C^\infty_{\mathsf{c}}(M,F) \\ C^{-\infty}_{\mathsf{c}}(M,E) &\to C^{-\infty}_{\mathsf{c}}(M,F) \\ C^\infty(M,E) &\to C^\infty(M,F) \\ C^{-\infty}(M,E) &\to C^{-\infty}(M,F) \quad . \end{aligned}$$

(6) $P \in \Psi^{-\infty}(M, \mathsf{Hom}(E,F))$ if and only if $P : C^{-\infty}_{\mathsf{c}}(M,E) \to C^\infty(M,F)$.

(1) says that differential operators are local operators which implies that any differential operator is properly supported due to (4.7). (3) implies that for $E = F = G$ the commutator $[Q,P]$ is a pseudo-differential operator of order $(r+s)$. The property on the right-handside of the equivalence in (6) is called *smoothing* and implies the smoothness of the Schwartz kernel and vice versa.

The *principal symbol* of a differential operator is defined through the highest order contribution in (4.8). An invariant definition is given by oscillatory testing: for any $\Phi \in C^\infty(M)$, such that $\xi = \mathrm{d}\Phi$, the principal symbol of the differential operator P of order k is defined by

$$\boldsymbol{\sigma}_k(P)(p,\xi)u = \lim_{\lambda \to \infty} \left(\frac{1}{\lambda}\right)^k \mathrm{e}^{-\mathrm{i}\lambda\Phi} P\left(\mathrm{e}^{\mathrm{i}\lambda\Phi}u\right)\Big|_p \tag{4.11}$$

for any smooth section u in E. It is an element in $C^\infty(T^*M, \pi^*_{T^*M}\mathsf{Hom}(E,F))$. The principal symbols for pseudo-differential operators of order r are denoted in the same way, but defined as symbols of order r modulo symbols of order $(r-1)$. If the symbol class is classical, i.e. the local symbol $a(P)$ of P has an asymptotic expansion

$$a(P)(p,\xi) \sim \sum_{j=0}^{\infty} a_j(p,\xi) \tag{4.12}$$

where each $a_j(p,\xi)$ is smooth on $\mathcal{U} \times (\mathbb{R}^n \setminus \{0\})$ and positive homogeneous of order $(r-j)$, the principal symbol is then given by the leading term $a_0(p,\xi)$. We denote the set of those pseudo-differential operators with the subscript cl. We collect some properties of principal symbols.

Lemma 4.1.3. *Let* $Q \in \Psi^s(M, \mathsf{Hom}(F,G))$ *and* $P \in \Psi^r(M, \mathsf{Hom}(E,F))$ *with principal symbols* $\boldsymbol{\sigma}_s(Q)$ *and* $\boldsymbol{\sigma}_r(P)$*;*

(1) $\boldsymbol{\sigma}_{r+s}(Q \circ P) = \boldsymbol{\sigma}_s(Q) \circ \boldsymbol{\sigma}_r(P)$ *if* P *or* Q *are properly supported.*

(2) $\boldsymbol{\sigma}_r(P^\dagger) = (\boldsymbol{\sigma}_r(P))^*$.

If the operators act between scalar sections, the principal symbols are $\mathbb{C}$-valued and the symbols in (1) commute.

The principal symbol of an arbitrary operator with distributional kernel is introduced within the concept of symbols and orders of distributions. The interested reader shall consult the two papers [Wei76] and [Wei78] of Weinstein for these concepts.

4.1.3. Sobolev spaces on manifolds

On a Riemannian manifold M there exists for any vector bundle $E \to M$ with Riemannian/Hermitian inner product and connection ∇^E as well as for any real number $s \in \mathbb{R}$ a properly supported, classical, elliptic pseudo-differential operator of order s with strictly positive principal symbol. A special choice is given by a power of the Laplace-type operator $(\nabla^E)^*\nabla^E + \mathbb{1} : C^\infty(M,E) \to C^\infty(M,E)$:

$$\Lambda^s := \left((\nabla^E)^*\nabla^E + \mathbb{1}\right)^{\frac{s}{2}} \quad . \tag{4.13}$$

This operator satisfies all stated features for $s \in \mathbb{R}$ and is furthermore essentially self-adjoint[16] on $L^2(M,E)$ with positive spectrum if M is compact or complete.

Sobolev spaces on compact manifolds M without boundary are going to be introduced first since they provide a bulding block for the non-compact case. The $s-$*Sobolev norm* of $u \in C^\infty(M,E)$ is defined as

$$\|u\|_{H^s(M,E)} := \|\Lambda^s u\|_{L^2(M,E)} \tag{4.14}$$

and the norm completion of $C^\infty(M,E)$ with respect to (4.14) defines the *Sobolev space of order* s which we designate with $H^s(M,E)$. The definition depends on neither the choice of the metric nor of the connection such that all Sobolev norms and spaces with different metrics and connections are equivalent. For $s = m \in \mathbb{N}_0$ the norm (4.14) is equivalent to

$$\|u\|^2_{\mathcal{H}^m(M,E)} := \sum_{j=0}^{m} \left\| \left(\nabla^E\right)^j u \right\|^2_{L^2(M,E)} \tag{4.15}$$

where we introduced another notation of this norm for distinguishing reasons. H^s can be extended to non-compact manifolds under additional conditions and concepts, see e.g. [Heb96] for details. We start with Sobolev sections of order m with respect to a vector bundle $E \to M$ which have a fixed support in $K \Subset M$. They are introduced as norm

[16] This follows from the essentially self-adjointness of the identity map and the Bochner Laplacian $(\nabla^E)^*\nabla^E$, see [BMS02] or [Yos92]. It holds true for any compact manifold since they are complete by the Hopf-Rinow theorem.

closure of smooth functions with compact support in K with respect to $\|\cdot\|_{\mathcal{H}^m(M,E)}$:

$$H^m_K(M,E) = \overline{C^\infty_K(M,E)}^{\|\cdot\|_{\mathcal{H}^m(M,E)}} \quad .$$

They carry the topology of a Hilbert space. The compactly supported Sobolev sections then follow by taking the union over all compact subsets:

$$H^m_{\mathsf{c}}(M,E) = \bigcup_{\substack{K\subset M\\ \text{compact}}} H^m_K(M,E) \tag{4.16}$$

which makes them an LH-space. We observe $H^0_{\mathsf{c}}(M,E) = L^2_{\mathsf{c}}(M,E)$. Local Sobolev sections are regarded as those distributional sections u such that $\phi u \in H^s_{\mathrm{supp}(\phi)}$ for all smooth and compactly supported function ϕ:

$$H^m_{\mathsf{loc}}(M,E) := \left\{u \in C^{-\infty}(M,E)\,|\,\phi u \in H^m_{\mathsf{c}}(M,E) \quad \forall\,\phi \in C^\infty_{\mathsf{c}}(M)\right\} \quad . \tag{4.17}$$

The norm in $H^m_{\mathrm{supp}(\phi)}$ induces a seminorm $\|\cdot\|_{H^m_{\mathsf{loc}}(M,E)}$ which makes H^m_{loc} a Fréchet space. Note that $H^0_{\mathsf{loc}}(M,E)$ corresponds to $L^2_{\mathsf{loc}}(M,E)$.

The extension of (4.16) and (4.17) to real powers $s \in \mathbb{R}$ can be performed with interpolation. We follow a more practical way of introducing Sobolev spaces on non-compact manifolds, based on [BTW15, Sec.1.6]. Sobolev sections of E with fixed compact support on a non-compact Riemannian manifold are reinterpreted as Sobolev sections on another Riemannian, but closed manifold. In order to do so, one extends all structures to the *double* of a suitable subset of M. Let K be a compact subset of M, containing the support of a function. Take another relatively compact subset $K_1 \subset M$ such that K_1 has a smooth boundary ∂K_1 and contains K inside $\mathring{K}_1$. The *closed double* of K_1 is constructed by taking a copy of K_1, denoted with K_2, and then one glues both copies together along their common boundary $\partial K_1 = \partial K_2$ which results in a closed manifold:

$$\widetilde{M} := K_1 \cup_{\partial K_1} K_2 := (K_1 \sqcup K_2) \setminus \partial K_1 \quad .$$

During this procedure everything on K is untouched. Any smooth vector bundle $\widetilde{E} \to \widetilde{M}$ can be considered as extension of $E|_{K_1}$ if $\widetilde{E}|_{K_1} = E|_{K_1}$. All bundle metrics on $K \subset \widetilde{M}$ can be extended to smooth bundle metrics on the whole closed double which is assured by additional assumptions[17] on K_1. The Levi-Civita connection with respect to the restricted metric and the Koszul connection of the restricted bundle $E|_{K_1}$ can be extended to a smooth metric and a smooth connection on $T\widetilde{M}$ and respectively $\widetilde{E}$. Any smooth section of E with compact support in K can then be viewed as smooth section of $\widetilde{E}$ over a closed manifold. The space $H^s_K(M,E)$ for real powers s is then defined as the completion of smooth, compactly supported functions with respect to the norm $\|\cdot\|_{H^s(\widetilde{M},\widetilde{E})}$ from (4.14):

$$H^s_K(M,E) := \overline{C^\infty_K(M,E)}^{\|\cdot\|_{H^s(\widetilde{M},\widetilde{E})}} \quad .$$

The spaces H^s_{c} and H^s_{loc} are then defined as in (4.16) und (4.17) for real orders s which completes the extension to arbitrary real Sobolev orders. The advantage of this definition

[17] K_1 has to be chosen in such a way that its boundary ∂K_1 is totally geodesic with normal vector field $\mathfrak{n}$ such that all normal derivatives of $\mathcal{R}(X,\mathfrak{n})\mathfrak{n}$ vanish; see [Mor91] for details

is that Sobolev sections with compact support can be treated in the same way as Sobolev sections on a closed manifold with all beneficial properties. If $K \subset K'$, one has the inclusion $C^\infty_K(M,E) \subset C^\infty_{K'}(M,E)$, inducing $H^s_K(M,E) \subset H^s_{K'}(M,E)$ as continuous linear inclusion. This implies the continuous inclusion $H^s_K(M,E) \hookrightarrow H^s_{\mathsf{c}}(M,E)$.

We list some properties of Sobolev spaces and properties of maps between those spaces.

Proposition 4.1.4 (pp.57-65 in [SA11]). *Given a manifold M with compact subsets K, E, F Hermitian vector bundles over M, and an operator $A \in \Psi^m(M, \mathsf{Hom}(E,F))$ for $m \in \mathbb{R}$, then the following properties are true for $s, t \in \mathbb{R}$:*

(1) (Localisation) let $a \in C^\infty(M)$, then

  a) $au \in H^s_{\mathsf{loc}}(M,E)$ for $u \in H^s_{\mathsf{loc}}(M,E)$;

  b) if $a \in C^\infty_{\mathsf{c}}(M)$ and $u \in H^s_{\mathsf{c}}(M,E)$, then $au \in H^s_{\mathsf{c}}(M,E)$.

(2) $H^s_K(M,E)$ is a Hilbert space with scalar product

$$\langle u \,|\, v \rangle_{H^s_K(M,E)} = \langle \Lambda^s u \,|\, \Lambda^s v \rangle_{L^2(K,E)} \quad \forall\, v \in H^s_K(M,E), u \in H^s_K(M,E) \quad ;$$

the same holds true for $H^s(M,E)$ if M is compact.

(3) (Continuous embeddings) $H^s_{\mathsf{c}}(M,E) \subset H^t_{\mathsf{c}}(M,E)$ and $H^s_{\mathsf{loc}}(M,E) \subset H^t_{\mathsf{loc}}(M,E)$ for $s > t$.

(4) (Rellich-Kontrachov theorem) the inclusion $H^s_K(M,E) \hookrightarrow H^t_K(M,E)$ is compact if M is compact.

(5) (Sobolev embedding theorem) if $k < s - \frac{\dim(M)}{2}$, then

  a) $H^s_{\mathsf{loc}}(M,E) \subset C^k(M,E)$ is continuous;

  b) $H^s_K(M,E) \subset C^k_{\mathsf{c}}(M,E)$ is compact.

(6) $\bigcap_{s\in\mathbb{R}} H^s_{\mathsf{loc}}(M,E) = C^\infty(M,E)$ and $\bigcup_{s\in\mathbb{R}} H^s_K(M,E) = C^{-\infty}_{\mathsf{c}}(M,E)$.

(7) (Duality) $(H^s_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M))^* = H^{-s}_{\mathsf{loc}}(M, \overline{E}^* \otimes |\Omega|^{\frac{1}{2}}_M)$ with respect to the dual pairing (4.6) and its distributional extension.

(8) (Restriction theorem) let $N \subset M$ be a submanifold of codimension d; if $s > \frac{d}{2}$, then the restriction of $u \in H^s_{\mathsf{loc}}(M,E)$ is a continuous map with $u|_N \in H^{s-\frac{d}{2}}(N, E|_N)$.

(9) (Regularity) the operator A maps between Sobolev spaces:

  a) $A : H^s_{\mathsf{c}}(M,E) \to H^{s-m}_{\mathsf{loc}}(M,F)$;

  b) $A : H^s_{\mathsf{c}}(M,E) \to H^{s-m}_{\mathsf{c}}(M,F)$ and $A : H^s_{\mathsf{loc}}(M,E) \to H^{s-m}_{\mathsf{loc}}(M,F)$ if A is properly supported.

4.2. Fourier integral operators

In this section we consider Fourier integral operators (FIO) as a class of operators between sections on possibly different manifolds which contains differential and pseudo-differential operators as subclasses. We will only focus on their global properties as we will only use them in the following analysis. The presented content is based on the papers of Hörmander and Hörmander-Duistermaat ([Hö71] and [DH72]) as well as the textbooks references [Dui10], [HÖ9] and [Tre22]. Further supporting notes are taken from [IS20, Sec.2.1] and [IS20, App.B] and further references which we will mention throughout this section.

In this section we take the manifolds M, N and the vector bundles E and F as introduced in subsection 4.1.2.

4.2.1. Lagrangian distributions

In order to introduce Lagrangian distributions, we recall the definition of a Lagrangian submanifold. The cotangent bundle T^*M is a symplectic manifold with respect to a non-degenerate and closed two-form ω_M. A subset $\Lambda \subset T^*M$ is a *Lagrangian submanifold* if ω_M vanishes on Λ and $\dim(\Lambda) = \frac{1}{2}\dim(T^*M) = m$. A Lagrangian submanifold Λ in the punctured cotangent bundle $\dot{T}^*M$ is called *conic* if

$$(p,\xi) \in \Lambda \Rightarrow (0,\xi) \in T_{(p,\xi)}\Lambda \quad .$$

A *clean phase function* is a real-valued smooth function ϕ on an open and conic subset $\mathcal{U} \subset M \times \dot{\mathbb{R}}^k$, $k \in \mathbb{N}_0$ which satisfies the following properties:

(a) ϕ is homogeneous of degree 1 with respect to the fibre variables $\xi \in \dot{\mathbb{R}}^k$:

$$\phi(p,\lambda\xi) = \lambda\phi(p,\xi)$$

for all $\lambda \in \dot{\mathbb{R}}$;

(b) $\mathrm{d}_{x,\xi}\phi \neq 0$;

(c) the fibre-critial set $C_\phi := \{(p,\theta) \in \mathcal{U} \,|\, (\mathrm{d}_\xi\phi)(p,\theta) = 0\}$ is a smooth manifold such that $T_{(p,\theta)}C_\phi = \ker\left([\mathrm{d}_{(x,\xi)}\partial_\xi\phi](p,\theta)\right)$ for all $(p,\theta) \in C_\phi$.

The number of linearly independent vectors in the tangent spaces is smaller or equal k; the difference is described by the *excess* $e := \dim(C_\phi) - \dim(M)$. If the excess is zero, the phase function is called *non-degenerate*. A clean phase function ϕ on an open subset in $\mathbb{R}^m$ defines an immersed conic Lagrangian submanifold Λ_ϕ in the cotangent bundle of this open subset which is defined by the map

$$C_\phi \ni (p,\xi) \quad \mapsto \quad (p, (\mathrm{d}_x\phi)(p,\xi)) \in \Lambda_\phi \quad . \tag{4.18}$$

With these concepts one is able to lift the notion of oscillatory integrals of order $r \in \mathbb{R}$ to manifolds via coordinate neighbourhoods. They are of the form

$$\left(\frac{1}{2\pi}\right)^{\mathfrak{e}_+(m,k,e)} \int_{\mathbb{R}^k} \mathrm{e}^{\mathrm{i}\phi(x,\xi)} a(x,\xi)\,\mathrm{d}\xi \tag{4.19}$$

where $\mathfrak{e}_\pm(m,k,e) = (m \pm 2k - 2e)/4$ and a is a symbol of order $(r + \mathfrak{e}_-(m,k,e))$ which is supported in the interior of a small conic neighbourhood of C_ϕ in the support of the clean phase function. Based on (4.19), Lagrangian distributions on manifolds can be defined as follows.

Definition 4.2.1 (Definition 3.2.2 in [HÖ3], Definition 2.2 in [IS20])**.** Let $\Lambda \subset \dot{T}^*M$ be a smooth, closed and conic Langrangian submanifold; a distribution $u \in C^{-\infty}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ is called a *Lagrangian distribution* of order $r \in \mathbb{R}$, associated to the Lagrangian submanifold Λ, if it can be represented as sum

$$u = \sum_j u_j$$

of locally finite supported oscillatory integrals u_j in a coordinate neighbourhood $\mathcal{U}_j$ of M which are characterised as follows:

(a) for each u_j there exists a clean phase function ϕ_j with critical set C_{ϕ_j} which defines the Lagrangian submanifold Λ_{ϕ_j} through (4.18),

(b) $\Lambda_{\phi_j} \subset \Lambda$ for all j,

(c) for each component $(u_j)^i$, $i \in \{1, ..., m_E\}$ of u_j with respect to local coordinates and to a bundle chart there exists a symbol $(a_j)^i$ of order $(r + \mathfrak{e}_-(m, k_j, e_j))$ with support in the interior of a sufficiently small conic neighbourhood of the critical set C_{ϕ_j} such that each component $(u_j)^i$ takes the form

$$(u_j)^i = \left(\frac{1}{2\pi}\right)^{\mathfrak{e}_+(m,k_j,e_j)} \int_{\mathbb{R}^k} \mathrm{e}^{\mathrm{i}\phi_j(x,\xi)} (a_j)^i(x,\xi)\,\mathrm{d}\xi$$

in the distributional sense.

We designate $I^r(M; \Lambda, E \otimes |\Omega|^{\frac{1}{2}}_M)$ as the set of Lagrangian distributions of order $r \in \mathbb{R}$, associated to the Lagrangian submanifold Λ.

4.2.2. Fourier integral operators

Loosely speaking, a map of the form $C^\infty_c(M, E \otimes |\Omega|^{\frac{1}{2}}_M) \to C^{-\infty}(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$ is a Fourier integral operator if its Schwartz kernel is a Lagrangian distribution on $N \times M$. In order to state this more precisely, let $\Lambda \subset \dot{T}^*(N \times M)$ be a closed conic Lagrangian submanifold with respect to the symplectic form $\omega_N \oplus \omega_M$ on $T^*(N \times M)$ where ω_M, ω_M are the symplectic forms for T^*M respectively T^*N. A *homogeneous canonical relation* from $\dot{T}^*M$ to $\dot{T}^*N$ is a closed conic Lagrangian submanifold C in $\dot{T}^*(N \times M)$ with respect to the symplectic form $\omega_N \oplus (-\omega_M)$ which is contained in $\dot{T}^*N \times \dot{T}^*M$. Its relation to the initial Lagrangian submanifold Λ is given by

$$\mathsf{C} = \left\{ (x,\xi,y,\eta) \in \dot{T}^*M \times \dot{T}^*N \,\middle|\, (x,\xi,y,-\eta) \in \Lambda \right\} \quad .$$

We follow the common literature and denote the corresponding Lagrangian submanifold Λ with C' to stress the homogeneous canonical relation.

Definition 4.2.2 (cf. Definition 25.2.1 in [Hö9]). Given $E \to M$ and $F \to N$ and a closed conic Lagrangian submanifold $\Lambda \subset \dot{T}^*(M \times N)$ with corresponding homogeneous canonical relation C from $\dot{T}^*N$ to $\dot{T}^*M$; a *Fourier integral operator of order* $r \in \mathbb{R}$ is an operator $P : C^\infty_\mathsf{c}(M, E \otimes |\Omega|^{\frac{1}{2}}_M) \to C^{-\infty}(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$ with Schwartz kernel $K \in I^r(N \times M; \Lambda, \mathbf{Hom}(E, F))$.

We will designate the space of those operators with $\mathcal{FIO}^r(M, N; \mathsf{C}'; \mathbf{Hom}(E, F))$. The local description of the kernels can be recapitulated in [IS20, Sec.2.1].

Before we list some properties, we introduce further notions and concepts. The maps

$$\begin{aligned} \mathsf{r} : T^*M &\to T^*M \\ (x, \xi) &\mapsto (x, -\xi) \end{aligned} \quad , \quad \begin{aligned} \mathsf{s} : N \times M &\to M \times N \\ (y, x) &\mapsto (x, y) \end{aligned}$$

are the reflection in the cotangent bundle respectively the interchanging of factors in the Cartesian product. We set $\Lambda^{-1} := \mathsf{r}^*\mathsf{s}^*(\Lambda)$ as inverse of the closed conic Lagrangian submanifold and C^{-1} denotes the corresponding inverse canonical relation which is itself a canonical relation from $\dot{T}^*N$ and $\dot{T}^*M$. Let W be another manifold; the composition of two homogeneous canonical relations C_1 from $\dot{T}^*N$ to $\dot{T}^*W$ and C_2 from $\dot{T}^*M$ to $\dot{T}^*N$ is

$$\mathsf{C}_1 \circ \mathsf{C}_2 := \Big\{ (x, \xi, z, \zeta) \in \dot{T}^*M \times \dot{T}^*W \,\Big|\, \exists\, (y, \eta) \in \dot{T}^*N \,:\, (x, \xi, y, \eta) \in \mathsf{C}_2 \\ \text{and} \quad (y, \eta, z, \zeta) \in \mathsf{C}_1 \Big\} \quad .$$

The composition is called *clean* if $\mathsf{C}_1 \times \mathsf{C}_2$ and $T^*M \times \operatorname{diag}(T^*N) \times T^*W$ intersect in a manifold $\widetilde{\mathsf{C}}$ such that

$$T_p\widetilde{\mathsf{C}} = T_p(\mathsf{C}_1 \times \mathsf{C}_2) \cap T_p(T^*M \times \operatorname{diag}(T^*N) \times T^*W)$$

for all points p in $\widetilde{\mathsf{C}}$. The composition is in contrast *transversal* if

$$T_p\widetilde{\mathsf{C}} = T_p(\mathsf{C}_1 \times \mathsf{C}_2) + T_p(T^*M \times \operatorname{diag}(T^*N) \times T^*W)$$

for all points p in $\widetilde{\mathsf{C}}$. Both cases are distinguished by the *(global) excess* e which is the codimension of fibres of the intersection. Transversality becomes equivalent to $\mathsf{e} = 0$. The composition is called *proper* if the projection $\widetilde{\mathsf{C}} \to \dot{T}^*(M \times W)$ is a proper map, and *connected* if the fibres of $\mathsf{C}_1 \circ \mathsf{C}_2 \to \mathsf{C}_1 \times \mathsf{C}_2$ are compact and connected.

A special situation arises for $m = n$. A homogeneous canonical relation from $\dot{T}^*M$ to $\dot{T}^*N$ will be called *local canonical graph* if both projections on $\dot{T}^*M$ and $\dot{T}^*N$ are local diffeomorphisms. The homogeneous canonical relation is locally the graph of a canonical transformation and a symplectic manifold on its own right. It is called *bijective* if in addition C^{-1} is a local canonical graph, too.

We first collect some algebraic properties of Fourier integral operators which are proven in and taken from [Hö9, Sec.25.2] and [Hö71, Chap.4], supported with additional details from [IS20, Sec.4.1].

Lemma 4.2.3. *Given $E \to M$ and $F \to N$, a vector bundle $G \to W$ and homogeneous canonical relations $\mathsf{C} = \mathsf{C}_1$ from $\dot{T}^*M$ to $\dot{T}^*N$ and C_2 from $\dot{T}^*N$ to $\dot{T}^*W$; the following properties hold for all $r, s \in \mathbb{R}$,*

(1) *(adjoint FIO) if $A \in \mathcal{FIO}^s(M, N; \mathsf{C}'; \mathbf{Hom}(E, F))$, then the adjoint/dual with respect to the pairing (4.6) satisfies*

$$A^\dagger \in \mathcal{FIO}^r(N, M; (\mathsf{C}^{-1})'; \mathbf{Hom}(\overline{F}^*, \overline{E}^*)) \quad ;$$

(2) *(composition) given two operators*

$$\begin{aligned} A_2 &\in \mathcal{FIO}^r(M, N; \mathsf{C}_2; \mathbf{Hom}(E, F)) \\ A_1 &\in \mathcal{FIO}^s(N, W; \mathsf{C}_1; \mathbf{Hom}(F, G)) \quad ; \end{aligned}$$

if the operators are properly supported and the composition $\mathsf{C}_1 \circ \mathsf{C}_2$ is clean, proper and connected with excess e, then $A_1 \circ A_2 \in \mathcal{FIO}^{r+s+\mathsf{e}/2}(M, W; (\mathsf{C}_1 \circ \mathsf{C}_2)'; \mathbf{Hom}(E, G))$; the same holds if the composition $\mathsf{C}_1 \circ \mathsf{C}_2$ is proper and transversal with $\mathsf{e} = 0$.

(3) *let $N = M$, then $\Psi^r(M, \mathsf{Hom}(E, F)) \subset \mathcal{FIO}^r(M, M; (N^*\mathrm{diag}(M))'; \mathsf{Hom}(E, F))$;*

(4) *$A \in \mathcal{FIO}^{-\infty}(M, N; \mathsf{C}; \mathbf{Hom}(E, F))$ if and only if A is smoothing.*

(5) *$(A^\dagger \circ A) \in \Psi^{2m}(M, \mathsf{End}(E))$ for A as in (1).*

Property (5) is a consequence of (1), (2) and (3) with

$$\mathsf{C}^{-1} \circ \mathsf{C} = N^*\mathrm{diag}(M) \quad \text{and} \quad \mathbf{Hom}(E, E) = \mathsf{End}(E) \otimes |\Omega|^{\frac{1}{2}}_{M\times M} \cong \mathsf{End}(E) \otimes |\Omega^n|_M \quad .$$

We denote the set of properly supported Fourier integral operators of order r with $\mathcal{FIO}^r_{\mathsf{prop}}$. Similarly we write Ψ^r_{prop} to indicate properly supported pseudo-differential operators of order r. To have a well-defined composition in (2), it is only necessary that the first operator is properly supported. But this does not assure that the resulting operator is again a FIO even if the other requirements are satisfied. If the homogeneous canonical relation is a local canonical graph, then one can show the following two regularity properties of Fourier integral operators which are also based on results, proven in [Hö09], [Hö71] and [Tre22, Sec.18.5.3].

Lemma 4.2.4. *Given two vector bundles $E \to M$, $F \to N$ and r as in Lemma 4.2.3; let C be a local canonical graph from $\dot{T}^*M$ to $\dot{T}^*N$ and $A \in \mathcal{FIO}^r(M, N; \mathsf{C}'; \mathbf{Hom}(E, F))$, then the following holds:*

(1) *(L^2-regularity) if $r = 0$, A becomes a continuous map from $L^2_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ to $L^2_{\mathsf{loc}}(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$; if*

$$\sup_{(x,y)\in K} \|\boldsymbol{\sigma}_0(A)(x, \xi; y, \eta)\|_{\mathbf{Hom}(E,F)} \quad \to \quad 0$$

for $|(\xi, \eta)| \to \infty$ for all $K \Subset M \times N$, then it maps as compact operator between $L^2(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ to $L^2(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$.

(2) (Sobolev regularity) the operator A maps continuously from $H^s_{\mathsf{c}}(M, E \otimes |\Omega|^{\frac{1}{2}}_M)$ to $H^{s-r}_{\mathsf{loc}}(N, F \otimes |\Omega|^{\frac{1}{2}}_N)$ for all $s \in \mathbb{R}$;

(3) if A is properly supported, then $A : H^s_{\mathsf{c}}(M, E) \to H^{s-r}_{\mathsf{c}}(N, F)$ and $A : H^s_{\mathsf{loc}}(M, E) \to H^{s-r}_{\mathsf{loc}}(N, F)$ are continuous linear maps for all $s \in \mathbb{R}$.

We end this subsection with some remarks about the principal symbol $\boldsymbol{\sigma}_r(A)$ of a Fourier integral operator A. The principal symbol of $A \in \mathcal{FIO}^r(M, N; \mathsf{C}'; \mathbf{Hom}(E, F))$ is an element in the equivalence class of symbols of order $(r + (n + m)/4)$ modulo symbols of order $(r + (n + m)/4 - 1)$ which are sections of the bundle $M_{\mathsf{C}} \otimes \pi^*_{\mathsf{C}}(\mathbf{Hom}(E, F)) \to \mathsf{C}$; π_{C} is the bundle projection $T^*(N \times M) \supset \mathsf{C} \to N \times M$, such that $\pi^*_{\mathsf{C}}(\mathbf{Hom}(E, F)) = \pi^*_{\mathsf{C}}(\mathsf{Hom}(E, F)) \otimes |\Omega|^{\frac{1}{2}}_{\mathsf{C}}$ and the bundle M_{C} is the *Keller-Maslov bundle* which is a complex trivial line bundle with respect to the structure group $\mathbb{Z}/4\mathbb{Z}$, describing the invariance of the amplitude and the half-density of an oscillatory integral under the change of the phase function due to coordinate transformations. The transition from one trivialisation to another is given by multiplying with $\mathbb{i}^l$ for any $l \in \mathbb{Z}$. More informations about the geometric nature of the Keller-Maslov bundle are given in [Dui10, Sec.4.1] and [Hö71, Sec.3.3].

The principal symbol of $A^\dagger$ is given by $\boldsymbol{\sigma}_r(A^\dagger) = \mathsf{s}^*((\boldsymbol{\sigma}_r(A))^*)$ by which it becomes a symbol section of the bundle $M_{\mathsf{C}^{-1}} \otimes \pi^*_{\mathsf{C}^{-1}}(\mathbf{Hom}(\overline{F}^*, \overline{E}^*)) \to \mathsf{C}^{-1}$ with the bundle projection $\pi_{\mathsf{C}^{-1}} : T^*(M \times N) \supset \mathsf{C}^{-1} \to M \times N$. The principal symbol of the composition of two Fourier integral operators with the required assumptions from Lemma 4.2.3 (2) is a more involved combination of the single principal symbols; see [Hö9, Thm.25.2.3] and [Hö71, Sec.3.2]. The situation becomes much easier if both homogeneous canonical relations of the composing Fourier integral operators are local canonical graphs. Then the composition of canonical relations is a graph of a composition of two symplectomorphisms and the principal symbol of the composition becomes

$$\boldsymbol{\sigma}_{r+s}(A_1 \circ A_2) = \boldsymbol{\sigma}_s(A_1) \circ \boldsymbol{\sigma}_r(A_2) \quad . \tag{4.20}$$

We refer to the literature and the content of [IS20, App.B] for details and section 2.1 in the same reference for the local expression of the principal symbol.

Having the notion of a principal symbol, we can define the notion of ellipticity of a FIO: a FIO, associated to a local canonical graph C as homogeneous canonical relation, is called *elliptic* if its principal symbol does not vanish on C.

Lemma 4.2.5. *Given two vector bundles $E \to M$, $F \to N$ and r as in Lemma 4.2.3; let C be a homogeneous canonical relation and $A \in \mathcal{FIO}^r_{\mathsf{prop}}(M, N; \mathsf{C}'; \mathbf{Hom}(E, F))$, then the following holds:*

(1) (exact sequence) there is a symbol exact sequence

$$0 \to \mathcal{FIO}^{r-1}_{\mathsf{prop}}(M, N; \mathsf{C}'; \mathbf{Hom}(E, F)) \hookrightarrow \mathcal{FIO}^r_{\mathsf{prop}}(M, N; \mathsf{C}'; \mathbf{Hom}(E, F)) \xrightarrow{\boldsymbol{\sigma}_r} C^\infty(\mathsf{C}, M_{\mathsf{C}} \otimes |\Omega|^{\frac{1}{2}}_{\mathsf{C}} \otimes \pi^*_{\mathsf{C}}\mathbf{Hom}(E, F)) \to 0$$

for all $r \in \mathbb{R}$; hence the operator has at least one order less if and only if its principal symbol vanishes.

(2) if A is a unitary map from $L^2_{\mathsf{c}}(M,E)$ to $L^2_{\mathsf{c}}(N,F)$ and C a local canonical graph, then A is elliptic.

Fact (2) is from [Tre22, Prop.18.5.28] for the scalar case; the proof carries over to the vector-bundle case. Another worth mentionable fact about principal symbols of Fourier integral operators is the Theorem of Egorov which is e.g. stated in [Hö9, Thm.25.3.5] for operators, mapping between scalar sections; a modified version for Fourier integral operators between vector-valued sections is presented in [IS20, App.D].

4.2.3. Restriction and corestriction as Fourier integral operators

Let $E \to M$ be as before and now N is an embedded submanifold of codimension k in M with inclusion $i : N \hookrightarrow M$. The pullback of the embedding defines the *restriction operator*

$$\mathsf{res}_N := i^* \,:\, C^\infty(M, E\otimes|\Omega|^{\frac{1}{2}}_M) \quad\to\quad C^\infty(N, E|_N\otimes|\Omega|^{\frac{1}{2}}_N) \tag{4.21}$$

which assigns each function its trace on the submanifold. The adjoint/dual of res_N with respect to the dual pairing $C^{-\infty}_{\mathsf{c}} \times C^\infty \to \mathbb{C}$ is the *corestriction operator*

$$\mathsf{res}^\dagger_N \,:\, C^{-\infty}_{\mathsf{c}}(N, E|_N\otimes|\Omega|^{\frac{1}{2}}_N) \quad\to\quad C^{-\infty}_{\mathsf{c}}(M, N\otimes|\Omega|^{\frac{1}{2}}_M) \quad . \tag{4.22}$$

As pushforward is the dual operator to pullback, one also writes i_* for $\mathsf{res}^\dagger_N$. We outline the important properties.

Proposition 4.2.6. *Let N be an embedded smooth submanifold of a smooth manifold M with codimension k and inclusion map $i : N \hookrightarrow M$ as well as a vector bundle $E \to M$; then the following holds.*

(1)

$$\begin{array}{rccc} \mathsf{res}_N : & H^s_{\mathsf{loc}}(M,E) & \to & H^{s-k/2}_{\mathsf{loc}}(N,E|_N) \\ \mathsf{res}^\dagger_N : & H^{-s+k/2}_{\mathsf{c}}(N,E|_N) & \to & H^{-s}_{\mathsf{c}}(M,E) \end{array} \tag{4.23}$$

for all $s\in\mathbb{R}$ with $s > k/2$.

(2) a) $i^* \in \mathcal{FIO}^{k/4}(M,N;\mathsf{C}'(i^*);\mathbf{Hom}(E,E|_N))$ with homogenous canonical relation

$$\mathsf{C}(i^*) := \left\{(y,\eta,x,\xi)\in \dot{T}^*(N\times M)\,\middle|\,(x,\xi)\in\dot{T}^*M \,:\, i^*(x,\xi)=(y,\eta)\right\}. \tag{4.24}$$

b) $i_* \in \mathcal{FIO}^{k/4}(N,M;\mathsf{C}'(i_*);\mathbf{Hom}(E|_N,E))$ with homogenous canonical relation

$$\mathsf{C}(i_*) := N^*\mathrm{graph}(i) = \mathsf{C}(i^*)^{-1} \quad .$$

(3) if N is a closed subset in M, then both operators (4.21) and (4.22) are properly supported.

Proof. (1) is a consequence of the restriction theorem of Sobolev spaces and the dual pairing; see (7) and (8) in Proposition 4.1.4. It has been shown in [Dui10, Sec.5.1] that the restriction operator is a Fourier integral operator of order $k/4$ with claimed homogeneous canonical relation in the scalar case. This carries over to the vector-valued case as this holds in any trivialisation of the vector bundle. The claim for the corestriction operator follows from Lemma 4.2.3 (1) which concludes the proof for (2).

The corestriction operator maps compactly supported distributions to compactly supported distributions by definition. Suppose $u \in C^\infty(M,E)$ has compact support in $K \subset M$. The support of i^*u is contained in $i(N) \cap K$ which is w.l.o.g. not empty or there is nothing to show otherwise. Since N is a closed subset in M, the embedding is a closed map as well. The intersection of the compact support with the closed subset $i(N)$ is a closed subset of K and thus itself compact. Thus the restriction maps compactly supported sections to compactly supported sections and due to (4.7) it is a properly supported operator in this situation. □

If M is a globally hyperbolic manifold with spacelike Cauchy hypersurface Σ, the closedness condition in Proposition 4.2.6 (3) is always satisfied because the Cauchy hypersurface of a globally hyperbolic manifold is always closed.

Corollary 4.2.7.

$$i^* \in \mathcal{FIO}^{1/4}_{\mathsf{prop}}(M,\Sigma;\mathsf{C}'(i^*);\mathbf{Hom}(E,E|_\Sigma)) \quad \textit{and} \quad i_* \in \mathcal{FIO}^{1/4}_{\mathsf{prop}}(\Sigma,M;\mathsf{C}'(i_*);\mathbf{Hom}(E|_\Sigma,E))\,.$$

4.3. Function spaces on globally hyperbolic manifolds

We recall some special function spaces on a globally hyperbolic manifold M with Cauchy hypersurface Σ and Cauchy temporal function $\mathcal{T} : M \to \mathbb{R}$. Further details and more examples of function spaces on globally hyperbolic manifolds are presented in [BTW15, Sec.1.7] and [Bä14, Chap.2].

The *space of spatially compactly supported sections* on E is defined via

$$C^\infty_{\mathsf{sc}}(M,E) := \bigcup_{\substack{A\subset M\\ A\,\text{spatially}\\ \text{compact}}} C^\infty_A(M,E) \quad .$$

This is an LF-space as the $C^\infty_A(M,E)$ are Fréchet spaces. We assume that the spacelike Cauchy hypersurface is complete and has no boundary such that the family of slices $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ foliates M. Fix $s \in \mathbb{R}$ and consider the family $\{H^s_{\mathsf{loc}}(E|_{\Sigma_t})\}_{t\in\mathcal{T}(M)}$ as Fréchet bundle over $\mathcal{T}(M) \subset \mathbb{R}$. The slices differ from each other only in the metric g_t, but not topologically. Since different metrics lead to equivalent Sobolev norms, each Sobolev space $H^s_K(E|_{\Sigma_t})$ for $K \Subset \Sigma$ and consequently $H^s_{\mathsf{c}}(E|_{\Sigma_t})$ and $H^s_{\mathsf{loc}}(E|_{\Sigma_t})$ are equivalent. We keep the extra t to mark the different metrics and furthermore keep notational compatability with the referred literature. $\{H^s_{\mathsf{loc}}(E|_{\Sigma_t})\}_{t\in\mathcal{T}(M)}$ can be globally trivialised as follows: for each $t \in \mathcal{T}(M)$ a section of this bundle becomes a section in $H^s_{\mathsf{loc}}(E|_{\Sigma_t})$. Sections of this bundle can be moved to $H^s_{\mathsf{loc}}(E|_{\Sigma_\tau})$ for a fixed $\tau \in \mathcal{T}(M)$ by parallel transport along the integral curves of the vector field $\mathrm{grad}(\mathcal{T})$. The support properties and the Sobolev regularity are preserved by this transport such that this bundle of Fréchet spaces becomes diffeomorphic to $\mathcal{T}(M)\times H^s_{\mathsf{loc}}(E|_{\Sigma_\tau})$; l-times continuously differentiable sections in the time parameter ($l \in \mathbb{N}_0$) are denoted by $C^l(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$. The elements can be considered as regular distribution on M:

$$\left[u \,\middle|\, \phi\right]_{C_c^\infty(M,\overline{E}^*)} := \int_{\mathcal{T}(M)} \left[u(t) \,\middle|\, (N\phi)|_{\Sigma_t}\right]_{C_c^\infty(\overline{E}^*|_{\Sigma_t})} \mathrm{d}t$$

for $u \in C^l(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ and $\phi \in C_c^\infty(M, \overline{E}^*)$. The lapse function $N \in C^\infty(M, \mathbb{R}_{>0})$ appears in the volume element dvol in (3.40). These distributional sections are locally integrable for $s \geq 0$ since $H^s_{\mathsf{loc}} \subset L^2_{\mathsf{loc}}$ by the continuous embedding of Sobolev spaces. The dual pairing in the integral can be expressed as regular distributional action of the form

$$\left[u(t) \,\middle|\, (N\phi)|_{\Sigma_t}\right]_{C_c^\infty(\overline{E}^*|_{\Sigma_t})} = \int_{\Sigma_t} \left[u \,\middle|\, (N\phi)|_{\Sigma_t}\right]_{(\overline{E}^*|_{\Sigma_t})^*} \mathsf{dvol}_{\Sigma_t}$$

and thus

$$\left[u \,\middle|\, \phi\right]_{C_c^\infty(M,\overline{E}^*)} = \int_{\mathcal{T}(M)} \left[\int_{\Sigma_t} \left[u \,\middle|\, (N\phi)|_{\Sigma_t}\right]_{(\overline{E}^*|_{\Sigma_t})^*} \mathsf{dvol}_{\Sigma_t}\right] \mathrm{d}t = \int_M \left[u \,\middle|\, \phi\right]_{\overline{E}^*} \mathsf{dvol} \quad .$$

This observation confirms that $C^l(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ is embedded into $C^{-\infty}(M, E)$. For any compact subinterval $I \subset \mathcal{T}(M)$ and any spatially compact $K \subset M$ one defines

$$C^l_K(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) := \left\{ u \in C^l(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \,\middle|\, \mathrm{supp}\,(u) \subset K \right\} \tag{4.25}$$

with the seminorm

$$\|u\|_{I,K,l,s} := \max_{k \in [0,l] \cap \mathbb{N}_0} \max_{t \in I} \left\| (\nabla_t)^k u \right\|_{H^s_{\mathsf{loc}}(E|_{\Sigma_t})} \quad . \tag{4.26}$$

Varying over all compact subsets $I \subset \mathcal{T}(M)$ shows for fixed l, K, and s that (4.25) is a Fréchet space. Taking the union over all spatially compact subset defines sections of this bundle which have support in any spatially compact subset of M:

$$C^l_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) := \bigcup_{\substack{K \subset M \\ K \text{ spatially} \\ \text{compact}}} C^l_K(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \quad . \tag{4.27}$$

This is again an LF-space. The inclusion $C^l_K(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \hookrightarrow C^l_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ is continuous and any linear map from $C^l_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ to any locally convex topological vector space is continuous if and only if the restriction to $C^l_K(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ is continuous for any spatially compact subset K. The case $l = 0$ will be of special interest.

Definition 4.3.1. Let $E \to M$ be a vector bundle over a globally hyperbolic manifold M with temporal function $\mathcal{T}$ and foliating family of spatial Cauchy hypersurfaces $\{\Sigma_t\}_{t \in \mathcal{T}(M)}$; for any $s \in \mathbb{R}$ we set

$$FE^s_{\mathsf{sc}}(M, \mathcal{T}, E) := C^0_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \tag{4.28}$$

to be the *space of finite s-energy sections.*

The differentiability in (4.27) with respect to the time coordinate can be weakened to local square-integrability:

Definition 4.3.2. Let $E \to M$ be a vector bundle over a globally hyperbolic manifold M with temporal function $\mathcal{T}$ and foliating family of spatial Cauchy hypersurfaces $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ and $K \subset M$ spatially compact; the space $L^2_{\mathsf{loc},K}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ consists of those sections u such that

(a) $\mathrm{supp}\,(u) \subset K \cap \Sigma_t$ for almost all $t \in \mathcal{T}(M)$;

(b) $t \mapsto \big[u \,\big|\, \phi|_{\Sigma_t}\big]_{C^\infty_{\mathsf{c}}((E|_{\Sigma_t})^*)}$ is measurable for any $\phi \in C^\infty_{\mathsf{c}}(M,E)$;

(c) $t \mapsto \|u\|_{H^s_{\mathsf{loc}}(E|_{\Sigma_t})}$ is in $L^2_{\mathsf{loc}}(\mathcal{T}(M))$

for any $s \in \mathbb{R}$.

One can prove with a similar argument as for $C^l_K(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ that the embedding

$$L^2_{\mathsf{loc},K}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \hookrightarrow C^{-\infty}(M,E)$$

is continuous. In order to topologise this space, one introduces the seminorms

$$\|u\|^2_{I,K,s} := \int_I \|u\|^2_{H^s_{\mathsf{loc}}(E|_{\Sigma_t})}\,\mathrm{d}t$$

for any compact subinterval I in $\mathcal{T}(M)$. This turns $L^2_{\mathsf{loc},K}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ into a Fréchet space from which one can define the LF-space

$$L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) = \bigcup_{\substack{K\subset M\\ K \text{ spatially}\\ \text{compact}}} L^2_{\mathsf{loc},K}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet})) \quad .$$

We recall one useful result from [BTW15].

Lemma 4.3.3 (Lemma 2 in [BTW15]). *Let M be a globally hyperbolic manifold with temporal function $\mathcal{T} : M \to \mathbb{R}$ such that the manifold is foliated by a family of spatial Cauchy hypersurfaces $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ and $E \to M$ a vector bundle; for any $s \in \mathbb{R}$ one has*

(1) $C^\infty_{\mathsf{sc}}(M,E)$ is a dense subset of $L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$;

(2) Any $u \in L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(E|_{\Sigma_\bullet}))$ with $\mathrm{supp}\,(u) \subset K$ can be approximated by smooth sections in $C^\infty_{K'}(M,E)$ if $K, K' \subset M$ are spatially compact with $K \subset \mathring{K}'$.

Other support systems for other types of compact subsets in globally hyperbolic manifolds are presented in [Bä14, Chap.1/2].

5. Galois coverings, von-Neumann algebras and Γ-operators

We recall the basic, but important concepts of functional algebra and analysis on Galois coverings with respect to a group Γ. The first section deals with topological properties of these kind of coverings which we specify to pseudo-Riemannian manifolds. The main objects for the functional analysis in this setting is given by *von Neumann algebras* and modules with respect to Γ. We introduce the basic background of these concepts and how Fredholm theory is implemented. The last section deals with analytic applications of the functional algebra and operators, acting between sections over Galois coverings, in regards to ellipticity and Fredholmness.

This chapter relies on [Shu, Chap.2] and [Shu, Chap.3] which is supported with additional material from [Bre68], [Bre69], [Vai08] as well as [Sch05].

5.1. Galois Coverings and Γ-manifolds

5.1.1. Basic facts about (Galois) coverings

Let $\mathcal{E}$ and $\mathcal{B}$ be topological spaces and $p : \mathcal{E} \to \mathcal{B}$ a continuous, surjective function. We call the triple $(\mathcal{E}, p, \mathcal{B})$ a *covering* of $\mathcal{B}$ if there exists for every point $b \in \mathcal{B}$ a neighbourhood $\mathcal{U}_b \subset \mathcal{B}$ such that the following holds:

(a) the preimage of $\mathcal{U}_b$ under p is a countable union of open and pairwise disjoint subsets in $\mathcal{O}_j \subset \mathcal{E}$:

$$p^{-1}(\mathcal{U}_b) = \bigsqcup_{j \in J} \mathcal{O}_j \quad (J \text{ index set})$$

and

(b) $p|_{\mathcal{O}_j} : \mathcal{O}_j \to \mathcal{U}_b$ is a homeomorphism for each $j \in J$.

Some authors presuppose in addition that the covering $\mathcal{E}$ has to be connected and locally path-connected. $\mathcal{E}$ is called *total space*, $\mathcal{B}$ *base* and p is the *projection/covering map*. If the base and the covering map is clear or fixed from the context, we refer the notion covering to the total space. The preimage of a point $b \in \mathcal{B}$ under p is the *fibre* $\mathcal{E}_b := p^{-1}(b)$. If for all $b \in \mathcal{B}$ each fibre has finite cardinality $\#\mathcal{E}_b =: l$, then the covering is called *finite* or more precisely *l-fold covering*.

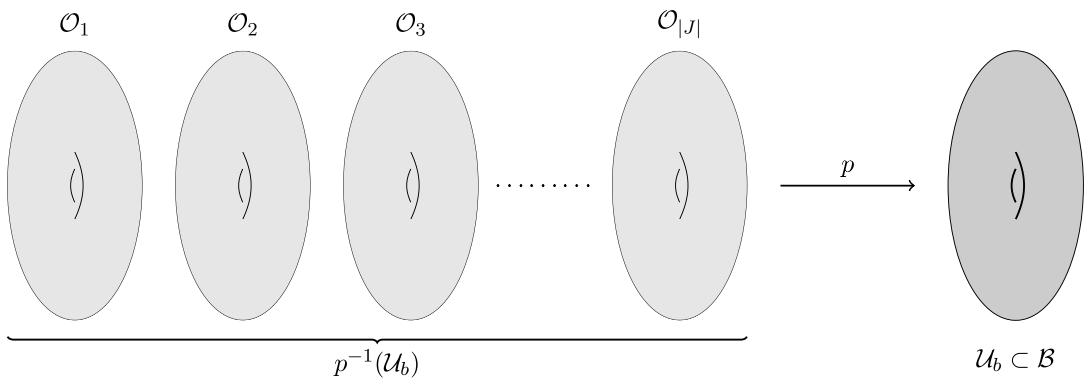

Figure 5.1.: Schematic illustration of a covering.

The *(first) fundamental group* $\Pi_1(\mathcal{B}, b)$ for fixed b is the homotopy class of curves, closing in b. It is a group with respect to concantenation of paths as group operation such that $\{b\}$ is the neutral element and the inverse element of a class of curve is given by the curve itself with reversed orientation. If G is a subgroup of $\Pi_1(\mathcal{B}, b)$, then there exists a covering with total space $\mathcal{E}$ such that $\mathcal{E}$ is connected and the image of the induced map $p_* : \Pi_1(\mathcal{E}, e) \to \Pi_1(\mathcal{B}, p(e))$ with $e \in \mathcal{E}_b$ coincides with G; see [Lee11, Thm.11.16/18]. If $p_*(\Pi_1(\mathcal{E}, e))$ is the neutral element in $\Pi_1(\mathcal{B}, p(e))$ and $\mathcal{E}$ connected, then the covering is called *universal*. A covering with connected total space is universal if and only if the total space is simple connected (see [Lee11, Thm.11.43]).

With this general background, we now consider group actions on manifolds and how they induce coverings. Suppose G is a topological group with group operation $*$ which acts from left or from the right on a topological space M:

$$\begin{aligned} \mathsf{G} \times M &\to M \\ (g, p) &\mapsto g \cdot p \end{aligned} \qquad\qquad \begin{aligned} M \times \mathsf{G} &\to M \\ (p, g) &\mapsto p \cdot g \end{aligned} \quad .$$

We denote with ϵ the neutral element in G and the action satisfies the associative law:

$$(g * h) \cdot x = g \cdot (h \cdot x)$$

for $x \in M$ and $g, h \in \mathsf{G}$ or

$$x \cdot (g * h) = (x \cdot g) \cdot h$$

for the right action. The space M becomes a G-*space*. It is sufficient to consider the left action on M as it automatically induces a right action by setting $x \cdot g := g^{-1} \cdot x$. The right action associative law is satisfied because of $(g * h)^{-1} = h^{-1} * g^{-1}$:

$$x \cdot (g * h) = (g * h)^{-1} \cdot x = (h^{-1} * g^{-1}) \cdot x = h^{-1} \cdot (g^{-1} \cdot x) = h^{-1} \cdot (x \cdot g) = (x \cdot g) \cdot h \, .$$

One can distinguish between different kinds of group actions (see [Lee13, Chap.7] for more information): the action of a group G is called

(a) *faithful/effective* if $g \cdot x = x$ for all $x \in M$ implies $g = \epsilon$.

(b) *(fix point) free* if $g \cdot x = x$ for some $x \in M$ implies $g = \epsilon$.

(c) *transitive* if for any $x, y \in M$ there exists a $g \in \mathsf{G}$, so that $y = g \cdot x$.

(d) *simple transitive/regular* if it is transitive and free, i.e. for any $x, y \in M$ exists a unique $g \in \mathsf{G}$, so that $y = g \cdot x$.

(e) *freely discontinuous* if every $x \in M$ has a neighbourhood $\mathcal{U}_x$ such that $g(\mathcal{U}_x) \cap \mathcal{U}_x = \emptyset$ for all $g \in \mathsf{G} \setminus \{\epsilon\}$.

(f) *continuous* if the left action is continuous.

(g) *proper* if $(g, x) \mapsto (x, g \cdot x)$ is a proper map: for two given compact subsets K_1, K_2 of M the set $\{g \in \mathsf{G} \,|\, g(K_1) \cap K_2 \neq \emptyset\}$ is compact.

(h) *properly discontinuous* if the set $\{g \in \mathsf{G} \,|\, g(K_1) \cap K_2 \neq \emptyset\}$ for every two compact subsets K_1, K_2 of M is finite.

Remarks 5.1.1.

(i) If G *is a discrete group, properness of* G *is equivalent to proper discontinuity of* G *(see [Lee13, Lem.21.11]).*

(ii) Free discontinuity of a group action in (e) can be formulated in terms of two group elements: a group action is (freely) discontinuous if every $x \in M$ *has a neighbourhood* $\mathcal{U}_x$ *such that* $g(\mathcal{U}_x) \cap h(\mathcal{U}_x) = \emptyset$ *for all* $g, h \in \mathsf{G}$ *with* $g \neq h$*.*

Let (M, p, X) be a covering and f an automorphism on M. This automorphism is called *deck transormation* if it preserves the covering map, i.e. $p \circ f = p$. We denote the set of deck transformations on M with respect to the covering p with $\mathsf{Deck}(M, p, X)$; this is a subgroup of the group of automorphisms on M. Following [Lee11, Chap.12] we can sum up the following properties of deck transformations:

(1) $\mathsf{Deck}(M, p, X)$ acts freely on M by homeomorphisms.

(2) $\mathsf{Deck}(M, p, X)$ acts transitively on each fibre of the covering if and only if the covering map is *normal*, i.e. for every $x \in X$ the subgroups $p_*(\Pi_1(M, m))$ are the same for all $m \in p^{-1}(x)$.

The *orbit* of a point $m \in M$ is the set of all points in M such that it can be related to m with an element in G through left action: $\mathsf{G} \cdot m := \{g \cdot m \,|\, g \in \mathsf{G}\} \subset M$. If the group action G is transitive, there exists only one orbit such that $\mathsf{G} \cdot m = M$ for all $m \in M$. The orbit space is then defined to be the quotient[18] of the action: $M/\mathsf{G} := \{\mathsf{G} \cdot m \,|\, m \in M\}$. The quotient map $M \to M/\mathsf{G}$ becomes a covering map in such a way that the group of deck transformations coincides with G.

Theorem 5.1.2 (Covering Space Quotient Theorem, cf. Theorem 12.14 in [Lee11])**.** *Let* M *be a connected and locally path-connected space with effective action of a group* G *on* M *by homeomorphisms. The quotient map* $q : M \to M/\mathsf{G}$ *is a normal covering map with* $\mathsf{Deck}(M, q, M/\mathsf{G}) = \mathsf{G}$ *if and only if the group acts properly discontinuous on* M *by homeomorphisms.*

[18] It is common to discern the orbits spaces for left and right actions with G/M respectively M/G. We don't distinguish between these two designations since we have restricted ourselves to left actions.

Quotient maps with respect to G, which are normal coverings as in this theorem, are called *Galois coverings* and the corresponding group G *Galois group*. We want to extend this to smooth manifolds for which we recapitulate some facts about smooth covering maps:

(1) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism. If the covering map is in addition injective, the covering map is a diffeomorphism.

(2) Suppose X is a smooth and connected manifold and $p : M \to X$ a topological covering map, then M is a topological manifold with unique smooth structure such that p is a smooth covering map. If X is furthermore a manifold with boundary ∂X, then also M is a topological manifold with boundary $\partial M = p^{-1}(\partial X)$.

(3) Given two non-empty connected smooth manifolds with or without boundary M and X and a smooth covering map $p : M \to X$. The deck transformation group $\mathsf{Deck}(M, p, X)$, equipped with the discrete topology, is a zero-dimensional Lie group, acting smoothly and freely on M. Furthermore, $\mathsf{Deck}(M, p, X)$ acts properly on M.

See [Lee13, Prop.4.33] for (1), [Lee13, Prop.4.40/41] for (2), and [Lee13, Prop.7.23] as well as [Lee13, Prop.21.12] for (3). Any group with discrete topology can be considered as discrete group. If in addition the group is finite or countably infinite, it becomes a zero-dimensional Lie group which defines the notion *discrete Lie group*. The following fact assures that for these groups G the orbit space is again a smooth manifold and the quotient map a Galois covering.

Proposition 5.1.3 (Quotient Manifold Theorem, discrete Lie group, cf. Theorem 21.13 in [Lee13]). *If M is a connected smooth manifold and G a discrete Lie group, acting smoothly, freely and properly on M, the orbit space is a topological manifold and has a unique smooth structure such that the quotient map becomes a smooth Galois covering.*

The main geometric objects in these thesis are Galois coverings which allow to extend index theory on compact manifolds to some non-compact manifolds under a certain additional requirement: a group action on M with respect to G is *cocompact* if there exists a subspace $A \subset M$ such that $M = \mathsf{G} \cdot A$. Cocompactness is equivalent to a compact orbit space M/G if the action is properly discontinuous.

5.1.2. Γ-manifolds

From now on we denote with Γ a discrete group which acts freely, freely discontinuous and cocompactly on a non-compact manifold M by diffeomorphisms. We reserve the notion of Galois groups only for these kind of discrete groups.

Definition 5.1.4. A connected manifold M is called Γ-*manifold* if it is a Galois covering with respect to a cocompactly Galois group Γ.

Proposition 5.1.3 assures that the orbit space M/Γ as base of the covering is again a smooth manifold. We also write M_Γ for M/Γ. Examples of such manifolds M are universal coverings of compact manifolds X where $\Gamma = \Pi_1(X)$ is the fundamental group which acts by deck transformations. A more concrete example is $M = \mathbb{R}^n$ and the n-Torus $X = \mathbb{T}^n$ with fundamental group $(\mathbb{Z}^n, +)$. From now on we designate the quotient map

$$\pi_\Gamma : M \to M_\Gamma \tag{5.1}$$

as covering map. If the Γ-manifold M is equipped with a smooth and complex vector bundle E with projection $\pi : E \to M$, one can define an action of Γ on the vector bundle E through the map

$$\pi_\Gamma|_p : E_p \to E_{\gamma p} \tag{5.2}$$

for all $p \in M, \gamma \in \Gamma$. We presuppose that this map is a linear isomorphism, such that the action on E covers the action on M, and that the projection π is Γ-*equivariant*, i.e. $\pi(\gamma p, v_{\gamma p}) = \gamma\pi(p, v_p)$ for all p, γ, and v_p vector at p. According to [Lee13, Exc.21-8] we can view E as normal covering of a smooth vector bundle over M/Γ which we denote by E/Γ. We follow the convention from our main reference [Shu] and call E a Γ-*vector bundle* if it is a vector bundle over a Γ-manifold such that E is the pullback bundle $\pi_\Gamma^*(E/\Gamma)$ of a vector bundle E/Γ over the compact base M/Γ. We can furthermore extend $\pi_\Gamma|_p$ to an isometry by introducing a Hermitian inner product $(\bullet \mid \bullet)_{E_p}$ on each fibre E_p for $p \in M$. This is either already given on E or it is induced as pullback of a bundle metric on E/Γ. In order to extend (5.2) to an isometry, we have to require that the bundle metric is Γ-invariant:

$$(\bullet \mid \bullet)_{E_{\gamma p}} = (\bullet \mid \bullet)_{E_p} \quad \forall \gamma \in \Gamma\,,\, p \in M \quad . \tag{5.3}$$

In a similar way we can equipp M with a positive, Γ-invariant smooth density $\mathrm{d}\mu$:

$$\mathrm{d}\mu(\gamma p) = \mathrm{d}\mu(p) \quad \forall \gamma \in \Gamma\,,\, p \in M \quad . \tag{5.4}$$

This can be either directly obtained by lifting a positive smooth density on M/Γ or with a metric on $T(M/\Gamma)$ which induces a Γ-invariant metric on TM and thus a Γ-invariant smooth density on M.

The action of a group G on a topological space by homeomorphisms implies the notion of a fundamental domain which is, loosely speaking, a set of representatives for the group action.

Definition 5.1.5. An open subset $\mathcal{F}$ of a topological set M, on which a group G acts on, is called *fundamental domain* of the action if it is disjoint from all its translates by G (i.e. $g(\mathcal{F}) \cap \mathcal{F} = \emptyset$ for all $g \in \mathsf{G} \setminus \{\epsilon\}$) and

$$M \setminus \bigcup_{g \in \mathsf{G}} g(\mathcal{F})$$

has measure zero.

The standard example of such a domain is the n-square $\mathcal{F} = [0,1)^n$ in $M = \mathbb{R}^n$ with transformation group $\mathsf{G} = (\mathbb{Z}^n, +)$. If the action is freely discontinuous, the fundamental domain contains the *free regular set*, being the largest subset in X on which the action is freely discontinuous. It is an open set which is transformed by G into disjoint copies and as good as the fundamental domain in representing the orbits. With this foreword and Remarks 5.1.1 (ii) we can reformulate the requirements for a fundamental domain $\mathcal{F}$ of Γ: $\mathcal{F}$ is an open subset in M such that

(a) $(\gamma_1\mathcal{F}) \cap (\gamma_2\mathcal{F}) = \emptyset \quad \forall \gamma_1, \gamma_2 \in \Gamma \quad \Rightarrow \quad \gamma_1 \neq \gamma_2$

(b) $M = \bigcup_{\gamma \in \Gamma} \gamma\overline{\mathcal{F}}$

(c) $\overline{\mathcal{F}} \setminus \mathcal{F}$ is a null set.

Property (a) reflects the property of containing or coinciding with the free regular set of the Γ-action. Moreover, these properties imply that

$$M \simeq \Gamma \times \mathcal{F} \quad , \tag{5.5}$$

i.e. M can be viewed as principal Γ-bundle.

The compactness of the orbit space allows to define a Γ-invariant locally finite open covering and a Γ-invariant partition of union subordinated to this open covering. Since M/Γ is compact, one can take a finite covering by open balls $\mathring{\mathbb{B}}_j$, $j \in J$ finite index set, such that

$$M/\Gamma = \bigcup_{j\in J} \mathring{\mathbb{B}}_j \quad .$$

These open balls can be lifted to M which defines an infinite, but countable covering for M due to the countability of Γ:

$$M = \bigcup_{\substack{j\in J\\ \gamma\in\Gamma}} \gamma(\mathring{\mathbb{B}}_j) \quad . \tag{5.6}$$

A subordinated partition of unity $(\gamma\mathring{\mathbb{B}}_j, \phi_{j,\gamma})_{\substack{j\in J\\ \gamma\in\Gamma}}$ can be constructed by taking compactly supported functions $\phi_{j,\gamma} \in C^\infty_c(\gamma\mathring{\mathbb{B}}_j, \mathbb{R}_{\geq 0})$ which satisfy $\phi_{j,\gamma}(p) = \phi_{j,e}(\gamma^{-1}p)$ as well as

$$\sum_{\substack{j\in J\\ \gamma\in\Gamma}} \phi_{j,\gamma} = 1 \quad . \tag{5.7}$$

Remarks 5.1.6.

(i) If we compare property (b) in the refined characterisation of the fundamental domain with (5.6), we observe that the closure $\overline{\mathcal{F}}$ of the fundamenmtal domain $\mathcal{F}$ of the group action is in fact the compact orbit space M/Γ. Thus, $\mathcal{F}$ is a dense subspace in M/Γ.

(ii) We observe from (5.6) by comparing with (3.47) that Γ-manifolds are indeed examples of manifolds of bounded geometry. The radii-condition on the functions for the partition of unity in Lemma 3.3.2 is replaced with the assumption on Γ to act freely discontinuous.

5.1.3. Galois coverings and pseudo-Riemannian manifolds

We give some few remarks about coverings and pseudo-Riemannian manifolds; we follow the material of [Bau81, Sec.2.3] and [O'N83, p.191]. We specify afterwards to globally hyperbolic coverings which are going to play an important role in the coming analysis.

Definition 5.1.7. A *pseudo-Riemannian covering map* is a covering map of pseudo-Riemannian manifolds that is a local isometry.

The isometry property is required to obtain a pseudo-Riemannian metric on the covering: if $p : (M_1, g_1) \to (M_2, g_2)$ is a covering map, then the exigence of p to be a local isometry implies that the pseudo-Riemannian metrics are related by the pullback metric ($g_1 = p^* g_2$) and both metrics have the same signatures. Every local isometry onto M_2 lifts to a local isometry onto M_1 through p, implying that any deck transformation is a local isometry since they are diffeomorphic to the lift of the identity. We have the following facts:

(1) If (M_2, g_2) is a connected and pseudo-Riemannian manifold and (M_1, g_1) a pseudo-Riemannian covering, the group $\mathsf{Deck}(M_1, p, M_2)$ becomes a properly discontinuous group of isometries of M_1.

(2) If (M_1, g_1) is simply connected, the pseudo-Riemannian metric g_1 is time and space-oriented if g_2 does.

(3) If G is a properly discontinuous group of isometries of a pseudo-Riemannian manifold M, the covering map $q : M \to M/\mathsf{G}$ is a pseudo-Riemannian covering with M/G as pseudo-Riemannian manifold with the same signature. If M is connected, then $\mathsf{Deck}(M, q, M/\mathsf{G}) = \mathsf{G}$ ([O'N83, Cor.7.12]).

Hence we don't need to impose further modifications than the assumption that the covering maps are local isometries.

We focus on a globally hyperbolic manifold M which is equipped with a Γ-invariant Lorentzian metric and a Cauchy hypersurface. As M is diffeomorphic to $\mathcal{T}(M) \times \Sigma$, one could consider Γ-actions on either the whole manifold or on the temporal and the spatial part separately. A *temporal* Γ*-manifold* is a Γ-manifold where the action is induced by a group Γ, acting on the time domain $\mathcal{T}(M)$ as Γ-manifold with closed base. Analogously a *spatial* Γ*-manifold* is a Γ-manifold where the action is induced by a group Γ, acting on the hypersurface as Γ-manifold. The Γ-invariance of the metric is then induced by the Γ-invariance of the Riemannian metric of the Cauchy hypersurface. The corresponding canonical projections are

$$\pi_{\Gamma,t} : M \to {}^{\mathcal{T}(M)}/_{\Gamma} \times \Sigma \quad \text{and} \quad \pi_{\Gamma,s} : M \to \mathcal{T}(M) \times {}^{\Sigma}/_{\Gamma} \quad . \tag{5.8}$$

Remark 5.1.8. *The temporal* Γ*-manifolds might come with conceptual problems, depending on* $\mathcal{T}(M)$*. If the the orbit space is closed, the resulting Lorentzian manifold is temporal closed, inducing closed timelike geodesics or the possibility of closed timelike curves such that one can not expect that the global hyperbolicity property is preserved. From the analytic point of view they cause non well-posed initial value problems and violate causality conditions such that they become unsuitable for applications in General Relativity, too.*

We will only consider the covering $\pi_{\Gamma,s}$ for this reason and we denote with π_Γ from now on the covering $\pi_{\Gamma,s}$; the orbit M/Γ for these spatial coverings coincides with $\mathcal{T}(M) \times (\Sigma/\Gamma)$. The facts and results at the beginning of this subsection assure that the covering is a time as well space-oriented Lorentzian manifold if and only if the base does. It is left to clarify how the property of global hyperbolicity is preserved under those spatial covering maps. The following result shows that in fact the covering M becomes globally hyperbolic if the base of the covering is already globally hyperbolic.

Lemma 5.1.9 (c.f Lemma 4.1. in [GH97]). *Let* M *be a globally hyperbolic spacetime and* $\widetilde{M}$ *its universal covering with metric* $\widetilde{g}$*, induced by the metric* g *on* M*, then* $\widetilde{M}$ *is also globally hyperbolic.*

The proof in the given reference shows that if Σ is a Cauchy hypersurface in M, the preimage of Σ under the covering map induces a Cauchy hypersurface $\widetilde{\Sigma} \subset \widetilde{M}$. Hence the Galois coverings $л_\Gamma$ ($л_{\Gamma,s}$ in (5.8)) indeed preserve the global hyperbolicity from the compact base.

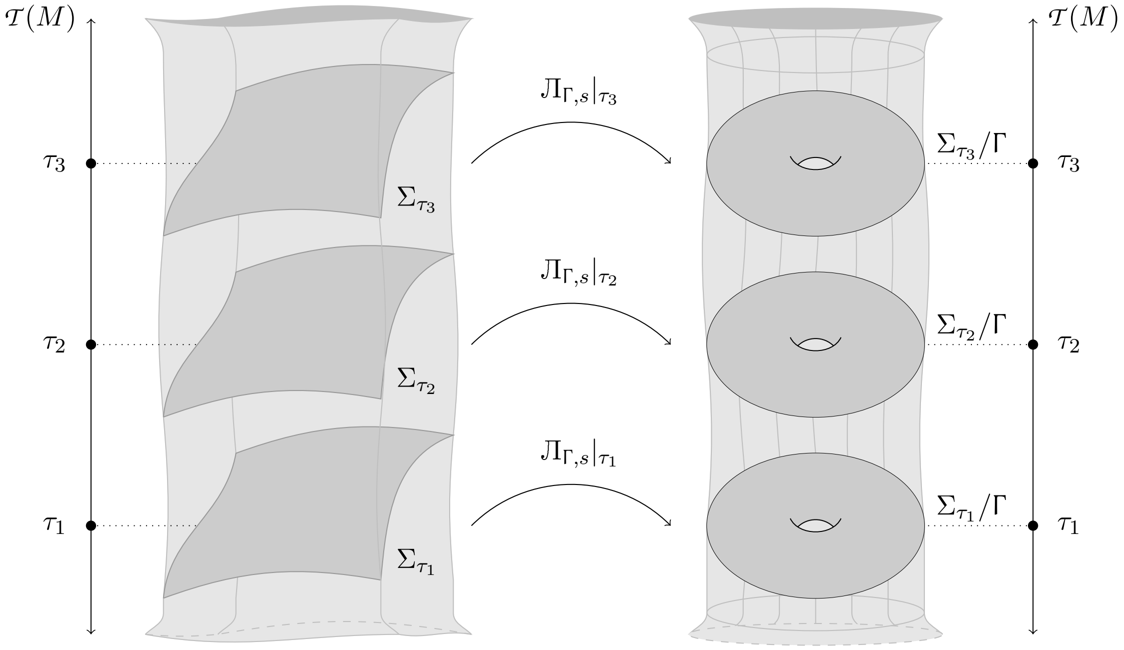

Figure 5.2.: Schematic illustration of a spatial Γ-manifold.

5.2. Von Neumann-algebra and Fredholm theory according to a Γ-action

We first introduce some general aspects of von Neumann algebras and specify later on those algebras which are associated with a discrete group. We introduce the modules and operators of our interest in this setting.

5.2.1. Von Neumann algebras - general aspects and in association with discrete groups

Next to our main reference [Shu] we refer to [Bla06] or [KR83] for the general aspects of Banach and von Neumann algebras.

We recall the definition of a C^*-*algebra*. Given a Banach space $(A, \|\bullet\|_A)$ and a $\mathbb{C}$-anti-linear map $* : A \to A$ as per $a \mapsto a^*$ which is an isometric involution, i.e. $a^{**} = a$, $(a \circ b)^* = b^* \circ a^*$ and $\|a^*\|_A = \|a\|_A$ for $a, b \in A$. If the norm on A is submultiplicative (i.e. $\|a \circ b\|_A \leq \|a\|_A \|b\|_A$ for $a, b \in A$) and satisfies $\|a^* \circ a\|_A = \|a\|_A^2$, the triple $(A, \|\bullet\|_A, *)$ is a C^*-algebra. If A contains a unit element, the algebra is called unital. Any quotient of a C^*-algebra with one of its two-sided ideals and any of its subalgebra, which is closed with respect to the involution, becomes a C^*-algebra on its own right.

The most prominent example of a unital C^*-algebra is the set of bounded linear operators $\mathscr{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with operator norm (2.2), identity $\mathbb{1}_\mathcal{H}$ and involution is given by adjoining the operator. In comparison, the set of bounded linear operators $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ between different Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ is **not** a C^*-algebra because adjoining an element in $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ gives an element in $\mathscr{B}(\mathcal{H}_2, \mathcal{H}_1)$ and thus defines no involution. The set of compact operators $\mathscr{K}(\mathcal{H})$ is a closed subalgebra of $\mathscr{B}(\mathcal{H})$ and due to the characterisation theorem of Schauder the adjoint of a compact operator is again compact. Hence it is another example of a C^*-algebra. As it is also a two-sided ideal, the Calkin-algebra $\mathscr{Q}(\mathcal{H}) := \mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ becomes a C^*-algebra, too. We fix the C^*-algebra to be $\mathscr{B}(\mathcal{H})$ and consider a subspace $\mathscr{N}$. The *commutant* of $\mathscr{N}$ is the subspace

$$\mathscr{N}' := \{A \in \mathscr{B}(\mathcal{H}) \,|\, AB = BA \quad \forall\, B \in \mathscr{N}\} \quad . \tag{5.9}$$

It is a closed and unital subalgebra of $\mathscr{B}(\mathcal{H})$. The *bicommutant* of $\mathscr{N}$ is $\mathscr{N}'' := (\mathscr{N}')'$. Like the question, how the dual of the dual of a space is related to the space itself, we could ask how the bicommutant of $\mathscr{N}$ is related to $\mathscr{N}$. This leads to the concept of a von Neumann algebra.

Definition 5.2.1. A subalgebra $\mathscr{N} \subset \mathscr{B}(\mathcal{H})$ with $\mathbb{1}_\mathcal{H} \in \mathscr{N}$ and $A^* \in \mathscr{N}$ for all $A \in \mathscr{N}$ is a *von Neumann algebra* if $\mathscr{N}'' = \mathscr{N}$.

The Bicommutant Theorem of von Neumann implies that a von Neumann algebra is weakly and thus strongly and norm closed. We collect some properties of these algebras.

Proposition 5.2.2 (see section 2.5 in [Shu])**.** *Let $\mathscr{N}$ be a von Neumann algebra.*

(1) A subset $\mathscr{M} \subset \mathscr{B}(\mathcal{H})$ is a von Neumann algebra if and only if it is a commutant of a subset $M \subset B(\mathcal{H})$ with $A \in M$ implying $A^* \in M$.

(2) If $A \in \mathscr{N}$ is self-adjoint (i.e. $A = A^* \in \mathscr{N}$), then the spectral projection $P_I(A) := \chi_I(A)$ are also elements in $\mathscr{N}$ for all intervals $I \subset \mathbb{R}$.

(3) $\mathscr{N}$ is the smallest von Neumann algebra, containing the set

$$P(\mathscr{N}) := \left\{P \in \mathscr{N} \,|\, P^2 = P = P^*\right\} \quad .$$

(4) $A \in \mathscr{N}$ if and only if $U^{-1}AU = A$ for all unitarities in $\mathscr{N}'$.

(5) Let $A \in \mathscr{B}(\mathcal{H})$ has polar decomposition $A = US$, then $A \in \mathscr{N}$ if and only if $U \in \mathscr{N}$ and $S \in \mathscr{N}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Taking an orthonormal basis $\{e_i\}_{i\in J}$ in $\mathcal{H}_2$ allows us to decompose the tensor product of Hilbert spaces into

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \bigotimes_{j \in J} \mathcal{H}_1 \otimes \{e_j\} \quad .$$

We can define matrix elements for an operator $A \in \mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with the help of projections $\pi_j : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \{e_j\}$ by setting $A_{jk} := \pi_j A \pi_k$ which are now viewed as bounded operators from $\mathcal{H}_1 \otimes \{e_k\}$ to $\mathcal{H}_1 \otimes \{e_j\}$. Hence the operator A can be represented as infinite block matrix $(A_{jk})_{j,k\in J}$ with entries in $\mathscr{B}(\mathcal{H}_1)$. The tensor product of a von Neumann algebra and $\mathscr{B}(\mathcal{H}_1)$ is defined as

$$\mathscr{N} \otimes \mathscr{B}(\mathcal{H}_2) = \{A \in \mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \,|\, A_{jk} \in \mathscr{N} \;\forall\, j,k \in J\} \quad . \tag{5.10}$$

A similar construction works if we decompose $\mathcal{H}_1 \otimes \mathcal{H}_2$ with an orthonormal basis of $\mathcal{H}_1$. The *Grassmannian* of a von Neumann algebra $\mathscr{N}$ is the set of all subspaces in $\mathcal{H}$ which are the range of projections in $\mathscr{N}$:

$$\mathrm{Gr}_{\mathscr{N}}(\mathcal{H}) := \{V \subset \mathcal{H} \,|\, V = \mathsf{ran}\,(P)\,\text{for a}\, P \in P(\mathscr{N})\} \quad .$$

The map $P \mapsto \mathsf{ran}\,(P)$ implies an isomorphism $P(\mathscr{N}) \cong \mathrm{Gr}_{\mathscr{N}}(\mathcal{H})$ which leads to the concept of a *dimension function* for elements in $\mathrm{Gr}_{\mathscr{N}}(\mathcal{H})$:

$$\dim_{\mathscr{N}} \,:\, \mathrm{Gr}_{\mathscr{N}}(\mathcal{H}) \to [0,\infty] \quad .$$

The construction of a unique dimension function is given by fixing a trace on a von Neumann algebra. Properties of these traces as well as this dimension function are going to be introduced later when we consider a special choice of a von Neumann algebra which we are going to use in the forthcoming analysis. We close this general point of view by recalling the classifications of factors: a *factor* is a von Neumann algebra $\mathscr{N}$ which satisfies

$$\mathscr{N} \cap \mathscr{N}' = \{B \in \mathscr{N} \,|\, AB = BA \quad \forall\, A \in \mathscr{N}\} = \{\lambda \mathbb{1}_{\mathcal{H}} \,|\, \lambda \in \mathbb{C}\} \quad .$$

According to the range of the dimension function, fixed by a trace, one can classify different types of factors.

Definition 5.2.3. Suppose $\mathscr{N}$ is a factor and $n \in \mathbb{N}$; $\mathscr{N}$ is of type

(a) I_n (finite and discrete) if $\dim_{\mathscr{N}}(\mathrm{Gr}_{\mathscr{N}}(\mathcal{H})) \in \{1, ..., n\}$;

(b) I_∞ (semifinite and discrete) if $\dim_{\mathscr{N}}(\mathrm{Gr}_{\mathscr{N}}(\mathcal{H})) \in \{1, ..., \infty\}$;

(c) II_1 (finite and continuous) if $\dim_{\mathscr{N}}(\mathrm{Gr}_{\mathscr{N}}(\mathcal{H})) \in [0,1]$;

(d) II_∞ (semifinite and continuous) if $\dim_{\mathscr{N}}(\mathrm{Gr}_{\mathscr{N}}(\mathcal{H})) \in [0,\infty]$;

(e) III (infinite) if there exists no trace on $\mathscr{N}$.

An example of a type I_n-factor is the set of square matrices in $\mathbb{C}^n$; $\mathscr{B}(\mathcal{H})$ itself is a type I_∞-factor if $\mathcal{H}$ is infinite-dimensional; otherwise[19] it is a type I_{n^2}-factor as $\mathscr{B}(\mathcal{H}) \cong \mathsf{Mat}(n,\mathbb{C})$ for $\dim_{\mathbb{C}}(\mathcal{H}) = n$. It is known that factors of type II_∞ are tensor products of type II_1-factors with $\mathscr{B}(\mathcal{H})$; these tensor products are finite type factors if $\mathcal{H}$ is finite-dimensional. A deeper insight of classifications is given in [Shu, Sec.2.28] or [Bla06, Sec.III.1.4]. The type III-factors can be even more subdivided into further classes. The interested reader shall consult [Tak77] or [Loi92] for more details. We now want to specify to a von Neumann algebra which we will use in the Γ-setting. Let u, v be two functions $\Gamma \to \mathbb{C}$; we define an inner product which is similar to the inner product on $l^2(\mathbb{N})$:

$$\langle u \,|\, v \rangle_{\ell^2(\Gamma)} := \sum_{g\in\Gamma} \overline{u(g)} v(g) \quad . \tag{5.11}$$

[19] $\mathsf{Mat}(n,\mathbb{C}) = \mathsf{Mat}_{n\times n}(\mathbb{C})$ is the algebra of complex $(n \times n)$-matrices.

Motivated from this resemblance, we define the Hilbert space of l^2-functions to be

$$\ell^2(\Gamma) := \left\{ u : \Gamma \to \mathbb{C} \,|\, \langle u \,|\, u \rangle_{\ell^2(\Gamma)} < \infty \right\} \quad . \tag{5.12}$$

The group action Γ induces *left and right translation operators* on $\ell^2(\Gamma)$, defined by

$$(l_{\gamma_1} u)(\gamma_2) = u(\gamma_1^{-1}\gamma_2) \quad \text{and} \quad (r_{\gamma_1} u)(\gamma_2) = u(\gamma_2\gamma_1) \tag{5.13}$$

for $\gamma_1, \gamma_2 \in \Gamma$ and a function $u : \Gamma \to \mathbb{C}$. They obey

$$\begin{aligned} (l_\gamma)^{-1} &= l_{\gamma^{-1}} \quad , \quad & (r_\gamma)^{-1} &= r_{\gamma^{-1}} \\ l_{\gamma_1\gamma_2} &= l_{\gamma_1} l_{\gamma_2} & r_{\gamma_1\gamma_2} &= r_{\gamma_1} r_{\gamma_2} \quad . \end{aligned} \tag{5.14}$$

The most important property is that (5.13) are unitary operators which follows from the inner product in $\ell^2(\Gamma)$ and (5.14):

$$(l_\gamma)^* = l_{\gamma^{-1}} \quad \text{and} \quad (r_\gamma)^* = r_{\gamma^{-1}} \quad .$$

Herewith, $\gamma \mapsto l_\gamma$ and $\gamma \mapsto r_\gamma$ become unitary representations of the discrete group in $\ell^2(\Gamma)$. We denote with $\mathscr{N}_r(\Gamma)$ the smallest von Neumann algebra in $\mathscr{B}(\ell^2(\Gamma))$ which contains the set $\{r_\gamma \,|\, \gamma \in \Gamma\}$ and $\mathscr{N}_l(\Gamma)$ as the smallest von Neumann algebra which contains $\{l_\gamma \,|\, \gamma \in \Gamma\}$. Since $\ell^2(\Gamma)$ is a Hilbert space, one can introduce an orthonormal basis $\{\delta_\gamma\}_{\gamma\in\Gamma}$ which satisfies $\delta_{\gamma_1}(\gamma_2) = 1$ if $\gamma_1 = \gamma_2$ and zero otherwise. Applying the left and right translation operators to this basis leads to

$$l_{\gamma_1}\delta_{\gamma_2} = \delta_{\gamma_1\gamma_2} \quad \text{and} \quad r_{\gamma_1}\delta_{\gamma_2} = \delta_{\gamma_2\gamma_1^{-1}} \tag{5.15}$$

which can be checked as follows: we compute (5.11) with $u = \delta_{\gamma_3}$ and $v = l_{\gamma_1}\delta_{\gamma_1}$ for $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, then

$$\langle \delta_{\gamma_3} \,|\, l_{\gamma_1}\delta_{\gamma_2} \rangle_{\ell^2(\Gamma)} = \sum_{g\in\Gamma} \overline{\delta_{\gamma_3}(g)} (l_{\gamma_1}\delta_{\gamma_2})(g) = (l_{\gamma_1}\delta_{\gamma_2})(\gamma_3) = \delta_{\gamma_2}(\gamma_1^{-1}\gamma_3) \quad ;$$

the inner product is non-zero for $\gamma_3 = \gamma_1\gamma_2$. As δ_γ is orthonormal and the left translation operators unitary, we have the first equivalence. The second relation with the right translation operator follows similarly. Each operator $A \in \mathscr{B}(\ell^2(\Gamma))$ can be related to a matrix element by $A_{\gamma_1,\gamma_2} = \langle A\delta_{\gamma_1} \,|\, \delta_{\gamma_2} \rangle_{\ell^2(\Gamma)}$ for $\gamma_1, \gamma_2 \in \Gamma$. In [Shu, Thm.2.10] it is proven that the von Neumann algebras $\mathscr{N}_l(\Gamma)$ and $\mathscr{N}_r(\Gamma)$ take the form

$$\begin{aligned} \mathscr{N}_l(\Gamma) &= \left\{ A \in \mathscr{B}(\ell^2(\Gamma)) \,|\, A_{a\gamma,b\gamma} = A_{a,b} \quad \forall\, a,b,\gamma \in \Gamma \right\} \quad , \\ \mathscr{N}_r(\Gamma) &= \left\{ A \in \mathscr{B}(\ell^2(\Gamma)) \,|\, A_{\gamma a,\gamma b} = A_{a,b} \quad \forall\, a,b,\gamma \in \Gamma \right\} \end{aligned} \tag{5.16}$$

with $(\mathscr{N}_l(\Gamma))' = \mathscr{N}_r(\Gamma)$ and $(\mathscr{N}_r(\Gamma))' = \mathscr{N}_l(\Gamma)$. It is proven in the same reference that these two von Neumann algebras are factors if and only if all conjugacy classes in $\Gamma/\{\epsilon\}$ are infinite. Those groups Γ, for which this is true, are called *i.c.c-groups* (infinite-conjugacy-class). For an operator A in either $\mathscr{N}_l(\Gamma)$ or $\mathscr{N}_l(\Gamma)$ one can define a *trace* by

$$\tau_\Gamma(A) := \langle A\delta_\gamma \,|\, \delta_\gamma \rangle_{\ell^2(\Gamma)} = \langle A\delta_\epsilon \,|\, \delta_\epsilon \rangle_{\ell^2(\Gamma)} \tag{5.17}$$

for $\gamma \in \Gamma$; the definition is independent of the choice of γ which explains the second equivalence.

Lemma 5.2.4 (Lemma 2.6.1 in [Shu]). *Let $\mathscr{N}_{r,l}^{+}(\Gamma) := \{A \in \mathscr{N}_{r,l}(\Gamma) \,|\, A \text{ positive definite}\}$; the trace τ_Γ as map from $\mathscr{N}_{r,l}^{+}(\Gamma)$ to $[0, \infty]$ satisfies the following conditions:*

(1) τ_Γ is linear: $\tau_\Gamma(aA + bB) = a\tau_\Gamma(A) + b\tau_\Gamma(B)$ for all $a, b \in \mathbb{R}_{\geq 0}$ and $A, B \in \mathscr{N}_{r,l}^{+}(\Gamma)$.

(2) τ_Γ is tracial: $\tau_\Gamma(A^*A) = \tau_\Gamma(AA^*)$ for $A \in \mathscr{N}_{r,l}(\Gamma)$.

(3) τ_Γ is faithful: $\tau_\Gamma(A) = 0$ for $A \in \mathscr{N}_{r,l}^{+}(\Gamma)$ implies $A = 0$.

(4) τ_Γ is normal: if $\{A_i\}_{i \in J}$ is a sequence of operators in $\mathscr{N}_{r,l}^{+}(\Gamma)$ with $A_i \to A \in \mathscr{N}_{r,l}^{+}(\Gamma)$, then $\tau_\Gamma(A_i) \to \tau_\Gamma(A)$.

(5) τ_Γ is invariant under conjugation with unitary operators: $\tau_\Gamma(A) = \tau_\Gamma(U^{-1}AU)$ for any $A \in \mathscr{N}_{r,l}^{+}(\Gamma)$ and unitary maps $U \in \mathscr{N}_{r,l}(\Gamma)$.

One observes that this trace shares the same properties as the trace in (2.6) if restricted on $\mathscr{B}^{+}(\mathcal{H}) = \{A \in \mathscr{B}(\mathcal{H}) \,|\, A \text{ is positive definite}\}$. For $\mathscr{N}_r(\Gamma)$ or $\mathscr{N}_l(\Gamma)$ the trace (5.17) gives rise to a dimension function $\dim_\tau$:

$$\dim_\tau(V) := \tau_\Gamma(P_V)$$

where $V \in \mathrm{Gr}_{\mathscr{N}_{r,l}(\Gamma)}(\ell^2(\Gamma))$ and $P_V \in P(\mathscr{N}_{r,l}(\Gamma))$. It is shown in [KR83, Prop.6.7.4] that $\mathscr{N}_r(\Gamma)$ and $\mathscr{N}_l(\Gamma)$ are finite von Neumann algebras[20]. If Γ is in addition an i.c.c.-group, they are factors of type II_1, according to Theorem 6.7.5 in the same reference.

Repeating the construction of the tensor product of a von Neumann algebra and the set of bounded operators, we take an ONB from $\ell^2(\Gamma)$ which then induces an orthogonal decomposition of the tensor product

$$\ell^2(\Gamma) \otimes \mathcal{H} = \bigotimes_{\gamma \in \Gamma} (\delta_\gamma \otimes \mathcal{H}) \quad .$$

This decomposition implies that any operator $A \in \mathscr{B}(\ell^2(\Gamma) \otimes \mathcal{H})$ can be represented with matrix entries $A_{\gamma_1,\gamma_2} \in \mathscr{B}(\mathcal{H})$ for $\gamma_1, \gamma_2 \in \Gamma$, defined by

$$\left\langle A_{\gamma_1,\gamma_2} x \,\middle|\, y \right\rangle_{\mathcal{H}} := \left\langle A(\delta_{\gamma_1} \otimes x) \,\middle|\, \delta_{\gamma_2} \otimes y \right\rangle_{\ell^2(\Gamma) \otimes \mathcal{H}} \tag{5.18}$$

for $x, y \in \mathcal{H}$. From [Shu, Thm.2.13.2] we use that

$$\begin{aligned} \mathscr{N}_l(\Gamma) \otimes \mathscr{B}(\mathcal{H}) &:= \left\{ A \in \mathscr{B}(\ell^2(\Gamma) \otimes \mathcal{H}) \,\middle|\, A_{a\gamma, b\gamma} = A_{a,b} \quad \forall\, a, b, \gamma \in \Gamma \right\} \quad , \\ \mathscr{N}_r(\Gamma) \otimes \mathscr{B}(\mathcal{H}) &:= \left\{ A \in \mathscr{B}(\ell^2(\Gamma) \otimes \mathcal{H}) \,\middle|\, A_{\gamma a, \gamma b} = A_{a,b} \quad \forall\, a, b, \gamma \in \Gamma \right\} \quad . \end{aligned} \tag{5.19}$$

are again von Neumann algebras. If $\mathscr{N}_l(\Gamma)$ and $\mathscr{N}_r(\Gamma)$ are factors, these tensor products are also factors, but now of type II_∞ as they are tensor products of type II_1-factors and the type I_∞-factor $\mathscr{B}(\mathcal{H})$ for an infinite-dimensional Hilbert space $\mathcal{H}$. As the traces in

[20] We also suggest the introduction of [Tay76] for more details as well as some analysis about the types if one replaces Γ with a suitable locally compact, but not necessarily discrete groups.

(5.17) and (2.6) share the same properties, the tensor product do have them as well and we define the Γ-*trace* to be the trace on tensor products (5.19):

$$\mathrm{Tr}_{\Gamma}(A) := (\tau_{\Gamma} \otimes \mathrm{Tr}(A)) = \mathrm{Tr}(A_{\epsilon,\epsilon}) \quad . \tag{5.20}$$

5.2.2. Hilbert Γ-modules and Γ-dimension

Given a Hilbert space $\mathscr{H}$ and a C^*-algebra A of operators, acting on $\mathscr{H}$. We call this space a (general) *Hilbert A-module* if $\mathscr{H}$ is a left or right A-module with A-valued inner product $\langle \cdot \,|\, \cdot \rangle_{\mathscr{H}} : \mathscr{H} \times \mathscr{H} \to A$ which is a positive definite and Hermitean sesquilinear form, and is conjugate-linear with respect to A:

$$\begin{aligned} \langle u \,|\, va \rangle_{\mathscr{H}} &= \langle u \,|\, v \rangle_{\mathscr{H}} a \quad \text{if} \quad \mathscr{H} \quad \text{is a right } A\text{-module, or} \\ \langle u \,|\, av \rangle_{\mathscr{H}} &= \langle u \,|\, v \rangle_{\mathscr{H}} a \quad \text{if} \quad \mathscr{H} \quad \text{is a left } A\text{-module.} \end{aligned}$$

for $u, v \in \mathscr{H}$ and $a \in A$. Any ordinary Hilbert space $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module. We refer to [LS95, Chap.1] for more informations. We focus on the C^*-algebra $\mathscr{B}(\ell^2(\Gamma)) \otimes \mathscr{B}(\mathcal{H})$. We abbreviate any left $[\mathscr{B}(\ell^2(\Gamma)) \otimes \mathscr{B}(\mathcal{H})]$-module as (general) Hilbert Γ-*module*. Thus, any Hilbert Γ-module carries a unitary *left action representation* L_γ of Γ which generalises the left translation operator, introduced for $\ell^2(\Gamma)$ in (5.13).

Definition 5.2.5. Let $\mathscr{H}$ and $\mathcal{H}$ be Hilbert spaces;

(a) A general Hilbert Γ-module $\mathscr{H}$ is called *free* if it is unitarily isomorphic to a Γ-module $\ell^2(\Gamma) \otimes \mathcal{H}$ and the representation of Γ is given by $\gamma \mapsto l_\gamma \otimes \mathbb{1}_{\mathcal{H}}$ for $\gamma \in \Gamma$.

(b) A general Hilbert Γ-module $\mathscr{H}$ is *projective* if it is unitarily isomorphic to a closed Γ-invariant subspace in $\ell^2(\Gamma) \otimes \mathcal{H}$.

The unitary isomorphisms are understood as unitary maps, commuting with the action of Γ. We collect some facts about Γ-modules (see in [Shu] the results Proposition 2.16, Corollary 2.16.1, Corollary 2.16.2).

Proposition 5.2.6.

(1) Let V_1, V_2 be general Hilbert Γ-modules and $A : V_1 \to V_2$ a linear topological Γ-isomorphism of Γ-modules, then there exists a unitary Γ-isomorphism $U : V_1 \to V_2$ of Hilbert Γ-modules.

(2) If V is a general Hilbert Γ-module and there exists a topological Γ-isomorphism onto a closed Γ-submodule of a free Hilbert Γ-module, then V is in fact a projective Hilbert Γ-module.

This is a consequence of Proposition 5.2.2 (5). The notion *topological Γ-isomorphism* means that the topological isomorphism commutes with the left action representation. Both properties (1) and (2) indicate that it is enough to consider topological isomorphisms in Definition 5.2.5. The focus will rely on free and projective Hilbert Γ-modules. Certain subspaces carry the structure of a Hilbert Γ-module which we will show for later purposes.

Lemma 5.2.7. *Let $\mathscr{H}, \mathscr{H}_1, \mathscr{H}_2$ be projective Hilbert Γ-modules; the subspaces $\mathscr{H}_1 \oplus \mathscr{H}_2$, $W^\perp$ and $\mathscr{H}/W$ for any closed, Γ-invariant subspace $W \subset \mathscr{H}$ are projective Hilbert Γ-(sub)modules.*

Proof. Let γ be any element in Γ. One needs to check that these subspaces are themselves Γ-invariant Hilbert spaces. The Hilbert space characters follow easily from functional analysis since any closed subspace of a Hilbert space is a Hilbert space. The direct sum of two Hilbert spaces is again a Hilbert space; the same holds true for the quotient of a Hilbert space with one of its closed subspaces and for the orthogonal complement.

Since $\mathscr{H}_1$ and $\mathscr{H}_2$ are both Hilbert Γ-modules, they could have in general different left action representations with respect to the same group: let $\{L_\gamma | \gamma \in \Gamma\}$ denote the left action representation on $\mathscr{H}_1$ and $\{\mathcal{L}_\gamma | \gamma \in \Gamma\}$ the left action representation on $\mathscr{H}_2$. The left action representation on the direct sum is the direct sum of the left action representation, i.e. Γ induces a diagonal action on both Hilbert Γ-modules. As the ranges of the left action representations are contained in their Hilbert Γ-module, we have

$$(L_\gamma \oplus \mathcal{L}_\gamma)(\mathscr{H}_1 \oplus \mathscr{H}_2) = L_\gamma(\mathscr{H}_1) \oplus \mathcal{L}_\gamma(\mathscr{H}_2) \subseteq \mathscr{H}_1 \oplus \mathscr{H}_2$$

for any $\gamma \in \Gamma$. Hence the direct sum is Γ-invariant and becomes a general Hilbert Γ-module. The orthogonal complement is also a Γ-invariant subpace: let $\{L_\gamma | \gamma \in \Gamma\}$ denote the left action representation on $\mathscr{H}$; suppose $v \in W^\perp$, i.e. $v \in \mathscr{H}$ such that $\langle v \,|\, u \rangle_{\mathscr{H}} = 0$ for all $u \in W$. Then $L_\gamma v \in W^\perp$ since the action of Γ is unitary:

$$\langle L_\Gamma v \,|\, L_\epsilon u \rangle_{\mathscr{H}} = \langle L_\gamma v \,|\, L_\gamma (L_\gamma)^{-1} u \rangle_{\mathscr{H}} = \langle v \,|\, (L_\gamma)^{-1} u \rangle_{\mathscr{H}} = 0 \quad \forall\, u \in W$$

and so $(L_\gamma)^{-1} u \in W$ since it is a Γ-submodule. One concludes that $L_\gamma(W^\perp) \subseteq W^\perp$, implying $W^\perp$ to be a general Hilbert-Γ-submodule. The quotient Hilbert space consists of equivalence classes for each element in $\mathscr{H}$ where two Hilbert vectors are equivalent if the difference is an element in W. We first check that this equivalence relation is also true for transformed Hilbert vectors: as $\mathscr{H}$ is a Hilbert Γ-module the elements $L_\gamma v_1, L_\gamma v_2$ are in $L_\gamma(\mathscr{H}) \subseteq \mathscr{H}$ for $v_1, v_2 \in \mathscr{H}$ and $L_\gamma(v_1 - v_2) \in L_\gamma(W) \subseteq W$ since W is a Γ-submodule. As the group action is linear, we have

$$L_\gamma(v_1 - v_2) = (L_\gamma v_1 - L_\gamma v_2) \in W \quad .$$

Thus each equivalence class is Γ-invariant: $L_\gamma\left(\mathscr{H}/W\right) \subseteq \mathscr{H}/W$.

Hence all Hilbert subspaces in the claim are Γ-invariant and thus carry the structure of a Hilbert Γ-module. It is left to check that they are projective. Let $\mathscr{H}, \mathscr{H}_1$ and $\mathscr{H}_2$ be projective Hilbert Γ-modules such that they are unitarily related to closed submodules of $\ell^2(\Gamma) \otimes \mathcal{H}, \ell^2(\Gamma) \otimes \mathcal{H}_1$ and respectively $\ell^2(\Gamma) \otimes \mathcal{H}_2$ where $\mathcal{H}, \mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. As any isomorphism is a closed map and these unitary isomorphisms commute with the Γ-action, we only need to check that the subspaces in the claim are isomorphic to a closed subset in $\ell^2(\Gamma) \otimes (\mathcal{H}_1 \oplus \mathcal{H}_2)$ and respectively $\ell^2(\Gamma) \otimes \mathcal{H}$

The direct sum of the two unitary Γ-isomorphisms implies a unitary Γ-isomorphism on the direct sum of the closed Γ-submodules which is again a closed Γ-submodule of

$$(\ell^2(\Gamma) \otimes \mathcal{H}_1) \oplus (\ell^2(\Gamma) \otimes \mathcal{H}_2) \cong \ell^2(\Gamma) \otimes (\mathcal{H}_1 \oplus \mathcal{H}_2) \quad .$$

Projectivity of $W^\perp$ follows from restricting the unitary Γ-isomorphism on $W^\perp$ which again maps to a closed subspace in $\ell^2(\Gamma) \otimes \mathcal{H}$. The commuting of the Γ-isomorphisms with the group action implies Γ-invariance of this closed subspace such that $W^\perp$ becomes a projective Hilbert Γ-submodule. Since $W^\perp$ and $\mathscr{H}/W$ are Γ-invariant, the known isomorphy $W^\perp \cong \mathscr{H}/W$ due to closedness of W is Γ-invariant and induces a unitary Γ-isomorphism according to Proposition 5.2.6 (1). As $W^\perp$ is already a projective Hilbert Γ-module, $\mathscr{H}/W$ becomes a projective Hilbert Γ-module due to Proposition 5.2.6 (2). □

Consider the space of bounded operators on a general Hilbert Γ-module $\mathscr{H}$ which commute with the left action representation L_γ for all $\gamma \in \Gamma$:

$$\mathscr{B}_\Gamma(\mathscr{H}) := \{A \in \mathscr{B}(\mathscr{H}) \,|\, AL_\gamma = L_\gamma A \quad \forall \gamma \in \Gamma\} \quad . \tag{5.21}$$

This is again a von Neumann algebra according to Proposition 5.2.2 (1) as $\mathscr{B}_\Gamma(\mathscr{H})$ is the commutant of the space $\{L_\gamma \,|\, \gamma \in \Gamma\}$ which is due to the unitarity of the left action representation closed under the involution: $L_g \in \{L_\gamma \,|\, \gamma \in \Gamma\}$ implies $(L_g)^* \in \{L_\gamma \,|\, \gamma \in \Gamma\}$ for all $g \in \Gamma$. If we replace $\mathscr{H}$ with the tensor product $\ell^2(\Gamma) \otimes \mathcal{H}$ with Hilbert space $\mathcal{H}$, we gain

$$\mathscr{B}_\Gamma(\ell^2(\Gamma) \otimes \mathcal{H}) = \left\{A \in \mathscr{B}(\ell^2(\Gamma) \otimes \mathcal{H}) \,|\, AL_\gamma = L_\gamma A \quad \forall \gamma \in \Gamma\right\} \quad . \tag{5.22}$$

The matrix representation $A_{g,h}$ of A for $g, h \in \Gamma$ and the unitarity of the left action representation show that

$$\begin{aligned} A_{\gamma g,\gamma h} &= \left\langle A\delta_{\gamma g} \,\middle|\, \delta_{\gamma h}\right\rangle_{\ell^2(\Gamma)} \overset{(5.15)}{=} \left\langle Al_\gamma \delta_g \,\middle|\, l_\gamma \delta_h\right\rangle_{\ell^2(\Gamma)} \overset{(*)}{=} \left\langle l_\gamma A\delta_g \,\middle|\, l_\gamma \delta_h\right\rangle_{\ell^2(\Gamma)} \\ &= \left\langle A\delta_g \,\middle|\, \delta_h\right\rangle_{\ell^2(\Gamma)} = A_{g,h} \end{aligned}$$

where we used the commuting of A with the left action representation in $(*)$. Comparing with (5.19) shows that in fact (5.22) reduces to

$$\mathscr{B}_\Gamma(\ell^2(\Gamma) \otimes \mathcal{H}) = \mathscr{N}_r(\Gamma) \otimes \mathscr{B}(\mathcal{H}) \quad . \tag{5.23}$$

Thus, if $\mathscr{H}$ is in fact a free Hilbert Γ-module which is unitarily isomorphic to $\ell^2(\Gamma) \otimes \mathcal{H}$ with Hilbert space $\mathcal{H}$, then this unitary Γ-isomorphism induces a unitary Γ-isomorphism between (5.21) and (5.23):

$$\mathscr{B}_\Gamma(\mathscr{H}) \cong \mathscr{N}_r(\Gamma) \otimes \mathscr{B}(\mathcal{H}) \quad . \tag{5.24}$$

Let V be a Hilbert Γ-module which is unitarily isomorphic to a closed Γ-invariant subspace $W \subset \ell^2(\Gamma) \otimes \mathcal{H}$. The orthogonal projection P_W commutes with $L_\gamma \otimes \mathbb{1}_{\mathcal{H}}$ for every $\gamma \in \Gamma$ such that $P_W \in \mathscr{B}_\Gamma(\ell^2(\Gamma) \otimes \mathcal{H})$ and thus $P_V \in \mathscr{B}_\Gamma(\mathscr{H})$. The trace on the von Neumann algebra (5.24) is given by (5.20). From [Shu, Thm.2.16] we recall that the trace does not depend on the inclusion of V in $\ell^2(\Gamma) \otimes \mathcal{H}$ via W such that the Γ-*dimension*

$$\dim_\Gamma(V) := \mathrm{Tr}_\Gamma\left(P_V\right) = \mathrm{Tr}_\Gamma\left(P_W\right) \tag{5.25}$$

is well-defined and has the following properties.

Proposition 5.2.8. *Let $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be (free or projective) Hilbert Γ-modules; the following properties are satisfied by (5.25):*

(1) $\dim_\Gamma(\mathscr{H})$ *is independent of the inclusion* $\mathscr{H} \subset \ell^2(\Gamma) \otimes \mathcal{H}$*;*

(2) $\dim_\Gamma(\ell^2(\Gamma)) = 1$, $\dim_\Gamma(\{0\}) = 0$ *and* $\dim_\Gamma(\ell^2(\Gamma) \otimes \mathcal{H}) = \dim_{\mathbb{C}}(\mathcal{H})$*;*

(3) $\dim_\Gamma(\mathscr{H}_1 \oplus \mathscr{H}_2) = \dim_\Gamma(\mathscr{H}_1) + \dim_\Gamma(\mathscr{H}_2)$ *for two orthogonal Γ-modules* $\mathscr{H}_1, \mathscr{H}_2$*;*

(4) $\mathscr{H}_1 \subset \mathscr{H}_2 \Rightarrow \dim_\Gamma(\mathscr{H}_1) \leq \dim_\Gamma(\mathscr{H}_2)$ *and* $\dim_\Gamma(\mathscr{H}_1) = \dim_\Gamma(\mathscr{H}_2) \Leftrightarrow \mathscr{H}_1 = \mathscr{H}_2$*;*

(5) $\dim_\Gamma(\mathscr{H}_1) = \dim_\Gamma(\mathscr{H}_2) \Leftrightarrow \mathscr{H}_1$ *and* $\mathscr{H}_2$ *are unitarily Γ-isomorphic to each other;*

(6) If Γ is finite with cardinality $|\Gamma|$*, then* $|\Gamma| \dim_\Gamma$ *coincides with* $\dim_{\mathbb{C}}$*; if* $\Gamma = \{\epsilon\}$*, then* $\dim_\Gamma$ *coincides with* $\dim_{\mathbb{C}}$*.*

The von Neumann algebra (5.21) motivates to define bounded operators between two different Hilbert Γ-modules in the same way.

Definition 5.2.9. Given two Hilbert Γ-modules $\mathscr{H}_1$, $\mathscr{H}_2$ with left action representations $\{L_\gamma | \gamma \in \Gamma\}$ respectively $\{\mathcal{L}_\gamma | \gamma \in \Gamma\}$ and $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, then we call A a

(a) Γ-*operator* if $AL_\gamma = \mathcal{L}_\gamma A$ for all $\gamma \in \Gamma$;

(b) Γ-*morphism* if A is a Γ-operator and $A \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$.

The second part in the definition contains operators in (5.21) for $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$ wherefore we will denote the space of Γ-morphisms with $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$. Note that in comparison to (5.21) this space is not a von Neumann algebra !

The von Neumann setting for linear unbounded operators becomes slightly more delicate. Let $A \in \mathscr{L}(\mathcal{H})$ has domain $\mathsf{dom}(A) := \mathsf{dom}_{\mathcal{H}}(A)$ and $\mathscr{N} \subset \mathscr{B}(\mathcal{H})$ a von Neumann algebra. We are looking for an algebra in $\mathscr{L}(\mathscr{H})$ such that it contains the von Neumann algebra $\mathscr{N}$ as subalgebra. In order to do so, we need to introduce the notion of *affiliation*.

Definition 5.2.10 (Definition 4.2.1 in [MvN36])**.** Let $\mathscr{M}$ be a subset of $\mathscr{L}(\mathcal{H})$ and $\mathscr{N} \subset \mathscr{B}(\mathcal{H})$ a von Neumann algebra; an operator $A \in \mathscr{M}$ is *affiliated to* $\mathscr{N}$ if $U^{-1}AU = A$ for all unitary operators $U \in \mathscr{N}'$

If $A \in \mathscr{B}(\mathcal{H})$, then A is also affiliated with $\mathscr{N}$. If A is a densely-defined, self-adjoint operator, which is affiliated to $\mathscr{N}$, and $f : \mathbb{R} \to \mathbb{C}$ a Borel function, then $f(A)$ is affiliated to $\mathscr{N}$ and $f(A) \in \mathscr{N}$ if f is bounded.

In order to find a suitable subset in $\mathscr{L}(\mathcal{H})$, which is a $*$-algebra with the affiliation property, we recall some basic facts about unbounded operators from Section 2.1. If we restrict to closed operators, then the domain of these operators becomes a Hilbert space with the graph norm as Hilbert space norm such that they map continuously between its domain and $\mathcal{H}$. Restricting to densely defined operators in addition implies that the domain is dense in the Hilbert space and an adjoint operator is well-defined, closed and densely defined which suggests a well-defined involution operation. For the affiliation property we need to understand the commutator of a linear densely defined and closed operators A and an operator $B \in \mathscr{B}(\mathcal{H})$. These two operators commute if for any $u \in \mathsf{dom}(A)$ also

$Bu \in \mathsf{dom}(A)$ such that ABu and thus $[AB, BA]u$ are meaningful. It becomes necessary to demand that B leaves the domain $\mathsf{dom}(A)$ invariant: $B(\mathsf{dom}(A)) \subset \mathsf{dom}(A)$. So if we take B to be unitary in $\mathscr{B}(\mathcal{H})$, we have $B^{-1}AB = A$ which gives us our affiliation condition if B is in the commutant of $\mathscr{N}$. This motivates the following definition.

Definition 5.2.11. Let $\mathscr{N} \subset \mathscr{B}(\mathcal{H})$ be a von Neumann algebra and $A \in \mathscr{L}(\mathcal{H})$; we say that A is a *$\mathscr{N}$-affiliated operator* if

(a) A is densely defined and closed;

(b) A is affiliated to $\mathscr{N}$.

We now come back to our Γ-setting. Let $\mathscr{H}$ be a free Hilbert Γ-module and $\mathscr{L}_\Gamma(\mathscr{H})$ is the set of linear, Γ-invariant operators[21] on $\mathscr{H}$. The commutant $\mathscr{B}'_\Gamma(\mathscr{H})$ of the von Neumann algebra (5.21) in $\mathscr{H}$ is the smallest von Neumann algebra which contains the set $\{L_\gamma \,|\, \gamma \in \Gamma\}$. Proposition 5.2.2 (4) implies that the unitary operators in $\mathscr{B}'_\Gamma(\mathscr{H})$ are exactly the unitary operators $\{L_\gamma \,|\, \gamma \in \Gamma\}$. Thus we can describe the $\mathscr{B}_\Gamma(\mathscr{H})$-affiliated operators as those elements in $\mathscr{L}(\mathscr{H})$ which are closed and densely defined Γ-operators and write

$$\mathscr{C}_\Gamma(\mathscr{H}) := \{A \in \mathscr{L}(\mathscr{H}) \,|\, A \,\text{closed and densely defined } \Gamma\text{-operator}\} \quad .$$

In fact, according to [MvN36, Theorem XV] this set is a $*$-algebra which contains the von Neumann algebra $\mathscr{B}_\Gamma(\mathscr{H})$. The commuting with the left action representation implies that the domain of operators in $\mathscr{C}_\Gamma(\mathscr{H})$ is invariant under the action of L_γ for each $\gamma \in \Gamma$; this is in fact already true for operators in $\mathscr{L}_\Gamma(\mathscr{H})$. The closedness of operators in $\mathscr{C}_\Gamma(\mathscr{H})$ implies that they are bounded on their domains. Their polar decomposition induces a unitary Γ-isomorphism from $\mathsf{dom}_{\mathscr{H}}(A)$ to $\mathscr{H}$ such that $\mathsf{dom}_{\mathscr{H}}(A)$ becomes a projective Hilbert Γ-module due to Proposition 5.2.6 (2). The graph of each operator in $\mathscr{C}_\Gamma(\mathscr{H})$ is a closed subspace in $\mathscr{H} \oplus \mathscr{H}$ which is a projective Hilbert Γ-module according to Lemma 5.2.7. The Γ-invariance of the operator $A \in \mathscr{C}_\Gamma(\mathscr{H})$ and the invariance of the domain $\mathsf{dom}(A)$ imply that

$$L_\gamma(\mathrm{Graph}(A)) \subset \mathrm{Graph}(A) \quad \forall \gamma \in \Gamma \quad .$$

Thus, the graph is Γ-invariant and in summary a projective Hilbert Γ-module.

These arguments for the graph and the domain do not depend on the $*$-algebra structure of $\mathscr{C}_\Gamma(\mathscr{H})$ and thus they carry over to the case where we could also take two different Hilbert Γ-modules $\mathscr{H}_1, \mathscr{H}_2$. We define the sets

$$\begin{aligned} \mathscr{L}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) &:= \{A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \,|\, AL_\gamma = \mathcal{L}_\gamma A \quad \forall \gamma \in \Gamma\} \\ &\text{and} \\ \mathscr{C}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) &:= \{A \in \mathscr{L}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \,|\, \mathsf{dom}_{\mathscr{H}_1}(A) \,\text{dense}\,,\ \mathrm{Graph}(A) \,\text{closed in}\, \mathscr{H}_1 \oplus \mathscr{H}_2\}\ . \end{aligned} \tag{5.26}$$

As long $\mathscr{H}_1 \neq \mathscr{H}_2$, the last set is not a $*$-algebra! All made observations for closed and densely-defined Γ-operators are then summarised as follows.

[21] Other authors use the terminology Γ-equivariant operator to stress that the operator intertwines the two left translation representations. We stick with the notion of Γ-invariant operator.

Lemma 5.2.12 (Lemma 3.8.4 in [Shu]). *Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert Γ-modules and $A \in \mathscr{C}(\mathscr{H}_1, \mathscr{H}_2)$, then*

(1) $A \in \mathscr{C}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ if and only if $\mathrm{Graph}(A)$ is Γ-invariant with respect to the diagonal action of Γ;

(2) If $A \in \mathscr{C}(\mathscr{H}_1, \mathscr{H}_2)$, the domain $(\mathsf{dom}_{\mathscr{H}_1}(A), \|\bullet\|_{\mathrm{Graph}(A)})$ becomes a projective Hilbert Γ-module.

Certain subspaces with respect to Γ-operators do carry a Hilbert Γ-module structure which we will show for later purposes.

Lemma 5.2.13. *Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert Γ-modules and $A \in \mathscr{C}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$, then*

(1) $\mathsf{ker}\,(A)$ is a projective Hilbert Γ-submodule of $\mathscr{H}_1$.

(2) If $\mathsf{ran}\,(A)$ is closed, then it is a projective Hilbert Γ-submodule of $\mathscr{H}_2$.

Proof. Let γ be any element in Γ and assume for simplicity that $A \in \mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$. We proceed as in the proof of Lemma 5.2.7. The kernel is always closed in $\mathscr{H}_1$ and the range of A is closed in $\mathscr{H}_2$ by assumption such that the kernel and the range are Hilbert spaces. We check that they are Γ-invariant due to the Γ-invariance of the operator A: let L_γ denote the left action representation of Γ on $\mathscr{H}_1$ and let $\mathcal{L}_\gamma$ the left action representation on $\mathscr{H}_2$;

(1) suppose $u \in \mathsf{ker}\,(A)$, then the intertwining of A and the left action representation shows that also $L_\gamma u \in \mathsf{ker}\,(A)$:

$$Au = 0 \quad \Rightarrow \quad AL_\gamma u = \mathcal{L}_\gamma A u = 0 \quad \Rightarrow \quad L_\gamma(\mathsf{ker}\,(A)) \subseteq \mathsf{ker}\,(A) \quad .$$

(2) if $v \in \mathsf{ran}\,(A)$, then there exists a $u \in \mathscr{H}_1$ such that $v = Au$. Applying $\mathcal{L}_\gamma$ from the left and using the commutation property of A gives

$$\mathcal{L}_\gamma v = \mathcal{L}_\gamma A u = A L_\gamma u \quad \Rightarrow \quad \mathcal{L}_\gamma(\mathsf{ran}\,(A)) \subseteq \{v \in \mathscr{H}_2 \,|\, \exists\, w \in \mathsf{ran}\,(L_\gamma) \,:\, v = Aw\} \quad .$$

But $\mathscr{H}_1$ is already a Hilbert Γ-module, so the unitary representation of Γ has range in $\mathscr{H}_1$ such that

$$\mathcal{L}_\gamma(\mathsf{ran}\,(A)) \subseteq \{v \in \mathscr{H}_2 \,|\, \exists\, u \in \mathscr{H}_1 \,:\, v = Au\} = \mathsf{ran}\,(A) \quad .$$

Consequently, the kernel and the range of a Γ-morphism are Γ-invariant and thus carry the structure of a Hilbert Γ-module. It is left to check that they are projective. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ as well as $\mathcal{H}_1$ and $\mathcal{H}_2$ be as in the proof of Lemma 5.2.7. As the unitary isomorphisms of $\mathscr{H}_1$ to $\ell^2(\Gamma) \otimes \mathcal{H}_1$ respectively $\mathscr{H}_2$ to $\ell^2(\Gamma) \otimes \mathcal{H}_2$ commute with the Γ-action and are closed maps, they map the closed subsets $\mathsf{ker}\,(A) \subset \mathscr{H}_1$ and $\mathsf{ran}\,(A) \in \mathscr{H}_2$ to closed subsets in $\ell^2(\Gamma) \otimes \mathcal{H}_1$ respectively $\ell^2(\Gamma) \otimes \mathcal{H}_2$ and stay Γ-invariant due to Γ-invariance of the kernel and the range of A; hence they become projective Hilbert Γ-submodules.

The arguments carry over to Γ-operators in $\mathscr{C}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ with little modification: from Section 2.1, the kernel and the range of unbounded operators are defined with their domains. According to Lemma 5.2.12 (2), the domain of A in $\mathscr{H}_1$ is a projective Hilbert Γ-module and the shown Γ-invariance holds true as the domain is Γ-invariant, too. □

5.2.3. Γ-operators and Γ-Fredholm theory

In the former subsection we already introduced the three spaces of operators $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ after Definition 5.2.9 as well as $\mathscr{L}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ and $\mathscr{C}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ in (5.26) for two Hilbert Γ-modules $\mathscr{H}_1$ and $\mathscr{H}_2$. Fredholm operators in Γ-setting are introduced with the help of Γ-*ideals* which we introduce first.

Definition 5.2.14 (Definition 3.10.4, 3.10.5 in [Shu], Definition 2.5 in [Vai08])**.** Given an operator $A \in \mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$; the following operator spaces are Γ-ideals:

(a) A is a *finite* Γ*-rank operator* (denoted by $A \in \mathscr{R}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$) if $\dim_\Gamma(\mathsf{ran}\,(A)) < \infty$.

(b) A is a Γ*-Hilbert-Schmidt operator* (denoted by $A \in \mathscr{S}^2_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$) if $\mathrm{Tr}_\Gamma\,(A^*A) < \infty$.

(c) A is a Γ*-trace class operator* (denoted by $A \in \mathscr{S}^1_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$) if $\mathrm{Tr}_\Gamma\,(|A|) < \infty$.

(d) A is a Γ*-compact operator* (denoted by $A \in \mathscr{K}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$) if A lies in the norm closure of $\mathscr{S}^1_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ in $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$.

Remarks 5.2.15.

(i) ([Shu, Lem.3.10.13(a)], [Vai08, Lem.2.6]) All introduced operators are right-ideals over $\mathscr{B}_\Gamma(\mathscr{H}_1)$ and left-ideals over $\mathscr{B}_\Gamma(\mathscr{H}_2)$. They become two-sided ideals if $\mathscr{H}_1 = \mathscr{H}_2$. We also have the following inclusions:

$$\mathscr{R}_\Gamma \subset \mathscr{S}^1_\Gamma \subset \mathscr{S}^2_\Gamma \subset \mathscr{K}_\Gamma \subset \mathscr{B}_\Gamma \subset \mathscr{C}_\Gamma \subset \mathscr{L}_\Gamma \quad .$$

(ii) ([Shu, Lem.3.10.13(b)]) An alternative definition for Γ-trace class operators is the representation as a finite sum of two Γ-Hilbert Schmidt operators: let J be a finite index set and $\{A_j\}_{j\in J}, \{B_j\}_{j\in J}$ collections of operators in $\mathscr{S}^2_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$, then

$$C \in \mathscr{S}^1_\Gamma(\mathscr{H}_1) \quad \Leftrightarrow \quad C = \sum_{j\in J} B_j^* \circ A_j \quad .$$

If $A \in \mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$, $B \in \mathscr{B}_\Gamma(\mathscr{H}_2, \mathscr{H}_1)$ and either one of them is Γ-trace class or both Γ-Hilbert-Schmidt, then $AB \in \mathscr{S}^1_\Gamma(\mathscr{H}_2)$, $BA \in \mathscr{S}^1_\Gamma(\mathscr{H}_1)$ and $\mathrm{Tr}_\Gamma\,(AB) = \mathrm{Tr}_\Gamma\,(BA)$ holds.

(iii) Γ-compact operators are in general not compact in the usual sense.

Let $\mathscr{H}_1 (= \mathscr{H})$ and $\mathscr{H}_2$ be free Hilbert Γ-modules which are isomorphically related to $\ell^2(\Gamma) \otimes \mathcal{H}_1$ and respectively $\ell^2(\Gamma) \otimes \mathcal{H}_2$ with Hilbert spaces $\mathcal{H}_1 (= \mathcal{H})$ and $\mathcal{H}_2$. We denote the unitary left action representation of $\mathscr{H}_1$ with L_γ while we take $\mathcal{L}_\gamma$ for the representation in $\mathscr{H}_2$.

$$\begin{array}{ccc}
\mathscr{H}_1 & \longleftrightarrow & \ell^2(\Gamma) \otimes \mathcal{H}_1 \\
A \big\downarrow & & \big\downarrow \mathbb{1}_{\ell^2(\Gamma)} \otimes \underline{A} \\
\mathscr{H}_2 & \longleftrightarrow & \ell^2(\Gamma) \otimes \mathcal{H}_2
\end{array}$$

Figure 5.3.: Depiction of the commutative diagrams for (5.27).

(5.24) indicates that an operator $A \in \mathscr{B}_\Gamma(\mathscr{H})$ corresponds to an operator of the form $\mathbb{1}_{\ell^2(\Gamma)} \otimes \underline{A}$ with $\underline{A} \in \mathscr{B}(\mathcal{H})$. As one can view any operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ as an operator on the direct sum $\mathscr{H}_1 \oplus \mathscr{H}_2$, we have in fact

$$\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \cong \mathcal{N}_r(\Gamma) \otimes \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2) \quad . \tag{5.27}$$

We want to show that (5.24) and thus (5.27) carry over to something similar for Γ-compact operators which is a more practical characterisation[22] of this operator class. We want to sketch this correspondence as there is no or at least not clearly written argument in the literature though it is used in practice[23]. For this we follow [LS95, pp.9-10] which gives a hint how to prove this correspondence. Let $X \in \mathscr{H}_2$ and $Y, Z \in \mathscr{H}_2$; we define a finite-rank projection operator

$$\Theta_{X,Y} : \mathscr{H}_1 \to \mathscr{H}_2 \qquad \text{as per} \qquad Z \mapsto \Theta_{X,Y}(Z) := \langle Y \,|\, Z \rangle_{\mathscr{H}_1} X \tag{5.28}$$

which is an analogue to rank-1-operators in Hilbert spaces (see (2.4)). Applying $\mathcal{L}_\gamma$ from the left shows

$$\mathcal{L}_\gamma \Theta_{X,Y}(Z) = \mathcal{L}_\Gamma(\langle Y \,|\, Z \rangle_{\mathscr{H}_1} X) = \langle Y \,|\, Z \rangle_{\mathscr{H}_1} (\mathcal{L}_\gamma X) = \Theta_{\mathcal{L}_\Gamma X, Y}(Z)$$

and the unitarity of L_γ implies

$$\Theta_{X,Y}(Z) = \langle Y \,|\, Z \rangle_{\mathscr{H}_1} X = \langle L_\gamma Y \,|\, L_\gamma Z \rangle_{\mathscr{H}_1} X = \Theta_{X, L_\gamma Y}(L_\gamma Z) \quad ;$$

we observe

$$\Theta_{X,Y}(Z) = \mathcal{L}_\gamma^* \mathcal{L}_\gamma \Theta_{X, L_\gamma Y}(L_\gamma Z) = \mathcal{L}_\gamma^* \Theta_{\mathcal{L}_\gamma X, L_\gamma Y}(L_\gamma Z)$$

and thus

$$\Theta_{\mathcal{L}_\gamma X, L_\Gamma Y} = \mathcal{L}_\gamma \circ \Theta_{X,Y} \circ L_\gamma^* = \mathcal{L}_\gamma \circ \Theta_{X,Y} \circ L_\gamma^{-1} \quad \forall \gamma \in \Gamma \quad , \tag{5.29}$$

i.e. the intertwining of the finite range projections with the left action representation. We can also consider the finite linear span of those operators: if we take finitely many elements $\{Y_k\}_{k \in L}$ in $\mathscr{H}_1$ and $\{X_l\}_{l \in L}$ in $\mathscr{H}_2$ for a finite index set L, the finite-rank projector takes the form

$$\sum_{k,l \in L} \Theta_{X_l, Y_k}(\cdot) = \sum_{k,l \in L} \langle Y_k \,|\, \cdot \,\rangle_{\mathscr{H}_1} X_l \quad . \tag{5.30}$$

Clearly, they are elements in $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$. As the Hilbert Γ-modules are free, any of their elements can be unitarily related to the following tensor products:

$$\begin{array}{llll} X & \text{is related to} & a \otimes x & \text{with} \quad a \in \ell^2(\Gamma), x \in \mathcal{H}_2 \quad , \\ Y & \text{is related to} & b \otimes y & \text{with} \quad b \in \ell^2(\Gamma), y \in \mathcal{H}_1 \quad , \\ Z & \text{is related to} & c \otimes z & \text{with} \quad c \in \ell^2(\Gamma), z \in \mathcal{H}_1 \quad . \end{array}$$

The action of (5.28) on tensor product elements is

$$\begin{aligned} \Theta_{a \otimes x, b \otimes y}(c \otimes z) &= \langle b \otimes y \,|\, c \otimes z \rangle_{\mathscr{H}_1} (a \otimes x) = \langle b \,|\, c \rangle_{\ell^2(\Gamma)} \langle y \,|\, z \rangle_{\mathcal{H}_1} (a \otimes x) \\ &= \langle b \,|\, c \rangle_{\ell^2(\Gamma)} a \otimes (\langle y \,|\, z \rangle_{\mathcal{H}_1} x) = \langle b \,|\, c \rangle_{\ell^2(\Gamma)} a \otimes \theta_{x,y}(z) \end{aligned} \tag{5.31}$$

[22] See [BR15, App.A] for the unitary identification of $\mathscr{K}_\Gamma(\mathscr{H})$ and $\mathscr{N}_r(\Gamma) \otimes \mathscr{K}(\mathcal{H})$.
[23] See e.g. [Vai08, Prop.4.1], [Vai08, Lem.6.5] or [Sch05, Thm.6.21].

where $\theta_{x,y}(z) := \langle y \,|\, z \rangle_{\mathcal{H}_1} x$ is now a rank-1 operator between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$: $\theta_{x,y} \in \mathscr{R}(\mathcal{H}_1, \mathcal{H}_2)$ (recall (2.4) from Section 2.1). The Γ-trace can be related to the ordinary trace in this case: let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases in $\mathcal{H}_1$ respectively $\mathcal{H}_2$, then

$$\begin{aligned}
\mathrm{Tr}_\Gamma\left(\Theta_{a\otimes x, b\otimes y}\right) &= \sum_{\substack{\gamma\in\Gamma \\ i\in I, J\in J}} \left\langle \Theta_{a\otimes x, b\otimes y}(\delta_\gamma \otimes e_i) \,\middle|\, \delta_\gamma \otimes f_j \right\rangle_{\ell^2(\Gamma)\otimes\mathcal{H}_2} \\
&\overset{(5.31)}{=} \sum_{\substack{\gamma\in\Gamma \\ i\in I, J\in J}} \left\langle b \,\middle|\, \delta_\gamma \right\rangle_{\ell^2(\Gamma)} \left\langle a \otimes \theta_{x,y}(e_j) \,\middle|\, \delta_\gamma \otimes f_j \right\rangle_{\ell^2(\Gamma)\otimes\mathcal{H}_2} \\
&= \sum_{\substack{\gamma\in\Gamma \\ i\in I, J\in J}} \left\langle b \,\middle|\, \delta_\gamma \right\rangle_{\ell^2(\Gamma)} \left\langle a \,\middle|\, \delta_\gamma \right\rangle_{\ell^2(\Gamma)} \left\langle \theta_{x,y}(e_j) \,\middle|\, f_j \right\rangle_{\mathcal{H}_2} \\
&= \sum_{\gamma\in\Gamma} \left\langle b \,\middle|\, \delta_\gamma \right\rangle_{\ell^2(\Gamma)} \left\langle a \,\middle|\, \delta_\gamma \right\rangle_{\ell^2(\Gamma)} \sum_{i\in I, J\in J} \left\langle \theta_{x,y}(e_j) \,\middle|\, f_j \right\rangle_{\mathcal{H}_2} \\
&\overset{(*)}{=} \left\langle a \,\middle|\, \bar{b} \right\rangle_{\ell^2(\Gamma)} \mathrm{Tr}\left(\theta_{x,y}\right)
\end{aligned}$$

where we have used both the Parseval equation for the Hilbert space $\ell^2(\Gamma)$ and the definition of the ordinary trace in $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ in $(*)$. As $\theta_{x,y}$ is a rank-1 operator between two Hilbert spaces, the ordinary trace is finite and $b \in \ell^2(\Gamma)$ imply $\bar{b} \in \ell^2(\Gamma)$. Hence $\Theta_{a\otimes x, b\otimes y}$ has finite Γ-trace. For $\mathscr{H}_1 = \mathscr{H}_2$ this implies that due to the unitary isomorphism $\mathscr{H}$ to $\ell^2(\Gamma) \otimes \mathcal{H}$, the traces of $\Theta_{X,Y}$ and $\Theta_{a\otimes x, b\otimes y}$ are the same if X, Y are unitarily related to $a \otimes x$ respectively $b \otimes y$. Consequently, any $\Theta_{X,Y}$ on a free Hilbert Γ-module is Γ-trace class and according to Definition 5.2.14 (a) is a finite Γ-rank operator. This carries over to any finite linear combination and it can be shown that any operator in $\mathscr{R}_\Gamma(\mathscr{H})$ and indeed any $\mathscr{R}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ can be written like (5.30). In summary, the unitary Γ-isomorphism of the free Hilbert Γ-module and (5.31) induce the isomorphism

$$\mathscr{R}_\Gamma(\mathscr{H}) \cong \mathcal{N}_r(\Gamma) \otimes \mathscr{R}(\mathcal{H}) \tag{5.32}$$

and with the same reasoning as applied for the Γ-morphisms they furthermore induce

$$\mathscr{R}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \cong \mathcal{N}_r(\Gamma) \otimes \mathscr{R}(\mathcal{H}_1, \mathcal{H}_2) \quad . \tag{5.33}$$

These isomorphisms are preserved under the norm closure with respect to the norm on $\mathscr{B}_\Gamma(\mathscr{H})$ respectively $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ which are induced by the norm on $\mathscr{B}(\mathscr{H})$ and respectively $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$. The norm closure of the space of finite Γ-rank operators as subset of the Γ-trace class operators is the space of Γ-compact operators due to Remarks 5.2.15 (i) and Definition 5.2.14 (d). The norm closure on the right-hand side of (5.32) respectively (5.33) decomposes into the norm closure of $\mathscr{N}_r(\Gamma)$ in $\mathscr{B}(\ell^2(\Gamma))$ and the norm closure of finite-rank operators in the space of bounded operators between the same Hilbert spaces. The latter one complies with ordinary compact operators. As $\mathscr{N}_r(\Gamma)$ is a von Neumann algebra, it is weakly closed and thus also already norm closed. In summary we get

$$\mathscr{K}_\Gamma(\mathscr{H}) \cong \mathcal{N}_r(\Gamma) \otimes \mathscr{K}(\mathcal{H}) \tag{5.34}$$

and also

$$\mathscr{K}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \cong \mathcal{N}_r(\Gamma) \otimes \mathscr{K}(\mathcal{H}_1, \mathcal{H}_2) \quad . \tag{5.35}$$

After this extensive introduction of Γ-ideals we can define Γ-Fredholm operators.

Definition 5.2.16 (Definition 3.10.3 in [Shu]). Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert Γ-modules and $A \in \mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$; A is a Γ-*Fredholm operator* (denoted with $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$) if there exists a $B \in \mathscr{B}_\Gamma(\mathscr{H}_2, \mathscr{H}_1)$ such that $(\mathbb{1}_{\mathscr{H}_1} - BA) \in \mathscr{S}^1_\Gamma(\mathscr{H}_1)$ and $(\mathbb{1}_{\mathscr{H}_2} - AB) \in \mathscr{S}^1_\Gamma(\mathscr{H}_2)$; this B is the Γ-*Fredholm parametrix* and the Γ-*index* is defined by the Atiyah-Bott formula

$$\mathrm{ind}_\Gamma(A) := \mathrm{Tr}_\Gamma\left(\mathbb{1}_{\mathscr{H}_1} - BA\right) - \mathrm{Tr}_\Gamma\left(\mathbb{1}_{\mathscr{H}_2} - AB\right) \quad . \tag{5.36}$$

The existence of one parametrix in this definition can be replaced with the existence of $B_1, B_2 \in \mathscr{B}_\Gamma(\mathscr{H}_2, \mathscr{H}_1)$ which are left- and right-parametrices for A: let $R_1 = \mathbb{1}_{\mathscr{H}_1} - B_1 A$ and $R_2 = \mathbb{1}_{\mathscr{H}_2} - AB_2$ be two remainders which are Γ-trace class, then

$$\begin{aligned} B_1 - B_2 &= B_1 \mathbb{1}_{\mathscr{H}_2} - \mathbb{1}_{\mathscr{H}_1} B_2 = B_1 A B_2 + B_1 R_2 - R_2 B_2 - B_1 A B_2 \\ &= B_1 R_2 - R_1 B_2 \in \mathscr{S}^1_\Gamma(\mathscr{H}_2, \mathscr{H}_1) \end{aligned}$$

and thus these parametrices differ in a Γ-trace class operator and we can replace the left Γ-Fredholm parametrix with the right Γ-Fredholm parametrix in Definition 5.2.16 and vice versa. Some properties of Γ-Fredholm operators and their Γ-indices are listed below:

Proposition 5.2.17 (section 3.10 in [Shu]).

(1) $\mathrm{ind}_\Gamma(A) : \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \to \mathbb{R}$ *is locally constant.*

(2) If $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*, its* Γ*-Fredholm parametrix* B *is in* $\mathscr{F}_\Gamma(\mathscr{H}_2, \mathscr{H}_1)$ *with* $\mathrm{ind}_\Gamma(B) = -\mathrm{ind}_\Gamma(A)$.

(3) If $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ *and* $B \in \mathscr{F}_\Gamma(\mathscr{H}_2, \mathscr{H}_3)$*, then* $BA \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_3)$ *and* $\mathrm{ind}_\Gamma(BA) = \mathrm{ind}_\Gamma(B) + \mathrm{ind}_\Gamma(A)$.

(4) If A *is an invertible element in* $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*, then* $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ *with* $\mathrm{ind}_\Gamma(A) = 0$.

(5) If $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*,* $b \in \mathbb{C}/\{0\}$ *and* C *is in one of the* Γ*-ideals in Definition 5.2.14, then* $\mathrm{ind}_\Gamma(bA + C) = \mathrm{ind}_\Gamma(A)$.

(6) If $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*, then* $A^* \in \mathscr{F}_\Gamma(\mathscr{H}_2, \mathscr{H}_1)$ *with* $\mathrm{ind}_\Gamma(A^*) = -\mathrm{ind}_\Gamma(A)$.

(7) If $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*, then* $\dim_\Gamma(\ker(A)) < \infty$ *and* $\dim_\Gamma(\ker(A^*)) < \infty$ *such that* $\mathrm{ind}_\Gamma(A) = \dim_\Gamma(\ker(A)) - \dim_\Gamma(\ker(A^*))$.

(8) $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ *if and only if* $\dim_\Gamma(\ker(A)) < \infty$ *and there exists a closed set* $W \in \mathscr{H}_2$ *such that* $W \subset \mathrm{ran}(A)$ *and* $\mathrm{codim}_\Gamma(W) := \dim_\Gamma\left(\mathscr{H}_2/_W\right) < \infty$.

Remarks 5.2.18.

(i) In Definition 5.2.16 it is required that the remainders are Γ*-trace class operators such that the* Γ*-index is well-defined. It can be shown that it is sufficient if we replace the ideal of* Γ*-trace class operators with* Γ*-compact operators and thus any* Γ*-ideal, presented in Definition 5.2.14. The* Γ*-index is then defined with Proposition 5.2.17 (7).*

(ii) *The* Γ*-ideal in Proposition 5.2.17 (5) can be replaced with norm continuous pertubations ([Shu95, p.492]): let* $A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$*, then there exists an* $\varepsilon > 0$ *such that, if* $C \in \mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ *with* $\|C\| < \varepsilon$*, then* $(A + C) \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ *with* Γ*-index* $\mathrm{ind}_\Gamma(A + C) = \mathrm{ind}_\Gamma(A)$.

A consequence of Proposition 5.2.17 (7) and (8) as well as Lemma 5.2.7 is

Corollary 5.2.19. *Given Hilbert* Γ*-modules* $\mathscr{H}_j$, $\mathscr{H}_j'$, $j \in \{1, 2\}$*, and two* Γ*-morphisms* $A_j : \mathscr{H}_j \to \mathscr{H}_j'$*, then*

$$(A_1 \oplus A_2) \in \mathscr{F}_\Gamma(\mathscr{H}_1 \oplus \mathscr{H}_2, \mathscr{H}_1' \oplus \mathscr{H}_2') \quad \textit{if and only if} \quad A_j \in \mathscr{F}_\Gamma(\mathscr{H}_j, \mathscr{H}_j')$$

and the Γ*-index of* $(A_1 \oplus A_2)$ *is*

$$\mathrm{ind}_\Gamma(A_1 \oplus A_2) = \mathrm{ind}_\Gamma(A_1) + \mathrm{ind}_\Gamma(A_2) \quad . \tag{5.37}$$

Proof. The kernel and the range of $(A_1 \oplus A_2)$ are unitarily related to the direct sum of kernels and ranges:

$$\mathsf{ker}\,(A_1 \oplus A_2) \cong \mathsf{ker}\,(A_1) \oplus \mathsf{ker}\,(A_2) \quad \text{and} \quad \mathsf{ran}\,(A_1 \oplus A_2) \cong \mathsf{ran}\,(A_1) \oplus \mathsf{ran}\,(A_2) \quad .$$

Lemma 5.2.7 and Lemma 5.2.13 imply that all occuring spaces are projective Hilbert Γ-modules such that each of these isomorphisms imply unitary Γ-isomorphisms. If $(A_1 \oplus A_2)$ is Γ-Fredholm, then the left-hand sides do have finite Γ-dimensions respectively Γ-codimension according to Proposition 5.2.17 (8) such that $\mathsf{ker}\,(A_i)$ have finite Γ-dimensions and $\mathsf{ran}\,(A_i)$ finite Γ-codimensions for $i \in \{1, 2\}$. Proposition 5.2.17 (8) again implies that also A_1 and A_2 are Γ-Fredholm. As the argument is symmetric, the other implication follows instantly. It is left to show that the Γ-index has the claimed form. Since all three operators are Γ-Fredholm, we can use Proposition 5.2.17 (7) to calculate the Γ-index:

$$\begin{aligned}
\mathrm{ind}_\Gamma(A_1 \oplus A_2) &= \dim_\Gamma \mathsf{ker}\,(A_1 \oplus A_2) - \dim_\Gamma \mathsf{ker}\,((A_1 \oplus A_2)^*) \\
&= \dim_\Gamma \mathsf{ker}\,(A_1 \oplus A_2) - \dim_\Gamma \mathsf{ker}\,(A_1^* \oplus A_2^*) \\
&= \dim_\Gamma \mathsf{ker}\,(A_1) - \dim_\Gamma \mathsf{ker}\,(A_1^*) + \dim_\Gamma \mathsf{ker}\,(A_2) - \dim_\Gamma \mathsf{ker}\,(A_2^*) \\
&= \mathrm{ind}_\Gamma(A_1) + \mathrm{ind}_\Gamma(A_2).
\end{aligned}$$

□

5.3. (Differential-)Operators and Sobolev spaces on Γ-manifolds

The geometric facts and the machinery of von Neumann algebras, associated to a Galois group, of the former sections now come together. Let $E \to M$ be a Γ-vector bundle over a Γ-manifold M. The left action representation of Γ on a smooth section u of E is then described by

$$(L^E_\gamma u)(p) = (\pi^E_\Gamma u)(\gamma^{-1} p) \tag{5.38}$$

where π^E_Γ is an isometry which maps the section at p to $(\gamma \cdot p)s$. Note that now $(L^E_\gamma)^{-1} \neq L^E_{\gamma^{-1}}$ as the isometry operator is in general not involutive. Hence we only have $(L^E_\gamma)^{-1} u(p) = ((\pi^E_\Gamma)^{-1} u)(\gamma p)$. We equip M with a Γ-invariant smooth density $\mathrm{d}\mu$, which is either defined by a Γ-invariant pseudo-Riemannian metric or otherwise, and E with a Γ-invariant inner product $(\,\cdot\mid\cdot\,)_{E_p}$ on each fiber E_p:

$$\mathrm{d}\mu(\gamma p) = \mathrm{d}\mu(p) \quad \text{and} \quad (\,\cdot\mid\cdot\,)_{E_{\gamma p}} = (\,\cdot\mid\cdot\,)_{E_p} \quad \forall\,\gamma \in \Gamma \quad .$$

We introduce the inner product

$$\langle u \mid v \rangle_{L^2_\Gamma(M,E)} = \int_M \langle u(p) \mid v(p) \rangle_{E_p} \, \mathrm{d}\mu(p) \quad . \tag{5.39}$$

The norm closure of $C^\infty_c(M,E)$ with respect to this inner product is the space of square-integrable sections of E on M which we denote with $L^2_\Gamma(M,E)$ in order to stress the Γ-invariant density and bundle metric. The Γ-invariance of the density and the bundle metric ensure that the left action representation (5.38) is unitary with respect to (5.39). As π^E is an isometry, the bundle metric in (5.39) for $\langle L^E_\gamma u \mid v \rangle_{L^2_\Gamma(M,E)}$ becomes

$$\begin{aligned}\langle (L^E_\gamma u)(p) \mid v(p) \rangle_{E_p} &= \langle \pi^E_\Gamma u(\gamma^{-1}p) \mid v(p) \rangle_{E_p} = \langle \pi^E_\Gamma u(\gamma^{-1}p) \mid \pi^E_\Gamma (\pi^E_\Gamma)^{-1} v(p) \rangle_{E_p} \\ &= \langle u(\gamma^{-1}p) \mid (\pi^E_\Gamma)^{-1} v(p) \rangle_{E_p}\end{aligned}$$

such that

$$\begin{aligned}\langle L^E_\gamma u \mid v \rangle_{L^2_\Gamma(M,E)} &= \int_M \langle (L_\gamma u)(p) \mid v(p) \rangle_{E_p} \, \mathrm{d}\mu(p) = \int_M \langle u(\gamma^{-1}p) \mid (\pi^E_\Gamma)^{-1} v(p) \rangle_{E_p} \, \mathrm{d}\mu(p) \\ &= \int_{\gamma^{-1}M} \langle u(q) \mid (\pi^E_\Gamma)^{-1} v(\gamma q) \rangle_{E_{\gamma q}} \, \mathrm{d}\mu(\gamma q) \\ &= \int_M \langle u(q) \mid (\pi^E_\Gamma)^{-1} v(\gamma q) \rangle_{E_q} \, \mathrm{d}\mu(q) = \langle u \mid (L^E_\gamma)^{-1} v \rangle_{L^2_\Gamma(M,E)} \quad .\end{aligned}$$

We observe $(L^E_\gamma)^* = (L^E_\gamma)^{-1}$ for each element in the group. Other function spaces of interest next to $L^2_\Gamma(M,E)$ are Sobolev spaces. We recall from [Shu, Sec.3.9] that the appropriate norm is defined with a Γ-invariant partition of unity (5.7):

$$\|u\|^2_{H^s_\Gamma(M,E)} := \sum_{\substack{j \in J \\ \gamma \in \Gamma}} \|\phi_{j,\gamma} u\|^2_{H^s(\mathrm{supp}(\phi_{j,\gamma}),E)} \tag{5.40}$$

where $H^s(\mathrm{supp}\,(\phi_{j,\gamma}), E) := H^s_{\mathrm{supp}(\phi_{j,\gamma})}(M,E)$. The corresponding Sobolev spaces are then defined for any $s \in \mathbb{R}$ via

$$H^s_\Gamma(M,E) := \left\{ u \in H^s_{\mathrm{loc}}(M,E) \,\middle|\, \|u\|_{H^s_\Gamma(M,E)} < \infty \right\} \tag{5.41}$$

as Γ-*Sobolev spaces*. We note from the main reference [Shu] that the definition does not depend on the choice of the Γ-invariant partition of unity and the choice of the discrete group, i.e. if we take a different discrete group Γ′, such that M is also a Γ′-manifold, the Sobolev spaces coincide. We also want to point out the resemblance of the Γ-Sobolev spaces with the Sobolev spaces on manifolds with bounded geometry from (3.48). An equivalent description of these Γ-Sobolev spaces is given in terms of operators which we need to introduce first.

Let $\mathcal{F}$ be the fundamental domain of the Γ-action. (5.5) implies that M is a principal Γ-bundle. The action of the group on the Γ-manifold becomes isomorphically related to an action on $\Gamma \times \mathcal{F}$: $\gamma_2(\gamma_1, p) = (\gamma_2\gamma_1, p)$ for each $p \in \mathcal{F}$ and $\gamma_1, \gamma_2 \in \Gamma$. This induces a unitary isomorphism between L^2-sections,

$$L^2_\Gamma(M, E) \cong \ell^2(\Gamma) \otimes L^2(\mathcal{F}, E|_\mathcal{F}) \quad , \tag{5.42}$$

which is given by

$$u \mapsto \sum_{\gamma \in \Gamma} \delta_\gamma \otimes (L^E_\gamma)^* u|_\mathcal{F} \tag{5.43}$$

for a $u \in L^2_\Gamma(M, E)$. The left action representation L^E_γ becomes $l_\gamma \otimes \mathbb{1}_\mathcal{F}$ on $\ell^2(\Gamma) \otimes L^2(\mathcal{F}, E|_\mathcal{F})$ where l_γ is the left translation operator (5.13). Moreover, (5.42) implies that $L^2_\Gamma(M, E)$ is a free Hilbert Γ-module according to Definition 5.2.5 (a) and hence any closed Γ-invariant subset of $L^2_\Gamma(M, E)$ becomes a projective Hilbert Γ-module. We can use Remarks 5.1.6 and replace $L^2(\mathcal{F}, E|_\mathcal{F})$ with $L^2(\overline{\mathcal{F}}, E|_{\overline{\mathcal{F}}}) = L^2(M_\Gamma, E_\Gamma)$ ($E_\Gamma := E/\Gamma$) as $\mathcal{F}$ and M_Γ differ in a set of measure zero and due to the Γ-invariance of the density on M the densities on X and $\mathcal{F}$ coincide such that both L^2-spaces become unitarily isomorphic and (5.42) becomes

$$L^2_\Gamma(M, E) \cong \ell^2(\Gamma) \otimes L^2(M_\Gamma, E_\Gamma) \quad . \tag{5.44}$$

(5.24) implies with $\mathscr{H} = L^2_\Gamma(M, E)$ and $\mathcal{H} = L^2(M_\Gamma, E_\Gamma)$ that the von Neumann algebra of operators in $\mathscr{B}_\Gamma(L^2_\Gamma(M, E))$ is naturally isomorphic to $\mathscr{N}_r(\Gamma) \otimes \mathscr{B}(L^2(M_\Gamma, E_\Gamma))$. This isomorphism extends to

$$\mathscr{B}_\Gamma(L^2_\Gamma(M', F), L^2_\Gamma(M, E)) \cong \mathscr{N}_r(\Gamma) \otimes \mathscr{B}(L^2(M'_\Gamma, F_\Gamma), L^2(M_\Gamma, E_\Gamma)) \tag{5.45}$$

according to (5.27) where M' is another Γ-manifold with $M'_\Gamma := M'/\Gamma$ and F a Γ-vector bundle over M' such that $F_\Gamma = F/\Gamma$ with respect to the same Γ-action. Similarly we have the descriptions (5.34) and (5.35) for Γ-compact operators. The formal trace $\mathrm{Tr}_\Gamma(\cdot)$ in (5.20) is independent of the choice of the fundamental domain, see [Shu, Thm.2.14].

The analytic expression of the trace $\mathrm{Tr}_\Gamma(\cdot)$ in terms of the Schwartz kernel of a Γ-trace class operator and properties of Γ-trace class and Γ-Hilbert-Schmidt operators in this analytic setting are explained in [Shu, Sec.2.19] and [Shu, Sec.2.23] as well as in [Ati76, §4] with some detail. We want to point out some facts and details which we are going to use at some point. Given an operator A which maps between sections of two Γ-vector bundles $E \to M$ and $F \to M'$ of two Γ-manifolds M and M' with respect to the same Γ-action, i.e. $A : C^\infty_c(M', F) \to C^{-\infty}(M, E)$. The Schwartz-Kernel K_A of A is in general a distribution in $C^{-\infty}(M \times M', E \boxtimes F^*)$ (for the purpose of this introduction we forgo the half-density description here). The action of A on $u \in C^\infty_c(M', F)$ then becomes

$$(Au)(p) = \int_{M'} K_A(p, q) u(q) \, \mathrm{d}\mu_{M'}(q) \quad .$$

If the Schwartz kernel is in $L^2_{\mathrm{loc}}(M \times M', E \boxtimes F^*)$, it becomes a regular distribution and Au becomes a function again. The following result shows how $A \in \mathscr{B}_\Gamma(L^2_\Gamma(M', F), L^2_\Gamma(M, E))$ implies Γ-equivariance of its Schwartz kernel function K_A.

Lemma 5.3.1 (cf. Lemma 2.24. in [Shu]). *Let $A \in \mathscr{B}_\Gamma(L^2_\Gamma(M',F), L^2_\Gamma(M,E))$ with Schwartz-kernel K_A, then*

$$K_A(\gamma p, \gamma q) = \left[\pi^E_\Gamma \circ K_A \circ (\pi^F_\Gamma)^{-1}\right](p,q) \tag{5.46}$$

for all $\gamma \in \Gamma$, $(p,q) \in M \times M'$ and isometries $\pi^E_\Gamma(p) : E_p \to E_{\gamma\cdot p}$, $\pi^F_\Gamma(q) : F_q \to F_{\gamma\cdot q}$.

Proof. The action of each $\gamma \in \Gamma$ on $M \times M'$ is given by the diagonal action: $\gamma(p,q) = (\gamma p, \gamma q)$. W.l.o.g. let $K_A \in L^2_{\mathsf{loc}}(M \times M', E \boxtimes F^*)$, otherwise we have to consider the proof in the paring picture which makes it cumbersome to follow. Applying L^F_γ to $u \in L^2_{\mathsf{c}}(M', F)$ gives $(\pi^F_\Gamma u)(\gamma^{-1}\cdot)$. The action of A on $L^F_\gamma u$ on the level of Schwartz kernels takes the form

$$K_A(p,q)(\pi^F_\Gamma u)(\gamma^{-1}q) = (K_A \circ \pi^F_\Gamma)(p, \gamma q)u(q) = (K_A \circ \pi^F_\Gamma)(p, \gamma q)u(q)$$

and composing with $(L^E_\gamma)^{-1}$ from the left leads to

$$(L^E_\gamma)^{-1} K_A(p,q)(\pi^F_\Gamma u)(\gamma^{-1}q) = \left[(\pi^E_\Gamma)^{-1} \circ K_A \circ \pi^F_\Gamma\right](\gamma p, \gamma q)u(q)$$

such that

$$(L^E_\gamma)^{-1} A(L^F_\gamma u)(p) = \int_{M'} \left[(\pi^E_\Gamma)^{-1} \circ K_A \circ \pi^F_\Gamma\right](\gamma p, \gamma q)u(q)\, \mathrm{d}\mu_{M'}(q) \quad .$$

Since A intertwines the actions on $L^2_\Gamma(M',F)$ and $L^2_\Gamma(M,E)$, we have $(L^E_\gamma)^{-1} A L^F_\gamma = A$ and thus

$$(L^E_\gamma)^{-1} A(L^F_\gamma u)(p) = (Au)(p) = \int_{M'} K_A(p,q)u(q)\, \mathrm{d}\mu_{M'}(q) \quad .$$

Thus we get

$$\left[(\pi^E_\Gamma)^{-1} \circ K_A \circ \pi^F_\Gamma\right](\gamma p, \gamma q) = K_A(p,q)$$

by coincidence of the operators which implies the coincidence of the Schwartz kernels because of the Schwartz Kernel Theorem 4.1.1. □

This Lemma implies that any element in $\mathscr{B}_\Gamma(L^2_\Gamma(M',F), L^2_\Gamma(M,E))$ has a Schwartz kernel which is a distribution on the compact orbit space $(M \times M')/\Gamma$ under the diagonal action of Γ.

From now on, we consider for simplicity the case $M = M'$. If we take an operator $A \in \mathscr{S}^1_\Gamma(L^2_\Gamma(M,E))$ or even Γ-Hilbert-Schmidt, the Schwartz-Kernel already satisfies $K_A \in L^2_\Gamma(M \times M, E \otimes E^*)$. The Γ-invariant density is the product of Γ-invariant densities on M. Let the Γ-trace of A be expressed with the trace over two Γ-Hilbert-Schmidt operators $C \in \mathscr{S}^2_\Gamma(L^2_\Gamma(M,E))$ and $B \in \mathscr{S}^2_\Gamma(L^2_\Gamma(M,E))$ such that $A = B \circ C$. Denote with K_C the Schwartz kernel of C which is a sections of $E \otimes E^* \to M \times M$; let K_B be the Schwartz kernel of B which is a section of $E \otimes E^* = \mathsf{End}(E) \to M$. With $\mathcal{F}$ as fundamental domain of the Γ-action on M and the fibrewise trace $\mathrm{tr}_p(\cdot)$ along E_p, $p \in \mathcal{F}$, the analytic expression of the Γ-trace of A becomes

$$\mathrm{Tr}_\Gamma(A) = \int_{\mathcal{F} \times M'} \mathrm{tr}_{E_p}\left(K_B(p,q) \circ K_C(q,p)\right) \mathrm{d}\mu_{M \times M}(p,q) \quad . \tag{5.47}$$

The equality follows from (5.42) such that

$$\left\langle A(\delta_\epsilon \otimes u) \,\middle|\, (\delta_\epsilon \otimes v)\right\rangle_{L^2_\Gamma(M,E)} = \left\langle A(\delta_\epsilon \otimes u) \,\middle|\, (\delta_\epsilon \otimes v)\right\rangle_{\ell^2(\Gamma)\otimes L^2(\mathcal{F},E|_\mathcal{F})} = \left\langle A_{\epsilon,\epsilon} u \,\middle|\, v\right\rangle_{L^2(\mathcal{F},E|_\mathcal{F})}$$

for $u, v \in L^2(\mathcal{F}, E|_\mathcal{F})$. If the Schwartz kernel of A is already continuous at the diagonal, the Schwartz kernels of B and C do as well and vice versa such that the inner integral can be performed, giving K_A and consequently

$$\mathrm{Tr}_\Gamma\,(A) = \int_\mathcal{F} \mathrm{tr}_{E_p}\,(K_A(p,p))\,\mathrm{d}\mu_M(p) \quad . \tag{5.48}$$

Lemma 5.3.1 implies

Corollary 5.3.2. *For $A \in \mathscr{S}^1_\Gamma(L^2_\Gamma(M,E))$ with Schwartz kernel K_A the fibrewise trace of (5.46) satisfies*

$$\mathrm{tr}_{E_{\gamma p}}\,(K_A(\gamma p, \gamma p)) = \mathrm{tr}_{E_p}\,(K_A(p,p)) \quad .$$

This can be proven by using the cyclic permutation of the fibrewise trace and that π^E_Γ is an isometry. As the fundamental domain differs from M_Γ by a set of measure zero and the integrands are Γ-invariant in both cases, we can rewrite the analytic Γ-traces as integral over M_Γ, too.

We want to consider differential operators, pseudo-differential and Fourier integral operators in the Γ-setting. We denote with

$$\begin{aligned}
\mathsf{Diff}^m_\Gamma(M, \mathsf{Hom}(E,F)) &:= \left\{A \in \mathsf{Diff}^m(M, \mathsf{Hom}(E,F)) \,|\, AL^E_\gamma = L^F_\gamma A \quad \forall\, \gamma \in \Gamma\right\} \quad , \\
\Psi^s_\Gamma(M, \mathsf{Hom}(E,F)) &:= \left\{A \in \Psi^s(M, \mathsf{Hom}(E,F)) \,|\, AL^E_\gamma = L^F_\gamma A \quad \forall\, \gamma \in \Gamma\right\} \quad , \\
\mathcal{FIO}^s_\Gamma(M, N; \Lambda; \mathbf{Hom}(E,G)) &:= \left\{A \in \mathcal{FIO}^s(M, N; \Lambda; \mathbf{Hom}(E,G)) \,|\, AL^E_\gamma = L^G_\gamma A \quad \forall\, \gamma \in \Gamma\right\}
\end{aligned}$$

the sets of Γ-differential, Γ-pseudo-differential and Γ-Fourier integral operators where M, N are Γ-manifolds, $E, F \to M$ and $G \to N$ are Γ-vector bundles, Λ a Lagrangian submanifold and $m \in \mathbb{N}_0$, $s \in \mathbb{R}$. These are in general unbounded operators, hence are elements in $\mathscr{L}_\Gamma$ between suitable Hilbert Γ-modules. If an operator in one of these operator spaces enjoys the property of being properly supported, its Schwartz kernel is compactly supported with support in $(M \times M)/\Gamma$ respectively $(M \times N)/\Gamma$. Thus, any Γ-invariant differential operator has compactly supported Schwartz kernel on the orbit space.

We are now in the position to present an alternative definition for Γ-Sobolev spaces with the help of Γ-differential operators:

$$H^s_\Gamma(M,E) := \Big\{u \in L^2_\Gamma(M,E) \,\Big|\, Au \in L^2_\Gamma(M,E) \\ \text{for any } A \in \mathsf{Diff}^m_\Gamma(M, \mathsf{End}(E)) \text{ with } m \le s\Big\} \tag{5.49}$$

for $s \geq 0$; $H^{-s}_\Gamma(M,E)$ for the same range of degrees is defined by duality, see Proposition 4.1.4 (7).

A certain subclass of Γ-invariant pseudo-differential operators are *classical* operators:

$$\begin{aligned} A \in \Psi^m_{\mathsf{cl},\Gamma}(M, \mathsf{Hom}(E,F)) \quad :\Leftrightarrow \quad & A \in \Psi^m_\Gamma(M, \mathsf{Hom}(E,F)) \,:\, A = \hat{A} + R \text{ where} \\ & \hat{A} \in \Psi^m_\Gamma(M, \mathsf{Hom}(E,F)) \text{ has classical symbol} \\ & \text{and } R \in \Psi^{-\infty}_\Gamma(M, \mathsf{Hom}(E,F)) \quad . \end{aligned}$$

We collect some known results.

Proposition 5.3.3 (Corollary 3.9.2, Theorem 3.9.1/2/3/4(a) in [Shu])**.**

(1) If $A \in \Psi^m_{\mathsf{prop},\Gamma}(M, \mathsf{Hom}(E,F))$, then A becomes a bounded map from $H^s_\Gamma(M,E)$ to $H^{s-m}_\Gamma(M,F))$ for all $s \in \mathbb{R}$ and commutes with the left action representation of Γ.

(2) Let $A \in \Psi^m_{\mathsf{cl},\Gamma}(M, \mathsf{Hom}(E,F))$ be properly supported and elliptic, and suppose $u \in H^{-r}_\Gamma(M,E)$ for some $r \in \mathbb{R}$ such that $Au \in H^{s-m}_\Gamma(M,F)$; then $u \in H^s_\Gamma(M,E)$. If $E = F$ is Hermitian and A symmetric in $L^2_\Gamma(M,E)$, then A becomes essentially self-adjoint and with $H^m_\Gamma(M,E)$ as domain of the closure.

(3) If $A \in \mathscr{B}_\Gamma(L^2_\Gamma(M,E), L^2_\Gamma(M,F))$ and $A \in \Psi^m_{\mathsf{prop},\Gamma}(M, \mathsf{Hom}(E,F))$ for $m < -\dim(M)$, then $A \in \mathscr{S}^1_\Gamma(L^2_\Gamma(M,E), L^2_\Gamma(M,F))$.

The results (2) and (3) have an important consequence: if we consider A as unbounded operator from $L^2_\Gamma(M,E)$ to $L^2_\Gamma(M,F)$ with order $m \geq 0$, one observes that A is closed and its domain coincides with $H^m_\Gamma(M,E)$. The identity operator from $H^m_\Gamma(M,E) \to \mathsf{dom}(A)$ is bounded due to (2) with respect to the graph norm. Thus, it is closed due to the Closed Graph theorem. As the identity operator is in addition bijective, it maps $\mathsf{dom}(A)$ to $H^m_\Gamma(M,E)$ by the bounded inverse theorem. Thus, the domain is dense and thus A densely defined! According to Lemma 5.2.12 (2) the domain becomes a Hilbert Γ-module and the introduced identity operator implies a topological isomorphism which is Γ-invariant and maps from $H^m_\Gamma(M,E)$ as general Hilbert Γ-module onto the closed domain such that Proposition 5.2.6 (2) indicates $H^m_\Gamma(M,E)$ to be a projective Hilbert Γ-module. The case $m < 0$ follows by duality. For more details concerning this fact, we refer to [Shu, Cor.3.9.2]. In summary: $H^m_\Gamma(M,E)$ are projective Hilbert Γ-modules and (1) then shows that in fact

$$A \in \mathscr{B}_\Gamma(H^s_\Gamma(M,E), H^{s-m}_\Gamma(M,F)) \tag{5.50}$$

for all $s \in \mathbb{R}$ if $A \in \Psi^m_{\mathsf{prop},\Gamma}(M, \mathsf{Hom}(E,F))$.

We note that the requirement of being properly supported for all wanted properties in Proposition 5.3.3 is always fulfilled for differential operators, but too restrictive for a general class of operators. We first introduce a wider class of Γ-operators such that they are smoothing in the sense that they map Γ-Sobolev spaces into Γ-Sobolev spaces.

Definition 5.3.4. Given $E \to M$ and $F \to M'$ Γ-vector bundles over Γ-manifolds M, M' with respect to the same Γ-action and let $A : C^\infty_{\mathsf{c}}(M,E) \to C^{-\infty}(M',F)$ be a Γ-operator; A is said to be *s-smoothing* if it extends to a continuous linear operator between Γ-Sobolev spaces for any orders r, p:

$$A \,:\, H^r_\Gamma(M,E) \to H^p_\Gamma(M',F) \quad .$$

This is a slight generalisation of s-smoothing operators, introduced in [Shu, Sec.3.11]; if $M = M'$ and A a pseudo-differential operator, we write $A \in S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F))$. We list some properties of the class $S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F))$.

Lemma 5.3.5 (Lemma 3.11.1/2 in [Shu]). *Given two Γ-vector bundles $E \to M$ and $F \to M'$ over Γ-manifolds M, M' with respect to the same Γ-action and let $A : C_c^\infty(M,E) \to C^{-\infty}(M',F)$ be a Γ-operator;*

(1) if A is a properly supported Γ-operator with smooth Schwartz kernel on $M' \times M$, then A is s-smoothing.

(2) if $M = M'$ and $A \in S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F))$, then its Schwartz kernel is smooth on $M \times M$.

(3) if $M = M'$ and $A \in S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F))$, then it is a Γ-trace class operator.

Especially for Γ-invariant pseudo-differential operators one can define a class of operators which differ from a properly supported Γ-invariant pseudo-differential operator in a s-smoothing pseudo-differential operator.

Definition 5.3.6 (Definition 3.11.2 in [Shu]). Let E, F be Γ-vector bundles over the Γ-manifold M; an operator $A \in \Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$ is called *s-regular* if there is an operator $\hat{A} \in \Psi_{\Gamma,\mathsf{prop}}^m(M, \mathsf{Hom}(E,F))$ such that $(A - \hat{A}) \in S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F))$. If $\hat{A} \in \Psi_{\mathsf{cl},\Gamma}^m(M, \mathsf{Hom}(E,F))$ and properly supported, then they are defined to be *classical s-regular*.

As in the reference we denote these spaces of operators with $S\Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$ and respectively $S\Psi_{\mathsf{cl},\Gamma}^m(M, \mathsf{Hom}(E,F))$. We also introduce the notation $S\mathcal{FIO}^m$ for $m \in \mathbb{R}$ to stress that an operator is the sum of a properly supported Γ-invariant Fourier integral operator and a s-smoothing remainder. We observe that Γ-invariant differential operators do belong to this class as they don't have any smoothing remainder and are always properly supported; the same holds for any properly supported Γ-invariant operator. We list some further important properties which can be found in our main reference [Shu] as Corollary 3.11.1/5, Lemma 3.11.3, Proposition 3.11.2/3/4, Theorem 3.11.2.

Proposition 5.3.7. *Let $E, F \to M$ be Γ-vector bundles over the Γ-manifold M and let $A : C_c^\infty(M,E) \to C^{-\infty}(M,F)$ be a Γ-operator.*

(1) $\mathsf{Diff}_\Gamma^m(M, \mathsf{Hom}(E,F)) \subset \Psi_{\mathsf{prop},\Gamma}^m(M, \mathsf{Hom}(E,F)) \subset S\Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$ and $\mathsf{Diff}_\Gamma^m(M, \mathsf{Hom}(E,F)) \subset \Psi_{\mathsf{cl},\Gamma}^m(M, \mathsf{Hom}(E,F)) \subset S\Psi_{\mathsf{cl},\Gamma}^m(M, \mathsf{Hom}(E,F))$.

(2) $S\Psi_\Gamma^{-\infty}(M, \mathsf{Hom}(E,F)) = \bigcap_{m\in\mathbb{R}} S\Psi_{(\mathsf{cl},)\Gamma}^m(M, \mathsf{Hom}(E,F))$.

(3) If $A \in \mathscr{B}_\Gamma(L_\Gamma^2(M,E), L_\Gamma^2(M,F))$ and $A \in S\Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$ for $m < -\dim(M)$, then $A \in \mathscr{S}_\Gamma^1(L_\Gamma^2(M,E), L_\Gamma^2(M,F))$.

(4) If $A \in S\Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$, then $A \in \mathscr{B}_\Gamma(H_\Gamma^s(M,E), H_\Gamma^{s-m}(M,F))$ for all $s \in \mathbb{R}$.

(5) (uniform elliptic regularity) If $A \in S\Psi_\Gamma^m(M, \mathsf{Hom}(E,F))$ is elliptic, $u \in H_\Gamma^{-r}(M,E)$ for some $r \in \mathbb{R}$ and $Au \in H_\Gamma^s(M,F)$ for some $s \in \mathbb{R}$, then $u \in H_\Gamma^{s+m}(M,E)$.

(6) *For any $s \in \mathbb{R}$ there exists a formally self-adjoint operator in $S\Psi^s_{\mathsf{cl},\Gamma}(M, \mathsf{Hom}(E,F))$ which is elliptic and maps from $H^s_\Gamma(M,E) \to L^2_\Gamma(M,F)$ as a topological isomorphism of Hilbert Γ-modules.*

(7) *If $A \in S\Psi^m_\Gamma(M, \mathsf{Hom}(E,F))$ is elliptic with $m > 0$ and formally self-adjoint, then A is essentially self-adjoint and $\chi_I(A) \in S\Psi^{-\infty}_\Gamma(M, \mathsf{Hom}(E,F))$ for any bounded Borel set $I \subset \mathbb{R}$.*

(8) *If E, F are Hermitean Γ-vector bundles and $A \in S\Psi^m_{\mathsf{cl},\Gamma}(M, \mathsf{Hom}(F,E))$ elliptic, then $A \in \mathscr{F}_\Gamma(H^s_\Gamma(M,F), H^{s-m}_\Gamma(M,E))$ for any $s \in \mathbb{R}$ with an s-independent Γ-index*

$$\mathrm{ind}_\Gamma(A) = \dim_\Gamma \mathsf{ker}\,(A) - \dim_\Gamma \mathsf{ker}\,(A^*)$$

where $A^* \in S\Psi^m_{\mathsf{cl},\Gamma}(M, \mathsf{Hom}(E,F))$ is the formal adjoint of A with respect to (5.39).

Properties (1) and (8) together imply that elliptic Γ-differential and elliptic, properly supported Γ-pseudo-differential operators are Γ-Fredholm. A generalisation of property (7) to unbounded intervals is presented in subsection 8.1.1. (6) implies together with Proposition 5.2.6 (1) that the topological isomorphism induces a unitary isomorphism of Hilbert Γ-modules and that in fact the Γ-Sobolev spaces are *free Hilbert Γ-modules* for all $s \in \mathbb{R}$ as $L^2_\Gamma(M,F)$ is a free Hilbert Γ-module. For $s > 0$ the unitary isomorphism is of the form

$$H^s_\Gamma(M,E) \cong \ell^2(\Gamma) \otimes H^s(\mathcal{F}, E|_\mathcal{F}) \cong \ell^2(\Gamma) \otimes H^s(M_\Gamma, E_\Gamma) \quad . \tag{5.51}$$

This as well as an extension of the Rellich–Kondrachov theorem from compact manifolds to Galois coverings with compact base can be proven where the identification (5.35) is exemplarily applied.

Proposition 5.3.8 (cf. Proposition 4.1 in [Vai08], Theorem 6.21 in [Sch05])**.** *Let $E \to M$ be a Γ-vector bundle over a Γ-manifold M and $s, r \in \mathbb{R}_{>0}$ with $s > r$; the embedding*

$$H^s_\Gamma(M,E) \hookrightarrow H^r_\Gamma(M,E)$$

is Γ-compact.

This is a consequence of the classical Rellich–Kondrachov theorem where the embedding $H^s(M_\Gamma, E_\Gamma) \hookrightarrow H^r(M_\Gamma, E_\Gamma)$ is compact and lifts due to (5.35) to a Γ-compact operator. Some other properties about Sobolev spaces on manifolds in Proposition 4.1.4 carry over to the Γ-setting:

(1) localisation: let $a \in C^\infty(M, \mathbb{R}_{>0})$ and $u \in H^s_\Gamma(M,E)$, then $au \in H^s_\Gamma(M,E)$; if $a \in C^\infty_{\mathsf{c}}(M)$, then the multiplication with $u \mapsto au \in H^s_\Gamma(M,E)$ is Γ-compact for $s > 0$.

(2) continuous embedding: $H^s_\Gamma(M,E) \hookrightarrow H^r_\Gamma(M,E)$ is a Γ-morphism for $s > r$.

Property (1) follows because $au \in H_{\mathsf{loc}}(M,E)$ is a consequence of the localisation property in Proposition 4.1.4 (1) and $\{a\phi_{i,\gamma}\}_{i\in I,\gamma\in\Gamma}$ is a Γ-invariant partition of unity, induced from the Γ-invariant partition of unity $\{\phi_{i,\gamma}\}_{i\in I,\gamma\in\Gamma}$, subordinated to a covering of M. The fact that the Γ-Sobolev norm does not depend on the concrete choice of the Γ-invariant partition of unity, concludes the argument. For the second part of (1) we refer to [Vai08, Prop.4.5]. (2) is a consequence of Proposition 4.1.4 (3) and (5.27) by identifying bounded maps between Sobolev spaces on the compact bases with Γ-morphisms on the Galois covering.

6. Dirac operators

In this last introductory chapter we focus on Dirac operators on pseudo-Riemannian manifolds and in particular on Lorentzian and Riemannian Dirac operators as both special cases will appear in our later analysis. We first repeat some basics about a spin structure and Dirac operators on pseudo-Riemannian manifolds. Afterwards, we consider the decomposition of the Dirac operator once along a Riemannian hypersurface in a time-oriented Lorentzian and once in a Riemannian manifold.

The content of this chapter relies on the detailed explainations in the textbooks [Bau81] and [LM16], with supporting notes from [BGM05], [vdD18] and [Gin09]. The sections about Dirac operators on hypersurfaces rely on [BGM05, Chap.3-5] as well as some additional notes from [vdD18, Sec.2.3] and [vdD18, Sec.3.1/2].

6.1. Spin structure for pseudo-Riemannian manifolds

We first start with the notions of Clifford algebra and spin groups on n-dimensional vector spaces V before considering the special case of $V = \mathbb{R}^n$ and how to implement these structure on a pseudo-Riemannian manifold.

Given a n-dimensional vector space V over the field $\mathbb{R}$ with quadratic form $q : V \to \mathbb{R}$ such that its polarisation defines a symmetric bilinear form $b : V \times V \to \mathbb{R}$ on V via:

$$2b(v, w) = q(v) + q(w) - q(v - w) \tag{6.1}$$

for $v, w \in V$. $\mathsf{Cl}(V, b)$ denotes the *Clifford algebra* of (V, b), which is an associative algebra over $\mathbb{K}$ with unit element 1, and linear mapping $\mathbf{c} : V \to \mathsf{Cl}(V, b)$ such that

(a) $\mathbf{c}\,(v)^2 := \mathbf{c}\,(v) \circ \mathbf{c}\,(v) = -q(v)1$.

(b) suppose $f : V \to A$ is a linear map into another associative algebra A with unit element 1_A and $f(v) \circ f(v) = -q(v)1_A$ for all $v \in V$, then this map extends uniquely to an algebra homomorphism $f' : \mathsf{Cl}(V, b) \to A$ such that $f = f' \circ \mathbf{c}$.

Condition (b) says that the Clifford algebra is uniquely determined up to isomorphisms. We follow the minus-convention in (a) and write "$\cdot$" for the composition; moreover, we can express (a) directly by means of b with (6.1) and linearity of $\mathbf{c}$:

$$\mathbf{c}\,(v) \cdot \mathbf{c}\,(w) + \mathbf{c}\,(w) \cdot \mathbf{c}\,(v) = -b(v, w)1 \quad . \tag{6.2}$$

This is known as *Clifford relation*. The vector space endomorphism $a : v \mapsto (-v)$ for $v \in V$ can be extended to an endomorphism in $\mathsf{Cl}(V, b)$ and induces a $\mathbb{Z}_2$-grading

$$\mathsf{Cl}(V, b) = \mathsf{Cl}^0(V, b) \oplus \mathsf{Cl}^1(V, b)$$

where $\mathsf{Cl}^i(V,b)$ for $i \in \{0,1\}$ are the eigenspaces

$$\mathsf{Cl}^i(V,b) = \left\{w \in \mathsf{Cl}(V,b) \,|\, aw = (-1)^i w\right\} \quad .$$

They satisfy $\mathsf{Cl}^i \circ \mathsf{Cl}^j \subset \mathsf{Cl}^{i+j}$ for $i, j, (i+j) \in \mathbb{Z}_2$. $\mathsf{Cl}(V,b)$ has dimension 2^n. The space $\mathsf{Cl}^\times(V,b)$ is a subgroup of elements in $\mathsf{Cl}(V,b)$ which have a left and right inverse. It defines two further subgroups:

(a) the *pin group* of (V,b) is

$$\mathsf{Pin}(V,b) := \{\mathbf{c}\,(v_1)\cdots\mathbf{c}\,(v_l) \in \mathsf{Cl}(V,b) \,|\, v_i \in V \;:\; q(v) = b(v,v) = \pm 1\} \quad ;$$

(b) the *spin group* of (V,b) is

$$\mathsf{Spin}(V,b) = \mathsf{Pin}(V,b) \cap \mathsf{Cl}^0(V,b) \quad .$$

The Cartan-Dieudonné theorem (see e.g in [LM16, Thm.2.7]) states that every element in the orthogonal group $\mathsf{O}(V,b) := \{A \in \mathsf{GL}(V,b) \,|\, b(Av,Aw) = b(v,w)\,\forall\, v,w \in V\}$ can be decomposed into maximal n concantenations of reflections which are defined by the adjoint representation of $\mathsf{Cl}^\times(V,b)$: for a $v \in V$, such that $q(v) \neq 0$, the reflection across the hyperplane $\{v\}^\perp$ is

$$V \ni w \quad \mapsto \quad v^{-1}wv = -w + 2\frac{b(v,w)}{q(v)}v \quad .$$

This implies a homomorphism $\mathsf{Cl}^\times(V,b) \to \mathsf{O}(V,b)$ which can be restricted to the pin- and spin-group. Both restrictions yield exact sequences for real vector spaces V with non-degenerate bilinear form b:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathsf{Pin}(V,b) \longrightarrow \mathsf{O}(V,b) \longrightarrow 1$$

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathsf{Spin}(V,b) \longrightarrow \mathsf{SO}(V,b) \longrightarrow 1$$

Figure 6.1.: Exact sequences for $\mathsf{Pin}(V,b)$ and $\mathsf{Spin}(V,b)$, adapted from [LM16, Thm.2.9].

Thus, the homomorphism, restricted to either the pin group or the spin group, maps to $\mathsf{O}(V,b)$ or $\mathsf{SO}(V,b) := \{A \in \mathsf{O}(V,b) \,|\, \det(A) = 1\}$. These are two-fold coverings of groups with kernel in $\mathbb{Z}_2 = \{1,-1\}$.

We now specify to $V = \mathbb{R}^n$ with $n = r+s$ and non-degenerate symmetric bilinear form (3.1). We denote the Clifford algebra with respect to this bilinear form on $\mathbb{R}^n$ with $\mathsf{CL}_{r,s}$. The even and odd subspaces of $\mathsf{CL}_{r,s}$ are then denoted with $\mathsf{CL}^0_{r,s}$ and respectively $\mathsf{CL}^1_{r,s}$. We designate the spin group with $\mathsf{Spin}(r,s)$ on which our attention is focused on. We also consider the *complexified Clifford algebra* $\mathbb{C}\mathsf{l}_{r,s} := \mathsf{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$ which satisfies the following isomorphy properties:

(1) $\mathbb{C}\mathrm{l}_{r,s} \cong \mathsf{Mat}(2^{n/2}, \mathbb{C})$ if n is even;

(2) $\mathbb{C}\mathrm{l}_{r,s} \cong \mathsf{Mat}(2^{(n-1)/2}, \mathbb{C}) \oplus \mathsf{Mat}(2^{(n-1)/2}\mathbb{C})$ if n is odd.

They imply representations of the spin group, known as *spinor representations*:

(1) $\mathsf{Spin}(r,s) \cong \mathsf{GL}(2^{n/2}, \mathbb{C})$ if n is even;

(2) $\mathsf{Spin}(r,s) \cong \mathsf{GL}(2^{(n-1)/2}, \mathbb{C}) \oplus \mathsf{GL}(2^{(n-1)/2}\mathbb{C})$ if n is odd.

The corresponding irreducible representation spaces are defined to be $\Delta_{r,s}$ for n even and $\Delta_{r,s}(1)$ and $\Delta_{r,s}(2)$ for the first and second summand in the representation for n odd. For even dimensions the *spinor module* $\Delta_{r,s}$ decomposes into two submodules $\Delta^{\pm}_{r,s}$ which are called *spinor modules of postive* respectively *negative chirality*:

$$\Delta_{r,s} = \Delta^{+}_{r,s} \oplus \Delta^{-}_{r,s} \quad .$$

Let $\{e_j\}_{j=1}^{n}$ be a positive-oriented orthonormal basis in $\mathbb{R}^n$; one can show that $\Delta^{\pm}_{r,s}$ can be characterised as eigenspaces of the *spinorial volume form*

$$\omega(r,s) := (-\mathbb{i})^{s+n(n+1)/2}\mathbf{c}(e_1)\cdot\mathbf{c}(e_2)\cdot\cdots\cdot\mathbf{c}(e_n) \in \mathsf{Cl}^0_{r,s} \tag{6.3}$$

such that

$$\Delta^{\pm}_{r,s} = \{w \in \Delta_{r,s} \,|\, \omega(r,s)w = \pm w\} \quad . \tag{6.4}$$

The Clifford algebra for odd dimensions n has two inequivalent spinor modules $\Delta_{r,s}(1)$ and $\Delta_{r,s}(2)$ which can be distinguished as "+"-eigenspace for $\Delta_{r,s}(1)$ and the "−"-eigenspace for $\Delta_{r,s}(2)$ of (6.3) as an element in $\mathsf{Cl}^1_{r,s}$. While restricting to $\mathsf{Cl}^0_{r,s}$, these two submodules coincide and we write $\Delta_{r,s}$ for both subbundles. Thus we can get rid of one summand in the representation of Spin such that we have the following irreducible *spinor representations*:

(1) $\mathsf{Spin}(r,s) \cong \mathsf{Aut}(\Delta^{+}_{r,s}) \oplus \mathsf{Aut}(\Delta^{-}_{r,s})$ if n is even;

(2) $\mathsf{Spin}(r,s) \cong \mathsf{Aut}(\Delta^{\pm}_{r,s})$ if n is odd.

All presented spinor modules carry a Hermitian form $\langle \cdot \,|\, \cdot \rangle_{\Delta_{r,s}}$ which is inherited from the standard inner product on $\mathbb{C}^{2^n}$ for $n = 2m$ or $n = 2m+1$. This inner product is invariant under the action of $\mathsf{Spin}(n) := \mathsf{Spin}(n,0)$, but for general signatures it is only invariant with respect to a compact subgroup of $\mathsf{Spin}(r,s)$ (see [Bau81, pp.55-57] for details). But one can find a canonical non-degenerate symmetric sesquiliniear form $(\cdot \,|\, \cdot)_{\Delta_{r,s}}$ which has the desired invariance property. However, it is possible to relate this non-degenerate form to the inner product on $\mathbb{C}^{2^m}$. In order to do so, we introduce the following element in $\mathbb{C}\mathrm{l}_{r,s}$: we denote with $\{e_{j+r}\}_{j=1}^{s}$ the timelike directions in $\{e_j\}_{j=1}^{n}$; we define

$$\mathsf{6} := \begin{cases} \mathbf{c}(e_{r+1})\cdot\mathbf{c}(e_{r+2})\cdots\mathbf{c}(e_{r+s}) & \text{for} \quad s \in \{0,1\} \bmod 4 \\ \mathbb{i}\mathbf{c}(e_{r+1})\cdot\mathbf{c}(e_{r+2})\cdots\mathbf{c}(e_{r+s}) & \phantom{\text{for}} \quad s \in \{2,3\} \bmod 4 \end{cases} ; \tag{6.5}$$

it is a spinorial volume form for all timelike directions and satisfies $\mathsf{6}\cdot\mathsf{6} = \mathsf{1}$ as well as

$$\langle \mathsf{6}\phi \,|\, \psi \rangle_{\Delta_{r,s}} = \langle \phi \,|\, \mathsf{6}\psi \rangle_{\Delta_{r,s}}$$

for $\phi, \psi \in \Delta_{r,s}$. This last equation implies how $\left(\cdot \mid \cdot \right)_{\Delta_{r,s}}$ and $\left\langle \cdot \mid \cdot \right\rangle_{\Delta_{r,s}}$ are related to each other:

$$\left(\phi \mid \psi\right)_{\Delta_{r,s}} = \left\langle \mathsf{6}\phi \mid \psi \right\rangle_{\Delta_{r,s}} \quad \text{for } \phi, \psi \in \Delta_{r,s} \tag{6.6}$$

which in addition satisfies for $v \in \mathbb{R}^n$ and $\phi, \psi \in \Delta_{r,s}$ the symmetry property

$$\left(\mathbf{c}\,(v)\,\phi \mid \psi\right)_{\Delta_{r,s}} + (-1)^s \left(\phi \mid \mathbf{c}\,(v)\,\psi\right)_{\Delta_{r,s}} = 0 \quad . \tag{6.7}$$

We now turn our attention on how to implement a spin structure on a time- and space-oriented n-dimensional pseudo-Riemannian manifold M with metric g of signature (r, s) such that $n = r + s$. There are two ways to implement a spin structure on a manifold. Let $\mathsf{GL}^+(n, \mathbb{R})$ be as in (2.14) and $P_{\mathsf{GL}^+}(M)$ the principal $\mathsf{GL}^+(n, \mathbb{R})$-bundle of positive-oriented tangent frames. Let τ be the connected double covering map with preimage $\widetilde{\mathsf{GL}}^+(n, \mathbb{R}) := \tau^{-1}(\mathsf{GL}^+(n, \mathbb{R}))$. A *topological spin structure* on M is a principal $\widetilde{\mathsf{GL}}^+(n, \mathbb{R})$-bundle $P_{\widetilde{\mathsf{GL}}^+}(M)$ over M with a principal bundle morphism $\Theta : P_{\widetilde{\mathsf{GL}}^+}(M) \to P_{\mathsf{GL}^+}(M)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
P_{\widetilde{\mathsf{GL}}^+}(M) \times \widetilde{\mathsf{GL}}^+(n, \mathbb{R}) & \longrightarrow & P_{\widetilde{\mathsf{GL}}^+}(M) & \searrow & \\
\Big\downarrow \Theta \times \tau & & \Big\downarrow \Theta & & M \\
P_{\mathsf{GL}^+}(M) \times \mathsf{GL}^+(n, \mathbb{R}) & \longrightarrow & P_{\mathsf{GL}^+}(M) & \nearrow &
\end{array}$$

Figure 6.2.: Commuting diagram for the definition of a topological spin structure according to [BGM05] & [vdD18].

This is a metric-independent spin structure on the oriented manifold M if it admits certain topological criterions[24]. We call such a manifold *spin manifold* or just *spin*. A *metric spin structure* in comparison is implemented as follows: let $P_{\mathsf{SO}}(M)$ be the principal $\mathsf{SO}(r, s)$-bundle of positive-oriented orthonormal tangent frames. Since $\mathsf{SO}(r, s) \subset \mathsf{GL}^+(n, \mathbb{R})$ we can restrict τ to the preimage of $\mathsf{SO}(r, s)$ which gives the twofold covering map $\uptau : \mathsf{Spin}(r, s) \to \mathsf{SO}(r, s)$. We define a *principal* $\mathsf{Spin}(r, s)$-bundle as the preimage of $P_{\mathsf{SO}}(M)$ under Θ: $P_{\mathsf{Spin}}(M) = \Theta^{-1}(P_{\mathsf{SO}}(M))$. The commutative diagram 6.2 then takes the form

$$\begin{array}{ccccc}
P_{\mathsf{Spin}}(M) \times \mathsf{Spin}(r, s) & \longrightarrow & P_{\mathsf{Spin}}(M) & \searrow & \\
\Big\downarrow \Theta \times \uptau & & \Big\downarrow \Theta & & M \\
P_{\mathsf{SO}}(M) \times \mathsf{SO}(r, s) & \longrightarrow & P_{\mathsf{SO}}(M) & \nearrow &
\end{array}$$

Figure 6.3.: Commuting diagram for the definition of a metric spin structure according to [BGM05] & [vdD18].

[24] Next to orientability (i.e the vanishing of the first Stiefel-Whitney class of TM) also its second Stiefel-Whitney class has to vanish.

The pair M with $P_{\mathsf{Spin}}(M)$ is a pseudo-Riemannian spin manifold[25]. Both spin structures can be reinterpreted in terms of each other; see [vdD18, Sec.2.2]. A *spin(or) bundle* of M is defined as complex vector bundle which is associated to the spinor representation ρ of the representation space $\Delta_{r,s}$:

$$\mathcal{S}(M) := P_{\mathsf{Spin}}(M) \times_\rho \Delta_{r,s} \quad . \tag{6.8}$$

In other words, the fibre $\mathcal{S}_p(M)$ of $\mathcal{S}(M)$ over a point $p \in M$ is a vector space of equivalence classes, given by pairs $[A, \phi]$ with $A \in P_{\mathsf{Spin}}(M)|_p$ and $\phi \in \Delta_{r,s}$, which satisfy

$$[A, \phi] = [A\mathsf{g}^{-1}, \mathsf{g}\phi] \quad \text{for} \quad \mathsf{g} \in \mathsf{Spin}(r,s)\,.$$

We call these elements *spinors* and sections of $\mathcal{S}(M)$ are defined to be *spinor fields*. We see that the spinor bundle construction depends on the metric. We recall that for even dimensions the spin representation space $\Delta_{r,s}$ decomposes into the two submodules $\Delta^{\pm}_{r,s}$. This chirality decomposition carries over to the spinor bundle such that $\mathcal{S}(M)$ splits into the two subbundles $\mathcal{S}^{\pm}(M) := P_{\mathsf{Spin}}(M) \times_{\rho_\pm} \Delta^{+}_{r,s}$ with $\rho_\pm$, denoting the spin representations for the submodules $\Delta^{\pm}_{r,s}$:

$$\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M) \quad . \tag{6.9}$$

These subbundles are called *positive* respectively *negative half-spin(or) bundles* and we call their sections spinor fields of positive respectively negative chirality.

The Clifford multiplication can also be lifted to the manifold. We recapitulate that the tangent bundle can be considered as associated vector bundle of the form $P_{\mathsf{SO}}(M) \times \mathbb{R}^n/\mathsf{SO}(r,s)$ if the tangent frames are oriented. The *Clifford bundle* is then considered as associated vector bundle of the form $\mathcal{C}\ell_{r,s}(M) := P_{\mathsf{SO}}(M) \times \mathsf{Cl}_{r,s}/\mathsf{SO}(r,s)$. A metric spin structure allows to rewrite the bundle $P_{\mathsf{SO}}(M)$ in terms of $P_{\mathsf{Spin}}(M) \times_\lambda \mathsf{SO}(r,s)$ where λ denotes the double cover $\mathsf{Spin}(r,s) \to \mathsf{SO}(r,s)$. The Clifford bundle can then be viewed as associated bundle of $P_{\mathsf{Spin}}(M)$ which takes the form

$$\mathcal{C}\ell_{r,s}(M) = P_{\mathsf{Spin}}(M) \times_{\mho} \mathsf{Cl}_{r,s}$$

where the representation map $\mho$ is defined by conjugating elements from $\mathsf{Cl}_{r,s}$ with elements in $\mathsf{Spin}(r,s)$.

Let $A \in P_{\mathsf{SO}}(M)$ and $v \in \mathbb{R}^n$ which define an element $[A, v]$ in TM as associated bundle. The *Clifford multiplication/representation* is then defined as map $\mathbf{c} : TM \to \mathcal{C}\ell_{r,s}(M)$ by

$$\mathbf{c}\left([A, v]\right) := [A, \imath_{\mathsf{Cl}_{r,s}}(v)]$$

where $\imath_{\mathsf{Cl}_{r,s}}$ denotes the subspace inclusion $\mathbb{R}^n \hookrightarrow \mathsf{Cl}_{r,s}$. In this way the Clifford multiplication inherits the Clifford relation of $\mathsf{Cl}_{r,s}$ such that $\mathbf{c}$ becomes a pointwise acting vector space homomorphism from T_pM to $\mathsf{End}(\mathcal{S}_p(M))$, satisfying

$$\mathbf{c}\,(X) \cdot \mathbf{c}\,(Y) + \mathbf{c}\,(Y) \cdot \mathbf{c}\,(X) = -2g_p(X,Y)\mathbb{1}_{\mathcal{S}_p(M)} \quad \forall\, X, Y \in T_pM \quad . \tag{6.10}$$

[25] Since M is time- and space-oriented, its tangent bundle obeys the decomposition (3.2). If the subbundles $T^{\pm}M$ admit themselves spin structures, TM has vanishing second Stiefel-Whitney class. For more details, see [Bau81, Sec.2.1].

The canonical non-degenerate form on $\Delta_{r,s}$ implies a non-degenerate Hermitian bundle product $\left(\cdot \mid \cdot\right)_{\mathcal{S}(M)}$ which is pointwise given by

$$\left(\cdot \mid \cdot\right)_{\mathcal{S}_p(M)} : \mathcal{S}_p(M) \times \mathcal{S}_p(M) \to \mathbb{C} \tag{6.11}$$

which has property (6.7) and is only positive definite in the Riemannian case ($(r,s) = (n,0)$). The action of $\mathbf{c}(X)$ on a spinor is formally self-adjoint with respect to (6.11): let $u, v \in \mathcal{S}_p(M)$;

$$\left(\mathbf{c}(X)\, u \,\middle|\, v\right)_{\mathcal{S}_p(M)} + (-1)^s \left(u \,\middle|\, \mathbf{c}(X)\, v\right)_{\mathcal{S}_p(M)} = 0 \quad \forall X \in T_pM \quad . \tag{6.12}$$

A consequence of (6.10) and (6.12) is that

$$\begin{aligned}\left(\mathbf{c}(X)\, u \,\middle|\, \mathbf{c}(X)\, v\right)_{\mathcal{S}_p(M)} &= -(-1)^s \left(u \,\middle|\, \mathbf{c}(X) \cdot \mathbf{c}(X)\, v\right)_{\mathcal{S}_p(M)} \\ &= (-1)^s g(X,X) \left(u \,\middle|\, v\right)_{\mathcal{S}_p(M)} .\end{aligned} \tag{6.13}$$

Integrating (6.11) over the manifold against the volume form dvol_g gives a pairing which is positive definite in the Riemannian case, but in general only non-degenerate:

$$\left(\phi \,\middle|\, \psi\right)_{C_c^\infty(\mathcal{S}(M))} := \int_M \left(\phi(p) \,\middle|\, \psi(p)\right)_{\mathcal{S}_p(M)} \mathsf{dvol}_g(p) \tag{6.14}$$

for $\phi, \psi \in C_c^\infty(\mathcal{S}(M))$. In order to construct L^2-spaces for spinor sections as Hilbert spaces, we have to consider the completion of $C_c^\infty(\mathcal{S}(M))$ with respect to (6.14) which only works for M being a Riemannian manifold. However, recalling the construction of the non-degenerate sesquilinear form (6.6) on $\Delta_{r,s}$, we are able to relate the non-degenerate bundle metric to a positive definite Hermitean sesquilinear form. For Lorentzian manifolds, which is the case of our interest, this works as follows: if M is space- and time-oriented, it admits a global unit and timelike vector field $\mathfrak{t}$ such that

$$\left\langle \cdot \mid \cdot \right\rangle_{\mathcal{S}(M)} := \left(\mathbf{c}(\mathfrak{t}) \cdot \mid \cdot\right)_{\mathcal{S}(M)} \tag{6.15}$$

is a positive-definite bundle metric on $\mathcal{S}(M)$ which implies (6.14) to be a (Hermitian) inner product after replacing (6.11) with (6.15). The completion of $C_c^\infty(\mathcal{S}(M))$ with respect to this inner product then induces a Krein space where $\mathbf{c}(\mathfrak{t})$ acts as fundamental symmetry, since $\mathbf{c}(\mathfrak{t})$ is self-adjoint by (6.12) ($s = 1$), and unitary from (6.13) because $g(\mathfrak{t},\mathfrak{t}) = -1$ holds:

$$\left(\mathbf{c}(\mathfrak{t})\, u \,\middle|\, \mathbf{c}(\mathfrak{t})\, v\right)_{\mathcal{S}(M)} = -g(\mathfrak{t},\mathfrak{t}) \left(u \,\middle|\, v\right)_{\mathcal{S}(M)} = \left(u \,\middle|\, v\right)_{\mathcal{S}(M)} \quad . \tag{6.16}$$

Remarks 6.1.1.

(i) *Let E be any vector bundle over M, then we can define the tensor product $\mathcal{S}_E(M) := \mathcal{S}(M) \otimes E$ as twisted spinor bundle. This modification becomes interesting if one wants to consider spinors, taking values in another vector space. The vector bundle appears to be a coefficient bundle, describing E-valued spinors.*

(ii) *Let $V = \mathbb{R}$ and b from (3.1). If we replace in Fig. 6.1 the group $\mathsf{SO}(r,s)$ with $\mathsf{SO}(r,s) \times \mathsf{U}(1)$ with $\mathsf{U}(1)$ as circle group, we get the group $\mathsf{Spin}^c(r,s)$ from the modified exact sequence:*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathsf{Spin}^c(r,s) \rightarrow \mathsf{SO}(r,s) \times \mathsf{U}(1) \longrightarrow 1$$

Figure 6.4.: Exact sequence for $\mathsf{Spin}^c(r,s)$

This exact-sequence property implies that $\mathsf{Spin}^c(r,s) = (\mathsf{Spin}(r,s) \times \mathsf{U}(1))/\mathbb{Z}_2$. Proceeding as for the ordinary spin group, one can define a (metric) spin^c-structure on M as principal $\mathsf{Spin}^c(r,s)$-bundle $P_{\mathsf{Spin}^c}(M)$ such that the bundle map $P_{\mathsf{Spin}^c}(M) \to P_{\mathsf{SO}}(M) \times P_{\mathsf{U}(1)}(M)$ is equivariant against $\mathsf{Spin}^c(r,s)$-actions. Such a structure exists on a manifold if M is orientable and if the third integral Stiefel-Whitney class is vanishing. The tangent frame bundle carries a Spin^c-structure and we call the manifold Spin^c-manifold. If M is already spin, then it is canonically a Spin^c-manifold with $P_{\mathsf{Spin}^c}(M) = (P_{\mathsf{Spin}}(M) \times \mathsf{U}(1))/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{\pm 1\}$ acts diagonally. The corresponding spinor bundles are then defined as in the spin case where $P_{\mathsf{Spin}}(M)$ is replaced by $P_{\mathsf{Spin}^c}(M)$ and the representation spaces of $\mathsf{Spin}(r,s)$ are replaced with the one for the group $\mathsf{Spin}^c(r,s)$. A more practical point of view is to treat these spinor bundles as twisted spinor bundles: as one can associate to a Spin^c-manifold a complex line bundle L, one can show that the spinor bundle in this situation is given by the tensor product $\mathcal{S}(M) \otimes L^{1/2} =: \mathcal{S}_L(M)$ where $\mathcal{S}(M)$ is a spinor bundle on M which exists only locally due to the possible lack of a spin structure on M. The square root $L^{1/2}$ of L does not need to exist globally either. But the tensor product of these two bundles exists globally. For more details we refer to [LM16, App.D] for the technical background and [Ike05] for the situation of pseudo-Riemannian manifolds.

(iii) *If a pseudo-Riemannian manifold M is spin, then its pseudo-Riemannian covering does admit a spin structure as well; Lemma 2.10 in [Bau81]. If M is a connected and pseudo-Riemannian manifold and $\widetilde{M}$ is its universal covering with respect to the deck transformations, e.g. a Galois covering, then the spin structure on $\widetilde{M}$ also implies a spin structure on M and vice versa; [Bau81, Satz 2.7]. We also refer to [Bau81, Sec.2.3] for more background informations. The Riemannian analogue is stated in [Gin09, Thm.1.4.2].*

6.2. Dirac operators

In order to define the spin- or Atiyah-Singer Dirac operator, we have to clarify how to implement a connection on $\mathcal{S}(M)$. One way is to lift the connection one-form of the Levi-Civita connection on TM via the covering map Θ to a connection one-form on $\mathcal{S}(M)$, inducing a covariant derivative for spinor fields. A less invariant, but equivalent and more concrete way to do so, is the a covariant derivative $\nabla^{\mathcal{S}(M)}$ on $\mathcal{S}(M)$ from the Levi-Civita connection ∇ is the following local description: a spinor field $\Psi \in C^\infty(\mathcal{S}(M))$ can be described as an equivalence class $[A, \psi]$ with A a local section of $P_{\mathsf{Spin}}(M)$ and ψ as element in the representation space of the spin group. Using the covering map Θ on A gives a local oriented pseudo-Riemannian-orthonormal tangent frame $\Theta(A) = \{e_i\}_{i=1}^n$, satisfying $g(e_i, e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = g(e_i, e_i) = \pm 1$. The Levi-Civita connection $\nabla := \nabla^{\mathrm{LC}}$ is locally given by (3.20):

$$\nabla_{e_i} e_j := \sum_{k=1}^{n} \Gamma_{ij}^k e_k \quad .$$

It is proven in [Bau81, Satz 3.2] that the local expression for the spin Levi-Civita connection $\nabla^{\mathcal{S}(M)}$ is given by

$$\nabla_X^{\mathcal{S}(M)}\Psi = \left[A, X(\Psi) + \frac{1}{2}\sum_{k<l}\epsilon_k\epsilon_l g(\nabla_X e_k, e_l)\mathbf{c}\,(e_k)\cdot\mathbf{c}\,(e_l)\,\psi\right] \quad . \tag{6.17}$$

The action of X on the local function ψ with values in $\Delta_{r,s}$ is defined as for scalar functions. Replacing X with an element e_j in the tangent frame allows to rewrite $g(\nabla_{e_j}e_k, e_l)$ with Christoffel symbols. As like any other Koszul connection, also $\nabla^{\mathcal{S}(M)}$ complies with the Leibniz rule which is inherited from the one of ∇. One can determine the connection further in such a way that the following compatability conditions are moreover satisfied:

$$X(u\,|\,v)_{\mathcal{S}(M)} = (\nabla_X^{\mathcal{S}(M)}u\,|\,v)_{\mathcal{S}(M)} + (u\,|\,\nabla_X^{\mathcal{S}(M)}v)_{\mathcal{S}(M)} \tag{6.18}$$

$$\nabla_X^{\mathcal{S}(M)}(\mathbf{c}\,(Y)\,u) = \mathbf{c}\,(\nabla_X Y)\,u + \mathbf{c}\,(Y)\,\nabla_X^{\mathcal{S}(M)}u \tag{6.19}$$

where $u, v \in C^\infty(\mathcal{S}(M))$ and $X, Y \in \mathfrak{X}(M)$. This makes $\mathcal{S}(M)$ a Dirac bundle with compatible bundle metric, fulfilling the Leibniz rule. Condition (6.19) implies that the connection satisfies $\nabla_X^{\mathsf{End}(\mathcal{S}(M))}\mathbf{c} = 0$ such that the spinorial volume form

$$\omega^{r,s}_{\mathcal{S}(M)} := (-\mathbb{i})^{s+n(n+1)/2}\mathbf{c}\,(e_1)\cdot\mathbf{c}\,(e_1)\cdot\dots\cdot\mathbf{c}\,(e_n) \tag{6.20}$$

becomes globally parallel: $\nabla_X^{\mathsf{End}(\mathcal{S}(M))}\omega^{r,s}_{\mathcal{S}(M)} = 0$. (6.4) carries over to the manifold situation such that for even dimensions n the subbundles can be rephrased as the eigenbundles:

$$\mathcal{S}^{\pm}(M) := \left\{u \in C^\infty(\mathcal{S}(M))\,\middle|\,\omega^{r,s}_{\mathcal{S}(M)}u = \pm u\right\} \quad ;$$

the global parallelity of the spinorial volume form implies that this connection preserves this eigenspaces and thus the eigenspace decomposition (6.9). The volume form moreover satisfies $(\omega^{r,s}_{\mathcal{S}(M)})^2 = (-1)^s\mathbb{1}_{\mathcal{S}(M)}$. Another consequence of (6.19) is the fact that Clifford multiplication commutes with the parallel transport: let $\Psi \in C^\infty(\mathcal{S}(M))$, $X, Y \in \mathfrak{X}(M)$ and $\gamma(s)$ be the integral curve of X. The spin covariant derivative on Ψ in direction of X can be expressed as parallel transport along γ with $p = \gamma(0)$:

$$\nabla_X^{\mathcal{S}(M)}\Psi\Big|_p = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}(\gamma)^0_\tau\Psi_{\gamma(\tau)}\Big|_{\tau=0}$$

(recall (2.17) for the notation). Global parallelity implies $\nabla_X^{\mathcal{S}(M)}(\mathbf{c}(Y)\Psi) = \mathbf{c}(Y)\nabla_X^{\mathcal{S}(M)}\Psi$ and thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}(\gamma)^0_\tau\mathbf{c}|_{\gamma(\tau)}(Y_{\gamma(\tau)})\Psi_{\gamma(\tau)} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{c}|_{\gamma(0)}(Y_{\gamma(0)})\mathcal{P}(\gamma)^0_\tau\Psi_{\gamma(\tau)}\\
\Leftrightarrow \quad 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\mathcal{P}(\gamma)^0_\tau\mathbf{c}|_{\gamma(\tau)}(Y_{\gamma(\tau)}) - \mathbf{c}|_{\gamma(0)}(Y_{\gamma(0)})\mathcal{P}(\gamma)^0_\tau\right)\Psi_{\gamma(\tau)}\right]\\
\Leftrightarrow \quad C &= \left(\mathcal{P}(\gamma)^0_\tau\mathbf{c}|_{\gamma(\tau)}(Y_{\gamma(\tau)}) - \mathbf{c}|_{\gamma(0)}(Y_{\gamma(0)})\mathcal{P}(\gamma)^0_\tau\right)\Psi_{\gamma(\tau)} \quad .
\end{aligned}$$

The right-hand side vanishes for $\tau = 0$ and because C is a constant we get

$$\mathcal{P}(\gamma)^0_\tau\mathbf{c}|_{\gamma(\tau)}(Y_{\gamma(\tau)})\Psi_{\gamma(\tau)} = \mathbf{c}|_{\gamma(0)}(Y_{\gamma(0)})\mathcal{P}(\gamma)^0_\tau\Psi_{\gamma(\tau)} \quad . \tag{6.21}$$

After clarifying the implementation of the connection, we are able to define the *Dirac operator* as the map $\mathcal{D} : C^\infty(\mathcal{S}(M)) \to C^\infty(\mathcal{S}(M))$ which is locally given by

$$\begin{aligned}\mathcal{D}\Psi &:= \sum_{j=1}^{n} \epsilon_j \mathbf{c}\left(e_j\right) \nabla^{\mathcal{S}(M)}_{e_j} \Psi \\ &\overset{(6.17)}{=} \left[A, \sum_{j=1}^{n} \left(\epsilon_j \mathbf{c}\left(e_j\right) e_j(\Psi) + \frac{1}{2} \sum_{k<l} \Gamma_{jk,l} \epsilon_j \epsilon_k \epsilon_l \mathbf{c}\left(e_j\right) \cdot \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) \psi \right) \right]\end{aligned} \tag{6.22}$$

for a local orthonormal pseudo-Riemannian tangent frame $e_1, ..., e_n$. One sees that $\mathcal{D} \in \mathsf{Diff}^1(\mathcal{S}(M))$ and its principal symbol can be calculated with (4.11):

$$\boldsymbol{\sigma}_1(\mathcal{D})(x,\xi)\Psi = \lim_{\lambda\to\infty} \left(\frac{1}{\lambda}\right) \mathrm{e}^{-\mathbb{i}\lambda\Phi} \mathcal{D}\left(\mathrm{e}^{\mathbb{i}\lambda\phi}\Psi\right)\Big|_x = \mathbb{i}\sum_{j=1}^{n} \epsilon_j \xi(e_j) \mathbf{c}\left(e_j\right)\Psi(x) = \mathbb{i}\mathbf{c}\left(\xi^\sharp\right)\Psi(x)\,;$$

the sharp isomorphism is taken with respect to g. It shows that $\mathcal{D}$ is not-elliptic if $s \neq 0$ as it can vanish for ξ being a lightlike covector. If M is odd-dimensional, the Dirac operator is purely defined by (6.22) and one does not need to distinguish between spinor fields of positive or negative chirality as the spin structure does not admit a chirality decomposition. But for n even, the $\mathbb{Z}_2$-grading of the spinor bundle (6.9) induces a decomposition of the Dirac operator itself. We first observe

$$\mathcal{D}\omega^{r,s}_{\mathcal{S}(M)} = (-1)^{n-1}\omega^{r,s}_{\mathcal{S}(M)}\mathcal{D} \tag{6.23}$$

which can be seen as follows: we choose a local pseudo-Riemannian-orthonormal tangent frame $\{e_j\}_{j=1}^n$ which is synchronous, i.e. $\nabla_{e_i} e_j = 0$ for all $i, j \in \{1, ..., n\}$; since

$$\begin{aligned}\mathbf{c}\left(e_i\right)\cdot\mathbf{c}\left(e_1\right)\cdots\mathbf{c}\left(e_i\right)\cdots\mathbf{c}\left(e_n\right) &= (-1)^{i-1}\mathbf{c}\left(e_1\right)\cdots\mathbf{c}\left(e_i\right)^2\cdots\mathbf{c}\left(e_n\right) \\ &= (-1)^{i-1+(n-i)}\mathbf{c}\left(e_1\right)\cdots\mathbf{c}\left(e_i\right)\cdots\mathbf{c}\left(e_n\right)\cdot\mathbf{c}\left(e_i\right) \\ &= (-1)^{n-1}\mathbf{c}\left(e_1\right)\cdots\mathbf{c}\left(e_i\right)\cdots\mathbf{c}\left(e_n\right)\cdot\mathbf{c}\left(e_i\right)\end{aligned} \tag{6.24}$$

we have $\mathbf{c}\left(e_i\right)\cdot\omega^{r,s}_{\mathcal{S}(M)} = (-1)^{n-1}\omega^{r,s}_{\mathcal{S}(M)}\cdot\mathbf{c}\left(e_i\right)$ and due to the chosen synchronous frame the relation (6.19) reduces to

$$\nabla^{\mathcal{S}(M)}_{e_i}(\mathbf{c}\left(e_j\right)u) = \mathbf{c}\left(\nabla^{\mathrm{LC}}_{e_i} e_j\right)u + \mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_i}u = \mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_i}u$$

for a section $u \in C^\infty(\mathcal{S}(M))$. Hence $\nabla^{\mathcal{S}(M)}_{e_i}$ commutes with the spinorial volume form and with (6.24) the equivalence (6.23) is proven; it shows that the Dirac operator respects the splitting (6.9). As we have focused on n even, we have in fact $\mathcal{D}\omega^{r,s}_{\mathcal{S}(M)} = -\omega^{r,s}_{\mathcal{S}(M)}\mathcal{D}$ and the Dirac operator maps sections of $\mathcal{S}^+(M)$ to sections in $\mathcal{S}^-(M)$ and vice versa. This allows to represent[26] $\mathcal{D}$ as

$$\mathcal{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \quad \text{with} \quad D_\pm \in \mathsf{Diff}^1(\mathcal{S}^\pm(M), \mathcal{S}^\mp(M)) \quad . \tag{6.25}$$

[26] Here and henceforth we use the convention that combinations of objects (A, B, g, f elements, C as set) like $A_\pm B_\mp$ or $g, f_\pm \in C^\pm$ are meant for the upper and lower signs separately if not otherwise stated, i.e. $A_\pm B_\mp$ is either A_+B_- or A_-B_+ whereas $A_\pm B_\pm$ stands for either A_+B_+ or A_-B_-. In a similar manner we write $g, f_\pm \in C^\pm$ for either $g, f_+ \in C^+$ or $g, f_- \in C^-$.

Both $D_\pm$ are also non-elliptic as long as $0 < s < n$.

The *Dirac-Laplacian* $\mathcal{D}^2$ is a differential operator of second order and its principal symbol can be calculated by Lemma 4.1.3 (1): $\boldsymbol{\sigma}_2(\mathcal{D}^2)(x,\xi) = \boldsymbol{\sigma}_1(\mathcal{D})(x,\xi) \circ \boldsymbol{\sigma}_1(\mathcal{D})(x,\xi) = g(\xi^\sharp,\xi^\sharp)\mathbb{1}_{\mathcal{S}(M)}$ which is also a non-elliptic operator apart from the Riemannian setting. In particular this shows that $\mathcal{D}^2$ is a normally hyperbolic operator in the Lorentzian setting; see (C.5) in Appendix C. $\mathcal{D}^2$ and (6.25) imply in particular that $D_\pm D_\mp$ are also normally hyperbolic.

One could ask how $\mathcal{D}^2$ and the Bochner-Laplacian $\nabla^{\mathcal{S}(M)^*}\nabla^{\mathcal{S}(M)}$ differ from each other. The Lichnerowicz formula answers this question in the Riemannian setting and remains true for pseudo-Riemannian manifolds:

$$\mathcal{D}^2 = \nabla^{\mathcal{S}(M)^*}\nabla^{\mathcal{S}(M)} + \frac{1}{4}\mathcal{R} \tag{6.26}$$

where $\mathcal{R}$ is the scalar curvature; we refer to [Bau81, Satz 3.4] and for more technical explainations to [LM16, Sec.II.8]. If we twist the spinor bundle with a vector bundle E over M and denote the corresponding Dirac operator with $\mathcal{D}^E$, the pseudo-Riemannian Lichnerowicz formula (6.26) gets an extra contribution from the curvature endomorphism $\mathfrak{R}^E$ which is defined as follows: let $\mathcal{R}^E_{e_i,e_j}$ be the curvature of the twisting bundle; an element in $C^\infty(\mathcal{S}_E(M))$ is given as a tensor product of elements $\psi \in C^\infty(\mathcal{S}(M))$ and $f \in C^\infty(E)$ such that the curvature endomorphism takes the form

$$\mathfrak{R}^E(\psi\otimes f) := \sum_{i,j=1}^{n}\left(\mathbf{c}\,(e_i)\,\mathbf{c}\,(e_j)\,\psi\otimes\left(\mathcal{R}^E_{e_i,e_j}f\right)\right) \quad . \tag{6.27}$$

(6.26) then generalises to

$$(\mathcal{D}^E)^2 = \nabla^{\mathcal{S}_E(M)^*}\nabla^{\mathcal{S}_E(M)} + \frac{1}{4}\mathcal{R} + \mathfrak{R}^E \quad . \tag{6.28}$$

If we moreover twist the spinor bundle with the square root of a complex line bundle L, we can consider Spin^c-structures instead and we formally replace the bundle E with $E_L := L^{1/2}\otimes E$. The curvature endomorphism (6.27) is replaced by $\mathfrak{R}^{E_L}$ in (6.28). As the curvature endomorphism factorises with respect to the tensor product, the curvature endomorphism $\mathfrak{R}^{E_L}$ splits into a sum of $\mathfrak{R}^E$ and a contribution from the line bundle curvature two form Ω^L, given by

$$\Omega^L := \sum_{j<k}\Omega^L(e_j,e_k)\mathbf{c}\,(e_j)\,\mathbf{c}\,(e_k) \quad .$$

The pseudo-Riemannian Lichnerowicz formula for the Spin^c-twisted Dirac operator $\mathcal{D}^{E_L}$, acting on sections of $\mathcal{S}_{L,E}(M) := \mathcal{S}_{E_L}(M)$, then takes the form

$$(\mathcal{D}^{E_L})^2 = \nabla^{\mathcal{S}_{L,E}(M)^*}\nabla^{\mathcal{S}_{L,E}(M)} + \frac{1}{4}\mathcal{R} + \mathfrak{R}^E + \frac{\mathbb{i}}{2}\Omega^L \quad . \tag{6.29}$$

For more informations about these modifications we refer to [LM16, Sec.II.8] and [LM16, App.D].

Now we want to show some formal self-adjointness properties of $\mathcal{D}$. In order to do so, we first consider a technical result.

Lemma 6.2.1. *Given $u, v \in C^\infty(\mathcal{S}(M))$ and a local pseudo-Riemannian-orthonormal tangent frame $\{e_j\}_{j=1}^n$; the Dirac operator satisfies*

$$\left(\mathcal{D}u \,\middle|\, v\right)_{\mathcal{S}(M)} + (-1)^{s+1}\left(u \,\middle|\, \mathcal{D}v\right)_{\mathcal{S}(M)} = (-1)^{s+1}\sum_{j=1}^{n} \epsilon_j \mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} e_j\right) . \tag{6.30}$$

Proof. We choose $\{e_j\}_{j=1}^n$ to be a synchronous frame. We use the adjoints of the Clifford multiplication and the spin connection:

$$\begin{aligned}
(-1)^s\left(\mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_j} u \,\middle|\, v\right)_{\mathcal{S}(M)} &\overset{(6.12)}{=} -\left(\nabla^{\mathcal{S}(M)}_{e_j} u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} \\
&\overset{(6.18)}{=} \left(u \,\middle|\, \nabla^{\mathcal{S}(M)}_{e_j}\mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} - e_j\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} \\
&\overset{(6.19)}{=} \left(u \,\middle|\, \mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_j} v\right)_{\mathcal{S}(M)} + \left(u \,\middle|\, \mathbf{c}\left(\nabla_{e_j} e_j\right)v\right)_{\mathcal{S}(M)} \\
&\qquad - e_j\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} \\
&\overset{(*)}{=} \left(u \,\middle|\, \mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_j} v\right)_{\mathcal{S}(M)} - e_j\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} .
\end{aligned} \tag{6.31}$$

We used in $(*)$ that the frame has been chosen to be synchronous. With the help of the product rule of divergence, the resulting term can be described as

$$e_j\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} = \mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} e_j\right) - \left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)}\mathrm{div}\left(e_j\right) \quad ,$$

but since e_j belongs to a synchronous frame, the latter divergence will vanish for all j. Plugging this into (6.31) shows

$$\left(\mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_j} u \,\middle|\, v\right)_{\mathcal{S}(M)} + (-1)^{s+1}\left(u \,\middle|\, \mathbf{c}\left(e_j\right)\nabla^{\mathcal{S}(M)}_{e_j} v\right)_{\mathcal{S}(M)} = (-1)^{s+1}\mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\left(e_j\right)v\right)_{\mathcal{S}(M)} e_j\right)$$

and after multiplying with ϵ_j and thus each summand in (6.30). □

We now specify M to be a globally hyperbolic spin manifold. We consider any compact, but fixed time interval $[t_1, t_2]$ such that M on this reduced time interval becomes temporal compact. We write $M|_{[t_1,t_2]}$ for this restriction. The global hyperbolicity implies the existence of a spacelike Cauchy hypersurface such that $M|_{[t_1,t_2]}$ becomes a manifold with boundary $\Sigma_1 \sqcup \Sigma_2$.

Proposition 6.2.2. *Suppose M is a globally hyperbolic spin manifold with spacelike Cauchy hypersurface Σ, Lorentzian metric (3.7) and $\mathcal{D}$ the Dirac operator; for any time interval $[t_1, t_2]$ and spinor fields $u, v \in C^\infty_{\mathsf{sc}}(\mathcal{S}(M))$ with temporal support in this time interval we have*

$$\begin{aligned}
\int_M &\left(\mathcal{D}u \,\middle|\, v\right)_{\mathcal{S}(M)} + \left(u \,\middle|\, \mathcal{D}v\right)_{\mathcal{S}(M)} \mathsf{dvol} \\
&= \int_{\Sigma_2} \left(u \,\middle|\, \mathbf{c}\left(\nu\right)v\right)_{\mathcal{S}(M)} \mathsf{dvol}_{\Sigma_t} - \int_{\Sigma_1} \left(u \,\middle|\, \mathbf{c}\left(\nu\right)v\right)_{\mathcal{S}(M)} \mathsf{dvol}_{\Sigma_t} .
\end{aligned} \tag{6.32}$$

Proof. We choose any, but a fixed time interval $[t_1, t_2]$ and two spinors u, v as in the claim. Recalling the preface of this claim it is enough to consider $M|_{[t_1,t_2]}$ due to the temporal support of the spinors. The boundary of $M|_{[t_1,t_2]}$ is given by $\Sigma_1 \sqcup \Sigma_2$. Applying Lemma 6.2.1 for $s = 1$ leads to

$$\left(\mathcal{D}u \,\middle|\, v\right)_{\mathcal{S}(M)} + \left(u \,\middle|\, \mathcal{D}v\right)_{\mathcal{S}(M)} = \sum_{j=0}^{n} \epsilon_j \mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\,(e_j)\, v\right)_{\mathcal{S}(M)} e_j\right) \quad . \tag{6.33}$$

Because M is time and space-oriented, we have a global unit timelike vector ν which we choose to be past-directed. We choose the future-oriented Lorentz-orthonormal tangent frame in such a way that e_0 is future-timelike, indicating $e_0 = -\nu$ and $\epsilon_0 = -1$; the other members in the frame are spacelike with $\epsilon_0 = 1$ for $j > 0$. We apply the divergence theorem for Lorentzian manifolds (see (A.7) in Appendix A) with a timelike unit normal vector $\mathfrak{n}$ where $\mathfrak{n} = -\nu$ is inwards-pointing on Σ_1 since it has been chosen to be a past-directed orthonormal to all hypersurfaces; $\mathfrak{n}$ coincides with ν on Σ_2 to assure that $\mathfrak{n}$ is inwards-pointing. Integrating the right-hand side of (6.33) gives

$$\sum_{j=1}^{n} \epsilon_j \int\limits_{M|_{[t_1,t_2]}} \mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\,(e_j)\, v\right)_{\mathcal{S}(M)} e_j\right) \,\mathsf{dvol} = -\sum_{j=1}^{n} \epsilon_j \int\limits_{\partial M} \left(u \,\middle|\, \mathbf{c}\,(e_j)\, v\right)_{\mathcal{S}(M)} g(e_j, \mathfrak{n}) \,\mathsf{dvol}_{\Sigma_t} .$$

As $g(e_j, \mathfrak{n}) = 0$ for $j > 1$, we gain

$$\begin{aligned}
\sum_{j=1}^{n} \epsilon_j \int\limits_{M|_{[t_1,t_2]}} \mathrm{div}\left(\left(u \,\middle|\, \mathbf{c}\,(e_j)\, v\right)_{\mathcal{S}(M)} e_j\right) \,\mathsf{dvol} &= -\int\limits_{\partial M|_{[t_1,t_2]}} \left(u \,\middle|\, \mathbf{c}\,(e_1)\, v\right)_{\mathcal{S}(M)} g(e_1, \mathfrak{n}) \,\mathsf{dvol}_{\partial M} \\
= -\int_{\Sigma_2} \left(u \,\middle|\, \mathbf{c}\,(e_1)\, v\right)_{\mathcal{S}(M)} g(e_1, \mathfrak{n}) \,\mathsf{dvol}_{\Sigma_t} &- \int_{\Sigma_1} \left(u \,\middle|\, \mathbf{c}\,(e_1)\, v\right)_{\mathcal{S}(M)} g(e_1, \mathfrak{n}) \,\mathsf{dvol}_{\Sigma_t} \\
\overset{(*)}{=} \int_{\Sigma_2} \left(u \,\middle|\, \mathbf{c}\,(\nu)\, v\right)_{\mathcal{S}(M)} g(e_1, \mathfrak{n}) \,\mathsf{dvol}_{\Sigma_t} &- \int_{\Sigma_1} \left(u \,\middle|\, \mathbf{c}\,(\nu)\, v\right)_{\mathcal{S}(M)} g(e_1, \mathfrak{n}) \,\mathsf{dvol}_{\Sigma_t} \quad .
\end{aligned}$$

In $(*)$ we have used $g(e_1, \mathfrak{n}) = -g(\nu, \mathfrak{n})$ which is just (-1) on Σ_1 and 1 on Σ_2. Some extra signs come from $\mathbf{c}\,(e_1) = -\mathbf{c}\,(\nu)$ such that we finally have proven (6.32). ☐

As a consequence we get

Corollary 6.2.3. *$\mathcal{D}$ is formally skew-adjoint if $u, v \in C_c^\infty(\mathcal{S}(M))$: $\mathcal{D}^\dagger = -\mathcal{D}$.*

Being compactly supported means that the spinor fields are spatially compact and temporal compact supported in the interior of M. We furthermore get for n even

Corollary 6.2.4. *For the same preassumptions one has formally $D_+^\dagger = -D_-$ and $D_-^\dagger = -D_+$.*

Remarks 6.2.5.

(i) *Proposition 6.2.2 remains true if we merely consider differetiable spinor fields with the same support properties. The results remain true if we replace $\mathcal{D}$ with its twisting version $\mathcal{D}^{E_L}$ if we take E and L to be Hermitian vector bundles (i.e. E has a positive definite bundle metric).*

(ii) *For M globally hyperbolic we have used that there exists a global unit timelike vector field ν. Recalling the construction of the inner product (6.15), we observe that we can define a bundle metric which is a positive definite and Hermitian: let $\beta := \mathbf{c}(\nu)$, then the bundle product*

$$\langle \cdot \mid \cdot \rangle_{\mathcal{S}(M)} := (\beta \cdot \mid \cdot)_{\mathcal{S}(M)} \tag{6.34}$$

is positive definite and induces an (Hermitian) inner product for $\phi, \psi \in C^\infty_{\mathrm{c}}(\mathcal{S}(M))$:

$$\langle \phi \mid \psi \rangle_{L^2(\mathcal{S}(M))} := \int_M \langle \phi \mid \psi \rangle_{\mathcal{S}(M)} \, \mathsf{dvol}_g = \int_M (\beta\phi \mid \psi)_{\mathcal{S}(M)} \, \mathsf{dvol}_g \quad . \tag{6.35}$$

The completion of $C^\infty_{\mathrm{c}}(\mathcal{S}(M))$ with respect to the norm, induced by (6.35), defines the space of square-integrable spinor fields $L^2(\mathcal{S}(M))$. With (6.16) we also have the property

$$\langle \beta\phi \mid \beta\psi \rangle_{\mathcal{S}(M)} = (\beta^2\phi \mid \beta\psi)_{\mathcal{S}(M)} = (\beta\phi \mid \psi)_{\mathcal{S}(M)} = \langle \phi \mid \psi \rangle_{\mathcal{S}(M)} \quad . \tag{6.36}$$

The Clifford multiplication on $\mathcal{S}_E(M)$ becomes $(\mathbf{c} \otimes \mathbb{1}_E)$ such that

$$\langle \cdot \mid \cdot \rangle_{\mathcal{S}_E(M)} := ((\beta \otimes \mathbb{1}_E) \cdot \mid \cdot)_{\mathcal{S}_E(M)} \tag{6.37}$$

is a (positive definite) Hermitian bundle metric, indicating a L^2-inner product

$$\langle \Phi \mid \Psi \rangle_{L^2(\mathcal{S}_E(M))} := \int_M \langle \Phi \mid \Psi \rangle_{\mathcal{S}_E(M)} \, \mathsf{dvol}_g \tag{6.38}$$

for $\Phi, \Psi \in C^\infty_{\mathrm{c}}(\mathcal{S}_E(M))$. The completion of $C^\infty_{\mathrm{c}}(\mathcal{S}_E(M))$ with respect to the norm, induced by this inner product, defines the space $L^2(\mathcal{S}_E(M))$. In addition, we get a twisted version of the isometry property (6.36):

$$\begin{aligned} \langle (\beta \otimes \mathbb{1}_E)\Phi \mid (\beta \otimes \mathbb{1}_E)\Psi \rangle_{\mathcal{S}_E(M)} &= ((\beta \otimes \mathbb{1}_E)^2\Phi \mid (\beta \otimes \mathbb{1}_E)\Psi)_{\mathcal{S}_E(M)} \\ &= ((\beta \otimes \mathbb{1}_E)\Phi \mid \Psi)_{\mathcal{S}_E(M)} = \langle \Phi \mid \Psi \rangle_{\mathcal{S}_E(M)} \quad . \end{aligned} \tag{6.39}$$

Both isometries (6.36) and (6.39) carry over to isometries on $L^2(\mathcal{S}(M))$ and respectively on $L^2(\mathcal{S}_E(M))$.

(iii) *It follows from Proposition 6.2.2 that the Riemannian Dirac operator is formally self-adjoint. We recall from [GL83, Thm.1.17] that in fact the Riemannian Dirac operator is essentially self-adjoint if the underlying Riemannian manifold is either compact or complete. If the underlying manifold is compact without boundary, the spectrum of the Dirac operator is discrete and unbounded on both sides of $\mathbb{R}$ with finite-dimensional eigenspaces; see [Gin09, Thm.1.3.7]. On non-compact, but complete manifolds, the spectrum of the Dirac operator decomposes into a, possibly non-discrete, point spectrum and a continuous spectrum; the eigenspaces of different eigenvalues are still orthogonal to each other, but may have infinite multiplicity. We refer to [Bä05, Sec.1.2] and [Bä09] for more informations. For Dirac operators on general pseudo-Riemannian manifolds the situation is somehow more complicated. We refer the interested reader to [Bau81, Sec.3.3.2/3/4] for more informations.*

6.3. Spin structures and Dirac operators along hypersurfaces

We start with a general pseudo-Riemannian hypersurface Σ in a pseudo-Riemannian manifold M. After this, we specify to even dimensions of M and the two cases that either M is Lorentzian with Σ as spacelike Cauchy hypersurface or M is a Riemannian topological product manifold which contains Σ as base hypersurface of the cylinder.

6.3.1. Spin structures on pseudo-Riemannian hypersurfaces

Suppose M is a $(n+1)$-dimensional spin manifold with pseudo-Riemannian metric g of signature (r,s) and $\Sigma \subset M$ is a pseudo-Riemannian hypersurface with trivial timelike normal bundle, i.e. there exists a global timelike unit vector field ν in M such that $g(\nu,\nu) = -1$ and $\{\nu\}^{\perp} = T\Sigma$. The hypersurface Σ then has signature $(r, s-1)$.

We first assume that the spin structure of M is of topological nature, i.e. $\Theta : P^{+}_{\widetilde{\mathsf{GL}}}(M) \to P^{+}_{\mathsf{GL}}(M)$ is the twofold covering map (see Fig. 6.2). We can restrict the topological spin structure two a metric spin structure by restricting $\mathsf{GL}^{+}(n+1)$ to $\mathsf{SO}(r,s)$ such that $P_{\mathsf{Spin}}(M) = \Theta^{-1}(P_{\mathsf{SO}}(M))$ and the covering map for the metric spin structure is given by $\Theta_g := \Theta|_{\Theta^{-1}(\mathsf{SO}(r,s))}$. The hypersurface inherits a spin structure in the following way: the tangent space TM along Σ decomposes into the tangent bundle $T\Sigma$ and the normal bundle $N\Sigma := T\Sigma^{\perp}$. Since the normal bundle is assumed to be trivial, we get

$$TM|_{\Sigma} = \mathbb{R} \oplus T\Sigma \quad .$$

Let $\{e_i\}_{i=1}^{n}$ be an oriented tangent frame for $T\Sigma$ with the corresponding frame bundle $P_{\mathsf{GL}^+}(\Sigma)$. A tangent frame for M can be established by adding ν as element e_0 to the frame such that $\{e_i\}_{i=0}^{n}$ becomes an oriented tangent frame for TM with the corresponding frame bundle $P_{\mathsf{GL}^+}(M)$. Thus, there is an embedding of frames, $i : \{e_i\}_{i=1}^{n} \hookrightarrow \{e_i\}_{i=0}^{n}$ such that the frame bundle $P_{\mathsf{GL}^+}(\Sigma)$ can be embedded into the restricted frame bundle $P_{\mathsf{GL}^+}(M)|_{\Sigma}$, defining the principal bundle

$$P_{\widetilde{\mathsf{GL}^+}}(\Sigma) := \Theta^{-1} \circ i(P_{\mathsf{GL}^+}(\Sigma))$$

for a topological spin structure on Σ. A corresponding metric spin structure with respect to the induced metric g_{Σ} on Σ can be found by restricting the inclusion[27] i of frames to $P_{\mathsf{SO}}(\Sigma)$, defining $i_{g_{\Sigma}} := i|_{P_{\mathsf{SO}}(\Sigma)}$; this is an embedding from $P_{\mathsf{SO}}(\Sigma)$ to $P_{\mathsf{SO}}(M)|_{\Sigma}$ where the latter frame bundle is given by the group $\mathsf{SO}(r,s)$ and thus with respect to the metric g on M. Hence

$$P_{\mathsf{Spin}}(\Sigma) := \Theta_g^{-1} \circ i_{g_{\Sigma}}(P_{\mathsf{SO}}(\Sigma))$$

becomes a metric spin structure for Σ with respect to g_{Σ}. Let ρ be the representation map of $\mathsf{Spin}(r,s)$ with representation space $\Delta_{r,s}$; we recall that the spinor bundle on M is given by (6.8). If n is even, i.e. M odd-dimensional, the restriction of spinor bundle do not decompose into a direct sum and we set

$$\mathcal{S}(\Sigma) := \mathcal{S}(M)|_{\Sigma}$$

[27] The group SO in the subscript of the frame bundle is an abbreviation for $\mathsf{SO}(r, s-1)$.

which admits a chirality decomposition as Σ is even-dimensional. For even-dimensional M (n odd), we already have a chirality decomposition of the spinor bundle such that each subbundle restricts to one and the same spinor bundle on Σ which is now an odd-dimensional hypersurface:

$$\mathcal{S}^{\pm}(M)|_{\Sigma} =: \mathcal{S}(\Sigma) \quad .$$

The Clifford representation with respect to spinors on M depend on the representation of $\mathbb{C}\mathsf{l}_{r,s}$ on $\Delta_{r,s}$ while the representation on Σ is depending on the representation of $\mathbb{C}\mathsf{l}_{r,s-1}$ on $\Delta_{r,s-1}$. Hence the concrete form of the Clifford multiplications depends implicitly on the signature. In the next subsection we focus on two representations which we are going to use in the forthcoming analysis.

We end this subsection by focusing on two special cases: we view M to be either a globally hyperbolic manifold with Cauchy hypersurface Σ, such that M is diffeomorphic to $\mathbb{R} \times \Sigma$ and the metric g as in (3.7), or M is a Riemannian product space with base Σ such that M becomes isomorphic to $\mathbb{R} \times \Sigma$ with metric $N^2\,\mathrm{d}t^{\otimes 2} + g_t$. In both cases we also consider Σ to be a smooth and Riemannian (thus spacelike) hypersurface in M with lapse function $N \in C^{\infty}(M, \mathbb{R}_{>0})$ and a smooth family of Riemannian metrics $\{g_t\}_{t\in\mathbb{R}}$. In both cases we consider M to be isomorphic to the topological product[28], thus both cases differs only by the choice of the metric. For the following implementations, which relies on [vdD18, Sec.3.1], we consider the metric (3.36). We set Σ_t to be a spacelike slice at time $t \in \mathbb{R}$ in the topological product space which is a Riemannian submanifold for each $t \in \mathbb{R}$ if we equip each of the slices with the Riemannian metric g_t.

Given a topological spin structure $P_{\widetilde{\mathsf{GL}}^+}(M)$ on M which induces a metric spin structure $P_{\mathsf{Spin}}(M)$ with respect to $g^{[\epsilon]}$. The made observations imply that a topological spin structure $P_{\widetilde{\mathsf{GL}}^+}(\Sigma_t)$ and a corresponding metric spin structure $P_{\mathsf{Spin}}(\Sigma_t)$ on Σ_t can be obtained for each t where the latter spin structure is defined with respect to the Riemannian metrics g_t. The topological spin structure depends on the inclusion $\Sigma_t \hookrightarrow M$ and therefore on the chosen $t \in \mathbb{R}$. However, because a topological spin structure does not depend on the metric, all topological spin strutures $P_{\widetilde{\mathsf{GL}}^+}(\Sigma_t)$ on Σ are equivalent and can be identified by parallel transport along t-lines $t \mapsto (t,p)$ ($p \in \Sigma$) between principal fibre bundles. This parallel transport extends to a principal bundle isomorphism $P_{\widetilde{\mathsf{GL}}^+}(\Sigma_1) \to P_{\widetilde{\mathsf{GL}}^+}(\Sigma_2)$ as any parallel transport on a principle bundle is compatible with the right action of the structure group. The converse startegy describes how a spin structure on Σ and thus each Σ_t implements a spin structure on M; see [BGM05, Chap.5] and supporting notes in [vdD18, Sec.3.1] for details.

Remarks 6.3.1.

(i) *As we will focus on even-dimensional ambient spaces (n odd), we abuse notation and write $\mathcal{S}^{\pm}(\Sigma)$ for $\mathcal{S}^{\pm}(M)|_{\Sigma}$ in the forthcoming analysis to point out which subbundle has been restricted. As long as we are in this case with Σ being a odd-dimensional manifold, there should be no danger of missunderstanding.*

(ii) *All made considerations carry over to twisted spinor bundles without further conceptual modifications.*

[28] In comparison: a *geometric product space* is the product space $\mathbb{R} \times \Sigma$ with product metric (6.8) where the hypersurface metric g_{Σ} is fixed for all $t \in \mathbb{R}$.

6.3.2. Special case: M globally hyperbolic

For this subsection we assume M to be an even dimensional globally hyperbolic spin manifold with spacelike Cauchy hypersurfaces Σ (of odd dimension n) with Lorentzian metric $g = g^{[-1]}$. There exists a global unit timelike vector field ν which we choose to be past-directed. The spinor bundle on M admits a chirality decomposition (6.9) and we denote Clifford multiplication with $\mathbf{c}$. Recalling Remarks 6.2.5 (ii), the vector field ν induces an isometry $\beta := \mathbf{c}(\nu)$ according to (6.36) (and (6.39) for the twisted case). (6.10) furthermore implies $\beta^2 = \mathbb{1}_{\mathcal{S}(M)}$ as well as the anti-commuting with $\mathbf{c}$ and it induces the (positive definite) Hermitian bundle metric (6.34) and thus the inner product (6.35).

In the last subsection we discussed how the spin structures of the hypersurfaces Σ and $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ are inherited from the one on M. As n is odd, we obtain a spinor bundle on each Σ_t:

$$\mathcal{S}(M)|_{\Sigma_t} = \mathcal{S}^+(M)|_{\Sigma_t} \oplus \mathcal{S}^-(M)|_{\Sigma_t} = \mathcal{S}(\Sigma_t) \oplus \mathcal{S}(\Sigma_t) =: \mathcal{S}(\Sigma_t)^{\oplus 2}$$

for all $t \in \mathcal{T}(M)$. In accordance with Remarks 6.3.1 (i), we write $\mathcal{S}^\pm(\Sigma_t)$ to stress which spinor eigenbundle on M is restricted to the hypersurface. Following the explainations in chapter 1 of [BS19], supported with section 2.3 of [vdD18], we can define Clifford multiplication for spinors on the hypersurfaces: for a vector field X on Σ_t we set

$$\mathbf{c}_t(X) := \mathbb{i}\beta\mathbf{c}(X) \quad ; \tag{6.40}$$

if it acts on sections of $\mathcal{S}(M)|_{\Sigma_t}$, it takes the form $\mathbf{c}_t(X) \oplus (-\mathbf{c}_t(X))$. If it only acts on sections of the restricted subbundles, it is either $+\mathbf{c}_t(X)$ for $\mathcal{S}^+(\Sigma_t)$ or $-\mathbf{c}_t(X)$ for $\mathcal{S}^-(\Sigma_t)$. A Clifford relation on the hypersurface is inherited from the one on M: choose two vector fields X, Y on Σ_t for fixed t, then

$$\begin{aligned}\mathbf{c}_t(X)\,\mathbf{c}_t(Y) &= -\beta\mathbf{c}(X)\,\beta\mathbf{c}(Y) = \mathbf{c}(X)\,\mathbf{c}(Y) \overset{(6.10)}{=} -\mathbf{c}(Y)\,\mathbf{c}(X) - 2g(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)}\\ &= \beta\mathbf{c}(Y)\,\beta\mathbf{c}(X) - 2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)} = -\mathbf{c}_t(Y)\,\mathbf{c}_t(X) - 2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)}\end{aligned}$$

and finally

$$\mathbf{c}_t(X)\,\mathbf{c}_t(Y) + \mathbf{c}_t(Y)\,\mathbf{c}_t(X) = -2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)} \quad . \tag{6.41}$$

In order to investigate further properties of this Clifford multiplication, we define a Hermitian bundle metric by restricting (6.34): let $u, v \in C^\infty_{\mathrm{c}}(\mathcal{S}(\Sigma_t))$, then

$$\langle u \,|\, v\rangle_{\mathcal{S}(\Sigma_t)} = (\beta u \,|\, v)_{\mathcal{S}(M)} \tag{6.42}$$

and hereby a L^2-inner product

$$\langle u \,|\, v\rangle_{L^2(\mathcal{S}(\Sigma_t))} := \int_{\Sigma_t} \langle u \,|\, v\rangle_{\mathcal{S}(\Sigma_t)} \,\mathrm{dvol}_{g_t} \quad . \tag{6.43}$$

We observe that $\mathbf{c}_t$ is formally skew-adjoint with respect to (6.42) for all t:

$$\langle \mathbf{c}_t(X)\,u \,|\, v\rangle_{\mathcal{S}(\Sigma_t)} = -\langle u \,|\, \mathbf{c}_t(X)\,v\rangle_{\mathcal{S}(\Sigma_t)} \quad ; \tag{6.44}$$

this skew-adjointness carries over to the inner product (6.43). β still acts as an isometry with respect to the induced bundle metric because of $\beta^2 = \mathbb{1}$ and its formal self-adjointness such that (6.36) implies

$$\left\langle \beta\phi \,\middle|\, \beta\psi \right\rangle_{\mathcal{S}(\Sigma_t)} = \left\langle \phi \,\middle|\, \psi \right\rangle_{\mathcal{S}(\Sigma_t)} \quad ; \tag{6.45}$$

the twisted case is considered with (6.39):

$$\left\langle (\beta \otimes \mathbb{1}_E)\Phi \,\middle|\, (\beta \otimes \mathbb{1}_E)\Psi \right\rangle_{\mathcal{S}_E(\Sigma_t)} = \left\langle \Phi \,\middle|\, \Psi \right\rangle_{\mathcal{S}_E(\Sigma_t)} \quad . \tag{6.46}$$

The connection on $\mathcal{S}(M)|_{\Sigma_t}$ is the direct sum connection $\nabla^{\mathcal{S}(\Sigma_t)} \oplus \nabla^{\mathcal{S}(\Sigma_t)}$ where $\nabla^{\mathcal{S}(\Sigma_t)}$ is induced by the covariant derivative along a vector in $T_p\Sigma_t$ from (3.30). If $e_0 = \nu, e_1, ..., e_n$ is a Lorentz-orthonormal tangent frame, then $e_1, ..., e_n$ becomes a Riemann-orthonormal frame for Σ_t. We first manipulate with (3.39) the sum in the local expression (6.17)

$$\begin{aligned}
\sum_{k<l} \Gamma_{jk,l}\epsilon_k\epsilon_l \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) &= \sum_{1\le k<l\le n} \Gamma_{jk,l}\epsilon_k\epsilon_l \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) + \sum_{0<l\le n} \Gamma_{j0,l}\epsilon_0\epsilon_l \mathbf{c}\left(e_0\right) \cdot \mathbf{c}\left(e_l\right) \\
&= \sum_{1\le k<l\le n} \Gamma^M_{jk,l} \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) + \sum_{0<l\le n} g(\mathcal{W}(e_j), e_l) \mathbf{c}\left(e_0\right) \cdot \mathbf{c}\left(e_l\right) \\
&= \sum_{1\le k<l\le n} \Gamma^M_{jk,l} \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) + \mathbf{c}\left(e_0\right) \sum_{0<l\le n} g(\mathcal{W}(e_j), e_l) \mathbf{c}\left(e_l\right) \\
&= \sum_{1\le k<l\le n} \Gamma^M_{jk,l} \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) + \mathbf{c}\left(e_0\right) \mathbf{c}\left(\mathcal{W}(e_j)\right) \quad .
\end{aligned}$$

With $e_0 = \nu$ we get

$$\frac{1}{2}\sum_{k<l} \Gamma_{jk,l}\epsilon_k\epsilon_l \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) = \frac{1}{2}\sum_{1\le k<l\le n} \Gamma^M_{jk,l} \mathbf{c}\left(e_k\right) \cdot \mathbf{c}\left(e_l\right) + \frac{1}{2}\beta \cdot \mathbf{c}\left(\mathcal{W}(e_j)\right)$$

and thus for a section u of $\mathcal{S}(M)|_{\Sigma_t}$

$$\nabla^{\mathcal{S}(M)}_X u\Big|_{\Sigma_t} = \nabla^{\mathcal{S}(\Sigma_t)}_X u\Big|_{\Sigma_t} + \frac{1}{2}\beta \mathbf{c}\left(\mathcal{W}(X)\right) u\Big|_{\Sigma_t} \quad . \tag{6.47}$$

This enables us to decompose the Dirac operator along a fixed hypersurface:

$$\begin{aligned}
\mathcal{D}u|_{\Sigma_t} &= \sum_{j=0}^{n} \varepsilon_j \, \mathbf{c}\left(e_j\right) \nabla^{\mathcal{S}(M)}_{e_j} u\Big|_{\Sigma_t} = -\, \beta \nabla^{\mathcal{S}(M)}_{\nu} u\Big|_{\Sigma_t} + \sum_{j=1}^{n} \mathbf{c}\left(e_j\right) \nabla^{\mathcal{S}(M)}_{e_j} u\Big|_{\Sigma_t} \\
&= -\beta\left(\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} u\Big|_{\Sigma_t} + \frac{1}{2}\beta \mathbf{c}\left(\mathcal{W}(\nu)\right) u\Big|_{\Sigma_t} \right) \\
&\quad + \sum_{j=1}^{n} \mathbf{c}\left(e_j\right)\left(\nabla^{\mathcal{S}(\Sigma_t)}_{e_j} u\Big|_{\Sigma_t} + \frac{1}{2}\beta \mathbf{c}\left(\mathcal{W}(e_j)\right) u\Big|_{\Sigma_t} \right) \\
&\overset{(*)}{=} -\beta \nabla^{\mathcal{S}(\Sigma_t)}_{\nu} u\Big|_{\Sigma_t} + \sum_{j=1}^{n} \mathbf{c}\left(e_j\right) \nabla^{\mathcal{S}(\Sigma_t)}_{e_j} u\Big|_{\Sigma_t} + \frac{1}{2}\sum_{j=1}^{n} \mathbf{c}\left(e_j\right) \beta \mathbf{c}\left(\mathcal{W}(e_j)\right) u\Bigg|_{\Sigma_t} \quad .
\end{aligned}$$

In $(*)$ we used Lemma 3.2.1 (2). In order to calculate the remaining triple Clifford multiplication, we use the following calculation: as each e_i is in $T_p\Sigma_t$ for all p, we have $\mathcal{W}(e_i) \in T_p\Sigma_t$

for all $p \in \Sigma_t$ and thus

$$\begin{aligned}-2\mathbf{c}(e_j)\,\beta\mathbf{c}(\mathcal{W}(e_j)) &= 2\beta\mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j)) = 2\mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j))\,\beta \\ &= \beta\mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j)) + \mathbf{c}(\mathcal{W}(e_j))\,\mathbf{c}(e_j)\,\beta \\ &= \beta\mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j)) - \mathbf{c}(\mathcal{W}(e_j))\,\beta\mathbf{c}(e_j) \\ &= [\beta\mathbf{c}(e_j), \mathbf{c}(\mathcal{W}(e_j))] \quad ;\end{aligned}$$

we observe on the other hand that

$$\begin{aligned}-2g_t(e_j, \mathcal{W}(e_j))\beta = -2g(e_j, \mathcal{W}(e_j))\beta &= \mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j))\,\beta + \mathbf{c}(\mathcal{W}(e_j))\,\mathbf{c}(e_j)\,\beta \\ &= \beta\mathbf{c}(e_j)\,\mathbf{c}(\mathcal{W}(e_j)) - \mathbf{c}(\mathcal{W}(e_j))\,\beta\mathbf{c}(e_j) = [\beta\mathbf{c}(e_j), \mathbf{c}(\mathcal{W}(e_j))]\end{aligned}$$

and thus

$$-[\beta\mathbf{c}(e_j), \mathbf{c}(\mathcal{W}(e_j))] = 2\mathbf{c}(e_j)\,\beta\mathbf{c}(\mathcal{W}(e_j)) = 2g_t(e_j, \mathcal{W}(e_j))\beta \quad . \tag{6.48}$$

The final expression for the decomposition of the Dirac operator then becomes

$$\begin{aligned}\mathcal{D}u\big|_{\Sigma_t} &= -\beta\nabla^{\mathcal{S}(\Sigma_t)}_{\nu}u\Big|_{\Sigma_t} \mp \mathbb{i}\beta\sum_{j=1}^{n}\mathbf{c}_t(e_j)\,\nabla^{\mathcal{S}(\Sigma_t)}_{e_j}u\Big|_{\Sigma_t} + \frac{1}{2}\beta\sum_{j=1}^{n}g_t(e_j, \mathcal{W}(e_j))u\Bigg|_{\Sigma_t} \\ &= -\left(\beta\nabla^{\mathcal{S}(\Sigma_t)}_{\nu}u\Big|_{\Sigma_t} \pm \mathbb{i}\beta\sum_{j=1}^{n}\mathbf{c}_t(e_j)\,\nabla^{\mathcal{S}(\Sigma_t)}_{e_j}u\Big|_{\Sigma_t} - \frac{1}{2}\beta\operatorname{tr}_{g_t}(\mathcal{W})\,u\Big|_{\Sigma_t}\right) \quad .\end{aligned}$$

With $nH_t = \operatorname{tr}_{g_t}(\mathcal{W})$ as mean curvature of the hypersurface Σ_t and

$$\mathcal{A}_t := \begin{pmatrix} A_t & 0 \\ 0 & -A_t \end{pmatrix} \quad \text{with} \quad A_t = \sum_{j=1}^{n}\mathbf{c}_t(e_j)\,\nabla^{\mathcal{S}(\Sigma_t)}_{e_j} \quad . \tag{6.49}$$

as the *hypersurface Dirac operator* on sections of $\mathcal{S}(M)|_{\Sigma_t}$ for an odd dimensional submanifold the Dirac operator finally becomes

$$\mathcal{D}u|_{\Sigma_t} = -\,\beta\left(\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} \pm \mathbb{i}\mathcal{A}_t - \frac{n}{2}H_t\right)u\Big|_{\Sigma_t} \tag{6.50}$$

and (6.25) implies

$$D_{\pm}u|_{\Sigma_t} = -\beta\left(\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} \pm \mathbb{i}A_t - \frac{n}{2}H_t\right)u\Big|_{\Sigma_t} = -\beta\left(\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} + B_{t,\pm}\right)u\Big|_{\Sigma_t} \tag{6.51}$$

where we introduced the abbreviation $B_{t,\pm} := \pm\mathbb{i}A_t - \frac{n}{2}H_t$ which is an operator of most first order, acting tangential to the hypersurface. If we further twist the Dirac operator with a Hermitian vector bundle E (either for a Spin^c-structure, a coefficient bundle or both), we replace in (6.49) $\nabla^{\mathcal{S}(\Sigma_t)}_{e_j}$ with the twisted spin connection $\nabla^{\mathcal{S}_E(\Sigma_t)}_{e_j}$ and $\mathbf{c}_t$ with $(\mathbf{c}_t \otimes \mathbb{1}_E)$; the result defines the *twisted hypersurface Dirac operator*

$$\mathcal{A}^E_t := \begin{pmatrix} A^E_t & 0 \\ 0 & -A^E_t \end{pmatrix} \quad \text{with} \quad A^E_t := \sum_{j=1}^{n}(\mathbf{c}_t(e_j) \otimes \mathbb{1}_E)\nabla^{\mathcal{S}_E(\Sigma_t)}_{e_j} \quad . \tag{6.52}$$

Hence for a section u of $\mathcal{S}^{\pm}_E(M)|_{\Sigma_t}$ we gain

$$\mathcal{D}^E u\big|_{\Sigma_t} = -\left(\beta \otimes \mathbb{1}_E\right)\left(\nabla^{\mathcal{S}_E(\Sigma_t)}_{\nu} \pm \mathrm{i}\mathcal{A}^E_t - \frac{n}{2} H_t \mathbb{1}_{\mathcal{S}_E(\Sigma_t)}\right) u\Big|_{\Sigma_t} \tag{6.53}$$

and consequently

$$\begin{aligned} D^E_{\pm} u\big|_{\Sigma_t} &= -(\beta \otimes \mathbb{1}_E)\left(\nabla^{\mathcal{S}_E(\Sigma_t)}_{\nu} \pm \mathrm{i}A^E_t - \frac{n}{2} H_t \mathbb{1}_{\mathcal{S}_E(\Sigma_t)}\right) u\Big|_{\Sigma_t} \\ &= -(\beta \otimes \mathbb{1}_E)\left(\nabla^{\mathcal{S}_E(\Sigma_t)}_{\nu} + B^E_{t,\pm}\right) u\Big|_{\Sigma_t} \end{aligned} \tag{6.54}$$

with $B^E_{t,\pm} := \pm\mathrm{i}A^E_t - \frac{n}{2} H_t \mathbb{1}_{\mathcal{S}_E(\Sigma_t)}$.

6.3.3. Special case: M Riemannian product space

Let M be isomorphic to $\mathbb{R} \times \Sigma$ with Riemannian metric $g^{[1]} =: \check{g}$ from (3.36) which is the flipped metric of g. We define $\check{M} := (M, \check{g})$ in order to stress that we consider M as Riemannian manifold with respect to $\check{g}$. The product structure implies that the normal vector field of Σ becomes a global normal vector field on M at each Σ_t. We also denote it with ν and we set it to be ouward-pointing, i.e. $\nu = -1/N\partial_t$ such that $\check{g}(\nu,\nu) = 1$. $\mathcal{S}(\check{M})$ is the spinor bundle of $\check{M}$ which also decomposes due to the even-dimensionality of the manifold according to (6.9), i.e.

$$\mathcal{S}(\check{M}) = \mathcal{S}^+(\check{M}) \oplus \mathcal{S}^-(\check{M}) \quad .$$

The bundle admits an a priori Hermitean bundle metric $\langle \cdot \mid \cdot \rangle_{\mathcal{S}(\check{M})}$ which is pointwise defined to be

$$\langle \cdot \mid \cdot \rangle_{\mathcal{S}_p(\check{M})} : \mathcal{S}_p(\check{M}) \times \mathcal{S}_p(\check{M}) \to \mathbb{C} \quad . \tag{6.55}$$

We write $\check{\mathbf{c}}$ for the Clifford multiplication. (6.10) takes the form

$$\check{\mathbf{c}}(X) \cdot \check{\mathbf{c}}(Y) + \check{\mathbf{c}}(Y) \cdot \check{\mathbf{c}}(X) = -2\check{g}_p(X,Y)\mathbb{1}_{\mathcal{S}_p(M)} \tag{6.56}$$

for $X, Y \in T_pM$. Formula (6.12) with $s = 0$ implies that $\check{\mathbf{c}}$ is skew-adjoint with respect to (6.55):

$$\langle \check{\mathbf{c}}(X)\, u \mid v \rangle_{\mathcal{S}_p(\check{M})} = -\langle u \mid \check{\mathbf{c}}(X)\, v \rangle_{\mathcal{S}_p(\check{M})} \quad \forall\, u, v \in \mathcal{S}_p(\check{M}) \quad . \tag{6.57}$$

A consequence of both relations is that

$$\langle \check{\mathbf{c}}(X)\, u \mid \check{\mathbf{c}}(X)\, v \rangle_{\mathcal{S}_p(\check{M})} = \check{g}_p(X,X) \langle u \mid v \rangle_{\mathcal{S}_p(\check{M})} . \tag{6.58}$$

We set $\check{\beta} := \check{\mathbf{c}}(\nu)$ which becomes an isometry according to (6.36)(and (6.39) in the twisted bundle case). (6.56) implies $\check{\beta}^2 = -\mathbb{1}_{\mathcal{S}(\check{M})}$ as well as the anti-commuting with $\check{\mathbf{c}}$.

We want to restrict the spinor bundle to the hypersurfaces Σ_t and set $\mathcal{S}(\check{\Sigma}_t) := \mathcal{S}(\check{M})|_{\Sigma_t}$. Because the metrics on the hypersurfaces are the same in the Lorentzian as well as Riemannian ambient space and the normal bundle of each hypersurface is trivial, Σ_t becomes spin and we can identify $\mathcal{S}(\check{\Sigma}_t)$ with $\mathcal{S}(\Sigma_t)$ for each t as the restricted spinor bundle can be identified with the intrinsic spinor bundle on Σ_t. This holds true for the restriction of subbundles $\mathcal{S}^{\pm}(\check{M})$ which we again write as $\mathcal{S}^{\pm}(\Sigma_t) = \mathcal{S}^{\pm}(\check{M})|_{\Sigma_t}$ to stress the restricted eigenbundle: (6.9) transfers to

$$\mathcal{S}(\check{M})|_{\Sigma_t} = \mathcal{S}^+(\Sigma_t) \oplus \mathcal{S}^-(\Sigma_t) \quad .$$

A Hermitian bundle metric can be defined by restricting (6.55) for spinor fields u, v on Σ_t:

$$\langle u \,|\, v \rangle_{\mathcal{S}(\Sigma_t)} := \langle u \,|\, v \rangle_{\mathcal{S}(\check{M})} \tag{6.59}$$

this induces a L^2-inner product:

$$\langle u \,|\, v \rangle_{L^2(\mathcal{S}(\Sigma_t))} := \int_{\Sigma_t} \langle u \,|\, v \rangle_{\mathcal{S}(\Sigma_t)} \, \mathsf{dvol}_{g_t} \quad . \tag{6.60}$$

In comparison to the Lorentzian case we define the Clifford multiplication for spinors on each Σ_t analogoues, but without the $\mathbb{i}$-factor like in (6.40): let X be a vector field on Σ_t, then the we set (cf. [Gin09, Prop.1.4.1])

$$\check{\mathbf{c}}_t\,(X) := \check{\mathbf{c}}\,(X)\,\check{\beta} \quad \text{for} \quad \mathcal{S}(\Sigma_t) = \mathcal{S}^{\pm}(\check{M})|_{\Sigma_t} \quad . \tag{6.61}$$

Depending on the chirality of the restricted spinor field, the restricted Clifford multiplication is $\check{\mathbf{c}}\,(X)\,|_{\Sigma_t} = \pm\check{\mathbf{c}}_t\,(X)$. It remains skew-adjoint with respect to (6.59):

$$\begin{aligned}\langle \check{\mathbf{c}}_t\,(X)\,u \,|\, v \rangle_{\mathcal{S}(\Sigma_t)} = \langle \check{\mathbf{c}}\,(X)\,\check{\beta}u \,|\, v \rangle_{\mathcal{S}(\check{M})} &= -\langle \check{\beta}u \,|\, \check{\mathbf{c}}\,(X)\,v \rangle_{\mathcal{S}(\check{M})} \\ &= \langle u \,|\, \check{\beta}\check{\mathbf{c}}\,(X)\,v \rangle_{\mathcal{S}(\check{M})} = -\langle u \,|\, \check{\mathbf{c}}_t\,(X)\,v \rangle_{\mathcal{S}(\check{M})} \quad .\end{aligned}$$

It also inherits a Clifford algebra: for X, Y vector fields tangent to Σ_t for fixed t we obtain

$$\begin{aligned}\check{\mathbf{c}}_t\,(X)\,\check{\mathbf{c}}_t\,(Y) &= \check{\mathbf{c}}\,(X)\,\check{\beta}\check{\mathbf{c}}\,(Y)\,\check{\beta} = \check{\mathbf{c}}\,(X)\,\check{\mathbf{c}}\,(Y) \overset{(6.56)}{=} -\check{\mathbf{c}}\,(Y)\,\check{\mathbf{c}}\,(X) - 2\check{g}(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)} \\ &= -\beta^2\mathbf{c}\,(Y)\,\mathbf{c}\,(X) - 2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)} = -\check{\mathbf{c}}\,(Y)\,\check{\beta}\check{\mathbf{c}}\,(X)\,\check{\beta} - 2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)} \\ &= -\check{\mathbf{c}}_t\,(Y)\,\check{\mathbf{c}}_t\,(X) - 2g_t(X,Y)\mathbb{1}_{\mathcal{S}(\Sigma_t)}\end{aligned}$$

and thus

$$\check{\mathbf{c}}_t\,(X)\,\check{\mathbf{c}}_t\,(Y) + \check{\mathbf{c}}_t\,(Y)\,\check{\mathbf{c}}_t\,(X) = -2g_t(X,Y)\mathbb{1}_{\mathcal{S}\Sigma_t} \quad . \tag{6.62}$$

If we compare with (6.41), we observe

$$\check{\mathbf{c}}_t\,(X) = \pm\mathbf{c}_t\,(X) \tag{6.63}$$

for each t and vectors X tangent to Σ_t. Both signs are possible, but for consistence we choose the negative one.

We indicate the Riemannian Dirac operator from (6.22) with the designation $\check{\mathcal{D}}$ as well as $\check{D}_{\pm}$ from (6.25) and want to proceed as in the former subsection. Let $e_0 = \nu, e_1, ..., e_n$ be a Riemann-orthonormal tangent frame with $e_1, ..., e_n$, becoming a Riemann-orthonormal frame for Σ_t. Using (3.39) with $\epsilon = 1$ in (6.17) shows

$$\begin{aligned}\sum_{k<l} \Gamma_{jk,l}\epsilon_k\epsilon_l \check{\mathbf{c}}\,(e_k)\cdot\check{\mathbf{c}}\,(e_l) &= \sum_{1\le k<l\le n} \Gamma_{jk,l}\check{\mathbf{c}}\,(e_k)\cdot\check{\mathbf{c}}\,(e_l) + \sum_{0<l\le n} \Gamma_{j0,l}\check{\mathbf{c}}\,(e_0)\cdot\check{\mathbf{c}}\,(e_l)\\ &= \sum_{1\le k<l\le n} \Gamma^M_{jk,l}\check{\mathbf{c}}\,(e_k)\cdot\check{\mathbf{c}}\,(e_l) - \sum_{0<l\le n} \check{g}(\mathcal{W}(e_j),e_l)\check{\mathbf{c}}\,(e_0)\cdot\check{\mathbf{c}}\,(e_l)\\ &= \sum_{1\le k<l\le n} \Gamma^M_{jk,l}\check{\mathbf{c}}\,(e_k)\cdot\check{\mathbf{c}}\,(e_l) - \check{\mathbf{c}}\,(\nu)\,\check{\mathbf{c}}\,(\mathcal{W}(e_j))\end{aligned}$$

and thus

$$\nabla_X^{\mathcal{S}(\check{M})} v\Big|_{\Sigma_t} = \nabla_X^{\mathcal{S}(\Sigma_t)} v\Big|_{\Sigma_t} - \frac{1}{2}\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(X))\,v\Big|_{\Sigma_t} \tag{6.64}$$

for $v \in C^\infty(\mathcal{S}(\check{M}))$. The Riemannian Dirac operator along a fixed hypersurface is then given as follows: for a spinor field $u \in C^\infty(\mathcal{S}^\pm(\check{M}))$ we obtain

$$\begin{aligned}\check{\mathcal{D}}u\Big|_{\Sigma_t} &= \sum_{j=0}^n \check{\mathbf{c}}\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\check{M})} u\Big|_{\Sigma_t} = \check{\beta}\nabla_\nu^{\mathcal{S}(\check{M})} u\Big|_{\Sigma_t} + \sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\check{M})} u\Big|_{\Sigma_t}\\ &= \check{\beta}\left(\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(\nu))\,u\Big|_{\Sigma_t}\right)\\ &\qquad + \sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\left(\nabla_{e_j}^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\right)\\ &\overset{(*)}{=} \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} + \sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\\ &= \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \check{\beta}^2\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\\ &\overset{(6.61)}{=} \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} \pm \check{\beta}\sum_{j=1}^n \check{\mathbf{c}}_t\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\\ &\overset{(6.63)}{=} \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} \mp \check{\beta}\sum_{j=1}^n \mathbf{c}_t\,(e_j)\,\nabla_{e_j}^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\\ &\overset{(6.49)}{=} \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} \mp \check{\beta}\,A_t u|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n \check{\mathbf{c}}\,(e_j)\,\check{\beta}\check{\mathbf{c}}\,(\mathcal{W}(e_j))\,u\Big|_{\Sigma_t}\\ &\overset{(**)}{=} \check{\beta}\,\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} \mp \check{\beta}\,A_t u|_{\Sigma_t} - \frac{1}{2}\sum_{j=1}^n g_t(\mathcal{W}(e_j),e_j)\check{\beta}u\Big|_{\Sigma_t}\\ &= -\check{\beta}\left(-\nabla_\nu^{\mathcal{S}(\Sigma_t)} u\Big|_{\Sigma_t} \pm A_t u|_{\Sigma_t} + \frac{1}{2}\mathrm{tr}_{g_t}(\mathcal{W})\,u\Big|_{\Sigma_t}\right)\quad .\end{aligned}$$

In $(*)$ we used that the integral curves of ν are geodesics, and in $(**)$ we used (6.48) which remains correct after replacing the Lorentzian Clifford multiplication with the Riemannian Clifford multiplication. According to (3.29), the Weingarten map in the Riemannian and Lorentzian setting differ in a minus sign as the normal vector ν is the same vector in both settings. Thus, $\mathrm{tr}_{g_t}(\mathcal{W}) = -nH_t$ and we obtain

$$\check{\mathcal{D}}u\big|_{\Sigma_t} = -\beta\left(-\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} \pm \mathcal{A}_t - \frac{n}{2}H_t\right)u\Big|_{\Sigma_t} \tag{6.65}$$

and consequently

$$\check{D}_{\pm}u\big|_{\Sigma_t} = -\beta\left(-\nabla^{\mathcal{S}(\Sigma_t)}_{\nu} \pm A_t - \frac{n}{2}H_t\right)u\Big|_{\Sigma_t} \tag{6.66}$$

Taking a twisting bundle E into account yields

$$\check{\mathcal{D}}^E u\big|_{\Sigma_t} = -(\beta\otimes\mathbb{1}_E)\left(-\nabla^{\mathcal{S}_E(\Sigma_t)}_{\nu} \pm \mathcal{A}^E_t - \frac{n}{2}H_t\mathbb{1}_{\mathcal{S}_E(\Sigma_t)}\right)u\Big|_{\Sigma_t} \tag{6.67}$$

and

$$\check{D}^E_{\pm}u\big|_{\Sigma_t} = -(\beta\otimes\mathbb{1}_E)\left(-\nabla^{\mathcal{S}_E(\Sigma_t)}_{\nu} \pm A^E_t - \frac{n}{2}H_t\mathbb{1}_{\mathcal{S}_E(\Sigma_t)}\right)u\Big|_{\Sigma_t} \tag{6.68}$$

for a section $u \in C^\infty(\mathcal{S}^{\pm}_E(\check{M}))$.

6.3.4. Parallel transport along t-lines

We close this chapter with a technicality which allows us to relate L^2-spaces for spinor bundles $\mathcal{S}(\Sigma_t)$ with different t. For simplicity, we only consider the untwisted spinor bundles; the results will transfer to the twisted spinor bundle case. Following the argumentation in [vdD18, Sec.3.2], we recall that any parallel transport acts as an isometry: taking the Hermitian bundle metric (6.59) and γ to be the integral curve of ν, the parallel transport map $\mathcal{P}^t_0(p) : \mathcal{S}_p(\Sigma_{t_0}) \to \mathcal{S}_p(\Sigma_t)$ for $p \in \Sigma$ and any $t_0, t \in \mathbb{R}$ satisfies

$$\left\langle \mathcal{P}^t_{t_0}(p)u \,\middle|\, \mathcal{P}^t_{t_0}(p)v \right\rangle_{\mathcal{S}(\Sigma_t)} = \left\langle u \,\middle|\, v\right\rangle_{\mathcal{S}(\Sigma_{t_0})} \tag{6.69}$$

for $u, v \in \mathcal{S}(\Sigma_{t_0})$. We write Σ_0 for Σ_{t_0} for simplicity. The parallel transport extends to a linear map between $L^2(\mathcal{S}(\Sigma_{t_0}))$ and $L^2(\mathcal{S}(\Sigma_t))$ with respect to (6.43) with either (6.42) or (6.55) as Hermitian bundle metric. In order to make it an isometry with respect to the L^2-inner product, we need to take the change of the volume form into account. The *volume function* in local coordinates is given by

$$\varrho_t := \left(\frac{\det(g_t)}{\det(g_{t_0})}\right)^{1/4} \quad . \tag{6.70}$$

It satisfies $\mathsf{dvol}_{\Sigma_t} = \varrho_t^2\,\mathsf{dvol}_{\Sigma_0}$. Due to the fact that the determinants of the occuring Riemannian metrics are positive, the function becomes positive and well-defined. The map

$$\begin{aligned} \mathcal{U}(t,0): \quad & L^2(\mathcal{S}(\Sigma_0)) \to L^2(\mathcal{S}(\Sigma_t)) \\ & \psi \mapsto (\varrho_t)^{-1}\mathcal{P}(\gamma)^t_0\psi \end{aligned} \tag{6.71}$$

takes spinorial L^2-sections from the reference hypersurface Σ_0 to L^2-sections on Σ_t for any t. It is a linear isomorphism with inverse $\mathcal{P}(\gamma)_t^0 \varrho_t$ which can be used to map spinor fields on Σ_t to the reference hypersurface. It is in fact an isometry and thus a bounded operator: let $u, v \in L^2(\mathcal{S}(\Sigma_0))$; with (6.59) as bundle metric we see that

$$\begin{aligned}
\langle \mathcal{U}(t,0)u \,|\, \mathcal{U}(t,0)v \rangle_{L^2(\mathcal{S}(\Sigma_t))} &= \int_\Sigma \langle \mathcal{U}(t,0)u \,|\, \mathcal{U}(t,0)v \rangle_{\mathcal{S}(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t} \\
&= \int_\Sigma \langle \mathcal{P}^t_{t_0}(p)u \,|\, \mathcal{P}^t_{t_0}(p)v \rangle_{\mathcal{S}(\Sigma_t)} \varrho_t^{-2} \,\mathsf{dvol}_{\Sigma_t} \\
&\overset{(6.69)}{=} \int_\Sigma \langle u \,|\, v \rangle_{\mathcal{S}(\Sigma_0)} \,\mathsf{dvol}_{\Sigma_0} = \langle u \,|\, v \rangle_{L^2(\mathcal{S}(\Sigma_0))} \quad .
\end{aligned}$$

With (6.42) as bundle metric we first observe that

$$\begin{aligned}
\langle \mathcal{U}(t,0)u \,|\, \mathcal{U}(t,0)v \rangle_{L^2(\mathcal{S}(\Sigma_t))} &= \int_\Sigma \langle \mathcal{U}(t,0)u \,|\, \mathcal{U}(t,0)v \rangle_{\mathcal{S}(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t} \\
&= \int_\Sigma \langle \mathcal{P}^t_{t_0}(p)u \,|\, \mathcal{P}^t_{t_0}(p)v \rangle_{\mathcal{S}(\Sigma_t)} \varrho_t^{-2} \,\mathsf{dvol}_{\Sigma_t} \\
&\overset{(6.70)}{=} \int_\Sigma \left(\beta \mathcal{P}^t_{t_0}(p)u \,|\, \mathcal{P}^t_{t_0}(p)v \right)_{\mathcal{S}(M)} \mathsf{dvol}_{\Sigma_0} \\
&= \int_\Sigma \left(\mathcal{P}^t_{t_0} \circ \mathcal{P}^{t_0}_t \beta \mathcal{P}^t_{t_0}(p)u \,|\, \mathcal{P}^t_{t_0}(p)v \right)_{\mathcal{S}(M)} \mathsf{dvol}_{\Sigma_0} \\
&\overset{(*)}{=} \int_\Sigma \left(\mathcal{P}^{t_0}_t \beta \mathcal{P}^t_{t_0}(p)u \,|\, v \right)_{\mathcal{S}(M)} \mathsf{dvol}_{\Sigma_0} \quad .
\end{aligned}$$

In $(*)$ we used the fact that parallel transport in the Lorentzian setting is an isometry with respect to $(\,\cdot \,|\, \cdot\,)_{\mathcal{S}(M)}$, restricted to spinors on Σ_t for each $t \in \mathcal{T}(M)$. As ν and Clifford multiplication are parallel, we conclude from (6.21) and $\mathcal{P}^t_{t_0}\nu = \nu\mathcal{P}^t_{t_0}$ (parallel transport in TM) that

$$\mathcal{P}^{t_0}_t \beta \mathcal{P}^t_{t_0} = \mathcal{P}^{t_0}_t \mathbf{c}(\nu)\, \mathcal{P}^t_{t_0} = \mathcal{P}^{t_0}_t \mathcal{P}^t_{t_0} \mathbf{c}(\nu) = \beta$$

and thus

$$\langle \mathcal{U}(t,0)u \,|\, \mathcal{U}(t,0)v \rangle_{L^2(\mathcal{S}(\Sigma_t))} = \int_\Sigma (\beta u \,|\, v)_{\mathcal{S}(M)} \,\mathsf{dvol}_{\Sigma_0}(p) \overset{(6.42)}{=} \langle u \,|\, v \rangle_{L^2(\mathcal{S}(\Sigma_0))} \quad .$$

We denote the inverse map of $\mathcal{U}(t,0)$ by $\mathcal{U}(0,t)$. The global parallelity of the spinorial volume form implies in addition that the parallel transport also preserves the grading for even-dimenional manifolds M, i.e.

$$\mathcal{P}^t_{t_0} : \mathcal{S}^{\pm}(\Sigma_0) \to \mathcal{S}^{\pm}(\Sigma_t) \quad \to \quad \mathcal{U}(t,0) : L^2(\mathcal{S}^{\pm}(\Sigma_0)) \to L^2(\mathcal{S}^{\pm}(\Sigma_t)) \quad .$$

Part III.

Results

General assumptions and settings

From now on we consider the following key assumptions in order to prove our Main Theorem 1, Main Theorem 2 and Corollary 1.0.7.

With the exception of the following first two sections, M will always denote a globally hyperbolic spin manifold of even dimension $(n+1)$ (n odd). The spacelike Cauchy hypersurface $\Sigma \subset M$ is assumed to be a Riemannian Galois covering with respect to a Galois group Γ which is a discrete, freely discontinuous, freely and cocompactly acting group with infinite conjugacy class. To be more precise Σ is a normal covering with deck transformation group Γ and compact quotient Σ/Γ. Thus Σ is from now on a Γ-manifold according to Definition 5.1.4. M becomes a spatial Γ-manifold with globally hyperbolic base such that it is isometric to $\mathcal{T}(M) \times \Sigma/\Gamma$ (see subsection 5.1.3 for the concrete definition). The group does not vary with the time parameter t. In the progress, we will also specify M to be temporal compact, i.e. there are times $t_1, t_2 \in \mathbb{R}$ such that the time domain is $\mathcal{T}(M) = [t_1, t_2]$. The base of M as covering then becomes compact. The manifold M gets two disjoint boundary hypersurfaces Σ_1 and Σ_2, induced by the Cauchy hypersurface Σ at times t_1, t_2: $\Sigma_1 := \Sigma_{t_1} = \{t_1\} \times \Sigma$ and $\Sigma_2 := \{t_2\} \times \Sigma$. M can therefore be viewed as intersection of a future and past light cone: $M = \mathcal{J}^+(\Sigma_1) \cap \mathcal{J}^-(\Sigma_2)$. These boundary hypersurfaces are Γ-manifolds with respect to the same Galois group Γ.

The metric g on M is given by Theorem 3.1.2 or rather (3.7). The flipped metric to g is denoted with $\breve{g}$. The smooth one-parameter family of hypersurface metrics $\{g_t\}$ on the slices Σ_t is assumed to be Γ-invariant which implies the spatial Γ-invariance of g and thus a spatial Γ-invariant volume form.

The assumptions on M to be globally hyperbolic and spin as well as on Σ to be a spin Cauchy hypersurface is justified as long these to properties are true on the compact bases M_Γ respectively Σ_Γ according to Lemma 5.1.9 and Remarks 6.1.1 (iii). The tangent bundles $T\Sigma$ and TM are Γ-vector bundles with respect to the covering $\Sigma \to \Sigma_\Gamma$ respectively $\pi_\Gamma : M \to M_\Gamma$. As the spinor bundles over Σ or M are naturally related to the tangent bundles over Σ respectively M, the bundles $\mathcal{S}^\pm(M)$, $\mathcal{S}(M)$, $\mathcal{S}^\pm(\Sigma) = \mathcal{S}(\Sigma)$ become Γ-vector bundles (recall Remarks 6.3.1 (i) for the notation). We consider the Lorentzian (Atiyah-Singer-)Dirac operator $\mathcal{D}$ from (6.22) and its decomposition into tangential and normal operators along a slice Σ_t, investigated in subsection 6.3.2. The Γ-invariance of the hypersurface metric implies that the tangential part of $\mathcal{D}$ commutes with the left action representation $L_\gamma^{\mathcal{S}(\Sigma_t)}$ on Σ_t at each time t and $\gamma \in \Gamma$. The left action representation on M acts as $L_\gamma^{\mathcal{S}(\Sigma_t)}$ in spatial direction and as identity in temporal direction, i.e. $(L_\gamma^{\mathcal{S}(M)} u)(t,p) = \pi_\Gamma^{\mathcal{S}(\Sigma_t)} u(t, \gamma^{-1} \cdot p)$ for $u \in C^\infty(\mathcal{S}(\Sigma_t))$. Hence the Dirac operator $\mathcal{D}$ becomes Γ-invariant with respect to this induced Γ-action and can be viewed as lift of a Dirac operator $\underline{\mathcal{D}}$ on the compact base. Consequently, also the Dirac operators $D_\pm$ from (6.25) become Γ-invariant with respect to the same action and the hypersurface Dirac operator A_t

from (6.49) becomes Γ-invariant with respect to the Γ-action on each slice Σ_t with respect to the left action representation $L_\gamma^{\mathcal{S}(\Sigma_t)}$ for each $t \in \mathcal{T}(M)$, $\gamma \in \Gamma$. Similar assumptions can be made for the Riemannian Dirac operators $\check{\mathcal{D}}$ and $\check{D}_\pm$ from subsection 6.3.3, induced by the flipped metric of (3.7). Thus also $D_\pm$, A_t, $\check{\mathcal{D}}$ and $\check{D}_\pm$ can be viewed as lifts of their corresponding Dirac operator $\underline{D}_\pm$, $\underline{A}_t$, $\underline{\check{\mathcal{D}}}$ respectively $\underline{\check{D}}_\pm$.

If we twist the Dirac operator, we will consider a Hermitian line bundle L with (possibly only existing) square-root $L^{\frac{1}{2}}$ and a twisting Hermitian vector bundle E. If we assume that M_Γ is Spin^c, we can take a possibly non-existing global spinor bundle $\mathcal{S}(M_\Gamma)$ on M_Γ and lift it to a possibly non-existing global spinor bundle $\mathcal{S}(M)$ on M; if we moreover assume that the twisting bundles are Γ-vector bundles over M, we can characterise the twisted spinor bundle $\mathcal{S}_{L,E}(M) = \mathcal{S}(M) \otimes E_L$ as Γ-vector bundle where E_L stands for $L^{\frac{1}{2}} \otimes E$. The same argument carries over for $\mathcal{S}^\pm_{L,E}(M)$ as well as for the restricted bundles $\mathcal{S}^\pm_{L,E}(\Sigma_t)$.

We will focus on L^2- and Sobolev-sections of the spinor bundles $\mathcal{S}^\pm_{L,E}(\Sigma_t) = \mathcal{S}_{L,E}(\Sigma_t)$ in the Γ-setting. We observe from (5.42) and (5.51) that due to the diffeomorphy of Σ to $\Gamma \times \mathcal{F}$ with $\mathcal{F}$ as fundamental domain of the Γ-action these spaces are free Hilbert Γ-modules:

$$
\begin{aligned}
L^2_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)) &\cong \ell^2(\Gamma) \otimes L^2(\mathcal{F}, \mathcal{S}^\pm_{L,E}(\Sigma_t)|_\mathcal{F}) \cong l^2(\Gamma) \otimes L^2(\mathcal{S}^\pm_{L,E}(\Sigma_t/\Gamma)) \quad , \\
H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)) &\cong \ell^2(\Gamma) \otimes H^s(\mathcal{F}, \mathcal{S}^\pm_{L,E}(\Sigma_t)|_\mathcal{F}) \cong l^2(\Gamma) \otimes H^s(\mathcal{S}^\pm_{L,E}(\Sigma_t/\Gamma))
\end{aligned}
$$

for all $s \in \mathbb{R}_{>0}$ and $t \in \mathcal{T}(M)$; the right isomorphisms in each line come from the density of $\mathcal{F}$ in the compact base. As Γ is assumed to have infinite conjugacy class, these Hilbert Γ-modules imply that

$$
\mathscr{B}_\Gamma(H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)), H^{s-m}_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))) \cong \mathscr{N}_r(\Gamma) \otimes \mathscr{B}(H^s(\mathcal{S}^\pm_{L,E}(\Sigma_t)), H^{s-m}(\mathcal{S}^\pm_{L,E}(\Sigma_t)))
$$

are type II_∞ von Neumann algebra factors for $s \geq m$.

As the (twisted) hypersurface Dirac operator A_t is a Riemannian Dirac operator, it is an elliptic, essentially self-adjoint Γ-operator. From Proposition 5.3.3 (1) and (2) it follows that it is an essentially self-adjoint operator with closure on the domain $H^1_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ and becomes a Γ-morphism from $H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ to $H^{s-1}_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$. From Proposition 5.3.7 (1) we could also view the hypersurface Dirac operators A_t as s-regular Γ-operator for each $t \in \mathcal{T}(M)$. Properties (4), (7) and (8) of Proposition 5.3.7 show that each A_t is a Γ-morphism betwenn Γ-Sobolev spaces and that ellipticity and (essentially) self-adjointness imply that the spectral projections onto L^2_Γ-subsets with respect eigenvalues in a bounded Borel set as well as the projection onto the kernel of each A_t in $L^2_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ are Γ-trace class operators. The most relevant conclusion is that each Γ-invariant hypersurface Dirac operator is Γ-Fredholm between Γ-Sobolev spaces and the Γ-index vanishes due to their self-adjointness (Proposition 5.3.7 (8)).

7. Well-posedness of the Cauchy problem for the Dirac equation

We deduce several well-posedness results in this chapter which become important in the progress of proving the Fredholmness property for the Lorentzian Dirac operator on Galois coverings in two ways: the well-posedness of the Cauchy problem for the Dirac equation allows us to identify certain spaces as Hilbert Γ-modules, where on the other hand the well-posedness of the homogeneous Cauchy problem in particular implies the existence of a wave evolution operator which we will analyse in great detail in this and coming chapters. This evolution operator will be they key ingredient to show Fredholmness in the Γ-setting.

The first subsection is dedicated to well-posedness results of the Cauchy problem for the Dirac equation in a more general situation that the Cauchy hypersurface Σ of the globally hyperbolic spacetime M is a non-compact, but complete submanifold. Since any two Cauchy hypersurfaces of one and the same spacetime are homeomorphic to each other, the completness is then given for any Cauchy hypersurface. The wave evolution operator is introduced in the ensuing section where its nature as Fourier integral operator is worked out. The content of the last section is a specification of these two results to the case of those Σ, being a Galois covering where M then becomes a spatial Γ-manifold.

7.1. Well-posedness of the Cauchy problem for the Dirac equation

The aim of this section is to prove well-posedness of the Cauchy problem for the Dirac operator $D^{E_L} := D_+^{E_L}$ on a globally hyperbolic Lorentzian spin manifold where each member in the foliating family $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ is a non-compact, but complete Cauchy hypersurface:

$$D^{E_L}u = f \quad \text{with} \quad u|_{\Sigma_t} = g \tag{7.1}$$

where u is a suitable weak solution as section of $\mathcal{S}^+_{L,E}(M)$, f a suitable section of $\mathcal{S}^-_{L,E}(M)$ and g a section of $\mathcal{S}^+_{L,E}(\Sigma_t)$. Similar results are then worked out for the Dirac operator $\tilde{D}^{E_L} := D_-^{E_L}$, acting on spinors with negative chirality.

7.1.1. Some preparatory results

In comparison to the approach in [BTW15], on which our proof strategy relies on, several special relations turn out to be useful in computing energy estimates.

Lemma 7.1.1. *The following relations hold for a vector field $X \in \mathfrak{X}(\Sigma_t)$ and a smooth section u of $\mathcal{S}(M)$ along a hypersurface Σ_t for each $t \in \mathcal{T}(M)$:*

(1) $\nabla_X^{\mathcal{S}(\Sigma_t)}(\beta u) = \beta \nabla_X^{\mathcal{S}(\Sigma_t)} u$ *and* $A_t(\beta u) = -\beta A_t u$ *;*

if moreover each Σ_t is complete, then

(2) $\Lambda_t^s \beta = \beta \Lambda_t^s$ *for $s \in \mathbb{R}$ where* $\Lambda_t^2 = \mathbb{1} + (\nabla^{\mathcal{S}(\Sigma_t)})^* \nabla^{\mathcal{S}(\Sigma_t)}$*;*

(3)

$$\langle B_{t,\pm} v \,|\, w \rangle_{L^2(\mathcal{S}(\Sigma_t))} + \langle v \,|\, B_{t,\pm} w \rangle_{L^2(\mathcal{S}(\Sigma_t))} = -nH_t \langle v \,|\, w \rangle_{L^2(\mathcal{S}(\Sigma_t))} \tag{7.2}$$

for all $v, w \in C^\infty(\mathcal{S}^\pm(\Sigma_t))$, sharing the same chirality.

Proof.

(1) The first commutativity is a consequence of the compatibility with Clifford multiplication (6.19) and $\beta \mathbf{c}_t(X) = -\mathbf{c}_t(X)\beta$ for all t, implying the anti-commuting with A_t.

(2) Denote with $\mathcal{R}_{\Sigma_t}$ the (Ricci) scalar curvature for Σ_t and $\mathcal{A}_t := A_t \oplus (-A_t)$. The Lichnerowicz formula (6.26) for the hypersurface Dirac operator and result (a) lead to

$$\begin{aligned}(\nabla^{\mathcal{S}(\Sigma_t)})^* \nabla^{\mathcal{S}(\Sigma_t)}(\beta u) &= \mathcal{A}_t^2(\beta u) - \frac{\mathcal{R}_{\Sigma_t}}{4}\beta u \\ &= \beta \mathcal{A}_t^2 u - \beta \frac{\mathcal{R}_{\Sigma_t}}{4} u = \beta (\nabla^{\mathcal{S}(\Sigma_t)})^* \nabla^{\mathcal{S}(\Sigma_t)} u\end{aligned} \tag{7.3}$$

and thus $\Lambda_t^2(\beta u) = \beta \Lambda_t^2 u$. This holds true for any positive even power Λ_t^{2k}, $k \in \mathbb{N}_0$, after applying (7.3) k times and thus for any polynomial in Λ_t^2. As Λ_t^2 is essentially self-adjoint on $L^2(\mathcal{S}^\pm(\Sigma_t))$ by completeness of the hypersurfaces, its spectrum is positive wherefore the function $f(x) = x^{s/2}$ is continuous on the spectrum of Λ_t^2. Consequently, $\Lambda_t^s = f(\Lambda_t^2)$ is defined by the limit of any sequence of polynomials in Λ_t^2, converging uniformly on the spectrum to f. $\Lambda_t^s(\beta u) = \beta \Lambda_t^s u$ then follows for any $s \in \mathbb{R}$ because the action of β commutes with the limit and with each element of the sequence.

(3) The left-hand side of the equation in the claim gives for both chiralities

$$\pm \left(\langle \mathbb{i} A_t v \,|\, w \rangle_{\mathcal{S}(\Sigma_t)} + \langle v \,|\, \mathbb{i} A_t w \rangle_{\mathcal{S}(\Sigma_t)} \right) - nH_t \langle v \,|\, w \rangle_{\mathcal{S}(\Sigma_t)} \quad .$$

A_t is formally self-adjoint with respect to the induced inner product on Σ_t since $\partial \Sigma_t = \emptyset$ for all $t \in \mathcal{T}(M)$ by assumption and by completeness of the hypersurface; the proof works similarly as the proof of Lemma 6.2.1. One only needs to be careful with the signs since Clifford multiplication is formally skew-adjoint (see (6.44)). The action of the covariant derivative on $\mathbf{c}_t(e_j)$ has two contributions, coming with $\mathbf{c}\left(\nabla_{e_j} \nu\right)$ after applying β and $\mathbf{c}\left(\nabla_{e_j} e_j\right)$. We choose the Riemann-orthonormal frame $e_1, ..., e_n$ in such a way that the Lorentz-orthonormal frame $\nu, e_1, ..., e_n$ becomes synchronous at a point. These Clifford multiplications won't contribute and the left boundary contribution vanishes. Thus, $\mathbb{i} A_t$ is formally skew-adjoint with respect to the same inner product and the term in the round brackets vanishes.

In order to extend this to a positive definite L^2-scalar product, we now use the essential self-adjointness of the Riemannian Dirac operator A_t on L^2-spaces which is justified[29] if each hypersurface is either compact or complete. The operator $B_{t,\pm}$ is then defined as in (6.51), but one takes the extension of A_t instead. □

This can be generalised to twisted Dirac operators. Let $B^{E_L}_{t,\pm} = \pm \mathrm{i} A^{E_L}_t - n/2 H_t$ be the tangential part of $D^{E_L}_{\pm}$ along Σ_t from (6.54).

Lemma 7.1.2. *The following relations hold for a vector field X and a smooth section u of $\mathcal{S}_{L,E}(M)$ along each hypersurface Σ_t without boundary for each $t \in \mathcal{T}(M)$:*

(1) $\Lambda^s_t(\beta \otimes \mathbb{1}_{E_L}) = (\beta \otimes \mathbb{1}_{E_L})\Lambda^s_t$ for $s \in \mathbb{R}$, where $\Lambda^2_t = \mathbb{1} + (\nabla^{\mathcal{S}_{L,E}(\Sigma_t)})^* \nabla^{\mathcal{S}_{L,E}(\Sigma_t)}$, if each Σ_t is complete;

(2) $\langle B^{E_L}_{t,\pm} u \,|\, v \rangle_{L^2(\mathcal{S}_{L,E}(\Sigma_t))} + \langle u \,|\, B^{E,L}_{t,\pm} v \rangle_{L^2(\mathcal{S}_{L,E}(\Sigma_t))} = -n \langle H_t u \,|\, v \rangle_{L^2(\mathcal{S}_{L,E}(\Sigma_t))}$ for all $u, v \in L^2(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ with the same chirality.

Proof. (1) Since β only acts on the pure spinor part and commutes with $\nabla^{\mathcal{S}(\Sigma_t)}$ for any $t \in \mathcal{T}(M)$, one has $\nabla^{\mathcal{S}_{L,E}(\Sigma_t)}_X(\beta \otimes \mathbb{1}_{E_L}) = (\beta \otimes \mathbb{1}_{E_L})\nabla^{\mathcal{S}_{L,E}(\Sigma_t)}_X$ for all vector fields X, tangent to Σ_t. Because of

$$(\mathbf{c}_t(e_j) \otimes \mathbb{1}_{E_L})(\beta \otimes \mathbb{1}_{E_L}) = (\mathbf{c}_t(e_j)\,\beta \otimes \mathbb{1}_{E_L}) = (-\beta \mathbf{c}_t(e_j) \otimes \mathbb{1}_{E_L}) \quad ,$$

the hypersurface Dirac operator $A^{E_L}_t$ with respect to the induced twisted Spin^c-bundle $\mathcal{S}_{L,E}(\Sigma_t)$ anti-commutes in the first factor, i.e.

$$(-\beta \otimes \mathbb{1}_{E_L})A^{E_L}_t = A^{E_L}_t(\beta \otimes \mathbb{1}_{E_L}) \quad ,$$

such that it commutes with its square. The modified Lichnerowicz-Weitzenböck formula (6.28) for the twisted spin Dirac operator on the hypersurface is

$$(\nabla^{\mathcal{S}_{L,E}(\Sigma_t)})^* \nabla^{\mathcal{S}_{L,E}(\Sigma_t)} = \left(A^{E_L}_t\right)^2 - \frac{\mathcal{R}_{\Sigma_t}}{4} - \mathfrak{R}^{E_L}_{\Sigma_t}$$

with $\mathfrak{R}^{E_L}_{\Sigma_t}$ as curvature endomorphism of E_L from (6.27). It is left to show that $(\beta \otimes \mathbb{1}_{E_L})$ commutes with this part in the Lichnerowicz formula. Since β anti-commutes with the restricted Clifford multiplication in the first tensor factor and thus commutes with $\mathbf{c}_t(e_i)\,\mathbf{c}_t(e_j)$, one observes $\mathfrak{R}^{E_L}_{\Sigma_t}(\beta \otimes \mathbb{1}_{E_L}) = (\beta \otimes \mathbb{1}_{E_L})\mathfrak{R}^{E_L}_{\Sigma_t}$ and finally $\Lambda^2_t(\beta \otimes \mathbb{1}_{E_L}) = (\beta \otimes \mathbb{1}_{E_L})\Lambda^2_t$. The rest of the proof works as in the proof of Lemma 7.1.1 (2).

(2) The density of C^∞_{c} in L^2 allows to restrict the proof to twisted spinor fields $u, v \in C^\infty_{\mathsf{c}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ where both spinors carry the same chirality. The same arguments as in the proof of Lemma 7.1.1 (3) can be applied where the commuting of the covariant derivative $\nabla^{\mathcal{S}_{L,E}(\Sigma_t)}_{e_i}$ with the product $(\mathbf{c}_t(e_i) \otimes \mathbb{1}_{E_L})$ has been used to rewrite the difference as a divergence of a vector field. The usual arguments for showing essential self-adjointness of Dirac operators show that only the part with the mean curvature is left after integration since $\partial\Sigma_t = 0$ for all $t \in \mathcal{T}(M)$. □

[29] See Remarks 6.2.5 (iii).

To distinguish between solutions of the homogeneous and inhomogeneous Cauchy problem in the space of finite energy sections one introduces the following two subspaces.

Definition 7.1.3. For any $s \in \mathbb{R}$ the set of *finite s-energy solutions of* $D_{\pm}^{E_L}$ is defined by

$$FE^s_{\mathsf{sc}}(M, \mathcal{T}, D_{\pm}^{E_L}) := \left\{ u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) \,\middle|\, D_{\pm}^{E_L} u \in L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet))) \right\} ; \quad (7.4)$$

the set of *finite s-energy kernel solutions of* $D_{\pm}^{E_L}$ is defined as

$$FE^s_{\mathsf{sc}}\left(M, \mathcal{T}, \ker\left(D_{\pm}^{E_L}\right)\right) := FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) \cap \ker\left(D_{\pm}^{E_L}\right) . \quad (7.5)$$

The kernel solutions come with an interesting property.

Lemma 7.1.4. *If* $u \in FE^s_{\mathsf{sc}}\left(M, \mathcal{T}, \ker\left(D_{\pm}^{E_L}\right)\right)$ *and* $s > \frac{n}{2} + 2$*, then* $u \in C^1_{\mathsf{sc}}(\mathcal{S}^{\pm}_{L,E}(M))$.

Proof. With (6.54) the equations $D_{\pm}^{E_L} u = 0$ along each hypersurface take the form

$$\nabla^{\mathcal{S}_{L,E}(M)}_{\partial_t} u\Big|_{\Sigma_t} = -N\, \nabla^{\mathcal{S}_{L,E}(M)}_{\mathsf{v}} u\Big|_{\Sigma_t} = N\left(\pm \mathbb{i} A^{E_L}_t - \frac{n}{2} H_t\right) u|_{\Sigma_t} \quad .$$

A section $u \in C^0_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$ has support inside a spatially compact subset in M by definition. For $u|_{\Sigma_t} \in H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ at each $t \in \mathcal{T}(M)$ and $\mathrm{supp}\,(u) \cap \Sigma_t$ compact by the spatial compactness of the support the right-hand side consists of differential operators at most order 1 along each Σ_t and thus the restriction of $\nabla^{\mathcal{S}_{L,E}(M)}_{\partial_t} u$ to Σ_t is in $\in H^{s-1}_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ and consequently $u \in C^1_{\mathsf{sc}}(\mathcal{T}(M), H^{s-1}_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$. The claim follows by the Sobolev embedding theorem for $s - 1 > \frac{n}{2} + 1$. □

We choose inhomogeneities $f \in L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet)))$ and initial data $u|_{\Sigma_t} \in H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ for any $s \in \mathbb{R}$ and $t \in \mathcal{T}(M)$ for each Cauchy problems of $D_{\pm}^{E_L}$ in (7.1). We start with the more stronger condition for each chirality that $f \in FE^{s-1}_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^{\mp}_{L,E}(M))$ which can be weakened later on, but does not affect the main proof. For the coming energy estimates a time reversal argument is going to be used, for which reason a closer look on the time reversed Cauchy problem needs to be taken. We denote with $M|_{[t_1,t_2]}$ the manifold M with restricted time domain to $[t_1, t_2]$. The *time reversal map*

$$\begin{array}{rccc} \mathcal{T} : & M|_{[t_1,t_2]} & \to & M|_{[t_1,t_2]} \\ & (t, x) & \mapsto & ((t_2 + t_1) - t, x) \end{array} \quad (7.6)$$

is smooth and acts as involution since $\mathcal{T}^2 = \mathbb{1}_M$. This makes it a diffeomorphism on $M|_{[t_1,t_2]}$. It implies in addition that $\mathcal{T}$ is formally self-adjoint. We will quote its inverse with the same letter as it is a self-inverse map. The pullback of a spinor field with respect to this diffeomorphism is well-defined as spinor field with respect to a Clifford algebra which is defined by the pullback metric $\mathcal{T}^* g$:

$$(\mathcal{T}^* u)(t, x) = u(\mathcal{T}(t, x))$$

for a smooth spinor field u; more details, concerning the structure of this scalar like transformation behaviour, can be found in [DP86]. We use these facts in the proof of the following statement which provides us a time reversal argument.

Lemma 7.1.5. *Given a globally hyperbolic manifold M with Cauchy temporal function $\mathcal{T}$, a spinor bundle $\mathcal{S}_{L,E}(M)$, $K \subset M$ compact and $s \in \mathbb{R}$; the following are equivalent for each time interval $[t_1, t_2] \subset \mathcal{T}(M)$:*

(1) $u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M))$ *solves the forward time Cauchy problem*

$$D^{E_L}_{\pm} u = f \in FE^{s-1}_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^{\mp}_{L,E}(M)) \quad , \quad u|_t =: u_0 \in H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$$

for the Dirac equation at fixed initial time $t \in \mathcal{T}(M)$.

(2) $\mathcal{T}^* u \in FE^s_{\mathsf{sc}}(\mathcal{T}^{-1}(M), \mathcal{T}(\mathcal{T}), (\mathcal{T}^{-1})^* \mathcal{S}^{\pm}_{L,E}(M))$ *solves the backward time Cauchy problem*

$$(\mathcal{T}^* \circ D^{E_L}_{\pm} \circ \mathcal{T}^*) u = \mathcal{T}^* f \quad , \quad (\mathcal{T}^* u)|_{\mathcal{T}(t)} = u_0 \in H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$$

with $\mathcal{T}^* f \in FE^{s-1}_{\mathsf{sc}}(\mathcal{T}^{-1}(M), \mathcal{T}(\mathcal{T}), (\mathcal{T}^{-1})^* \mathcal{S}^{\mp}_{L,E}(M))$ *for the Dirac equation at fixed initial time* $\mathcal{T}(t) := (t_2 + t_1 - t) \in \mathcal{T}(M)$.

Moreover, $(\mathcal{T}^* \circ D^{E_L}_{\pm} \circ \mathcal{T}^*)$ *is the twisted Dirac operator for reversed time orientation and takes the form*

$$(\mathcal{T}^* \circ D^{E_L}_{\pm} \circ \mathcal{T}^*) v\Big|_{\Sigma_t} = -\left(\widetilde{\beta} \otimes \mathbb{1}_E\right) \left(\nabla_{\widetilde{\nu}} \pm \mathrm{i} \widetilde{A}^{E_L}_t v - \frac{n}{2} \widetilde{H}_t\right) v\Big|_{\Sigma_t}$$

where $\widetilde{\nu} = \mathcal{T}_* \nu$, $\widetilde{\beta} = \mathbf{c}(\widetilde{\nu})$, $\widetilde{H}_t$ *is the mean curvature with respect to the normal vector* $\widetilde{\nu}$ *and* $\widetilde{A}^{E_L}_t$ *the hypersurface Dirac operator, defined as in (6.52) with a Riemann-orthonormal tangent frame with respect to* $\mathcal{T}^* g_t$.

Proof. We note that the pullback with the time reversion map $\mathcal{T}$ commutes with the tensor product and $\mathcal{T}^* \mathbb{1}_{E_L} \mathcal{T}^* = \mathbb{1}_{E_L}$; we can stick to the untwisted case for the sake of legibility.

We choose any, but a fixed time interval $[t_1, t_2] \in \mathcal{T}(M)$ and define $\mathcal{T}$ as in (7.6). To keep the proof legible, we assume M to be temporal compact with $\mathcal{T}(M) = [t_1, t_2]$ such that we don't need to restrict the time domain. Both Dirac equations are formally the same if one applies a pullback by $\mathcal{T}$ to both sides and uses the self-inverse property $\mathcal{T}^2 = \mathbb{1}_M$ between $D_\pm$ and the spinor u with appropriate chirality. The claim will be proven for smooth initial data and inhomogeneities. Because finite energy sections are embedded in the set of distributional sections, the claim follows from the proof of this reduction by duality: consider the dual pairing $[\cdot|\cdot]$ of any operator, applied to a spin-valued distribution $u \in C^{-\infty}(\mathcal{S}(M))$, with a smooth compactly supported section $\phi \in C^\infty_{\mathsf{c}}(\mathcal{S}^*(M))$, i.e. a smooth compactly supported cospinor field. This is equivalent with the dual pairing of u with the formal adjoint operator, now acting on ϕ. Thus, the forward and backward time Dirac equations for distributions are

$$\begin{aligned} \left[D_\pm u \,\middle|\, \phi\right]_{\mathcal{S}(M)} &= -\left[u \,\middle|\, D_\mp \phi\right]_{\mathcal{S}(M)} \\ \text{and} \quad \left[(\mathcal{T}^* \circ D_\pm \circ \mathcal{T}^*) u \,\middle|\, \phi\right]_{\mathcal{S}(M)} &= -\left[u \,\middle|\, (\mathcal{T}^* \circ D_\mp \circ \mathcal{T}^*)\phi\right]_{\mathcal{S}(M)} \end{aligned}$$

where we have used the formal self-adjointness of $\mathcal{T}$ and Corollary 6.2.4. Since the support of u is contained in the future and past light cone, $\mathcal{T}$ only swaps these two cones wherefore the support satisfies $\mathrm{supp}\,(\mathcal{T}^* u) \subset \mathcal{J}(K)$. Suppose $u|_{\Sigma_t} \in C^\infty_{\mathsf{c}}(\mathcal{S}^{\pm}(\Sigma_t))$ and

$f \in C_c^\infty(\mathcal{S}^\mp(M))$. The statement [AB18, Thm.4] implies the existence of a unique section $u \in C^\infty(\mathcal{S}^\pm(M))$ with support $\operatorname{supp}(u) \subset \mathcal{J}(K)$ for $K \subset M$ compact, solving $D_\pm u = f$ on M with initial condition $u|_{\Sigma_t}$. $\mathcal{T}^*u$ and $\mathcal{T}^*f$ are defined and smooth; the latter is compactly supported. The initial value has to be imposed at time $\mathcal{T}(t) = (t_2 + t_1 - t_0 - t)$ if t is the time for the initial value for the foreward time Cauchy problem. Hence $(\mathcal{T}^*u)|_{\mathcal{T}(t)}$ coincides with u_0 as (7.6) is an involution. The same holds true for any initial value with Sobolev regularity since only the metric is influenced by the time reversal, but different metrics leads to equivalent Sobolev norms. $v = \mathcal{T}^*u$ for a solution u is defined and again smooth with $\operatorname{supp}(v) \subset \mathcal{J}(K)$. The formal equivalence of the forward and backward time Dirac equation implies that v solves $\mathcal{T}^*D_\pm\mathcal{T}^*v = \mathcal{T}^*f$ if and only if u solves $D_\pm u = f$. Thus, one only needs to check that $\mathcal{T}^*D_\pm\mathcal{T}^*$ along any hypersurface Σ_t are also Dirac operators, given as in (6.51). The pullback spin-structure is determined by the pullback metric such that the pullback on any Clifford multiplication $\mathbf{c}(X)$ with respect to a vector field X becomes the Clifford multiplication with respect to the pushforward $\mathcal{T}_*X$ at each point:

$$\mathcal{T}^* \circ \mathbf{c}(X) = \mathbf{c}(\mathcal{T}_*X) \quad .$$

We can apply to each component of (6.51) the pullback on the Dirac operator along any spatial hypersurface by linearity: if a Riemann-orthonormal tangent frame $\{e_j\}_{j=1}^n$ with respect to g_t for each leaf is given, $\{\mathcal{T}_*e_j\}_{j=1}^n = \{\mathcal{T}^*e_j\}_{j=1}^n$ becomes a Riemann-orthonormal tangent frame with respect to $\mathcal{T}^*g_t$ for each leaf $\Sigma_{\mathcal{T}(t)}$. Using all these ingredients shows

$$\begin{aligned}
\mathcal{T}_*\nu &= -\frac{1}{N\circ\mathcal{T}}\mathcal{T}_*\partial_t =: \widetilde{\nu} \\
\mathcal{T}^* \circ (\beta\nabla_\nu) \circ \mathcal{T}^*v &= \mathbf{c}(\mathcal{T}_*\nu)\,\mathcal{T}^*\nabla_\nu(\mathcal{T}^*v) = \mathbf{c}(\widetilde{\nu})\,\nabla_{\widetilde{\nu}}\mathcal{T}^*\mathcal{T}^*v = \widetilde{\beta}\nabla_{\widetilde{\nu}}v \\
\mathcal{T}^* \circ (\beta A_t) \circ \mathcal{T}^*v &= \widetilde{\beta}\sum_{j=1}^n \mathbf{c}(\mathcal{T}_*e_j)\,\mathcal{T}^*\nabla_{e_j}\mathcal{T}^*v = \widetilde{\beta}\sum_{j=1}^n \mathbf{c}(\mathcal{T}_*e_j)\,\nabla_{\mathcal{T}_*e_j}v = \widetilde{\beta}\widetilde{A}_t v \\
\mathcal{T}^* \circ (\beta H_t) \circ \mathcal{T}^*v &= \widetilde{\beta}\sum_{j=1}^n \mathcal{T}^*g_t\left(\widetilde{\mathcal{W}}(e_j), e_j\right) v = \widetilde{\beta}\widetilde{H}_t v \\
\Rightarrow\ (\mathcal{T}^* \circ D_\pm \circ \mathcal{T}^*)v &= -\widetilde{\beta}\left(\nabla_{\widetilde{\nu}} + \mathrm{i}\widetilde{A}_t v - \frac{n}{2}\widetilde{H}_t\right) v
\end{aligned} \tag{7.7}$$

where the tilded quantities are Clifford multiplication, the Weingarten map and the mean curvature with respect to the future poining vector $\widetilde{\nu}$, being orthonormal to each hypersurface as well. All quantities along a hypersurface have been lifted to the manifold M before (without extra notation) in order to compute the pullback.

As the whole proof does not depend on the concrete choice of the subinterval in the time domain, it holds for all time intervals in the (possible unbounded) time domain. □

The third line in (7.7) implies that $A_t^{E_L}$ is invariant under this time-reversing, so $(A_t^{E_L})^2$ and $(\mathcal{A}_t^{E_L})^2$ do as well. The Lichnerowicz formula then shows that the spinorial Laplacian fulfills

$$\mathcal{T}^* \circ (\nabla^{\mathcal{S}_{L,E}(\Sigma_t)})^*\nabla^{\mathcal{S}_{L,E}(\Sigma_t)} \circ \mathcal{T}^* = (\nabla^{\mathcal{S}_{L,E}(\Sigma_{\mathcal{T}(t)})})^*\nabla^{\mathcal{S}_{L,E}(\Sigma_{\mathcal{T}(t)})}.$$

In an analogous way as in the proof of Lemma 7.1.1 (2) one gains with $\mathcal{T}(t)$ as t under reversed time orientation

$$\Lambda^s_{\mathcal{T}(t)} = \mathcal{T}^* \Lambda^s_t \mathcal{T}^* \quad \forall\, s \in \mathbb{R}\,, t \in \mathcal{T}(M) \tag{7.8}$$

where Λ^s_t is given by (4.13) with the induced spin connection on each hypersurface Σ_t.

7.1.2. Energy estimates

Roughly speaking, one defines the energy along an initial spatial hypersurface as sum of the norms of all values which has to be fixed on this leaf. Suppose M is spatially compact, then the Cauchy hypersurface Σ is compact and one defines the *s-energy along* Σ of a sufficiently differentiable section u of a vector bundle $E \to M$ as

$$\mathcal{E}_s(u, \Sigma) := \|u|_\Sigma\|^2_{H^s(E|_\Sigma)} \quad .$$

For a non-compact hypersurface Σ one needs to apply the presented doubling procedure in order to reduce to the above considered compact case. Let (M, g) be a globally hyperbolic spacetime and $E \to M$ a Riemannian or Hermitian vector bundle. Choose a connection which is compatible with the bundle metric. Let u be again a sufficiently differentiable section of this bundle, but has compact support $\operatorname{supp}(u) \subset \mathcal{J}(K)$ with $K \Subset M$. Since the support is defined to be a closed set, one observes that it is spatially compact by this assumption. As a consequence one obtains that $\mathfrak{K} := (\operatorname{supp}(u) \cap \Sigma) \subset \Sigma$ is compact for every Cauchy hypersurface. Without changing $\mathfrak{K}$ we choose a compact subset K_1 as in the description of Sobolev spaces via the double and receive $\widetilde{\Sigma}$ as double of K_1. With this procedure we get a corresponding extended vector bundle $\widetilde{E}$ of $E|_{K_1}$ and a zero-extended section $\tilde{u}$ of $u|_\Sigma$. This allows to consider the *s-energy* along Σ of a sufficiently differentiable section u in a similar manner:

$$\mathcal{E}_s(u, \Sigma) := \|\tilde{u}\|^2_{H^s(\widetilde{\Sigma}, \widetilde{E})} \tag{7.9}$$

for any $s \in \mathbb{R}$. The following statement is the pendant of [BTW15, Thm.8] for the Dirac equation acting for spinor sections of positive chirality. The proof contains a similar argumentation, but since only one initial value is given and no constraint on the mean curvature is proposed, we had to show the equations in (2) and (3) of Lemma 7.1.1 in advance.

Proposition 7.1.6. *Let $I \subset \mathcal{T}(M)$ be a closed interval, $K \subset M$ compact and $s \in \mathbb{R}$; there exists a constant $C > 0$, depending on K and s such that*

$$\mathcal{E}_s(u, \Sigma_{t_1}) \leq \mathcal{E}_s(u, \Sigma_{t_0}) \mathrm{e}^{C(t_1 - t_0)} + \int_{t_0}^{t_1} \mathrm{e}^{C(t_1 - \tau)} \left\| D^{E_L}_\pm u|_{\Sigma_\tau} \right\|^2_{H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}(\Sigma_\tau))} \mathrm{d}\tau \tag{7.10}$$

applies for all $t_0, t_1 \in I$ with $t_0 < t_1$ and for all $u \in FE^{s+1}_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M))$ with support $\operatorname{supp}(u) \subset \mathcal{J}(K)$ and $D^{E_L}_\pm u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^\mp_{L,E}(M))$.

Proof. It is enough to prove this Proposition in detail for the untwisted case and for spinor solutions of positive chirality. The necessary changes for the twisted case and for solutions of negative chirality will be mentioned.

W.l.o.g. we assume M to be spatially compact, i.e. every leaf Σ_t is closed; otherwise, one applies the doubling procedure of each complete, non-compact hypersurfaces and starts with (7.9) - the difference in the following calculation is only of notational nature. The Dirac operator is decomposed into a tangential and normal part with respect to the hypersurface Σ_t: $D = -\beta\,(\nabla_\nu + B_t)$ with $B_t := B_{t,+}$ as in Lemma 7.1.1 (3) (Lemma 7.1.2 (2) for the twisted case). Rewriting this as the covariant derivative $\nabla_{\partial_t} = -N_t\nabla_\nu$ leads to

$$\nabla_{\partial_t} u|_{\Sigma_t} = N_t \beta D u|_{\Sigma_t} + N_t B_t u|_{\Sigma_t} \quad .$$

B_t is a differential operator of order at most 1, acting in tangential direction; the preassumption $u \in FE^{s+1}_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^+(M))$ implies $u|_{\Sigma_t} \in H^{s+1}_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_t))$ and thus $N_t B_t u|_{\Sigma_t} \in H^s_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_t))$, implying $N_t B_t u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^+(M))$. Since $Du \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^-(M))$ by preassumption the first part of the right-hand side satisfies $\beta N_t D u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^+(M))$. Thus, the covariant derivative with respect to ν along any hypersurface Σ_t is a Sobolev section in $H^s_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_t))$ and therefore $\nabla_{\partial_t} u \in C^0(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_\bullet))$, implying $u \in C^1(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_\bullet))$. This time-differentiability and the continuity of the norm shows that the map $t \mapsto \mathcal{E}_s(u, \Sigma_t)$ is differentiable. Differentiation of $\mathcal{E}_s(u, \Sigma_t)$ with respect to t and Lemma 3.2.2 together imply

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u, \Sigma_t) &= \int_{\Sigma_t} n H_t \langle \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{\mathcal{S}(\Sigma_t)} - \nu\langle \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{\mathcal{S}(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t} \\
&= n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} - \int_{\Sigma_t} \nu\langle \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{\mathcal{S}(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t}
\end{aligned}$$

where u is evaluated on the hypersurface Σ_t and one has choosen $\phi = 1$ in Lemma 3.2.2 since every hypersurface is an artificially closed submanifold. Choose the connection to be compatible with the bundle metric and one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u, \Sigma_t) &= n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} - 2\mathfrak{Re}\left\{\langle \Lambda^s_t u \,|\, \nabla_\nu \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\} \\
&= n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} - 2\mathfrak{Re}\left\{\langle \Lambda^s_t u \,|\, [\nabla_\nu, \Lambda^s_t] u\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\} \\
&\quad - 2\mathfrak{Re}\left\{\langle u \,|\, \nabla_\nu u\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} \\
&= n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} + \|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|[\nabla_\nu, \Lambda^s_t] u\|^2_{L^2(\mathcal{S}(\Sigma_t))} \\
&\quad - \|(\Lambda^s_t + [\nabla_\nu, \Lambda^s_t]) u\|^2_{L^2(\mathcal{S}(\Sigma_t))} - 2\mathfrak{Re}\left\{\langle u \,|\, \nabla_\nu u\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} \\
&\leq n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} + (1 + c_1)\,\|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} - 2\mathfrak{Re}\left\{\langle u \,|\, \nabla_\nu u\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} \\
&= c_2\,\|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + n\langle H_t \Lambda^s_t u \,|\, \Lambda^s_t u\rangle_{L^2(\mathcal{S}(\Sigma_t))} \\
&\quad + 2\mathfrak{Re}\left\{\langle u \,|\, \beta D u\rangle_{H^s(\mathcal{S}(\Sigma_t))} + \langle u \,|\, B_t u\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\}
\end{aligned}$$

where one has used polarisation identities of the real parts and a Sobolev estimate for ∇_ν as first order operator along Σ_t which has led to the first inequality. The compactness[30] of the hypersurface justifies the use of Lemma 7.1.1 (2) (or Lemma 7.1.2 (1) for the twisted case). With (7.2) (or its twisted analogue in Lemma 7.1.2 (2)) and another polarisation identity the calculation goes on as follows:

[30] This is still justified in the non-compact case because the Cauchy hypersurfaces are assumed to be complete.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u,\Sigma_t) &= c_2\,\|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + n\big\langle H_t\Lambda^s_t u \,\big|\, \Lambda^s_t u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))} + 2\mathfrak{Re}\left\{\big\langle \Lambda^s_t u \,\big|\, \beta\Lambda^s_t D u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\}\\ &\quad + 2\mathfrak{Re}\left\{\big\langle \Lambda^s_t u \,\big|\, B_t\Lambda^s_t u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))} + \big\langle \Lambda^s_t u \,\big|\, [\Lambda^s_t, B_t]u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\}\\ &\leq (c_2+2)\,\|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|\beta\Lambda^s_t D u\|^2_{L^2(\mathcal{S}(\Sigma_t))} + \|[\Lambda^s_t, B_t]u\|^2_{L^2(\mathcal{S}(\Sigma_t))}\\ &\leq c\,\|u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|Du\|^2_{H^s(\mathcal{S}(\Sigma_t))} = c\mathcal{E}_s(u,\Sigma_t) + \|Du\|^2_{H^s(\mathcal{S}(\Sigma_t))} \quad .\end{aligned}$$

Formula (6.45) has been used in the last step which carries over to the twisted case with (6.46). The remaining commutator acts as pseudo-differential operator of order $(s+1)$ which generates together with the continuous embedding of Sobolev spaces (Proposition 4.1.4 (3)) the last inequality.

A closer look on the constant c, coming from the Sobolev estimates, needs to be taken before applying Grönwall's inequality: it is known from the local theory that these constants depend on compact supersets of $\operatorname{supp}(u)$ (either for Σ_t closed or $\mathfrak{K}$ for Σ_t non-compact, depending on K via the support of u in the light cone) and compact subsets, appearing in the symbol estimation of the acting pseudo-differential operator. Further dependencies on the Sobolev regularity degree s and on finitely many derivatives on the product of the section u with the volume form prefactor are coming from partial integration (for this we take s to be in $\mathbb{N}_0$ first). By Jacobi's formula the derivatives on the volume form prefactor generate derivatives of the metric g_t under recreation of the volume form prefactor. This makes the constant c time-dependent and, since the product rule generates products of derivatives on the section and on the metric, it depends smoothly on t because g_t does. The derivatives on the section can be extracted which generates the Sobolev norms in terms of (4.15) and finally the known embedding and operator estimates in the end of the above calculation. Since on a (here possibly auxiliary) closed manifold the space H^{-s} is dual to H^s for $s \in \mathbb{N}_0$, the estimate also holds for Sobolev degrees $s \in \mathbb{Z}$. Interpolating between H^s and $H^{s\pm 1}$ for any of these degrees allows an extension to real-valued orders. In summary, the computed estimate of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u,\Sigma_t)$ is precisely

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u,\Sigma_t) \leq c(\|g_t\|_{\mathfrak{K}(K),m(s)})\mathcal{E}_s(u,\Sigma_t) + \|Du\|^2_{H^s(\mathcal{S}(\Sigma_t))}$$

with the (spatial) seminorm (4.1) and $m(s) \in \mathbb{N}_0$ such that $|s| \leq m$. Grönwall's Lemma gives

$$\mathcal{E}_s(u,\Sigma_{t_1}) \leq \mathcal{E}_s(u,\Sigma_{t_0})\mathrm{e}^{\int_{t_0}^{t_1} c(\|g_t\|_{\mathfrak{K}(K),m(s)})\,\mathrm{d}t} + \int_{t_0}^{t_1} \mathrm{e}^{\int_{\tau}^{t_1} c(\|g_t\|_{\mathfrak{K}(K),m(s)})\,\mathrm{d}t}\,\|Du|_{\Sigma_\tau}\|^2_{H^s(\mathcal{S}(\Sigma_\tau))}\,\mathrm{d}\tau\,.$$

The extreme value theorem on closed (sub-)intervals in I, applied on c, leads to the stated result where $C = C(\|g_\bullet\|_{\mathcal{J}(K),m(s)})$ is the maximum on I. Going back to non-compact hypersurfaces, the same procedure can be applied where the duality between H^s_{loc} and H^{-s}_{c} has to be used instead. This explains the norm for local Sobolev sections in the claim.

For the case u having negative chirality we observe that the only influence of the chirality appears in expressing the spinorial covariant derivative with respect to ν in terms of D_- and operators on the hypersurfaces where B_t has to be replaced by $B_{t,-}$. But since the result of Lemma 7.1.1 (2), (3) (Lemma 7.1.2 (1) and (2) for the twisted case) are independent of the chirality, the proof carries over for this case as well. □

One can conclude several consequences and technicalities for the next subsection. The following two corollaries are the equivalent results from [BTW15, Cor.10/11], specified for the Dirac equation.

Corollary 7.1.7. *Given $[t_0, t_1] \subset \mathcal{T}(M)$, $\tau \in \mathcal{T}(M)$, $K \subset M$ compact and $s \in \mathbb{R}$; there exists a $C > 0$, depending on K and s such that*

$$\mathcal{E}_s(u, \Sigma_t) \leq C \left(\mathcal{E}_s(u, \Sigma_\tau) + \left\| D_\pm^{E_L} u \right\|^2_{[t_0,t_1],\mathcal{J}(K),s} \right)$$

is valid for all $t \in [t_0, t_1]$, for all $u \in FE^{s+1}_{\mathsf{sc}}(M, \mathcal{T}, D_\pm^{E_L})$ with $\operatorname{supp}(u) \subset \mathcal{J}(K)$ and $D_\pm^{E_L} u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^\mp_{L,E}(M))$.

Proof. As in the proof of Proposition 7.1.6 we focus on the untwisted Dirac operator to keep notation simple, but take both chiralities into account:

Assume again that each leaf is closed, otherwise one extends again everything to a suitable double. Choose $\tau \in [t_0, t_1]$, otherwise take $t_0, t_1 \in \mathcal{T}(M)$ in such a way that it is true. Suppose first $t \in [\tau, t_1]$; since $u \in FE^{s+1}_{\mathsf{sc}}(M, \mathcal{T}, D_\pm) \subset FE^{s+1}_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^\pm(M))$ all preassumptions from this corollary coincides with the one from Proposition 7.1.6 such that (7.10) holds. By assumption $D_\pm u \in L^2_{\mathsf{loc},J(K)}(\mathcal{T}, H^s(\mathcal{S}^\mp(\Sigma_\bullet)))$ implies integrability of the map $\lambda \mapsto \|D_\pm u|_{\Sigma_\lambda}\|_{H^s(\mathcal{S}(\Sigma_\lambda))}$ such that

$$\begin{aligned}
\mathcal{E}_s(u, \Sigma_t) &\leq c\mathcal{E}_s(u, \Sigma_\tau) \mathrm{e}^{c(t-\tau)} + \int_\tau^t \mathrm{e}^{c(t-\lambda)} \|D_\pm u|_{\Sigma_\lambda}\|^2_{H^s(\mathcal{S}(\Sigma_\lambda))} \, \mathrm{d}\lambda \\
&\leq c\mathcal{E}_s(u, \Sigma_\tau) + \int_\tau^t \|D_\pm u|_{\Sigma_\lambda}\|^2_{H^s(\mathcal{S}(\Sigma_\lambda))} \, \mathrm{d}\lambda \\
&\leq \max\{c, 1\} \left(\mathcal{E}_s(u, \Sigma_\tau) + \int_{t_0}^{t_1} \|D_\pm u|_{\Sigma_\lambda}\|^2_{H^s(\mathcal{S}(\Sigma_\lambda))} \, \mathrm{d}\lambda \right) \\
&\leq C \left(\mathcal{E}_s(u, \Sigma_\tau) + \|D_\pm u\|^2_{[t_0,t_1],\mathcal{J}(K),s} \right) \quad .
\end{aligned}$$

We use a time-reversal argument with $\mathcal{T}$ from (7.6), applied on $[t_0, \tau]$. Lemma 7.1.5 ensures that $\mathcal{T}^* u$ solves the backward time Dirac equation with time-reversed data such that the proof of Proposition 7.1.6 can be repeated and applied for this situation:

$$\begin{aligned}
\mathcal{E}_s(\mathcal{T}^* u, \Sigma_{\mathcal{T}(t)}) &\leq c\mathcal{E}_s(\mathcal{T}^* u, \Sigma_{\mathcal{T}(\tau)}) \mathrm{e}^{c(\mathcal{T}(t) - \mathcal{T}(\tau))} \\
&\quad + \int_{\mathcal{T}(\tau)}^{\mathcal{T}(t)} \mathrm{e}^{c(\mathcal{T}(t)-\lambda)} \left\| (\widetilde{D}_\pm \mathcal{T}^* u)|_{\Sigma_\lambda} \right\|^2_{H^s(\mathcal{S}(\Sigma_\lambda))} \, \mathrm{d}\lambda \\
&\leq C \left(\mathcal{E}_s(\mathcal{T}^* u, \Sigma_{\mathcal{T}(\tau)}) + \int_{\mathcal{T}(\tau)}^{\mathcal{T}(t)} \left\| (\widetilde{D}_\pm \mathcal{T}^* u)|_{\Sigma_\lambda} \right\|^2_{H^s(\mathcal{S}(\Sigma_\lambda))} \, \mathrm{d}\lambda \right)
\end{aligned}$$

where $\widetilde{D}_\pm = \mathcal{T}^* \circ D_\pm \circ \mathcal{T}^*$. Because $\mathcal{T}$ is an involution, the s-energy and the last integral over the inhomogeneity are invariant under this time-orientation reversion according to (7.8):

$$\begin{aligned}\mathcal{E}_s(\mathcal{T}^*u, \Sigma_{\mathcal{T}(t)}) &= \int_{\Sigma_{\mathcal{T}(t)}} \langle \mathcal{T}^*(\Lambda_t^s u) \,|\, \mathcal{T}^*(\Lambda_t^s u)\rangle_{\mathcal{T}^*S(\Sigma_t)} \mathcal{T}^* \,\mathsf{dvol}_{\Sigma_t} \\ &= \int_{\Sigma_{\mathcal{T}(t)}} \mathcal{T}^* \left(\langle \Lambda_t^s u \,|\, \Lambda_t^s u\rangle_{S(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t}\right) = \int_{\Sigma_t} \langle \Lambda_t^s u \,|\, \Lambda_t^s u\rangle_{S(\Sigma_t)} \,\mathsf{dvol}_{\Sigma_t} \\ &= \mathcal{E}_s(u, \Sigma_t)\end{aligned}$$

where the pullback-description of the integral transformation law has been used. A similar integration shows the latter invariance by substitution:

$$\begin{aligned}&\int_{\mathcal{T}(\tau)}^{\mathcal{T}(t)} \left\| (\widetilde{D}_\pm \mathcal{T}^* u)|_{\Sigma_\lambda} \right\|^2_{H^s(S(\Sigma_\lambda))} \,\mathrm{d}\lambda \\ &= \int_{\mathcal{T}(\tau)}^{\mathcal{T}(t)} \int_{\Sigma_\lambda} \langle \Lambda_t^s(\widetilde{D}_\pm \mathcal{T}^* u)|_{\Sigma_\lambda} \,|\, \Lambda_t^s(\widetilde{D}_\pm \mathcal{T}^* u)|_{\Sigma_\lambda}\rangle_{S(\Sigma_\lambda)} \,\mathsf{dvol}_{\Sigma_\lambda} \,\mathrm{d}\lambda \\ &= \int_{\mathcal{T}(\tau)}^{\mathcal{T}(t)} \int_{\Sigma_\lambda} \mathcal{T}^* \langle \Lambda_t^s(D_\pm u)|_{\Sigma_{\mathcal{T}(\lambda)}} \,|\, \Lambda_t^s(D_\pm u)|_{\Sigma_{\mathcal{T}(\lambda)}}\rangle_{S(\Sigma_{\mathcal{T}(\lambda)})} \,\mathsf{dvol}_{\Sigma_{\mathcal{T}(\lambda)}} \,\mathrm{d}\lambda \\ &= \int_{\mathcal{T}([\tau,t]\times\Sigma)} \mathcal{T}^* \left(\langle \Lambda_t^s(D_\pm u)|_{\Sigma_\rho} \,|\, \Lambda_t^s(D_\pm u)|_{\Sigma_\rho}\rangle_{S(\Sigma_\rho)} \,\mathsf{dvol}_{\Sigma_\rho} \,\mathrm{d}\rho\right) \\ &= \int_\tau^t \left\| (D_\pm u)|_{\Sigma_\rho} \right\|^2_{H^s(S(\Sigma_\rho))} \,\mathrm{d}\rho \quad .\end{aligned}$$

Consequently, we get the same inequality as in the case without time reversion. □

The following conclusion can be interpreted as *uniqueness of the Cauchy problem for the Dirac equation.*

Corollary 7.1.8. *$u \in FE^s_{\mathrm{sc}}(M, \mathcal{T}, D_\pm^{E_L})$ is uniquely determined by the inhomogeneity $D_\pm^{E_L} u$ and the initial condition $u|_{\Sigma_t}$ on a hypersurface Σ_t for any $t \in \mathcal{T}(M)$.*

Proof. The argumentation carries over literally from the proof of [BTW15, Cor.11] where Corollary 10 in the same reference is replaced by Corollary 7.1.7: we suppose that there are two sections u_1 and u_2 in $FE^s_{\mathrm{sc}}(M, \mathcal{T}, D_\pm^{E_L})$ which solves the Dirac equation with the same inhomogeneity $D_\pm^{E_L} u_1 = f = D_\pm^{E_L} u_2$ and the same initial condition $u_1|_{\Sigma_t} = u_0 = u_2|_{\Sigma_t}$ for all $t \in \mathcal{T}(M)$. Since the space of finite energy spinors is a topological vector space and $D_\pm^{E_L}$ are linear, one observes that $(u_1 \pm u_2)$ is a section in $FE^s_{\mathrm{sc}}(M, \mathcal{T}, D_\pm^{E_L})$ as well. The difference $U = (u_1 - u_2)$ then solves the homogeneous Dirac equation with initial condition $U|_{\Sigma_t} = 0$ on all leaves. Using Corollary 7.1.7 with a shifted value s and fixed initial condition on Σ_τ leads to

$$\mathcal{E}_{s-1}(U, \Sigma_t) \leq C \left(\mathcal{E}_{s-1}(U, \Sigma_\tau) + \left\| D_\pm^{E_L} U \right\|^2_{[t_0,t_1],\mathcal{J}(K),s} \right) = 0$$

and thus $\mathcal{E}_{s-1}(U, \Sigma_t) = 0$ for all t. Since the s-energy is defined by a s-Sobolev norm (either for Σ_t closed or interpreted on the double), one obtains $U|_{\Sigma_t} = 0$ for any t. Thus, the solution U is identically vanishing. □

7.1.3. Well-posedness of $D_{\pm}^{E_L}$

After all preparations the well-posedness of the inhomogeneous Cauchy problem for the Dirac equation for spinor fields with positive chirality can be proven in the same way as it has been done for the wave equation in [BTW15]. The proof from this reference carries over literally to the setting of our interest where the used corollaries 10 and 11 from [BTW15] are replaced by Corollary 7.1.7 as well as by the uniqueness of solutions of the Dirac equation in Corollary 7.1.8. But for the sake of completeness and of later use we repeat the argument. We introduced the restriction operator res_t in (4.21). The map which generates the initial value problem onto any hypersurface Σ_t, associated to $D_{\pm}^{E_L}$, is

$$\begin{array}{rccc} \mathsf{res}_t \oplus D_{\pm}^{E_L} : & C^\infty(\mathcal{S}_{L,E}^{\pm}(M)) & \to & C^\infty(\mathcal{S}_{L,E}^{\pm}(\Sigma_t)) \oplus C^\infty(\mathcal{S}_{L,E}^{\mp}(M)) \\ & u & \mapsto & (u|_{\Sigma_t}, D_{\pm}^{E_L} u) \quad . \end{array} \tag{7.11}$$

Theorem 7.1.9. *For a fixed $t \in \mathcal{T}(M)$ and $s \in \mathbb{R}$ the map (7.11) extends to*

$$\mathsf{res}_t \oplus D_{\pm}^{E_L} \ : \ FE^s_{\mathsf{sc}}(M, \mathcal{T}, D_{\pm}^{E_L}) \ \to \ H^s_{\mathsf{c}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_t)) \oplus L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\mp}(\Sigma_\bullet))) \tag{7.12}$$

which is an isomorphism of topological vector spaces.

Proof. This result has been proven in [Dam21, Thm 4.7,Thm 4.13] for the untwisted Dirac operator. But since the proof is of functional-analytic nature, we can present it here in the full case with just notational modifications. One first checks the continuity of the map $\mathsf{res}_t \oplus D_{\pm}^{E_L}$, induced by the continuity of both summands: by definition $FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}_{L,E}^{\pm}(M))$ is the union of all continuous functions from $\mathcal{T}(M)$ to $H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet))$ with spatially compact support in $\mathcal{K} \subset M$. An intersection of $\mathcal{K}$ with any Cauchy hypersurface in the foliation of M is a compact subset and, since $H^s_{\mathsf{c}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet))$ is also defined as union over all compact subsets in any slice, it is enough to consider the restriction onto a fixed hypersurface Σ_t as map between $C^0_{\mathcal{K}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet)))$ and $H^s_{\mathcal{K}\cap\Sigma_t}(\mathcal{S}_{L,E}^{\pm}(\Sigma_t))$. The continuity follows immediatly from the estimate

$$\|\mathsf{res}_t u\|_{H^s(\mathcal{K}\cap\Sigma_t, \mathcal{S}_{L,E}(\Sigma_t))} \leq \max_{\tau \in \mathcal{T}(M)} \left\{ \|u|_{\Sigma_\tau}\|_{H^s(\mathcal{K}\cap\Sigma_\tau, \mathcal{S}_{L,E}(\Sigma_t))} \right\} = \|u\|_{\mathcal{T}(M), \mathcal{K}, 0, s}$$

with the norm on $C^0_{\mathcal{K}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet)))$ as defined in (4.26). The two inclusion mappings $C^0_{\mathcal{K}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet))) \hookrightarrow C^0_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet)))$ and $H^s_{\mathcal{K}\cap\Sigma_\bullet}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet)) \hookrightarrow H^s_{\mathsf{c}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_\bullet))$ are continuous and the restriction map between $FE^s_{\mathsf{sc}}(M, \mathcal{T}, D_{\pm}^{E_L})$ as subset of $FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}_{L,E}^{\pm}(M))$ and $H^s_{\mathsf{c}}(\mathcal{S}_{L,E}^{\pm}(\Sigma_t))$ for a fixed Σ_t becomes continuous. $D_{\pm}^{E_L}$ as map from $FE^s_{\mathsf{sc}}(M, \mathcal{T}, D_{\pm}^{E_L})$ onto the range $L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}^{\mp}(\Sigma_\bullet)))$ is continuous on $FE^s_{\mathsf{sc}}(M, \mathcal{T}, D_{\pm}^{E_L})$, implying the whole map to be continuous.

It is left to show that the map is bijective with continuous inverse. For this step one constructs the inverse map of (7.12). We take a $K \subset \Sigma_t$ compact for $t \in \mathcal{T}(M)$ fixed. The well-posedness of the Cauchy problem for the Dirac equation with smooth and compactly supported initial data on Σ_t (see Proposition B.1) states that for given $u_0 \in C^\infty_K(\mathcal{S}_{L,E}^{\pm}(\Sigma_t))$ and $f \in C^\infty_{\mathcal{J}(K)}(\mathcal{S}_{L,E}^{\mp}(M))$ there exists a solution $u \in C^\infty(\mathcal{S}_{L,E}^{\pm}(M))$ of the Dirac equation with inhomogeneity f and initial value $u_0 = u|_{\Sigma_t}$ which has support in $\mathcal{J}(K)$ by finite propagation speed. Since $C^\infty(\mathcal{S}_{L,E}^{\pm}(M)) \subset FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}_{L,E}^{\pm}(M))$ for all $s \in \mathbb{R}$, one can apply Corollary 7.1.7 in order to estimate the norm of $C^0_{\mathsf{sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$: let $I \subset \mathcal{T}(M)$ be a subinterval and $t \in I$ a fixed initial time, then

$$\begin{aligned} \|u\|^2_{I,\mathcal{J}(K),0,s} &= \max_{\tau\in I}\left\{\|u|_{\Sigma_\tau}\|^2_{H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}(\Sigma_t))}\right\} = \max_{\tau\in I}\left\{\mathcal{E}(u,\Sigma_\tau)\right\} \\ &\leq C\max_{\tau\in I}\left\{\|u|_{\Sigma_t}\|^2_{H^s_{\mathsf{loc}}(\mathcal{S}_{L,E}(\Sigma_t))} + \left\|D^{E_L}_{\pm}u\right\|^2_{I,\mathcal{J}(K),s}\right\} \\ &= C\left(\|u_0\|^2_{H^s(\mathcal{S}_{L,E}(\Sigma_t))} + \|f\|^2_{I,\mathcal{J}(K),s}\right) \end{aligned}$$

where the estimation constant comes from the used corollary and thus it is not depending on the smooth data of the Cauchy problem. This result implies that the continuous map $(u_0, f) \mapsto u$ from Proposition B.1 can be extended to a continuous map

$$H^s_K(\mathcal{S}^{\pm}_{L,E}(\Sigma_t)) \oplus L^2_{\mathsf{loc},\mathcal{J}(K)}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet))) \ \to\ C^0_{J(K)}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet))) \quad .$$

Since $\mathcal{J}^{\pm}(K)$ is closed for $K \subset \Sigma_t \subset M$ compact, also $\mathcal{J}(K)$ is closed and thus spatially compact. As a consequence, the continuity of the map

$$H^s_{\mathsf{c}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t)) \oplus L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet))) \ \to\ FE^s_{\mathsf{sc}}(M,\mathcal{T},\mathcal{S}^{\pm}_{L,E}(M))$$

is proven as the inclusions $H^s_K \hookrightarrow H^s_{\mathsf{c}}$, $L^2_{\mathsf{loc},\mathcal{J}(K)} \hookrightarrow L^2_{\mathsf{loc,sc}}$ and $C^0_{\mathcal{J}(K)} \hookrightarrow C^0_{\mathsf{sc}}$ are continuous. The formal inverse is the map

$$H^s_{\mathsf{c}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t)) \oplus L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet))) \ \to\ FE^s_{\mathsf{sc}}(M,\mathcal{T},D^{E_L}_{\pm}) \quad . \tag{7.13}$$

The composition (7.12) after (7.13) clearly gives the identity after fixing one hypersurface for the initial data. The converse composition starts with a solution from which one extracts the initial data and the inhomogeneity and solves again by (7.13). From Corollary 7.1.8 the solution is unique and the result coincides with the input. This implies that also this composition gives the identity and thus bijectivity of (7.12). The continuity of the inverse follows from the composition of $D^{E_L}_{\pm}$ with (7.13) which is a restriction on the second summand. Thus, the composition is continuous and because $D^{E_L}_{\pm}$ is continuous on $FE^s_{\mathsf{sc}}(M,\mathcal{T},D^{E_L}_{\pm})$ the claim follows. Summarising all results shows that (7.12) is indeed an isomorphism. □

The well-posedness of the homogeneous Cauchy problem for the Dirac equation follows immediately.

Corollary 7.1.10. *For a fixed $t \in \mathcal{T}(M)$ and $s \in \mathbb{R}$ the map*

$$\mathsf{res}_t \ :\ FE^s_{\mathsf{sc}}\left(M,\mathcal{T},\ker\left(D^{E_L}_{\pm}\right)\right) \ \to\ H^s_{\mathsf{c}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$$

is an isomorphism of topological vector spaces.

These two results lead to the following consequences.

Corollary 7.1.11. *For any $s \in \mathbb{R}$*

(1) $C^\infty_{\mathsf{sc}}(\mathcal{S}^{\pm}_{L,E}(M)) \subset FE^s_{\mathsf{sc}}(M,\mathcal{T},D^{E_L}_{\pm})$ is dense,

(2) $C^\infty_{\mathsf{sc}}(\mathcal{S}^{\pm}_{L,E}(M)) \cap \ker(D) \subset FE^s_{\mathsf{sc}}\left(M,\mathcal{T},\ker\left(D^{E_L}_{\pm}\right)\right)$ is dense.

Proof. The proof can be taken from [BTW15, Cor.15] with the only difference that Theorem 13 and Corollary 14 in [BTW15] are replaced by the well-posedness for the homogeneous and inhomogeneous Cauchy problem: Proposition B.1 says that smooth Cauchy data $u_0 \in C^\infty_K(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ and $f \in C^\infty_{\mathcal{J}(K)}(M, \mathcal{S}^\mp_{L,E}(M))$ give smooth solutions (Σ_t initial hypersurface, $K \subset \Sigma_t$ compact). Since $C^\infty_{\mathsf{c}}(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ is a dense subset in $H^s_{\mathsf{c}}(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ for all $s \in \mathbb{R}$ and the inclusion $C^\infty_{\mathsf{sc}}(M, \mathcal{S}^\pm_{L,E}(M)) \subset L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet)))$ is dense by Lemma 4.3.3 (1), the isomorphisms in either Theorem 7.1.9 or Corollary 7.1.10 map the smooth Cauchy data onto smooth data with support in the spatially compact set $\mathcal{J}(K)$ by the map (7.13). Since this map is surjective and continuous, it maps the mentioned dense subsets to another dense subset in $FE^s_{\mathsf{sc}}(M, \mathcal{T}, D^{E_L}_\pm)$ for (1) and in $FE^s_{\mathsf{sc}}\left(M, \mathcal{T}, \ker\left(D^{E_L}_\pm\right)\right)$ for (2). □

A consequence is that solutions of the Dirac equation have *finite propagation speed.*

Corollary 7.1.12. *A solution u with $u|_{\Sigma_t} = u_0$ for any leaf Σ_t, $t \in \mathcal{T}(M)$, and $D^{E_L}_\pm u = f$ satisfies* $\operatorname{supp}(u) \subset \mathcal{J}(K)$ *for a compact subset $K \subset M$ which satisfies the support relation* $\operatorname{supp}(u_0) \cup \operatorname{supp}(f) \subset K$.

Proof. Since $C^\infty_{\mathsf{sc}}(\mathcal{S}^+(M))$ is dense in $FE^s_{\mathsf{sc}}(M, \mathcal{T}, D)$, the finiteness of propagation speed from Corollary B.2 for smooth and spatially compact spinor fields can be extended to finite s-energy spinors. □

Another consequence of Corollary 7.1.11 is an optimisation of the assumed regularity in Proposition 7.1.6 as it has been done in [BTW15, Cor.17] for the wave equation.

Corollary 7.1.13. *The energy estimate (7.10) already holds for all $t_0, t_1 \in I \subset \mathcal{T}(M)$ with $t_0 < t_1$ and for all $u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, D^{E_L}_\pm)$ with support* $\operatorname{supp}(u) \subset \mathcal{J}(K)$.

Proof. Because of Corollary 7.1.11 (1) the energy estimate in Proposition 7.1.6 holds for all $u \in C^\infty_{\mathsf{sc}}(M, \mathcal{S}^\pm_{L,E}(M))$ which has support in $\mathcal{J}(K)$ with $K \subset M$ compact. By continuity of the restriction map in the proof of Theorem 7.1.9 the first term in (7.10) can be expressed with the norm in $FE^s_{\mathsf{sc}}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M)) \supset FE^s_{\mathsf{sc}}(M, \mathcal{T}, D^{E_L}_\pm)$. The second term can be estimated by a $L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet)))$ seminorm by which the estimate can be extended to $u \in FE^s_{\mathsf{sc}}(M, \mathcal{T}, D^{E_L}_\pm)$ with $\operatorname{supp}(u) \subset \mathcal{J}(K)$. □

The last conclusion from the density of smooth sections in finite energy spinors is the independence of the Cauchy temporal function $\mathcal{T}$ for those finite energy spinors which are solutions of the homogeneous Dirac equation.

Corollary 7.1.14. *Given two Cauchy temporal functions $\mathcal{T}$ and $\mathcal{T}'$ on M, then for all $s \in \mathbb{R}$*

$$FE^s_{\mathsf{sc}}\left(M, \mathcal{T}, \ker\left(D^{E_L}_\pm\right)\right) = FE^s_{\mathsf{sc}}\left(M, \mathcal{T}', \ker\left(D^{E_L}_\pm\right)\right) \quad .$$

Proof. The detailed, but involved proof can be taken from [BTW15, Cor.18] with the shown uniqueness of the Cauchy problem for the Dirac equation in Corollary 7.1.8. □

Thus, we can simplify notation to $FE^s_{\mathsf{sc}}\left(M, \ker\left(D^{E_L}_\pm\right)\right)$ to stress this independence of the temporal function.

7.2. (Dirac-)Wave evolution operators

The well-posedness of the homogeneous Cauchy problem in Corollary 7.1.10 motivates to define several evolution operators which we denote as *(Dirac-)wave evolution operators*. This section is dedicated to study several properties of these wave evolution operators and in particular their property of being a Fourier integral operator.

7.2.1. General properties

For latter reasons we will first consider the wave-evolution operators for the untwisted Dirac operators.

Definition 7.2.1. For a globally hyperbolic manifold M and $t_1, t_2 \in \mathcal{T}(M)$ the (Dirac-) wave evolution operators for positive and negative chirality are the following isomorphisms of topological vector spaces

$$\begin{aligned} Q(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &:& \quad H^s_{\mathsf{c}}(\mathcal{S}^+(\Sigma_1)) \;\to\; H^s_{\mathsf{c}}(\mathcal{S}^+(\Sigma_2)) \;, \\ \tilde{Q}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &:& \quad H^s_{\mathsf{c}}(\mathcal{S}^-(\Sigma_1)) \;\to\; H^s_{\mathsf{c}}(\mathcal{S}^-(\Sigma_2)) \;. \end{aligned}$$

Both operators close their commutative part in the following diagram where $\mathcal{S}^{\oplus 2}(\Sigma_j)$ denotes the direct sum $\mathcal{S}^+(\Sigma_j) \oplus \mathcal{S}^-(\Sigma_j)$

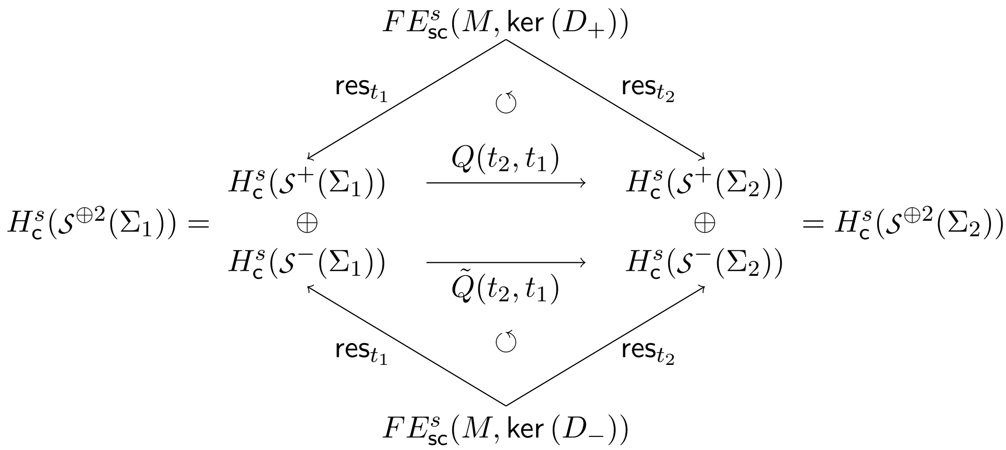

Figure 7.1.: Commuting diagrams for defining $Q(t_2,t_1)$ and $\tilde{Q}(t_2,t_1)$.

The operator $Q(t_2,t_1)$ occurs in [BS19] for compact hypersurfaces and in [BS20] for square-integrable sections on non-compact hypersurfaces. The wave evolution operators in our setting act between compactly supported Sobolev sections of any degree over non-compact, but complete hypersurfaces. We can define the evolution operators for the twisted Dirac operators analogously.

Definition 7.2.2. For a globally hyperbolic manifold M and $t_1, t_2 \in \mathcal{T}(M)$ the twisted (Dirac-)wave evolution operators for positive and negative chirality are the following isomorphisms of topological vector spaces

$$\begin{aligned} Q^{E_L}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &:& \quad H^s_{\mathsf{c}}(\mathcal{S}^+_{L,E}(\Sigma_1)) \;\to\; H^s_{\mathsf{c}}(\mathcal{S}^+_{L,E}(\Sigma_2)) \;, \\ \tilde{Q}^{E_L}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &:& \quad H^s_{\mathsf{c}}(\mathcal{S}^-_{L,E}(\Sigma_1)) \;\to\; H^s_{\mathsf{c}}(\mathcal{S}^-_{L,E}(\Sigma_2)) \;. \end{aligned}$$

The same properties of Q, as shown in [BS19], are given for all introduced operators as well.

Lemma 7.2.3. *The following properties hold for any $s \in \mathbb{R}$ and $t, t_1, t_2, t_3 \in \mathcal{T}(M)$:*

(1) $Q^{E_L}(t_3, t_2) \circ Q^{E_L}(t_2, t_1) = Q^{E_L}(t_3, t_1)$ *;*

(2) $Q^{E_L}(t, t) = \mathbb{1}_{H^s_c(\mathcal{S}^+_{L,E}(\Sigma_t))}$ *and* $Q^{E_L}(t_1, t_2) = (Q^{E_L})^{-1}(t_2, t_1)$ *;*

(3) $\tilde{Q}^{E_L}(t_3, t_2) \circ \tilde{Q}^{E_L}(t_2, t_1) = \tilde{Q}^{E_L}(t_3, t_1)$ *;*

(4) $\tilde{Q}^{E_L}(t, t) = \mathbb{1}_{H^s_c(\mathcal{S}^-_{L,E}(\Sigma_t))}$ *and* $\tilde{Q}^{E_L}(t_1, t_2) = (\tilde{Q}^{E_L})^{-1}(t_2, t_1)$ *;*

(5) For any $[t_1, t_2] \subset \mathcal{T}(M)$ *the operators* $Q^{E_L}(t_2, t_1)$ *and* $\tilde{Q}^{E_L}(t_2, t_1)$ *are unitary for* $s = 0$*.*

For the untwisted case these properties carry over to Q and $\tilde{Q}$.

Proof. To keep notation simple, we will focus on the untwisted case since the twist only modifies the complexity, but not the following arguments. (1) to (4) follow by the same reasoning as in [BS19]: because the composition of two suitable wave evolution operators is again an isomorphism of topological vector spaces, one obtains another wave evolution operator of the same kind respectively; hence (1) and (3) are clear.

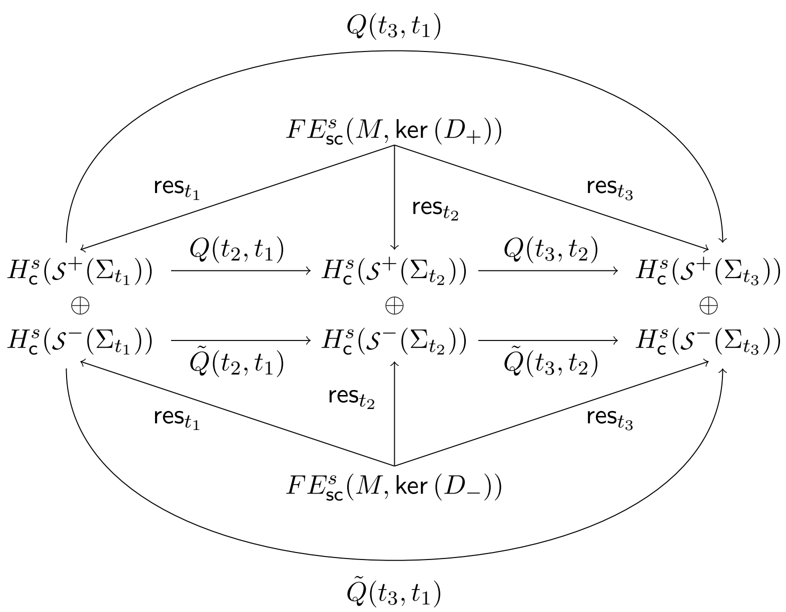

Figure 7.2.: Diagram for the proof of 7.2.3 (1)-(4) for the untwisted Dirac-wave evolution operators.

The identity properties in (2) and (4) are trivial: since Q is an isomorphism, we use (2) with $Q(t_2, t_1) \circ Q(t_1, t_2) = Q(t_2, t_2) = \mathbb{1}_{H^s_c(\mathcal{S}^+(\Sigma_2))}$ and apply the inverse from the left. The same argument transfer to the same properties of $\tilde{Q}$.

The well-posedness of the homogeneous Cauchy problem implies that any initial value $u_\pm \in H^s_{\mathsf{c}}(\mathcal{S}^\pm(\Sigma_t))$ for $t \in \mathcal{T}(M)$ is uniquely related to a finite energy spinor $\psi_\pm \in FE^s_{\mathsf{sc}}(M, \ker(D_\pm))$ for both chiralities such that $\psi_\pm|_{\Sigma_t} = u_\pm$. Lemma 7.1.4 ensures that one even has $\psi_\pm \in C^1_{\mathsf{sc}}(\mathcal{S}^\pm(M))$ for fixed $s > \frac{n}{2} + 2$. Restricting to any compact time interval $[t_1, t_2]$ implies that $\psi_\pm \in C^1_{\mathsf{c}}(\mathcal{S}^\pm(M))$ and Proposition 6.2.2 leads to the claim: for $\psi_\pm \in C^1_{\mathsf{c}}(\mathcal{S}^\pm(M)) \cap \ker(D_\pm))$ we have

$$
\begin{aligned}
0 &= \int_{M|_{[t_2,t_1]}} \left(\mathcal{D}\psi_\pm \,\middle|\, \psi_\pm\right)_{\mathcal{S}(M)} + \left(\psi_\pm \,\middle|\, \mathcal{D}\psi_\pm\right)_{\mathcal{S}(M)} \mathsf{dvol} \\
&\overset{(6.32),(6.42)}{=} \int_{\Sigma_2} \left\langle \psi_\pm|_{\Sigma_2} \,\middle|\, \psi_\pm|_{\Sigma_2} \right\rangle_{\mathcal{S}(\Sigma_2)} \mathsf{dvol}_{\Sigma_2} - \int_{\Sigma_1} \left\langle \psi_\pm|_{\Sigma_1} \,\middle|\, \psi_\pm|_{\Sigma_1} \right\rangle_{\mathcal{S}(\Sigma_1)} \mathsf{dvol}_{\Sigma_1} \quad .
\end{aligned}
$$

Using $\psi_+|_{\Sigma_2} = Q(t_2, t_1)\psi_+|_{\Sigma_1} = Q(t_2, t_1)u_+$ and $\psi_-|_{\Sigma_2} = \tilde{Q}(t_2, t_1)u_-$, we get

$$
\begin{aligned}
0 &= \int_{\Sigma_2} \left\langle Q(t_2, t_1)u_+ \,\middle|\, Q(t_2, t_1)u_+ \right\rangle_{\mathcal{S}(\Sigma_2)} \mathsf{dvol}_{\Sigma_2} - \int_{\Sigma_{t_1}} \left\langle u_+ \,\middle|\, u_+ \right\rangle_{\mathcal{S}(\Sigma_1)} \mathsf{dvol}_{\Sigma_1} \\
&= \|Q(t_2, t_1)u_+\|^2_{L^2(\mathcal{S}^+(\Sigma_2))} - \|u_+\|^2_{L^2(\mathcal{S}^+(\Sigma_1))}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \int_{\Sigma_2} \left\langle \tilde{Q}(t_2, t_1)u_- \,\middle|\, \tilde{Q}(t_2, t_1)u_- \right\rangle_{\mathcal{S}(\Sigma_2)} \mathsf{dvol}_{\Sigma_2} - \int_{\Sigma_{t_1}} \left\langle u_- \,\middle|\, u_- \right\rangle_{\mathcal{S}(\Sigma_1)} \mathsf{dvol}_{\Sigma_1} \\
&= \left\| \tilde{Q}(t_2, t_1)u_- \right\|^2_{L^2(\mathcal{S}^-(\Sigma_2))} - \|u_-\|^2_{L^2(\mathcal{S}^-(\Sigma_1))} \quad .
\end{aligned}
$$

Here we assumed w.l.o.g. that the compact supports of $\psi_\pm$ intersect both boundary hypersurfaces Σ_1 and Σ_2; otherwise one gets the trivial identifications $\|Q(t_2, t_1)u_+\|_{L^2(\mathcal{S}^+(\Sigma_2))} = 0 = \|u_+\|_{L^2(\mathcal{S}^+(\Sigma_1))}$ and $\|\tilde{Q}(t_2, t_1)u_-\|_{L^2(\mathcal{S}^-(\Sigma_2))} = 0 = \|u_-\|_{L^2(\mathcal{S}^-(\Sigma_1))}$. Hence (5) has been shown. □

Remark 7.2.4. *The map $t \mapsto \mathsf{res}_t = \mathsf{res}_{\Sigma_t}$ is continuous for all $t \in \mathcal{T}(M)$. This implies the continuity of the map $t \mapsto Q(t, \cdot)$. Corollary 7.1.10 states that the solution depends continuously on the initial data and is continuous in t due to the definition of finite energy spinors. Hence $t \mapsto \mathsf{res}_t^{-1}$ is continuous for all $t \in \mathcal{T}(M)$ such that $t \mapsto Q(\cdot, t)$ is continuous, too. Thus, the map*

$$(t, s) \mapsto Q(t, s)$$

is continuous for all $s, t \in \mathcal{T}(M)$. The same reasoning transfers to the other introduced Dirac-wave evolution operators.

7.2.2. Evolution operators as FIO

The proof of the following result is the main task of this subsection and generalises a part of [BS19, Lem.2.6].

Theorem 7.2.5. *For all $s \in \mathbb{R}$ the operators Q, $\tilde{Q}$, Q^{E_L} and $\tilde{Q}^{E_L}$ satisfy*

$$Q(t_2,t_1) \in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathsf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2)))$$
$$\tilde{Q}(t_2,t_1) \in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathsf{Hom}(\mathcal{S}^-(\Sigma_1),\mathcal{S}^-(\Sigma_2)))$$
$$Q^{E_L}(t_2,t_1) \in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathsf{Hom}(\mathcal{S}^+_{L,E}(\Sigma_1),\mathcal{S}^+_{L,E}(\Sigma_2)))$$
$$\tilde{Q}^{E_L}(t_2,t_1) \in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathsf{Hom}(\mathcal{S}^-_{L,E}(\Sigma_1),\mathcal{S}^-_{L,E}(\Sigma_2)))$$

for any fixed time interval $[t_1,t_2] \subset \mathcal{T}(M)$ and with canonical graphs

$$\begin{aligned} \mathsf{C}_{1\to 2} &= \mathsf{C}(i^*) \circ \mathcal{C} = \mathsf{C}_{1\to 2|+} \sqcup \mathsf{C}_{1\to 2|-} \quad \textit{where} \\ \mathsf{C}_{1\to 2|\pm} &= \left\{ ((x_\pm,\xi_\pm),(y,\eta)) \in \dot{T}^*\Sigma_2 \times \dot{T}^*\Sigma_1 \,\middle|\, (x_\pm,\xi_\pm) \sim (y,\eta) \right\} \end{aligned} \tag{7.14}$$

with respect to the lightlike (co-)geodesic flow as canonical relation; their principal symbols are

$$\begin{aligned} \boldsymbol{\sigma}_0(Q^{E_L})(x,\xi_\pm;y,\eta) &= \boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta) \otimes \left[\mathbb{1}_{E_L|\Sigma_2} \circ \mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \circ \mathbb{1}_{E_L|\Sigma_1} \right] \\ \textit{with} \qquad \boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta) &= \pm\frac{1}{2}\,\|\eta\|^{-1}_{g_{t_1}(y)} \left(\mp \|\xi_\pm\|_{g_{t_2}(x)}\,\beta + \mathbf{c}_{t_2}\left(\xi^\sharp_\pm\right)\right) \\ &\quad \circ \mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \circ \beta \end{aligned} \tag{7.15}$$

and

$$\begin{aligned} \boldsymbol{\sigma}_0(\tilde{Q}^{E_L})(x,\xi_\pm;y,\eta) &= \boldsymbol{\sigma}_0(\tilde{Q})(x,\xi_\pm;y,\eta) \otimes \left[\mathbb{1}_{E_L|\Sigma_2} \circ \mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \circ \mathbb{1}_{E_L|\Sigma_1} \right] \\ \textit{with} \qquad \boldsymbol{\sigma}_0(\tilde{Q})(x,\xi_\pm;y,\eta) &= \pm\frac{1}{2}\,\|\eta\|^{-1}_{g_{t_1}(y)} \left(\mp \|\xi_\pm\|_{g_{t_2}(x)}\,\beta - \mathbf{c}_{t_2}\left(\xi^\sharp_\pm\right)\right) \\ &\quad \circ \mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \circ \beta \end{aligned} \tag{7.16}$$

where $(y,\zeta_\pm) \in T^*_{\Sigma_1}M$ and $(x,\varsigma_\pm) \in T^*_{\Sigma_2}M$ restrict to (y,η) and $(x,\xi_\pm)$ respectively.

$\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}$ and $\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}$ denote the parallel transport from $(y,\zeta_\pm)$ to $(x,\varsigma_\pm)$ with respect to the spinorial and respectively twisting bundle covariant derivative, $\|\bullet\|_{g_t}$ a norm for covectors, induced by the dual metric of g_t for fixed t, and $T^*_{\Sigma_t}M := T^*_pM$ for any $p \in \Sigma_t$.

Proof. W.l.o.g. we assume that M is temporal compact with $\mathcal{T}(M) = [t_1,t_2]$; otherwise we have to replace M with the temporal restriction $M|_{[t_1,t_2]}$. In order to describe Q^{E_L} and $\tilde{Q}^{E_L}$ as FIOs, one uses the fact that $(\mathcal{D}^{E_L})^2$ and thus $D^{E_L}_\mp D^{E_L}_\pm$ are normally hyperbolic. Theorem C.2 assures that the homogeneous Cauchy problems

$$\begin{aligned} D^{E_L}_+ D^{E_L}_- v = 0\ , &\quad \mathsf{res}_{\Sigma_1} v = g_0^-\ , \quad \mathsf{res}_{\Sigma_1}(-\nabla^{\mathcal{S}_{L,E}(M)}_{\mathsf{v}})v = g_1^- \\ D^{E_L}_- D^{E_L}_+ w = 0\ , &\quad \mathsf{res}_{\Sigma_1} w = g_0^+\ , \quad \mathsf{res}_{\Sigma_1}(-\nabla^{\mathcal{S}_{L,E}(M)}_{\mathsf{v}})w = g_1^+ \end{aligned} \tag{7.17}$$

have unique solutions v and w as spinor field with negative and respectively positive chirality where Σ_1 is chosen to be the initial hypersurface with initial values $g_0^\pm$ and $g_1^\pm$ as spinor fields on this hypersurface. The solutions can be expressed with the solution operators from Theorem C.2:

$$v = \mathcal{G}_0^-(t_1)g_0^- + \mathcal{G}_1^-(t_1)g_1^- \quad \text{and} \quad w = \mathcal{G}_0^+(t_1)g_0^+ + \mathcal{G}_1^+(t_1)g_1^+$$

where $\mathcal{G}_j^\pm(t_1) \in \mathcal{FIO}^{-j-1/4}(\Sigma, M; \mathcal{C}'; \mathsf{Hom}(\mathcal{S}^\pm_{L,E}(\Sigma_1), \mathcal{S}^\pm_{L,E}(M)))$ with canonical relation $\mathcal{C}$ from (C.6). Consider the spinor fields $u_+ = D_-^{E_L} v$ and $u_- = D_+^{E_L} w$ which satisfy $D_+^{E_L} u_+ = 0 = D_-^{E_L} u_-$ by construction. The boundary conditions for v and w imply boundary conditions for $u_\pm$:

$$\begin{aligned} u_+|_{\Sigma_1} &= D_-^{E_L} v|_{\Sigma_1} \overset{(6.54)}{=} -(\beta \otimes \mathbb{1}_{E_L}) \left[\nabla_\nu^{\mathcal{S}(M)} v\Big|_{\Sigma_1} + B^{E_L}_{t_1,-} v\Big|_{\Sigma_1} \right] \\ &\overset{(7.17)}{=} (\beta \otimes \mathbb{1}_{E_L})|_{\Sigma_1} g_1^- - (\beta \otimes \mathbb{1}_{E_L})\, B^{E_L}_{t_1,-} v\Big|_{\Sigma_1} \end{aligned}$$

and

$$\begin{aligned} u_-|_{\Sigma_1} &= D_+^{E_L} w|_{\Sigma_1} \overset{(6.54)}{=} -(\beta \otimes \mathbb{1}_{E_L}) \left[\nabla_\nu^{\mathcal{S}(M)} w\Big|_{\Sigma_1} + B^{E_L}_{t_1,+} w\Big|_{\Sigma_1} \right] \\ &\overset{(7.17)}{=} (\beta \otimes \mathbb{1}_{E_L})|_{\Sigma_1} g_1^+ - (\beta \otimes \mathbb{1}_{E_L})\, B^{E_L}_{t_1,+} w\Big|_{\Sigma_1} . \end{aligned}$$

The spinorial covariant derivatives $\nabla_\nu^{\mathcal{S}_{L,E}(M)}$ and $\nabla_\nu^{\mathcal{S}_{L,E}(\Sigma_1)}$ differ in a term which contains a Clifford multiplication with respect to $\nabla_\nu \nu$. This vanishes as the t-lines are geodesics. Because the initial conditions for the Cauchy problem of the normally hyperbolic equations are free to choose, it is suitable to take $g_0^\pm = 0$ and $g_1^\pm$ as any initial spinor fields on the initial hypersurface with positive or negative chirality. Both $B^{E_L}_{t_1,\pm}$ act tangential to the hypersurface Σ_1 and due to the chosen boundary conditions both v and w vanish along Σ_1. This implies that the terms with $B^{E_L}_{t_1,-} v\Big|_{\Sigma_1}$ and $B^{E_L}_{t_1,+} w\Big|_{\Sigma_1}$ vanish as well. We obtain $u_\pm|_{\Sigma_1} = (\beta \otimes \mathbb{1}_{E_L})|_{\Sigma_1} g_1^\pm$. The spinor fields $u_\pm$ then solve the homogeneous initial value problems

$$D_\pm^{E_L} u_\pm = 0 \,, \quad \mathsf{res}_{\Sigma_1} u_\pm = (\beta \otimes \mathbb{1}_{E_L})|_{\Sigma_1} g_1^\pm = (\beta \otimes \mathbb{1}_{E_L|_{\Sigma_1}}) g_1^\pm \quad .$$

The solution operator is then formally given by $D_\mp^{E_L} \circ \mathcal{G}^\mp(t_1)$ where we write $\mathcal{G}^\pm(t_1)$ for $\mathcal{G}_1^\pm(t_1)$. The Clifford multiplication in the initial condition changes the chirality and has to be added to the solution operators in order to get rid of the Clifford multiplication in the initial condition: $D_\mp^{E_L} \circ \mathcal{G}^\mp(t_1) \circ (\beta \otimes \mathbb{1}_{E_L|_{\Sigma_1}})$. Choosing compactly supported initial values $g_1^\pm \in C_c^\infty(\mathcal{S}^\mp_{L,E}(\Sigma_1)$, one can represent the Dirac wave evolution operators $Q^{E_L}(t_2, t_1)$ and $\tilde{Q}^{E_L}(t_2, t_1)$ by lifting each spinor field $u_\pm|_{\Sigma_1}$ on the initial hypersurface to a solution on M via $\mathcal{G}^\pm(t_1) \circ (\beta \otimes \mathbb{1}_{E_L|_{\Sigma_1}})$, apply $D_\mp^{E_L}$ and then restrict to Σ_2, i.e.

$$Q^{E_L}(t_2, t_1) = \mathsf{res}_{\Sigma_2} \circ D_-^{E_L} \circ \mathcal{G}^-(t_1) \circ (\beta \otimes \mathbb{1}_{E_L|_{\Sigma_1}}) \tag{7.18}$$

and

$$\tilde{Q}^{E_L}(t_2, t_1) = \mathsf{res}_{\Sigma_2} \circ D_+^{E_L} \circ \mathcal{G}^+(t_1) \circ (\beta \otimes \mathbb{1}_{E_L|_{\Sigma_1}}) \quad . \tag{7.19}$$

In order to show that (7.18) and (7.19) are well defined on non-compact manifolds, the compositions of the two canonical relations $\mathsf{C}(i^*)$ from (4.24) with $\mathcal{C}$ and of the operators are well defined:

(1) res_{Σ_2} is properly supported;

(2) $\mathcal{G}^{\pm}(t_1)$ is properly supported;

(3) $D_{\pm}^{E_L} \circ \mathcal{G}^{\pm}(t_1) \in \mathcal{FIO}_{\mathsf{prop}}^{-1/4}(M; \mathcal{C}'; \mathsf{Hom}(\mathcal{S}_{L,E}^{\pm}(\Sigma_1), \mathcal{S}_{L,E}^{\pm}(M)))$;

(4) $\mathsf{C}(i^*) \circ \mathcal{C}$ transversal and proper.

(1) follows from Corollary 4.2.7.

(2) We show that the projections $\pi_M : M \times \Sigma_1 \to M$ and $\pi_{\Sigma_1} : M \times \Sigma_1 \to \Sigma_1$ are proper maps on the support of (the Schwartz kernels of) $\mathcal{G}^{\pm}$. The support is according to Theorem C.2 given by

$$S := \left\{ (p, x) \in M \times \Sigma_1 \,|\, x \in \mathcal{J}^-(p) \cap \Sigma_1 \right\} \quad ,$$

i.e. for every point p in M only those points on the initial hypersurface Σ_1 contribute which are lying inside its causal past $\mathcal{J}^-(p) = \mathcal{J}^-(\{p\})$. Since $\mathcal{J}^-(\{p\})$ is spatially compact and Σ_1 a Cauchy hypersurface, $\mathcal{J}^-(\{p\}) \cap \Sigma_1$ is compact and so only points from this subset contribute to the solution at p. Since M is temporally compact and Σ_1 closed, M and the product $M \times \Sigma_1$ are closed as well such that S becomes closed, too. Let K be a compact subset in M; its preimage under the first projection $(\pi_M)^{-1}(K) = K \times \Sigma_1$ is closed. The intersection $S \cap (\pi_M)^{-1}(K)$ contains only points in $K \times (\mathcal{J}^-(K) \cap \Sigma_1)$:

$$S \cap (\pi_M)^{-1}(K) \subset K \times (\mathcal{J}^-(K) \cap \Sigma_1) \quad . \tag{7.20}$$

As $\mathcal{J}^-(K) \subset \mathcal{J}(K)$ is again spatially compact, it implies $\mathcal{J}^-(K) \cap \Sigma_1$ to be compact. As $S \cap (\pi_M)^{-1}(K)$ is closed and contained in a compact set, it is compact as well, showing that π_M is proper on S.

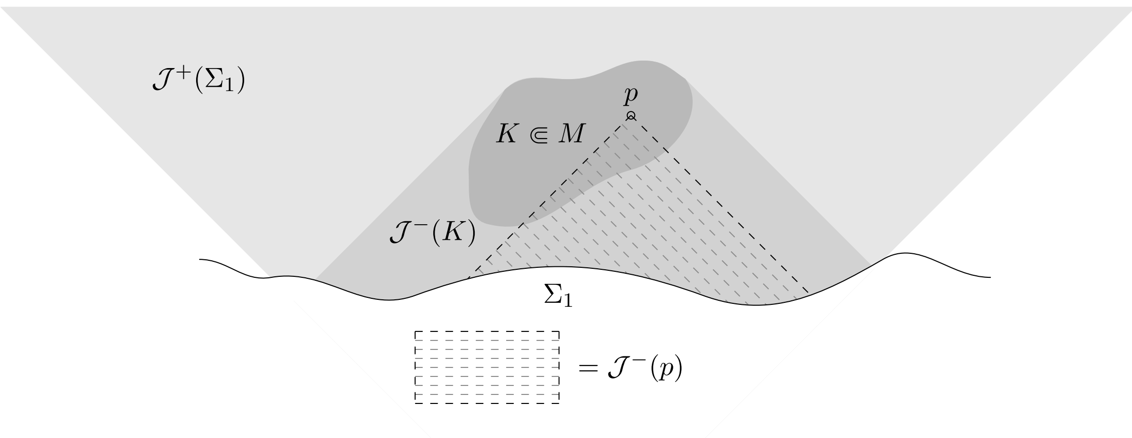

Figure 7.3.: Illustration for the intersection (7.20).

Suppose that K is now a compact subset in Σ_1. The preimage of K under π_{Σ_1} is $(\pi_{\Sigma_1})^{-1}(K) = M \times K$. The intersection is

$$\begin{aligned} S \cap (\pi_{\Sigma_1})^{-1}(K) = S \cap (M \times K) = \big\{(p,x) \in M \times K \,|\, x \in \mathcal{J}^-(p) \cap K\big\} \\ \subset (\mathcal{J}^+(K) \cap M) \times K \quad . \end{aligned} \tag{7.21}$$

$(\pi_{\Sigma_1})^{-1}(K) = M \times K$ is closed as M is closed and K a compact subset in a Hausdorff space. Thus, K and finally the intersection are closed, too. Temporal compactness of M implies that $\mathcal{J}^+(K) \cap M$ is temporal and spatial compact and thus fully compact. The intersection consequently becomes a closed set inside a compact set for which reason it becomes compact, showing the properness of π_{Σ_1} on S and finally properly supportness of $\mathcal{G}^\pm$.

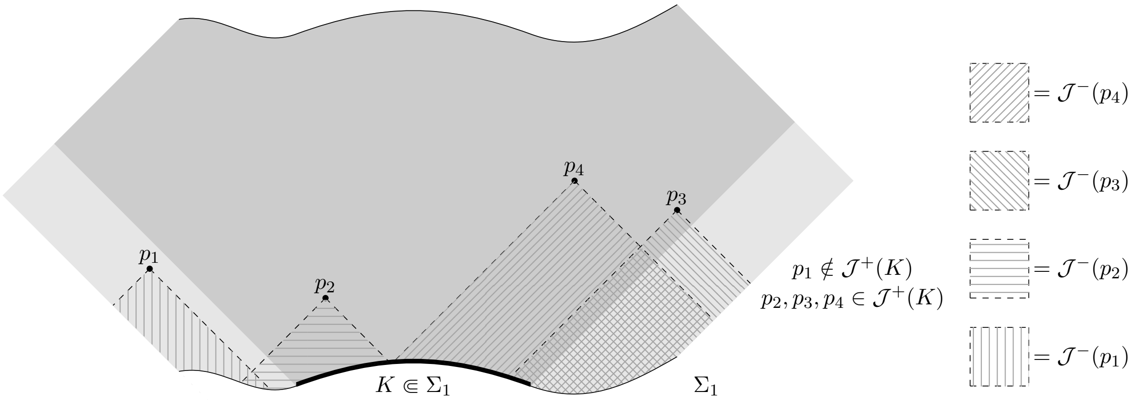

Figure 7.4.: Illustration for the intersection (7.21).

Hence the first compositions from the right in (7.18) and (7.19) is well-defined and a properly supported Fourier integral operator of order 0 with the canonical relation $\mathcal{C}$ from $\mathcal{G}^\pm(t_1)$ because $(\beta \otimes \mathbb{1}_{E_L}|_{\Sigma_1})$ is a properly supported operator of order zero with canonical relation $N^*\mathrm{diag}(\Sigma_1)$.

(3) Since $D^{E_L}_\pm \in \mathsf{Diff}^1(M, \mathsf{Hom}(\mathcal{S}^\pm_{L,E}(M), \mathcal{S}^\mp_{L,E}(M)))$ is properly supported, one can take the composition with $\mathcal{G}^\pm(t_1) \in \mathcal{FIO}^{-5/4}_{\mathsf{prop}}(\Sigma, M; \mathcal{C}'; \mathsf{Hom}(\mathcal{S}^\pm_{L,E}(\Sigma_1), \mathcal{S}^\mp_{L,E}(M)))$: since differential operators on M can be interpreted as FIO from M to M with the conormal bundle of the diagonal in M (see (3) in Lemma 4.2.3), the composition $N^*\mathrm{diag}(M) \circ \mathcal{C}$ is proper and transversal and results in $\mathcal{C}$. Since both operators are properly supported, part (2) from Lemma 4.2.3 implies that $(D^{E_L}_\mp \circ \mathcal{G}^\mp(t_1)) \in \mathcal{FIO}^{-1/4}_{\mathsf{prop}}(\Sigma, M; \mathcal{C}'; \mathsf{Hom}(\mathcal{S}^\mp_{L,E}(\Sigma_1), \mathcal{S}^\pm_{L,E}(M)))$.

(4) The construction of the solution operators as FIO has been already done in such a way that the canonical relation $\mathsf{C}(i^*) \circ \mathcal{C}$ as composition is transversal and proper. We refer to the explainations in Appendix C and in particular [Dui10, Chap.5] for details. The Dirac operators $D^{E_L}_\pm$ do not affect the argument since its canonical relation corresponds to the conormal bundle of the diagonal as explained in (3).

Lemma 4.2.3 (2) implies that the compositions (7.18) and (7.19) are indeed well-defined properly supported Fourier integral operators of order 0. Since H^s_c is the completion of

C^∞_c with respect to the Sobolev norm on each fixed compact support, these representations extend to initial values in $H^s_\mathsf{c}(\mathcal{S}^\mp_{L,E}(\Sigma_1)$ and maps to $H^s_\mathsf{c}(\mathcal{S}^\mp_{L,E}(\Sigma_2)$ by properly supportness (see Lemma 4.2.4 (3)).

The composition $\mathsf{C}(i^*) \circ \mathcal{C}$ of the canonical relations and its interpretation are given in accordance to [BS19]. For an element $(y, \eta) \in \dot{T}^*\Sigma_1$ there are two lightlike covectors $\zeta_\pm \in \dot{T}^*_y M$ such that $\mathsf{res}_{\Sigma_1}\zeta_\pm = \eta$ according to (C.7). Assigning ζ_+ to be future- and ζ_- to be past-directed, the geodesic flows along each lightlike initial covector intersect Σ_2 in at most one point in $\dot{T}^*_{\Sigma_2} M$ for each direction due to transversality:

$$(x_\pm, \xi_\pm) \sim (y, \zeta_\pm)$$

where the notation is explained after the definiton of (C.6). The pullback via restriction to Σ_2 then gives two points in $\dot{T}^*\Sigma_2$ which are also denoted as $(x_\pm, \xi_\pm)$. Summing up, the composition $\mathsf{C}(i^*) \circ \mathcal{C}_\pm$ for each connected component $\mathcal{C}_\pm$ of $\mathcal{C}$ belongs to those pairs $((x_\pm, \xi_\pm), (y, \eta)) \in \dot{T}^*\Sigma_2 \times \dot{T}^*\Sigma_1$ which are related by lightlike geodesic flows in future and past direction. Because of this two connected components we also write

$$\begin{aligned} &\mathsf{C}_{1\to 2} = \mathsf{C}_{1\to 2|+} \sqcup \mathsf{C}_{1\to 2|-} \quad \text{where} \\ &\mathsf{C}_{1\to 2|\pm} = \mathsf{C}(i^*) \circ \mathcal{C}_\pm = \left\{ ((x_\pm, \xi_\pm), (y, \eta)) \in \dot{T}^*\Sigma_2 \times \dot{T}^*\Sigma_1 \,\middle|\, (x_\pm, \xi_\pm) \sim (y, \eta) \right\} \quad . \end{aligned}$$

The canonical relation is a union of canonical graphs for each component: the geodesic flow on TM can be interpreted as Hamiltonian/cogeodesic flow on T^*M since the Hamiltonian for (co-)geodesics is given by half of the principal symbol of normally hyperbolic operators. Hamiltonian flows preserve the Hamiltonian equations by which they are canonical transformations. Each of the point $(x_\pm, \xi_\pm)$ is then related to (y, η) by these canonical transformations $\Phi_\pm$: $(x_\pm, \xi_\pm) = \Phi_\pm(y, \eta)$ such that

$$\mathsf{C}_{1\to 2|\pm} = \left\{ ((y, \eta), \chi_\pm(y, \eta)) \in \dot{T}^*\Sigma_1 \times \dot{T}^*\Sigma_2 \,\middle|\, (y, \eta) \in \dot{T}^*\Sigma_1 \right\} = \operatorname{graph}(\Phi_\pm)$$

and thus the claim.

This observation allows to calculate the principal symbol of Q by multiplying the principal symbols of each occuring operator (see (4.20) or recall [Hö71, Thm.4.2.2/3] for the details of this fact). The principal symbols of the solution operators follow from the initial value problem for the normally hyperbolic operator from Theorem C.2. With $\mathsf{res}_\Sigma \circ (-(\nabla_\nu)^j) \circ \mathcal{G}_k = \delta_{jk} \mathbb{1}$ up to smoothing operators one yields for the covectors $\zeta_\pm \in \dot{T}^*_y M$

$$\begin{aligned} \boldsymbol{\sigma}_0(\mathcal{G}_0^\pm|_{\Sigma_1})(y, \zeta_\pm; y, \eta) = \mathbb{1} \quad &, \quad \boldsymbol{\sigma}_{-1}(\mathcal{G}_1^\pm|_{\Sigma_1})(y, \zeta_\pm; y, \eta) = \mathbb{O} \quad , \\ \boldsymbol{\sigma}_1((-\nabla_\nu) \circ \mathcal{G}_0^\pm|_{\Sigma_1})(y, \zeta_\pm; y, \eta) = \mathbb{O} \quad &, \quad \boldsymbol{\sigma}_0((-\nabla_\nu) \circ \mathcal{G}_1^\pm|_{\Sigma_1})(y, \zeta_\pm; y, \eta) = \mathbb{1} \quad . \end{aligned}$$

The left equation in the first line determines the principal symbols of $\mathcal{G}_0^\pm$, restricted to the initial hypersurface Σ_1. The multiplication of symbols is then used in the second line in order to extract the symbol of the covariant derivative on the initial hypersurface which is

$$\begin{aligned} \boldsymbol{\sigma}_1(-\nabla_\nu)(y, \zeta_\pm)u &= -\lim_{\lambda\to\infty} \left(\frac{1}{\lambda}\right) \mathrm{e}^{-\mathbb{i}\lambda\Phi} \nabla_\nu \left(\mathrm{e}^{\mathbb{i}\lambda\phi} u\right)\Big|_y = -\lim_{\lambda\to\infty} \left(\frac{1}{\lambda}\right) (\mathbb{i}\lambda \,\mathrm{d}\phi(\nu) + \nabla_\nu u)\Big|_y \\ &= -\mathbb{i}\zeta_\pm(\nu)\,. \end{aligned}$$

This system of equations becomes

$$\begin{aligned} \boldsymbol{\sigma}_0(\mathcal{G}_0^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \mathbb{1} \quad &, \quad \boldsymbol{\sigma}_{-1}(\mathcal{G}_1^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \mathbb{O} \quad , \\ -\mathbb{i}\zeta_{\pm}(\nu)\boldsymbol{\sigma}_0(\mathcal{G}_0^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \mathbb{O} \quad &, \quad -\mathbb{i}\zeta_{\pm}(\nu)\boldsymbol{\sigma}_{-1}(\mathcal{G}_1^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \mathbb{1} \quad . \end{aligned}$$

Since $\zeta_{\pm}$ is non-zero and in general $\zeta_{\pm}(\nu) \neq 0$, we can divide in the lower left equation and add it to the upper left equation, giving

$$\mathbb{1} = \mathbb{1} + \mathbb{O} = 2\boldsymbol{\sigma}_0(\mathcal{G}_0^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) \quad \Leftrightarrow \quad \boldsymbol{\sigma}_0(\mathcal{G}_0^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \frac{1}{2}\mathbb{1} \quad .$$

Multiplying the upper equation on the right side with $(-\mathbb{i}\zeta_{\pm}(\nu))$ and adding this to the equation below gives

$$\mathbb{1} = \mathbb{1} + \mathbb{O} = -2\mathbb{i}\zeta_{\pm}(\nu)\boldsymbol{\sigma}_{-1}(\mathcal{G}_1^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) \Leftrightarrow \boldsymbol{\sigma}_{-1}(\mathcal{G}_1^{\pm}|_{\Sigma_1})(y,\zeta_{\pm};y,\eta) = \frac{\mathbb{i}}{2}\zeta_{\pm}(\nu)^{-1}\mathbb{1} \,.$$

If $(x,\varsigma_{\pm}) \in \dot{T}^*_{\Sigma_2}M$ lies on the orbit of the lightlike cogeodesic flow and restricts to $(x,\xi_{\pm})$ on Σ_2, the parallel transport operator $\mathcal{P}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}$ with respect to $\nabla = \nabla^{\mathcal{S}_{L,E}(M)}$ along this geodesic relates these two points such that the pure principal symbols of both solution operators are

$$\boldsymbol{\sigma}_{-1/4}(\mathcal{G}_0)(x,\varsigma_{\pm};y,\eta) = \frac{1}{2}\mathcal{P}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}$$

and

$$\boldsymbol{\sigma}_{-5/4}(\mathcal{G}_1)(x,\varsigma_{\pm};y,\eta) = \frac{\mathbb{i}}{2}\zeta_{\pm}(\nu)^{-1}\mathcal{P}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}$$

where the restriction operator res_{t_1} as FIO of order 1/4 has been taken out, which is again possible, as both operators are properly supported and their canonical relations are constructed in such a way that their composition is proper, transversal and a graph of a canonical transformation. The choice of the sign depends on the initial velocity covector $\zeta_{\pm}$, influencing the direction of the geodesic flow in future or past timelike direction. The prefactor can be rewritten as follows: $\zeta_{\pm}$ is a lightlike covector which restricts to η on Σ_1. Since ν was chosen to be past-directed and orthonormal to each hypersurface, the projection of $\zeta_{\pm}^{\sharp}$ in temporal direction is $\mp\nu$. The lightlike property and the ansatz $\zeta_{\pm}^{\sharp} = \mp\alpha\nu + \eta$ along Σ_1 allows to compute $\zeta_{\pm}(\nu)|_y$:

$$\begin{aligned} 0 \;&=\; g|_y(\zeta_{\pm}^{\sharp},\zeta_{\pm}^{\sharp}) = \alpha^2 g|_y(\nu,\nu) + g_{t_1}|_y(\eta,\eta) = -\alpha^2(y) + g_{t_1}|_y(\eta,\eta) \\ \Leftrightarrow &\qquad \alpha(y) = g_{t_1}|_y(\eta,\eta) =: \|\eta\|^2_{g_{t_1}(y)} \neq 0 \quad \forall\, y \in \Sigma_1 \\ \Rightarrow &\qquad \zeta_{\pm}(\nu)|_y = g|_y(\zeta_{\pm}^{\sharp},\nu) = \mp\alpha(y)g|_y(\nu,\nu) + 0 = \pm\alpha(y) = \pm\,\|\eta\|_{g_{t_1}(y)} \quad . \end{aligned}$$

The parallel transport on the tensor product is naturally defined to be the tensor product of the parallel transport of each factor:

$$\mathcal{P}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}(\psi\otimes f) = \mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}(\psi)\otimes\mathcal{P}^{E_L}_{(x,\varsigma_{\pm})\leftarrow(y,\zeta_{\pm})}(f)$$

with $\mathcal{P}^{\mathcal{S}(M)}$ and $\mathcal{P}^{E_L}$, denoting the parallel transport operator for each factor in $\mathcal{S}_{L,E}(M)$, and ψ as well as f sections of the untwisted spinor bundle and respectively the twisting bundle. By $C^\infty(M)$-linearity this extends to any section of the twisted spinor bundle.

Thus, both Fourier integral operators have the principal symbols

$$\begin{aligned}\boldsymbol{\sigma}_{-1/4}(\mathcal{G}_0)(x,\varsigma_\pm;y,\eta) &= \left[\frac{1}{2}\mathcal{P}^{S(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\right]\otimes\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \qquad \text{and} \\ \boldsymbol{\sigma}_{-5/4}(\mathcal{G}_1)(x,\varsigma_\pm;y,\eta) &= \left[\pm\frac{\mathbb{i}}{2\left\|\eta\right\|_{g_{t_1}(y)}}\mathcal{P}^{S(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\right]\otimes\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \quad .\end{aligned} \tag{7.22}$$

The last part comes from the principal symbols of $D^{E_L}_\pm$ at $(x,\varsigma=\varsigma_\pm)$, restricted to $(x,\xi=\xi_\pm)$ on the hypersurface Σ_2: let $\{e_i\}_{i=0}^n$ be a future-oriented Lorentz-orthonormal frame with $e_0=-\nu$, then

$$\begin{aligned}D^{E_L}_\pm u &= \sum_{i=0}^n \epsilon_i(\mathbf{c}\,(e_i)\otimes\mathbb{1}_{E_L})\nabla_{e_i}u = -(\beta\otimes\mathbb{1}_{E_L})\nabla_{-\nu}u+\sum_{i=1}^n(\mathbf{c}\,(e_i)\otimes\mathbb{1}_{E_L})\nabla_{e_i}u \\ \Rightarrow \boldsymbol{\sigma}_1(D^{E_L}_\pm)((t,x),\varsigma)u &= (\beta\otimes\mathbb{1}_{E_L})\boldsymbol{\sigma}_1(-\nabla_{-\nu})(x,\varsigma)u+\sum_{i=1}^n(\mathbf{c}\,(e_i)\otimes\mathbb{1}_{E_L})\boldsymbol{\sigma}_1(\nabla_{e_i})(x,\varsigma)u \\ &= \mathbb{i}\varsigma(\nu)(\beta\otimes\mathbb{1}_{E_L})u+\sum_{i=1}^n\mathbb{i}\varsigma(e_i)(\mathbf{c}\,(e_i)\otimes\mathbb{1}_{E_L})u \\ &= \mathbb{i}\left[\varsigma(\nu)\beta\otimes+\sum_{i=1}^n\mathbb{i}\varsigma(e_i)\mathbf{c}\,(e_i)\right]\otimes\mathbb{1}_{E_L}u\end{aligned}$$

which implies

$$\boldsymbol{\sigma}_{5/4}(D^{E_L}_\pm)|_{\Sigma_2}(x,\xi) := \boldsymbol{\sigma}_{5/4}(\mathsf{res}_{\Sigma_2}\circ D^{E_L}_\pm)(x,\xi) = \mathbb{i}\left[\varsigma|_x(\nu)\beta\pm\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right]\otimes\mathbb{1}_{E_L|_{\Sigma_2}} \quad ;$$

the difference in the signs occurs since the choice of the Clifford multiplication on the hypersurface depends on the chirality of the spinor on which the Dirac operators $D^{E_L}_\pm$ are acting on. The order $5/4$ comes by composing $D^{E_L}_\pm$ with the restriction as FIO of order $1/4$. Since $\varsigma=\varsigma_\pm$ is lightlike, the same calculations lead to $\varsigma_\pm(\nu)|_x=\pm\left\|\xi_\pm\right\|_{g_{t_2}}(x)$. Composing everything shows the claimed principal symbols:

$$\begin{aligned}&\boldsymbol{\sigma}_0(Q^{E_L})(x,\xi_\pm;y,\eta) = \boldsymbol{\sigma}_{5/4}(D^{E_L}_-)|_{\Sigma_2}(x,\xi_\pm)\boldsymbol{\sigma}_{-5/4}(\mathcal{G}^-(t_1))(x,\xi_\pm;y,\eta)\circ(\beta\otimes\mathbb{1}_{E_L|_{\Sigma_1}}) \\ &\quad= \frac{\pm\mathbb{i}^2}{2\left\|\eta\right\|_{g_{t_1}(y)}}\left[\left(\pm\left\|\xi_\pm\right\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\otimes\mathbb{1}_{E_L|_{\Sigma_2}}\right]\circ\left[\mathcal{P}^{S(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\otimes\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\right] \\ &\qquad\qquad \circ(\beta\otimes\mathbb{1}_{E_L|_{\Sigma_1}}) \\ &\quad= \pm\frac{1}{2}\left\|\eta\right\|^{-1}_{g_{t_1}(y)}\left[\left(\mp\left\|\xi_\pm\right\|_{g_{t_2}(x)}\beta+\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\circ\mathcal{P}^{S(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\beta\right] \\ &\qquad\qquad \otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\sigma}_0(\tilde{Q}^{E_L})(x,\xi_\pm;y,\eta) &= \boldsymbol{\sigma}_{5/4}(D_+^{E_L})|_{\Sigma_2}(x,\xi_\pm)\boldsymbol{\sigma}_{-5/4}(\mathcal{G}^+(t_1))(x,\xi_\pm;y,\eta)\circ(\beta\otimes\mathbb{1}_{E_L|_{\Sigma_1}})\\ &= \frac{\pm\mathrm{i}^2}{2\|\eta\|_{g_{t_1}(y)}}\left[\left(\pm\|\xi_\pm\|_{g_{t_2}(x)}\beta+\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\otimes\mathbb{1}_{E_L|_{\Sigma_2}}\right]\circ\left[\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\otimes\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\right]\\ &\qquad\circ(\beta\otimes\mathbb{1}_{E_L|_{\Sigma_1}})\\ &= \pm\frac{1}{2}\|\eta\|^{-1}_{g_{t_1}(y)}\left[\left(\mp\|\xi_\pm\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\circ\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\beta\right]\\ &\qquad\otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]\quad.\end{aligned}$$

The same procedure can be done analogously for the untwisted case where (7.18) and (7.19) reduce to

$$Q^{E_L}(t_2,t_1) = \mathsf{res}_{\Sigma_2}\circ D_-\circ\mathcal{G}^-(t_1)\circ\beta \tag{7.23}$$

and

$$\tilde{Q}^{E_L}(t_2,t_1) = \mathsf{res}_{\Sigma_2}\circ D_+\circ\mathcal{G}^+(t_1)\circ\beta\quad. \tag{7.24}$$

The character of the opertors does not explicitly depend on the twisting such that

$$\begin{aligned}Q &\in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to2};\mathsf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2)))\\ \tilde{Q} &\in \mathcal{FIO}^0_{\mathsf{prop}}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to2};\mathsf{Hom}(\mathcal{S}^-(\Sigma_1),\mathcal{S}^-(\Sigma_2)))\end{aligned}$$

with the same canonical relations and support properties as stated. The principal symbols of each component in the composition take the form

$$\boldsymbol{\sigma}_{-5/4}(\mathcal{G}_1)(x,\varsigma_\pm;y,\eta) = \frac{\pm\mathrm{i}}{2\|\eta\|_{g_{t_1}}(y)}\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}$$

and

$$\boldsymbol{\sigma}_{5/4}(D_\pm)|_{\Sigma_2}(x,\xi) = \mathrm{i}\left(\varsigma|_x(\nu)\beta\pm\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)$$

such that

$$\boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta) = \pm\frac{1}{2}\|\eta\|^{-1}_{g_{t_1}(y)}\left(\mp\|\xi_\pm\|_{g_{t_2}(x)}\beta+\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\circ\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\beta \tag{7.25}$$

$$\boldsymbol{\sigma}_0(\tilde{Q})(x,\xi_\pm;y,\eta) = \pm\frac{1}{2}\|\eta\|^{-1}_{g_{t_1}(y)}\left(\mp\|\xi_\pm\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi^\sharp\right)\right)\circ\mathcal{P}^{\mathcal{S}(M)}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\beta\ . \tag{7.26}$$

These are exactly the first factors in the tensor product for the twisting case. □

Remark 7.2.6. *The properly supportness of the Dirac-wave evolution operators imply that they are extendable as map from local Sobolev sections to local Sobolev sections:*

$$\begin{aligned}Q^{E_L}(t_2,t_1)\,:\,H^s_{\mathsf{loc}}(\mathcal{S}^+_{L,E}(\Sigma_1)) &\to H^s_{\mathsf{loc}}(\mathcal{S}^+_{L,E}(\Sigma_2))\ ,\\ \tilde{Q}^{E_L}(t_2,t_1)\,:\,H^s_{\mathsf{loc}}(\mathcal{S}^-_{L,E}(\Sigma_1)) &\to H^s_{\mathsf{loc}}(\mathcal{S}^-_{L,E}(\Sigma_2))\ ;\end{aligned}\tag{7.27}$$

recall Lemma 4.2.4 (3). This becomes important for Sobolev spaces in the setting of Galois coverings, defined by local Sobolev sections.

7.3. Well-posedness in Γ-setting

From now on we implement all assumptions and settings, we made at the beginning of Part III. M is a globally hyperbolic, spatial Γ-manifold with spin structure. Later we will also consider the case, were M is also temporal compact. All coming vector bundles are understood as Γ-vector bundles. The Γ-invariances of A_t, $\mathcal{D}$ and $D_\pm$ have to be understood as intertwining with the left action representations:

$$A_t L_\gamma^{\mathcal{S}(\Sigma_t)} = L_\gamma^{\mathcal{S}(\Sigma_t)} A_t \qquad \mathcal{D} L_\gamma^{\mathcal{S}(M)} = L_\gamma^{\mathcal{S}(M)} \mathcal{D} \qquad D_\pm L_\gamma^{\mathcal{S}^\pm(M)} = L_\gamma^{\mathcal{S}^\mp(M)} D_\pm \tag{7.28}$$

for any $\gamma \in \Gamma$ and $t \in \mathcal{T}(M)$. We assume similar for the twisted Dirac operators.

We define the following spaces for $H_\Gamma^s \subset H_{\mathsf{loc}}^s$ and any $s \in \mathbb{R}$:

(a) $FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M)) := C^0_{\mathsf{sc}}(\mathcal{T}(M), H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet)))$;

(b) $FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, D^{E_L}_\pm) :=$
$$\left\{ u \in FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M)) \,\middle|\, D^{E_L}_\pm u \in L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_\Gamma(\mathcal{S}^\mp_{L,E}(\Sigma_\bullet))) \right\};$$

(c) $FE^s_{\mathsf{sc},\Gamma}\left(M, \ker\left(D^{E_L}_\pm\right)\right) := \left\{ u \in FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M)) \,\middle|\, D^{E_L}_\pm u = 0 \right\}$.

The seminorm on $FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, \mathcal{S}^\pm_{L,E}(M))$ is defined as in (4.26) where the seminorm of local Sobolev sections is replaced by the norm for Γ-Sobolev spaces:

$$\|u\|_{I,K,l,s,\Gamma} := \max_{k \in [0,l] \cap \mathbb{N}_0} \max_{t \in I} \left\| (\nabla_t)^k u \right\|_{H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet))} \quad . \tag{7.29}$$

We can formulate with this modifications Γ-versions of Theorem 7.1.9 and Corollary 7.1.10 where the isomorphisms become Γ-invariant.

Proposition 7.3.1. *For a fixed $t \in \mathcal{T}(M)$ with M a globally hyperbolic spatial Γ-manifold and any $s \in \mathbb{R}$ the maps*

$$\mathsf{res}_t \oplus D^{E_L}_\pm \; : \; FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, D^{E_L}_\pm) \; \to \; H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)) \oplus L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_\Gamma(\mathcal{S}^\mp_{L,E}(\Sigma_\bullet))) \tag{7.30}$$

are Γ-isomorphisms of topological vector spaces.

Corollary 7.3.2. *For a fixed $t \in \mathcal{T}(M)$ with M a globally hyperbolic spatial Γ-manifold and any $s \in \mathbb{R}$ the maps*

$$\mathsf{res}_t \; : \; FE^s_{\mathsf{sc},\Gamma}\left(M, \ker\left(D^{E_L}_\pm\right)\right) \; \to \; H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)) \tag{7.31}$$

are Γ-isomorphisms of topological vector spaces.

Proof. W.l.o.g. we will prove this result for both untwisted Γ-invariant Dirac operators and label those places where a modification for the twisted case is needed.

The continuity of $D_\pm$ as map from $FE^s_{\mathsf{sc},\Gamma}(M, \mathcal{T}, D_\pm)$ to $L^2_{\mathsf{loc,sc}}(\mathcal{T}(M), H^s_\Gamma(\mathcal{S}^\mp(\Sigma_\bullet)))$ follows by construction of the domain. The continuity of the restriction is given by the following modified argument: for all $t \in \mathcal{T}(M)$ the diffeomorphism between M and the product manifold $\mathcal{T}(M) \times \Sigma$ by Theorem 3.1.2 implies that for each Γ-hypersurface one and the same time independent Γ-invariant partition of unity $\{\phi_{i,\gamma}\}_{\substack{i\in I\\ \gamma\in\Gamma}}$, subordinated to a covering of Σ, can be chosen. Thus, every slice in $\{\Sigma_t\}_{t\in\mathcal{T}(M)}$ has the same partition of unity. As in the the proof of Theorem 7.1.9 with $\mathcal{K}\cap\Sigma_t$ replaced by $K(t,i,\gamma) := \mathcal{K}\cap\Sigma_t\cap\operatorname{supp}(\phi_{i,\gamma})$ for each $i\in I$, $\gamma\in\Gamma$ and $\mathcal{K}$ spatially compact we gain

$$\begin{aligned}
\|\mathsf{res}_t u\|^2_{H^s_\Gamma(\mathcal{S}^\pm(\Sigma_t))} &= \sum_{\substack{i\in I\\ \gamma\in\Gamma}} \|\phi_{i,\gamma}\mathsf{res}_t u\|^2_{H^s(K(t,i,\gamma),\mathcal{S}^\pm(\Sigma_t))} \\
&\leq \max_{\tau\in\mathcal{T}(M)} \sum_{\substack{i\in I\\ \gamma\in\Gamma}} \|\phi_{i,\gamma}\mathsf{res}_\tau u\|^2_{H^s(K(\tau,i,\gamma),\mathcal{S}^\pm(\Sigma_\tau))} \\
&= \max_{\tau\in\mathcal{T}(M)} \left\{\|\mathsf{res}_\tau u\|^2_{H^s_\Gamma(\mathcal{S}^\pm(\Sigma_\tau))}\right\} \overset{(7.29)}{\leq} \|u\|^2_{\mathcal{T}(M),\mathcal{K},0,s,\Gamma} \\
\Leftrightarrow \quad \|\mathsf{res}_t u\|_{H^s_\Gamma(\mathcal{S}^\pm(\Sigma_t))} &\leq \|u\|_{\mathcal{T}(M),\mathcal{K},0,s,\Gamma} \quad .
\end{aligned}$$

The continuous inclusion of $C^0_{\mathcal{K}}$ into C^0_{c} then leads to the wanted feature. The second estimate in the proof of Theorem 7.1.9 can be modified after introducing a suitable energy in the Γ-setting: define the *Γ-s-energy* as square of the Γ-Sobolev norm:

$$\mathcal{E}_{s,\Gamma}(u,\Sigma_t) = \|u\|^2_{H^s_\Gamma(\mathcal{S}(\Sigma_t))} = \sum_{\substack{i\in I\\ \gamma\in\Gamma}} \|\phi_{i,\gamma}u\|^2_{H^s(K(t,i,\gamma),\mathcal{S}(\Sigma_t))} = \sum_{\substack{i\in I\\ \gamma\in\Gamma}} \mathcal{E}_s(\phi_{i,\gamma}u,\Sigma_t) \quad .$$

From the last equality we can see that the defined Γ-s-energy can be rewritten as the usual s-energy. We abbreviate $\phi := \phi_{i,\gamma}$ and $\tilde{u} = \phi u$ and perform the estimate of the energy $\mathcal{E}_s(\phi_{i,\gamma}u,\Sigma_t)$ from the proof of Proposition 7.1.6 up to the fourth line where one makes use of the time independence of the partition of unity:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(\tilde{u},\Sigma_t) &\leq n\big\langle H_t\Lambda^s_t\tilde{u}\,\big|\,\Lambda^s_t\tilde{u}\big\rangle_{L^2(\mathcal{S}(\Sigma_t))} + c_2\,\|\tilde{u}\|^2_{H^s(\mathcal{S}(\Sigma_t))} - 2\mathfrak{Re}\left\{\big\langle\tilde{u}\,\big|\,\phi\nabla_{\mathsf{v}}u\big\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} \\
&= n\big\langle H_t\Lambda^s_t\tilde{u}\,\big|\,\Lambda^s_t\tilde{u}\big\rangle_{L^2(\mathcal{S}(\Sigma_t))} + c_2\,\|\tilde{u}\|^2_{H^s(\mathcal{S}(\Sigma_t))} \\
&\quad + 2\mathfrak{Re}\left\{\big\langle\tilde{u}\,\big|\,\beta\phi D_\pm u\big\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} + 2\mathfrak{Re}\left\{\big\langle\tilde{u}\,\big|\,\phi B_{t,\pm}u\big\rangle_{H^s(\mathcal{S}(\Sigma_t))}\right\} \\
&\overset{(*)}{\leq} n\left|\big\langle H_t\Lambda^s_t\tilde{u}\,\big|\,\Lambda^s_t\tilde{u}\big\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right| + c_2\,\|\tilde{u}\|^2_{H^s(\mathcal{S}(\Sigma_t))} \\
&\quad + 2\mathfrak{Re}\left\{\big\langle\Lambda^s_t\tilde{u}\,\big|\,\beta\Lambda^s_t\phi D_\pm u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\} + 2\mathfrak{Re}\left\{\big\langle\Lambda^s_t\tilde{u}\,\big|\,\Lambda^s_t\phi B_{t,\pm}u\big\rangle_{L^2(\mathcal{S}(\Sigma_t))}\right\} \\
&\overset{(**)}{\leq} n\,\|H_t\Lambda^s_t\tilde{u}\|^2_{L^2(\mathcal{S}(\Sigma_t))} + c_3\,\|\tilde{u}\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|\beta\Lambda^s_t\phi D_\pm u\|^2_{L^2(\mathcal{S}(\Sigma_t))} \\
&\quad + \|\Lambda^s_t\phi B_{t,\pm}u\|^2_{L^2(\mathcal{S}(\Sigma_t))} \\
&\leq c_4\,\|\tilde{u}\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|\phi D_\pm u\|^2_{H^s(\mathcal{S}(\Sigma_t))} + \|\phi B_{t,\pm}u\|^2_{H^s(\mathcal{S}(\Sigma_t))} \quad .
\end{aligned}$$

We used in $(*)$ Lemma 7.1.1 (2) (Lemma 7.1.2 (1) for the twisted case after replacing β with $\beta\otimes\mathbb{1}_{E_L}$) and in $(**)$ the Cauchy-Schwarz inequality and polarisation identities. Since $H_t\Lambda^s_t\in\Psi^s(\Sigma_t,\mathcal{S}^\pm(\Sigma_t))$, we have $H_t\Lambda^s_t\tilde{u}\in L^2(\mathcal{S}^\pm(\Sigma_t))$ and the estimate for Sobolev norms; furthermore, we used the isometry of β from (6.45) ((6.46) for the twisted case) in the last step. Since $B_{t,\pm}$ are properly supported, we have $B_{t,\pm}u|_{\Sigma_t}\in H^s_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_t))$ for

$u|_{\Sigma_t} \in H^{s+1}_{\mathsf{loc}}(\mathcal{S}^+(\Sigma_t))$. Summing over all covering balls and Γ-actions gives

$$\sum_{\substack{i\in I\\ \gamma\in\Gamma}} \|\phi_{i,\gamma} B_{t,\pm} u\|^2_{H^s(\mathcal{S}(\Sigma_t))} \leq c \sum_{\substack{i\in I\\ \gamma\in\Gamma}} \|\phi_{i,\gamma} u\|^2_{H^{s+1}(\mathcal{S}(\Sigma_t))} \leq \tilde{c}\mathcal{E}_{s,\Gamma}(u,\Sigma_t) \quad .$$

The second inequality is a result of the continuous inclusion of Sobolev spaces. One finally yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(u,\Sigma_t) \leq c_5 \mathcal{E}_{s,\Gamma}(u,\Sigma_t) + \|D_\pm u\|^2_{H^s_\Gamma(\mathcal{S}(\Sigma_t))}$$

and repeating all further steps from the proof in the general case as well as from Corollary 7.1.7 shows that for $\tau \in \mathcal{T}(M)$, $K \subset M$ compact and $s \in \mathbb{R}$ there exists a $C > 0$ such that

$$\mathcal{E}_{s,\Gamma}(u,\Sigma_t) \leq C\left(\mathcal{E}_{s,\Gamma}(u,\Sigma_\tau) + \|D_\pm u\|^2_{\mathcal{T}(M),\mathcal{J}(K),s,\Gamma}\right) \tag{7.32}$$

is valid for all t in any $[t_1,t_2] \subset \mathcal{T}(M)$ and for all $u \in FE^{s+1}_{\mathsf{sc},\Gamma}(M,\mathcal{T},D_\pm)$ with $D_\pm u \in FE^s_{\mathsf{sc},\Gamma}(M,\mathcal{T},\mathcal{S}^\mp(M))$ and $\operatorname{supp}(u) \subset \mathcal{J}(K)$. In comparison to the statements in Proposition 7.1.6 and Corollary 7.1.7 the spatial compact support has been ensured by the Γ-invariant partition of unity, which has been used in order to carry over the proof up to these modifications. Hence the constant C depends on the projection of the support onto Σ/Γ, but since this base is compact by our general preassumption, the constant C is now independent of the support of u. The Γ-versions of Corollary 7.1.7 and Corollary 7.1.8 then follow with identical arguments and can be used for the well-posedness of the Cauchy problem on Γ-manifolds.

Any finite energy section in $FE^s_{\mathsf{sc},\Gamma}(M,\mathcal{T}(M),D_\pm)$ can be estimated with an initial value $u_0 \in H^s_\Gamma(\Sigma_1,\mathcal{S}^\pm(\Sigma_1))$ and inhomogeneity $f = D_\pm u \in L^2_{\mathsf{loc,sc}}(\mathcal{T}(M),H^s_\Gamma(\mathcal{S}^\mp(\Sigma_\bullet)))$ with (7.32) at initial time $t = t_1$:

$$\begin{aligned}
\|u\|^2_{\mathcal{T}(M),\mathcal{J}(K),0,s,\Gamma} &\leq \max_{\tau\in\mathcal{T}(M)}\left\{\|u\|^2_{H^s_\Gamma(\mathcal{S}^\pm(\Sigma_\tau))}\right\} = \max_{\tau\in\mathcal{T}(M)}\left\{\mathcal{E}_{s,\Gamma}(u,\Sigma_\tau)\right\}\\
&\leq C\left(\max_{\tau\in\mathcal{T}(M)}\left\{\mathcal{E}_{s,\Gamma}(u,\Sigma_1) + \|D_\pm u\|^2_{\mathcal{T}(M),\mathcal{J}(K),s,\Gamma}\right\}\right)\\
&= C\left(\mathcal{E}_{s,\Gamma}(u_0,\Sigma_1) + \|f\|^2_{\mathcal{T}(M),\mathcal{J}(K),s,\Gamma}\right) \quad .
\end{aligned}$$

The rest of the proof works analogously such that one yields an isomorphism between the topological vector spaces $FE^s_{\mathsf{sc},\Gamma}(M,\mathcal{T},D_\pm)$ and $H^s_\Gamma(\mathcal{S}^\pm(\Sigma_t)) \oplus L^2_{\mathsf{loc,sc}}(\mathcal{T}(M),H^s_\Gamma(\mathcal{S}^\pm(\Sigma_\bullet)))$. It is left to show that the isomorphisms are Γ-invariant. $D_\pm$ and $D^{E_L}_\pm$ are Γ-invariant by assumption and since the restriction operator res_t just fixes a slice at time t it intertwines the Γ-action:

$$L^{\mathcal{S}^\pm(\Sigma_t)}_\gamma \mathsf{res}_t = \mathsf{res}_t L^{\mathcal{S}^\pm(M)}_\gamma \qquad \forall\, t \in \mathcal{T}(M), \forall\, \gamma \in \Gamma \quad . \tag{7.33}$$

Consequently, the direct sum is Γ-invariant. The proof of the Corollary follows easily. □

If M is moreover temporal compact, any spatially compact subset of M is itself compact since it is a closed subset in M and $\mathcal{J}(K) \subset M$ for any $K \Subset M$ becomes compact. Thus,

$$\begin{aligned}
C^l_{\mathsf{sc}}(\mathcal{T}(M),H^s_{\mathsf{loc}}(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet))) &\longrightarrow C^l(\mathcal{T}(M),H^s_{\mathsf{loc}}(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet)))\\
L^2_{\mathsf{loc,sc}}(\mathcal{T}(M),H^s_{\mathsf{loc}}(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet))) &\longrightarrow L^2(\mathcal{T}(M),H^s_{\mathsf{loc}}(\mathcal{S}^\pm_{L,E}(\Sigma_\bullet)))
\end{aligned}$$

for $l \in \mathbb{N}_0$ and $s \in \mathbb{R}$. We define

$$FE^s_{\mathsf{c}}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) := C^0(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet))) \quad .$$

The definition of the subspaces $FE^s_{\mathsf{c}}(M, \mathcal{T}, D^{E_L}_{\pm})$ and $FE^s_{\mathsf{c}}\left(M, \mathsf{ker}\left(D^{E_L}_{\pm}\right)\right)$ are clear from the general case. Theorem 7.1.9 and Corollary 7.1.10 are still valid in this setting since the isomorphism, restricted to the subset $FE^s_{\mathsf{c}}(M, \mathcal{T}, D^{E_L}_{\pm}) \subset FE^s_{\mathsf{sc}}(M, \mathcal{T}, D^{E_L}_{\pm})$, maps isomorphically to $L^2(\mathcal{T}(M), H^s_{\mathsf{loc}}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$. In the Γ-setting we then define the spaces

$$\begin{aligned}
&\text{(a)} \quad FE^s_{\Gamma}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) := C^0(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet))) \quad ;\\
&\text{(b)} \quad FE^s_{\Gamma}(M, \mathcal{T}, D^{E_L}_{\pm}) := \Big\{u \in FE^s_{\Gamma}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) \,\Big|\, \\
&\qquad\qquad D^{E_L}_{\pm} u \in L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet)))\Big\} \quad ;\\
&\text{(c)} \quad FE^s_{\Gamma}\left(M, \mathsf{ker}\left(D^{E_L}_{\pm}\right)\right) := \Big\{u \in FE^s_{\Gamma}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M)) \,\Big|\, D^{E_L}_{\pm} u = 0\Big\} \quad .
\end{aligned} \tag{7.34}$$

The seminorm on $FE^s_{\Gamma}(M, \mathcal{T}, \mathcal{S}^{\pm}_{L,E}(M))$ is defined as in (7.29). $L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$ is a bundle of free Hilbert Γ-modules; the left action representation $L^{\mathcal{S}^{\pm}_{L,E}(\Sigma_t)}$ on $H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ for each time induces a left action $L^{\mathcal{S}^{\pm}_{L,E}(M)}$ on $L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$ by applying $L^{\mathcal{S}^{\pm}_{L,E}(\Sigma_t)}$ at each time $t \in \mathcal{T}(M)$. Thus, $L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$ becomes a free Hilbert Γ-module on its own right. We point out that

$$L^2(\mathcal{T}(M), H^0_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet))) = L^2(\mathcal{T}(M), L^2_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet))) = L^2_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(M)) \quad .$$

After this clarification further well-posedness results for $D^{E_L}_{\pm}$ can be proven:

Theorem 7.3.3. *For a fixed $t \in \mathcal{T}(M)$ with M a temporal compact globally hyperbolic spatial Γ-manifold and any $s \in \mathbb{R}$ the maps*

$$\mathsf{res}_t \oplus D^{E_L}_{\pm} \ : \ FE^s_{\Gamma}(M, \mathcal{T}, D^{E_L}_{\pm}) \ \to \ H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t)) \oplus L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\mp}_{L,E}(\Sigma_\bullet))) \tag{7.35}$$

are Γ-isomorphisms of Hilbert Γ-modules.

Corollary 7.3.4. *For a fixed $t \in \mathcal{T}(M)$ with M a temporal compact globally hyperbolic spatial Γ-manifold and any $s \in \mathbb{R}$ the maps*

$$\mathsf{res}_t \ : \ FE^s_{\Gamma}\left(M, \mathsf{ker}\left(D^{E_L}_{\pm}\right)\right) \ \to \ H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t)) \tag{7.36}$$

are Γ-isomorphisms of Hilbert Γ-modules.

Proof. Proposition 7.3.1 already imply that $\mathsf{res}_t \oplus D^{E_L}_{\pm}$ respectively res_t are Γ-isomorphisms as the reduction to a bounded time interval has no effect. We clarified beforehand that $L^2(\mathcal{T}(M), H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_\bullet)))$ are free Hilbert Γ-modules for all $s \in \mathbb{R}$. Since $H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ is also a free Hilbert Γ-module, its direct sum in (7.30) becomes a free Hilbert Γ-module due to Lemma 5.2.7. $FE^s_{\Gamma}(M, \mathcal{T}, D^{E_L}_{\pm})$ is a general Hilbert Γ-module because the isomorphism implies a Hilbert space structure and the space admits a left action representation, induced from the one on each $H^s_{\Gamma}(\mathcal{S}^{\pm}_{L,E}(\Sigma_t))$ for each time t. Proposition 5.2.6 (2) finally ensures that $FE^s_{\Gamma}(M, \mathcal{T}, D^{E_L}_{\pm})$ becomes a (free) Hilbert Γ-module. □

A similar reasoning, based on Corollary 7.3.2, applies to $FE^s_\Gamma\left(M, \ker\left(D^{E_L}_\pm\right)\right)$ which shows Corollary 7.3.4. As in the general setting we can extract several Dirac-wave evolution operators from Corollary 7.3.4.

Definition 7.3.5. For a globally hyperbolic manifold M and $t_1, t_2 \in \mathcal{T}(M)$ the (Dirac-) wave evolution operators for positive and negative chirality are the following isomorphisms of Hilbert Γ-modules

$$\begin{aligned} Q(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &: \quad H^s_\Gamma(\mathcal{S}^+(\Sigma_1)) &\to H^s_\Gamma(\mathcal{S}^+(\Sigma_2)) \;, \\ \tilde{Q}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &: \quad H^s_\Gamma(\mathcal{S}^-(\Sigma_1)) &\to H^s_\Gamma(\mathcal{S}^-(\Sigma_2)) \;. \end{aligned}$$

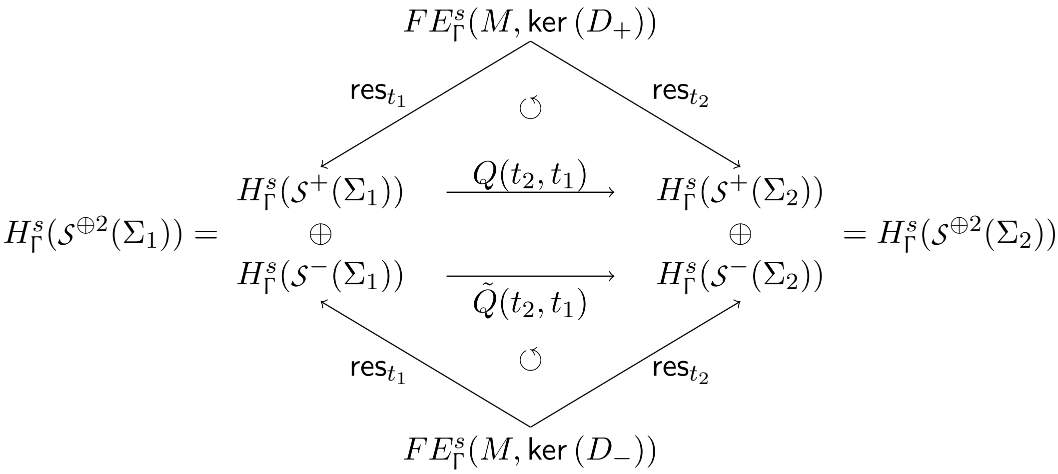

Figure 7.5.: Commuting diagrams for defining $Q(t_2, t_1)$ and $\tilde{Q}(t_2, t_1)$ in Γ-setting.

Analogously we can define the evolution operators for the twisted Dirac operators.

Definition 7.3.6. For a globally hyperbolic manifold M and $t_1, t_2 \in \mathcal{T}(M)$ the twisted (Dirac-)wave evolution operators for positive and negative chirality are the following isomorphisms of Hilbert Γ-modules

$$\begin{aligned} Q^{E_L}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &: \quad H^s_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_1)) &\to H^s_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)) \;, \\ \tilde{Q}^{E_L}(t_2,t_1) &:= \mathsf{res}_{t_2} \circ (\mathsf{res}_{t_1})^{-1} &: \quad H^s_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_1)) &\to H^s_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)) \;. \end{aligned}$$

The properties of the wave evolution operators in Definition 7.2.1 and Definition 7.2.2 transfer to the Γ-case.

Lemma 7.3.7. *For any $s \in \mathbb{R}$ and $t, t_1, t_2, t_3 \in \mathcal{T}(M)$ the following hold*

(1) $Q^{E_L}(t_3,t_2) \circ Q^{E_L}(t_2,t_1) = Q^{E_L}(t_3,t_1)$ *;*

(2) $Q^{E_L}(t,t) = \mathbb{1}_{H^s_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_t))}$ *and* $Q^{E_L}(t_1,t_2) = (Q^{E_L})^{-1}(t_2,t_1)$ *;*

(3) $\tilde{Q}^{E_L}(t_3,t_2) \circ \tilde{Q}^{E_L}(t_2,t_1) = \tilde{Q}^{E_L}(t_3,t_1)$ *;*

(4) $\tilde{Q}^{E_L}(t,t) = \mathbb{1}_{H^s_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_t))}$ *and* $\tilde{Q}^{E_L}(t_1,t_2) = (\tilde{Q}^{E_L})^{-1}(t_2,t_1)$ *;*

(5) *For any* $[t_1, t_2] \subset \mathcal{T}(M)$ *the operators* $Q^{E_L}(t_2, t_1)$ *and* $\tilde{Q}^{E_L}(t_2, t_1)$ *are unitary for* $s = 0$*;*

(6) $Q^{E_L} \in \mathcal{FIO}^0_\Gamma(\Sigma_1, \Sigma_2; \mathsf{C}'_{1\to 2}; \mathsf{Hom}(\mathcal{S}^+_{L,E}(\Sigma_1), \mathcal{S}^+_{L,E}(\Sigma_2)))$ *and* $\tilde{Q}^{E_L} \in \mathcal{FIO}^0_\Gamma(\Sigma_1, \Sigma_2; \mathsf{C}'_{1\to 2}; \mathsf{Hom}(\mathcal{S}^+_{L,E}(\Sigma_1), \mathcal{S}^+_{L,E}(\Sigma_2)))$ *as in Theorem 7.2.5.*

As before these properties carry over to Q and $\tilde{Q}$ for the untwisted case.

Proof. The claims (1) to (5) follow as in the proof of Lemma 7.2.3. Claim (6) has been proven in Theorem 7.2.5 for a more general setting. It is left to show that the Dirac-wave evolution operators are Γ-invariant, i.e. intertwine the Γ action on $H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_1))$ and $H^s_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_2))$. We just consider the untwisted case for spinors of positive chirality: as the restriction operator is Γ-invariant and bijective on $H^s_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_t))$ for each $t \in \mathcal{T}(M)$, (7.33) implies

$$\begin{aligned} L^{\mathcal{S}^+(\Sigma_2)}_\gamma Q(t_2, t_1)\mathsf{res}_{t_1} &= L^{\mathcal{S}^+(\Sigma_2)}_\gamma \mathsf{res}_{t_2} = \mathsf{res}_{t_2} L^{\mathcal{S}^+(M)}_\gamma = Q(t_2, t_1)\mathsf{res}_{t_1} L^{\mathcal{S}^+(M)}_\gamma \\ &= Q(t_2, t_1) L^{\mathcal{S}^+(\Sigma_1)}_\gamma \mathsf{res}_{t_1} \end{aligned}$$

and thus the claim; the other cases follow with the same reasoning such that

$$\begin{aligned} L^{\mathcal{S}^+(\Sigma_2)}_\gamma Q(t_2, t_1) &= Q(t_2, t_1) L^{\mathcal{S}^+(\Sigma_1)}_\gamma \quad , \\ L^{\mathcal{S}^-(\Sigma_2)}_\gamma \tilde{Q}(t_2, t_1) &= \tilde{Q}(t_2, t_1) L^{\mathcal{S}^-(\Sigma_1)}_\gamma \quad , \\ L^{\mathcal{S}^+_{L,E}(\Sigma_2)}_\gamma Q^{E_L}(t_2, t_1) &= Q^{E_L}(t_2, t_1) L^{\mathcal{S}^+_{L,E}(\Sigma_1)}_\gamma \quad , \\ L^{\mathcal{S}^-_{L,E}(\Sigma_2)}_\gamma \tilde{Q}^{E_L}(t_2, t_1) &= \tilde{Q}^{E_L}(t_2, t_1) L^{\mathcal{S}^-_{L,E}(\Sigma_1)}_\gamma \quad . \end{aligned} \tag{7.37}$$

□

8. Projectors and Spectral flow in the Γ-setting

This chapter contains more technical questions concerning the role of projections in von Neumann algebras, their regularity as a Γ-pseudo-differential operator, and to define boundary conditions. As the second major task in this chapter we classify the definition of spectral flow in Γ-setting and extend the definition to the case where the family of operators is not defined on a common domain.

8.1. Projectors and g(a)APS-boundary conditions in the Γ-setting

In this section we introduce the boundary conditions which we need to show Γ-Fredholmness for $\mathcal{D}^{E_L}$. Moreover, we clarify how the pseudo-differential operator property of projectors for compact manifolds transfer to Galois coverings. To be more precise, we are looking for a version of Seeley's theorem for complex powers of pseudo-differential operators from [See69] on Galois coverings.

8.1.1. Projections as Γ-pseudo-differential operators

The aim of this subsection is a Γ-capable description of spectral projectors as pseudo-differential operators with certain regularity properties. We already know from Proposition 5.3.7 (7) that elliptic and formally self-adjoint Γ-pseudo-differential operators of positive order are essential self-adjoint and their spectral projectors onto any bounded Borel set in the spectrum are s-smoothing Γ-operators. In particular, the projection onto the kernel of the operator becomes s-smoothing, too. Lemma 5.3.5 (3) implies that these spectral projections are Γ-trace class between two Hilbert Γ-modules.

We want to generalise Proposition 5.3.7 (7) to unbounded intervals in $\mathbb{R}$. In order to do so, we consider an elliptic and self-adjoint operator $A \in \Psi^m_{\Gamma,\mathsf{prop}}(\Sigma, E)$ of order $m > 0$ which acts between smooth sections of the (Hermitian) Γ-vector bundle E over the (Riemannian) manifold Σ. The characteristic function of such an operator can be defined by means of unbounded functional calculus for self-adjoint operators or the Browder-Garding Theorem [Bro54, Thm.1]. These approaches have the disadvantage that one can cannot extract enough analytic informations in order to understand its interaction with other differential operators. To bypass this, we rewrite the characteristic function χ on $(0, \infty)$ and $(-\infty, 0)$ with the signum-function which in turn can be expressed as quotient $x/\left|x\right|$ if x is the variable:

$$\begin{aligned}\chi_{>0}(x) &:= \chi_{(0,\infty)} = \frac{1}{2}\left(1 + \frac{x}{|x|} - \chi_{\{0\}}(x)\right)\\ \chi_{<0}(x) &:= \chi_{(-\infty,0)} = \frac{1}{2}\left(1 - \frac{x}{|x|} - \chi_{\{0\}}(x)\right) \quad .\end{aligned} \tag{8.1}$$

Hence the idea is to replace $\chi_{\gtrless 0}(A)$ with (8.1) where the variable is replaced with the operator A. The one in these formulas has to be replaced with the unity operator; the expression $\chi_{\{0\}}(A)$ already makes sense due to Proposition 5.3.7 (7) and is a s-smoothing Γ-pseudo-differential operator. It is left to show that the expressions

$$A \circ |A|^{-1}$$

and in particular $|A|^{-1}$ are meaningful. The latter can be rephrased as operator $A^*A = A^2$ to the power $(-1/2)$. If Σ is closed, Seeley's theorem for complex powers states that under further conditions on A any of its complex powers become meaningful as pseudo-differential operator and their principal symbols can be calculated from the one of A. We want something similar for the Γ-setting, thus our aim is a Γ-invariant pendent of Seeley's theorem for complex powers of elliptic pseudo-differential operators.

Fortunately, such a result is known for manifolds of bounded geometry. As Γ-manifolds are special cases of these type of manifolds, we hope that this result transfers to our situation. We need to take a little detour through some theoretic aspects of pseudo-differential operators on manifolds of bounded geometry. We give some details and clarify its pendent in the Γ-setting. The interested reader is referred to [Kor00], [Shu92] and [Kor91] for more details.

Let Σ be a manifold of bounded geometry and $E \to \Sigma$ a vector bundle of bounded geometry as described in Section 3.3. An element $B \in \mathsf{B\Psi}^m_{\mathsf{prop}}(\Sigma, E)$ is a pseudo-differential operator of order $m \in \mathbb{R}$ with uniformly bounded symbol if the following requirements are satisfied:

(a) In any coordinate system the operator is of the form $B = \mathsf{Op}(a) + R$ where

   1. $$(\mathsf{Op}(a)u)(x) := \iint \mathrm{e}^{\mathrm{i}\langle x-y \,|\, \xi\rangle} a(x,\xi)u(y)\,\mathrm{d}y\,đ\xi$$

      for any $u \in C^\infty_c(\mathring{\mathbb{B}}^n(0))$, such that $a(x,\xi) \in C^\infty(\mathring{\mathbb{B}}^n(0) \times \mathbb{R}^n)$ is a complete symbol and there exists a constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0$ with

      $$\left|\partial_x^{\boldsymbol{\alpha}}\partial_\xi^{\boldsymbol{\beta}} a(x,\xi)\right| \leq C_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1+|\xi|)^{m-|\boldsymbol{\beta}|}$$

      for $x \in \mathring{\mathbb{B}}^n(0)$, $\xi \in \mathbb{R}^n$ and $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \mathbb{N}_0^n$.
   2. R is a smoothing and uniformly bounded operator on $\mathring{\mathbb{B}}^n(0)$.

(b) (c-locality): there exists a constant $c > 0$ such that the Schwartz kernel K_B of B satisfies $K_B(x,y) = 0$ if the geodesic distance satisfies $\rho(x,y) > c$.

(c) For any $\epsilon > 0$ all covariant derivatives of the Schwartz kernel are uniformly bounded on $(\Sigma \times \Sigma) \setminus \mathcal{U}_\epsilon$ where $\mathcal{U}_\epsilon$ is an ϵ-neighbourhood of the diagonal in Σ.

If the complete symbol admits a uniformly asymptotic expansion of the form (4.12) in any coordinate system, the related pseudo-differential operator is classical. The operators R in (a) has a complete uniformly bounded symbol and maps between $C^\infty_{\mathsf{c}}(M,E)$ and $C^\infty(M,E)$; moreover, it is continuous mapping between any Sobolev spaces $H^s(\Sigma,E)$ of bounded geometry. We denote the class of such operators with $\mathsf{B\Psi}^{-\infty}(\Sigma,E)$. Any uniformly bounded pseudo-differential operator of order $m\in\mathbb{R}$ can be represented as a sum of uniformly bounded pseudo-differential operators where one is properly supported with the same order and the other is smoothing: for $B\in\mathsf{B\Psi}^m(\Sigma,E)$ exists a $B'\in\mathsf{B\Psi}^m_{\mathsf{prop}}(\Sigma,E)$ and a $R\in\mathsf{B\Psi}^{-\infty}(\Sigma,E)$ such that

$$B = B' + R \quad ;$$

if B' is classical, B becomes also classical.

From now on, let $B\in\mathsf{B\Psi}^m_{\mathsf{prop,cl}}(\Sigma,E)$ be a uniformly elliptic and positive operator with scalar positive definite principal symbol $\boldsymbol{\sigma}_m(B)(p,\xi)$ which lies inside a sector. Following [Kor00], we construct a parametrix $C(\lambda)$ for the operator $(\lambda-B)$ for $\lambda\in\mathbb{C}\setminus\mathbb{R}_+$ with principal symbol $a(p,\xi;\lambda):=(\lambda-\boldsymbol{\sigma}_m(B))$ as for these values of λ the sum $(\lambda-B)$ stays elliptic. It is shown that

$$\begin{aligned}
(\lambda-B)C(\lambda) &= \mathbb{1}+R(\lambda) \\
&\text{with} \\
\mathsf{B\Psi}^{-\infty}(\Sigma,E)\ni R(\lambda) &: \exists C>0 \quad \text{with} \quad \|R(\lambda)\|_{\mathscr{B}(H^s(\Sigma,E),H^t(\Sigma,E))}\leq \frac{C}{1+|\lambda|} \\
\mathsf{B\Psi}^{-m}(\Sigma,E)\ni C(\lambda) &: \exists C>0 \quad \text{with} \quad \|C(\lambda)\|_{\mathscr{B}(H^s(\Sigma,E),H^{s+\alpha m}(\Sigma,E))}\leq C(1+|\lambda|)^{\alpha-1}
\end{aligned}$$

hold for $\lambda\in\Lambda_\theta:=\{\lambda\in\mathbb{C}\,|\,|\arg(\lambda)|>\theta\}$ with $\theta\in(0,\pi/2)$, $s,t\in\mathbb{R}$ and $\alpha\in(0,1)$. This is a sector in the complex plane with vertex at $0\in\mathbb{C}$. W.l.o.g. we assume that Λ_θ does not contain any point λ^* such that the principal symbol $a(p,\xi;\lambda^*)$ is vanishing for all $\xi\neq 0$. The construction of the parametrix works as usual by inverting the principal symbol in the asymptotic expansion:

$$b_{-m}(p,\xi;\lambda):=\frac{1}{a(p,\xi;\lambda)}=\frac{1}{\lambda-\boldsymbol{\sigma}_m(B)} \quad . \tag{8.2}$$

This value defines the principal symbol $\boldsymbol{\sigma}_{-m}(C(\lambda))(p,\xi)$ for $C(\lambda)$ while the subprincipal symbols b_{-m+l} for $l\in\mathbb{N}_0$ are defined recursively via a transport equation. It's asymptotic summation yields a local parametrix and by gluing all coordinate patches with the partition of unity from Lemma 3.3.2, it gives a global parametrix. W.l.o.g. we assume that Λ_θ does not intersect with the spectrum $\sigma(B)$ of B; otherwise we shrink Λ_θ to a smaller subset such that this condition holds. This implies for $\Lambda_\theta\neq\emptyset$ that the resolvent set $\rho(B)$ is not void and $\Lambda_\theta\subset\rho(B)$. Then $(\lambda-B)$ becomes invertible as unbounded operator with resolvent $\mathsf{R}(B,\lambda)=(\lambda-B)$ which is bounded on $H^s(\Sigma,E)$ due to the fact that the non-empty resolvent set of B implies closedness of B. Thus, for any $\lambda\in\Lambda_{\theta,r}:=\Lambda_\theta\cap\{\lambda\in\mathbb{C}\,|\,r<|\lambda|\}$ with $r>0$ the resolvent can be expressed by the parametrix: applying $C(\lambda)$ from the right gives

$$\begin{aligned}\mathbb{1} = \mathsf{R}(B,\lambda)(\lambda - B) \quad &\Rightarrow \quad C(\lambda) = \mathsf{R}(B,\lambda)\circ(\mathbb{1} + R(\lambda)) = \mathsf{R}(B,\lambda) + R(B,\lambda)\circ R(\lambda) \\ \Leftrightarrow \quad \mathsf{R}(B,\lambda) \quad &= \quad C(\lambda) - \mathsf{R}(B,\lambda)\circ R(\lambda) \quad .\end{aligned}$$

Since smoothing operators extend to continuous operators from any Sobolev space to any other Sobolev space and the resolvent is bounded on any Sobolev space, the composition on the right-hand side is again a smoothing operator and finally $\mathsf{R}(B,\lambda) \in \mathsf{B\Psi}^{-m}(\Sigma, E)$. In addition, the following norm estimates hold:

$$\begin{aligned}\exists\, C(s) > 0 \quad &: \quad \|\mathsf{R}(B,\lambda)\|_{\mathscr{B}(H^s(\Sigma,E),H^s(\Sigma,E))} \leq \frac{C(s)}{1+|\lambda|} \quad , \\ \exists\, C(s) > 0 \quad &: \quad \|\mathsf{R}(B,\lambda)\|_{\mathscr{B}(H^s(\Sigma,E),H^{s+m}(\Sigma,E))} \leq C(s)\end{aligned}$$

for any $s \in \mathbb{R}$ and $\lambda \in \Lambda_{\theta,r}$ with $r > 0$ and $\theta \in (0, \pi/2)$. One observes that B is a sectorial operator according to Definition 2.1.5 with sector S_θ such that $\Lambda_\theta = \mathbb{C} \setminus \overline{S_\theta}$ (see also footnote 6 on page 23).

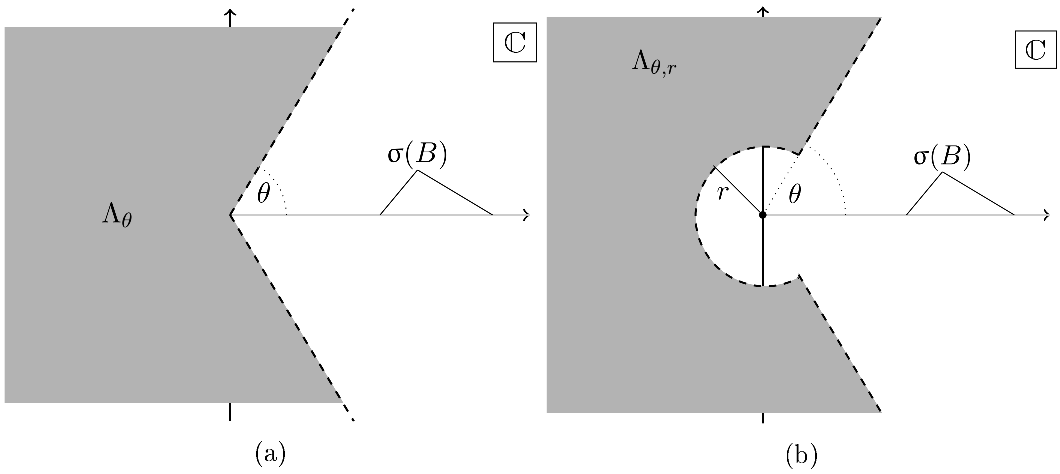

Figure 8.1.: Sector Λ_θ in (a) and the keyhole-sector $\Lambda_{\theta,r}$ in (b).

Now we are allowed to use the holomorphic functional calculus for sectorial operators with (2.10): let f be a function, which can be extended to an entire function such that for any $\eta \in \mathbb{R}$ the function $\mathbb{R}_+ \ni x \mapsto f(x + \mathbb{i}y)$ is a Schwartz function with uniformly bounded seminorms on compact subsets in $\mathbb{R}$. $f(B)$ can be expressed as Cauchy integral

$$f(B) := \frac{1}{2\pi\mathbb{i}} \int_\gamma f(\lambda)(\lambda - B)^{-1}\,\mathrm{d}\lambda = \frac{1}{2\pi\mathbb{i}} \int_\gamma f(\lambda)\mathsf{R}(B,\lambda)\,\mathrm{d}\lambda \tag{8.3}$$

where γ is a Hankel-like contour (2.12) which acts as boundary curve of the keyhole-sector $\Lambda_{\theta,r}$ in Fig. 8.1 (b). This allows us for example to consider the functions $f(B) = \mathrm{e}^{-tB}$ for $t \in \mathbb{R}_+$ because the resolvent estimates and the closedness of B imply that B is a generator of a parabolic semigroup. According to [Kor00, Prop.1] this can even be extended to a holomorphic semigroup. Another examples of our interest are complex powers with $z \in \mathbb{C}$ which can be either defined directly by setting $f(\lambda) = \lambda^z$ with the branch chosen, such that $\lambda^z = \mathrm{e}^{z \log \lambda}$ for $\lambda > 0$, or with the parabolic semigroup via

$$B^z := \frac{1}{\Gamma(-z)} \int_0^\infty x^{-(z+1)} \mathrm{e}^{-xB}\, \mathrm{d}x \tag{8.4}$$

for $\mathfrak{Re}\{z\} < 0$ and the Gamma function $\Gamma(-z)$ which is holomorphic for complex arguments with positive real part. The semigroup property then carries over to the case with complex powers: let $z, w \in \mathbb{C}$ have negative real parts. One can calculate the composition for $\mathfrak{Re}\{z + w\} < 0$ as follows:

$$\begin{aligned}
B^w \circ B^z &= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_{x=0}^\infty \int_{y=0}^\infty x^{-(w+1)} y^{-(z+1)} \mathrm{e}^{-xB} \mathrm{e}^{-yB}\, \mathrm{d}y\, \mathrm{d}x \\
&= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_{x=0}^\infty \int_{y=0}^\infty x^{-(w+1)} y^{-(z+1)} \mathrm{e}^{-(x+y)B}\, \mathrm{d}y\, \mathrm{d}x \\
&= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_{x=0}^\infty x^{-(w+1)} \int_{\sigma=x}^\infty (\sigma - x)^{-(z+1)} \mathrm{e}^{-\sigma B}\, \mathrm{d}\sigma\, \mathrm{d}x \\
&= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_{\varrho=0}^1 \varrho^{-(w+1)} (1 - \varrho)^{-(z+1)} \int_{\sigma=0}^\infty \sigma^{-(z+w+1)} \mathrm{e}^{-\sigma B}\, \mathrm{d}\sigma\, \mathrm{d}\varrho \\
&= \frac{1}{\Gamma(-z)\Gamma(-w)} \int_{\varrho=0}^1 \varrho^{-(w+1)} (1 - \varrho)^{-(z+1)}\, \mathrm{d}\varrho \int_{\sigma=0}^\infty \sigma^{-(z+w+1)} \mathrm{e}^{-\sigma B}\, \mathrm{d}\sigma \\
&= \frac{\Gamma(-w)\Gamma(-z)}{\Gamma(-z)\Gamma(-w)} \frac{1}{\Gamma(-(z+w))} \int_{\sigma=0}^\infty \sigma^{-(z+w+1)} \mathrm{e}^{-\sigma B}\, \mathrm{d}\sigma = B^{z+w} \quad .
\end{aligned} \tag{8.5}$$

We used that e^{-tB} is a semigroup and we have substituted $\sigma = (x + y)$ as well as $x = \sigma\varrho$ in the third and fourth step with $\varrho \in [0, 1]$. We used both depictions of the Beta functions in terms of its defining integral and as combination of Gamma functions. The semigroup property (8.5) can be extended to powers with positive real part: let $\mathfrak{Re}\{z\} > 0$; we choose a $(-k) \in \mathbb{N}_0$ such that $\mathfrak{Re}\{z + k\} < 0$. Hence B^{-k} and B^{z+k} are defined and moreover its composition such that we can define complex powers of B for any $z \in \mathbb{C}$ with positive real part to be

$$B^z := B^{z+k} \circ B^{-k} \quad . \tag{8.6}$$

It is clear from its definitions in (8.4) and (8.6) that B^z is a uniformly bounded pseudo-differential operator. The principal symbol of B^z for $\mathfrak{Re}\{z\} < 0$ is defined by the same integral in (8.3) where the resolvent is replaced with $(\boldsymbol{\sigma}_m(B)(p, \xi) - \lambda)^{-1}$:

$$\boldsymbol{\sigma}_{m\mathfrak{Re}\{z\}}(B^z)(p, \xi) := \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p, \xi)}\, \mathrm{d}\lambda = (\boldsymbol{\sigma}_m(B)(p, \xi))^z \quad . \tag{8.7}$$

The second equality follows with a contour integration argument: for any, but fixed $p \in \Sigma$, $\xi \in \dot{T}^*\Sigma$ we choose a $\rho \in (r, \infty)$. A parametrisation of the path γ is given by (2.12). We cut the two rays of the path at length ρ, defining the path γ_ρ which satisfies $\lim_{\rho\to\infty} \gamma_\rho = \gamma$. We close γ_p to a counterclockwise contour Γ_ρ with an arc of radius ρ inside the resolvent set: $\Gamma_\rho = \gamma_\rho \cup \mathsf{arc}_\rho$. By assumption, the scalar principal symbol of B lies on the positive half-line in the complex plane and thus outside the closed contour Γ_p. Since B is assumed to be a positive operator, its spectrum lies in $[0, \infty)$ which is not intersected by the closed contour for any value ρ. Moreover, the branch cut of λ^z at $\lambda = 0$ is bypassed by the closed contour such that $\lambda \mapsto \lambda^z(\boldsymbol{\sigma}_m(B)(p, \xi) - \lambda)^{-1}$ is holomorphic in the domain, rimmed by

Γ_ρ, for all $\rho > 0$, $\mathfrak{Re}\{z\} < 0$ and $p \in M$, $\xi \in \dot{T}_p^*\Sigma$. Applying the Cauchy integral formula yields

$$\begin{aligned}\int_{\gamma_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\lambda &= \int_{\Gamma_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\lambda - \int_{\mathsf{arc}_\rho} \frac{\lambda^z}{\boldsymbol{\sigma}_m(B)(p,\xi) - \lambda}\,\mathrm{d}\lambda \\ &= 2\pi\mathbb{i}\boldsymbol{\sigma}_m^z(B)(p,\xi) - \int_{\mathsf{arc}_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\lambda \quad .\end{aligned}$$

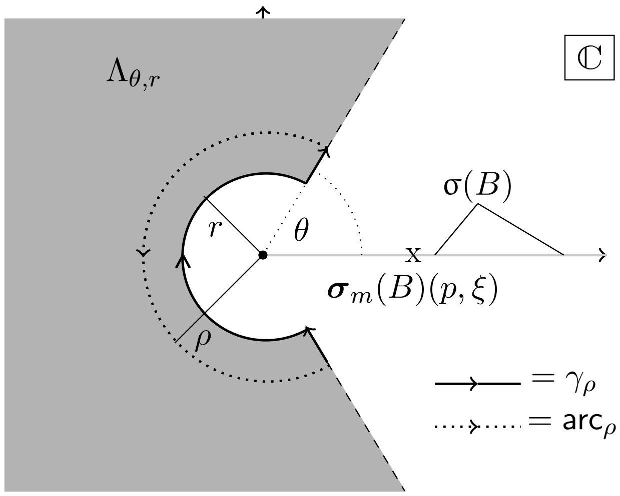

Figure 8.2.: Position of $\boldsymbol{\sigma}_m(B)(p,\xi)$ compared to the closed path Γ_ρ.

The arc-contribution can be estimated:

$$\begin{aligned}\int_{\mathsf{arc}_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\lambda &= \mathbb{i}\int_{\alpha=\theta}^{2\pi-\theta} \frac{\rho^{z+1}\mathrm{e}^{(1+z)\mathbb{i}\alpha}}{\rho\mathrm{e}^{\mathbb{i}\alpha} - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\alpha \\ &= \mathbb{i}\int_{\alpha=\theta}^{2\pi-\theta} \frac{\rho^{z+1}\mathrm{e}^{(1+\mathfrak{Re}\{z\})\mathbb{i}\alpha}\mathrm{e}^{-\mathfrak{Im}\{z\}\alpha}}{\rho\mathrm{e}^{\mathbb{i}\alpha} - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\alpha \\ \Rightarrow \left|\int_{\mathsf{arc}_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)}\,\mathrm{d}\lambda\right| &\leq \int_{\alpha=\theta}^{2\pi-\theta} \frac{\rho^{\mathfrak{Re}\{z\}+1}\mathrm{e}^{-\mathfrak{Im}\{z\}\alpha}}{\sqrt{(\boldsymbol{\sigma}_m(B)(p,\xi) - \rho\cos(\alpha))^2 + \rho^2\sin^2(\alpha)}}\,\mathrm{d}\alpha \\ &\leq \int_{\alpha=\theta}^{2\pi-\theta} \frac{C(\theta,z)\rho^{\mathfrak{Re}\{z\}+1}}{\sqrt{\boldsymbol{\sigma}_m^2(B)(p,\xi) + \rho^2 - 2\rho\boldsymbol{\sigma}_m(B)(p,\xi)\cos(\alpha)}}\,\mathrm{d}\alpha\end{aligned}$$

where $C(\theta, z)$ is the maximum of the exponential function on the domain of integration. If we choose $\rho > \boldsymbol{\sigma}_m(B)(p,\xi)$ for fixed p, ξ, we observe

$$\begin{aligned}\boldsymbol{\sigma}_m^2(B)(p,\xi) + \rho^2 - 2\rho\boldsymbol{\sigma}_m(B)(p,\xi)\cos(\alpha) &\geq \boldsymbol{\sigma}_m^2(B)(p,\xi) + \rho^2 - 2\rho\boldsymbol{\sigma}_m(B)(p,\xi) \\ &= (\rho - \boldsymbol{\sigma}_m(B)(p,\xi))^2 > 0\end{aligned}$$

and thus

$$\begin{aligned}&\Leftrightarrow \quad \sqrt{\boldsymbol{\sigma}_m^2(B)(p,\xi) + \rho^2 - 2\rho\boldsymbol{\sigma}_m(B)(p,\xi)\cos(\alpha)} \geq \rho - \boldsymbol{\sigma}_m(B)(p,\xi) > 0 \\ &\Leftrightarrow \quad (\boldsymbol{\sigma}_m^2(B)(p,\xi) + \rho^2 - 2\rho\boldsymbol{\sigma}_m(B)(p,\xi)\cos(\alpha))^{-\frac{1}{2}} \leq (\rho - \boldsymbol{\sigma}_m(B)(p,\xi))^{-1} \quad .\end{aligned}$$

Hence the arc-contribution vanishes for $\rho \to +\infty$ for $\Re\mathfrak{e}\{z\} < 0$:

$$\left| \int_{\mathrm{arc}_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)} \,\mathrm{d}\lambda \right| \leq C(\theta) \int_{\alpha=\theta}^{2\pi-\theta} \frac{\rho^{\Re\mathfrak{e}\{z\}+1}}{\rho - \boldsymbol{\sigma}_m(B)(p,\xi)} \,\mathrm{d}\alpha$$
$$= C(\theta) 2(\pi - \theta) \frac{\rho^{\Re\mathfrak{e}\{z\}+1}}{\rho - \boldsymbol{\sigma}_m(B)(p,\xi)} \overset{\rho\to\infty}{\longrightarrow} 0 \quad .$$

So we get with $\{\gamma_\rho\}$ as compact exhaustion of γ

$$\frac{1}{2\pi\mathbb{i}} \int_\gamma \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)} \,\mathrm{d}\lambda = \lim_{\rho\to\infty} \frac{1}{2\pi\mathbb{i}} \int_{\gamma_\rho} \frac{\lambda^z}{\lambda - \boldsymbol{\sigma}_m(B)(p,\xi)} \,\mathrm{d}\lambda = \boldsymbol{\sigma}_m^z(B)(p,\xi)$$

and so the second equality in (8.7). The factorising property of the principal symbol and the semigroup-property (8.6) of complex powers allows to extend the equality to all $z \in \mathbb{C}$. We sum up the results with all necessary details.

Proposition 8.1.1 (cf. Proposition 1 in [Kor00]). *Let $B \in \mathsf{B}\Psi^m_{\mathsf{prop,cl}}(\Sigma, E)$ be a positive and uniformly elliptic operator with scalar positive definite principal symbol $a(p,\xi) := \boldsymbol{\sigma}_m(B)(p,\xi)$ such that $(a(p,\xi) - \lambda)$ is not vanishing for λ in a sector $\Lambda_\theta \subset \rho(B)$ for $\theta \in (0, \pi/2)$ and $\xi \in \dot{T}_p\Sigma$. Then the following holds:*

(1) there exists a value $r > 0$ such that for $\lambda \in \Lambda_{\theta,r} \subset \mathbb{C}$ the operator B becomes sectorial and the resolvent satisfies $\mathsf{R}(B,\lambda) \in \mathsf{B}\Psi^{-m}(\Sigma, E)$ with principal symbol (8.2); moreover $\mathsf{R}(B,\lambda)$ becomes a bounded operator from $H^s(\Sigma, E)$ to $H^{s+m}(\Sigma, E)$ for any $s \in \mathbb{R}$;

(2) B generates a holomorphic semigroup e^{-zB} for $\Re\mathfrak{e}\{z\} > 0$ in $L^2(\Sigma, E)$;

(3) $B^z \in \mathsf{B}\Psi^{m\Re\mathfrak{e}\{z\}}(\Sigma, E)$ for all $z \in \mathbb{C}$ with principal symbol (8.7).

The last assertion is an equivalent of Seeley's theorem for complex powers in the setting of manifolds and vector bundles with bounded geometry.

For the purpose of this detour we specify Proposition 8.1.1 to such manifolds of bounded geometry which come from a Galois covering of a compact manifold. We have already pointed out in Remarks 5.1.6 (2) that a Galois covering with compact base can be viewed as manifold of bounded geometry. The role of the vector bundle of bounded geometry E reduces to a Γ-vector bundle. Any pseudo-differential operator, which acts between sections of a Γ-vector bundles, can be considered as uniformly bounded pseudo-differential operator, acting between vector bundles of bounded geometry. Elements in $\Psi^*_{\Gamma,\mathsf{prop}}(\Sigma, E)$ commute with the left representation action L^E_γ for all $\gamma \in \Gamma$. They correspond to uniformly bounded pseudo-differential operators which commute with L^E_γ. The c-locality property is implied by Γ-invariance and properly supportness. The subordinated partition of unity is also uniformly bounded due to Γ-invariance such that the Sobolev spaces of bounded geometry transfer to the Γ-Sobolev spaces in the Γ-setting. The space $\mathsf{B}\Psi^{-\infty}(\Sigma, E)$ corresponds to s-smoothing operators $S\Psi^{-\infty}_\Gamma(\Sigma, E)$ such that $\mathsf{B}\Psi^m_{(\mathsf{cl})}(\Sigma, E)$ corresponds to (classical) s-regular operators $S\Psi^m_{\Gamma,(\mathrm{cl})}(\Sigma, E)$.

Σ has bounded geometry	$\to$	Σ Γ-manifold
E has bounded geometry	$\to$	E Γ-vector bundle
$H^s(\Sigma, E)$	$\to$	$H^s_\Gamma(\Sigma, E)$
$\mathsf{B}\Psi_{\mathsf{prop}}(\Sigma, E)$	$\to$	$\Psi_{\Gamma,\mathsf{prop}}(\Sigma, E)$
$\mathsf{B}\Psi^{-\infty}(\Sigma, E)$	$\to$	$S\Psi^{-\infty}_\Gamma(\Sigma, E)$
$\mathsf{B}\Psi^m(\Sigma, E)$	$\to$	$S\Psi^m_\Gamma(\Sigma, E)$

Table 8.1.: Correspondences between quantities in the bounded geometry setting to the one in the Γ-setting.

After clarifying all correspondences we can extract from Proposition 8.1.1.

Corollary 8.1.2. *Let $B \in \Psi^m_{\Gamma,\mathsf{prop,cl}}(\Sigma, E)$ be an elliptic and positive operator with scalar positive definite principal symbol $a(p, \xi) := \boldsymbol{\sigma}_m(B)(p, \xi)$ such that $(a(p, \xi) - \lambda)$ is not vanishing for λ in a sector $\Lambda_\theta \subset \rho(B)$ for $\theta \in (0, \pi/2)$ and $\xi \in \dot{T}_p\Sigma$. Then the following holds:*

(1) there exists a value $r > 0$ such that for $\lambda \in \Lambda_{\theta,r} \subset \mathbb{C}$ the operator B becomes sectorial and the resolvent satisfies $\mathsf{R}(B, \lambda) \in S\Psi^{-m}_\Gamma(M, E)$ with principal symbol (8.2); moreover $\mathsf{R}(B, \lambda)$ becomes a bounded operator from $H^s_\Gamma(\Sigma, E)$ to $H^{s+m}_\Gamma(M, E)$ for any $s \in \mathbb{R}$;

(2) B generates a holomorphic semigroup e^{-zB} for $\mathfrak{Re}\{z\} > 0$ in $L^2_\Gamma(\Sigma, E)$;

(3) $B^z \in S\Psi^{m\mathfrak{Re}\{z\}}_\Gamma(\Sigma, E)$ for all $z \in \mathbb{C}$ with principal symbol (8.7).

The third point in this result is thereby an analogue of Seeley's theorem for complex powers in the setting of Galois coverings. It is only left to show that the resolvent commutes with the left action representation if B does: the commuting $L^E_\gamma B = B L^E_\gamma$ implies

$$(\lambda - B)L^E_\gamma = L^E_\gamma(\lambda - B)$$

for all $\lambda \in \mathbb{C}$ and any $\gamma \in \Gamma$. For $\lambda \in \rho(B)$ we can conclude from

$$\begin{aligned}(\lambda - B)^{-1}L^E_\gamma &= (\lambda - B)^{-1}L^E_\gamma(\lambda - B)(\lambda - B)^{-1} = (\lambda - B)^{-1}(\lambda - B)L^E_\gamma(\lambda - B)^{-1}\\ &= L^E_\gamma(\lambda - B)^{-1}\end{aligned}$$

the commuting of the resolvent with the left action representation. Thus, $f(B)$ in (8.3) commutes with L^E_Γ for all γ and allowable functions f.

We now come back to the (essentially) self-adjoint and elliptic operator $A \in \Psi^m_{\Gamma,\mathsf{prop,cl}}(\Sigma, E)$, $m > 0$; the operator $B := A^* \circ A = A^2 \in \Psi^{2m}_{\Gamma,\mathsf{prop,cl}}(\Sigma, E)$ is again self-adjoint and its spectrum lies on the positive half-line. Hence B becomes positive and altogether sectorial. The ellipticity of B is also inherited from the ellipticity of A and the principal symbol is positive definite if the principal symbol of A is positive definite. Under this assumptions, the expression $|A|^{-1}$ becomes well-defined as s-regular pseudo-differential operator of order $(-m)$ and finally one can consider projectors as s-regular Γ-pseudo-differential operators.

Corollary 8.1.3. *Let $A \in \Psi^m_{\Gamma,\mathsf{prop,cl}}(M,E)$ be elliptic and (essentially) self-adjoint with positive order $m>0$ such that A^2 has a positive definite principal symbol; the spectral projections onto eigenspaces of (the closure of) A in the spectral range $(0,\infty)$*

$$P_{>0}(A) := P_+(A) := \chi_{(0,\infty)}(A) = \frac{1}{2}\left(\mathbb{1} + A\circ |A|^{-1} - P_0(A)\right) \tag{8.8}$$

and in the spectral range $(-\infty,0)$

$$P_{<0}(A) := P_-(A) := \chi_{(-\infty,0)}(A) = \frac{1}{2}\left(\mathbb{1} - A\circ |A|^{-1} - P_0(A)\right) \tag{8.9}$$

are well-defined and satisfy $P_\pm(A) \in S\Psi^0_\Gamma(\Sigma,E)$ with principal symbols

$$\boldsymbol{\sigma}_0(P_\pm)(p,\xi) = \frac{1}{2}\left(1 \pm \boldsymbol{\sigma}_m(A)(p,\xi)\boldsymbol{\sigma}_{-m}\left(|A|^{-1}\right)(p,\xi)\right) \tag{8.10}$$

for all $p\in\Sigma$ and $\xi \in \dot{T}^*\Sigma$.

The composition $A\circ|A|^{-1}$ is well-defined due to properly supportness of A and is an element in $S\Psi^0_\Gamma(\Sigma,E)$.

Remarks 8.1.4.

(i) One has $P_\pm = p_\pm + r_\pm$ where $p_\pm \in \Psi^0_{\Gamma,\mathsf{prop,cl}}(\Sigma,E)$ and $r_\pm$ is s-smoothing. As the principal symbol is defined modulo smoothing terms, one has $\boldsymbol{\sigma}_m(P_\pm) = \boldsymbol{\sigma}_m(p_\pm)$.

(ii) $P_{\geq 0}(A) := \chi_{[0,\infty)}(A)$ and $P_{\leq}(A) := \chi_{(-\infty,0]}(A)$ differ from $P_+(A)$ respectively $P_-(A)$ in $P_0 := \chi_{\{0\}}$ which is s-smoothing. Hence we also have $P_{\geq 0}(A), P_{\leq 0}(A) \in S\Psi^0_\Gamma(\Sigma,E)$ with principal symbols (8.10).

(iii) Since A has been chosen to be self-adjoint, the complex power z in the construction of B^z does not depend on the angle of the rays in the Hankel-like contour. The same treatment could be applied to non-self-adjoint operators A since the self-adjoint operator A^*A occurs in this framework, but now each complex power z depends on the choice of the ray by fixing an angle via $\arg(z)$. This observation has been used in [BS20] for the index theorem with compact Cauchy boundaries, but a-priori non-self-adjoint Dirac operators.

8.1.2. g(a)APS-boundary conditions

In order to introduce boundary conditions, we need some kind of product structure near the boundary. The collar neighbourhood theorem states that the manifold is diffeomorphic to a product structure near each Cauchy boundary. The metric near the boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$ can be deformed in such a way that it becomes *ultra-static*: $g = -\,\mathrm{d}t^{\otimes 2} + g_{t_j}$ near Σ_j for $j\in\{1,2\}$; and each mean curvature H_{t_j} of the spacelike boundary hypersurfaces is vanishing identically. Both Dirac operators are then given by

$$D^{E_L}_\pm|_{\Sigma_j} := (\beta\otimes \mathbb{1}_{E_L})\,(\partial_t \mp \mathrm{i}A_j) = -(\beta\otimes\mathbb{1}_{E_L})\,(-\partial_t \pm \mathrm{i}A_j) \tag{8.11}$$

along Σ_j with past-directed timelike vector $\nu = -\partial_t$, $\beta = \mathbf{c}(\nu)$ and $A_j = A_{t_j}$ where we supressed the superscript for the twisting bundle. Recalling Remarks 6.2.5 (iii), the hypersurface Dirac operators A_1 and A_2 are essentially self-adjoint and the spectrum of their unique self-adjoint extensions, still denoted as A_1 and A_2, decomposes disjointly into a point and continuous spectrum. The eigenvalues with their multiplicities in the point spectrum are real and their eigenspaces are orthogonal to each other, but these spaces have in general infinitely many dimensions (multiplicites) and the point spectrum does not need to be discrete as for example on closed manifolds. The continuous spectrum is real-valued as well and their eigensections are smooth, but not square-integrable.

Recapitulating our general assumptions, the (twisted) hypersurface Dirac operators A_t along each slice are Γ-invariant differential operators of first order: $A_t \in \mathsf{Diff}^1_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$ for each $t \in \mathcal{T}(M)$. Hence they can be viewed as properly supported Γ-invariant pseudo-differential operators of order 1. Moreover, they are elliptic and essentially self-adjoint such that A_t^2 is an elliptic, positive and properly supported Γ-invariant pseudo-differential operator for each $t \in \mathcal{T}(M)$. In order to apply Corollary 8.1.3, we need to analyse the principal symbol of A_t which we calculate for the untwisted Dirac operator: fix any $t \in \mathcal{T}(M)$; let $w \in \Sigma_t$ and Φ is a smooth function on Σ_t such that $\rho := \mathrm{d}\Phi|_w$. We lift ρ to M where it becomes a one form ϑ, such that $\vartheta|_{\Sigma_t} = \rho$. We choose an orthonormal tangent frame $\{e_i\}_{i=1}^n$ on Σ_t which we also lift to a spacelike orthonormal tangent frame in M. The action of A_t of a spinor field $u \in C^\infty(\mathcal{S}^\pm(M))$ becomes

$$\sum_{i=1}^n \mathbf{c}\left(e_i\right) \nabla_{e_i} u = \mathbb{i}\beta \sum_{i=1}^n \mathbf{c}_t\left(e_i\right) \nabla_{e_i} u = \mathbb{i}\beta A_t u$$

which implies

$$\mathbb{i}\beta\boldsymbol{\sigma}_1(A_t)(w,\vartheta)u = \sum_{i=1}^n \boldsymbol{\sigma}_1(\mathbf{c}\left(e_i\right)\nabla_{e_i})(w,\vartheta)u = \mathbb{i}\sum_{i=1}^n \vartheta(e_i)\mathbf{c}\left(e_i\right)u = \mathbf{c}\left(\rho^\sharp\right)u \quad .$$

Restricted to the hypersurface, we gain

$$\mathbb{i}\beta\boldsymbol{\sigma}_1(A_t)(w,\rho) = \mathbb{i}\beta\boldsymbol{\sigma}_1(A_t)|_{\Sigma_t}(w,\rho) = \pm\mathbb{i}\mathbf{c}_t\left(\rho^\sharp\right)$$

and finally

$$\boldsymbol{\sigma}_1(A_t)(w,\rho) = \mp\beta\mathbf{c}_t\left(\rho^\sharp\right) \tag{8.12}$$

where the upper sign is for spinors with positive chirality and the lower sign for spinors with negative chirality; the sharp isomorphism is taken with respect to the Riemannian metric g_t. The principal symbol of A_t^2 becomes

$$\boldsymbol{\sigma}_2(A_t^* \circ A_t)(w,\rho) = \boldsymbol{\sigma}_2(A_t^2)(w,\rho) = \beta\mathbf{c}_t\left(\rho^\sharp\right)\beta\mathbf{c}_t\left(\rho^\sharp\right) = g_t(\rho^\sharp,\rho^\sharp)\mathbb{1}_{\mathcal{S}(\Sigma_t)} \tag{8.13}$$

which is a scalar times identity and positive definite because the metric does. Hence Corollary 8.1.3 and Proposition 5.3.7 (7) imply that $P_I(t) := \chi_I(A_t)$ is well-defined for all $t \in \mathcal{T}(M)$ and all measurable intervals $I \subset \mathbb{R}$, and in particular that

$$P_\pm(A_t) := \frac{1}{2}\left(\mathbb{1} \pm A_t \circ |A_t|^{-1} - P_0(A_t)\right) \in S\Psi^0_\Gamma(\mathcal{S}^\pm(\Sigma)) \quad ; \tag{8.14}$$

if I is bounded, we have $P_I(t) \in S\Psi_\Gamma^{-\infty}(\mathcal{S}^\pm(\Sigma_t))$, e.g. $P_0(t) := P_{\ker(A_t)}$. (8.7), (8.10), (8.12) and (8.13) then imply

$$\boldsymbol{\sigma}_{-1}\left(|A_t|^{-1}\right)(w,\rho) = (\boldsymbol{\sigma}_2(A_t^*A_t)(w,\rho))^{-\frac{1}{2}} = \left(g_t|_w(\rho^\sharp,\rho^\sharp)\right)^{-\frac{1}{2}} = \frac{1}{\|\rho\|_{g_t(w)}}\mathbb{1}_{\mathcal{S}(\Sigma_t)} \quad ;$$

and thus

$$\begin{aligned}\boldsymbol{\sigma}_0(P_{\gtrless 0}(t))(w,\rho) = \boldsymbol{\sigma}_0(P_\pm(t))(w,\rho) &= \frac{1}{2}\left(1 \pm \boldsymbol{\sigma}_1(A_t)(w,\rho)\boldsymbol{\sigma}_{-1}\left(|A_t|^{-1}\right)(w,\rho)\right) \\ &= \frac{1}{2}\left(1 \mp (\|\rho\|_{g_t(w)})^{-1}\beta\mathbf{c}_t\left(\rho^\sharp\right)\right) \quad .\end{aligned} \tag{8.15}$$

(8.14) carries over to the twisted case such that

$$P^{E_L}_{\gtrless 0}(t) = P^{E_L}_\pm(t) = \frac{1}{2}\left(\mathbb{1} \pm A_t^{E_L} \circ \left|A_t^{E_L}\right|^{-1} - P_{\ker\left(A_t^{E_L}\right)}\right) \in S\Psi^0_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t)) \quad . \tag{8.16}$$

Because the principal symbol of a twisted covariant derivative becomes the product of the principal symbols times the tensor product of the identities for each bundle, the principal symbol of the twisted hypersurface Dirac operator becomes $\boldsymbol{\sigma}_1(A_t^{E_L})(w,\rho) = \boldsymbol{\sigma}_1(A_t)(w,\rho) \otimes \mathbb{1}_{E_L|_{\Sigma_t}}$ such that its principal symbol becomes $\mp(\beta\mathbf{c}_t\left(\rho^\sharp\right) \otimes \mathbb{1}_{E_L|_{\Sigma_t}})$ and

$$\begin{aligned}\boldsymbol{\sigma}_2((A_t^{E_L})^*A_t^{E_L})(w,\rho) &= (\beta\mathbf{c}_t\left(\rho^\sharp\right) \otimes \mathbb{1}_{E_L|_{\Sigma_t}}) \circ (\beta\mathbf{c}_t\left(\rho^\sharp\right) \otimes \mathbb{1}_{E_L|_{\Sigma_t}}) \\ &= (\beta\mathbf{c}_t\left(\rho^\sharp\right)\beta\mathbf{c}_t\left(\rho^\sharp\right) \otimes \mathbb{1}_{E_L|_{\Sigma_t}}) = g_t|_w(\rho^\sharp,\rho^\sharp)\mathbb{1}_{\mathcal{S}_{L,E}(\Sigma_t)} \quad .\end{aligned}$$

With the same arguments as for the untwisted hypersurface Dirac operator we gain

$$\boldsymbol{\sigma}_{-1}\left(\left|A_t^{E_L}\right|^{-1}\right)(w,\rho) = \left(\boldsymbol{\sigma}_2((A_t^{E_L})^*A_t^{E_L})(w,\rho)\right)^{-\frac{1}{2}} = \frac{1}{\|\rho\|_{g_t(w)}}\mathbb{1}_{\mathcal{S}_{L,E}(\Sigma_t)} \quad .$$

One observes that the projectors also factorise with respect to the tensor product:

$$\begin{aligned}\boldsymbol{\sigma}_0\left(P^{E_L}_\pm(t)\right)(w,\rho) &= \frac{1}{2}\left(\mathbb{1}_{\mathcal{S}_{L,E}(\Sigma_t)} \pm \boldsymbol{\sigma}_{-1}\left(\left|A_t^{E_L}\right|^{-1}\right)(w,\rho) \circ \boldsymbol{\sigma}_1(A_t^{E_L})(w,\rho)\right) \\ &= \frac{1}{2}\left(\mathbb{1}_{\mathcal{S}_{L,E}(\Sigma_t)} \mp (\|\rho\|_{g_t(w)})^{-1}(\beta\mathbf{c}_t\left(\rho^\sharp\right) \otimes \mathbb{1}_{E_L|_{\Sigma_t}})\right) \\ &= \boldsymbol{\sigma}_0\left(P_\pm(t)\right)(w,\rho) \otimes \mathbb{1}_{E_L|_{\Sigma_t}} \quad .\end{aligned} \tag{8.17}$$

If we cut the spectral range at another point $a \in \mathbb{R}$ than zero, we will also write

$$\begin{aligned} P_{>a}(t) &:= P_{(a,\infty)}(t) = \chi_{(a,\infty)}(A_t) \quad \text{and} \quad P_{<a}(t) := P_{(-\infty,a)}(t) = \chi_{(-\infty,a)}(A_t) \\ P_{\geq a}(t) &:= P_{>a}(t) + P_{\{a\}}(t) \quad \text{and} \quad P_{\leq a}(t) := P_{<a}(t) + P_{\{a\}}(t)\,. \end{aligned}$$

These projectors are also s-regular Γ-pseudo-differential operators: depending on whether the spectral cutpoint $a \in \mathbb{R}$ is bigger or smaller then 0, they can be related to $P_{\lesseqgtr 0}$: for a fixed $t \in \mathcal{T}(M)$ the differences

$$P_{\leq 0}(t) - P_{\leq a}(t) = \begin{cases} P_{(a,0]}(t) & a < 0 \\ 0 \quad \text{for} & a = 0 \\ -P_{(0,a]}(t) & a > 0 \end{cases} \tag{8.18}$$

and

$$P_{\geq 0}(t) - P_{\geq a}(t) = \begin{cases} P_{[a,0)}(t) & a < 0 \\ 0 \quad \text{for} & a = 0 \\ -P_{[0,a)}(t) & a > 0 \end{cases} \tag{8.19}$$

as well as all other possibilities via

$$P_{\leq a}(t) - P_{<a}(t) = P_{\{a\}}(t) = P_{\geq a}(t) - P_{>a}(t) \quad . \tag{8.20}$$

The differences are projectors with bounded intervals in the spectrum of A_t wherefore they are s-smoothing. Consequently,

$$P_{\gtrless a}(t), P_{\geq a}(t), P_{\leq a}(t) \in S\Psi^0_\Gamma(\mathcal{S}^\pm(\Sigma_t)) \quad . \tag{8.21}$$

Knowing this, one can easily see that the *Kato dual* $P_I^\perp(t) := \mathbb{1}_{\mathcal{S}^\pm(\Sigma_t)} - P_I(t) = P_{I^\complement}(t)$ is s-regular of order 0 for all $I \subset \mathbb{R}$ and all $t \in \mathcal{T}(M)$. All these properties of course transfer to any Γ-invariant operator A as in Corollary 8.1.3:

$$P_{\gtrless b}(A), P_{\geq b}(A), P_{\leq b}(A), P_I(A) \in S\Psi^0_\Gamma(\Sigma, E) \tag{8.22}$$

for all $b \in \mathbb{R}$ and unbounded $I \subset \mathbb{R}$. After clarifying these details, we are able to introduce the boundary conditions of our interest. The *Atiyah-Patodi-Singer (APS) boundary conditions* are defined as follows:

$$\begin{array}{rcl} P_{[0,\infty)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(-\infty,0]}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for positive chirality}} \end{array} \quad \text{and} \quad \begin{array}{rcl} P_{(-\infty,0)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(0,\infty)}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for negative chirality}} \end{array} \quad . \tag{8.23}$$

Another set of boundary conditions are the *anti Atiyah-Patodi-Singer (aAPS or anti-APS) boundary conditions* which are orthogonal to the APS boundary conditions:

$$\begin{array}{rcl} P_{(-\infty,0)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(0,\infty)}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for positive chirality}} \end{array} \quad \text{and} \quad \begin{array}{rcl} P_{[0,\infty)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(-\infty,0]}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for negative chirality}} \end{array} \quad . \tag{8.24}$$

The boundary conditions for the negative chirality are chosen to be the adjoint (a)APS-boundary conditions for positive chirality because $D_- = -D_+^\dagger$ by Corollary 6.2.4. Thus, they are defined by the hypersurface boundary operator of D_-. Since D_+ acts with A_j along the hypersurface, we need to take $(-A_j)$ at the boundary Σ_j. Following the general treatment in [BB11, Sec.7.2] and the concrete construction in [Shi, Sec.4.14], the adjoint boundary condition of $P_I(A_j)u|_{\Sigma_j} = 0$ becomes with $\chi_I(-A_t) = \chi_{-I}(A_t)$ and $(-I) := \{-x \,|\, x \in I\}$

$$0 = P_{-I^\complement}(-A_j)u|_{\Sigma_j} = P_{I^\complement}(A_j)u|_{\Sigma_j}$$

where $I \subset \mathbb{R}$ is any Borel set in the spectrum of A_j. This is based on the observation that the involution β (see $\beta^2 = \mathbb{1}_{\mathcal{S}(M)}$) anti-commutes with A_t for each slice such that $P_I(A_t)\beta = \beta P_I(-A_t) = \beta P_{-I}(A_t)$. The complement follows from the definition of adjoint boundary conditions as an orthogonality relation. Hence the adjoint boundary conditions for positive chirality are indeed the orthogonal boundary conditions.

All introduced boundary conditions induce an orthogonal splitting of L^2_Γ-spaces. We denote the range of the projectors on L^2_Γ-spaces by $L^2_{\Gamma,I}(\mathcal{S}^\pm_{L,E}(\Sigma_j)) = P_I(t_j)[L^2_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_j))] =: \mathsf{ran}\,(P_I(t_j))$, then we can decompose as follows:

$$\begin{array}{rcl} L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_1)) &=& L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \\ L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)) &=& L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_2)) \oplus L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+_{L,E}(\Sigma_2)) \\ L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_1)) &=& L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \\ L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)) &=& L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_2)) \oplus L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-_{L,E}(\Sigma_2)) \end{array} \quad . \tag{8.25}$$

The choice of the cut in the spectrum around zero is somehow arbitrary. In order to relax this choice one introduces generalised boundary conditions: let $a_1, a_2 \in \mathbb{R}$; the *generalised Atiyah-Patodi-Singer (gAPS) boundary conditions* are defined by

$$\begin{array}{rcl} P_{[a_1,\infty)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(-\infty,a_2]}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for positive chirality}} \end{array} \quad \text{and} \quad \begin{array}{rcl} P_{(-\infty,a_1)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(a_2,\infty)}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for negative chirality}} \end{array} \quad . \tag{8.26}$$

Another set of boundary conditions is given by the *generalised anti-Atiyah-Patodi-Singer (gaAPS) boundary conditions*:

$$\begin{array}{rcl} P_{(-\infty,a_1)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(a_2,\infty)}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for positive chirality}} \end{array} \quad \text{and} \quad \begin{array}{rcl} P_{[a_1,\infty)}(t_1)(u|_{\Sigma_1}) &=& 0 \\ P_{(-\infty,a_2]}(t_2)(u|_{\Sigma_2}) &=& 0 \\ \multicolumn{3}{c}{\text{for negative chirality}} \end{array} \quad , \tag{8.27}$$

cf. [BH18, Sec.4.2]. Since our projectors are orthogonal, we can introduce similar L^2_Γ-orthogonal splittings:

$$\begin{array}{rcl} L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_1)) &=& L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \\ L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)) &=& L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_2)) \oplus L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+_{L,E}(\Sigma_2)) \\ L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_1)) &=& L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \\ L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)) &=& L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_2)) \oplus L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^-_{L,E}(\Sigma_2)) \end{array} \quad . \tag{8.28}$$

Because of $\mathsf{ran}\,(P_I(t)) = \mathsf{ker}\,(\mathbb{1} - P_I(t))$, all subspaces in (8.28) are closed for all a_1, a_2. Since $P_I(t)$ is s-regular for all $I \subset \mathbb{R}$, it is a Γ-morphism on L^2_Γ-spaces in regards to Proposition 5.3.7 (4) such that all the ranges and hence all subspaces in the orthogonal splittings are projective Hilbert Γ-modules. In particular, we can show that these closed subspaces are free Hilbert Γ-modules.

Lemma 8.1.5. *Let $I \subset \mathbb{R}$ and $E \to \Sigma$ the Γ-vector bundle over the Γ-manifold Σ with compact base Σ_Γ, then there exists a unitary isomorphism such that*

$$L^2_{\Gamma,I}(E) \cong \ell^2(\Gamma) \otimes L^2_I(E_\Gamma) \quad .$$

Proof. We already noticed that $L^2_{\Gamma,I}(E)$ are closed and Γ-invariant subsets of the free Hilbert Γ-module $L^2_\Gamma(E)$ and are therefore unitarily related to a closed Γ-invariant subset of $\ell^2(\Gamma)\otimes L^2(E_\Gamma)$; hence they are projective Hilbert Γ-modules. In order to show that all $L^2_{\Gamma,I}$ are free Hilbert Γ-modules, we recall the isomorphism (5.43) which we denote with $\mathcal{J}$. We want to show that the restriction of $\mathcal{J}$ to $L^2_{\Gamma,I}(E)$ for any, but fixed interval I is an isomorphism with range $\ell^2(\Gamma)\otimes L^2_I(E_\Gamma)$.

Let $v\in L^2_{\Gamma,I}(E)$; the action of $\mathcal{J}$ on v is

$$\mathcal{J}v=\sum_{\gamma\in\Gamma}\delta_\gamma\otimes L^E_{-\gamma}v|_{\mathcal{F}}$$

with $\mathcal{F}$ as fundamental domain of Σ. We denote with $P_I : L^2_\Gamma(E)\to L^2_{\Gamma,I}(E)$ the Γ-invariant spectral projection on the level of Γ-modules and $\underline{P_I} : L^2(E|_{\mathcal{F}})\to L^2_I(E|_{\mathcal{F}})$ the spectral projection on the level of Hilbert spaces over the base manifold. $P_I v=v$ implies that

$$\mathcal{J}v=\mathcal{J}(P_Iv)=\sum_{\gamma\in\Gamma}\delta_\gamma\otimes L^E_{-\gamma}(P_Iv)|_{\mathcal{F}}\overset{(*)}{=}\sum_{\gamma\in\Gamma}\delta_\gamma\otimes P_I(L^E_{-\gamma}(v)|_{\mathcal{F}})=(\mathbb{1}\otimes\underline{P_I})\mathcal{J}v$$

where we used in $(*)$, that P_I commutes with the left action representation and with the restriction to the fundamental domain. We observe that $\mathcal{J}v\in\ell^2(\Gamma)\otimes L^2_I(E|_{\mathcal{F}})$ and thus $\mathcal{J}(L^2_{\Gamma,I}(E))\subset\ell^2(\Gamma)\otimes L^2_I(E|_{\mathcal{F}})$. As $\mathcal{F}$ is dense in Σ_Γ, we moreover have $\mathcal{J}(L^2_{\Gamma,I}(E))\subset\ell^2(\Gamma)\otimes L^2_I(E_\Gamma)$.

Let $w\in\ell^2(\Gamma)\otimes L^2_I(E_\Gamma)$. The unitary isomorphism implies that there exists a unique $u\in L^2_\Gamma(E)$ such that $w=\mathcal{J}u$. On the other hand we have $(\mathbb{1}\otimes\underline{P_I})w=w$ and herewith

$$w=(\mathbb{1}\otimes\underline{P_I})w=(\mathbb{1}\otimes\underline{P_I})\mathcal{J}u=\sum_{\gamma\in\Gamma}\delta_\gamma\otimes\underline{P_I}(L^E_{-\gamma}(u)|_{\mathcal{F}})\overset{(**)}{=}\sum_{\gamma\in\Gamma}\delta_\gamma\otimes L^E_{-\gamma}(P_Iu)|_{\mathcal{F}}$$

where in $(**)$ we can argue as for $(*)$. If we take the difference with w on both sides and apply the inner product with a basis element $(\delta_g\otimes e_i)$ of $\ell^2(\Gamma)\otimes L^2_I(E_\Gamma)$ for $g\in\Gamma$ and $e_i\in L^2_I(E_\Gamma)$, we get

$$\begin{aligned}
0&=w-\textstyle\sum_{\gamma\in\Gamma}\delta_\gamma\otimes L^E_{-\gamma}(P_Iu)|_{\mathcal{F}}=\sum_{\gamma\in\Gamma}\delta_\gamma\otimes\left[L^E_{-\gamma}(u)|_{\mathcal{F}}-L^E_{-\gamma}(P_Iu)|_{\mathcal{F}}\right]\\
\Rightarrow\ 0&=\textstyle\sum_{\gamma\in\Gamma}\left\langle\delta_\gamma\otimes\left[L^E_{-\gamma}(u)|_{\mathcal{F}}-L^E_{-\gamma}(P_Iu)|_{\mathcal{F}}\right]\,\middle|\,\delta_g\otimes e_i\right\rangle_{\ell^2(\Gamma)\otimes L^2_I(E_\Gamma)}\\
\Leftrightarrow\ 0&=\textstyle\sum_{\gamma\in\Gamma}\left\langle\left[L^E_{-\gamma}(u)|_{\mathcal{F}}-L^E_{-\gamma}(P_Iu)|_{\mathcal{F}}\right]\,\middle|\,e_i\right\rangle_{L^2_I(E_\Gamma)}\left\langle\delta_\gamma\,\middle|\,\delta_g\right\rangle_{\ell^2(\Gamma)}\\
\Leftrightarrow\ 0&=\left\langle\left[L^E_{-g}(u)|_{\mathcal{F}}-L^E_{-g}(P_Iu)|_{\mathcal{F}}\right]\,\middle|\,e_i\right\rangle_{L^2_I(E_\Gamma)}
\end{aligned}$$

and thus $L^E_{-g}(u)|_{\mathcal{F}}=L^E_{-g}(P_Iu)|_{\mathcal{F}}$. Since the left action representation is unitary, this implies $u|_{\mathcal{F}}=(P_Iu)|_{\mathcal{F}}$. As $\mathcal{F}$ and $\overline{\mathcal{F}}$ differ in a set of measure zero, we have $u|_{\overline{\mathcal{F}}}=(P_Iu)|_{\overline{\mathcal{F}}}$ almost everywhere. This carries over to all translations of $\overline{\mathcal{F}}$ with Γ:

$$u|_{\gamma\overline{\mathcal{F}}}=(P_Iu)|_{\gamma\overline{\mathcal{F}}}$$

almost everywhere for all $\gamma\in\Gamma$. As $\Sigma=\bigcup_{\gamma\in\Gamma}\gamma(\overline{\mathcal{F}})$, we have $u=P_Iu$ almost everywhere and so $u \in L^2_{\Gamma,I}(E)$. Consequently, we have shown $\mathcal{J}(L^2_{\Gamma,I}(E)) = \ell^2(\Gamma)\otimes L^2_I(E_\Gamma)$ and (5.43) restricts to an isomorphism as stated in the claim. This restriction does not affect the Γ-invariance of this map and a unitary isomorphism can be taken from its polar decomposition due to Proposition 5.2.6 (1). □

Similar splittings can be obtained if we replace the L^2_Γ-spaces with Γ-Sobolev spaces:

$$H^s_{\Gamma,I}(\mathcal{S}^\pm(\Sigma_j)) := P_I(t_j)\left(H^s_\Gamma(\mathcal{S}^\pm(\Sigma_j))\right) \quad .$$

Motivated from the functional-analytic treatment of spectral flow in [Ron19] and [vdDR20], we also introduce projectors with domains on the range of another projector: let $a, b \in \mathbb{R}$ and $\blacktriangle, \blacktriangledown$ any symbol in $\{<, >, \leq, \geq\}$; we define as *restricted projectors* the maps

$$\overline{P}^{\blacktriangle a}_{\blacktriangledown b} := P_{\blacktriangledown b} \,:\, \mathsf{ran}\,(P_{\blacktriangle a}) \to \mathsf{ran}\,(P_{\blacktriangledown b}) \quad . \tag{8.29}$$

One is tempted to write $P_{[a,b]}$ for $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}$ for $a < b$. However, the former is a bounded operator on L^2_Γ-spaces, but in general not necessarily Fredholm. The latter operator is the restriction of the former to a closed subspace of L^2_Γ. It turns out that (8.29) is in fact Γ-Fredholm under certain conditions.

Lemma 8.1.6. *Let $a, b \in \mathbb{R}$, A as in Corollary 8.1.3 and $P_{\blacktriangle a}$, $P_{\blacktriangledown b}$ spectral projections of A in $L^2_\Gamma(E)$; if* $\mathrm{codim}_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a})\right) < \infty$ *and* $\dim_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangle a}) \cap \mathsf{ran}\left((P_{\blacktriangledown b})^\perp\right)\right) < \infty$, *then $\overline{P}^{\blacktriangle a}_{\blacktriangledown b} \in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{\blacktriangle a}), \mathsf{ran}\,(P_{\blacktriangledown b}))$ with*

$$\mathrm{ind}_\Gamma(\overline{P}^{\blacktriangle a}_{\blacktriangledown b}) = \dim_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangle a}) \cap \mathsf{ran}\left((P_{\blacktriangledown b})^\perp\right)\right) - \mathrm{codim}_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a})\right) \tag{8.30}$$

where

$$\mathrm{codim}_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangledown b})\right) = \dim_\Gamma\left(\mathsf{ran}\,(P_{\blacktriangledown b}) \big/ \mathsf{ran}\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a})\right) \quad . \tag{8.31}$$

Proof. The range and the kernel of $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}$ can be calculated directly:

$$\begin{aligned} \ker\left(\overline{P}^{\blacktriangle a}_{\blacktriangledown b}\right) &= \ker\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a}) = \mathsf{ran}\left((P_{\blacktriangledown b})^\perp\right) \cap \mathsf{ran}\,(P_{\blacktriangle a}) \\ \mathsf{ran}\left(\overline{P}^{\blacktriangle a}_{\blacktriangledown b}\right) &= \mathsf{ran}\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a}) \end{aligned}$$

because an element in $\mathsf{ran}\,(P_{\blacktriangle a})$ might have non-trivial intersection with $\mathsf{ran}\,(P_{\blacktriangledown b})$ and $(\mathsf{ran}\,(P_{\blacktriangledown b}))^\perp$, but only functions in the former intersection contribute to the range. By preassumption the kernel has finite Γ-dimension. Since all ranges are closed subsets of $L^2_\Gamma(\mathcal{S}^\pm_{L,E}(\Sigma_t))$, the range of $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}$ as intersection of closed sets does as well and has finite Γ-codimension by assumption.

In order to calculate the Γ-index, it is left to compute the kernel of the formal adjoint of $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}$. As $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}$ has closed range by construction, the closed range theorem implies that exactly this kernel is given by

$$\left(\mathsf{ran}\,(P_{\blacktriangledown b}) \cap \mathsf{ran}\,(P_{\blacktriangle a})\right)^\perp$$

where the orthogonal complement has to be taken in $\mathsf{ran}\,(P_{\blacktriangledown b})$ as ambient Hilbert space. The intersection of Hilbert Γ-modules is in general not a Hilbert Γ-module. But since the intersection of ranges results again in a range of a spectral projection, the intersection

$$\mathsf{ran}\left(P_{I(a,b)}\right) := \mathsf{ran}\left(P_{\blacktriangledown b}\right) \quad \text{with} \quad I(a,b) \subset \mathbb{R}$$

is again a projective Hilbert Γ-module. Lemma 5.2.7 implies that the orthogonal complement of $\mathsf{ran}\left(P_{I(a,b)}\right)$ is a projective Hilbert Γ-module, too. The same holds true for the quotient

$$\mathsf{ran}\,(P)_{\blacktriangledown b} \Big/ \mathsf{ran}\left(P_{I(a,b)}\right) \quad .$$

As $\mathsf{ran}\left(P_{I(a,b)}\right)$ is closed, the canonical isomorphy between the quotient and the orthogonal complement implies a unitary Γ-isomorphism such that

$$\dim_\Gamma \mathsf{ran}\left(P_{I(a,b)}\right)^\perp = \dim_\Gamma \left(\mathsf{ran}\,(P)_{\blacktriangledown b} \Big/ \mathsf{ran}\left(P_{I(a,b)}\right) \right)$$

according to Proposition 5.2.8 (5). The right-hand side coincides with the Γ-codimension of $\mathsf{ran}\left(P_{I(a,b)}\right)$ which shows the claimed Γ-index formula. □

8.2. Γ-spectral flow

The spectral flow is used in [BS19] to connect the algebraic index with geometric quantities. We introduce the algebraic definition of spectral flow in our setting and give an analytic expression at the end of this section. We refer to the achievements, made in [BCP+06] for the case of a general von Neumann algebra, and specify as well as modify the results to our case.

8.2.1. Idea of spectral flow

Let $I \subset \mathbb{R}$ be a closed interval and $\{S_t\}_{t\in I}$ a continuous path of self-adjoint Fredholm operators on a Hilbert space $\mathcal{H}$. The heuristical idea of the spectral flow is that it counts the net number of eigenvalues with multiplicities which pass through zero from negative to positive along the path. If $\mathcal{H}$ is finite-dimensional, the path of operators coincides with a continuous path of Hermitian maps/matrices; this path defines in turn a path of quadratic forms

$$s_t(u,v) := \left\langle S_t u \,\middle|\, v \right\rangle_{\mathcal{H}}$$

for $u, v \in \mathcal{H}$ and $t \in I$. The spectral flow for $I = [t_1, t_2]$ is half the difference of the signatures of these quadratic forms at $t = t_1$ and $t = t_2$:

$$\mathrm{sf}\,\{S_t\}_{t\in I} = \frac{1}{2}\left[\mathrm{Sign}(s_{t_2}) - \mathrm{Sign}(s_{t_1})\right] \quad .$$

A topological definition of spectral flow for operators on an infinite-dimensional Hilbert space with discrete spectrum is presented in [APS76, Sec.7] where one needs to look at the graph of eigenvalues $t \mapsto \lambda(t)$ and perturbs the t-axis into broken lines with alternating horizontal and vertical segments. The segments are chosen in such a way that the multiplicites are correctly counted and the horizontal segments do not contain points of the spectrum. An analytic definition has been given by John Phillips in [Phi96] which we are going to present in this subsection. $\mathscr{Q}(\mathcal{H})$ denotes the Calkin algebra $\mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ with Calkin map $\Pi : \mathscr{B}(\mathcal{H}) \to \mathscr{Q}(\mathcal{H})$. As Fredholm operators are characterised by operators on

$\mathcal{H}$, which are invertible modulo compact operators, the Calkin map of those operators are invertible in $\mathscr{Q}(\mathcal{H})$ and the set of Fredholm operators can be defined as

$$\mathscr{F}(\mathcal{H}) := \{A \in \mathscr{B}(\mathcal{H}) \,|\, \Pi(A) \,\text{invertible in}\, \mathscr{Q}(\mathcal{H})\} \quad . \tag{8.32}$$

The subset $\mathscr{F}^{\text{sa}}(\mathcal{H})$ contains all self-adjoint elements in $\mathscr{F}(\mathcal{H})$. The essential spectrum σ_{ess} of these operators induces a decomposition of this space into three different subsets:

$$\mathscr{F}^{\text{sa}}(\mathcal{H}) = \mathscr{F}^{\text{sa}}_{+}(\mathcal{H}) \sqcup \mathscr{F}^{\text{sa}}_{-}(\mathcal{H}) \sqcup \mathscr{F}^{\text{sa}}_{*}(\mathcal{H})$$

with $(\mathbb{R}_+ := (0,\infty)$, $\mathbb{R}_- := (-\infty,0))$

$$\begin{aligned}\mathscr{F}^{\text{sa}}_{\pm}(\mathcal{H}) &:= \{A \in \mathscr{F}^{\text{sa}}(\mathcal{H}) \,|\, \Pi(A) \gtrless 0\} = \{A \in \mathscr{F}^{\text{sa}}(\mathcal{H}) \,|\, \sigma_{\text{ess}}(\Pi(A)) \subset \mathbb{R}_{\pm}\} \quad , \\ \mathscr{F}^{\text{sa}}_{*}(\mathcal{H}) &:= \{A \in \mathscr{F}^{\text{sa}}(\mathcal{H}) \,|\, \sigma_{\text{ess}}(\Pi(A)) \nsubseteq (\mathbb{R}_+ \sqcup \mathbb{R}_-)\} \quad ,\end{aligned}$$

denoting the spaces of essentially positive and respectively negative operators in $\mathscr{F}^{\text{sa}}(\mathcal{H})$ and their unified complement. It is a classical result[31] that the subspaces $\mathscr{F}^{\text{sa}}_{\pm}(\mathcal{H})$ are contractible to $\{\pm\mathbb{1}_{\mathcal{H}}\}$. The introduced continuous path of operators can then be viewed as a path in $\mathscr{F}^{\text{sa}}_{*}(\mathcal{H})$ as the only non-trivial contribution in $\mathscr{F}^{\text{sa}}(\mathcal{H})$. Suppose there is a continuous family $\{P_t\}$ of finite-rank projections such that P_t is the spectral projection of S_t for each $t \in I$ inside an interval $[-a,a]$ for $a \in \mathbb{R}$. The spectral flow is then described as the difference of eigenspace dimensions for non-negative eigenvalues of S_tP_t at the endpoints of the path, i.e.

$$\dim \mathsf{Eig}(S_{t_2}P_{t_2}) - \dim \mathsf{Eig}(S_{t_1}P_{t_1}) \quad .$$

As the path of finite projections is assumed to be continuous, any leakage of eigenvalues is precluded. The existence of such a global path is in general not possible, but locally constructable by partioning the time interval I into subintervals and adding up each spectral flow contribution over the subintervals. This construction is assured by the following result.

Lemma 8.2.1 (Lemma in [Phi96]). *Let $S \in \mathscr{F}^{sa}_{*}(\mathcal{H})$; there exists a number $a \in \mathbb{R}_+$ and a neighbourhood $\mathcal{U}_S \subset \mathscr{F}^{sa}_{*}(\mathcal{H})$ of S such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous finite-rank projection on $\mathcal{U}_S$.*

Hence the initial analytic construction of spectral flow can be made rigorous.

Definition 8.2.2 (cf. [Phi96]). Let $\{S_t\}_{t\in I}$ be a continuous path in $\mathscr{F}^{\text{sa}}_{*}(\mathcal{H})$; choose an $N \in \mathbb{N}$ and a partition of the time interval $I = [t_1,t_2]$, $t_1 = \tau_0 < \tau_1 < \ldots < \tau_N = t_2$, and positive real numbers $a_0, a_1, \ldots, a_N$ such that $t \mapsto \chi_{[-a_i,a_i]}(S_t)$ is a continuous finite-rank projection on $[\tau_{i-1},\tau_i]$ for each $i \in \{1,\ldots,N\}$, then the spectral flow of the path is defined to be the number

$$\text{sf}\,\{S_t\}_{t\in I} := \sum_{i=1}^{N} \left[\dim \mathsf{ran}\left(\chi_{[0,a_i]}(S_{\tau_i})\right) - \dim \mathsf{ran}\left(\chi_{[0,a_i]}(S_{\tau_{i-1}})\right)\right] \quad . \tag{8.33}$$

We collect some properties of this quantity.

[31] See [AS69, Thm.B] or one recalls [Phi96, Prop.1].

Proposition 8.2.3 (Theorem and Proposition 2,3 in [Phi96]). *Let $\{S_t\}_{t\in I}$ be a continuous path in $\mathscr{F}_*^{sa}(\mathcal{H})$ and I_1, I_2 disjoint subintervals of I.*

(1) $\operatorname{sf}\{S_t\}_{t\in I}$ *is well-defined and depends only on the continuous path.*

(2) If the path $\{S_t\}_{t\in I}$ is a concatenation of the paths $\{S_t\}_{t\in I_1}$ and $\{S_t\}_{t\in I_2}$, such that $I = I_1 \sqcup I_2$, the spectral flows add to

$$\operatorname{sf}\{S_t\}_{t\in I} = \operatorname{sf}\{S_t\}_{t\in I_1} + \operatorname{sf}\{S_t\}_{t\in I_2} \quad .$$

(3) $\operatorname{sf}\{S_t\}_{t\in I}$ *is a homotopy invariant: if $\{S'_t\}_{t\in I}$ is another continuous path in $\mathscr{F}_*^{sa}(\mathcal{H})$ on $I = [t_1, t_2]$ such that $\{S_t\}_{t\in I}$ and $\{S'_t\}_{t\in I}$ are homotopic with fixed endpoints $S'_{t_1} = S_{t_1}$ and $S'_{t_2} = S_{t_2}$, then $\operatorname{sf}\{S_t\}_{t\in I} = \operatorname{sf}\{S'_t\}_{t\in I}$.*

(4) $\operatorname{sf}\{S_t\}_{t\in I}$ *is $\mathbb{Z}$-valued.*

Fact (1) can be reformulated by saying that the spectral flow is independent of the choice of the partition and of the choice of the real numbers in Definition 8.2.2. Next to paths of self-adjoint Fredholm operators, one can consider *real skew-adjoint Fredholm operators*

$$\mathscr{F}^{\mathrm{rsa}}(\mathcal{H}) := \left\{A \in \mathscr{F}(\mathcal{H}) \,|\, \overline{T} = T \quad \text{and} \quad T^* = -T\right\}$$

which can be considered in the known framework by multiplying with $\mathbb{i}$, inducing a self-adjoint operator. The definition and properties of the spectral flow carry over besides that the spectral flow becomes $\mathbb{Z}_2$-valued with additivity modulo 2. The interested reader should consult [CPSB19, Sec.3/4] for more details about this setting. Another modification of current research is the extension to unbounded self-adjoint Fredholm operators which has been intensively studied in [Les04] and [BBLP05] as well as [BCLR20, Sec.6] for the skew-adjoint case. A key element is the restriction to self-adjoint Fredholm operators in $\mathscr{C}(\mathcal{H}) \subset \mathscr{B}(\mathcal{H})$. Definition 8.2.2 and the properties of the spectral flow remain correct if one replaces $\mathscr{F}_*^{\mathrm{sa}}(\mathcal{H})$ with the set

$$\mathscr{C}\mathscr{F}^{\mathrm{sa}}(\mathcal{H}) := \{A \in \mathscr{C}(\mathcal{H}) \,|\, A \text{ is Fredholm, closed and self-adjoint}\} \quad ;$$

see [BBLP05, Prop.2.10] and [BBLP05, Def.2.12]. We note that in this case the decomposition into disjoint subsets as for $\mathscr{F}^{\mathrm{sa}}(\mathcal{H})$ does not lead to similar contractability results; the interested reader should consult [BBLP05, Sec.1.2].

Before we discuss an analytic expression, we introduce a slight extension of the spectral flow formula. Denote with

$$\operatorname{ind}(P|Q) := \operatorname{ind}(Q : \mathsf{ran}\,(P) \to \mathsf{ran}\,(Q)) = \operatorname{ind}(QP : \mathsf{ran}\,(P) \to \mathsf{ran}\,(Q))$$

the relative index of two projections P and Q on $\mathcal{H}$ such that $(P - Q) \in \mathscr{K}(\mathcal{H})$. Such a pair of projections is a *Fredholm pair*. We denote with $P_{\geq\lambda}(t)$ the spectral projection of the self-adjoint Fredholm operator S_t of the path $\{S_t\}_{t\in I}$ which maps onto the subspace, spanned by eigenvectors with eigenvalues bigger equal λ. Relying on the implementations in [Les04, Sec.3], the existence of finite-rank projections in Definition 8.2.2 can be relaxed to the condition that the path $\{P_{\geq\lambda}(t)\}_{t\in I}$ of projections satisfies

$$\left\| P_{>a_j}(t_s) - P_{>a_j}(t_r) \right\|_{\mathscr{B}(\mathcal{H})} < 1 \tag{8.34}$$

for all $r, s \in [\tau_{j-1}, \tau_j]$ and in each subinterval $[\tau_{j-1}, \tau_j]$ of the partition. In particular, [Les04, Thm.3.6] and [Les04, Cor.3.8] motivate the following definition of spectral flow which we formulate for the bounded case.

Definition 8.2.4. Let $\{S_t\}_{t\in[t_1,t_2]}$ be a continuous path of bounded self-adjoint Fredholm operators and choose a subdivision $t_1 = \tau_0 < \tau_1 < ... < \tau_N = t_2$ of I fine enough such that

$$\left\| P_{>0}(t) - P_{>0}(t') \right\|_{\mathscr{B}(\mathcal{H})} < 1$$

is satisfied for $t, t' \in [\tau_{j-1}, \tau_j]$, then we have

$$\operatorname{sf}\{S_t\}_{t\in I} := \sum_{i=1}^{N} \operatorname{ind}(P_{\geq 0}(\tau_j) \,|\, P_{\geq 0}(\tau_{j-1})) \quad . \tag{8.35}$$

This definition is the key ingredient in defining a spectral flow in von Neumann algebras and in particular in the Γ-setting. If there exist positive real numbers $a_0, a_1, ..., a_N$ with $\pm a_j \notin \sigma(S_t)$ and $\sigma_{\mathrm{ess}}(S_t) \cap [-a_j, a_j] = \emptyset$ for t in every subinterval $[t_{j-1}, t_j]$, we get from this definition the formula (8.33) and thus Definition 8.2.2 follows as the family of projectors $\left\{P_{>a_j}\right\}$ becomes continuous for $\pm a_j \notin \sigma(S_t)$, and

$$\operatorname{ind}(P_{\geq 0}(\tau_j) \,|\, P_{\geq 0}(\tau_{j-1})) = \dim \mathsf{ran}\left(P_{[0,a_i]}(\tau_i)\right) - \dim \mathsf{ran}\left(P_{[0,a_i]}(\tau_{i-1})\right) \quad .$$

This definition of spectral flow transfers to operators in $\mathscr{CF}^{\mathrm{sa}}(\mathcal{H})$ with the additional assumptions that $(1 + S_t^2)^{-1/2} \in \mathscr{K}(\mathcal{H})$ and $\mathrm{e}^{-sS_t^2} \in \mathscr{S}^1(\mathcal{H})$ for each t and $s > 0$. The spectral flow is then defined by replacing the path $\{S_t\}_{t\in[t_1,t_2]}$ with the path

$$t \mapsto S_t(1 + S_t^2)^{-1/2}$$

of bounded self-adjoint Fredholm operators. We refer the interested reader for further details to [Les04, Thm.3.6] for the unbounded case as well as [vdDR20, Lem.3.4] and [vdDR20, Lem.3.5], based on arguments from [BS19, Sec.4.2].

We now come back to the formal relation of spectral flow with the notion of signature for infinite-dimensional Hilbert spaces. We consider an elliptic and self-adjoint (pseudo-)differential operator A of order $m \geq 0$ on a closed manifold M of dimension n which is equivalent to A, being self-adjoint and Fredholm and with discrete spectrum $\sigma(A) = \{\lambda_j \,|\, j \in \mathbb{N}_0\}$. We recapitulate the definition of the ζ-function: if A is in addition a positive operator, then

$$\zeta(z; A) = \sum_{\lambda\in\sigma(A)\setminus\{0\}} \lambda^{-z} = \operatorname{Tr}\left(A^{-z}\right) \quad . \tag{8.36}$$

It is an analytic function for $\mathfrak{Re}\{z\} > \frac{n}{m}$ and has a meromorphic extension with simple poles at $z \in (-1/m)\mathbb{Z} \setminus \{0\}$. The positivity condition can be relaxed by introducing the eta-function of a self-adjoint and elliptic operator A:

$$\eta(z; A) := \sum_{\lambda\in\sigma(A)\setminus\{0\}} \frac{\operatorname{sign}(\lambda)}{|\lambda|^z} = \operatorname{Tr}\left(A \circ |A|^{-(z+1)}\right) \quad . \tag{8.37}$$

It is used to study eigenvalues of a self-adjoint operator and the asymmetry of its spectrum (see introduction of [APS75a]). It is holomorphic on the same domain as the zeta-function and can can be expressed with (8.36): the authors of [APS76] introduced with Seeley's theorem for complex powers of positive and elliptic pseudo-differential operators the pseudo-differential operators

$$A_{\pm} = \frac{3}{2}\,|A| \pm \frac{1}{2}A$$

where $|A| = \sqrt{A^*A}$. These operators are again elliptic, self-adjoint and positive. Herewith they have shown that

$$\eta(z;A) = \frac{\zeta(z;A_+) - \zeta(z;A_-)}{2^{-z}-1}$$

which implies that $z = 0$ is a simple pole as the zeta-functions are holomorphic at $z = 0$. The residue of $\eta(z;A)$ at $z = 0$ is determined by the complete symbol of the operator A (see [APS76, Prop.2.8]), implying that it is in general non-zero. However, the same authors has shown that the residue is vanishing for n odd (see [APS76, Thm.4.5]); the same results hold for the case of even-dimensional manifolds which has been studied in [Gil81] (Theorem 0.1, Lemma 1.1/2). Thus, $z = 0$ is a removeable singularity and $\eta(0;A)$ the well-defined *eta-invariant*. It mimics the signature in infinite-dimensional Hilbert spaces as (8.37) for $z = 0$ can be viewed a sum of infinitely many signs of eigenvalues.

As in complex analysis, the zeta- and eta- functions of an operator each have an integral expression where the integral formula of the Gamma function $\Gamma(z)$ has been used to express the summands. Instead of using the geometric series, one ends up with the trace of the operator $A\mathrm{e}^{-sA^2}$ where s is an integration parameter. The eta-function of A then becomes (see [Goe00])

$$\eta(z;A) = \frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty s^{\frac{z-1}{2}}\,\mathrm{Tr}\left(A\mathrm{e}^{-sA^2}\right)\,\mathrm{d}s \quad . \tag{8.38}$$

This presentation of the eta-function allows to give an explicit formula for the eta-invariant:

$$\eta(A) := \eta(0,A) = \frac{1}{\sqrt{\pi}}\int_0^\infty s^{-\frac{1}{2}}\mathrm{Tr}\left(A\mathrm{e}^{-sA^2}\right)\,\mathrm{d}s \quad . \tag{8.39}$$

If one adds the dimension of the kernel of A, one gains the *xi-invariant* (see [APS75b])

$$\xi(A) := \frac{\eta(A) + \dim\ker(A)}{2} \tag{8.40}$$

as another invariant which occurs in the APS index Theorem 1.0.1 for compact manifolds with boundary. The connection of the spectral flow and the eta-invariant as infinite-dimensional pendent of the signature is studied in [APS76] in section 2,4,6,7, and 8 where K-theoretic as well as analytic methods has been used. The following fact sums up their result.

Proposition 8.2.5. *Let $\{S_t\}_{t\in[t_1,t_2]}$ be a smooth family of elliptic, self-adjoint operators; the function $t \mapsto \eta(S_t)$ is piecewise smooth and admits a decomposition of the form*

$$\xi(S_{t_2}) - \xi(S_{t_1}) = \mathrm{sf}\,\{S_t\}_{t\in[t_1,t_2]} + \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\eta(S_t)\,\mathrm{d}t \tag{8.41}$$

where the spectral flow is piecewise constant in t and the integral expression smooth in the parameter t.

(8.41) on the other hand can be used to determine the spectral flow in terms of the (reduced) eta-invariants:

$$\mathrm{sf}\,\{S_t\}_{t\in[t_1,t_2]} = \xi(S_{t_2}) - \xi(S_{t_1}) - \frac{1}{2}\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\eta(S_t)\,\mathrm{d}t \quad . \tag{8.42}$$

Remark 8.2.6. *A modification of the eta- and xi-invariant for possibly non-self-adjoint Dirac-type operators A has been studied in [BS20, Sec.4/5]: let*

$$\Delta_\theta^s(A) := \exp\left(s\log(A^*A + P_{\mathsf{ker}(A)})\right)\circ(1 - P_{\mathsf{ker}(A)})$$

for $s\in\mathbb{R}$; the kernel projection is viewed as

$$P_{\mathsf{ker}(A)} = \frac{1}{2\pi\mathrm{i}}\int_\gamma (A^*A - \lambda\mathbb{1})^{-1}\,\mathrm{d}\lambda \quad .$$

The angle θ controls a sector, containing the spectrum. If A is self-adjoint, the zero-eigenvalue lies in the resolvent set. The projectors onto positive and negative parts of the spectrum of A are defined by

$$p_{>0} := \frac{1}{2}\left(\mathbb{1} - P_{\mathsf{ker}(A)} + \Delta_\theta^{-1/2}(A)A\right) \quad ,$$
$$p_{<0} := \frac{1}{2}\left(\mathbb{1} - P_{\mathsf{ker}(A)} - \Delta_\theta^{-1/2}(A)A\right) \quad .$$

The attentive reader observes the conceptual resemblance of these expressions for the spectral projections and the one in Corollary 8.1.3. The eta-function is then introduced via

$$\eta(z;A) := \mathrm{Tr}\left((p_{>0} - p_{<0})\Delta_\theta^{-z/2}(A)\right)$$

which is regular at $z = 0$, defining the eta-invariant.

8.2.2. Γ-Fredholm pairs and their Γ-indices

We will introduce the spectral flow in the Γ-setting through Definition 8.2.4. In order to do so, we need to transfer some further concepts of relative Fredholm theory to von Neumann algebras in association with a Galois group Γ. The introduced framework for this and the following sections is based on the papers of [BCP+06] for general semifinite (or type II) von Neumann algebras $\mathscr{N}$ which refers to previous work by Carey and Phillips (e.g. [CP99]). We translate the concepts to our setting with $\mathscr{N} = \mathscr{B}_\Gamma(\mathscr{H})$ for a free Hilbert Γ-module $\mathscr{H}$.

We first deal with Fredholm operators in a Hilbert Γ-module where their domains and images are restricted to the range of certain projectors. Given two free Hilbert Γ-modules $\mathscr{H}_1$, $\mathscr{H}_2$ and two projections P and Q with $\mathsf{ran}\,(P)\subset\mathscr{H}_2$ and $\mathsf{ran}\,(Q)\subset\mathscr{H}_1$. We can define the following extended notion of Fredholmness in the Γ-setting

Definition 8.2.7. Given two projectors P and Q in Hilbert Γ-modules $\mathscr{H}_1$ $\mathscr{H}_2$ such that $\mathsf{ran}\,(Q)\subset\mathscr{H}_1$ and $\mathsf{ran}\,(P)\subset\mathscr{H}_2$; an operator $A\in\mathscr{B}_\Gamma(P(\mathscr{H}_1),Q(\mathscr{H}_2))$ is called (P,Q)-Γ-Fredholm if $A\in\mathscr{F}_\Gamma(Q(\mathscr{H}_1),P(\mathscr{H}_2))$.

The belonging Γ-index is defined as

$$\begin{aligned}\underset{(P,Q)}{\mathrm{ind}_\Gamma}(A) &:= \mathrm{ind}_\Gamma\left(A|_{\mathsf{ran}(Q)\to\mathsf{ran}(P)}\right)\\ &= \dim_\Gamma(\ker(A)\cap Q(\mathscr{H}_1)) - \dim_\Gamma(\ker(A^*)\cap P(\mathscr{H}_2)) \quad .\end{aligned} \tag{8.43}$$

Several properties of ordinary Γ-Fredholm operators carry over to this new concept.

Proposition 8.2.8. *Given projectors Q, P, P_1, P_2, P_3 on Hilbert Γ-modules $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ such that* $\mathsf{ran}(Q) \subset \mathscr{H}_1$, $\mathsf{ran}(P) \subset \mathscr{H}_2$ *and* $\mathsf{ran}(P_i) \subset \mathscr{H}_i$ *for* $i \in \{1,2,3\}$;

(1) *if* $A \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$ *with parametrix* B, *then* $B \in \mathscr{F}_\Gamma(Q(\mathscr{H}_2), P(\mathscr{H}_1))$ *with* $\underset{(Q,P)}{\mathrm{ind}_\Gamma}(B) = -\underset{(P,Q)}{\mathrm{ind}_\Gamma}(A)$.

(2) *if* $A \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$, *then* $A^* \in \mathscr{F}_\Gamma(Q(\mathscr{H}_2), P(\mathscr{H}_1))$ *with* $\underset{(Q,P)}{\mathrm{ind}_\Gamma}(A^*) = -\underset{(P,Q)}{\mathrm{ind}_\Gamma}(A)$.

(3) *if* $A \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$ *and* $C \in \mathscr{K}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$, *then the sum* $(A + C) \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$ *with* $\underset{(P,Q)}{\mathrm{ind}_\Gamma}(A + C) = \underset{(P,Q)}{\mathrm{ind}_\Gamma}(A)$.

(4) *if* $A \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$, *then there exists a* $\delta > 0$ *such that*

$$\|A - B\|_{\mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)} < \delta \quad \textit{for} \quad B \in \mathscr{B}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2)) \Rightarrow B \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$$

with $\underset{(P,Q)}{\mathrm{ind}_\Gamma}(B) = \underset{(P,Q)}{\mathrm{ind}_\Gamma}(A)$.

(5) *if* $A \in \mathscr{F}_\Gamma(P_1(\mathscr{H}_1), P_2(\mathscr{H}_2))$ *and* $B \in \mathscr{F}_\Gamma(P_2(\mathscr{H}_2), P_3(\mathscr{H}_3))$, *then the composition satisfies* $BA \in \mathscr{F}_\Gamma(P_1(\mathscr{H}_1), Q(\mathscr{H}_3))$ *with* $\underset{(P_3,P_1)}{\mathrm{ind}_\Gamma}(BA) = \underset{(P_3,P_2)}{\mathrm{ind}_\Gamma}(B) + \underset{(P_2,P_1)}{\mathrm{ind}_\Gamma}(A)$.

(6) *if* $A \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$ *has polar decomposition* $A = U|A|$, *then* $|A| \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1))$ *and* $U \in \mathscr{F}_\Gamma(Q(\mathscr{H}_1), P(\mathscr{H}_2))$ *with* $\underset{(P,Q)}{\mathrm{ind}_\Gamma}(U) = 0$ *and* $\underset{(P,Q)}{\mathrm{ind}_\Gamma}(A) = \underset{(Q,Q)}{\mathrm{ind}_\Gamma}(|A|)$.

An example of such operators are the restricted projections (8.29) with $P = P_{\blacktriangledown b}$ and $Q = P_{\blacktriangle a}$ and Hilbert Γ-modules $\mathscr{H}_1 = \mathscr{H}_2 = L^2_\Gamma(M, E)$.

A Fredholm pair of projections P and Q is another concept, related to the question, under which conditions the composition PQ, restricted as map from $\mathsf{ran}(Q)$ to $\mathsf{ran}(P)$ becomes Fredholm. The has been studied in [ASS94] for the ordinary Hilbert space setting. We follow the implementations in [BCP+06] and convert the framework into the Hilbert Γ-module setting. We introduce the Γ-Calkin-map

$$\Pi_\Gamma : \quad \mathscr{B}_\Gamma(\mathscr{H}) \quad \longrightarrow \quad {}^{\mathscr{B}_\Gamma(\mathscr{H})}\Big/_{\mathscr{K}_\Gamma(\mathscr{H})} \quad .$$

An operator is Γ-Fredholm if and only if its Γ-Calkin map is invertible. We formulate a condition under which the composition of two projectors $P, Q \in \mathscr{B}_\Gamma(\mathscr{H})$ is (P,Q)-Γ-Fredholm.

Lemma 8.2.9 (cf. Lemma 4.1 in [BCP+06])**.** *Given two projections* $P, Q \in \mathscr{B}_\Gamma(\mathscr{H})$, *then*

$$PQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H})) \quad \textit{if and only if} \quad \|\Pi_\Gamma(P) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} < 1 \quad .$$

In this case we call these projectors Γ-*Fredholm pair* and we define the corresponding index as

$$\mathrm{ind}_\Gamma(P|Q) := \underset{(P,Q)}{\mathrm{ind}_\Gamma}(PQ) = \mathrm{ind}_\Gamma(P|_{\mathsf{ran}(Q)\to\mathsf{ran}(P)}) \quad . \tag{8.44}$$

We can conclude the following properties of this index from [BCP⁺06].

Proposition 8.2.10. *For orthogonal projections $P, Q, R \in \mathscr{B}_\Gamma(\mathscr{H})$ the following holds:*

(1) if $PQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H}))$, then $\dim_\Gamma \mathsf{ker}\,(P - Q \pm \mathbb{1}) < \infty$ and

$$\mathrm{ind}_\Gamma(P|Q) = \dim_\Gamma \mathsf{ker}\,(P - Q + \mathbb{1}) - \dim_\Gamma \mathsf{ker}\,(P - Q - \mathbb{1}) \quad . \tag{8.45}$$

(2)

$$PQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H})) \quad \textit{if and only if} \quad QP \in \mathscr{F}_\Gamma(QP(\mathscr{H}))$$

with $\mathrm{ind}_\Gamma(Q|P) = -\mathrm{ind}_\Gamma(P|Q)$.

(3) The Kato duals $P^\perp$ and $Q^\perp$ satisfy

$$PQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H})) \quad \textit{if and only if} \quad P^\perp Q^\perp \in \mathscr{F}_\Gamma(P^\perp Q^\perp(\mathscr{H}))$$

with $\mathrm{ind}_\Gamma(P^\perp|Q^\perp) = -\mathrm{ind}_\Gamma(P|Q)$.

(4) if $PR \in \mathscr{F}_\Gamma(PR(\mathscr{H}))$ and $RQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H}))$ with

$$\|\Pi_\Gamma(R) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} < \frac{1}{2} \quad \textit{and} \quad \|\Pi_\Gamma(P) - \Pi_\Gamma(R)\|_{\mathscr{B}(\mathscr{H})} < \frac{1}{2} \quad ,$$

then $(P \circ Q) \in \mathscr{F}_\Gamma(PQ(\mathscr{H}))$ with

$$\mathrm{ind}_\Gamma(P|Q) = \mathrm{ind}_\Gamma(P|R) + \mathrm{ind}_\Gamma(R|Q) \quad .$$

(5) if $\|P - Q\|_{\mathscr{B}(\mathscr{H})} < 1$, then $\mathrm{ind}_\Gamma(P|Q) = 0$.

Proof. As the condition for Fredholmness in Lemma 8.2.9 is symmetric in P and Q, the equivalence (2) follows easily. As the identity operator is invertible, the linear Γ-Calkin map gives $\Pi_\Gamma(\mathbb{1}) = \mathbb{1}$ wherefore

$$\left\|\Pi_\Gamma(P^\perp) - \Pi_\Gamma(Q^\perp)\right\|_{\mathscr{B}(\mathscr{H})} = \|\mathbb{1} - \Pi_\Gamma(P) - \mathbb{1} + \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} = \|\Pi_\Gamma(P) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})}$$

shows the equivalence (3).

Statement (1) can be proven purely algebraically as in the proof of [ASS94, Prop.3.1]:

$$\begin{aligned}
\mathsf{ker}\left(P|_{\mathsf{ran}(Q)\to\mathsf{ran}(P)}\right) &= \{x \in \mathsf{ran}\,(Q) \mid PQx = 0\} = \{x \in \mathsf{ran}\,(Q) \mid Px = 0\} \\
&= \mathsf{ker}\,(P - Q + \mathbb{1}) \quad , \\
\mathsf{ker}\left(Q|_{\mathsf{ran}(P)\to\mathsf{ran}(Q)}\right) &= \{x \in \mathsf{ran}\,(P) \mid QPx = 0\} = \{x \in \mathsf{ran}\,(P) \mid Qx = 0\} \\
&= \mathsf{ker}\,(P - Q - \mathbb{1})
\end{aligned}$$

where the second equivalence in each line comes from the fact that $P^2 = P$ and $Q^2 = Q$

for $x \in \mathsf{ran}\,(P)$ and respectively $x \in \mathsf{ran}\,(Q)$. The last equivalences follow from

$$\begin{aligned} Px = 0 \quad &= \quad x - x = Qx - x = (Q - \mathbb{1})x \quad \Leftrightarrow \quad (P - Q + \mathbb{1})x = 0 \quad \forall x \in \mathsf{ran}\,(Q) \\ &\quad\ \text{and} \\ Qx = 0 \quad &= \quad x - x = Px - x = (P - \mathbb{1})x \quad \Leftrightarrow \quad (P - Q - \mathbb{1})x = 0 \quad \forall x \in \mathsf{ran}\,(P) \quad . \end{aligned}$$

Since $PQ \in \mathscr{F}_\Gamma(PQ(\mathscr{H}))$, the range of $P|_{\mathsf{ran}(Q)\to\mathsf{ran}(P)}$ is closed and

$$\begin{aligned} \dim_\Gamma \mathsf{ker}\,(P - Q + \mathbb{1}) &= \dim_\Gamma \mathsf{ker}\left(P|_{\mathsf{ran}(Q)\to\mathsf{ran}(P)}\right) < \infty \quad , \\ \dim_\Gamma \mathsf{ker}\,(P - Q - \mathbb{1}) &= \dim_\Gamma \mathsf{ker}\left(Q|_{\mathsf{ran}(P)\to\mathsf{ran}(Q)}\right) < \infty \end{aligned}$$

such that the index formula (8.45) and the relation between the indices in (2) and (3) follow.

(4) and (5) are proven in [BCP+06] as Lemma 4.3 and Remark 4.4 for general semifinite von Neumann algebras. We only point out that by

$$\begin{aligned} 1 &> \|\Pi_\Gamma(R) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} + \|\Pi_\Gamma(P) - \Pi_\Gamma(R)\|_{\mathscr{B}(\mathscr{H})} \\ &\geq \|\Pi_\Gamma(P) - \Pi_\Gamma(R) + \Pi_\Gamma(R) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} = \|\Pi_\Gamma(P) - \Pi_\Gamma(Q)\|_{\mathscr{B}(\mathscr{H})} \end{aligned}$$

the Fredholm-property follows easily such that all indices are defined. □

8.2.3. Γ-spectral flow - algebraic definition

We consider a family $\{S_t\}_{t\in[a,b]}$ of self-adjoint operators in $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ for two fixed Hilbert Γ-modules such that the path $t \mapsto S_t$ is continuous and $\Pi_\Gamma(S_t)$ is invertible modulo any norm-closed two-sided ideal in $\mathscr{B}_\Gamma(\mathscr{H}_1, \mathscr{H}_2)$ for any $t \in [a, b]$. We define

$$\mathscr{F}_\Gamma^{\mathsf{sa}}(\mathscr{H}_1, \mathscr{H}_2) := \{A \in \mathscr{F}_\Gamma(\mathscr{H}_1, \mathscr{H}_2) \,|\, A \text{ self-adjoint}\} \tag{8.46}$$

as set of all self-adjoint Γ-Fredholm operators between the two Hilbert Γ-modules $\mathscr{H}_1$ and $\mathscr{H}_2$.

The heuristic idea of spectral flow from subsection 8.2.1 is that it measures the amount of spectrum gained minus the amount of spectrum lost while moving along the path. The mathematical tool behind this quantity is provided by the Γ-index of a Γ-Fredholm pair of projections $P_{\geq 0}(S_t)$ (see (8.44)):

$$\mathrm{ind}_\Gamma(P_{\geq 0}(S_b)|P_{\geq 0}(S_a)) = \mathrm{ind}_\Gamma\left(P_{\geq 0}(S_b)|_{\mathsf{ran}\left(P_{\geq 0}(S_a)\right)\to\mathsf{ran}\left(P_{\geq 0}(S_b)\right)}\right) \quad .$$

One would define the Γ-spectral flow to be

$$\mathrm{ind}_\Gamma(P_{\geq 0}(S_b)|P_{\geq 0}(S_a))$$

if the right-hand side is well-defined. The question, whether a pair of projections with respect to a continuous path is a Γ-Fredholm pair, depends on the path. If we choose the path $t \mapsto S_t$ to lie entirely in $\mathscr{F}_\Gamma^{\mathsf{sa}}(\mathscr{H}_1, \mathscr{H}_2)$, the path of projections $P_{\geq 0}(S_t)$ initially becomes a path in $\mathscr{B}_\Gamma(\mathscr{H})$ which is discontinuous in t. In order to get around these problems, we mimic the approach for spectral flow in any von Neumann setting ([CP99] and [BCP+06]) and adapt it to our setting: we choose an $L \in \mathbb{N}$ big enough and a corresponding partition $a = t_0 < t_1 < \cdots < t_L = b$ of the time interval $[a, b]$ such that

$$\|\Pi_\Gamma(P_{\geq 0}(S_t)) - \Pi_\Gamma(P_{\geq 0}(S_r))\|_{\mathscr{B}(\mathscr{H})} < 1 \quad \forall\, t, r \in [t_{j-1}, t_j] \quad .$$

Roughly speaking, we approximate the path by a polygonal chain in such a way that the projectors and the endpoints of each segment of the chain are a Γ-Fredholm pair and thus $\mathrm{ind}_\Gamma(P_{\geq 0}(S_t)|P_{\geq 0}(S_r))$ is well-defined for each t, r in each subinterval of the partition. Taking the finite sum over all contributions gives a well-defined quantity which we identify with the spectral flow.

Definition 8.2.11. Let $\{S_t\}_{t\in[a,b]}$ be a continuous path in $\mathscr{F}_\Gamma^{\mathrm{sa}}(\mathscr{H}_1, \mathscr{H}_2)$ and $\{P_{\geq 0}(t)\}_{t\in[a,b]}$ the corresponding path of projections in $\mathscr{B}_\Gamma(\mathscr{H}_2)$, defined by $P_{\geq 0}(t) := \chi(S_t)$ for each $t \in [a, b]$; for an $L \in \mathbb{N}$ consider the partition $a = t_0 < t_1 < \cdots < t_L = b$ such that

$$\|\Pi_\Gamma(P_{\geq 0}(t)) - \Pi_\Gamma(P_{\geq 0}(r))\|_{\mathscr{B}(\mathscr{H}_2)} < 1 \quad \forall\, t, r \in [t_{j-1}, t_j]$$

is satisfied for each $j \in \{1, 2, .., L\}$. The **Γ-spectral flow** of the path $\{S_t\}_{t\in[a,b]}$ is defined as the number

$$\mathrm{sf}_\Gamma \{S_t\}_{t\in[a,b]} := \sum_{j=1}^{L} \mathrm{ind}_\Gamma(P_{\geq 0}(t_{j-1})|P_{\geq 0}(t_j)) \quad . \tag{8.47}$$

Remarks 8.2.12.

(i) *The definition is independent of the chosen partition and works equally well for all von Neumann algebras of type I and II; see [BCP⁺06, Thm.2.1]. In the first case the Γ-spectral flow is integer-valued while in the semifinite case the Γ-spectral flow becomes a real number due to the fact that the Γ-indices are in general real numbers.*

(ii) *If the path of projections $\{P_{\geq 0}(t)\}$ is already continuous, we can choose a fine enough partition such that already*

$$\|P_{\geq 0}(t)) - P_{\geq 0}(s))\|_{\mathscr{B}(\mathscr{H}_1)} < 1 \quad \forall\, t, s \in [t_{j-1}, t_j]$$

is fulfilled. Proposition 8.2.10 (5) then implies that each two projections are a Γ-Fredholm pair with $\mathrm{ind}_\Gamma(P_{\geq 0}(r)|P_{\geq 0}(s)) = 0$ for each $r, s \in [t_{j-1}, t_j]$ in the partition. As a consequence $\mathrm{sf}_\Gamma \{S_t\}_{t\in[a,b]}$ vanishes. This fits with the heuristical expectation that the spectral flow is only non-trivial if the path of projections has discontinuities.

(iii) *If $\{S_t\}_{t\in[a,b]}$ and $\{\bar{S}_t\}_{t\in[a,b]}$ are two continuous paths and homotopic in $\mathscr{F}_\Gamma^{sa}(\mathscr{H}_1, \mathscr{H}_2)$, such that $\bar{S}_a = S_a$ and $\bar{S}_b = S_b$, the Γ-spectral flow of these paths coincide. Hence the Γ-spectral flow is a homotopy invariant. This is proven for general von Neumann algebras of type I and II in [BCP⁺06, Prop.4.7].*

So far we considered the case that the domain $\mathscr{H}_1$ and the range $\mathscr{H}_2$ of each member in the continuous path are the same for each $t \in [a, b]$. In view of applying the Γ-spectral flow concept to the path of Γ-invariant hypersurface Dirac operators $\{A_t\}$, we have to deal with time-depending domains and ranges. We need to consider a reference hypersurface, in order to fix this for such kind of operators, and an operation which maps everything to this hypersurface in such a way that it does not influence the index and thus the Γ-spectral flow. For our special interest we have such an operator, given by the Dirac wave evolution operator $Q(\tau_2,\tau_1)$ for $\tau_1,\tau_2 \in [a,b]$. The following ideas are inspired by the treatment of the Lorentzian index theorem in [Ron19] and [vdDR20] which we will use and adapt for our situation. We assume that there exists a unitary Γ-morphism

$$\mathcal{U}(\tau_2,\tau_2) \,:\, \mathscr{H}(\tau_1) \to \mathscr{H}(\tau_2) \tag{8.48}$$

between two time-different Hilbert Γ-modules $\mathscr{H}(\tau_1)$ and $\mathscr{H}(\tau_2)$ for each $\tau_1,\tau_2 \in [a,b]$. We denote with $\mathscr{D}(\tau)$ the domain $\mathsf{dom}(S_\tau)$ of each S_τ; we furthermore assume that each domain is a dense subspace in $\mathscr{H}(\tau)$ for each $\tau \in [a,b]$ separately. This and the (essential) self-adjointness imply that each (closure of) S_t becomes closed and densely defined. Recalling Lemma 5.2.12, the domain $\mathscr{D}(\tau)$ becomes a Hilbert Γ-module with respect to the graph norm if the domain is not already a Hilbert Γ-module on its own right. We introduce the *evolved operator*

$$\hat{S}_t := \mathcal{U}(a,t) S_t \mathcal{U}(t,a)$$

as operator from $\mathscr{D}(a)$ to $\mathscr{H}(a)$ for each $t \in [a,b]$ and consider the Γ-spectral flow with respect to this path. We need to check that this path is well-defined to connect it with a spectral flow.

Lemma 8.2.13. *Given $J \subset \mathbb{R}$ and a bundle $\{\mathscr{H}(t)\}_{t\in[a,b]}$ of Hilbert Γ-modules and a bundle of domains $\{\mathscr{D}(t)\}_{t\in[a,b]}$ such that for each fixed time $\tau \in [a,b]$ the domain $\mathscr{D}(\tau)$ is dense in $\mathscr{H}(\tau)$; we moreover assume that there exists a unitary Γ-morphism (8.48) which is continuous in both time arguments. For $S_\bullet \in C^0([a,b], \mathscr{F}^{sa}_\Gamma(\mathscr{D}(\bullet),\mathscr{H}(\bullet))$ the evolved operator satisfies*

(1) $[a,b] \ni t \mapsto \hat{S}_t \in \mathscr{F}^{sa}_\Gamma(\mathscr{D}(a),\mathscr{H}(a))$ is continuous.

(2) $\hat{P}_J(t) = \mathcal{U}(a,t)P_J(t)\mathcal{U}(t,a)$ for all $t \in [a,b]$ is an orthogonal projection with $\mathsf{ran}\left(\hat{P}_J(t)\right) = \mathcal{U}(a,t)(\mathsf{ran}\,(P_J(t)))$.

Proof. Because S_t and $\mathcal{U}(t,a)|_{\mathscr{D}(a)\to\mathcal{U}(t,a)(\mathscr{D}(a))}$ are Γ-Fredholm, it is clear that the evolved operator satisfies $\hat{S}_t \in \mathscr{F}_\Gamma(\mathscr{D}(a),\mathscr{H}(a))$. Furthermore, the Γ-indices of S_t and $\hat{S}_t$ coincide for each t. The self-adjointness of S_t and the unitarity of $\mathcal{U}(t,a)$ imply

$$(\hat{S}_t)^* = \mathcal{U}(t,a)^*(S_t)^*\mathcal{U}(a,t)^* = \mathcal{U}(a,t)S_t\mathcal{U}(t,a) = \hat{S}_t \quad .$$

Hence $\hat{S}_t$ is symmetric[32] and thus $\hat{S}_t \in \mathscr{F}^{\mathrm{sa}}_\Gamma(\mathscr{D}(a),\mathscr{H}(a))$ for each t. The continuity in t follows from the continuous dependence of S_t and $\mathcal{U}(t,\cdot)$ on t which finally shows the first assertion.

The second assertion follows by proving $\chi_J(\hat{S}_t) = \mathcal{U}(a,t)\chi_J(S_t)\mathcal{U}(t,a)$ for all $t \in [a,b]$ which we show for any real-valued Borel function f: $f(\hat{S}_t) = \mathcal{U}(a,t)f(S_t)\mathcal{U}(t,a)$. Fix a $t \in [a,b]$. Since S_t and $\hat{S}_t$ are self-adjoint for each t, their domains $\mathscr{D}(t)$ and $\mathscr{D}(a)$ are dense in $\mathscr{H}(t)$ and respectively $\mathscr{H}(a)$ and so we can consider the self-adjointness in these ranges and use the spectral theorem for unbounded self-adjoint operators: for each S_t there exists a measure space (X,μ) and a unitary operator $U(t)$ from $\mathscr{H}(t)$ to $L^2(X,\mu)$

[32] We can check that the closedness of S_t implies the closedness of $\hat{S}_t$ for each fixed t due to the fact that the wave evolution operator is bounded. The condition $\ker\left(\hat{S}_t{}^* \pm \mathrm{i}\mathbb{1}_{\mathscr{H}_2(a)}\right) = \{0\}$ is implied by the preassumption $\ker\left(S_t^* \pm \mathrm{i}\mathbb{1}_{\mathscr{H}_2(t)}\right) = \{0\}$ since $Q(t,a)$ and $Q(a,t)$ are isomorphisms.

and a measurable function $s_t : X \to \mathbb{R}$ such that $S_t = U^{-1}(t)\mathscr{M}_{s_t}U(t)$ where $\mathscr{M}$ denotes multiplication with this measurable function. The Γ-invariance of S_t for each t forces $U(t)$ and $\mathscr{M}_{s_t}$ to be Γ-invariant: the unitary left action representation $\mathcal{L}_\gamma$ on $\mathscr{H}(t)$ does not depend on time because we have generally assumed that the Γ-action is the same on each hypersurface. As $\mathscr{D}(t)$ is dense in $\mathscr{H}(t)$, the left action representation can be applied to elements in the domain such that

$$\begin{aligned} & \mathcal{L}_\gamma S_t = S_t \mathcal{L}_\gamma \\ \Leftrightarrow \quad & \mathcal{L}_\gamma U(t)^{-1}\mathscr{M}_{s_t}U(t) = U(t)^{-1}\mathscr{M}_{s_t}U(t)\mathcal{L}_\gamma \\ \Leftrightarrow \quad & U(t)^{-1}\mathscr{M}_{s_t}U(t) = (\mathcal{L}_\gamma)^{-1}U(t)^{-1}\mathscr{M}_{s_t}U(t)\mathcal{L}_\gamma \\ \Leftrightarrow \quad & U(t)^{-1}\mathscr{M}_{s_t}U(t) = (U(t)\circ\mathcal{L}_\gamma)^{-1}\mathscr{M}_{s_t}(U(t)\circ\mathcal{L}_\gamma) \quad . \end{aligned}$$

This shows that $U(t)$ is right Γ-invariant: $U(t) = U(t)\circ\mathcal{L}_\gamma$ for all $\gamma \in \Gamma$. On the other hand we can insert $L_\epsilon = (\mathcal{L}_\gamma)^{-1}\mathcal{L}_\gamma$ and observe

$$\begin{aligned} U(t)^{-1}\mathscr{M}_{s_t}U(t) &= U(t)^{-1}\circ(\mathcal{L}_\gamma)^{-1}\mathcal{L}_\gamma\circ\mathscr{M}_{s_t}\circ(\mathcal{L}_\gamma)^{-1}\mathcal{L}_\gamma\circ U(t) \\ &= (\mathcal{L}_\gamma\circ U(t))^{-1}\circ\mathcal{L}_\gamma\mathscr{M}_{s_t}(\mathcal{L}_\gamma)^{-1}\circ(\mathcal{L}_\gamma\circ U(t)) \quad . \end{aligned}$$

Thus, $U(t)$ is also left-Γ-invariant and finally Γ-invariant. We observe that the multiplication operator is Γ-invariant as well: $\mathcal{L}_\gamma\mathscr{M}_{s_t}(\mathcal{L}_\gamma)^{-1} = \mathscr{M}_{s_t}$. Thus, for any real-valued Borel function f the operator $f(S_t)$ is defined by $U(t)^{-1}\mathscr{M}_{f\circ s_t}U(t)$ and is Γ-invariant. Since $\hat{S}_t$ is the conjugation of S_t, we observe

$$\begin{aligned} \hat{S}_t &= \mathcal{U}(a,t)\circ S_t\circ\mathcal{U}(t,a) = \mathcal{U}(a,t)\circ U^{-1}(t)\mathscr{M}_{s_t}U(t)\circ\mathcal{U}(t,a) \\ &= (U\circ\mathcal{U}(t,a))^{-1}(t)\mathscr{M}_{s_t}(U(t)\circ\mathcal{U}(t,a)) \end{aligned} \tag{8.49}$$

which gives a spectral representation of $\hat{S}_t$ with the same multiplication operator, but with another Γ-invariant unitary operator $U(t)\circ\mathcal{U}(t,a)$ from $\mathscr{H}(a)$ to $L^2(X,\mu)$. Applying the spectral theorem, we get

$$\begin{aligned} f(\hat{S}_t) &= (U\circ\mathcal{U}(t,a))^{-1}(t)\mathscr{M}_{f\circ s_t}(U(t)\circ\mathcal{U}(t,a)) \\ &= \mathcal{U}(a,t)\circ U^{-1}(t)\mathscr{M}_{f\circ s_t}U(t)\circ\mathcal{U}(t,a) = \mathcal{U}(a,t)f(S_t)\mathcal{U}(t,a) \end{aligned} \tag{8.50}$$

and the claim is proven for $f(S_t) = \chi_J(S_t) = P_J(t)$. The part about the range follows from the fact that $P_J(t) = \chi_J(S_t)$ is an orthogonal projection which holds true for $\hat{P}_J(t)$ by the unitarity of $\mathcal{U}(t,a)$:

$$\begin{aligned} (\hat{P}_J(t))^2 &= \mathcal{U}(a,t)P_J(t)\mathcal{U}(t,a)\mathcal{U}(a,t)P_J(t)\mathcal{U}(t,a) = \mathcal{U}(a,t)(P_J(t))^2\mathcal{U}(t,a) = \hat{P}_J(t) \\ (\hat{P}_J(t))^* &= \mathcal{U}(a,t)(P_J(t))^*\mathcal{U}(t,a) = \mathcal{U}(a,t)P_J(t)\mathcal{U}(t,a) = \hat{P}_J(t) \quad . \end{aligned}$$

The ranges and the kernels of $P_J(t)$ and its evolved operator can be related to each other by using the Kato dual for $P_J(t)$:

$$\begin{aligned} \mathsf{ker}\,(P_J(t)) &= \mathsf{ran}\left((P_J(t))^\perp\right) = \mathsf{ran}\,(\mathbb{1} - P_J(t)) = \mathsf{ran}\left(P_{J^\complement}(t)\right) \\ \mathsf{ran}\,(P_J(t)) &= \mathsf{ran}\left(\mathbb{1} - (P_J(t))^\perp\right) = \mathsf{ker}\left((P_J(t))^\perp\right) = \mathsf{ker}\left(P_{J^\complement}(t)\right) \end{aligned}$$

and the same for $\hat{P}_J(t)$ with its Kato-dual $(\hat{P}_J(t))^\perp = \mathcal{U}(a,t)(P_J(t))^\perp\mathcal{U}(t,a)$. As the isometry has trivial kernel, we get

$$\begin{aligned}\mathsf{ran}\left(\hat{P}_J(t)\right) &= \mathsf{ker}\left((\hat{P}_J(t))^\perp\right) = (\mathcal{U}(t,a))^{-1}\left\{((P_J(t))^\perp)^{-1}\{0\}\right\} \\ &= (\mathcal{U}(t,a))^{-1}\left\{\mathsf{ker}\left((P_J(t))^\perp\right)\right\} = \mathcal{U}(a,t)(\mathsf{ran}\,(P_J(t))) \quad . \quad \square\end{aligned}$$

Remark 8.2.14. *One observes directly that* $\hat{S}_t^2 = \mathcal{U}(a,t)S_t^2\mathcal{U}(t,a)$. *Alternatively, we could have also used the holomorphic functional calculus for sectorial operators from Corollary 8.1.2 to define* $f(S_t^2)$ *if* S_t^2 *is sectorial:*

$$f(S_t^2) = \frac{1}{2\pi\mathbb{i}}\int_\gamma f(\lambda)R(\lambda,S_t^2)\,\mathrm{d}\lambda$$

with $R(\lambda,S_t^2)$ *as resolvent and* γ *a Hankel-like curve along the rays of the sector (see again figure Fig. 8.1). Since the resolvent of* $\hat{S}_t^2$ *obeys*

$$\begin{aligned}R(\lambda,\hat{S}_t^2) &= \left(\lambda-\hat{S}_t^2\right)^{-1} = \left(\mathcal{U}(a,t)(\lambda-S_t^2)\mathcal{U}(t,a)\right)^{-1} = (\mathcal{U}(t,a))^{-1}R(\lambda,S_t^2)(\mathcal{U}(t,a))^{-1} \\ &= \mathcal{U}(a,t)R(\lambda,S_t^2)\mathcal{U}(t,a) \quad ,\end{aligned}$$

the calculus implies

$$f(\hat{S}_t^2) = \mathcal{U}(a,t)f(S_t^2)\mathcal{U}(t,a) \quad . \tag{8.51}$$

The advantage in using the spectral theorem for unbounded operators is that it also works for non-sectorial operators. However, the calculation here is going to be useful in subsection 8.2.5.

After proving all necessary ingredients, the Γ-spectral flow of the path $\{\hat{S}_t\}$ is well-defined and we introduce the Γ-spectral flow of a path of operators, acting on different Hilbert Γ-modules in a Hilbert Γ-module bundle as follows: let $\{\mathscr{H}(t)\}_{t\in[a,b]}$ and $\{\mathscr{D}(t)\}_{t\in[a,b]}$ be given as introduced and let $\mathcal{U}$ in (8.48) be time-continuous.

Definition 8.2.15. Given a continuous path $\{S_t\}_{t\in[a,b]}$, such that $S_t\in\mathscr{F}_\Gamma^{\mathrm{sa}}(\mathscr{D}(t),\mathscr{H}(t))$ for each t, and let $\{P_{\geq 0}(t)\}_{t\in[a,b]}$ be the path of spectral projections such that $P_{\geq 0}(t) = \chi_{(0,\infty)}(S_t)$ and $P_{\geq 0}(t)\in\mathscr{B}_\Gamma(\mathscr{H}(t))$ for each t; let $\{\hat{S}_t\}_{t\in[a,b]}$ be the corresponding continuous path of evolved operators in $\mathscr{F}_\Gamma^{\mathrm{sa}}(\mathscr{D}(a),\mathscr{H}_2(a))$ and $\{\hat{P}_{\geq 0}(t)\}_{t\in[a,b]}$ the path of projections in $\mathscr{B}_\Gamma(\mathscr{H}(a))$ such that $\hat{P}_{\geq 0}(t) = \chi_{(0,\infty)}(\hat{S}_t)$ for each t. For an $L\in\mathbb{N}$ consider the partition $a = t_0 < t_1 < \cdots < t_L = b$ such that

$$\left\|\Pi_\Gamma(\hat{P}_{\geq 0}(s)) - \Pi_\Gamma(\hat{P}_{\geq 0}(r))\right\|_{\mathscr{B}(\mathcal{U}(a,t)(\mathscr{H}_2(t)))} < 1 \quad \forall\, s,r\in[t_{j-1},t_j]$$

is satisfied for each $j\in\{1,2,..,L\}$. The **modified Γ-spectral flow** of the path $\{S_t\}_{t\in[a,b]}$ is the number

$$\widetilde{\mathrm{sf}}_\Gamma\,\{S_t\}_{t\in[a,b]} := \mathrm{sf}_\Gamma\left\{\hat{S}_t\right\}_{t\in[a,b]} = \sum_{j=1}^L \mathrm{ind}_\Gamma(\hat{P}_{\geq 0}(t_{j-1})|\hat{P}_{\geq 0}(t_j)) \quad . \tag{8.52}$$

We will show that in fact $\widetilde{\mathrm{sf}}_\Gamma \{S_t\}_{t\in[a,b]} = \mathrm{sf}_\Gamma \{S_t\}_{t\in[a,b]}$ by showing that the Γ-spectral flow is invariant under conjugation with isometries. Because of the previous lemma, we could rewrite this with the original projectors and the evolution operator between two points in time of the partition:

$$\begin{aligned}
\mathrm{ind}_\Gamma(\hat{P}_{\geq 0}(t_{j-1})|\hat{P}_{\geq 0}(t_j)) &= \mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t_{j-1})\hat{P}_{\geq 0}(t_j)|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(t_j)\right)\to\mathsf{ran}\left(\hat{P}_{\geq 0}(t_{j-1})\right)}\right)\\
&= \mathrm{ind}_\Gamma\left(\mathcal{U}(a,t_{j-1})P_{\geq 0}(t_{j-1})\mathcal{U}(t_{j-1},t_j)P_{\geq 0}(t_j)\mathcal{U}(t_j,a)\Big|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(t_j)\right)\to\mathsf{ran}\left(\hat{P}_{\geq 0}(t_{j-1})\right)}\right)\\
&= \mathrm{ind}_\Gamma\left(P_{\geq 0}(t_{j-1})\mathcal{U}(t_{j-1},t_j)P_{\geq 0}(t_j)\Big|_{\mathsf{ran}\left(P_{\geq 0}(t_j)\right)\to\mathsf{ran}\left(P_{\geq 0}(t_{j-1})\right)}\right)
\end{aligned}$$

where we have used that $\mathsf{ran}\left(\hat{P}_{\gtrless}\right) = \mathcal{U}(a,\tau)(\mathsf{ran}\,(P_{\gtrless}(\tau)))$. Thus, (8.52) becomes

$$\widetilde{\mathrm{sf}}_\Gamma \{S_t\}_{t\in[a,b]} = \sum_{j=1}^{L} \mathrm{ind}_\Gamma\left(P_{\geq 0}(t_{j-1})\mathcal{U}(t_{j-1},t_j)P_{\geq 0}(t_j)\Big|_{\mathsf{ran}\left(P_{\geq 0}(t_j)\right)\to\mathsf{ran}\left(P_{\geq 0}(t_{j-1})\right)}\right) \quad .$$

We see that the rough heuristical picture of spectral flow as measure of the amount of spectrum gained minus the amount of spectrum lost while moving along the path, is kept, but in order to compare the spectra at two different times the amount has to be transported isometrically to the spectrum of a former point along the path.

8.2.4. Eta- and Rho-invariants in Γ-setting

M. Ramachandran introduced in [Ram93] in his proof of Theorem 1.0.4 the Γ-pendent for the eta function (8.38) where the trace is replaced with the Γ-trace; more precisely, for any Γ-invariant and geometric Dirac operator[33] the Γ-*eta function* is defined via

$$\eta_\Gamma(z;A) = \frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty s^{\frac{z-1}{2}}\mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)\mathrm{d}s \quad . \tag{8.53}$$

Ramachandran has also studied the Γ-*eta invariant* which is the value of $\eta_\Gamma(z;A)$ at $z=0$. He has shown that it is well-defined and given by

$$\eta_\Gamma(A) := \frac{1}{\sqrt{\pi}}\int_0^\infty s^{-\frac{1}{2}}\mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)\mathrm{d}s \quad . \tag{8.54}$$

To show well-definedness, he introduced the *lower and upper truncated* Γ-*eta invariants*

$$\eta_\Gamma^{>\epsilon}(A) = \frac{1}{\sqrt{\pi}}\int_\epsilon^\infty s^{-1/2}\mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)\mathrm{d}s \quad \text{and}$$
$$\eta_\Gamma^{<\epsilon}(A) = \frac{1}{\sqrt{\pi}}\int_0^\epsilon s^{-1/2}\mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)\mathrm{d}s \quad ; \tag{8.55}$$

he studied the limit $\lim_{\epsilon\to\infty}\eta_\Gamma^{<\epsilon}(0;A)$ and moreover proved that the Γ-invariant fibrewise trace of the Γ-equivariant Schwartz kernel $K_\Gamma^A(p,q;s)$ $(p,q\in\Sigma)$ of $A\mathrm{e}^{-sA^2}$ behaves like

$$\mathrm{tr}_{\mathcal{E}_p}\left(K_\Gamma^A(p,p;s)\right) = \mathcal{O}(s^{\frac{1}{2}}) \qquad \text{for} \quad s\to 0^+ \quad . \tag{8.56}$$

[33] E.g. the spin-Dirac, signature- or Gauss-Bonnet operator.

Hence the asymptotic expansion of $s^{-1/2}\mathrm{tr}_{\mathcal{E}_p}\left(K^A_\Gamma(p,p;s)\right)$ near $s=0$ does not contain any singular terms. The coefficients of the asymptotic expansion are also Γ-invariant such that integrating over the fundamental domain of Σ does not change the asymptotic behaviour. Thus, $s^{-1/2}\mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)=\mathcal{O}(1)$ and it becomes integrable near 0. This observation from (8.56) is known as *Bismut-Freed-cancellation property* (see [Ram93, Lem.3.1.1] or [BF86]) and the upper truncated Γ-eta invariant becomes well-defined and

$$\lim_{\epsilon\to 0^+}\eta^{<\epsilon}_\Gamma(A)=0 \quad . \tag{8.57}$$

The well-definedness of the lower truncated Γ-eta invariant exists because the integrand is rapidly decreasing for $s\to\infty$ which has been used for the proof in [Ram93]. He showed that

$$\lim_{\epsilon\to\infty}\eta^{<\epsilon}_\Gamma(A)=\eta_\Gamma(A) \tag{8.58}$$

such that the integral (8.54) and also the Γ-*xi invariant*

$$\xi_\Gamma(A):=\frac{\eta_\Gamma(A)+\dim_\Gamma\ker(A)}{2} \tag{8.59}$$

are well-defined. The latter appears to be the boundary contribution of the Γ-index (1.3) in Theorem 1.0.4. The difference of $\eta_\Gamma(A)$ and the ordinary eta-invariant (8.39) for the operator $\underline{A}$ on the compact base, which lifts to A on the covering, is studied for several geometric operators and is defined as the *relative* Γ-*eta /Cheeger-Gromov rho invariant* ([CG85], [BR15]):

$$\rho_\Gamma(A,\underline{A}):=\eta_\Gamma(A)-\eta(\underline{A}) \quad . \tag{8.60}$$

The difference

$$\eta_\Gamma(z;A)-\eta(z;\underline{A}) \tag{8.61}$$

is regular at $z=0$ if both eta functions $\eta_\Gamma(z;A),\eta(z;\underline{A})$ are regular at $z=0$, i.e. both eta-invariants are well-defined. If regularity of the eta-functions is a priori not given, the short time asymptotics ($s\to 0^+$) of the traces

$$\mathrm{Tr}\left(\underline{A}\mathrm{e}^{-s\underline{A}^2}\right) \quad\text{and}\quad \mathrm{Tr}_\Gamma\left(A\mathrm{e}^{-sA^2}\right)$$

cancel and (8.61) becomes regular at $z=0$, implying the well-definedness of (8.60).

If A_t and $\underline{A}_t$ are Riemannian spin-Dirac operators, we observe that they depend smoothly on the parameter t as Clifford multiplication $\mathbf{c}_t(\cdot)$ and the Riemannian Levi-Civita connection ∇^{Σ_t} depend smoothly on t due to (6.62) and the Koszul formula. The dependence of ρ_Γ on the parameter t has been studied for the signature and the spin-Dirac operator for which the following result is known.

Proposition 8.2.16 (cf. Theorem 1.11 in [PS07]). *If the path of smooth Riemannian metrics $\{g_t\}_{t\in I}$ has positive scalar curvature (i.e. $\mathcal{R}_{\Sigma_t}>0$ for all $t\in I$), then $\rho_\Gamma(A_t,\underline{A_t})$ is constant.*

According to [PS07, Rem.1.12], this result has been proven by Nigel Higson and Thomas Schick in an unpublished work. We refer to the given reference for more informations. We end this subsection by focusing on the special case that Γ is finite and thus Σ is compact and an ordinary $|\Gamma|$-covering where $|\Gamma|$ denotes the finite cardinality of Γ. Following

Proposition 5.2.8 (6) and [Shu, Sec.3.17] the Γ-trace of a Γ-trace-class operator C reduce to

$$\mathrm{Tr}_\Gamma\,(C) = \frac{1}{|\Gamma|}\mathrm{Tr}\,(C) \quad .$$

The Γ-dimension of a Hilbert Γ-submodule $\mathscr{H}$ is defined by the Γ-trace over a Γ-trace class projector $P_{\mathscr{H}}$ and reduces to

$$\dim_\Gamma(\mathscr{H}) = \frac{1}{|\Gamma|}\mathrm{Tr}\,(P_{\mathscr{H}}) = \frac{1}{|\Gamma|}\dim_{\mathbb{C}}(\mathscr{H}) \quad .$$

If $\mathscr{H}$ is already a free Hilbert Γ-module, it is unitarily isomorphic to $l^2(\Gamma)\otimes\mathcal{H}$ with Hilbert space $\mathcal{H}$. In this case we have $|\Gamma|\cdot\dim_\Gamma(\mathscr{H}) = \dim_{\mathbb{C}}(\mathcal{H})$. The Γ-eta and Γ-xi functions and correspondingly their invariants at $z=0$ for a Γ-invariant operator A in (8.54) and (8.59) are then given by

$$\eta_\Gamma(z;A) = \frac{1}{|\Gamma|}\eta(z;A) \quad \text{and} \quad \xi_\Gamma(z;A) = \frac{1}{|\Gamma|}\xi(z;A) \tag{8.62}$$

where each right-hand side is defined as in (8.39) and (8.40). We define the difference in (8.61) as the *relative eta function* $\rho_{|\Gamma|}(z;A,\underline{A})$:

$$\rho_\Gamma(z;A,\underline{A}) = \frac{1}{|\Gamma|}\eta(z;A) - \eta(z;\underline{A}) = \frac{1}{|\Gamma|}\left(\eta(z;A) - |\Gamma|\,\eta(z;\underline{A}\right) =: \frac{1}{|\Gamma|}\rho_{|\Gamma|}(z;A,\underline{A}) \quad . \tag{8.63}$$

The well-definedness of the Cheeger-Gromov rho invariant implies that also the reduced Γ-eta function is well-defined at $z=0$ which defines the *reduced eta-invariant* $\rho_{|\Gamma|}(A,\underline{A}) := \rho_{|\Gamma|}(0;A,\underline{A})$. (8.63) vanishes for $|\Gamma|=1$ because the cover coincides with the base and thus A with $\underline{A}$.

8.2.5. Analytic expression for the Γ-spectral flow

As mentioned in the introduction of this chapter, we want an analytic expression of the Γ-spectral flow by means of the Γ-eta invariant as in (8.42). As a starting point we use the result [AW11, Prop.3.3]. Based on the same algebraic definition of spectral flow, the authors have proven an analytic spectral flow formula, similar to (8.42), for any type II von Neumann algebra setting. We only need to adapt their result to our setting.

$$\begin{aligned}
\mathrm{sf}_\Gamma\,\{A_t\}_{t\in[t_1,t_2]} &= \frac{\eta_\Gamma^{>\epsilon}(A_2)+\dim_\Gamma\mathsf{ker}\,(A_2)}{2} - \frac{\eta_\Gamma^{>\epsilon}(A_1)+\dim_\Gamma\mathsf{ker}\,(A_1)}{2} \\
&\quad +\sqrt{\frac{\epsilon}{\pi}}\int_{t_1}^{t_2}\mathrm{Tr}_\Gamma\left(\dot{A}_t\mathrm{e}^{-\epsilon A_t^2}\right)\,\mathrm{d}t \\
&= \xi_\Gamma^{>\epsilon}(A_2) - \xi_\Gamma^{>\epsilon}(A_1) + \sqrt{\frac{\epsilon}{\pi}}\int_{t_1}^{t_2}\mathrm{Tr}_\Gamma\left(\dot{A}_t\mathrm{e}^{-\epsilon A_t^2}\right)\,\mathrm{d}t
\end{aligned} \tag{8.64}$$

for all $\epsilon>0$ where $\xi_\Gamma^{>\epsilon}(A_i)$ are the lower truncated Γ-xi invariants $\xi_\Gamma^{>\epsilon}(A_i)$ from (8.59) where the Γ-eta invariants are replaced by the lower truncated Γ-eta invariant (8.55). $\dot{A}_t$ is the derivative of A_t with respect to t.

The truncation parameter ϵ has been introduced to bypass possible irregularities of the trace of $A\mathrm{e}^{-sA^2}$ at $s=0$ for any von Neumann algebra setting. In view of (8.57) we can get rid of the dependence on ϵ in the Γ-setting. We rewrite (8.64) with the well-defined upper truncated Γ-eta invariants and consider the limit $\epsilon \to 0^+$:

$$\begin{aligned} \mathrm{sf}_\Gamma \{A_t\}_{t\in[t_1,t_2]} = \xi_\Gamma(A_2) - \xi_\Gamma(A_1) + \frac{\eta_\Gamma^{<\epsilon}(A_1) - \eta_\Gamma^{<\epsilon}(A_2)}{2} \\ + \sqrt{\frac{\epsilon}{\pi}} \int_{t_1}^{t_2} \mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-\epsilon A_t^2} \right) \mathrm{d}t \quad . \end{aligned} \tag{8.65}$$

The integral part can be expressed as the derivative of $\eta_\Gamma^{\geq\epsilon}(A_t)$.

Lemma 8.2.17.

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma^{\geq\epsilon}(A_t) = -2\sqrt{\frac{\epsilon}{\pi}} \mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-\epsilon A_t^2} \right) \quad .$$

Proof. The main part of the proof consists of justifying the interchange of derivatives and integration over the fundamental domain of Σ_t. To simplify the calculation we consider again the evolved operators $\hat{A}_t := \mathcal{U}(t_1,t) A_t \mathcal{U}(t,t_1)$ and $\mathrm{e}^{-s\hat{A}_t^2} = \mathcal{U}(t_1,t)\mathrm{e}^{-sA_t^2}\mathcal{U}(t,t_1)$ where $\mathcal{U}(t,t_1)$ is the natural isometry on a Hilbert Γ-module which contains the domain of A_t as dense subset and is defined by parallel transport (see subsection 6.3.4). This allows us to consider the fibrewise trace of the heat kernel and the Γ-volume measure with respect to the reference hypersurface Σ_1. Like in the construction of the modified Γ-spectral flow it ensures that the operator $\hat{A}_t$ acts on one and the same domain. This simplifies the calculation because then the only t- and s-dependence comes from $s^{-1/2}\mathrm{tr}_{\mathcal{E}_p} \left(\hat{K}_\Gamma(p,p,t;s) \right)$ where $\hat{K}_\Gamma(p,p,t;s)$ is the Schwartz kernel of $\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2}$ along the diagonal in $\Sigma_1 \times \Sigma_1$. We want to show that

(1) $\partial_t \mathrm{Tr}_\Gamma \left(\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2} \right) = \mathrm{Tr}_\Gamma \left(\partial_t (\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2}) \right) \quad ,$

(2) $\partial_s s^{1/2} \mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-sA_t^2} \right) = \mathrm{Tr}_\Gamma \left(\partial_s (s^{1/2} \dot{A}_t \mathrm{e}^{-sA_t^2}) \right) \quad .$

Denote by $\dot{K}_\Gamma(p,p,t;s)$ the Schwartz kernel of $\dot{A}_t \mathrm{e}^{-sA_t^2}$. The map $p \mapsto \mathrm{tr}_{\mathcal{E}_p} \left(\hat{K}_\Gamma(p,p,t;s) \right)$ is integrable for all t over the fundamental domain $\mathcal{F}$ of Σ since $A_t \mathrm{e}^{-sA_t^2}$ and thus $\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2}$ are Γ-trace class for all t. $\dot{A}_t$ is an unbounded operator, but since $\mathrm{e}^{-sA_t^2}$ is Γ-trace class and smoothing, the composition $\dot{A}_t \mathrm{e}^{-sA_t^2}$ becomes Γ-trace class for all $s \geq 0$ and all t which won't change if we multiply with $s^{1/2}$. So the map $p \mapsto s^{1/2}\mathrm{tr}_{\mathcal{E}_p} \left(\dot{K}_\Gamma(p,p,t;s) \right)$ is also integrable over $\mathcal{F}$ for all $s \geq 0$ and t. The smoothness of A_t, $\mathrm{e}^{-sA_t^2}$ and differentiability of $\mathcal{U}(\cdot,t)$ and $\mathcal{U}(t,\cdot)$ with respect to t imply differentiability of $\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2}$. In terms of the Schwartz kernel, this is inherited by the fact that near $s=0$ (choose $\epsilon > 0$ small enough) $\hat{K}_\Gamma(p,p,t;s)$ has an asymptotic expansion of the form

$$\mathrm{tr}_{\mathcal{E}_p} \left(\hat{K}_\Gamma(p,p,t;s) \right) \underset{s\to 0^+}{\sim} \sum_{j \geq \frac{n+3}{2}}^{\infty} \hat{a}_j(p,t) s^{-\frac{n-2j+1}{2}} \quad ;$$

with coefficients $\hat{a}_j(p,t)$ which are differentiable with respect to t because they are locally computed by the fibrewise trace over t-differentiable curvature expressions which are in turn expressions in $\mathfrak{g}_t$. The cancellation property (8.56) forces the sum to start at the first positive exponent in s which explains the starting index of the asymptotic summation.

Thus, the map $t \mapsto \hat{K}_\Gamma(p,p,t;s)$ is differentiable for all s and $p \in \mathcal{F}$ with derivative given by the derivative of the expansion coefficients. Hence the asymptotic expansion of the fibrewise trace of the Schwartz kernel $\dot{K}_\Gamma(p,p,t;s)$ takes the form

$$s^{1/2}\mathrm{tr}_{\mathcal{E}_p}\left(\dot{K}_\Gamma(p,p,t;s)\right) \underset{s\to 0^+}{\sim} \sum_{j\geq\frac{n+3}{2}}^{\infty} b_j(p,t)s^{-\frac{n-2j}{2}} \quad . \tag{8.66}$$

The coefficients $b_j(p,t)$ are smooth in t due to their local definitions via curvature expressions. We also observe that $s^{1/2}\mathrm{tr}_{\mathcal{E}_p}\left(\dot{K}_\Gamma(p,p,t;s)\right)$ is differentiable in s and its derivative $\partial_s\left[s^{1/2}\mathrm{tr}_{\mathcal{E}_p}\left(\dot{K}_\Gamma(p,p,t;s)\right)\right]$ exists for all $p \in \mathcal{F}$. The expressions

$$\partial_t \mathrm{tr}_{\mathcal{E}_p}\left(\hat{K}_\Gamma(p,p,t;s)\right) \quad \text{and} \quad \partial_s\left[s^{1/2}\mathrm{tr}_{\mathcal{E}_p}\left(\dot{K}_\Gamma(p,p,t;s)\right)\right]$$

are continuous in t and respectively s, so restricting to any subinterval $I_t \subset \mathbb{R}$ for t and $I_s \subset [0,\infty)$ gives an integrable majorant in each case. The two claims (1) and (2) are then justified by the dominated convergence theorem and so we can compute the terms as follows:

$$\begin{aligned}
\partial_t \mathrm{Tr}_\Gamma\left(\hat{A}_t e^{-s\hat{A}_t{}^2}\right) &\overset{(1)}{=} \mathrm{Tr}_\Gamma\left(\partial_t(\hat{A}_t e^{-s\hat{A}_t^2})\right) = \mathrm{Tr}_\Gamma\left(\partial_t[\mathcal{U}(t_1,t)A_t e^{-sA_t^2}\mathcal{U}(t,t_1)]\right) \\
&= \mathrm{Tr}_\Gamma\left(\dot{\mathcal{U}}(t_1,t)A_t e^{-sA_t^2}\mathcal{U}(t,t_1) + \mathcal{U}(t_1,t)A_t e^{-sA_t^2}\dot{\mathcal{U}}(t,t_1)\right) \\
&\quad + \mathrm{Tr}_\Gamma\left(\mathcal{U}(t_1,t)\partial_t(A_t e^{-sA_t^2})\mathcal{U}(t,t_1)\right) \\
&\overset{(*)}{=} \mathrm{Tr}_\Gamma\left(\dot{A}_t e^{-sA_t^2} + A_t\partial_t e^{-sA_t^2}\right) = \mathrm{Tr}_\Gamma\left(\dot{A}_t e^{-sA_t^2} - sA_t\dot{A}_tA_t e^{-sA_t^2}\right) \\
&\overset{(**)}{=} \mathrm{Tr}_\Gamma\left(\dot{A}_t(e^{-sA_t^2} - sA_t^2 e^{-sA_t^2})\right) = \mathrm{Tr}_\Gamma\left(\dot{A}_t\left(1+2s\partial_s\right)e^{-sA_t^2}\right) \quad .
\end{aligned}$$

In $(*)$ we used that by unitarity we have $\dot{\mathcal{U}}(t_1,t)\mathcal{U}(t,t_1) = -\mathcal{U}(t_1,t)\dot{\mathcal{U}}(t,t_1)$ and since $\mathcal{U}$ and $\dot{\mathcal{U}}A_t e^{-sA_t^2}$ are bounded, we can rearrange under the Γ-trace in cyclic order such that the first two terms cancel each other. For the same reason the isometries cancel each other in the remaining term. For $(**)$ we used, that A_t is unbounded, but A_t^2 is sectorial such that $e^{-sA_t^2}$ and $A_t^k e^{-sA_t^2}$ are bounded for all k, since $e^{-sA_t^2}$ is smoothing, and A_t commutes with $e^{-sA_t^2}$. This allows us to commute cyclically under the trace as follows:

$$\begin{aligned}
\mathrm{Tr}_\Gamma\left(A_t\dot{A}_tA_t e^{-sA_t^2}\right) &= \mathrm{Tr}_\Gamma\left(A_t\dot{A}_tA_t e^{-\frac{s}{2}A_t^2}e^{-\frac{s}{2}A_t^2}\right) = \mathrm{Tr}_\Gamma\left(e^{-\frac{s}{2}A_t^2}A_t\dot{A}_tA_t e^{-\frac{s}{2}A_t^2}\right) \\
&= \mathrm{Tr}_\Gamma\left(A_t e^{-\frac{s}{2}A_t^2}\dot{A}_t e^{-\frac{s}{2}A_t^2}A_t\right) = \mathrm{Tr}_\Gamma\left(\dot{A}_t e^{-\frac{s}{2}A_t^2}A_t^2 e^{-\frac{s}{2}A_t^2}\right) \\
&= \mathrm{Tr}_\Gamma\left(\dot{A}_tA_t^2 e^{-sA_t^2}\right) \quad .
\end{aligned}$$

We can multiply with $s^{-1/2} = 2\partial_s s^{1/2}$ which leads to

$$\begin{aligned}
\partial_t s^{-1/2}\mathrm{Tr}_\Gamma\left(\hat{A}_t e^{-s\hat{A}_t^2}\right) &= \mathrm{Tr}_\Gamma\left(\dot{A}_t\left(s^{-1/2}+2s^{1/2}\partial_s\right)e^{-sA_t^2}\right) = 2\mathrm{Tr}_\Gamma\left(\partial_s\left(\dot{A}_t s^{1/2}e^{-sA_t^2}\right)\right) \\
&\overset{(2)}{=} 2\partial_s\mathrm{Tr}_\Gamma\left(\dot{A}_t s^{1/2}e^{-sA_t^2}\right) \quad .
\end{aligned}$$

We perform the integration over $s \in (\epsilon, \infty)$: on one hand we get

$$\int_\epsilon^\infty \partial_s \mathrm{Tr}_\Gamma \left(\dot{A}_t s^{1/2} \mathrm{e}^{-sA_t^2} \right) \mathrm{d}s = -\sqrt{\epsilon} \mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-\epsilon A_t^2} \right) \quad .$$

On the other hand $s^{-1/2} \mathrm{Tr}_\Gamma \left(\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2} \right)$ is integrable for all t and the partial derivative with respect to t exists, so we can conclude again with the dominated convergence theorem that

$$\begin{aligned} \int_\epsilon^\infty \partial_s \mathrm{Tr}_\Gamma \left(\dot{A}_t s^{1/2} \mathrm{e}^{-sA_t^2} \right) \mathrm{d}s &= \frac{1}{2} \int_\epsilon^\infty \partial_t \left[s^{-1/2} \mathrm{Tr}_\Gamma \left(\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2} \right) \right] \mathrm{d}s \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\epsilon^\infty s^{-1/2} \mathrm{Tr}_\Gamma \left(\hat{A}_t \mathrm{e}^{-s\hat{A}_t^2} \right) \mathrm{d}s = \frac{\sqrt{\pi}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma^{>\epsilon}(A_t) \\ &= \frac{\sqrt{\pi}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma(A_t) - \frac{\sqrt{\pi}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma^{<\epsilon}(A_t) \end{aligned}$$

and finally the claim. □

With this observations we can rewrite (8.65) further into

$$\begin{aligned} \mathrm{sf}_\Gamma \left\{ A_t \right\}_{t \in [t_1, t_2]} &= \xi_\Gamma(A_2) - \xi_\Gamma(A_1) + \frac{\eta_\Gamma^{<\epsilon}(A_1) - \eta_\Gamma^{<\epsilon}(A_2)}{2} - \frac{1}{2} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma(A_t) \, \mathrm{d}t \\ &\quad + \frac{1}{2} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma^{<\epsilon}(A_t) \, \mathrm{d}t \quad . \end{aligned}$$

We repeat the last step from the former proof, but perform the integration for $s \in [0, \epsilon)$ giving us on one hand

$$\int_0^\epsilon \partial_s \mathrm{Tr}_\Gamma \left(\dot{A}_t s^{1/2} \mathrm{e}^{-sA_t^2} \right) \mathrm{d}s = \sqrt{\epsilon} \mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-\epsilon A_t^2} \right)$$

and on the other hand with a similar argument

$$\int_0^\epsilon \partial_s \mathrm{Tr}_\Gamma \left(\dot{A}_t s^{1/2} \mathrm{e}^{-sA_t^2} \right) \mathrm{d}s = \frac{\sqrt{\pi}}{2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma^{<\epsilon}(A_t) \quad .$$

The asymptotic expansion in ϵ near 0 for $\mathrm{Tr}_\Gamma \left(\dot{A}_t \mathrm{e}^{-\epsilon A_t^2} \right)$ can be calculated with (8.66) and shows that the last expression is $\mathcal{O}(\epsilon^{3/2})$ and thus

$$\mathrm{sf}_\Gamma \left\{ A_t \right\}_{t \in [t_1, t_2]} = \xi_\Gamma(A_2) - \xi_\Gamma(A_1) - \frac{1}{2} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma(A_t) \, \mathrm{d}t + \frac{\eta_\Gamma^{<\epsilon}(A_1) - \eta_\Gamma^{<\epsilon}(A_2)}{2} + \mathcal{O}(\epsilon^{3/2}) \quad .$$

The left-hand side does not depend on the truncating parameter, so we can consider the limit $\epsilon \to 0^+$ for which the latter three terms vanish according to (8.57). As a result, we have worked out a Γ-version of (8.42):

$$\mathrm{sf}_\Gamma \left\{ A_t \right\}_{t \in [t_1, t_2]} = \xi_\Gamma(A_2) - \xi_\Gamma(A_1) - \frac{1}{2} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \eta_\Gamma(A_t) \, \mathrm{d}t \quad . \tag{8.67}$$

In subsection 8.2.3 we have defined the modified Γ-spectral flow as the ordinary Γ-spectral flow of the evolved family of Dirac operators. Because any functional calculus is invariant under conjugation with isometries, we have

$$\mathrm{e}^{-s\hat{A}_t^2} = \mathrm{e}^{-s\mathcal{U}(t_1,t)A_t^2\mathcal{U}(t,t_1)} = \mathcal{U}(t_1,t)\mathrm{e}^{-sA_t^2}\mathcal{U}(t,t_1)$$

and with cyclic permutation under the Γ-trace

$$\begin{aligned}\mathrm{Tr}_\Gamma\left(\hat{A}_t\mathrm{e}^{-s\hat{A}_t^2}\right) &= \mathrm{Tr}_\Gamma\left(\mathcal{U}(t_1,t)A_t\mathrm{e}^{-sA_t^2}\mathcal{U}(t,t_1)\right)\\ &= \mathrm{Tr}_\Gamma\left(\mathcal{U}(t,t)A_t\mathrm{e}^{-sA_t^2}\right) = \mathrm{Tr}_\Gamma\left(A_t\mathrm{e}^{-sA_t^2}\right) \quad .\end{aligned}$$

(8.67) then shows that our modified Γ-spectral flow coincides with the ordinary Γ-spectral flow:

$$\mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} = \mathrm{sf}_\Gamma\left\{\hat{A}_t\right\}_{t\in[t_1,t_2]} = \widetilde{\mathrm{sf}}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} \quad . \tag{8.68}$$

All of these results of course transfer to the smooth family of twisted hypersurface Dirac operators $\{A_t^{E_L}\}$.

9. Γ-Fredholmness and Γ-indices of the Dirac-wave evolution operators

We present regularity and Fredholmness of Q and $\tilde{Q}$ as well as their spectral decomposits with respect to the splitting due to boundary conditions. We first focus on Γ-Fredholmness under (a)APS boundary conditions in the first section. These results are then extended to g(a)APS boundary conditions in the second section. The last section is dedicated to the computation of the Γ-indices which will turn out to be related to the Γ-spectral flow.

9.1. Γ-Fredholmness for (a)APS boundary conditions

We start with (a)APS boundary conditions and consider the decomposition of L^2_Γ-spaces for spinors on the lower and upper Cauchy boundaries Σ_1 and Σ_2 due to these boundary conditions. These imply a decomposition of Q^{E_L} and $\tilde{Q}^{E_L}$ like (1.6). We investigate how the Fourier integral operator character of the wave evolution operators transfers to each spectral entry. The Γ-Fredholmness of Q^{E_L} and $\tilde{Q}^{E_L}$ as unitary Γ-morphisms between L^2_Γ-spaces likewise motivates the question how unitarity and Fredholmness carry over to the spectral entries.

9.1.1. Q and $\tilde{Q}$ under (a)APS boundary conditions

If we apply the splittings in (8.25) for positive chirality to $Q(t_2,t_1)$, it allows us to rewrite the operator as a (2×2)-matrix

$$Q(t_2,t_1) = \begin{pmatrix} Q_{++}(t_2,t_1) & Q_{+-}(t_2,t_1) \\ Q_{-+}(t_2,t_1) & Q_{--}(t_2,t_1) \end{pmatrix} \tag{9.1}$$

like in (1.6). We define the entries as maps of the form

$$\begin{array}{rcccc} Q_{++}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ & & u & \mapsto & P_{>0}(t_2)\circ Q(t_2,t_1)\circ P_{\geq 0}(t_1)u \\ Q_{--}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \\ & & u & \mapsto & P_{\leq 0}(t_2)\circ Q(t_2,t_1)\circ P_{<0}(t_1)u \\ Q_{+-}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ & & u & \mapsto & P_{>0}(t_2)\circ Q(t_2,t_1)\circ P_{<0}(t_1)u \\ Q_{-+}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \\ & & u & \mapsto & P_{\leq 0}(t_2)\circ Q(t_2,t_1)\circ P_{\geq 0}(t_1)u \end{array} . \tag{9.2}$$

These matrix entries will be referred on as *spectral* or rather *matrix entries*. We call $Q_{\pm\pm}$ the *diagonal*[34] and $Q_{\pm\mp}$ the *off-diagonal entries* of Q. We will also need these operators,

[34] We follow the convention in footnote 26 and write $Q_{\pm\pm}$ for either Q_{++} or Q_{--} while $Q_{\pm\mp}$ is either Q_{+-} or Q_{-+}. We apply this to all coming quantities with similar subscripts.

acting as maps on spectral subspaces of $L^2_\Gamma(\mathcal{S}^+(\Sigma_1))$:

$$\begin{array}{rcl} Q_{++}(t_2,t_1) = P_{>0}(t_2)\circ Q(t_2,t_1): & L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ Q_{--}(t_2,t_1) = P_{\leq 0}(t_2)\circ Q(t_2,t_1): & L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \\ Q_{+-}(t_2,t_1) = P_{>0}(t_2)\circ Q(t_2,t_1): & L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ Q_{-+}(t_2,t_1) = P_{\leq 0}(t_2)\circ Q(t_2,t_1): & L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \end{array} . \tag{9.3}$$

Any spectral entry in (9.3) is a bounded and Γ-invariant operator because $Q(t_2,t_1)$ is a Γ-isomorphism between H^s_Γ-spaces due to Definition 7.3.5; the projectors are equally Γ-morphisms as maps between L^2_Γ-subspaces due to (8.14) and Proposition 5.3.7 (4). As the spectral subspaces of L^2_Γ are closed and free Hilbert Γ-modules (recall Lemma 8.1.5), all entries in (9.3) have closed range. The commuting with the left action representation is clear. Hence these observations prove the following result.

Lemma 9.1.1. *All spectral entries in (9.3) are Γ-morphism between Hilbert Γ-modules, i.e.*

$$\begin{array}{rcl} Q_{++}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2))) \\ Q_{--}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2))) \\ Q_{+-}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2))) \\ Q_{-+}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2))) \end{array} .$$

Furthermore, all ranges are closed.

The unitarity property on L^2-sections in Lemma 7.2.3 (5) implies that the off-diagonal entries are isomorphisms between the kernels of the diagonal entries and their adjoints.

Lemma 9.1.2. *The operators $Q_{+-}(t_2,t_1)$ and $Q_{-+}(t_2,t_1)$ restrict to Γ-isomorphisms*

$$\begin{array}{rcl} Q_{+-}(t_2,t_2) & : & \ker(Q_{--}(t_2,t_1)) \to \ker((Q_{++}(t_2,t_1))^*) \\ Q_{-+}(t_2,t_2) & : & \ker(Q_{++}(t_2,t_1)) \to \ker((Q_{--}(t_2,t_1))^*) \end{array} .$$

Proof. The result is shown in [BS19, Lem.2.5] which can be taken without any further modifications because the proof is purely algebraic. For the sake of completeness we repeat their arguments.

The matrix representation (9.1) is done with respect to sections in $L^2_\Gamma(\mathcal{S}^+(\Sigma_j))$ for $j\in\{1,2\}$. From $(Q(t_2,t_1))^*Q(t_2,t_1) = \mathbb{1}_{L^2_\Gamma(\mathcal{S}^+(\Sigma_1))}$ and $Q(t_2,t_1)(Q(t_2,t_1))^* = \mathbb{1}_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))}$ we get the following system of equations:

$$\begin{array}{rcl} (Q_{++}(t_2,t_1))^*Q_{++}(t_2,t_1) + (Q_{-+}(t_2,t_1))^*Q_{-+}(t_2,t_1) & = & \mathbb{1} \\ (Q_{+-}(t_2,t_1))^*Q_{+-}(t_2,t_1) + (Q_{--}(t_2,t_1))^*Q_{--}(t_2,t_1) & = & \mathbb{1} \\ (Q_{++}(t_2,t_1))^*Q_{+-}(t_2,t_1) + (Q_{-+}(t_2,t_1))^*Q_{--}(t_2,t_1) & = & 0 \\ (Q_{+-}(t_2,t_1))^*Q_{++}(t_2,t_1) + (Q_{--}(t_2,t_1))^*Q_{-+}(t_2,t_1) & = & 0 \end{array} \tag{9.4}$$

and

$$\begin{array}{rcl} Q_{++}(t_2,t_1)(Q_{++}(t_2,t_1))^* + Q_{+-}(t_2,t_1))(Q_{+-}(t_2,t_1))^* & = & \mathbb{1} \\ Q_{-+}(t_2,t_1)(Q_{-+}(t_2,t_1))^* + Q_{--}(t_2,t_1))(Q_{--}(t_2,t_1))^* & = & \mathbb{1} \\ Q_{++}(t_2,t_1)(Q_{-+}(t_2,t_1))^* + Q_{+-}(t_2,t_1))(Q_{--}(t_2,t_1))^* & = & 0 \\ Q_{-+}(t_2,t_1)(Q_{++}(t_2,t_1))^* + Q_{--}(t_2,t_1))(Q_{+-}(t_2,t_1))^* & = & 0 \end{array} . \tag{9.5}$$

Suppose $u \in L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) \cap \ker(Q_{--}(t_2,t_1))$; applying the third equation of (9.4) to u shows $(Q_{++}(t_2,t_1))^* Q_{+-}(t_2,t_1)u = 0$ and thus $Q_{+-}(t_2,t_1)u \in L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) \cap \ker((Q_{++}(t_2,t_1)^*)$. If one takes $u \in L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) \cap \ker(Q_{++}(t_2,t_1))$, one can observe from the fourth equation in (9.4) that $Q_{-+}(t_2,t_1)u \in L^2_\Gamma(\mathcal{S}^+(\Sigma_1)) \cap \ker((Q_{--}(t_2,t_1)^*)$. Thus, the operators, restricted to the given kernels, map to the corresponding kernels as given in the Lemma to show. If vice versa u lies in the kernel of $(Q_{++}(t_2,t_1))^*$, the fourth equation in (9.5) implies that $(Q_{+-}(t_2,t_1))^* u$ lies in $\ker(Q_{--}(t_2,t_1))$. Using the third equation in (9.5) for a section u in $\ker((Q_{--}(t_2,t_1)^*)$ shows $(Q_{-+}(t_2,t_1))^* u \in \ker(Q_{++}(t_2,t_1))$. Thus, the adjoints of the operators map between the same kernels in reversed direction. This observation and the non-vanishing equations in (9.4) and (9.5) lead to the conclusion that the restricted operators are isomorphisms because taking again u in the kernel of $Q_{--}(t_2,t_1)$ implies from the second equation in (9.4) that $(Q_{+-}(t_2,t_1))^* Q_{+-}(t_2,t_1)u = u$; $Q_{+-}(t_2,t_1)(Q_{+-}(t_2,t_1))^* v = v$ is implied from the first equation in (9.5) if v lies in the kernel of $(Q_{++}(t_2,t_1))^*$. So the compositions $(Q_{+-}(t_2,t_1))^* Q_{+-}(t_2,t_1)$ and $Q_{+-}(t_2,t_1)(Q_{+-}(t_2,t_1))^*$ act as identities on the kernels and thus $(Q_{+-}(t_2,t_1))^*$ becomes the inverse operator of $Q_{+-}(t_2,t_1)$. One proves from the so far unused equations in (9.4) and (9.5) with an analogue argument that $(Q_{-+}(t_2,t_1))^*$ is the inverse operator of $Q_{-+}(t_2,t_1)$.

All isomorphisms act between kernels of Γ-invariant operators. As those kernels are projective Hilbert Γ-modules and in particular Γ-invariant, the isomorphisms become Γ-invariant as well. □

If we apply the splittings in (8.25) for negative chirality to $\tilde{Q}(t_2,t_1)$, we get a similar description as (2x2)-matrix:

$$\tilde{Q}(t_2,t_1) = \begin{pmatrix} \tilde{Q}_{++}(t_2,t_1) & \tilde{Q}_{+-}(t_2,t_1) \\ \tilde{Q}_{-+}(t_2,t_1) & \tilde{Q}_{--}(t_2,t_1) \end{pmatrix} \tag{9.6}$$

where the matrix entries are analogously defined via

$$\begin{array}{rcccc} \tilde{Q}_{++}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^-(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)) \\ & & u & \mapsto & P_{>0}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{\geq 0}(t_1)u \\ \tilde{Q}_{--}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^-(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)) \\ & & u & \mapsto & P_{\leq 0}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{<0}(t_1)u \\ \tilde{Q}_{+-}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^-(\Sigma_1)) & \to & L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)) \\ & & u & \mapsto & P_{>0}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{<0}(t_1)u \\ \tilde{Q}_{-+}(t_2,t_1) & : & L^2_\Gamma(\mathcal{S}^-(\Sigma_1)) & \to & L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)) \\ & & u & \mapsto & P_{\leq 0}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{\geq 0}(t_1)u \end{array} . \tag{9.7}$$

We also need to consider the restrictions to certain spectral subspaces as in the case for positive chirality.

$$\begin{array}{rcl} \tilde{Q}_{++}(t_2,t_1) = P_{>0}(t_2) \circ \tilde{Q}(t_2,t_1) : & L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_1)) & \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)) \\ \tilde{Q}_{--}(t_2,t_1) = P_{\leq 0}(t_2) \circ \tilde{Q}(t_2,t_1) : & L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)) & \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)) \\ \tilde{Q}_{+-}(t_2,t_1) = P_{>0}(t_2) \circ \tilde{Q}(t_2,t_1) : & L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)) & \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)) \\ \tilde{Q}_{-+}(t_2,t_1) = P_{\leq 0}(t_2) \circ \tilde{Q}(t_2,t_1) : & L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_1)) & \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)) \end{array} . \tag{9.8}$$

Since $\tilde{Q}$ is equally a unitary Γ-morphism on L^2_Γ, its matrix representation (9.6) implies the following set of equations:

$$\begin{array}{rcl}(\tilde{Q}_{++}(t_2,t_1))^*\tilde{Q}_{++}(t_2,t_1)+(\tilde{Q}_{-+}(t_2,t_1))^*\tilde{Q}_{-+}(t_2,t_1) & = & \mathbb{1}\\ (\tilde{Q}_{+-}(t_2,t_1))^*\tilde{Q}_{+-}(t_2,t_1)+(\tilde{Q}_{--}(t_2,t_1))^*\tilde{Q}_{--}(t_2,t_1) & = & \mathbb{1}\\ (\tilde{Q}_{++}(t_2,t_1))^*\tilde{Q}_{+-}(t_2,t_1)+(\tilde{Q}_{-+}(t_2,t_1))^*\tilde{Q}_{--}(t_2,t_1) & = & 0\\ (\tilde{Q}_{+-}(t_2,t_1))^*\tilde{Q}_{++}(t_2,t_1)+(\tilde{Q}_{--}(t_2,t_1))^*\tilde{Q}_{-+}(t_2,t_1) & = & 0\end{array} \tag{9.9}$$

and

$$\begin{array}{rcl}\tilde{Q}_{++}(t_2,t_1)(\tilde{Q}_{++}(t_2,t_1))^*+\tilde{Q}_{+-}(t_2,t_1))(\tilde{Q}_{+-}(t_2,t_1))^* & = & \mathbb{1}\\ \tilde{Q}_{-+}(t_2,t_1)(\tilde{Q}_{-+}(t_2,t_1))^*+\tilde{Q}_{--}(t_2,t_1))(\tilde{Q}_{--}(t_2,t_1))^* & = & \mathbb{1}\\ \tilde{Q}_{++}(t_2,t_1)(\tilde{Q}_{-+}(t_2,t_1))^*+\tilde{Q}_{+-}(t_2,t_1))(\tilde{Q}_{--}(t_2,t_1))^* & = & 0\\ \tilde{Q}_{-+}(t_2,t_1)(\tilde{Q}_{++}(t_2,t_1))^*+\tilde{Q}_{--}(t_2,t_1))(\tilde{Q}_{+-}(t_2,t_1))^* & = & 0\end{array}\ . \tag{9.10}$$

The pendent of Lemma 9.1.2 for negative chirality can be proven equally.

Lemma 9.1.3. *The operators $\tilde{Q}_{+-}(t_2,t_1)$ and $\tilde{Q}_{-+}(t_2,t_1)$ restrict to Γ-isomorphisms*

$$\begin{array}{rcccl}\tilde{Q}_{+-}(t_2,t_2) & : & \ker\left(\tilde{Q}_{--}(t_2,t_1)\right) & \to & \ker\left((\tilde{Q}_{++}(t_2,t_1))^*\right)\\ \tilde{Q}_{-+}(t_2,t_2) & : & \ker\left(\tilde{Q}_{++}(t_2,t_1)\right) & \to & \ker\left((\tilde{Q}_{--}(t_2,t_1))^*\right)\end{array}$$

We also get a pendent for Lemma 9.1.1

Lemma 9.1.4. *All spectral entries in (9.8) are Γ-morphisms between Hilbert Γ-modules, i.e.*

$$\begin{array}{rcl}\tilde{Q}_{++}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_1)),L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)))\\ \tilde{Q}_{--}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)),L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)))\\ \tilde{Q}_{+-}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)),L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)))\\ \tilde{Q}_{-+}(t_2,t_1) & \in & \mathscr{B}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_1)),L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)))\end{array}$$

with closed ranges.

All made results remain true if we twist the graded spinor bundles with E_L. We just have to replace Q with Q^{E_L} and $\tilde{Q}$ with $\tilde{Q}^{E_L}$; the projectors are then defined with respect to the twisted hypersurface Dirac operator.

9.1.2. Regularity properties of the matrix entries

We have seen how unitarity of the Dirac-wave evolution operators as Γ-morphisms on L^2_Γ carry over to the matrix entries in (9.1) and (9.6). We now want to clarify how their regularity as Fourier integral operator transfers to the spectral entries. To do so, we need to investigate whether the compositions of the wave evolution operators with the projectors as s-regular pseudo-differential operators are meaningful on the level of Fourier integral operators. We will show that these compositions are s-regular Fourier integral operators of the same order and with same canonical relation as for Q and $\tilde{Q}$; hence they differ from a properly supported Fourier integral operator in a s-smoothing operator.

Proposition 9.1.5. *The operators in (9.2) and (9.7) as well as their twisted versions are Γ-invariant Fourier integral operators of order* 0 *modulo s-smoothing remainders, i.e.*

(1) $Q_{\pm\pm}(t_2,t_1), Q_{\pm\mp}(t_2,t_1) \in \mathcal{SFIO}^0_\Gamma(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2)))$;

(2) $\tilde{Q}_{\pm\pm}(t_2,t_1), \tilde{Q}_{\pm\mp}(t_2,t_1) \in \mathcal{SFIO}^0_\Gamma(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^-(\Sigma_1),\mathcal{S}^-(\Sigma_2)))$;

(3) $Q^{E_L}_{\pm\pm}(t_2,t_1), Q^{E_L}_{\pm\mp}(t_2,t_1) \in \mathcal{SFIO}^0_\Gamma(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+_{L,E}(\Sigma_1),\mathcal{S}^+_{L,E}(\Sigma_2)))$;

(4) $\tilde{Q}^{E_L}_{\pm\pm}(t_2,t_1), \tilde{Q}^{E_L}_{\pm\mp}(t_2,t_1) \in \mathcal{SFIO}^0_\Gamma(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^-_{L,E}(\Sigma_1),\mathcal{S}^-_{L,E}(\Sigma_2)))$.

Proof. We prove (1) of this assertion as the following arguments can be transferred to the proof of (2) by replacing Q with $\tilde{Q}$; (3) as well (4) follow with the same reasoning.

Let $P_\pm(t_j)$ be either $P_{>0}(t_j)$ for the plus case or $P_{<0}(t_j)$ for the minus case; these projectors are elements in $\mathcal{S}\Psi^0_\Gamma(\Sigma_j,\mathcal{S}^+(\Sigma_j))$ ((8.14)). According to Remarks 8.1.4 (i) they can be decomposed as $P_\pm(t_j) = p_\pm(t_j) + r_\pm(t_j)$ where $r_\pm(t_j) \in \mathcal{S}\Psi^{-\infty}_\Gamma(\Sigma_j,\mathcal{S}^+(\Sigma_j))$ and $p_\pm(t_j) \in \Psi^0_{\Gamma,\mathsf{cl}}(\Sigma_j,\mathcal{S}^+(\Sigma_j))$ are properly supported and classical Γ-pseudo-differential operators. The other projectors $P_{\geq 0}(t_j)$ and $P_{\leq 0}(t_j)$ differ from $P_{>0}(t_j)$ respectively $P_{<0}(t_j)$ in an s-smoothing projector $P_0(t_j)$. We write $\tilde{r}_\pm(t_j)$ for $r_\pm(t_j) + P_0(t_j)$ such that

$$P_{\geq 0}(t_j) = p_+(t_j) + \tilde{r}_+(t_j) \quad \text{and} \quad P_{\leq 0}(t_j) = p_-(t_j) + \tilde{r}_-(t_j) \quad .$$

Each spectral entry in (9.2) can be split up into a sum of a properly supported Γ-Fourier integral operator and a s-smoothing operator:

$$\begin{aligned} Q_{\pm\pm}(t_2,t_1) &= q_{\pm\pm}(t_2,t_1) + R_{\pm\pm}(t_2,t_1) \quad \text{with} \quad q_{\pm\pm}(t_2,t_1) := p_\pm(t_2)\circ Q \circ p_\pm(t_1) \\ &\text{and} \quad R_{\pm\pm}(t_2,t_1) := \tilde{r}_\pm(t_2)\circ Q\circ p_\pm(t_1) + p_\pm \circ Q \circ r_\pm(t_1) + \tilde{r}_\pm(t_2)\circ Q\circ r_\pm(t_1)\,; \\ Q_{\pm\mp}(t_2,t_1) &= q_{\pm\mp}(t_2,t_1) + R_{\pm\mp}(t_2,t_1) \quad \text{with} \quad q_{\pm\mp}(t_2,t_1) := p_\pm(t_2)\circ Q \circ p_\mp(t_1) \\ &\text{and} \quad R_{\pm\mp}(t_2,t_1) := \tilde{r}_\pm(t_2)\circ Q\circ p_\mp(t_1) + p_\pm \circ Q \circ r_\mp(t_1) + \tilde{r}_\pm(t_2)\circ Q\circ r_\mp(t_1)\,. \end{aligned}$$

The triple compositions $q_{\pm\pm}$ and $q_{\pm\mp}$ of properly supported operators are properly supported Fourier integral operators of order 0 with canonical relation (7.14) since the composition of $N^*\mathrm{diag}(\Sigma_j)$ and $\mathsf{C}_{1\to 2}$ is proper, transversal, and results in $\mathsf{C}_{1\to 2}$:

$$q_{\pm\pm}(t_2,t_1), q_{\pm\mp}(t_2,t_1) \in \mathcal{FIO}^0_{\Gamma,\mathsf{prop}}(\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2))) \quad . \tag{9.11}$$

The remainders $R_{\pm\pm}(t_2,t_1)$ and $R_{\pm\mp}(t_2,t_1)$ are sums of compositions between s-smoothing pseudo-differential operators and the properly supported Fourier integral operator Q. As $p_\pm$ is a properly supported pseudo-differential operator of order 0, it maps $H^s_\Gamma(\mathcal{S}^+(\Sigma_1))$ to $H^s_\Gamma(\mathcal{S}^+(\Sigma_1))$ for any $s \in \mathbb{R}$ (see Proposition 5.3.3 (1)). Composing with Q is on one hand well-defined as Fourier integral operator, but on the other hand gives a map from $H^s_\Gamma(\mathcal{S}^+(\Sigma_1))$ to $H^s_\Gamma(\mathcal{S}^+(\Sigma_2))$ for any $s\in\mathbb{R}$. Since $r_\pm$ and $\tilde{r}_\pm$ are s-smoothing, they map between any two Γ-Sobolev spaces. As this holds true for the first two summands in $R_{\pm\pm}$ and $R_{\pm\mp}$, they become s-smoothing, too. $Q(t_2,t_1)$ on the other hand only maps $H^*_\Gamma(\mathcal{S}^+(\Sigma_1))$ to $H^*_\Gamma(\mathcal{S}^+(\Sigma_2))$ without affecting the order. Hence the remaining triple compositions of the form $\tilde{r}\circ Q\circ r$ are equally s-smoothing wherefore $R_{\pm\pm}$ and $R_{\pm\mp}$ become s-smoothing.

The intertwining of the left action representations is clear since every part in the composition is a Γ-invariant operator on its own right. Summing up, this shows that $Q_{\pm\pm}$ and $Q_{\pm\mp}$ are sums of properly supported Γ-Fourier integral operators and s-smoothing operators and thus elements in $\mathcal{SFIO}^0_\Gamma(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2)))$. □

Now we are able to compute the principal symbols of the compositions by multiplication since the character of $\mathsf{C}_{1\to 2}$, being a disjoint union of graphs of symplectomorphisms, has been preserved under the above calculated composition. In doing so, we observe that we can improve the result for the off-diagonal entries.

Proposition 9.1.6.

(1) $Q_{\pm\mp}(t_2,t_1) \in S\mathcal{FIO}_{\Gamma}^{-1}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+(\Sigma_1),\mathcal{S}^+(\Sigma_2)))$;

(2) $\tilde{Q}_{\pm\mp}(t_2,t_1) \in S\mathcal{FIO}_{\Gamma}^{-1}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^-(\Sigma_1),\mathcal{S}^-(\Sigma_2)))$;

(3) $Q^{E_L}_{\pm\mp}(t_2,t_1) \in S\mathcal{FIO}_{\Gamma}^{-1}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^+_{L,E}(\Sigma_1),\mathcal{S}^+_{L,E}(\Sigma_2)))$;

(4) $\tilde{Q}^{E_L}_{\pm\mp}(t_2,t_1) \in S\mathcal{FIO}_{\Gamma}^{-1}(\Sigma_1,\Sigma_2;\mathsf{C}'_{1\to 2};\mathbf{Hom}(\mathcal{S}^-_{L,E}(\Sigma_1),\mathcal{S}^-_{L,E}(\Sigma_2)))$.

Proof. We will first prove (1) and (2). For (3) and (4) we will observe that the former claims become important and their use simplify the argumentation.

We start with (1): let $q_{\pm\mp}(t_2,t_1)$ be the properly supported part of $Q_{\pm\mp}(t_2,t_1)$ from (9.11); their principal symbols are the same up to smoothing terms which are not contributing; the same holds for the symbol of the s-regular projectors. The fact, that the resulting canonical relation and thus its nature as graph of a symplectomorphism are preserved under the composition, allows us to compute the principal symbol of the compositions by just composing the principal symbol of each occuring operator (see (4.20)):

$$\begin{aligned} \mathcal{q}_{\pm\mp}(x,\xi_\pm;y,\eta) &:= \boldsymbol{\sigma}_0(q_{\pm\mp})(x,\xi_\pm;y,\eta) = \boldsymbol{\sigma}_0(Q_{\pm\mp})(x,\xi_\pm;y,\eta) \\ &= \boldsymbol{\sigma}_0(p_\pm)(x,\xi_\pm)\circ\boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta)\circ\boldsymbol{\sigma}_0(p_\mp)(y,\eta) \end{aligned}$$

for $(x,\xi_\pm)\in T^*_x\Sigma_2$ and $(y,\eta)\in T^*_y\Sigma_2$. The principal symbol (7.25) of the wave evolution operator comes with future- and past-pointing lightlike covectors $\varsigma_\pm$ which restrict to $\xi_\pm$ respectively at Σ_2. We need to distinguish between these two directions in the principal symbol calculation. For the future-pointing lightlike covector ς_+ we have

$$\begin{aligned} \mathcal{q}_{\pm\mp}(x,\xi_+;y,\eta) = &\frac{\|\eta\|^{-1}_{\mathcal{g}_{t_1}(y)}}{8}\left(1\mp\|\xi_+\|^{-1}_{\mathcal{g}_{t_2}(x)}\,\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{\mathcal{g}_{t_2}(x)}\,\upbeta+\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right) \\ &\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\upbeta\circ\left(1\pm\|\eta\|^{-1}_{\mathcal{g}_{t_1}(y)}\,\upbeta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \\ = &\frac{\|\eta\|^{-2}_{\mathcal{g}_{t_1}(y)}\,\|\xi_+\|^{-1}_{\mathcal{g}_{t_2}(x)}}{8}\left(\|\xi_+\|_{\mathcal{g}_{t_2}(x)}\mp\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{\mathcal{g}_{t_2}(x)}\,\upbeta+\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\upbeta^2\right) \\ &\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\upbeta\circ\left(\|\eta\|_{\mathcal{g}_{t_1}(y)}\pm\upbeta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \\ = &\frac{\|\eta\|^{-2}_{\mathcal{g}_{t_1}(y)}\,\|\xi_+\|^{-1}_{\mathcal{g}_{t_2}(x)}}{8}\left(\|\xi_+\|_{\mathcal{g}_{t_2}(x)}\mp\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{\mathcal{g}_{t_2}(x)}-\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right) \\ &\circ\upbeta\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\upbeta\circ\left(\|\eta\|_{\mathcal{g}_{t_1}(y)}\pm\upbeta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \\ = -&\frac{\|\eta\|^{-2}_{\mathcal{g}_{t_1}(y)}\,\|\xi_+\|^{-1}_{\mathcal{g}_{t_2}(x)}}{8}\left(\|\xi_+\|_{\mathcal{g}_{t_2}(x)}\mp\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(\|\xi_+\|_{\mathcal{g}_{t_2}(x)}+\upbeta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right) \\ &\circ\upbeta\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\upbeta\circ\left(\|\eta\|_{\mathcal{g}_{t_1}(y)}\pm\upbeta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \quad . \end{aligned}$$

We used the fact that the parallel transport acts as linear isomorphism such that for fixed $(x, \xi_\pm; y, \eta)$ the shift of the norms from one side of the composition to the other side of the parallel transport operator does not affect the norm. A similar calculation shows a similar formula for the past-pointing lightlike covector ς_- which restricts to ξ_- at Σ_2:

$$\begin{aligned}
q_{\pm\mp}(x,\xi_-;y,\eta) = &-\frac{\|\eta\|^{-1}_{g_{t_1}(y)}}{8}\left(1 \mp \|\xi_-\|^{-1}_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}\,\beta + \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\circ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)} \circ \beta \circ \left(1 \pm \|\eta\|^{-1}_{g_{t_1}(y)}\,\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
= &-\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\,\|\xi_-\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_-\|_{g_{t_2}(x)} \mp \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}\,\beta + \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta^2\right)\\
&\circ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)} \circ \beta \circ \left(\|\eta\|_{g_{t_1}(y)} \pm \beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
= &-\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\,\|\xi_-\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_-\|_{g_{t_2}(x)} \mp \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\circ \beta \circ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)} \circ \beta \circ \left(\|\eta\|_{g_{t_1}(y)} \pm \beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
= &\ \frac{\|\eta\|^{-2}_{g_{t_1}(y)}\,\|\xi_-\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_-\|_{g_{t_2}(x)} \mp \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(-\|\xi_-\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\circ \beta \circ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)} \circ \beta \circ \left(\|\eta\|_{g_{t_1}(y)} \pm \beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \quad .
\end{aligned}$$

The composition of Clifford multiplications on the left-hand side of the parallel transport in $q_{\pm\mp}(x,\xi_+;y,\eta)$ can be further analysed:

$$\begin{aligned}
&\left(\|\xi_+\|_{g_{t_2}(x)} \mp \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta\right)\circ\left(\|\xi_+\|_{g_{t_2}(x)} + \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta\right)\\
= &\ \|\xi_+\|^2_{g_{t_2}(x)} + \|\xi_+\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right) \pm \|\xi_+\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right) \mp \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\\
= &\ \|\xi_+\|^2_{g_{t_2}(x)} + (1\mp 1)\,\|\xi_+\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right) \pm \beta^2\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)^2\\
= &\ \|\xi_+\|^2_{g_{t_2}(x)} + (1\mp 1)\,\|\xi_+\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right) \mp g_{t_2}(\xi_+^\sharp,\xi_+^\sharp)
\end{aligned}$$

which vanishes for the upper signs; the same composition of Clifford multiplications in $q_{\pm\mp}(x,\xi_-;y,\eta)$ on the other hand becomes

$$\begin{aligned}
&\left(\|\xi_-\|_{g_{t_2}(x)} \mp \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta\right)\circ\left(-\|\xi_-\|_{g_{t_2}(x)} + \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta\right)\\
= &\ -\|\xi_-\|^2_{g_{t_2}(x)} + \|\xi_-\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right) \pm \|\xi_-\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right) \mp \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\\
= &\ -\|\xi_-\|^2_{g_{t_2}(x)} + (1\pm 1)\,\|\xi_-\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right) \pm \beta^2\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)^2\\
= &\ -\|\xi_-\|^2_{g_{t_2}(x)} + (1\pm 1)\,\|\xi_-\|_{g_{t_2}(x)}\,\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right) \mp g_{t_2}(\xi_-^\sharp,\xi_-^\sharp)
\end{aligned}$$

which vanishes for the lower signs. Equation (6.10) has been used in each last line and

$$\beta\mathbf{c}_t\left(X\right) = \mathbb{i}\beta^2\mathbf{c}\left(X\right) = -\mathbb{i}\beta\mathbf{c}\left(X\right)\beta = -\mathbf{c}_t\left(X\right)\beta \tag{9.12}$$

for any spacelike vector field X along Σ_t. So $q_{+-}(x,\xi_+;y,\eta) = q_{-+}(x,\xi_-;y,\eta) = 0$. In order to show the vanishing of the other combinations, we need to use the following fact.

Claim.

$$\mathcal{P}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\left(\mp\|\eta\|_{g_{t_1}(y)}\,\beta+\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \\ =\left(\mp\|\xi_\pm\|_{g_{t_2}(x)}\,\beta+\mathbf{c}_{t_2}\left(\xi_\pm^\sharp\right)\right)\mathcal{P}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \quad . \tag{9.13}$$

In order to show this, we make use of the global parallelity of the Clifford multiplication in the form of (6.21). We choose the path γ in (6.21) to be a lightlike geodesic, connecting $(y,\zeta_\pm^\sharp)$ and $(x,\varsigma_\pm^\sharp)$; the parametrisation is chosen in such a way that $\gamma(0)=x$, $\dot\gamma(0)=\varsigma_\pm^\sharp$ and $\gamma(s)=y$, $\dot\gamma(s)=\zeta_\pm^*$ are satisfied for a fixed parameter s. We gain

$$\mathcal{P}(\gamma)_s^0\mathbf{c}(\zeta_\pm^*)=\mathbf{c}(\varsigma_\pm^*)\mathcal{P}(\gamma)_s^0 \quad\text{with}\quad \mathcal{P}(\gamma)_s^0=\mathcal{P}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \quad .$$

$\zeta_\pm$ and $\varsigma_\pm$ restrict to η at $y\in\Sigma_1$ and respectively to $\xi_\pm$ at $x\in\Sigma_2$; both are (anti-)parallel[35] to ν such that one can make the ansatz

$$\zeta_\pm^*|_{\Sigma_1}=\mp a\nu+\eta^\sharp \quad , \\ \varsigma^*_{\pm|\Sigma_2}=\mp b\nu+\xi_\pm^\sharp$$

where the positive normalisation factors a,b can be determined by the constraint that both vectors are lightlike:

$$0=g_y(\zeta_\pm)=-a^2+g_t|_y(\eta^\sharp,\eta^\sharp)=-a^2+\|\eta\|^2_{g_{t_1}(y)} \quad\Rightarrow\quad a=\|\eta\|_{g_{t_1}(y)}$$

and analogously $b=\|\xi_\pm\|_{g_{t_2}(x)}$. The corresponding Clifford-multiplications are

$$\mathbf{c}\left(\zeta_\pm|_{\Sigma_1}\right)=\mp\|\eta\|_{g_{t_1}(y)}\,\mathbf{c}\left(\nu\right)+\mathbf{c}_{t_1}\left(\eta^\sharp\right)=\mp\|\eta\|_{g_{t_1}(y)}\,\beta+\mathbf{c}_{t_1}\left(\eta^\sharp\right) \quad , \\ \mathbf{c}\left(\varsigma_\pm|_{\Sigma_1}\right)=\mp\|\xi_\pm\|_{g_{t_2}(x)}\,\mathbf{c}\left(\nu\right)+\mathbf{c}_{t_2}\left(\xi_\pm^\sharp\right)=\mp\|\xi_\pm\|_{g_{t_2}(x)}\,\beta+\mathbf{c}_{t_2}\left(\xi_\pm^\sharp\right)$$

and with (6.21) the claim (9.13) is proven.

The other not already vanishing principal symbols also vanish by changing the order of parallel transport and Clifford multiplication: denote as a shorthand notation the prefactor

$$\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\,\|\xi_+\|^{-1}_{g_{t_2}(x)}}{8}=:C(\eta,\xi_\pm) \quad .$$

We have to rearrange the left- and right-hand side of the parallel transport operators in $q_{-+}(x,\xi_+;y,\eta)$ and $q_{+-}(x,\xi_-;y,\eta)$ by perfoming the Clifford multiplications with β in order to apply (9.13):

[35] Since ν is chosen to be globally past-directed, the temporal projection of ζ_+ is anti-parallel to ν whereas the projection of ζ_- in temporal direction is parallel; the same holds for $\varsigma_\pm$.

$$\begin{aligned}
q_{-+}(x,\xi_+;y,\eta) &= -C(\eta,\xi_+)\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\
&\qquad\circ\beta\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)} - \beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= -C(\eta,\xi_+)\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(\|\xi_+\|_{g_{t_2}(x)}\beta + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta - \beta^2\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= -C(\eta,\xi_+)\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(\|\xi_+\|_{g_{t_2}(x)}\beta - \beta^2\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta - \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= C(\eta,\xi_+)\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{g_{t_2}(x)}\beta + \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta - \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&\overset{(9.13)}{=} C(\eta,\xi_+)\left(\|\xi_+\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)} \qquad (9.14)\\
&\qquad\circ\left(-\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\circ\left(\|\eta\|_{g_{t_1}(y)}\beta - \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) \quad .
\end{aligned}$$

The composition of the round brackets in (9.14) vanishes:

$$\begin{aligned}
&\left[-\|\eta\|^2_{g_{t_1}(y)}\beta^2 + \|\eta\|_{g_{t_1}(y)}\left(\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right) + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\beta\right) - \mathbf{c}_{t_1}\left(\eta^\sharp\right)^2\right]\\
&= \left[-\|\eta\|^2_{g_{t_1}(y)}\beta^2 + g_{t_1}(\eta^\sharp,\eta^\sharp)\right] = 0 \quad .
\end{aligned}$$

We can perform the same calculations for $q_{+-}(x,\xi_-;y,\eta)$:

$$\begin{aligned}
q_{+-}(x,\xi_-;y,\eta) &= C(\eta,\xi_-)\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(-\|\xi_-\|_{g_{t_2}(x)} + \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\qquad\circ\beta\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)} + \beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= C(\eta,\xi_-)\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(-\|\xi_-\|_{g_{t_2}(x)}\beta + \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \beta^2\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= C(\eta,\xi_-)\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(-\|\xi_-\|_{g_{t_2}(x)}\beta - \beta^2\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&= -C(\eta,\xi_-)\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}\beta + \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&\qquad\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&\overset{(9.13)}{=} -C(\eta,\xi_-)\left(\|\xi_-\|_{g_{t_2}(x)} - \beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)} \qquad (9.15)\\
&\qquad\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)
\end{aligned}$$

such that the round brackets in (9.15) yield

$$\begin{aligned}&\left[\|\eta\|^2_{g_{t_1}(y)}\,\beta^2+\|\eta\|_{g_{t_1}(y)}\left(\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)+\mathbf{c}_{t_1}\left(\eta^\sharp\right)\beta\right)+\mathbf{c}_{t_1}\left(\eta^\sharp\right)^2\right]\\&=\left[\|\eta\|^2_{g_{t_1}(y)}-g_{t_1}(\eta^\sharp,\eta^\sharp)\right]=0\quad.\end{aligned}$$

Hence $q_{-+}(x,\xi_+;y,\eta)$ and $q_{+-}(x,\xi_-;y,\eta)$ vanish, too.

The calculation of the principal symbols with respect to the matrix entries of $\tilde{Q}$ can be performed likewise. We again use the fact that the resulting canonical relation and thus its nature as graph of a symplectomorphism are preserved under the composition such that

$$\begin{aligned}\tilde{q}_{\pm\mp}(x,\xi_\pm;y,\eta)&:=\boldsymbol{\sigma}_0(\tilde{q}_{\pm\mp})(x,\xi_\pm;y,\eta)=\boldsymbol{\sigma}_0(\tilde{Q}_{\pm\mp})(x,\xi_\pm;y,\eta)\\&=\boldsymbol{\sigma}_0(p_\pm)(x,\xi_\pm)\circ\boldsymbol{\sigma}_0(\tilde{Q})(x,\xi_\pm;y,\eta)\circ\boldsymbol{\sigma}_0(p_\mp)(y,\eta)\end{aligned}$$

with $(x,\xi_\pm)\in T^*_x\Sigma_2$ and $(y,\eta)\in T^*_y\Sigma_2$. The principal symbol of $\tilde{Q}$ is given as in (7.26) of Theorem 7.2.5 and again we need to distinguish between the future- and past-pointing lightlike covector ς_+ and ς_- which restrict to ξ_+ respectively ξ_- at Σ_2:

$$\begin{aligned}\tilde{q}_{\pm\mp}(x,\xi_+;y,\eta)&=\frac{\|\eta\|^{-1}_{g_{t_1}(y)}}{8}\left(1\mp\|\xi_+\|^{-1}_{g_{t_2}(x)}\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\&\qquad\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\beta\circ\left(1\pm\|\eta\|^{-1}_{g_{t_1}(y)}\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\&=\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\|\xi_+\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_+\|_{g_{t_2}(x)}\mp\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta^2\right)\\&\qquad\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)}\pm\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\&=\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\|\xi_+\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_+\|_{g_{t_2}(x)}\mp\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\left(-\|\xi_+\|_{g_{t_2}(x)}+\beta\mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\&\qquad\circ\beta\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)}\pm\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\end{aligned}$$

for the future-pointing lightlike covector ς_+ and

$$\begin{aligned}\tilde{q}_{\pm\mp}(x,\xi_-;y,\eta)&=-\frac{\|\eta\|^{-1}_{g_{t_1}(y)}}{8}\left(1\mp\|\xi_-\|^{-1}_{g_{t_2}(x)}\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\&\qquad\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\beta\circ\left(1\pm\|\eta\|^{-1}_{g_{t_1}(y)}\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\&=-\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\|\xi_-\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_-\|_{g_{t_2}(x)}\mp\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta^2\right)\\&\qquad\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)}\pm\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\&=-\frac{\|\eta\|^{-2}_{g_{t_1}(y)}\|\xi_-\|^{-1}_{g_{t_2}(x)}}{8}\left(\|\xi_-\|_{g_{t_2}(x)}\mp\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\left(\|\xi_-\|_{g_{t_2}(x)}+\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\&\qquad\circ\beta\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)}\pm\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\end{aligned}$$

for the past-pointing lightlike covector ς_-. We use the same techniques to show that these principal symbols also vanish: from $\tilde{q}_{\pm\mp}(x,\xi_+;y,\eta)$ we have

$$\begin{aligned}
&\left(\|\xi_+\|_{g_{t_2}(x)} \mp \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right) \circ \left(-\|\xi_+\|_{g_{t_2}(x)} + \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\\
&= -\|\xi_+\|^2_{g_{t_2}(x)} + \|\xi_+\|_{g_{t_2}(x)}\left(\beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right) \pm \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right) \mp \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\\
&= -\|\xi_+\|^2_{g_{t_2}(x)} + \|\xi_+\|_{g_{t_2}(x)}\beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)(1\pm 1) \pm \beta^2 \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)^2\\
&= -\|\xi_+\|^2_{g_{t_2}(x)} + \|\xi_+\|_{g_{t_2}(x)}\beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)(1\pm 1) \mp g_{t_2}|_x(\xi_+^\sharp,\xi_+^\sharp) \quad ;
\end{aligned}$$

this vanishes for the lower signs, implying $\tilde{q}_{-+}(x,\xi_+;y,\eta)=0$. The same composition in the symbol $\tilde{q}_{\pm\mp}(x,\xi_-;y,\eta)$ takes the form

$$\begin{aligned}
&\left(\|\xi_-\|_{g_{t_2}(x)} \mp \beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right) \circ \left(\|\xi_-\|_{g_{t_2}(x)} + \beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\\
&= \|\xi_-\|^2_{g_{t_2}(x)} + \|\xi_-\|_{g_{t_2}(x)}\left(\beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right) \mp \beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right) \mp \beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\\
&= \|\xi_-\|^2_{g_{t_2}(x)} + \|\xi_-\|_{g_{t_2}(x)}\beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)(1\mp 1) \pm \beta^2 \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)^2\\
&= \|\xi_-\|^2_{g_{t_2}(x)} + \|\xi_-\|_{g_{t_2}(x)}\beta \mathbf{c}_{t_2}\left(\xi_-^\sharp\right)(1\mp 1) \mp g_{t_2}|_x(\xi_-^\sharp,\xi_-^\sharp)
\end{aligned}$$

and vanishes for the upper signs; hence $\tilde{q}_{+-}(x,\xi_-;y,\eta)=0$. The vanishing of the two remaining symbols can be again shown with (9.13) where we point out that the plus sign in front of the restricted Clifford multiplication is somewhat arbitrary and can be replaced by a minus sign:

$$\begin{aligned}
&\mathcal{P}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\left(\mp\|\eta\|_{g_{t_1}(y)}\beta - \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
&\qquad = \left(\mp\|\xi_\pm\|_{g_{t_2}(x)}\beta - \mathbf{c}_{t_2}\left(\xi_\pm^\sharp\right)\right)\mathcal{P}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)} \quad . \qquad (9.16)
\end{aligned}$$

In $\tilde{q}_{+-}(x,\xi_+;y,\eta)$ we have

$$\begin{aligned}
&\left(-\|\xi_+\|_{g_{t_2}(x)} + \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\beta\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)} + \beta \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=\;&\left(-\|\xi_+\|_{g_{t_2}(x)}\beta + \beta \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\beta\right)\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=\;&\left(-\|\xi_+\|_{g_{t_2}(x)}\beta - \mathbf{c}_{t_2}\left(\xi_+^\sharp\right)\right)\circ\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
\overset{(9.16)}{=}\;&-\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\circ\left(\|\eta\|_{g_{t_1}(y)}\beta + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=\;&-\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left[\|\eta\|^2_{g_{t_1}(y)}\beta^2 + \|\eta\|_{g_{t_1}(y)}\left(\beta \mathbf{c}_{t_1}\left(\eta^\sharp\right) + \mathbf{c}_{t_1}\left(\eta^\sharp\right)\beta\right) + \mathbf{c}_{t_1}\left(\eta^\sharp\right)^2\right]\\
=\;&-\mathcal{P}_{(x,\varsigma_+)\leftarrow(y,\zeta_+)}\circ\left[\|\eta\|^2_{g_{t_1}(y)} + \|\eta\|_{g_{t_1}(y)}\left(\beta \mathbf{c}_{t_1}\left(\eta^\sharp\right) - \beta \mathbf{c}_{t_1}\left(\eta^\sharp\right)\right) - g_{t_1}|_y(\eta^\sharp,\eta^\sharp)\right] = 0
\end{aligned}$$

and consequently $\tilde{q}_{+-}(x,\xi_+;y,\eta)=0$; we equally have in $\tilde{q}_{-+}(x,\xi_-;y,\eta)$ the expression

$$\begin{aligned}
&\left(\|\xi_-\|_{g_{t_2}(x)}+\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\beta\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\beta\circ\left(\|\eta\|_{g_{t_1}(y)}\pm\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=&\left(\|\xi_-\|_{g_{t_2}(x)}\beta+\beta\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\beta\right)\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta-\beta^2\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=&\left(\|\xi_-\|_{g_{t_2}(x)}\beta-\mathbf{c}_{t_2}\left(\xi_-^\sharp\right)\right)\circ\mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta-\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=&\ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left(\|\eta\|_{g_{t_1}(y)}\beta-\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\circ\left(\|\eta\|_{g_{t_1}(y)}\beta-\mathbf{c}_{t_1}\left(\eta^\sharp\right)\right)\\
=&\ \mathcal{P}_{(x,\varsigma_-)\leftarrow(y,\zeta_-)}\circ\left[\|\eta\|^2_{g_{t_1}(y)}\beta^2-\|\eta\|_{g_{t_1}(y)}\left(\beta\mathbf{c}_{t_1}\left(\eta^\sharp\right)+\mathbf{c}_{t_1}\left(\eta^\sharp\right)\beta\right)+\mathbf{c}_{t_1}\left(\eta^\sharp\right)^2\right]=0
\end{aligned}$$

and finally also $\tilde{q}_{-+}(x,\xi_-;y,\eta)=0$. Hence the principal symbols of $Q_{\pm\mp}(t_2,t_1)$ and $\tilde{Q}_{\pm\mp}(t_2,t_1)$ are identically vanishing. The exact sequence property in Lemma 4.2.5 (1) then implies that the order of the properly supported part is (-1) which proves (1) and (2).

The vanishing of the principal symbols of order zero for (3) and (4) can be explained as follows: (7.15), (7.16), and (8.17) show that the principal symbols of the wave evolution operators and the projectors for both chiralities decompose into a tensor product of the principal symbols where the left factors are equivalent to the principal symbols of the corresponding objects in the untwisted case. Let $\blacktriangle,\blacktriangledown$ denote either $+$ or $-$ in the following argument. Combining all calculated principal symbols shows that the principal symbols in the twisted case factorise with respect to the tensor product:

$$\begin{aligned}
\boldsymbol{\sigma}_0(Q^{E_L}_{\blacktriangle,\blacktriangledown})(x,\xi_\pm;y,\eta)&=\boldsymbol{\sigma}_0\left(P^{E_L}_{\blacktriangle}(t_2)\right)(x,\xi_\pm)\circ\boldsymbol{\sigma}_0(Q^{E_L})(x,\xi_\pm;y,\eta)\circ\boldsymbol{\sigma}_0\left(P^{E_L}_{\blacktriangle}(t_1)\right)(y,\eta)\\
&=\left[\boldsymbol{\sigma}_0\left(P_\blacktriangle(t_2)\right)(x,\xi)\otimes\mathbb{1}_{E_L|_{\Sigma_2}}\right]\\
&\qquad\circ\boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta)\otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]\\
&\qquad\circ\left[\boldsymbol{\sigma}_0\left(P_\blacktriangledown(t_1)\right)(y,\eta)\otimes\mathbb{1}_{E_L|_{\Sigma_1}}\right]\\
&=\left[\boldsymbol{\sigma}_0\left(P_\blacktriangle(t_2)\right)(x,\xi)\circ\boldsymbol{\sigma}_0(Q)(x,\xi_\pm;y,\eta)\circ\boldsymbol{\sigma}_0\left(P_\blacktriangledown(t_1)\right)(y,\eta)\right]\\
&\qquad\otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]\\
&=\boldsymbol{\sigma}_0(Q_{\blacktriangle,\blacktriangledown})(x,\xi_\pm;y,\eta)\otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]
\end{aligned}$$

and

$$\boldsymbol{\sigma}_0(\tilde{Q}^{E_L}_{\blacktriangle,\blacktriangledown})(x,\xi_\pm;y,\eta)=\boldsymbol{\sigma}_0(\tilde{Q}_{\blacktriangle,\blacktriangledown})(x,\xi_\pm;y,\eta)\otimes\left[\mathbb{1}_{E_L|_{\Sigma_2}}\circ\mathcal{P}^{E_L}_{(x,\varsigma_\pm)\leftarrow(y,\zeta_\pm)}\circ\mathbb{1}_{E_L|_{\Sigma_1}}\right]\quad.$$

The vanishing of the principal symbols in the untwisted case then implies the vanishing of these tensor products. The exact sequence property in Lemma 4.2.5 (1) finally proves the proposition. □

9.1.3. Γ-Fredholmness of the matrix entries

We have proven in Section 7.3 that Q and $\tilde{Q}$ as well as Q^{E_L} and $\tilde{Q}^{E_L}$ are unitary Γ-morphisms on L^2_Γ-spaces. Hence these four operators are Γ-Fredholm with identical vanishing Γ-indices.

Corollary 9.1.7.

(1) $Q \in \mathscr{F}_\Gamma(L^2_\Gamma(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$ *with* $\mathrm{ind}_\Gamma(Q) = 0$ *;*

(2) $\tilde{Q} \in \mathscr{F}_\Gamma(L^2_\Gamma(\mathcal{S}^-(\Sigma_1)), L^2_\Gamma(\mathcal{S}^-(\Sigma_2)))$ *with* $\mathrm{ind}_\Gamma(\tilde{Q}) = 0$ *;*

(3) $Q^{E_L} \in \mathscr{F}_\Gamma(L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)))$ *with* $\mathrm{ind}_\Gamma(Q^{E_L}) = 0$ *;*

(4) $\tilde{Q}^{E_L} \in \mathscr{F}_\Gamma(L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_1)), L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)))$ *with* $\mathrm{ind}_\Gamma(\tilde{Q}^{E_L}) = 0$ *.*

Now the question arises how the Γ-Fredholm property carries over to their matrix entries for (a)APS boundary conditions. The following result gives an answer.

Theorem 9.1.8.

(1) $Q_{\pm\pm}$ *and* $\tilde{Q}_{\pm\pm}$ *are Γ-Fredholm as maps from*

$$\begin{aligned} Q_{++}(t_2,t_1) &: L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)) \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ Q_{--}(t_2,t_1) &: L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)) \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_2)) \\ \tilde{Q}_{++}(t_2,t_1) &: L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_1)) \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_2)) \\ \tilde{Q}_{--}(t_2,t_1) &: L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)) \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_2)) \end{aligned} \tag{9.17}$$

with Γ-indices

$$\mathrm{ind}_\Gamma(Q_{++}(t_2,t_1)) = -\mathrm{ind}_\Gamma(Q_{--}(t_2,t_1))$$

and

$$\mathrm{ind}_\Gamma(\tilde{Q}_{++}(t_2,t_1)) = -\mathrm{ind}_\Gamma(\tilde{Q}_{--}(t_2,t_1)) \quad .$$

(2) $Q^{E_L}_{\pm\pm}$ *and* $\tilde{Q}^{E_L}_{\pm\pm}$ *are Γ-Fredholm as maps from*

$$\begin{aligned} Q^{E_L}_{++}(t_2,t_1) &: L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+_{L,E}(\Sigma_2)) \\ Q^{E_L}_{--}(t_2,t_1) &: L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+_{L,E}(\Sigma_1)) \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+_{L,E}(\Sigma_2)) \\ \tilde{Q}^{E_L}_{++}(t_2,t_1) &: L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \to L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_2)) \\ \tilde{Q}^{E_L}_{--}(t_2,t_1) &: L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-_{L,E}(\Sigma_1)) \to L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-_{L,E}(\Sigma_2)) \end{aligned} \tag{9.18}$$

with Γ-indices

$$\mathrm{ind}_\Gamma(Q^{E_L}_{++}(t_2,t_1)) = -\mathrm{ind}_\Gamma(Q^{E_L}_{--}(t_2,t_1))$$

and

$$\mathrm{ind}_\Gamma(\tilde{Q}^{E_L}_{++}(t_2,t_1)) = -\mathrm{ind}_\Gamma(\tilde{Q}^{E_L}_{--}(t_2,t_1)) \quad .$$

Proof. As in the former subsection we focus on the spectral entries of the Dirac-wave evolution operators with respect to $D_\pm$ without twisting bundle E_L.

Based on Proposition 9.1.5 (1), a precise analysis of the compositions $Q^*_{\pm\mp} \circ Q_{\pm\mp}$ with idempotence and self-adjointness of the projections shows that

$$\begin{aligned}
Q^*_{\pm\mp} \circ Q_{\pm\mp} &= (P_\pm(t_2) \circ Q(t_2,t_1) \circ P_\mp(t_1))^* \circ (P_\pm(t_2) \circ Q(t_2,t_1) \circ P_\mp(t_1)) \\
&= P_\mp(t_1) \circ Q^*(t_1,t_2) \circ P_\pm(t_2) \circ P_\pm(t_2) \circ Q(t_2,t_1) \circ P_\mp(t_1) \\
&= P_\mp(t_1) \circ Q^*(t_1,t_2) \circ P_\pm(t_2) \circ Q(t_2,t_1) \circ P_\mp(t_1) \\
&= P_\mp(t_1) \circ Q^*(t_1,t_2) \circ Q_{\pm\mp}(t_2,t_1) \\
&= (p_\mp(t_1) + \tilde{r}_\mp(t_1)) \circ Q^*(t_1,t_2) \circ (q_{\pm\mp}(t_2,t_1) + R_{\pm\mp}(t_2,t_1)) \\
&= p_\mp(t_1) \circ Q^*(t_1,t_2) \circ q_{\pm\mp}(t_2,t_1) + \tilde{r}_\mp(t_1) \circ Q^*(t_1,t_2) \circ q_{\pm\mp}(t_2,t_1) \\
&\quad + p_\mp(t_1) \circ Q^*(t_1,t_2) \circ R_{\pm\mp}(t_2,t_1) + \tilde{r}_\mp(t_1) \circ Q^*(t_1,t_2) \circ R_{\pm\mp}(t_2,t_1) \quad .
\end{aligned} \tag{9.19}$$

The first triple composition is a properly supported operator as it is a composition of properly supported operators; the same follows for $p_\pm \circ Q^*$ and $Q^* \circ q_{\pm\mp}$. Moreover, they are Γ-morphisms between Hilbert Γ-modules. As $\tilde{r}_\mp$ is a s-smoothing Γ-invariant pseudo-differential operator, it is Γ-trace class due to Lemma 5.3.5 (3) and consequently $\tilde{r}_\mp(t_1) \circ Q^*(t_1,t_2) \circ q_{\pm\mp}(t_2,t_1)$, too. Recalling the definitions of $R_{\pm\mp}(t_2,t_1)$ from the proof of Proposition 9.1.5, i.e.

$$R_{\pm\mp}(t_2,t_1) = \tilde{r}_\pm(t_2) \circ Q \circ p_\mp(t_1) + p_\pm \circ Q \circ r_\mp(t_1) + \tilde{r}_\pm(t_2) \circ Q \circ r_\mp(t_1) \quad ,$$

we observe that the first two compositions are Γ-trace class as they are s-smoothing. Q^* is a (bounded) Γ-isomorphism and the s-smoothing pseudo-differential operators $r_\mp(t_1)$ and $\tilde{r}_\pm(t_2)$ are Γ-trace class (Lemma 5.3.5 (3)) such that the left triple compositions $\tilde{r}_\pm(t_2) \circ Q \circ r_\mp(t_1)$ are Γ-trace class as left ideal (via $r_\mp(t_1)$) and as right ideal (via $\tilde{r}_\pm(t_2)$). Thus, $R_{\pm\mp}(t_2,t_1)$ and finally the last remainder in (9.19) are Γ-trace class.

We equally notice that $p_\mp(t_1) \circ Q^* \circ q_{\pm\mp}(t_2,t_1)$ coincides with $q^*(t_1,t_2) \circ q_{\pm\mp}(t_2,t_1)$ modulo a Γ-trace class operator: recalling $q_{\pm\mp}$ from the proof of Proposition 9.1.5, we get from the idempotency of the projectors $p_\pm^2 = p_\pm$ modulo a s-smoothing pseudo-differential operator and thus

$$\begin{aligned}
p_\mp(t_1) \circ Q^*(t_1,t_2) \circ q_{\pm\mp}(t_2,t_1) &= p_\mp(t_1) \circ Q^* \circ q_{\pm\mp}(t_2,t_1) \\
&= p_\mp(t_1) \circ Q^* \circ p_\pm(t_2) \circ Q \circ p_\mp(t_1) \\
&= p_\mp(t_1) \circ Q^* \circ p_\pm^2(t_2) \circ Q \circ p_\mp(t_1) \\
&= (p_\pm(t_2) \circ Q \circ p_\mp(t_1))^* \circ p_\pm(t_2) \circ Q \circ p_\mp(t_1) \\
&= q^*_{\pm\mp}(t_1,t_2) q_{\pm\mp}(t_2,t_1)
\end{aligned}$$

modulo an s-smoothing and thus Γ-trace class operator such that

$$Q^*_{\pm\mp}(t_1,t_2) \circ Q_{\pm\mp}(t_2,t_1) \equiv q^*_{\pm\mp}(t_1,t_2) q_{\pm\mp}(t_2,t_1)$$

where we use '≡' to mark the equivalence up to Γ-trace class operators. The compositions $(q^*_{\pm\mp} \circ q_{\pm\mp})(t_1)$ are in fact properly supported Γ-pseudo-differential operators of order (-2) due to Lemma 4.2.3 (5). Hence $Q^*_{\pm\mp}(t_1,t_2) \circ Q_{\pm\mp}(t_2,t_1)$ is the sum of an element $B \in \Psi^{-2}_{\Gamma,\mathsf{prop}}(\mathcal{S}^+(\Sigma_1))$ with a Γ-trace class remainder.

The first two equations in (9.4) show the important observation that Q^*_{++} and Q^*_{--} can be used as initial parametrices of Q_{++} respectively Q_{--} with remainders of the form $Q^*_{\mp\pm} \circ Q_{\mp\pm}$. We can construct an even better parametrix for each operator with a Neumann series argument, so that the order of the error becomes sufficiently negative. Ellipticity would become an important property in constructing an initial parametrix, but becomes irrelevant here as the unitarity of Q replaces this step. We define $\mathcal{Q}_{\pm\mp} := -Q^*_{\pm\mp} \circ Q_{\pm\mp}$. Proposition 5.3.3 (1) implies that B is a Γ-morphism. Hence any power $l \in \mathbb{N}_0$ of $\mathcal{Q}_{\pm\mp}$ is the sum of $B^l \in \Psi^{-2l}_{\mathsf{prop}}(\mathcal{S}^+(\Sigma_1))$ with different combinations of powers of B and Γ-trace class remainders, i.e.

$$(-\mathcal{Q}_{\pm\mp})^l \equiv (-1)^l B^l \quad . \tag{9.20}$$

We first construct a left and a right parametrix for $Q_{--}(t_2, t_1)$. Given an operator $\mathcal{Q}_N \in \Psi^{-2N}_{\Gamma,\mathsf{prop}}(\mathcal{S}^+(\Sigma_1))$ for an $N \in \mathbb{N}$, we define

$$\mathcal{P}_{+-} := \left(\sum_{l=0}^{N-1} (-1)^l \mathcal{Q}^l_{+-} + \mathcal{Q}_N \right) Q^*_{--} \quad . \tag{9.21}$$

Because $Q_{--} \equiv q^*_{--}$, we deduce from (9.20) that

$$\mathcal{P}_{+-} \equiv \left(\sum_{l=0}^{N-1} (-1)^l B^l + \mathcal{Q}_N \right) q^*_{--} \quad . \tag{9.22}$$

Since q^*_{--} is an element in $\mathcal{FIO}^0_{\Gamma,\mathsf{prop}}(\Sigma_1, \Sigma_2; (\mathsf{C}_{1\to 2})^{-1\prime}; \mathbf{Hom}(\mathcal{S}^+(\Sigma_1), \mathcal{S}^+(\Sigma_2)))$, the implications

$$B^l \circ q^*_{--} \in \mathcal{FIO}^{-2l}_{\Gamma,\mathsf{prop}}(\Sigma_1, \Sigma_2; (\mathsf{C}_{1\to 2})^{-1\prime}; \mathbf{Hom}(\mathcal{S}^+(\Sigma_1), \mathcal{S}^+(\Sigma_2))) \quad \text{for } l \in \{0, 1, .., N-1\}$$

and

$$\mathcal{Q}_N \circ q^*_{--} \in \mathcal{FIO}^{-2N}_{\Gamma,\mathsf{prop}}(\Sigma_1, \Sigma_2; (\mathsf{C}_{1\to 2})^{-1\prime}; \mathbf{Hom}(\mathcal{S}^+(\Sigma_1), \mathcal{S}^+(\Sigma_2)))$$

follow; hence (9.22) is a properly supported Γ-Fourier integral operator of order $(-2N)$ with canonical relation $(\mathsf{C}_{1\to 2})^{-1}$ modulo Γ-trace class operators. We apply (9.22) to Q_{--} from the left and use the second equation in (9.4):

$$\begin{aligned}
\mathcal{P}_{+-} Q_{--} &= \left(\sum_{l=0}^{N-1} (-1)^l \mathcal{Q}^l_{+-} + \mathcal{Q}_N \right) Q^*_{--} Q_{--} = \left(\sum_{l=0}^{N-1} (-1)^l \mathcal{Q}^l_{+-} + \mathcal{Q}_N \right) (\mathbb{1} + \mathcal{Q}_{+-}) \\
&= \sum_{l=0}^{N-1} (-1)^l \mathcal{Q}^l_{+-} + \mathcal{Q}_N (\mathbb{1} + \mathcal{Q}_{+-}) + \sum_{l=0}^{N-1} (-1)^l \mathcal{Q}^{l+1}_{+-} \\
&= \mathbb{1} + (-1)^{N+1} \mathcal{Q}^N_{+-} + \mathcal{Q}_N (\mathbb{1} + \mathcal{Q}_{+-})
\end{aligned}$$

where an index shift has been applied to get the last equation. We conclude from here that

$$(\mathcal{P}_{+-} Q_{--} - \mathbb{1}) \in \Psi^{-2N}_{\Gamma,\mathsf{prop}}(\mathcal{S}^+(\Sigma_1)) \tag{9.23}$$

modulo Γ-trace class operators. If we choose $N > \dim(\Sigma)/2$ and recall Lemma 9.1.1, then Proposition 5.3.3 (3) already characterises $(\mathcal{P}_{+-} Q_{--} - \mathbb{1})$ as Γ-trace class operator and thus $\mathcal{P}_{+-}$ becomes a suitable left parametrix. The second equation in (9.5) implies for the same reason that

$$\mathcal{T}_{+-} := Q^*_{--}\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^l + \mathcal{Q}^*_N\right) \tag{9.24}$$

is a properly supported Γ-Fourier integral operator of order $(-2N)$ with canonical relation $(\mathsf{C}_{1\to 2})^{-1}$ modulo Γ-trace class operators. We apply (9.24) to Q_{--} and use the second equation in (9.5):

$$\begin{aligned}
Q_{--}\mathcal{T}_{+-} &= Q_{--}Q^*_{--}\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^l + \mathcal{Q}^*_N\right) = (\mathbb{1} + \mathcal{Q}^*_{+-})\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^l + \mathcal{Q}^*_N\right)\\
&= \left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^l + (\mathbb{1} + \mathcal{Q}^*_{+-})\mathcal{Q}^*_N\right) + \sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^{l+1}\\
&= \left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{+-})^l + (\mathbb{1} + \mathcal{Q}^*_{+-})\mathcal{Q}^*_N\right) - \sum_{l=1}^{N}(-1)^k(\mathcal{Q}^*_{+-})^k\\
&= \mathbb{1} + (\mathbb{1} + \mathcal{Q}^*_{+-})\mathcal{Q}^*_N + (-1)^{N+1}(\mathcal{Q}^*_{+-})^N \quad .
\end{aligned}$$

This implicates

$$(Q_{--}\mathcal{T}_{+-} - \mathbb{1}) \in \Psi^{-N}_{\Gamma,\mathsf{prop}}(\mathcal{S}^+(\Sigma_2)) \tag{9.25}$$

up to a Γ-trace class pertubation and becomes itself Γ-trace class if we choose $N > \dim(\Sigma)/2$. (9.24) becomes a right parametrix and consequently Q_{--} is Γ-Fredholm.

Suitable left and right parametrices for Q_{++} are

$$\mathcal{P}_{-+} := \left(\sum_{l=0}^{N-1}(-1)^l\mathcal{Q}^l_{-+} + \mathcal{Q}_N\right)Q^*_{++} \tag{9.26}$$

and

$$\mathcal{T}_{-+} := Q^*_{++}\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^l + \mathcal{Q}^*_N\right) \quad . \tag{9.27}$$

(9.26) and (9.27) are properly supported Γ-Fourier integral operator of order $(-2N)$ with canonical relation $(\mathsf{C}_{1\to 2})^{-1}$ modulo Γ-trace class operators for the same reasons as (9.22) and (9.24). We apply $\mathcal{P}_{-+}$ from the left to Q_{++} which shows

$$\begin{aligned}
\mathcal{P}_{-+}Q_{++} &= \left(\sum_{l=0}^{N-1}(-1)^l\mathcal{Q}^l_{-+} + \mathcal{Q}_N\right)Q^*_{++}Q_{++} = \left(\sum_{l=0}^{N-1}(-1)^l\mathcal{Q}^l_{-+} + \mathcal{Q}_N\right)(\mathbb{1} + \mathcal{Q}_{-+})\\
&= \sum_{l=0}^{N-1}(-1)^l\mathcal{Q}^l_{-+} + \mathcal{Q}_N\,(\mathbb{1} + \mathcal{Q}_{-+}) + \sum_{l=0}^{N-1}(-1)^l\mathcal{Q}^{l+1}_{-+}\\
&= \mathbb{1} + (-1)^{N+1}\mathcal{Q}^N_{-+} + \mathcal{Q}_N\,(\mathbb{1} + \mathcal{Q}_{-+})
\end{aligned}$$

with the first equation in (9.4); similarly applying $\mathcal{P}_{-+}$ to Q_{++} from the right shows

$$
\begin{aligned}
Q_{++}\mathcal{P}_{-+} &= Q_{++}Q^*_{++}\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^l + \mathcal{Q}^*_N\right) = (\mathbb{1}+\mathcal{Q}^*_{-+})\left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^l + \mathcal{Q}^*_N\right)\\
&= \left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^l + (\mathbb{1}+\mathcal{Q}^*_{-+})\mathcal{Q}^*_N\right) + \sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^{l+1}\\
&= \left(\sum_{l=0}^{N-1}(-1)^l(\mathcal{Q}^*_{-+})^l + (\mathbb{1}+\mathcal{Q}^*_{-+})\mathcal{Q}^*_N\right) - \sum_{l=1}^{N}(-1)^k(\mathcal{Q}^*_{-+})^k\\
&= \mathbb{1} + (\mathbb{1}+\mathcal{Q}^*_{-+})\mathcal{Q}^*_N + (-1)^{N+1}(\mathcal{Q}^*_{-+})^N
\end{aligned}
$$

with the first equation in (9.5). Choosing $N > \dim(\Sigma)/2$ shows that $(\mathcal{P}_{-+}Q_{++} - \mathbb{1})$ and $(Q_{++}\mathcal{P}_{-+} - \mathbb{1})$ are Γ-trace class operators and finally Q_{++} becomes Γ-Fredholm.

Lemma 9.1.2 and Proposition 5.2.6 (1) imply that there are unitary Γ-isomorphisms such that

$$\dim_\Gamma \ker\left(Q_{\pm\pm}\right) = \dim_\Gamma \ker\left(Q^*_{\mp\mp}\right) \quad . \tag{9.28}$$

The index formula in Proposition 5.2.17 (7) completes the proof for (1) because

$$
\begin{aligned}
\mathrm{ind}_\Gamma(Q_{--}) &= \dim_\Gamma \ker\left(Q_{--}\right) - \dim_\Gamma \ker\left(Q^*_{--}\right)\\
&\overset{(9.28)}{=} \dim_\Gamma \ker\left(Q^*_{++}\right) - \dim_\Gamma \ker\left(Q_{++}\right) = \mathrm{ind}_\Gamma(Q_{++}) \quad .
\end{aligned}
$$

The argument carries over to $\tilde{Q}_{\pm\pm}(t_2,t_1)$ with the help of (9.9), (9.10) and Lemma 9.1.3. If we repeat the argumentation with the twisted versions of (9.4), (9.5), Lemma 9.1.2, (9.9), (9.10) and Lemma 9.1.3, the second assertion follows. □

The presented proof has been published in [Dam21]. At that time there were some caveats concerning the correctness of (5.35) and its use as proof method. As we have clarified this characterisation of Γ-compact operators in subsection 5.2.3, we can give an alternative and more direct proof of Theorem 9.1.8, based on the following fact.

Lemma 9.1.9.

(1) $Q_{+-}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$ *and* $Q_{-+}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$.

(2) $\tilde{Q}_{+-}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_1)), L^2_\Gamma(\mathcal{S}^-(\Sigma_2)))$ *and* $\tilde{Q}_{-+}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$.

(3) $Q^{E_L}_{+-}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)))$ *and* $Q^{E_L}_{-+}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_\Gamma(\mathcal{S}^+_{L,E}(\Sigma_2)))$.

(4) $\tilde{Q}^{E_L}_{+-}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-_{L,E}(\Sigma_1)), L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)))$ *and* $\tilde{Q}^{E_L}_{-+}(t_2,t_1) \in \mathscr{K}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-_{L,E}(\Sigma_1)), L^2_\Gamma(\mathcal{S}^-_{L,E}(\Sigma_2)))$.

(1) then implies that $Q^*_{\pm\mp}$ and the compositions $Q^*_{\pm\mp}Q_{\pm\mp}$ are equally Γ-compact such that Γ-Fredholmness of $Q_{\pm\pm}$ follow directly from (9.4) and (9.5). The Γ-Fredholm parametrices are then fully given by the adjoints of $Q_{\pm\pm}$. The second assertion imply with (9.9) and (9.10) the Γ-Fredholmness of $\tilde{Q}_{\pm\pm}$ with Γ-Fredholm parametrices $\tilde{Q}^*_{\pm\pm}$. The reasoning for the twisted cases are similar.

Proof. We focus on (1) in this proof; (2), (3) and (4) follow with the same argumentation.

$Q_{\pm\mp}$ are unitarily related to $(\mathbb{1} \otimes \underline{Q_{\pm\mp}})$ according to the following commutative diagram.

$$\begin{array}{ccc} L^2_{\Gamma,I_\pm}(\mathcal{S}^+(\Sigma_1)) & \xrightarrow{\quad\cong\quad} & \ell^2(\Gamma)\otimes L^2_{I_\pm}(\mathcal{S}^+(\Sigma_1/\Gamma)) \\ {\scriptstyle Q_{\pm\mp}(t_2,t_1)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbb{1}_{\ell^2(\Gamma)}\otimes \underline{Q_{\pm\mp}}(t_2,t_1)} \\ L^2_\Gamma(\mathcal{S}^+(\Sigma_2)) & \xrightarrow{\quad\cong\quad} & \ell^2(\Gamma)\otimes L^2(\mathcal{S}^+(\Sigma_2/\Gamma)) \end{array}$$

Figure 9.1.: Commutative diagram for $Q_{\pm\mp}(t_2, t_1)$.

$L^2_{\Gamma,I}$ are free Hilbert Γ-modules for any interval $I \subset \mathbb{R}$ as per Lemma 8.1.5. The proof in [BS19, Lem.2.6] shows that $\underline{Q_{\pm\mp}}(t_2, t_1)$ are compact operators on the compact bases:

$$\begin{aligned} \underline{Q_{+-}}(t_2,t_1) &\in \mathscr{K}(L^2_{(-\infty,0)}(\mathcal{S}^+(\Sigma_1/\Gamma)), L^2(\mathcal{S}^+(\Sigma_2/\Gamma))) \\ \underline{Q_{-+}}(t_2,t_1) &\in \mathscr{K}(L^2_{[0,\infty)}(\mathcal{S}^+(\Sigma_1/\Gamma)), L^2(\mathcal{S}^+(\Sigma_2/\Gamma))) \quad . \end{aligned}$$

We recall the argument: $\underline{Q_{\pm\mp}}$ are (properly supported) Fourier integral operators of order 0. The vanishing of their principal symbols (of order 0) imply that these tend to 0 outside any compact subset such that Lemma 4.2.4 (1) implies compactness of $\underline{Q_{\pm\mp}}$ as proclaimed. Thus

$$\begin{aligned} \mathbb{1}\otimes\underline{Q_{+-}}(t_2,t_1) &\in \mathscr{N}_r(\Gamma)\otimes\mathscr{K}(L^2_{(-\infty,0)}(\mathcal{S}^+(\Sigma_1/\Gamma)), L^2(\mathcal{S}^+(\Sigma_2/\Gamma))) \\ \mathbb{1}\otimes\underline{Q_{-+}}(t_2,t_1) &\in \mathscr{N}_r(\Gamma)\otimes\mathscr{K}(L^2_{[0,\infty)}(\mathcal{S}^+(\Sigma_1/\Gamma)), L^2(\mathcal{S}^+(\Sigma_2/\Gamma))) \end{aligned}$$

and (5.35) implies Γ-compactness of $Q_{\pm\mp}$. □

The Γ-indices then follow with the same trick as presented in the proof of Theorem 9.1.8.

9.2. Generalised (a)APS-boundary conditions

The aim of this subsection is to extend the Γ-Fredholm results to generalised (a)APS boundary conditions, introduced in subsection 8.1.2. According to the splittings in (8.28), we can represent $Q(t_2, t_1)$ and $\tilde{Q}(t_2, t_1)$ as (2x2)-matrices:

$$Q(t_2,t_1) = \begin{pmatrix} Q^{>a_2}_{\geq a_1}(t_2,t_1) & Q^{>a_2}_{<a_1}(t_2,t_1) \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) & Q^{\leq a_2}_{<a_1}(t_2,t_1) \end{pmatrix} \tag{9.29}$$

and

$$\tilde{Q}(t_2,t_1) = \begin{pmatrix} \tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1) & \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1) \\ \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) & \tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1) \end{pmatrix} \tag{9.30}$$

where the entries are given by

$$\begin{aligned} Q^{>a_2}_{\geq a_1}(t_2,t_1) &:= P_{(a_2,\infty)}(t_2) \circ Q(t_2,t_1) \circ P_{[a_1,\infty)}(t_1) \\ Q^{\leq a_2}_{<a_1}(t_2,t_1) &:= P_{(-\infty,a_2]}(t_2) \circ Q(t_2,t_1) \circ P_{(-\infty,a_1)}(t_1) \\ Q^{>a_2}_{<a_1}(t_2,t_1) &:= P_{(a_2,\infty)}(t_2) \circ Q(t_2,t_1) \circ P_{(-\infty,a_1)}(t_1) \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) &:= P_{(-\infty,a_2]}(t_2) \circ Q(t_2,t_1) \circ P_{[a_1,\infty)}(t_1) \end{aligned} \tag{9.31}$$

and

$$\begin{aligned} \tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1) &:= P_{(a_2,\infty)}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{[a_1,\infty)}(t_1) \\ \tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1) &:= P_{(-\infty,a_2]}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{(-\infty,a_1)}(t_1) \\ \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1) &:= P_{(a_2,\infty)}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{(-\infty,a_1)}(t_1) \\ \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) &:= P_{(-\infty,a_2]}(t_2) \circ \tilde{Q}(t_2,t_1) \circ P_{[a_1,\infty)}(t_1) \end{aligned} \; . \tag{9.32}$$

We again refer to each first two as diagonal entries and the remaining are the off-diagonal matrix entries. For $a_1 = a_2 = 0$ we have the already known matrix entries

$$\begin{array}{rclrcl} Q_{++}(t_2,t_1) &=& Q^{>0}_{\geq 0}(t_2,t_1) & Q_{+-}(t_2,t_1) &=& Q^{>0}_{<0}(t_2,t_1) \\ Q_{--}(t_2,t_1) &=& Q^{\leq 0}_{<0}(t_2,t_1) & Q_{-+}(t_2,t_1) &=& Q^{\leq 0}_{\geq 0}(t_2,t_1) \end{array}$$

and

$$\begin{array}{rclrcl} \tilde{Q}_{++}(t_2,t_1) &=& \tilde{Q}^{>0}_{\geq 0}(t_2,t_1) & \tilde{Q}_{+-}(t_2,t_1) &=& \tilde{Q}^{>0}_{<0}(t_2,t_1) \\ \tilde{Q}_{--}(t_2,t_1) &=& \tilde{Q}^{\leq 0}_{<0}(t_2,t_1) & \tilde{Q}_{-+}(t_2,t_1) &=& \tilde{Q}^{\leq 0}_{\geq 0}(t_2,t_1) \end{array}$$

from (9.2) and (9.7). We conclude as in subsection 9.1.1 the following fact.

Lemma 9.2.1. *All spectral entries in (9.3) are Γ-morphism between Hilbert Γ-modules, i.e.*

$$\begin{array}{rcl} Q^{>a_2}_{\geq a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2))) \\ Q^{\leq a_2}_{<a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+(\Sigma_2))) \\ Q^{>a_2}_{<a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2))) \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)), L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+(\Sigma_2))) \\ \tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^-(\Sigma_1)), L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^-(\Sigma_2))) \\ \tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^-(\Sigma_1)), L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^-(\Sigma_2))) \\ \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^-(\Sigma_1)), L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^-(\Sigma_2))) \\ \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) &\in& \mathscr{B}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^-(\Sigma_1)), L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^-(\Sigma_2)) \end{array}$$

with closed ranges.

As for the ordinary splitting due to (a)APS boundary conditions, unitarity of $Q(t_2,t_1)$ and $\tilde{Q}(t_2,t_1)$ implies that the off-diagonal matrix entries are isomorphisms if they are restricted to the kernels of the diagonal entries where each one maps onto the kernel of the adjoint of the other entry.

Lemma 9.2.2. *The operators $Q^{>a_2}_{<a_1}(t_2,t_1)$, $Q^{\leq a_2}_{\geq a_1}(t_2,t_1)$, $\tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)$ and $\tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1)$ restrict to Γ-isomorphisms*

$$\begin{aligned} Q^{>a_2}_{<a_1}(t_2,t_1) &: \mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) \to \mathsf{ker}\left((Q^{>a_2}_{\geq a_1}(t_2,t_1))^*\right) \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) &: \mathsf{ker}\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) \to \mathsf{ker}\left((Q^{\leq a_2}_{<a_1}(t_2,t_1))^*\right) \\ \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1) &: \mathsf{ker}\left(\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\right) \to \mathsf{ker}\left((\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1))^*\right) \\ \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) &: \mathsf{ker}\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right) \to \mathsf{ker}\left((\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1))^*\right) \end{aligned} .$$

Since the proof for the ordinary splitting is fully algebraic and do not depend on the cut in the spectrum, we skip the proof here since it is just exchanging 0 by a_1 and a_2. The unitarity of $Q(t_2,t_1)$ and $\tilde{Q}(t_2,t_1)$ implies that the non-vanishing elements of $(Q(t_2,t_1))^*Q(t_2,t_1) = \mathbb{1}$ and $Q(t_2,t_1)\,(Q(t_2,t_1))^* = \mathbb{1}$ as well as $(\tilde{Q}(t_2,t_1))^*\tilde{Q}(t_2,t_1) = \mathbb{1}$ and $\tilde{Q}(t_2,t_1)\left(\tilde{Q}(t_2,t_1)\right)^* = \mathbb{1}$ satisfy

$$\begin{aligned} \left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right)^* Q^{>a_2}_{\geq a_1}(t_2,t_1) &= \mathbb{1} - \left(Q^{>a_2}_{<a_1}(t_2,t_1)\right)^* Q^{\leq a_2}_{\geq a_1}(t_2,t_1) \\ \left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right)^* Q^{\leq a_2}_{<a_1}(t_2,t_1) &= \mathbb{1} - \left(Q^{>a_2}_{<a_1}(t_2,t_1)\right)^* Q^{\leq a_2}_{\geq a_1}(t_2,t_1) \\ Q^{>a_2}_{\geq a_1}(t_2,t_1)\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right)^* &= \mathbb{1} - Q^{>a_2}_{<a_1}(t_2,t_1)\left(Q^{\leq a_2}_{\geq a_1}(t_2,t_1)\right)^* \\ Q^{\leq a_2}_{<a_1}(t_2,t_1)\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right)^* &= \mathbb{1} - Q^{>a_2}_{<a_1}(t_2,t_1)\left(Q^{\leq a_2}_{\geq a_1}(t_2,t_1)\right)^* \end{aligned} \tag{9.33}$$

and

$$\begin{aligned} \left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right)^* \tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1) &= \mathbb{1} - \left(\tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)\right)^* \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) \\ \left(\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\right)^* \tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1) &= \mathbb{1} - \left(\tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)\right)^* \tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1) \\ \tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right)^* &= \mathbb{1} - \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)\left(\tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1)\right)^* \\ \tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\left(\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\right)^* &= \mathbb{1} - \tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)\left(\tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1)\right)^* \end{aligned} . \tag{9.34}$$

We could follow the same strategy as in subsection 9.1.3 and view the adjoints of the diagonal entries $Q^{>a_2}_{\geq a_1}(t_2,t_1)$,$Q^{\leq a_2}_{<a_1}(t_2,t_1)$, $\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)$, and $\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)$ as (initial) parametrices if we show that the error terms are s-regular pseudodifferential operators with suitable negative order such that we can construct full parametrices and conclude Γ-Fredholmness. The following result shows that this in fact true, but instead of repeating this procedure, we consider the restricted projection operator (8.29) which relates our known results to this generalised situation. We recall that $\mathsf{ran}\,(P_I(t))$ stands for $L^2_{\Gamma,I}(\mathcal{S}^{\pm}(\Sigma_t))$ for $t \in \mathcal{T}(M)$ and $I \subset \sigma(A_t)$.

Theorem 9.2.3.

$$\begin{aligned}
Q^{>a_2}_{\geq a_1}(t_2,t_1) &\in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{\geq a_1}(t_1))\,,\mathsf{ran}\,(P_{>a_2}(t_2)))\\
Q^{\leq a_2}_{<a_1}(t_2,t_1) &\in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{<a_1}(t_1))\,,\mathsf{ran}\,(P_{\leq a_2}(t_2)))\\
\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1) &\in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{\geq a_1}(t_1))\,,\mathsf{ran}\,(P_{>a_2}(t_2)))\\
\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1) &\in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{<a_1}(t_1))\,,\mathsf{ran}\,(P_{\leq a_2}(t_2)))
\end{aligned}$$

for $a_1, a_2 \in \mathbb{R}$ with Γ-indices

$$\begin{aligned}
\mathrm{ind}_\Gamma\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) = &\quad \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right) - \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right)\\
&+\chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right) - \chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right)\\
&+\mathrm{ind}_\Gamma\left(Q_{++}(t_2,t_1)\right) \quad , \qquad (9.35)\\
\mathrm{ind}_\Gamma\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) = &\quad \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right) - \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right)\\
&+\chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right) - \chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right)\\
&+\mathrm{ind}_\Gamma\left(Q_{--}(t_2,t_1)\right) \quad , \qquad (9.36)\\
\mathrm{ind}_\Gamma\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right) = &\quad \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right) - \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right)\\
&+\chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right) - \chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right)\\
&+\mathrm{ind}_\Gamma\left(\tilde{Q}_{++}(t_2,t_1)\right) \quad , \qquad (9.37)\\
\mathrm{ind}_\Gamma\left(\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\right) = &\quad \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right) - \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right)\\
&+\chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right) - \chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right)\\
&+\mathrm{ind}_\Gamma\left(\tilde{Q}_{--}(t_2,t_1)\right) \quad . \qquad (9.38)
\end{aligned}$$

$\chi_{a>0}$ and $\chi_{a<0}$ are abbreviations for the characteristic functions $\chi_{(0,\infty)}(a)$ respectively $\chi_{(-\infty,0)}(a)$.

Proof. We show that $Q^{>a_2}_{\geq a_1}(t_2,t_1)$ and $Q^{\leq a_2}_{<a_1}(t_2,t_1)$ are compositions of Γ-Fredholm operators; the results for $\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)$ and $\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)$ follow by replacing $Q(t_2,t_1)$ with $\tilde{Q}(t_2,t_1)$. We already know for $a_1 = a_2 = 0$ that the entries $Q_{++}(t_2,t_1)$ and $Q_{--}(t_2,t_1)$ are Γ-Fredholm. We can rewrite the projections in (9.31) as spectral projections with spectral cut at 0 with the help of (8.29) such that $Q_{++}(t_2,t_1)$ and $Q_{--}(t_2,t_1)$ can be recovered. We have for all $a_1, a_2 \in \mathbb{R}$

$$\begin{aligned}
P_{>a_2}(t_2) &= P_{>a_2}(t_2)\mathbb{1} = P_{>a_2}(t_2)\left(P_{>0}(t_2) + P_{\leq 0}(t_2)\right)\\
&= P_{>a_2}(t_2)\circ P_{>0}(t_2) + P_{>a_2}(t_2)\circ P_{\leq 0}(t_2)\\
&= \overline{P}^{>0}_{>a_2}(t_2)P_{>0}(t_2) + P_{(a_2,\infty)\cap(-\infty,0]}(t_2) \quad ,
\end{aligned}$$

$$\begin{aligned}
P_{\geq a_1}(t_1) &= \mathbb{1}P_{\geq a_1}(t_1) = \left(P_{\geq 0}(t_1) + P_{<0}(t_1)\right)P_{\geq a_1}(t_1)\\
&= P_{\geq 0}(t_1)\circ P_{\geq a_1}(t_1) + P_{<0}(t_1)\circ P_{\geq a_1}(t_1)\\
&= \overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1) + P_{[a_1,\infty)\cap(-\infty,0)}(t_1) \quad .
\end{aligned}$$

The projectors

$$P_{(a_2,\infty)\cap(-\infty,0]}(t_2) = \begin{cases} P_\emptyset(t_2) & a_2 \geq 0 \\ & \text{for} \\ P_{(a_2,0]}(t_2) & a_2 < 0 \end{cases}$$

and

$$P_{[a_1,\infty)\cap(-\infty,0)}(t_1) = \begin{cases} P_\emptyset(t_1) & a_1 \geq 0 \\ & \text{for} \\ P_{[a_1,0)}(t_1) & a_1 < 0 \end{cases}$$

are s-smoothing and therefore Γ-trace class because they are spectral projections on a bounded Borel set. Hence we get

$$\begin{aligned} Q^{>a_2}_{\geq a_1}(t_2,t_1) &= \overline{P}^{>0}_{>a_2}(t)P_{>0}(t_2)Q(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1) + \mathscr{S}^1_\Gamma \\ &= \overline{P}^{>0}_{>a_2}(t)Q_{++}(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1) + \mathscr{S}^1_\Gamma \end{aligned}$$

for all $a_1, a_2 \in \mathbb{R}$. Likewise, we can relate $Q^{\leq a_2}_{<a_1}(t_2,t_1)$ to $Q_{--}(t_2,t_1)$: we have

$$\begin{aligned} P_{\leq a_2}(t_2) &= P_{\leq a_2}(t_2)\mathbb{1} = P_{\leq a_2}(t_2) \circ (P_{\leq 0}(t_2) + P_{>0}(t_2)) \\ &= P_{\leq a_2}(t_2) \circ P_{\leq 0}(t_2) + P_{\leq a_2}(t_2) \circ P_{>0}(t_2) \\ &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)P_{\leq 0}(t_2) + P_{(-\infty,a_2]\cap(0,\infty)}(t_2) \quad , \end{aligned}$$

$$\begin{aligned} P_{<a_1}(t_1) &= \mathbb{1}P_{<a_1}(t_1) = (P_{<0}(t_1) + P_{\geq 0}(t_1)) \circ P_{<a_1}(t_1) \\ &= P_{<0}(t_1) \circ P_{<a_1}(t_1) + P_{\geq 0}(t_1) \circ P_{<a_1}(t_1) \\ &= \overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1) + P_{[0,\infty)\cap(-\infty,a_1)}(t_1) \end{aligned}$$

with s-smoothing and thus Γ-trace class remainders

$$P_{(-\infty,a_2]\cap(0,\infty)}(t_2) = \begin{cases} P_\emptyset(t_2) & a_2 \leq 0 \\ & \text{for} \\ P_{(0,a_2]}(t_2) & a_2 > 0 \end{cases}$$

and

$$P_{[0,\infty)\cap(-\infty,a_1)}(t_1) = \begin{cases} P_\emptyset(t_1) & a_1 \leq 0 \\ & \text{for} \\ P_{[0,a_1)}(t_1) & a_1 > 0 \end{cases} \quad .$$

We obtain for all $a_1, a_2 \in \mathbb{R}$

$$\begin{aligned} Q^{\leq a_2}_{<a_1}(t_2,t_1) &= \overline{P}^{\leq 0}_{\leq a_2}(t)P_{\leq 0}(t_2)Q_(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1) + \mathscr{S}^1_\Gamma \\ &= \overline{P}^{\leq 0}_{\leq a_2}(t)Q_{--}(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1) + \mathscr{S}^1_\Gamma \quad . \end{aligned}$$

If we consider $Q^{>a_2}_{\geq a_1}$ as operator on $\mathsf{ran}\,(P_{\geq a_1}(t_1))$ and $Q^{\leq a_2}_{<a_1}$ as operator on $\mathsf{ran}\,(P_{<a_1}(t_1))$, we gain

$$\begin{aligned} Q^{>a_2}_{\geq a_1}(t_2,t_1) &= \overline{P}^{>0}_{>a_2}(t_2)Q_{++}(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1) + \mathscr{S}^1_\Gamma \quad , \\ Q^{\leq a_2}_{<a_1}(t_2,t_1) &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)Q_{--}(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1) + \mathscr{S}^1_\Gamma \quad . \end{aligned} \tag{9.39}$$

As we would like to show Γ-Fredholmness of $Q^{>a_2}_{\leq a_1}$ and $Q^{\geq a_2}_{<a_1}$ with the help of the known Γ-Fredholmness of $Q_{\pm\pm}$ from Theorem 9.1.8, we are left with the task to show that the restricted projectors $\overline{P}^{>0}_{>a_2}(t_2)$, $\overline{P}^{\geq a_1}_{\geq 0}(t_1)$, $\overline{P}^{\leq 0}_{\leq a_2}(t_2)$ and $\overline{P}^{<a_1}_{<0}(t_1)$ are Γ-Fredholm. To do so we calculate the Γ-dimensions and -codimensions from Lemma 8.1.6:

$$\begin{aligned}
\dim_\Gamma\left(\mathsf{ran}\,(P_{>0})\cap(\mathsf{ran}\,(P_{>a_2}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{>0})\cap\mathsf{ran}\,(P_{\leq a_2})\right)\\
&= \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{(0,a_2]}\right)\right)\ ,\\
\dim_\Gamma\left(\mathsf{ran}\,(P_{\geq a_1})\cap(\mathsf{ran}\,(P_{\geq 0}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{\geq a_1})\cap\mathsf{ran}\,(P_{<0})\right)\\
&= \chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{[a_1,0)}\right)\right)\ ,\\
\dim_\Gamma\left(\mathsf{ran}\,(P_{\leq 0})\cap(\mathsf{ran}\,(P_{\leq a_2}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{\leq 0})\cap\mathsf{ran}\,(P_{>a_2})\right)\\
&= \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{(a_2,0]}\right)\right)\ ,\\
\dim_\Gamma\left(\mathsf{ran}\,(P_{<a_1})\cap(\mathsf{ran}\,(P_{<0}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{<a_1})\cap\mathsf{ran}\,(P_{\geq 0})\right)\\
&= \chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{[0,a_1)}\right)\right)\quad ;
\end{aligned}$$

$$\begin{aligned}
\operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{>0})\cap\mathsf{ran}\,(P_{>a_2})\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{>a_2})\big/\mathsf{ran}\,(P_{>0})\cap\mathsf{ran}\,(P_{>a_2})\right)\\
&= \left\{\begin{matrix}\dim_\Gamma\left(\mathsf{ran}\,(P_{>a_2})\big/\mathsf{ran}\,(P_{>a_2})\right) & & a_2\geq 0\\ & \text{for} & \\ \dim_\Gamma\left(\mathsf{ran}\,(P_{>a_2})\big/\mathsf{ran}\,(P_{>0})\right) & & a_2<0\end{matrix}\right\}\\
&= \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{(a_2,0]}\right)\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{\geq 0})\cap\mathsf{ran}\,(P_{\geq a_1})\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{\geq 0})\big/\mathsf{ran}\,(P_{\geq 0})\cap\mathsf{ran}\,(P_{\geq a_1})\right)\\
&= \left\{\begin{matrix}\dim_\Gamma\left(\mathsf{ran}\,(P_{\geq 0})\big/\mathsf{ran}\,(P_{\geq a_1})\right) & & a_1>0\\ & \text{for} & \\ \dim_\Gamma\left(\mathsf{ran}\,(P_{\geq 0})\big/\mathsf{ran}\,(P_{\geq 0})\right) & & a_1\leq 0\end{matrix}\right\}\\
&= \chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{[0,a_1)}\right)\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{\leq 0})\cap\mathsf{ran}\,(P_{\leq a_2})\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{\leq a_2})\big/\mathsf{ran}\,(P_{\leq 0})\cap\mathsf{ran}\,(P_{\leq a_2})\right)\\
&= \left\{\begin{matrix}\dim_\Gamma\left(\mathsf{ran}\,(P_{\leq a_2})\big/\mathsf{ran}\,(P_{\leq a_2})\right) & & a_2\leq 0\\ & \text{for} & \\ \dim_\Gamma\left(\mathsf{ran}\,(P_{\leq a_2})\big/\mathsf{ran}\,(P_{\leq 0})\right) & & a_2>0\end{matrix}\right\}\\
&= \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\,\left(P_{(0,a_2]}\right)\right)\ ,
\end{aligned}$$

$$\begin{aligned}\operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{<0})\cap\mathsf{ran}\,(P_{<a_1})\right) &= \dim_\Gamma\left({}^{\mathsf{ran}\,(P_{<0})}\big/{}_{\mathsf{ran}\,(P_{<0})\cap\mathsf{ran}\,(P_{<a_1})}\right)\\ &= \left\{\begin{matrix}\dim_\Gamma\left({}^{\mathsf{ran}\,(P_{<0})}\big/{}_{\mathsf{ran}\,(P_{<a_1})}\right) & & a_1<0\\ & \text{for} & \\ \dim_\Gamma\left({}^{\mathsf{ran}\,(P_{<0})}\big/{}_{\mathsf{ran}\,(P_{<0})}\right) & & a_1\geq 0\end{matrix}\right\}\\ &= \chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{[a_1,0)})\right)\quad .\end{aligned}$$

All occuring Γ-dimensions are finite since the spaces are images of spectral projections for eigenvalues in bounded Borel sets. Hence they are Γ-trace class. This shows that the restricted projectors $\overline{P}^{>0}_{>a_2}(t_2)$, $\overline{P}^{\geq a_1}_{\geq 0}(t_1)$, $\overline{P}^{\leq 0}_{\leq a_2}(t_2)$ and $\overline{P}^{<a_1}_{<0}(t_1)$ are Γ-Fredholm with Γ-indices

$$\begin{aligned}\operatorname{ind}_\Gamma\left(\overline{P}^{>0}_{>a_2}(t_2)\right) &= \chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{(0,a_2]})\right)-\chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{(a_2,0]})\right) \quad ,\\ \operatorname{ind}_\Gamma\left(\overline{P}^{\geq 0}_{\geq a_1}(t_1)\right) &= \chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{[a_1,0)})\right)-\chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{[0,a_1)})\right) \quad ,\\ \operatorname{ind}_\Gamma\left(\overline{P}^{\leq 0}_{\leq a_2}(t_2)\right) &= \chi_{\{a_2<0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{(a_2,0]})\right)-\chi_{\{a_2>0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{(0,a_2]})\right) \quad ,\\ \operatorname{ind}_\Gamma\left(\overline{P}^{<0}_{<a_1}(t_1)\right) &= \chi_{\{a_1>0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{[0,a_1)})\right)-\chi_{\{a_1<0\}}\dim_\Gamma\left(\mathsf{ran}\,(P_{[a_1,0)})\right) \quad .\end{aligned}$$

Hence $Q^{>a_2}_{\geq a_1}(t_2,t_1)$ and $Q^{\leq a_2}_{<a_1}(t_2,t_1)$ in (9.39) are Γ-Fredholm as compositions of Γ-Fredholm operators with Γ-trace class pertubations. The invariance with respect to Γ-compact pertubations and the additivity of the Γ-index with respect to compositions leads to the given formulas in the claim. □

We can identify further Γ-index relations between different spectral entries with the help of those in Theorem 9.1.8 (1).

$$\begin{gathered}\operatorname{ind}_\Gamma\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) = -\operatorname{ind}_\Gamma\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right)\\ \text{and}\\ \operatorname{ind}_\Gamma\left(\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)\right) = -\operatorname{ind}_\Gamma\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right)\quad .\end{gathered}\tag{9.40}$$

Remark 9.2.4. *We focused on the untwisted case for the sake of readability, but the results and proofs carry over to the matrix entries of the wave evolution operators with respect to the twisted Dirac operators.*

With regard to Lemma 9.1.9, we could have also proven Theorem 9.2.3 without Theorem 9.1.8 by showing Γ-compactness of the off-diagonal terms in(9.29) and respectively (9.30).

Lemma 9.2.5.

(1) $Q^{>a_2}_{<a_1}(t_2,t_1)\in\mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)),L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$ *and* $Q^{\leq a_2}_{\geq a_1}(t_2,t_1)\in\mathscr{K}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)),L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$.

(2) $\tilde{Q}^{>a_2}_{<a_1}(t_2,t_1)\in\mathscr{K}_\Gamma(L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^-(\Sigma_1)),L^2_\Gamma(\mathcal{S}^-(\Sigma_2)))$ *and* $\tilde{Q}^{\leq a_2}_{\geq a_1}(t_2,t_1)\in\mathscr{K}_\Gamma(L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)),L^2_\Gamma(\mathcal{S}^+(\Sigma_2)))$.

Proof. We proceed as in the first proof of Theorem 9.2.3 and recapitulate the representations of the projectors $P_{>a_2}$, $P_{\leq a_2}$, $P_{\geq a_1}$ and $P_{<a_1}$ with restricted projectors (8.29):

$$\begin{aligned} P_{>a_2}(t) &= \overline{P}^{>0}_{>a_2}(t_2)P_{>0}(t_2)\,, & P_{\geq a_1}(t) &= \overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1)\,, \\ P_{\leq a_2}(t) &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)P_{\leq 0}(t_2)\,, & P_{<a_1}(t) &= \overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1)\,, \end{aligned}$$

each modulo Γ-trace class operators. $Q^{>a_2}_{<a_1}(t_2,t_1)$ and $Q(t_2,t_1)$ are related to $Q_{+-}(t_2,t_1)$ and respectively $Q^{\leq a_2}_{\geq a_1}(t_2,t_1)$ as follows:

$$\begin{aligned} Q^{>a_2}_{<a_1}(t_2,t_1) &= \overline{P}^{>0}_{>a_2}(t_2)P_{>0}(t_2)Q(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1) + \mathscr{S}^1_\Gamma \\ &= \overline{P}^{>0}_{>a_2}(t_2)Q_{+-}(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1)P_{<a_1}(t_1) + \mathscr{S}^1_\Gamma \quad , \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)P_{\leq 0}(t_2)Q(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1) + \mathscr{S}^1_\Gamma \\ &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)Q_{-+}(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1)P_{\geq a_1}(t_1) + \mathscr{S}^1_\Gamma \quad . \end{aligned}$$

If we apply $Q^{>a_2}_{<a_1}$ to sections in the domains $L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma))$ and $Q^{\leq a_2}_{\geq a_1}$ to sections in the domain $L^2_{\Gamma,[a_1,\infty)]}(\mathcal{S}^+(\Sigma))$, we obtain

$$\begin{aligned} Q^{>a_2}_{<a_1}(t_2,t_1) &= \overline{P}^{>0}_{>a_2}(t_2)Q_{+-}(t_2,t_1)\overline{P}^{<a_1}_{<0}(t_1) + \mathscr{S}^1_\Gamma \quad , \\ Q^{\leq a_2}_{\geq a_1}(t_2,t_1) &= \overline{P}^{\leq 0}_{\leq a_2}(t_2)Q_{-+}(t_2,t_1)\overline{P}^{\geq a_1}_{\geq 0}(t_1) + \mathscr{S}^1_\Gamma \quad . \end{aligned}$$

We have proven in Theorem 9.2.3 that the occuring restricted projections are Γ-Fredholm and thus Γ-morphisms. As $Q_{\pm\mp}$ are Γ-compact due to Lemma 9.1.9 (1), the Γ-ideal property of $\mathscr{K}_\Gamma$ shows that $Q^{>a_2}_{<a_1}(t_2,t_1)$ and $Q^{\leq a_2}_{\geq a_1}(t_2,t_1)$ are Γ-compact operators as claimed. The second assertion follows analogously by replacing Q with $\tilde{Q}$. □

9.3. Γ-indices

For simplicity and legibility we neglect the superscript E_L everywhere by focussing on the untwisted case. The concepts, results and observations from all former subsections now come together to express the Γ-indices of $Q_{\pm\pm}$ and $\tilde{Q}_{\pm\pm}$ with the modified spectral flow; the Γ-indices of $Q^{>a_2}_{\geq a_1}$, $Q^{\leq a_2}_{<a_1}(t_2,t_1)$, $\tilde{Q}^{>a_2}_{\geq a_1}$ and $\tilde{Q}^{\leq a_2}_{<a_1}(t_2,t_1)$ follow with (9.35), (9.36), (9.37) respectively (9.38).

Theorem 9.3.1. *The Γ-indices of $Q_{\pm\pm}(t_2,t_1)$ and $\tilde{Q}_{\pm\pm}(t_2,t_1)$ from Theorem 9.1.8 are*

$$\operatorname{ind}_\Gamma\left(Q_{\pm\pm}(t_2,t_1)\right) = \mp\operatorname{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} \pm \dim_\Gamma \mathsf{ker}\left(A_2\right) = \operatorname{ind}_\Gamma\left(\tilde{Q}_{\pm\pm}(t_2,t_1)\right) \quad . \tag{9.41}$$

Before we start with the concrete proof, we will prepare certain conclusions and observations.

Lemma 9.3.2. *For a fixed $t \in \mathcal{T}(M)$ we have $Q_{++}(t,t) \in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{\geq 0}(t))\,, \mathsf{ran}\,(P_{>0}(t)))$ and $Q_{--}(t,t) \in \mathscr{F}_\Gamma(\mathsf{ran}\,(P_{<0}(t))\,, \mathsf{ran}\,(P_{\leq 0}(t)))$ with Γ-indices*

$$\operatorname{ind}_\Gamma\left(Q_{\pm\pm}(t,t)\right) = \pm\dim_\Gamma(\mathsf{ker}\,(A_t)) = \operatorname{ind}_\Gamma\left(\tilde{Q}_{\pm\pm}(t,t)\right) \quad . \tag{9.42}$$

Proof. (Lemma 9.3.2) Fix a $t \in \mathcal{T}(M)$. The Γ-Fredholm-property is a consequence of Theorem 9.1.8 (1) (or (2) for the twisted case). The concrete forms of $Q_{\pm\pm}(t,t)$ become

$$\begin{aligned} Q_{++}(t,t) &= \tilde{Q}_{++}(t,t) = P_{>0}(t)P_{\geq 0}(t) = \overline{P}^{\geq 0}_{>0}(t)P^2_{\geq 0}(t) = \overline{P}^{\geq 0}_{>0}(t)P_{\geq 0}(t) \quad , \\ Q_{--}(t,t) &= \tilde{Q}_{--}(t,t) = P_{\leq 0}(t)P_{<0}(t) = \overline{P}^{<0}_{\leq 0}(t)P_{<0}(t) \quad . \end{aligned} \tag{9.43}$$

If $Q_{++}(t,t)$ and $\tilde{Q}_{++}(t,t)$ are viewed as operators on $\mathsf{ran}\,(P_{\geq 0}(t))$, they coincide with $\overline{P}^{\geq 0}_{>0}(t)$; likewise, if $Q_{--}(t,t)$ and $\tilde{Q}_{--}(t,t)$ are viewed as operators on $\mathsf{ran}\,(P_{<0}(t))$, they coincide with $\overline{P}^{<0}_{\leq 0}(t)$. Because $Q_{\pm\pm}(t,t)$ and $\tilde{Q}_{\pm\pm}(t,t)$ are Γ-Fredholm, $\overline{P}^{<0}_{\leq 0}(t)$ and $\overline{P}^{\geq 0}_{>0}(t)$ become Γ-Fredholm on the same domain and range. Their Γ-indices can be computed as in Lemma 8.1.6 at each t:

$$\begin{aligned} \dim_\Gamma\left(\mathsf{ran}\,(P_{\geq 0}) \cap (\mathsf{ran}\,(P_{>0}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{\geq 0}) \cap \mathsf{ran}\,(P_{\leq 0})\right) = \dim_\Gamma(\mathsf{ran}\,(P_0)) \quad , \\ \dim_\Gamma\left(\mathsf{ran}\,(P_{<0}) \cap (\mathsf{ran}\,(P_{\leq 0}))^\perp\right) &= \dim_\Gamma\left(\mathsf{ran}\,(P_{<0}) \cap \mathsf{ran}\,(P_{>0})\right) = \dim_\Gamma(\mathsf{ran}\,(P_\emptyset)) = 0 \quad , \\ \operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{\geq 0}) \cap \mathsf{ran}\,(P_{>0})\right) &= \dim_\Gamma\left({\mathsf{ran}\,(P_{>0})}/{\mathsf{ran}\,(P_{\geq 0}) \cap \mathsf{ran}\,(P_{>0})}\right) \\ &= \dim_\Gamma\left(\mathsf{ran}\,(P_\emptyset)\right) = 0 \quad , \\ \operatorname{codim}_\Gamma\left(\mathsf{ran}\,(P_{<0}) \cap \mathsf{ran}\,(P_{\leq 0})\right) &= \dim_\Gamma\left({\mathsf{ran}\,(P_{\leq 0})}/{\mathsf{ran}\,(P_{<0}) \cap \mathsf{ran}\,(P_{\leq 0})}\right) \\ &= \dim_\Gamma\left(\mathsf{ran}\,(P_0)\right) \end{aligned}$$

and thus

$$\operatorname{ind}_\Gamma\left(Q_{\pm\pm}(t,t)\right) = \pm\dim_\Gamma(\mathsf{ran}\,(P_0(t))) \quad .$$

The range of $P_0(t)$ has finite Γ-dimension because $P_0(t)$ is the projector onto the kernel of A_t which is a Γ-Fredholm operator for all t. □

The isometry between M and $\mathbb{R}\times\Sigma$ suggests that at each $t \in \mathcal{T}(M)$ the globally hyperbolic manifold M locally has product structure around the slice Σ_t. The used boundary conditions therefore makes sense on some artificial boundary hypersurface Σ_t for each t. The unitarity of $Q(\tau_2,\tau_1)$ and $\tilde{Q}(\tau_2,\tau_1)$ remains true since they are unitary for any bounded time interval in $\mathcal{T}(M)$. Hence $Q(\tau_2,\tau_1)$ and $\tilde{Q}(\tau_2,\tau_1)$ remain Γ-Fredholm with vanishing Γ-indices. Then the argument of the proof of Theorem 9.1.8 stays valid if we replace the interval $[t_1,t_2]$ with a smaller subinterval $[\tau_1,\tau_2] \subset [t_1,t_2]$ such that

$$\begin{aligned} Q_{++}(\tau_2,\tau_1) &\in \mathscr{F}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^+(\Sigma_{\tau_1})), L^2_{\Gamma,(0,\infty)}(\mathcal{S}^+(\Sigma_{\tau_2}))) \quad , \\ Q_{--}(\tau_2,\tau_1) &\in \mathscr{F}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^+(\Sigma_{\tau_1})), L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^+(\Sigma_{\tau_2}))) \quad , \\ \tilde{Q}_{++}(\tau_2,\tau_1) &\in \mathscr{F}_\Gamma(L^2_{\Gamma,[0,\infty)}(\mathcal{S}^-(\Sigma_{\tau_1})), L^2_{\Gamma,(0,\infty)}(\mathcal{S}^-(\Sigma_{\tau_2}))) \quad , \\ \tilde{Q}_{--}(\tau_2,\tau_1) &\in \mathscr{F}_\Gamma(L^2_{\Gamma,(-\infty,0)}(\mathcal{S}^-(\Sigma_{\tau_1})), L^2_{\Gamma,(-\infty,0]}(\mathcal{S}^-(\Sigma_{\tau_2}))) \end{aligned} \tag{9.44}$$

with the same Γ-index relations as in Theorem 9.1.8. This of course carries over to the twisted case (claim (2) in Theorem 9.1.8).

We notice the following facts for any $a, b \in \mathbb{R}$ and $\blacktriangle, \blacktriangledown$, being one of the relations in $\{<, >, \leq, \geq\}$:

- $$\hat{P}_{\blacktriangledown b}(t) = \widehat{\overline{P}^{\blacktriangle a}_{\blacktriangledown b}}(t)\hat{P}_{\blacktriangle a}(t) \tag{9.45}$$

  with $\widehat{\overline{P}^{\blacktriangle a}_{\blacktriangledown b}}(t) = Q(t_1, t)\overline{P}^{\blacktriangle a}_{\blacktriangledown b}(t)Q(t, t_1)$ which is Γ-Fredholm if only if $\overline{P}^{\blacktriangle a}_{\blacktriangledown b}(t)$ is Γ-Fredholm and they have the same Γ-index.

- $$\mathsf{ran}\,(P_{\blacktriangle 0}(t)) = Q^{\blacktriangle 0}_{\blacktriangledown 0}(t, t_1)[\mathsf{ran}\,(P_{\blacktriangledown 0}(t_1))] \quad . \tag{9.46}$$

These observations result in the following claim.

Lemma 9.3.3. *Given* $\tau, t \in \mathcal{T}(M)$*;*

$$\hat{P}_{>0}(t) : \mathsf{ran}\left(\hat{P}_{\geq 0}(\tau)\right) \to \mathsf{ran}\left(\hat{P}_{>0}(t)\right) \quad \textit{and} \quad \hat{P}_{\leq 0}(t) : \mathsf{ran}\left(\hat{P}_{<0}(\tau)\right) \to \mathsf{ran}\left(\hat{P}_{\leq 0}(t)\right)$$

are Γ*-Fredholm with* Γ*-indices*

$$\begin{aligned} \mathrm{ind}_\Gamma\left(\hat{P}_{>0}(t)\right) &= \mathrm{ind}_\Gamma\left(Q_{++}(t, \tau)\right) = \mathrm{ind}_\Gamma\left(\tilde{Q}_{++}(t, \tau)\right) \\ &\textit{and} \\ \mathrm{ind}_\Gamma\left(\hat{P}_{\leq 0}(t)\right) &= \mathrm{ind}_\Gamma\left(Q_{--}(t, \tau)\right) = \mathrm{ind}_\Gamma\left(\tilde{Q}_{--}(t, \tau)\right) \quad . \end{aligned} \tag{9.47}$$

Proof. Let $\tau, t \in \mathcal{T}(M)$. We express each evolved projector as composition of Γ-Fredholm operators: let $v \in \mathsf{ran}\,(P_{\geq 0}(\tau))$ and $u = Q(t_1, \tau)v \in \mathsf{ran}\left(\hat{P}_{\geq 0}(\tau)\right)$, then

$$\begin{aligned} \hat{P}_{>0}(t)u &= Q(t_1, t)P_{>0}(t)Q(t, t_1)u = Q(t_1, t)P_{>0}(t)Q(t, t_1)Q(t_1, \tau)v = Q(t_1, t)Q_{++}(t, \tau)v \\ &= Q(t_1, t)Q_{++}(t, \tau)Q(\tau, t_1)u \quad . \end{aligned}$$

If we take $v \in \mathsf{ran}\,(P_{<0}(\tau))$ instead such that $u = Q(t_1, \tau)v \in \mathsf{ran}\left(\hat{P}_{<0}(\tau)\right)$, we get

$$\begin{aligned} \hat{P}_{\leq 0}(t)u &= Q(t_1, t)P_{\leq 0}(t)Q(t, t_1)u = Q(t_1, t)P_{\leq 0}(t)Q(t, t_1)Q(t_1, \tau)v = Q(t_1, t)Q_{--}(t, \tau)v \\ &= Q(t_1, t)Q_{--}(t, \tau)Q(\tau, t_1)u \quad . \end{aligned}$$

Similar things hold if we replace Q with $\tilde{Q}$.

Because $Q_{\pm\pm}$ and $\tilde{Q}_{\pm\pm}$ are Γ-Fredholm for all subintervals in $[t_1, t_2]$ as well as Q and $\tilde{Q}$ are unitary for all time intervals with vanishing Γ-index, we observe that the evolved projectors are indeed Γ-Fredholm with claimed Γ-indices. □

This and the corresponding relations for the case of negative chirality are going to be used to prove the main result of this subsection.

Proof. (Theorem 9.3.1) We start with the wave evolution operator of D_+. We apply Definition 8.2.15 with the isometry $\mathcal{U}(\tau_2, \tau_1) = Q(\tau_2, \tau_1)$ to L^2_Γ-spaces for computing the modified Γ-spectral flow of $\{A_t\}_{t\in[t_1,t_2]}$ which is a smooth path of (essentially) self-adjoint Riemannian Dirac operators on each Σ_t. Since every A_t is Γ-invariant and elliptic, each

A_t of the path is Γ-Fredholm between the Hilbert Γ-modules $H^1_\Gamma(\mathcal{S}^+(\Sigma_t)$ and $L^2_\Gamma(\mathcal{S}^+(\Sigma_t))$; hence

$$[t_1,t_2] \ni t \mapsto A_t \in \mathscr{F}^{\mathrm{sa}}_\Gamma(H^1_\Gamma(\mathcal{S}^+(\Sigma_t), L^2_\Gamma(\mathcal{S}^+(\Sigma_t)))$$

is continuous. Lemma 8.2.13 and Remark 7.2.4 imply continuity of

$$[t_1,t_2] \ni t \mapsto \hat{A}_t \in \mathscr{F}^{\mathrm{sa}}_\Gamma(H^1_\Gamma(\mathcal{S}^+(\Sigma_1), L^2_\Gamma(\mathcal{S}^+(\Sigma_1))) \quad .$$

We choose a large enough L such that the partition $t_1 = \tau_0 < \tau_1 < \cdots < \tau_L = t_2$ of $[t_1,t_2]$ implies

$$\left\| \Pi_\Gamma(\hat{P}_{\geq 0}(s)) - \Pi_\Gamma(\hat{P}_{\geq 0}(r)) \right\| < 1 \quad \forall\, s,r \in [\tau_{j-1},\tau_j]$$

to be satisfied for all $j \in \{1,2,..,L\}$. Likewise, we could argue that by Lemma 9.3.3 the evolved projectors at any point in the time interval are Γ-Fredholm with well-defined Γ-index. Starting with any of these arguments shows that the sum

$$\widetilde{\mathrm{sf}}_\Gamma \left\{A_t\right\}_{t\in[t_1,t_2]} = \mathrm{sf}_\Gamma \left\{\hat{A}_t\right\}_{t\in[t_1,t_2]} = \sum_{j=1}^{L} \mathrm{ind}_\Gamma \left(\hat{P}_{\geq 0}(\tau_{j-1})|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(\tau_j)\right)\to \mathsf{ran}\left(\hat{P}_{\geq 0}(\tau_{j-1})\right)} \right)$$

from formula (8.52) is well-defined. Unfortunately, the Γ-index relations in Lemma 9.3.3 are not useful in this form, even after correcting the spectral range of the projectors. We rewrite the evolved projector $\hat{P}_{\geq 0}(t)$ in the following way to continue with the proof: let $t,\tau \in [\tau_{j-1},\tau_j]$; if we apply $\hat{P}_{\geq 0}(t)$ to an element in $\mathsf{ran}\,(P_{\geq 0}(t_1))$, we get

$$\begin{aligned}
\hat{P}_{\geq 0}(t)|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(\tau)\right)} &:= \hat{P}_{\geq 0}(t)|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(\tau)\right)\to\mathsf{ran}\left(\hat{P}_{\geq 0}(t)\right)} \\
&= \hat{P}_{\geq 0}(t)\widehat{\overline{P}^{>0}_{\geq 0}}(\tau)|_{\mathsf{ran}\left(\hat{P}_{>0}(\tau)\right)} \overset{(*)}{=} \hat{P}_{\geq 0}(t)\widehat{\overline{P}^{>0}_{\geq 0}}(\tau)Q(t_1,\tau)|_{\mathsf{ran}(P_{>0}(\tau))} \\
&= Q(t_1,t)P_{\geq 0}(t)Q(t,t_1)Q(t_1,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q(\tau,t_1)Q(t_1,\tau)|_{\mathsf{ran}(P_{>0}(\tau))} \\
&\overset{(**)}{=} Q(t_1,t)P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)|_{\mathsf{ran}(P_{>0}(\tau))} \\
&\overset{(9.46)}{=} Q(t_1,t)P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)
\end{aligned} \tag{9.48}$$

where we used $\mathsf{ran}\left(\hat{P}_J(t)\right) = Q(t_1,t)[\mathsf{ran}\,(P_J(t))]$ in $(*)$ and the evolution operator properties from Lemma 7.3.7 (1) and (2) in $(**)$. The Γ-index then takes the form

$$\begin{aligned}
\mathrm{ind}_\Gamma \left(\hat{P}_{\geq 0}(t)|_{\mathsf{ran}\left(\hat{P}_{\geq 0}(\tau)\right)}\right) &= \mathrm{ind}_\Gamma \left(Q(t_1,t)P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right) \\
&= \mathrm{ind}_\Gamma \left(P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right) \\
&\quad + \mathrm{ind}_\Gamma \left(Q(t_1,t)|_{\mathsf{ran}\left(P_{\geq 0}(t)\right)\to\mathsf{ran}\left(\hat{P}_{\geq 0}(t)\right)}\right) \\
&= \mathrm{ind}_\Gamma \left(P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right)
\end{aligned}$$

after applying the composition rule of the Γ-index with the restriction of the Γ-isomorphism Q to its range which is still a Γ-isomorphism with identically vanishing Γ-index. We extend this expression by introducing a zero in the form of the Γ-index of

$$P_{>0}(t_1)Q(t_1,t)|_{\mathsf{ran}\left(P_{\geq 0}(t)\right)} = Q_{++}(t_1,t)$$

to eliminate the extra projector; this step cancels afterwards:

$$\begin{aligned}\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t)|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau))}\right) &= \mathrm{ind}_\Gamma\left(P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right)\\ &\quad + \mathrm{ind}_\Gamma\left(P_{>0}(t_1)Q(t_1,t)|_{\mathsf{ran}(P_{\geq 0}(t))}\right) - \mathrm{ind}_\Gamma\left(Q_{++}(t_1,t)\right)\\ &= \mathrm{ind}_\Gamma\left(P_{>0}(t_1)Q(t_1,\tau)P_{\geq 0}(t)Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right)\\ &\quad - \mathrm{ind}_\Gamma\left(Q_{++}(t_1,t)\right)\\ &= \mathrm{ind}_\Gamma\left(Q(t,\tau)\overline{P}^{>0}_{\geq 0}(\tau)Q_{++}(\tau,t_1)\right) \quad .\end{aligned}$$

Each of the remaining three operators is Γ-Fredholm such that we finally gain

$$\begin{aligned}\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t)|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau))}\right) &= \mathrm{ind}_\Gamma\left(Q(t,\tau)|_{\mathsf{ran}(P_{>0}(\tau))\to Q(t,\tau)[\mathsf{ran}(P_{>0}(t))]}\right) + \mathrm{ind}_\Gamma\left(\overline{P}^{>0}_{\geq 0}(\tau)\right)\\ &\quad + \mathrm{ind}_\Gamma\left(Q_{++}(\tau,t_1)\right) = \mathrm{ind}_\Gamma\left(\overline{P}^{>0}_{\geq 0}(\tau)\right) + \mathrm{ind}_\Gamma\left(Q_{++}(\tau,t_1)\right) \quad .\end{aligned}$$

We implicitly used the fact that in the proof of the Γ-Fredholmness of $Q_{\pm\pm}$ the time-orientation is soley determined by the wave evolution operator which does not affect the Γ-Fredholmness of Q_{++}. In order to calculate the Γ-index of $\overline{P}^{>0}_{\geq 0}(\tau)$ we proceed as in the proof of Lemma 9.3.2 and observe

$$\mathrm{ind}_\Gamma\left(\overline{P}^{>0}_{\geq 0}(\tau)\right) = -\mathrm{ind}_\Gamma\left(\overline{P}^{\geq 0}_{>0}(\tau)\right) \overset{(9.43)}{=} -\mathrm{ind}_\Gamma\left(Q_{++}(\tau,\tau)\right) \overset{(9.42)}{=} -\dim_\Gamma \mathsf{ker}\left(A_\tau\right) \quad .$$

The results of the whole calculation becomes

$$\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t)|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau))}\right) = \mathrm{ind}_\Gamma\left(Q_{++}(\tau,t_1)\right) - \dim_\Gamma \mathsf{ker}\left(A_\tau\right) \quad . \tag{9.49}$$

It is left to show that the modified spectral flow can be related to the Γ-indices of $Q_{\pm\pm}$. We add a zero in the form of

$$\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(\tau_j)|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))\to\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))}\right) = \mathrm{ind}_\Gamma\left(\mathbb{1}_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))}\right) = 0$$

to each summand in the Γ-spectral flow formula and perform the telescope sum afterwards:

$$\begin{aligned}\widetilde{\mathrm{sf}}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} &= \sum_{j=1}^{L}\Big[\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(\tau_{j-1})|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))\to\mathsf{ran}(\hat{P}_{\geq 0}(\tau_{j-1}))}\right)\\ &\qquad -\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(\tau_j)|_{\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))\to\mathsf{ran}(\hat{P}_{\geq 0}(\tau_j))}\right)\Big]\\ &= -\mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t_2)|_{\mathsf{ran}(\hat{P}_{\geq 0}(t_2))}\right) + \mathrm{ind}_\Gamma\left(\hat{P}_{\geq 0}(t_1)|_{\mathsf{ran}(\hat{P}_{\geq 0}(t_1))}\right) \quad .\end{aligned}$$

(9.42) and (9.49) imply

$$\begin{aligned}\widetilde{\mathrm{sf}}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} &= -\mathrm{ind}_\Gamma\left(Q_{++}(t_2,t_1)\right) + \dim_\Gamma \mathsf{ker}\left(A_2\right) + \mathrm{ind}_\Gamma\left(Q_{++}(t_1,t_1)\right)\\ &\quad - \dim_\Gamma \mathsf{ker}\left(A_1\right) = -\mathrm{ind}_\Gamma\left(Q_{++}(t_2,t_1)\right) + \dim_\Gamma \mathsf{ker}\left(A_2\right)\\ &= \mathrm{ind}_\Gamma\left(Q_{--}(t_2,t_1)\right) + \dim_\Gamma \mathsf{ker}\left(A_2\right) \quad ,\end{aligned}$$

We can express our modified Γ-spectral flow with the ordinary Γ-spectral flow due to

(8.68). The argument carries over to the case of negative chirality by replacing the isometry $\mathcal{U}(\tau_2, \tau_1)$ with $\tilde{Q}(\tau_2, \tau_1)$. □

(9.42) and the Γ-index relations in (9.17) implicate

$$\mathrm{ind}_\Gamma(Q_{--}(t_2,t_1)) = -\mathrm{ind}_\Gamma(Q_{++}(t_2,t_1)) = \mathrm{ind}_\Gamma(\tilde{Q}_{--}(t_2,t_1)) = -\mathrm{ind}_\Gamma(\tilde{Q}_{++}(t_2,t_1)) \quad .$$

Combining (9.42) with (9.35), (9.36), (9.37) and (9.38) yields the following Γ-index formulas for the matrix entries of $Q(t_2,t_1)$ and $\tilde{Q}(t_2,t_1)$ according to the splitting with respect to generalised (anti-)APS boundary conditions:

$$\begin{aligned}
\mathrm{ind}_\Gamma\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) &= \mathrm{ind}_\Gamma\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right) = -\mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} + \dim_\Gamma \mathsf{ker}\left(A_2\right) \\
&\quad + \chi_{\{a_2>0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right) - \chi_{\{a_2<0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right) \\
&\quad + \chi_{\{a_1<0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right) - \chi_{\{a_1>0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right) \\
\text{and} & \\
\mathrm{ind}_\Gamma\left(Q^{\leq a_2}_{< a_1}(t_2,t_1)\right) &= \mathrm{ind}_\Gamma\left(\tilde{Q}^{\leq a_2}_{< a_1}(t_2,t_1)\right) = \mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} - \dim_\Gamma \mathsf{ker}\left(A_2\right) \\
&\quad + \chi_{\{a_2<0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{(a_2,0]}\right)\right) - \chi_{\{a_2>0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{(0,a_2]}\right)\right) \\
&\quad + \chi_{\{a_1>0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{[0,a_1)}\right)\right) - \chi_{\{a_1<0\}} \dim_\Gamma\left(\mathsf{ran}\left(P_{[a_1,0)}\right)\right) \quad .
\end{aligned} \tag{9.50}$$

We recall that we introduced the notion $\mathcal{S}^\pm(\Sigma_t)$ to stress which subbundle $\mathcal{S}^\pm(M)$ has been restricted to Σ_t. As Σ_t is odd-dimensional, there is no chirality decomposition, so we have $\mathcal{S}^\pm(\Sigma_t) = \mathcal{S}(\Sigma_t)$ and all ranges of projectors in (9.50) due to shifted boundary conditions are in fact the same for both chiralities. Thus, we can point out the following Γ-index relations:

$$\begin{aligned}
-\mathrm{ind}_\Gamma\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) &= \mathrm{ind}_\Gamma\left(Q^{\leq a_2}_{< a_1}(t_2,t_1)\right) \\
&= \mathrm{ind}_\Gamma\left(\tilde{Q}^{\leq a_2}_{< a_1}(t_2,t_1)\right) = -\mathrm{ind}_\Gamma\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right) \quad .
\end{aligned} \tag{9.51}$$

10. Γ-Fredholmness and Γ-index of the Lorentzian Dirac operator

This chapter finally presents the proof of the Γ-Fredholmness of $D_{\pm}^{E_L}$ with generalised (a)APS boundary conditions from Main Theorem 2 and in particular the special case Main Theorem 1 with Corollary 1.0.7. We will show that the Fredholm-property to show is related to the Γ-Fredholmness of the diagonal spectral entries of Q and $\tilde{Q}$. In order to do such a comparison, we furthermore need another technical result in the Γ-setting. This tool is presented in the first section, before we finally show the Γ-Fredholmness of $D_{\pm}^{E_L}$ in the following section. The Γ-indices are then determined by the Γ-spectral flow of the smooth family of twisted hypersurface Dirac operators. To compare with the Riemannian Γ-index formula, we need to rewrite the Γ-spectral flow with geometric quantities. This procedure and results will be explained in the third section. We end this chapter by considering the case of finite coverings as a special case of Galois coverings.

10.1. An important Lemma

We want to provide an important technical result in order to apply it for proving the Γ-Fredholm part of Main Theorem 2. It is a Hilbert Γ-module version of [BB11, Lem.A.1] which has been used in the proof of the Fredholmness in [BS19] for compact Cauchy boundary.

Lemma 10.1.1. *Let $\mathscr{H}, \mathscr{H}_1, \mathscr{H}_2$ be (projective) Hilbert Γ-modules, $A \in \mathscr{B}_\Gamma(\mathscr{H}, \mathscr{H}_1)$, and $B \in \mathscr{B}_\Gamma(\mathscr{H}, \mathscr{H}_2)$ which is onto; define $C = A|_{\mathsf{ker}(B)} \oplus \mathbb{1}_{\mathscr{H}_2}$, then*

(1) $\dim_\Gamma \mathsf{ker}\,(C) = \dim_\Gamma \mathsf{ker}\,(A \oplus B)$;

(2) $\mathsf{ran}\,(C)$ *is closed if and only if* $\mathsf{ran}\,(A \oplus B)$ *is closed and*

$$\mathrm{codim}_\Gamma(\mathsf{ran}\,(C)) = \mathrm{codim}_\Gamma(\mathsf{ran}\,(A \oplus B)) \quad ;$$

(3) C *is Γ-Fredholm if and only if* $A \oplus B$ *is Γ-Fredholm and* $\mathrm{ind}_\Gamma(C) = \mathrm{ind}_\Gamma(A \oplus B)$.

Proof. Let $\gamma \in \Gamma$; we denote with $\mathcal{L}_\gamma$ and $L_{(i,\gamma)}$ the left action representations on $\mathscr{H}$ and respectively $\mathscr{H}_i$ for $i \in \{1, 2\}$. The left action representation on $\mathscr{H}_1 \oplus \mathscr{H}_2$ is $L_{(1,\gamma)} \oplus L_{(2,\gamma)}$.

Since A and B are bounded, also $A|_{\mathsf{ker}(B)}$, C and $A \oplus B$ are bounded and their Γ-invariance follows trivially in each summand: $C, (A \oplus B) \in \mathscr{B}_\Gamma(\mathscr{H}^{\oplus 2}, \mathscr{H}_1 \oplus \mathscr{H}_2)$.

(1) One equally concludes from the proof in [BB11] that there exist two Γ-isomorphisms $\mathcal{I} \in \mathscr{B}_\Gamma(\mathscr{H}^{\oplus 2})$ and $\mathcal{J} \in \mathscr{B}_\Gamma(\mathscr{H}_1 \oplus \mathscr{H}_2)$ such that

$$C = \mathcal{J} \circ (A \oplus B) \circ \mathcal{I} \quad . \tag{10.1}$$

$B|_{\mathsf{ker}(B)^\perp}$ is injective and thus bijective due to the assumed surjectivity of B. The open mapping theorem implies that $B|_{\mathsf{ker}(B)^\perp}$ is open and consequently the inverse $(B|_{\mathsf{ker}(B)^\perp})^{-1}$ of $B|_{\mathsf{ker}(B)^\perp}$ is bounded by the bounded inverse theorem. By the fact, that $\mathsf{ker}\,(B)$ and $\mathsf{ker}\,(B)^\perp$ are Γ-invariant subspaces (see Lemma 5.2.13), the restriction $B|_{\mathsf{ker}(B)^\perp}$ is equally Γ-invariant with Γ-invariant inverse $(B|_{\mathsf{ker}(B)^\perp})^{-1}$:

$$\begin{aligned}\mathcal{L}_\gamma(B|_{\mathsf{ker}(B)^\perp})^{-1} &= \left(B|_{\mathsf{ker}(B)^\perp}\circ(\mathcal{L}_\gamma)^*\right)^{-1} = \left((L_{(2,\gamma)})^* B|_{\mathsf{ker}(B)^\perp}\right)^{-1}\\ &= (B|_{\mathsf{ker}(B)^\perp})^{-1}L_{(2,\gamma)} \quad .\end{aligned}$$

Thus, $(B|_{\mathsf{ker}(B)^\perp})^{-1}$ intertwines the Γ-action on $\mathscr{H}_2$ and $\mathscr{H}$. We used the fact that $(L_{(2,\gamma)})^*$ and $(\mathcal{L}_\gamma)^*$ are also left action representations. We define

$$\mathcal{I} = \mathbb{1}_{\mathsf{ker}(B)} \oplus \left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \quad \text{and} \quad \mathcal{J} = \begin{pmatrix} \mathbb{1}_{\mathscr{H}_1} & -A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix} \quad .$$

$\mathcal{I}$ is clearly a Γ-isomorphism with Γ-invariant inverse

$$\mathcal{I}^{-1} = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & (B|_{\mathsf{ker}(B)^\perp})^{-1}\end{pmatrix} \quad .$$

The inverse of $\mathcal{J}$ is

$$\mathcal{J}^{-1} = \begin{pmatrix} \mathbb{1}_{\mathscr{H}_1} & A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix}$$

which can be seen from

$$\begin{aligned}&\begin{pmatrix} \mathbb{1}_{\mathscr{H}_1} & \mp A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix}\begin{pmatrix} \mathbb{1}_{\mathscr{H}_1} & \pm A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix}\\ &= \begin{pmatrix} \mathbb{1}_{\mathscr{H}_1} & \mp A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \pm A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix} = \mathbb{1}_{\mathscr{H}_1\oplus\mathscr{H}_2}\,.\end{aligned}$$

$\mathcal{J}$ and $\mathcal{J}^{-1}$ are Γ-morphisms because its matrix entries are Γ-morphisms. Hence $\mathcal{J}$ is a Γ-isomorphism. (10.1) can be checked by multiplying the matrices of operators with the matrix representation of the direct sum of operators, decomposed by $\mathscr{H} = \mathsf{ker}\,(B)\oplus\mathsf{ker}\,(B)^\perp$: with $B|_{\mathsf{ker}(B)} = \mathbb{O}$ we have

$$\begin{aligned}(A\oplus B)\circ\mathcal{I} &= \begin{pmatrix} A|_{\mathsf{ker}(B)} & A|_{\mathsf{ker}(B)^\perp} \\ \mathbb{O} & B|_{\mathsf{ker}(B)^\perp}\end{pmatrix}\begin{pmatrix} \mathbb{1}_{\mathsf{ker}(B)} & \mathbb{O} \\ \mathbb{O} & \left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1}\end{pmatrix}\\ &= \begin{pmatrix} A|_{\mathsf{ker}(B)} & A|_{\mathsf{ker}(B)^\perp}\circ\left(B|_{\mathsf{ker}(B)^\perp}\right)^{-1} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix} = \mathcal{J}^{-1}\circ C\\ \Leftrightarrow\ \mathcal{J}\circ(A\oplus B)\circ\mathcal{I} &= \mathcal{J}\mathcal{J}^{-1}C = C = \begin{pmatrix} A|_{\mathsf{ker}(B)} & \mathbb{O} \\ \mathbb{O} & \mathbb{1}_{\mathscr{H}_2}\end{pmatrix} \quad .\end{aligned}$$

The null-spaces of the left- and right-hand side of (10.1) are projective Hilbert Γ-modules. Proposition 5.2.8 (4) then implies

$$\dim_\Gamma \mathsf{ker}\,(C) = \dim_\Gamma(\mathsf{ker}\,(\mathcal{J}\circ(A\oplus B)\circ\mathcal{I})) \quad .$$

$\mathcal{J}$ and $\mathcal{I}^{-1}$ restrict to Γ-isomorphisms between the kernels as projective Γ-modules: if $u \in \mathsf{ker}\,(A\oplus B)$, then $\mathcal{I}^{-1}u \in \mathsf{ker}\,((A\oplus B)\circ\mathcal{I})$. If on the other hand $v \in \mathsf{ker}\,((A\oplus B)\circ\mathcal{I})$, then $\mathcal{I}v \in \mathsf{ker}\,((A\oplus B))$ whereof one concludes

$$\mathcal{I}^{-1} \ : \ \mathsf{ker}\,(A\oplus B) \ \leftrightarrow \ \mathsf{ker}\,((A\oplus B)\circ\mathcal{I}) \quad .$$

The Γ-isomorphism $\mathcal{J}$ maps $\mathsf{ker}\,((A\oplus B)\circ\mathcal{I})$ to $\mathsf{ker}\,(\mathcal{J}\circ(A\oplus B)\circ\mathcal{I})$. Thus, the composition

$$\mathcal{J}\circ\mathcal{I}^{-1} \ : \ \mathsf{ker}\,(A\oplus B) \ \leftrightarrow \ \mathsf{ker}\,(\mathcal{J}\circ(A\oplus B)\circ\mathcal{I}) \quad .$$

is a Γ-isomorphism. Proposition 5.2.6 (1) guarantees the existence of a unitary Γ-isomorphism between the same Γ-modules such that

$$\mathsf{ker}\,(A\oplus B) \cong \mathsf{ker}\,(\mathcal{J}\circ(A\oplus B)\circ\mathcal{I}) \quad .$$

The invariance of the Γ-dimension between unitary isomorphic spaces (Proposition 5.2.8 (5)) shows the claim.

(2) Since closedness does not involve any property with respect to the group Γ, this claim follows as in [BB11, Lem.A.1] from the closed range theorem. We repeat the argument for the sake of completeness: suppose $\mathsf{ran}\,(A\oplus B)$ is closed. By the closed range theorem this is equivalent with a closed range of the adjoint of $A\oplus B$ and its range equals the annihilator of the kernel of $A\oplus B$ which corresponds to the orthogonal complement in the Hilbert space setting: $\mathsf{ran}\,((A\oplus B)^*) = \mathsf{ker}\,(A\oplus B)^\perp$. Because the constructed isomorphism maps between the kernels of $A\oplus B$ and C, it induces another isomorphism between the orthogonal complements since the kernels and their orthogonal complements are closed, Γ-invariant, and decompose the same Hilbert space:

$$\begin{array}{ccccc} \mathscr{H}^{\oplus 2} = & \mathsf{ker}\,(C) & \oplus & \mathsf{ker}\,(C)^\perp \\ & \updownarrow\cong & & \updownarrow\cong \\ \mathscr{H}^{\oplus 2} = & \mathsf{ker}\,(A\oplus B) & \oplus & \mathsf{ker}\,(A\oplus B)^\perp \end{array}$$

The orthogonal complement of the kernel with respect to C is isomorphic to the dual of the quotient space $\mathscr{H}^{\oplus 2}/\mathsf{ker}\,(C)$ and since it is a Hilbert space, it is isomorphic to the quotient. The homomorphism theorem then implies that

$$\mathsf{ran}\,((A\oplus B)^*) \cong \mathsf{ker}\,(C)^\perp \cong \mathscr{H}^{\oplus 2}\big/\mathsf{ker}\,(C) \cong \mathsf{ran}\,(C) \quad .$$

The closedness of $\mathsf{ran}\,((A\oplus B)^*)$ is preserved under isomorphisms such that $\mathsf{ran}\,(C)$ is closed. Starting with $\mathsf{ran}\,(C)$ being closed, the same argumentation backwards shows that $\mathsf{ran}\,((A \oplus B)^*)$ is closed. The closed range theorem finally implies the closedness of $\mathsf{ran}\,(A \oplus B)$.

Lemma 5.2.7 and closedness imply that

$$(\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(C)} \quad \text{and} \quad (\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(A \oplus B)}$$

are projective Hilbert Γ-submodules. A unitary Γ-isomorphism between these two spaces needs to be found. Since both quotient spaces are again Hilbert spaces, they are isomorphic to their dual spaces by the Frechet-Riesz theorem. The dual of a quotient of a Hilbert space and a closed subset is isomorphic to the orthogonal complement of the closed subset:

$$(\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(C)} \cong \left((\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(C)}\right)' \cong \mathsf{ran}\,(C)^\perp$$

and analogous for the other quotient. The closed range theorem implies again that the orthogonal complement of the ranges are equal to the null space of their adjoint operators such that

$$(\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(C)} \cong \mathsf{ker}\,(C^*) \quad \text{and} \quad (\mathscr{H}_1 \oplus \mathscr{H}_2)\big/_{\mathsf{ran}\,(A \oplus B)} \cong \mathsf{ker}\,((A \oplus B)^*)\ .$$

These isomorphisms become Γ-invariant because the spaces on both sides of the isomorphy are Γ-invariant subspaces with respect to their unitary left action representation. It is left to show that the kernels of the adjoint operators are isomorphic to each other. Adjoining (10.1) yields

$$C^* = \mathcal{I}^* \circ (A \oplus B)^* \circ \mathcal{J}^*$$

where $\mathcal{I}^* \in \mathscr{B}_\Gamma(\mathscr{H}^{\oplus 2})$ and $\mathcal{J}^* \in \mathscr{B}_\Gamma(\mathscr{H}_1 \oplus \mathscr{H}_2)$ are again Γ-isomorphisms with inverses, given by the adjoints of the inverses of $\mathcal{I}$ and $\mathcal{J}$. A similar argument as in (1) shows that the kernels of C^* and $(A \oplus B)^*$ are indeed (unitarily) isomorphic to each other:

$$\mathsf{ker}\,(C^*) \cong \mathsf{ker}\,((A \oplus B)^*) \quad .$$

Thus, the composition with the other isomorphisms, used to reduce the quotients, implies again a topological and thus a unitary Γ-isomorphism between projective Γ-submodules which finally leads to the claim:

$$\begin{aligned} \operatorname{codim}_\Gamma(\mathsf{ran}\,(C)) &= \dim_\Gamma \mathsf{ran}\,(C)^\perp = \dim_\Gamma \mathsf{ker}\,(C^*) = \dim_\Gamma \mathsf{ker}\,((A \oplus B)^*) \\ &= \dim_\Gamma \mathsf{ran}\,(A \oplus B)^\perp = \operatorname{codim}_\Gamma(\mathsf{ran}\,(A \oplus B)) \quad . \end{aligned}$$

(3) Γ-Fredholmness of C implicates $\dim_\Gamma \mathsf{ker}\,(C) < \infty$, $\mathsf{ran}\,(C)$ is closed, and that the Γ-codimension satisfies $\operatorname{codim}_\Gamma(\mathsf{ran}\,(C)) < \infty$. (1) and (2) imply $\dim_\Gamma \mathsf{ker}\,(A \oplus B) < \infty$, $\mathsf{ran}\,(A \oplus B)$ is closed and $\operatorname{codim}_\Gamma(\mathsf{ran}\,(A \oplus B)) < \infty$ and thus Γ-Fredholmness of $A \oplus B$. We can on the other hand conclude the Γ-Fredholmness of C from the assumed Γ-Fredholmness of $A \oplus B$ as the above arguments are symmetric in terms of both operators. The equivalence of the Γ-indices follows from the equivalence of

the Γ-dimensions in this proof and (7) of Proposition 5.2.17:

$$\begin{aligned}\mathrm{ind}_\Gamma(C) &= \dim_\Gamma \ker(C) - \dim_\Gamma \ker(C^*) = \dim_\Gamma \ker(A \oplus B) - \dim_\Gamma \ker((A \oplus B)^*) \\ &= \mathrm{ind}_\Gamma(A \oplus B) \quad .\end{aligned}$$

□

10.2. Γ-Fredholmness of $\mathcal{D}^{E_L}$ under generalised (a)APS boundary conditions

We are now in the position to prove the Fredholm part of our main theorem. In doing so, we focus on $D^{E_L} = D_+^{E_L}$ first and give a detailed proof. In the next subsection we consider $\tilde{D}^{E_L} = D_-^{E_L}$ and finally show the expected result for the full Dirac operator $\mathcal{D}^{E_L}$.

10.2.1. Γ-Fredholmness of $D_+^{E_L}$ with generalised (a)APS boundary conditions

The following statement is a generalisation of [Dam21, Thm.7.6] to twisting Dirac operators with generalised (a)APS boundary conditions. We first define the finite energy spinors of the Dirac equation which satisfy either gAPS or gaAPS boundary conditions for positive chirality in the setting of non-compact manifolds, coming from a Galois covering: based on (7.34) (b), we define

$$\begin{aligned}&FE^s_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \\ &\quad := \left\{u \in FE^s_\Gamma(M,\mathcal{T},D^{E_L}) \,\middle|\, P^{E_L}_{[a_1,\infty)}(t_1) \circ \mathsf{res}_{\Sigma_1} u = 0 = P^{E_L}_{(-\infty,a_2]}(t_2) \circ \mathsf{res}_{\Sigma_2} u\right\} \quad ,\end{aligned}$$

$$\begin{aligned}&FE^s_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \\ &\quad := \left\{u \in FE^s_\Gamma(M,\mathcal{T},D^{E_L}) \,\middle|\, P^{E_L}_{(-\infty,a_1)}(t_1) \circ \mathsf{res}_{\Sigma_1} u = 0 = P^{E_L}_{(a_2,\infty)}(t_2) \circ \mathsf{res}_{\Sigma_2} u\right\} \quad .\end{aligned}$$

They coincide with the spaces $FE^s_{\Gamma,\mathrm{APS}}(M,\mathcal{T},D)$ respectively $FE^s_{\Gamma,\mathrm{aAPS}}(M,\mathcal{T},D)$ for $a_1 = a_2 = 0$ in the untwisted case, introduced in [Dam21, Sec.7.3]. All these spaces are closed subspaces of $FE^s_\Gamma(M,\mathcal{T},D^{E_L})$. Since $FE^s_\Gamma(M,\mathcal{T},D^{E_L})$ is a free Hilbert Γ-module due to Theorem 7.3.3, its left action representation transfers to $FE^s_{\Gamma,\mathrm{(a)APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L})$; moreover, it is Γ-invariant as it is defined with spectral projectors and restrictions as Γ-invariant operators. Proposition 5.2.6 (2) then implies that they are projective Hilbert Γ-submodules on its own right. The choice $s = 0$ is of particular interest as the related spaces then appear to be the correct domain for Γ-Fredholmness. We thus define the *Dirac operator under gAPS boundary conditions*

$$D^{E_L}_{\mathrm{APS}(a_1,a_2)} : FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \to L^2_\Gamma(\mathcal{S}^-_{L,E}(M))$$

and the *Dirac operator under gaAPS boundary conditions*

$$D^{E_L}_{\mathrm{aAPS}(a_1,a_2)} : FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}) \to L^2_\Gamma(\mathcal{S}^-_{L,E}(M)) \quad .$$

We show their Γ-Fredholm property.

Theorem 10.2.1. *Let $a_1, a_2 \in \mathbb{R}$, M a temporal compact, globally hyperbolic spatial Γ-manifold with compact base M_Γ, $\mathcal{S}^+_{L,E}(M) \to M$ the Γ-spin bundle of positive chirality which is twisted with a Hermitian Γ-vector bundle $E \to M$ and twisted with the square-root of a Hermitian Γ-line bundle $L \to M$ for a* Spin^c*-structure. The Γ-invariant Dirac operators $D^{E_L}_{\mathrm{APS}(a_1,a_2)}$ and $D^{E_L}_{\mathrm{aAPS}(a_1,a_2)}$ as lifts of Dirac operators on the base manifold are Γ-Fredholm with Γ-indices*

$$\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{APS}(a_1,a_2)}) = \mathrm{ind}_\Gamma\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right)$$

and

$$\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) = \mathrm{ind}_\Gamma\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) \quad .$$

Proof. We suppress the super- and subscripts, denoting the twisting character, for the sake of readability. Besides, a proof in the untwisted case is sufficient since it turns out that it is purely algebraic.

We denote with

$$\mathbb{P}_+ := (P_{[a_1,\infty)}(t_1) \circ \mathsf{res}_{\Sigma_1}) \oplus (P_{(-\infty,a_2]}(t_2) \circ \mathsf{res}_{\Sigma_2}) \tag{10.2}$$

the boundary condition operator for gAPS boundary conditions and with

$$\mathbb{P}_- := (P_{(-\infty,a_1)}(t_1) \circ \mathsf{res}_{\Sigma_1}) \oplus (P_{(a_2,\infty)}(t_2) \circ \mathsf{res}_{\Sigma_2}) \tag{10.3}$$

the operator for gaAPS boundary conditions. The main task is to show that

$$\begin{aligned}&\mathbb{P}_+ \oplus D :\\ &\quad FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},D) \to \left[L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+(\Sigma_2))\right] \oplus L^2_\Gamma(\mathcal{S}^-(M))\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}_- \oplus D :\\ &\quad FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},D) \to \left[L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2))\right] \oplus L^2_\Gamma(\mathcal{S}^-(M))\end{aligned}$$

are Γ-Fredholm with the claimed Γ-indices. We want to apply Lemma 10.1.1 with

$$\begin{aligned}\mathscr{H} &= FE^0_\Gamma(M,\mathcal{T},D) & \mathscr{H}_1 &= L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+(\Sigma_2))\\ & & \mathscr{H}_2 &= L^2_\Gamma(\mathcal{S}^-(M))\\ B &= D = D_+ & A &= \mathbb{P}_+\end{aligned} \tag{10.4}$$

to prove Γ-Fredholmness of $\mathbb{P}_+ \oplus D$ by checking that $C = A|_{\mathsf{ker}(B)} \oplus \mathbb{1}_{\mathscr{H}_2} = \mathbb{P}_+|_{\mathsf{ker}(D)} \oplus \mathbb{1}_{\mathscr{H}_2}$ is Γ-Fredholm. The use is legitimated, because $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert Γ-modules and $\mathscr{H}$ is a Hilbert Γ-module by Theorem 7.3.3 and Proposition 5.2.6 (2); boundness in the spririt of Γ-morphisms and surjectivity of D_+ follow from Theorem 7.3.3 from which also follows that the restriction to the hypersurfaces are Γ-morphisms on $\mathscr{H}$; the s-regularity of the spectral projections then imply that $\mathbb{P}_+$ is equally a Γ-morphism. We furthermore use the shorthand notations $\mathscr{H}_1(t_1) = L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1))$ and $\mathscr{H}_1(t_2) := L^2_{\Gamma,(-\infty,a_2]}(\mathcal{S}^+(\Sigma_2))$ which imply

$$\mathscr{H}_1 = \mathscr{H}_1(t_1) \oplus \mathscr{H}_1(t_2)$$
$$\mathscr{H}_1^{\perp}(t_1) = L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)) \qquad \mathscr{H}_1^{\perp}(t_2) = L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_1)) \quad .$$

The kernel of C consists of spinors $u \in FE^0_\Gamma(M, \mathsf{ker}\,(D))$ such that $u|_{\Sigma_1} \in \mathscr{H}_1^{\perp}(t_1)$ and $u|_{\Sigma_2} \in \mathscr{H}_1^{\perp}(t_2)$; $u|_{\Sigma_1} \in \mathscr{H}_1^{\perp}(t_1)$ already satisfies the boundary condition on the initial hypersurface. In order to assure the boundary condition on the final hypersurface Σ_2, the evolution of $u|_{\Sigma_1}$ with $Q(t_2,t_1)$ has to map to $\mathscr{H}_1^{\perp}(t_2)$, i.e.

$$P_{(-\infty,a_2]}(t_2)(Q(t_2,t_1)(u|_{\Sigma_1})) = Q^{\leq a_2}_{<a_1}(t_2,t_1)(u|_{\Sigma_1})) = 0 \quad .$$

Thus, the kernel of C becomes

$$\begin{aligned}\mathsf{ker}\,(C) &= \left\{ u \in FE^0_\Gamma(M, \mathsf{ker}\,(D)) \,\middle|\, u|_{\Sigma_1} \in \mathscr{H}_1^{\perp}(t_1)\,,\, P_{(-\infty,a_2]}(t_2) u|_{\Sigma_2} = 0 \right\} \oplus \{0_{\mathscr{H}_2}\} \\ &= \mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) \oplus \{0_{\mathscr{H}_2}\} \quad ;\end{aligned}$$

$\{0_{\mathscr{H}_2}\}$ is the only element in the kernel of $\mathbb{1}_{\mathscr{H}_2}$. The right-hand side has finite Γ-dimension due to Γ-Fredholmness of $Q^{\leq a_2}_{<a_1}(t_2,t_1)$ such that

$$\begin{aligned}\dim_\Gamma \mathsf{ker}\,(C) &= \dim_\Gamma \left[\mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) \oplus \{0\}\right] \\ &= \dim_\Gamma \mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) + \dim_\Gamma \{0_{\mathscr{H}_2}\} = \dim_\Gamma \mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) < \infty\end{aligned}$$

where Proposition 5.2.8 (3) and (4) have been used. The image of C can be computed as follows: let $u_1 = u|_{\Sigma_1} \in \mathscr{H}_1(t_1)$; the action of C on any $u \in FE^0_\Gamma(M, \mathsf{ker}\,(D))$ with $u|_{\Sigma_1} = u_1$ is given by (u_1, u_2) where $u_2 = u|_{\Sigma_2} \in \mathscr{H}_1(t_2)$ can be decomposed into a part, coming from u_1, and one from an element $v \in \mathscr{H}_1^{\perp}(t_1)$ which is mapped by the Dirac wave evolution operator to the second Cauchy boundary:

$$\begin{aligned}u_2 &= P_{(-\infty,a_2]}(t_2)(u|_{\Sigma_2}) = P_{(-\infty,a_2]}(t_2)(Q(t_2,t_1)u_1 + Q(t_2,t_1)v) \\ &= Q^{\leq a_2}_{\geq a_1}(t_2,t_1)u_1 + Q^{\leq a_2}_{<a_1}(t_2,t_1)v\,.\end{aligned} \tag{10.5}$$

Thus

$$\mathsf{ran}\,(C) = \left\{ \left[(u_1, Q^{\leq a_2}_{\geq a_1}(t_2,t_1)u_1 + Q^{\leq a_2}_{<a_1}(t_2,t_1)v), f\right] \in \mathscr{H}_1 \oplus \mathscr{H}_2 \,\middle|\, v \in \mathscr{H}_1^{\perp}(t_1)\,,\, f \in \mathscr{H}_2 \right\}.$$

As the spaces $\mathscr{H}_1^{\perp}(t_1)$, $\mathscr{H}_1(t_1)$ and $\mathscr{H}_1(t_2)$ are closed Hilbert spaces, any sequences $(u_{1,i})$ in $\mathscr{H}_1^{\perp}(t_1)$, $(u_{2,i})$ in $\mathscr{H}_1(t_1)$ and (v_i) in $\mathscr{H}_1(t_2)$ converge in their belonging spaces where the latter sequence has to be chosen such that

$$(u_{1,i}, Q^{\leq a_2}_{\geq a_1}(t_2,t_1)u_{1,i} + Q^{\leq a_2}_{<a_1}(t_2,t_1)v_i) \quad \longrightarrow \quad (u_1, u_2) \quad .$$

$Q^{\leq a_2}_{<a_1}(t_2,t_1)$ are Γ-morphisms between the L^2_Γ-spaces of interest and each has closed range (see Lemma 9.2.1). Either implies that the images converge in $\mathscr{H}_1(t_2)$,

$$Q^{\leq a_2}_{\geq a_1}(t_2,t_1)u_{1,i} \to Q^{\leq a_2}_{\geq a_1}(t_2,t_1)u_1$$

and

$$Q^{\le a_2}_{<a_1}(t_2,t_1)v_i = u_{2,i} - Q^{\le a_2}_{\ge a_1}(t_2,t_1)u_{1,i} \;\to\; u_2 - Q^{\le a_2}_{\ge a_1}(t_2,t_1)u_1 = Q^{\le a_2}_{<a_1}(t_2,t_1)v$$

from (10.5). The limit $(u_1, Q^{\le a_2}_{\ge a_1}(t_2,t_1)u_1 + Q^{\le a_2}_{<a_1}v)$ thus lies in the image and consequently implies the closedness of $\mathsf{ran}\,(C)$. This closedness then implies that we can consider the orthogonal complement instead of the quotient:

$$\begin{aligned}\mathsf{ran}\,(C)^\perp =&\Big\{((v_1,v_2),0) \in \mathscr{H}_1 \oplus \mathscr{H}_2 \,\Big|\, \big\langle (v_1,v_2)\,\big|\,(u_1,u_2)\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))^{\oplus 2}} = 0 \\ &\qquad \forall\,(u_1,u_2) \in \mathsf{ran}\,(C)\Big\} \\ =&\Big\{((v_1,v_2) \in \mathscr{H}_1 \,\Big|\, \big\langle (v_1,v_2)\,\big|\,(u_1,u_2)\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))^{\oplus 2}} = 0 \\ &\qquad \forall\,(u_1,u_2) \in \mathsf{ran}\,(C)\Big\} \oplus \{0_{\mathscr{H}_2}\} \quad .\end{aligned}$$

As $(u_1,u_2) \in \mathsf{ran}\,(C)$, we have $u_1 \in \mathscr{H}_1(t_1)$ and with a $v \in \mathscr{H}_1^\perp(t_1)$

$$u_2 = Q^{\le a_2}_{\ge a_1}(t_2,t_1)u_1 + Q^{\le a_2}_{<a_1}(t_2,t_1)v \quad .$$

The inner product then becomes

$$\begin{aligned}0 &= \big\langle (v_1,v_2)\,\big|\,(u_1,u_2)\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))^{\oplus 2}} = \big\langle v_1\,\big|\,u_1\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} + \big\langle v_2\,\big|\,u_2\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} \\ &= \big\langle v_1\,\big|\,u_1\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} + \big\langle v_2\,\big|\,Q^{\le a_2}_{\ge a_1}(t_2,t_1)u_1 + Q^{\le a_2}_{<a_1}(t_2,t_1)v\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} \\ &= \big\langle v_1 + (Q^{\le a_2}_{\ge a_1}(t_2,t_1))^* v_2\,\big|\,u_1\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} + \big\langle Q^{\le a_2}_{<a_1}(t_2,t_1)^* v_2\,\big|\,v\big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))}\end{aligned}$$

such that

$$\mathsf{ran}\,(C)^\perp = \Big\{(-(Q^{\le a_2}_{\ge a_1}(t_2,t_1))^* v_2, v_2) \,\Big|\, v_2 \in \mathsf{ker}\,\Big((Q^{\le a_2}_{<a_1})^*(t_1,t_2)\Big)\Big\} \oplus \{0_{\mathscr{H}_2}\} \quad .$$

Lemma 9.2.2 implies that $(Q^{\le a_2}_{\ge a_1})^*(t_2,t_1)$ is an Γ-isomorphism on $\mathsf{ker}\,\Big((Q^{\le a_2}_{<a_1})^*(t_1,t_2)\Big)$. Hence the map

$$(v_2,v_2) \quad\mapsto\quad \Big(-(Q^{\le a_2}_{\ge a_1}(t_2,t_1))^* v_2, v_2\Big)$$

is an Γ-isomorphism for $v_2 \in \mathsf{ker}\,\Big((Q^{\le a_2}_{<a_1})^*(t_1,t_2)\Big)$. The embedding $V \hookrightarrow V^{\oplus 2}$ for any vector space V is an isomorphism if it is restricted onto its image. As the kernel of any Γ-morphism is invariant under the action of the unitary left action representation, the embedding intertwines the left action on the kernel with the diagonal action on $\mathsf{ker}\,\Big((Q^{\le a_2}_{<a_1})^*(t_1,t_2)\Big)^{\oplus 2}$ and becomes a Γ-isomorphism, too. Consequently, the map

$$v_2 \quad\mapsto\quad (v_2,v_2) \quad\mapsto\quad \Big(-(Q^{\le a_2}_{\ge a_1}(t_2,t_1))^* v_2, v_2\Big)$$

is a Γ-isomorphism such that

$$\mathsf{ran}\,(C)^\perp \cong \mathsf{ker}\,\Big((Q^{\le a_2}_{<a_1})^*(t_1,t_2)\Big) \oplus \{0_{\mathscr{H}_2}\} \quad .$$

The quotient $\mathscr{H}_1 \oplus \mathscr{H}_2/\mathsf{ran}\,(C)$ is isomorphic to $\mathsf{ran}\,(C)^\perp$, as $\mathsf{ran}\,(C)$ is closed, and since the quotient and the orthogonal complement are Γ-invariant subspaces of Hilbert Γ-modules (see Lemma 5.2.7), this isomorphism becomes a Γ-isomorphism such that

$$\mathscr{H}_1 \oplus \mathscr{H}_2 / \mathsf{ran}\,(C) \cong \mathsf{ker}\left((Q^{\leq a_2}_{<a_1})^*(t_1,t_2)\right) \oplus \{0_{\mathscr{H}_2}\} \quad .$$

We use Proposition 5.2.6 (1) to extract a unitary Γ-isomorphism and Proposition 5.2.8 (5) such that

$$\begin{aligned}\dim_\Gamma\left(\mathscr{H}_1 \oplus \mathscr{H}_2 / \mathsf{ran}\,(C)\right) &= \dim_\Gamma\left[\mathsf{ker}\left((Q^{\leq a_2}_{<a_1})^*(t_2,t_1)\right) \oplus \{0_{\mathscr{H}_2}\}\right] \\ &= \dim_\Gamma \mathsf{ker}\left((Q^{\leq a_2}_{<a_1})^*(t_2,t_1)\right) = \dim_\Gamma \mathsf{ker}\left((Q^{\leq a_2}_{<a_1}(t_2,t_1))^*\right) < \infty\end{aligned}$$

since $Q^{\leq a_2}_{<a_1}(t_2,t_1)$ is Γ-Fredholm. We conclude that the Γ-codimension of the range of C is finite from which we conclude Γ-Fredholmness of C and $\mathbb{P}_+ \oplus D$ with coinciding Γ-index

$$\begin{aligned}\mathrm{ind}_\Gamma(\mathbb{P}_+ \oplus D) &= \mathrm{ind}_\Gamma(C) = \dim_\Gamma \mathsf{ker}\,(C) - \dim_\Gamma \mathsf{coker}\,(C) \\ &= \dim_\Gamma \mathsf{ker}\left(Q^{\leq a_2}_{<a_1}(t_2,t_1)\right) - \dim_\Gamma \mathsf{ker}\left((Q^{\leq a_2}_{<a_1}(t_2,t_1))^*\right) = \mathrm{ind}_\Gamma(Q^{\leq a_2}_{<a_1}(t_2,t_1)).\end{aligned}$$

The second statement follows in the same way by choosing

$$\mathscr{H}_1 = L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2)) \quad \text{and} \quad A = \mathbb{P}_-$$

in (10.4) with the same legimitations. Now we replace the shorthand notations from the upper case to $\mathscr{H}_1(t_1) = L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1))$ and $\mathscr{H}_1(t_2) := L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2))$ such that $\mathscr{H}_1 = \mathscr{H}_1(t_1) \oplus \mathscr{H}_1(t_2)$ and $\mathscr{H}_1^\perp(t_1) = L^2_{\Gamma,[a_1,\infty)}(\mathcal{S}^+(\Sigma_1))$.

It turns out that the Γ-Fredholmness can be related to the Γ-Fredholm-property of $Q^{>a_2}_{\geq a_1}(t_2,t_1)$: the kernel of $C = \mathbb{P}_-|_{\mathsf{ker}(D)} \oplus \mathbb{1}_{\mathscr{H}_2}$ consists of sections $u \in FE^0_\Gamma(M, \mathsf{ker}\,(D))$ which restriction onto the initial hypersurface Σ_1 is a section in $\mathscr{H}_1^\perp(t_1)$ and its restriction onto Σ_2, transported by $Q(t_2,t_1)$, vanishes:

$$0 = P_{(a_2,\infty)}(t_2)(v) = P_{(a_2,\infty)}(t_2) \circ Q(t_2,t_1) \circ P_{[a_1,\infty)}(t_1)u|_{\Sigma_1} = Q^{>a_2}_{\geq a_1}(t_2,t_1)u|_{\Sigma_1} \quad .$$

Thus, the kernel of C is isomorphic to the kernel of $Q^{>a_2}_{\geq a_1}(t_2,t_1)$:

$$\mathsf{ker}\,(C) \cong \mathsf{ker}\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) \oplus \{0_{\mathscr{H}_2}\} \quad .$$

Because $Q^{>a_2}_{\geq a_1}(t_2,t_1)$ is Γ-Fredholm, we gain with the same reasoning

$$\dim_\Gamma \mathsf{ker}\,(C) = \dim_\Gamma \mathsf{ker}\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) + \dim_\Gamma(\{0\}) = \dim_\Gamma \mathsf{ker}\left(Q^{>a_2}_{\geq a_1}(t_2,t_1)\right) < \infty \quad .$$

We compute the image in a similar way: let $((u_1,u_2),f) \in \mathscr{H}_1 \oplus \mathscr{H}_2$ are given by elements $u_1 \in \mathscr{H}_1(t_1)$ and $u_2 \in \mathscr{H}_1(t_2)$. The action of C on $u \in FE^0_\Gamma(M, \mathsf{ker}\,(D))$ leads to an element $(u_1,u_2) \in \mathscr{H}_1 \oplus \mathscr{H}_2$ where $u_1 = u|_{\Sigma_1} \in \mathscr{H}_1(t_1)$ and $u_2 = u|_{\Sigma_2} \in \mathscr{H}_1(t_2)$. The latter one can be decomposed into a part, coming from u_1, and one coming from an element $w \in \mathscr{H}_1^\perp(t_1)$

such that

$$\begin{aligned} u_2 &= P_{(a_1,\infty)}(t_2)(u|_{\Sigma_2}) = P_{(a_2,\infty)}(t_2)(Q(t_2,t_1)u_1 + Q(t_2,t_1)w) \\ &= Q^{>a_2}_{<a_1}(t_2,t_1)u_1 + Q^{>a_2}_{\geq a_1}(t_2,t_1)w \quad . \end{aligned} \tag{10.6}$$

Because $Q^{>a_2}_{<a_1}(t_2,t_1)$ and $Q^{>a_2}_{\geq a_1}(t_2,t_1)$ are Γ-morphisms, the same argument from the case with gAPS boundary conditions applies such that

$$\mathsf{ran}\,(C) = \left\{ \left[u_1, (Q^{>a_2}_{<a_1}(t_2,t_1)u_1 + (Q^{>a_2}_{\geq a_1}(t_2,t_1)w), f\right] \in \mathscr{H}_1 \oplus \mathscr{H}_2 \,\middle|\, w \in \mathscr{H}_1(t_2), f \in \mathscr{H}_2 \right\}$$

is closed. We can similarly compute the Γ-codimension after determining the orthogonal complement:

$$\begin{aligned} \mathsf{ran}\,(C)^\perp =& \Big\{ ((v_1,v_2),0) \in \mathscr{H}_1 \oplus \mathscr{H}_2 \,\Big|\, \big\langle (v_1,v_2) \,\big|\, (u_1,u_2) \big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))^{\oplus 2}} = 0 \\ & \qquad \forall\, (u_1,u_2) \in \mathscr{H}_1 \oplus \mathscr{H}_2 \Big\} \\ =& \Big\{ (v_1,v_2) \in \mathscr{H}_1 \,\Big|\, \big\langle v_2 \,\big|\, Q^{>a_2}_{<a_1}(t_2,t_1)u_1 + Q^{>a_2}_{\geq a_1}(t_2,t_1)w \big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} = 0 \\ & \text{and } \big\langle v_1 \,\big|\, u_1 \big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} = 0 \; \forall\, u_1 \in \mathscr{H}_1(t_1),\, w \in \mathscr{H}_1^\perp(t_1) \Big\} \oplus \{0_{\mathscr{H}_2}\} \\ =& \Big\{ (v_1,v_2) \in \mathscr{H}_1 \,\Big|\, \big\langle v_1 + (Q^{>a_2}_{<a_1}(t_2,t_1))^* v_2 \,\big|\, u_1 \big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} = 0 \\ & \text{and } \big\langle (Q^{>a_2}_{\geq a_1}(t_2,t_1))^* v_2 \,\big|\, w \big\rangle_{L^2_\Gamma(\mathcal{S}^+(\Sigma_2))} = 0 \; \forall\, u_1 \in \mathscr{H}_1(t_1),\, w \in \mathscr{H}_1^\perp(t_1) \Big\} \oplus \{0_{\mathscr{H}_2}\} \\ =& \Big\{ (-(Q^{>a_2}_{<a_1}(t_2,t_1))^* v_2, v_2) \,\Big|\, v_2 \in \mathsf{ker}\left((Q^{>a_2}_{\geq a_1}(t_2,t_1))^* \right) \Big\} \oplus \{0_{\mathscr{H}_2}\} \quad . \end{aligned}$$

We can argue as for $\mathbb{P}_+|_{\mathsf{ker}(D)}$ and observe that

$$\mathscr{H}_1 \oplus \mathscr{H}_2 \big/ \mathsf{ran}\,(C) \cong \mathsf{ker}\left((Q^{>a_2}_{\geq a_1}(t_2,t_1))^* \right) \oplus \{0_{\mathscr{H}_2}\}$$

and consequently

$$\dim_\Gamma \left(\mathscr{H}_1 \oplus \mathscr{H}_2 \big/ \mathsf{ran}\,(C) \right) = \dim_\Gamma \mathsf{ker}\left((Q^{>a_2}_{\geq a_1}(t_2,t_1))^* \right) < \infty$$

which concludes the proof that $\mathbb{P}_-|_{\mathsf{ker}(D)}$ is Γ-Fredholm. Hence $\mathbb{P}_- \oplus D$ is Γ-Fredholm with index

$$\begin{aligned} \mathrm{ind}_\Gamma(\mathbb{P}_- \oplus D) &= \mathrm{ind}_\Gamma(C) = \dim_\Gamma \mathsf{ker}\left(Q^{>a_2}_{\geq a_1}(t_2,t_1) \right) - \dim_\Gamma \mathsf{ker}\left((Q^{>a_2}_{\geq a_1}(t_2,t_1))^* \right) \\ &= \mathrm{ind}_\Gamma(Q^{>a_2}_{\geq a_1}(t_2,t_1)) \quad . \end{aligned}$$

Γ-Fredholmness of $\mathbb{P}_\pm \oplus D$ implies Γ-Fredholmness of $D \oplus \mathbb{P}_\pm$ as this property does not depend on the order of the direct sum. The Dirac operators of interest are related to $\mathbb{P}_\pm$ via

$$D_{\mathrm{APS}(a_1,a_2)} = D|_{\mathsf{ker}(\mathbb{P}_+)} \quad \text{and} \quad D_{\mathrm{aAPS}(a_1,a_2)} = D|_{\mathsf{ker}(\mathbb{P}_-)} \quad . \tag{10.7}$$

The main claim thus follows by applying Lemma 10.1.1 with

$$\begin{aligned} \mathscr{H} &= FE^0_\Gamma(M,\mathcal{T},D) & \mathscr{H}_2 &= L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ & & \mathscr{H}_1 &= L^2_\Gamma(\mathcal{S}^-(M)) \\ A &= D = D_+ & B &= \mathbb{P}_+ \quad , \end{aligned}$$

such that $D_{\mathrm{APS}(a_1,a_2)}$ is Γ-Fredholm with index $\mathrm{ind}_\Gamma(Q^{\leq a_2}_{<a_1}(t_2,t_1))$, and

$$\begin{aligned} \mathscr{H} &= FE^0_\Gamma(M,\mathcal{T},D) & \mathscr{H}_2 &= L^2_{\Gamma,(-\infty,a_1)}(\mathcal{S}^+(\Sigma_1)) \oplus L^2_{\Gamma,(a_2,\infty)}(\mathcal{S}^+(\Sigma_2)) \\ & & \mathscr{H}_1 &= L^2_\Gamma(\mathcal{S}^-(M)) \\ A &= D = D_+ & B &= \mathbb{P}_- \quad , \end{aligned}$$

proving Γ-Fredholmness of $D_{\mathrm{aAPS}(a_1,a_2)}$. The application is justified since D and $\mathbb{P}_\pm$ are Γ-morphism for the known reasons and $\mathbb{P}_\pm$ are onto between the correct spaces as the spectral projections are surjective. The stated Γ-indices follow from (9.50). □

The proof is an adapted version of the one given in [Dam21] for $a_1 = a_2 = 0$ which we state as corollary.

Corollary 10.2.2. *Let M be a temporal compact, even-dimensional globally hyperbolic spatial Γ-manifold, $\mathcal{S}^+(M) \to M$ the Γ-spin bundle of positive chirality; the Γ-invariant Dirac operators D_{APS} and D_{aAPS} as lifts of Dirac operators on the base manifold are Γ-Fredholm with Γ−indices*

$$\mathrm{ind}_\Gamma(D_{\mathrm{APS}}) = \mathrm{ind}_\Gamma\left(Q_{--}(t_2,t_1)\right) = \mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} - \dim_\Gamma \mathsf{ker}\,(A_2)$$

and

$$\mathrm{ind}_\Gamma(D_{\mathrm{aAPS}}) = \mathrm{ind}_\Gamma\left(Q_{++}(t_2,t_1)\right) = -\mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} + \dim_\Gamma \mathsf{ker}\,(A_2) \quad .$$

10.2.2. Γ-Fredholmness of $D^{E_L}_-$ and $\mathcal{D}^{E_L}$ with generalised (a)APS boundary conditions

We can prove in a similar fashion Γ-Fredholmness for the Dirac operator $\tilde{D}^{E_L} := D^{E_L}_-$, acting on spinor fields with negative chirality. We define

$$\begin{aligned} &FE^s_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \\ &\quad := \left\{ u \in FE^s_\Gamma(M,\mathcal{T},\tilde{D}^{E_L}) \,\middle|\, P^{E_L}_{(-\infty,a_1)}(t_1)\circ \mathsf{res}_{\Sigma_1} u = 0 = P^{E_L}_{(a_2,\infty)}(t_2)\circ \mathsf{res}_{\Sigma_2} u \right\} \quad , \end{aligned}$$

$$\begin{aligned} &FE^s_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \\ &\quad := \left\{ u \in FE^s_\Gamma(M,\mathcal{T},\tilde{D}^{E_L}) \,\middle|\, P^{E_L}_{[a_1,\infty)}(t_1)\circ \mathsf{res}_{\Sigma_2} u = 0 = P^{E_L}_{(-\infty,a_2]}(t_2)\circ \mathsf{res}_{\Sigma_1} u \right\} \quad . \end{aligned}$$

These spaces have been introduced in [Dam21, Sec.7.4] for the special case $a_1 = a_2 = 0$ and without twisting bundle ($FE^s_{\Gamma,\mathrm{APS}}(M,\mathcal{T},\tilde{D})$ and $FE^s_{\Gamma,\mathrm{APS}}(M,\mathcal{T},\tilde{D})$). For the same reasons as in the last subsection these spaces are all Hilbert Γ-modules. Restricting $\tilde{D}^{E_L}$ to these domains for $s = 0$ defines the Dirac operators for g(a)APS-boundary conditions with respect to negative chirality:

$$\begin{aligned}\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)} &: FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}^+_{L,E}(M)) \quad ,\\ \tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)} &: FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}^+_{L,E}(M)) \quad .\end{aligned}$$

The proof for their Γ-Fredholmness works out as the one for Theorem 10.2.1 and generalises [Dam21, Thm.7.9] to twisting Dirac operators and generalised (a)APS boundary conditions:

Theorem 10.2.3. *Let $a_1, a_2 \in \mathbb{R}$, M a temporal compact, globally hyperbolic spatial Γ-manifold with compact base M_Γ, $\mathcal{S}^-_{L,E}(M) \to M$ the Γ-spin bundle of negative chirality which is twisted with a Hermitian Γ-vector bundle $E \to M$ and twisted with the square-root of a Hermitian Γ-line bundle $L \to M$ for a Spin^c-structure. The Γ-invariant Dirac operators $\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}$ and $\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}$ as lifts of Dirac operators on the base manifold are Γ-Fredholm with Γ-indices*

$$\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) = \mathrm{ind}_\Gamma\left(\tilde{Q}^{>a_2}_{\geq a_1}(t_2,t_1)\right)$$

and

$$\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) = \mathrm{ind}_\Gamma\left(\tilde{Q}^{\leq a_2}_{< a_1}(t_2,t_1)\right) \quad .$$

Proof. The diagonal matrix entries of $\tilde{Q}$ with respect to the used boundary conditions are just defined as those for Q where Q is replaced with $\tilde{Q}$ (see (9.31) and (9.32)), the space $FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L})$ coincides with $FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},D^{E_L})$ and the space $FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M,\mathcal{T},\tilde{D}^{E_L})$ coincides with $FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L})$. The same proof strategy shows that $\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}$ and $\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}$ are Γ-Fredholm with claimed Γ-indices. □

If we reduce to the untwisted case with $a_1 = a_2 = 0$, we get the mentioned result from [Dam21].

Corollary 10.2.4. *Let M be a temporal compact, even-dimensional globally hyperbolic spatial Γ-manifold, $\mathcal{S}^-(M) \to M$ the Γ-spin bundle of negative chirality; the Γ-invariant Dirac operators $\tilde{D}_{\mathrm{APS}}$ and $\tilde{D}_{\mathrm{aAPS}}$ as lifts of Dirac operators on the base manifold are Γ-Fredholm with Γ-indices*

$$\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{APS}}) = \mathrm{ind}_\Gamma\left(\tilde{Q}_{++}(t_2,t_1)\right) = -\mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} + \dim_\Gamma \mathsf{ker}\,(A_2)$$

and

$$\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{aAPS}}) = \mathrm{ind}_\Gamma\left(\tilde{Q}_{--}(t_2,t_1)\right) = \mathrm{sf}_\Gamma\left\{A_t\right\}_{t\in[t_1,t_2]} - \dim_\Gamma \mathsf{ker}\,(A_2) \quad .$$

(9.51) shows

$$\begin{aligned}-\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) &= \mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{APS}(a_1,a_2)})\\ &= -\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) = \mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) \quad .\end{aligned} \tag{10.8}$$

We define $FE^0_{\Gamma,\mathrm{(a)APS}(a_1,a_2)}(M,\mathcal{T},\mathcal{D}^{E_L})$ to be the direct sums

$$FE^0_{\Gamma,\mathrm{(a)APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}_+) \oplus FE^0_{\Gamma,\mathrm{(a)APS}(a_1,a_2)}(M,\mathcal{T},D^{E_L}_-) \quad .$$

In view of Corollary 5.2.19, Theorem 10.2.1 and Theorem 10.2.3 imply the following result which is expected due to skew-adjointness of $\mathcal{D}$.

Corollary 10.2.5. *Let $a_1, a_2 \in \mathbb{R}$, M a temporal compact, globally hyperbolic spatial Γ-manifold with compact base M_Γ, $\mathcal{S}_{L,E}(M) \to M$ a Γ-spin bundle which is twisted with a Hermitian Γ-vector bundle $E \to M$ and twisted with the square-root of a Hermitian Γ-line bundle $L \to M$ for a Spin^c-structure. Under these assumptions, the Γ-invariant Dirac operators*

$$\mathcal{D}^{E_L}_{\mathrm{APS}(a_1,a_2)} : FE^0_{\Gamma,\mathrm{APS}(a_1,a_2)}(M, \mathcal{T}, \mathcal{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}_{L,E}(M))$$

and

$$\mathcal{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)} : FE^0_{\Gamma,\mathrm{aAPS}(a_1,a_2)}(M, \mathcal{T}, \mathcal{D}^{E_L}) \to L^2_\Gamma(\mathcal{S}_{L,E}(M)) \quad ,$$

which are lifts of Dirac operators on the base manifold and equipped with gAPS and respectively gaAPS boundary conditions on the Cauchy boundary hypersurfaces $\Sigma_1 = \Sigma_{t_1}$ and $\Sigma_2 = \Sigma_{t_2}$, are Γ-Fredholm with Γ-indices

$$\mathrm{ind}_\Gamma(\mathcal{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) = 0 = \mathrm{ind}_\Gamma(\mathcal{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) \quad .$$

Proof. The Γ-Fredholmness and the Γ-indices are consequences of Corollary 5.2.19 and the fact that

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

is an invertible element in $\mathscr{B}_\Gamma$:

$$\begin{aligned}
\mathrm{ind}_\Gamma(\mathcal{D}^{E_L}|_{\mathrm{(a)APS}(a_1,a_2)}) &= \mathrm{ind}_\Gamma\left(\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}(D^{E_L}_+|_{\mathrm{(a)APS}(a_1,a_2)} \oplus D^{E_L}_-|_{\mathrm{(a)APS}(a_1,a_2)})\right) \\
&= \mathrm{ind}_\Gamma\left(\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\right) + \mathrm{ind}_\Gamma\left(D^{E_L}_+|_{\mathrm{(a)APS}(a_1,a_2)}\right) \\
&\quad + \mathrm{ind}_\Gamma\left(D^{E_L}_-|_{\mathrm{(a)APS}(a_1,a_2)}\right) \\
&= \mathrm{ind}_\Gamma\left(D^{E_L}_+|_{\mathrm{(a)APS}(a_1,a_2)}\right) + \mathrm{ind}_\Gamma\left(D^{E_L}_-|_{\mathrm{(a)APS}(a_1,a_2)}\right) \\
&= \mathrm{ind}_\Gamma\left(D^{E_L}_{\mathrm{(a)APS}(a_1,a_2)}\right) + \mathrm{ind}_\Gamma\left(\tilde{D}^{E_L}_{\mathrm{(a)APS}(a_1,a_2)}\right) \overset{(10.8)}{=} 0 \quad . \qquad \square
\end{aligned}$$

10.3. Geometric Γ-index formulas for $D^{E_L}_{\pm,(\mathrm{a})\mathrm{APS}(a_1,a_2)}$

Now we tackle the geometric description of the Γ-index formula for which we need to recapitulate the Γ-index of the Riemannian Dirac operator $\check{D}^{E_L}_{\mathrm{APS}}$. The aim is to express this index by means of the Γ-spectral flow of the smooth family of hypersurface Dirac operators $\{A_t\}_{t\in[t_1,t_2]}$. The role of the Γ-spectral flow then becomes important as a connecting element of the geometric Γ-index expression and the Lorentzian Γ-index of $D^{E_L}_{(\mathrm{a})\mathrm{APS}(a_1,a_2)}$ which allow us to express the latter one in geometric terms. This task is presented in the first two subsections. The special case of finite coverings is covered in the last subsection.

10.3.1. Γ-index of $\check{D}^{E_L}_{\mathrm{APS}}$ as Γ-spectral flow

We consider an auxiliary Riemannian situation with product structure near the boundary M, introduced in subsection 6.3.3. We assume the base hypersurfaces to be a Γ-manifold such that M becomes a Galois covering with boundary. We introduce a product structure on the Riemannian Dirac operator of interest by choosing the auxiliary Γ-invariant Riemannian metric $\check{g}$ appropiately such that it takes the form

$$\check{g}|_{[0,\epsilon)\times\Sigma_j} = \mathrm{d}t^{\otimes 2} + g_{t_j} \quad (\epsilon > 0, j \in \{1,2\})$$

in the collar neighbourhood of the boundary hypersurfaces Σ_1 and Σ_2. The twisted Γ-invariant Riemannian Dirac operator $\check{D}^{E_L}$ then has the desired product structure

$$\check{D}^{E_L}|_{[0,\epsilon)\times\Sigma_j} = (\beta\otimes\mathbb{1}_{E_L})(\partial_t - A_j^{E_L}) = -(\beta\otimes\mathbb{1}_{E_L})(\nu + A_j^{E_L})$$

near the boundary due to (6.68) with $N = 0$, $H_{t_j} = 0$, and the boundary hypersurface Dirac operators $A_j := A_{t_j}$. This brings us in the position to apply Theorem 1.0.4 to $\check{D}^{E_L}$ which is a Γ-Fredholm operator from $H^1_{\Gamma,\mathrm{APS}}(\mathcal{S}^+_{L,E}(\check{M}))$ to $L^2_\Gamma(\mathcal{S}^-(\check{M}))$ where the Γ-Sobolev spaces, subject to APS boundary conditions, takes the form

$$H^1_{\Gamma,\mathrm{APS}}(\mathcal{S}^+_{L,E}(\check{M})) := \left\{ u \in H^1_\Gamma(\mathcal{S}^+_{L,E}(\check{M})) \,\middle|\, P_{\geq 0}(t_1)\circ \mathsf{res}_{t_1} u = 0 = P_{<0}(t_2)\circ\mathsf{res}_{t_2} u \right\} \quad .$$

Since M is closed by two boundary hypersurfaces, we have to be careful while evaluating the boundary contribution. Since we have chosen ν to be an outwards-pointing normal vector to each hypersurface, it points outwards at Σ_1, but inwards at Σ_2. To apply Theorem 1.0.4 correctly, we have to take $(-\nu)$ as normal vector at Σ such that the sign in front of $(-A_1^{E_L})$ flips. The index formula (1.3) thus takes the form

$$\mathrm{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) = \int_{M_\Gamma} a(\check{M}_\Gamma, E_\Gamma, L_\Gamma) - \xi_\Gamma(A_1^{E_L}) - \xi_\Gamma(-A_2^{E_L}) \tag{10.9}$$

where the integrand can be related to certain characteristic classes from the proof of the compact case (see e.g. [LM16, Thm.III.13.10] and [LM16, Thm.IV.1.3]):

$$a(\check{M}_\Gamma, E_\Gamma, L_\Gamma) = \hat{\mathrm{A}}\left(T(\check{M}/\Gamma)\right)\wedge \mathrm{ch}\,(E_L/\Gamma) = \hat{\mathrm{A}}\left(T(\check{M}/\Gamma)\right)\wedge \mathrm{e}^{\mathrm{c}_1(L/\Gamma)/2}\wedge\mathrm{ch}\,(E/\Gamma) \quad ;$$

$\hat{\mathrm{A}}\left(\check{M}/\Gamma\right)$ is the (total) $\hat{A}$-genus with respect to the Riemannian Levi-Civita connection on the tangent bundle, $\mathrm{c}_1\,(L/\Gamma)$ is the first Chern class of a Hermitian line bundle L/Γ and $\mathrm{ch}\,(E/\Gamma)$ is the Chern character of the Hermitian vector bundle E/Γ. Though the concrete definition of these characteristic classes won't be important for the rest of this thesis, we present them for completeness: let Ω^L and Ω^E be the curvature two-forms of the connections of L/Γ and E/Γ, then the first Chern form and the (total) Chern character form are

$$\mathrm{c}_1(L/\Gamma) := \frac{\mathbb{i}}{2\pi}\mathrm{tr}\left(\Omega^L\right) \quad \text{and} \quad \mathrm{ch}\left(E/\Gamma\right) := \mathrm{tr}\left(\exp\left(\frac{\mathbb{i}}{2\pi}\Omega^E\right)\right)$$

each modulo exact forms. The concrete definition of the $\hat{A}$-genus will be recalled in the coming subsection. The wedge product of these characteristic classes is in general a direct sum of all even forms since all classes are defined via formal power series in the curvature two-form. For powers higher than half of the dimension of M/Γ the sum cancels, consequently the wedge product contains all possible and non-trivial combinations of even degree forms where only those of total degree $(n+1)$ contribute to the integral.

The boundary contribution in (10.9) can be rewritten as follows:

$$\begin{aligned}\xi_\Gamma(A_1^{E_L}) + \xi_\Gamma(-A_2^{E_L}) &= \frac{\eta_\Gamma(A_1^{E_L}) + \dim_\Gamma \mathsf{ker}\left(A_1^{E_L}\right) + \eta_\Gamma(-A_2^{E_L}) + \dim_\Gamma \mathsf{ker}\left(-A_2^{E_L}\right)}{2} \\ &= \frac{\eta_\Gamma(A_1^{E_L}) + \dim_\Gamma \mathsf{ker}\left(A_1^{E_L}\right)}{2} - \frac{\eta_\Gamma(A_2^{E_L}) + \dim_\Gamma \mathsf{ker}\left(A_2^{E_L}\right)}{2} \\ &\quad + \dim_\Gamma \mathsf{ker}\left(A_2^{E_L}\right) = \xi_\Gamma(A_1^{E_L}) - \xi_\Gamma(A_2^{E_L}) + \dim_\Gamma \mathsf{ker}\left(A_2^{E_L}\right) \quad .\end{aligned}$$

The difference of Γ-xi invariants can be expressed by means of the analytic Γ-spectral flow formula (8.67):

$$\begin{aligned}\mathrm{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) = &\int_{M_\Gamma} \hat{\mathbb{A}}\left(\check{M}_\Gamma\right) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\left(E_\Gamma\right) + \mathrm{sf}_\Gamma\left\{A_t^{E_L}\right\}_{t\in[t_1,t_2]} \\ &+ \frac{1}{2}\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\eta_\Gamma(A_t^{E_L})\,\mathrm{d}t - \dim_\Gamma \mathsf{ker}\left(A_2^{E_L}\right) \quad .\end{aligned} \tag{10.10}$$

We want to rewrite the Γ-spectral flow by means of geometric and spectral invariants. The Γ-index and the Γ-spectral flow as finite sum of Γ-indices are locally constant functions from $\mathscr{F}_\Gamma$ to $\mathbb{R}$. Besides the geometric contribution with the $\hat{A}$-genus, taking values in $\mathbb{Z}$, all other contributions are real-valued. In order to split the equation into an integer-valued part and a part which is continuous in t_1, t_2, we consider the index of the Dirac operator $\underline{\check{D}}^{E_L}_{\mathrm{APS}}$ on the compact base M/Γ from [APS75a] and [APS76]: we denote with $\underline{A}_t^{E_L}$ the twisted hypersurface Dirac operator on Σ_t/Γ for each $t \in [t_1, t_2]$ and $\eta(\underline{A}_t^{E_L})$ from (8.39) (we recall that the spectrum of Riemannian Dirac operators on compact manifolds is discrete), then

$$\begin{aligned}\mathrm{ind}\left(\underline{\check{D}}^{E_L}_{\mathrm{APS}}\right) &= \int_{M_\Gamma} \hat{\mathbb{A}}\left(\check{M}_\Gamma\right) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\left(E_\Gamma\right) + \xi(\underline{A}_2^{E_L}) - \xi(\underline{A}_1^{E_L}) - \dim \mathsf{ker}\left(\underline{A}_2^{E_L}\right) \\ &\overset{(8.42)}{=} \int_{M_\Gamma} \hat{\mathbb{A}}\left(\check{M}_\Gamma\right) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\left(E_\Gamma\right) + \mathrm{sf}\left\{\underline{A}_t^{E_L}\right\}_{t\in[t_1,t_2]} - \dim \mathsf{ker}\left(\underline{A}_2^{E_L}\right) \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\eta(\underline{A}_t^{E_L})\,\mathrm{d}t \quad .\end{aligned} \tag{10.11}$$

Apart from the last term every summand is integer-valued, so we can compare this ordinary index with the Γ-index and arrange the terms as proposed, provided that the integral over the characteristic classes functions as connecting element between (10.11) and (10.11):

$$\begin{aligned}
&\operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\eta_\Gamma(A^{E_L}_t)\,\mathrm{d}t + \dim_\Gamma \ker\left(A^{E_L}_2\right)\\
&= \int_{M_\Gamma}\hat{\mathrm{A}}\left(\check{M}_\Gamma\right)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\left(E_\Gamma\right)\\
&= \operatorname{ind}\left(\underline{\check{D}}^{E_L}_{\mathrm{APS}}\right) - \operatorname{sf}\left\{\underline{A}^{E_L}_t\right\}_{t\in[t_1,t_2]} - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\eta(\underline{A}^{E_L}_t)\,\mathrm{d}t + \dim \ker\left(\underline{A}^{E_L}_2\right) \quad .
\end{aligned}$$

We add the integral over the time derivative of $\eta(\underline{A}^{E_L}_t)$ and use (8.60):

$$\begin{aligned}
&\operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta_\Gamma(A^{E_L}_t) - \eta(\underline{A}^{E_L}_t)\right)\,\mathrm{d}t + \dim_\Gamma \ker\left(A^{E_L}_2\right)\\
&= \int_{M_\Gamma}\hat{\mathrm{A}}\left(\check{M}_\Gamma\right)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\left(E_\Gamma\right) + \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\eta(\underline{A}^{E_L}_t)\,\mathrm{d}t \qquad (10.12)\\
&= \operatorname{ind}\left(\underline{\check{D}}^{E_L}_{\mathrm{APS}}\right) - \operatorname{sf}\left\{\underline{A}^{E_L}_t\right\}_{t\in[t_1,t_2]} + \dim \ker\left(\underline{A}^{E_L}_2\right) \quad .
\end{aligned}$$

If we compare the first and the third line in (10.12), we get

$$\begin{aligned}
&\operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\Gamma(A^{E_L}_t, \underline{A}^{E_L}_t)\,\mathrm{d}t + \dim_\Gamma \ker\left(A^{E_L}_2\right)\\
= \quad &\operatorname{ind}\left(\underline{\check{D}}^{E_L}_{\mathrm{APS}}\right) - \operatorname{sf}\left\{\underline{A}^{E_L}_t\right\}_{t\in[t_1,t_2]} + \dim \ker\left(\underline{A}^{E_L}_2\right) \quad \in \mathbb{Z} \quad .
\end{aligned}$$

Because the right-hand side is integer-valued, the left-hand side also becomes integer-valued. If we compare the first and second line in (10.12), we can moreover show that it is already zero: starting from

$$\operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\Gamma(A^{E_L}_t, \underline{A}^{E_L}_t)\,\mathrm{d}t + \dim_\Gamma \ker\left(A^{E_L}_2\right) \qquad (10.13)$$

$$= \int_{M_\Gamma}\hat{\mathrm{A}}\left(\check{M}_\Gamma\right)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\left(E_\Gamma\right) + \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\eta(\underline{A}^{E_L}_t)\,\mathrm{d}t \quad , \qquad (10.14)$$

we see that the right-hand side of this equation is continuous in t_1 and t_2. We fix t_1 and consider (10.14) as function of t_2. (10.13) of the same equation implies that also (10.14) is integer-valued. By continuity in t_2 of (10.14), (10.13) however becomes locally constant in t_2. Because t_1, t_2 are from a connected interval, the side (10.13) becomes even constant in t_2. Consequently, the limit $t_2 \to t_1$ does not affect (10.13) while (10.14) vanishes because $\hat{\mathrm{A}}_\Gamma\left(\check{M}\right)\wedge \mathrm{e}^{\mathrm{c}^\Gamma_1(L)}\wedge \mathrm{ch}_\Gamma\left(E\right)$ is a sum of even degree forms up to degree $(n+1)$ and thus has no contribution while integrated over the remaining odd-dimensional submanifold Σ_1/Γ.

The constancy of (10.13) implies in the limit that it already equals zero:

$$\operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} \\ - \frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\rho_\Gamma(A^{E_L}_t,\underline{A}^{E_L}_t)\,\mathrm{d}t + \dim_\Gamma\mathsf{ker}\left(A^{E_L}_2\right) = 0 \quad . \tag{10.15}$$

The remaining integral can be perfomed by the fact that $t\mapsto\rho_\Gamma(A^{E_L}_t,\underline{A}^{E_L}_t)$ is differentiable from the proof of Proposition 8.2.16. This gives us the difference of two Cheeger-Gromov-invariants at different boundary hypersurfaces. (10.15) enables us to express the Γ-spectral flow by means of geometric and spectral invariants:

$$\begin{aligned}\operatorname{sf}_\Gamma\left\{A^{E_L}_t\right\}_{t\in[t_1,t_2]} &= \operatorname{ind}_\Gamma(\check{D}^{E_L}_{\mathrm{APS}}) - \frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2},\underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1},\underline{A}^{E_L}_{t_1})\right) + \dim_\Gamma\mathsf{ker}\left(A^{E_L}_2\right)\\ &= \int_{M_\Gamma}\hat{\mathrm{A}}\left(\check{M}_\Gamma\right)\wedge \mathrm{e}^{c_1(L_\Gamma)/2}\wedge\operatorname{ch}(E_\Gamma) - \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2) \qquad (10.16)\\ &\quad -\frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2},\underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1},\underline{A}^{E_L}_{t_1})\right) \quad .\end{aligned}$$

10.3.2. Geometric Γ-index formulas for $D^{E_L}_{\pm,(\mathrm{a})\mathrm{APS}}$

Before we relate (10.16) to the Γ-spectral flow of $\{A_t\}_{t\in\mathcal{T}(M)}$, we further have to manipulate (10.16) in such a way that the geometric elements are purely expressed by means of Lorentzian quantities. For this we need to analyse the dependency on the auxiliary Riemannian metric.

Apart from the integral over the wedge product of characteristic classes in (10.16) the other terms in the Γ-spectral flow formula are coming from the boundary hypersurfaces Σ_1 and Σ_2 which are the same in both the Lorentzian and the auxiliary Riemannian situation. The only leftover of the auxiliary Riemannian structure is just the first integral over the whole manifold. Since the Hermiticity of E and a possibly necessary line bundle L has been already considered in the Lorentzian case, the only piece, carrying the auxiliary Riemannian structure, comes from the $\hat{A}$-genus which is evaluated with the curvature two-form $\check{\Omega}$ of the Levi-Civita connection with respect to $\check{g}$. The $\widehat{A}$-genus is defined as element in $\bigoplus_{m\in4\mathbb{N}_0}\Omega^m(M,\mathbb{R})$ and has the form

$$\widehat{\mathrm{A}}\left(\check{M}_\Gamma\right) = \sqrt{\det\left(\frac{\check{\Omega}/2}{\sinh(\check{\Omega}/2)}\right)} =: \mathscr{P}(\check{\Omega}) \quad , \tag{10.17}$$

see [BGV03, Sec.1.5]. We use $\mathscr{P}(\check{\Omega})$ as abbreviation and describes a formal, but terminating power series in the curvature two-form. The determinant is an invariant towards the adjoint action of $\mathsf{GL}(n+1,\mathbb{R})$, so $\mathscr{P}(\check{\Omega})$ is an invariant polynomial with respect to invertible matrices, i.e.

$$\mathscr{P}(U^{-1}\check{\Omega}U) = \mathscr{P}(\check{\Omega}) \quad \forall U\in\mathsf{GL}(n+1,\mathbb{R}) \quad .$$

The second Bianchi identity and the definition of the curvature two-form via the connection one-form matrix $\check{\omega}$ for $\check{\nabla}$ imply that $\mathscr{P}(\check{\Omega})$ is a closed form. The Chern-Weil theorem states that the $\widehat{A}$-genus is independent of the choice of the connection and this fact allows us to relate this genus to the one with respect to ∇, induced by g: let $\{\nabla(s)\}_{s\in[0,1]}$ be a smooth path in the space of connections on M_Γ which is a connected set. This path has endpoints at $\nabla(1)=\check{\nabla}$ and $\nabla(0)=\nabla$. It induces smooth paths of connection one-forms $\{\omega(s)\}_{s\in[0,1]}$ and curvature two-forms $\{\Omega(s)\}_{s\in[0,1]}$ with endpoints

$$\omega(0)=\omega, \quad \omega(1)=\check{\omega}, \quad \Omega(0)=\Omega \quad \text{and} \quad \Omega(1)=\check{\Omega} \quad .$$

The cohomology class of $\mathscr{P}$ does not depend on the concrete choice of the connection, so the difference of $\mathscr{P}(\Omega)$ and $\mathscr{P}(\check{\Omega})$ are in the same cohomology class, i.e. they differ in an exact form:

$$\mathscr{P}(\check{\Omega})-\mathscr{P}(\Omega)= \mathrm{d}\left[\int_0^1 \dot{\nabla}(s)\mathscr{P}'(\Omega(s))\,\mathrm{d}s\right] =: \mathrm{d}T\widehat{\mathbb{A}}(\check{\nabla},\nabla) \quad .$$

The form $T\widehat{\mathbb{A}}(\check{\nabla},\nabla)$ is called *transgression form* of the $\widehat{A}$-genus; $\mathscr{P}'(\Omega(s))$ is the derivative of $\mathscr{P}(\Omega(s))$ with respect to the curvature entries. The homotopy invariance of the de Rham cohomology allows us to consider the concrete path of connections $\nabla(s)=\nabla+s(\check{\nabla}-\nabla)$ which implies $\omega(s)=\omega+s(\check{\omega}-\omega)$. The transgression form then becomes

$$T\widehat{\mathbb{A}}(\check{\nabla},\nabla)=\int_0^1(\check{\nabla}-\nabla)\mathscr{P}'(\Omega(s))\,\mathrm{d}s \quad . \tag{10.18}$$

Since the Chern classes and Chern characters are equally invariant polynomials, they are also closed forms such that we can rewrite the wedge product $\widehat{\mathbb{A}}\left(\check{M}_\Gamma\right)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)$ as follows:

$$\begin{aligned}
&\widehat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)+\mathrm{d}T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)\\
=&\widehat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)+\mathrm{d}\left[T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)\right]\\
&\mp T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{d}(\mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2})\wedge \mathrm{ch}\,(E_\Gamma)\mp T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{d}(\mathrm{ch}\,(E_\Gamma))\\
=&\widehat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)+\mathrm{d}\left[T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)\right] \quad ;
\end{aligned}$$

the integral formula over M_Γ becomes

$$\begin{aligned}
&\int_{M_\Gamma}\hat{\mathbb{A}}\left(\check{M}\right)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)=\\
&\qquad\int_{M_\Gamma}\widehat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)+\int_{\partial M_\Gamma} i^*\left[T\widehat{\mathbb{A}}(\check{\nabla},\nabla)\wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma)\right]
\end{aligned}$$

where we have used Stokes-Cartan (A.1) with the embedding $i:\partial M_\Gamma\hookrightarrow M_\Gamma$. Since the boundary is the disjoint union $\Sigma_1/\Gamma\sqcup\Sigma_2/\Gamma$, where each Γ-hypersurface has opposite orientation, the embedding coincides with $i_j:\Sigma_j/\Gamma\to M_\Gamma$ on each disjoint Γ-hypersurface ($j\in\{1,2\}$). Since pullbacks distribute over wedge products and the Chern classes and

Chern characters are natural with respect to pullbacks, the boundary contribution becomes

$$\int_{\partial M_\Gamma} i^* \left[T\widehat{\mathbb{A}}(\check{\nabla}, \nabla)) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}_\Gamma\,(E_{\partial M})\right] =: \int_{\partial M_\Gamma} \mathcal{T}\mathcal{G}_\Gamma(\check{\nabla}, \nabla)$$

with

$$\mathcal{T}\mathcal{G}_\Gamma(\check{\nabla}, \nabla) = i^*(T\widehat{\mathbb{A}}(\check{\nabla}, \nabla)) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) \tag{10.19}$$

as *twisted transgression form*; we introduced the abbreviations $c_1\,(L_\Gamma|_{\partial M_\Gamma})$ for $c_1\,(i^* L_\Gamma)$ and $\mathrm{ch}\,(E_\Gamma|_{\partial M})$ for $\mathrm{ch}\,(i^* E_\Gamma)$. The only possible leftovers from the auxiliary Riemannian structure in (10.19) might appear in $i^* T\widehat{\mathbb{A}}(\check{\nabla}, \nabla)$ which we are going to investigate further. We apply (D.3) in Appendix D to $\mathtt{c} = \widehat{\mathbb{A}}$, $\mathtt{w} = T\widehat{\mathbb{A}}$ and $f = i$. This results in $i^* T\widehat{\mathbb{A}}(\check{\nabla}, \nabla) = T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla)$ plus an exact form $\mathrm{d}\mathtt{v}$. The closedness of the other charateristic classes then shows that the twisted transgression forms $i^*(T\widehat{\mathbb{A}}(\check{\nabla}, \nabla))$ and $T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla)$ differ in an exact form:

$$\begin{aligned}
i^*&(T\widehat{\mathbb{A}}(\check{\nabla}, \nabla)) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) \\
&= T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) + \mathrm{d}\mathtt{v} \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) \\
&= T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) + \mathrm{d}\left[\mathtt{v} \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M})\right] \\
&\quad \pm \mathtt{v} \wedge \mathrm{d}\left[\mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2}\right] \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) \pm \mathtt{v} \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{d}\left[\mathrm{ch}\,(E_\Gamma|_{\partial M})\right] \\
&= T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) + \mathrm{d}\left[\mathtt{v} \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M})\right].
\end{aligned}$$

But since we integrate over the boundary, the exact part does not contribute due to Stokes-Cartan theorem. It finally remains to analyse how

$$T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla) = \int_0^1 (i^*\check{\nabla} - i^*\nabla)\mathscr{P}'((i^*\Omega)(s))\,\mathrm{d}s$$

in

$$\mathcal{T}\mathcal{G}_\Gamma(\check{\nabla}, \nabla) = T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla) \wedge \mathrm{e}^{c_1(L_\Gamma|_{\partial M})/2} \wedge \mathrm{ch}\,(E_\Gamma|_{\partial M}) \tag{10.20}$$

depends on $\check{g}$ with product structure near the boundary hypersurfaces. Let X be a vector field, tangent to Σ_i/Γ $(i \in \{1, 2\})$. The formulas in (3.35) describe how the connection coefficients $(i^*\omega)$ and $(i^*\check{\omega})$ of $i^*\nabla$ and respectively $i^*\check{\nabla}$ are determined:

$$-(i^*\omega)^j_0(X) = (i^*\omega)^0_j(X) = -\mathrm{II}_g(e_j, X), \quad (i^*\omega)^k_j(X) = \omega^{T\Sigma_\Gamma}(X) = (i^*\check{\omega})^k_j(X), \tag{10.21}$$

$$-(i^*\check{\omega})^j_0(X) = (i^*\check{\omega})^0_j(X) = \mathrm{II}_{\check{g}}(e_j, X), \qquad (i^*\omega)^0_0 = 0 = (i^*\check{\omega})^0_0 \tag{10.22}$$

for $k, j \in \{1, ..., n\}$. Since $\check{g}$ has product structure near the boundary, the boundary hypersurfaces become totally geodesic, implying $\mathrm{II}_{\check{g}}(e_j, X)$ to vanish for all j and X. The difference of the first two equations in (10.21) is thus

$$(i^*\check{\omega})^0_j(X) - (i^*\omega)^0_j(X) = [i^*(\check{\omega} - \omega)]^0_j(X) = \mathrm{II}_g(e_j, X) \tag{10.23}$$

for all $j \in \{1, ..., n\}$. This reflects that $i^*(\check{\nabla} - \nabla)$ does not depend on $\check{g}$. The path of curvature two-forms is the covariant derivative with respect to the path of connection one-forms:

$$\begin{aligned}\Omega(s) &= \nabla(s)\omega(s) = \mathrm{d}\omega(s) + \frac{1}{2}\left[\omega(s) \wedge \omega(s)\right] \\ &= \mathrm{d}\omega + s\,\mathrm{d}\left(\breve{\omega} - \omega\right) + \frac{1}{2}\left(\left[\omega \wedge \omega\right] + 2s\left[\omega \wedge (\breve{\omega} - \omega)\right] + s^2\left[(\breve{\omega} - \omega) \wedge (\breve{\omega} - \omega)\right]\right) \\ &= \Omega + \Omega^{s(\breve{\omega}-\omega)} + s\left[\omega \wedge (\breve{\omega} - \omega)\right] \quad .\end{aligned}$$

$\Omega^{s(\breve{\omega}-\omega)}$ is the curvature two-form expression, evaluated with the coefficients of $s(\breve{\omega} - \omega)$. Because the difference of connection one-forms (10.23) is independent of the metric $\breve{g}$, the curvature two-form of the path is also independent of the metric $\breve{g}$ and is fully determined by the curvature with respect to ∇ and the second fundamental form $\mathrm{I\!I}_g$ as well as its derivatives in $\Omega^{s(\breve{\omega}-\omega)}$. Consequently, we have shown that $T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla)$ in fact do not depend on the choice of the auxiliary metric, but only on informations from g and $\mathrm{I\!I}_g$. We set $T\widehat{\mathbb{A}}(g)$ for $T\widehat{\mathbb{A}}(i^*\check{\nabla}, i^*\nabla)$ as well as $\mathcal{TG}(g, E_\Gamma, L_\Gamma)$ for $\mathcal{TG}_\Gamma(\check{\nabla}, \nabla)$ to stress this observation. The Γ-spectral flow formula (10.16) becomes completely determined by the geometry of (M, g) and spectral invariants on the Cauchy hypersurface. Inserting this into the formulas in Theorem 10.2.1 and Theorem 10.2.3 gives

$$\begin{aligned}\operatorname{ind}_\Gamma(D^{E_L}_{\mathrm{APS}(a_1,a_2)}) &= \int_{M_\Gamma} \hat{\mathbb{A}}\left(M_\Gamma\right) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\left(E_\Gamma\right) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma) \\ &\quad - \frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) - \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2) \\ &\quad - \chi_{\{a_2>0\}} \dim_\Gamma\left(L^2_{\Gamma,(0,a_2]}(\mathcal{S}^+_{L,E}(\Sigma_2))\right) + \chi_{\{a_2<0\}} \dim_\Gamma\left(L^2_{\Gamma,(a_2,0]}(\mathcal{S}^+_{L,E}(\Sigma_2))\right) \\ &\quad - \chi_{\{a_1<0\}} \dim_\Gamma\left(L^2_{\Gamma,[a_1,0)}(\mathcal{S}^+_{L,E}(\Sigma_1))\right) + \chi_{\{a_1>0\}} \dim_\Gamma\left(L^2_{\Gamma,[0,a_1)}(\mathcal{S}^+_{L,E}(\Sigma_1))\right) \\ &= -\operatorname{ind}_\Gamma(D^{E_L}_{\mathrm{aAPS}(a_1,a_2)})\end{aligned} \tag{10.24}$$

and

$$\begin{aligned}\operatorname{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}(a_1,a_2)}) &= \int_{M_\Gamma} \hat{\mathbb{A}}\left(M_\Gamma\right) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\left(E_\Gamma\right) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma) \\ &\quad - \frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) - \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2) \\ &\quad - \chi_{\{a_2>0\}} \dim_\Gamma\left(L^2_{\Gamma,(0,a_2]}(\mathcal{S}^-_{L,E}(\Sigma_2))\right) + \chi_{\{a_2<0\}} \dim_\Gamma\left(L^2_{\Gamma,(a_2,0]}(\mathcal{S}^-_{L,E}(\Sigma_2))\right) \\ &\quad - \chi_{\{a_1<0\}} \dim_\Gamma\left(L^2_{\Gamma,[a_1,0)}(\mathcal{S}^-_{L,E}(\Sigma_1))\right) + \chi_{\{a_1>0\}} \dim_\Gamma\left(L^2_{\Gamma,[0,a_1)}(\mathcal{S}^-_{L,E}(\Sigma_1))\right) \\ &= -\operatorname{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{APS}(a_1,a_2)})\end{aligned} \tag{10.25}$$

and thus the geometric Γ-indices from Main Theorem 2. They satisfy (10.8) and conclusively show all the Γ-index formulas in Main Theorem 2. We recall that we have $\mathcal{S}^+_{L,E}(\Sigma_j) = \mathcal{S}^-_{L,E}(\Sigma_j)$ for $j \in \{1, 2\}$ as Σ_1 and Σ_2 are odd-dimensional and thus admit no chirality decomposition. For ordinary APS and aAPS boundary conditions $(a_1 = 0 = a_2)$

these Γ-index formulas reduce to

$$\begin{aligned}\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{APS}}) = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{c_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma)\\ &-\frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) - \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2)\\ =&\,\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(\tilde{D}^{E_L}_{\mathrm{APS}}) \quad ;\end{aligned}$$

if we consider no twisting bundles L and E, we gain

$$\begin{aligned}&\mathrm{ind}_\Gamma(D_{\mathrm{APS}}) = \mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(D_{\mathrm{aAPS}}) = -\mathrm{ind}_\Gamma(\tilde{D}_{\mathrm{APS}})\\ = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma) + \int_{\partial M_\Gamma} T\hat{\mathbb{A}}(g) - \frac{1}{2}\left(\rho_\Gamma(A_{t_2}, \underline{A}_{t_2}) - \rho_\Gamma(A_{t_1}, \underline{A}_{t_1})\right) - \xi_\Gamma(A_1) - \xi_\Gamma(-A_2)\end{aligned}$$

which proves the Γ-index formulas in Corollary 1.0.7.

10.3.3. Special case: finite coverings

We now want to consider finite coverings as special case of Galois coverings. We assume that Γ is a finite discrete Galois group. As a consequence that the Cauchy hypersurface Σ has been assumed to be a Galois covering with closed base, Σ becomes itself compact because it is covered by finitely many open balls. Consequently, M becomes compact with boundary due to its temporal compactness. The Galois covering map becomes an ordinary l-fold covering map in this situation.

We write l for the cardinality of Γ: $|\Gamma| = l \in \mathbb{N}$. The Γ-trace of a Γ-trace class operator A then becomes $\mathrm{Tr}_\Gamma\,(A) = \frac{1}{l}\mathrm{Tr}\,(A)$ such that the Γ-dimension, Γ-eta invariant and the Γ-xi invariant reduce to

$$\dim_\Gamma = \frac{1}{l}\dim \quad , \; \eta_\Gamma = \frac{1}{l}\eta \quad \Rightarrow \quad \xi_\Gamma = \frac{1}{l}\xi \quad .$$

The Γ-index of a Γ-Fredholm operator A then becomes $\mathrm{ind}_\Gamma(A) = \frac{1}{l}\mathrm{ind}(A)$. We rewrite (10.24) to

$$\begin{aligned}\mathrm{ind}_\Gamma(D^{E_L}_{\mathrm{APS}(a_1,a_2)}) = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma)\wedge \mathrm{e}^{c_1(L_\Gamma)/2}\wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma)\\ &- \mathrm{P}_\Gamma - \Xi_\Gamma + S_\Gamma\end{aligned} \tag{10.26}$$

where

$$\begin{aligned}\mathrm{P}_\Gamma &:= \frac{1}{2}\left(\rho_\Gamma(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_\Gamma(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) \quad ,\\ \Xi_\Gamma &:= \xi_\Gamma(A^{E_L}_1) - \xi_\Gamma(-A^{E_L}_2) \quad ,\\ S_\Gamma &:= -\chi_{\{a_2>0\}}\dim_\Gamma\left(L^2_{\Gamma,(0,a_2]}(\mathcal{S}^+_{L,E}(\Sigma_2))\right) + \chi_{\{a_2<0\}}\dim_\Gamma\left(L^2_{\Gamma,(a_2,0]}(\mathcal{S}^+_{L,E}(\Sigma_2))\right)\\ &\quad -\chi_{\{a_1<0\}}\dim_\Gamma\left(L^2_{\Gamma,[a_1,0)}(\mathcal{S}^+_{L,E}(\Sigma_1))\right) + \chi_{\{a_1>0\}}\dim_\Gamma\left(L^2_{\Gamma,[0,a_1)}(\mathcal{S}^+_{L,E}(\Sigma_1))\right)\end{aligned}$$

are shorthand notations for the boundary contributions. We recall the reduced eta function from (8.63) and introduce further abbreviations:

$$\begin{aligned}
\mathrm{P}_l &:= \frac{1}{2}\left(\rho_l(A^{E_L}_{t_2}, \underline{A}^{E_L}_{t_2}) - \rho_l(A^{E_L}_{t_1}, \underline{A}^{E_L}_{t_1})\right) \quad,\\
\Xi_l &:= \xi(A^{E_L}_1) - \xi(-A^{E_L}_2) \quad,\\
S &:= -\chi_{\{a_2>0\}} \dim\left(L^2_{(0,a_2]}(\mathcal{S}^+_{L_\Gamma,E_\Gamma}(\Sigma_2/\Gamma))\right) + \chi_{\{a_2<0\}} \dim\left(L^2_{(a_2,0]}(\mathcal{S}^+_{L_\Gamma,E_\Gamma}(\Sigma_2/\Gamma))\right)\\
&\quad -\chi_{\{a_1<0\}} \dim\left(L^2_{[a_1,0)}(\mathcal{S}^+_{L_\Gamma,E_\Gamma}(\Sigma_1/\Gamma))\right) + \chi_{\{a_1>0\}} \dim\left(L^2_{[0,a_1)}(\mathcal{S}^+_{L_\Gamma,E_\Gamma}(\Sigma_1/\Gamma))\right) \quad,\\
\Xi &:= \xi(\underline{A}^{E_L}_1) - \xi(-\underline{A}^{E_L}_2) \quad.
\end{aligned}$$

Hence for finite covering we gain from (10.26)

$$\begin{aligned}
gammaindexshort\frac{1}{l}\mathrm{ind}(D^{E_L}_{\mathrm{APS}(a_1,a_2)}) &= \int_{M_\Gamma} \hat{\mathrm{A}}\,(M_\Gamma) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma)\\
&\quad - \frac{1}{l}\mathrm{P}_l - \frac{1}{l}\Xi_l + \frac{1}{l}S \quad, \qquad (10.27)
\end{aligned}$$

We multiply (10.27) with l and add several zeroes such that the index formula (1.5) for the Dirac operator $\underline{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}$ on the compact base, i.e.

$$\mathrm{ind}(\underline{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) = \int_{M_\Gamma} \hat{\mathrm{A}}\,(M_\Gamma) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma) - \Xi + S \quad,$$

can be recovered:

$$\begin{aligned}
\mathrm{ind}(D^{E_L}_{\mathrm{APS}(a_1,a_2)}) &= l \int_{M_\Gamma} \hat{\mathrm{A}}\,(M_\Gamma) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\,(E_\Gamma) + l \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma)\\
&\quad - \mathrm{P}_l - \Xi_l + S\\
&= l\left(\int_{M_\Gamma} \hat{\mathrm{A}}\,(M_\Gamma) \wedge \mathrm{e}^{\mathrm{c}_1(L_\Gamma)/2} \wedge \mathrm{ch}\,(E_\Gamma) + \int_{\partial M_\Gamma} \mathcal{TG}(g, E_\Gamma, L_\Gamma) - \Xi + S\right)\\
&\quad + l\Xi - lS - \mathrm{P}_l - \Xi_l + S\\
&= \quad l\mathrm{ind}(\underline{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) + l\Xi - \Xi_l + (1-l)S - \mathrm{P}_l \quad. \qquad (10.28)
\end{aligned}$$

We already observed for $l = 1$ (no covering) that due to $\underline{A}^{E_L}_t = A^{E_L}_t$ the differences of ξ-invariants coincide, $\Xi_1 = \Xi$, and the reduced eta-invariant ρ_1 vanishes such that $\mathrm{P}_1 = 0$. Hence (10.28) reduces to

$$\mathrm{ind}\left(D^{E_L}_{\mathrm{gAPS}(a_1,a_2)}\right) = \mathrm{ind}\left(\underline{D}^{E_L}_{\mathrm{gAPS}(a_1,a_2)}\right) \qquad (10.29)$$

and we recover the index formulas from [BH18] and from [BS19] if we furthermore set $a_1 = a_2 = 0$. For any $l > 1$, we can rearrange the additional contribution $(l\Xi - \Xi_l - \mathrm{P}_l)$ a little bit more:

$$\begin{aligned}
l\Xi - \Xi_l - \mathrm{P}_l = \quad & l\xi(\underline{A}_1^{E_L}) + l\xi(-\underline{A}_2^{E_L}) - \xi(A_1^{E_L}) - \xi(-A_2^{E_L}) - \mathrm{P}_l \\
= & \frac{l}{2}\left[\eta(\underline{A}_1^{E_L}) + \dim\ker\left(\underline{A}_1^{E_L}\right) + \eta(-\underline{A}_2^{E_L}) + \dim\ker\left(-\underline{A}_2^{E_L}\right)\right] \\
& - \frac{1}{2}\left[\eta(A_1^{E_L}) + \dim\ker\left(A_1^{E_L}\right) + \eta(-A_2^{E_L}) + \dim\ker\left(-A_2^{E_L}\right)\right] - \mathrm{P}_l \\
= & \frac{l}{2}\left[\eta(\underline{A}_1^{E_L}) + \dim\ker\left(\underline{A}_1^{E_L}\right) - \eta(\underline{A}_2^{E_L}) + \dim\ker\left(\underline{A}_2^{E_L}\right)\right] \\
& - \frac{1}{2}\left[\eta(A_1^{E_L}) + \dim\ker\left(A_1^{E_L}\right) - \eta(A_2^{E_L}) + \dim\ker\left(A_2^{E_L}\right)\right] - \mathrm{P}_l \\
= & \frac{1}{2}\left[l\dim\ker\left(\underline{A}_1^{E_L}\right) - \dim\ker\left(A_1^{E_L}\right) + l\dim\ker\left(\underline{A}_2^{E_L}\right) - \dim\ker\left(A_2^{E_L}\right)\right] \\
& - \frac{1}{2}\left[\eta(A_1^{E_L}) - l\eta(\underline{A}_1^{E_L}) - (\eta(A_2^{E_L}) - l\eta(\underline{A}_2^{E_L}))\right] - \mathrm{P}_l \\
\overset{(*)}{=} & \frac{1}{2}\left[l\dim\ker\left(\underline{A}_1^{E_L}\right) - \dim\ker\left(A_1^{E_L}\right) + l\dim\ker\left(\underline{A}_2^{E_L}\right) - \dim\ker\left(A_2^{E_L}\right)\right] \\
& + \frac{1}{2}\left[\rho_l(A_2^{E_L}, \underline{A}_2^{E_L}) - \rho_l(A_1^{E_L}, \underline{A}_1^{E_L})\right] - \mathrm{P}_l \\
= & \frac{1}{2}\left[l\dim\ker\left(\underline{A}_1^{E_L}\right) - \dim\ker\left(A_1^{E_L}\right) + l\dim\ker\left(\underline{A}_2^{E_L}\right) - \dim\ker\left(A_2^{E_L}\right)\right] \\
=: & \, \mathrm{DIM}_l
\end{aligned}$$

whereby we used (8.63) in $(*)$. Hence the contribution from the reduced eta-invariant ρ_l is eliminated in the index for finite coverings. We rewrite l into $|\Gamma|$ and get

$$\operatorname{ind}\left(D^{E_L}_{\mathrm{APS}(a_1,a_2)}\right) = |\Gamma| \operatorname{ind}(\underline{D}^{E_L}_{\mathrm{APS}(a_1,a_2)}) + (1 - |\Gamma|)S + \mathrm{DIM}_{|\Gamma|} \tag{10.30}$$

and thus a *Lorentzian finite covering index formula*. The same observations can be made for the other indices in (10.24) and (10.25). For $|\Gamma| = l = 1$ we also observe (10.29).

11. Open questions and further tasks

Our results Main Theorem 1, Main Theorem 2 and Corollary 1.0.7 extend [BS19, Thm.4.1], [BS19, Thm.7.1] and the index formulas from [BH18, Sec.4.1/2] to a spatial non-compact situation by replacing the compact Cauchy hypersurfaces with Galois coverings, associated to a Galois group Γ. All quantities in the known index formulas transfer to their pendent in the Γ-setting apart from one other contribution which is given by the difference of Cheeger-Gromov rho invariants for the hypersurface Dirac operators along Σ_2 and Σ_1: we recall from (10.24) the Γ-index of $D_{\mathrm{APS}(a_1,a_2)}$:

$$\begin{aligned}\mathrm{ind}_\Gamma(D_{\mathrm{APS}(a_1,a_2)}) = &\int_{M_\Gamma} \hat{\mathbb{A}}\,(M_\Gamma) + \int_{\partial M_\Gamma} T\hat{\mathbb{A}}_\Gamma(g) - \xi_\Gamma(A_1) - \xi_\Gamma(-A_2)\\ &+ \chi_{\{a_2<0\}} \dim_\Gamma \left(L^2_{\Gamma,(a_2,0]}(\mathcal{S}(\Sigma_2))\right) - \chi_{\{a_2>0\}} \dim_\Gamma \left(L^2_{\Gamma,(0,a_2]}(\mathcal{S}(\Sigma_2))\right)\\ &+ \chi_{\{a_1>0\}} \dim_\Gamma \left(L^2_{\Gamma,[0,a_1)}(\mathcal{S}(\Sigma_1))\right) - \chi_{\{a_1<0\}} \dim_\Gamma \left(L^2_{\Gamma,[a_1,0)}(\mathcal{S}(\Sigma_1))\right)\\ &- \frac{1}{2}\left(\rho_\Gamma(A_{t_2},\underline{A}_{t_2}) - \rho_\Gamma(A_{t_1},\underline{A}_{t_1})\right) \quad ;\end{aligned}$$

the new input is the last line. As long as the smooth family of (Riemannian) hypersurface metrics on each Σ_t has non-positive scalar curvature, this additional term is a non-trivial extension of the results from the compact setting; otherwise Proposition 8.2.16 implies that the Cheeger-Gromov rho invariant is constant in the parameter t and the extra contribution vanishes. We have seen in subsection 10.3.3 that this additional term cancels for finite coverings and the expected covering version of [BS19, Thm.7.1] with generalised (a)APS boundary conditions shows up. This observation affirms that the difference of Cheeger-Gromov rho invariants is indeed not an artefact and may influences the use of our results in geometrical as well as application-oriented fields, e.g. extending the rigorous treatment of the chiral anomaly effect from [BS16] to our non-compact setting.

Several possible modifications can be discussed or even proposed within or beyond our setting.

[A] Spatial and temporal non-compactness

The other so far known extension of the treatment and results in [BS19] to spatial non-compact *odd*-dimensional globally hyperbolic manifolds is presented in [Bra20] for strongly Callias-Dirac operators. There, a Callias potential is added to the Dirac operator and controls the regularity outside any compact subset of the Cauchy hypersurface. The most obvious wanted extension of our result is to replace the hypersurfaces as Galois coverings with general manifolds of bounded geometry. Our treatment already provided several necessary ingredients for this situation, e.g. existence and properties of the wave evolution operators which has been shown for general complete Cauchy hypersurfaces. As we have already used results about function spaces on manifolds with bounded geometry to characterise the projectors as suitable Γ-pseudo differential operators, the regularity results for showing Fredholmness can be adapted within this framework. A suitable spectral flow concept as well as an index theorem for the Riemannian Dirac operator on manifolds with boundary and bounded geometry has to be worked out in order to derive an index formula in the same way. But this can be circumvented as a local index theorem for any Dirac-type operator, provided in [BS20], makes the necessity of the spectral flow and a Riemannian index theorem redundant whether the hypersurface is compact or not. Another non-compactness result is presented in [SW22] wherein the temporal compactness is relaxed and hypersurfaces at infinity can be taken into account if the metric is of asymptotically static type, i.e. the metric decays to a static and product type metric for $(t \to \pm\infty)$. Thus, one could think of relaxing the temporal compactness in our situation as well where for example one needs to consider the mentioned metrics and boundary conditions at infinity in the Γ-setting. There might be some issues in showing that the Dirac-wave evolution operators are Fourier integral operators for a possibly unbounded time domain $\mathcal{T}(M)$ because temporal compactness has been a crucial point in this step.

[B] Other groups Γ

The group Γ has been taken to be a discrete group of deck transformations with compact quotient Σ/Γ. One could think of replacing the group Γ with a locally compact unimodular group G which then acts properly, cocompactly and as isometry on the complete hypersurfaces Σ. The corresponding L^2-index theory is presented in e.g. [Wan11, Sec.2/3] and is quite similar to the one we used for our Γ-setting apart from a cutoff-function along the orbits which comes with a proper and cocompact G-action and contributes to the G-index. Because of this resemblance, one could have the impression that the Fredholm part in our proof can be extended to this situation which is in fact clear up to the part where we extended Seeley's theorem of complex powers to Galois coverings. Our Γ-spectral flow concept can be carried over as we introduced it as a special case of spectral flow in a general semifinite von Neumann algebra. A lower truncated eta-invariant $\eta_{\mathsf{G}}^{\geq\epsilon}$ can be defined as in [AW11, Prop.3.3] with the G-trace, introduced in [Wan11, Sec.3.1]: let A be an elliptic and G-invariant operator such that e^{-sA^2} becomes G-trace class for all $s > 0$, then

$$\eta_{\mathsf{G}}^{\geq\epsilon}(A) = \frac{1}{\sqrt{\pi}} \int_{\epsilon}^{\infty} s^{-1/2} \mathrm{Tr}_{\mathsf{G}} \left(A\mathrm{e}^{-sA^2} \right) \mathrm{d}s$$

is well-defined for $\epsilon > 0$. The spectral flow expression with $\eta_{\mathsf{G}}^{\geq\epsilon}$ becomes similar to (8.64). It is left to check that the limit $\epsilon \to 0^+$ and thus the full G-eta invariant is well-defined. In order to express the G-index with geometric data, a corresponding G-index for the Riemannian Dirac operator is needed. There are known equivariant APS-index theorems for the latter case with G being a compact Lie group which has been studied for example in [Don78], [Goe00] and [BM17] for compact manifolds with boundary and non-product type Dirac operators near the boundary.

Instead of replacing the group, one could also fix an element $\mathsf{g} \in \Gamma \setminus \{\mathsf{e}\}$ which have finitely many conjugates in Γ and take the finite conjugacy class $< \mathsf{g} >$ instead of Γ. In other words, we relax the condition that the discrete group is an i.c.c. group. The case for non-compact manifolds has been studied in [Lüc02]. The extension to manifolds with boundary could be worked out as in [Ram93] and the discussed Γ-eta invariant reduces to Lott's delocalised eta-invariant: let $\mathcal{F}$ be the fundamental domain and $K(p, p; s)$ the

Schwartz kernel of $A\mathrm{e}^{-sA^2}$ for an elliptic Γ-invariant operator A, then

$$\eta_{<\mathsf{g}>}(A) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} \mathrm{Tr}_{<\mathsf{g}>} \left(A\mathrm{e}^{-sA^2} \right) \mathrm{d}s$$
$$\text{with} \quad \mathrm{Tr}_{<\mathsf{g}>} \left(A\mathrm{e}^{-sA^2} \right) = \sum_{\gamma \in \mathsf{g}} \int_{\mathcal{F}} \mathrm{tr}_p \left(K(p, \gamma p; s) \right) \mathsf{dvol}\,(p)$$

is well-defined for finite $< \mathsf{g} >$ or of polynomial growth if $|< \mathsf{g} >| = \infty$, see [Lot99]. One could also think about relaxing the condition that the Γ-action is the same on every slice, i.e. does depend on the time parameter. Let $\{\Gamma(t)\}_{t\in\mathcal{T}(M)}$ a smooth one-parameter family of groups of deck transformations; this change implies that the Γ-invariant partition of unity becomes time-dependent. The possible time-dependence of the left action representation on t has been already used in this thesis because the Dirac-wave evolution operators and all their following operators already have intertwined the Γ-actions on Hilbert Γ-modules for different times. It is an open question whether a time-depending Γ-invariant partition of unity can be used without further constraints which might be necessary for the proof of Proposition 7.3.1.

[C] Further possible modifications

Another conceivable generalisation of our result is to consider any other, not necessary self-adjoint Dirac type operator than just the Atiyah-Singer Dirac operator; hence a spatially L^2-index version of [BS20, Thm.5.4] becomes very attractive. This attempt would allow to introduce any complex twisting bundles. Besides the already proposed generalisations in section 7 of [BS19], one could also think about an odd-dimensional version, inspired by [Zad08, Thm.1] and based on [Fre96, Thm.B]. A geometric expression becomes redundant since the heat kernel expression in the integral over the odd-dimensional manifold and the eta-invariants on the even-dimensional boundary hypersurfaces vanish. Using another connection than the Levi-Civita connection would allow to extend the physical appplications from a general relativistic point of view to any modified geometric theory of gravity, based on other possible choices of a metric-affine connections. This kind of modifications would lead to another transgression form of the $\hat{A}$-genus as additional boundary integral in the non-product case. It is an open task how Fredholmness of the Dirac operator is affected.

At the end we want to point out that there is an alternative way in proving L^2-index theorems with respect to a Galois covering $M \to M_\Gamma$. Instead of working with the lifted Dirac operators on the covering, one can consider the operator on the compact base M_Γ, but one needs to twist the operator with the *Mishchenko-Fomenko bundle* which is the line bundle

$$M \times C^*(\Gamma)/\Gamma \to M_\Gamma$$

where $C^*(\Gamma)$ is the (reduced) C^*-algebra of Γ which is defined as follows: let $\mathbb{C}\Gamma$ denote the space of complex-valued functions on Γ with finite support which is a subset of $\ell^2(\Gamma)$. The algebra $C^*(\Gamma)$ is the norm closure of the image $l_\gamma(\mathbb{C}\Gamma)$ in the C^*-algebra $\mathscr{B}(\ell^2(\Gamma))$. This treatment is suitable for a K-theoretic approach and proof of an index theorem as one can work on compact manifolds. On the other hand it provides a base for *higher index theories* which has topological and geometric applications as the question of the existence of positive-scalar-curvature metrics on M wherefore the Cheeger-Gromov invariant becomes an important tool. We refer to [Lot92] for more background informations.

Backmatter

A. Gauss divergence theorem on pseudo-Riemannian manifolds

We give a brief recapitulation about the divergence theorem on pseudo-Riemannian manifolds. Let M be a $(n+1)$-dimensional manifold with smooth boundary $\mathfrak{d}M$ and pseudo-Riemannian metric g of any signature (r,s) such that $r+s=n+1$. If M is assumed to be fully oriented, it admits a $(n+1)$-form ω as volume form. An orientation on $\mathfrak{d}M$ can be introduced as follows: denote with $i:\mathfrak{d}M\hookrightarrow M$ the embedding of the boundary into the manifold. As the boundary is a smooth manifold, there exists a transverse vector field $\mathfrak{n}\in C^\infty(i^*(TM))$ on $\mathfrak{d}M$, i.e. for all $p\in\mathfrak{d}M$ the vector $\mathfrak{n}(p)$ lies in $T_p\mathfrak{d}M\subset T_pM$ and is not entirely tangent to $\mathfrak{d}M$. Thus a transverse vector field is roughly speaking a vector field in M which is everywhere inwards- or outwards-pointing on any connected component of $\mathfrak{d}M$. Conventionally, one chooses the outwards-pointing orientation. Such a vector field induces an orientation on the boundary by the interior product $\iota_\mathfrak{n}$:

$$\omega_{\mathfrak{d}M}=\iota_\mathfrak{n}\omega \quad .$$

If M has no orientation, one can only introduce a smooth density $\mathrm{d}\mu$ on M which induces a smooth density on the boundary: let $\mathfrak{n}$ be a transverse vector field and $\{e_j\}_{j=1}^n$ a basis on $T_p\mathfrak{d}M$ for $p\in\mathfrak{d}M$. The interior product on the density is then defined by

$$(\iota_\mathfrak{n}\,\mathrm{d}\mu)|_p(e_1,...,e_n):=\mathrm{d}\mu(\mathfrak{n}(p),e_1,...,e_n)$$

and is again a smooth density. It is positive, if $\mathrm{d}\mu$ does, and $C^\infty(M)$-linear: $\iota_\mathfrak{n}(f\,\mathrm{d}\mu)=(i^*f)\iota_\mathfrak{n}\,\mathrm{d}\mu$ for all $f\in C^\infty(M)$ and $\iota_{g\mathfrak{n}}\,\mathrm{d}\mu=|g|\,\iota_\mathfrak{n}\,\mathrm{d}\mu$ for all $g\in C^\infty(\mathfrak{d}M)$. As the whole construction does not involve any information about the pseudo-Riemannian metric we can fix the first step in direction to a divergence theorem on a pseudo-Riemannian manifold in the shape of the Stokes-Cartan theorem.

Theorem A.1 (Theorem 16.11 in [Lee13]). *Let M be an oriented smooth manifold of dimension $(n+1)$ with boundary inclusion $i:\mathfrak{d}M\hookrightarrow M$ and ν a compactly supported smooth $(n-1)$-form on M, then*

$$\int_M \nu=\int_{\mathfrak{d}M} i^*\nu \quad . \tag{A.1}$$

The assumption on orientability of M can be relaxed if we replace forms with smooth densities (see [Bot82, Thm.7.7]).

Now we take the pseudo-Riemannian metric g on M into account. One can distinguish different vector fields due to its causal character through this metric which induces a decomposition of the boundary into three pairwise disjoint components: let $\mathfrak{n}$ be a transverse vector field along $\mathfrak{d}M$; one defines

$$\begin{aligned} \mathfrak{d}M_{\gtrless 0} &:= \{p \in \mathfrak{d}M \,|\, g(\mathfrak{n}(p), \mathfrak{n}(p)) \gtrless 0\} \quad , \\ \mathfrak{d}M_0 &:= \{p \in \mathfrak{d}M \,|\, g(\mathfrak{n}(p), \mathfrak{n}(p)) = 0\} \quad , \\ \mathfrak{d}M &:= \mathfrak{d}M_{>0} \sqcup \mathfrak{d}M_{<0} \sqcup \mathfrak{d}M_0 \quad . \end{aligned} \tag{A.2}$$

$\mathfrak{d}M_{\gtrless 0}$ are both open subsets and $\mathfrak{d}M_0$ is closed (see [Ün95, Lem.3.3]). We only consider the case of a *non-degenerate boundary* where $\mathfrak{d}M_0 = \emptyset$. Thus, $g(\mathfrak{n}(p), \mathfrak{n}(p))$ is either positive or negative on each connected component. A metric allows to characterise transverse vectors to be orthonormal vectors to $\mathfrak{d}M$. As $\mathfrak{d}M$ is a codimension one submanifold, the normal space $N_p\mathfrak{d}M := (T_p\mathfrak{d}M)^\perp$ is one-dimensional at each point $p \in \mathfrak{d}M$. As non-degeneracy of the boundary implies that $N_p\mathfrak{d}M$ does not contain any lightlike vector, the transverse vector field is the unique outwards-pointing vector field, orthonormal to $\mathfrak{d}M$, which we refer on as normal vector field. A pseudo-Riemannian metric on the boundary can be induced in a unique way by pulling back the metric on M via the embedding: $g_{\mathfrak{d}M} := i^*g$. The pseudo-Riemannian volume density in local coordinates $x^0, x^1, ..., x^n$ is

$$\mathrm{d}\mu_g = \sqrt{|\det(g)|}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge x^n$$

and the normal vector field induces a unique pseudo-Riemannian volume density by means of

$$\mathrm{d}\mu_{g_{\mathfrak{d}M}} = \mathrm{d}\mu_{i^*g} = \iota_{\mathfrak{n}}\, \mathrm{d}\mu_g \quad . \tag{A.3}$$

We first focus on the orientable case. Let X be a compactly supported vector field on M. We want to apply Stokes theorem for $\nu = \iota_X\, \mathrm{d}\mu_g$. We can express the exterior derivative with the Lie derivative $\mathcal{L}_X$ with the help of Cartan's magic formula:

$$\mathrm{d}\nu = \mathrm{d} \circ \iota_X\, \mathrm{d}\mu_g = \mathcal{L}_X\, \mathrm{d}\mu_g - \iota_X \circ \mathrm{d}\, \mathrm{d}\mu_g = \mathcal{L}_X\, \mathrm{d}\mu_g = \mathrm{div}(X)\, \mathrm{d}\mu_g \quad .$$

Formula (A.1) then implies

$$\int_M \mathrm{div}(X)\, \mathrm{d}\mu_g = \int_{\mathfrak{d}M} i^*(\iota_X\, \mathrm{d}\mu_g) \quad . \tag{A.4}$$

The integrand on the right-hand side can be calculated as for Riemannian metrices (e.g. [Lee13, Lem.16.30]) and one obtains

$$i^*\iota_X\, \mathrm{d}\mu_g = \epsilon g(X, \mathfrak{n})\, \mathrm{d}\mu_{g_{\mathfrak{d}M}}$$

where $\epsilon = g(\mathfrak{n}, \mathfrak{n})$ is constant on $\mathfrak{d}M$ such that (A.4) transfers to

Proposition A.2 (cf. Theorem 3.7 in [Ün95]). *Let M be an oriented manifold with non-degenerate boundary $\mathfrak{d}M$ and pseudo-Riemannian metric g which induces a volume form $\mathrm{d}\mu_g$ on M. For a compactly supported vector field X one obtains*

$$\int_M \mathrm{div}(X)\, \mathrm{d}\mu_g = \epsilon \int_{\mathfrak{d}M} g(X, \mathfrak{n})\, \mathrm{d}\mu_{g_{\mathfrak{d}M}} \tag{A.5}$$

where $\mathfrak{n}$ is the outwards-pointing unit normal vector field along $\mathfrak{d}M$ and $\mathrm{d}\mu_{g_{\mathfrak{d}M}}$ the induced volume form on the boundary.

The decomposition $\partial M = \partial M_{>0} \sqcup \partial M_{<0}$ implies a splitting of the normal vector field into normal vector fields on each connected components:

$$\mathfrak{n} = \begin{cases} \mathfrak{n}_+ & \\ & \text{on} \\ \mathfrak{n}_- & \end{cases} \begin{matrix} \partial M_{>0} \\ \\ \partial M_{<0} \end{matrix} \quad \text{with} \quad g(\mathfrak{n}_\pm, \mathfrak{n}_\pm) = \pm 1 \quad \text{and} \quad g(\mathfrak{n}_\pm, \mathfrak{n}_\mp) = 0 \quad .$$

(A.5) then takes the form

$$\int_M \operatorname{div}(X)\, \mathrm{d}\mu_g = \int_{\partial M_{>0}} g(X, \mathfrak{n}_+)\, \mathrm{d}\mu_{g_{\partial M}} - \int_{\partial M_{<0}} g(X, \mathfrak{n}_-)\, \mathrm{d}\mu_{g_{\partial M}} \quad . \tag{A.6}$$

The extension to non-oriented pseudo-Riemannian manifolds can be justified as in the Riemannian case by using an orientation covering, which is a local isometry, and apply Proposition A.2 to the covering; see e.g. [Lee13, Thm.16.48] for the technical details which carry over to the pseudo-Riemannian case.

The Lorentzian case of our interest is covered as follows. M is a time- and space-oriented Lorentzian manifold which contains a spacelike Cauchy hypersurface, i.e. M is globally hyperbolic. Hence M is isometric to a topological cylinder manifold $\mathbb{R} \times \Sigma$. If we restrict the temporal domain from $\mathbb{R}$ to $[T, \infty)$ for a $T \in \mathbb{R}$, then M becomes a manifold with boundary $\partial M = \Sigma_T := \{T\} \times \Sigma$. We write dvol_M as volume form on M and dvol_{Σ_T} as volume form on the boundary. The global hyperbolicity of the Lorentzian manifold implies the existence of a global timelike unit vector field $\mathfrak{t}$. The divergence theorem (A.5) for a compactly supported vector field X on M becomes

$$\int_M \operatorname{div}(X)\, \mathrm{d}\mu_g = -\int_{\partial M_{<0}} g(X, \mathfrak{t})\, \mathsf{dvol}_{\Sigma_T} = -\int_{\Sigma_T} g(X, \mathfrak{t})\, \mathsf{dvol}_{\Sigma_T} \quad . \tag{A.7}$$

Up to this point there is still a freedome to choose which timelike direction (future, past) corresponds with the property of pointing outwards.

B. Well-posedness of the Cauchy problem for smooth initial data

This chapter provides some known results about the well-posedness of the Cauchy problem for the Dirac equation with smooth and compactly supported initial data for Section 7.1. We formulate the results for any operator P which is of Dirac-type, i.e. P^2 is normally hyperbolic (see (C.5))

Proposition B.1 (cf. Theorem 4 in [AB18])**.** *Let E and F be a smooth vector bundles over a globally hyperbolic manifold M with spacelike Cauchy hypersurface Σ and P a Dirac-type operator*

$$P : C^\infty(M,E) \to C^\infty(M,F) \quad .$$

For each $u_0 \in C^\infty_c(\Sigma, E|_\Sigma)$ and for each $f \in C^\infty_c(M,F)$ there exists a unique $u \in C^\infty(M,E)$ which solves the Cauchy problem

$$Pu = f \quad \text{on } M \quad \text{with} \quad u|_\Sigma = u_0 \quad ; \tag{B.1}$$

the solution depends continuously on the data (u_0, f).

If we take the foliation of M with a smooth familiy of Cauchy hypersurfaces $\{\Sigma_t\}$ into account, we can rephrase this result as follows: the Cauchy problem (B.1) is well-posed, i.e. for any, but fixed $t \in \mathcal{T}(M)$ the map

$$\begin{array}{rccc} \mathsf{res}_t \oplus P : & C^\infty(M,E) & \to & C^\infty_c(\Sigma_t, E_{\Sigma_t}) \oplus C^\infty_c(M,F) \\ & u & \mapsto & (u|_{\Sigma_t}, Pu) \end{array} \tag{B.2}$$

extends to an isomorphism of topological vector spaces. This proposition is a consequence of the well-posedness of the Cauchy problem for the linear wave equation of normally hyperbolic operators which is applied to P^2; see e.g. [AB18, Thm.2] or the main source [BGP07, Thm.3.2.11].

A finite propagation speed result can be deduced from the one of the wave equation.

Corollary B.2. *The solution u in the situation of Proposition B.1 satisfies* $\mathrm{supp}\,(u) \subset \mathcal{J}(K)$ *where K is the compact subset* $\mathrm{supp}\,(u_0) \cup \mathrm{supp}\,(f)$*.*

We will apply these results to the spin-Dirac operators $P = D_\pm$, $E = \mathcal{S}^\pm(M)$, $F = \mathcal{S}^\mp(M)$ and $E|_{\Sigma_t} = \mathcal{S}^\pm(\Sigma_t)$ which we will also equip with a twisting bundle.

C. Solution operators for initial value problems of normally hyperbolic operators

This appendix is dedicated to solutions of the Cauchy Problem

$$Pu = 0 \quad \text{and} \quad \mathsf{res}_\Sigma Q_j u = Q_j u|_\Sigma = g_j \tag{C.1}$$

where $P \in \Psi^m(M)$, $Q_j \in \Psi^{m_j}(M)$ and $g_j \in C_\mathsf{c}^{-\infty}(\Sigma)$, $j \in \{0, ..., m-1\}$, are initial values on the initial hypersurface Σ of a manifold M. The Q_j are operators which generates the intial values. The aim is to express u with solution operators, applied to the initial values g_j in (C.1). These solution operators are represented as FIOs and we recapitulate some facts and results from [Dui10, Chap.5]. In order to do so, one needs to impose several conditions on P and Q_j: the bicharacteristics from the principal symbol $\boldsymbol{\sigma}_m(P)$ needs to be transversal to the initial hypersurface with $\dim(\mathsf{ker}\,(\boldsymbol{\sigma}_m(P)(x_0, \xi_0))) = \mu$ for all $(x_0, \xi_0) \in \dot{T}^*\Sigma$. Such an operator is called *strictly hyperbolic* of multiplicity μ with respect to Σ. The value μ counts the number of solutions of

$$\boldsymbol{\sigma}_m(P)(x_0, \xi) = 0 \quad \text{and} \quad \xi|_{T_{x_0}\Sigma} = \xi_0 \quad . \tag{C.2}$$

The transversality of the bicharacteristics is equivalent to the condition that zeroes of the principal symbol at (x_0, ξ) are simple and non-vanishing on the orthogonal complement of $T^*_{x_0}\Sigma$; in other words: the initial hypersurface is non-characteristic and the number of simple zeroes is finite. Further assumptions has to be made to guarantee well-defined compositions of the solution operators with Q_j as pseudo-differential operators on the hypersurface. The solution operators should have a canonical relation of the form

$$\mathsf{C} := \left\{ (y, \eta, x_0, \xi_0) \in \dot{T}^*M \times \dot{T}^*\Sigma \,\middle|\, (x_0, \xi) \rightsquigarrow (y, \eta) \;:\; \text{(C.2) holds} \right\} \tag{C.3}$$

where $(x_0, \xi) \rightsquigarrow (y, \eta)$ denotes that (y, η) is connected with a bicharacteristic through (x_0, ξ). It is an embedded submanifold of $\dot{T}^*(M \times \Sigma)$ if in addition the following conditions hold:

(1) (transversality) every bicharacteristic curve of P intersects Σ at most once;

(2) (properness) for all $K \Subset M$ exists a $K_0 \Subset \Sigma$ such that each bicharacteristic curve, starting in K, hits Σ in K_0;

(3a) no bicharacteristic curve, starting on Σ, stays in a compact region in M;

(3b) (pseudo-convexity) for all $K_0 \Subset \Sigma$ and $K \Subset M$ exists a $K' \Subset M$ such that a segment of the bicharacteristic curve, connecting one point in Σ and one in M, lies inside K'.

Another necessary condition takes the initial value operators Q_j into account:

(4) the principal symbols of Q_j are non-singular for any $(x_0, \xi^{(j)}(x_0,\xi_0)) \in \dot{T}^*\Sigma$ such that (C.2) holds for any solution $\xi^{(j)}$, $j \in \{1, ..., \mu\}$.

[Dui10, Thm.5.1.2] states that there exist solution operators $\mathcal{G}_k \in \mathcal{FIO}^{-m_k-1/4}(\Sigma, M; \mathsf{C}')$ with canonical relation (C.3) for strictly hyperbolic pseudo-differential operators, satisfying conditions (1) up to (4). A solution u of (C.1) can be expressed by means of $u = \sum_{j=1}^{\mu} \mathcal{G}_j g_j$.

We restrict our attention to scalar-valued differential operators P of order m and (C.1) as inhomogenous initial value problem, i.e. $Q_j = (\nabla_{\partial_t})^j$ are the initial value operators for $j \in \{1, ..., m\}$:

$$\begin{aligned} Pu &= f \quad \text{in} \quad M \\ \mathsf{res}_{\Sigma_0}(\nabla_{\partial_t})^j u &= g_j \quad \text{for} \quad j \in \{1, ..., m\} \end{aligned} \tag{C.4}$$

where Σ_0 is the initial hypersurface, $g_j \in C^\infty(\Sigma_0)$ the initial values and $f \in C^\infty(M)$ the inhomogeneity. The manifold M is an open subset in $\mathbb{R} \times \Sigma$ and $\Sigma_t \subset M$ is a slice of the form $\Sigma_t = (\{t\} \times \Sigma) \cap M$ for any $t \in \mathbb{R}$. A differential operator is called *strictly hyperbolic* (of multiplicity $\mu = m$) with respect to Σ_t if (1) up to (3) hold. The principal symbols of the initial value operators are given by $q_j(x,\xi) := \boldsymbol{\sigma}_j(Q_j)(x,\xi) = (\mathbb{i}\xi(\partial_t))^j|_x$ for $x \in M, \xi \in \dot{T}^*_x M$. The matrix $q_j(x_0, \xi^{(k)}(x_0,\xi_0))$ is a Vandermonde matrix for $(x_0, \xi^{(k)})$ which solves (C.2). Its determinant vanishes if all $\xi^{(k)}(\partial_t)$ for $k \in \{1, ..., m\}$ are the same. But since P is a differential operator of order m, its principal symbol is a polynomial of order m in ξ and by strict hyperbolicity the multiplicity equals the order of the polynom wherefore all roots of the principal symbol are different. Thus, (4) is satisfied.

[Dui10, Lem.5.1.3] and [Dui10, Lem.5.1.4] assure the existence of a neighbourhood $\mathcal{U}_p$ of any point $p \in M$ and a family of continuous mappings $\mathcal{G}(t)$, depending smoothly on t, and solution operators $\mathcal{G}_j$ for Q_j such that

$$\begin{aligned} P\mathcal{G}(t)f &= 0 \quad \text{in} \quad \mathcal{U}_p \\ \mathsf{res}_{\Sigma_t}(\nabla_{\partial_t})^j \mathcal{G}(t) &= 0 \quad \text{for} \quad j \in \{0, ..., m-1\}\,, \, t \in \mathbb{R}\,. \end{aligned}$$

This leads to the following result where $\mathscr{D}(p)$ denotes a neighbourhood at $p \in M$, containing all points, which can be reached from p with curves γ such that $\dot{\gamma}$ is a tangent vector of the bicharacteristics through p, pointing downwards along a cone in T_pM.

Theorem C.1 (cf. Theorem 5.1.6 in [Dui10]). *Let $M \mathring{\subset} \mathbb{R} \times \Sigma$ and $P \in \mathsf{Diff}^m(M)$ be strictly hyperbolic with respect to Σ_t for all $t \in \mathbb{R}$ and suppose $\mathscr{D}(p) \cap \Sigma_0$ is compact for every $p \in M$, then (C.4) has a unique solution u for every $f \in C^\infty_{\mathsf{c}}(M)$ and $g_j \in C^\infty(\Sigma_0)$ such that*

$$u = \mathcal{G}f + \sum_{j=0}^{m} \mathcal{G}_j g_j$$

where the solution operators $\mathcal{G}$ and $\mathcal{G}_j$ are continuous mappings from $C^\infty(M)$ to $C^\infty(M)$ and respectively $C^\infty_{\mathsf{c}}(\Sigma_0)$ to $C^\infty(M)$ with the properties

(1) $\operatorname{supp}(\mathcal{G}) \subset \{(p,q) \in M \times M \mid q \in \mathscr{D}(p)\}$,

(2) $\operatorname{supp}(\mathcal{G}_j) \subset \{(p,x) \in M \times \Sigma_0 \mid x \in \mathscr{D}(p) \cap \Sigma_0\}$ *and*

(3) $\mathcal{G}_j \in \mathcal{FIO}^{-j-1/4}(\Sigma_0, M; \mathsf{C}')$ *with* C *as in* (C.3).

supp $(\mathcal{G}_j)$ and supp $(\mathcal{G})$ refer to the support of the corresponding kernels. Theorem C.1 can be extended to differential operators with scalar-valued principal symbol or to (any system of) differential operators of real principal type where the principal symbol is real and homogeneous of order m (= order of the differential operator) and no (complete) bicharacteristic strip stays over a compact set. If the differential operator acts between two vector bundles on M with the same rank r or is given as r-quadratic system of scalar-valued differential operators, the principal symbol is an r-quadratic matrix and the determinant defines the characteristic symbol.

A special situation arises if M is globally hyperbolic with Lorentzian metric g, $E \to M$ a vector bundle, Σ a Cauchy hypersurface and $P \in \mathsf{Diff}^2(M, \mathsf{End}(E))$ is *normally hyperbolic*, i.e. its principal symbol is determined by the metric:

$$\boldsymbol{\sigma}_2(P)(p,\xi) = \pm g_p(\xi^\sharp, \xi^\sharp)\mathbb{1}_E \quad . \tag{C.5}$$

The vanishing of the principal symbol corresponds to the vanishing of $g_p(\xi^\sharp, \xi^\sharp)$ at each point $p \in M$ which is fulfilled for $\xi^\sharp$, being a lightlike vector at x. The bicharacteristic strips are determined by Hamilton's equations for the Hamilton function $1/2 g_p(\xi^\sharp, \xi^\sharp)$ which can be reduced to the geodesic equation on M. Thus, the bicharacteristic curves (projection of the bicharacteristic strip on M) are given by lightlike geodesics. Since normally hyperbolic operators have scalar-valued principal symbol times identity, one can try to apply Theorem C.1: the domain $\mathscr{D}(p)$ corresponds to the past causal domain $\mathcal{J}^-(p)$ at p. As $\mathcal{J}^-(p) \subset \mathcal{J}(p)$, it is spatial compact for any point in M and the intersection with any Cauchy hypersurface is compact wherefore $\mathcal{J}^-(p) \cap \Sigma_0$ is compact for every $p \in M$. The global hyperbolicity of M is equivalent to M being causal and strongly causal, see Theorem 3.1.3. The strong causality impose that, if any causal geodesic is confined inside a compact region of M, it has already its endpoints inside this compact region. Otherwise the Picard-Lindelöf theorem would imply that the time-domain for the lightlike geodesics is $\mathbb{R}$ and thus the geodesics become complete. Their tangent vectors tend to zero if the parametrisation tends to $\pm\infty$. Since any geodesic parallel transport is determined by its own tangent vector, it implies that the tangent vectors are already vanishing for all times wherefore the geodesics must be constant and thus a contradiction to the assumption. Theorem 3.1.3 also says that the intersection of a future and past light cone of different points is compact: $\mathcal{J}^+(p) \cap \mathcal{J}^-(q)$ compact for all $p, q \in M$. This holds true for any two compact subsets of M as well wherefore one has for $K \Subset M$ also $\mathcal{J}^+(K) \cap \mathcal{J}^-(K) =: K'$ compact. If a segment of a lightlike geodesic has its endpoints in a compact region K which contains a compact domain $K_0 \subset \Sigma$, from which it starts, then the curve stays inside K'. This proves strict hyperbolicity of normally hyperbolic operators and thus (3a) and pseudo convexity (3b) are satisfied; see [Rad96, Prop.4.3/4.4] for these facts. Transversality (1) follows from the global hyperbolicity of M which implies a Cauchy temporal function whose level sets are Cauchy hypersurfaces and any inextendable causal curve crosses any Σ_t once. Thus, the bicharacteristic curves as lightlike geodesic curves intersect the initial slice Σ_0 at most once. Properness (2) follows from the same causal diamond argument: if we take any $K \Subset M$, then also $\mathcal{J}^+(K) \cap \mathcal{J}^-(K)$ is compact. Define $K_0 := (\mathcal{J}^+(K) \cap \mathcal{J}^-(K)) \cap \Sigma_0$ which is compact as $\mathcal{J}^+(K) \cap \mathcal{J}^-(K))$ is compact and Σ_0 is closed. Any lightlike geodesic starting in K will hit Σ_0 in K_0.

Thus, all conditions from (1) to (3) on the bicharacteristics are proven; we have already seen that (4) is satisfied for initial value problems of differential operators. The canonical relation in (C.3) for the solution operators can be rewritten to

$$\mathcal{C} := \left\{ (x, \xi, y, \eta) \in \dot{T}^*M \times \dot{T}^*\Sigma_0 \,\middle|\, (x, \xi) \sim (y, \eta) \right\} \tag{C.6}$$

where $(x, \xi) \sim (y, \eta)$ means that there is a lightlike vector $\zeta \in T^*_y M$ such that the points (x_0, ξ) and (y, ζ) are on the same lightlike geodesic strip (i.e. are connected by the same orbit of the null geodesic flow in M) with $\mathsf{res}^*_{\Sigma_0}\zeta = \eta$. This lightlike vector ζ is either future- or past-directed if $g(\zeta, \partial_t) < 0$ respectively $g(\zeta, \partial_t) > 0$. Its pullback to the initial hypersurface gives the same covector η. In order to distinguish both directions one considers ξ to be a future- or past-directed lightlike vector such that (x, ξ) and (y, ζ) are connected by a null geodesic from past to future or vice versa. This allows to decompose $\mathcal{C}$ into two connected components $\mathcal{C}^{\pm}$:

$$\mathcal{C}^+ = \{(x, \xi, y, \eta) \in \mathcal{C} \,|\, \xi \rhd 0\} \quad \text{and} \quad \mathcal{C}^- = \{(x, \xi, y, \eta) \in \mathcal{C} \,|\, \xi \lhd 0\} \tag{C.7}$$

where $\xi \rhd 0$ means that $(+\xi)$ is a future-directed lightlike covector and $\xi \lhd 0$ means that ξ past-directed lightlike covector or equivalently $(-\xi)$ a future-directed. This decomposition can be used to trivialise the Keller-Maslov line bundle and the half-density bundle over the canonical relation which is explained in the appendix of [BS19].

After all this details and conclusions we have finally recovered [BS19, Thm.A.1] with further needed details.

Theorem C.2. *Let M be a globally hyperbolic manifold with Cauchy hypersurface Σ and $E \to M$ a vector bundle; the Cauchy problem for a normally hyperbolic operator $P \in \mathsf{Diff}^2(M, \mathsf{End}(E))$ with initial hypersurface Σ_0,*

$$\begin{aligned} Pu &= 0 \quad \textit{in} \quad M \\ \mathsf{res}_{\Sigma_0}(\nabla_{\partial_t})^j u &= g_j \quad \textit{for} \quad j \in \{0, 1\}, \end{aligned}$$

has a unique solution u for every $g_j \in C^\infty_{\mathsf{c}}(\Sigma_0)$ such that $u = \sum_{j=0}^m \mathcal{G}_j g_j$ where the solution operators $\mathcal{G}_j$ are continuous mappings from $C^\infty_{\mathsf{c}}(\Sigma_0, E|_{\Sigma_0})$ to $C^\infty(M, E)$ with the properties

(1) $\operatorname{supp}(\mathcal{G}_j) \subset \{(p, x) \in M \times \Sigma_0 \,|\, x \in \mathcal{J}^-(p) \cap \Sigma_0\}$ *and*

(2) $\mathcal{G}_j \in \mathcal{FIO}^{-j-1/4}(\Sigma_0, M; \mathcal{C}'; \mathbf{Hom}(E|_{\Sigma_0}, E))$ *with $\mathcal{C}$ as in (C.6).*

D. Auxiliary calculation

Let $\mathtt{c}(E,\nabla)$ be any characteristic class of a vector bundle $E \to M'$ over a manifold M' with connection ∇ such that it is defined by the curvature Ω_∇ with respect to ∇. Given another connection ∇' on E, the difference of $\mathtt{c}(E,\nabla)$ and $\mathtt{c}(E,\nabla')$ is in the same cohomology class and thus there exists a form $\mathtt{w}(\nabla,\nabla')$ (transgression form) with

$$\mathtt{c}(E,\nabla') - \mathtt{c}(E,\nabla) = \mathrm{d}\mathtt{w}(\nabla,\nabla') \quad .$$

Let $f : M \to M'$ be any continuous map between M and M'. We can define the pullback bundle f^*E and the pullback connection $f^*\nabla$ according to (2.13) and (2.15). The naturality of characteristic classes, i.e.

$$f^*\mathtt{c}(E,\nabla) = \mathtt{c}(f^*E, f^*\nabla) \quad ,$$

implies

$$f^*\left[\mathtt{c}(E,\nabla') - \mathtt{c}(E,\nabla)\right] = \mathtt{c}(f^*E, f^*\nabla') - \mathtt{c}(f^*E, f^*\nabla) = \mathrm{d}(f^*\mathtt{w}(\nabla,\nabla')) \quad . \tag{D.1}$$

The curvature $\Omega_{f^*\nabla}$ with respect to the pullback connection is the pullback of the curvature with respect to ∇, i.e.

$$\Omega_{f^*\nabla}(X,Y)f^*u = f^*\left[\Omega_\nabla(f_*X, f_*Y)u\right]$$

where X, Y are vectors on M' and u a differentiable section of $E \to M'$. It indicates

$$f^*\left[\mathtt{c}(E,\nabla') - \mathtt{c}(E,\nabla)\right] = \mathtt{c}(f^*E, f^*\nabla') - \mathtt{c}(f^*E, f^*\nabla) = \mathrm{d}\mathtt{w}(f^*\nabla, f^*\nabla') \quad . \tag{D.2}$$

The difference of (D.1) and (D.2) shows that

$$\mathrm{d}\left(f^*\mathtt{w}(\nabla,\nabla') - \mathtt{w}(f^*\nabla, f^*\nabla')\right) = 0 \quad ;$$

hence there exists a form $\mathtt{v}$ such that

$$f^*\mathtt{w}(\nabla,\nabla') - \mathtt{w}(f^*\nabla, f^*\nabla') = \mathrm{d}\mathtt{v} \quad . \tag{D.3}$$

Bibliography

[AB18] L. Andersson and C. Bär. Wave and Dirac equations on manifolds. *Space – Time – Matter*, page 324–348, Apr 2018.

[ABP73] M. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem. *Inventiones mathematicae*, 19(4):279–330, Dec 1973.

[AG83] L. Alvarez-Gaumé. Supersymmetry and the Atiyah-Singer index theorem. *Communications in Mathematical Physics*, 90(2):161 – 173, 1983.

[AIMSII68] M. F. Atiyah, M. F. Atiyah I. M. Singer (I, III), and G. B. Segal (II). The Index of Elliptic Operators: I,II,III. *Annals of Mathematics*, 87(3):484–604, 1968.

[APS75a] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian Geometry. I. *Mathematical Proceedings of the Cambridge Philosophical Society*, 77(1):43–69, 1975.

[APS75b] M. F Atiyah, V. K Patodi, and I. M Singer. Spectral asymmetry and Riemannian geometry. II. *Mathematical proceedings of the Cambridge Philosophical Society*, 78(3):405–432, 1975.

[APS76] M. F Atiyah, V. K Patodi, and I. M Singer. Spectral asymmetry and Riemannian geometry. III. *Mathematical proceedings of the Cambridge Philosophical Society*, 79(1):71–99, 1976.

[AS63] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bulletin of the American Mathematical Society*, 69(3):422 – 433, 1963.

[AS69] M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators. *Publications Mathématiques de l'IHÉS*, 37:5–26, 1969.

[AS71] M. F. Atiyah and I. M. Singer. The Index of Elliptic Operators: IV,V. *Annals of Mathematics*, 93(1):119–149, 1971.

[ASS94] J. Avron, R. Seiler, and B. Simon. The Index of a Pair of Projections. *Journal of Functional Analysis*, 120(1):220–237, 1994.

[Ati76] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque analyse et topologie*, number 32-33 in Astérisque. Société mathématique de France, 1976.

[AW11] S. Azzali and C. Wahl. Spectral flow, index and the signature operator. *Journal of Topology and Analysis*, 03(01):37–67, 2011.

[Bau81] H. Baum. *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*. Seminarbericht: Sektion Mathematik. Teubner, 1981.

[BB11] C. Bär and W. Ballmann. Boundary Value Problems for Elliptic Differential Operators of First Order. *arXiv e-prints*, page arXiv:1101.1196, January 2011.

[BBLP05] B. Booss-Bavnbek, M. Lesch, and J. Phillips. Unbounded Fredholm Operators and Spectral Flow. *Canadian Journal of Mathematics*, 57(2):225–250, 2005.

[BCLR20] C. Bourne, A. L. Carey, M. Lesch, and A. Rennie. The KO-valued spectral flow for skew-adjoint Fredholm operators. *Journal of Topology and Analysis*, 14(02):505–556, Sep 2020.

[BCP$^+$06] M. T. Benameur, A. Carey, J. Phillips, A. Rennie, F. Sukochev, and K. Wojciechowski. *An Analytic Approach to Spectral Flow in von Neumann Algebras*, pages 297–352. Jan 2006.

[BF86] J. M. Bismut and D. S. Freed. The analysis of elliptic families. *Communications in Mathematical Physics*, 107(1):103–163, Mar 1986.

[BGM05] C. Bär, P. Gauduchon, and A. Moroianu. Generalized cylinders in semi-Riemannian and spin geometry. *Mathematische Zeitschrift*, 249(3):545–580, Jan 2005.

[BGP07] C. Bär, N. Ginoux, and F. Pfäffle. *Wave Equations on Lorentzian Manifolds and Quantization*. ESI lectures in Mathematics and Physics. European Mathematical Society, 2007.

[BGV03] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. Springer Berlin Heidelberg, 2003.

[BH18] C. Bär and S. Hannes. Boundary Value Problems for the Lorentzian Dirac Operator. In *Geometry and Physics: Volume I: A Festschrift in honour of Nigel Hitchin*. Oxford University Press, Oct 2018.

[Bla06] B. Blackadar. *Operator Algebras : Theory of C*-Algebras and von Neumann Algebras*. Encyclopaedia of Mathematical Sciences 122. Berlin, Heidelberg : Springer, 2006.

[BM17] M. Braverman and G. Maschler. Equivariant APS index for Dirac operators of non-product type near the boundary. *Indiana University Mathematics Journal*, 68, Feb 2017.

[BMS02] M. Braverman, O. Milatovic, and M. Shubin. Essential self-adjointness of Schrödinger-type operators on manifolds. *Russian Mathematical Surveys*, 57(4):641–692, Aug 2002.

[Bot82] R. Bott. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics, 82. 1st ed. 1982. edition, 1982.

[BR15] M. T. Benameur and I. Roy. The Higson–Roe exact sequence and ℓ^2-eta invariants. *Journal of Functional Analysis*, 268, Feb 2015.

[Bra20] M. Braverman. An index of strongly Callias operators on Lorentzian manifolds with non-compact boundary. *Mathematische Zeitschrift*, 294(1):229–250, Feb 2020.

[Bre68] M. Breuer. Fredholm theories in von Neumann algebras. I. *Mathematische Annalen*, 178(3):243–254, Sep 1968.

[Bre69] M. Breuer. Fredholm theories in von Neumann algebras. II. *Mathematische Annalen*, 180(4):313–325, Dec 1969.

[Bro54] F. E. Browder. The Eigenfunction Expansion Theorem for the General Self-Adjoint Singular Elliptic Partial Differential Operator. I. The Analytical Foundation. *Proceedings of the National Academy of Sciences of the United States of America*, 40(6):454–459, 1954.

[BS78] R. Bott and R. Seeley. Some remarks on the paper of Callias: "Axial anomalies and index theorems on open spaces. *Communications in mathematical physics*, 62(no. 3):235–245, 1978.

[BS05] A. N. Bernal and M. Sánchez. Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes. *Communications in Mathematical Physics*, 257(1):43–50, 2005.

[BS06] A. N. Bernal and M. Sánchez. Further Results on the Smoothability of Cauchy Hypersurfaces and Cauchy Time Functions. *Letters in Mathematical Physics*, 77(2):183–197, 2006.

[BS07] A. N. Bernal and M. Sánchez. Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal'. *Classical and Quantum Gravity*, 24(3):745–749, Jan 2007.

[BS16] C. Bär and A. Strohmaier. A Rigorous Geometric Derivation of the Chiral Anomaly in Curved Backgrounds. *Communications in Mathematical Physics*, 347(3):703–721, May 2016.

[BS19] C. Bär and A. Strohmaier. An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary. *American Journal of Mathematics*, 141:1421–1455, Oct 2019.

[BS20] C. Bär and A. Strohmaier. Local Index Theory for Lorentzian Manifolds. *arXiv e-prints*, page arXiv:2012.01364, Dec 2020.

[BTW15] C. Bär and R. Tagne Wafo. Initial Value Problems for Wave Equations on Manifolds. *Mathematical Physics Analysis and Geometry*, 18, Aug 2015.

[Bä05] C. Bär. In *Dirac operators: yesterday and today*. International Press of Boston, Inc., Jan 2005.

[Bä09] C. Bär. Spectral bounds for Dirac operators on open manifolds. *Annals of Global Analysis and Geometry*, 36(1):67–79, Jan 2009.

[Bä14] C. Bär. Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes. *Communications in Mathematical Physics*, 333(3):1585–1615, Jul 2014.

[Cal78] C. Callias. Axial anomalies and index theorems on open spaces. *Communications in mathematical physics*, 62(3):213–234, 1978.

[CG85] J. Cheeger and M. Gromov. Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds. *Journal of Differential Geometry*, 21(1):1 – 34, 1985.

[CG91] J. Cheeger and M. Gromov. Chopping Riemannian manifolds. pages 85–94, Berlin, Heidelberg, 1991.

[CP99] A. L. Carey and J. Phillips. Spectral flow in Breuer-Fredholm modules. In *Joint Australian-Taiwanese workshop on analysis and applications, Brisbane, Australia, March 31 – April 04, 1997*, pages 99–108. Canberra: Australian National University, Centre for Mathematics and its Applications, 1999.

[CPSB19] A. Carey, J. Phillips, and H. Schulz-Baldes. Spectral flow for skew-adjoint Fredholm operators. *Journal of spectral theory*, 9(1):137–170, 2019.

[Dam21] O. Damaschke. Atiyah-Singer Dirac Operator on spacetimes with non-compact Cauchy hypersurface. *arXiv:2107.08532*, 2021.

[DH72] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta mathematica*, 128:183–269, 1972.

[Don78] H. Donnelly. Eta Invariants for G-Spaces. *Indiana University Mathematics Journal*, 27(6):889–918, 1978.

[DP86] L. Dąbrowski and R. Percacci. Spinors and diffeomorphisms. *Comm. Math. Phys.*, 106(4):691–704, 1986.

[Dui10] J.J. Duistermaat. *Fourier Integral Operators*, volume 130 of *Modern Birkhäuser Classics*. Birkhäuser, Boston, 1. aufl. edition, 2010.

[EM98] C. Epstein and R. Melrose. Contact degree and the index of Fourier integral operators. *Mathematical research letters*, 5(3):363–381, 1998.

[Fre96] D. S. Freed. Two nontrivial index theorems in odd dimensions. *arXiv:dg-ga/9601005*, Jan 1996.

[FS04] K. Fujikawa and H. Suzuki. *Path Integrals and Quantum Anomalies.* Oxford University Press, Apr 2004.

[Ger70] R. P. Geroch. Domain of Dependence. *Journal of Mathematical Physics*, 11(2):437–449, 1970.

[Get83] E. Getzler. Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. *Communications in mathematical physics*, 92(2):163–178, 1983.

[GH97] D. Garfinkle and S. G. Harris. Ricci fall-off in static and stationary, globally hyperbolic, non-singular spacetimes. *Classical and Quantum Gravity*, 14(1):139–151, Jan 1997.

[Gil81] P. B. Gilkey. The residue of the global η-function at the origin. *Advances in Mathematics*, 40(3):290–307, 1981.

[Gin09] N. Ginoux. *The Dirac Spectrum*, volume 1976 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[GL83] M. Gromov and H. B. Lawson. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Publications Mathématiques de l'IHÉS*, 58:83–196, 1983.

[Goe00] S. Goette. Equivariant η-invariants and η-forms. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2000:181–236, Jan 2000.

[HÖ3] L. Hörmander. *The Analysis of Linear Partial Differential Operators I : Distribution Theory and Fourier Analysis.* Classics in Mathematics 256. Springer Berlin Heidelberg, second edition edition, 2003.

[HÖ7] L. Hörmander. *The Analysis of Linear Partial Differential Operators III : Pseudo-Differential Operators.* Classics in Mathematics. Springer Berlin Heidelberg, 2nd ed. edition, 2007.

[HÖ9] L. Hörmander. *The Analysis of Linear Partial Differential Operators IV : Fourier Integral Operators.* Classics in Mathematics. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[Haa06] M. Haase. *The Functional Calculus for Sectorial Operators.* Operator Theory: Advances and Applications 169. 2006.

[Heb96] E. Hebey. *Sobolev Spaces on Riemannian Manifolds.* Number Nr. 1635 in Lecture Notes in Mathematics. Springer, 1996.

[Hö71] L. Hörmander. Fourier integral operators. I. *Acta mathematica*, 127:79–183, 1971.

[Ike05] A. Ikemakhen. Parallel Spinors on Lorentzian Spin^c Manifolds. 2005. Accessed: 20-11-2022.

[IS20] O. Islam and A. Strohmaier. On microlocalization and the construction of Feynman propagators for normally hyperbolic operators. *arXiv:2012.09767*, 2020.

[Kad83] R. V. Kadison. *Fundamentals of the theory of operator algebras.: Elementary theory.* New York : Academic Press, New York, 1983. Includes bibliographical references and indexes.

[Kat76] T. Kato. *Perturbation Theory for Linear Operators.* Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics 132. 2 edition, 1976.

[Kor91] Y. Kordyukov. L^p-Theory of elliptic differential operators on manifolds of bounded geometry. *Acta Applicandae Mathematicae*, 23:223–260, Jan 1991.

[Kor00] Y. Kordyukov. L^p-Estimates for Functions of Elliptic Operators on Manifolds of Bounded Geometry. *Russian Journal of Mathematical Physics, v.7, 216-229 (2000)*, 7, Apr 2000.

[KR83] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras : Volume 2. Advanced theory.* Advanced theory. New York: Academic Press, 1983.

[Lee11] J. M. Lee. *Introduction to Topological Manifolds.* Graduate texts in Mathematics,202. Springer, 2011.

[Lee13] J. M. Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics ; 218. New York Heidelberg Dordrecht London: Springer, 2nd ed. edition, 2013.

[Lee19] J. M. Lee. *Introduction to Riemannian Manifolds.* Graduate Texts in Mathematics. Springer International Publishing, 2019.

[Les04] M. Lesch. The uniqueness of the spectral flow on spaces of unbounded self–adjoint Fredholm operators. *Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds, AMS Contemporary Math Proceedings*, 366, Feb 2004.

[LM16] H. B. Lawson and M. L. Michelson. *Spin Geometry (PMS-38).* Princeton mathematical series 38. Princeton, NJ : Princeton University Press, 2016.

[LNT01] E. Leichtnam, R. Nest, and B. Tsygan. Local Formula for the Index of a Fourier Integral Operator. *Journal of Differential Geometry*, 59(2):269 – 300, 2001.

[Loi92] P. H. Loi. On the theory of index for type III factors. *Journal of Operator Theory*, 28(2):251–265, 1992.

[Lot92] J. Lott. Superconnections and higher index theory. *Geometric & Functional Analysis GAFA*, 2(4):421–454, Dec 1992.

[Lot99] J. Lott. Delocalized L^2-Invariants. *Journal of Functional Analysis*, 169(1):1–31, 1999.

[LS95] E. C. Lance and London Mathematical Society. *Hilbert C*-Modules: A Toolkit for Operator Algebraists.* Lecture note series / London Mathematical Society. Cambridge University Press, 1995.

[Lüc02] W. Lück. The relation between the Baum-Connes Conjecture and the Trace Conjecture. *Inventiones mathematicae*, 149(1):123–152, Jul 2002.

[McC92] J. D. McCrea. Irreducible decompositions of nonmetricity, torsion, curvature and Bianchi identities in metric-affine spacetimes. *Classical and Quantum Gravity*, 9(2):553, Feb 1992.

[Mor91] H. Mori. The Riemannian double of a manifold and eigenspaces of the Laplacian. *Mathematics journal of Toyama University*, 14:67–80, 1991.

[MvN36] F. J. Murray and J. v. Neumann. On Rings of Operators. *Annals of Mathematics*, 37(1):116–229, 1936.

[NS] T. M. W. Nye and M. A. Singer. An L^2-Index Theorem for Dirac Operators on $S^1 \times R^3$. *arXiv:math/0009144.*

[O'N83] B. O'Neill. *Semi-Riemannian geometry : with applications to relativity.* Pure and applied mathematics 103. Academic Press, San Diego New York Boston London Sydney Tokyo Toronto, 1983.

[Phi96] J. Phillips. Self-Adjoint Fredholm Operators And Spectral Flow. *Canadian Mathematical Bulletin*, 39(4):460–467, 1996.

[Pia93] P. Piazza. On the Index of Elliptic Operators on Manifolds with Boundary. *Journal of Functional Analysis*, 117(2):308–359, 1993.

[PS07] P. Piazza and T. Schick. Bordism, rho-invariants and the Baum–Connes conjecture. *Journal of Noncommutative Geometry*, pages 27–111, 2007.

[Rad96] M. J. Radzikowski. Micro-local approach to the Hadamard condition in Quantum Field Theory on curved space-time. *Comm. Math. Phys.*, 179(3):529–553, 1996.

[Ram93] M. Ramachandran. von Neumann index theorems for manifolds with boundary. *J. Differential Geom.*, 38(2):315–349, 1993.

[Roe88a] J. Roe. An index theorem on open manifolds. I. *Journal of Differential Geometry*, 27(1):87 – 113, 1988.

[Roe88b] J. Roe. An index theorem on open manifolds. II. *Journal of Differential Geometry*, 27(1):115 – 136, 1988.

[Roe99] J. Roe. *Elliptic Operators, Topology, and Asymptotic Methods, Second Edition*. Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis, 1999.

[Ron19] L. Ronge. Index Theory for Globally Hyperbolic Spacetimes. *arXiv:1910.10452*, 2019.

[SA11] M. A. Shubin and S. I. Andersson. *Pseudodifferential Operators and Spectral Theory*. Springer Berlin Heidelberg, 2011.

[Sch05] T. Schick. L^2-index theorems, KK-theory, and connections, New York. *New York J. Math.*, 11:387–443, 01 2005.

[See69] R. Seeley. The Resolvent of an Elliptic Boundary Problem. *American Journal of Mathematics*, 91(4):889–920, 1969.

[Shi] P. Shi. Boundary value problems for Dirac-type operators and an application. https://ckottke.ncf.edu/neu/7376_sp16/shi.pdf.

[Shu] M. A. Shubin. *Von Neumann algebras and L^2-techniques in Geometry and Topology*. book in preparation.

[Shu92] M. A. Shubin. Spectral theory of elliptic operators on non-compact manifolds. In *Méthodes semi-classiques Volume 1 - École d'Été (Nantes, juin 1991)*, number 207 in Astérisque, pages 35–108. Société mathématique de France, 1992.

[Shu95] M. A. Shubin. *L^2-Riemann-Roch Theorem for Elliptic Operators*, pages 482–527. Birkhäuser Basel, Basel, 1995.

[SW22] D. Shen and M. Wrochna. An index theorem on asymptotically static spacetimes with compact Cauchy surface. *Analysis*, 4:727–766, 2022.

[Tak77] M. Takesaki. Factors of type III. In Bernard B. Morrel and I. M. Singer, editors, *K-Theory and Operator Algebras*, pages 139–156, Berlin, Heidelberg, 1977. Springer Berlin Heidelberg.

[Tay76] K. F. Taylor. The type structure of the regular representation of a locally compact group. *Mathematische Annalen*, 222(3):211–224, Oct 1976.

[Tre06] F. Treves. *Topological Vector Spaces, Distributions and Kernels*. Dover books on Mathematics. Dover Publications, 2006.

[Tre22] F. Treves. *Fourier Integral Operators*, pages 685–760. Springer International Publishing, Cham, 2022.

[Vai08] B. Vaillant. Index Theory for Coverings. *arXiv:0806.4043*, 2008.

[vdD18] K. van den Dungen. Families of spectral triples and foliations of space(time). *Journal of Mathematical Physics*, 59(6):063507, 2018.

[vdDR20] K. van den Dungen and L. Ronge. The APS-index and the spectral flow. *arXiv:2004.01085*, 2020.

[Wan11] H. Wang. L^2-index formula for proper cocompact group actions. *Journal of Noncommutative Geometry*, 8, Jun 2011.

[Wei76] A. Weinstein. The principal symbol of a distribution. *Bulletin of the American Mathematical Society*, 82(4):548 – 550, 1976.

[Wei78] A. Weinstein. The Order and Symbol of a Distribution. *Transactions of the American Mathematical Society*, 241:1–54, 1978.

[Yos92] N. Yoshida. Sobolev spaces on a Riemannian manifold and their equivalence. *Journal of Mathematics of Kyoto University*, 32(3):621–654, 1992.

[Yos95] K. Yosida. *Functional Analysis*. Classics in Mathematics. 6th ed. 1995. edition, 1995.

[Zad08] M. E. Zadeh. Heat equation approach to index theorems on odd dimensional manifolds. *arXiv:0801.1093*, 2008.

[Ün95] B. Ünal. Divergence theorems in semi-Riemannian geometry. *Acta applicandae mathematicae*, 40(2):173–178, 1995.

Orville Damaschke

Curriculum Vitae

Education

2019-2023 **PhD studies**, *Mathematics*, Carl von Ossietzky Universität Oldenburg.
Supervised by Prof.Dr.Boris Vertman
Thesis "L^2-equivariant index theorem for the Atiyah-Singer-Dirac operator on globally hyperbolic spacetimes"

2014–2019 **Masters of Science**, *Physics*, Carl von Ossietzky Universität.

2011–2014 **Bachelor of Science**, *Physics*, Universität Bremen.

June 2011 **Abitur**, Ganderkesee.

Research Visits

2018 **Visiting Student**, *Laboratory of Theoretical Physics*, University of Tartu (Tartu Ülikooli), Tartu/Estonia.
Hosted by Ass. Prof. Dr. Manuel Hohmann

Work Experience

2019-2022 **Research Assistent**, *IfM*, Carl von Ossietzky Universität Oldenburg.

2012-2018 **Tutor**, Universität Bremen.

Institut für Mathematik, Carl von Ossietzky Universität Oldenburg
Carl-von-Ossietzky-Straße 9-11, D-26129 Oldenburg (Oldb)
✉ *orville.damaschke@uol.de*

Eidesstattliche Erklärung

Hiermit versichere ich an Eides statt, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Außerdem versichere ich, dass ich die allgemeinen Prinzipien wissenschaftlicher Arbeit und Veröffentlichung, wie sie in den Leitlinien guter wissenschaftlicher Praxis der Carl von Ossietzky Universität Oldenburg festgelegt sind, befolgt habe.

Oldenburg, 27-02-2023
Ort, Datum

Orville Damaschke