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Recursive Towers over Finite Fields

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Abstract

An infinite sequence $\mathcal{F} = (F_0, F_1, \dots)$ of function fields $F_n/\mathbb{F}_q$ of transcendence degree one with full constant field $\mathbb{F}_q$ is called a tower of function fields if all extensions F_{n+1}/F_n are finite separable and the genus $g(F_n)$ tends to infinity as $n \to \infty$. A tower is called asymptotically good if its limit $\lambda(\mathcal{F}) := \lim\limits_{n\to\infty} \frac{N(F_n)}{g(F_n)}$ is a positive real number where $N(F_n)$ denotes the number of $\mathbb{F}_q$-rational places in F_n.

Good towers can be used to construct Goppa codes with good parameters. Unfortunately, many of the known good towers are constructed with methods which involve class field theory or modular curves and these constructions do not provide explicit presentations of the function fields F_n. However, a special type of towers are recursive towers which are recursively defined by bivariate polynomials $f(X, Y)$ over $\mathbb{F}_q$ and provide a sequence of elements $x_n \in F_n$ such that F_n is of the explicit form $\mathbb{F}_q(x_0, \dots, x_n)$ where these elements satisfy the equation $f(x_n, x_{n+1}) = 0$ for all $n \in \mathbb{N}_0$.

The main tool of this thesis is a directed graph which is associated with the recursive tower $\mathcal{F}$, namely the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$. Its vertex set $V(\Gamma_\mathcal{F})$ is the set $\mathbb{P}_{F_0}$ of all places in F_0, its edge set $E(\Gamma_\mathcal{F})$ the set $\mathbb{P}_{F_1}$ of all places in F_1 and an edge Q in $\Gamma_\mathcal{F}$ goes from $Q \cap F_0$ to $\sigma^{-1}(Q \cap \sigma(F_0))$ where σ is the $\mathbb{F}_q$-algebra morphism $F_0 = \mathbb{F}_q(x_0) \to \sigma(F_0) = \mathbb{F}_q(x_1)$ with $\sigma(x_0) = x_1$.

In directed graphs, circles are defined as closed paths without further repetitions and weakly connected components are connected components if the directions of the edges are neglected. Moreover, an edge Q in $\Gamma_\mathcal{F}$ is called ramified if one of the ramification indices $e(Q|Q \cap F_0)$ and $e(Q|Q \cap \sigma(F_0))$ is greater than one. Paths $\mathcal{P}$ have balanced ramification indices if the products $\prod_{i=1}^n e(Q_i|Q_i \cap F_0)$ and $\prod_{i=1}^n e(Q_i|\sigma^{-1}(Q_i) \cap F_0)$ are equal for the edge sequence $(Q_1, \dots, Q_n)$ of $\mathcal{P}$. Then a weakly connected component only containing circles which have balanced ramification indices is called balanced.

The ramification subgraph $\Gamma_\mathcal{F}^{\mathrm{ram}}$ is defined as the union of all weakly connected components of $\Gamma_\mathcal{F}$ which contain ramified edges. Finally, a subgraph Γ stops ramifying from some level on if there is some $m \in \mathbb{N}_0$ such that $e(Q|Q \cap F_m) = 1$ for all $n \geq m$ and all $Q \in \mathbb{P}_{F_n}$ with $Q \cap F_0 \in V(\Gamma)$, then .

In 2005, Beelen-Garcia-Stichtenoth conjectured that a good recursive tower $\mathcal{F} = (F_\nu)_\nu$ has to have rational splitting in F_0, i.e. there is a rational place in F_0 which splits completely in all extensions F_n/F_0 with $n \in \mathbb{N}_0$. This conjecture is false. But in 2022, Beelen confirmed that the following weaker version of this conjecture is still open: For every good recursive tower, there is some $m \in \mathbb{N}_0$ such that $\mathcal{F}$ has rational splitting in F_m, i.e. there is a rational place in F_m which splits completely in all extensions F_n/F_m with $n \geq m$.

The main result of this thesis is an almost complete answer to this conjecture. More concretely, it is concluded that this conjecture holds true for a recursive tower if its tower graph only has finite balanced weakly connected components which stop ramifying from some level on. There is only one recursive tower known to the author which does not fulfill this condition, namely the CNT-tower.

As two further results, it is shown that the tower graph has at most one finite balanced weakly connected component and that the limit of a recursive tower cannot increase after

a finite constant field extension. These results are improvements of results from Beelen in 2004 and from Hallouin-Perret in 2012 and they also improve a result from Chara-Navarro-Toledano in 2018.

As a fourth result the precise limits of all tame recursive towers are determined such that the ramification subgraph is finite and only has balanced weakly connected components which stop ramifying from some level on. Moreover, the precise limits of all wild recursive towers are also determined which additionally are α-weakly ramified are also determined, i.e. there is a map $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ such that the set of the elements $a(Q \cap F_0)e(Q|Q \cap F_0) - d(Q|Q \cap F_0)$ with $Q \in \bigcup_{\nu \in \mathbb{N}_0} \mathbb{P}_{F_\nu}$ can only attain finitely many values. Again, the only recursive tower known to the author to which these methods cannot be applied is the CNT-tower.

As a consequence of being able to compute precise limits for most recursive towers in the literature, implications for several important results are derived, e.g. from Bezerra-Garcia-Stichtenoth in 2005 (the BezGS-towers do not supply a better lower bound for Ihara's constant for cubic q than already established), from Bassa-Beelen-Garcia-Stichtenoth in 2015 (the BBGS-towers do not supply a better lower bound for Ihara's constant for non prime q than already established), from Stichtenoth-Tutdere in 2015 (there are no good quadratic recursive towers over $\mathbb{F}_2$) and from Bassa-Ritzenthaler in 2020 (the BR-towers do not improve any known lower bounds for Ihara's constant).

Finally, as a fifth result, a method with an implementation in the computer algebra system Magma is derived which computes genus formulas for tame recursive towers with so called separating power ramification subgraphs. The author is not aware of any tame recursive tower not having a separating power ramification subgraph. For instance, the implementation works on all recursive towers from Maharaj-Wulftange in 2002.

Kurzfassung

Eine unendliche Folge von Funktionenkörpern $\mathcal{F} = (F_0, F_1, \dots)$ mit Transzendenzgrad eins über einem endlichen Konstantenkörper $\mathbb{F}_q$ wird als Funktionenköperturm bezeichnet, wenn alle Erweiterungen F_{n+1}/F_n endlich und separabel sind und das Geschlecht $g(F_n)$ gegen unendlich strebt, wenn $n \to \infty$. Ein Funktionenkörperturm wird als asymptotisch gut bezeichnet, wenn sein Grenzwert $\lambda(\mathcal{F}) := \lim_{n\to\infty} \frac{N(F_n)}{g(F_n)}$ eine positive reelle Zahl ist, wobei $N(F_n)$ die Anzahl der $\mathbb{F}_q$-rationalen Stellen in F_n bezeichnet.

Gute Türme können zur Konstruktion von Goppa-Codes mit guten Parametern verwendet werden. Leider werden viele bekannte gute Türme mit Methoden konstruiert, die Klassenkörpertheorie oder modulare Kurven involvieren, und diese Konstruktionen liefern keine expliziten Darstellungen der Funktionenkörper F_n. Es gibt jedoch eine spezielle Art von Türmen, nämlich rekursive Türme, die durch bivariate Polynome $f(X, Y)$ über $\mathbb{F}_q$ rekursiv definiert werden und eine Folge von Elementen $x_n \in F_n$ liefern, sodass F_n die explizite Form $\mathbb{F}_q(x_0, \dots, x_n)$ hat, wobei diese Elemente die Gleichung $f(x_n, x_{n+1}) = 0$ für alle $n \in \mathbb{N}_0$ erfüllen.

Das Hauptwerkzeug dieser Arbeit ist ein gerichteter Graph, der mit dem rekursiven Turm $\mathcal{F}$ assoziiert ist, nämlich der Turmgraph $\Gamma_{\mathcal{F}}$ von $\mathcal{F}$. Seine Eckenmenge $V(\Gamma_{\mathcal{F}})$ besteht aus allen Stellen in F_0, seine Kantenmenge $E(\Gamma_{\mathcal{F}})$ besteht aus allen Stellen in F_1 und eine Kante Q in $\Gamma_{\mathcal{F}}$ führt von $Q \cap F_0$ zu $\sigma^{-1}(Q \cap \sigma(F_0))$, wobei σ der $\mathbb{F}_q$-Algebrenhomomorphismus $F_0 = \mathbb{F}_q(x_0) \to \sigma(F_0) = \mathbb{F}_q(x_1)$ mit $\sigma(x_0) = x_1$ ist.

In gerichteten Graphen werden Kreise als geschlossene Pfade ohne weitere Wiederholungen definiert und schwach zusammenhängende Komponenten sind zusammenhängende Komponenten, wenn die Richtungen der Kanten vernachlässigt werden. Darüber hinaus wird eine Kante Q in $\Gamma_{\mathcal{F}}$ als verzweigt bezeichnet, wenn einer der Verzweigungsindizes $e(Q|Q \cap F_0)$ und $e(Q|Q \cap \sigma(F_0))$ größer als eins ist. Pfade $\mathcal{P}$ haben balancierte Verzweigungsindizes, wenn die Produkte $\prod_{i=1}^{n} e(Q_i|Q_i \cap F_0)$ und $\prod_{i=1}^{n} e(Q_i|\sigma^{-1}(Q_i) \cap F_0)$ für die Kantensequenz $(Q_1, \dots, Q_n)$ von $\mathcal{P}$ gleich sind. Eine schwach zusammenhängende Komponente, die nur Kreise enthält, deren Verzweigungsindizes balanciert sind, wird als balanciert bezeichnet.

Der Verzweigungsuntergraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ ist definiert als die Vereinigung aller schwach zusammenhängenden Komponenten von $\Gamma_{\mathcal{F}}$, die verzweigte Kanten enthalten. Schließlich stoppt die Verzweigung eines Untergraphs Γ ab einer Stufe, wenn es ein $m \in \mathbb{N}_0$ gibt, für das $e(Q|Q \cap F_m) = 1$ für alle $n \geq m$ und alle $Q \in \mathbb{P}_{F_n}$ mit $Q \cap F_0 \in V(\Gamma)$ gilt.

Im Jahr 2005 haben Beelen-Garcia-Stichtenoth die Vermutung aufgestellt, dass ein guter rekursiver Turm $\mathcal{F} = (F_\nu)_\nu$ in F_0 rationale Zerlegung haben muss, d.h., es gibt eine rationale Stelle in F_0, die sich in allen Erweiterungen F_n/F_0 mit $n \in \mathbb{N}_0$ vollständig zerlegt. Diese Vermutung ist falsch. Aber im Jahr 2022 bestätigte Beelen, dass die folgende schwächere Version dieser Vermutung immer noch offen ist: Für jeden guten rekursiven Turm gibt es ein $m \in \mathbb{N}_0$, sodass $\mathcal{F}$ in F_m rationale Zerlegung hat, d.h., es gibt eine rationale Stelle in F_m, die sich in allen Erweiterungen F_n/F_m mit $n \geq m$ vollständig zerlegt.

Das Hauptergebnis dieser Arbeit ist eine nahezu vollständige Antwort auf diese Vermutung. Konkret wird gezeigt, dass diese Vermutung für einen rekursiven Turm wahr ist,

wenn sein Turmgraph nur endliche balancierte schwach zusammenhängende Komponenten hat, die ab einer bestimmten Stufe die Verzweigung stoppen. Dem Autor ist nur ein rekursiver Turm bekannt, der diese Bedingung nicht erfüllt, nämlich der CNT-Turm.

Zwei weitere Ergebnisse zeigen, dass der Turmgraph nicht mehr als eine endliche, balancierte schwach zusammenhängende Komponente hat und dass der Grenzwert eines rekursiven Turms nach einer endlichen Konstantenkörpererweiterung nicht steigen kann. Diese Ergebnisse verbessern die Ergebnisse von Beelen aus dem Jahr 2004 und von Hallouin-Perret aus dem Jahr 2012 sowie ein Ergebnis von Chara-Navarro-Toledano aus dem Jahr 2018.

Als ein viertes Ergebnis werden die genauen Grenzwerte aller zahmen rekursiven Türme bestimmt, bei denen der Verzweigungsgraph endlich ist und nur balancierte schwach zusammenhängende Komponenten enthält, die ab einer Stufe aufhören zu verzweigen. Darüber hinaus werden auch die genauen Grenzwerte aller wilden rekursiven Türme bestimmt, die zusätzlich α-schwach verzweigt sind. Das bedeutet, es gibt eine Abbildung $\alpha : \mathcal{P}_{F_0} \to \mathbb{R}$, so dass die Menge der Elemente $a(Q \cap F_0)e(Q|Q \cap F_0) - d(Q|Q \cap F_0)$ mit $Q \in \bigcup_{\nu \in \mathbb{N}_0} \mathcal{P}_{F_\nu}$ nur endlich viele Werte annehmen kann. Der einzige bekannte rekursive Turm, auf den diese Methoden nicht angewendet werden können, ist der CNT-Turm.

Als Konsequenz, genaue Grenzwerte für die meisten rekursiven Türme in der Literatur berechnen zu können, ergeben sich Implikationen auf mehrere wichtige Ergebnisse. Zum Beispiel auf die Ergebnisse von Bezerra-Garcia-Stichtenoth im Jahr 2005 (die BezGS-Türme liefern keine bessere untere Schranke für Iharas Konstante für kubische q als bereits etabliert), von Bassa-Beelen-Garcia-Stichtenoth im Jahr 2015 (die BBGS-Türme liefern keine bessere untere Schranke für Iharas Konstante für nicht-prime q als bereits etabliert), von Stichtenoth-Tutdere im Jahr 2015 (es gibt keine guten quadratischen rekursiven Türme über $\mathbb{F}_2$) und von Bassa-Ritzenthaler im Jahr 2020 (die BR-Türme verbessern keine bekannten unteren Schranken für Iharas Konstante).

Schließlich wird als ein fünftes Ergebnis eine Methode mit einer Implementierung im Computeralgebrasystem Magma hergeleitet, die Geschlechterformeln für zahme rekursive Türme mit sogenannten separierenden Potenz-Verzweigungsgraphen berechnet. Dem Autor ist kein zahmer rekursiver Turm bekannt, der keinen separierenden Potenz-Verzweigungsgraphen hat. Die Implementierung lässt sich zum Beispiel auf alle rekursiven Türmen von Maharaj-Wulftange aus dem Jahr 2002 anwenden.

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Contents

1 Introduction

Curves with many points. Smooth geometrically integral projective curves C over finite fields $k = \mathbb{F}_q$ with many $\mathbb{F}_q$-rational points for a given genus $g = g(C)$ are of interest for several reasons. For example because they can be used to construct error-correcting Goppa Codes with good parameters (see [Sti08, p. 243, Chapter 7]).

The Hasse-Weil-bound $q + 1 + \lfloor 2\sqrt{q} \rfloor g(C)$ provides a first non-trvial upper bound for the number $N(C)$ of $\mathbb{F}_q$-rational points on C. However, in [Iha82], Ihara established that the Hasse-Weil-bound cannot be attained for large genera $g(C)$ and also introduced the number

$$A(q) := \limsup_{g \to \infty} \frac{\max\{N(C) : C \text{ curve of genus } g \text{ over } \mathbb{F}_q \text{ as above}\}}{g}$$

which was later called **Ihara's constant**. Then, in [VD83], it was shown that Ihara's constant $A(q)$ has the Drinfeld-Vladut-bound $\sqrt{q} - 1$ as an upper bound. On the other hand, lower bounds $b(q)$ for Ihara's constant $A(q)$ ensure the existence of sequences $(C_\nu)_{\nu \in \mathbb{N}_0}$ of smooth geometrically integral projective curves over $\mathbb{F}_q$ with many rational points and small genera. More precisely, we have $\lim_{\nu \to \infty} N(C_\nu)/g(C_\nu) \geq b(q)$ where $g(C_\nu) \to \infty$ as $\nu \to \infty$. A **tower of curves** is a sequence $(C_\nu)_\nu$ as above with additional separable surjective morphisms $C_{n+1} \to C_n$ for all $n \in \mathbb{N}_0$. The following lower bounds for $A(q)$ were established via constructing such towers of curves:

- q square: $A(q) = \sqrt{q} - 1$ ([TVZ82] modular curves, [GS95] recursive towers)
- $q = p^{2m+1}$ with p prime and $m \geq 1$: $A(q) \geq \frac{2(p^{m+1}-1)}{p+1+\frac{p-1}{p^m-1}}$ ([BBGS15] recursive towers)
- $q \neq 2, 3$: $A(q) \geq \frac{2}{q-2}$ ([BR20] recursive towers)
- q, l arbitrary: $A(q^l) \geq c\frac{l^2 \log(q)^2}{l+\log(q)}$ ([Tem01] class field towers)
- $A(2) \geq 0.316999$, $A(3) \geq 0.492876$ ([DM13] class field towers)

Recursive towers. The constructions of the towers from above which use class field theory or modular techniques have the disadvantage that they do not come with explicit presentations for the curves C_n. On the other hand, the curves C_n in **recursive towers** $(C_\nu)_\nu$ are constructed via one geometrically irreducible bivariate polynomial $f = f(X, Y)$ in the following more explicit way: Let $\tilde{C}_0 = \mathbb{A}^1$ and let $\tilde{C}_n$ be the possibly singular affine geometrically integral curve in $\mathbb{A}^{n+1}$ for all $n \in \mathbb{N}$ which is defined by the equations $f(x_{i-1}, x_i) = 0$ for all $i = 1, \ldots, n$. If the sequence $(C_\nu)_\nu$ of the normalizations C_n of the projective closures of $\tilde{C}_n$ with the canonical morphisms $C_{n+1} \to C_n$ is a tower of curves, then $(C_\nu)_\nu$ is called a **recursive tower of curves** which is defined by the polynomial f.

Geometrically, we can think of $\tilde{C}_n$ as the intersection of the same but slightly rotated hypersurfaces $H_i(f) := \{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1} : f(x_{i-1}, x_i) = 0\}$ for all $i = 1, \ldots, n$. In Figure 1.1, we visualized the construction of the first three curves $\tilde{C}_0$, $\tilde{C}_1$ and $\tilde{C}_2$ for the defining polynomial $f = Y^2 - X(X^2 + 1) \in \mathbb{R}[X, Y]$ using [Inc].

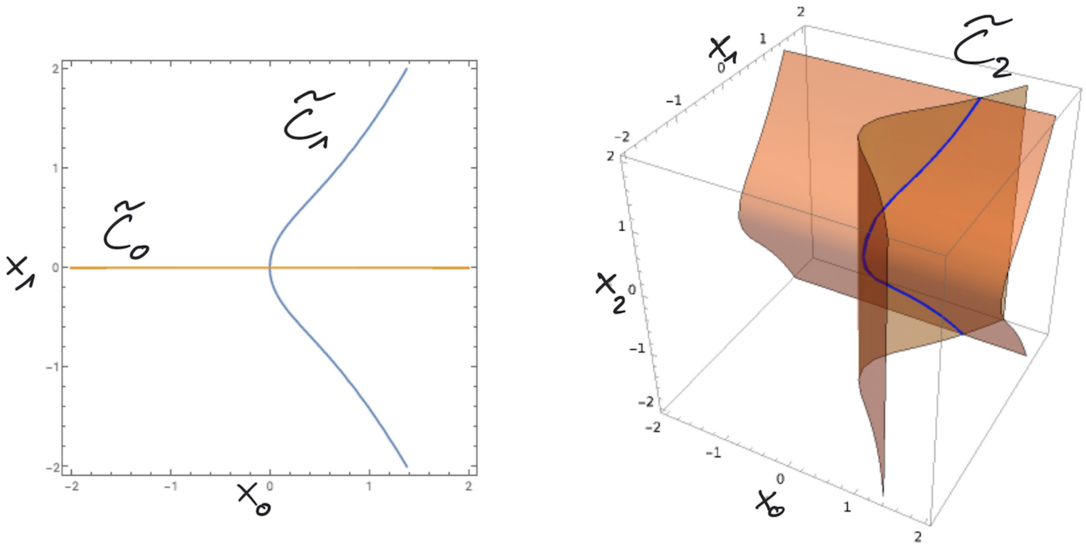


Figure 1.1: First three levels of a recursive tower of curves.

As we are only interested in the normalizations C_n of the projective closures of $\tilde{C}_n$, we can equivalently consider the towers $\mathcal{F} = (F_\nu)_\nu$ of function fields. Such towers are given via proper finite separable extensions F_{n+1}/F_n of function fields where the genus $g(F_\nu)$ tends to infinity as $\nu \to \infty$. Then a tower $\mathcal{F} = (F_\nu)_\nu$ of function fields is called **recursively defined** by the polynomial $f = f(X, Y)$ if there are k-transcendental elements x_n such that $F_n = k(x_0, \ldots, x_n)$ and $f(x_n, x_{n+1}) = 0$ hold for all $n \in \mathbb{N}_0$. We will call such a tower $\mathcal{F}$ of function fields a **recursive tower of function fields**. In this thesis, we will only work with towers of function fields and therefore switch to the function field language from now on.

History of finding good recursive towers. The **limit** of a tower over a finite field $\mathbb{F}_q$ is the defined as

$$\lambda(\mathcal{F}) := \lim_{\nu\to\infty} N(F_\nu)/g(F_\nu) = \nu(\mathcal{F})/\gamma(\mathcal{F})$$

where

$$\nu(\mathcal{F}) := \lim_{\nu\to\infty} N(F_\nu)/[F_\nu : F_0] \quad \text{and} \quad \gamma(\mathcal{F}) := \lim_{\nu\to\infty} g(F_\nu)/[F_\nu : F_0]$$

are called the **splitting rate** and the **genus** of $\mathcal{F}$, respectively. Then a tower $\mathcal{F}$ is called **good** if $\lambda(\mathcal{F}) > 0$ and otherwise **bad**. If $\lambda(\mathcal{F})$ even attains the Vladut-Drinfeld-bound $\sqrt{q} - 1$, then $\mathcal{F}$ is called **optimal**.

The first appearance of good recursive towers $\mathcal{F} = (F_\nu)_\nu$ of function fields over finite fields $\mathbb{F}_q$ was in [GS95] from 1995. There the authors provided optimal recursive towers $\mathcal{F}$ for all square q.

After several interim results for at least cubic q, the recursive **BBGS-towers** $\mathcal{F}$ were introduced in [BBGS15]. To this day, these BBGS-towers $\mathcal{F}$ supply the best known lower bounds for Ihara's constant $A(p^m)$ for all primes p and $m \geq 2$. More concretely, the BBGS-towers supply the estimate

$$2 \cdot \left(\frac{1}{p^{\lfloor m/2 \rfloor} - 1} + \frac{1}{p^{\lceil m/2 \rceil} - 1} \right)^{-1} \leq A(p^m)$$

where $\lfloor m/2 \rfloor$ (resp. $\lceil m/2 \rceil$) denotes $m/2$ rounded down (resp. up).

On the other hand, for prime $q = p$, just finding any good recursive towers was more difficult. It was only in [BR20] from 2020 that the first good recursive towers over prime fields $\mathbb{F}_p$ for all prime $p \neq 2, 3$ were found. We will call these recursive towers the **BR-towers**. Although these BR-towers did not improve any known lower bounds for Ihara's constant $A(q)$, they at least established that good recursive towers over prime fields actually exist. This was in question because 25 years no examples could be found.

How to construct or find good recursive towers? There is no clear path on how to construct or find good recursive towers. Nonetheless, there are several approaches. The most classical approach is to somehow come up with a polynomial f having a special structure which ensures that f defines a good recursive towers. For instance, the first recursive towers in [GS95] and the BBGS-towers in [BBGS15] were constructed in this way.

Another approach is to search for good tame recursive towers with the computer for small q and small $\deg(f)$. For instance, this was done in [MW05], [Wul02] and [Lö07].

There are also two newer and more instructive approaches which are promising for the future: On the one hand, in [BR20], the authors used Galois theory to construct the BR-towers. On the other hand, in [HP16], the authors demonstrated how good recursive towers can be derived from certain finite directed graphs.

Finally, for most of the good recursive towers in the literature, a modular interpretation of the towers had been found. Hence, Elkies [Elk01, p. 9] and others conjectured that there is a higher meaning to good recursive towers. This conjecture is often referred to as **Elkies' modularity Fantasia**.

Objectives - Properties of good recursive towers and computing limits and genus formulas. In this thesis, we will not try to construct or find good recursive towers. Instead we will prove three important and closely related properties of good recursive towers and we will provide methods to determine precise limits of recursive towers and genus formulas of tame recursive towers.

Before we can accurately formulate all five major results, we will have to define and elaborate on the main tool in the next paragraph.

The main tool - The tower graph of a recursive tower. In [Lö07, p. 20, Definition 2.21], the definition of a recursive tower $\mathcal{F} = (F_\nu)_\nu$ of function fields over a field k was generalized via the so called **multi-step towers**. These multi-step towers are recursively defined by more than one polynomial and therefore also allow non-rational function fields at the bottom level F_0. We will also work with a generalized, but different notion of recursive towers, namely **pair-recursive towers** (see Definition 5(ii)). These pair-recursive towers $\mathcal{F} = (F_\nu)_\nu = (\prod_{i=0}^{\nu} \sigma^i(F_0))_\nu$ will be defined by a pair (σ, F_0) where σ is an automorphism of k-algebras on an algebraic closure of $\bigcup_{\nu=1}^{\infty} F_\nu$. If $\mathcal{F} = (F_\nu)_\nu = (k(x_0, \ldots, x_\nu))_\nu$ is recursively defined by a polynomial, then we will have $\sigma(x_n) = x_{n+1}$ for all $n \in \mathbb{N}_0$. Although the definition of pair-recursive towers will be equivalent to the definition of multi-step towers, it will be a better fit for our more abstract purposes.

In the following, we will use the term recursive towers for the more general **pair-recursive towers** and the term **polynomial-recursive tower** for the classical recursive towers which are defined by polynomials as above (see also [BGS04] for instance). But in order to avoid technicalities in this introduction and because there are yet no relevant examples of pair-recursive towers which are not already polynomial-recursive, the subsequent explanations will only be formulated for polynomial-recursive towers.

In [Bee04, p. 221, Definition 2.2] and [HP12, p. 15, Definition 10], certain directed graphs were introduced which are associated with recursive towers. We will call these graphs the **Beelen-graph** and the **HP-graph**, respectively. The main tool of this thesis will be yet another directed graph associated with recursive towers $\mathcal{F}$, namely the **tower**

graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F} = (F_\nu)_\nu$ (see Definition 74). Essentially, the tower graph will be a modification of the Beelen-graph, where the modification will be unavoidable because of the more general definition of pair-recursive towers. In fact, up to a canonical isomorphism of directed graphs, the Beelen-graph will be a subgraph of the tower graph.

Formally, the vertex set $V(\Gamma_{\mathcal{F}})$ of the tower graph $\Gamma_{\mathcal{F}}$ will be defined as the set $\mathbb{P}_{F_0}$ of all places in F_0, the edge set $E(\Gamma_{\mathcal{F}})$ of $\Gamma_{\mathcal{F}}$ will be defined as the set $\mathbb{P}_{F_1}$ of all places in F_1 and an edge Q in $\Gamma_{\mathcal{F}}$ will go from $Q \cap F_0$ to $\sigma^{-1}(Q) \cap F_0$. Informally, we can think of all these three directed graphs in the following way: The vertices in these graphs can be seen as the points on $\tilde{C}_0 = \mathbb{A}^1$ and the edges as the points (x_0, x_1) on $\tilde{C}_1$ where an edge (x_0, x_1) goes from x_0 to x_1. This means that we have an edge $x_0 \to x_1$ if and only if $f(x_0, x_1) = 0$ (see Figure 1.1).

Furthermore, for the tower graph $\Gamma_{\mathcal{F}}$, the extensions F_1/F_0 and $F_1/\sigma(F_0)$ of function fields attribute ramification indices $e(Q|Q \cap F_0)$ and $e(Q|\sigma^{-1}(Q) \cap F_0)$ to the edges $Q \in \mathbb{P}_{F_1}$ in $\Gamma_{\mathcal{F}}$. In particular, the following property of paths $\mathcal{P}$ in $\Gamma_{\mathcal{F}}$ will be key for many of the results in this thesis. We will say that a path $\mathcal{P}$ in $\Gamma_{\mathcal{F}}$ with the edge sequence $(Q_1, \ldots, Q_n)$ has **balanced ramification indices** if the products $\prod_{i=1}^{n} e(Q_i|Q_i \cap F_0)$ and $\prod_{i=1}^{n} e(Q_i|\sigma^{-1}(Q_i) \cap F_0)$ are equal. Otherwise, we will say that $\mathcal{P}$ has **unbalanced ramification indices**.

Finally, note that a **circle** will be a closed path without further repetitions. A ***d*-regular** directed graph will be a directed graph which has d ingoing and d outgoing edges at every vertex. A **weakly connected** subgraph of a directed graph will be a connected subgraph if we forget about the directions of the edges. A **weakly connected component** of a directed graph will be a connected component if we again forget about the directions of the edges. A **balanced** weakly connected component will be a weakly connected component which only has circles with balanced ramification indices.

Five Major Results. Now all necessary notions are available to accurately formulate the five major results of this thesis in the following five paragraphs.

Main result - An almost complete answer to Conjecture 1(iii). A remarkable property of every good recursive tower $\mathcal{F} = (F_\nu)_\nu$ in the literature is that F_0 always has rational places which split completely on every level F_n/F_0. We will call the subset $\mathrm{Split}(\mathcal{F}/F_0) \subset \mathbb{P}_{F_0}$ of all such rational places the **splitting locus** of $\mathcal{F}$.

In [BGS04, p. 7, Conjecture 1], it was conjectured that every good recursive tower has a non-empty splitting locus (see also Conjecture 1(i)). Although we will give a counterexample to this conjecture in Example 129, the following weaker version of this conjecture is still open:

> There is always some $m \in \mathbb{N}_0$ such that F_m contains a rational place
> which splits on every level F_n/F_m with $n \geq m$ (see Conjecture 1(iii)).

This weaker conjecture was formulated in [Sti10, p. 5, Problem 1] for the first time and it was confirmed in [Bee22, p. 10] to be still open.

The **main and chronologically second result** will be our almost complete answer to this Conjecture 1(iii) in Corollary 184. Up to finite constant fields extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example known to the author, it will come out that a recursive tower $\mathcal{F}$ satisfies $\mathrm{Split}(\mathcal{F}/F_0) = \nu(\mathcal{F})$ if and only if every finite weakly connected component of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ contains circles with unbalanced ramification indices.

Existence of at most one finite balanced weakly connected component. In [Bee04, p. 238, Theorem 5.5] and in [HP12, p. 27, Theorem 23], it was shown that most of the Beelen-graphs and all of the HP-graphs have at most one finite d-regular weakly connected component, respectively, where $d = \deg_X(f(X,Y)) = \deg_Y(f(X,Y))$. As the chronologically **first major result**, we will show in Theorem 155 that the tower graph not only has at most one finite d-regular weakly connected component but even at most one finite balanced weakly connected component.

On the one hand, by Corollary 156, this will especially imply that the Beelen-graph also has at most one finite balanced weakly connected component. On the other hand, in Theorem 154, we will also present a simplified proof of [HP12, p. 27, Theorem 23] which will also work in more general settings.

This part is joint work with Florian Heß.

Limits of good recursive towers are stable under finite constant field extensions. In [Bee04, p. 238, Corollary 5.6] and in [HP12, p. 27, Theorem 24], it was shown that the limit of a good recursive tower cannot increase after a finite constant field extension if some technical conditions are met.

As the **third major result**, in Theorem 188, we will show that these technical conditions can even be dropped. This means that the limit of a good recursive tower can never increase after a finite constant field extension.

Determining precise limits for most recursive towers in the literature. A priori, determining the precise limit $\lambda(\mathcal{F}) = \lim_{\nu\to\infty} N(F_\nu)/g(F_\nu)$ of a recursive tower $\mathcal{F}$ is a delicate problem. Fortunately, most of the times, [Sti08, p. 249, Theorem 7.2.10] (see Theorem 4) is applicable and provides at least a lower bound for the limit $\lambda(\mathcal{F})$. But up to utilizing some structure or some modular interpretation of the concrete recursive tower at hand, there were no known methods for determining the precise limits of recursive towers which are not specific to narrow classes of recursive towers.

As the **fourth major result** of this thesis, we will provide such methods in Corollary 195 and Corollary 200. These corollaries will be sharp versions of the above mentioned [Sti08, p. 249, Theorem 7.2.10] (see Theorem 4). Hence, they will even yield the precise limits $\lambda(\mathcal{F})$ and not only lower bounds. These corollaries will work on all α-weakly ramified (see Definition 199) recursive towers with a finite ramification subgraph which only have unbalanced weakly connected components. Again, the CNT-tower in Examples 8(v) is the only example in the literature known to the author which does not meet these requirements.

Being able to compute the precise limits of most recursive towers will have immediate implications for several important results from the literature: For instance, in Corollary 203, it will come out that the lower bound for the limit $\lambda(\mathcal{F})$ of the BBGS-towers $\mathcal{F}$ which was established in [BBGS15, p. 3, Theorem 1.1] is already equal to the precise limit $\lambda(\mathcal{F})$. In particular, this will imply that the BBGS-towers do not provide an even larger lower bound for Ihara's constant $A(p^m)$ than already shown in [BBGS15, p. 3, Theorem 1.1].

Moreover, in Corollary 202, it will also come out that there are no good polynomial-recursive towers over $\mathbb{F}_2$ of degree $2 = \deg_X(f(X,Y))$. This will conlcude the endeavour of finding all such good recursive towers, which was started in [ST15]. There the authors reduced the potential recursive towers to the four candidates in [ST15, p. 667, Theorem 1.4] and [ST15, p. 680, Theorem 2.14]. We will call these four recursive towers the **ST-towers**.

Finally, in Corollary 205, we will also show that the lower bounds for the limits $\lambda(\mathcal{F})$ of the BR-towers $\mathcal{F}$ which were established in [BR20, p. 4, Theorem 2.3] are equal to the precise limits $\lambda(\mathcal{F})$. Hence, we can be sure that the BR-towers do not improve any known lower bound for Ihara's constant $A(q)$.

Computing genus formulas for tame recursive towers. In [HP16, p. 12, Proposition 12], by hand, the authors computed a formula for the genus sequence

$$\begin{aligned} g(F_n) &= 2^n + 1 - (2 + n \bmod 2) \cdot 2^{\lfloor \frac{n}{2} \rfloor} \\ &= 2^n + 1 - \frac{(4+3\sqrt{2})}{4} \cdot \sqrt{2}^n - \frac{(4-3\sqrt{2})}{4} \cdot (-\sqrt{2})^n \end{aligned}$$

for all $n \in \mathbb{N}_0$ of the tame recursive tower $\mathcal{F} = (F_\nu)_\nu$ which is defined by the polynomial $f = Y^2(3X-1) - (X^2+X) \in \mathbb{F}_q[X,Y]$.

As the **fifth major result** of this thesis, in Corollary 246, we will prove that the genus sequence $g(F_n)$ of certain tame recursive towers $\mathcal{F} = (F_\nu)_\nu$ is of the form

$$g(F_n) = \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n$$

for all $n \geq c(\mathcal{F})$ with some $c(\mathcal{F}) \in \mathbb{N}_0$, some finite subset $\Lambda \subset \overline{\mathbb{Q}}$ and some polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ for all $\lambda \in \Lambda$. This Corollary 246 will work on all tame recursive towers only having **finite separating power ramification subgraphs** (see Definition 229(iii) and Definition 239). The author is not aware of a single tame recursive tower not meeting this condition.

Moreover, in the proof of Corollary 246, we will even construct these lower bound $c(\mathcal{F}) \in \mathbb{N}_0$, finite subset $\Lambda \subset \overline{\mathbb{Q}}$ and polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ for all $\lambda \in \Lambda$. Consequently, we will be able to automatize this construction and, in Subsection 8.3.2, provide a first naive implementation which can compute genus formulas for all tame recursive towers in the literature of which the author is aware.

Further Results. There will be further results in this thesis which are not as significant as the five above mentioned major results but still worth mentioning, namely Key Lemma 36(i), Key Lemma 36(iii), Proposition 39, Corollary 51 and Theorem 225. Nonetheless, in order to avoid getting too technical in this introduction, we will leave a more detailed summary of these further results to the respective chapters.

Structure of the thesis. In the first preliminary Chapter 2, we will introduce all basic notions for recursive towers of function fields and make some preparations for the other chapters. Moreover, we will introduce the more general pair-recursive towers $\mathcal{F} = (F_\nu)_\nu$. Finally, we will also formulate and start discussing the main subject of this thesis, which is Conjecture 1(i)-(iv).

In Chapter 3, we will prove the first more involved result of this thesis, which is Corollary 51. This Corollary will supply a first useful upper bound for the number $N(F_n)$ of rational places in F_n. Moreover, we will also make further preparations for the remaining chapters by proving Key Lemma 36, Proposition 39 and defining Abhyankar ramification indices in Definition 41.

In the second preliminary Chapter 4, we will introduce the main tool of this thesis, which is the tower graph of a recursive tower. Moreover, we will also define the rational, splitting and ramification subgraphs of the tower graph. In most cases, these subgraphs will already carry all information which are necessary to determine the limit of a recursive tower. Finally, after making further preparations for the remaining chapters, we will relate the tower graph to the Beelen-graph and HP-graph via epimorphisms of directed graphs.

In Chapter 5, we will deduce the first major result of this thesis, which is Theorem 155. Moreover, we will introduce two new concepts for directed graphs associated with recursive towers, namely constant field extensions and truncations of subgraphs (see Definition 107 and Definition/Lemma 128, respectively). These two concepts will be crucial for the main result in Corollary 184.

In Chapter 6, we will exhibit three of the five major results. First, we will prove the main result, which is the almost complete answer to Conjecture 1(iii) in Corollary 184. Second, we will derive the third major result in Theorem 188, which provides that the limits of good recursive towers are stable under finite constant field extensions. Third, we will show the fourth major result in Corollary 195 and Corollary 200, which will enable us to determine the precise limits of most recursive towers in the literature.

Finally, we will deduce the above mentioned implications of Corollary 195 and Corollary 200 for several important results in the literature, for instance the BBGS-towers, BR-towers and ST-towers.

In Chapter 7, we will derive the three interim results Theorem 168, Corollary 170 and Corollary 171, which will already be formulated and used in the preceding Chapter 6. These three interim results will form the core of the proof of the main result. Moreover, we will derive degree bounds for the places in recursive towers in Theorem 225.

In Chapter 8, we will conclude the fifth and last major result, which are Corollary 246 and its first naive implementation in Subsection 8.3.2. This will enable us to automatically compute genus formulas for tame recursive towers. Finally, in Examples 250, we will list genus formulas for some representative tame recursive towers from the literature which were computed using the above mentioned implementation.

Finally, in Chapter 9, we will list prospects for possible future directions.

Mathematical notation. Here we will fix some mathematical notation which is not specific to function fields. First, we will denote the non-negative integers by $\mathbb{N}_0$ and the natural numbers by $\mathbb{N} = \mathbb{N}_0 \backslash \{0\}$. We will denote proper inclusions by $\subset$ or $\subsetneq$ and inclusions which can also be equalities by $\subseteq$. Analogously, we will use the symbols $<$, $\lneqq$ and $\leq$ for estimates.

Second, let R be a ring and $\mathbb{P}$ be the set of prime numbers. Then we will denote the set of sequences with elements in R by $R^{\mathbb{N}_0}$, $R^{\mathbb{N}}$ and $R^{\mathbb{P}}$ with the index sets $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{P}$, respectively. More generally, we will denote the set of maps $M \to R$ with R^M. Moreover, we will denote the sequence which only consists of ones by $\mathbf{1} = (1, 1, 1, \dots)$ and the sequence of primes by $\mathbf{P} = (2, 3, 5, ...)$ where the index sets should always be clear from the context. We will denote the set of sequences in R^M where almost all elements are zero by $(R^M)'$ for $M \in \{\mathbb{N}_0, \mathbb{N}, \mathbb{P}\}$.

Third, let R be a ring and $(\alpha_p)_p \in R^{\mathbb{P}}$. If nothing different is indicated, the multiplication in a product of the form $\prod_{p \in \mathbb{P}} a_p$ runs in the ascending order. For instance, we will also use the multi-index notation

$$\alpha^e = \prod_{p \in \mathbb{P}} \alpha_p^{e_p} \tag{1}$$

for all $e = (e_p)_p \in (\mathbb{N}_0^{\mathbb{P}})'$. Moreover, for all $(\alpha_p)_p, (\beta_p)_p \in R^{\mathbb{P}}$, we will denote the componentwise product of the sequences by

$$(\alpha_p)_p * (\beta_p)_p := (\alpha_p \cdot \beta_p)_p. \tag{2}$$

.

Fourth, we will denote the set of matrices with size m times n by $R^{m \times n}$ and we define the map

$$v_{\mathbf{P}} : \mathbb{Q} \backslash \{0\} \to (\mathbb{Z}^{\mathbb{P}})' \text{ via } a \mapsto (v_p(a))_p \tag{3}$$

where $v_p(a)$ is the usual prime exponent of p in the prime factorization of a.

2 Preliminaries I - Recursive Towers

Purpose of this chapter. The main purpose of this chapter is to introduce all basic notions in regards to the main objects of this thesis, which are recursive towers of function fields, and to make some preparations for the remaining chapters.

Here, we will work with a more general definition of recursive towers $\mathcal{F} = (F_\nu)_\nu$ than the usual definition via bivariate polynomials $f(X, Y)$ (compare [BGS04] for instance and Definition 5(i)). These more general recursive towers $\mathcal{F} = (F_\nu)_\nu$ will be defined by a pair (σ, F_0) where F_0 is a function field, σ an automorphism on an algebraic closure of F_0 and $F_n = \prod_{i=0}^{n} \sigma^i(F_0)$ for all $n \in \mathbb{N}_0$ (see Definition 5(ii)). As F_0 does not need to be a rational function field in this definition, these pair-recursive towers are more general than the polynomial-recursive towers.

In this chapter, we will also formulate and start discussing the main subject of this thesis, namely Conjecture 1 which originates in [BGS04, p. 7, Conjecture 1]. The main result of this thesis will be the almost complete answer to Conjecture 1(iii) in Corollary 184.

Structure of this chapter. In Section 2.1, we will fix notation and repeat some fundamental statements in regards to function fields.

In Section 2.2, we will define towers, polynomial-recursive and pair-recursive towers of function fields and introduce crucial concepts like pyramids of function fields, pyramids of places and paths of places. This involves preparations for the remaining chapters. Moreover, we will also formulate and start discussing the main subject, which is Conjecture 1, and provide an extensive list of examples for recursive towers in Figure 2.3.

In Section 2.3 and Section 2.4, we will define constant field extensions and truncations of recursive towers and draw connections between them and the initial recursive towers. Both concepts will be important for the main result.

In Section 2.5 and 2.6, we will define and briefly discuss locally Galois and dual recursive towers as they will only play a side role in this thesis. These sections can be skipped if the reader is only interested in the major results.

2.1 Function Fields

Purpose of this section. In this section, we will fix some function fields specific notation and fundamental statements. Most of the notation will be standard and can be found in [Sti08].

Definition 1. *(i) An algebraic **function field** F/k of one variable over k is an extension field $F \supset k$ such that F is a finite algebraic extension of $k(x)$ for some element $x \in F$ which is transcendental over k.*

*The field k is called the **constant field** of F/k and the algebraic closure k' of k in F is called the **full constant field** of F/k.*

Since all algebraic function fields F/k considered in this thesis will be of this kind and have full constant field k, we will simply refer to the F/k as the function field F over the full constant field k.

(ii) Let F' and F be function fields over the full constant fields k' and k, respectively, such that $F' \supseteq F$ and $k' \supseteq k$. Then the extension F'/F of fields is called an ***(algebraic) extension of function fields*** *if F'/F is algebraic.*

Moreover, if the extension F'/F of fields satisfies a property $\mathcal{P}$ (e.g. being finite or separable), then we also say that the extension F'/F of function fields satisfies $\mathcal{P}$.

Assumption 1. *The full constant fields of all function fields will be perfect.*

The genus of F will be denoted by $g(F)$ and the set of places of F by $\mathbb{P}_F$. Moreover, we define

$$\mathbb{P}_F^{(r)} := \{P \in \mathbb{P}_F : \deg(P) = r\} \text{ for all } r \in \mathbb{N} \text{ and } N(F) := \#\mathbb{P}_F^{(1)} \tag{4}$$

and the places of degree one are called **rational**.

Let E/F be a finite separable extension of function fields. We will write Q/P for all $Q \in \mathbb{P}_E$ and $P \in \mathbb{P}_F$ if Q lies over P and call Q/P an **extension of places** in E/F. Moreover, we will denote the ramification index of Q/P by $e(Q|P)$, the relative degree of Q/P by $f(Q|P)$ and the different exponent of Q/P by $d(Q|P)$.

Let F/F_i be a finite separable extension of function fields for all $i \in \mathbb{N}$. For all $\mathcal{P} := (P_i)_i \in \prod_{i \in I} \mathbb{P}_{F_i}$ and $I \subseteq \mathbb{N}$, we define

$$\mathbb{P}_F(\mathcal{P}) := \{P \in \mathbb{P}_F : P \text{ lies over } P_i \text{ for all } i = 1, \ldots, r\}$$

and, for all $A \subseteq \bigcup_{I \subseteq \mathbb{N}} \prod_{i \in I} \mathbb{P}_{F_i}$, we define

$$\mathbb{P}_F(A) := \bigcup_{\mathcal{P} \in A} \mathbb{P}_F(\mathcal{P}), \quad \mathbb{P}_F^{(1)}(A) := \mathbb{P}_F(A) \cap \mathbb{P}_F^{(1)}, \quad N(F, A) := \#\mathbb{P}_F^{(1)}(A). \tag{5}$$

For all finite $A = \{\mathcal{P}_1, \ldots, \mathcal{P}_s\} \subseteq \bigcup_{I \subseteq \mathbb{N}} \prod_{i \in I} \mathbb{P}_{F_i}$, we will also write $\mathbb{P}_F(\mathcal{P}_1, \ldots, \mathcal{P}_s) := \mathbb{P}_F(A)$ and $N(F, \mathcal{P}_1, \ldots, \mathcal{P}_s) := N(F, A)$ and neglect the brackets for all elements $\mathcal{P}_j = (P_j)$ which only consist of one place P_j.

Let E/F_i and F_i/F be separable extensions of function fields and Q/P_i and P_i/P be extensions of places in E/F_i and F_i/F, respectively, or all $i = 1, 2$. Then we call

$$(E, F_1, F_2, F) \text{ (resp. } (Q, P_1, P_2, P)) \tag{6}$$

a diamond of function fields (resp. places). Moreover, a diamond of function fields (resp. of places) is called **non-flat** if all involved function fields (resp. places) are pairwise distinct. Otherwise the diamond is called **flat**.

Some fundamental statements for extensions of function fields. Here, we will list some fundamental statements for function field extensions which we will apply frequently.

Let $E/F/L$ (i.e. E/F and F/L) be finite extensions of function fields. Then we call the identities

$$e(Q|R) = e(Q|P)e(P|R) \text{ resp. } f(Q|R) = f(Q|P)f(P|R) \tag{7}$$

which hold for all extensions $Q/P/R$ of places in $E/F/L$ (i.e. Q/P in E/F and P/R in F/L) the **multiplicative transitivity rules** for the ramification indices and relative degrees (see [Sti08, p. 71, Proposition 3.1.6]), respectively.

Let E/F be a finite separable (resp. finite Galois) extension of function fields. Then the identity

$$[E : F] = \sum_{Q \in \mathbb{P}_E(P)} e(Q|P)f(Q|P) \text{ (resp. } [E : F] = \#\mathbb{P}_E(P) \cdot e(Q|P) \cdot f(Q|P)) \tag{8}$$

which holds for all $P \in \mathbb{P}_F$ is called the **fundamental equality** (see [Sti08, p. 74, Theorem 3.1.11]) and if E and F have the same full constant field, the identity

$$2g(E) - 2 = (2g(F) - 2)[E : F] + \sum_{Q \in \mathbb{P}_E} d(Q|Q \cap F) \deg(Q) \tag{9}$$

is called the **Hurwitz Genus Formula** (see [Sti08, p. 99, Theorem 3.4.13]).

Let $E/F_i/F$ be finite separable extensions of function fields over the same full constant field for all $i = 1, 2$ such that $E = F_1 \cdot F_2$ is the compositum of F_1 and F_2. Let P_i/P be an extension of places in F_i/F for all $i = 1, 2$ such that at least one of these extensions is tame. Then the identities

$$e(Q|P) = \operatorname*{lcm}_{k=1,2} e(P_k|P) \text{ and } e(Q|P_i) = \frac{e(P_j|P)}{\operatorname*{gcd}_{k=1,2} e(P_k|P)} \tag{10}$$

hold for all $Q \in \mathbb{P}_E((P_1, P_2))$ and $\{i, j\} = \{1, 2\}$ by **Abhyankar's Lemma** (see [Sti08, p. 137, Lemma 3.9.1]).

Some invariance properties of the ramification indices, degree and relative degrees. Let F and E be k-algebras and F_1/F_0 be an extension of function fields over the same constant field k such that $F_1 \subseteq F$. Then any monomorphism $\phi : F \to E$ of k-algebras restricts to an isomorphism $F_i \to \phi(F_i)$ of k-algebras for all $i = 1, 2$. Moreover, for all extension Q/P of places in F_1/F_0, we get the extension $\phi(Q)/\phi(P)$ of places in $\phi(F_1)/\phi(F_0)$ and the equalities

$$e(Q|P) = e(\phi(Q)|\phi(P)) \text{ and } f(Q|P) = f(\phi(Q)|\phi(P)) \text{ and } \deg(Q) = \deg(\phi(Q)) \tag{11}$$

hold. We call the first (resp. second; resp. third) equality the **invariance of the ramification indices (resp. relative degrees; resp. degrees of places) under the action of the isomorphism ϕ.**

Let E/F be an extension of function fields over the same full constant field k and k' an algebraic extension field over k which is contained in an algebraic closure C of E. Let $F' := k' \cdot F$ (resp. $E' := k' \cdot E$) be the compositum of k' and F (resp. k' and E) in C. Then we have the equalities

$$e(Q'|Q) = 1 \text{ and } e(P'|P) = 1 \text{ and } e(Q|P) = e(Q'|P') \tag{12}$$

for all $Q' \in P_{E'}$, $Q := Q \cap E$, $P' := Q \cap F'$ and $P := Q \cap F$ by [Sti08, p. 114, Theorem 3.6.3(a)] and the multiplicative transitivity rule for ramification indices in (7) implying the identities $e(Q'|P') = e(Q'|P')e(P'|P) = e(Q'|P) = e(Q'|Q)e(Q|P) = e(Q|P)$. We call the last identity the **invariance of the ramification indices under constant field extensions**.

2.2 Recursive Towers

Purpose of this section. In this section, we will define the basic concepts with which we will work for the rest of this thesis.

Structure of Section 2.2. In Subsection 2.2.1, we will define towers of function fields and its limit for finite constant fields. Then we will recite a sufficient criterion which provides lower bounds for the limits of towers over finite fields.

In Subsection 2.2.2, we will define recursive towers of function fields. Here, we will first give one of the usual definitions via bivariate polynomials and then the more general definition via pairs. In the rest of this thesis, we will work with the more general definition of pair-recursive towers.

In Subsection 2.2.3, we will formulate the main subject of this thesis which is Conjecture 1 and elaborate on all of its four different versions.

In Subsection 2.2.4, we will give an extensive list of examples of recursive towers from the literature.

Finally, in the Subsections 2.2.5, 2.2.6 and 2.2.7, we will introduce crucial concepts like pyramids of function fields, pyramids of places and paths of places and make preparations for the following chapters.

2.2.1 Towers of Function Fields

Purpose of this subsection. In the following Definition 2, we will first define towers of function fields and their limits. Then, in Theorem 4, we will recite a sufficient criterion for the existence of some lower bound for the limit of a tower of function fields over a finite field from [Sti08, p. 249, Theorem 7.2.10].

Towers of function fields. For the following definition of towers of function fields in Definition/Lemma 2, remember Assumption 1, i.e. all function fields F will be defined over perfect full constant fields k.

Definition/Lemma 2. *Let $\mathcal{F} = (F_\nu)_\nu = (F_0, F_1, \dots)$ be a sequence of function fields F_n over the full constant field k. Then we define the following:*

*(i) The sequence $\mathcal{F}$ is called a **tower of function fields over** k if the extensions F_{n+1}/F_n are finite, proper, separable for all $n \in \mathbb{N}_0$ and $g(F_n) \to \infty$ as $n \to \infty$.*

*(ii) We call d the **degree** of the tower $\mathcal{F}$ if $d = [F_{n+1} : F_n]$ holds for all large $n \in \mathbb{N}_0$ and we say that $\mathcal{F}$ has **constant degree** d if $d = [F_{n+1} : F_n]$ holds for all $n \in \mathbb{N}_0$.*

Moreover, we define $\mathbb{P}_{\mathcal{F}}$ (resp. $\mathbb{P}_{\mathcal{F}}(A)$) as the set of all places in $\mathcal{F}$ (resp. in $\mathcal{F}$ which also lie above one of the places in $A \subseteq \mathbb{P}_{\mathcal{F}}$), i.e.

$$\mathbb{P}_{\mathcal{F}} := \coprod_{n \in \mathbb{N}_0} \mathbb{P}_{F_n} \qquad (\textit{resp. } \mathbb{P}_{\mathcal{F}}(A) := \coprod_{n \in \mathbb{N}_0} \mathbb{P}_{F_n}(A)).$$

(iii) Suppose that k is a finite field. Then the sequence $(\frac{N(F_n)}{[F_n:F_0]})_n$ is monotonically decreasing and, hence, converges in $\mathbb{R}_{\geq 0}$ as $n \to \infty$. We call the limit

$$\nu(\mathcal{F}/F_m) := \lim_{n \to \infty} \frac{N(F_n)}{[F_n : F_m]}$$

*the **splitting rate** of $\mathcal{F}$ over F_m and, more specifically, we call $\nu(\mathcal{F}) := \nu(\mathcal{F}/F_0)$ the **splitting rate** of $\mathcal{F}$.*

Moreover, the sequence $(\frac{g(F_n)-1}{[F_n:F_0]})_n = (\frac{g(F_n)}{[F_n:F_0]} - \frac{1}{[F_n:F_0]})_n$ is monotonically increasing and, hence, converges in $\mathbb{R}_{>0} \cup \{\infty\}$ as $n \to \infty$. Consequently, the sequence $(\frac{g(F_n)}{[F_n:F_0]})_n$ also converges in $\mathbb{R}_{>0} \cup \{\infty\}$ as $n \to \infty$. We call the limit

$$\gamma(\mathcal{F}) := \lim_{n \to \infty} \frac{g(F_n)}{[F_n : F_m]}$$

*the **(asymptotic) genus** of $\mathcal{F}$ over F_m and we call $\gamma(\mathcal{F}) := \gamma(\mathcal{F}/F_0)$ the **(asymptotic) genus** of $\mathcal{F}$.*

Finally, the nonnegative real number

$$\lambda(\mathcal{F}) := \frac{\nu(\mathcal{F})}{\gamma(\mathcal{F})} = \lim_{n\to\infty} \frac{N(F_n)}{g(F_n)}$$

is called the ***limit*** *of* $\mathcal{F}$*.*

(iv) *Let us denote* ***Ihara's Constant*** *as* $A(q)$*, i.e.*

$$A(q) := \limsup_{g\to\infty} \frac{N_q(g)}{g} \text{ with } N_q(g) := \max_{g(F)=g} N(F)$$

where the maximum runs over all function fields F *over* $\mathbb{F}_q$ *with genus* g*. For,* $k = \mathbb{F}_q$*, we have the estimates*

$$0 \leq \lambda(\mathcal{F}) \leq A(q) \leq \sqrt{q} - 1$$

where $\sqrt{q}-1$ *is called the* ***Drinfeld-Vladut-Bound*** *and* $\mathcal{F}$ *is called* ***bad*** *if* $\lambda(\mathcal{F}) = 0$*,* ***good*** *if* $\lambda(\mathcal{F}) > 0$ *and* ***optimal*** *if* $\lambda(\mathcal{F}) = A(q)$*.*

Proof. See [Sti08, p. 246, Lemma 7.2.3] for a proof of (iii) and [Sti08, p. 244, Theorem 7.1.3] for a proof of (iv). □

Different types of towers of function fields. There are several types of towers of function fields: Some towers are constructed by using class field theory and others by using modular techniques. The reader is referred to [Bee22] for more on these constructions. In this thesis, we will only be concerned with recursive towers.

Lower bounds for the limits of towers. Determining the values $\nu(\mathcal{F})$, $\gamma(\mathcal{F})$ and $\lambda(\mathcal{F})$ or even finding non-trivial estimates can be quite challenging. Here the following criterion in Theorem 4, which comes from [Sti08, p. 249, Theorem 7.2.10], provides a simple way to find estimates for these values. It provides a lower (resp. upper) bound for $\nu(\mathcal{F})$ (resp. $\gamma(\mathcal{F})$) in terms of the splitting locus $\mathrm{Split}(\mathcal{F}/F_0)$ (resp. ramification locus $\mathrm{Ram}(\mathcal{F}/F_0)$) of $\mathcal{F}$ over F_0 (see Definition 3).

The fourth major result of this thesis, which consists of Corollary 196 and Corollary 200, will provide sharp versions of this criterion for recursive towers satisfying conditions which we will state later. This means that we will get the precise limits and not only lower bounds. The only recursive tower in the literature known to the author for which these corollaries are not applicable is the CNT-tower in the following Examples 8(v).

Definition 3. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a tower. Then we define the following:*

(i) *The set*

$$\mathrm{Split}(\mathcal{F}/F_m) = \{P \in \mathbb{P}^{(1)}_{F_m} : P \textit{ splits completely in } F_n/F_m \textit{ for all } n \in \mathbb{N}_0\}$$

is called the ***splitting locus*** *of* $\mathcal{F}$ *over* F_m *for all* $m \in \mathbb{N}_0$*.*

(ii) *The set*

$$\mathrm{Ram}(\mathcal{F}/F_m) = \{P \in \mathbb{P}_{F_m} : P \textit{ is ramified in } F_n/F_m \textit{ for some } n \in \mathbb{N}\}$$

is called the ***ramification locus*** *of* $\mathcal{F}$ *over* F_m *for all* $m \in \mathbb{N}_0$*.*

Theorem 4. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a tower over a finite field. Then the following hold:*

(i) We have the estimate

$$\nu(\mathcal{F}) \geq \#\operatorname{Split}(\mathcal{F}/F_0).$$

(ii) Suppose that $\operatorname{Ram}(\mathcal{F}/F_0)$ *is finite and that there are* $\alpha(P) \in \mathbb{R}_{\geq 0}$ *such that*

$$\alpha(P) \cdot e(Q|P) - d(Q|P) \geq 0$$

for all $P \in \operatorname{Ram}(\mathcal{F}/F_0)$*, all* $Q \in \mathbb{P}_{F_n}(P)$ *and all* $n \in \mathbb{N}_0$ *Then we have the estimate*

$$\gamma(\mathcal{F}) \leq \mathrm{g}(F_0) - 1 + \frac{1}{2} \cdot \sum_{P \in \operatorname{Ram}(\mathcal{F}/F_0)} \alpha(P)\deg(P).$$

(iii) If $\operatorname{Split}(\mathcal{F}/F_0)$ *is non-empty,* $\operatorname{Ram}(\mathcal{F}/F_0)$ *is finite and there are positive real numbers* $\alpha(P)$ *as in (ii), then* $\mathcal{F}$ *is a good tower and we have the following lower bound*

$$\lambda(\mathcal{F}) \geq \frac{\#\operatorname{Split}(\mathcal{F}/F_0)}{g(F_0) - 1 + \frac{1}{2} \cdot \sum_{P \in \operatorname{Ram}(\mathcal{F}/F_0)} \alpha(P)\deg(P)}.$$

2.2.2 Polynomial- and Pair-Recursive Towers.

The main objects of this thesis: Recursive towers of function fields. Next, we will come to the definition of the main objects of this thesis, namely recursive towers of function fields. In Definition 5(i), we will give one of the usual definitions of a recursive tower, which is defined by a bivariate polynomial. For instance, [BGS04] uses this definition. In this thesis, we will mainly work with the definition in Definition 5(ii) of a recursive tower which is defined by a pair. We prefer this definition because of the following two reasons: First, we will prove in Lemma 7 that this definition is more general and, second, there are also practical reasons which we will elaborate on in the items (ii) and (iii) of Remark 6 and in the last paragraph 'Advantages of the introduction of pair-recursive towers' of this subsection.

Definition 5. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a tower of constant degree* d *over a field* k.

(i) The tower $\mathcal{F}$ *is called* ***recursively defined by a polynomial*** $f = f(X, Y) \in k[X, Y]$ *if* f *is geometrically irreducible, separable in both variables,* $\deg_Y(f) = d$ *and there exist elements* $x_n \in F_n$ *for all* $n \in \mathbb{N}_0$ *such that*

$$F_0 = k(x_0), \quad F_{n+1} = F_n(x_{n+1}) = k(x_0, \dots, x_n, x_{n+1}), \quad f(x_n, x_{n+1}) = 0.$$

(ii) Let $\overline{\mathbf{F}}$ *be an algebraic closure of* $\mathbf{F} = \bigcup_{n=0}^{\infty} F_n$ *and let* $\overline{k}$ *be the algebraic closure of* k *which is contained in* $\overline{\mathbf{F}}$*. We call the tower* $\mathcal{F}$ ***recursively defined by the pair*** (σ, F_0) *with* $\sigma \in \operatorname{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ *if* $F_1/\sigma^i(F_0)$ *is separable for all* $i = 0, 1$ *and*

$$F_{n+1} := F_n \cdot \sigma^{n+1}(F_0) = \prod_{i=0}^{n} \sigma^i(F_0)$$

for all $n \in \mathbb{N}_0$ *(see Figure 2.1). If we just say that a tower is* ***recursively defined*** *or a* ***recursive*** *tower, then we mean that the tower is recursively defined by a pair* (σ, F_0).

Notice that there could be multiple σ *which make* $\mathcal{F}$ *to a recursive tower. Most of the concepts, which we will define for a recursive tower* $\mathcal{F}$ *in this thesis, will also depend on the concrete* σ*. However, we will mostly neglect this dependence in our formulations as this should never lead to confusions. Correspondingly, we will also call* σ *'the'* ***tower map*** *of* $\mathcal{F}$.

Moreover, we say that the recursive tower $\mathcal{F}$ *has* ***balanced degree*** d *if* $[F_1 : \sigma(F_0)] = d$.

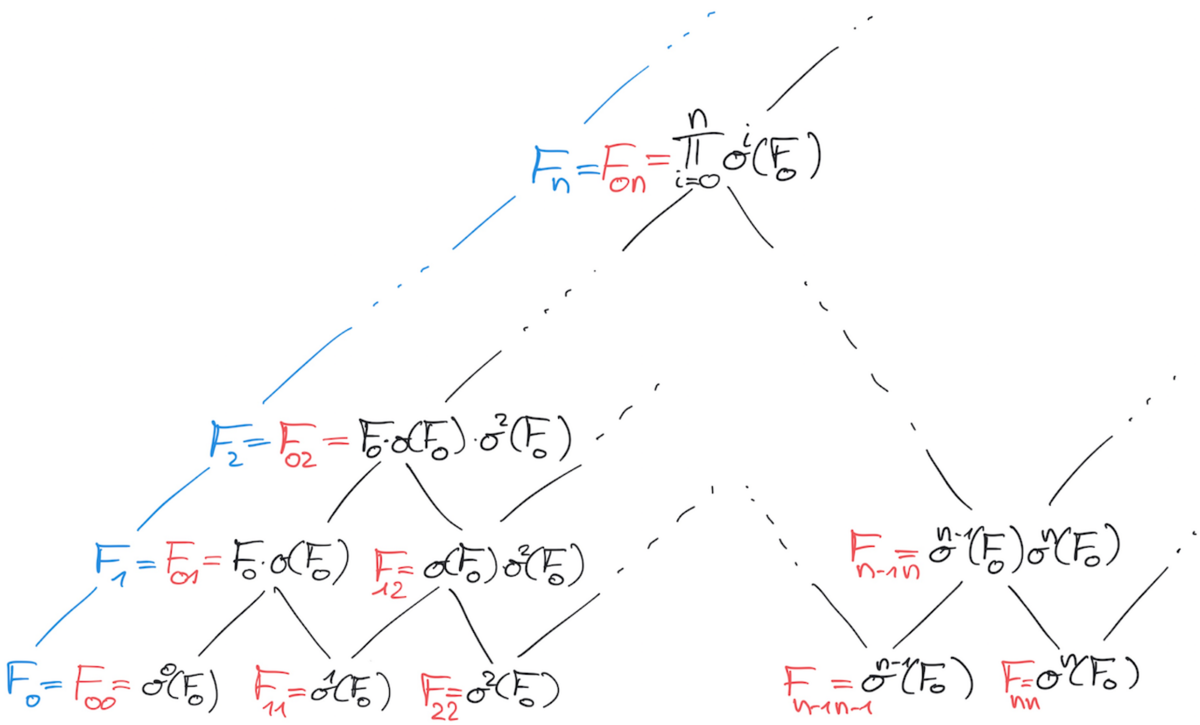


Figure 2.1: Pair-recursive tower and the involved composita

Remark 6. *(i) There are other definitions of recursive towers which are not of constant degree (see [Sti08, p. 251, Definition 7.2.12]). But then there is an index m such that $(F_{m+\nu})_\nu$ is a pair-recursive tower as in Definition 5(ii).*

Moreover, in other definitions, it is also not required that the defining polynomial $f(X,Y)$ is separable in Y. However, this more general definition, does essentially not provide more recursive towers: In this case, we can write $f(X,Y) = g(X, Y^{p^s})$ with a polynomial $g(X,Y)$ which is separable in both variables. As the inseparable part of the extension $k(x_0,x_1)/k(x_1)$ does not effect $N(F_1)$ or $g(F_1)$, we can just replace $f(X,Y)$ with $g(X,Y)$ and obtain isomorphic function fields on every level of the recursive towers.

(ii) An equivalent way of defining a pair-recursive tower in Definition 5(ii) is to extend the definition of a polynomial-recursive tower in Definition 5(i) and allow finitely many polynomials. For instance, the multi-step towers in [Lö07, p. 17, Definition 2.19, p. 21 Definition 2.21] use such a definition.

However, we chose the 'pair' definition over the 'multi-step' definition for practical reasons: We will mainly work abstractly with recursive tower and often with the geometric tower $\overline{\mathcal{F}}$ (see Definition 21), the pyramid $\mathrm{Pyr}(\mathcal{F})$ (see Definition 9) and the tower graph $\Gamma_{\mathcal{F}}$ (see Definition 74) of $\mathcal{F}$ which are easier to define and deal with if we already have the automorphism $\sigma \in \mathrm{Aut}_{\overline{k}}(\overline{\mathbf{F}})$.

(iii) Notice that, in Definition 5(ii), it would also be sufficient to define σ to only be a monomorphism $\sigma' : \mathbf{F} \to \mathbf{F}$ of k-algebras. However, we extended σ to an automorphism of some algebraic closure $\overline{\mathbf{F}}$ of $\mathbf{F}$, again, for practical reasons:

At many points, it will be useful to apply σ^{-1} on places $Q \in \mathbb{P}_{F_n}$. Then the definition with the tower map $\sigma \in \mathrm{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ avoids extending σ' to an automorphism or laboriously circumventing the application of σ^{-1} on Q.

Moreover, if we want change the constant field k of $\mathcal{F}$ via an algebraic extension

k'/k of fields, then $\overline{\mathbf{F}}$ already contains an isomorphic copy of k'. Consequently, we can restrict ourselves to algebraic extension fields k' of k which are contained in $\overline{\mathbf{F}}$ and work with composita of subfields inside of $\overline{\mathbf{F}}$.

Polynomial-recursive towers are pair-recursive. As we already proclaimed, the definition of a pair-recursive tower in Definition 5(ii) is more general than the definition of a polynomial-recursive tower in Definition 5(i), i.e. a tower which is recursively defined by a polynomial is also recursively defined by a pair. This will be the subject of the following Lemma 7.

Lemma 7. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by a polynomial f, let x_n be the generator of F_n in Definition 5(i) for all $n \in \mathbb{N}_0$, let $\overline{\mathbf{F}}$ be an algebraic closure of $\mathbf{F} := \bigcup_{n=0}^{\infty} F_n = k((x_n)_n) = k(x_0, x_1, \dots)$ and let $\overline{k}$ be the algebraic closure of k which is contained in $\overline{\mathbf{F}}$. Then the following hold:*

(i) Then

$$\sigma' : \mathbf{F} \to \mathbf{F} \text{ via } h(x_0, \dots, x_n) \mapsto h(x_1, \dots, x_{n+1})$$

is a well defined k-algebra monomorphism (see Figure 2.2).

(ii) There is an extension $\sigma \in \operatorname{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ of σ'.

(iii) For all extensions $\sigma \in \operatorname{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ of σ', the tower $\mathcal{F}$ is recursively defined by the pair (σ, F_0).

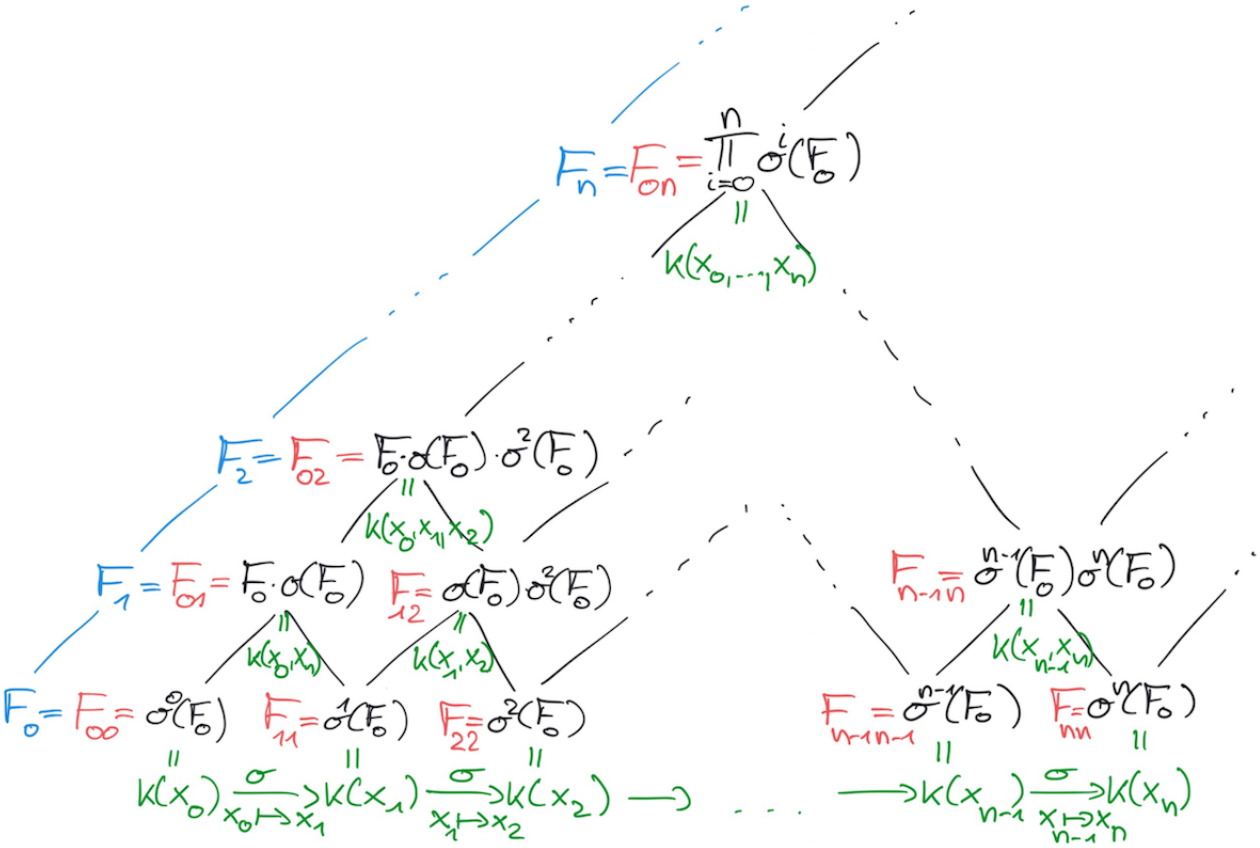


Figure 2.2: Construction of the tower map for polynomial-recursive towers

Proof. For (i): First, we show that $\sigma'_n := \sigma'_{|F_n}$ is a well defined morphism of k-algebras via induction over $n \in \mathbb{N}_0$.

Let $n = 0$. Then $F_0 = k(x_0)$ is a rational function field over k. Since $\deg_y(f(x,y)) = d \geq 2$ by Definition 2(i), we have a representation $f(x,y) = \sum_{i=0}^{d} a_i(x)y^i$ with $a_i(x) \in k[x]$ for all $i = 0, \dots, d$ and $a_d(x) \neq 0$. Now, choosing $\alpha \in \overline{k}$ with $a_d(\alpha) \neq 0$ yields $\deg_y(f(\alpha, y)) = d$. But, this implies that

$$f(x, x_1) \in k(x_1)[x] \backslash \{0\} \tag{13}$$

by the following reasoning: Otherwise $f(\alpha, x_1) = 0$ yields that x_1 is algebraic over k because α is algebraic over k and $\deg_y(f(\alpha, y)) \geq 2$. Then we even conclude $x_1 \in \overline{k} \backslash k$ by $[k(x_0, x_1) : k(x_0)] = [F_0(x_1) : F_0] = [F_1 : F_0] = d \geq 2$ and $k \subseteq F_0$. But this contradicts that k is the full constant field of F_1.

Hence, by (13), we obtain

$$[F_1 : k(x_1)] = [k(x_0, x_1) : k(x_1)] \leq \deg_x(f(x, x_1)) < \infty \tag{14}$$

by Definition 5(i) providing the identity $f(x_0, x_1) = 0$. Then we compute the equalities

$$[k(x_1) : k] = \frac{[F_1 : k]}{[F_1 : k(x_1)]} = \frac{[F_1 : F_0][F_0 : k]}{[F_1 : k(x_1)]} = \infty \tag{15}$$

where the first two equalities hold by the well known rule $[K_3 : K_1] = [K_3 : K_2][K_2 : K_1]$ for extensions $K_3/K_2/K_1$ of fields and the last equality holds by the estimate $[F_1 : k(x_1)] < \infty$ in (14) and the fact that $F_0 := k(x_0)$ being a rational function field over k implies the equality $[F_0 : k] = \infty$. Now, the identity in (15) provides that x_1 is transcendental over k and, consequently, $\sigma'_0 : F_0 := k(x_0) \to \mathbf{F}$, $h(x_0) \mapsto h(x_1)$ is a well defined morphism of k-algebras.

Next, let $n \geq 1$. On the one hand, because we have the identities $f(x_{n-1}, x_n) = 0$ and $\deg_y(f(x,y)) = d = [F_n : F_{n-1}]$ in Definition 2(i), the polynomial $f(x_{n-1}, t) \in F_{n-1}[t]$ is the minimal polynomial of x_n over F_{n-1} (up to multiplication with an element in F^*_{n-1}). On the other hand, the induction hypothesis yields that $\sigma'_{n-1} : F_{n-1} \to \mathbf{F}$, $h(x_0, \dots, x_{n-1}) \mapsto h(x_1, \dots, x_n)$ is a well defined morphism of k-algebras and $f(x_n, x_{n+1}) = 0$ and we obtain the identity $\widehat{\sigma'_{n-1}}(f(x_{n-1}, t)) = f(x_n, t)$ for the canonical extension $\widehat{\sigma'_{n-1}} : F_{n-1}[t] \to \overline{\mathbf{F}}[t]$ of σ'_{n-1}. Combining these two conclusions and the fact that F_n/F_{n-1} is a simple algebraic field extension with primitive element x_n supplies that the extension $\sigma'_n : F_n \to \mathbf{F}$, $h(x_0, \dots, x_n) \mapsto h(x_1, \dots, x_{n+1})$ of σ'_{n-1} is a well defined morphism of k-algebras.

Finally, we notice that $\mathbf{F}$ (with the inclusion maps $\iota_i : F_i \hookrightarrow \mathbf{F}$ for all $i \in \mathbb{N}_0$) is a direct limit of the ordered system $(F_\nu)_\nu$ (with the inclusion maps $F_i \hookrightarrow F_j$ for all $i \leq j$). Then, by the universal property of direct limits, there is a morphism $\tau : \mathbf{F} \to \mathbf{F}$ of k-algebras such that $\sigma'_n = \tau \circ \iota_n$ for all $n \in \mathbb{N}_0$. But this morphism τ of k-algebras has to be σ'. Moreover, σ' is a monomorphism because $\mathbf{F}$ is a field and because σ' is a non-zero morphism of rings.

For (ii): As F_n has full constant field k, we have $F_n \cap \overline{k} = k$ for all $n \in \mathbb{N}_0$ and, therefore, $\mathbf{F} \cap \overline{k} = k$ by the definition of the full constant field of a function field in [Sti08, p. 1, Definition 1.1.1]. As k is a perfect field, $\overline{k}/k$ is a Galois extension. Hence, combining the identity $\mathbf{F} \cap \overline{k} = k$ and [Coh91, p. 188, Theorem 5.5] supplies that $\overline{k}/k$ and $\mathbf{F}/k$ are linearly disjoint. But this implies that $\overline{k} \cdot F$ is a tensor product of $\overline{k}$ and F of k-algebras (with the inclusion morphisms) by [Coh91, p. 188, Proposition 5.4]. Consequently, by the universal property of tensor products, there is a morphism $\tilde{\sigma} : \overline{k} \cdot F \to \overline{F}$ of k-algebras which restricts to the identity on $\overline{k}$ and to σ' on $\mathbf{F}$. Hence, $\tilde{\sigma}$ is a morphism of $\overline{k}$-algebras. Now, it is well known that any morphism $\tilde{\sigma} : \mathbf{F} \to \overline{\mathbf{F}}$ of $\overline{k}$-algebras has an extension to some $\overline{k}$-algebra automorphism $\sigma : \overline{\mathbf{F}} \to \overline{\mathbf{F}}$. Therefore, we deduce the existence of an extension

$\sigma \in \mathrm{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ of σ'.

For (iii): Let $\sigma \in \mathrm{Aut}_{\overline{k}}(\overline{\mathbf{F}})$ be an extension of σ'. Then $\sigma^n(F_0) = k(x_n)$ and, therefore, $\prod_{i=0}^{n} \sigma^i(F_0) = k(x_0, \dots, x_n) =: F_n$. Hence, $\mathcal{F}$ is recursively defined by the pair (σ, F_0) by Definition 2(ii). ☐

Advantages of the introduction of pair-recursive towers. Although, except for some multi-step towers in [Lö07] (e.g. [Lö07, p. 114, Example 6.1.3.1.1] and [Lö07, p. 117, Example 6.1.3.1.2]), there are no examples of pair-recursive towers in the literature which are not already polynomial-recursive towers, the introduction of the more general concept of pair-recursive towers has still some advantages:

First, it is more general and thus includes more towers of function fields, which can maybe be interesting in the more general pursue of finding good towers over finite fields.

Second, as we are interested in the more general versions of Conjecture 1, namely Conjecture 1(iii) and Conjecture 1(iv), it will make sense to consider the level m truncation $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) := (F_{m+\nu})_\nu$ of the recursive tower $\mathcal{F} = (F_\nu)_\nu$ (see Definition 27) and to also apply the introduced concepts to the truncated tower $\mathcal{F}_{\geq m}$. But, for large level m, $\mathcal{F}_{\geq m}$ is not a polynomial-recursive tower anymore. However, it is still a pair-recursive tower with the pair (σ, F_m).

2.2.3 Conjecture 1

The main subject of this thesis: Conjecture 1. In the following, we will formulate the main subject of this thesis, namely Conjecture 1 where Conjecture 1(i) originates from [BGS04, p. 7, Conjecture 1]. In more descriptive words, in [BGS04, p. 7, Conjecture 1], it is conjectured that any recursive tower $\mathcal{F} = (F_\nu)_\nu$ with a positive splitting rate must also have a rational place P in F_0 which splits completely on every level F_n/F_0.

Conjecture 1. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field of balanced degree. Then we have the following four conjectures:*

(i) If $\nu(\mathcal{F}) > 0$, then $\# \mathrm{Split}(\mathcal{F}/F_0) > 0$.

(ii) $\nu(\mathcal{F}) = \# \mathrm{Split}(\mathcal{F}/F_0)$.

(iii) If $\nu(\mathcal{F}) > 0$, then there is some $m \in \mathbb{N}_0$ such that $\# \mathrm{Split}(\mathcal{F}/F_m) > 0$.

(iv) There is some $m \in \mathbb{N}_0$ such that $\nu(\mathcal{F}/F_m) = \# \mathrm{Split}(\mathcal{F}/F_m)$.

Clearly, (ii) implies (i) (resp. (iv)) and, moreover, (i) (resp. (iv)) implies (iii).

More on Conjecture 1(i). Conjecture 1(i) is the original Conjecture 1 in [BGS04, p. 7, Conjecture 1] (2005). Although there are no counterexamples published in the literature, Conjecture 1(i) was never mentioned again: In [Sti10, p. 5, Problem 1] (2010) only the weaker Conjecture 1(iii) was formulated and, in [Bee22, p. 10, p. 24] (2022), it was confirmed that the weaker Conjecture 1(iii) is still open. Thus, we speculate that the authors of [Sti10] and [Bee04] actually found counterexamples to Conjecture 1(i) but have not published them. Nonetheless, we will still give a counterexample to Conjecture 1(i) in Example 129.

More on Conjecture 1(ii). In [BGS04, p. 7, Conjecture 1], the authors even formulated a refinement of Conjecture 1(i), which is Conjecture 1(ii). In more descriptive words, it conjectures that the only places P in F_0 which contribute to the splitting rate $\nu(\mathcal{F})$, i.e. for which $\lim_{\nu\to\infty} N(F_\nu, P)/d^\nu > 0$ holds, are the places P which split completely on every level F_n/F_0. Conjecture 1(ii) was also never mentioned again. There are many recursive towers in [MW05] which we observed to be counterexamples to Conjecture 1(ii), e.g. the recursive tower $\mathcal{F}_{MW,11}$ over $\mathbb{F}_9$ which is defined by the polynomial $f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$ in [MW05, p. 212, f_{11}].

More on Conjecture 1(iii). As already mentioned, in [Sti10, p. 5, Problem 1] (2010), the weaker Conjecture 1(iii) was formulated for the first time and, in [Bee22, p. 10, p. 24] (2022), it is confirmed that Conjecture 1(iii) is still open. Indeed, all the counterexamples to Conjecture 1(i) and Conjecture 1(ii) still satisfy Conjecture 1(iii).

However, although the author is convinced that Conjecture 1(iii) is true for most recursive towers, he is not convinced that it is true for all recursive towers. More specifically, in the paragraph 'A sufficient criterion to disprove Conjecture 1(iii)' of Subsection 6.3, we will present a strategy to possibly produce a counterexample to Conjecture 1(iii).

More on Conjecture 1(iv). Finally, Conjecture 1(iv) is new and it is a refinement of Conjecture 1(iii). It also provides the splitting rates and is therefore useful. Nonetheless, in Example 181, we will show that the wild CNT-tower in Examples 8(v) is a counterexample to this conjecture. It is the only counterexample to this conjecture known to the author.

Main result - An almost complete answer to Conjecture 1(iii). Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example known to the author, Corollary 183 will characterize all recursive towers $\mathcal{F} = (F_\nu)_\nu$ which satisfy Conjecture 1(ii), i.e. $\nu(\mathcal{F}) = \mathsf{Split}(\mathcal{F}/F_0)$. More specifically, in a less technical manner, Corollary 184 will reformulate the criterion for satisfying Conjecture 1(ii) as a sufficient criterion.

This criterion will be very mild and the CNT-tower will be the only recursive tower $\mathcal{F} = (F_\nu)_\nu$ known to the author which does not have some truncation (that is the recursive tower $\mathcal{F}_{\geq m} = (F_{m+\nu})_n$ for some $m \in \mathbb{N}_0$; see also Definition/Lemma 27) satisfying this criterion. Hence, we will call Corollary 184 our almost complete answer to Conjecture 1(iii).

2.2.4 Examples of Recursive Towers

Purpose of this subsection. In this section, we will give an extensive list of polynomial-recursive towers which appear in the literature. Most of the examples will reoccur at some points in this thesis.

First, in Examples 8, we will introduce the examples, specify the involved parameters, discuss in which contexts these examples appear in the literature and mention the implications which the results of this thesis have on these examples. Second, in Figure 2.3, we will provide a compact list of all the recursive towers from Examples 8.

Two further mentionable references which are not represented in Examples 8 are [Wul02] and [Lö07]. There we can find many further examples of recursive towers.

Examples 8. *(i) For small quadratic q, in [MW05], the authors found about one hundred good tame recursive towers $\mathcal{F}_{MW,i}$ of degree two via a computational search. In the subsequent list in Figure 2.3, some of the corresponding quadratic polynomials $f_{MW,i}$ are included. Note that our indexing of these polynomials is the same as the indexing in [MW05].*

In the following, we will call the polynomials $f_{MW,i}$ the ***MW-polynomials*** *and the good quadratic tame recursive towers $\mathcal{F}_{MW,i}$ the* ***MW-towers****.*

For most of the MW-towers the limit is known. This comes mainly from the fact that the lower bounds which are computed for the limits in [MW05] already attain the Drinfeld-Vladut-Bound in Definition 2(iv). However, for some of these MW-towers only lower bounds for their limits were known and as a consequence of Corollary 195, we will be able to determine the precise limits of these remaining MW-towers in Subsection 6.4.5. The subsequent list in Figure 2.3 contains all these remaining recursive MW-towers. They are defined by the polynomials $f_{MW,i}$ for $i = 8, 12, 14, 15, 16, 20, 21$. Also note that, in [MW05, p. 213], the authors explicitly asked whether the computed lower bounds for the limits are already the precise limits.

(ii) For quadratic q, in chapter seven of [Sti08], the author provides several examples of good recursive towers. We want to include the following three:

First, in [Sti08, p. 262, Proposition 7.3.2] (resp. Second, in [Sti08, p. 261, Proposition 7.3.3]), it is shown that, for $l \geq 3$ and $q := l^2$ (resp. $l \geq 2$, $e \geq 2$, $q := l^e$ and $m := \frac{q-1}{l-1}$), the polynomial

$$f_{GS,1} := Y^{l-1} + (X+1)^{l-1} - 1 \quad (\textit{resp. } f_{GS,2} := Y^m + (X+1)^m - 1\ )$$

defines a good tame recursive tower $\mathcal{F}_{GS,1}$ (resp. $\mathcal{F}_{GS,2}$) over $\mathbb{F}_q$ and its limit has the lower bound $\lambda(\mathcal{F}_{GS,1}) \geq \frac{2}{l-1}$ (resp. $\lambda(\mathcal{F}_{GS,1}) \geq \frac{2}{q-2}$).

Third, in [Sti08, p. 262, Definition 7.4.1] it is shown that, for $q := l^2$, the polynomial

$$f_{GS,3} := (Y^l - Y)(1 - X^{l-1}) - X^l$$

defines a good wild recursive tower $\mathcal{F}_{GS,3}$ over $\mathbb{F}_q$ and its limit attains the Drinfeld-Vladut-Bound $\lambda(\mathcal{F}_{GS,3}) = l - 1$.

In the following, we will call the polynomials $f_{GS,i}$ the ***GS-polynomials*** *and the recursive towers $\mathcal{F}_{GS,i}$ the* ***GS-towers****.*

As a consequence of Corollary 195, in Subsection 6.4.5 we will be able to prove that the limits of the GS-towers $\mathcal{F}_{GS,i}$ are actually equal to the lower bounds from above for $i = 1, 2$.

(iii) For cubic $q = l^3$, the best known lower bound for Ihara's constant $A(q)$ is given by the estimate

$$\frac{2(l^2-1)}{l+2} \leq A(q).$$

For instance, these lower bound are realized via the limits of the recursive ***BezGS-towers*** *$\mathcal{F}_{BezGS,l}$ in [BGS05, p. 161, Main Theorem] which are defined by the* ***BezGS-polynomials***

$$f_{BezGS,l} := Y^l(X^l + X - 1) - (1 - Y)X.$$

In [BGS05, p. 161, Main Theorem], it was shown that the limit $\lambda(\mathcal{F}_{BezGS,l})$ of $\mathcal{F}_{BezGS,l}$ has the lower bound

$$\frac{2(l^2-1)}{l+2} \leq \lambda(\mathcal{F}_{BezGS,l}).$$

As a consequence of our Corollary 200, we will derive that the lower bound $2(l^2 - 1)/(l + 2)$ is actually equal to the limit $\lambda(\mathcal{F}_{BezGS,l})$ of $\mathcal{F}_{BezGS,l}$ in Corollary 204.

For $l = 2$ and after a change of coordinates, the BezGS-tower $\mathcal{F}_{BezGS,2}$ can be identified with the recursive ***GV-tower*** *$\mathcal{F}_{GV}$ which was introduced in [vdGvdV02, p. 292] and is defined by the* ***GV-polynomial***

$$f_{GV} := (Y^2 + Y)X + X^2 + X + 1.$$

In [vdGvdV02, p. 292], the equality $\lambda(\mathcal{F}_{GV}) = 3/2$ was already proven.

(iv) In [ST15, p. 680, Theorem 2.14], up to isomorphisms, the authors reduced the potential candidates for quadratic polynomials over $\mathbb{F}_2$ which can potentially define good recursive towers to the four polynomials

$$f_{ST,1} := Y^2X + Y + X^2 + 1, \qquad f_{ST,3} := X^2Y^2 + XY^2 + Y + X,$$
$$f_{ST,2} := X^2 + XY^2 + X + Y, \qquad f_{ST,4} := X^2Y^2 + XY^2 + Y + X^2 + 1 \quad (16)$$

In the following, we will call the polynomials $f_{ST,i}$ the ***ST-polynomials*** *and the wild recursive towers $\mathcal{F}_{ST,i}$ which are defined by the ST-polynomials $f_{ST,i}$ the* ***ST-towers****.*

As a consequence of the Main Theorem 177, we will be able to prove in Corollary 202 that all four ST-towers are asymptotically bad. Thus, there is no good quadratic recursive tower over $\mathbb{F}_2$.

(v) In [BGS06, p. 69, Theorem6.4], the authors classified all potentially good polynomial-recursive towers of so called ***Kummer-type*** *and* ***Artin-Schreier-type****. For polynomials of degree two with coefficients in $\mathbb{F}_2$, their classification reduced the potentially good polynomal-recursive towers of Artin-Schreier-type to four candidates. Essentially, these were the GS-tower $\mathcal{F}_{GS,3}$, two times the GV-tower $\mathcal{F}_{GV}$ after coordinate changes and the CNT-tower $\mathcal{F}_{CNT,s}$ which is defined by the* ***CNT-polynomial***

$$f_{CNT} := (Y^2 + Y)(X^2 + X + 1) + X.$$

Twelve years later in [CNT18, p. 19, Corollary 4.13], the authors finally concluded that $\mathcal{F}_{CNT,s}$ is good over $\mathbb{F}_{2^s}$ for even s, bad for odd s and optimal for $s = 2$, i.e. $\lambda(\mathcal{F}_{CNT,2}) = 1 = \sqrt{4} - 1$.

The CNT-tower $\mathcal{F}_{CNT,s}$ is a very unique recursive tower because it is the only example of a recursive tower known to the author satisfying the following property where all occurring notions will be defined later: The ramification subgraph of every truncation of the CNT-tower has a finite balanced weakly connected component with ramified edges. This will be proven in Example 181.

On the one hand, this will have the consequence that we cannot apply the main result in Corollary 184 to $\mathcal{F}_{CNT,s}$ and, by that, conclude $\nu(\mathcal{F}/F_m) = \mathrm{Split}(\mathcal{F}/F_m)$ for some $m \in \mathbb{N}_0$. In fact, we will derive the estimate $\nu(\mathcal{F}/F_m) > \mathrm{Split}(\mathcal{F}/F_m)$ for all $m \in \mathbb{N}_0$ in Example 181. Therefore, the CNT-tower is a counterexample to Conjecture 1(iv).

On the other hand, this will also mean that we cannot apply the fourth major result in Corollary 195 and Corollary 200 to $\mathcal{F}_{CNT,s}$ and, by that, obtain the precise limit $\lambda(\mathcal{F}_{CNT,2}) = 1$. This equality and the estimates $\lambda(\mathcal{F}_{CNT,s}) \geq 1$ for all even s were the contribution of [CNT18, p. 19, Corollary 4.13, Corollary 4.14].

Nonetheless, as a consequence of the third major result in Theorem 188, we will be able to show that the equality $\lambda(\mathcal{F}_{CNT,2}) = 1$ holds for all even s in Corollary 207.

(vi) In [BBGS15, p. 3], the authors introduced the ***BBGS-towers*** *$\mathcal{F}_{BBGS,q,i,j}$ over $\mathbb{F}_{q^m}$ for all prime powers q and all $m := i + j$ with $i, j \in \mathbb{N}$ and $\gcd(i, j) = 1$. Note that,*

in [BBGS15], it was only proven that $\mathcal{F}$ does not get stationary and not that it has constant degree. However, in [CCH21, p. 3, Main Theorem] and [CCH21, p. 10, Lemma 2.4], it was shown that $\mathcal{F}$ has constant degree. Hence, the BBGS-towers $\mathcal{F}$ are indeed recursive towers in sense of Definition 5(ii).

Then, in [BBGS15, p. 4, Theorem 1.2], they could prove the lower bound

$$\lambda(\mathcal{F}_{BBGS,q,i,j}) \geq 2 \cdot \left(\frac{1}{q^j - 1} + \frac{1}{q^i - 1}\right)^{-1} \tag{17}$$

for the limit of $\mathcal{F}_{BBGS,q,i,j}$. By that, in [BBGS15, p. 3, Theorem 1.1], the authors provided the lower bound

$$A(p^m) \geq 2 \cdot \left(\frac{1}{p^{\lfloor m/2 \rfloor} - 1} + \frac{1}{p^{\lceil m/2 \rceil} - 1}\right)^{-1} \tag{18}$$

for all $p \in \mathbb{P}$ and all $m \geq 2$ which is currently the largest known lower bound for Ihara's constant $A(p^m)$ for all $m \geq 2$.

More concretely, the BBGS-towers are the wild recursive towers $\mathcal{F}_{BBGS,q,i,j}$ which are defined by the equation

$$\mathrm{Tr}_j\left(\frac{Y}{X^{q^i}}\right) + \mathrm{Tr}_i\left(\frac{Y^{q^j}}{X}\right) = 1 \tag{19}$$

over $\mathbb{F}_{q^m}$ for all prime powers q and all $m := i + j$ with $i, j \in \mathbb{N}$ and $\gcd(i,j) = 1$ where $\mathrm{Tr}_a(t) := \sum_{\nu=0}^{a-1} t^{q^\nu}$ for all $a \in \mathbb{N}_0$.

Let $\varepsilon_{q,i,j}$ be the main denominator of the left side of the defining equation in (19). Then we will call the polynomials

$$f_{BBGS,q,i,j} = \varepsilon_{q,i,j} \cdot \left(\mathrm{Tr}_j\left(\frac{Y}{X^{q^i}}\right) + \mathrm{Tr}_i\left(\frac{Y^{q^j}}{X}\right) - 1\right) \in \mathbb{F}_q[X,Y]$$

the ***BBGS-polynomials****.*

As a consequence of our Corollary 200, we will be able to prove in Corollary 203 that the limit of $\mathcal{F}_{BBGS,q,i,j}$ is actually equal to the already established lower bound in (357).

(vii) *Let*

$$f_{HP,q} := Y^2(3X - 1) - (X^2 + X) \in \mathbb{F}_q[X,Y]$$

for $q = p^s$ with $p \in \mathbb{P}\backslash\{2,3\}$ and $s \in \mathbb{N}$. In [HP16, p. 12, Proposition 12], it is shown that $f_{HP,q}$ defines a recursive tower $\mathcal{F}_{HP,q}$ over $\mathbb{F}_q$.

In the following, we will call the polynomials $f_{HP,q}$ the ***HP-polynomials*** *and the tame recursive towers $\mathcal{F}_{HP,q}$ the* ***HP-towers****.*

(viii) *Let*

$$f_{BR,5} := (X^6 + X + 2)(Y^5 - Y) - (X^5 - X)(Y^6 + Y^5 + 2Y + 3) \in \mathbb{F}_5[X,Y]$$

and

$$f_{BR,q} := (X^{q+1} + b)(b + n)(Y^q - Y) - 2b(Y^{q+1} + n)(X^q - X) \in \mathbb{F}_q[X,Y]$$

for all $q \notin \{2,3,5\}$ and non-squares $-b, -n \in \mathbb{F}_q^\times$ such that $n \neq \pm b$. Then $f_{BR,q}$ defines a recursive tower $\mathcal{F}_{BR,q}$ over $\mathbb{F}_q$ in [BR20, p. 4].

In the following, we will call the polynomials $f_{BR,q}$ the ***BR-polynomials*** *and the tame recursive towers $\mathcal{F}_{BR,q}$ the* ***BR-towers****.*

In [BR20, p. 3], it is shown that $\mathcal{F}_{BR,q}$ is a good tame recursive tower and that its limit has the lower bound $\lambda(\mathcal{F}_{BR,q}) \geq \frac{2}{q-2}$. Up to this point, $\mathcal{F}_{BR,q}$ is the only good recursive towers over $\mathbb{F}_q$ for all primes $q \notin \{2,3\}$.

As a consequence of Corollary 195, we will be able to prove in Corollary 205 that the limit of $\mathcal{F}_{BR,q}$ is actually equal to $\lambda(\mathcal{F}_{BR,q}) = \frac{2}{q-2}$ for all $q \notin \{2,3\}$.

A compact list of the examples. Any member of the following list in Figure 2.3 consists of the following data: A reference for the appearance of the recursive tower $\mathcal{F}$ in the literature. Further requirements on q. A defining polynomial $f \in \mathbb{F}_q[X,Y]$. The precise limit $\lambda(\mathcal{F})$. Finally, if the lower bound b which was established in the reference differs from the final precise limit, then $(\geq b)$ is also included in this list.

2.2.5 Pyramids of Recursive Towers

Purpose of this subsection. In this subsection, we will introduce the crucial concept of pyramids of function fields and prove some first simple properties of pyramids of function fields.

Pyramids of function fields. As we already saw in Figure 2.1, the pair (σ, F_0) of a recursive tower $\mathcal{F}$ provides a pyramidal structure. Formalizing this pyramid will be the subject of the following Definition 9.

Definition 9. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and define*

$$F_{i,j} := \prod_{k=i}^{j} \sigma^k(F_0)$$

for all $i,j \in \mathbb{N}_0$ with $i \leq j$.

(i) We call

$$\mathrm{Pyr}(\mathcal{F}) := (F_{i,j})_{i,j}$$

the ***pyramid of function fields*** *or* ***pyramid*** *of $\mathcal{F}$ (see Figure 2.4 where $\mathcal{F}$ has also balanced degree d) and mean with the notation $(F_{i,j})_{i,j}$ that the indices i,j run over all $i,j \in \mathbb{N}_0$ with $i \leq j$.*

(ii) We call $j-i$ the ***level*** *of the function field $F_{i,j}$ (resp. place $Q \in \mathbb{P}_{F_{i,j}}$). Notice that, for a recursive tower $\mathcal{F}$ over k which is defined by a polynomial and the corresponding sequence $(x_n)_n$ in Definition 5(i) and for $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$, we have the identity $F_{i,j} = k(x_i, x_{i+1}, \ldots, x_j)$ for all $i \leq j$.*

Moreover, for $k \leq i \leq j \leq l$, we call $(l-k)-(i,j)$ the ***height*** *of the extension $F_{k,l}/F_{i,j}$ (resp. Q/P in $F_{k,l}/F_{i,j}$).*

(iii) We call the extension $F_{k,l}/F_{i,j}$ of function fields an ***(elementary) extension in*** $\mathrm{Pyr}(\mathcal{F})$ *(if $(l-k)-(j-i)=1$) and a diamond (E, F_1, F_2, F) of function fields a* ***(elementary) diamond in*** $\mathrm{Pyr}(Q)$ *if E/F_i and F_i/F are (elementary) extension in* $\mathrm{Pyr}(\mathcal{F})$ *for all $i = 1,2$.*

(iv) We define the set

$$\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})} := \coprod_{i \leq j} \mathbb{P}_{F_{i,j}}$$

of all places which appear in one of the function fields of the pyramid of $\mathcal{F}$.

Reference	q	$f \in \mathbb{F}_q[X,Y]$	$\lambda(\mathcal{F})$ $(\geq b)$
[MW05, p. 212, f_2]	9	$f_{MW,2} = Y^2 + XY + 2X^2 + 1$	$2/3$
[MW05, p. 214, f_3]	49	$f_{MW,3} = Y^2 + X^2Y + 5X^2 + 5$	6 (≥ 4)
[MW05, p. 212, f_4]	9	$f_{MW,4} = Y^2 + X^2Y + 1$	2
[MW05, p. 212, f_6]	9	$f_{MW,6} = Y^2 + (X^2+1)Y + 2X^2$	$2/3$
[MW05, p. 213, f_8]	25	$f_{MW,8} = Y^2 + (X^2+3)Y + 4X^2$	1
[MW05, p. 212, f_{11}]	9	$f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$	2
[MW05, p. 213, f_{12}]	25	$f_{MW,12} = X^2Y^2 + (X^2+3X+3)Y + 4$	4 (≥ 3)
[MW05, p. 213, f_{14}]	25	$f_{MW,14} = X^2Y^2 + (X^2+4X+2)Y + 4X^2 + 2$	4 (≥ 3)
[MW05, p. 213, f_{15}]	25	$f_{MW,15} = X^2Y^2 + (X^2+4X+4)Y + 4X^2 + 3X + 2$	4 (≥ 3)
[MW05, p. 213, f_{16}]	25	$f_{MW,16} = (X^2+1)Y^2 + (X+1)Y + 2X^2 + 4X + 1$	1
[MW05, p. 213, f_{20}]	25	$f_{MW,20} = Y^2 + 2XY + 4X^2 + 1$	1
[MW05, p. 213, f_{21}]	25	$f_{MW,21} = Y^2 + 2XY + 4X^2 + 2$	1
[MW05, p. 213, f_{22}]	25	$f_{MW,22} = Y^2 + 4XY + X^2 + X$	4
[Sti08, p. 260]	l^2	$f_{GS,1} = Y^{l-1} + (X+1)^{l-1} - 1$ $l \geq 3$	$2/(l-1)$
[Sti08, p. 261]	l^e	$f_{GS,2} = Y^m + (X+1)^m - 1,$ $e \geq 2,\ m := \frac{q-1}{l-1}$	$2/(q-2)$
[Sti08, p. 261]	l^2	$f_{GS,3} = (Y^l - Y)(1 - X^{l-1}) - X^l$	$l-1$
[vdGvdV02, p. 292]	8	$f_{GV} = (Y^2+Y)X + X^2 + X + 1$	$2/3$
[BGS05, p. 161]	l^3	$f_{BezGS,l} = Y^l(X^l + X - 1) - (1-Y)X$	$2(l^2-1)/(l+2)$
[ST15, p. 680]	2	$f_{ST,1} = Y^2X + Y + X^2 + 1$	0
[ST15, p. 680]	2	$f_{ST,2} = X^2 + XY^2 + X + Y$	0
[ST15, p. 680]	2	$f_{ST,3} = X^2Y^2 + XY^2 + Y + X$	0
[ST15, p. 680]	2	$f_{ST,4} = X^2Y^2 + XY^2 + Y + X^2 + 1$	0
[BBGS15, p. 4]	q^m	$f_{BBGS,q,i,j} = \varepsilon_{q,i,j} \cdot (\mathrm{Tr}_j(Y/X^{q^i}) + \mathrm{Tr}_i(Y^{q^j}/X) - 1)$ $m := i + j$ with $i, j \in \mathbb{N}$ and $\gcd(i,j) = 1$	$2/\left(\frac{1}{q^j-1} + \frac{1}{q^i-1}\right)$
[HP16, p. 12]	5	$f_{HP,5} = Y^2(3X-1) - (X^2+X)$	0
[CNT18, p. 19]	2^s	$f_{CNT,s} = (Y^2+Y)(X^2+X+1) + X$ s even	1
[BR20, p. 4]	5	$f_{BR,5} = (X^6 + X + 2)(Y^5 - Y)$ $-(X^5 - X)(Y^6 + Y^5 + 2Y + 3)$	$2/(5-2)$
[BR20, p. 4]	q	$f_{BR,q} = (X^{q+1} + b)(b+n)(Y^q - Y)$ $-2b(Y^{q+1} + n)(X^q - X)$	$2/(q-2)$

Figure 2.3: Table with examples of recursive towers from references and their precise limits.

Properties of the pyramid of function fields. All the function fields $F_{i,j}$ in the pyramid $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ on the same level $l = j - i$ are isomorphic via powers of the tower map σ of $\mathcal{F}$. In the following Lemma 10, we will list this and some other immediate properties of pyramids of function fields.

Lemma 10. *Let $\mathcal{F}$ be a recursive tower of degree d which is defined by a pair (σ, F_0) and let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ by the pyramid of $\mathcal{F}$. Then the following hold:*

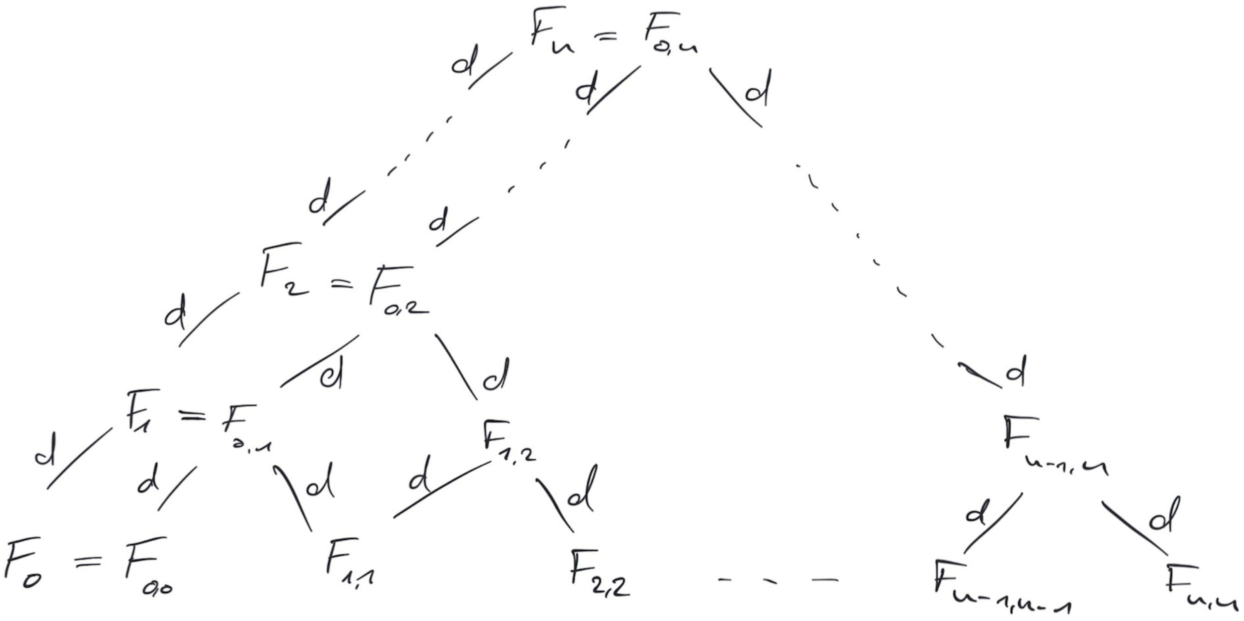


Figure 2.4: Pyramid of function fields

(i) *We have $F_n = F_{0,n}$ and $F_{i,j} \subseteq F_{m,n}$ for all $m \leq i \leq j \leq n$.*

(ii) *The automorphism σ^k restricts to an isomorphism $F_{i,j} \to F_{i+k,j+k}$ for all $i, j, k \in \mathbb{Z}$ with $0 \leq i \leq j$ and $0 \leq i+k$.*

(iii) *We have the estimate $\hat{d} := [F_1 : \sigma(F_0)] > 1$ and the extensions $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ are linearly disjoint extensions of degree d^{l-j} and $\hat{d}^{i-k}$, respectively, and satisfy the identities $F_{k,j} \cdot F_{i,l} = F_{k,l}$ and $F_{k,j} \cap F_{i,l} = F_{i,j}$ for all $k \leq i \leq j \leq l$*

Moreover, for all $k \leq i \leq j \leq l$, the extension $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ are separable.

(iv) *In particular, if $\mathcal{F}$ is a good tower over a finite field, then $[F_1 : \sigma(F_0)] = d$ and, therefore, $\mathcal{F}$ has the balanced degree d.*

Proof. For (i): By Definition 5(ii) and Definition 9, we have $F_n = \prod_{k=0}^{n} \sigma^k(F_0) = F_{0,n}$ and $F_{i,j} = \prod_{k=i}^{j} \sigma^k(F_0) \subseteq \prod_{k=m}^{n} \sigma^k(F_0) = F_{m,n}$. Hence, (i) follows.

For (ii): We obtain the equalities $F_{i+k,j+k} = \prod_{m=i+k}^{j+k} \sigma^m(F_0) = \sigma^k(\prod_{m=i}^{j} \sigma^m(F_0)) = \sigma^k(F_{i,j})$ where the first and last equalities hold because of Definition 9 and the second equality holds because σ is a morphism of algebras and because of the definition of the compositum of fields. Hence, (ii) follows.

For the 'degree'-statements in (iii): We get the equalities $\sigma^i(F_{0,j-i}) = F_{i,j}$, $\sigma^i(F_{0,j-i-1}) = F_{i,j-1}$ and $\sigma^i(F_{1,j-i}) = F_{i+1,j}$ for all $0 \leq i < j$ by Lemma 10(ii). Thus, combining these equalities and the fact that the degrees of extensions of fields are invariant under the action of the isomorphism σ^i yields the equalities

$$[F_{i,j} : F_{i,j-1}] = [F_{0,j-i} : F_{0,j-i-1}] \text{ and } [F_{i,j} : F_{i+1,j}] = [F_{0,j-i} : F_{1,j-i}]. \qquad (20)$$

As recursive towers have constant degree by Definition 5 on page 28, we have the equality $[F_{0,j-i} : F_{0,j-i-1}] = d$. Hence, this equality and the first equality in (20) provide the equalities

$$[F_{i,j} : F_{i,j-1}] = [F_{0,j-i} : F_{0,j-i-1}] = d \qquad (21)$$

for all $0 \leq i < j$.

Next, we show the equality $[F_{i,j} : F_{i+1,j}] = [F_1 : \sigma(F_0)] = \hat{d}$ for all $0 \leq i < j$ by induction over the level $j - i \in \mathbb{N}$ of $F_{i,j}$: For $j - i = 1$, the second equality in (20) supplies this desired equality

$$[F_{i,j} : F_{i+1,j}] = [F_{i,i+1} : F_{i+1,i+1}] = [F_{0,1} : F_{1,1}] = [F_1 : \sigma(F_0)] = \hat{d}.$$

Now, for $i - j \geq 2$, we obtain the equalities

$$\begin{aligned}[F_{0,j-i} : F_{1,j-i}] = \frac{[F_{0,j-i} : F_{1,j-i-1}]}{[F_{1,j-i} : F_{1,j-i-1}]} &= \frac{[F_{0,j-i} : F_{0,j-i-1}][F_{0,j-i-1} : F_{1,j-i-1}]}{[F_{1,j-i} : F_{1,j-i-1}]} \\ = [F_{0,j-i-1} : F_{1,j-i-1}] & \end{aligned} \tag{22}$$

(see Figure 2.13) where the first and second equalities hold since Lemma 10(i) implies

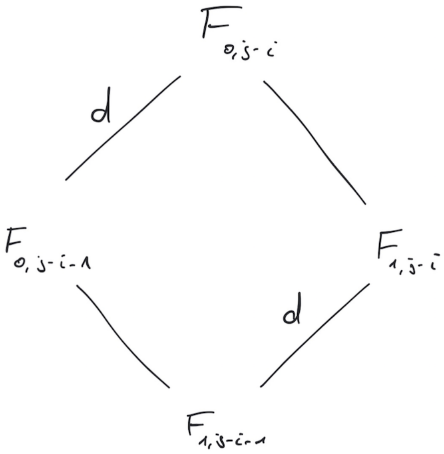


Figure 2.5: Elementary diamond of function fields in a proof

the inclusions $F_{0,j-i} \supseteq F_{1,j-i} \supseteq F_{1,j-i-1}$ and $F_{0,j-i} \supseteq F_{0,j-i-1} \supseteq F_{1,j-i-1}$ and the third equality holds since the first desired statement provides the identity $[F_{0,j-i} : F_{0,j-i-1}] = [F_{1,j-i} : F_{1,j-i-1}] = d$. Then we conclude the equalities

$$[F_{i,j} : F_{i+1,j}] = [F_{0,j-i} : F_{1,j-i}] = [F_{0,j-i-1} : F_{1,j-i-1}] = \hat{d} \tag{23}$$

for all $0 \leq i < j$ where the first equality holds by second identity in (20), the second equality holds by the equality in (22) and the third equality holds by the induction hypothesis.

Second to last, we notice the equalities

$$[F_{k,l} : F_{i,j}] = [F_{k,l} : F_{k,j}][F_{k,j} : F_{i,j}] = \prod_{\nu=j}^{l-1} [F_{k,\nu+1} : F_{k,\nu}] \prod_{\nu=k}^{i-1} [F_{\nu,j} : F_{\nu+1,j}] = d^{l-j}\hat{d}^{i-k} \tag{24}$$

for all $k \leq i \leq j \leq l$ where the first and second equality holds since, by the inclusion in Lemma 10(i), the involved function fields are indeed exactly contained in each other as the formulas indicate and the third equality holds by the equalities in (21) and (23). In particular, the equality in (24) provides that, for all $k \leq i \leq j \leq l$, the extensions $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ indeed have the desired degrees d^{l-j} and $\hat{d}^{i-k}$, respectively.

Finally, the latter also implies the desired estimate $\hat{d} > 1$ by the following reasoning: Assume $\hat{d} = 1$. Then we compute

$$[F_n : \sigma^n(F_0)] = [F_{0,n} : F_{n,n}] = 1 \tag{25}$$

where the first equality holds since Lemma 10(i) implies the equality $F_n = F_{0,n}$ and since the definition of $\mathrm{Pyr}(\mathcal{F}) = (F_{i,j})_{i,j}$ in Definition 9(i) implies the equality $\sigma^n(F_0) = F_{n,n}$, and the third equality holds because we already shown that $F_{0,n}/F_{n,n}$ has degree $\hat{d}^n$ and because of the assumption $\hat{d} = 1$.

Consequently, we conclude the equalities

$$g(F_n) = g(\sigma^n(F_0)) = g(F_0) \tag{26}$$

for all $n \in \mathbb{N}_0$ where the first equality holds since the equality in (25) implies the equality $F_n = \sigma^n(F_0)$ and the second equality holds because the genus is invariant under the action of isomorphisms. But that equality in (26) is impossible since $g(F_n) \to \infty$ as $n \to \infty$ by the definition of the tower $\mathcal{F} = (F_\nu)_\nu$ in Definition 2(i). Hence, we indeed deduce $\hat{d} > 1$.

For the compositum'-, 'intersection'- and 'linearly disjointness'-statements in (iii): We already obtain the desired compositum-statement by the equalities

$$F_{k,j} \cdot F_{i,l} = \prod_{\nu=k}^{j} \sigma^\nu(F_0) \cdot \prod_{\nu=i}^{l} \sigma^\nu(F_0) = \prod_{\nu=k}^{l} \sigma^\nu(F_0) = F_{k,l} \tag{27}$$

for all $k \leq i \leq j \leq l$ where the first and last equalities hold by the definition of $\mathrm{Pyr}(\mathcal{F}) = (F_{r,s})_{r,s}$ in Definition 9(i) and the second equality holds because the estimate $i \leq j$ implies that we have repeating factors which can be neglected in the definition of composite fields. But then the combination of the equalities in (24) and (27) yield that the extensions $(F_{k,j} \cdot F_{i,l})/F_{i,l}$ and $F_{k,j}/F_{i,j}$ have the same degree $\hat{d}^{i-k}$ and, thus, the extensions $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ are indeed linearly disjoint for all $k \leq i \leq j \leq l$.

Finally, it is well known that, for all linearly disjoint extensions L/K and M/K such that the fields L and M which are contained in some common extension field, we have the identity $M \cap L = K$. Hence, combining the above conclusion that $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ are linearly disjoint and this well known identity provides the final desired 'intersection'-statement, namely the identity $F_{i,l} \cap F_{k,j} = F_{i,j}$ for all $k \leq i \leq j \leq l$.

For the 'separability'-statements in (iii): By the definition of towers in Definition 2, the extension F_1/F_0 (resp. $F_1/\sigma(F_0)$) is separable and, moreover, by the invariance of the separability-property under the action of isomorphisms, the extension

$$\sigma^{n-1}(F_1)/\sigma^{n-1}(F_0) \text{ (resp. that } F_1/\sigma(F_0)) \text{ is also separable.} \tag{28}$$

Then we compute

$$\begin{aligned} F_{\varepsilon,n} &= \prod_{l=\varepsilon}^{n} \sigma^l(F_0) = \prod_{l=\varepsilon}^{1} \sigma^l(F_0) \cdot \prod_{l=1}^{n} \sigma^l(F_0) = \sigma^\varepsilon(\prod_{l=0}^{1-\varepsilon} \sigma^l(F_0)) \cdot \prod_{l=1}^{n} \sigma^l(F_0) \\ &= \sigma^\varepsilon(F_{0,1-\varepsilon}) \cdot F_{1,n} = \sigma^\varepsilon(F_{1-\varepsilon}) \cdot F_{1,n} \end{aligned} \tag{29}$$

and

$$\begin{aligned} F_{0,n-1+\varepsilon} &= \prod_{l=0}^{n-1+\varepsilon} \sigma^l(F_0) = \prod_{l=0}^{n-1} \sigma^l(F_0) \cdot \prod_{l=n-1}^{n-1+\varepsilon} \sigma^l(F_0) = \prod_{l=0}^{n-1} \sigma^l(F_0) \cdot \sigma^{n-1}(\prod_{l=0}^{\varepsilon} \sigma^l(F_0)) \\ &= F_{0,n-1} \cdot \sigma^{n-1}(F_{0,\varepsilon}) = F_{0,n-1} \cdot \sigma^{n-1}(F_\varepsilon) \end{aligned} \tag{30}$$

for all $n \in \mathbb{N}_0$ and all $\varepsilon = 0, 1$ where the first and second to last equalities hold by the definition of $\mathrm{Pyr}(\mathcal{F}) = (F_{i,j})_{i,j}$ in Definition 9(i), the second equalities holds as we only split the product in two, the third equality holds because σ is an isomorphism of algebras and because of the definition of composite fields and the last equality holds by the identity in Lemma 10(i).

Now, we remember the well known fact that the extension $(K \cdot M)/(K \cdot L)$ is separable for all separable extensions M/L of fields and all fields K and M which are contained in some common extension field. Thus, combining (28), the equalities in (30) (resp. (29)) for all $\varepsilon = 0, 1$ and this well known fact yields that the extensions

$$F_{0,n}/F_{0,n-1} \text{ (resp. } F_{0,n}/F_{1,n}\text{) are separable} \tag{31}$$

for all $n \in \mathbb{N}_0$.

Next, for all $0 \leq i < j$, we choose $n := j - i > 1$. Then combining the conclusions in (31), Lemma 10(ii), the invariance of separability under the action of the isomorphism σ^i and Lemma 10(ii) also yields that the extensions

$$F_{i,j}/F_{i,j-1} \text{ (resp. } F_{i,j}/F_{i+1,j}\text{) are separable.} \tag{32}$$

for all $0 \leq i < j$.

Finally, the inclusion in Lemma 10(i) supplies the inclusions

$$F_{i,l} \supseteq F_{i,l-1} \supseteq \cdots \supseteq F_{i,j} \text{ and } F_{k,j} \supseteq F_{k+1,j} \supseteq \ldots \supseteq F_{i,j} \tag{33}$$

for all $k \leq i \leq j \leq l$. Consequently, because of the conclusions in (32), because of the inclusions in (33) and because the separability-property is also transitive, we deduce the desired 'separability'-statements, namely that the extensions $F_{i,l}/F_{i,j}$ and $F_{k,j}/F_{i,j}$ are separable for all $k \leq i \leq j \leq l$.

For (iv): Let $\hat{d} := [F_1 : \sigma(F_0)] = [F_{0,1} : F_{1,1}]$. Since $g(F_{0,n}) = g(F_n) \to \infty$ as $n \to \infty$ by the definition of a tower of function fields in Definition 2(i), there is an index $i \in \mathbb{N}_0$ such that $g(F_{0,i}) \geq 2$. One the one hand, for all $n \geq i$, we have the estimates $N(F_{0,n}) \leq N(F_{0,i})d^{n-i}$ and $N(F_{0,n}) \leq N(F_{n-i,n})\hat{d}^{n-i}$ by applying the fundamental equality in (8) on the degrees $d^{n-i} = [F_{0,n} : F_{0,i}]$ and $\hat{d}^{n-i} = [F_{0,n} : F_{n-i,n}]$ where the equalities hold by Lemma 10(iii). On the other hand, for all $n \geq i$ we have the estimates $g(F_{0,n}) - 1 \geq (g(F_{0,i}) - 1)d^n$ and $g(F_{0,n}) - 1 \geq (g(F_{n-i,n}) - 1)\hat{d}^n$ by applying the Hurwitz Genus Formula in (9) on the same extensions. Then, we estimate

$$\lambda(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_{0,n})}{g(F_{0,n})} = \lim_{n\to\infty} \frac{N(F_{0,n})}{g(F_{0,n}) - 1} \leq \frac{N(F_{0,i})}{g(F_{n-i,n}) - 1} \cdot \lim_{n\to\infty} \left(\frac{d}{\hat{d}}\right)^n$$

and, analogously,

$$\lambda(\mathcal{F}) \leq \frac{N(F_{n-i,n})}{g(F_{0,i}) - 1} \cdot \lim_{n\to\infty} \left(\frac{\hat{d}}{d}\right)^n$$

Thus, the equality $\lambda(\mathcal{F}) = 0$ holds if $d \neq \hat{d}$. Consequently, the equality $d = \hat{d}$ needs to hold if $\mathcal{F}$ is a good tower. Hence, (iv) follows. □

2.2.6 Pyramids of Places

Purpose of this subsection. In this subsection, we will introduce the crucial concept of pyramids of places and prove some first simple properties of pyramids of places.

Pyramids of places. For any recursive tower $\mathcal{F}$ with the pyramid $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ and the place Q in $F_{m,n}$ for $m \leq n$, we also obtain a pyramidal structure $\mathrm{Pyr}(Q)$ for Q by intersecting Q with the function fields $F_{i,j}$ for all $m \leq i \leq j \leq n$ (see Figure 2.6). This will be the subject of the following Definition 11.

Definition 11. *Let $\mathcal{F}$ be a recursive tower, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$ and let $Q \in \mathbb{P}_{F_{m,n}}$ for some $m \leq n$. Then we call the family*

$$\mathrm{Pyr}(Q) := (Q \cap F_{i,j})_{i,j} := (Q \cap F_{i,j})_{m \leq i \leq j \leq n} \in \prod_{m \leq i \leq j \leq n} \mathbb{P}_{F_{i,j}}$$

*the **pyramid** of places of Q (see Figure 2.6).*

Most of the time and as long as this does not lead to confusions, we will use the short notation $(Q \cap F_{i,j})_{i,j}$ for the pyramid of Q and mean by this that the indices i, j run over all $i, j \in \mathbb{N}_0$ with $m \leq i \leq j \leq n$.

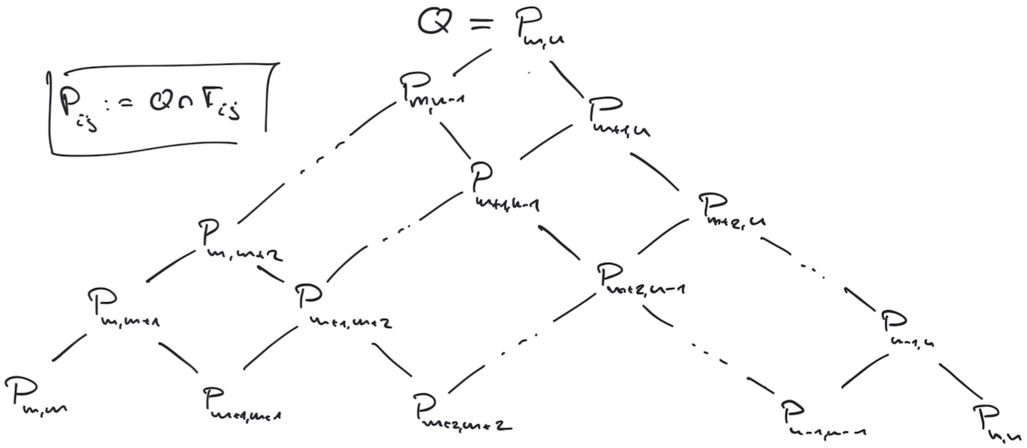


Figure 2.6: Pyramid of places

Examples 12. *Consider the recursive towers $\mathcal{F}'_{MW,2} = (F'_\nu)_\nu$ over $\mathbb{F}_9$ and $\mathcal{F}_{MW,2} = (F_\nu)_\nu$ over $\mathbb{F}_3$ which are defined by the polynomial $f_{MW,2} = Y^2 + XY + 2X^2 + 1$ (see Figure 2.3). Let $a \in \mathbb{F}_9$ be a zero of the polynomial $t^2 + 1 \in \mathbb{F}_3[t]$. In the figures 2.7 and 2.8, we have the pyramids of two places $Q \in \mathbb{P}_{F_4}$ and $Q' \in \mathbb{P}_{F'_4}$ such that Q'/Q in F'_4/F_4 where the blue numbers are the ramification indices of the corresponding elementary extensions.*

We will derive the existence of such a place Q' in Examples 77(i). Then the pyramid of $Q := Q' \cap F_4$ is automatically of the form in Figure 2.7.

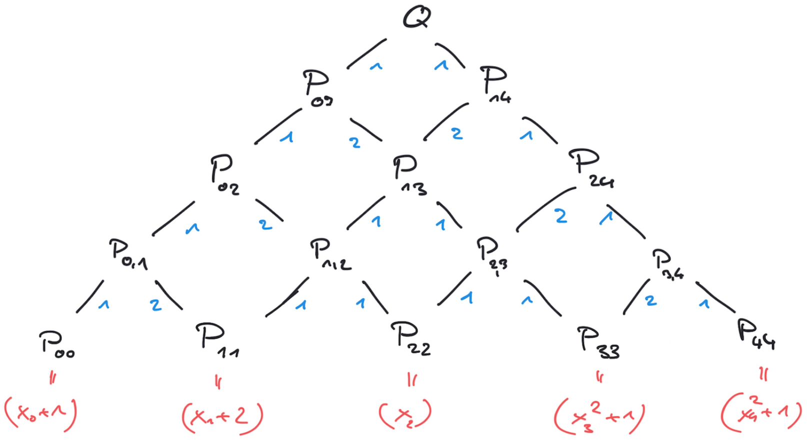


Figure 2.7: Example of a pyramid of places

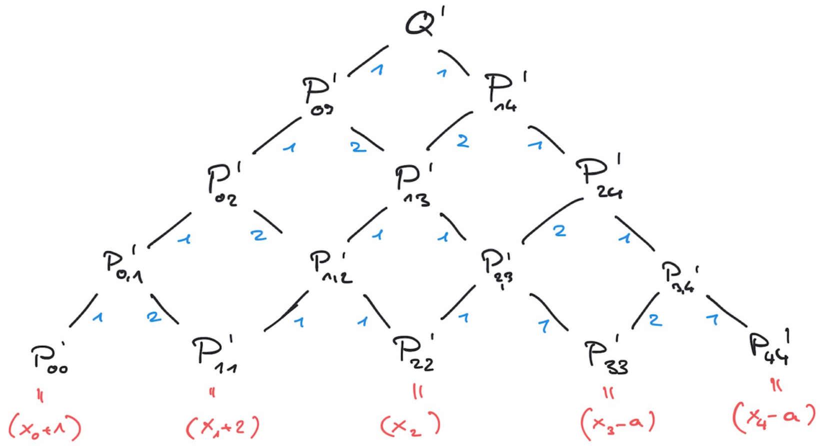


Figure 2.8: Another example of a pyramid of places

Extensions in pyramids of places. Analogously, to the (elementary) extensions and diamonds of function fields in the pyramid $\mathrm{Pyr}(\mathcal{F})$ of function fields, we will also speak of (elementary) extensions and diamonds of places in the pyramid $\mathrm{Pyr}(Q)$ of places. This will be the subject of the following Lemma 13 and Definition 14.

Lemma 13. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$, let $Q \in \mathbb{P}_{F_{m,n}}$ for some $m \leq n$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ be the pyramid of Q. Then we have*

$$P_{k,l}/P_{i,j}$$

for all $m \leq k \leq i \leq j \leq l \leq n$ and

$$\mathrm{Pyr}(P_{k,l}) = (P_{i,j})_{k \leq i \leq j \leq l}$$

for all $m \leq k \leq l \leq n$.

Proof. The first desired statement immediately follows from the equalities

$$P_{k,l} \cap F_{i,j} = (Q \cap F_{k,l}) \cap F_{i,j} = Q \cap F_{i,j} = P_{i,j}$$

for all $m \leq k \leq i \leq j \leq l \leq n$ where the first and last equalities hold because of the definition of $\mathrm{Pyr}(Q) = (Q \cap F_{r,s})_{r,s}$ in Definition 11 and the second equality holds because Lemma 10(i) implies the inclusion $F_{i,j} \subseteq F_{k,l}$.

The desired equality also already follows from the equalities

$$\begin{aligned}\mathrm{Pyr}(P_{k,l}) &= (P_{k,l} \cap F_{i,j})_{k \leq i \leq j \leq l} = ((Q \cap F_{k,l}) \cap F_{i,j})_{k \leq i \leq j \leq l} \\ &= (Q \cap F_{i,j})_{k \leq i \leq j \leq l} = (P_{i,j})_{k \leq i \leq j \leq l}\end{aligned}$$

where the first equality holds by the definition of Pyr in Definition 11, the second and last equalities hold by the assertion $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ and the third equality holds by Lemma 10(i). ☐

Definition 14. *Let $\mathcal{F}$ be a recursive function field, $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$, $Q \in \mathbb{P}_{F_{m,n}}$ and $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$.*

(i) We define the sets

$$\mathrm{Ext}(\mathrm{Pyr}(Q)) := \{(P_{k,l}, P_{i,j}) : i, j, k, l \in \mathbb{N}_0 \text{ and } m \leq k \leq i \leq j \leq l \leq n\},$$

and

$$\mathrm{ElemExt}(\mathrm{Pyr}(Q)) := \{(P_{k,l}, P_{i,j}) \in \mathrm{Ext}(\mathrm{Pyr}(Q)) : (l-k)-(j-i) = 1\},$$

of all ***extensions*** *and* ***elementary extensions*** *of places in* $\mathrm{Pyr}(Q)$*, respectively. We will often directly write* R/P *for any extension* (R, P) *of* $\mathrm{Pyr}(Q)$*.*

Notice that the elementary extensions are the one-step-extensions in $\mathrm{Pyr}(Q)$ *from one level* $m = j - i$ *to the next* $m + 1 = k - l$*.*

(ii) We call a diamond (R, P_1, P_2, P) *of places a* ***(elementary) diamond in*** $\mathrm{Pyr}(Q)$ *if* R/P_i *and* P_i/P *are (elementary) extensions in* $\mathrm{Pyr}(Q)$ *for all* $i = 1, 2$*.*

Extending the action of the tower map to pyramids of places. By Lemma 10(ii), the tower map σ acts on the function fields in the pyramid $\mathrm{Pyr}(\mathcal{F}) = (F_{i,j})_{i,j}$ of function fields by basically shifting the function fields $F_{i,j}$ one to the right in $\mathrm{Pyr}(\mathcal{F})$, which is the function field $F_{i+1,j+1}$. Correspondingly, σ also acts on the places in the pyramid $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$.

Because we will use this action all the time and because there is a lot of potential to repeatedly get lost in the indices, we will formalize this action more abstractly in the following Definition/Lemma 15.

Definition/Lemma 15. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a recursive tower which is defined by the pair* (σ, F_0)*, let* $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ *be the pyramid of* $\mathcal{F}$*, let* $Q \in \mathbb{P}_{F_{m,n}}$ *for some* $m \leq n$ *and let* $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ *be the pyramid of* Q*.*

(i) For all $k \in \mathbb{Z}$ *with* $0 \leq m + k$*, we define*

$$\sigma^k(\mathrm{Pyr}(Q)) := (\sigma^k(P_{i-k,j-k}))_{i,j} = \mathrm{Pyr}(\sigma^k(Q))$$

where $i, j \in \mathbb{N}_0$ *run over all* $m + k \leq i \leq j \leq n + k$*.*

(ii) For all $k, l \in \mathbb{Z}$ *with* $0 \leq m + k$ *and* $0 \leq m + k + l$*, we have the identity*

$$\sigma^l(\sigma^k(\mathrm{Pyr}(Q))) = \sigma^{k+l}(\mathrm{Pyr}(Q)).$$

Proof. For (i): For all $m + k \leq i \leq j \leq n + k$, we compute

$$\sigma^k(Q) \cap F_{i,j} = \sigma^k(Q) \cap \sigma^k(F_{i-k,j-k}) = \sigma^k(P_{i-k,j-k}) \tag{34}$$

where the first equality holds by Lemma 10(ii) and the second equality holds because σ^k is a bijection and because of the definition of $P_{i-k,j-k} = Q \cap F_{i-k,j-k}$ in Definition 11. Then the desired identity in (i), namely $(\sigma^k(P_{i-k,j-k}))_{i,j} = \mathrm{Pyr}(\sigma^k(Q))$, follows from the definition of the pyramid $\mathrm{Pyr}(\sigma^k(Q)) = (\sigma^k(Q) \cap F_{i,j})_{i,j}$ of $\sigma^k(Q)$ in Definition 11 and from the equality in (34).

For (ii): For all $k, l \in \mathbb{Z}$ with $0 \leq m + k$ and $0 \leq m + k + l$, we immediately derive the desired identity in (ii) by the equalities

$$\sigma^l(\sigma^k(\mathrm{Pyr}(Q))) = \mathrm{Pyr}(\sigma^l(\sigma^k(Q))) = \sigma^{k+l}(\mathrm{Pyr}(Q))$$

where the equalities hold by Definition/Lemma 15(i). □

2.2.7 Paths of Places

Purpose of this subsection. In this subsection, we will introduce the crucial concept of paths of places and prove some first simple properties of paths of places.

Motivating paths of places. Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field which is defined by the pair (σ, F_0). The main objects of concern are the value $N(F_n)$ which is the number of rational places in F_n and the genus $g(F_n)$ which, by the Hurwitz Genus Formula in (9), only depends on the ramified places in F_n/F_0. Thus, in order to make statements about $N(F_n)$ and $g(F_n)$, we will have to extract information of these two special kinds of places.

Now, let $Q \in \mathbb{P}_{F_n}$ and $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$. Then the lower levels $l = j - i$ of $\mathrm{Pyr}(Q)$ already carry significant amounts of information in regards to the rationality of Q and the ramification index of Q in F_n/F_0. Additionally, we notice that gathering information for the zeroth and first levels $l = j - i$ in $\mathrm{Pyr}(Q)$ is quite approachable because the extensions $P_{k,k+1}/P_{k,k}$ (resp. $P_{k,k+1}/P_{k+1,k+1}$) of places in $F_{k,k+1}/F_{k,k}$ (resp. $F_{k,k+1}/F_{k+1,k+1}$) can be pulled back to extensions $\sigma^{-k}(P_{k,k+1})/\sigma^{-k}(P_{k,k})$ (resp. $\sigma^{-k}(P_{k,k+1})/\sigma^{-k}(P_{k+1,k+1})$) of places in F_1/F_0 (resp. $F_1/\sigma(F_0)$) via the isomorphisms σ^{-k} for all $k = 0, \dots, n-1$.

Thus, the **paths of the places**, which are the zeroth and first levels of their pyramids, can be solely generated via extensions of places in F_1/F_0 and $F_1/\sigma(F_0)$. This will ultimately motivate to assign a directed graph to $\mathcal{F}$, namely the **tower graph** $\Gamma_{\mathcal{F}}$ in Definition 74. This tower graph will capture the information of the extensions of places in F_1/F_0 and $F_1/\sigma(F_0)$. Then the paths in $\mathcal{F}$ can be generated via the bijective **tower graph map** $\sigma_{\Gamma_{\mathcal{F}}}$ from the paths in $\Gamma_{\mathcal{F}}$ to the paths in $\mathcal{F}$ (see Definition/Lemma 76).

Paths of places and some first properties. In the following Definition 16, we will define paths of places in $\mathcal{F}$.

Definition 16. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$.*

(i) Let $m, n \in \mathbb{N}_0$ with $m \leq n$ and $P_{i,j} \in \mathbb{P}_{F_{i,j}}$ for all $m \leq i \leq j \leq n$ with $j - i \leq 1$. We call the family

$$\mathcal{P} := (P_{i,j})_{j-i\leq 1} := (P_{i,j})_{\substack{m\leq i\leq j\leq n\\ j-i\leq 1}} \in \prod_{\substack{j-i\leq 1\\ m\leq i\leq j\leq n}} \mathbb{P}_{F_{i,j}}$$

*an (m,n)-**path** in $\mathcal{F}$ if $P_{i,i+1}/P_{i,i}$ in $F_{i,i+1}/F_{i,i}$ and $P_{i,i+1}/P_{i+1,i+1}$ in $F_{i,i+1}/F_{i+1,i+1}$ for all $i = m, \dots, n-1$ (see Figure 2.6).*

Most of the time and as long as it does not lead to confusions, we will use the short notation $(P_{i,j})_{j-i\leq 1}$ for the path $\mathcal{P}$ and mean by this that the indices run over all $i, j \in \mathbb{N}_0$ with $m \leq i \leq j \leq n$ and $j - i \leq 1$.

*Moreover, we define $W(\mathcal{F}, m, n)$ to be the set of all (m,n)-paths in $\mathcal{F}$, call any path $\mathcal{P} \in W(\mathcal{F}, n) := W(\mathcal{F}, 0, n)$ a **path** n in $\mathcal{F}$ and define $W(\mathcal{F}) := \coprod_{m\leq n} W(\mathcal{F}, m, n)$.*

*(ii) We call $(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, k, l)$ the (k,l)-**subpath** of the path $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ for all $m \leq k \leq l \leq n$.*

*(iii) Let $m, n, r \in \mathbb{N}_0$ with $m \leq n \leq r$. Let $\mathcal{P}_1 = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ and $\mathcal{P}_2 = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n, r)$. Then we call $\mathcal{P}_1$ and $\mathcal{P}_2$ **composable** since the last place of $\mathcal{P}_1$ and the first place of $\mathcal{P}_2$ are the same place $P_{n,n}$ and we call $\mathcal{P}_1\mathcal{P}_2 := \mathcal{P}_1 \cdot \mathcal{P}_2 := (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, r)$ the **composition** of $\mathcal{P}_1$ and $\mathcal{P}_2$.*

*(iv) We call a path $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ **rational** if $P_{i,j}$ is rational for all $m \leq i \leq j \leq n$ with $j - i \leq 1$ and denote the corresponding subsets of rational paths by $W_{\mathrm{rat}}(\mathcal{F}, m, n) \subseteq W(\mathcal{F}, m, n)$, $W_{\mathrm{rat}}(\mathcal{F}, n) \subseteq W(\mathcal{F}, n)$ and $W_{\mathrm{rat}}(\mathcal{F}) \subseteq W(\mathcal{F})$.*

(v) *We say that a path $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ has* ***balanced ramification indices*** *if the equality $\prod_{i=m}^{n} e(P_{i-1,i}|P_{i-1,i-1}) = \prod_{i=m}^{n} e(P_{i-1,i}|P_{i,i})$ holds. Otherwise, we say that $\mathcal{P}$ has* ***unbalanced ramification indices****.*

(vi) *We call the path $\mathcal{P}$* ***tame*** *if, for all $m+1 \leq i \leq j \leq n-1$, at least one of the two extensions $P_{i-1,i}/P_{i,i}$ and $P_{j,j+1}/P_{j,j}$ is tame. Otherwise, we call $\mathcal{P}$* ***wild****.*

(vii) *We define the set*

$$\operatorname{Ext}(\mathcal{P}) := \{(P_{i-1,i}, P_{i-1,i-1}), (P_{i-1,i}, P_{i,i}) : i = m+1, \ldots, n\}$$

of all ***extensions of places*** *in the path $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$.*

The path map. In the following Definition/Lemma 17(i), we will define the path map $\operatorname{Path} : \mathbb{P}_{\operatorname{Pyr}(\mathcal{F})} \to W(\mathcal{F})$ which maps a place Q in $\mathcal{F}$ to the zeroth and first level of its pyramid $\operatorname{Pyr}(Q)$. Moreover, we will prove that the map Path is surjective. Hence, any path $\mathcal{P}$ of places in $\mathcal{F}$ is a path $\operatorname{Path}(Q)$ of a place in $\mathcal{F}$ or, in more suggestive words, there lies a place Q over any path $\mathcal{P}$ in $\mathcal{F}$.

For rational paths $\mathcal{P}$ in $\mathcal{F}$ which only have unramified extensions, Lemma 17(iv) will even provide that there lies exactly one place Q over $\mathcal{P}$ and that this place Q is rational.

Definition/Lemma 17. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $(F_{i,j})_{i,j} := \operatorname{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$ and let $m, n \in \mathbb{N}_0$ with $m \leq n$. Then the following hold:*

(i) *The map*

$$\operatorname{Path}_{m,n} : \mathbb{P}_{F_{m,n}} \to W(\mathcal{F}, m, n) \text{ via } Q \mapsto (Q \cap F_{i,j})_{j-i\leq 1}$$

is a well defined surjection and, by taking the disjoint unions of the domain and codomain of $\operatorname{Path}_{i,j}$ for all $0 \leq i \leq j$, we obtain an extension map

$$\operatorname{Path}_{\mathcal{F}} := \operatorname{Path} : \mathbb{P}_{\operatorname{Pyr}(\mathcal{F})} = \coprod_{0\leq i\leq j} \mathbb{P}_{F_{i,j}} \to W(\mathcal{F}).$$

Moreover, by the definition of $\mathbb{P}_{F_{m,n}}(\mathcal{P})$ in (5) for all $\mathcal{P} \in W(\mathcal{F}, m, n)$, we also have the identity $\operatorname{Path}^{-1}(\mathcal{P}) = \mathbb{P}_{F_{m,n}}(\mathcal{P})$ and we say that any place $Q \in \operatorname{Path}^{-1}(\mathcal{P})$ ***lies over the path*** *$\mathcal{P}$ (see Figure 2.6).*

(ii) *On the one hand, if $\operatorname{Pyr}(Q) = (P_{i,j})_{i,j}$, then we have $\operatorname{Path}(Q) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$.*

On the other hand, if $\operatorname{Path}(Q) = (P_{i,j})_{j-i\leq 1}$, then $\operatorname{Path}(P_{k,l}) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, k, l)$ for all $m \leq k \leq l \leq n$.

(iii) *If $Q \in \mathbb{P}_{\operatorname{Pyr}(\mathcal{F})}$ is rational, then $\operatorname{Path}(Q)$ is also rational.*

(iv) *Let $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W_{\mathrm{rat}}(\mathcal{F}, m, n)$. If all extensions in $\operatorname{Ext}(\mathcal{P})$ are unramified, i.e. $e(P_{k,k+1}|P_{k,k}) = e(P_{k,k+1}|P_{k+1,k+1}) = 1$ holds for all $k = m, \ldots, n-1$, then there is exactly one place $Q \in \mathbb{P}_{F_{m,n}}$ which lies over $\mathcal{P}$ and this place Q is rational.*

(v) *Let $Q \in \mathbb{P}_{F_{m,n}}$. Then $\operatorname{Path}(Q)$ is tame if and only if, for all elementary diamonds (R, P_1, P_2, P) in $\operatorname{Pyr}(Q)$, at least one of the two elementary extensions P_1/P and P_2/P is tame.*

In the following proof, we will already use Key Lemma 36(iv), which will only be proved later. However, there will be no circular reasoning.

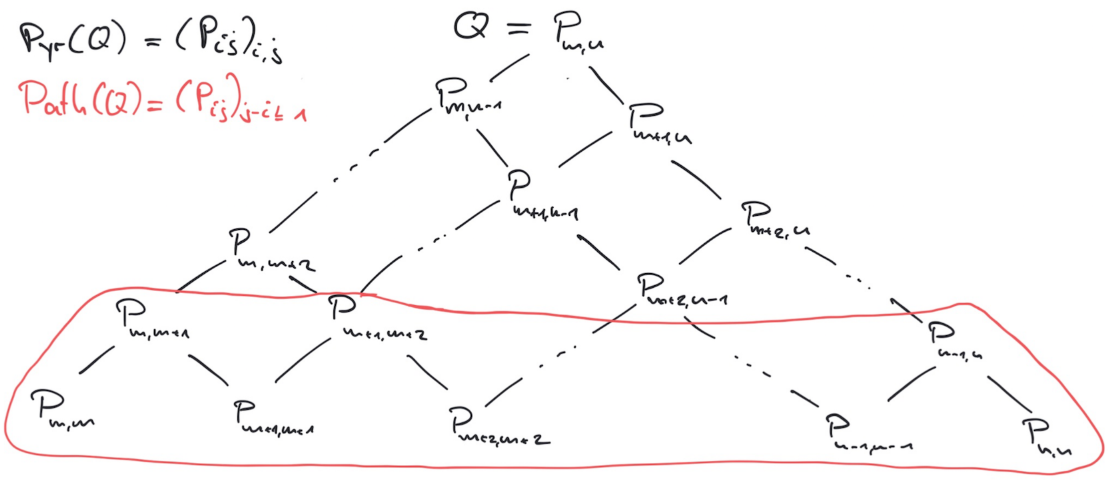


Figure 2.9: Path and Pyramid of Places

Proof. For (i): Let σ be the tower map of $\mathcal{F}$. Let $k \in \{m, \dots, n-1\}$. Then the equalities $Q \cap F_{k,k} = (Q \cap F_{k,k+1}) \cap F_{k,k}$ and $Q \cap F_{k+1,k+1} = (Q \cap F_{k,k+1}) \cap F_{k+1,k+1}$ follow because Lemma 10(i) implies the inclusions $F_{k,k} \subseteq F_{k,k+1}$ and $F_{k+1,k+1} \subseteq F_{k,k+1}$. Thus, $Q \cap F_{k,k+1}$ lies over $Q \cap F_{k,k}$ and $Q \cap F_{k+1,k+1}$. Hence, $\mathrm{Path}_{m,n}$ is well defined.

We first show the surjectivity of $\mathrm{Path}_{m,n}$ for $m = 0$ via induction over $n \in \mathbb{N}_0$. Let $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n)$. For $n \in \{0, 1\}$, we find a preimage of $\mathcal{P}$ by the equalities

$$\mathrm{Path}(P_{0,0}) = (P_{0,0} \cap F_{0,0}) = (P_{0,0}) = \mathcal{P}$$

for $n = 0$ and

$$\mathrm{Path}(P_{0,1}) = (P_{0,1} \cap F_{0,0}, P_{0,1} \cap F_{0,1}, P_{0,1} \cap F_{1,1}) = (P_{0,0}, P_{0,1}, P_{1,1}) = \mathcal{P}$$

for $n = 1$ where the first and last equalities hold because of the definitions of Path and $\mathcal{P}$ for $n \in \{0, 1\}$ and the second equalities hold because the assumption $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n)$ in Definition 16(i) implies $P_{0,i} \in \mathbb{P}_{F_{0,i}}$ for all $i = 1, 2$ and $P_{0,1}/P_{0,0}$ and $P_{0,1}/P_{1,1}$.

Now, let $n \geq 2$. Then, by applying the induction hypothesis to $\mathcal{P}' := (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n-1)$, there is a place $Q' \in \mathbb{P}_{F_{n-1}}$ such that $\mathrm{Path}(Q') = \mathcal{P}'$. Moreover, Lemma 10(iii) yields that the finite extensions $F_{n-1}/F_{n-1,n-1}$ and $F_{n-1,n}/F_{n-1,n-1}$ are linearly disjoint. Consequently, the 'moreover'-part of Key Lemma 36(i) supplies that there is a place $Q \in \mathbb{P}_{F_n}$ such that Q/Q' and $Q/P_{n-1,n}$. But, combining this and the identities $(Q' \cap F_{i,j})_{j-i\leq 1} = \mathrm{Path}(Q') = \mathcal{P}' = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n-1)$ then yield the equalities $\mathrm{Path}(Q) = (Q \cap F_{i,j})_{j-i\leq 1} = (P_{i,j})_{j-i\leq 1} = \mathcal{P}' \in W(\mathcal{F}, n)$. Hence, $\mathrm{Path}_{0,n}$ is surjective for all $n \in \mathbb{N}_0$.

Next, let m be arbitrary and $\mathcal{P} = (P_{i,j})_{i,j} \in W(\mathcal{F}, m, n)$. Define $\mathcal{P}' = (\sigma^{-m}(P_{i,j}))_{i,j} \in W(\mathcal{F}, n-m)$. On the one hand, by the surjectivity of $\mathrm{Path}_{0,n-m}$, which we already proved in the preceding part, there is a place $Q' \in \mathbb{P}_{F_{n-m}}$ with $(Q' \cap F_{i,j})_{j-i\leq 1} = \mathrm{Path}(Q') = \mathcal{P}' = (\sigma^{-m}(P_{i,j}))_{j-i\leq 1} \in W(\mathcal{F}, 0, n-m)$. On the other hand, Lemma 15(i) provides the identity $\mathrm{Pyr}(\sigma^m(Q')) = (\sigma^m(Q' \cap F_{i-m,j-m}))_{i,j}$. Then combining these two identities and the fact that elements of $\mathrm{Pyr}(\sigma^m(Q'))$ with $j-i \leq 1$ are exactly the elements of $\mathrm{Path}(\sigma^m(Q'))$ yield the equalities $\mathrm{Path}(\sigma^m(Q')) = (\sigma^m(Q' \cap F_{i-m,j-m}))_{j-i\leq 1} = (P_{i,j})_{j-i\leq 1} = \mathcal{P} \in W(\mathcal{F}, m, n)$. Hence, $\mathrm{Path}_{m,n}$ is surjective and (i) follows.

For (ii): The first desired identity immediately follows from the definitions of Pyr in Definition 11 and of Path in Definition/Lemma 17(i).

The second desired identity immediately follows from the first desired identity and the identity in Lemma 13.

For (iii): This immediately follows since the places in $\mathrm{Path}(Q)$ are places which lie under Q by its definition in Definition/Lemma 17(i).

For (iv): We show this by induction over $n-m \in \mathbb{N}_0$. For $n-m \leq 1$, we have $Q = P_{m,n}$ and (iv) holds trivially.

Now, let $n-m \geq 2$. Then the induction hypothesis implies that there is exactly one place $Q' \in \mathbb{P}_{F_{m,n-1}}$ which lies over the path $\mathcal{P}' = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n-1)$ and this place Q' is rational. Thus, we get the equality $\mathbb{P}_{F_{m,n}}(\mathcal{P}') = \mathbb{P}_{F_{m,n}}(Q')$ and even the equalities

$$\mathbb{P}_{F_{m,n}}(\mathcal{P}) = \mathbb{P}_{F_{m,n}}((Q', P_{n-1,n}, P_{n,n})) = \mathbb{P}_{F_{m,n}}((Q', P_{n-1,n})). \tag{35}$$

where the first equality holds by the identity $\mathbb{P}_{F_{m,n}}(\mathcal{P}') = \mathbb{P}_{F_{m,n}}(Q')$ and by the fact that $\mathcal{P}'$ contains all places in $\mathcal{P}$ except for $P_{n-1,n}$ and $P_{n,n}$ and the second equality holds by the definition of a path in Definition 16(i) which implies $P_{n-1,n}/P_{n,n}$.

Next, let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q')$. Then we conclude the equality

$$e(Q'|P_{n-1,n-1}) = \prod_{i=m+1}^{n-1} e(P_{i-1,n-1}|P_{i,n-1}) = \prod_{i=m+1}^{n-1} e(P_{i-1,i}|P_{i,i}) = 1 \tag{36}$$

where the equalities and estimate hold by the following reasonings: The first equality holds because of the identity $Q' = P_{m,n-1}$, because Lemma 13 implies that we have the extensions $P_{m,n-1}/P_{m-1,n-1}/\ldots/P_{n-1,n-1}$ of places and because of the multiplicative transitivity rule for the ramification indices in (7) to these extensions. The estimate holds because Key Lemma 36(iv) implies the estimates $e(P_{i-1,n-1}|P_{i,n-1}) \leq e(P_{i-1,i}|P_{i,i})$ for all $i = m+1, \ldots, n-1$. The second equality hold by the assertion that the extensions in $\mathcal{P} = (P_{i,j})_{j-i\leq 1}$ are unramified.

Finally, we obtain the desired equality $1 = \#\mathbb{P}_{F_{m,n}}(\mathcal{P})$ by the estimates

$$\begin{aligned} 1 &= e(Q'|P_{n-1,n-1})f(Q'|P_{n-1,n-1})e(P_{n-1,n}|P_{n-1,n-1})f(P_{n-1,n}|P_{n-1,n-1}) \\ &\geq \sum_{Q\in\mathbb{P}_{F_{m,n}}(\mathcal{P})} e(Q|P_{n-1,n-1})f(Q|P_{n-1,n-1}) \geq \#\mathbb{P}_{F_{m,n}}(\mathcal{P}) \geq 1 \end{aligned} \tag{37}$$

where the equality and estimates hold by the following reasonings: The equality holds by the assumption $e(P_{n-1,n}|P_{n-1,n-1}) = 1$, by the assumption that $P_{n-1,n}$ is rational, by the identity in (36) and because Q' is rational too. The first estimate holds by Key Lemma 36(i) and the identity in (35). The second estimate holds as the ramification indices and relative degrees are positive natural numbers. The last estimate holds because Lemma 17(i) provides that the map Path is surjective and hence the estimates $\#\mathbb{P}_{F_{m,n}}(\mathcal{P}) = \#\,\mathrm{Path}^{-1}(\mathcal{P}) \geq 1$.

Moreover, from these estimates in (37), we even derive

$$1 \geq e(Q|P_{n-1,n-1})f(Q|P_{n-1,n-1}) \geq f(Q|P_{n-1,n-1}) \geq 1$$

for $\mathbb{P}_{F_{m,n}}(\mathcal{P}) = \{Q\}$ and, thus, $\deg(Q) = f(Q|P_{n-1,n-1})\deg(P_{n-1,n-1}) = 1$ where the equality $\deg(P_{n-1,n-1}) = 1$ holds by assumption. Hence, (iv) follows.

For (v): We show the desired equivalence in (v) by induction over $n-m \in \mathbb{N}_0$. For $n-m \in \{0,1\}$, both statements of the desired equivalence are always true because the conditions are empty in these cases. For $n-m=2$, the desired equivalence is exactly the definition of $\mathrm{Path}(Q)$ being tame in Definition 16(vi) and, thus, holds trivially.

Now, let $n-m \geq 3$. For the 'if'-part, suppose that, for all elementary diamonds (R, P_1, P_2, P) in $\mathrm{Pyr}(Q) =: (P_{i,j})_{i,j}$, at least one of the two elementary extensions P_1/P and P_2/P is tame. Especially, this supplies that

$$\text{at least one of the extensions } P_{m,n-1}/P_{m+1,n-1} \text{ and } P_{m+1,n}/P_{m+1,n-1} \text{ is tame.} \tag{38}$$

Moreover, as the elementary diamonds in $\mathrm{Pyr}(P_{m,n-1})$ and $\mathrm{Pyr}(P_{m+1,n})$ are elementary diamonds in $\mathrm{Pyr}(Q)$, this condition also holds for all elementary diamonds in $\mathrm{Pyr}(P_{m,n-1})$ and $\mathrm{Pyr}(P_{m+1,n})$. Then applying the induction hypothesis to $P_{m,n-1}$ and $P_{m+1,n}$ yields that the paths

$$\mathrm{Path}(P_{m,n-1}) \text{ and } \mathrm{Path}(P_{m+1,n}) \text{ are tame} \tag{39}$$

which means that, for all $m+2 \leq i \leq j \leq n-1$ and $m+1 \leq i \leq j \leq n-2$, at least one of the two extensions $P_{i-1,i}/P_{i,i}$ and $P_{j,j+1}/P_{i,j}$ is tame. Thus, by the definition of $\mathrm{Path}(Q)$ being tame in Definition 16(vi), the 'if'-part of (v) follows if we show that at least one of the two extensions $P_{m,m+1}/P_{m+1,m+1}$ and $P_{n-1,n}/P_{n-1,n-1}$ is tame.

Now, assume the contrary, i.e. both extensions $P_{m,m+1}/P_{m+1,m+1}$ and $P_{n-1,n}/P_{n-1,n-1}$ are wild. Then, by (39), all extensions $P_{i-1,i}/P_{i,i}$ and $P_{j,j+1}/P_{i,j}$ with $m+2 \leq i \leq j \leq n-2$ are tame and, therefore, we can iteratively apply Abhyankar's Lemma to all elementary diamonds in $\mathrm{Pyr}(P_{m,n-1})$ and $\mathrm{Pyr}(P_{m+1,n})$. But, this then yields that both extensions $P_{m,n-1}/P_{m+1,n-1}$ and $P_{m+1,n}/P_{m+1,n-1}$ are wild which contradicts (38). Hence, the 'if'-part of (v) follows.

For the 'only if'-part of (v), suppose that $\mathrm{Path}(Q)$ is tame. This especially implies that $\mathrm{Path}(P_{m+1,n})$ and $\mathrm{Path}(P_{m,n-1})$ are tame. Consequently, applying the induction hypothesis to $P_{m+1,n}$ and $P_{m,n-1}$ yields that, for all elementary diamonds $(R, P_1, P_2, P) \neq (P_{m,n}, P_{m,n-1}, P_{m+1,n}, P_{m+1,n-1})$ in $\mathrm{Pyr}(Q)$, at least one of the two elementary extensions P_1/P and P_2/P is tame.

For the remaining diamond $(P_{m,n}, P_{m,n-1}, P_{m+1,n}, P_{m+1,n-1})$, we consider the diamonds $D_1 := (P_{m,n}, P_{m,m+1}, P_{m+1,n}, P_{m+1,m+1})$ and $D_2 := (P_{m,n}, P_{m,n-1}, P_{n-1,n}, P_{n-1,n-1})$. Since $\mathrm{Path}(Q)$ is tame, we conclude by its definition in Definition 17(v) that at least one of the elementary extensions $P_{m,m+1}/P_{m+1,m+1}$ and $P_{n-1,n}/P_{n-1,n-1}$ is tame. But, then Abhyankar's Lemma in (10) is applicable to at least one of the diamonds D_1 and D_2 and, thus, yields that at least one of the extensions $P_{m,n-1}/P_{m+1,n-1}$ and $P_{m+1,n}/P_{m+1,n-1}$ is tame as well. Hence, the 'only if'-part of (v) follows. □

Examples 18. *In Examples 12, we considered the recursive towers* $\mathcal{F}'_{MW,2} = (F'_\nu)_\nu$ *over* $\mathbb{F}_9$ *and* $\mathcal{F}_{MW,2} = (F_\nu)_\nu$ *over* $\mathbb{F}_3$ *which are defined by the polynomial* $f_{MW,2} := Y^2 + XY + 2X^2 + 1$ *(see Figure 2.3) and two pyramids* $\mathrm{Pyr}(Q')$ *and* $\mathrm{Pyr}(Q)$ *of places in the figures 2.7 and 2.8. The zeroth and first levels of the pyramids form the paths* $\mathrm{Path}(Q')$ *and* $\mathrm{Path}(Q)$ *of* Q' *and* Q*, respectively.*

Here, both paths $\mathrm{Path}(Q')$ *and* $\mathrm{Path}(Q)$ *are tame. Moreover,* $\mathrm{Path}(Q)$ *is not rational as* $P_{3,3} = (x_3^2 + 1)$ *has degree two. But, because we have not specified the places* $P_{i,i+1}$ *in Figure 2.8, we cannot tell whether* $\mathrm{Path}(Q')$ *is rational or not yet. However, in Examples 106, it will come out that* $\mathrm{Path}(Q')$ *is indeed rational.*

Places lying over paths with further requirements. Sometimes, we will not only need places R which lie over a path $\mathcal{P}$ but which also lie over further places $Q_1, \ldots, Q_r$. The following Lemma 19 ensures that such places R exist as long as $Q_1, \ldots, Q_r$ also lie over compatible subpaths of $\mathcal{P}$.

Lemma 19. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$, let $m_i, n_i \in \mathbb{N}_0$ for all $i = 0, \dots, r+1$ such that $m_0 = n_0 = 0$ and $n_{i-1} \le m_i \le n_i$ for all $i = 1, \dots, r+1$, let $Q_i \in \mathbb{P}_{F_{m_i,n_i}}$ for all $i = 1, \dots, r$ and let $\mathcal{P}_i \in W(\mathcal{F}, n_i, m_{i+1})$ be paths for all $i = 0, \dots, r$ such that the composition*

$$\mathcal{P} := \mathcal{P}_0 \cdot \prod_{i=1}^{r} \mathrm{Path}(Q_i) \cdot \mathcal{P}_i \in W(\mathcal{F}, 0, m_{r+1})$$

is well defined. Then the set $\mathbb{P}_{F_{m_{r+1}}}(\mathcal{P}) \cap \mathbb{P}_{F_{m_{r+1}}}((Q_1, \dots, Q_r))$ is non-empty (see Figure 2.10).

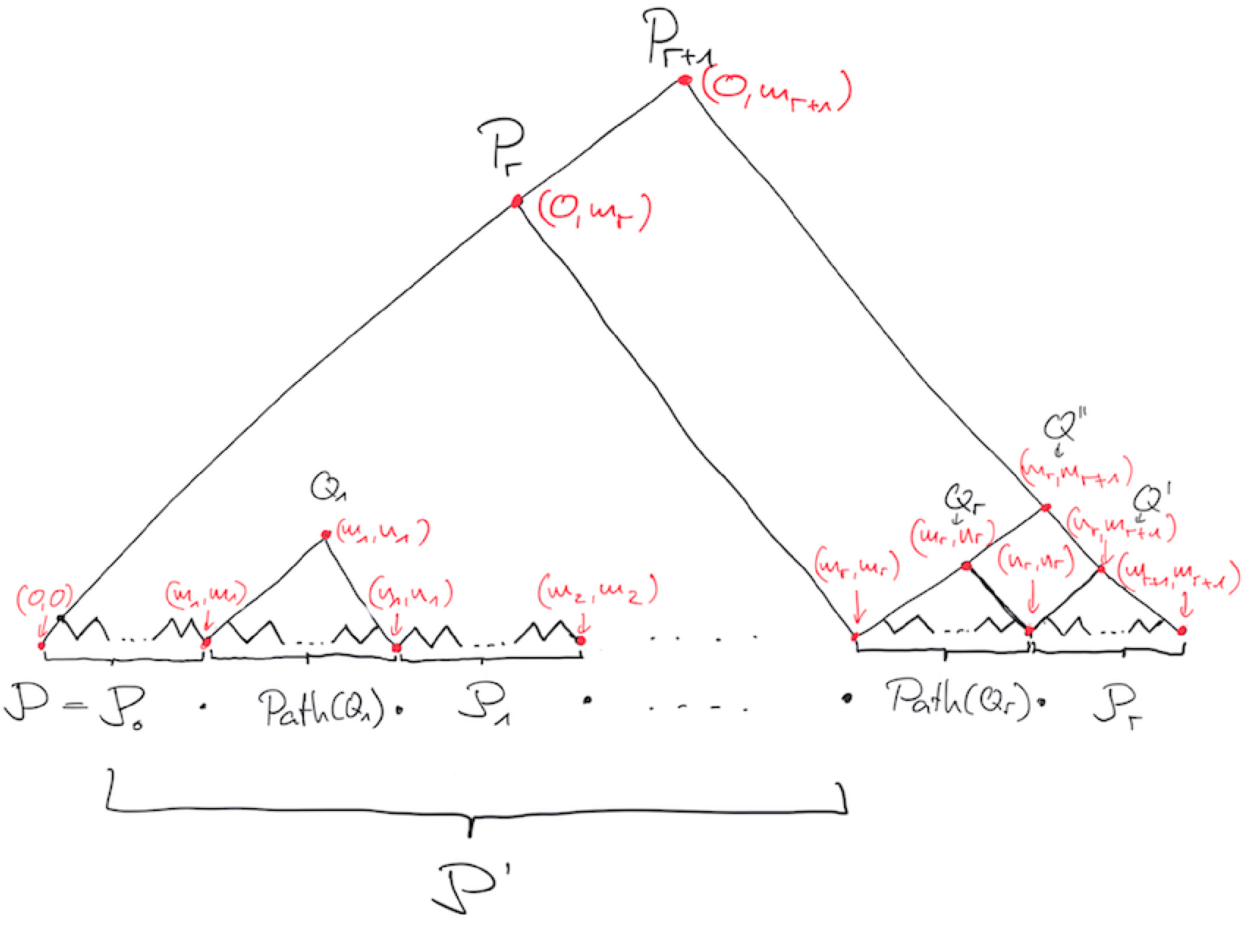


Figure 2.10: Pyramid of places in a proof

Proof. We will find a place $P_r \in \mathbb{P}_{F_{m_{r+1}}}(\mathcal{P}) \cap \mathbb{P}_{F_{m_{r+1}}}((Q_1, \dots, Q_r))$ by induction over $r \in \mathbb{N}_0$. For $n = 0$, we only have $\mathcal{P} = \mathcal{P}_0$ for some path $\mathcal{P}_0 \in W(\mathcal{F}, 0, m_1)$. Then the surjectivity of the map Path in Definition/Lemma 17(i) provides some place $P_0 \in \mathbb{P}_{F_{m_1}}(\mathcal{P})$.

Now, let $n \ge 1$ and define the path

$$\mathcal{P}' := \mathcal{P}_0 \cdot \prod_{i=1}^{r-1} \mathrm{Path}(Q_i) \cdot \mathcal{P}_i \in W(\mathcal{F}, 0, m_r)$$

(see Figure 2.10). Then the induction hypothesis provides some place

$$P_{r-1} \in \mathbb{P}_{F_{m_r}}(\mathcal{P}') \cap \mathbb{P}_{F_{m_r}}((Q_1 \dots, Q_{r-1})). \tag{40}$$

Especially, for $(P'_{i,j})_{j-i\leq 1} := \mathcal{P}'$, we conclude the equality

$$P_{r-1} \cap F_{m_r,m_r} = P'_{m_r,m_r} \tag{41}$$

by the definition of $\mathbb{P}_{\cdot}(\cdot)$ in (5).

Next, since $\mathcal{P} = \mathcal{P}' \cdot \mathrm{Path}(Q_r) \cdot \mathcal{P}_r$ is well defined, the paths $\mathrm{Path}(Q_r) = (Q_r \cap F_{i,j})_{j-i} \in W(\mathcal{F}, m_r, n_r)$ and $\mathcal{P}_r =: (P_{i,j})_{j-i\leq 1} \in W(n_r, m_{r+1})$ must be composable. By its definition in Definition 16(iii), this implies the equality $Q_r \cap F_{n_r,n_r} = P_{n_r,n_r}$. Moreover, again by the surjectivity of Path, we obtain some place $Q' \in \mathbb{P}_{F_{n_r,n}}(\mathcal{P}_r)$. In particular, Q' satisfies the equalties $Q' \cap F_{n_r,n_r} = P_{n_r,n_r} = Q_r \cap F_{n_r,n_r}$. Consequently, because of this equality and because $F_{m_r,n_r}/F_{n_r,n_r}$ and $F_{n_r,m_{r+1}}/F_{n_r,n_r}$ are linearly disjoint extensions with $F_{m_r,n_r} \cdot F_{n_r,m_{r+1}} = F_{m_r,m_{r+1}}$ by Lemma 10(iii), the 'moreover'-part of Key Lemma 36(i) even provides some place

$$Q'' \in \mathbb{P}_{F_{m_r,m_{r+1}}}((Q_r, Q')) \subseteq \mathbb{P}_{F_{m_r,m_{r+1}}}(Q_r) \cap \mathbb{P}_{F_{m_r,m_{r+1}}}(\mathcal{P}_r) \tag{42}$$

Especially, Q'' satisfies the equality

$$Q'' \cap F_{m_r,m_r} = Q_r \cap F_{m_r,m_r}. \tag{43}$$

Second to last, we get the equalities

$$P_{r-1} \cap F_{m_r,m_r} = P'_{m_r,m_r} = Q_r \cap F_{m_r,m_r} = Q'' \cap F_{m_r,m_r} \tag{44}$$

where the first equality holds by the equality in (41), the second equality holds since the composition $\mathcal{P} = \mathcal{P}' \cdot \mathrm{Path}(Q_r) \cdot \mathcal{P}_r$ is well defined and, thus, $\mathcal{P}'$ and $\mathrm{Path}(Q_r)$ must be composable and the third equality holds by the equality in (43).

Finally, because $F_{0,m_r}/F_{m_r,m_r}$ and $F_{m_r,m_{r+1}}/F_{m_r,m_r}$ are also linearly disjoint with $F_{0,m_r} \cdot F_{m_r,m_{r+1}} = F_{0,m_{r+1}}$ by Lemma 10(iii) and because of the equality in (44), the 'moreover'-part in Key Lemma 36(i) supplies some desired place P_r in

$$\begin{aligned}\mathbb{P}_{F_{0,m_{r+1}}}((P_{r-1}, Q'')) &\subseteq \Big(\mathbb{P}_{F_{m_{r+1}}}(\mathcal{P}') \cap \mathbb{P}_{F_{m_{r+1}}}((Q_1, \dots, Q_{r-1}))\Big) \\ &\quad \cap \Big(\mathbb{P}_{F_{m_{r+1}}}(\mathcal{P}_r) \cap \mathbb{P}_{F_{m_{r+1}}}(Q_r)\Big) \\ &= \mathbb{P}_{F_{m_{r+1}}}(\mathcal{P}) \cap \mathbb{P}_{F_{m_{r+1}}}((Q_1, \dots, Q_r))\end{aligned}$$

where the inclusion holds because of the choices of P_{r-1} in (40) and Q'' in (42) and because of the definition of $\mathbb{P}_{\cdot}(\cdot)$ in (5) and the equality holds by the equality $\mathcal{P} = \mathcal{P}'' \cdot \mathrm{Path}(Q_r) \cdot \mathcal{P}_r$. □

Extending the action of the tower map to paths of places. Analogously to the definition of the action of the tower map σ on pyramids $\mathrm{Pyr}(Q)$ of places in Definition/Lemma 15, we will also define the following action of σ on the paths $\mathrm{Path}(Q)$ of places in Definition/Lemma 20.

Definition/Lemma 20. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ for some $m \leq n$.*

(i) For all $k \in \mathbb{Z}$ with $0 \leq m + k$, we define

$$\sigma^k(\mathcal{P}) := (\sigma^k(P_{i-k,j-k}))_{j-i\leq 1} \in W(\mathcal{F}, m+k, n+k)$$

(ii) Then, for all $Q \in \mathbb{P}_{F_{m,n}}$ and all $k \in \mathbb{Z}$ with $0 \leq m + k$, we have the identity

$$\sigma^k(\mathrm{Path}(Q)) = \mathrm{Path}(\sigma^k(Q)) \in W(\mathcal{F}, m+k, n+k).$$

(iii) For all $k, l \in \mathbb{Z}$ with $0 \leq m+k$ and $0 \leq m+k+l$, we have the identity

$$\sigma^l(\sigma^k(\mathcal{P})) = \sigma^{k+l}(\mathcal{P}) \in W(\mathcal{F}, m+k+l, n+k+l).$$

Proof. For (ii): This immediately follows from Definition/Lemma 15(i) as $\mathrm{Path}(R)$ only contains the places of $\mathrm{Pyr}(R)$ at the positions (i,j) with $j-i \leq 1$ for all places $R \in \mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}$.

For (iii): For all $k, l \in \mathbb{Z}$ with $0 \leq m+k$ and $0 \leq m+k+l$, we immediately derive the desired identity in (iii) by the equalities

$$\sigma^l(\sigma^k(\mathcal{P})) = (\sigma^l(\sigma^k(P_{i-k-l,j-k-l})))_{j-i \leq 1} = \sigma^{k+l}(\mathcal{P})$$

where the equalities hold by Definition/Lemma 20(i). □

2.3 Constant Field Extensions of Recursive Towers

Purpose of this section. In this subsection, we will rigorously formalize constant field extensions of towers of function fields, and prove some first simple properties of constant field extensions of recursive towers.

Relevance of constant field extensions of recursive towers for the almost complete answer to Conjecture 1(iii) in Corollary 184. Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a field k which is defined by the pair (σ, F_0) and let $\overline{k}$ be the algebraic closure of k which is contained in the domain of the tower map σ (see Definition 5(ii)). For determining or even estimating the limit

$$\lambda(\mathcal{F}) := \frac{\nu(\mathcal{F})}{\gamma(\mathcal{F})} = \frac{\lim\limits_{n\to\infty} N(F_n)/d^n}{\lim\limits_{n\to\infty} g(F_n)/d^n},$$

of a tower, we need to analyze the asymptotic behavior of the quotients $N(F_n)/d^n$ and $g(F_n)/d^n$ as $n \to \infty$.

Now, let $\overline{F}_n := \overline{k} \cdot F_n$ be the compositum of $\overline{k}$ and F_n for all $n \in \mathbb{N}_0$. On the one hand, we have the equality $g(\overline{F}_n) = g(F_n)$ by [Sti08, p. 114, Theorem 3.6.3(b)] and, on the other hand, because there lies exactly one place $\overline{Q}$ in $\overline{F}_n$ over any rational place Q in F_n by [Sti08, p. 119, Lemma 3.6.5], we have the identity $N(F_n) = \#\mathbb{P}_{\overline{F}_n}(\mathbb{P}^{(1)}_{F_n}) = N(\overline{F}_n, \mathbb{P}^{(1)}_{F_n})$ for all $n \in \mathbb{N}_0$. Thus, instead of analyzing $N(F_n)/d^n$ and $g(F_n)/d^n$, we can also analyze $N(\overline{F}_n, \mathbb{P}^{(1)}_{F_n})/d^n$ and $g(\overline{F}_n)/d^n$. We will do this at many crucial points (e.g Corollary 51, Proposition 175). Therefore, we want to formalize some relations between the function fields $\overline{F}_n$ and F_n which will enable us to conveniently transfer results from $\overline{F}_n$ to F_n.

More concretely, the almost complete answer to Conjecture 1(iii) in Corollary 184 will basically prove that the number of places $\overline{Q}$ in $\overline{F}_n$ which lie over the 'decisive' rational places P in F_0 is already negligible in face of d^n as $n \to \infty$. Therefore, we will not even need to take into account that many of these places $\overline{Q}$ do not lie over rational places $Q = \overline{Q} \cap F_n$ in F_n.

Constant field extensions of towers. In the following Definition/Lemma 21, we will define constant field extensions $\mathcal{F} = k' \cdot \mathcal{F}$ of recursive towers $\mathcal{F}$ for algebraic extension fields k' of k.

Definition/Lemma 21. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower of function fields, let $\mathbf{F}$ be the union of the function fields F_n for all $n \in \mathbb{N}_0$, let $\overline{\mathbf{F}}$ be some algebraic closure of $\mathbf{F}$, let k' be an algebraic extension field of k which is contained in $\overline{\mathbf{F}}$ and let $\overline{k}$ be the algebraic closure of k in $\overline{\mathbf{F}}$. Then the following hold:*

(i) The sequence $k' \cdot \mathcal{F} := (k' \cdot F_\nu)_\nu$ of function fields is a tower of function fields over k'. Moreover, we have the identities $[k'F_{n+1} : k'F_n] = [F_{n+1} : F_n]$ and $g(k' \cdot F_n) = g(F_n)$ for all $n \in \mathbb{N}_0$.

*We call $k' \cdot \mathcal{F}$ the k'-**constant field extension** of the tower $\mathcal{F}$ and $\overline{k} \cdot \mathcal{F}$ a **geometric tower** of $\mathcal{F}$.*

(ii) Suppose that $\mathcal{F}$ is a recursive tower which is defined by the pair (σ, F_0), let $\overline{\mathbf{F}}$ be the domain of σ and let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Then $k' \cdot \mathcal{F}$ is a recursive tower which is defined by the pair $(\sigma, k' \cdot F_0)$ and we have the identity $\mathrm{Pyr}(k' \cdot \mathcal{F}) = (k' \cdot F_{i,j})_{i,j}$.

In the following, if $\mathcal{F}$ is a recursive tower with tower map σ, then we will always implicitly choose $\overline{\mathbf{F}}$ as the domain of σ and, hence, will always speak of 'the' geometric tower $\overline{k} \cdot \mathcal{F}$ of $\mathcal{F}$.

Proof. For (i): First, the definition that $\mathcal{F} = (F_\nu)_\nu$ is a tower of function fields over k in Definition 2(i) provides that k is the full constant field of the function field F_n for all $n \in \mathbb{N}_0$, the extensions F_{n+1}/F_n are finite, proper and separable for all $n \in \mathbb{N}_0$ and $g(F_n) \to \infty$ as $n \to \infty$.

Now, as k is a perfect field and as k'/k is an algebraic extension, the field k' is perfect too. By [Sti08, p. 120, Corollary 3.6.7], we then obtain that k' is the full constant field of $k'F_n$ and the identity $[k'F_{n+1} : k'F_n] = [F_{n+1} : F_n]$ for all $n \in \mathbb{N}_0$.

Next, for all $n \in \mathbb{N}_0$, the separability of the extension $k'F_{n+1}/k'F_n$ follows from the separability of the field extension F_{n+1}/F_n and from the fact that separability is invariant under translation.

Finally, from [Sti08, p. 114, Theorem 3.6.3(b)], we derive the equality $g(k'F_n) = g(F_n)$ and, therefore, $g(k' \cdot F_n) = g(F_n) \to \infty$ as $n \to \infty$.

All together, $k' \cdot \mathcal{F}$ satisfies all conditions of a tower of function fields over k' in Definition 2(i). By the way, we also established the desired identities in (i), namely $g(k'F_n) = g(F_n)$ and $[k'F_{n+1} : k'F_n] = [F_{n+1} : F_n]$ for all $n \in \mathbb{N}_0$.

For (ii): For all $0 \leq i \leq j$, we compute

$$\prod_{m=i}^{j} \sigma^m(k' \cdot F_0) = k' \cdot \prod_{m=i}^{j} \sigma^m(F_0) = k' \cdot F_{i,j} \tag{45}$$

where the first equality holds because σ is an an isomorphism of $\overline{k}$-algebras in Definition 5(ii), because the algebraic closure $\overline{k}$ of k in $\overline{\mathbf{F}}$ contains the algebraic extension field k' of k and because of the definition of composite fields.

For $i = 0$, the equalities in (45) yield that $k' \cdot \mathcal{F} = (k' \cdot F_\nu)_\nu = (k' \cdot F_{0,\nu})_\nu$ is indeed recursively defined by the pair $(\sigma, k' \cdot F_0)$ because of the definition of recursive towers in Definition 5(ii).

Moreover, for all $0 \leq i \leq j$, the equalities in (45) then yield the desired identity $\mathrm{Pyr}(k' \cdot \mathcal{F}) = (k' \cdot F_{i,j})_{i,j}$ by the definition of Pyr in Definition 9. ☐

Example 22. *In Examples 12 and Examples 18 the recursive tower $\mathcal{F}'_{MW,2}$ is the constant field extension $\mathbb{F}_9 \cdot \mathcal{F}_{MW,2}$.*

Projection maps from constant field extensions. Let $\mathcal{F}' = (F'_n)_n$ be a constant fields extension of the recursive tower $\mathcal{F} = (F_\nu)_\nu$. In the following Lemma 76, we will continue the formalization of relations between $\mathcal{F}$ and $\mathcal{F}'$. Here the first two simple items (i) and (ii) will be useful later and formulate relations between the pyramids and paths of places Q and Q' where Q'/Q in F'_n/F_n. The last two items (iii) and (iv) first extend the canonical restriction map of extensions of places to maps $\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')} \to \mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}$ and $W(\mathcal{F}') \to W(\mathcal{F})$ and then clarifies how these maps interplay with the path maps $\mathrm{Path}_{\mathcal{F}'}$ and $\mathrm{Path}_{\mathcal{F}}$.

Lemma 23. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0) and let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Let $\mathcal{F}'$ be a constant field extension of $\mathcal{F}$ and let $(F'_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F}')$ be the pyramid of $\mathcal{F}'$. Let $Q' \in \mathbb{P}_{F'_{m,n}}$ for some $m \leq n$ and let $Q := Q' \cap F_{m,n}$. Then the following hold:*

(i) We have the identity $\mathrm{Pyr}(Q) = (Q' \cap F_{i,j})_{i,j}$ and $\mathrm{Path}(Q) = (Q' \cap F_{i,j})_{j-i\leq 1}$.

(ii) Let $(P'_{i,j})_{i,j} := \mathrm{Pyr}(Q')$ and $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$. For all $k = m, \ldots, n-1$, we then have the identity $\sigma^{-k}(P'_{k,k+1}) \cap F_1 = \sigma^{-k}(P_{k,k+1})$.

(iii) We have the surjection

$$\pi_{i,j} : \mathbb{P}_{F'_{i,j}} \to \mathbb{P}_{F_{i,j}} \text{ via } Q' \mapsto Q' \cap F_{i,j}$$

for all $i \leq j$ and $\pi_{i,j}$ extends to a surjection

$$\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}} : \mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')} = \coprod_{i\leq j} \mathbb{P}_{F'_{i,j}} \to \mathbb{P}_{\mathrm{Pyr}(\mathcal{F})} = \coprod_{i\leq j} \mathbb{P}_{F_{i,j}}.$$

(iv) We have the surjection

$$\pi_{W(\mathcal{F}')/W(\mathcal{F})} : W(\mathcal{F}') \to W(\mathcal{F}) \text{ via } (P'_{i,j})_{j-i\leq 1} \mapsto (P'_{i,j} \cap F_{i,j})_{j-i\leq 1}$$

which even restricts to a surjection $W(\mathcal{F}', m, n) \to W(\mathcal{F}, m, n)$ for all $m \leq n$.

Moreover, if we denote the path maps in Lemma 17(i) as $\mathrm{Path}_{\mathcal{F}'}$ for $\mathcal{F}'$ and as $\mathrm{Path}_{\mathcal{F}}$ for $\mathcal{F}$, then we also have the identity

$$\mathrm{Path}_{\mathcal{F}} \circ \pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}} = \pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \mathrm{Path}_{\mathcal{F}'}.$$

Proof. For (i): We immediately compute

$$\mathrm{Pyr}(Q) = (Q \cap F_{i,j})_{i,j} = ((Q' \cap F_{m,n}) \cap F_{i,j})_{i,j} = (Q' \cap F_{i,j})_{i,j}$$

where the first equality holds because of the definition of $\mathrm{Pyr}(Q)$ in Definition 11, the second equality holds because of the assumption $Q = Q' \cap F_{m,n}$ and the third equality holds because Lemma 10(i) implies the inclusion $F_{m,n} \supseteq F_{i,j}$. Hence, the first statement $\mathrm{Pyr}(Q) = (Q' \cap F_{i,j})_{i,j}$ in (i) follows from this identity and the second statement $\mathrm{Path}(Q) = (Q' \cap F_{i,j})_{j-i\leq 1}$ in (i) follows because the definitions of $\mathrm{Pyr}(Q)$ in Definition 11 and of Path in Definition 17(i) provide that $\mathrm{Pyr}(Q)$ contains the elements of $\mathrm{Path}(Q)$.

For (ii): We obtain the equalities

$$\sigma^{-k}(P'_{k,k+1}) \cap F_1 = \sigma^{-k}(P'_{k,k+1} \cap F_{k,k+1}) = \sigma^{-k}(P_{k,k+1})$$

for all $k = m, \ldots, n-1$ where the first equality holds because Lemma 10(i) and Lemma 10(ii) imply the identities $F_1 = F_{0,1} = \sigma^{-k}(F_{k,k+1})$ and because σ^{-k} is a bijection and the second equality holds because the definition of Pyr in Definition 11 implies the identities $P'_{i,j} = Q' \cap F'_{i,j}$ and $P_{i,j} = Q \cap F_{i,j}$ for all $m \leq i \leq j \leq n$ and by then applying Lemma 76(i). Hence, (ii) follows.

For (iii): The well definedness of $\pi_{i,j}$ follows because $F'_{i,j}/F_{i,j}$ is a constant field extension by Definition/Lemma 21 and the surjectivity of $\pi_{i,j}$ follows from [Sti08, p. 71, Proposition 3.1.7]. Moreover, the second desired statement immediately follows from the first one. Hence, (iv) follows.

For (iv): Let $m, n \in \mathbb{N}_0$ with $m \leq n$. By Lemma 17(i), $\mathrm{Path}_{\mathcal{F}'}$ restricts to a surjection $\mathbb{P}_{F'_{m,n}} \to W(\mathcal{F}', m, n)$ and, thus, for all $\mathcal{P}' = (P'_{i,j})_{j-i\leq 1} \in W(\mathcal{F}', m, n)$, there is some place $Q' \in \mathbb{P}_{F'_{m,n}}$ such that

$$(P'_{i,j})_{j-i\leq 1} = \mathcal{P}' = \mathrm{Path}_{\mathcal{F}'}(Q') = (Q' \cap F_{i,j})_{j-i\leq 1}. \tag{46}$$

Consequently, we compute

$$\begin{aligned}\pi_{W(\mathcal{F}')/W(\mathcal{F})}(\mathcal{P}') &= ((Q' \cap F'_{i,j}) \cap F_{i,j})_{j-i\leq 1} = ((Q' \cap F_{m,n}) \cap F_{i,j})_{j-i\leq 1} \\ &= \mathrm{Path}_{\mathcal{F}}(Q' \cap F_{m,n}) = \mathrm{Path}_{\mathcal{F}}(\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}}(Q')) \in W(\mathcal{F}, m, n)\end{aligned} \tag{47}$$

where the equalities hold by the following reasonings: The first equality holds by the identity in (46) and the definition of $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$. The second equality holds because Definition/Lemma 21 implies the inclusion $F'_{i,j} \supseteq F_{i,j}$ and because Lemma 10(i) implies the inclusion $F_{m,n} \supseteq F_{i,j}$. The third equality holds by the definition of $\mathrm{Path}_{\mathcal{F}}(Q' \cap F_{m,n})$ in Definition 17(i). The last equality holds by the definition of $\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}}$ in Lemma 76(iii).

Consequently, the identity in (47) provides that $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ is a well defined map which restricts to a map $W(\mathcal{F}', m, n) \to W(\mathcal{F}, m, n)$.

For the surjectivity of $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$, let $\mathcal{P} \in W(\mathcal{F}, m, n)$. Then combining the surjectivity of $\mathrm{Path}_{m,n}$ in Definition/Lemma 17(i) yields that there is some $Q \in F_{m,n}$ such that $\mathrm{Path}_{\mathcal{F}}(Q) = \mathcal{P}$. Now, choosing $Q' \in \mathbb{P}_{F'_{m,n}}(Q) = \pi^{-1}_{m,n}(Q)$ arbitrary and $\mathcal{P}' := \mathrm{Path}_{\mathcal{F}'}(Q') \in W(\mathcal{F}', m, n)$ yields the same equalities in (47). This means that $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ restricts to a surjection $W(\mathcal{F}', m, n) \to W(\mathcal{F}, m, n)$ and, hence, $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ must also be a surjection $W(\mathcal{F}') := \coprod_{m\leq n} W(\mathcal{F}', m, n) \to W(\mathcal{F}) := \coprod_{m\leq n} W(\mathcal{F}, m, n)$.

Finally, if we start with $\mathcal{P}' := \mathrm{Path}_{\mathcal{F}'}(Q')$ for any $Q \in \mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}$ in the equalities in (47), this provides the last desired equality $\mathrm{Path}_{\mathcal{F}} \circ \pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}} = \pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \mathrm{Path}_{\mathcal{F}'}$. Hence, (iv) follows. ☐

Example 24. *As already mentioned in Example 22, the recursive tower $\mathcal{F}'_{MW,2}$ is the constant field extension $\mathbb{F}_9 \cdot \mathcal{F}_{MW,2}$. Moreover, the extension Q'/Q of places in Example 12 provides the identity $Q = \pi_{\mathbb{P}_{\mathcal{F}'_{MW,2}}/\mathbb{P}_{\mathcal{F}_{MW,2}}}(Q')$ and, thus, Lemma 76(iv) supplies the diagram in figure 2.11.*

Galois group of the constant field extensions with the algebraic closure. The following Lemma 25 and Lemma 26 will be useful later on.

Lemma 25. *Let F be a function fields over the full constant field k, let Ω be a field which contains F and an algebraic closure $\overline{k}$ of k and let $\overline{F} := \overline{k} \cdot F$. Then $\overline{F}/F$ is Galois and, for all $P \in \mathbb{P}_{\overline{F}}$ and all $\overline{P}_1, \overline{P}_2 \in \mathbb{P}_F(P)$, there is some $\tau \in \mathrm{Gal}(\overline{F}/F)$ and some $t \in \mathbb{N}$ such that $\tau(\overline{P}_1) = \overline{P}_2$ and $\tau^t(\overline{P}_1) = \overline{P}_1$.*

Proof. As k is a perfect field, $\overline{k}/k$ is a Galois extension and, hence, its translation $\overline{F}/F$ via F is also Galois which is the first desired statement.

Now, let $P \in \mathbb{P}_F$, let $\overline{P}_i \in \mathbb{P}_{\overline{F}}(P)$ for all $i = 1, 2$, let k' be the Galois closure of the residue field of P over k inside of $\overline{k}$ and define $F' := k' \cdot F$. Then k'/k and F'/F are finite Galois extensions such that F' is contained in $\overline{F}$ and $\deg(P') = 1$ holds for all

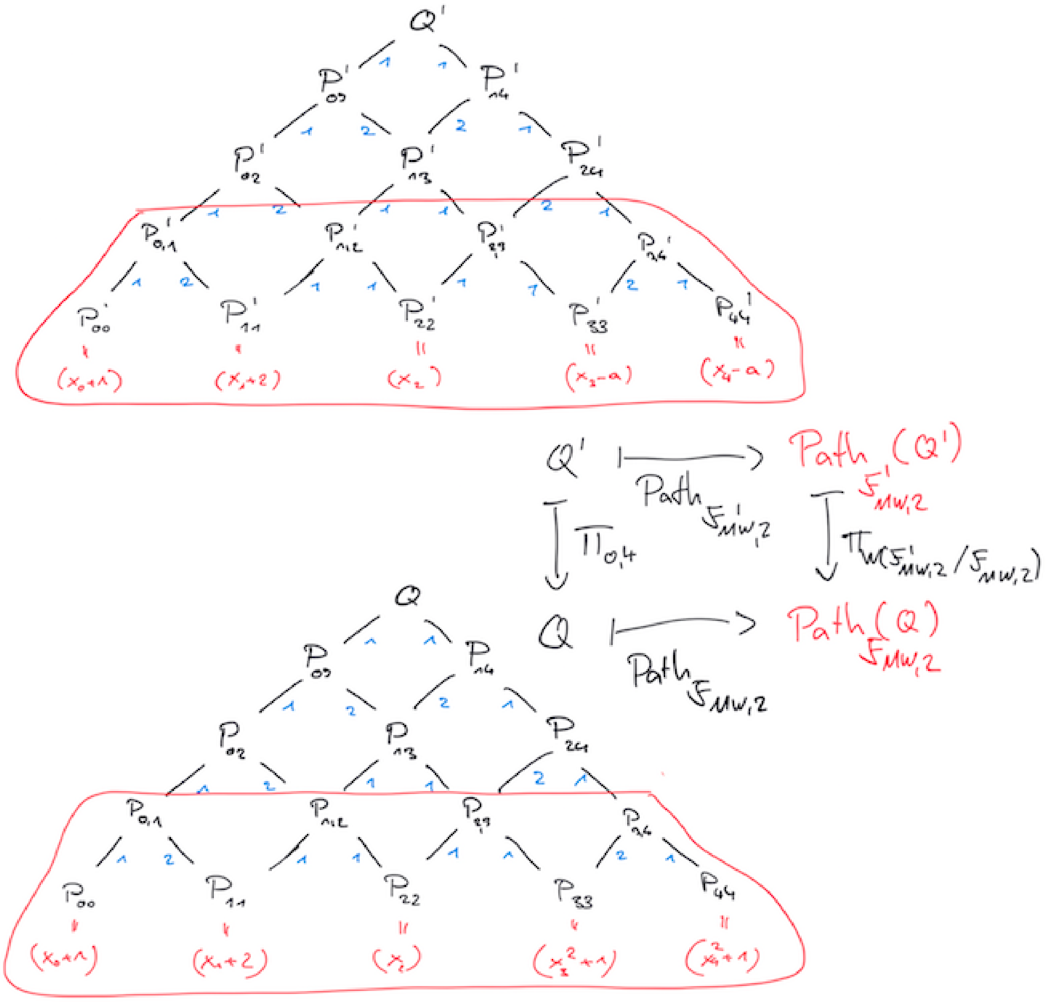


Figure 2.11: Example for the projection map of paths for constant field extensions

$P' \in \mathbb{P}_{F'}(P)$ by [Sti08, p. 114, Theorem 3.6.3(g)]. Thus, [Sti08, p. 119, Lemma 3.6.5] also implies that $\mathbb{P}_{\overline{F}}(P')$ is a singleton for all $P' \in \mathbb{P}_{F'}(P)$. Consequently, for all $i = 1, 2$ and for $P'_i := \overline{P}_i \cap F' \in \mathbb{P}_{F'}(P)$, we get the equality

$$\mathbb{P}_{\overline{F}}(P'_i) = \{\overline{P}_i\}. \tag{48}$$

Next, because F'/F is a finite Galois extension, say of degree $t \in \mathbb{N}$, [Sti08, p. 121, Theorem 3.7.1] supplies an automorphism $\tau' \in \mathrm{Gal}(F'/F)$ such that

$$\tau'(P'_1) = P'_2 \text{ and } (\tau')^t(P'_1) = \mathrm{id}_{F'}(P'_1) = P'_1. \tag{49}$$

Finally, let $\tau \in \mathrm{Gal}(\overline{F}/F)$ be any extension of τ. By this choice and the equalities in (48) and (49), we then obtain the equalities

$$\tau(\overline{P}_1) \in \mathbb{P}_{\overline{F}}(\tau'(P'_1)) = \mathbb{P}_{\overline{F}}(P'_2) = \{\overline{P}_2\} \tag{50}$$

and

$$\tau^t(\overline{P}_1) \in \mathbb{P}_{\overline{F}}((\tau')^t(P'_1)) = \mathbb{P}_{\overline{F}}(P'_1) = \{\overline{P}_1\}. \tag{51}$$

Hence, τ is the desired automorphism in $\mathrm{Gal}(\overline{F}/F)$. □

Lemma 26. *Let E/F be a finite extension of function fields over the same full constant field k and Ω be a field which contains E and an algebraic closure $\overline{k}$ of k. Moreover, let k' be an intermediate field of the extension $\overline{k}/k$ and define the constant field extensions $E' := k' \cdot E$ and $F' := k' \cdot F$. Then we have*

$$\mathbb{P}_{E'}((Q, P')) \neq \emptyset$$

for all $Q \in \mathbb{P}_E$ and all $P' \in \mathbb{P}_{F'}$ with $Q \cap F = P' \cap F$.

Proof. Let $Q \in \mathbb{P}_E$ and let $P' \in \mathbb{P}_{F'}$ with $P := Q \cap F = P' \cap F$ and define the constant field extensions $\overline{F} := \overline{k} \cdot F$ and $\overline{E} := \overline{k} \cdot E$. Then $\overline{F} = \overline{k} \cdot F = \overline{k} \cdot (k' \cdot F) = \overline{k} \cdot F'$ is also a constant field extension of F' (see Figure 2.12).

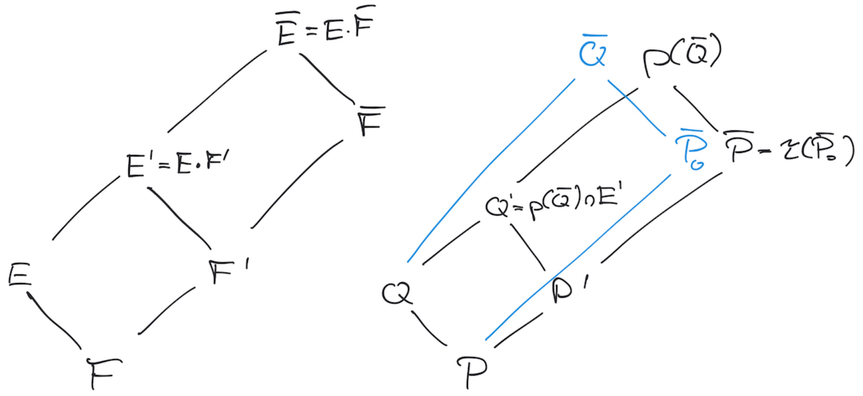


Figure 2.12: Diagram with extensions in a proof

First, choose $\overline{Q} \in \mathbb{P}_{\overline{E}}(Q)$ and $\overline{P} \in \mathbb{P}_{\overline{F}}(P')$ arbitrary and define $\overline{P}_0 := \overline{Q} \cap \overline{F}$. Then the equalities

$$\overline{P} \cap F = (\overline{P} \cap F') \cap F = P' \cap F = P$$

and

$$\overline{P}_0 \cap F = (\overline{Q} \cap \overline{F}) \cap F = (\overline{Q} \cap E) \cap F = Q \cap F = P$$

provide that $\overline{P}$ and $\overline{P}_0$ are both places in $\mathbb{P}_{\overline{F}}(P)$.

Second, Lemma 25 supplies some automorphism $\tau \in \mathrm{Gal}(\overline{F}/F)$ satisfying the equality $\tau(\overline{P}_0) = \overline{P}$.

Third, we also obtain the equality $[\overline{E} : \overline{F}] = [E : F]$ by [Sti08, p. 119, Proposition 3.6.6] and, hence, E/F and $\overline{F}/F$ are linearly disjoint. Then the proof of [Coh91, p. 188, Proposition 5.4] implies that $E \cdot \overline{F} = \overline{k} \cdot E = \overline{E}$ is a tensor product of the F-algebras E and $\overline{F}$ (with the inclusion morphisms). Consequently, by the universal property of tensor products, we obtain some F-automorphism ρ on $\overline{E}$ which restricts to id_E on E and to τ on $\overline{F}$. Therefore, we have $\rho(\overline{Q}) \in \mathbb{P}_{\overline{E}}((Q, \overline{P}))$.

Finally, as $\overline{E} = \overline{k} \cdot E = \overline{k} \cdot (k' \cdot E) = \overline{k} \cdot E'$ is also a constant field extension of E', the intersection $Q' := \rho(\overline{Q}) \cap E'$ is a place in E' and the equalities

$$Q' \cap E = (\rho(\overline{Q}) \cap E') \cap E = \rho(\overline{Q}) \cap E = Q$$

and

$$Q' \cap F' = (\rho(\overline{Q}) \cap E') \cap F' = (\rho(\overline{Q}) \cap \overline{E}) \cap F' = \overline{P} \cap F' = P'$$

even provide that Q' is a place in $\mathbb{P}_{E'}((Q, P'))$. Hence, Lemma 26 follows. ☐

2.4 Truncations of Recursive Towers

Purpose of this section. As we are interested in the more general versions of Conjecture 1, namely Conjecture 1(iii) and Conjecture 1(iv), in this section, we will define the level m truncation $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) := (F_{m+\nu})_\nu$ of the recursive tower $\mathcal{F} = (F_\nu)_\nu$ and prove some first properties of truncations. Then one of the advantages of the definition of pair-recursive towers will be that the truncated recursive tower $\mathcal{F}_{\geq m}$ of $\mathcal{F}$ will still be a pair-recursive tower which is defined by the pair (σ, F_m). Hence, we will also be able to apply all results to $\mathcal{F}_{\geq m}$.

Furthermore, in the Reduction Lemma 30, we will also provide a sufficient criterion for a polynomial-recursive tower $\mathcal{F}_{\geq 1} = (F_{1+\nu})_\nu$ to be a level one truncation of some polynomial-recursive tower $\mathcal{F} = (F_\nu)_\nu$ which starts with two rational function fields F_0 and F_1. Many polynomial-recursive towers in the literature can be reduced to such recursive towers (e.g. $\mathcal{F}_{GS,i}$ for all i, $\mathcal{F}_{HP,q}$ for all q and $\mathcal{F}_{BR,q}$ for all q in Examples 8).

When we will later work with the tower graphs of recursive towers (see Definition 74), this reduction step will simplify the tower graphs.

Truncations of towers. In the following Definition/Lemma 27, we will define truncations of towers and ensure that the truncations of a recursive tower is still a recursive tower.

Definition/Lemma 27. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower of function fields. Then*

$$\mathcal{F}_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F}) := (F_{m+\nu})_\nu$$

*is also a tower over the same field and of the same degree (see Figure 2.13). We call $\mathcal{F}_{\geq m}$ the **level** m **truncation** of $\mathcal{F}$.*

Moreover, if $\mathcal{F}$ is a recursive tower which is defined by the pair (σ, F_0), then $\mathcal{F}_{\geq m}$ is also a recursive tower which is defined by the pair (σ, F_m).

Proof. From the definition of towers of function fields in Definition 2(i), it immediately follows that $\mathcal{F}_{\geq m} = (F_{m+\nu})_\nu$ is also a tower of function fields over the same field and of the same degree as $\mathcal{F}$.

Next, suppose that $\mathcal{F}$ is a recursive tower which is defined by the pair (σ, F_0). Then $\mathcal{F}$ has a constant degree and, thus, $\mathcal{F}_{\geq m}$ also has the same constant degree. Moreover, we compute

$$\prod_{i=0}^{n} \sigma^i(F_m) = \prod_{i=0}^{n} \sigma^i(\prod_{j=0}^{m} \sigma^j(F_0)) = \prod_{i=0}^{n}\prod_{j=0}^{m} \sigma^{i+j}(F_0) = \prod_{i=0}^{n+m} \sigma^i(F_0) = F_{m+n} \tag{52}$$

for all $n \in \mathbb{N}_0$ where the first (resp. last) equality holds by the equality of $F_m = \prod_{j=0}^{m} \sigma^j(F_0)$ (resp. $F_{n+m} = \prod_{j=0}^{m+n} \sigma^j(F_0)$) in the definition of a recursive tower in Definition 5(ii), the second equality holds by the definition of composite fields and by the fact that σ is an isomorphism of algebras and the third equality holds because repeating factors can be neglected in the definition of a composite field.

Finally, the fact that $\mathcal{F}_{\geq m}$ has constant degree and the equality (52) yields that $\mathcal{F}_{\geq 0}$ is recursively defined via the pair (σ, F_m) by Definition 5(ii). ☐

Truncations and their pyramids of function fields.

Lemma 28. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$ and let $\mathcal{F}_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Then we have the identity*

$$\mathrm{Pyr}(\mathcal{F}_{\geq m}) = (F_{i,m+j})_{i,j}$$

(see Figure 2.13).

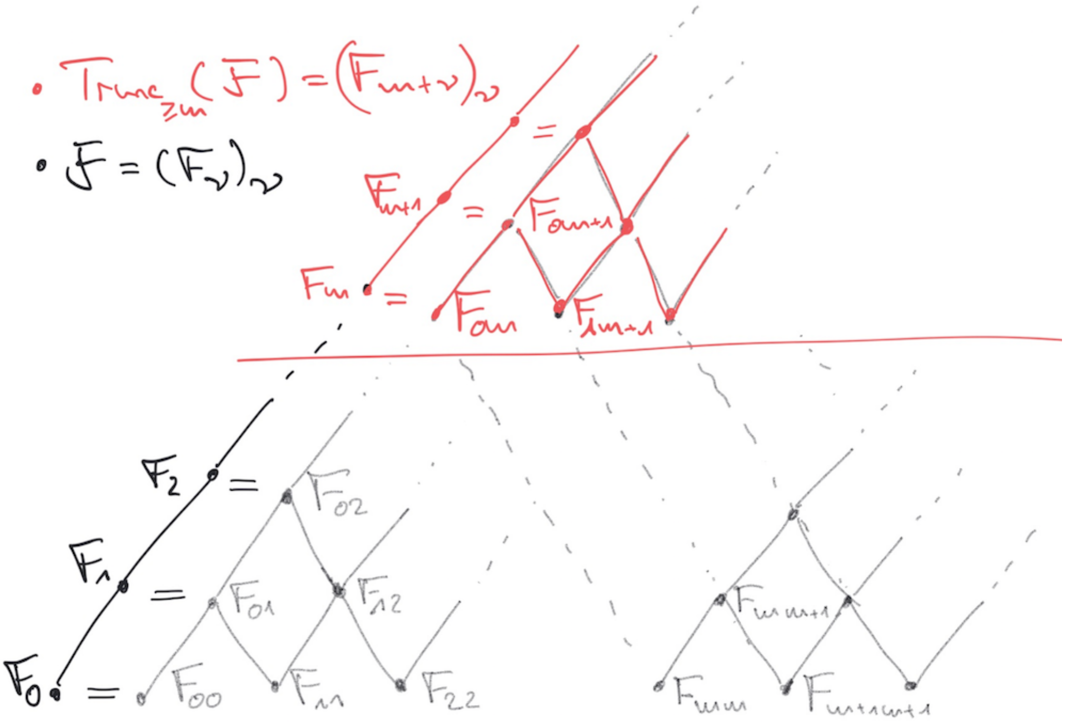


Figure 2.13: Pyramid of a truncated recursive tower

Proof. Let σ be the tower map of $\mathcal{F}$. Then we already obtain the desired equality from that fact that, by the definition of $\mathrm{Pyr}(\mathcal{F}_{\geq m})$ in Definition 9(i), the function field on the (k,l)-th position in $\mathrm{Pyr}(\mathcal{F}_{\geq m})$ is equal to

$$\prod_{i=k}^{l} \sigma^i(F_m) = \prod_{i=k}^{l} \sigma^i(\prod_{j=0}^{m} \sigma^j(F_0)) = \prod_{i=k}^{l} \prod_{j=0}^{m} \sigma^{i+j}(F_0) = \prod_{i=k}^{m+l} \sigma^i(F_0) = F_{k,m+l}$$

where the equalities holds by the following reasonings: The first equality holds by the equality of $F_m = \prod_{j=0}^{m} \sigma^j(F_0)$ in the definition of recursive towers in Definition 5(ii). The second equality holds because σ is an isomorphism of algebras and because of the definition of composite fields. The third equality holds since we can leave out repeating factors in the definition of a composite field. The last equality holds by the definition of $\mathrm{Pyr}(\mathcal{F}) = (F_{i,j})_{i,j}$ in Definition 9(i). □

Truncations and constant field extensions commute.

Lemma 29. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower of function fields over the field k. Moreover, let k' be an algebraic extension field of k which is contained a common extension field of all F_n and let $m \in \mathbb{N}_0$. Then we have the identity*

$$\mathrm{Trun}_{\geq m}(k' \cdot \mathcal{F}) = k' \cdot \mathrm{Trun}_{\geq m}(\mathcal{F}).$$

Proof. By the definitions of truncations of towers in Definition/Lemma 27 and of constant field extensions of towers in Definition/Lemma 21(i), we obtain the equalities $\mathrm{Trun}_{\geq m}(k' \cdot \mathcal{F}) = (k' \cdot F_{m+\nu})_\nu = k' \cdot \mathrm{Trun}_{\geq m}(\mathcal{F})$. □

2.4.1 Reduction Lemma

Many of the recursive towers $\mathcal{F}_{\geq 1} = (F_{1+\nu})_\nu$ in the literature are defined by polynomials f which separate variables, i.e. $f = g_1(X)h_2(Y) - g_2(X)h_1(Y)$ with univariate polynomials g_i, h_i for all $i = 1, 2$. In other words, $\mathcal{F}_{\geq 1}$ is defined by the equation $g_1(X)/g_2(X) = h_1(Y)/h_2(Y)$. For instance, see the defining polynomials $f_{GS,i}$ for all i, $f_{HP,q}$ for all q and $f_{BR,q}$ for all q in Examples 8.

The following Reduction Lemma 30 provides that, under some conditions (e.g. if $d = \deg_Y(f)$ is prime), we can add a zeroth level F_0 to such recursive towers $\mathcal{F}_{\geq 1}$.

Lemma 30 (Reduction Lemma)**.** *Let k be a perfect field, let g_i, h_i be non-zero univariate polynomials over k for all $i = 1, 2$, let $g := g_1/g_2$ and $h := h_1/h_2$, suppose that* $\gcd(g_1, g_2) = \gcd(h_1, h_2) = 1$ *and suppose that $f := g_1(X)h_2(Y) - g_2(X)h_1(Y)$ is a bivariate polynomial of degree $d \geq 2$ in both variables X and Y.*

Moreover, let $\mathcal{F}_{\geq 1} = (F_{1+\nu})_\nu$ be a recursive tower which is defined by the polynomial f and let σ be the tower map of $\mathcal{F}_{\geq 1}$ in Lemma 7. Finally, define $z_0 := h(x_1)$ and $F_0 := k(z_0)$. Then the following hold:

- *(i) If $k(x_1) = k(h(x_1), g(x_1))$, then $\mathcal{F} := (F_\nu)_\nu$ is a recursive tower which is defined by the pair (σ, F_0). Moreover, we then even have the identity* $\mathrm{Trun}_{\geq 1}(\mathcal{F}) = \mathcal{F}_{\geq 1}$.
- *(ii) If k is finite and d is a prime number, then the 'if'-condition in (i) is satisfied.*
- *(iii) If there is a place Q in $F_1 = k(x_1)$ which is totally ramified in one of the extensions $k(x_1)/k(g(x_1))$ and $k(x_1)/k(h(x_1))$ but is unramified in the other, then the 'if'-condition in (i) is satisfied.*

Proof. For (i): Suppose that the equality

$$k(x_1) = k(h(x_1), g(x_1)) \tag{53}$$

holds. First, we notice that since f has degree d in both variables and since the polynomials g_i, h_i are non-zero for all $i = 1, 2$, the maximal degree of the numerator and denominator in $g = g_1/g_2$ (resp. $h = h_1/h_2$) is equal to d. But, it is well known that this implies that the degree of $k(x_1)/k(g(x_1))$ (resp. $k(x_1)/k(h(x_1))$) is equal to d.

Second, since $\mathcal{F}_{\geq 1} = (F_\nu)_{\nu \geq 1}$ is a tower of function fields, the definition of towers of function fields in Definition 2(i) supplies that the full constant field of F_1 is k. But, since $F_0 = k(z_0) = k(h(x_1))$ is a subfield of $F_1 = k(x_1)$, the same must be true for F_0.

Third, because of the definition of (pair-) recursive towers in Definition 5(ii) and because $\mathcal{F}_{\geq 1} = (F_\nu)_{\nu \geq 1}$ is a recursive tower which is defined by the pair (σ, F_1) by the choice of σ in Lemma 7(iii), the only remaining property left to check for $\mathcal{F}$ being a (σ, F_0)-recursive tower is the equality $F_1 = F_0 \cdot \sigma(F_0)$. And this desired equality follows from the equalities

$$F_0 \cdot \sigma(F_0) = k(z_0) \cdot \sigma(k(z_0)) = k(h(x_1)) \cdot k(h(\sigma(x_1))) = k(h(x_1), g(x_1)) = k(x_1) = F_1$$

where the equalities hold by the following reasonings: The first and second equalities hold because of the definitions of $z_0 = h(x_1)$ and $F_0 = k(z_0)$ in the assumptions and because σ is a morphism of k-algebras. The third equality holds since $\mathcal{F}_{\geq 1}$ is recursively defined by the polynomial $f = g_1(X)h_2(Y) - g_2(X)h_1(Y)$ and since this implies the equalities $g(x_1) = h(x_2) = h(\sigma(x_1))$. The fourth equality holds by the assertion in (53). The last equality holds by the definition of $F_1 = k(x_1)$ in the assumptions.

Finally, the 'moreover'-part in (i) is obvious.

For (ii): Suppose that k is finite and d a prime number. Since $k(h(x_1), g(x_1))$ is an intermediate field of the extension $k(x_1)/k(h(x_1))$ of prime degree d, we only have to check that $k(h(x_1))$ and $k(g(x_1))$ are not equal for the desired identity $k(x_1) = k(h(x_1), g(x_1))$.

Now, assume the contrary, i.e. $k(g(x_1)) = k(h(x_2))$. Then we conclude the equalities

$$k(z_0) = k(h(x_1)) = k(g(x_1)) = k(h(\sigma(x_1)) = \sigma(k(z_0)) \tag{54}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $z_0 = h(x_1)$ in the assumptions. The second equality is the assertion of the contradiction. The third equality holds since $\mathcal{F}_{\geq 1}$ is recursively defined by the polynomial $f = g_1(X)h_2(Y) - g_2(X)h_1(Y)$ and since this implies the equalities $g(x_1) = h(x_2) = h(\sigma(x_1))$. The last equality holds since σ is a morphism of k-algebras. Hence, (54) implies that σ restricts to an automorphism on F_0.

Next, we notice that since k is finite, the automorphism group G of F_0 is finite. Consequently, the fixed field E of G in F is also a function field with full constant field k (see Figure 2.14). Moreover, since σ restricts to an element in G, we deduce that σ must fix all elements in E.

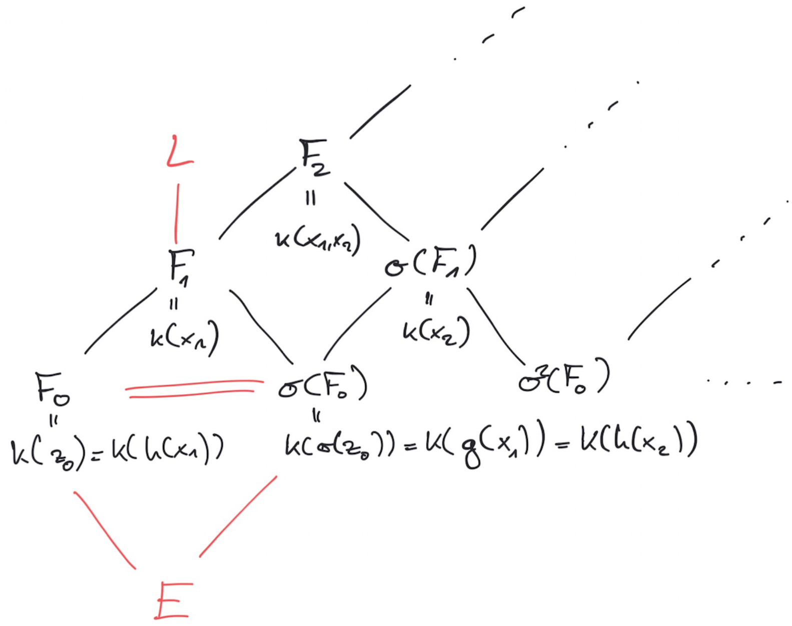


Figure 2.14: Recursive tower with separated variables in a proof

Finally, consider the Galois closure L of F_1/E in the domain of σ. Then the restriction of σ on L is an element of the finite Galois group of L/E and, thus, has finite order $m \in \mathbb{N}$. But this implies the equalities $x_m = \sigma^m(x_1) = x_1$ which is impossible because the tower $(F_\nu)_{\nu \geq 1} = (k(x_1, \ldots, x_\nu))_{\nu \geq 1}$ is a non-stationary sequence of function fields. Hence, the desired inequality $k(g(x_1)) \neq k(h(x_2))$ must indeed be the case.

For (iii): Suppose that Q is a place in $F_1 = k(x_1)$ which is totally ramified in one of the extensions $k(x_1)/k(g(x_1))$ and $k(x_1)/k(h(x_1))$ but is unramified in the other. Moreover, let $F := k(g(x_1)) \cdot k(h(x_1)) = k(g(x_1), h(x_1))$ and let $Q' := Q \cap F$.

Now, on the one hand, because w.l.o.g Q is totally ramified in $F_1/k(h(x_1))$ and because of the multiplicative transitivity rule of ramification indices in (7), we conclude that the ramification index of Q/Q' is equal to the degree of F_1/F.

But, on the other hand, because Q is unramified in $F_1/k(g(x_1))$ and because of the multiplicative transitivity rule of ramification indices in (7), we also conclude that the ramification index of Q/Q' is equal to one.

Consequently, we established the desired identity $k(x_1) = F_1 = F = k(g(x_1), h(x_1))$ in the 'if'-condition of (i). ☐

The tower map cannot fix subfields.

Remark 31. *Note that the proof of Lemma 30(i) can be modified to show that* σ *has to satisfy* $\sigma(K) \neq K$ *for all finite extensions* F_n/K *and all* $n \in \mathbb{N}_0$ *if* $\mathcal{F}$ *is a recursive tower.*

2.5 Locally Galois Recursive Towers

Purpose of this section. In this subsection, we will define locally Galois recursive towers $\mathcal{F}$ (see Definition 32). All elementary extensions in the pyramid $\mathrm{Pyr}(\mathcal{F})$ of a locally Galois recursive tower are Galois (see Lemma 33). Later, in Section 7.3, locally Galois recursive towers will play a side role. More concretely, the degree bounds which are provided in Section 7.3 can be sharpened for locally Galois recursive towers. If the reader is only interested in the major results, then this section can skipped.

Locally Galois recursive towers. In the following Definition 32, we will define locally Galois recursive towers. Many recursive towers in the literature are locally Galois (e.g. [MW05], [ST15]), [BR20]). In fact, all recursive towers of balanced degree two are locally Galois.

Definition 32. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a tower. Then we call* $\mathcal{F}$ ***Galois*** *if the extensions* F_n/F_0 *are Galois for all* $n \in \mathbb{N}$ *and we call* $\mathcal{F}$ ***Galois in every step*** *if the extensions* F_n/F_{n-1} *are Galois for all* $n \in \mathbb{N}$*. Moreover, if* $\mathcal{F}$ *is a recursive tower which is defined by the pair* (σ, F_0) *such that* F_1/F_0 *and* $F_1/\sigma(F_0)$ *are Galois, then we call* $\mathcal{F}$ ***locally Galois****.*

Lemma 33. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a locally Galois recursive tower which is defined by the pair* (σ, F_0) *and let* $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ *be the pyramid of* $\mathcal{F}$*.*

Then the extensions $F_{i,j}/F_{i,j-1}$ *and* $F_{i,j}/F_{i+1,j}$ *are Galois for all* $0 \leq i \leq j$ *with* $j - i \geq 1$*. In particular,* $\mathcal{F}$ *is also Galois in every step.*

Moreover, the extensions $F_{i,j}/F_{i+1,j-1}$ *are also Galois for all* $0 \leq i \leq j$ *with* $j - i \geq 2$*.*

Proof. Let $\varepsilon := 0$ (resp. $\varepsilon := 1$). First, let $j - i = 1$, i.e. $j = i + 1$. Then Lemma 10(i) provides that $F_1/\sigma^\varepsilon(F_0)$ is equal to $F_{0,1}/\sigma^\varepsilon(F_{0,0})$ and Lemma 10(ii) even provides that the latter is isomorphic to $F_{i,i+1}/F_{i+\varepsilon,i+\varepsilon}$. But, because of this isomorphism and because $F_1/\sigma^\varepsilon(F_0)$ is Galois by the assertion that $\mathcal{F}$ is locally Galois, we conclude that the extension

$$F_{i,i+1}/F_{i+\varepsilon,i+\varepsilon} \text{ is indeed Galois.} \tag{55}$$

Next, let $j - i \geq 2$. Then Lemma 10(iii) supplies the equalities $F_{i,j} = F_{i,j-1} \cdot F_{j-1,j}$ and $F_{i,j-1} = F_{i,j-1} \cdot F_{j-1,j-1}$ (resp. $F_{i,j} = F_{i,i+1} \cdot F_{i+1,j}$ and $F_{i+1,j} = F_{i+1,i+1} \cdot F_{i+1,j}$). Consequently, from these equalities, from (55) and from the well known fact that translations of Galois extensions are again Galois, we already conclude that the extension $F_{i,j}/F_{i+\varepsilon,j-1+\varepsilon}$ is Galois. Hence, the 'main'-part follows.

The 'in particular'-part immediately follows from the 'main'-part, from the fact that Lemma 10(i) implies the equality $F_{0,n} = F_n$ for all $n \in \mathbb{N}_0$ and from the definition of Galois in every step towers in Definition 32.

Finally, we deduce the 'moreover'-part because the extensions $F_{i+1,j}/F_{i+1,j-1}$ and $F_{i,j-1}/F_{i+1,j-1}$ are Galois by the 'main'-part, because Lemma 10(iii) implies the equality $F_{i+1,j} \cdot F_{i,j-1} = F_{i,j}$ and because of the well known fact that composita of Galois extension fields are again Galois extension fields. ☐

Lemma 34. *Let $\mathcal{F} = (F_\nu)_\nu$ be a locally Galois recursive tower which is defined by the pair (σ, F_0). Then the constant field extension $k' \cdot \mathcal{F} = (F'_\nu)_\nu$ of $\mathcal{F}$ for any algebraic extension k'/k is also locally Galois.*

Proof. First, we notice that, by the definition of the constant field extension $k' \cdot \mathcal{F}$ of $\mathcal{F}$ in Definition/Lemma 21, it is recursively defined by the pair $(\sigma, k' \cdot F_0)$ and we have the equality $F'_n = k' \cdot F_n$ for all $n \in \mathbb{N}_0$.

Second, we notice that since $\mathcal{F}$ is locally Galois, the extensions $F_1/\sigma^\varepsilon(F_0)$ are Galois for all $\varepsilon = 0, 1$ by its definition in Definition 32.

Third, we also notice that σ is an automorphism of k'-algebras by its definition in Definition 5(ii) and, thus, the equalities $k' \cdot \sigma^\varepsilon(F_0) = \sigma^\varepsilon(k' \cdot F_0) = \sigma^\varepsilon(F'_0)$ for all $\varepsilon = 0, 1$.

Finally, combining these three conclusions and the well known fact that translations of Galois extensions are again Galois yields that the extension $F'_1/\sigma^\varepsilon(F'_0)$ (which are equal to $k \cdot F_1/k' \cdot \sigma^\varepsilon(F_0)$) is Galois for all $\varepsilon = 0, 1$. Hence, $k' \cdot \mathcal{F}$ is indeed a locally Galois recursive tower. ☐

2.6 Dual Recursive Towers

Purpose of this section. In this subsection, we will define (pair-)dual recursive towers (see Definition/Lemma 35(i)). The only purpose of this section is to provide the proper background to connect Beelen's Graphs and the tower graphs (see Definition 74) in Subsection 4.4.1. Except for this connection, dual recursive towers will play no role in this thesis. Hence, if the reader is only interested in the major results, then this section can be skipped.

Dual recursive towers. In the following Definition/Lemma 35, we will define polynomial- and pair-dual recursive towers and connect them.

Definition/Lemma 35. *(i) Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by a pair (σ, F_0). Then the pair (σ^{-1}, F_0) also defines a recursive tower $\hat{\mathcal{F}}$ (see Figure 2.15). We call $\hat{\mathcal{F}}$ the* ***pair-dual recursive tower*** *of $\mathcal{F}$. Often, we will just call $\hat{\mathcal{F}}$ the dual recursive tower of $\mathcal{F}$.*

Also, notice the identity $\hat{\hat{\mathcal{F}}} = \mathcal{F}$.

(ii) Let $\mathcal{F}$ be a recursive tower over the field k which is defined by a polynomial $f = f(X, Y)$, then we call a recursive tower which is defined by the polynomial $g = g(X, Y) := f(Y, X)$ a ***polynomial-dual recursive tower*** *of $\mathcal{F}$.*

Moreover, consider the pair (σ, F_0) in Lemma 7(i) which is induced by the polynomial f. Then $\mathcal{F}$ is recursively defined by this pair (σ, F_0) and the pair-dual recursive tower $\hat{\mathcal{F}}$ of $\mathcal{F}$ is also a polynomial-dual recursive tower of $\mathcal{F}$.

Proof. For (i): Let k be the field over which $\mathcal{F}$ is defined, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Moreover, define $\hat{\mathcal{F}} = (\hat{F}_\nu)_\nu$ as the sequence of function fields over k which is defined by

$$\hat{F}_n := \prod_{l=0}^{n} \sigma^{-l}(F_0). \tag{56}$$

for all $n \in \mathbb{N}_0$ (see Figure 2.15). We will show that $\hat{\mathcal{F}}$ is a tower. Then the definition of recursive towers supplies that $\hat{\mathcal{F}}$ is even a recursive tower which is defined by the pair (σ^{-1}, F_0).

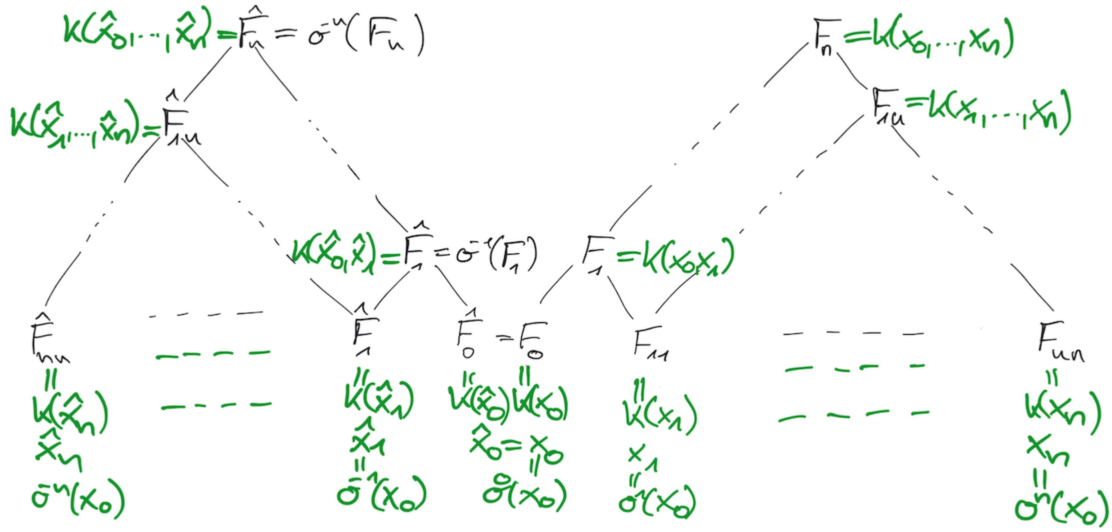


Figure 2.15: Dual recursive tower

For that, we first notice the equalities

$$\sigma^{-j}(F_{i,j}) = \sigma^{-j}(\prod_{l=i}^{j}\sigma^{l}(F_0)) = \prod_{l=i-j}^{0}\sigma^{l}(F_0) = \prod_{l=0}^{j-i}\sigma^{-l}(F_0) = \hat{F}_{j-i} \tag{57}$$

for all $0 \leq i \leq j$ where the first equality holds by the definition of the pyramid $(F_{i,j})_{i,j} = \mathrm{Pyr}(\mathcal{F})$ in Definition 9(i), the second equality holds by the fact that σ is an isomorphism of algebras and by the definition of composite fields, the third equality holds by changing the indexing and the last equality holds by the definition of $\hat{F}_{j-i}$ in (56).

On the one hand, this equality in (57) yields the equality $\hat{F}_n = \sigma^{-n}(F_{0,n}) = \sigma^{-n}(F_n)$ for all $n \in \mathbb{N}_0$. Combining this equality and the fact that the full constant field k and the genus are invariant under the k-automorphism σ provides that $\hat{F}_n$ also has full constant field k and the sequence $(g(\hat{F}_\nu))_\nu$ tends to ∞.

On the other hand, the equality in (57) also implies that the extension $\hat{F}_{n+1}/\hat{F}_n$ is isomorphic to $F_{0,n+1}/F_{1,n+1}$ via $\sigma^{-(n+1)}$. Thus, combining the that $F_1/\sigma(F_0)$ is separable and Lemma 10(iii) supplies that $\hat{F}_{n+1}/\hat{F}_n$ is a separable extension of degree $\hat{d} := [F_1 : \sigma(F_0)] > 1$.

Consequently, $\hat{\mathcal{F}}$ is a tower of constant degree $\hat{d}$ by its definition in Definition 2(i). Hence, (i) follows.

For (ii): Let $(x_\nu)_\nu$ be the sequence by which the tower $\mathcal{F} = (F_\nu)_\nu$ is recursively defined via the polynomial $f = f(X, Y)$ in Definition 5(i). Then, by its definition in Definition 5(i), we have the equalities $F_1 = k(x_0, x_1)$, $\sigma(F_0) = \sigma(k(x_0)) = k(x_1)$ and $f(x_0, x_1) = 0$. Moreover, because f is geometrically irreducible, $f(t, x_1) \in \sigma(F_0)[t]$ is the minimal polynomial of x_0 over $\sigma(F_0)$. But, because $k(x_1)$ is a rational function field and because f is separable in both variables, we deduce that $f(t, x_1)$ is separable. This means that x_0 is separable over $\sigma(F_0)$ and, therefore, the first desired statement follows, namely that $F_1 = \sigma(F_0)(x_0)$ is separable over $\sigma(F_0)$. Moreover, this also supplies that the pair-dual recursive tower $\hat{\mathcal{F}} = (\hat{F}_\nu)_\nu$ of $\mathcal{F}$ in Definition/Lemma 35(i) is well defined by the pair (σ^{-1}, F_0) and that

$$\hat{\mathcal{F}} \text{ has constant degree } \hat{d} := [F_1 : \sigma(F_0)] = \deg_X(f) = \deg_Y(g). \tag{58}$$

Next, we define the sequence $(\hat{x}_\nu)_\nu := (\sigma^{-\nu}(x_0))_{\nu\in\mathbb{N}_0}$. On the one hand, we obtain the equalities

$$\begin{aligned} g(\hat{x}_n, \hat{x}_{n+1}) &= f(\hat{x}_{n+1}, \hat{x}_n) = f(\sigma^{-(n+1)}(x_0), \sigma^{-n}(x_0)) = \sigma^{-(n+1)}(f(x_0, \sigma(x_0)) \\ &= \sigma^{-(n+1)}(f(x_0, x_1)) = 0 \end{aligned} \tag{59}$$

for all $n \in \mathbb{N}_0$ where the equalities hold by the following reasonings: The first equality holds by the choice of $g = g(X,Y) = f(Y,X)$. The second equality holds by the choice of the sequence $(\hat{x}_\nu)_\nu = (\sigma^{-\nu}(x_0))_\nu$. The third equality holds as σ is a k-automorphism of algebras. The fourth equality holds since the definition of σ in Lemma 7(i) provides the equality $\sigma(x_0) = x_1$. The last equality holds by the definition of the sequence $(x_\nu)_\nu$ in Definition 5(i).

On the other hand, we also obtain the equalities

$$\begin{aligned} \hat{F}_n &= \prod_{l=0}^{n} \sigma^{-l}(F_0) = \sigma^{-n}(\prod_{l=0}^{n} \sigma^{l}(F_0)) = \sigma^{-n}(F_n) = \sigma^{-n}(k(x_0, x_1, \ldots, x_n)) \\ &= k(\sigma^{-n}(x_0), \sigma^{-n}(x_1), \ldots, \sigma^{-n}(x_n))) = k(\sigma^{-n}(x_0), \sigma^{-(n-1)}(x_0), \ldots, \sigma^{0}(x_0))) \\ &= k(\hat{x}_n, \hat{x}_{n-1}, \ldots, \hat{x}_0) \end{aligned} \tag{60}$$

for all $n \in \mathbb{N}_0$ where the equalities hold by the following reasonings: The first (resp. third) equality holds by the definition of $\hat{F}_n$ (resp. F_n) in Definition 5(ii). The second equality holds by the fact that σ is an isomorphism of algebras, by the definition of composite fields and by reversing the indexing of the product. The fourth equality holds by the choice of the sequence $(x_\nu)_\nu$ in Definition 5(i). The fifth equality holds since σ is a morphism of k-algebras. The sixth equality holds since the definition of σ in Lemma 7(i) provides the equality $x_i = \sigma^i(x_0)$ for all $i \in \mathbb{N}_0$. The last equality holds by the choice of the sequence $(\hat{x}_\nu)_\nu = (\sigma^{-\nu}(x_0))_\nu$.

Finally, combining the equalities in (59) and (60) and the conclusion in (58) yields that the tower $\hat{\mathcal{F}} = (\hat{F}_\nu)_\nu$ is recursively defined by the geometrically irreducible polynomial g. Hence, (ii) follows. □

3 First Upper Bound for the Splitting Rate

Summary of the results of this chapter. First, in this chapter, we will prove the first more involved result of this thesis, which is Corollary 51. This corollary will have the following consequence: Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu$ be its geometric tower and define $V := \mathbb{P}_{\overline{F}_0}(\mathbb{P}^{(1)}_{F_0})$, which is a finite set if the constant field of $\mathcal{F}$ is finite. Then Corollary 51 will supply an upper bound $\tilde{N}(\overline{F}_n, V)$ for $N(\overline{F}_n, V) \geq N(F_n)$ for all $n \in \mathbb{N}_0$. Consequently, the limit $\lim_{n\to\infty} \tilde{N}(\overline{F}_n, V)/d^n$ (which will exist) yields an upper bound for the splitting rate $\nu(\mathcal{F}) = \lim_{n\to\infty} N(F_n)/d^n$. This upper bound for $\nu(\mathcal{F})$ will prove to be useful in the following chapters. Moreover, it will also be essential for the proof of the main result of this thesis in Corollary 184.

Second, a further result of this chapter is Key Lemma 36. Here, the 'moreover'-part in Key Lemma 36(i) generalizes the following statement of [Sti08, p. 141, Lemma 3.9.6] for linearly disjoint extensions: Let (E, F_1, F_2, F) be a diamond of function fields with $E = F_1 \cdot F_2$ and suppose that F_i/F is a finite separable extension for all $i = 1, 2$. If $P \in \mathbb{P}_F$ splits completely in F_1/F, then all places $P_2 \in \mathbb{P}_{F_2}(P)$ also split completely in E/F_2.

In Key Lemma 36(i), the 'moreover'-part states that if F_1/F and F_2/F are also linearly disjoint, then we have the equality

$$\sum_{Q \in \mathbb{P}_E((P_1, P_2))} e(Q|P)f(Q|P) = e(P_1|P)f(P_1|P)e(P_2|P)f(P_2|P)$$

for all $P_i \in \mathbb{P}_{F_i}(P)$, $i = 1, 2$. In particular, if P splits completely in F_1/F, then the $[F_1 : F]$ many choices of $P_1 \in \mathbb{P}_{F_1}(P)$ yield exactly $[F_1 : F] = [E : F_2]$ places $Q \in \mathbb{P}_E((P_1, P_2))$ for all $P_2 \in \mathbb{P}_{F_2}(P)$. Hence, all places $P_2 \in \mathbb{P}_{F_2}(P)$ also split completely in E/F_2, which was the statement in [Sti08, p. 141, Lemma 3.9.6].

Moreover, if the constant field of E is a finite field (but F_1/F and F_2/F do not need to be linearly disjoint anymore), then Key Lemma 36(iii) will also provide the degree bounds

$$\begin{aligned}\deg(Q) &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \frac{e(P_1|P) \cdot e(P_2|P)}{e(Q|P)} \\ &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \gcd(e(P_1|P), e(P_2|P))\end{aligned}$$

for all diamonds (Q, P_1, P_2, P) of places in (E, F_1, F_2, F). The first degree bound will be the key ingredient for the degree bounds in Theorem 225. This Theorem 225 will then provide upper bounds for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{F_n}$ which can be expressed solely in terms of the degree d of $\mathcal{F}$, of the ramification indices of extensions in $\operatorname{Pyr}(Q)$ and the degrees of the places in $\operatorname{Path}(Q)$.

Third, as an application to recursive towers, we will use the 'moreover'-part of Key Lemma 36(i) to prove Proposition 39. This Proposition 39 is one of the two keys in the proof of the main result of this chapter, which is Corollary 51, and it includes the statement that if $\mathcal{F} = (F_\nu)_\nu$ is a recursive tower over an algebraically closed field, then we have the identity

$$\sum_{Q \in \mathbb{P}_{F_n}(\mathcal{P})} e(Q|P_{0,0}) = \prod_{i=0}^{n-1} e(P_{i,i+1}|P_{i,i})$$

for all $n \in \mathbb{N}_0$ and all $\mathcal{P} = (P_{i,j})_{j-i \leq 1} \in W(\mathcal{F}, n)$. This Proposition 39 will also be essential for the proof of another major result, which is Theorem 155.

Fourth, in this chapter, we will define the Abhyankar ramification indices $\tilde{e}(Q|P)$ of extensions of places Q/P in recursive towers $\mathcal{F}$ (see Definition 41). These Abhyankar ramification indices $\tilde{e}(Q|P)$ will be equal to the usual ramification indices $e(Q|P)$ if $\mathcal{F}$ is tame but, in wild recursive towers $\mathcal{F}$, they will mimic the behavior of the usual ramification indices for tame recursive towers. This means that if $\mathcal{F} = (F_\nu)_\nu$ is tame and $Q \in \mathbb{P}_{F_n}$, then we can iteratively apply Abhyankar's Lemma in (10) to the elementary diamonds in the pyramid $\mathrm{Pyr}(Q)$ to compute $e(Q|Q \cap F_0)$. But, in wild recursive towers $\mathcal{F}$, Abhyankar's Lemma is not always applicable. However, if we apply Abhyankar's lemma regardless, we obtain the Abhyankar ramification index $\tilde{e}(Q|Q \cap F_0)$.

Moreover, Theorem 47 will show that the Abhyankar ramification index $\tilde{e}(Q|Q \cap F_0)$ divides the usual ramification index $e(Q|Q \cap F_0)$. This property will be the second key in the proof of the main result of this chapter, which is Corollary 51.

Finally, also note that the results of this chapter will imply the main results of [Kuh17] which are [Kuh17, p. 22, Theorem 2.3(f)] and the estimates in [Kuh17, p. 54, Theorem 3.20(c),(j)]. We will expand on this implication in Remark 54.

Strategy for finding the upper bound $\tilde{N}(\overline{F}_n, V)$ of $N(F_n)$. In the following, we will elaborate on finding the upper bound $\tilde{N}(\overline{F}_n, V)$ for $N(\overline{F}_n, V) \geq N(F_n)$ with $V := \mathbb{P}_{\overline{F}_0}(\mathbb{P}^{(1)}_{F_0})$ in Corollary 51. This will be the most important result of this chapter since it will be essential for the main result of this thesis in Corollary 184.

Because the estimate $N(\overline{F}_n, V) \geq N(F_n)$ is clear, we only have to find the upper bound $\tilde{N}(\overline{F}_n, V)$ of $N(\overline{F}_n, V)$ and, thus, we only need to consider recursive towers $\mathcal{F} = (F_\nu)_\nu$ over algebraically closed constant fields.

Now, let $(P_{i,j})_{j-i \leq 1} = \mathcal{P} \in W(\mathcal{F}, n)$, $Q \in \mathbb{P}_{F_n}(\mathcal{P})$, $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ and $P := P_{0,0}$. First, we will prove Key Lemma 36 and then, as a consequence, in Proposition 39, we will derive the crucial equality

$$\prod_{i=0}^{n-1} e(P_{i,i+1}|P_{i,i}) = \sum_{Q' \in \mathbb{P}_{F_n}(\mathcal{P})} e(Q'|P). \tag{61}$$

This equality (61) already carries some information about $N(F_n, \mathcal{P}) = \#\mathbb{P}^{(1)}_{F_n}(\mathcal{P}) = \mathbb{P}_{F_n}(\mathcal{P})$. If $\mathcal{P} = \mathrm{Path}(Q)$ is tame, then the sum on the right side of (61) simplifies because Lemma 17(v) ensures that Abhyankar's Lemma in (10) is applicable to all elementary diamonds of places in $\mathrm{Pyr}(Q')$ for all $Q' \in \mathbb{P}_{F_n}(\mathcal{P})$. Hence, in the tame case, $e(Q|P)$ is uniquely determined by the ramification indices of the extensions in $\mathcal{P}$ and we get the equalities $e(Q|P) = e(Q'|P)$ for all $Q' \in \mathbb{P}_{F_n}(\mathcal{P})$. Consequently, this again yields the equalities

$$N(F_n, \mathcal{P}) = \sum_{Q' \in \mathbb{P}_{F_n}(\mathcal{P})} 1 = \sum_{Q' \in \mathbb{P}_{F_n}(\mathcal{P})} \frac{e(Q'|P)}{e(Q|P)} = \frac{\prod_{i=0}^{n-1} e(P_{i,i+1}|P_{i,i})}{e(Q|P)} =: \tilde{N}(F_n, \mathcal{P})$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $N(F_n, \mathcal{P}) = \#\mathbb{P}^{(1)}_{F_n}(\mathcal{P})$ in (5) and by the fact that $\mathcal{F}$ is defined over an algebraically closed field which then again implies the equality $P^{(1)}_{F_n}(\mathcal{P}) = \mathbb{P}_{F_n}(\mathcal{P})$. The second equality holds by the equalities $e(Q|P) = e(Q'|P)$ for all $Q' \in \mathbb{P}_{F_n}(\mathcal{P})$. The second to last equality holds by the equality in (61). This will already be sufficient for the tame case and thus provide the equality $\tilde{N}(F_n, \mathcal{P}) = N(F_n, \mathcal{P})$.

Only for a wild path $\mathcal{P}$ the situation is more difficult: Although the equality in (61) also holds in the wild case, the ramification index $e(Q|P)$ is not uniquely determined by the path $\mathcal{P}$ and, thus, the sum $\sum_{Q' \in \mathbb{P}_{F_n}(\mathcal{P})} e(Q'|P)$ can be more complicated. However, we will use the tame case as a model to at least find an upper bound $\tilde{N}(F_n, \mathcal{P})$ for $N(F_n, \mathcal{P})$ in the wild case. For that, we will introduce **Abhyankar ramification indices** $\tilde{e}(Q'|P')$ on the elementary extensions Q'/P' of places in $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$. These will be defined as the usual ramification indices $e(Q'|P')$ for the extensions Q'/P' in $\mathrm{Path}(Q) = (P_{i,j})_{j-i \le 1}$ and, else, will be recursively defined as

$$\tilde{e}(Q'|P_r) := \frac{\tilde{e}(P_s|P')}{\gcd(\tilde{e}(P_1|P'), \tilde{e}(P_2|P'))} \tag{62}$$

for all $\{r, s\} = \{1, 2\}$ and elementary diamonds (Q', P_1, P_2, P') in $\mathrm{Pyr}(Q)$ which are $(Q', P_1, P_2, P') = (P_{i-1,j+1}, P_{i-1,j}, P_{i,j+1}, P_{i,j})$ with $1 \le i \le j \le n-1$. Furthermore, we will extend $\tilde{e}$ on all extensions in $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$, i.e. $P_{k,l}/P_{i,j}$ with $0 \le k \le i \le j \le l \le n$, by the well defined (see Lemma 44(ii)) unique extension satisfying the multiplicative transitivity rule

$$\tilde{e}(P_1|P_3) = \tilde{e}(P_1|P_2)\tilde{e}(P_2|P_3).$$

for all $(P_1, P_2, P_3) = (P_{r,s}, P_{k,l}, P_{i,j})$ with $0 \le r \le k \le i \le j \le l \le s \le n$. In Lemma 44(i), we will derive that $\tilde{e}$ satisfies the equality in (62) more generally, i.e. for all diamonds (Q', P_1, P_2, P') of places in $\mathrm{Pyr}(Q)$. We will call this equality the **$\tilde{e}$-version of Abhyankar's Lemma** or **virtual Abhyankar's Lemma** and we can apply it to any diamond of places in $\mathrm{Pyr}(Q)$. For tame paths, we will also define $\tilde{e}$ in the same way and, by Abhyankar's Lemma, obtain the identity $\tilde{e} = e$. Then, in the 'in particular'-part of Theorem 47, we will deduce that $\tilde{e}(Q|P)$ divides $e(Q|P)$.

Finally, we will find the desired upper $\tilde{N}(F_n, \mathcal{P})$ for $\tilde{N}(F_n, \mathcal{P})$ by the estimate and equalities

$$N(F_n, \mathcal{P}) \le \sum_{Q' \in \mathbb{P}_{F_n}(\mathcal{P})} \frac{e(Q'|P)}{\tilde{e}(Q|P)} = \frac{\prod_{i=0}^{n-1} \tilde{e}(P_{i,i+1}|P_{i,i})}{\tilde{e}(Q|P)} =: \tilde{N}(F_n, \mathcal{P}) \tag{63}$$

where the estimate holds by the estimate $\tilde{e}(Q|P) \le e(Q|P)$ and the first equality holds because of the equality in (61) and because the definition of $\tilde{e}$ provides the equality $\tilde{e} = e$ on the extensions in $\mathcal{P}$, i.e. $\tilde{e}(P_{i,i+1}|P_{i,i}) = e(P_{i,i+1}|P_{i,i})$ for all $i = 0, \ldots, n-1$.

For our purposes, the error $\tilde{N}(F_n, \mathcal{P}) - N(F_n, \mathcal{P})$ of the estimate $\tilde{N}(F_n, \mathcal{P}) \ge N(F_n, \mathcal{P})$ will be negligible.

Structure of this chapter. In the Section 3.1, we will first prove Key Lemma 36, and then apply it to pyramids of places in recursive towers to prove the crucial identity (61) in Proposition 39.

In Section 3.2, we will first introduce the Abhyankar ramification indices in Definition 41 and then prove some of their properties in Lemma 44. Consequently, we will be able to relate Abhyankar ramification indices and usual ramification indices via Theorem 47. As a consequence, we will obtain the desired upper bound $\tilde{N}(F_n, \mathcal{P})$ for $N(F_n, \mathcal{P})$ in Corollary 51.

3.1 Key Lemma I

Summary of the results of this section. In this Section 3.1, we will prove Key Lemma 36 (Key Lemma I) and then, from this, derive the already proclaimed identity

(61) in Proposition 39. This identity is essential for the rest of this thesis since it will provide the desired upper bound $\lim_{n\to\infty} \tilde{N}(\overline{F}_n, V)/d^n$ for the splitting rate $\nu(\mathcal{F})$ and since it will be used in the first major result Theorem 155. Moreover, in Secion 7.3, we will apply Key Lemma 36(iii) to find upper bounds for the degrees of places in $\mathcal{F}$.

Key Lemma I For the following Key Lemma 36, recall Assumption 1: In this thesis, all function fields F are defined over perfect full constant fields k.

Lemma 36 (Key Lemma I)**.** *Let F, F_i and E be function fields over the the same full constant field k such that $F \subseteq F_i \subseteq E$ for all $i = 1, 2$. Let F_1/F and F_2/F be finite separable extensions of function fields such that $E = F_1 \cdot F_2$ is the compositum of F_1 and F_2 and let P be a place in F. Then the following hold:*

(i) Then we have the estimate

$$\sum_{Q\in\mathbb{P}_E((P_1,P_2))} e(Q|P)f(Q|P) \leq e(P_1|P)f(P_1|P)e(P_2|P)f(P_2|P)$$

for all places $P_1 \in \mathbb{P}_{F_1}(P)$ and $P_2 \in \mathbb{P}_{F_2}(P)$ (see the first two diagrams in Figure 3.1). Moreover, if F_1/F and F_2/F are linearly disjoint, then this estimate is even an equality.

(ii) We have the estimate

$$\#\mathbb{P}_E((P_1,P_2)) \leq \gcd(e(P_1|P), e(P_2|P)) \gcd(f(P_1|P), f(P_2|P))$$

for all places $P_1 \in \mathbb{P}_{F_1}(P)$ and $P_2 \in \mathbb{P}_{F_2}(P)$.

Moreover, if F_1/F and F_2/F are linearly disjoint, at least one of the extensions P_1/P and P_2/P of places is tamely ramified and the identity $f(Q|P) = 1$ holds for all $Q \in \mathbb{P}_E((P_1,P_2))$, then we have the identity

$$\#\mathbb{P}_E((P_1,P_2)) = \gcd(e(P_1|P), e(P_2|P)).$$

In particular, if F_1/F and F_2/F are linearly disjoint, at least one of the extensions F_1/F and F_2/F is tame and k is algebraically closed, then this identity holds.

(iii) Let k be a finite field. Then we have the estimates

$$\begin{aligned}\deg(Q) &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \frac{e(P_1|P)\cdot e(P_2|P)}{e(Q|P)} \\ &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \gcd(e(P_1|P), e(P_2|P))\end{aligned}$$

for all $P_1 \in \mathbb{P}_{F_1}(P)$, $P_2 \in \mathbb{P}_{F_2}(P)$ and $Q \in \mathbb{P}_E((P_1,P_2))$ (see the first and third diagrams in Figure 3.1).

(iv) Let k be a finite field. Then we have the estimate

$$e(Q|P_j) \leq e(P_i|P)$$

for all $P_1 \in \mathbb{P}_{F_1}(P)$, $P_2 \in \mathbb{P}_{F_2}(P)$, $Q \in \mathbb{P}_E((P_1,P_2))$ and $\{i,j\} = \{1,2\}$.

Remark 37. *We can also Key Lemma 36(iv) without implicitly applying Key Lemma 36(i). See [Tut12, p. 14, Lemma 2.2.5] for instance.*

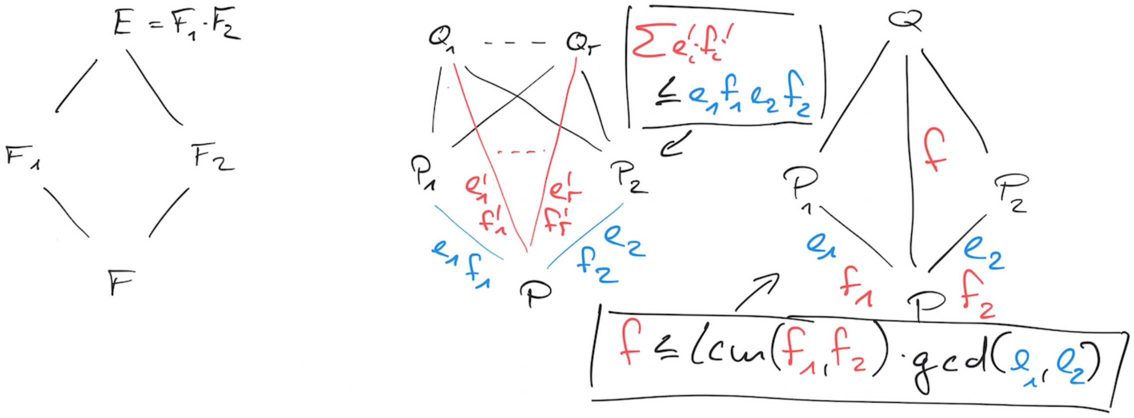


Figure 3.1: Diamonds of function fields and places for Key Lemma I

Examples 38. *(i) All the claims in the following example were verified via Magma.*

Let $F := \mathbb{F}_3(z)$ be a rational function field and define the polynomials $f := t^3 - zt - 1$ and $g := t^3 - zt - t^2 - 1$ over F. Moreover, let x and y be roots of f and g, respectively, which are contained in the same algebraic closure of F and define $F_1 := F(x)$ and $F_2 := F(y)$. Then F_1/F and F_2/F are linearly disjoint extensions of function fields of degree 3 such that $E := F_1 \cdot F_2 = F(x, y)$. Let $P := (t)$.

Then there are exactly one place P_i in the set $\mathbb{P}_{F_i}(P)$ for all $i = 1, 2$ and exactly two places Q and Q' in the set $\mathbb{P}_E(P) = \mathbb{P}_E((P_1, P_2))$ where the equality holds by the equality $\mathbb{P}_{F_i}(P) = \{P_i\}$. Moreover, we obtain the equalities $e(P_i|P) = 3$ and $\{e(Q|P), e(Q'|P)\} = \{3, 6\}$. Hence, combining these equalities and the fact that all involved places are rational yields the equalities

$$\sum_{R \in \mathbb{P}_E((P_1,P_2))} e(R|P)f(R|P) = 9 = e(P_1|P)f(P_1|P)e(P_2|P)f(P_2|P)$$

which are in accordance to the 'moreover'-part of Key Lemma 36(i).

(ii) For linearly disjoint tame Galois extensions F_1/F and F_2/F of prime degree p, the 'moreover'-part in Key Lemma 36(i) can also be derived from the fundamental equality in (8). However, already for linearly disjoint tame Galois extensions of degree p^2, this is not possible anymore. For instance, if the constant field is algebraically closed (only for simplicity) and there is a place P and $p = 2$ places in $\mathbb{P}_{F_i}(P) = \{P_{i,1}, P_{i,2}\}$ with ramification indices equal to 2 for all $i = 1, 2$, then we have 8 places in $\mathbb{P}_E(P)$ with ramification indices all equal to 2 over P because of Abhyankar's Lemma and the fundamental equality. Now, the 'moreover'-part in Key Lemma 36(i) yields that these 8 places distribute equally

$$[2, 2, 2, 2] = [\#P_E((P_{1,i}, P_{2,j})) : (i, j) = (1, 1), (1, 2), (2, 1), (2, 2)]$$

over all 4 possible combinations $(P_{1,i}, P_{2,j})$. But, by only using the the fundamental equality, a distribution of the form $[3, 1, 1, 3]$ cannot be ruled out.

Notation in the proof of Key Lemma 36. In the proof of Key Lemma 36, we will use the following notation regarding completions: For any function field K and place R in K, we will fix a single completion of K to R and denote this fixed completion as $\hat{K}_R$. Clearly, for all extension L/K of function fields over the same full constant field and any extension S/R of places in L/K, there is a unique completion K' to R inside of $\hat{L}_S$ by

[Lor08, p.47, Definition] (just take the closure of K with the induced topology on L_R). But, as completions are unique up to isomorphisms by [Lor08, p.48, Theorem 2], there is an isomorphism $\hat{K}_R \to K'$. Thus, we will write $\iota_{R,S} : \hat{K}_R \hookrightarrow \hat{L}_S$ for the composition of this isomorphism and the inclusion map $K' \hookrightarrow \hat{L}_S$. This embedding makes L_S to a K_R-vector space and, in the special case $K = F$, $R = P$, $x \in \hat{L}_S$ and $y \in \hat{F}_P$, we will use the notation $x \cdot y := x \cdot \iota_{S,P}(y)$.

Sketch of the proof of Key Lemma 36(i). Before we come to the actual proof of Key Lemma 36(i), we want to sketch it briefly: The crucial point of this proof is to show for all $P_i \in \mathbb{P}_{F_i}(P)$ and $i = 1, 2$ that the map

$$\phi'_{P_1,P_2} : \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \to \prod_{Q \in \mathbb{P}_E((P_1,P_2))} \hat{E}_Q \text{ via } x \otimes_{\hat{F}_P} y \mapsto (\iota_{P_1,Q}(x) \cdot \iota_{P_2,Q}(y))_Q$$

is a well defined epimorphism of $\hat{F}_P$-vector spaces and even an isomorphism if F_1/F and F_2/F are linearly disjoint. Then the desired estimate in the 'main'-part and the desired equality in the 'moreover'-part will be immediate consequences of this fact and of the fact that $e(S|P)f(S|P)$ is the $\hat{F}_P$-dimension of the involved completions $\hat{L}_S$.

Concretely, we will derive these properties for ϕ'_{P_1,P_2} from first deducing that the canonical extension

$$\phi' : \prod_{\substack{P_1 \in \mathbb{P}_{F_1}(P) \\ P_2 \in \mathbb{P}_{F_2}(P)}} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \to \prod_{Q \in \mathbb{P}_E(P)} \hat{E}_Q$$

of these maps ϕ'_{P_1,P_2} is a well defined epimorphism and even an isomorphism if F_1/F and F_2/F are linearly disjoint.

Finally, we will conclude these properties for ϕ' by proving that the domain of ϕ' is a tensor product $F_1 \otimes_F F_2 \otimes_F \hat{F}_P$, that its codomain is a tensor product $E \otimes_F \hat{F}_P$ and that, with these representatives, the morphism ϕ' is the canonical epimorphism $\phi : F_1 \otimes_F F_2 \otimes_F \hat{F}_P \to E \otimes_F \hat{F}_P$. Moreover, as ϕ is even an isomorphism if F_1/F and F_2/F are linearly disjoint, this will be sufficient.

Proof of Key Lemma 36. For (i): First, [Lor08, p.60, Theorem 5] provides that the map

$$F_i \otimes_F \hat{F}_P \to \prod_{P_i \in \mathbb{P}_{F_i}(P)} \widehat{(F_i)}_{P_i} \text{ via } x \otimes_F y \mapsto (x \cdot y)_{P_i}$$

is an isomorphism of $\hat{F}_P$-vector spaces for all $i = 1, 2$. Then these isomorphisms and some basic rules for tensor products yield an isomorphism

$$\begin{aligned} F_1 \otimes_F F_2 \otimes_F \hat{F}_P &\cong (F_1 \otimes_F \hat{F}_P) \otimes_{\hat{F}_P} (F_2 \otimes_F \hat{F}_P) \\ &\cong \prod_{P_1 \in \mathbb{P}_{F_1}(P)} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \prod_{P_2 \in \mathbb{P}_{F_2}(P)} \widehat{(F_2)}_{P_2} \\ &\cong \prod_{\substack{P_1 \in \mathbb{P}_{F_1}(P) \\ P_2 \in \mathbb{P}_{F_2}(P)}} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \end{aligned} \tag{64}$$

of $\hat{F}_P$-vector spaces. Therefore, $\prod_{P_1,P_2} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2}$ is a tensor product of F_1, F_2 and $\hat{F}_P$ of F-vector spaces where the F-multilinear map $\psi_1 : F_1 \times F_2 \times \hat{F}_P \to \prod_{P_1,P_2} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P}$

$\widehat{(F_2)}_{P_2}$ is the composition of the canonical multilinear map $F_1 \times F_2 \times \hat{F}_P \to F_1 \otimes_F F_2 \otimes_F \hat{F}_P$ and the isomorphisms in (64). If we go through the isomorphisms in (64), we obtain

$$\psi_1 : F_1 \times F_2 \times \hat{F}_P \to \prod_{\substack{P_1 \in \mathbb{P}_{F_1}(P) \\ P_2 \in \mathbb{P}_{F_2}(P)}} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \text{ via } (x, y, z) \mapsto ((x \otimes_{\hat{F}_P} y) \cdot z)_{P_1, P_2}.$$

Second, by applying the universal property of the tensor product $(F_1 \otimes_F F_2) \otimes_F \hat{F}_P$ to the canonical epimorphism $F_1 \otimes_F F_2 \to E$, $x \otimes_F y \mapsto x \cdot y$ of F-algebras, we obtain an epimorphism

$$(F_1 \otimes_F F_2) \otimes_F \hat{F}_P \to E \otimes_F \hat{F}_P \tag{65}$$

of $\hat{F}_P$-vector spaces. Moreover, if F_1/F and F_2/F are linearly disjoint, then the canonical epimorphism $F_1 \otimes_F F_2 \to E = F_1 \cdot F_2$ even becomes an isomorphism. Because of that and because of the functor $\otimes_F \hat{F}_P$ from the category of F-vector spaces to the category of $\hat{F}_P$-vector spaces being an exact functor by the flatness of the F-vector space $\hat{F}_P$, the epimorphism in (65) becomes an isomorphism too. Next, [Lor08, p.60, Theorem 5] also provides that the map

$$E \otimes_F \hat{F}_P \to \prod_{Q \in \mathbb{P}_E(P)} \hat{E}_Q \text{ via } x \otimes_F y \mapsto (x \cdot y)_Q \tag{66}$$

is an isomorphism of $\hat{F}_P$-vector spaces. Then, by taking the composition of the morphisms in (64), (65) and (66), we get an epimorphism

$$\phi : \prod_{\substack{P_1 \in \mathbb{P}_{F_1}(P) \\ P_2 \in \mathbb{P}_{F_2}(P)}} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \to \prod_{Q \in \mathbb{P}_E(P)} \hat{E}_Q \tag{67}$$

of $\hat{F}_P$-vector spaces which is an isomorphism if F_1/F and F_2/F are linearly disjoint. Furthermore, by the universal property of tensor products,

$$\phi \text{ is the unique morphism of } F\text{-vector spaces such that } \phi \circ \psi_1 = \psi_2 \tag{68}$$

for the F-multilinear map

$$\psi_2 := \phi \circ \psi_1 : F_1 \times F_2 \times \hat{F}_P \to \prod_{Q \in \mathbb{P}_E(P)} \hat{E}_Q \text{ via } (x, y, z) \mapsto (x \cdot y \cdot z)_Q. \tag{69}$$

Third, we consider the morphism

$$\phi'_{P_1, P_2} : \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \to \prod_{Q \in \mathbb{P}_E((P_1, P_2))} \hat{E}_Q \text{ via } x \otimes_{\hat{F}_P} y \mapsto (\iota_{P_1, Q}(x) \cdot \iota_{P_2, Q}(y))_Q$$

of $\hat{F}_P$-vector spaces for all $P_i \in \mathbb{P}_{F_i}(P)$ and $i = 1, 2$. Let

$$\phi' : \prod_{\substack{P_1 \in \mathbb{P}_{F_1}(P) \\ P_2 \in \mathbb{P}_{F_2}(P)}} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2} \to \prod_{Q \in \mathbb{P}_E(P)} \hat{E}_Q$$

be the canonical extension of the morphisms ϕ'_{P_1, P_2} on the product $\prod_{P_1, P_2} \widehat{(F_1)}_{P_1} \otimes_{\hat{F}_P} \widehat{(F_2)}_{P_2}$. Then we notice that

$$\phi' \text{ is a morphism of } \hat{F}_P\text{-vector spaces and } \phi' \circ \psi_1 = \psi_2 \text{ holds} \tag{70}$$

by the equalities

$$\begin{aligned}(\phi'\circ\psi_1)(x,y,z) &= \phi'(((x\otimes_{\hat{F}_P} y)\cdot z)_{P_1,P_2}) = ((\iota_{P_1,Q}(x)\cdot\iota_{P_2,Q}(y))\cdot z)_Q \\ &= (x\cdot y\cdot z)_Q =: \psi_2(x,y,z)\end{aligned}$$

for all $x\in F_1$, $y\in F_2$ and $z\in\hat{F}_P$ where the first equality holds by the definition of ψ_1, the second equality holds by the $\hat{F}_P$-linearity of ϕ', by the definitions of the $\hat{F}_P$-scalar multiplications in the domain and codomain of ϕ', by the definition of ϕ' and by the definition of ϕ'_{P_1,P_2}, the third equality holds as the embedding $\iota_{P_i,Q}$ fixes the subfield F_i for all $i=1,2$ and the last equality holds by the definition of ψ_2 in (69). Combining (70) and (68) yields the equality $\phi=\phi'$ and, thus, ϕ' is an epimorphism of $\hat{F}_P$-vector space and even an isomorphism if F_1/F and F_2/F are linearly disjoint. Consequently, the

$$\text{epimorphism } \phi'_{P_1,P_2} \text{ is also an isomorphism if } F_1/F \text{ and } F_2/F \text{ are linearly disjoint} \tag{71}$$

$P_i\in\mathbb{P}_{F_i}(P)$ and $i=1,2$ as ϕ' is an extension of ϕ'_{P_1,P_2}.

Fourth and finally, we obtain the equalities and estimate

$$\begin{aligned}\sum_{Q\in\mathbb{P}_E((P_1,P_2))} e(Q|P)f(Q|P) &= \sum_{Q\in\mathbb{P}_E((P_1,P_2))} \dim_{\hat{F}_P}(\hat{E}_Q) = \dim_{\hat{F}_P}(\prod_{Q\in\mathbb{P}_E((P_1,P_2))} \hat{E}_Q) \\ &\leq \dim_{\hat{F}_P}(\widehat{(F_1)}_{P_1}\otimes_{\hat{F}_P}\widehat{(F_2)}_{P_2}) = \dim_{\hat{F}_P}(\widehat{(F_1)}_{P_1})\cdot\dim_{\hat{F}_P}(\widehat{(F_2)}_{P_2}) \\ &= e(P_1|P)f(P_1|P)e(P_2|P)f(P_2|P)\end{aligned}$$

$P_i\in\mathbb{P}_{F_i}(P)$ and $i=1,2$ where the first and last equalities hold by [Lor08, p.69, Theorem 1], the second equality holds by the additivity of $\dim_{\hat{F}_P}$ for direct sums, the estimate holds as ϕ'_{P_1,P_2} is an epimorphism of $\hat{F}_P$-vector spaces and the third equality holds by the multiplicativity of $\dim_{\hat{F}_P}$ for tensor products of $\hat{F}_P$-vector spaces. Moreover, if F_1/F and F_2/F are linearly disjoint, the only estimate is even an equality because ϕ'_{P_1,P_2} is an isomorphism in this case by (71). Hence, (i) follows.

For the 'main'-part of (ii): Let $P_1\in\mathbb{P}_{F_1}(P)$ and $P_2\in\mathbb{P}_{F_2}(P)$. Then, for all $Q\in\mathbb{P}_E((P_1,P_2))$ and $a\in\{e,f\}$, $\operatorname{lcm}(a(P_1|P),a(P_2|P))$ is a divisor of $a(Q|P)$ as we have the equalities $a(Q|P)=a(Q|P_i)a(P_i|P)$ for all $i=1,2$ by the multiplicative transitivity rule of a in (7). Thus, we obtain the estimate

$$1\leq\frac{a(Q|P)}{\operatorname{lcm}(a(P_1|P),a(P_2|P))} \tag{72}$$

for all $Q\in\mathbb{P}_E((P_1,P_2))$ and $a\in\{e,f\}$. Consequently, we estimate

$$\begin{aligned}\#\mathbb{P}_E(P_1,P_2) &\leq \sum_{Q\in\mathbb{P}_E((P_1,P_2))}\frac{e(Q|P)}{\operatorname{lcm}(e(P_1|P),e(P_2|P))}\cdot\frac{f(Q|P)}{\operatorname{lcm}(f(P_1|P),f(P_2|P))} \\ &\leq \frac{e(P_1|P)e(P_2|P)}{\operatorname{lcm}(e(P_1|P),e(P_2|P))}\cdot\frac{f(P_1|P)f(P_2|P)}{\operatorname{lcm}(f(P_1|P),f(P_2|P))} \\ &= \gcd(e(P_1|P),e(P_2|P))\cdot\gcd(f(P_1|P),f(P_2|P))\end{aligned} \tag{73}$$

where the first estimate holds by the estimates in (72), the second estimate holds by Key Lemma 36(i) and the equality holds by the well known identity $ab=\operatorname{lcm}(a,b)\gcd(a,b)$ for all $a,b\in\mathbb{N}$. Hence, the 'main'-part of (ii) follows from the estimate in (73).

For the 'moreover'- and 'in particular'-part of (ii): Let $P_1\in\mathbb{P}_{F_1}(P)$ and $P_2\in\mathbb{P}_{F_2}(P)$. One the one hand, if at least one of the extensions P_1/P and P_2/P of places is tamely ramified, we have the equality $e(Q|P) = \operatorname{lcm}(e(P_1|P), e(P_2|P))$ for all $Q \in \mathbb{P}_E((P_1, P_2))$ by Abhankar's Lemma in (10). On the other hand, if $f(Q|P) = 1$ for all $Q \in \mathbb{P}_E((P_1, P_2))$, then also $f(P_i|P) = 1$ holds for all $i = 1, 2$ by the equalities $1 = f(Q|P) = f(Q|P_i)f(P_i|P)$. In particular, we get the equalities $f(Q|P) = 1 = \operatorname{lcm}(f(P_1|P), f(P_2|P))$ in this case too. Both together yield that the first estimate in (73) is even an equality in this cases. Moreover, if F_1/F and F_2/F are linearly disjoint, the second estimate in (73) is also an equality by the 'moreover'-part of Key Lemma 36(i). Hence, the moreover'-part and, by that, the 'in particular'-part of (ii) follow.

For (iii): Let $k = \mathbb{F}_q$, $r \in \mathbb{N}$ and define $F' := \mathbb{F}_{q^r} \cdot F$, $F_i' := \mathbb{F}_{q^r} \cdot F_i$ for all $i = 1, 2$ and $E' := \mathbb{F}_{q^r} \cdot E$. Then F_1'/F' and F_2'/F' are finite separable extensions of function fields over the full constant fields $\mathbb{F}_{q^r}$ such that $E' = \mathbb{F}_{q^r} \cdot (F_1 \cdot F_2) = F_1' \cdot F_2'$. Moreover, let $Q' \in \mathbb{P}_{E'}(Q)$ and define $P' := Q \cap F'$ and $P_i' := Q \cap F_i'$ for all $i = 1, 2$. Then P'/P in F'/F and P_i'/P_i in F_i'/F_i for all $i = 1, 2$. Also, [Sti08, p.190, Lemma 5.1.9] provides the equalities

$$\deg(R) = \deg(R') \gcd(\deg(R), r) \tag{74}$$

for all $(R', R) \in \{(P', P), (P_1', P_1), (P_2', P_2), (Q', Q)\}$. Now, choose

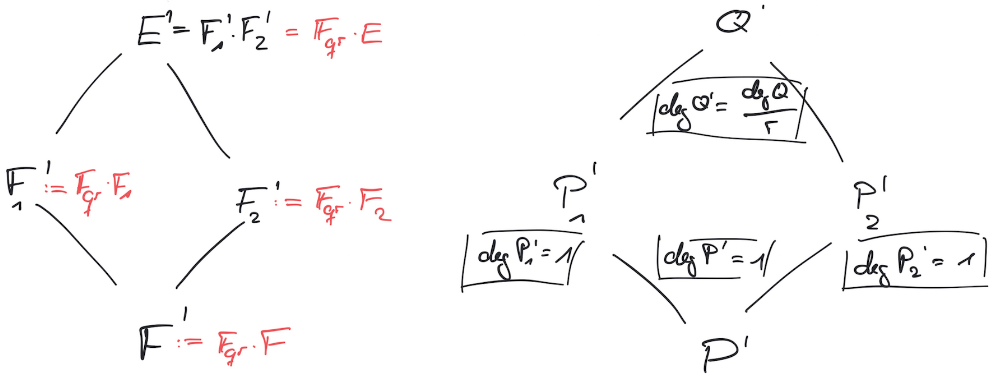


Figure 3.2: Diamonds for constant field extensions in a proof

$$r := \operatorname{lcm}(\deg(P_1), \deg(P_2)) = \deg(P) \cdot \operatorname{lcm}(f(P_1|P), f(P_2|P)). \tag{75}$$

Then r is a multiple of $\deg(P)$, $\deg(P_i) = f(P_i|P)\deg(P)$ for all $i = 1, 2$ and a divisor of $\deg(Q)$ by the equality $\deg(Q) = f(Q|P_i)\deg(P_i)$ for all $i = 1, 2$. Thus, combining these facts and the identities in (74) yields the equalities

$$\deg(P') = \deg(P_i') = 1 \text{ and } \deg(Q') = \deg(Q)/r. \tag{76}$$

Moreover, we estimate

$$\begin{aligned} e(Q|P)\deg(Q') = e(Q'|P')f(Q'|P')\deg(P') &\leq \sum_{Q'' \in \mathbb{P}_{E'}((P_1', P_2'))} e(Q''|P)f(Q''|P) \\ &\leq e(P_1'|P')e(P_2'|P') = e(P_1|P)e(P_2|P) \end{aligned} \tag{77}$$

where the equalities hold by the invariance of the ramification indices under constant field extensions in (12), the first estimate holds as $e(Q'|P')f(Q'|P')$ is one of the non negative summands in the $\sum_{Q''} e(Q''|P)f(Q''|P)$ and as $\deg(P') = 1$ by (76) and the last estimate holds because of Key Lemma 36(i) and because (76) implies the equality $f(P_i'|P') = \deg(P_i')/\deg(P') = 1$ for all $i = 1, 2$.

Finally, we obtain the desired estimate in (iii) by the equality and estimates

$$\begin{aligned}\deg(Q) = r \cdot \deg(Q') &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \frac{e(P_1|P)e(P_2|P)}{e(Q|P)} \\ &\leq \operatorname{lcm}(\deg(P_1), \deg(P_2)) \cdot \gcd(e(P_1|P), e(P_2|P))\end{aligned}$$

where the equality holds by the second equality in (76), the first estimate holds by the definition of r in (75) and by the estimate in (77) and the second estimate holds because $\operatorname{lcm}(e(P_1|P), e(P_2|P))$ is a divisor of $e(Q|P) = e(Q|P_i)e(P_i|P)$ for all $i = 1, 2$ and by the well known identity $ab = \operatorname{lcm}(a, b) \gcd(a, b)$ for all $a, b \in \mathbb{N}$. Hence, the estimates in (iii) hold.

For (iv): By the equality $\deg(Q) = f(P_i|P) \deg(P_i)$ for all $i = 1, 2$, the degree $\deg(Q)$ must be a multiple of $\operatorname{lcm}(\deg(P_1), \deg(P_2))$ and, especially, we obtain the estimate

$$\operatorname{lcm}(\deg(P_1), \deg(P_2)) \leq \deg(Q) \tag{78}$$

Consequently, we derive the desired estimate in (iv) from the estimate and equalities

$$1 \leq \frac{e(P_1|P) \cdot e(P_2|P)}{e(Q|P)} = \frac{e(P_1|P) \cdot e(P_2|P)}{e(Q|P_j) \cdot e(P_j|P)} = \frac{e(P_i|P)}{e(Q|P_j)}$$

for all $\{i, j\} = \{1, 2\}$ where the estimate holds by combining the estimate in (78) and the first estimate in Key Lemma 36(iii), the first equality holds because the multiplicative transitivity rule for e in (7) implies the equality $e(Q|P) = e(Q|P_j) \cdot e(P_j|P)$ and the second equality holds by then canceling the factors $e(P_j|P)$ in the numerator and denominator. □

3.1.1 Application to Recursive Towers

A generalization of Key Lemma 36 for recursive towers. The following already proclaimed Proposition 39 can be seen as a generalization of the 'moreover'-part of Key Lemma 36(i) to paths of recursive towers $\mathcal{F}$ of balanced degree over any algebraically closed fields k. Instead of having a diamond of function fields over the full constant field k and summing up the ramification indices $e(Q|P)$ over all diamonds of places (Q, P_1, P_2, P) with fixed places P_1, P_2 and P, we sum up the ramification indices $e(Q|P)$ over all places $Q \in \mathbb{P}_{F_{m,n}}(\mathcal{P})$ for a fixed path $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ with $P := P_{0,0}$. In fact, the following Proposition 39 will follow from recursively applying Key Lemma 36(i) to the elementary diamonds of function fields in $\operatorname{Pyr}(\mathcal{F})$.

Proposition 39. *Let $\mathcal{F}$ be a recursive tower over an algebraically closed field and $(F_{i,j})_{i,j} :=$ $\operatorname{Pyr}(\mathcal{F})$. Then we have the identity*

$$\sum_{Q \in \mathbb{P}_{F_{m,n}}(\mathcal{P})} e(Q|P_{m,m}) = \prod_{i=m}^{n-1} e(P_{i,i+1}|P_{i,i})$$

for all $m \leq n$ and $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$ (see Figure 3.3 for $m = 0$).

Proof. We show this by induction over $n - m \in \mathbb{N}_0$. Let $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m, n)$. For $m = n$, we have the identities $\mathcal{P} = (P_{m,m})$ and $\mathbb{P}_{F_{m,m}}(\mathcal{P}) = \{P_{m,m}\}$ and, thus, the desired equality $\sum_{Q \in \mathbb{P}_{F_{m,n}}(\mathcal{P})} e(Q|P_{m,m}) = \prod_{i=m}^{n-1} e(P_{i,i+1}|P_{i,i})$ holds in this case as both sides equal one.

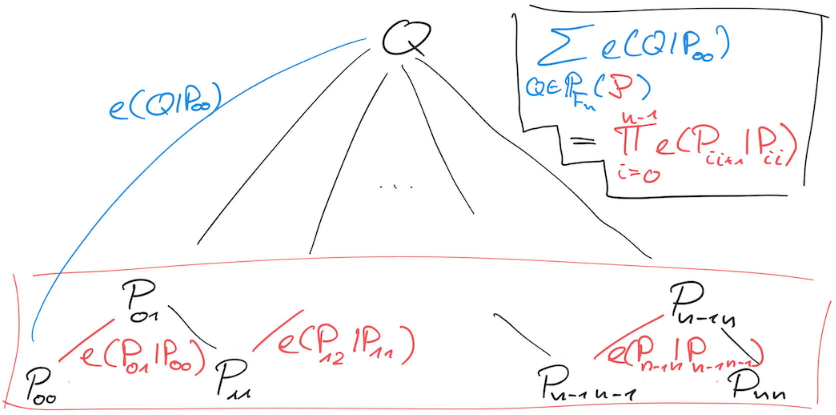


Figure 3.3: Places over a path and their ramification indices

Now, let $m < n$ and $\mathcal{P}' := (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, m+1, n)$. Then we already obtain the desired equality by the equalities

$$\begin{aligned}
\sum_{Q\in\mathbb{P}_{F_{m,n}}(\mathcal{P})} e(Q|P_{m,m}) &= \sum_{Q'\in\mathbb{P}_{F_{m+1,n}}(\mathcal{P}')} \; \sum_{Q\in\mathbb{P}_{F_{m,n}}((P_{m,m+1},Q'))} e(Q|P_{m,m}) \\
&= \sum_{Q'\in\mathbb{P}_{F_{m+1,n}}(\mathcal{P}')} \frac{e(P_{m,m+1}|P_{m,m})}{e(P_{m,m+1}|P_{m+1,m+1})} \sum_{Q\in\mathbb{P}_{F_{m,n}}((P_{m,m+1},Q'))} e(Q|P_{m+1,m+1}) \\
&= \sum_{Q'\in\mathbb{P}_{F_{m+1,n}}(\mathcal{P}')} \frac{e(P_{m,m+1}|P_{m,m})}{e(P_{m,m+1}|P_{m+1,m+1})} e(P_{m,m+1}|P_{m+1,m+1}) e(Q'|P_{m+1,m+1}) \\
&= e(P_{m,m+1}|P_{m,m}) \sum_{Q'\in\mathbb{P}_{F_{m+1,n}}(\mathcal{P}')} e(Q'|P_{m+1,m+1}) \\
&= \prod_{i=m}^{n-1} e(P_{i,i+1}|P_{i,i})
\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds since the definition of $\mathcal{P}'$ implies that, on the one hand, any place $Q \in \mathbb{P}_{F_{m,n}}(\mathcal{P})$ clearly lies over $P_{m,m+1}$ and $Q' := Q \cap F_{m+1,n} \in \mathbb{P}_{F_{m+1,n}}(\mathcal{P}')$ and that, on the other hand, any place in $Q \in \mathbb{P}_{F_{m,n}}(P_{m,m+1}, Q')$ for some $Q' \in \mathbb{P}_{F_{m+1,n}}(\mathcal{P}')$ also clearly lies over $\mathcal{P}$. The second equality holds since the multiplicative transitivity rule for ramification indices in (7) implies the identities $e(Q|P_{m,m}) = e(Q|P_{m,m+1}) \cdot e(P_{m,m+1}|P_{m,m})$ and $e(Q|P_{m,m+1}) \cdot e(P_{m,m+1}|P_{m+1,m+1}) = e(Q|P_{m+1,m+1})$. The third equality holds by applying Key Lemma 36(i) to the sum $\sum_Q e(Q|P_{m+1,m+1})$. The fourth equality holds by elementary arithmetics. The last equality holds by applying the induction hypothesis to $\mathcal{P}'$. □

Examples 40. *Consider the geometric tower* $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu := \overline{\mathbb{F}}_3 \cdot \mathcal{F}_{MW,2}$ *of* $\mathcal{F}_{MW,2}$ *and the place* $Q \in \mathbb{P}_{F_4}$ *in Examples 12. For any place* $Q' \in \mathbb{P}_{\overline{F}_4}(Q)$*, we obtain the same pyramid* $(P'_{i,j})_{i,j} := \mathrm{Pyr}(Q')$ *which is depicted in Figure 2.8. In particular, all these places* Q' *lie over the same path* $\mathcal{P}'$*. Consequently, Proposition 39 provides the equalities*

$$N(\overline{F}_4, \mathcal{P}') = \sum_{Q'\in\mathbb{P}_{\overline{F}_4}(\mathcal{P}')} e(Q'|P'_{0,0}) = \prod_{i=0}^{3} e(P'_{i,i+1}|P'_{i,i}) = 2$$

3.2 Upper Bound for the Splitting Rate

Summary of the results of this section. In this section, motivated by Abhyankar's Lemma, we will first define the Abhyankar ramification indices $\tilde{e}(R|P)$ of extensions R/P in pyramids $\mathrm{Pyr}(Q)$ of places in recursive towers $\mathcal{F} = (F_\nu)_\nu$. Then, in Theorem 47, we will prove that $\tilde{e}(Q|Q \cap F_0)$ divides $e(Q|Q \cap F_0)$. Finally, we will be able to define the desired upper bound $\tilde{N}(F_n, \mathcal{P})$ of $N(F_n, \mathcal{P})$ for any path $\mathcal{P}$ of $\mathcal{F}$ in Definition 50 and to prove the estimate $N(F_n, \mathcal{P}) \leq \tilde{N}(F_n, \mathcal{P})$ in Corollary 51. This estimate will be essential for the rest of this thesis and it will be an equality for tame recursive towers $\mathcal{F}$.

3.2.1 Abhyankar Ramification Indices

Purpose of this subsection. In this subsection, we will define the Abhyankar ramification indices and prove some first properties of Abhyankar ramification indices.

Abhyankar ramification indices. In the following Definition 41, we will define the Abhyankar ramification indices $\tilde{e}(R|P)$ of extensions R/P in pyramids $\mathrm{Pyr}(Q)$ of places in recursive towers $\mathcal{F}$. This definition will be motivated by Abhyankar's Lemma in (10) and correspondingly yield the usual ramification indices $e(R|P)$ if Abhyankar's Lemma is applicable to all elementary diamonds in $\mathrm{Pyr}(Q)$. For instance, this will be the case if $\mathcal{F}$ is tame or if the path $\mathrm{Path}(Q)$ is tame.

Definition 41. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, $n \in \mathbb{N}_0$, $Q \in \mathbb{P}_{F_n}$ and $(P_{i,j})_{i,j} :=$ $\mathrm{Pyr}(Q)$. Then we define the map*

$$\tilde{e} : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$$

*recursively in the following way where we use the notation $\tilde{e}(P_{k,l}|P_{i,j}) := \tilde{e}(P_{k,l}, P_{i,j})$ for all $(P_{k,l}, P_{i,j}) \in \mathrm{Ext}(\mathrm{Pyr}(Q))$ and call $\tilde{e}(P_{i,j}|P_{k,l})$ the **Abhyankar ramification index** of the extension $P_{k,l}/P_{i,j}$:*

(i) For the extensions in $\mathrm{Path}(Q) = (P_{i,j})_{j-i\leq 1}$, we define $\tilde{e}$ to be the usual map e of ramification indices, i.e.

$$\tilde{e}(P_{k,l}|P_{i,j}) := e(P_{k,l}|P_{i,j})$$

for all $0 \leq k \leq i \leq j \leq l \leq n$ with $k - l \leq 1$.

(ii) For the other elementary extensions in $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$

$$\tilde{e}(R|P_r) := \frac{\tilde{e}(P_s|P)}{\gcd(\tilde{e}(P_1|P), \tilde{e}(P_2|P))} \tag{79}$$

for all $\{r, s\} = \{1, 2\}$ and $(R, P_1, P_2, P) := (P_{i-1,j+1}, P_{i-1,j}, P_{i,j+1}, P_{i,j})$ with $1 \leq i \leq j \leq n-1$ (see Figure 3.4).

Notice that this recursive definition provides a well defined $\tilde{e}$-value for any elementary extension in $\mathrm{Pyr}(Q)$ outside of $\mathrm{Path}(Q)$ which is also natural number as the denominator is a divisor of the numerator in (79).

(iii) Finally, for any extension $P_{i,j}/P_{k,l}$ of places in $\mathrm{Pyr}(Q)$ there are paths $\mathcal{P}$ in the directed graph $\mathrm{Graph}(\mathrm{Pyr}(Q))$ of elementary extensions from $P_{i,j}$ to $P_{k,l}$. Then we choose the unique such path $\mathcal{P}$ containing the place $P_{i,l}$ and define the $\tilde{e}$-value as the product of the $\tilde{e}$-values of the elementary extensions in $\mathcal{P}$, i.e.

$$\tilde{e}(P_{k,l}|P_{i,j}) := \tilde{e}(P_{k,l}|P_{i,l}) \cdot \tilde{e}(P_{i,l}|P_{i,j}) := \prod_{\nu=k}^{i-1} \tilde{e}(P_{\nu,l}|P_{\nu+1,l}) \cdot \prod_{\nu=j}^{l-1} \tilde{e}(P_{i,\nu+1}|P_{i,\nu})$$

for all $0 \leq k \leq i \leq j \leq l \leq n$ (see Figure 3.4).

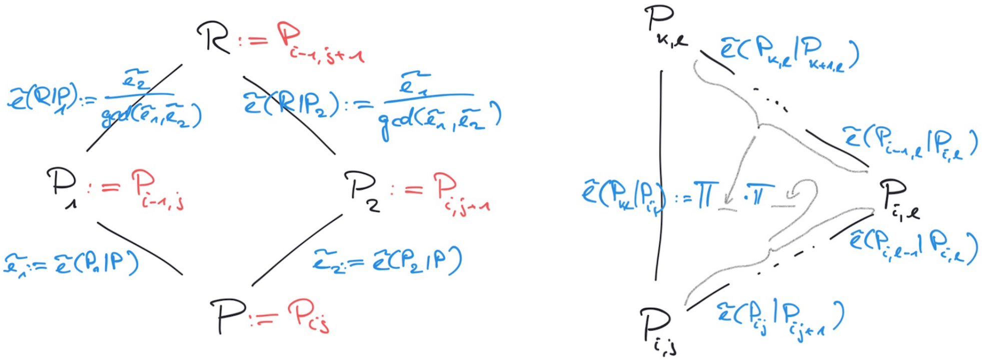


Figure 3.4: Definition of the virtual Abhyankar's Lemma and the multiplicative transitivity rule for $\tilde{e}$

Examples 42. *(i) On the extensions in the pyramids* $\mathrm{Pyr}(Q)$ *and* $\mathrm{Pyr}(Q')$ *for the MW-tower* $\mathcal{F}_{MW,2}$ *in Examples 12, we have the equality* $e = \tilde{e}$ *because the towers are tame and because of Lemma 44(iii).*

(ii) Consider the recursive ST-tower $\mathcal{F} = \mathcal{F}_{ST,3} = (F_\nu)_\nu$ *over* $\mathbb{F}_2$ *in Examples 8(iv) which is defined by the polynomial* $f_{ST,3} = X^2Y^2 + XY^2 + X + Y$.

In Examples 77(ii), we will show that there is a place $Q \in \mathbb{P}_{F_3}$ *such that* $\mathrm{Pyr}(Q)$ *is of the form which is depicted in Figure 3.5 where the blue numbers are the ramification indices of the elementary extensions in* $\mathrm{Pyr}(Q)$.

Here the Abhyankar ramification indices of the elementary extensions in $\mathrm{Pyr}(Q)$ *are equal to the usual ramification indices everywhere except for the extensions* $Q/P_{0,2}$ *and* $Q/P_{1,3}$. *There we have* $e(Q|P_{0,2}) = e(Q|P_{1,3}) = 2$ *(blue) and* $\tilde{e}(Q|P_{0,2}) = \tilde{e}(Q|P_{1,3}) = 1$ *(green). This is only possible because the extensions* $P_{0,2}/P_{1,2}$ *and* $P_{1,3}/P_{1,2}$ *are both wild and, thus, Abyankar's Lemma is not applicable to the diamond* $(Q, P_{0,2}, P_{1,3}, P_{1,2})$.

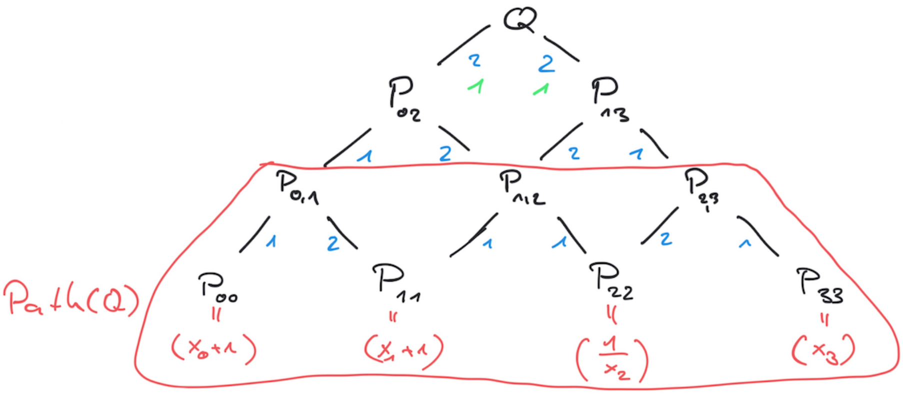


Figure 3.5: Example of a pyramid of places with Abhyankar ramification indices

Remark 43. *(i) For the following remark, we already use the definition of the directed graph* $\mathrm{Graph}(\mathrm{Pyr}(Q))$ *in Definition 72(i). However, this definition only interprets the pyramid* $\mathrm{Pyr}(Q)$ *(see Figure 2.6) as a directed graph* $\mathrm{Graph}(\mathrm{Pyr}(Q))$ *where the places in the pyramid are the vertices and the elementary extensions* R/P *are the edges* $R \to P$.

Because of the directed graph structure of $\mathrm{Graph}(\mathrm{Pyr}(Q))$, *we could have also defined the map* $\tilde{e}$ *by first defining* $\tilde{e}$ *as a weight function on* $\mathrm{Graph}(\mathrm{Pyr}(Q))$. *As the edges* *of* Graph(Pyr(Q)) *are the elementary extensions in* Pyr(Q)*, this is basically what happens in the items (i) and (ii) of Definition 41. Then we could have defined the final map* $\tilde{e}$ *as the extension of this weight function on all paths in* Graph(Pyr(Q)) *which includes Definition 41(iii). Finally, we would also have needed to check that the* $\tilde{e}$*-values of the paths only depend on the initial and terminal vertices. This will be proven in following Lemma 44(ii).*

(ii) In Definition 41, we defined the Abhyankar ramification indices depending on n *and* Q*. It immediately follows from this definition for all* $R := P_{k,l}$ *and* $P := P_{i,j}$ *with* $0 \leq k \leq i \leq j \leq l \leq n$ *that the value* $\tilde{e}(R|P)$ *only depends on the subpath* $\text{Path}(Q) = (P_{r,s})_{s-r\leq 1}$ *consisting of the places* $P_{r,s}$ *with* $k \leq r \leq s \leq l$*. But these places equal* $P_{r,s} = R \cap F_{r,s}$ *and thus* $\tilde{e}(R|P)$ *only depends on the extension* R/P*. Consequently, we do not need to specify* Q *and* n *before we refer to* $\tilde{e}(R|P)$*.*

First properties of the Abhyankar ramification indices. In the following Lemma 44, we will show some first properties of Abhyankar ramification indices: First, the virtual Abhyankar's Lemma does not only hold for elementary diamonds but for all diamonds in Pyr(Q). Second, the Abhyankar ramification indices satisfy the multiplicative transitivity rule. Third, the Abhyankar ramification indices are equal to the usual ramification indices if Path(Q) is tame.

Lemma 44. *In the situation of Definition 41, we have the following:*

(i) The map $\tilde{e}$ *satisfies the* ***$\tilde{e}$-version of Abhyankar's Lemma (or virtual Abhyankar Lemma)****, i.e.*

$$\tilde{e}(R|P) = \text{lcm}(\tilde{e}(P_1|P), \tilde{e}(P_2|P)) \text{ and } \tilde{e}(R|P_r) = \frac{\tilde{e}(P_s|P)}{\gcd(\tilde{e}(P_1|P), \tilde{e}(P_2|P))}$$

for all $\{r, s\} = \{1, 2\}$ *and diamonds* $(R, P_1, P_2, P) := (P_{k,l}, P_{k,j}, P_{i,l}, P_{i,j})$ *of places in* Pyr(Q) *with* $0 \leq k \leq i \leq j \leq l \leq n$ *(see Figure 3.6).*

(ii) The map $\tilde{e}$ *satisfies the* ***multiplicative transitivity rule*** *, i.e.*

$$\tilde{e}(R|P) = \tilde{e}(R|P_1)\tilde{e}(P_1|P)$$

for any $R := P_{r,s}$*,* $P_1 := P_{k,l}$*,* $P := P_{i,j}$ *where* $0 \leq r \leq k \leq i \leq j \leq l \leq s \leq n$ *(see Figure 3.6).*

(iii) If Path(Q) *is tame, then* $e = \tilde{e}$ *on the extensions in* Pyr(Q)*.*

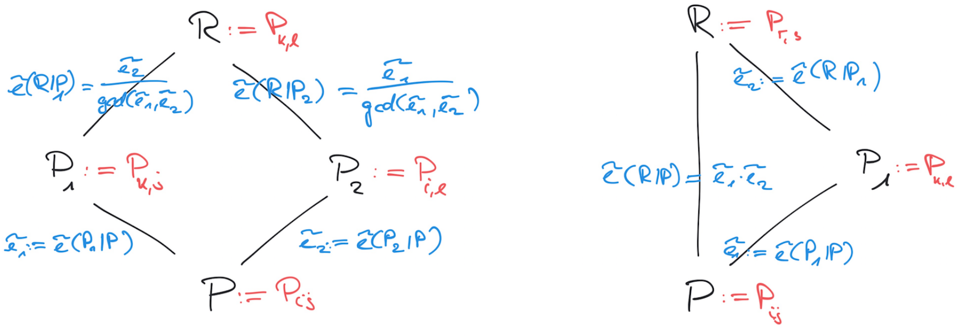


Figure 3.6: General virtual Abhyankar's Lemma and multiplicative transitivity rule for $\tilde{e}$

Proof. For the second equality in (i): First, we consider flat diamonds $(R, P_1, P_2, P) := (P_{k,l}, P_{k,j}, P_{i,l}, P_{i,j})$, i.e. $k = i$ or $l = j$. For $k = i$ (resp. $l = j$), the second equality in (i) holds trivially as then $(\tilde{e}(P_1|P), \tilde{e}(P_2|P)) = (1, \tilde{e}(R|P_1))$ (resp. $(\tilde{e}(P_1|P), \tilde{e}(P_2|P)) = (\tilde{e}(R|P_2), 1)$) holds by the equalities $(R, P_1) = (P_2, P)$ (resp. $(R, P_2) = (P_1, P)$) and by the definition of $e(P|P)$ being an empty product in Definition 41(iii). Consequently, the denominator on the right side of the desired equality equals one and the numerator $\tilde{e}(P_s|P)$ equals the left side $\tilde{e}(R|P_r)$ for all $\{r, s\} = \{1, 2\}$. Hence, the second equality in (i) holds for $k = i$ (resp. $l = j$).

Next, we consider non-flat diamonds $(R, P_1, P_2, P) := (P_{k,l}, P_{k,j}, P_{i,l}, P_{i,j})$, i.e. $0 \leq k < i \leq j < l \leq n$. We show the second equality in (i) by induction over $m := (l-k)-(j-i) \in \mathbb{N}$ with $2 \leq m$, which can be interpreted as the hight of the diamond (see Figure 3.7): For $2 = m = (j - i) - (l - k)$, we have the identities $k = i + 1$ and $l = j - 1$ by the assumption $0 \leq k < i \leq j < l \leq n$ and, thus, the extensions R/P_r and P_r/P in the diamond (R, P_1, P_2, P) are elementary for all $r = 1, 2$. Hence, the desired second equality in (i) follows from Definition 41(ii) for $2 = m = (l - k) - (j - i)$.

Now, let $3 \leq m := (l - k) - (j - i) = (l - j) + (i - k)$, then at least one of the estimates $l - j \geq 2$ and $i - k \geq 2$ needs to hold. First, suppose that the estimate $i - k \geq 2$ holds. Then we set $R' := P_{k+1,l}$ and $P_1' = P_{k+1,j}$ (see Figure 3.7). Due to the equality $(l - (k + 1)) - (j - i) = m - 1$, the induction hypothesis can be applied to the diamond $(R', P_1', P_2, P) = (P_{k+1,l}, P_{k+1,j}, P_{i,l}, P_{i,j})$ and, thus, we obtain the identites

$$\tilde{e}(R'|P_1') = \frac{\tilde{e}(P_2|P)}{\gcd(\tilde{e}(P_1'|P), \tilde{e}(P_2|P))} \text{ and } \tilde{e}(R'|P_2) = \frac{\tilde{e}(P_1'|P)}{\gcd(\tilde{e}(P_1'|P), \tilde{e}(P_2|P))}. \tag{80}$$

Moreover, due to the estimate $k+1 < i$ and the consequent estimate $(l-k)-(j-(k+1)) < (l - k) - (j - i) =: m$, the induction hypothesis can also be applied to the diamond $(R, P_1, R', P_1') = (P_{k,l}, P_{k,i}, P_{k+1,l}, P_{k+1,j})$ and, thus, we obtain the identities

$$\tilde{e}(R|P_1) = \frac{\tilde{e}(R'|P_1')}{\gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1'))} \text{ and } \tilde{e}(R|R') = \frac{\tilde{e}(P_1|P_1')}{\gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1'))}. \tag{81}$$

Let $g := \gcd(\tilde{e}(P_1'|P), \tilde{e}(P_2|P))$. Then we deduce the desired second equality in (i) for $r = 1$ and $s = 2$ by the equalities

$$\begin{aligned}\tilde{e}(R|P_1) &= \frac{\tilde{e}(R'|P_1')}{\gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1'))} = \frac{\frac{\tilde{e}(P_2|P)}{g}}{\gcd(\tilde{e}(P_1|P_1'), \frac{\tilde{e}(P_2|P)}{g})} = \frac{\tilde{e}(P_2|P)}{\gcd(\tilde{e}(P_1|P_1') \cdot g, \tilde{e}(P_2|P))}\\ &= \frac{\tilde{e}(P_2|P)}{\gcd(\tilde{e}(P_1|P_1') \cdot \tilde{e}(P_1'|P), \tilde{e}(P_2|P))} = \frac{\tilde{e}(P_2|P)}{\gcd(\tilde{e}(P_1|P), \tilde{e}(P_2|P))}\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by the first identity in (81). The second equality holds by first identity in (80). The third equality holds by the identity $\gcd(ac, bc) = \gcd(a, b)c$ for any $a, b, c \in \mathbb{N}$, The fourth equality holds by Lemma 45. The last equality holds because Definition 41(iii) implies the equalities

$$\begin{aligned}\tilde{e}(P_1|P_1')\tilde{e}(P_1'|P) = \tilde{e}(P_{k,j}|P_{k+1,j})\tilde{e}(P_{k+1,j}|P_{i,j}) = \prod_{\nu=k}^{i-1} \tilde{e}(P_{\nu,j}|P_{\nu+1,j}) =: \tilde{e}(P_{k,j}|P_{i,j})\\ = \tilde{e}(P_1|P).\end{aligned} \tag{82}$$

Furthermore, for proving the desired second equality in (i) for $r = 2$ and $s = 1$, we first compute

$$\tilde{e}(R|P_2) = \tilde{e}(R|R')\tilde{e}(R'|P_2) = \frac{\tilde{e}(P_1|P_1') \cdot \tilde{e}(P_1'|P)}{\gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1')) \cdot g}$$

$$= \frac{\tilde{e}(P_1|P)}{\gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1')) \cdot g} \tag{83}$$

where the first equality holds since Definition 41(iii) implies this equality analogously to the equality in (82), the second equality holds by the second identites in (80) and (81) and the last equality holds by the identity in (82). Moreover, we compute

$$\begin{aligned} \gcd(\tilde{e}(P_1|P_1'), \tilde{e}(R'|P_1')) \cdot g &= \gcd(\tilde{e}(P_1|P_1') \cdot g, \tilde{e}(R'|P_1') \cdot g) = \gcd(\tilde{e}(P_1|P_1') \cdot g, \tilde{e}(P_2|P)) \\ &= \gcd(\tilde{e}(P_1|P_1') \cdot \tilde{e}(P_1'|P), \tilde{e}(P_2|P)) = \gcd(\tilde{e}(P_1|P), \tilde{e}(P_2|P)) \end{aligned} \tag{84}$$

where the first equality holds by the identity $\gcd(ac, bc) = \gcd(a, b)c$ for any $a, b, c \in \mathbb{N}$, the second equality holds by the first identity in (80), the third equality holds by Lemma 45 and the last equality holds by the identity in (82).

Finally, combining (83) and (84) yields the desired second equality in (i) for $r = 2$ and $s = 1$.

Second, suppose that the estimate $j - l \geq 2$ holds. In this case, we define $R' := P_{k,l-1}$ and $P_2' := P_{i,l-1}$. Then it is obvious that the reasonings in the first case can be mirrored and, hence, the desired second equality in (i) follows in any case.

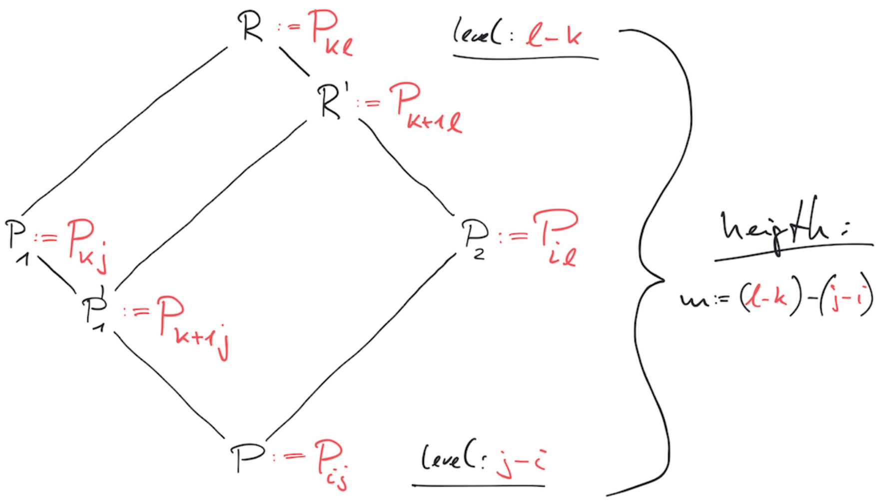


Figure 3.7: Diagram for the $\tilde{e}$-version of Abhyankar's Lemma in a proof

For (ii): In Definition 41(iii), we defined the Abhyankar ramification indices $\tilde{e}(R|P)$, $\tilde{e}(R|P_1)$ and $\tilde{e}(P_1|P)$ as the products of the $\tilde{e}$-values of the elementary extensions along the paths $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, which are depicted in Figure 3.8. Thus, we basically have to show that the product of the $\tilde{e}$-values along the path $\mathcal{P}$ equals the product of the $\tilde{e}$-values along the path $\mathcal{P}_1$ and $\mathcal{P}_2$. For that, Figure 3.8 already suggests that this should follow from showing that the product of the $\tilde{e}$-values along the left path $\mathcal{P}_l$ equals the product of the $\tilde{e}$-values along the right path $\mathcal{P}_r$.

First, the lastly mentioned equality for the paths $\mathcal{P}_l$ and $\mathcal{P}_r$ follows from the equalities

$$\tilde{e}(P_{k,s}|P_{k,l})\tilde{e}(P_{k,l}|P_{i,l}) = \frac{\tilde{e}(P_{i,s}|P_{i,l})\tilde{e}(P_{k,l}|P_{i,l})}{\gcd(\tilde{e}(P_{i,s}|P_{i,l}), \tilde{e}(P_{k,l}|P_{i,l}))} = \tilde{e}(P_{k,s}|P_{i,s})\tilde{e}(P_{i,s}|P_{i,l}). \tag{85}$$

for the diamond $(P_{k,s}, P_{k,l}, P_{i,s}, P_{i,l})$ of places (see Figure 3.8) where the first (resp. second) equality holds by the second equality in Lemma 44(i) implying the equality $\tilde{e}(P_{k,s}|P_{k,l}) = \frac{\tilde{e}(P_{i,s}|P_{i,l})}{\gcd(\tilde{e}(P_{i,s}|P_{i,l}), \tilde{e}(P_{k,l}|P_{i,l}))}$ (resp. $\tilde{e}(P_{k,s}|P_{i,s}) = \frac{\tilde{e}(P_{k,l}|P_{i,l})}{\gcd(\tilde{e}(P_{i,s}|P_{i,l}), \tilde{e}(P_{k,l}|P_{i,l}))}$). Then we already conclude the desired equality in (ii), i.e. the firstly mentioned equality for the paths $\mathcal{P}$, $\mathcal{P}_1$

and $\mathcal{P}_2$, by the equalities

$$\begin{aligned}\tilde{e}(R|P_1)\tilde{e}(P_1|P) &= \tilde{e}(P_{r,s}|P_{k,l})\tilde{e}(P_{k,l}|P_{i,j}) = \tilde{e}(P_{r,s}|P_{k,s})\tilde{e}(P_{k,s}|P_{k,l})\tilde{e}(P_{k,l}|P_{i,l})\tilde{e}(P_{i,l}|P_{i,j}) \\ &= \tilde{e}(P_{r,s}|P_{k,s})\tilde{e}(P_{k,s}|P_{i,s})\tilde{e}(P_{i,s}|P_{i,l})\tilde{e}(P_{i,l}|P_{i,j}) = \tilde{e}(P_{r,s}|P_{i,s})\tilde{e}(P_{i,s}|P_{i,j}) \\ &= \tilde{e}(P_{r,s}|P_{i,j}) = \tilde{e}(R|P)\end{aligned}$$

where the equalities hold by the following reasonings: The first and last equalities hold by the definitions of R, P_1 and P. The second and second to last equalities hold by the definitions of $\tilde{e}(P_{r,s}|P_{k,l})$ and $\tilde{e}(P_{k,l}|P_{i,j})$ in Definition 41(iii). The third equality holds by the identity in (85). The fourth equality holds since Definition 41(iii) implies the equalities $\tilde{e}(P_{r,s}|P_{k,s})\tilde{e}(P_{k,s}|P_{i,s}) = \tilde{e}(P_{r,s}|P_{i,s})$ and $\tilde{e}(P_{i,s}|P_{i,l})\tilde{e}(P_{i,l}|P_{i,j}) = \tilde{e}(P_{i,s}|P_{i,j})$.

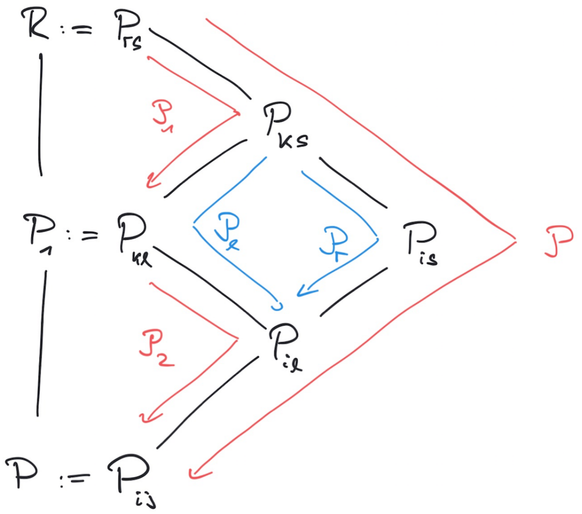


Figure 3.8: Diagram for the multiplicative transitivity rule for $\tilde{e}$ in a proof

For the first equality in (i): We immediately obtain the desired first equality in (i) by the equalities

$$\tilde{e}(R|P) = \tilde{e}(R|P_1)\tilde{e}(P_1|P) = \frac{\tilde{e}(P_2|P)\tilde{e}(P_1|P)}{\gcd(\tilde{e}(P_1|P), \tilde{e}(P_2|P))} = \operatorname{lcm}(\tilde{e}(P_1|P), \tilde{e}(P_2|P))$$

where the first equality holds by the multiplicative transitivity rule for $\tilde{e}$ in Lemma 44(ii), the second equality holds by the second equality in Lemma 44(i) and the last equality holds by the identity $ab = \gcd(a,b)\operatorname{lcm}(a,b)$ for all $a, b \in \mathbb{N}$.

For (iii): On the extensions in Path(Q), the desired identity $\tilde{e} = e$ holds independently of Path(Q) being tame by the definition of $\tilde{e}$ in Definition 41(i).

Next, we notice that, by the tameness of Path(Q), Lemma 17(v) supplies that we can apply Abhyankar's Lemma to all extensions in Pyr(Q). But since we start with the same values on the extensions in Path(Q) for e and $\tilde{e}$ and since Abhyankar's Lemma in (9) and its $\tilde{e}$-version in Definition 41(ii) clearly provide the same values, we obtain the desired identity $\tilde{e} = e$ on all elementary extensions in Pyr(Q).

Finally, on all extensions in Pyr(Q), the desired identity $\tilde{e} = e$ follows from the definition of the Abhyankar ramification indices in Definition 41(iii), from the validity of the desired identity $\tilde{e} = e$ on all elementary extensions and from the multiplicative transitivity rules for the ramification indices in (7). □

Lemma 45. *We have the identity* $\gcd(a \cdot \gcd(b,c), c) = \gcd(a \cdot b, c)$ *for all* $a, b, c \in \mathbb{N}$.

Proof. On the one hand, because $\gcd(b,c)$ is a divisor of b, it follows that $\gcd(a\cdot\gcd(b,c),c)$ is a divisor of $\gcd(a\cdot b,c)$. On the other hand, let p^{k+l} be a prime power which is a divisor of $a\cdot b$ and c such that p^k is a divisor of a and p^l is a divisor of b. Then p^l is common divisor of b and c and, therefore, a divisor of $\gcd(b,c)$. Thus, p^{k+l} is a common divisor $a\cdot\gcd(b,c)$ and c and, hence, a divisor of $\gcd(a\cdot\gcd(b,c),c)$. Both together yield the statement. ☐

Invariance of Abhyankar ramification indices under the action of the tower map. Analogously to usual ramification indices in (11), in the following Lemma 46, we will prove that the Abhyankar ramification indices are also invariant under the action of the tower map σ. This property will be useful later in Section 7.2.

Lemma 46. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ,F_0), let $(F_{i,j})_{i,j}:=$ $\mathrm{Pyr}(\mathcal{F})$ be its pyramid, let $P\in\mathbb{P}_{F_{k,l}}$ and let $Q\in\mathbb{P}_{F_{m,n}}$ for some $m\leq k\leq l\leq n$. Then we have the identity $\tilde{e}(\sigma^s(Q)|\sigma^s(P))=\tilde{e}(Q|P)$ for all $s\in\mathbb{Z}$ with $0\leq m+s$.*

We call this property the ***invariance of the Abhyankar ramification indices under the action of*** *σ.*

Proof. Let $(P_{i,j})_{i,j}:=\mathrm{Pyr}(Q)$ where i,j run over all $i,j\in\mathbb{N}_0$ with $m\leq i\leq j\leq n$. Then Lemma 15(i) supplies the identity $\mathrm{Pyr}(\sigma^s(Q))=(\sigma^s(P_{i'-s,j'-s}))_{i',j'}$ for all $s\in\mathbb{Z}$ with $0\leq m+s$ where i',j' run over all $i',j'\in\mathbb{N}_0$ with $m+s\leq i'\leq j'\leq n+s$. In particular, this implies the equality $\mathrm{Path}(\sigma^s(Q))=(\sigma^s(P_{i'-s,j'-s}))_{j'-i'\leq 1}\in W(\mathcal{F},m+s,n+s)$ by the definitions of Pyr in Definition 11 and of Path in Definition/Lemma 17(i).

Now, for all $m+s+1\leq i'\leq n+s$, all $\varepsilon=0,1$ and $i:=i'-s$, we compute

$$\begin{aligned}\tilde{e}(\sigma^s(P_{i-1,i})|\sigma^s(P_{i-\varepsilon,i-\varepsilon}))&=e(\sigma^s(P_{i-1,i})|\sigma^s(P_{i-\varepsilon,i-\varepsilon}))=e(P_{i-1,i}|P_{i-\varepsilon,i-\varepsilon})\\&=\tilde{e}(P_{i-1,i}|P_{i-\varepsilon,i-\varepsilon})\end{aligned}\tag{86}$$

where the first (resp. last) equality holds by the definition of $\tilde{e}=e$ on the extensions in the path $\mathrm{Path}(\sigma^s(Q))$ (resp. $\mathrm{Path}(Q)$) in Definition 41(i) and the second equality holds by the invariance of the ramification indices under the action of isomorphisms in (11).

Thus, the equalities in (86) provide that the Abhyankar ramification indices in the extensions in the paths $\mathrm{Path}(\sigma^s(Q))$ and $\mathrm{Path}(Q)$ up to translation of the indices. But, as the definition of $\tilde{e}$ in Definition 41 implies that $\tilde{e}$ only depends on its values on the extensions in the path, we conclude the desired equality by the equalities

$$\tilde{e}(\sigma^s(Q)|\sigma^s(P))=\tilde{e}(\sigma^s(P_{m,n})|\sigma^s(P_{k,l}))=\tilde{e}(P_{m,n}|P_{k,l})=\tilde{e}(Q|P).$$

☐

3.2.2 Relation between Abhyankar and Usual Ramification Indices

Summary of the results of this subsection. In the following Theorem 47, we will prove the estimate $\tilde{e}(Q|Q\cap F_0)\leq e(Q|Q\cap F_0)$. Moreover, we will even show that $\tilde{e}(Q|Q\cap F_0)$ divides $e(Q|Q\cap F_0)$.

Intuition for Theorem 47. Although Theorem 47 is non-trivial, it is not surprising: On the one hand, both maps e and $\tilde{e}$ are maps $\mathrm{Ext}(\mathrm{Pyr}(Q))\to\mathbb{N}$ which satisfy the multiplicative transitivity rule and agree on $\mathrm{Ext}(\mathrm{Path}(Q))$. On the other hand, $\tilde{e}$ even satisfies the $\tilde{e}$-version of Abhyankar's Lemma which, for any diamond (R,P_1,P_2,P) of places in $\mathrm{Pyr}(Q)$, provides the smallest value $\tilde{e}(R|P)=\mathrm{lcm}(\tilde{e}(P_1|P),\tilde{e}(P_2|P))$ for given values $\tilde{e}(P_1|P)$ and $\tilde{e}(P_2|P)$ which is still compatible with the multiplicative transitivity rule for $\tilde{e}$. Consequently, $\tilde{e}(R|P)=\mathrm{lcm}(\tilde{e}(P_1|P),\tilde{e}(P_2|P))$ even happens to be a divisor of any other possible value for R/P being compatible with the multiplicative transitivity rule.

This means that as $\tilde{e}$ and e agree on $\operatorname{Ext}(\operatorname{Path}(Q))$ applying the above reasoning on the elementary diamonds consisting of places in the zeroth, first and second levels supplies that $\tilde{e}(R|P)$ divides $e(R|P)$ for all extensions R/P of places in $\operatorname{Pyr}(Q)$ such that R is on the second and P on the zeroth level.

Now, for elementary diamonds on higher levels, i.e. consisting of places on the levels l, $l+1$ and $l+2$ for some $l = 1, \ldots, n-2$, it gets more complicated, because we start with different (although tightly related) values for e and $\tilde{e}$. Although, for $l \geq 2$, the value $\tilde{e}(R|P)$ can actually be larger than $e(R|P)$, Theorem 47 ensures that this is impossible if we take P from the zeroth level.

Theorem 47. *In the situation of Definition 41, we have*

$$\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} = \frac{e(P_{0,m}|P_{m,m})}{\tilde{e}(P_{0,m}|P_{m,m})} \in \mathbb{N}$$

for all $0 \leq m \leq n$. *In particular, we have the estimates*

$$\tilde{e}(Q|P_{0,0}) \leq e(Q|P_{0,0}) \text{ and } \tilde{e}(Q|P_{n,n}) \leq e(Q|P_{n,n}).$$

Proof. For the identity of the quotients: First, we show the desired identity

$$\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} = \frac{e(P_{0,m}|P_{m,m})}{\tilde{e}(P_{0,m}|P_{m,m})}$$

by induction over $m \in \mathbb{N}_0$. For $m = 0$, both involved extensions $P_{0,m}/P_{0,0}$ and $P_{0,m}/P_{m,m}$ are trivial and, thus, their e- and $\tilde{e}$-values and are even equal to one. For $m = 1$, both involved extensions $P_{0,m}/P_{0,0}$ and $P_{0,m}/P_{m,m}$ are elementary and, thus, we obtain the equalities

$$\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} = 1 = \frac{e(P_{0,m}|P_{m,m})}{\tilde{e}(P_{0,m}|P_{m,m})}$$

since Definition 41(i) implies the identity $\tilde{e} = e$ on all elementary extensions in $\operatorname{Path}(Q)$.

Now, let $m \geq 2$. First, we notice the equalities

$$\frac{e(P_{m-1,m}|P_{m,m})}{\tilde{e}(P_{m-1,m}|P_{m,m})} = 1 = \frac{e(P_{m-1,m}|P_{m-1,m-1})}{\tilde{e}(P_{m-1,m}|P_{m-1,m-1})} \tag{87}$$

since Definition 41(i) again implies the identity $\tilde{e} = e$ on all elementary extensions in $\operatorname{Path}(Q)$. Then we already conclude the desired identity by the computation (see Figure 3.9)

$$\begin{aligned}\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} &= \frac{e(P_{0,m}|P_{0,m-1})}{\tilde{e}(P_{0,m}|P_{0,m-1})}\frac{e(P_{0,m-1}|P_{0,0})}{\tilde{e}(P_{0,m-1}|P_{0,0})} = \frac{e(P_{0,m}|P_{0,m-1})}{\tilde{e}(P_{0,m}|P_{0,m-1})}\frac{e(P_{0,m-1}|P_{m-1,m-1})}{\tilde{e}(P_{0,m-1}|P_{m-1,m-1})}\\ &= \frac{e(P_{0,m}|P_{m-1,m-1})}{\tilde{e}(P_{0,m}|P_{m-1,m-1})} = \frac{e(P_{0,m}|P_{m-1,m})}{\tilde{e}(P_{0,m}|P_{m-1m})}\frac{e(P_{m-1,m}|P_{m-1,m-1})}{\tilde{e}(P_{m-1,m}|P_{m-1,m-1})}\\ &= \frac{e(P_{0,m}|P_{m-1,m})}{\tilde{e}(P_{0,m}|P_{m-1m})}\frac{e(P_{m-1,m}|P_{m,m})}{\tilde{e}(P_{m-1,m}|P_{m,m})} = \frac{e(P_{0,m}|P_{m,m})}{\tilde{e}(P_{0,m}|P_{m,m})}\end{aligned} \tag{88}$$

where the first, third, fourth and last equalities hold by the multiplicative transitivity rules for e in (7) and for $\tilde{e}$ in Lemma 44(ii), the second equality holds by the induction hypothesis and the fifth equality holds by the equalities in (87).

For the quotients being natural numbers: We will show $\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} \in \mathbb{N}$ by induction over $m \in \mathbb{N}_0$. For $m = 0, 1$, we already noticed in the first part of the proof that the equality $\tilde{e}(P_{0,m}|P_{0,0}) = e(P_{0,m}|P_{0,0})$ holds.

Now, let $m \geq 2$. We first compute

$$\begin{aligned}\tilde{e}(P_{0,m}|P_{m-1,m-1}) &= \operatorname{lcm}(\tilde{e}(P_{0,m-1}|P_{m-1,m-1}), \tilde{e}(P_{m-1,m}|P_{m-1,m-1})) \\ &= \operatorname{lcm}(\tilde{e}(P_{0,m-1}|P_{m-1,m-1}), e(P_{m-1,m}|P_{m-1,m-1})) \qquad (89)\end{aligned}$$

where the first equality holds by the $\tilde{e}$-version of Abhyankar's Lemma in Lemma 44(i) and the second equality holds because again implies the identity $\tilde{e} = e$ on all elementary extensions in Path(Q). Next, since $\tilde{e}(P_{0,m-1}|P_{m-1,m-1})$ divides $e(P_{0,m-1}|P_{m-1,m-1})$ by the induction hypothesis, we derive from (89) that

$$\tilde{e}(P_{0,m}|P_{m-1,m-1}) \text{ divides } \operatorname{lcm}(e(P_{0,m-1}|P_{m-1,m-1}), e(P_{m-1,m}|P_{m-1,m-1})). \qquad (90)$$

Then, because the multiplicative transitivity rule of e in (7) implies the identities

$$\begin{aligned}e(P_{0,m}|P_{0,m-1}) \cdot e(P_{0,m-1}|P_{m-1,m-1}) &= e(P_{0,m}|P_{m-1,m-1}) \\ &= e(P_{0,m}|P_{m-1,m}) \cdot e(P_{m-1,m}|P_{m-1,m-1}),\end{aligned}$$

we obtain that $\operatorname{lcm}(e(P_{0,m-1}|P_{m-1,m-1}), e(P_{m-1,m}|P_{m-1,m-1}))$ divides $e(P_{0,m}|P_{m-1,m-1})$. Combining this and (90) yields that $\tilde{e}(P_{0,m}|P_{m-1,m-1})$ divides $e(P_{0,m}|P_{m-1,m-1})$ and, thus, the quotient $\frac{e(P_{0,m}|P_{m-1,m-1})}{\tilde{e}(P_{0,m}|P_{m-1,m-1})}$ is a natural number. Finally, this again supplies the desired statement by the equality

$$\frac{e(P_{0,m}|P_{0,0})}{\tilde{e}(P_{0,m}|P_{0,0})} = \frac{e(P_{0,m}|P_{m-1,m-1})}{\tilde{e}(P_{0,m}|P_{m-1,m-1})} \in \mathbb{N}.$$

where the equality holds by the first three equalities in (88).

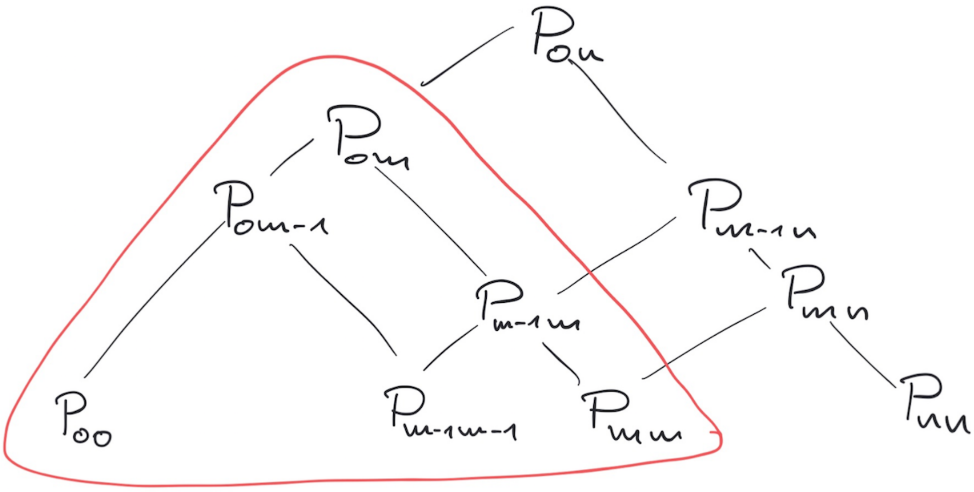


Figure 3.9: Pyramid of places with a sub pyramid

For the 'in particular'-part: The estimates in the 'in particular'-part immediately follow from the quotients in the 'main'-part being natural numbers for $m = n$. ☐

Examples 48. *In Examples 42(ii), the quotients in Theorem 47 for* $m = n = 3$ *equal* 2.

Remark 49. *In Theorem 47, we proved that, after reordering the quotients in the identity there, we have the identity*

$$\frac{e(Q|P_{0,0})}{e(Q|P_{n,n})} = \frac{\tilde{e}(Q|P_{0,0})}{\tilde{e}(Q|P_{n,n})}.$$

In the following, we will elaborate on the fact that the quotient $\frac{e(Q|P_{0,0})}{e(Q|P_{n,n})}$ *turns out to be an invariant of all maps* $\phi : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *which satisfy the multiplicative transitivity rule and agree with* e *on* $\mathrm{Ext}(\mathrm{Path}(Q))$ *and that* $\tilde{e}(Q|P_{k,k})$ *divides* $\phi(Q|P_{k,k})$ *for all such maps* $\phi : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *and* $k = 0, n$*.*

Indeed, for the corresponding statements with $\tilde{e}$ *and* e *in the proof of Theorem 47, we only used the fact that* $e, \tilde{e} : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *satisfy those two properties. Consequently, any such map* $\phi : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *satisfies the equality*

$$\frac{e(Q|P_{0,0})}{e(Q|P_{n,n})} = \frac{\phi(Q|P_{0,0})}{\phi(Q|P_{n,n})} \tag{91}$$

and the property that $\tilde{e}(Q|P_{k,k})$ *divides* $\phi(Q|P_{k,k})$ *for* $k = 0, n$*.*

In particular, (91) also holds for the unique such map $\mathrm{Const} : \mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *satisfying* $\mathrm{Const}(P_{i,j+1}|P_{i,j}) = e(P_{j,j+1}|P_{j,j})$ *and* $\mathrm{Const}(P_{i,j}|P_{i-1,j}) = e(P_{i,i-1}|P_{i,i})$ *for all* $0 \le i \le j \le n$*. Consequently, we obtain the identities*

$$\Delta(\mathrm{Path}(Q)) := \frac{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1})}{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i})} = \frac{\mathrm{Const}(P_{0,n}|P_{0,0})}{\mathrm{Const}(P_{0,n}|P_{n,n})} = \frac{e(Q|P_{0,0})}{e(Q|P_{n,n})}. \tag{92}$$

which even imply that the invariant $\Delta(\mathrm{Path}(Q))$ *of those maps* $\mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *only depends on the ramification indices of the extensions in* $\mathrm{Path}(Q)$*.*

Finally, we want give some intuition for why this invariant $\Delta(\mathrm{Path}(Q))$ *even exists. For all such maps* ϕ *and elementary diamonds* (R, P_1, P_2, P) *in* $\mathrm{Pyr}(Q)$*, we have the equalities* $\phi(R|P_1) \cdot \phi(P_1|P) = \phi(R|P) = \phi(R|P_2) \cdot \phi(P_2|P)$ *and, thus,*

$$\frac{\phi(R|P_1)}{\phi(R|P_2)} = \frac{\phi(P_2|P)}{\phi(P_1|P)}. \tag{93}$$

Therefore, if we first consider the quotient

$$\frac{\phi(Q|P_{0,0})}{\phi(Q|P_{n,n})} = \frac{\prod_{j=0}^{n-1} \phi(P_{0,j+1}|P_{0,j})}{\prod_{i=1}^{n} \phi(P_{i-1,n}|P_{i,n})} = \frac{\phi(P_{0,n}|P_{0,n-1})}{\phi(P_{0,n}|P_{1,n})} \cdot \frac{\prod_{j=0}^{n-2} \phi(P_{0,j+1}|P_{0,j})}{\prod_{i=2}^{n} \phi(P_{i-1,n}|P_{i,n})},$$

then apply the equality in (93) to the diamond $(R, P_1, P_2, P) = (P_{0,n}, P_{0,n-1}, P_{1,n}, P_{1,n-1})$ *at the top level and then iteratively continue applying this equality to all diamonds for decreasing levels, we end up with the desired equality*

$$\frac{\phi(Q|P_{0,0})}{\phi(Q|P_{n,n})} = \frac{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1})}{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i})} = \Delta(\mathrm{Path}(Q)) \tag{94}$$

In other words, the local invariance, i.e. the invariance of the quotients in (93) for all elementary diamonds in $\mathrm{Pyr}(Q)$*, provides that the quotient* $\frac{\phi(Q|P_{0,0})}{\phi(Q|P_{n,n})}$ *only depends on the starting values in* $\mathrm{Path}(Q)$ *via the equality in (94).*

3.2.3 The Upper Bound

Summary of the results of this subsection. We are now able to define the desired upper bound $\tilde{N}(F_n, \mathcal{P})$ of $N(F_n, \mathcal{P})$ in the following Definition 50 and to prove the estimate $N(F_n, \mathcal{P}) \le \tilde{N}(F_n, \mathcal{P})$ in Corollary 51 as a consequence of Proposition 39 and Theorem 47.

Definition 50. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, $n \in \mathbb{N}_0$, $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n)$ and $Q \in \mathbb{P}_{F_n}(\mathcal{P})$. Then we define*

$$\tilde{N}(F_n, \mathcal{P}) := \frac{\prod_{i=0}^{n-1} \tilde{e}(P_{i,i+1}|P_{i,i})}{\tilde{e}(Q|P_{0,0})} \quad \text{and} \quad \tilde{N}(F_n, V) := \sum_{P\in V} \sum_{\mathcal{P}'\in W(\mathcal{F},n,P)} \tilde{N}(F_n, \mathcal{P}')$$

for all finite $V \subseteq \mathbb{P}_{F_0}$.

Corollary 51. *In the situation of Definition 50, let $\mathcal{F}$ be defined over an algebraically closed field. Then we have the estimate $N(F_n, \mathcal{P}) \leq \tilde{N}(F_n, \mathcal{P})$, which is even an equality if $\mathcal{P}$ is tame.*

In particular, we obtain the estimate $N(F_n, V) \leq \tilde{N}(F_n, V)$ for all finite $V \subseteq \mathbb{P}_{F_0}$ which is even an equality if all paths in $W(\mathcal{F}, n, V)$ are tame.

Proof. On the one hand, the second desired estimate (resp. equality) follows from the first estimate (resp. equality) by the identities $N(F_n, V) = \sum_{P\in V} \sum_{\mathcal{P}\in W(\mathcal{F},n,P)} N(F_n, \mathcal{P})$ in (5) and $\tilde{N}(F_n, V) = \sum_{\mathcal{P}\in W(\mathcal{F},n,V)} \tilde{N}(F_n, \mathcal{P})$ in Definition 50.

On the other hand, for $P := P_{0,0}$, we obtain the first desired estimate by the equalities and estimate

$$\begin{aligned}\tilde{N}(F_n, \mathcal{P}) \cdot \tilde{e}(Q|P) &= \prod_{i=0}^{n-1} e(P_{i,i+1}|P_{i,i}) = \sum_{Q'\in\mathbb{P}_{F_n}(\mathcal{P})} e(Q'|P) \geq \sum_{Q'\in\mathbb{P}_{F_n}(\mathcal{P})} \tilde{e}(Q'|P) \\ &= N(F_n, \mathcal{P}) \cdot \tilde{e}(Q|P) \qquad (95)\end{aligned}$$

where the equalities and estimate hold by the following reasoning: The first equality holds by the definition of $\tilde{N}(F_n, \mathcal{P})$ in Definition 50 and by the definition of $\tilde{e}(P_{i,i+1}|P_{i,i}) = e(P_{i,i+1}|P_{i,i})$ for all $i = 0, \ldots, n-1$ in Definition 41(i). The second equality holds by Proposition 39. The estimate holds by the 'in particular'-part of Theorem 47. The last equality holds since the definition of $\tilde{e}$ in Definition 41 implies that $\tilde{e}(Q'|P_{0,0})$ is the same for all $Q' \in \mathbb{P}_{F_n}(\mathcal{P})$.

Moreover, since Lemma 44(iii) implies the equality $\tilde{e}(Q'|P) = e(Q'|P)$ for all $Q' \in \mathbb{P}_{F_n}(\mathcal{P})$ if $\mathcal{P}$ is tame, the only estimate in (95) becomes an equality and, thus, the first desired equality also follows. □

Examples 52. *The reasoning in Examples 42(ii) can also be applied to the geometric tower $\overline{\mathcal{F}} = (\overline{F}_n)_n$ of the recursive ST-tower $\mathcal{F}_{ST,3}$. The corresponding path $\mathcal{P}$ in $\overline{\mathcal{F}}$ consequently satisfies the equality $\tilde{N}(\overline{F}_3, \mathcal{P}) = 2$. On the other hand, Proposition 39 provides the equality $N(\overline{F}_3, \mathcal{P}) = 1$. Thus, we obtain the estimate $1 = N(\overline{F}_3, \mathcal{P}) \leq \tilde{N}(\overline{F}_3, \mathcal{P}) = 2$ which is in accordance with Corollary 51.*

Remark 53. *In the situation of Corollary 51 and for any finite set $V \subset \mathbb{P}_{F_0}$ containing* Ram$(\mathcal{F}/F_0)$*, we also obtain the lower bound*

$$\tilde{g}(F_n) := d^n \cdot \left(g(F_0) - 1 + \frac{\#V}{2}\right) - \frac{\tilde{N}(F_n, V)}{2} + 1$$

for the genus $g(F_n)$ of F_n by the equalities and estimates

$$\begin{aligned}2 \cdot ((g(F_n) - 1) - d^n \cdot (g(F_0) - 1)) &= \sum_{Q\in\mathbb{P}_{F_n}(V)} d(Q|Q \cap F_0) \geq \sum_{Q\in\mathbb{P}_{F_n}(V)} e(Q|Q \cap F_0) - 1 \\ &= d^n \cdot \#V - N(F_n, V) \geq d^n \cdot \#V - \tilde{N}(F_n, V)\end{aligned}$$

where the equalities and estimates hold by the following reasonings: The first equality holds by the Hurwitz Genus Formula in (9) and by the fact that V contains Ram$(\mathcal{F}/F_0)$*. The* *first estimate holds since the different exponent is at least the ramification indices minus one by [Sti08, p. 106, Corollary 3.5.5]. For the second equality, we first notice that since* V *and, thus,* $\mathbb{P}_{F_n}(V)$ *are finite sets and since* $\mathcal{F}$ *is defined over an algebraically closed field, applying the fundamental equality in (8) to all places in* V *yields the identities*

$$\sum_{Q \in \mathbb{P}_{F_n}(V)} e(Q|Q \cap F_0) = [F_n : F_0] \cdot \#V = d^n \cdot \#V. \tag{96}$$

Moreover, Definition (5) and the fact that $\mathcal{F}$ *is defined over an algebraically closed field imply the identities*

$$\sum_{Q \in \mathbb{P}_{F_n}(V)} 1 = \#\mathbb{P}_{F_n}(V) = \#\mathbb{P}^{(1)}_{F_n}(V) = N(F_n, V). \tag{97}$$

The second equality then follows from combining these two identities in (96) and (97). The last estimate holds by Corollary 51.

It is not difficult to show that this lower bound $\tilde{g}(F_n)$ *is independent of the concrete choice of* V*. However, we will not prove this, because we will not use this statement in the rest of this thesis.*

Remark 54. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a recursive tower over the field* k*,* $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n)$*. As already announced in the introduction of this chapter, we want to elaborate on the fact that the results of this chapter imply the main results in [Kuh17] which are [Kuh17, p. 22, Theorem 2.3(f)] and the estimates in [Kuh17, p. 54, Theorem 3.20(c),(j)].*

In short, [Kuh17, p. 22, Theorem 2.3(f)] follows from the 'moreover'-part of Key Lemma 36(i). Also, if k *is algebraically closed, the estimates in [Kuh17, p. 54, Theorem 3.20(c),(j)] follow from the estimates* $\tilde{N}(F_n, \mathcal{P}) \leq N(F_n, \mathcal{P})$ *in Corollary 51 and* $\tilde{g}(F_n) \geq g(F_n)$ *in Remark 53.*

Moreover, the additional assumption of k *being algebraically closed in Corollary 51 and Remark 53 is not really necessary. It only simplifies the definitions, statements and proofs due to the fact that all places become rational. By making some cumbersome adjustments, we could even avoid this additional assumption. However, we already pointed out in the introduction of this Chapter 3 that, for our purposes, it is sufficient to only conclude the estimate* $N(F_n, \mathcal{P}) \leq \tilde{N}(F_n, \mathcal{P})$ *for algebraically closed full constants fields* k*.*

4 Preliminaries II - Tower Graphs

Purpose of this chapter. In this second preliminary chapter, we will introduce the main tool of this thesis, which is the tower graph of a recursive tower $\mathcal{F} = (F_\nu)_\nu$. It will capture the information about the extensions of places in F_1/F_0 and $F_1/\sigma(F_0)$ for the tower map σ of $\mathcal{F}$. Moreover, we will also define the rational, splitting and ramification subgraphs of the tower graph. These subgraphs will already carry all information which are necessary to determine the limit of a recursive tower in most cases.

We will also relate the tower graph to the Beelen-graph in [BGS04, p. 10, Definition 4.2] and the HP-graph in [HP12, p. 15, Definition 10] via epimorphisms of directed graphs in Proposition 95 and in Proposition 104, respectively.

Finally, in Subsection 4.4.2, we will also make some preparations for the proof of the first major result of this thesis in the next chapter, namely Theorem 155.

Structure of this chapter. In Section 4.1, we will formulate our adjusted definitions of directed graphs, weighted adjacency matrices and further related concepts with which we will work for the rest of this thesis.

In Section 4.2, we will define the tower graph $\Gamma_\mathcal{F}$ of a recursive tower $\mathcal{F}$ and its tower graph map $\sigma_{\Gamma_\mathcal{F}}$ which connects the paths in $\Gamma_\mathcal{F}$ and the paths in $\mathcal{F}$ bijectively. Moreover, we will define the rational, splitting and ramification subgraphs of the tower graph which carry the crucial information for the desired values $N(F_n)$ and $g(F_n)$.

In Section 4.3, we will relate the tower graph of a recursive tower $\mathcal{F}$ to the tower graphs of its dual tower $\hat{\mathcal{F}}$.

In Section 4.4, we will connect the tower graph to two other graphs which are also attributed to recursive towers in the literature, namely the Beelen-graph in [BGS04, p. 10, Definition 4.2] and the HP-graph in [HP12, p. 15, Definition 10].

4.1 Directed Graphs

Purpose of this section. In this Subsection 4.1, we will collect some basic definitions and facts in regards to directed graphs and we will also add some new definitions for our purposes. Most of the definitions can either be directly found or are closely related to the definitions in [GR01] and [Sun13].

4.1.1 Weakly and Strongly Connected Directed Graphs

Purpose of this subsection. In this subsection, we will give several standard definitions with regards to directed graphs, e.g. paths, weakly and strongly connected directed graphs.

Weakly and strongly connected directed graphs. In the following Definition 55(i), the definition of a directed graph is the same as in [BGS04, p. 10].

Definition 55. *Let* V *and* E *be sets, let* $\alpha : E \to V^2$ *be a map and let* $\pi_i : V^2 \to V$ *be the canonical projection from* V^2 *to its* i*-th component for all* $i = 1, 2$*. Then we define the following:*

(i) *We call* $\Gamma := (V, E, a)$ *a* ***directed graph*** *and write* $V(\Gamma) := V$, $E(\Gamma) := E$, $v_{\text{init}} := \pi_1 \circ \alpha$ *and* $v_{\text{term}} := \pi_2 \circ \alpha$.

Moreover, we call any element $v \in V(\Gamma)$ *a* ***vertex****, any element* $e \in E(\Gamma)$ *an* ***edge*** *of* Γ*, the map* α *the* ***edge map****,* v_{init} *the* ***initial vertex map*** *and* v_{term} *the* ***terminal vertex map*** *on* Γ.

Finally, for all edges $e \in E(\Gamma)$*, we call the vertex* $v_{\text{init}}(e)$ *the* ***initial vertex*** *and the vertex* $v_{\text{term}}(e)$ *the* ***terminal vertex*** *of* e.

(ii) *A finite sequence* $\mathcal{P} := (e_1, \dots, e_n) \in E(\Gamma)^n$ *is called a* ***(directed) path of length*** $n \in \mathbb{N}$ *in* Γ *if* $v_{\text{term}}(e_{i-1}) = v_{\text{init}}(e_i)$ *for all* $i = 1, \dots, n$ *and we also call any vertex* $v \in V(\Gamma)$ *a* ***path of length*** 0.

If $\mathcal{P} = v$ *is a path of length* 0*, we define* $\mathrm{Vertex}_0(\mathcal{P}) := v$ *and if* $\mathcal{P}$ *is a path of length* $n \in \mathbb{N}$*, we define*

$$\mathrm{Vertex}_0(\mathcal{P}) := v_{\text{init}}(e_1),\ \mathrm{Vertex}_i(\mathcal{P}) := v_{\text{term}}(e_i),\ \mathrm{Edge}_i(\mathcal{P}) := e_i$$

for all $i = 1, \dots, n$ *and call* $\mathrm{Vertex}_i(\mathcal{P})$ *the* i***-th vertex*** *of* $\mathcal{P}$ *for all* $i = 0, \dots, n$ *and* $\mathrm{Edge}_i(\mathcal{P})$ *the* i***-th edge*** *of* $\mathcal{P}$ *for all* $i = 1, \dots, n$.

Moreover, we define $v_{\text{init}}(\mathcal{P}) := \mathrm{Vertex}_0(\mathcal{P})$ *(resp.* $v_{\text{term}}(\mathcal{P}) := \mathrm{Vertex}_n(\mathcal{P})$*) and* $\mathrm{Length}(\mathcal{P}) = n$ *and call* $v_{\text{init}}(\mathcal{P})$ *(resp.* $v_{\text{term}}(\mathcal{P})$*) the* ***initial vertex*** *(resp.* ***terminal vertex****) of* $\mathcal{P}$.

We will also use the notation

$$\mathcal{P} = \left(\mathrm{Vertex}_0(\mathcal{P}) \xrightarrow{\mathrm{Edge}_1(\mathcal{P})} \mathrm{Vertex}_1(\mathcal{P}) \xrightarrow{\mathrm{Edge}_2(\mathcal{P})} \dots \xrightarrow{\mathrm{Edge}_n(\mathcal{P})} \mathrm{Vertex}_n(\mathcal{P})\right).$$

See Notation 56 for another non-standard notation $\mathcal{P} = [P_{i,j}]_{j-i \leq 1}$ *which we will mostly use for a path* $\mathcal{P}$ *in* Γ.

(iii) *We define* $W(\Gamma, n)$ *as the set of all paths of length* $n \in \mathbb{N}_0$ *in* Γ *and*

$$W(\Gamma) := \coprod_{n \in \mathbb{N}_0} W(\Gamma, n).$$

Thus, we have the identity $W(\Gamma, 0) = V(\Gamma)$ *and, furthermore, since we can identify any path* $\mathcal{P} = (e)$ *of length one with its one and only edge* e*, we also have the identity* $W(\Gamma, 1) = E(\Gamma)$.

For all $n \in \mathbb{N}_0$ *and* $v_1, v_2 \in V(\Gamma)$*, we also define*

$$W(\Gamma, n, v_1, v_2) := \{\mathcal{P} \in W(\Gamma, n) : (v_{\text{init}}(\mathcal{P}), v_{\text{term}}(\mathcal{P})) = (v_1, v_2)\}$$

and

$$E(\Gamma, v_1, v_2) := W(\Gamma, 1, v_1, v_2) \text{ and } W(\Gamma, v_1, v_2) := \coprod_{m \in \mathbb{N}_0} W(\Gamma, m, v_1, v_2).$$

(iv) *A path* $\mathcal{P} \in W(\Gamma)$ *is called* ***closed*** *if* $v_{\text{init}}(\mathcal{P}) = v_{\text{term}}(\mathcal{P})$ *and a closed path* $\mathcal{P} \in W(\Gamma, n)$ *is called a* ***circle*** *if* $\#\{\mathrm{Vertex}_i(\mathcal{P}) : i = 0, \dots, n\} = n$*. Hence, a circle is a closed path without further repeating vertices.*

(v) *For all paths $\mathcal{P}_1 \in W(\Gamma, m)$ and $\mathcal{P}_2 \in W(\Gamma, n)$ with $v_{\text{term}}(\mathcal{P}_1) = v_{\text{init}}(\mathcal{P}_2)$, we call $\mathcal{P}_1$ and $\mathcal{P}_2$* ***composable (in the order** $(\mathcal{P}_1, \mathcal{P}_2)$**)** *and call the path $\mathcal{P}_1\mathcal{P}_2 = \mathcal{P}_1 \cdot \mathcal{P}_2 \in W(\Gamma, m+n)$ with*

$$\operatorname{Vertex}_i(\mathcal{P}_1\mathcal{P}_2) = \begin{cases} \operatorname{Vertex}_i(\mathcal{P}_1) & i = 0, \ldots, m \\ \operatorname{Vertex}_{i-m}(\mathcal{P}_2) & i = m, \ldots, m+n \end{cases}$$

the ***composition*** *of $\mathcal{P}_1$ and $\mathcal{P}_2$.*

(vi) *A directed graph Γ is called* ***weakly connected*** *if there is some* ***undirected path*** *between all $v, v' \in V(\Gamma)$, i.e. a finite sequence $(v_0, \ldots, v_n)$ of vertices in $V(\Gamma)$ with $v_0 = v$ and $v_n = v'$ and a finite sequence $(e_1, \ldots, e_n)$ of edges in $E(\Gamma)$ for some $n \in \mathbb{N}_0$ such that $\alpha(e_i) \in \{(v_i, v_{i-1}), (v_{i-1}, v_i)\}$ for all $i = 1, \ldots, n$.*

A directed graph Γ is called ***strongly connected*** *if there is some directed path from v to v' in Γ (i.e. $W(\Gamma, v, v') \neq \emptyset$) for all $v, v' \in V(\Gamma)$.*

(vii) *We define*

$$E_+(\Gamma, v) := \coprod_{v' \in V(\Gamma)} E(\Gamma, v, v') = \{e \in E(\Gamma) : v_{\text{init}}(e) = v\}$$

and

$$E_-(\Gamma, v) := \coprod_{v' \in V(\Gamma)} E(\Gamma, v', v) = \{e \in E(\Gamma) : v_{\text{term}}(e) = v\}$$

for all $v \in V(\Gamma)$. The number $\deg_+(v) := \#E_+(\Gamma, v)$ is called the ***out-degree*** *and the number $\deg_-(v) := \#E_-(\Gamma, v)$ is called the* ***in-degree*** *of $v \in V(\Gamma)$.*

*Moreover, Γ is called d-**regular** if $\deg_+(v) = \deg_-(v) = d$ holds for all $v \in V(\Gamma)$.*

Another notation for paths. In our applications, we will have canonical maps between paths $\mathcal{P}$ in directed graphs and paths $(P_{i,j})_{j-i\leq 1}$ in recursive towers (see Definition 16(i)). Therefore, in Notation 56, we define the non-standard notation $\mathcal{P} = [P_{i,j}]_{j-i\leq 1}$ for a path $\mathcal{P}$ in a directed graph. This notation is closer to the notation $(P_{i,j})_{j-i\leq 1}$ of paths in recursive towers.

Notation 56. *We will use the notation $\mathcal{P} = [P_{i,j}]_{j-i\leq 1}$ for a path $\mathcal{P} \in W(\Gamma)$ where $P_{i,i} := \operatorname{Vertex}_i(\mathcal{P})$ for all $i = 0, \ldots, \operatorname{Length}(\mathcal{P})$ and $P_{i-1,i} := \operatorname{Edge}_i(\mathcal{P})$ for all $i = 1, \ldots, \operatorname{Length}(\mathcal{P})$.*

Lemma 57. *Let Γ be a weakly connected directed graph. Then Γ is strongly connected if and only if, for all edges Q in Γ, there is some path $\mathcal{P}_Q$ from $v_{\text{term}}(Q)$ to $v_{\text{init}}(Q)$.*

Proof. The 'only if'-part follows immediately. For the 'if'-part, suppose that, for all edges Q in Γ, there is some path $\mathcal{P}_Q$ from $v_{\text{term}}(Q)$ to $v_{\text{init}}(Q)$. Then, since Γ is weakly connected, there is an undirected path $\mathcal{P}$ between any two vertices P_0 and P_1 in Γ. Now, we can replace any of the edges Q in $\mathcal{P}$ which go in the wrong direction with the path $\mathcal{P}_Q$ and, consequently, obtain a directed path from P_0 to P_1. □

4.1.2 Weight Functions and Adjacency Matrices

Purpose of this subsection. In this subsection, we will define weights on directed graphs and weighted adjacency matrices. Then adjacency matrices for suitable weights on the tower graphs of recursive towers will play the main role in all major results of this thesis.

Moreover, we will also prove some statements in regards to quadratic matrices which will be useful later.

Definition 58. *Let Γ be a directed graph and let R be a (not necessarily commutative) ring.*

*(i) We call any map $w : E(\Gamma) \to R$ a **weight function** on Γ and define $w(\mathcal{P}) := \prod_{i=1}^{n} w(\mathrm{Edge}_i(\mathcal{P}))$ for all path $\mathcal{P} \in W(\Gamma, n)$ with $n \in \mathbb{N}_0$.*

In particular, we have the identity $w(\mathcal{P}_1 \cdot \mathcal{P}_2) = w(\mathcal{P}_1) \cdot w(\mathcal{P}_2)$ for all composable paths $\mathcal{P}_1$ and $\mathcal{P}_2$.

*(ii) If Γ is a **finite** directed graph (i.e. has finitely many vertices and edges) and $v := (v_1, \ldots, v_m)$ is some enumeration of the vertices of Γ, then we call the matrix*

$$A = \left(\sum_{e \in E(\Gamma, v_i, v_j)} w(e) \right)_{i,j} \in R^{m \times m}$$

*the w**-adjacency matrix of** Γ **for the enumeration** v.*

Powers of adjacency matrices. A standard adjacency matrix A_0 of a finite directed graph Γ is the adjacency matrix of Γ with the constant weight function $w_0 : E(\Gamma) \to \mathbb{Z}$, $e \mapsto 1$ for any enumeration v of the vertices in Γ. Now, it is well known that the (i, j)-th entry of A_0^n is the number of paths of length $n \in \mathbb{N}_0$ from the i-th vertex to the j-th vertex in Γ.

Moreover, in the next Lemma 59, we will prove a generalization of this statement to an arbitrary weight function $w : E(\Gamma) \to R$ where R is a ring. Here, more generally, the entries in A^n turn out to be the weighted sums $\sum_{\mathcal{P} \in W(\Gamma, n, v_i, v_j)} w(\mathcal{P})$ of the paths of length n from v_i to v_j. In our applications, this Lemma 59 will come in handy.

Lemma 59. *Let R be a ring, Γ be a finite directed graph with a weight-function $w : E(\Gamma) \to R$ and A be the w-adjacency matrix of Γ for some enumeration $v = (v_1, \ldots, v_m)$ of the vertices of Γ. For all $n \in \mathbb{N}_0$, we then have the identity*

$$A^n = \left(\sum_{\mathcal{P} \in W(\Gamma, n, v_i, v_j)} w(\mathcal{P}) \right)_{i,j}.$$

Proof. We show this statement by induction over the exponent $n \in \mathbb{N}_0$. Let $n = 0$. On the one hand, by the definition of paths of length 0 being the vertices of Γ in Definition 55(iii), we conclude that $W(\Gamma, 0, v_i, v_j)$ consists of the path $\mathcal{P} = v_i$ if $i = j$ and is empty otherwise. On the other hand, we have the identity $w(v_i) = 1$ by Definition 58. Both together yield the second equality

$$A^0 = (\delta_{i,j})_{i,j} = \left(\sum_{\mathcal{P} \in W(\Gamma, 0, v_i, v_j)} w(\mathcal{P}) \right)_{i,j}$$

where $\delta_{i,j}$ is the Kronecker delta symbol.

Next, let $n \geq 1$ and define $(a_{i,j})_{i,j} := A$ and $(a_{i,j}^{(k)})_{i,j} := A^k$ for all $k \in \mathbb{N}_0$. Then we obtain the desired equality by the equalities

$$\begin{aligned} a_{i,j}^{(n)} &= \sum_{k=1}^{m} a_{i,k}^{(n-1)} \cdot a_{k,j} = \sum_{k=1}^{m} (\sum_{\mathcal{P}' \in W(\Gamma, n-1, v_i, v_k)} w(\mathcal{P}')) \cdot a_{k,j} \\ &= \sum_{k=1}^{m} \sum_{\mathcal{P}' \in W(\Gamma, n-1, v_i, v_k)} \sum_{Q \in E(\Gamma, v_k, v_j)} w(\mathcal{P}') \cdot w(Q) = \sum_{\mathcal{P} \in W(\Gamma, n, v_i, v_j)} w(\mathcal{P}) \end{aligned}$$

for all $i, j \in \{1, \ldots, m\}$ where the first equality holds by the definition of the matrix multiplication, the second equality holds by applying the induction hypothesis to the (i, k)-th entry $a_{i,k}^{(n-1)}$ of A^{n-1} for all $i, k \in \{1, \ldots, m\}$, the third equality holds by the definition of the (k, j)-th entry $a_{k,j}$ of A for all $k, j \in \{1, \ldots, m\}$ in Definition 58 and the last equality holds by the identity $E(\Gamma, v_k, v_j) = W(\Gamma, 1, v_k, v_j)$ in Definition 55(iii) and by the map

$$\coprod_{k=1}^{m} W(\Gamma, n-1, v_i, v_k) \times W(\Gamma, 1, v_k, v_j) \to W(\Gamma, n, v_i, v_j) \text{ via } (\mathcal{P}', \mathcal{P}'') \to \mathcal{P}' \cdot \mathcal{P}''$$

clearly being bijective and satisfying the identity $w(\mathcal{P}') \cdot w(\mathcal{P}'') = w(\mathcal{P}' \cdot \mathcal{P}'')$ for all $(\mathcal{P}', \mathcal{P}'') \in W(\Gamma, n-1, v_i, v_k) \times W(\Gamma, 1, v_k, v_j)$ and $k = 1, \ldots, m$ in Definition 58. ☐

Quadratic complex matrices. For the almost complete answer to Conjecture 1(iii), we will analyze the asymptotic behavior of the sums of the entries in the n-powers A^n of certain quadratic complex matrices A as $n \to \infty$. Thus, in Definition 60, we will formalize the map N which takes the sum of these entries and, in Lemma 61, we will describe the entries in A^n in terms of the n-th powers of the eigenvalues of A.

Definition 60. *We define the morphism*

$$N : \mathbb{C}^{m \times m} \to \mathbb{C} \text{ via } (a_{i,j})_{i,j} \mapsto \sum_{i,j} a_{i,j}$$

of $\mathbb{C}$-vector spaces for all $m \in \mathbb{N}$.

Lemma 61. *Let K be a field, let $A \in K^{m \times m}$, let $\lambda_1, \ldots, \lambda_r \in K$ be the pairwise distinct eigenvalues of A, let m_k be the size of the largest Jordan block of A for the eigenvalue λ_k for all $k = 1, \ldots, r$.*

Then there are elements $c_{i,j,k,l} \in K$ for all $i, j \in \{1, \ldots, m\}$, all $k = 1, \ldots, r$ and all $l = 0, \ldots, m_k - 1$ such that we have the identity

$$A^n = \left(\sum_{k=1}^{r} \sum_{l=0}^{m_k - 1} c_{i,j,k,l} \binom{n}{l} \lambda_k^{n-l} \right)_{i,j} \tag{98}$$

for all $n \in \mathbb{N}_0$ where $\binom{n}{l} := 0$ if $n < l$.

Moreover, for all $n \geq c(A) := \max_{k=1,\ldots,r} m_k - 1$, there are even polynomials $f_{i,j,k} \in K[n]$ with $\deg(f_{i,j,k}) \leq m_k - 1$ for all $i, j \in \{1, \ldots, m\}$ and all $k = 1, ..., r$ such that we have the identity

$$A^n = \left(\sum_{k=1}^{r} f_{i,j,k}(n) \lambda_k^n \right)_{i,j} .$$

In particular, for $K = \mathbb{C}$ and the spectral radius

$$\rho(A) := \max_{k=1,\ldots,r} |\lambda_k|$$

of A, we have the identity

$$N(A^n) = \mathcal{O}((\rho(A) + \varepsilon)^n)$$

for all $\varepsilon > 0$ as $n \to \infty$.

Proof. For the 'main'-part: Let

$$J = \begin{pmatrix} J_{m_{1,1}}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{m_{r,s_r}}(\lambda_r) \end{pmatrix} \tag{99}$$

be the canonical Jordan form of A with the Jordan blocks $J_{m_{k,\nu}}(\lambda_k) \in K^{m_{k,\nu} \times m_{k,\nu}}$ for all $k = 1, ..., r$ and all $\nu = 1, \ldots, s_k$. Moreover, let $T \in \mathrm{GL}(m, K)$ be a transformation matrix such that $A = TJT^{-1}$, i.e. we have the equalities

$$A^n = TJ^nT^{-1} = T \begin{pmatrix} J_{m_{1,1}}(\lambda_1)^n & & 0 \\ & \ddots & \\ 0 & & J_{m_{r,s_r}}(\lambda_r)^n \end{pmatrix} T^{-1}$$

for all $n \in \mathbb{N}_0$. Also, for all $k = 1, ..., r$ and $\nu = 1, \ldots, s_k$, let $I_{m_{k,\nu}}$ be the identity matrix of size $m_{k,\nu}$ and let $N_{m_{k,\nu}}$ is the quadratic matrix of size $m_{k,\nu}$ with ones on the superdiagonal and zeroes else. Then $N^l_{m_{k,\nu}} = 0$ for all $l \geq m_{k,\nu}$ by [HJ90, p.122, Lemma 3.1.4].

Now, we notice that we can write $J_{m_{k,\nu}}(\lambda_k) = \lambda_k I_{m_{k,\nu}} + N_{m_{k,\nu}}$ for all $k = 1, ..., r$ and all $\nu = 1, \ldots, s_r$. Then as the identity matrix $I_{m_{k,\nu}}$ commutes with $N_{m_{k,\nu}}$, we obtain

$$J_{m_{k,\nu}}(\lambda_k)^n = (\lambda_k I_{m_{k,\nu}} + N_{m_{k,\nu}})^n = \sum_{l=0}^{m_{k,\nu}-1} \binom{n}{l} \lambda_k^{n-l} N^l_{m_{k,\nu}} \tag{100}$$

by the Binomial Theorem for commuting matrices for all $n \in \mathbb{N}_0$ where $\binom{n}{l} = 0$ if $n < l$.

Consequently, combining the equalities in (99) and (100) yields the desired presentation of A^n in the 'main'-part.

For the 'moreover'-part: Suppose $n \geq c(A) = \max\{m_k : k = 1, \ldots r\} = \max\{m_{k,\nu} : k = 1, \ldots r \text{ and } \nu = 1, \ldots, s_k\}$. Then the binomial coefficients in (98) do not vanish and thus are even polynomials in n.

Consequntly, on the hand, for $\lambda_k \neq 0$, the sum $\sum_{l=0}^{m_k-1} c_{i,j,k,l} \binom{n}{l} \lambda_k^{-l} \in K[n]$ can be chosen as the desired polynomial $f_{i,j,k}(n)$ in the 'moreover'-part.

On the other hand, for $\lambda_k = 0$, in (100), we see that $J_{m_{k,\nu}}(\lambda_k)^n = N^n_{m_{k,\nu}}$ does not vanish if and only if $m_{k,\nu} - 1 = m_k - 1 = c(A) = n$. Thus, we can choose $f_{i,jk} = c_{i,j,k,c(A)} \in K$ if $m_k - 1 = c(A)$ and $f_{i,jk} = 0$ else.

For the 'in particular'-part: The 'in particular'-part immediately follows from the 'moreover'-part. □

Strongly connected graphs and irreducible matrices. In the following Definition 62 and Lemma 63, we will translate the properties of being weakly and strongly connected for finite directed graphs into the properties of being connected and irreducible, respectively, for the corresponding weighted adjacency matrices.

Definition 62. *Let R be a commutative ring which is not the zero ring. Then we define the following.*

*(i) We call a matrix $A \in R^{m \times m}$ **disconnected** if there is a permutation matrix $P \in \{0,1\}^{m \times m}$ such that PAP^t is a block diagonal matrix with two diagonal blocks. Otherwise we call A **connected**.*

(ii) We extend the definition of irreducible and reducible quadratic complex matrices in [HJ90, p. 360, Definition 6.2.21] by just replacing $\mathbb{C}$ with R: We call a matrix

*$A \in R^{m\times m}$ **reducible** if there is a permutation matrix $P \in \{0,1\}^{m\times m}$ such that PAP^t is an upper block triangular matrix with two diagonal blocks. Otherwise we call A **irreducible**.*

Lemma 63. *Let R be a commutative ring which is not the zero ring, let Γ be a finite directed graph, let $w : E(\Gamma) \to R$ be a weight function on Γ such that $\sum_{Q\in E(\Gamma,v,v')} w(Q) \neq 0$ for all $v, v' \in V(\Gamma)$ with $E(\Gamma, v, v') \neq \emptyset$ and let $A = (\sum_{Q\in E(\Gamma,v_i,v_j)} w(Q))_{i,j} \in R^{m\times m}$ be the w-adjacency matrix of Γ for some enumeration $(v_1, \dots, v_m)$ of the vertices in Γ. Then the following hold.*

(i) Γ is weakly connected if and only if A is connected.

(ii) Γ is strongly connected if and only if A is irreducible.

Proof. For (i): If $P_\sigma \in \{0,1\}^{m\times m}$ is the permutation matrix to the permutation σ on $\{1, \dots, m\}$, then $P_\sigma AP_\sigma^t$ is the w-adjacency matrix of Γ for the enumeration $(v_{\sigma(1)}, \dots, v_{\sigma(m)})$ of the vertices in Γ.

First, by the definition of disconnected matrices in Definition 62, we obtain that A is disconnected if and only if there are a permutation matrix $P_\sigma \in \{0,1\}^{m\times m}$ and matrices $B \in \mathbb{R}^{k\times k}$ and $C \in \mathbb{R}^{m-k\times m-k}$ for some $1 \leq k \leq m-1$ such that

$$P_\sigma AP_\sigma^t = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}. \tag{101}$$

But, because $P_\sigma AP_\sigma^t$ is the w-adjacency matrix of Γ for the enumeration $(v_{\sigma(1)}, \dots, v_{\sigma(m)})$ and because of the assumption $\sum_{Q\in E(\Gamma,v,v')} w(Q) \neq 0$ for all $v, v' \in V(\Gamma)$ with $E(\Gamma, v, v') \neq \emptyset$, this identity in (101) is equivalent to the fact that there is no edge between the vertices in the set $S_1 := \{v_{\sigma(1)}, \dots, v_{\sigma(k)}\} \neq \emptyset$ and the vertices in the set $S_2 := \{v_{\sigma(k+1)}, \dots, v_{\sigma(m)}\} \neq \emptyset$.

On the one hand, the lack of edges between the non-empty sets S_1 and S_2 implies that Γ cannot not be weakly connected. Hence, the ’only if’-part of the desired equivalence in (i) follows.

On the other hand, if Γ is not weakly connected, it is clear that there are two non-empty sets M_1 and M_2 of vertices in Γ such that there is no edge between the vertices in M_1 and the vertices in M_2. Then we can choose $k := \#M_1$ and a suitable permutation σ on $\{1, \dots, m\}$ such that $S_1 = M_1$ and $S_2 = M_2$. Then this yields that A must be disconnected by the equivalences from above. Hence, the other implication of the desired equivalence in (i) follows too.

For (ii): Let $(a_{i,j})_{i,j} := A$ and define $B := (b_{i,j})_{i,j} \in \{0,1\}^{m\times m} \subset \mathbb{C}^{m\times m}$ via $b_{i,j} := 0$ if $a_{i,j} = 0$ and $b_{i,j} := 1$ else. Since the multiplication of a matrix with a permutation matrix from left or right only reorders its entries, it is clear that the irreducibility of a matrix only depends on the positions of its non-zero entries and not on the concrete non-zero entries. Therefore, we conclude that

$$A \text{ is irreducible if and only if } B \text{ is irreducible.} \tag{102}$$

Then, by the assumption that $\sum_{Q\in E(\Gamma,v,v')} w(Q) \neq 0$ for all $v, v' \in V(\Gamma)$ with non-empty $E(\Gamma, v, v')$, we notice that up to removing multiple edges and renaming the vertices and edges in Γ, the directed graph $\Gamma(B)$ in [HJ90, p. 357, Definition 6.2.11] is equal to Γ and, thus, $\Gamma(B)$ is strongly connected if and only if Γ is strongly connected. Consequently, the equivalence of the items (a) and (d) in [HJ90, p. 362, Theorem 6.2.24] provides that

$$B \text{ is irreducible if and only if } \Gamma \text{ is strongly connected.} \tag{103}$$

Finally, combining the two equivalences in (102) and (103) yields the desired equivalence in (ii). □

Quadratic matrices with nonnegative real entries. The following Lemma 64(ii) will be useful in Section 7.1.

Lemma 64. *Let $A \in \mathbb{R}_{\geq 0}^{m \times m}$ be irreducible.*

(i) For all non-zero $B \in \mathbb{R}_{\geq 0}^{m \times m}$, the sum $A + B$ is also irreducible and satisfies the estimate $\rho(A) < \rho(A + B)$.

(ii) If all row (resp. column) sums in A are at most $d \in \mathbb{R}_{>0}$, but at least one of the row (resp. column) sums is less than d, then we have the estimate $\rho(A) < d$.

Proof. For (i): Let $(a_{i,j})_{i,j} := A$ and $(b_{i,j})_{i,j} := B$. On the one hand, as A and B only have non-negative real entries, we get the estimates $0 \leq a_{i,j} \leq a_{i,j} + b_{i,j}$ for all $i, j \in \{1, \ldots, m\}$. Then the irreducibility of $A + B$ already follows from this, from the irreducibility of A and from the definition of irreducible matrices in Definition 62(ii).

On the other hand, by [HJ90, p. 491, Theorem 8.1.18], we obtain the estimate $\rho(A) \leq \rho(A + B)$. Now, assume that we have the equality $\rho(A) = \rho(A + B)$. Then, as suggested in [HJ90, p. 515, Exercise 15], we notice that [HJ90, p. 509, Theorem 8.4.5] supplies some real numbers $\varphi, \theta_1, \ldots, \theta_m$ such that the equalities

$$(a_{k,l})_{k,l} = A = (e^{i(\varphi + \theta_k - \theta_l)} \cdot (a_{k,l} + b_{k,l}))_{k,l} \tag{104}$$

hold. Consequently, if we take the entrywise absolute value of the equality in (104), we derive the equalities $a_{k,l} = a_{k,l} + b_{k,l}$ or all $k, l \in \{1, \ldots, m\}$ from the equality $|e^{i(\varphi + \theta_k - \theta_l)}| = 1$ and from $a_{k,l}, b_{k,l} \in \mathbb{R}_{\geq 0}$. But these equalities yield $B = 0$ which is impossible because B is non-zero by assumption. Hence, we conclude the desired estimate $\rho(A) < \rho(A + B)$ in (i).

For (ii): Suppose that all row (resp. column) sums in A are at most d but at least one of the row (resp. column) sums is less than d. Then there is some non-zero matrix $B \in \mathbb{R}_{\geq 0}^{m \times m}$ such that all rows (resp. column) sums in $A + B$ are equal to d. Consequently, we obtain the desired estimate in (ii) by the estimate and equality

$$\rho(A) < \rho(A + B) = d$$

where the estimate holds by Lemma 64(i) and the equality holds by [HJ90, p.492, Theorem 8.1.22]. □

4.1.3 Morphisms

Purpose of this subsection. In the literature, there is no standard definition of a morphism of directed graphs. Often, it is a map ϕ between the sets of vertices such that if two vertices v_1 and v_2 in the domain are adjacent via an edge $v_1 \to v_2$, then the images $\phi(v_1)$ and $\phi(v_2)$ are also adjacant via an edge $\phi(v_1) \to \phi(v_2)$.

However, our definition is closer to the definition of a morphism of undirected graphs in [Sun13, p. 24] which additionally requires a map ϕ_E between the sets of edges and is similar to the definition of a functor in category theory. In our applications, we will always have natural maps ϕ_V between the sets of vertices and ϕ_E between the sets of edges and, thus, we will define a morphism ϕ of directed paths as the pair (ϕ_V, ϕ_E) of both these maps.

Definition 65. *Let Γ and Γ' be directed graphs. Then a pair $\phi = (\phi_V, \phi_E)$ of two maps $\phi_V : V(\Gamma) \to V(\Gamma')$ and $\phi_E : E(\Gamma) \to E(\Gamma')$ is called a* ***covariant (resp. contravariant) morphism of directed graphs*** *and we write $\phi : \Gamma \to \Gamma'$ if $\phi_V(v_1) \xrightarrow{\phi_E(e)} \phi_V(v_2)$ (resp. $\phi_V(v_1) \xleftarrow{\phi_E(e)} \phi_V(v_2)$) holds for all $v_1 \xrightarrow{e} v_2$. If we just write that ϕ is morphism of directed graphs, then we mean that ϕ is covariant.*

Moreover, ϕ extends to the map

$$\phi_W : W(\Gamma') \to W(\Gamma) \text{ via } [P_{i,j}]_{j-i\leq 1} \mapsto [\phi(P_{i,j})]_{j-i\leq 1}$$

and we will often just write $\phi(v) := \phi_V(v)$ for all $v \in V(\Gamma)$, $\phi(e) := \phi_E(e)$ for all $e \in E(\Gamma)$ and $\phi(\mathcal{P}) := \phi_W(\mathcal{P})$ for all $\mathcal{P} \in W(\Gamma)$.

Analogously to ϕ_W, we can also extend ϕ to a map from the set of undirected paths in Γ' to the set of undirected paths in Γ.

*Finally, we call ϕ a **monomorphism (resp. epimorphism; resp. isomorphism)** if ϕ_V and ϕ_E are both injective (resp. surjective; resp. bijective).*

4.1.4 Subgraphs

Purpose of this subsection. In this subsection, we will give several definitions with regards to subgraphs of directed graphs. Some of these definitions will be non-standard, e.g. forward and backward complete subgraphs. Moreover, we will also prove some simple properties.

Subgraphs of directed graphs. In Definition 66, we want to introduce some notions with respect to subgraphs which we will use frequently. Most of the notions are standard. Only the definition of forward (resp. backward) complete subgraphs in (iii) is non-standard. It is motivated by the definition of forward (reps. backward) complete subsets of places in [Bee04, p. 238, Theorem 5.5] and in [HP16, p. 4, Definition 5]. Moreover, although the chosen definition of weakly connected components in (v) is non-standard, in Lemma 68(i), we will prove that it is equivalent to one of the standard definitions.

Definition 66. *Let Γ and Γ' be directed graphs with edge maps α and α', respectively. Then we define the following:*

*(i) We call Γ' a **subgraph** of Γ if the inclusions $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$ and the identity $\alpha(e') = \alpha'(e')$ hold for all $e' \in E(\Gamma')$. Moreover, we call Γ' a **proper** subgraph of Γ if $\Gamma' \neq \Gamma$.*

Notice that if subsets $V \subseteq V(\Gamma)$ and $E \subseteq E(\Gamma)$ satisfy $v_{\text{init}}(e), v_{\text{term}}(e) \in V$ for all $e \in E$, then there is a unique subgraph Γ' of Γ with $V(\Gamma') = V$ and $E(\Gamma') = E$.

(ii) Let M be a set of subgraphs of some directed graph Γ_0 with edge map $\alpha_0 : E(\Gamma_0) \to V(\Gamma_0)^2$. Then, for all $\Phi \in \{\bigcap, \bigcup\}$, the map α_0 restricts to a map

$$\alpha_\Phi : \underset{\Gamma\in M}{\Phi} E(\Gamma) \to \left(\underset{\Gamma\in M}{\Phi} V(\Gamma)\right)^2$$

and, thus, the directed graph

$$\underset{\Gamma\in M}{\Phi} \Gamma := \left(\underset{\Gamma\in M}{\Phi} V(\Gamma), \underset{\Gamma\in M}{\Phi} E(\Gamma), \alpha_\Phi\right)$$

*is a subgraph Γ_0. Moreover, $\bigcap_{\Gamma\in M} \Gamma$ is even a subgraph of any subgraph in M and all subgraphs in M are subgraphs of $\bigcup_{\Gamma\in M} \Gamma$. We call $\bigcap_{\Gamma\in M} \Gamma$ (resp. $\bigcup_{\Gamma\in M} \Gamma$) the **intersection (resp. union) subgraph** of the subgraphs in M.*

*Subgraphs Γ and Γ' of Γ_0 are called **disjoint** if their intersection subgraph is the **empty graph**, i.e. has empty vertex and edge sets.*

Finally, for the disjoint union, we will use the symbol $\coprod$ and, for the intersection, union and disjoint unions of finitely many subgraphs, we will also use the usual symbols $\cap, \cup, \sqcup$.

(iii) *A subgraph* Γ' *of* Γ *is called* ***forward (resp. backward) complete*** *at the vertex* $v \in V(\Gamma')$ *if*

$$E_+(\Gamma, v) = E_+(\Gamma', v) \text{ (resp. } E_-(\Gamma, v) = E_-(\Gamma', v)\text{).}$$

Moreover, a subgraph Γ' *of* Γ *is called* ***forward (resp. backward) complete*** *if* Γ' *is forward (resp. backward) complete at all* $v \in V(\Gamma')$.

Notice that any forward (resp. backward) complete subgraph Γ' *of* Γ *satisfies the equality* $E(\Gamma, v', v) = \emptyset$ *(resp.* $E(\Gamma, v, v') = \emptyset$*) for all* $v' \in V(\Gamma')$ *and* $v \in V(\Gamma)\backslash V(\Gamma')$.

Also, notice that intersection and union subgraphs of forward (resp. backward) complete subgraphs are again forward (resp. backward) complete.

(iv) *Let* Γ' *be a forward and backward complete subgraph of* Γ. *Then* α *restricts to a map*

$$\alpha'' : E(\Gamma)\backslash E(\Gamma') \to (V(\Gamma)\backslash V(\Gamma'))^2$$

by the second to last comment in Definition 66(iii) and therefore the triple

$$\Gamma\backslash\Gamma' := (V(\Gamma)\backslash V(\Gamma'), E(\Gamma)\backslash E(\Gamma'), \alpha'')$$

is directed graph too. We call $\Gamma\backslash\Gamma'$ *the* ***complementary*** *or* ***difference subgraph*** *of* Γ' *in* Γ.

Notice that $\Gamma\backslash\Gamma'$ *is also a forward and backward complete subgraph of* Γ *by the second to last comment in Definition 66(iii).*

(v) *Any non-empty weakly connected forward and backward complete subgraph* Γ' *of* Γ *is called a* ***weakly connected component*** *of* Γ.

(vi) *Any non-empty subgraph* Γ' *of* Γ *is called a* ***strongly connected component*** *of* Γ *if* Γ' *is a maximal strongly connected subgraph of* Γ*, i.e.* Γ' *is not a proper subgraph of another strongly connected subgraph* Γ'' *of* Γ.

Remark 67. *The definition of forward (resp. backward) complete subgraphs in Definition 66(iii) is unrelated to the often used notion of complete undirected graphs which means that all distinct vertices in a graph are adjacent.*

Moreover, we remark that, in [BGS04, p. 10], forward and backward complete subgraphs are called components and that [HP12, p. 15, Definition 10(ii)] defines a weakly connected component to be a maximal weakly connected subgraph which is equivalent to our definition in Definition 66(v) by Lemma 68(i).

Forward and backward complete subgraphs and weakly connected components. As already announced in Lemma 68(i), we will show that our non-standard definition of weakly connected components is equivalent to one of the standard definitions.

Moreover, in Lemma 68(iii), we will also prove that any forward and backward complete subgraph is a union of weakly connected components.

Lemma 68. *Let* Γ' *and* Γ'' *be subgraphs of the directed graph* Γ. *Then the following hold:*

(i) *Suppose that* Γ' *is non-empty and weakly connected. Then* Γ' *is a weakly connected component of* Γ *if and only if* Γ' *is a maximal weakly connected subgraph of* Γ.

(ii) *If* Γ' *and* Γ'' *are weakly connected components of* Γ*, then* Γ' *and* Γ'' *are either equal or disjoint.*

(iii) Γ' *is a forward and backward complete subgraph of* Γ *if and only if* Γ' *is a disjoint union of all the weakly connected components of* Γ *which are also subgraphs of* Γ'.

Proof. For (i): We first notice that Γ' is not a maximal weakly connected subgraph of Γ if and only if there is an edge $e \in E(\Gamma)\backslash E(\Gamma')$ with $v_{\text{init}}(e) \in V(\Gamma')$ or $v_{\text{term}}(e) \in V(\Gamma')$. This again is equivalent to the proper inclusion $E_+(\Gamma', v_{\text{init}}(e)) \subset E_+(\Gamma, v_{\text{init}}(e))$ or $E_-(\Gamma', v_{\text{term}}(e)) \subset E_-(\Gamma, v_{\text{term}}(e))$, respectively. But the validity of at least one of these inclusions for some $e \in E(\Gamma)$ with $v_{\text{init}}(e) \in V(\Gamma')$ or $v_{\text{term}}(e) \in V(\Gamma')$, respectively, is exactly the definition of Γ' not being a forward and backward complete subgraph of Γ' in Definition 66(iii). Hence, the equivalence in (i) follows.

For (ii): Assume that Γ' and Γ'' are not disjoint, i.e. there is a vertex $v_0 \in V(\Gamma')\cap V(\Gamma'')$ by Definition 66(ii).

Now, the definition of forward complete subgraphs in Definition 66(iii) provides the equalities $E_+(\Gamma', v) = E_+(\Gamma, v) = E_+(\Gamma'', v)$ for all $v \in V(\Gamma')\cap V(\Gamma'')$. By these equalities, we obtain the equalities $E(\Gamma') = \coprod_{v\in V(\Gamma')} E_+(\Gamma, v)$ and $E(\Gamma'') = \coprod_{v\in V(\Gamma'')} E_+(\Gamma, v)$ and, thus, it is enough to show the equality

$$V(\Gamma') = V(\Gamma''). \tag{105}$$

For the inclusion from left to right, let $v \in V(\Gamma')$. By the definition of the weakly connectedness of Γ' in Definition 55(vi), there is an undirected path $\mathcal{P}$ between v_0 and v in Γ'. Since Γ' is a subgraph of Γ, the undirected path $\mathcal{P}$ is especially contained in Γ. Thus, starting at $v_0 \in V(\Gamma'')$, we can iteratively apply the definition of the fact that Γ'' is a forward and backward complete subgraph of Γ to the vertices of $\mathcal{P}$. Finally, this supplies $v \in V(\Gamma'')$ and, hence, the inclusion from left to right in (105) follows.

For the other inclusion, we just notice that the roles of Γ' and Γ'' can be swapped. Hence, we obtain the desired equality in (105). and (ii) follows.

For the 'if'-part in (iii): By the definition of a weakly connected component of Γ in Definition 66(iii), it is forward and backward complete subgraph of Γ. As the union of forward and backward complete subgraphs is again a forward and backward complete subgraph by the last comment in Definition 66(iii), the 'if'-part in (iii) follows.

For the 'only if'-part in (iii): On the one hand, for all $v \in V(\Gamma')$, the vertex set $\{v\} \subseteq V(\Gamma')$ and edge set $\emptyset \subseteq E(\Gamma')$ induce a weakly connected subgraph of Γ' which contains v as a vertex. On the other hand, the union of any number (possibly infinite) of weakly connected subgraphs of Γ' containing v as a vertex is again a weakly connected subgraph of Γ' containing v as a vertex. Therefore, combining these two facts and Zorn's Lemma supplies the existence of a unique maximal weakly connected subgraph Γ_v of Γ' containing v as a vertex.

Consequently, Lemma 68(i) supplies that Γ_v is the unique weakly connected component of Γ' which contains v and, hence, $M := \{\Gamma_v : v \in V(\Gamma')\}$ is the set of all weakly connected components of Γ'. As Lemma 68(ii) also provides that two distinct weakly connected components of Γ' are disjoint, we obtain that $\Gamma' = \coprod_{G\in M} G$ is the disjoint union of its weakly connected components.

Next, the definition of weakly connected components in Definition 66(v) imply that all $G \in M$ are also forward and backward complete subgraphs of Γ'. But, the property of being a forward (resp. backward) complete subgraph is clearly transitive by its definition in Definition 66(iii), i.e. if G satisfies this property as a subgraph of Γ' and Γ' satisfies this property as a subgraph of Γ, then G also satisfies this property as a subgraph of Γ. Thus, all $G \in M$ are also non-empty pairwise disjoint weakly connected forward and backward complete subgraphs of Γ. This again means that all $G \in M$ are even pairwise disjoint weakly connected components of Γ and they are also subgraphs of Γ'.

Finally, because any maximal weakly connected subgraph of Γ which is also a subgraph of Γ' must especially be a maximal weakly connected subgraph of Γ', Lemma 68(i) implies that M is even the set of all weakly connected components of Γ which are also subgraphs of Γ'.

Hence, we established the desired statement, namely that $\Gamma' = \coprod_{G \in M} G$ is a disjoint union of all weakly connected components which are also subgraphs of Γ'. $\square$

4.1.5 Image and Preimage Graphs

Purpose of this subsection. In the following Definition/Lemma 69, we will define image and preimage graphs for morphism of directed graphs. Then, in Lemma 70, we will explore which properties of subgraphs of directed graphs are stable under morphisms of directed graphs, i.e. which properties still hold for the image or preimage graphs of these subgraphs.

Image and preimage graphs.

Definition/Lemma 69. *Let $\phi : \Gamma_0' \to \Gamma_0$ be a morphism of directed graphs, let Γ' be a subgraph of Γ_0' and let Γ be a subgraph of Γ_0.*

(i) The vertex and edge sets

$$V(\phi(\Gamma')) := \phi(V(\Gamma')) \text{ and } E(\phi(\Gamma')) := \phi(E(\Gamma'))$$

*define a subgraph $\phi(\Gamma')$ of Γ_0 and we call $\phi(\Gamma')$ the **ϕ-image graph** of Γ'.*

(ii) The vertex and edge sets

$$V(\phi^{-1}(\Gamma)) := \phi^{-1}(V(\Gamma)) \text{ and } E(\phi^{-1}(\Gamma)) := \phi^{-1}(E(\Gamma))$$

*define a subgraph $\phi^{-1}(\Gamma)$ of Γ_0'. We call $\phi^{-1}(\Gamma_0')$ the **ϕ-preimage graph** of Γ_0'.*

Proof. Let us write v'_{init} (resp. v'_{term}) for the initial (resp. terminal) vertex map on Γ_0' and v_{init} (resp. v_{term}) for the initial (resp. terminal) vertex map on Γ_0.

For (i): Let $e \in E(\Gamma')$. Then $v'_{\text{init}}(e)$ and $v'_{\text{term}}(e)$ are vertices in Γ' because Γ' is a subgraph of Γ_0'. Combining this and the definition of morphisms of directed graphs in Definition 65 yields that $v_{\text{init}}(\phi(e)) = \phi(v'_{\text{init}}(e))$ and $v_{\text{term}}(\phi(e)) = \phi(v'_{\text{term}}(e))$ are elements in $\phi(V(\Gamma')) = V(\phi(\Gamma'))$. Hence, $\phi(\Gamma')$ is indeed a subgraph of Γ_0 by the 'notice'-part in Definition 66(i).

For (ii): Let $e \in \phi^{-1}(E(\Gamma))$, i.e. $e \in E(\Gamma_0')$ with $\phi(e) \in E(\Gamma)$. Then combining the definition of morphisms of directed graphs in Definition 65 and the assumption that Γ is a subgraph of Γ_0 yields that $\phi(v'_{\text{init}}(e)) = v_{\text{init}}(\phi(e))$ and $\phi(v'_{\text{term}}(e)) = v_{\text{term}}(\phi(e))$ are elements in $V(\Gamma)$. This again implies that $v'_{\text{init}}(e)$ and $v'_{\text{term}}(e)$ are elements in $\phi^{-1}(V(\Gamma)) = V(\phi^{-1}(\Gamma))$. Hence, $\phi^{-1}(\Gamma)$ is indeed a subgraph of Γ_0' by the 'notice'-part in Definition 66(i). $\square$

Properties stable under image and preimage graphs.

Lemma 70. *Let $\phi : \Gamma_0' \to \Gamma_0$ be a morphism of directed graphs, let Γ' be a subgraph of Γ_0' and let Γ be a subgraph of Γ_0. Then the following hold:*

(i) If G' (resp. G) is also a subgraph of Γ'_0 (resp. Γ_0), then we have the identity $\phi(\Gamma' \cup G') = \phi(\Gamma') \cup \phi(G')$ (resp. $\phi^{-1}(\Gamma' \cup G') = \phi^{-1}(\Gamma') \cup \phi^{-1}(G')$).

(ii) Γ' is a subgraph of $\phi^{-1}(\phi(\Gamma'))$. Moreover, if ϕ is a monomorphism, then we even have the identity $\Gamma' = \phi^{-1}(\phi(\Gamma'))$.

(iii) $\phi(\phi^{-1}(\Gamma))$ is a subgraph of Γ. Moreover, if ϕ is an epimorphism, then we even have the identity $\Gamma' = \phi(\phi^{-1}(\Gamma'))$.

(iv) If Γ is a forward (resp. backward) complete subgraph of Γ_0, then $\phi^{-1}(\Gamma)$ is also a forward (resp. backward) complete subgraph of Γ'_0.

Moreover, if ϕ is an epimorphism, then this is even an equivalence.

(v) If Γ' is weakly (resp. strongly) connected, then $\phi(\Gamma')$ is also weakly (resp. strongly) connected.

Proof. For (i), (ii) and (iii): All desired statements immediately follow from the definitions of the image and preimage subgraphs in Definition 69.

For the 'main'-part in (iv): Suppose that Γ is a forward (resp. backward) complete subgraph of Γ_0 and let us write $E_0 := E_+$ (resp. $E_1 := E_-$), v'_0 (resp. v'_1) for the initial (resp. terminal) vertex map on Γ'_0 and v_0 (resp. v_1) for the initial (resp. terminal) vertex map on Γ_0.

Now, let $i := 0$ (resp. $i := 1$) and $e \in E(\Gamma'_0)$ with $v'_i(e) \in V(\phi^{-1}(\Gamma))$. Then we obtain

$$\phi(e) \in E_i(\Gamma_0, v_i(\phi(e))) = E_i(\Gamma_0, \phi(v'_i(e))) = E_i(\Gamma, \phi(v'_i(e))) \subseteq E(\Gamma) \tag{106}$$

where the containment-statement and inclusion hold by the definition of E_i in Definition 55(vii), the first equality holds since we have the equality $v_i(\phi(e)) = \phi(v'_i(e))$ by the definition of morphism of directed graphs in Definition 65 and the second equality holds because of $v'_i(e) \in V(\phi^{-1}(\Gamma)) = \phi^{-1}(V(\Gamma))$, because Γ is a forward (resp. backward) complete subgraph of Γ_0 and because of its definition in Definition 66(iii).

Consequently, (106) yields that e is an edge in $\phi^{-1}(E(\Gamma)) = E(\phi^{-1}(\Gamma))$ and, hence, this again yields that $\phi^{-1}(\Gamma)$ is a forward (resp. backward) complete subgraph of Γ'_0.

For the 'moreover'-part in (iv): Suppose that ϕ is an epimorphism and that $\phi^{-1}(\Gamma)$ is a forward (resp. backward) complete subgraph of Γ_0 and let us use the same notation as in the proof of the main'-part in (iv).

Now, let $i = 0$ (resp. $i := 1$) and let $e \in E(\Gamma_0)$ with $v_i(e) \in V(\Gamma)$. As ϕ is an epimorphism, there is some $e' \in E(\Gamma'_0)$ such that $\phi(e') = e$. In particular, we then have the equalities $\phi(v'_i(e')) = v_i(\phi(e')) = v_i(e)$ by the definition of morphisms of directed graphs in Definition 65. Therefore, we deduce $v'_i(e') \in \phi^{-1}(V(\Gamma)) = V(\phi^{-1}(\Gamma))$. Moreover, because $\phi^{-1}(\Gamma)$ be a forward (resp. backward) complete subgraph of Γ_0, we conclude $e' \in E_i(\Gamma'_0, v'_i(e')) = E_i(\phi^{-1}(\Gamma), v'_i(e'))$ by its definition in Definition 66(iii). But this again implies $e = \phi(e') \in E(\Gamma)$ and, hence, Γ is also a forward (resp. backward) complete subgraph of Γ_0.

For (v): Let v_0 and v_1 be vertices in $V(\phi(\Gamma')) = \phi(V(\Gamma'))$. Moreover, there are some $v'_i \in V(\Gamma')$. with $\phi(v'_i) = v_i$ for all $i = 0, 1$. Since Γ' is weakly (resp. strongly) connected, its definition in Definition 55(vi) supplies some undirected (resp. directed) path $\mathcal{P}'$ from v'_0 to v'_1 in Γ' by Definition 55(vi). Then $\phi(\mathcal{P}')$ is an undirected (resp. directed) path from $\phi(v'_0) = v_0$ to $\phi(v'_1) = v_1$ by the definition of the extension of ϕ on undirected (resp. directed) paths in Definition 65. This again yields that $\phi(\Gamma')$ is also weakly (resp. strongly) connected. Hence, (v) follows. □

Example 71. *In Lemma 70(iv), it is shown that the preimage graph of a forward (resp. backward) complete subgraph is again a forward (resp. backward) complete subgraph. For the image graph, this does not hold in general:*

For instance, consider a directed graph Γ *with three vertices* v_1, v_2, v_3 *and two edges* $e_{1,2}, e_{3,2}$ *from* v_1 *to* v_2 *and from* v_3 *to* v_2*, respectively. Moreover, let* Γ' *be a directed graph which only consists of one vertex* v *and the morphism* $\phi : \Gamma' \to \Gamma$ *of directed graphs which just map* v *to* v_2*. Although* Γ' *is clearly a forward and backward complete subgraph of itself, its image graph* $\phi(\Gamma')$ *which only consists of the vertex* v_2 *is neither a forward or backward complete subgraph of* Γ*.*

Furthermore, if we add a copy of Γ *to* Γ' *and extend* ϕ *via the identity on the copy of* Γ *in* Γ'*, then* ϕ *even becomes an epimorphism. But the image graph of the weakly connected component of* Γ' *which only consists of* v *still only consists of the vertex* v_2 *and, thus, is neither a forward or a backward complete subgraph of* Γ*.*

4.1.6 Pyramidal Graph

Purpose of this subsection. In this subsection, we will attribute a graph structure to the pyramids of places and interpret the ramification indices and relative degrees as weights on the edges.

Pyramidal graph. As it is obvious from Figure 2.6, we can attribute a directed graph to the pyramid of a place in a recursive tower. Here, the places in the pyramid are the vertices and the elementary extensions are the edges of the directed graph.

Definition 72. *Let* $\mathcal{F}$ *be a recursive function field, let* $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ *be the pyramid of* $\mathcal{F}$*, let* $Q \in \mathbb{P}_{F_{m,n}}$ *for some* $m \leq n$ *and let* $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$*. Let* $V := \{P_{i,j} : m \leq i \leq j \leq n\}$*, let* $E := \mathrm{ElemExt}(\mathrm{Pyr}(Q)) \subseteq V^2$ *and let* $\alpha : E \to V^2$ *be the inclusion map* $(R, P) \mapsto (R, P)$*.*

- *(i) Then we call the directed graph* $\mathrm{Graph}(\mathrm{Pyr}(Q))$ *the* ***pyramidal graph*** *of* $\mathrm{Pyr}(Q)$*.*
- *(ii) The maps* e *for the ramification indices and* f *for the relative degrees restrict to weight functions* $e_{|E}, f_{|E} : E := \mathrm{ElemExt}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *on* $\mathrm{Graph}(\mathrm{Pyr}(Q))$*. Let us write* e' *and* f'*, respectively, for the canonical extensions of these weight functions on all paths in* $\mathrm{Graph}(\mathrm{Pyr}(Q))$*. Then, for all* $\alpha \in \{e, \tilde{e}, f\}$ *and paths* $\mathcal{P}$ *in* $\mathrm{Graph}(\mathrm{Pyr}(Q))$*, we obtain the identity* $\alpha'(\mathcal{P}) = a(v_{\mathrm{init}}(\mathcal{P})|v_{\mathrm{term}}(\mathcal{P}))$ *because* $\alpha'(\mathcal{P})$ *is defined as the product of the* $\alpha_{|E}$*-values of the edges in* $\mathcal{P}$ *by Definition 58 and because* α *satisfies the multiplicative transitivity rule in (7) or Lemma 44(ii).*

  Therefore, we can interpret the restrictions of e*,* $\tilde{e}$ *and* f *to maps* $\mathrm{Ext}(\mathrm{Pyr}(Q)) \to \mathbb{N}$ *as maps on the paths in* $\mathrm{Graph}(\mathrm{Pyr}(Q))$*.*

Example 73. *In figures 2.7 and 2.8, the two pyramidal graphs* $\mathrm{Graph}(\mathrm{Pyr}(Q))$ *and* $\mathrm{Graph}(\mathrm{Pyr}(Q'))$ *of the places* Q *and* Q' *in Example 12 are already depicted up to the directions of the edges. In particular, as explained in Definition 72(ii) the blue numbers which are the ramification indices of the elementary extensions in the pyramids can be interpreted as weights on the pyramidal graphs.*

4.2 Tower Graphs

The main tool of this thesis: The tower graph. In this subsection, we will define the main tool of this thesis which is the tower graph $\Gamma_{\mathcal{F}}$ of a recursive tower $\mathcal{F} = (F_\nu)_\nu$. It will capture information about all extensions of places in F_1/F_0 and $F_1/\sigma(F_0)$. More specifically, we will also define the rational, splitting and ramification subgraphs of $\Gamma_{\mathcal{F}}$. From these crucial subgraphs, we will derive information for the desired values $N(F_n)$ and $g(F_n)$ for all $n \in \mathbb{N}$.

Structure of this section. First, we will define the tower graph and the tower graph map and then, in Figure 4.1, we will add figures of finite subgraphs of the corresponding tower graphs to our list of examples in Figure 2.3.

In Subsection 4.2.1, we will attribute the properties of being rational, being tame and having balanced ramification indices to paths in the tower graph. Here, the last property will yield the decisive criterion for the almost complete answer to Conjecture 1(iii) in Corollary 184.

In Subsection 4.2.2, we will define the sets of places in the tower which lie over subgraphs of the tower graph.

In Subsection 4.2.3, we will define the rational, splitting and ramification subgraphs of a tower graph and prove some first properties of these subgraphs.

Tower graphs and Beelen-graphs. In the following Definition 74, we will define the tower graphs $\Gamma_{\mathcal{F}}$ of recursive tower $\mathcal{F}$. The tower graph is basically a slight modification of the Beelen-graph in [BGS04, p. 10, Definition 4.2]. For polynomial-recursive towers and over algebraically closed constant fields, these two graphs are canonically isomorphic (see Proposition 95).

However, there are two reasons for why we will work with towers graphs instead of the Beelen-graphs: First, our definition of pair-recursive towers is not compatible with the Beelen-graph. Second, the Beelen-graph only includes the rational places of F_0 and F_1 as the vertices and edges, respectively. Thus, if we want to have the information of all extensions of places in F_1/F_0 and $F_1/\sigma(F_0)$, we need to switch to the geometric tower $\overline{\mathcal{F}}$ of $\mathcal{F}$. But by doing so, this inflates the graph since every single extension of places of higher degrees then yields many copies of the same extension but with rational places.

Definition 74. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0). Then we define the map*

$$\alpha : \mathbb{P}_{F_1} \to \mathbb{P}_{F_0} \times \mathbb{P}_{F_0} \text{ via } Q \mapsto (Q \cap F_0, \sigma^{-1}(Q) \cap F_0).$$

*and we call the directed graph $\Gamma_{\mathcal{F}} := (\mathbb{P}_{F_0}, \mathbb{P}_{F_1}, \alpha)$ the **tower graph** of $\mathcal{F}$.*

Examples 75. *For all recursive towers $\mathcal{F} = (F_\nu)_\nu$, we call the subgraph Γ_c of $\Gamma_{\mathcal{F}}$ which only consists of the edges Q with $\deg(Q) \leq c$ and all their initial and terminal vertices P the **degree c subgraph** of $\Gamma_{\mathcal{F}}$.*

In Figure 4.1, to our list of examples of recursive towers $\mathcal{F} = (F_\nu)_\nu$, we added references to figures displaying degree c subgraphs Γ_c of the corresponding tower graphs in Figure 2.3. All the data for these figures of subgraphs were automatically computed with Magma [BCP97] and are collected in Chapter B. Each of these figures also displays a table with the following data:

First, the defining polynomial f of the function field F_1. Second, the balanced degree d of $\mathcal{F}$. Third, the upper bound $c \in \{1, 2\}$ for the degrees of the edges in the displayed degree c subgraph Γ_c. Fourth, abbreviations Dr_Is and F_0-generators of the vertices in Γ_c where r denotes the degree of the corresponding place and s the index for some automatically generated enumeration of the places of degree r. Finally, the variables $z_1 = x$ and $z_2 = y$ just repeat the chosen generators of F_1.

Moreover, the weights which are displayed on all edges $P_0 \xrightarrow{Q} P_1$ are the values $(e(Q|P_0), f(Q|P_0)), (e(Q|\sigma(P_1)), f(Q|\sigma(P_1)))$.

Finally, we also attached comprehensive texts with information to the considered recursive towers. Thus, in the following, if we want to give examples, we will often just refer to the subgraph in the corresponding figure in Chapter B. Further information on the considered recursive tower can then be extracted from the attached text.

Reference	q	$f \in \mathbb{F}_q[X,Y]$	Figure	$\lambda(\mathcal{F})$ $(\geq b)$
[MW05, p. 212]	9	$f_{MW,2} = Y^2 + XY + 2X^2 + 1$	B.1	2/3
[MW05, p. 212]	3	$f_{MW,2} = Y^2 + XY + 2X^2 + 1$	B.2	0
[MW05, p. 214]	49	$f_{MW,3} = Y^2 + X^2Y + 5X^2 + 5$	B.3	6 (≥ 4)
[MW05, p. 212]	9	$f_{MW,4} = Y^2 + X^2Y + 1$	B.4	2
[MW05, p. 212]	9	$f_{MW,6} = Y^2 + (X^2+1)Y + 2X^2$	B.5	2/3
[MW05, p. 213]	25	$f_{MW,8} = Y^2 + (X^2+3)Y + 4X^2$	B.6	1
[MW05, p. 212]	9	$f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$	B.7	2
[MW05, p. 212]	3	$f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$	B.8	0
[MW05, p. 213]	25	$f_{MW,12} = X^2Y^2 + (X^2+3X+3)Y + 4$	B.9	4 (≥ 3)
[MW05, p. 213]	25	$f_{MW,14} = X^2Y^2 + (X^2+4X+2)Y + 4X^2 + 2$	B.10	4 (≥ 3)
[MW05, p. 213]	25	$f_{MW,15} = X^2Y^2 + (X^2+4X+4)Y + 4X^2 + 3X + 2$	B.11	4 (≥ 3)
[MW05, p. 213]	25	$f_{MW,16} = (X^2+1)Y^2 + (X+1)Y + 2X^2 + 4X + 1$	B.12	1
[MW05, p. 213]	25	$f_{MW,20} = Y^2 + 2XY + 4X^2 + 1$	B.13	1
[MW05, p. 213]	25	$f_{MW,21} = Y^2 + 2XY + 4X^2 + 2$	B.14	1
[MW05, p. 213]	25	$f_{MW,22} = Y^2 + 4XY + X^2 + X$	B.15	4
[Sti08, p. 260]	l^2	$f_{GS,1} = Y^{l-1} + (X+1)^{l-1} - 1$ $l \geq 3$	B.16	$2/(l-1)$
[Sti08, p. 261]	l^e	$f_{GS,2} = Y^m + (X+1)^m - 1,$ $e \geq 2,\ m := \frac{q-1}{l-1}$	B.17 B.18	$2/(q-2)$
[Sti08, p. 261]	l^2	$f_{GS,3} = (Y^l - Y)(1 - X^{l-1}) - X^l$	B.19	$l-1$
[vdGvdV02, p. 292]	8	$f_{GV} = (Y^2+Y)X + X^2 + X + 1$	B.20	2/3
[BGS05, p. 161]	l^3	$f_{BezGS,l} = Y^l(X^l + X - 1) - (1-Y)X$		$2(l^2-1)/(l+2)$
[ST15, p. 680]	2	$f_{ST,1} = Y^2X + Y + X^2 + 1$	B.21	0
[ST15, p. 680]	2	$f_{ST,2} = X^2 + XY^2 + X + Y$	B.22	0
[ST15, p. 680]	2	$f_{ST,3} = X^2Y^2 + XY^2 + Y + X$	B.23	0
[ST15, p. 680]	2	$f_{ST,4} = X^2Y^2 + XY^2 + Y + X^2 + 1$	B.24	0
[BBGS15, p. 4]	q^m	$f_{BBGS,q,i,j} = \varepsilon_{q,i,j} \cdot (\mathrm{Tr}_j(Y/X^{q^i}) + \mathrm{Tr}_i(Y^{q^j}/X) - 1)$ $i,j \in \mathbb{N}$ with $\gcd(i,j) = 1, m := i+j$	B.25	$2/\left(\frac{1}{q^j-1} + \frac{1}{q^i-1}\right)$
[HP16, p. 12]	5	$f_{HP,5} = Y^2(3X-1) - (X^2+X)$	B.26	0
[CNT18, p. 19]	4	$f_{CNT} = (Y^2+Y)(X^2+X+1) + X$ s even	B.27	1
[BR20, p. 4]	5	$f_{BR,5} = (X^6+X+2)(Y^5-Y)$ $-(X^5-X)(Y^6+Y^5+2Y+3)$	B.28	$2/(5-2)$
[BR20, p. 4]	q	$f_{BR,q} = (X^{q+1}+b)(b+n)(Y^q-Y)$ $-2b(Y^{q+1}+n)(X^q-X)$		$2/(q-2)$

Figure 4.1: Extended table with examples for recursive towers, with references to the literature, with references to figures displaying the degree one or two subgraphs of the corresponding tower graphs and with the precise limits of the towers.

Tower graph maps. In the following Definition/Lemma 76, we connect the paths in $\Gamma_{\mathcal{F}}$ and the paths of places in $\mathcal{F}$ via the bijective tower graph map $\sigma_{\Gamma_{\mathcal{F}}}$.

Definition/Lemma 76. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0) and let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$. Then the map*

$$\sigma_{\Gamma_{\mathcal{F}}} : W(\Gamma_{\mathcal{F}}) \to W(\mathcal{F}) \text{ via } [P_{i,j}]_{j-i\leq 1} \mapsto (\sigma^i(P_{i,j}))_{j-i\leq 1}.$$

is a well defined bijection and, for all $n \in \mathbb{N}_0$, all $[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n)$, all $i = 1, \ldots, n$ and all $\varepsilon = 0, 1$, we have the identity

$$e(P_{i-1,i}|\sigma^{\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon})) = e(\sigma^{i-1}(P_{i-1,i})|\sigma^{i-1+\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon})).$$

*Moreover, $\sigma_{\Gamma_{\mathcal{F}}}$ restricts to a bijection $W(\Gamma_{\mathcal{F}}, n) \to W(\mathcal{F}, n)$ for all $n \in \mathbb{N}_0$. We call $\sigma_{\Gamma_{\mathcal{F}}}$ the **tower graph map** of $\mathcal{F}$.*

Proof. By the definitions of $W(\Gamma_{\mathcal{F}}, n)$ in Definition 55(iii) and $W(\mathcal{F}, n)$ in Definition 16(i) and by the invariance of ramification indices under the action of isomorphisms in (11), it immediately follows that $\sigma_{\Gamma_{\mathcal{F}}}$ restricts to a well defined map $W(\Gamma_{\mathcal{F}}, n) \to W(\mathcal{F}, n)$ for all $n \in \mathbb{N}_0$ which also satisfies the desired equalities $e(P_{i-1,i}|\sigma^{\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon})) = e(\sigma^{i-1}(P_{i-1,i})|\sigma^{i-1+\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon}))$ for all $i = 1, \ldots, n$ and all $\varepsilon = 0, 1$.

On the other hand, $\sigma_{\Gamma_{\mathcal{F}}}$ clearly has the inverse map

$$\sigma_{\Gamma_{\mathcal{F}}}^{-1} : W(\mathcal{F}) \to W(\Gamma_{\mathcal{F}}) \text{ via } (P_{i,j})_{j-i\leq 1} \mapsto [\sigma^{-i}(P_{i,j})]_{j-i\leq 1}$$

which also restricts to a map $W(\mathcal{F}, n) \to W(\Gamma_{\mathcal{F}}, n)$. Hence, Lemma 76 follows. □

Examples 77. *(i) Consider the recursive MW-towers $\mathcal{F} = \mathcal{F}_{MW,2} = (F_\nu)_\nu$ over $\mathbb{F}_3$ and $\mathcal{F}' = \mathcal{F}'_{MW,2} = (F'_\nu)_\nu$ over $\mathbb{F}_9$ which are defined by the polynomial $f_{MW,2} = Y^2 + XY + 2X^2 + 1$. Their degree two and degree one subgraphs Γ and Γ' are depicted in the figures B.1 and B.2, respectively.*

In Example 12, we claimed that there are places Q and Q' having pyramids which are of the forms as in the figures 2.7 and *2.8.*

Indeed, in Γ there is a path $\mathcal{P}$ as in Figure 4.2 and in Γ' there is a path $\mathcal{P}'$ as in Figure 4.3. The bijectivities of the tower maps $\sigma_{\Gamma_{\mathcal{F}}}$ and $\sigma_{\Gamma_{\mathcal{F}'}}$ in Definition/Lemma 76 and the surjectivities of Path$_{\mathcal{F}}$ *and* Path$_{\mathcal{P}'}$ *in Definition/Lemma 17(i) then supply the desired places $Q \in \mathbb{P}_{F_3}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ and $Q' \in \mathbb{P}_{F'_3}(\sigma_{\Gamma_{\mathcal{F}'}}(\mathcal{P}'))$.*

(ii) Consider the recursive ST-tower $\mathcal{F} = \mathcal{F}_{ST,3} = (F_\nu)_\nu$ over $\mathbb{F}_2$ in Examples 8(iv) which is defined by the polynomial $f_{ST,3} = X^2Y^2 + XY^2 + X + Y$. The degree one subgraph Γ of $\mathcal{F}$ is depicted in Figure B.23.

In Examples 42(ii), we claimed that there is a place $Q \in \mathbb{P}_{F_3}$ such that Pyr(Q) *is of the form which is depicted in Figure 3.5 where the blue numbers are the ramification indices of the elementary extensions in* Pyr(Q).

Indeed, in Γ there is a path $\mathcal{P}$ as in Figure 4.4. The bijectivity of the tower map $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and the surjectivity of Path *in Definition/Lemma 17(i) then supply the desired place $Q \in \mathbb{P}_{F_3}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$.*

However, the ramification indices in the top diamond $(Q, P_{0,2}, P_{1,3}, P_{1,2})$ cannot be obtained from Abhyankar's Lemma and therefore we used Magma [BCP97].

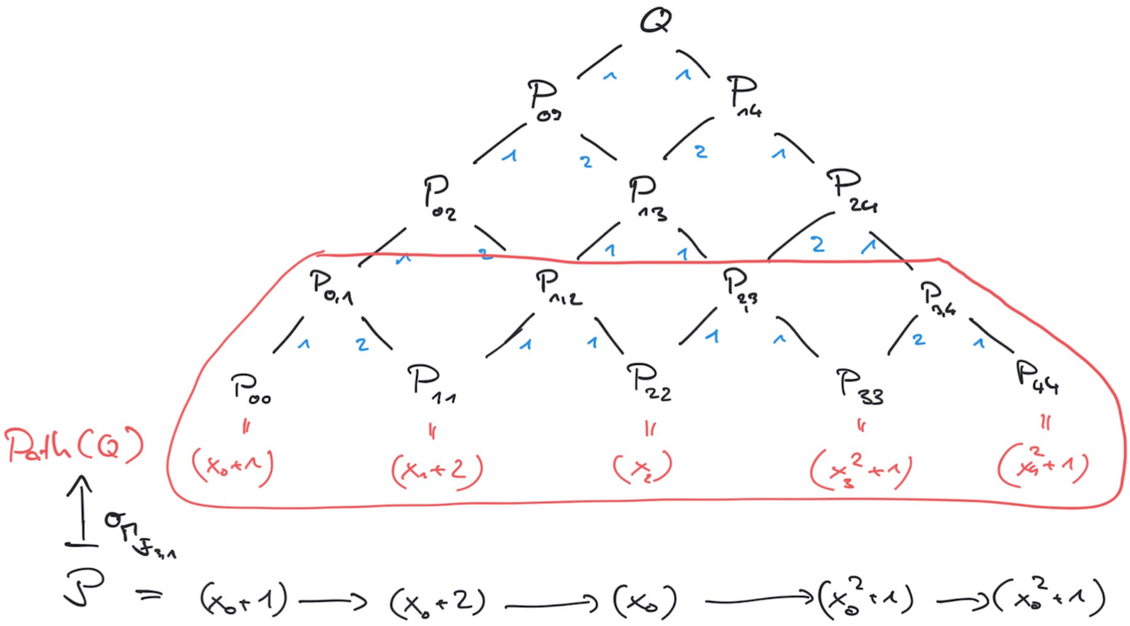


Figure 4.2: First example of the tower graph map and a pyramid of places with Abhyankar ramification indices

Tower graph map on compositions of paths. The following rule in Lemma 78 will often be helpful to avoid cumbersome index chasings for compositions of paths.

Lemma 78. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\Gamma_\mathcal{F}$ be the tower graph of $\mathcal{F}$ and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be composable paths in $\Gamma_\mathcal{F}$ of the lengths l_1 and l_2, respectively. Then we have the identity*

$$\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_1 \cdot \mathcal{P}_2) = \sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_1) \cdot \sigma^{l_1}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_2)) \in W(\mathcal{F}, l_1 + l_2).$$

Proof. Let $[P_{i,j}]_{j-i\leq 1} := \mathcal{P}_1 \cdot \mathcal{P}_2 \in W(\Gamma_\mathcal{F}, l_1 + l_2)$. Then we have the equalities

$$\mathcal{P}_1 = [P_{i,j}]_{j-i\leq 1} = [P_{i,j}]_{\substack{0\leq i\leq j\leq l_1 \\ j-i\leq 1}} \in W(\Gamma_\mathcal{F}, l_1) \tag{107}$$

and

$$\mathcal{P}_2 = [P_{l_1+i,l_1+j}]_{j-i\leq 1} = [P_{l_1+i,l_1+j}]_{\substack{0\leq i\leq j\leq l_2 \\ j-i\leq 1}} \in W(\Gamma_\mathcal{F}, l_2). \tag{108}$$

Let us also use the double subscript notation for paths in $\mathcal{F}$. Then we already obtain the desired identity by the computation

$$\begin{aligned}
\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_1 \cdot \mathcal{P}_2) &= (\sigma^i(P_{i,j}))_{\substack{0\leq i\leq j\leq l_1+l_2 \\ j-i\leq 1}} = (\sigma^i(P_{i,j}))_{\substack{0\leq i\leq j\leq l_1 \\ j-i\leq 1}} \cdot (\sigma^{l_1}(\sigma^{i-l_1}(P_{i,j})))_{\substack{l_1\leq i\leq j\leq l_1+l_2 \\ j-i\leq 1}} \\
&= (\sigma^i(P_{i,j}))_{\substack{0\leq i\leq j\leq l_1 \\ j-i\leq 1}} \cdot \sigma^{l_1}((\sigma^i(P_{l_1+i,l_1+j}))_{\substack{0\leq i\leq j\leq l_2 \\ j-i\leq 1}}) \\
&= \sigma_{\Gamma_\mathcal{F}}([P_{i,j}]_{\substack{0\leq i\leq j\leq l_1 \\ j-i\leq 1}}) \cdot \sigma^{l_1}(\sigma_{\Gamma_\mathcal{F}}([P_{l_1+i,l_1+j}]_{\substack{0\leq i\leq j\leq l_2 \\ j-i\leq 1}})) \\
&= \sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_1) \cdot \sigma^{l_1}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_2))
\end{aligned}$$

where the equalities hold by the following reasonings: The first and second to last equalities hold by the definition of $\sigma_{\Gamma_\mathcal{F}}$ in Definition/Lemma 76. The second equality holds as we just split the path in two composable paths at the position (l_1, l_1). The third equality holds by Definition/Lemma 20(i). The last equality holds by the equalities in (107) and (108) □

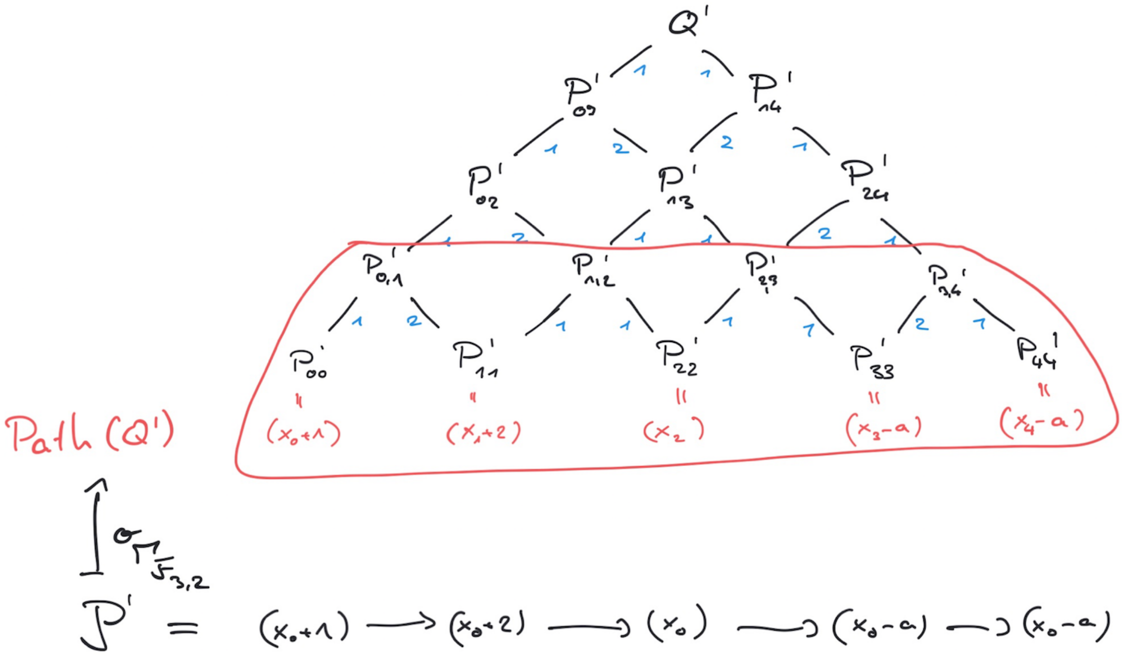


Figure 4.3: Second example of the tower graph map and a pyramid of places with Abhyankar ramification indices

4.2.1 Properties of Paths

Analogously to the properties of paths in recursive towers $\mathcal{F}$ in Definition 16, i.e. being rational, being tame and having balanced ramification indices in Definition 16, we will attribute the same properties to the paths in tower graphs $\Gamma_{\mathcal{F}}$. Here, our definitions will ensure that these properties are invariant under the action of the tower graph map $\sigma_{\Gamma_{\mathcal{F}}}$.

Rational paths. In the following Definition 79, we will define rational paths in $\Gamma_{\mathcal{F}}$ and, in Lemma 80, we will show that the rational paths in $\Gamma_{\mathcal{F}}$ already generate all rational paths in $\mathcal{F}$ via the bijective tower graph map $\sigma_{\Gamma_{\mathcal{F}}}$.

Definition 79. *Let $\mathcal{F}$ be a recursive tower and let $\Gamma_{\mathcal{F}}$ be its tower graph. Then we call a path $\mathcal{P}$ in $\Gamma_{\mathcal{F}}$* ***rational*** *if all vertices and edges of $\mathcal{P}$ are rational.*

Moreover, for all $n \in \mathbb{N}_0$ and all subgraphs Γ in $\Gamma_{\mathcal{F}}$, we denote the set of all rational paths in Γ by $W_{\mathrm{rat}}(\Gamma)$ and the set of all paths in Γ of length n by $W_{\mathrm{rat}}(\Gamma, n)$.

Lemma 80. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\Gamma_{\mathcal{F}}$ be its tower graph. Then $\sigma_{\Gamma_{\mathcal{F}}}$ restricts to a bijection $W_{\mathrm{rat}}(\Gamma) \to W_{\mathrm{rat}}(\mathcal{F})$. In particular, for all $n \in \mathbb{N}_0$ and all rational places $Q \in \mathbb{P}_{F_n}$, the path $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q))$ in $\Gamma_{\mathcal{F}}$ is rational.*

Proof. On the one hand, from the definitions of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76, of $W_{\mathrm{rat}}(\Gamma)$ in Definition 79 and of $W_{\mathrm{rat}}(\mathcal{F})$ in Definition 16(iv), we immediately derive the 'main'-part, namely that $\sigma_{\Gamma_{\mathcal{F}}}$ indeed restricts to a bijection $W_{\mathrm{rat}}(\Gamma) \to W_{\mathrm{rat}}(\mathcal{F})$.

On the other hand, combining this 'main'-part and Definition/Lemma 17(iii) also immediately yields the 'in particular'-part. ☐

Tame and wild paths. In the following Definition/Lemma 81, we will define tame and wild paths in tower graphs. Here, we should think of tame paths $\mathcal{P}$ in the tower graph as the paths which ensure that Abhyankar's Lemma can be applied to all diamonds in the

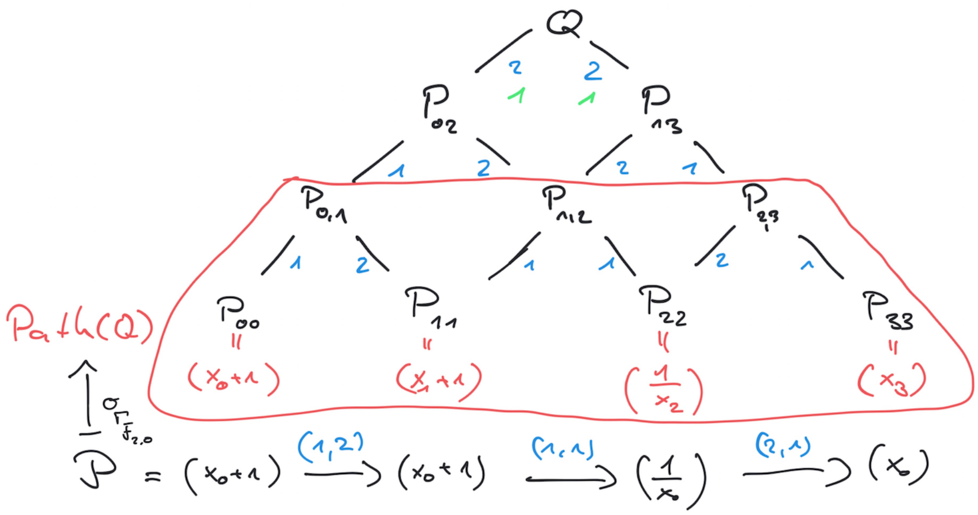


Figure 4.4: Third example of the tower graph map and a pyramid of places with Abhyankar ramification indices

pyramid $\mathrm{Pyr}(Q)$ with Q lying over $\mathcal{P}$. This property was also already captured in Lemma 17(v) for the tame paths in the recursive tower.

In our applications, the different exponents $d(Q|P)$ will be completely irrelevant. Therefore, in wild recursive towers, our only additional problem will be that Abhyankar's Lemma will not always be applicable. But even in wild recursive towers $\mathcal{F}$, most of the paths are tame and, thus, we can also apply Abhyankar's Lemma to most paths in $\mathcal{F}$.

Definition/Lemma 81. *Let $\mathcal{F}$ be a recursive tower and let $\Gamma_{\mathcal{F}}$ be its tower graph. Then we call a path $\mathcal{P} = [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n)$ **tame** if one of the following two equivalent properties is satisfied:*

- *(i) The path $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) \in W(\mathcal{F}, n)$ is tame in the sense of Definition 16(vi).*
- *(ii) For all $1 \leq i \leq j \leq n-1$, at least one of the extensions $P_{i-1,i}/\sigma(P_{i,i})$ and $P_{j,j+1}/P_{j,j}$ is tame.*

*Otherwise, we call $\mathcal{P}$ **wild**.*

Proof. Let σ be the tower map of $\mathcal{F}$. By the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76, we have the equality $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) = (\sigma^i(P_{i,j}))_{j-i\leq 1}$. Thus, by the invariance of the ramification indices under the action of isomorphisms in (11), we obtain that, for all $i = 1, \dots, n-1$ and all $\varepsilon = 0, 1$, the extension $P_{i-1,i}/\sigma^{\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon})$ is tame if and only if the extension $\sigma^i(P_{i-1,i})/\sigma^{i+\varepsilon}(P_{i-1+\varepsilon,i-1+\varepsilon})$ is tame. Combining these equivalences with the definition of the tameness of $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$ in Definition 16(vi) supplies the desired equivalence. ☐

Paths with balanced ramification indices. The following property of having balanced ramification indices will be crucial for the almost complete answer to Conjecture 1(iii) in Corollary 184 and all its implications:

Up to finite constant field extension and up to some wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author, it will come out that a recursive tower satisfies the stronger Conjecture 1(ii) if and only if its ramification subgraph only has finite weakly connected components which contain circles with unbalanced ramification indices.

Moreover, as a further major result of this thesis, we will also show in Theorem 155 that the tower graph has at most one finite weakly connected component which only contains circles with balanced ramification indices. In particular, Theorem 155 will improve the results in [Bee04, p. 238, Theorem 5.5] and [HP12, p. 27, Theorem 23]. There, it was shown that most of the Beelen-graphs (see Definition 94(i)) and all HP-graphs (see Definition 101(i)) have at most one finite d-regular weakly connected component.

Definition/Lemma 82. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\Gamma_\mathcal{F}$ be its tower graph. We say that a path $\mathcal{P} = [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_\mathcal{F}, n)$ has **balanced (resp. unbalanced) ramification indices** if one of the following two equivalent properties is satisfied:*

- *(i) The path $\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}) \in W(\mathcal{F}, n)$ has balanced ramification indices in the sense of Definition 16(v).*
- *(ii) The equality $\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1}) = \prod_{i=1}^{n} e(P_{i-1,i}|\sigma(P_{i,i}))$ holds.*

*Otherwise, we say that $\mathcal{P}$ has **unbalanced ramification indices**.*

*For brevity, we will sometimes call a subgraph which only contains circles with balanced ramification indices **balanced**.*

Proof. The invariance of the ramification indices under the action of isomorphisms in (11) provides the equality

$$\frac{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1})}{\prod_{i=1}^{n} e(P_{i-1,i}|\sigma(P_{i,i}))} = \frac{\prod_{i=1}^{n} e(\sigma^{i-1}(P_{i-1,i})|\sigma^{i-1}(P_{i-1,i-1}))}{\prod_{i=1}^{n} e(\sigma^{i-1}(P_{i-1,i})|\sigma^{i}(P_{i,i}))} \tag{109}$$

Now, we notice that $\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}) = (\sigma^i(P_{i,j}))_{j-i\leq 1} \in W(\mathcal{F}, n))$ has balanced ramification indices if and only if the right side of the equality in (109) is equal to one. Hence, we obtain the desired equivalence. □

Lemma 83. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let Γ be a subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$. If all circles in Γ have balanced ramification indices, then also all closed paths in Γ have balanced ramification indices.*

Proof. Let σ be the tower map of $\mathcal{F}$ and let $\mathcal{C} = [P_{i,j}]_{j-i\leq 1}$ be a closed path of length n in Γ. We will show that $\mathcal{C}$ has balanced ramification indices by induction over its length $n \in \mathbb{N}_0$. For $n = 0$, the statement holds trivially as $\mathcal{C}$ is only a vertex in this case.

Next, let $n \geq 1$. Either $\mathcal{C}$ is a circle and we are done by the assumption that all circles have balanced ramification indices. Or there are a repeating vertices $P_{i,i} = P_{j,j}$ with $0 \leq i < j < n$. Now, choose k and l such that l is minimal with this property. Then, by the minimality of this choice, the subpath $\mathcal{C}' := [P_{i+k,j+k}]_{j-i\leq 1} \in W(\Gamma, l-k)$ of $\mathcal{C}$ cannot have repeating vertices except for the initial and terminal vertices (see Figure 4.5). Thus,

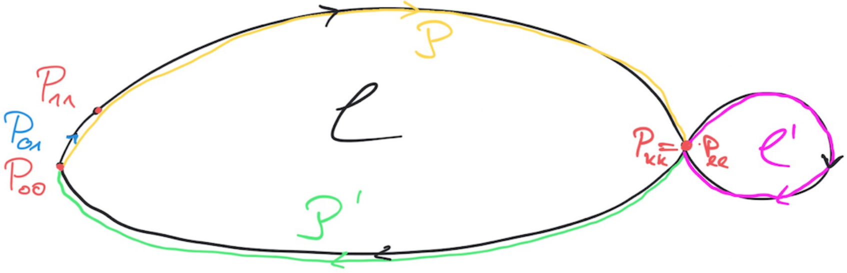


Figure 4.5: A closed path in a proof

$\mathcal{C}'$ is a circle of length $l - k \geq 1$ and we have the equality $\mathcal{C} = \mathcal{P}\mathcal{C}'\mathcal{P}'$ for the composable subpaths $\mathcal{P} = [P_{i,j}]_{j-i\leq 1} \in W(\Gamma, k)$ and $\mathcal{P}' = [P_{i+l,j+l}]_{j-i\leq 1} \in W(\Gamma, n-l)$ of $\mathcal{C}$.

On the one hand, we notice that $\mathcal{P}\mathcal{P}'$ is also a closed path in Γ. But $\mathcal{P}\mathcal{P}'$ has length $n-(l-k) \leq n-1$. Hence, applying the induction hypothesis to $\mathcal{P}\mathcal{P}'$ supplies that $\mathcal{P}\mathcal{P}'$ has balanced ramification indices. Thus, we obtain the equalities

$$\begin{aligned}\prod_{i=1}^{k} e(P_{i-1,i}|P_{i-1,i-1}) \prod_{i=l}^{n} e(P_{i-1,i}|P_{i-1,i-1}) \\ &= \prod_{i=1}^{k} e(P_{i-1,i}|P_{i-1,i-1}) \prod_{i=1}^{n-l} e(P_{i-1+l,i+l}|P_{i-1+l,i-1+l}) \\ &= \prod_{i=1}^{k} e(P_{i-1,i}|\sigma(P_{i,i})) \prod_{i=1}^{n-l} e(P_{i-1+l,i+l}|\sigma(P_{i+l,i+l})) \\ &= \prod_{i=1}^{k} e(P_{i-1,i}|\sigma(P_{i,i})) \prod_{i=l}^{n} e(P_{i-1,i}|\sigma(P_{i,i})). \end{aligned} \tag{110}$$

where the first and last equalities hold as we only changed the indexings and the second equality holds by the definition of paths with balanced ramification indices in Definition/Lemma 82(ii) On the other hand, since $\mathcal{C}'$ is a circle, it also has balanced ramification indices and, thus, we analogously obtain the equalities

$$\begin{aligned}\prod_{i=k}^{l} e(P_{i-1,i}|P_{i-1,i-1}) &= \prod_{i=1}^{l-k} e(P_{i-1+k,i+k}|P_{i-1+k,i-1+k}) = \prod_{i=1}^{l-k} e(P_{i-1+k,i+k}|\sigma(P_{i+k,i+k})) \\ &= \prod_{i=k}^{l} e(P_{i-1,i}|\sigma(P_{i,i})). \end{aligned} \tag{111}$$

Finally, we conclude the equalities

$$\begin{aligned}\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1}) &= \prod_{i=1}^{k} e(P_{i-1,i}|P_{i-1,i-1}) \prod_{i=k}^{l} e(P_{i-1,i}|P_{i-1,i-1}) \prod_{i=l}^{n} e(P_{i-1,i}|P_{i-1,i-1}) \\ &= \prod_{i=1}^{k} e(P_{i-1,i}|\sigma(P_{i,i})) \prod_{i=k}^{l} e(P_{i-1,i}|\sigma(P_{i,i})) \prod_{i=l}^{n} e(P_{i-1,i}|\sigma(P_{i,i})) \\ &= \prod_{i=1}^{n} e(P_{i-1,i}|\sigma(P_{i,i})) \end{aligned} \tag{112}$$

where the first and second equalities hold as we only split the products in three parts at the positions k and l and the second equality holds by (110) and (111).

Hence, the desired statement, namely that $\mathcal{C}$ has balanced ramification indices, follows from the equality in (111) and the definition of paths with balanced ramification indices.

□

Examples 84. *(i) Consider the paths $\mathcal{P}$ and $\mathcal{P}'$ in Example 77(i) which are depicted in the figures 4.2 and 4.3, respectively. There, $\mathcal{P}$ is not rational but $\mathcal{P}'$ is rational.*

(ii) In all tame recursive towers, every path is tame. But the path $\mathcal{P}$ in Example 77(ii), which is depicted in Figure 4.4, is wild.

(iii) In general, all paths which only have unramified edges (in F_1/F_0 and $F_1/\sigma(F_0)$) have balanced ramification indices.

In Figure B.8, the first weakly connected component has circles with unbalanced ramification indices but the second weakly connected component only has circles with balanced ramification indices.

4.2.2 Places over Paths

For our almost complete answer to Conjecture 1(iii) in Corollary 184, we will have to estimate the number of places which lie over paths in fixed subgraphs of the tower graph. In the following Definition 85, we will define the set of these places.

Definition 85. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let Γ be a subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$. For all $n \in \mathbb{N}_0$, we define the set*

$$\begin{aligned}\mathbb{P}_{F_n}[\Gamma] &:= \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(W(\Gamma, n))) = \{Q \in \mathbb{P}_{F_n} : \sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma, n)\}\\ &= \{Q \in \mathbb{P}_{F_n} : \sigma^{-i}(Q) \cap F_1 \in E(\Gamma) \text{ for all } i = 0, \ldots, n-1\}\end{aligned}$$

*of places which **lie over a path of length** n **in** Γ, the set*

$$\mathbb{P}^{(1)}_{F_n}[\Gamma] := \mathbb{P}_{F_n}[\Gamma] \cap \mathbb{P}^{(1)}_{F_n} = \{Q \in \mathbb{P}_{F_n}[\Gamma] : Q \text{ is rational}\}$$

of rational places which lie over a path of length n in Γ and the set

$$\mathbb{P}^{\circ}_{F_n}[\Gamma] := \{Q \in \mathbb{P}_{F_n}[\Gamma] : \sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}(Q)) \text{ is a circle}\}.$$

of places which lie over a circle of length n in Γ. Moreover, we also define the number

$$N[F_n, \Gamma] := \#\mathbb{P}^{(1)}_{F_n}[\Gamma],$$

and the disjoint unions

$$\mathbb{P}_\mathcal{F}[\Gamma] := \coprod_{n \in \mathbb{N}_0} \mathbb{P}_{F_n}[\Gamma] \text{ and } \mathbb{P}^{\circ}_\mathcal{F}[\Gamma] := \coprod_{n \in \mathbb{N}_0} \mathbb{P}^{\circ}_{F_n}[\Gamma].$$

Paths starting at a vertex in a subgraph and paths being completely contained in a subgraph. For every forward complete subgraph Γ, a path which starts in Γ is completely contained in Γ. Correspondingly and up to technicalities, this will yields the 'moreover'-part in the following Lemma 86.

Lemma 86. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let Γ be a subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$. Then we have the inclusions*

$$\mathbb{P}_{F_n}[\Gamma] \subseteq \mathbb{P}_{F_n}(V(\Gamma)) \text{ and } \mathbb{P}_{F_{n+1}}[\Gamma] \subseteq \mathbb{P}_{F_{n+1}}(E(\Gamma)).$$

for all $n \in \mathbb{N}_0$. Moreover, if Γ is a forward complete subgraph of $\Gamma_\mathcal{F}$, then the inclusions are even identities.

Proof. Let $\varepsilon := 0$ (resp. $\varepsilon := 1$) and define $M_0 := V(\Gamma)$ (resp. $M_1 := E(\Gamma)$). We immediately obtain the desired inclusion in the 'main'-part by the equality and inclusion

$$\mathbb{P}_{F_{n+\varepsilon}}[\Gamma] = \mathbb{P}_{F_{n+\varepsilon}}(\sigma_{\Gamma_\mathcal{F}}(W(\Gamma, n+\varepsilon))) \subseteq \mathbb{P}_{F_{n+\varepsilon}}(M_\varepsilon)$$

for all $n \in \mathbb{N}_0$ where the equality holds by the definition of $\mathbb{P}_{F_{n+\varepsilon}}[\Gamma] = \mathbb{P}_{F_{n+\varepsilon}}(\sigma_{\Gamma_\mathcal{F}}(W(\Gamma, n+\varepsilon)))$ in Definition 85 and the inclusion holds as the initial vertex (resp. edge) of any path in $\sigma_{\Gamma_\mathcal{F}}(W(\Gamma, n+\varepsilon))$ is a vertex (resp. edge) in Γ by the definitions of $W(\Gamma, n+\varepsilon)$ in Definition 55(iii) and of $\sigma_{\Gamma_\mathcal{F}}$ in Definition/Lemma 76.

Now, for the 'moreover'-part, suppose that Γ is a forward complete subgraph of $\Gamma_\mathcal{F}$. Then we obtain the equalities

$$\begin{aligned}\sigma_{\Gamma_\mathcal{F}}(W(\Gamma, n+\varepsilon)) &= \sigma_{\Gamma_\mathcal{F}}(\{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_\mathcal{F}, n+\varepsilon) : P_{0,\varepsilon} \in M_\varepsilon\})\\ &= \{(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n+\varepsilon) : P_{0,\varepsilon} \in M_\varepsilon\} \qquad (113)\end{aligned}$$

for all $n \in \mathbb{N}_0$ where the equalities hold by the following reasonings: The inclusion from left to right of the first equality holds by the inclusion

$$W(\Gamma, n+\varepsilon) \subseteq \{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n+\varepsilon) : P_{0,\varepsilon} \in M_\varepsilon\}. \tag{114}$$

The other inclusion of the first equality holds because the definition of the forward completeness of Γ in Definition 66(iii) implies that any path in $\Gamma_{\mathcal{F}}$ which starts at a vertex (resp. edge) in Γ must be completely contained in Γ and, thus, the inclusion in (114) is even an equality. The second equality in (113) holds by the definition of the bijection $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76.

Finally, we deduce the first (resp. second) desired identity in the 'moreover'-part by the equalities

$$\begin{aligned}\mathbb{P}_{F_{n+\varepsilon}}[\Gamma] &= \mathbb{P}_{F_{n+\varepsilon}}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, n+\varepsilon))) = \mathbb{P}_{F_{n+\varepsilon}}(\{(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n+\varepsilon) : P_{0,\varepsilon} \in M_\varepsilon\}) \\ &= \mathbb{P}_{F_{n+\varepsilon}}(M_\varepsilon)\end{aligned}$$

for all $n \in \mathbb{N}_0$ where the first equality holds by the definition of

$$\mathbb{P}_{F_{n+\varepsilon}}[\Gamma] = \mathbb{P}_{F_{n+\varepsilon}}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, n+\varepsilon)))$$

in Definition 85, the second equality holds by the equality in (113), the inclusion from left to right of the third equality holds trivially and the other inclusion of the third equality holds because any place $Q \in \mathbb{P}_{F_n}(M_\varepsilon)$ has a path $(P'_{i,j})_{j-i\leq 1} := \mathrm{Path}(Q) \in W(\mathcal{F}, n+\varepsilon)$ with $P'_{0,\varepsilon} \in M_\varepsilon$. □

4.2.3 The Rational, Splitting and Ramification Subgraphs

Purpose of this subsection. In this subsection, we will define the rational, splitting and ramification subgraphs of $\Gamma_{\mathcal{F}}$. From these crucial subgraphs, we will derive information for the desired values $N(F_n)$ and $g(F_n)$ for all $n \in \mathbb{N}$.

Fundamental equality for the vertices in the tower graph. In the following Lemma 87, we will translate the fundamental equality for the places in F_0 into terms of the vertices in the tower graph. Here, the 'in particular'-part in Lemma 87 will imply that the row and column sums of the standard adjacency matrix for any finite subgraph of the tower graph have the upper bound d.

Lemma 87 (Fundamental Equality for the Vertices in Tower Graphs)**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d which is defined by the pair (σ, F_0) and let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$. Then we have the identities*

$$d = \sum_{Q \in E_+(\Gamma_{\mathcal{F}}, P)} e(Q|P)f(Q|P) = \sum_{Q \in E_-(\Gamma_{\mathcal{F}}, P)} e(Q|\sigma(P))f(Q|\sigma(P))$$

for all $P \in V(\Gamma_{\mathcal{F}})$. We call these identities the ***fundamental equality for the vertices in tower graphs****.*

In particular, the in- and out-degrees of all vertices in $\Gamma_{\mathcal{F}}$ are positive and have the upper bound d.

Proof. The desired identities in the 'main'-part immediately follow from the equalities

$$d = [F_1 : F_0] = \sum_{Q \in \mathbb{P}_{F_1}(P)} e(Q|P)f(Q|P) = \sum_{Q \in E_+(\Gamma_{\mathcal{F}}, P)} e(Q|P)f(Q|P)$$

and

$$d = [F_1 : \sigma(F_0)] = \sum_{Q \in \mathbb{P}_{F_1}(\sigma(P))} e(Q|\sigma(P)) f(Q|\sigma(P)) = \sum_{Q \in E_-(\Gamma_{\mathcal{F}}, P)} e(Q|\sigma(P)) f(Q|\sigma(P))$$

for all $P \in V(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_0}$ where the equalities hold by the following reasonings: The first equalities hold because $\mathcal{F}$ has balanced degree d which then implies the equalities $d = [F_1 : F_0] = [F_1 : \sigma(F_0)]$ by Definition 5(ii). The second equalities hold by applying the fundamental equality in (8) to $P \in \mathbb{P}_{F_0}$ in F_1/F_0 and to $\sigma(P) \in \mathbb{P}_{\sigma(F_0)}$ in $F_1/\sigma(F_0)$. The third equalities hold because the definitions of $E_+(\Gamma_{\mathcal{F}}, P)$ and $E_-(\Gamma_{\mathcal{F}}, P)$ in Definition 55(vii) imply the identities

$$E_+(\Gamma_{\mathcal{F}}, P) = \mathbb{P}_{F_1}(P)$$

and

$$\begin{aligned} E_-(\Gamma_{\mathcal{F}}, P) &= \{Q \in \mathbb{P}_{F_1} : \sigma^{-1}(Q) \cap F_0 = P\} = \{Q \in \mathbb{P}_{F_1} : Q \cap \sigma(F_0) = \sigma(P)\} \\ &= \mathbb{P}_{F_1}(\sigma(P)). \end{aligned}$$

Finally, the 'in particular'-part immediately follows from the identities in the 'main'-part. □

Rational, splitting, ramification subgraph. Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree over a finite field. Not all places in F_n are relevant for the splitting rate $\nu(\mathcal{F})$ and the asymptotic genus $\gamma(\mathcal{F})$. Only the rational places contribute to $N(F_n) = \#\mathbb{P}^{(1)}_{F_n}$ and only the ramified places contribute to $g(F_n)$ due to the Hurwitz Genus Formula in (9). Thus, only the paths in $\Gamma_{\mathcal{F}}$ which are images of the rational and ramified places under the surjection $\sigma^{-1}_{\Gamma_{\mathcal{F}}} \circ \mathrm{Path} : \mathbb{P}_{\mathcal{F}} \to W(\Gamma_{\mathcal{F}})$ are relevant for our purposes.

Correspondingly, in the following Definition 88, the finite rational subgraph $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ of $\Gamma_{\mathcal{F}}$ contains all paths contributing to $N(F_n)$ and the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is the smallest forward and backward complete subgraph containing all paths contributing to $g(F_n)$. Moreover, in Lemma 92(i), we will show that the vertices of the splitting subgraph $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ of $\Gamma_{\mathcal{F}}$ are contained in $\mathrm{Split}(\mathcal{F}/F_0)$.

Definition 88. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$.*

*(i) We define $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ as the subgraph $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ of $\Gamma_{\mathcal{F}}$ with $V(\Gamma^{\mathrm{rat}}_{\mathcal{F}}) := \mathbb{P}^{(1)}_{F_0} \subseteq V(\Gamma)$ and $E(\Gamma^{\mathrm{rat}}_{\mathcal{F}}) := \mathbb{P}^{(1)}_{F_1} \subseteq E(\Gamma)$ and call $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ the **rational subgraph** of $\Gamma_{\mathcal{F}}$.*

Notice that $v_{\mathrm{init}}(Q) = Q \cap F_0$ and $v_{\mathrm{term}}(Q) = \sigma^{-1}(Q) \cap F_0$ are rational places for all $Q \in E(\Gamma^{\mathrm{rat}}_{\mathcal{F}}) = \mathbb{P}^{(1)}_{F_1}$ and, thus, $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ is indeed a well defined subgraph of Γ by Definition 66(i).

*(ii) Suppose that $\mathcal{F}$ has balanced degree d. Then we define the subgraph $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ as the union of all d-regular subgraphs of $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ and call $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ the **(rational) splitting subgraph** of $\Gamma_{\mathcal{F}}$.*

Notice that $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ is also d-regular because the 'in particular'-part in Lemma 87 implies that d is an upper bound for the in- and out-degree of all vertices in $\Gamma_{\mathcal{F}}$. Therefore, $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ is the largest d-regular subgraph of $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$.

Moreover, also notice that the 'in particular'-part in Lemma 87 also implies that all d-regular subgraphs of $\Gamma_{\mathcal{F}}$ are even forward and backward complete. Therefore, $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ is also the largest d-regular forward and backward complete subgraph of $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ and still a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$.

(iii) We call an edge Q in $\Gamma_{\mathcal{F}}$ ***ramified*** *if it is ramified in F_1/F_0 or in $F_1/\sigma(F_0)$. Otherwise, we call Q* ***unramified****.*

Moreover, we define $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ as the intersection subgraph of all forward and backward complete subgraphs of $\Gamma_{\mathcal{F}}$ containing all the ramified edges of $\Gamma_{\mathcal{F}}$ and call $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ the ***ramification subgraph*** *of $\Gamma_{\mathcal{F}}$.*

Notice that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is also a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by the last comment in Definition 66(iii). Therefore, it is the smallest (i.e. unique minimal with respect to the subgraph relation) forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ containing all the ramified edges.

Notice that any of the subgraphs $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$, $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ and $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ of $\Gamma_{\mathcal{F}}$ can be the ***empty graph****, i.e. $V(\Gamma) = E(\Gamma) = \emptyset$.*

Example 89. *In Figure B.5, the rational subgraph for the recursive MW-tower $\mathcal{F} = \mathcal{F}_{MW,6}$ over $\mathbb{F}_9$ which is defined by the MW-polynomial $f_{MW,6} = Y^2 + (X^2+1)Y + 2X^2$ is displayed. Here, the first weakly connected component is the ramification subgraph and the second subgraph is the splitting subgraph. This comes out from the following reasoning:*

The displayed subgraph Γ_1 is the degree one subgraph. Therefore, it only contains the rational edges in $\Gamma_{\mathcal{F}}$. Moreover, as there are exactly ten rational places in F_0, Γ_1 also contains all rational vertices and, thus, is equal to the rational subgraph $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$.

Moreover, by the fundamental equality for the vertices in tower graphs in Lemma 87, both displayed weakly connected components of $\Gamma_1 = \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ are already weakly connected components of $\Gamma_{\mathcal{F}}$.

On the one hand, the second weakly connected component is even some 2-regular weakly connected component of $\Gamma_{\mathcal{F}}$ and, thus, is contained in the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$. But since the first weakly connected component is not 2-regular and since there are no other rational vertices in $\Gamma_{\mathcal{F}}$, the second weakly connected component must already be equal to $\Gamma_{\mathcal{F}}^{\mathrm{split}}$.

On the other hand, the first weakly connected component Γ contains eight ramified edges and, consequently, must be contained in the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Moreover, because the defining polynomial $f_{MW,6}$ of $\mathcal{F}$ has degree two in both variables and because of Riemann's inequality in [Sti08, p. 148, Corollary 3.11.4], we deduce that the genus of $F_1 = \mathbb{F}_9(x,y)$ is at most one. In particular, if we consider Hurwitz genus formula in (9) and the fact that $\mathcal{F}$ is tame, then we conclude that F_1 has genus one and all of the maximal eight ramified edges of $\Gamma_{\mathcal{F}}$ are contained in Γ. Hence, Γ must also already be equal to $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Remark 90. *In Theorem 155, it will come out that $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ is already a weakly connected component of $\Gamma_{\mathcal{F}}$.*

d-regular subgraphs. The following Lemma 91 will be useful to prove Proposition 92. First, in Lemma 91(i), we will show that any finite subgraph Γ of $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ is d-regular if and only if it is forward (resp. backward) complete.

Then, in Lemma 91(ii), we will consider the surjective composition $\tau_{\mathcal{F}} := \sigma_{\Gamma_{\mathcal{F}}}^{-1} \circ \mathrm{Path} : \mathbb{P}_{\mathcal{F}} \to W(\Gamma_{\mathcal{F}})$ and prove that any path $\mathcal{P}$ in $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ has exactly one place $Q \in \mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}$ which lies above it, i.e. $\tau_{\mathcal{F}}(Q) = \mathcal{P}$, and that this place Q is rational.

Lemma 91. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d which is defined by the pair (σ, F_0), let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$ and let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph and $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ the rational subgraph of $\Gamma_{\mathcal{F}}$. Then the following hold:*

(i) For all finite subgraphs Γ of $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$, the following statements are equivalent:

(a) Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$.

(b) Γ *is a backward complete subgraph of* $\Gamma_{\mathcal{F}}$*.*

(c) Γ *is d-regular.*

(d) Γ *is a union of disjoint weakly connected components of* $\Gamma_{\mathcal{F}}$*.*

In (d), the weakly connected components are even strongly connected.

(ii) Let Γ *be a subgraph of* $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$*. Then the surjection*

$$\tau_{\mathcal{F}} := \sigma_{\Gamma_{\mathcal{F}}}^{-1} \circ \mathrm{Path} : \mathbb{P}_{\mathcal{F}} \to W(\Gamma_{\mathcal{F}}) \text{ via } Q \mapsto [\sigma^{-i}(Q) \cap F_{j-i}]_{j-i \leq 1}$$

restricts to a bijection $\tau_{\mathcal{F}}^{-1}(W(\Gamma)) \to W(\Gamma)$ *and all places in* $\tau_{\mathcal{F}}^{-1}(W(\Gamma))$ *are rational.*

Proof. For (i): First, we notice the equalities and estimates

$$\#E(\Gamma) = \sum_{P \in V(\Gamma)} \#E_+(\Gamma, P) = \sum_{P \in V(\Gamma)} \#E_-(\Gamma, P) \leq d \cdot \#V(\Gamma) \tag{115}$$

where the equalities and estimates hold by the following reasonings: The first two equalities hold as the definitions of $E_+(\Gamma, P)$ and $E_-(\Gamma, P)$ in Definition 55(vii) imply the identities $E(\Gamma) = \coprod_{P \in V(\Gamma)} E_+(\Gamma, P) = \coprod_{P \in V(\Gamma)} E_-(\Gamma, P)$. The estimate hold as the 'in particular'-part of Lemma 87 implies the estimate

$$\#E_+(\Gamma, P) \leq \#E_+(\Gamma_{\mathcal{F}}, P) \leq d \text{ and } \#E_-(\Gamma, P) \leq \#E_-(\Gamma_{\mathcal{F}}, P) \leq d. \tag{116}$$

In particular, combining the equalities and estimates in (115) and the estimates in (116) imply the equivalence of the statements

$$\#E_+(\Gamma, P) = d \text{ for all } P \in V(\Gamma) \text{ and } \#E_-(\Gamma, P) = d \text{ for all } P \in V(\Gamma). \tag{117}$$

Second, we notice that since $E(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ contains all places $Q \in \mathbb{P}_{F_1}$ which are ramified in F_1/F_0 or in $F_1/\sigma(F_0)$ by Definition 88(iii), the set $E(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) = E(\Gamma_{\mathcal{F}}) \backslash E(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ contains none of these ramified places. Therefore, we have the equalities $e(Q|Q \cap F_0) = 1$ and $e(Q|Q \cap \sigma(F_0)) = e(\sigma^{-1}(Q)|\sigma^{-1}(Q) \cap F_0) = 1$ for all $Q \in E(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}})$. On the one hand, combining these equalities, Lemma 87 and the equalities $f(Q|P) = \deg(Q) = f(Q|\sigma(P))$ for all vertices $P \in V(\Gamma) \subseteq V(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$ and all edges $Q \in E_+(\Gamma_{\mathcal{F}}, P) \cup E_-(\Gamma_{\mathcal{F}}, P)$ provides the equalities

$$\sum_{Q \in E_+(\Gamma_{\mathcal{F}}, P)} \deg(Q) = d = \sum_{Q \in E_-(\Gamma_{\mathcal{F}}, P)} \deg(Q) \tag{118}$$

for all $P \in V(\Gamma) \subseteq V((\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$.

Now, on the one hand, if Γ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$, its definition in Definition 66(iii) provides $E_+(\Gamma, P) = E_+(\Gamma_{\mathcal{F}}, P)$ (resp. $E_-(\Gamma, P) = E_-(\Gamma_{\mathcal{F}}, P)$). Thus, as any place in $E(\Gamma) \subseteq E(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$ is rational, the first (resp. second) equality in (118) yields the equality $d = \#E_+(\Gamma, P)$ (resp. $d = \#E_-(\Gamma, P)$). On the other hand, if the equality $d = \#E_+(\Gamma, P)$ (resp. $d = \#E_-(\Gamma, P)$) holds, we obtain the equalities and estimates

$$d = \#E_+(\Gamma, P) \leq \#E_+(\Gamma_{\mathcal{F}}, P) \leq \sum_{Q \in E_+(\Gamma_{\mathcal{F}}, P)} \deg(Q) = d$$

$$(\text{resp. } d = \#E_-(\Gamma, P) \leq \#E_-(\Gamma_{\mathcal{F}}, P) \leq \sum_{Q \in E_-(\Gamma_{\mathcal{F}}, P)} \deg(Q) = d)$$

where the last equality hold by the first (resp. second) equality in (118). Therefore, all estimates are equalities which especially supplies that Γ is forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$ in this case. Consequently, combining these equivalences, the equivalences in (117) and the definition of d-regularity in Definition 55(vii), we already obtain the desired equivalences of (a), (b) and (c).

Finally, by Lemma 68(iii), we obtain that the combination of (a) and (b) is equivalent to (d) without the 'strongly connected'-part. In particular, we obtain that (d) without the 'strongly connected'-part is also equivalent to (c).

Let Γ_0 be any of these weakly connected components of $\Gamma_{\mathcal{F}}$. Then (c) implies that Γ_0 is d-regular. Hence, combining this conclusion, the 'in particular'-part in Lemma 87 and [Sti08, p. 119, Proposition 3.6.6] yields that Γ_0 is also strongly connected.

All together, we established (i).

For (ii): First, for all $Q \in \mathbb{P}_{\mathcal{F}}$, the desired equality $\tau_{\mathcal{F}}(Q) = [\sigma^{-i}(Q) \cap F_{j-i}]_{j-i\leq 1}$ follows from the equalities

$$\tau_{\mathcal{F}}(Q) = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) = [\sigma^{-i}(Q \cap F_{i,j})]_{j-i\leq 1} = [\sigma^{-i}(Q) \cap F_{j-i}]_{j-i\leq 1} \tag{119}$$

where the first equality holds by the definition of $\tau_{\mathcal{F}} = \sigma_{\Gamma_{\mathcal{F}}}^{-1} \circ \mathrm{Path}$, the second equality holds by the definitions of the maps Path in Definition 17(i) and $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition 76 and the last equality holds because Lemma 10(i) and Lemma 10(ii) imply the equalities

$$\sigma^{-i}(Q \cap F_{i,j}) = \sigma^{-i}(Q) \cap \sigma^{-i}(F_{i,j}) = \sigma^{-i}(Q) \cap F_{j-i}.$$

Next, for showing that $\tau_{\mathcal{F}}$ restricts to a bijection $\tau_{\mathcal{F}}^{-1}(W(\Gamma)) \to W(\Gamma)$, we first notice that it is enough to show that the map Path restricts to a bijection $\tau_{\mathcal{F}}^{-1}(W(\Gamma)) = \mathrm{Path}^{-1}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma))) \to \sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma))$ because $\sigma_{\Gamma_{\mathcal{F}}}^{-1}$ is a bijection by Definition/Lemma 76.

Let $\mathcal{P} = (P_{i,j})_{j-i\leq 1} \in \sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, n)) \subseteq W(\mathcal{F}, n)$ for some $n \in \mathbb{N}_0$. We will show that $\mathcal{P}$ has exactly one preimage under the map $\tau_{\mathcal{F}}$ and that this preimage is rational.

Now, since $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathcal{P}) = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1}$ is a path in Γ and, hence, in $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$, all places $\sigma^{-i}(P_{i,j})$ are rational by the definition of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ in Definition 88(i) and the equalities $e(\sigma^{-k}(P_{k,k+1})|\sigma^{-k}(P_{k,k})) = e(\sigma^{-k}(P_{k,k+1})|\sigma^{-k}(P_{k+1,k+1})) = 1$ hold for all $k = 0, \ldots, n-1$ by the definitions of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ in Definition 88(iii) and of $\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ in Definition 66(iv). Combining these conclusions and the invariance of the degrees of places and ramification indices under the action of isomorphisms in (11) and then applying Lemma 17(iv) to $\mathcal{P} = (P_{i,j})_{j-i\leq 1}$ supplies that there is exactly one preimage of $\mathcal{P}$ under the map Path and that this place is rational. Consequently, the map Path indeed restricts to a bijection $\mathrm{Path}^{-1}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma))) \to \sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma))$ and all places in $\tau_{\mathcal{F}}^{-1}(W(\Gamma)) = \mathrm{Path}^{-1}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma)))$ are rational. Hence, (ii) follows. □

Connecting the splitting and ramification loci with the splitting and ramification subgraphs. The following Proposition 92 will connect the splitting and ramification loci with the splitting and ramification subgraphs. In particular, this will provide an effective way for applying the sufficient criterion for positive limits of towers in Theorem 4 to tame recursive towers with finite ramification subgraphs and non-empty splitting subgraphs.

For many of the examples of recursive towers in Chapter B (e.g. figure B.1), we can combine Proposition 92, Theorem 4 and the information about the splitting and ramification subgraphs in the attached texts to obtain the lower bounds for the limits in Figure 4.1.

Proposition 92. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d over a finite field, let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph, $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ the rational subgraph and $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ the splitting subgraph of $\Gamma_{\mathcal{F}}$. Then the following hold:*

(i) We have the inclusions

$$V(\Gamma_{\mathcal{F}}^{\text{split}}) \subseteq \operatorname{Split}(\mathcal{F}/F_0) \subseteq V(\Gamma_{\mathcal{F}}^{\text{split}}) \cup V(\Gamma_{\mathcal{F}}^{\text{ram}}).$$

(ii) We have the inclusion

$$\operatorname{Ram}(\mathcal{F}/F_0) \subseteq V(\Gamma_{\mathcal{F}}^{\text{ram}}).$$

Proof. Let σ be the tower map of $\mathcal{F}$ and let $(F_{i,j})_{i,j}$ be the pyramid of $\mathcal{F}$.

For the first inclusion in (i): Let $P \in V(\Gamma_{\mathcal{F}}^{\text{split}})$. Then we obtain the equalities and estimates

$$\begin{aligned} d^n &= [F_n : F_0] \geq \#\mathbb{P}_{F_n}(P) \geq \#\{(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n) : P_{0,0} = P\} \\ &= \#\{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n) : P_{0,0} = P\} \geq d^n \end{aligned} \tag{120}$$

for all $n \in \mathbb{N}_0$ where the equalities and estimates hold by the following reasonings: The first equality holds because $\mathcal{F}$ has constant degree d in Definition 5(ii). The first estimate holds by applying the fundamental equality in (8) to the place $P \in V(\Gamma_{\mathcal{F}}^{\text{split}}) \subseteq \mathbb{P}_{F_0}$ in F_n/F_0. The second estimate holds because Lemma 17(i) implies the surjectivity of the map $\operatorname{Path}_{0,n} : \mathbb{P}_{F_{0,n}} = \mathbb{P}_{F_n} \to W(\mathcal{F}, n)$ and because the definition of $\operatorname{Path}_{0,n}(Q) = (Q \cap F_{i,j})_{j-i\leq 1}$ for all $Q \in \mathbb{P}_{F_n}$ implies the equality $\mathbb{P}_{F_n}(P) = \operatorname{Path}_{0,n}^{-1}(\{(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n) : P_{0,0} = P\})$. The second equality holds since Definiton/Lemma 76 implies that $\sigma_{\Gamma_{\mathcal{F}}}$ is a bijection satisfying $\sigma_{\Gamma_{\mathcal{F}}}(\{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n) : P_{0,0} = P\}) = \{(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n) : P_{0,0} = P\}$. The last estimate holds because $\Gamma_{\mathcal{F}}^{\text{split}}$ is d-regular by its definition in Definition 88(ii) and because this then implies that there are d^n paths in $\Gamma_{\mathcal{F}}^{\text{split}}$ of length n which start at P.

Consequently, all estimates in (123) must be equalities and, in particular, the equality $\#P_{F_n}(P) = d^n$ follows. But this means that the rational place $P \in V(\Gamma_{\mathcal{F}}^{\text{split}}) \subseteq V(\Gamma_{\mathcal{F}}^{\text{rat}}) = \mathbb{P}_{F_1}^{(1)}$ splits completely in F_n/F_0 and, thus, is contained in the splitting locus $\operatorname{Split}(\mathcal{F}/F_0)$ by its definition in Definition 3(i).

Hence, the first inclusion in (i) follows.

For the second inclusion in (i): Let $P \in \operatorname{Split}(\mathcal{F}/F_0)\backslash V(\Gamma_{\mathcal{F}}^{\text{ram}})$ and define Γ_P as the subgraph of $\Gamma_{\mathcal{F}}$ which consists of all vertices and edges which can be reached via a path in $\Gamma_{\mathcal{F}}$ starting at P, i.e. we have the vertex set

$$V(\Gamma_P) := \bigcup_{n\in\mathbb{N}_0} \{P_{i,i} : [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n) \text{ with } P_{0,0} = P \text{ and } i = 0, \ldots, n\} \tag{121}$$

and the edge set

$$E(\Gamma_P) := \bigcup_{n\in\mathbb{N}_0} \{P_{i-1,i} : [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n) \text{ with } P_{0,0} = P \text{ and } i = 1, \ldots, n\}. \tag{122}$$

This subgraph Γ_P of course contains the vertex P due to trivial path $P = [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, 0) = V(\Gamma_{\mathcal{F}})$.

We will show that Γ_P is a d-regular forward and backward complete subgraph of $\Gamma_{\mathcal{F}}^{\text{rat}}$ and, by the definition of $\Gamma_{\mathcal{F}}^{\text{split}}$ as the largest such subgraph of $\Gamma_{\mathcal{F}}^{\text{rat}}$ in Definition 88(ii), then obtain $P \in V(\Gamma_{\mathcal{F}}^{\text{split}})$.

For that, we first notice that Γ_P is clearly weakly connected since the definition of Γ_P in (121) and (122) always supplies an undirected path between two vertices in Γ_P through P.

Second, we notice that the assumption $P \notin V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ implies that

$$\Gamma_P \text{ is a subgraph of } \Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}} \tag{123}$$

because Γ_P is clearly weakly connected, because $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ (resp. $\Gamma_{\mathcal{F}}/\Gamma_{\mathcal{F}}^{\mathrm{ram}}$) is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by Definition 88(iii) (resp. by the last comment in Definition 66(iv)) and because of the definition of forward and backward complete subgraphs in Definition 66(iii).

Third, we also notice the equalities

$$\begin{aligned}\tau_{\mathcal{F}}(\mathbb{P}_{F_n}(P)) &= \{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n) : P_{0,0} = P\} \\ &= \{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_P, n) : P_{0,0} = P\} \end{aligned} \tag{124}$$

for all $n \in \mathbb{N}_0$ where the first equality holds by the surjectivity of $\tau_{\mathcal{F}}$ in Lemma 91(ii) and the second equality holds since the definition of Γ_P implies that Γ_P contains all paths in $\Gamma_{\mathcal{F}}$ which start at P.

Moreover, because the assumption $P \in \mathrm{Split}(\mathcal{F}/F_0)$ implies that $\mathbb{P}_{F_n}(P)$ exactly consists of d^n rational places for all $n \in \mathbb{N}_0$ and because of the invariance of the degree of places under isomorphisms in (11), we conclude that the path $\tau_{\mathcal{F}}(Q) = [\sigma^{\ -i}(Q) \cap F_{j-i}]_{j-i\leq 1}$ only consists of rational places for all $Q \in \mathbb{P}_{F_n}(P)$ and all $n \in \mathbb{N}_0$. Hence, combining this conclusion, the identity in (124), the definition of Γ_P in (121) and (122) and the definitions $V(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) = \mathbb{P}_{F_0}^{(1)}$ and $E(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) = \mathbb{P}_{F_1}^{(1)}$ in Definition 88(i) provides that

$$\Gamma_P \text{ is a subgraph of } \Gamma_{\mathcal{F}}^{\mathrm{rat}}. \tag{125}$$

Consequently, (123) and (125) together yield that

$$\Gamma_P \text{ is a subgraph of } (\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}. \tag{126}$$

Next, combining (126), the identity in (124) and Lemma 91(ii) provides that $\tau_{\mathcal{F}}$ even restricts to a bijection $\mathbb{P}_{F_n}(P) \to \{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma, n) : P_{0,0} = P\}$ for all $n \in \mathbb{N}_0$. Therefore, because of this bijectivity and because $P \in \mathrm{Split}(\mathcal{F}/F_0)$ implies that $\mathbb{P}_{F_n}(P)$ consists of d^n rational places, we get the equalities

$$\#\{[P_{i,j}]_{j-i\leq 1} \in W(\Gamma_P, n) : P_{0,0} = P\} = \#\mathbb{P}_{F_n}(P) = d^n \tag{127}$$

for all $n \in \mathbb{N}_0$. Then applying the 'in particular'-part of Lemma 87 supplies the estimates $E_+(\Gamma_P, P') \leq E_+(\Gamma_{\mathcal{F}}, P') \leq d$ for all $P' \in V(\Gamma_P)$. But, for all $P' \in V(\Gamma_P)$, we get the equality

$$E_+(\Gamma_P, P') = d \tag{128}$$

because the definition of Γ_P in (121) and (122) ensures that there is a path in Γ_P from P to P' and because a proper estimate $E_+(\Gamma_P, P') < d$ would contradict the identity in (127).

Thus, the identity in (128) supplies that Γ_P is a forward complete subgraph of the intersection graph $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ by Definition 66(iii). Moreover, because $\mathcal{F}$ is defined over a finite field, we obtain that the sets $\mathbb{P}_{F_0}^{(1)} = V(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) \supseteq V(\Gamma_P)$ and $\mathbb{P}_{F_1}^{(1)} = E(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) \supseteq E(\Gamma_P)$ are finite sets and, therefore, that Γ_P is a finite directed graph. Then applying Lemma 91(i) yields that Γ_P is a d-regular forward and backward complete subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$.

Hence, the second inclusion in (i) follows.

For (ii): Let $P \in \mathrm{Ram}(\mathcal{F}/F_0)$ and $Q \in \mathbb{P}_{F_n}(P)$ such that Q/P is ramified and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q) = (Q \cap F_{i,j})_{i,j}$ be the pyramid of Q.

Then, by Lemma 10(i), we have the equalities $Q = Q \cap F_n = Q \cap F_{0,n} = P_{0,n}$ and $P = Q \cap F_0 = Q \cap F_{0,0} = P_{0,0}$ and, by Lemma 13, we also have the extensions $P_{0,n}/P_{0,n-1}/\ldots,/P_{0,0}$. Thus, combining these equalities and estimates, the transitive multiplicativity rule for the ramification indices and the choice of Q/P as a ramified extension yields

$$2 \leq e(Q|P) = \prod_{j=0}^{n-1} e(P_{0,j+1}|P_{0,j}).$$

Therefore, due to this estimate, there is some index $0 \leq l \leq n-1$ such that $2 \leq e(P_{0,l+1}|P_{0,l})$.

Next, we can apply Key Lemma 36(iv) to the diamond $(P_{0,l+1}, P_{0,l}, P_{l,l+1}, P_{l,l})$ of places in $\mathrm{Pyr}(Q)$ and obtain the estimates $2 \leq e(P_{0,l+1}|P_{0,l}) \leq e(P_{l,l+1}|P_{l,l})$. More so, because of the invariance of ramification indices under isomorphisms in (11) and because Lemma 10(ii) and Lemma 10(i) supply the equalities $\sigma^{-l}(F_{l,l+\varepsilon}) = F_{0,\varepsilon} = F_\varepsilon$ for all $\varepsilon \in \{0,1\}$, we even obtain that $\sigma^{-l}(P_{l,l+1})/\sigma^{-l}(P_{l,l})$ is ramified in F_1/F_0 and, thus,

$$\sigma^{-l}(P_{l,l}) \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}). \tag{129}$$

Finally, we notice that $\sigma^{-l}(P_{l,l})$ is the l-th vertex of the path

$$\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}})$$

where the equality holds by the definitions of Path in Definition 17(i), of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q) = (Q \cap F_{i,j})_{i,j}$ and of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition 76. Consequently, as $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a backward complete subgraph of $\Gamma_{\mathcal{F}}$, the first vertex $\sigma^{-0}(P_{0,0}) = Q \cap F_{0,0} = P$ must also be contained in $V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ and, hence, (ii) follows. □

4.3 Tower Graphs of Dual Recursive Towers

Purpose of this section. In this section, we will relate the tower graphs of recursive towers to the towers graphs of their dual recursive towers.

As we already pointed out in Section 2.6, dual recursive towers only play a side role in this thesis. Thus, if the reader is only interested in the main results of this thesis, then this section can be skipped.

Tower graphs of dual recursive towers In the following Lemma 93, we will prove that $\Gamma_{\mathcal{F}}$ and $\Gamma_{\hat{\mathcal{F}}}$ are isomorphic via a contravariant isomorphism which respects the ramification indices.

Lemma 93. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\hat{\mathcal{F}} = (\hat{F}_\nu)_\nu$ be the dual tower of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\hat{\mathcal{F}}}$) be the tower graph of $\mathcal{F}$ (resp. $\hat{\mathcal{F}}$). Finally, let σ_0 be the restriction of σ to an isomorphism*

$$\sigma_0 : \hat{F}_1 = F_0 \cdot \sigma^{-1}(F_0) \to F_1 = F_0 \cdot \sigma(F_0).$$

Then the pair

$$\phi := (\mathrm{id}_{\mathbb{P}_{F_0}}, \sigma_0^{-1}) : \Gamma_{\mathcal{F}} \to \Gamma_{\hat{\mathcal{F}}}$$

is a contravariant isomorphism of directed graphs which satisfies the identity

$$e(Q|\sigma^i(P_i)) = e(\phi(Q)|\sigma^{i-1}(\phi(P_i)))$$

for all edges $P_0 \xrightarrow{Q} P_1 \in E(\Gamma_{\mathcal{F}})$ and all $i = 0, 1$.

In particular, ϕ restricts to a contravariant isomorphism from the ramification subgraph of $\Gamma_{\mathcal{F}}$ to the ramification subgraph of $\Gamma_{\hat{\mathcal{F}}}$.

Proof. Let $(P_0 \xrightarrow{Q} P_1) \in E(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_1}$. Then, by the definition of $\Gamma_{\mathcal{F}}$ in Definition 74 and by the choice of σ_0, we obtain the equalities

$$P_0 = Q \cap F_0 = \sigma(\sigma_0^{-1}(Q)) \cap F_0 \text{ and } P_1 = \sigma^{-1}(Q) \cap F_0 = \sigma_0^{-1}(Q) \cap F_0.$$

But these equalities imply $(P_0 \xleftarrow{\sigma_0^{-1}(Q)} P_1) = (\phi(P_0) \xleftarrow{\phi(Q)} \phi(P_1)) = \phi(P_0 \xrightarrow{Q} P_1) \in E(\Gamma_{\hat{\mathcal{F}}})$ by the definition of $\Gamma_{\hat{\mathcal{F}}}$. Therefore, ϕ is a well defined contravariant morphism of directed graphs. Moreover, as $\mathrm{id}_{\mathbb{P}_{F_0}}$ and σ_0^{-1} are bijections, we conclude the first desired statement, namely that ϕ is a contravariant isomorphism.

Next, we obtain the desired equality by the equalities

$$e(Q|\sigma^i(P_i)) = e(\sigma^{-1}(Q)|\sigma^{i-1}(P_i)) = e(\phi(Q)|\sigma^{i-1}(\phi(P_i))) \tag{130}$$

for all $i = 0, 1$ where the first equality holds by the invariance of ramification indices under the action of isomorphisms in (11) and the second equality holds by the definition of ϕ.

On the one hand, the equalities in (130) also provide that

$$\phi(P_0 \xrightarrow{Q} P_1) = (P_0 \xleftarrow{\sigma_0^{-1}(Q)} P_1)$$

is a ramified edge in $\Gamma_{\hat{\mathcal{F}}}$ if and only if $P_0 \xrightarrow{Q} P_1$ is a ramified edge in $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(iii). On the other hand, by Lemma 70(iv), isomorphisms of directed graphs map forward and backward complete subgraphs again on forward and backward complete subgraphs. Hence, combining the above equivalence, this fact and the definition of ramification subgraphs in Definition 88(iii) yields the last desired statement, namely that ϕ restricts to a contravariant isomorphism from the ramification subgraph of $\Gamma_{\mathcal{F}}$ to the ramification subgraph of $\Gamma_{\hat{\mathcal{F}}}$. □

4.4 Connection to other Graphs Associated to Recursive Towers

Purpose of this section. In this section, we will draw connections from the tower graph to two other directed graphs which are also associated with recursive towers, namely the Beelen-graph in [BGS04, p. 10, Definition 4.2] and the HP-graph in [HP12, p. 15, Definition 10]. More concretely, the connections will be realized via epimorphisms of directed graphs in Proposition 95 for the Beelen-graph and in Proposition 104 for the HP-graph.

Moreover, in Subsection 4.4.2, we will also make some preparations for the proof of the first major result of this thesis in the next chapter, namely Theorem 155.

.

4.4.1 Beelen-Graph of Polynomial-Recursive Towers

Purpose of this subsection. In this subsection, we will connect the tower graph to the Beelen-graph in [BGS04, p. 10, Definition 4.2] (see Definition 94(i)). Essentially, the tower graph is a slight modification of the Beelen-graph.

We will also introduce the values $\rho(\mathcal{F})$ of polynomial-recursive towers $\mathcal{F}$. This value will measure the contribution of the vertices in the ramification subgraph to the splitting rate (see the last identity in Proposition 95). Moreover, in [BGS04, p. 15, Theorem

4.10], which is Theorem 96, it was shown for polynomial-recursive towers that the only places $P \in \mathbb{P}_{F_0} \setminus \operatorname{Split}(\mathcal{F}/F_0)$ which can contribute to the splitting rate, i.e. which satisfy $\lim_{\nu\to\infty} N(F_\nu, P)/d^\nu > 0$, are the vertices in the ramification subgraph.

First, we will extend this insight to pair-recursive towers in Proposition 180. Then, in Corollary 184 and Corollary 185, we will characterize the recursive towers $\mathcal{F}$ which satisfy $\rho(\mathcal{F}) = 0$ up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author. Here, $\rho(\mathcal{F}) = 0$ will hold if and only if every finite weakly connected component of the ramification subgraph contains circles with unbalanced ramification indices.

For convenience, in Definition 94 and Theorem 96, we will summarize some of the definitions and theorems from [BGS04] which are relevant here. Note that the definitions and theorems in this summary are not taken verbatim but with adaptions to the notions which we already introduced. However, all adaptions are immediate and should not lead to confusions.

Beelen-graph. In the following Definition 94(i), we will formulate the definition of Beelen-graphs in [BGS04, p. 10, Definition 4.2] but only with notions which were already introduced in this thesis.

For the definition of $W(\mathcal{F})$ in Definition 94(ii) and for the last two equalities in (135) of the proof of Proposition 95, we will require two more parts from the next Chapter 5: First, the definition of the CFE-projection morphism $\pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}} : \Gamma_{\overline{\mathcal{F}}} \to \Gamma_{\mathcal{F}}$ in Definition/Lemma 105(i) and, second, Lemma 124. But Definition/Lemma 105(i) needs no further context and, thus, we should already be able to grasp it without any problems. Moreover, we should also have no problems applying Lemma 124 to the last two equalities in (135) as a black box.

Definition 94. *Let $f \in \mathbb{F}_q[X,Y]$ be an absolutely irreducible polynomial, let $\mathbb{F}/\mathbb{F}_q$ be an algebraic extension, let $F_1' := \mathbb{F}(x_0, x_1)$ be the function field which is defined by $f(x_0, x_1) = 0$ over the full constant field $\mathbb{F}$, let F_0' be the rational function field $\mathbb{F}(x_0)$, let $F_1 := \mathbb{F}_q(x_0, x_1) \subseteq F_1'$ and let $F_0 := \mathbb{F}_q(x_0) \subseteq F_0'$.*

(i) In [BGS04, p. 10, Definition 4.2], the directed graph $\Gamma(f, \mathbb{F}) := (V, E, \alpha)$ is defined via

$$V := \mathbb{F} \cup \{\infty\}, \quad E := \mathbb{P}^{(1)}_{F_1'}, \quad \alpha(Q) := (x_0(Q), x_1(Q))$$

where, for all $i = 0, 1$, the element $x_i(Q)$ either denotes the unique element in $\mathbb{F}$ which is contained in the residue class $x_i + Q$ or ∞.

Moreover, V can be identified with the set $\mathbb{P}^{(1)}_{F_0'}$ of rational places in F_0' via the bijection

$$\phi_{\mathbb{F}} : \mathbb{P}^{(1)}_{F_0'} \to V, \ P' \mapsto x_0(P').$$

*We will often call $\Gamma(f, \mathbb{F})$ the **Beelen-graph** for f and $\mathbb{F}$.*

(ii) Suppose that the polynomial f defines a recursive tower $\mathcal{F} = (F_\nu)_\nu$ over $\mathbb{F}_q$ of balanced degree d, let (σ, F_0) be the pair in Lemma 7(iii), i.e. $\mathcal{F}$ is a recursive tower which is defined by the pair (σ, F_0), let $\overline{\mathcal{F}} := \overline{\mathbb{F}}_q \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$ and let $\overline{\mathcal{G}}$ be the dual recursive tower of $\overline{\mathcal{F}}$. Then [BGS04, p. 14] defines

$$V(\overline{\mathcal{F}}) := \phi_{\overline{\mathbb{F}}_q}(\operatorname{Ram}(\overline{\mathcal{F}}/\overline{F}_0)), \quad V(\overline{\mathcal{G}}) := \phi_{\overline{\mathbb{F}}_q}(\operatorname{Ram}(\overline{\mathcal{G}}/\overline{F}_0))$$

$\overline{W}(\overline{\mathcal{F}})$ as the vertex set of the smallest forward and backward complete subgraph of $\Gamma(f, \overline{\mathbb{F}}_q)$ which contains $V(\overline{\mathcal{F}})$ and $V(\overline{\mathcal{G}})$ and

$$W(\mathcal{F}) := \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\phi_{\overline{\mathbb{F}}_q}^{-1}(\overline{W}(\overline{\mathcal{F}}))) \subseteq \mathbb{P}_{F_0}.$$

Finally, [BGS04, p. 15, Definition 4.9] defines

$$\rho(\mathcal{F}) := \lim_{n\to\infty} \frac{N(F_n, W(\mathcal{F}))}{d^n}.$$

Connection between the Beelen-graph and the tower graph. The following Proposition 95 proves that the Beelen-graph $\Gamma(f, \mathbb{F})$ is basically the rational subgraph $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ of the constant field extension $\mathcal{F}' = \mathbb{F} \cdot \mathcal{F}$. In particular, the third identity in Proposition 95 translates the value $\rho(\mathcal{F})$, which will be crucial for the almost complete answer to Conjecture 1(iii) in Corollary 184, into terms of the ramification subgraph. Moreover, in Definition 179, we will extend the definition of $\rho(\mathcal{F})$ to pair-recursive towers via this identity.

Proposition 95. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field $\mathbb{F}_q$ of degree d which is defined by a polynomial f, let (σ, F_0) be the pair in Lemma 7(iii), i.e. $\mathcal{F}$ is a recursive tower which is defined by this pair (σ, F_0), let $(\mathbb{F}'_\nu)_\nu = \mathcal{F}' := \mathbb{F} \cdot \mathcal{F}$ be a constant field extension of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and let $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ be the rational subgraph of the tower graph $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$. Then we have the monomorphism*

$$\psi_{\mathbb{F}} := (\iota_0 \circ \phi_{\mathbb{F}}^{-1}, \iota_1) : \Gamma(f, \mathbb{F}) \to \Gamma_{\mathcal{F}'}$$

where $\iota_i : \mathbb{P}_{F'_i}^{(1)} \to \mathbb{P}_{F'_i}$ denotes the inclusion map for all $i = 0, 1$. Moreover, we have the identities

$$\psi_{\mathbb{F}}(\Gamma(f, \mathbb{F})) = \Gamma_{\mathcal{F}'}^{\mathrm{rat}}, \quad W(\mathcal{F}) = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}), \quad \rho(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{d^n}.$$

Proof. For the 'main'-part: Let $x_0(Q) \xrightarrow{Q} x_1(Q)$ be an edge in $\Gamma(f, \mathbb{F})$, define $P_i := \phi_{\mathbb{F}}^{-1}(x_i(Q))$ and let $i \in \{0, 1\}$.

If $x_i(Q) \in \mathbb{F}$, then P_i is the unique place which contains $x_0 - x_i(Q)$ in $\mathbb{F}(x_0) = F'_0$ and Q contains the element $x_i - x_i(Q)$. Consequently, in this case, the place $\sigma^{-i}(Q)$ contains the element

$$\sigma^{-i}(x_i - x_i(Q)) = \sigma^{-i}(x_i) - x_i(Q) = x_0 - x_i(Q)$$

where the first equality holds as σ is a morphism of $\mathbb{F}$-algebras and the second equality holds by the choice of σ in Lemma 7(iii). Hence, in this case, the place $\sigma^{-i}(Q)$ lies over the place P_i in F_1/F_0.

Otherwise, if $x_i(Q) = \infty$, then P_i is the place at infinity and Q does not contain any element of the form $x_i - \alpha$ with $\alpha \in \mathbb{F}$. Consequently, in this case, the place $\sigma^{-i}(Q)$ does not contain any element of the element $\sigma^{-i}(x_i - \alpha) = x_0 - \alpha$ with $\alpha \in \mathbb{F}$. Hence, because $\sigma^{-i}(Q)$ is also rational, it must lie over the place P_i at infinity in F_1/F_0.

Then combining both cases and the definition of $\Gamma_{\mathcal{F}}$ in Definition 74 yields that $(P_0 \xrightarrow{Q} P_1) = \psi_{\mathbb{F}}(x_0(Q) \xrightarrow{Q} x_1(Q))$ is indeed an edge in $\Gamma_{\mathcal{F}}$ and, therefore, by the definition of morphisms of directed graphs in Definition 65, we therefore conclude that $\psi_{\mathbb{F}}$ is a well defined morphism. Moreover, as $\phi_{\mathbb{F}}^{-1}$ and ι are injections, this even supplies the desired statement in the 'main'-part, namely that $\psi_{\mathbb{F}}$ is a monomorphism.

For first identity $\psi_{\mathbb{F}}(\Gamma(f, \mathbb{F})) = \Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ in the 'moreover'-part: This desired identity immediately follows because we have the equalities $V(\Gamma_{\mathcal{F}'}^{\mathrm{rat}}) = \mathbb{P}_{F'_0}^{(1)}$ and $E(\Gamma_{\mathcal{F}'}^{\mathrm{rat}}) = \mathbb{P}_{F'_1}^{(1)}$ by the definition of the rational subgraph in Definition 88(i), because these sets are the images of maps $\iota_0 \circ \phi_{\overline{\mathbb{F}}}^{-1}$ and ι_1, respectively, and because of the definition of the image graph in Definition/Lemma 69(i).

For the second identity $W(\mathcal{F}) = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ in the ’moreover’-part: Let $\mathcal{G}$ be the dual tower of $\mathcal{F}$. Then, by the definitions of dual towers in Definition/Lemma 35(i) and of constant field extensions of towers in Definition/Lemma 21, we get that

$$\text{the geometric tower } \overline{\mathcal{G}} \text{ of } \mathcal{G} \text{ is also the dual tower of } \overline{\mathcal{F}}. \tag{131}$$

First, we notice that the monomorphism $\psi_{\overline{\mathbb{F}}_q}$ is even an isomorphism because all places in $\overline{F}_0$ and $\overline{F}_1$ are rational. Thus, by Lemma 70(iv), the smallest forward and backward complete subgraph G which contains $V(\overline{\mathcal{F}}) \cup V(\overline{\mathcal{G}})$ is isomorphically mapped via $\psi_{\overline{\mathbb{F}}_q}$ to the

$$\begin{aligned} &\text{smallest forward and backward complete subgraph } \psi_{\overline{\mathbb{F}}_q}(G) \\ &\text{which contains } \psi_{\overline{\mathbb{F}}_q}(V(\overline{\mathcal{F}}) \cup V(\overline{\mathcal{G}})) = \mathrm{Ram}(\overline{\mathcal{F}}/\overline{F}_0) \cup \mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0) \end{aligned} \tag{132}$$

where the equality holds by the definitions of $V(\overline{\mathcal{F}}) = \phi_{\overline{\mathbb{F}}_q}(\mathrm{Ram}(\overline{\mathcal{F}}/\overline{F}_0))$ and $V(\overline{\mathcal{G}}) = \phi_{\overline{\mathbb{F}}_q}(\mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0))$ in Definition 94(ii).

Now, on the one hand, for all edges $Q \in E(\Gamma_{\overline{\mathcal{F}}})$ such that $Q/Q \cap \overline{F}_0$ (resp. $Q/Q \cap \sigma(\overline{F}_0)$) is ramified in $\overline{F}_1/\overline{F}_0$ (resp. $\overline{F}_1/\sigma(\overline{F}_0)$), we conclude that $v_{\mathrm{init}}(Q) = Q \cap \overline{F}_0$ (resp. $v_{\mathrm{term}}(Q)) = \sigma^{-1}(Q) \cap \overline{F}_0$ is a place in $\mathrm{Ram}(\overline{\mathcal{F}}/\overline{F}_0)$ (resp. $\mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0)$). Consequently, by the conclusion in (132) and the definition of ramified edges in Definition 88(iii), all ramified edges of $\Gamma_{\mathcal{F}}$ are contained in the forward and backward complete subgraph $\psi_{\overline{\mathbb{F}}_q}(G)$ of $\Gamma_{\mathcal{F}}$. In particular, because of this conclusion and because of the definition of the ramification subgraph $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ of $\Gamma_{\overline{\mathcal{F}}}$ in Definition 88(iii) as the smallest forward and backward complete subgraph containing these ramified edges, we deduce that $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ is a subgraph of $\psi_{\overline{\mathbb{F}}_q}(G)$.

On the other hand, Lemma 92(ii) supplies that $\mathrm{Ram}(\overline{\mathcal{F}}/\overline{F}_0)$ is contained in $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ and that $\mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0)$ is contained in the ramification subgraph $\Gamma_{\overline{\mathcal{G}}}^{\mathrm{ram}}$ of the tower graph $\Gamma_{\overline{\mathcal{G}}}$ of $\overline{\mathcal{G}}$. But, due to (131), the morphism ϕ in Lemma 93 restricts to a contravariant isomorphism $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} \to \Gamma_{\overline{\mathcal{G}}}^{\mathrm{ram}}$ which is the identity on the vertices. Thus, $\mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0) = \phi^{-1}(\mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0))$ is also contained in $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$. Moreover, because $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ is a forward and backward complete subgraph which contains $\mathrm{Ram}(\overline{\mathcal{F}}/\overline{F}_0) \cup \mathrm{Ram}(\overline{\mathcal{G}}/\overline{F}_0)$ and because of (132), we deduce that $\psi_{\overline{\mathbb{F}}_q}(G)$ is a subgraph of $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$. Combining these two inclusions yields the equality

$$\psi_{\overline{\mathbb{F}}_q}(G) = \Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}. \tag{133}$$

Moreover, we then obtain the equalities

$$\overline{W}(\overline{\mathcal{F}}) = V(G) = \psi_{\overline{\mathbb{F}}_q}^{-1}(V(\psi_{\overline{\mathbb{F}}_q}(G)) = \phi_{\overline{\mathbb{F}}_q}(V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})) \tag{134}$$

where the first equality holds by the choice of G and the definition of $\overline{W}(\overline{\mathcal{F}})$ in Definition 94(ii), the second equality holds as $\psi_{\overline{\mathbb{F}}_q}$ is an isomorphism and the third equality holds by the definition of $\psi_{\overline{\mathbb{F}}_q}$ and by the equality in (133).

Finally, we derive the desired identity $W(\mathcal{F}) = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ from the equalities

$$\begin{aligned} W(\mathcal{F}) &= \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\phi_{\overline{\mathbb{F}}_q}^{-1}(\overline{W}(\overline{\mathcal{F}}))) = \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\phi_{\overline{\mathbb{F}}_q}^{-1}(\phi_{\overline{\mathbb{F}}_q}(V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))) \\ &= \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(V(\overline{\mathbb{F}}_q \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}})) = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \end{aligned} \tag{135}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $W(\mathcal{F}) = \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\phi_{\overline{\mathbb{F}}_q}^{-1}(\overline{W}(\overline{\mathcal{F}})))$ in Definition 94(ii). The second equality holds by the equality in (133). The last two equalities hold because Lemma 124 provides that $\pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} = \overline{\mathbb{F}}_q \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}} \to \Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

For the last identity in the 'moreover'-part: This desired identity immediately follows from the definition of $\rho(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_n, W(\mathcal{F}))}{d^n}$ in Definition 94(ii) and from the equality $W(\mathcal{F}) = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$. □

A sufficient criterion for Conjecture 1(ii). The following Theorem 96 gives a sufficient criterion for Conjecture 1(ii) and it was proven in [BGS04, p. 15, Theorem 4.10]. It states that a polynomial-recursive tower $\mathcal{F} = (F_\nu)_\nu$ over a finite field of balanced degree d satisfies Conjecture 1(ii), i.e. $\nu(\mathcal{F}) = \mathrm{Split}(\mathcal{F}/F_0)$, if $\rho(\mathcal{F})$ vanishes.

This sufficient criterion in Theorem 96 exactly identifies the decisive set $V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \subseteq \mathbb{P}_{F_0}$ of places P which do not need to split on every level F_n/F_0 but can still contribute to the splitting rate $\nu(\mathcal{F})$, i.e. satisfy $\lim_{\nu\to\infty} N(F_\nu, P)/d^\nu > 0$.

In Corollary 184 and Corollary 185, we will even conclude that, up to finite constant field extensions and up to some very specific wild recursive towers for which there are no examples in the literature, the cases in which $\rho(\mathcal{F}) = 0$ holds can be characterized by the appearance of circles with unbalanced ramification ramification indices in all finite weakly connected components of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Theorem 96. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field of balanced degree which is defined by a polynomial. If $\rho(\mathcal{F}) = 0$, then $\nu(\mathcal{F}) = \#\,\mathrm{Split}(\mathcal{F}/F_0)$.*

4.4.2 HP-Graph of correspondences

Purpose of this subsection. In this subsection, we will connect the tower graph with the **HP-graph** in [HP12, p. 15, Definition 10] (see Definition 101(i)). There, the HP-graph is called the **geometric graph**. We will also mostly use the term geometric graph. Only if we talk about the geometric graph together with the Beelen-graph, we will prefer the term HP-graph. More concretely, the connection will be realized via some epimorphism of directed graphs in Proposition 104.

Different than in this thesis, [HP12, p. 3, Hypothesis] and [HP16] start with special correspondences of curves over a finite field k and, from there, construct recursive towers of smooth geometrically integral curves over k. Then the geometric graphs is associated with these correspondence of curves. Here, we will consider more general correspondences of curves than the ones in [HP12, p. 3, Hypothesis]. For instance, we will also include the normalizations of the correspondences from [HP12, p. 3, Hypothesis]. For these normalizations, the epimorphism from the tower graph to the geometric graph will even be an isomorphism in Proposition 104.

Moreover, in this subsection, we will also make preparations for Subsection 5.3 of the next chapter. There we will prove the first major result of this thesis in Theorem 155, which states that the tower graph has at most one finite balanced weakly connected component.

Correspondences and their geometric graphs. We will need some algebraic geometry on curves in this subsection and, for that, the reader is referred to [Liu02]. Let us only briefly recall the definitions of varieties and curves from [Liu02, p. 55, Definition 2.3.47] and [Liu02, p. 75, Definition 2.5.29].

Definition 97. *Let k be a field. An* ***affine variety over*** *k is an affine scheme associated with a finitely generated k-algebra.*

A ***variety over*** *k is a k-scheme which is covered by a finite number of affine open subschemes which are affine varieties over k.*

A ***curve over*** *k is a variety over k whose irreducible components all have dimension one.*

Moreover, we will denote the function field of an integral curve C over k by $K(C)$ and the pullback of a finite morphism $\varphi : D \to C$ of integral curves over k by $\varphi^* : K(X) \to K(D)$.

Correspondences. In the following, we will consider cases of prime self-correspondences of curves. But as our goal is to construct correspondences from recursive towers in Proposition 102(i), we will already include properties of these correspondences in the definition (e.g. X and Y are geometrically integral in Definition 98). Nonetheless, this will already cover the correspondences in [HP12, p. 3, Hypothesis].

Definition 98. *Let X and Y be projective geometrically integral curves over the perfect field k, suppose that X is smooth and let $\pi_i : Y \to X$ be finite morphisms of curves for all $i = 1, 2$. Moreover, let π'_1 and π'_2 be the canonical projection morphisms $X \times_k X \to X$ for the fiber product of $X \times_k X$ and let $\pi' : Y \to X \times_k X$ be the canonical morphism of varieties over k such that $\pi_i = \pi'_i \circ \pi'$ for all $i = 1, 2$.*

Then we call (Y, π_1, π_2) a ***correspondence on*** *X* ***of type*** *(d_1, d_2) if π_i are separable finite morphisms of degree d_i for all $i = 1, 2$ and $\pi' : Y \to X \times_k X$ restricts to a birational morphism $\pi : Y \to \pi'(Y)$ of curves over k.*

In [HP12, p. 3, Hypothesis], it is required that Y is a subset of the surface $X \times_k X$ and that $\pi_i : Y \to X$ are the restrictions of the canonical projection morphisms $\pi'_i : X \times_k X \to X$ for all $i = 1, 2$. For instance, $\pi'(Y) \subset X \times_k X$ in Definition 98 is a correspondence in the sense of [HP12, p. 3, Hypothesis].

Singular-recursive towers. In [HP12, p. 4], there is a recursive tower of curves associated with a correspondence, namely the singular-recursive tower.

Notice that the definition of the singular-recursive tower $(C_\nu)_\nu$ in Definition 99 differs slightly from the original definition in [HP12, p. 4]. We will discuss the differences after Definition 99.

Definition 99. *Let X be a smooth projective geometrically integral curve over the perfect field k and let (Y, π_1, π_2) be a correspondence on X of type (d, d) with the canonical projection morphisms $\pi_i : Y \to X$ for $i = 1, 2$. Then we define*

$$C_0 := X \text{ and } C_n := Y \times_X Y \times_X \cdots \times_X Y$$

where the fiber products runs over $n \in \mathbb{N}$ factors and where we take $\pi_2 : Y \to X$ for the left factors $Y \times_X$ and $\pi_1 : Y \to X$ for the right factors $\times_X Y$

Moreover, the sequence $(C_\nu)_\nu$ is called the ***singular-recursive tower*** *of the correspondence (Y, π_1, π_2) if C_n is an integral curve over k for all $n \in \mathbb{N}_0$, the canonical projection morphism $\rho_n : C_{n+1} \to C_n$ are finite separable of degree d for all $n \in \mathbb{N}_0$ and $g(C_n) \to \infty$ as $n \to \infty$.*

Finally, for a singular-recursive tower $(C_\nu)_\nu$, we also get the commutative diagram in Figure 4.6 with the canonical projection morphisms $\pi_{i,j,e} : C_i \to C_j$ on the factor in the middle of $C_i = C_e \times_X C_j \times_X C_{i-j-e}$ for all $0 \leq e \leq i - j$. Note that we have $\rho_n = \pi_{n+1,n,0}$, $\rho_0 = \pi_{1,0,0} = \pi_1$ and $\pi_{1,0,1} = \pi_2$.

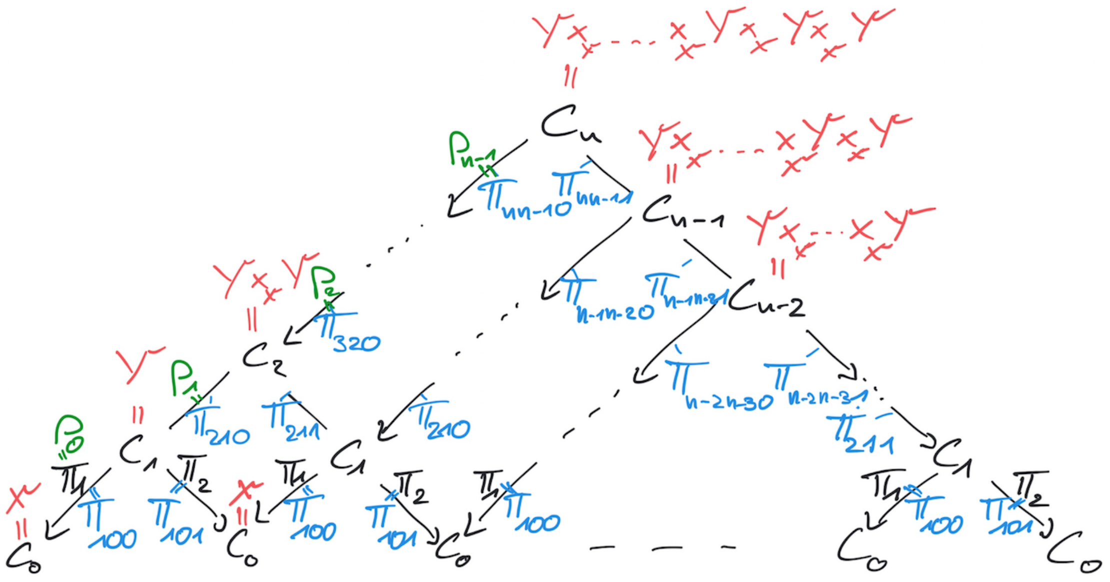


Figure 4.6: Pyramid of the singular-recursive tower

The differences. The definition in Definition 99 differs from the definition in [HP12, p. 4] in the following ways:

First, in accordance with our indexing of towers of function fields, the sequence of curves in Definition 99 already starts at the index zero and not at one as in [HP12, p. 4].

Second, we defined the curves C_n as fiber products which is isomorphic to the definition in [HP12, p. 4].

Third and finally, we suppose that the canonical morphisms have degree d and that $g(C_n) \to \infty$ as $n \to \infty$. Hence, the definition in Definition 99 is more restrictive. However, in the case that the genus does not tend to infinity, the tower becomes repetitive. Moreover, from some level m on, the morphisms in the singular-recursive tower $(C_{m+\nu})_\nu$ from [HP12, p. 4] must have some constant degree d too. Consequently, up to isomorphisms, the singular recursive towers $(C_{m+\nu})_\nu$ in [HP12, p. 4] are also singular-recursive towers in the sense of Definition 99. Hence, the definition in Definition 99 covers all the essential singular-recursive towers in [HP12, p. 4].

Etaleness. In the following, we will associate the commutative diagram in Figure 4.7with the correspondence (Y, π_1, π_2): Let $\overline{k}$ be an algebraic closure of the perfect field k. For any

$$\begin{tikzcd}
\overline{X} \arrow[r, "\pi_X"'] & X \\
\overline{Y} \arrow[u, "\overline{\pi}_2"] \arrow[d, "\overline{\pi}_1"] \arrow[r, "\pi_Y"'] & Y \arrow[u, "\pi_2"] \arrow[d, "\pi_1"] \\
\overline{X} \arrow[r, "\pi_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{X}^c \arrow[r, "\sim", "\Theta_X"'] & \overline{X}(\overline{k}) \arrow[r, "\sim", "\pi_{X,\overline{k}}"'] & X(\overline{k}) \\
\overline{Y}^c \arrow[u, "\overline{\pi}_2^c"] \arrow[d, "\overline{\pi}_1^c"] \arrow[r, "\sim", "\Theta_Y"'] & \overline{Y}(\overline{k}) \arrow[u, "\overline{\pi}_{2,\overline{k}}"] \arrow[d, "\overline{\pi}_{1,\overline{k}}"] \arrow[r, "\sim", "\pi_{Y,\overline{k}}"'] & Y(\overline{k}) \arrow[u, "\pi_{2,\overline{k}}"] \arrow[d, "\pi_{1,\overline{k}}"] \\
\overline{X}^c \arrow[r, "\sim", "\Theta_X"'] & \overline{X}(\overline{k}) \arrow[r, "\sim", "\pi_{X,\overline{k}}"'] & X(\overline{k})
\end{tikzcd}$$

Figure 4.7: Commutative diagrams for rational and closed points on curves

curves C and D over k and finite morphism $\varphi : D \to C$, let us denote the $\overline{k}$-base changes of C and φ by $\overline{C} := C \times_k \overline{k}$ and $\overline{\varphi} := \varphi \times_k \overline{k} : \overline{D} \to \overline{C}$, respectively, and let π_C be the canonical projection morphism $\pi_C : \overline{C} \to C$.

By the universal property of the fiber product, we then obtain that $\times_k \overline{k}$ is a functor from the category of projective geometrically integral curves over k to the category of projective integral curves over $\overline{k}$. Therefore, the first of the two diagrams in Figure 4.7 is commutative.

Although, $\overline{C}$ and $\overline{D}$ are no varieties over k, they are schemes over k. The $\overline{k}$-rational points $C(\overline{k})$ of a k-scheme C can be identified with the morphisms $\operatorname{Spec}(\overline{k}) \to C$ of k-schemes. Thus, for any morphism $\varphi : D \to C$ of k-schemes, the $\operatorname{Hom}(\operatorname{Spec}(\overline{k}), \cdot)$-functor provides a morphism $\varphi_{\overline{k}} : D(\overline{k}) \to C(\overline{k})$ of sets making the right two squares of the second diagram in Figure 4.7 commutative too. Moreover, $\pi_{C,\overline{k}}$ is a bijection by [Liu02, p.92, Proposition 2.18(a)].

Next, let C^c be the closed points of the curve C. Any finite morphism $\varphi : D \to C$ of projective integral curves over k maps closed points to closed points by Lemma 251 and, hence, restricts to a map $\varphi^c : C^c \to D^c$. Also, by [Liu02, p. 92, Proposition 2.18], there is a canonical bijection $\theta_C : \overline{C}^c \to \overline{C}(\overline{k})$ for any geometrically integral curve C over k such that the rest of the second diagram in 4.7 is commutative too.

Finally, let us denote the composition of $\pi_{C,\overline{k}}$ and θ_C by

$$\gamma_C := \pi_{C,\overline{k}} \circ \theta_C. \tag{136}$$

Definition 100. *Let X be a smooth projective geometrically integral curve over the perfect field k and let (Y, π_1, π_2) be a correspondence on X of type (d, d). Moreover, let $\overline{k}$ be an algebraic closure of k.*

*Then the morphism π_i is called **etale** at the $\overline{k}$-rational point $y \in Y(\overline{k})$ if π_i is etale at the closed point $\gamma_Y^{-1}(y)$ in the sense of [Liu02, p. 139, Definition 3.17].*

Geometric graph. With any correspondence (Y, π_1, π_2) on X, [HP12, p. 15, Definition 10] associates its geometric graph $\mathcal{G}_\infty$.

Definition 101. *Let X be a smooth projective geometrically irreducible curve over the perfect field k and let (Y, π_1, π_2) be a correspondence on X of type (d, d). Moreover, let $\overline{k}$ be an algebraic closure of k.*

*(i) The **geometric graph** or **HP-graph** $\mathcal{G}_\infty$ of (Y, π_1, π_2) is defined as the directed graph*

$$\mathcal{G}_\infty := \left(X(\overline{k}), Y(\overline{k}), \pi_{1,\overline{k}} \times \pi_{2,\overline{k}}\right)$$

i.e. the vertex set $X(\overline{k})$ is the set of $\overline{k}$-rational points on X, the edge set $Y(\overline{k})$ is the set of $\overline{k}$-rational points on Y and the edge map is

$$\pi_{1,\overline{k}} \times \pi_{2,\overline{k}} : Y(\overline{k}) \to X(\overline{k})^2 \text{ via } Q \mapsto \left(\pi_{1,\overline{k}}(Q), \pi_{2,\overline{k}}(Q)\right)$$

where $\pi_{i,\overline{k}} : Y(\overline{k}) \to X(\overline{k})$ denotes the canonical map which is induced by the morphism π_i for all $i = 1, 2$.

*(ii) We call an edge $y \in E(\mathcal{G}_\infty) = Y(\overline{k})$ **etale** if π_1 and π_2 are etale at y. Otherwise, we call $y \in E(\mathcal{G}_\infty)$ a **non-etale** edge.*

*The **singular part** $\mathcal{G}_{\text{sing}}$ is defined as the intersection subgraph of all the forward and backward complete subgraphs of $\mathcal{G}_\infty$ which contains all the non-etale edges y.*

Notice that, by the last comment in Definition 66(iii), $\mathcal{G}_{\text{sing}}$ is again a forward and backward complete subgraph of $\mathcal{G}_\infty$ which contains all the non-etale edges y. Hence, $\mathcal{G}_{\text{sing}}$ the smallest (i.e. unique minimal with respect to the subgraph relation) forward and backward complete subgraph of $\mathcal{G}_\infty$ which contains all the non-etale edges y.

Correspondences constructed from recursive towers of function fields. Next, we will construct correspondences of curves from recursive towers of function fields. For that, remember that all constant fields are perfect in this thesis by Assumption 1.

Proposition 102. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the (perfect) field k of balanced degree d which is defined by the pair (σ, F_0) and let $\overline{k}$ be the algebraic closure of k which is contained in the domain of σ in Definition 5(ii). Moreover, let $\iota : F_0 \to F_1$ be the inclusion morphism of k-algebras. Then the following hold:*

(i) There is some smooth projective geometrically integral curve X over k and some correspondence (Y, π_1, π_2) on X of type (d, d) and isomorphisms $\phi_X : K(X) \to F_0$ and $\phi_Y : K(Y) \to F_1$ of k-algebras making the first diagram in Figure 4.8 commutative.

(ii) In (i), the correspondence (Y, π_1, π_2) can be chosen as a closed subvariety of $X \times_k X$.

(iii) In (i), the correspondence (Y, π_1, π_2) can be chosen as a smooth curve over k.

(iv) For any choice of (Y, π_1, π_2) in (i), the sequence $(C_\nu)_\nu$ in Definition 99 satisfies the requirements of the singular-recursive tower of (Y, π_1, π_2) and there are isomorphisms $\psi_n : F_n \to K(C_n)$ of k-algebras for all $n \in \mathbb{N}$ making the second diagram in Figure 4.8 commutative where ι_n denotes the inclusion morphism $F_n \to F_{n+1}$.

Moreover, let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the finite subcategories of k-algebras consisting of the k-algebras and morphisms which are depicted in the commutative diagrams in Figure 4.9 and Figure 4.10, respectively. Then the isomorphisms $\psi_{j-i} \circ \sigma^{-i} : F_{i,j} \to K(C_{j-i})$ with $0 \le i \le j$ induce an isomorphism $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ of categories.

Also note that the diagram in Figure 4.10 is the image of the diagram in Figure 4.6 under the functor which maps an integral curve C to its function field $K(C)$.

$$\begin{tikzcd}
F_0 \arrow[d, "\sigma\circ\iota"'] & K(X) \arrow[l, "\sim", "\phi_X"'] \arrow[d, "\pi_2^*"] \\
F_1 & K(Y) \arrow[l, "\sim", "\phi_Y"'] \\
F_0 \arrow[u, "\iota"] & K(X) \arrow[l, "\sim", "\phi_X"'] \arrow[u, "\pi_1^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{n+1} \arrow[r, "\psi_{n+1}", "\sim"'] & K(C_{n+1}) \\
F_n \arrow[u, "\iota_n"] \arrow[r, "\psi_n", "\sim"'] & K(C_n) \arrow[u, "\rho_n^*"']
\end{tikzcd}$$

Figure 4.8: Commutative diagrams connecting recursive towers of function fields and singular-recursive towers

Proof. For (i), (ii) and (iii): We will find a curve $Y \subset X \times_k X$ as in (ii) and a smooth curve $\tilde{Y}$ as in (iii) simultaneously. This will then especially imply (i).

First, let $\mathcal{C}_1$ be the category of normal projective integral curves over k with finite morphisms and $\mathcal{C}_2$ be the category of function fields over k with monomorphisms of k-algebras. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent by [Liu02, p.278, Remark 3.14]. More precisely, the covariant functor

$$\Psi : \mathcal{C}_1 \to \mathcal{C}_2 \text{ via } \Psi(C) := K(C) \text{ and } \Psi(\varphi) := \varphi^* \tag{137}$$

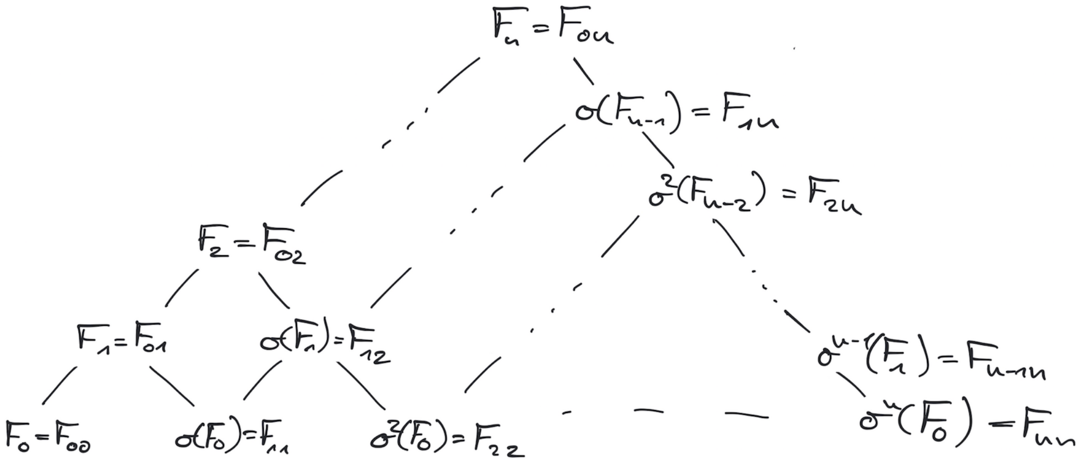


Figure 4.9: Pyramid of function fields

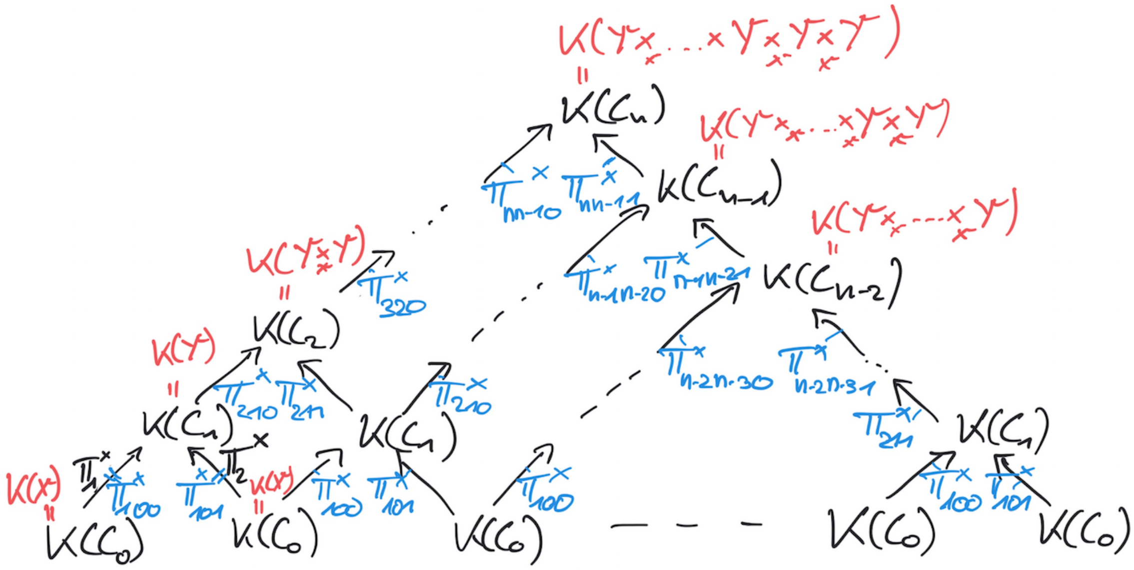


Figure 4.10: Pyramid of function fields of singular-recursive towers

induces this categorial equivalence.

Now, we may also replace 'normal' with 'smooth' in $\mathcal{C}_1$ because any integral projective curve over the perfect field k is normal if and only if it is smooth by [Liu02, p.128, Example 2.9] and [Liu02, p.142, Corollary 3.33]. Consequently, this categorial equivalence provides

smooth projective integral curves X and $\tilde{Y}$ over k,
finite morphisms $\tilde{\pi}_1 : \tilde{Y} \to X$ and $\tilde{\pi}_2 : \tilde{Y} \to X$,
isomorphisms $\phi_X : K(X) \to F_0$ and $\phi_{\tilde{Y}} : K(\tilde{Y}) \to F_1$ of k-algebras (138)

such that the two squares in the first diagram in Figure 4.11 are commutative.

Second, let π'_1 and π'_2 be the canonical projection morphisms $X \times_k X \to X$ of the fiber product $X \times_k X$ over k (see the second diagram in Figure 4.11). The universal property of the fiber product then supplies some unique morphism $\pi_0 : \tilde{Y} \to X \times_k X$ satisfying the equalities $\tilde{\pi}_1 = \pi'_1 \circ \pi_0$ and $\tilde{\pi}_2 = \pi'_2 \circ \pi_0$. As projective varieties are complete, the image $Y = \pi_0(\tilde{Y})$ of the irreducible projective curve $\tilde{Y}$ over k is an irreducible closed subset of the projective variety $X \times_k X$ over k. Therefore, Y can be equipped with a structure sheaf

$$\begin{tikzcd}
F_0 \arrow[d, "\sigma\circ\iota"'] & K(X) \arrow[l, "\sim", "\phi_X"'] \arrow[d, "\tilde{\pi}_2^*"] \arrow[dr, bend left, "\pi_2^*"] & \\
F_1 & K(\tilde{Y}) \arrow[l, "\sim", "\phi_{\tilde{Y}}"'] & K(Y) \arrow[l, "\pi^*"'] \\
F_0 \arrow[u, "\iota"] & K(X) \arrow[l, "\sim", "\phi_X"'] \arrow[u, "\tilde{\pi}_1^*"'] \arrow[ur, "\pi_1^*"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X & & & \\
\tilde{Y} \arrow[u, "\tilde{\pi}_2"] \arrow[d, "\tilde{\pi}_1"', "\sim"] \arrow[r, "\pi"] & Y \arrow[ul, "\pi_2"'] \arrow[dl, "\pi_1"] \arrow[r, phantom, "\subseteq"] & X \times_k X \arrow[ull, "\pi_2'"'] \arrow[dll, "\pi_1'"] \\
X & & &
\end{tikzcd}$$

Figure 4.11: Commutative diagrams connecting the base extensions in recursive towers and correspondences

such

$$Y \text{ becomes an irreducible closed subvariety of } X \times_k X \text{ over } k \tag{139}$$

and π_0 factorizes through a morphism $\pi : \tilde{Y} \to Y$ and the morphism which is induced by the inclusion $Y \subseteq X \times_k X$. Moreover, as $\tilde{Y}$ is reduced, we may choose the structure sheaf of Y such that Y becomes reduced by [Liu02, p.60, Proposition 4.2(c),(d)]. Thus, if we choose $\pi_i : Y \to X$ to be the restriction of the canonical projection morphisms $\pi_i' : X \times_k X \to X$ for all $i = 1, 2$, then the rest of the diagrams in Figure 4.11 becomes commutative too.

Third, we derive the equalities and inclusion

$$\begin{aligned}
K(\tilde{Y}) &= \phi_{\tilde{Y}}^{-1}(F_1) = \phi_{\tilde{Y}}^{-1}(F_0 \cdot \sigma(F_0)) = \phi_{\tilde{Y}}^{-1}(F_0) \cdot \phi_{\tilde{Y}}^{-1}(\sigma(F_0)) \\
&= (\tilde{\pi}_1^* \circ \phi_X^{-1})(F_0) \cdot (\tilde{\pi}_2^* \circ \phi_X^{-1})(F_0) = ((\pi^* \circ \pi_1^*) \circ \phi_X^{-1})(F_0) \cdot ((\pi^* \circ \pi_2^*) \circ \phi_X^{-1})(F_0) \\
&= \pi^*((\pi_1^* \circ \phi_X^{-1})(F_0) \cdot (\pi_2^* \circ \phi_X^{-1})(F_0)) \subseteq \pi^*(K(Y))
\end{aligned} \tag{140}$$

where the equalities and inclusion hold by the following reasonings: The first and third equalities hold because $\phi_{\tilde{Y}} : F_1 \to K(\tilde{Y})$ is an isomorphism of rings by its choice in (138), The second equality holds because the definition that $\mathcal{F}$ is recursively defined by the pair (σ, F_0) in Definition 5(ii) implies the equality $F_1 = F_0 \cdot \sigma(F_0)$. The fourth equality holds because of the equalities $\phi_{\tilde{Y}}^{-1}(F_0) = (\phi_{\tilde{Y}}^{-1} \circ \iota)(F_0)$ and $\phi_{\tilde{Y}}^{-1}(\sigma(F_0)) = (\phi_{\tilde{Y}}^{-1} \circ (\sigma \circ \iota))(F_0)$ and because of the commutativity of the two squares in the first diagram in Figure 4.11. The fifth equality holds because we chose π to satisfy the equalities $\tilde{\pi}_i = \pi_i \circ \pi$ for all $i = 1, 2$. The last equality holds because π^* is a morphism of rings. The inclusion holds because $(\pi_1^* \circ \phi_X^{-1})(F_0)$ and $(\pi_2^* \circ \phi_X^{-1})(F_0)$ are subfields of $K(Y)$.

Therefore, this inclusion in (140) implies that the monomorphism $\pi^* : K(Y) \to K(\tilde{Y})$ is also surjective and, hence, an isomorphism. In particular, because of this and because $\tilde{Y}$ is a curve over k, we conclude that the integral variety Y over k is also a curve over k. Moreover, by the definition of birational morphisms in [Sta22, Tag 01RO], this also implies that

$$\pi : \tilde{Y} \to Y \text{ is a birational morphism of integral curves over } k. \tag{141}$$

Fourth, define

$$\phi_Y : K(Y) \to F_1 \text{ as the isomorphism } \phi_Y := \pi^* \circ \phi_{\tilde{Y}} \text{ of } k\text{-algebras.} \tag{142}$$

By the definition of a tower in Definition 2(i), the function field F_i has full constant field k for all $i = 0, 1$ and, thus, satisfies the equality $F_i \cap \overline{k} = k$. Because of these equalities, because k is a perfect field by assumption and because $\overline{k}/k$ is therefore a Galois extension, [Coh91, p. 188, Theorem 5.5] provides that $\overline{k}/k$ and F_i/k are linearly disjoint for all $i = 0, 1$. Thus, any tensor product $\overline{k} \otimes_k F_i$ of k-algebras (with the inclusion morphism) is a field for all $i = 0, 1$. Furthermore, because $\phi_X : K(X) \to F_0$ (resp. $\phi_{\tilde{Y}} : K(\tilde{Y}) \to F_1$; resp. $\phi_Y : K(Y) \to F_1$) is an isomorphism by its choice in (137) (resp. in (137); resp. in (142)), because of the universal property of tensor products and because the field extension $\overline{k}/k$ is flat, we also obtain that $\overline{k} \otimes_k K(X) \cong \overline{k} \otimes_k F_0$ (resp. $\overline{k} \otimes_k K(\tilde{Y}) \cong \overline{k} \otimes_k F_1$; resp $\overline{k} \otimes_k K(Y) \cong \overline{k} \otimes_k F_1$) are fields and, therefore, that

$$K(X)/k \text{ (resp. } K(\tilde{Y})/k\text{; resp. } K(Y)/k\text{) and } \overline{k}/k \text{ are also linearly disjoint.} \tag{143}$$

In particular, because of this and because X, $\tilde{Y}$ and Y are integral curves over k, [Liu02, p.91, Corollary 2.14(c)] then supplies that

$$X, \tilde{Y} \text{ and } Y \text{ are geometrically integral curves over } k. \tag{144}$$

Next, we notice that $\pi_i : Y \to X$ is non-constant because it factorizes the finite and, hence, non-constant morphism $\tilde{\pi}_i$ for $i = 1, 2$. Thus [Liu02, p.277, Lemma 3.10(i),(ii)] implies that

$$\pi_i \text{ is a finite morphism.} \tag{145}$$

Moreover, we have the equalities

$$\phi_{\tilde{Y}}^{-1}(\sigma^{i-1}(F_0)) = (\phi_{\tilde{Y}}^{-1} \circ (\sigma^{i-1} \circ \iota))(F_0) = (\tilde{\pi}_i^* \circ \phi_X^{-1})(F_0) = \tilde{\pi}_i^*(K(X)) \tag{146}$$

and

$$\phi_Y^{-1}(\sigma^{i-1}(F_0)) = (\phi_Y^{-1} \circ (\sigma^{i-1} \circ \iota))(F_0) = (\pi_i^* \circ \phi_X^{-1})(F_0) = \pi_i^*(K(X)) \tag{147}$$

for all $i = 1, 2$ where the first equalities hold by the definition of the inclusion morphism $\iota : F_0 \to F_1$, the second equalities hold by the commutativity of the first diagram in 4.11 and the third equalities hold since $\phi_X : K(X) \to F_0$ is an isomorphism by its choice in (138).

Consequently, the equality in (146) (resp. in (147)) implies that the function field extensions $F_1/\sigma^{i-1}(F_0)$ and $K(\tilde{Y})/\tilde{\pi}_i^*(K(X))$ (resp. $K(Y)/\pi_i^*(K(X))$) are isomorphic for all $i = 1, 2$. In particular, because of this, because the definition of recursive towers in Definition 5(ii) provides that $F_1/\sigma^{i-1}(F_0)$ is a separable extension of degree d and because of (138) (resp. (145)), we deduce that

$$\tilde{\pi}_i \text{ (resp. } \pi_i\text{) is a separable finite morphism of degree } d \tag{148}$$

for all $i = 1, 2$.

In summary, by (138), by (139), by (141), by (144), by (148), by the definition of correspondences in Definition 98 and by the commutativity of the left diagram in Figure 4.11, we found the desired X, ϕ_X and $\tilde{Y}$, $\phi_{\tilde{Y}}$ in (iii) (resp. Y, ϕ_Y in (ii)).

For (iv): Let (Y, π_1, π_2) and X be as in (i) and let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. First, by Lemma 10(i) and Lemma 10(ii), we obtain the equalities $\sigma^i(F_\varepsilon) = \sigma^i(F_{0,\varepsilon}) = F_{i,i+\varepsilon}$ for all $\varepsilon = 0, 1$ and $i \in \mathbb{N}_0$. Thus, the two lowest rows in the diagrams in Figure 4.9 and Figure 4.12 are isomorphic (as subcategories of k-algebras) via the restrictions of the isomorphisms σ^i to isomorphisms $F_\varepsilon \to \sigma^i(F_\varepsilon) = F_{i,i+\varepsilon}$ for all $\varepsilon = 0, 1$ and all $i \in \mathbb{N}_0$.

Now, by Lemma 10(iii), the extensions $F_{i-1,j}/F_{i,j}$ and $F_{i,j+1}/F_{i,j}$ are linearly disjoint for all $1 \leq i \leq j$ and, thereby, the composite field $F_{i-1,j} \cdot F_{i,j+1} = F_{i-1,j+1}$ is a tensor product $F_{i-1,j} \otimes_{F_{i,j}} F_{i,j+1}$ of k-algebras (with the inclusion morphisms). Consequently, we iteratively obtain that the complete diagrams in Figure 4.9 and Figure 4.12 are isomorphic. In particular, this means that all the tensor products in the diagram in Figure 4.12 are fields.

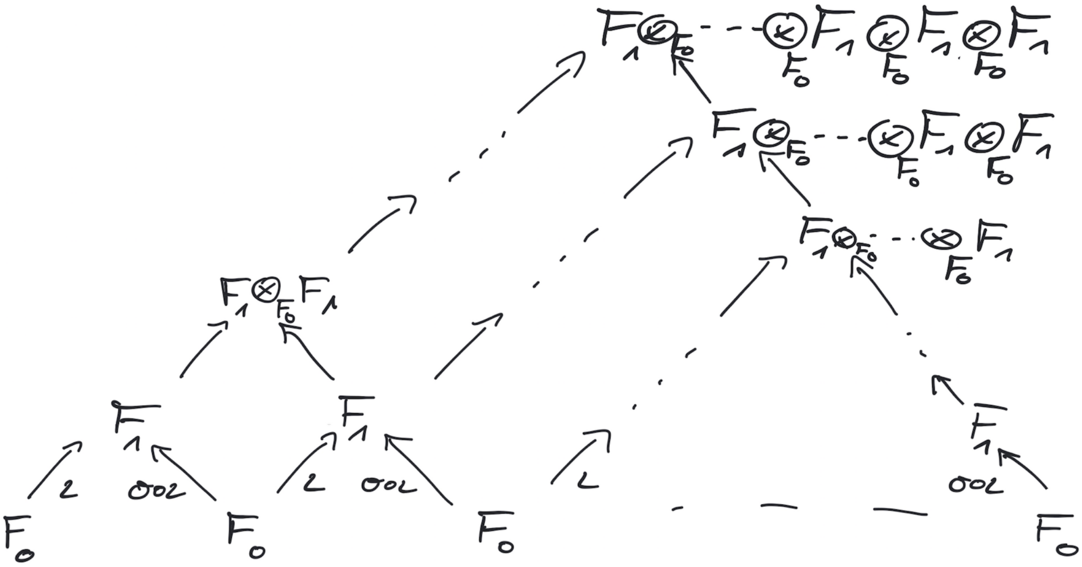


Figure 4.12: Pyramid of tensor products of function fields

Second, the isomorphisms ϕ_X and ϕ_Y in (i) and the commutativity of the first diagram in Figure 4.8 ensure that the diagrams in Figure 4.12 and Figure 4.13 are isomorphic too. In particular, this again means that

$$\text{all the tensor products in the diagram in Figure 4.13 are fields.} \tag{149}$$

Third, we consider the morphism $\pi_{n,0,n} : C_n \to C_0 = X$. For all $n \in \mathbb{N}$, (149) then implies that we can iteratively apply the equivalence in Lemma 255(ii) to $\pi_{n,0,n} : C_n \to X$ and $\pi_1 = \pi_{1,0,0} : Y \to X$. By that and the other two items in Lemma 255, we conclude that C_{n+1} is also an integral curve over k, that $\rho_n = \pi_{n+1,n,0} : C_{n+1} \to C_n$ is finite for all $n \in \mathbb{N}_0$ and that the diagrams in Figure 4.13 and Figure 4.10 are isomorphic as well. Hence, combining the three isomorphisms of the diagrams from above yields the desired isomorphism $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ in the 'moreover'-part.

Second to last, the definition of the assertion that $\mathcal{F} = (F_\nu)_\nu$ is a recursive tower of function fields of constant degree d over k in Definition 5(ii) especially provides that $\mathcal{F} = (F_\nu)$ is a tower of function fields of constant degree d. Then Definition 2(i) implies that, for all $n \in \mathbb{N}_0$, the function field F_n has full constant field k, that the extension $F_{n+1} \,/F_n$ is separable of degree d and that $g(F_n) \to \infty$ as $n \to \infty$.

Finally, if we apply the isomorphism Ψ, we derive the same statements for the function fields $K(C_n)$ and the corresponding extensions in the diagram in Figure 4.10. Hence, this supplies that the canonical morphisms $\rho_n : C_{n+1} \to C_n$ are finite separable morphisms of degree d.

All together, by the definition of singular-recursive towers in Definition 99, we established that $(C_\nu)_\nu$ is indeed a singular-recursive tower and that there are isomorphisms $\psi_n : F_n \to K(C_n)$ of k-algebras making the second diagram in 4.8 commutative for all $n \in \mathbb{N}_0$. Hence, the 'main'-part in (iv) also follows. $\square$

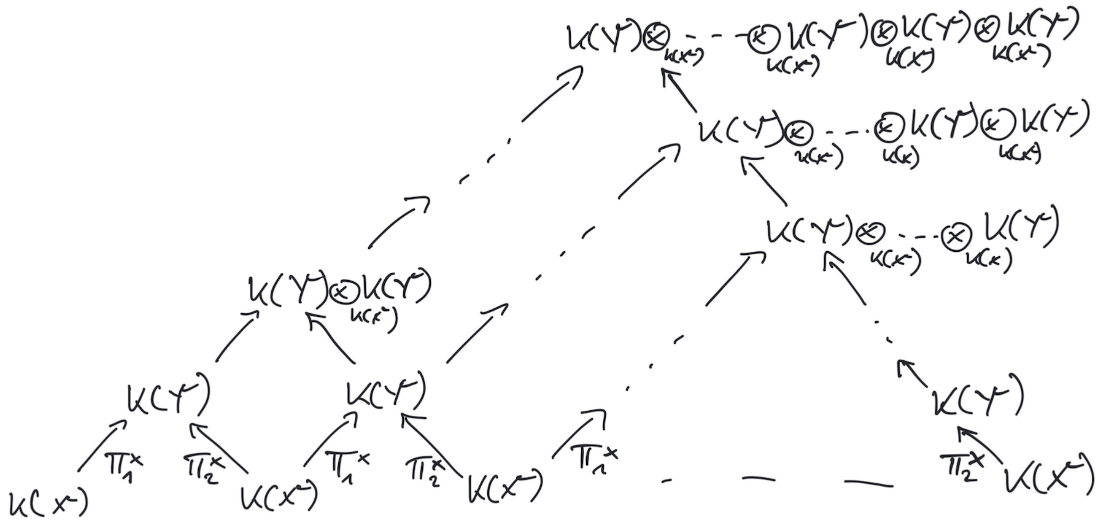


Figure 4.13: Pyramid of tensor products of function fields of curves

Epimorphism from the tower graph to the geometric graph. Next, via the epimorphism ψ in Proposition 104, we will relate the tower graph of a recursive tower to the geometric graphs of the correspondences which we constructed in Proposition 102. Moreover, if the correspondence is chosen to be smooth as in Proposition 102(iii), ψ will even be an isomorphism. For that, we will first prove the following Lemma 103.

Note that, for all monomorphism $\tau : \overline{F} \to \overline{E}$ of $\overline{k}$-algebras, we will denote the preimage map $\mathbb{P}_{\overline{E}} \to \mathbb{P}_{\overline{F}}$, $\overline{Q} \mapsto \tau^{-1}(\overline{Q})$ by τ^{-1}.

Lemma 103. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over k of balanced degree d and let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$. Let $\bar{\iota} : \overline{F}_0 \to \overline{F}_1$ be the inclusion morphism of k-algebras.*

Moreover, as in in Proposition 102(i), let X be a smooth projective geometrically integral curve X over k, let (Y, π_1, π_2) be a correspondence on X of type (d, d) and let $\phi_X : K(X) \to F_0$ and $\phi_Y : K(Y) \to F_1$ be isomorphisms of k-algebras. Let $\gamma_X : \overline{X}^c \to X(k)$ and $\gamma_Y : \overline{Y}^c \to Y(k)$ be the bijections in (136) and let $\overline{\pi}_i^c$ and $\pi_{i,\overline{k}}$ be the maps in the second commutative diagram in Figure 4.7 for all $i = 1, 2$.

Finally, let A_i be the set of places in $\overline{F}_1$ which are ramified in $\overline{F}_1/\sigma^{i-1}(\overline{F}_0)$ and let B_i be the set of $\overline{k}$-rational points on Y at which π_i is not etale for all $i = 1, 2$. Then the following hold:

(i) There are bijections α_X, α_Y, β_X and a surjection β_Y making the diagram in Figure 4.14 commutative.

(ii) For all $i = 1, 2$, the composition $\phi := \gamma_Y \circ \beta_Y \circ \alpha_Y$ restricts to a

map $A_i \to B_i$ and even a bijection $\mathbb{P}_{\overline{F}_1} \backslash \phi^{-1}(B_i) \to Y(\overline{k}) \backslash B_i$.

(iii) If Y is smooth, then ϕ is even a bijection which also restricts to bijections $A_i \to B_i$ and $\mathbb{P}_{\overline{F}_1} \backslash A_i \to Y(\overline{k}) \backslash B_i$ for all $i = 1, 2$.

Proof. For (i): First, we notice that the tensor product $\otimes_k \overline{k}$ is a functor on the category of k-algebras which preserves isomorphisms since the extension $\overline{k}/k$ is flat. Thus, the middle two squares of the diagram in are commutative by the commutativity of the left two squares of the diagram in .

$$\begin{tikzcd}
\mathbb{P}_{\overline{F}_0} \arrow[r, "\sim", "\alpha_X"'] & \mathbb{P}_{K(\overline{X})} \arrow[r, "\sim", "\beta_X"'] & \overline{X}^c \arrow[r, "\sim", "\tau_X"'] & X(\overline{k}) \\
\mathbb{P}_{\overline{F}_1} \arrow[u, "(\sigma\circ\overline{\iota})^{-1}"] \arrow[d, "\overline{\iota}^{-1}"'] \arrow[r, "\sim", "\alpha_Y"'] & \mathbb{P}_{K(\overline{Y})} \arrow[u, "(\overline{\pi}_2^*)^{-1}"'] \arrow[d, "(\overline{\pi}_1^*)^{-1}"] \arrow[r, two heads, "\beta_Y"'] & \overline{Y}^c \arrow[u, "\overline{\pi}_2^c"'] \arrow[d, "\overline{\pi}_1^c"] \arrow[r, "\sim", "\tau_Y"'] & Y(\overline{k}) \arrow[u, "\pi_{2,\overline{k}}"'] \arrow[d, "\pi_{1,\overline{k}}"] \\
\mathbb{P}_{\overline{F}_0} \arrow[r, "\sim", "\alpha_X"'] & \mathbb{P}_{K(\overline{X})} \arrow[r, "\sim", "\beta_X"'] & \overline{X}^c \arrow[r, "\sim", "\tau_X"'] & X(\overline{k})
\end{tikzcd}$$

Figure 4.14: Commutative diagram connecting places and rational points

$$\begin{tikzcd}
\overline{F}_0 \arrow[d, hook, "\sigma\circ\overline{\iota}"'] & F_0\otimes_k\overline{k} \arrow[l, "\sim"'] \arrow[d, hook, "(\sigma\circ\iota)\otimes_k\overline{k}"] & K(X)\otimes_k\overline{k} \arrow[l, "\sim"', "\phi_X\otimes_k\overline{k}"] \arrow[d, hook, "\pi_2^*\otimes_k\overline{k}"] & K(\overline{X}) \arrow[l, "\sim"'] \arrow[d, hook, "\overline{\pi}_2^*"] \\
\overline{F}_1 & F_1\otimes_k\overline{k} \arrow[l, "\sim"'] & K(Y)\otimes_k\overline{k} \arrow[l, "\sim"', "\phi_Y\otimes_k\overline{k}"] & K(\overline{Y}) \arrow[l, "\sim"'] \\
\overline{F}_0 \arrow[u, hook, "\overline{\iota}"] & F_0\otimes_k\overline{k} \arrow[l, "\sim"'] \arrow[u, hook, "\iota\otimes_k\overline{k}"'] & K(X)\otimes_k\overline{k} \arrow[l, "\sim"', "\phi_X\otimes_k\overline{k}"] \arrow[u, hook, "\pi_1^*\otimes_k\overline{k}"'] & K(\overline{X}) \arrow[l, "\sim"'] \arrow[u, hook, "\overline{\pi}_1^*"']
\end{tikzcd}$$

Figure 4.15: Commutative diagram connecting the base extensions in the geometric tower and the base field extension of the correspondences

Second, since we already established in (278) that $K(X)/k$ (resp. $K(Y)/k$) and $\overline{k}/k$ are linearly disjoint, [Liu02, p.91, Corollary 2.14] provides $K(\overline{X}) \cong K(X) \otimes_k \overline{k}$ (resp. $K(\overline{Y}) \cong K(Y) \otimes_k \overline{k}$) and, hence, the right two squares of the diagram in Figure 4.15 are commutative as well.

Third, as $F_0 \otimes_k \overline{k} \cong K(\overline{X})$ (resp. $F_1 \otimes_k \overline{k} \cong K(\overline{Y})$) is a field, it must be isomorphic to the compositum $\overline{F}_0 = \overline{k} \cdot F_0$ (resp. $\overline{F}_0 = \overline{k} \cdot F_0$) and, hence, the left two squares of the diagram in Figure 4.15 is commutative too.

Fourth, in the diagram in Figure 4.7, let us denote the composition of the isomorphisms in the first (and last) row by $\varphi_X : K(\overline{X}) \to \overline{F}_0$ and of the second row as $\varphi_Y : K(\overline{Y}) \to \overline{F}_1$. Then their inverse isomorphisms φ_X^{-1} and φ_Y^{-1} induce the desired bijections

$$\alpha_X : \mathbb{P}_{\overline{F}_0} \to \mathbb{P}_{K(\overline{X})},\ \overline{P} \mapsto \varphi_X^{-1}(\overline{P}) \quad \text{and} \quad \alpha_Y : \mathbb{P}_{\overline{F}_1} \to \mathbb{P}_{K(\overline{Y})},\ \overline{Q} \mapsto \varphi_Y^{-1}(\overline{Q}) \tag{150}$$

at the rows in the left two squares of the diagram in Figure 4.14. In particular, we get that these two squares are commutative.

Moreover, for all $i = 1, 2$ and all $\overline{Q} \in \mathbb{P}_{\overline{F}_1}$, we compute

$$\begin{aligned}\alpha_Y(\overline{Q}) \cap \overline{\pi}_i^*(K(\overline{X})) &= \overline{\pi}_i^*\left((\overline{\pi}_i^*)^{-1}\left(\alpha_Y(\overline{Q})\right)\right) = \overline{\pi}_i^*\left(\alpha_X\left((\sigma^{i-1} \circ \overline{\iota})^{-1}(\overline{Q})\right)\right) \\ &= \alpha_y\left((\sigma^{i-1} \circ \overline{\iota})\left((\sigma^{i-1} \circ \overline{\iota})^{-1}(\overline{Q})\right)\right) = \alpha_y\left(\overline{Q} \cap \sigma^{i-1}(\overline{F}_0)\right)\end{aligned} \tag{151}$$

where the equalities hold by the following reasonings: The first equality holds because π_i^* is a map $K(X) \to K(Y)$ with image $\pi_i^*(K(X))$ and because of a quick straight forward computation. The second equality holds by the commutativity of the left two squares in the diagram in Figure 4.14. The third equality holds by the choices of α_X and α_Y in (150) and by the commutativity of the outer square in the diagram in Figure 4.15. The last equality holds because $\sigma^{i-1} \circ \overline{\iota}$ is a map $\overline{F}_0 \to \overline{F}_1$ with image $\sigma^{i-1}(\overline{F}_0)$ and because of a quick straight forward computation.

Consequently, because of this equality in (151), because φ_Y^{-1} is an isomorphism of $\overline{k}$-algebras and because of the choice of α_Y in (150),

$$\begin{aligned}&\overline{Q}/\overline{Q} \cap \sigma^{i-1}(F_0) \text{ is unramified in } \overline{F}_1/\sigma^{i-1}(F_0) \text{ if and only if}\\ &\alpha_Y(\overline{Q})/\alpha_Y(\overline{Q}) \cap \overline{\pi}_i^*(K(\overline{X})) \text{ is unramified in } K(Y)/\overline{\pi}_i^*(K(\overline{X}))\end{aligned} \tag{152}$$

for all $i = 1, 2$.

Fifth, it is well known that, for any integral projective curve C over $\overline{k}$, the discrete valuation ring $\mathcal{O}_P$ of any place P in $K(C)$ contains the local ring $\mathcal{O}_{C,p}$ of some unique closed point p in C and, moreover,

$$\text{if } C \text{ is smooth at } p, \text{ then we even have the equality } \mathcal{O}_P = \mathcal{O}_{C,p}. \tag{153}$$

Because it is also well known that any of these local rings is contained in some of the discrete valuation rings of places, this provides surjective maps $\beta_X : \mathbb{P}_{K(\overline{X})} \to \overline{X}^c$ and $\beta_Y : \mathbb{P}_{K(\overline{Y})} \to \overline{Y}^c$ such that the middle two squares of the diagram in Figure 4.14 are commutative and, moreover,

$$\text{the } \beta_X\text{- and } \beta_Y\text{-fibers of smooth points are singletons.} \tag{154}$$

Since smooth morphisms are stable under base change by [Liu02, p. 143, Proposition 3.39], $\overline{X}$ is also a smooth curve over $\overline{k}$ and, therefore by (154), β_X is even bijective. All in all, we established (i).

For (ii): Let $j \in \{1, 2\}$, let $\overline{Q} \in A_j$, i.e. $\overline{Q}$ be a place in $\overline{F}_1$ which is ramified in $F_1/\sigma^{j-1}(F_0)$, let $\overline{Q}_0 := \alpha_Y(\overline{Q}) \in \mathbb{P}_{K(\overline{Y})}$, let $\overline{y} := \beta_Y(\overline{Q}_0) = \beta_Y(\alpha_Y(\overline{Q}))$ and let $\overline{x} := \overline{\pi}_j(\overline{y})$.

On the one hand, (152) implies that $\overline{Q}_0$ is ramified in $K(Y)/\overline{\pi}_j^*(K(\overline{X}))$. On the other hand, by the choice of the map β_Y, the place $\overline{Q}_0$ contains the maximal ideal of the local ring of the closed point $\beta_Y(\overline{Q}_0) = \overline{y}$. Then applying Lemma 252 to the morphism $\overline{\pi}_j : \overline{Y} \to \overline{X}$ and the place $\overline{Q}_0$ provides that $\overline{\pi}_j$ is not etale at $\overline{y}$. But, by the definition of etaleness of π_j at $\overline{k}$-rational points in Definition 100, the latter implies that π_j is not etale at $\gamma_Y(\overline{y}) = (\gamma_Y \circ \beta_Y \circ \alpha_Y)(\overline{Q}) = \phi(\overline{Q})$. Hence, we established the first desired statement in (ii), namely that ϕ restricts to a map $A_j \to B_j$.

Finally, combining the 'only if'-part in Lemma 252 and (154) even provides that β_Y restricts to a bijection $\beta_Y^{-1}(\alpha_Y^{-1}(Y(\overline{k})\backslash B_j)) \to \alpha_Y^{-1}(Y(\overline{k})\backslash B_j)$. But since α_Y and γ_Y are bijections anyways, we deduce the last desired statement in the 'in particular'-part, namely that ϕ restricts to a bijection $\mathbb{P}_{\overline{F}_1}\backslash\phi^{-1}(B_i) \to Y(\overline{k})\backslash B_i$.

For (iii): Suppose that Y is smooth. Then the same last part of the proof of (i) from (154) on also works for Y instead of X. Consequently, this provides that β_Y is also bijective. Hence, since α_Y and γ_Y are also bijections by Lemma 103(i), we obtain the first desired statement in (iii), namely that $\phi = \gamma_Y \circ \beta_Y \circ \alpha_Y$ is even a bijection.

In particular, by Lemma 103(ii), we deduce that ϕ restricts to an injection $A_j \to B_j$. Now, let $j \in \{1, 2\}$ and let $y \in B_j$. Then, by the definitions of B_j in the assumptions and of etaleness in Definition 100, the morphism $\overline{\pi}_j$ is not etale at $\overline{y} = \gamma_Y^{-1}(y)$. Moreover, since $\overline{Y}$ is regular at $\overline{y}$, the 'if'-part in Lemma 252 implies that $\beta_Y^{-1}(\overline{y}) =: \overline{Q}_0$ is ramified in $K(Y)/\overline{\pi}_j^*(K(\overline{X}))$. In particular, this means that the place $\phi^{-1}(y) = \alpha_Y^{-1}(\overline{Q}_0)$ is ramified in $\overline{F}_1/\sigma^{j-1}(F_0)$ and thus an element in A_j. This implies that ϕ restricts to a surjection $A_j \to B_j$.

Therefore, we also obtain the last two desired statements in (iii), namely that ϕ restricts to bijections $A_j \to B_j$ and $\mathbb{P}_{\overline{F}_1}\backslash A_j \to Y(\overline{k})\backslash B_j$. $\square$

Proposition 104. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the finite field k of balanced degree d, let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$.*

Moreover, let X be a smooth projective geometrically integral curve X over k, let (Y, π_1, π_2) be a correspondence on X of type (d, d) and let $\alpha_X, \beta_X, \gamma_X, \alpha_Y, \beta_Y, \gamma_Y$ be the maps in Proposition 103.

Finally, let $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ be the ramification subgraph of the tower graph $\Gamma_{\overline{\mathcal{F}}}$ of $\overline{\mathcal{F}}$ and let $\mathcal{G}_{\mathrm{sing}}$ be the singular part of the geometric graph $\mathcal{G}_\infty$ of (Y, π_1, π_2). Then the following hold:

(i) The pair

$$\psi := (\gamma_X \circ \beta_X \circ \alpha_X,\ \gamma_Y \circ \beta_Y \circ \alpha_Y) : \Gamma_{\overline{\mathcal{F}}} \to \mathcal{G}_\infty$$

is an epimorphism of directed graphs which is even a bijection on the vertex sets.

(ii) Moreover, ψ restricts to a morphism $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} \to \mathcal{G}_{\mathrm{sing}}$ and even to an isomorphism $\Gamma_{\overline{\mathcal{F}}}\backslash\psi^{-1}(\mathcal{G}_{\mathrm{sing}}) \to \mathcal{G}_\infty\backslash\mathcal{G}_{\mathrm{sing}}$.

(iii) If Y is smooth, then ψ is even an isomorphism which also restricts to isomorphisms $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} \to \mathcal{G}_{\mathrm{sing}}$ and $\Gamma_{\overline{\mathcal{F}}}\backslash\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} \to \mathcal{G}_\infty\backslash\mathcal{G}_{\mathrm{sing}}$.

Proof. For the (i): It immediately follows from the commutativity of the diagram in Figure 4.14 and from the definitions of $\Gamma_{\overline{\mathcal{F}}}$ in Definition 74, of $\mathcal{G}_\infty$ in Definition 101(i) and of morphisms of directed graphs in Definition 65 that ψ is a well defined morphism of directed graphs. Moreover, since the maps $\alpha_X, \beta_X, \gamma_X$ are all bijections and the maps $\alpha_Y, \beta_Y, \gamma_Y$ are all at least surjections, we already obtain the desired statement in the 'main'-part, namely that ψ is an epimorphism of directed graphs which is even a bijection on the vertex sets.

For the (ii): First, let A be the set of all ramified edges in $\Gamma_{\overline{\mathcal{F}}}$, let B be the set of all non-etale edges in $\mathcal{G}_\infty$, and consider the sets A_i and B_i for all $i = 1, 2$ in Proposition 103.

From the definitions of the ramified edges in $\Gamma_{\overline{\mathcal{F}}}$ in Definition 88(iii) and of non-etale edges in $\mathcal{G}_\infty$ in Definition 101(ii), we derive the equalities

$$A = A_1 \cup A_2 \text{ and } B = B_1 \cup B_2.$$

Consequently, by these equalities and by Proposition 103(ii), we conclude that ψ restricts to a

$$\text{map } A \to B \text{ and even a bijection } E(\Gamma_{\overline{\mathcal{F}}})\backslash\psi^{-1}(B) \to Y(\overline{k})\backslash B. \tag{155}$$

Next, we remember that, in Definition 101(ii), the singular part $\mathcal{G}_{\mathrm{sing}}$ is defined as the smallest forward and backward complete subgraph of $\mathcal{G}_\infty$ containing B. Because of this definition, because ψ restricts to a map $A \to B$ by (155) and because of Lemma 70(iv), we deduce that $\psi^{-1}(\mathcal{G}_{\mathrm{sing}})$ is even a forward and backward complete subgraph of $\Gamma_{\overline{\mathcal{F}}}$ which contains A. In particular, by the definition of $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ in Definition 88(iii) as the smallest forward and backward complete subgraph of $\Gamma_{\overline{\mathcal{F}}}$ which contains A, we deduce that $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $\psi^{-1}(\mathcal{G}_{\mathrm{sing}})$. Hence, the first desired statement in the 'moreover'-part follows, namely that ψ indeed restricts to a morphism $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} \to \mathcal{G}_{\mathrm{sing}}$.

Finally, for the second desired statement in the 'moreover'-part, combining the equalities

$$\begin{aligned}\mathbb{P}_{\overline{F}_1}\backslash\psi^{-1}(E(\mathcal{G}_{\text{sing}})) = E(\Gamma_{\overline{\mathcal{F}}})\backslash\psi^{-1}(E(\mathcal{G}_{\text{sing}})) &= E(\Gamma_{\overline{\mathcal{F}}})\backslash E(\psi^{-1}(\mathcal{G}_{\text{sing}})) \\ &= E(\Gamma_{\overline{\mathcal{F}}}\backslash\psi^{-1}(\mathcal{G}_{\text{sing}}))\end{aligned}$$

and

$$Y(\overline{k})\backslash E(\mathcal{G}_{\text{sing}}) = E(\mathcal{G}_\infty)\backslash E(\mathcal{G}_{\text{sing}}) = E(\mathcal{G}_\infty\backslash\mathcal{G}_{\text{sing}}),$$

the inclusion $B \subseteq E(\mathcal{G}_{\text{sing}})$ and the second conclusion in (155) yields that ψ restricts to a bijection $E(\Gamma_{\overline{\mathcal{F}}}\backslash\psi^{-1}(\mathcal{G}_{\text{sing}})) \to E(\mathcal{G}_\infty\backslash\mathcal{G}_{\text{sing}})$. But since ψ is a already a bijections on the vertex sets, we obtain the second desired statement in the 'moreover'-part, namely that ψ restricts to an isomorphism $\Gamma_{\overline{\mathcal{F}}}\backslash\psi^{-1}(\mathcal{G}_{\text{sing}}) \to \mathcal{G}_\infty\backslash\mathcal{G}_{\text{sing}}$.

For the (iii): Suppose that Y is smooth. Then we first notice that the map ϕ in Lemma 103(ii) is exactly the restriction of the morphism ψ on the edge sets, i.e. $E(\Gamma_{\overline{\mathcal{F}}}) \to E(\mathcal{G}_\infty)$. Consequently, the first part of Lemma 103(iii) supplies that ψ is a bijection on the edge sets. As Proposition 104(i) already supplies that ψ is a morphism which restricts to a bijection on the vertex sets, we conclude the first desired statement in (iii), namely that ψ is an isomorphism.

Moreover, the second part of Lemma 103(iii) even provides that, in (155), the map ψ restricts to a bijection $A \to B$. Hence, because of this, because $\psi : \Gamma_{\overline{\mathcal{F}}} \to \mathcal{G}_\infty$ is an isomorphism, because $\Gamma_{\overline{\mathcal{F}}}^{\text{ram}}$ is the smallest forward and backward complete subgraph which contains A and because $\mathcal{G}_{\text{sing}}$ is the smallest forward complete subgraph which contains B, we derive that the last two desired statements in (iii), namely that ψ even restricts to isomorphisms $\Gamma_{\overline{\mathcal{F}}}^{\text{ram}} \to \mathcal{G}_{\text{sing}}$ and $\Gamma_{\overline{\mathcal{F}}}\backslash\Gamma_{\overline{\mathcal{F}}}^{\text{ram}} \to \mathcal{G}_\infty\backslash\mathcal{G}_{\text{sing}}$. □

5 Structure of Tower Graphs

Summary of the results of this chapter. In [Bee04, p. 238, Theorem 5.5] and in [HP12, p. 27, Theorem 23], it was shown that most of the Beelen-graphs and all of the HP-graphs have at most one finite d-regular weakly connected component, respectively, where d is the balanced degree of the given recursive tower. As the first major result of this thesis, in Theorem 155 of this chapter, we will show that the tower graph not only has at most one finite d-regular weakly connected component but even at most one finite balanced weakly connected component.

On the one hand, by Corollary 156, this will especially imply that the Beelen-graph also has at most one finite balanced weakly connected component. On the other hand, in Theorem 154, we will present a simplified proof of [HP12, p. 27, Theorem 23] which will also work on the more general definition of correspondences in Definition 98. This part is joint work with Florian Heß.

Moreover, this first major result will be one of the two main ingredients to prove the third major result in Theorem 188. There it will come out that the limit of a good recursive tower is stable under finite constant field extensions.

Purpose of this chapter. This chapter has two goals. On the one hand, as already mentioned above, we will prove the first major result of this thesis in Theorem 155. On the other hand, we will also make preparations for the subsequent chapters by further studying the structure of tower graphs. More concretely, we will introduce two new concepts for subgraphs of the tower graph, namely constant field extensions and truncations of subgraphs.

Structure of this chapter. In the Sections 5.1 and 5.2, we will relate the tower graph of a recursive tower $\mathcal{F}$ to the tower graphs of its constant field extensions $k' \cdot \mathcal{F}$ and truncations $\mathrm{Trun}_{\geq m}(\mathcal{F})$, respectively. Here, we will also have relations which are non-trivial and need more elaborated proofs, e.g. Lemma 120 and Lemma 138.

In Section 5.3, we will then prove the first major result of this thesis, which is Theorem 155.

5.1 Tower Graphs of Constant Field Extensions

Purpose of this section. In this section, we will connect the tower graph of a recursive tower $\mathcal{F}$ with the tower graph of its constant field extension $\mathcal{F}' = k' \cdot \mathcal{F}$ and introduce the new concept of constant field extensions of subgraphs of tower graphs.

Structure of this section. First, we will connect the tower graph $\Gamma_{\mathcal{F}}$ of a recursive tower $\mathcal{F}$ to the tower graph $\Gamma_{\mathcal{F}'}$ of its constant field extension $\mathcal{F}' = k' \cdot \mathcal{F}$ via the CFE-projection morphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} : \Gamma_{\mathcal{F}'} \to \Gamma_{\mathcal{F}}$.

In Subsection 5.1.1, we will define constant field extensions of subgraphs and prove some first properties.

In Subsection 5.1.2, we will relate the places which lie over a subgraph Γ to the places which lie over the constant field extension of Γ.

In Subsection 5.1.3, we will discuss how automorphisms of k-algebras on k' induce automorphisms of directed graphs on $\Gamma_{\mathcal{F}'}$ where k denotes the constant field of $\mathcal{F}$.

In Subsection 5.1.4, we will first provide the Path Lifting Lemma 119 and then list properties which are in some sense invariant under constant field extensions of subgraphs.

In Subsection 5.1.5, we will relate the rational, splitting and ramification subgraphs of $\mathcal{F}$ to the rational, splitting and ramification subgraphs of $\mathcal{F}'$.

The CFE-projection morphism. In the following Definition/Lemma 105, we will connect the tower graph $\Gamma_{\mathcal{F}'}$ of a constant field extension $\mathcal{F}' = k' \cdot \mathcal{F}$ of a recursive tower $\mathcal{F}$ and the tower graph $\Gamma_{\mathcal{F}}$ via the CFE-projection morphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} : \Gamma_{\mathcal{F}'} \to \Gamma_{\mathcal{F}}$.

Definition/Lemma 105. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\mathcal{F}'$ be a constant field extension of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$) and define*

$$\pi_V : V(\Gamma_{\mathcal{F}'}) \to V(\Gamma_{\mathcal{F}}) \text{ via } P' \to P' \cap F_0 \text{ and } \pi_E : E(\Gamma_{\mathcal{F}'}) \to E(\Gamma_{\mathcal{F}}) \text{ via } Q' \to Q' \cap F_1$$

and

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} := (\pi_V, \pi_E) : \Gamma_{\mathcal{F}'} \to \Gamma_{\mathcal{F}}.$$

(i) Then $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ is a well defined epimorphism of directed graphs satisfying the identity

$$e(Q'|\sigma^i(P'_i)) = e(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q')|\sigma^i(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P'_i)))$$

for all edges $P'_0 \xrightarrow{Q'} P'_1$ in $\Gamma_{\mathcal{F}'}$ and all $i = 0, 1$.

*We call $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ the **CFE-projection morphism** of $\mathcal{F}$ for k'.*

(ii) The extension $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} : W(\Gamma_{\mathcal{F}'}) \to W(\Gamma_{\mathcal{F}})$ of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ in Definition 65 satisfies the identity

$$\sigma_{\Gamma_{\mathcal{F}}} \circ \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} = \pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \sigma_{\Gamma_{\mathcal{F}'}}.$$

(iii) Let $\mathcal{F}''$ be a constant field extension of $\mathcal{F}'$ and let $\Gamma_{\mathcal{F}''}$ be its tower graph, Then $\pi_{\Gamma_{\mathcal{F}''}/\Gamma_{\mathcal{F}}}$ factorizes through $\Gamma_{\mathcal{F}'}$, i.e. we have the identity

$$\pi_{\Gamma_{\mathcal{F}''}/\Gamma_{\mathcal{F}}} = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} \circ \pi_{\Gamma_{\mathcal{F}''}/\Gamma_{\mathcal{F}'}}.$$

Proof. For (i): Let us write v_0 (resp. v'_0) for the initial vertex map on $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) and v_1 (resp. v'_1) for the terminal vertex map on $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$).

First, we compute

$$\begin{aligned}\pi_V(v'_i(Q')) &= \pi_V(\sigma^{-i}(Q') \cap F'_0) = (\sigma^{-i}(Q') \cap F'_0) \cap F_0 = (\sigma^{-i}(Q') \cap \sigma^{-i}(F_1)) \cap F_0 \\ &= \sigma^{-i}(Q' \cap F_1) \cap F_0 = \sigma^{-i}(\pi_E(Q')) \cap F_0 = v_i(\pi_E(Q')) \qquad (156)\end{aligned}$$

for all $Q' \in E(\Gamma_{\mathcal{F}'})$ and all $i = 0, 1$ where the equalities hold by the following reasonings: The first (resp. last) equality holds by the definition of the tower graph $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) in Definition 74. The second (resp. second to last) equality holds by the definition of π_V (resp. π_E). The third equality holds since Definition 5(ii) and Definition 21 imply the inclusions $F_0 \subseteq \sigma^{-i}(F_0) \cdot \sigma^{1-i}(F_0)) = \sigma^{-i}(F_0 \cdot \sigma(F_0)) = \sigma^{-i}(F_1)$ and $F_0 \subseteq F'_0$. The fourth equality holds since σ is a bijection.

Consequently, the equalities in (156) provide that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ is a well defined morphism of directed graphs by the definition of those morphisms in Definition 65.

Second, by the definitions of the tower graphs $\Gamma_{\mathcal{F}'}$ and $\Gamma_{\mathcal{F}}$ in Definition 74, we have the identities $V(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_0}$, $V(\Gamma_{\mathcal{F}'}) = \mathbb{P}_{F_0'}$, $E(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_1}$ and $E(\Gamma_{\mathcal{F}'}) = \mathbb{P}_{F_1'}$. Combining these identities, the fact that F_1'/F_1 and F_0'/F_0 are extensions of functions fields by Definition 21, the definitions of π_V and π_E and [Sti08, p. 71, Proposition 3.1.7(b)] yields that the maps π_V and π_E are surjective. Therefore, we conclude the first desired statement in (i), namely that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ is a well defined epimorphism.

Finally, for all edges $P_0' \xrightarrow{Q'} P_1'$ in $\Gamma_{\mathcal{F}'}$ and all $i = 0, 1$, we also obtain the desired identity $e(Q'|\sigma^i(P_i')) = e(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q')|\sigma^i(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_i')))$ by the equalities

$$e(Q'|\sigma^i(P_i')) = e(Q' \cap F_1|\sigma^i(P_i' \cap F_0)) = e(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q')|\sigma^i(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_i'))) \tag{157}$$

where the first equality holds because of the equalities $F_1' = k' \cdot F_1$, $\sigma(F_0') = \sigma(k' \cdot F_0) = k' \cdot \sigma(F_0)$, and $\sigma^i(P_i') \cap \sigma^i(F_0) = \sigma^i(P_i' \cap F_0)$ and because of the invariance of the ramification indices under constant field extensions in (12) and the second equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$.

All together, (i) follows.

For (ii): The desired identity in (ii) already follows from the equalities

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}([P_{i,j}']_{j-i\leq 1})) &= \sigma_{\Gamma_{\mathcal{F}}}([P_{i,j}' \cap F_{j-i}]_{j-i\leq 1}) = (\sigma^i(P_{i,j}' \cap F_{j-i}))_{j-i\leq 1} \\ &= (\sigma^i(P_{i,j}') \cap F_{i,j})_{j-i\leq 1} = \pi_{W(\mathcal{F}')/W(\mathcal{F})}((\sigma^i(P_{i,j}'))_{j-i\leq 1}) \\ &= \pi_{W(\mathcal{F}')/W(\mathcal{F})}(\sigma_{\Gamma_{\mathcal{F}'}}([P_{i,j}']_{j-i\leq 1}))\end{aligned}$$

for all $[P_{i,j}']_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}'})$ where the equalities hold by the following reasonings: The first equality holds by the definition of the extension $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} : W(\Gamma_{\mathcal{F}'}) \to W(\Gamma_{\mathcal{F}})$ in Definition 65 and by the definitions of π_V and π_E. The second and last equalities hold by the definitions of $\sigma_{\Gamma_{\mathcal{F}}}$ and $\sigma_{\Gamma_{\mathcal{F}'}}$ in Definition/Lemma 76. The third equality holds because σ^i is bijective and because Lemma 10(i) and Lemma 10(ii) imply the identities $\sigma^i(F_{j-i}) = \sigma^i(F_{0,j-i}) = F_{i,j}$. The fourth equality holds by the definition of the map $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ in Lemma 76(iii).

For (iii): This immediately follows from the obvious fact that $\mathcal{F}''$ is also a constant field extension of $\mathcal{F}$ and from the definitions of the involved morphisms. □

Example 106. *Let us again consider the recursive MW-towers* $\mathcal{F} = \mathcal{F}_{MW,2} = (F_\nu)_\nu$ *over* $\mathbb{F}_3$ *and* $\mathcal{F}' = \mathcal{F}_{MW,2}' = (F_\nu')_\nu$ *over* $\mathbb{F}_9$ *from Example 77 which are defined by the polynomial* $f_{MW,2} = Y^2 + XY + 2X^2 + 1$. *There the paths* $\mathcal{P}$ *and* $\mathcal{P}'$ *which are depicted in the figures B.1 and B.2 clearly satisfy the identity* $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \mathcal{P}$. *Thus, we obtain the equalities*

$$\begin{aligned}(\sigma_{\Gamma_{\mathcal{F}}} \circ \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}})(\mathcal{P}') &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) = \mathrm{Path}(Q) = \pi_{W(\mathcal{F}')/W(\mathcal{F})}(\mathrm{Path}(Q')) \\ &= (\pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \sigma_{\Gamma_{\mathcal{F}'}})(\mathcal{P}')\end{aligned}$$

which are in accordance to the identity in Definition/Lemma 105.

Diagram with all the defined maps. Let $n \in \mathbb{N}_0$. Then we collected all the maps Path, Pyr, $\sigma_{\Gamma_{\mathcal{F}}}$, $\tau_{\mathcal{F}}$, $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$, $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ in the commutative diagram in Figure 5.1. We also added the evident projection map $\mathrm{Proj} : \mathrm{Pyr}(\mathbb{P}_{F_n}) \to W(\mathcal{F}, n)$ satisfying $\mathrm{Proj} \circ \mathrm{Pyr} = \mathrm{Path}$ on $\mathbb{P}_{F_n}$ to this diagram.

$$\begin{tikzcd}
\mathbb{P}_{F_u} \arrow[r, two heads, "\tau_{\mathcal{F}}"] \arrow[d, hook, "P_{Y\Gamma}"'] \arrow[dr, two heads, "Path"] & W(\Gamma_{\mathcal{F}}, u) \arrow[d, "\sigma_{\mathcal{F}}", "\wr"'] & W(\Gamma_{\mathcal{F}'}, u) \arrow[l, two heads, "\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}"'] \arrow[d, "\sigma_{\mathcal{F}'}", "\wr"'] \\
P_{Y\Gamma}(\mathbb{P}_{F_u}) \arrow[r, two heads, "Proj"] & W(\mathcal{F}, u) & W(\mathcal{F}', u) \arrow[l, two heads, "\pi_{W(\mathcal{F}')/W(\mathcal{F})}"']
\end{tikzcd}$$

Figure 5.1: Commutative diagram with the path map, the tower graph map and the CFE-projection morphism

5.1.1 Constant Field Extensions of Subgraphs

Purpose of this subsection. In this subsection, we will define constant field extensions of subgraphs of tower graphs and prove some first properties.

Later in the Subsection 5.1.5, we will use constant field extensions of subgraphs to connect the rational, splitting and ramification subgraphs of $\mathcal{F}$ with the rational, splitting and ramification subgraphs of its constant field extensions $\mathcal{F}' = k' \cdot \mathcal{F}$. This will play an important role in the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Constant field extensions of subgraphs.

Definition 107. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let $\mathcal{F}' := k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be a constant field extension of $\mathcal{F}$ for some algebraic extension k'/k, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$) and let Γ be a subgraph of $\Gamma_{\mathcal{F}}$. Then we define*

$$k' \cdot \Gamma := \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma)$$

*and call $k' \cdot \Gamma$ the k'-**constant field extension** of Γ. Notice that $k' \cdot \Gamma$ is a subgraph of $\Gamma_{\mathcal{F}'}$ which has the vertex set*

$$V(k' \cdot \Gamma) = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma)) = \mathbb{P}_{F'_0}(V(\Gamma))$$

and edge set

$$E(k' \cdot \Gamma) = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(E(\Gamma)) = \mathbb{P}_{F'_1}(E(\Gamma)).$$

Examples 108. *(i) The first (resp. second) weakly connected component in Figure B.1 is the $\mathbb{F}_9$-constant field extension of the second (resp. first) weakly connected component in Figure B.2.*

(ii) The first (resp. second; resp third) weakly connected component in Figure B.7 is the $\mathbb{F}_9$-constant field extension of the first (resp. second; resp third) weakly connected component in Figure B.8.

Constant field extensions of subgraphs and the CFE-projection morphism.

Lemma 109. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k and let $\mathcal{F}' := k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be a constant field extension of $\mathcal{F}$ for some algebraic extension k'/k of fields. Moreover, let Γ be a subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$.*

Then the morphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}$ restricts to an epimorphism $k' \cdot \Gamma \to \Gamma$ and we have the identity $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}(W(\Gamma)) = W(k' \cdot \Gamma)$.

Proof. Both desired statements immediately follow from the definition of $k' \cdot \Gamma$ as the preimage graph $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}(\Gamma)$ and from the fact that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}$ is an epimorphism by Definition/Lemma 105(i). □

Transitivity rule for constant field extensions of subgraphs.

Lemma 110. *Let $\mathcal{F}$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let k' and k'' be algebraic extension fields of k which are contained in the domain of σ. Moreover, let Γ be a subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$.*

If k'' is an extension field of k, then we have the identity

$$k'' \cdot (k' \cdot \Gamma) = k'' \cdot \Gamma.$$

Moreover, if k'' is not an extension field of k' and Γ' is a subgraph of the tower graph of $k' \cdot \mathcal{F}$, then we also define $k'' \cdot \Gamma' := (k'' \cdot k') \cdot \Gamma'$.

Proof. Let $\mathcal{F}' := k' \cdot \mathcal{F}$ and $\mathcal{F}'' := k'' \cdot \mathcal{F}$. Then we immediately obtain the desired identity since Definition 107 and Definition/Lemma 105(iii) supply the equalities

$$\begin{aligned} k'' \cdot (k' \cdot \Gamma) &= \pi^{-1}_{\Gamma_{\mathcal{F}''}/\Gamma_{\mathcal{F}'}}(\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}(\Gamma)) = (\pi_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}} \circ \pi_{\Gamma_{\mathcal{F}''}/\Gamma_{\mathcal{F}'}})^{-1}(\Gamma) = \pi^{-1}_{\Gamma_{\mathcal{F}''}/\Gamma_\mathcal{F}}(\Gamma) \\ &= k'' \cdot \Gamma. \end{aligned}$$

□

Intersections and unions of constant field extensions of subgraphs.

Lemma 111. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0). Moreover, let k' be an algebraic extension field of k which is contained in the domain of σ. Finally, let M be a set of subgraphs of $\Gamma_\mathcal{F}$.*

For all $\Phi \in \{\bigcap, \bigcup\}$, we then have the identity

$$k' \cdot \underset{\Gamma \in M}{\Phi} \Gamma = \underset{\Gamma \in M}{\Phi} k' \cdot \Gamma.$$

Moreover, if the subgraphs Γ in M are pairwise disjoint, then their constant field extensions $k' \cdot \Gamma$ are also pairwise disjoint and we have the identity

$$k' \cdot \coprod_{\Gamma \in M} \Gamma = \coprod_{\Gamma \in M} k' \cdot \Gamma.$$

Proof. Every desired statement in immediately follows from the definition of constant field extensions of subgraphs as the $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_\mathcal{F}}$-preimage graphs in Definition 107, from the definition of preimage graphs in Definition 69(ii) and from the corresponding statements for sets and maps, i.e. the preimage of the intersection/union/disjoint union of sets is the intersection/union/disjoint union of the preimages of these sets. □

Constant field extensions of subgraphs and ramification.

Lemma 112. *Let $\mathcal{F}$ be a recursive tower over the field k which is defined by the pair (σ, F_0) and let k' be an algebraic extension fields of k which is contained in the domain of σ. Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$.*

Then Γ contains ramified edges if and only if $k' \cdot \Gamma$ contains ramified edges.

Proof. This immediately follows from the fact that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $k' \cdot \Gamma \to \Gamma$ in Lemma 109, from the invariance of the ramification indices under the action of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ in Definition/Lemma 105(i) and from the definition of ramified edges in Definition 88(iii). ☐

5.1.2 Places over Constant Field Extensions of Subgraphs

Purpose of this subsection. In this subsection, we will relate the places Q which lie over a subgraph Γ to the places Q' which lie over its constant field extension $k' \cdot \Gamma$. Here, the first Lemma 113 will provide that the places Q' are exactly the places which lie over the places Q. Then the second Lemma 114 will deepen this relationship via the projection maps π_1 and π_2 which satisfy the degree property in (160).

In particular, we will need the estimates and equalities of the N-values in Lemma 114 for the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Places over constant field extensions of subgraphs.

Lemma 113. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k, let $\mathcal{F}' := k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be a constant field extension of $\mathcal{F}$ for some algebraic extension k'/k, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$), let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and let $\Gamma' := k' \cdot \Gamma$. Then we have the identities*

$$\mathbb{P}_{F'_n}[\Gamma'] = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma]) \quad \textit{and} \quad \mathbb{P}^{(1)}_{F'_n}[\Gamma'] = \mathbb{P}^{(1)}_{F'_n}(\mathbb{P}_{F_n}[\Gamma])$$

for all $n \in \mathbb{N}_0$.

Proof. Let $n \in \mathbb{N}_0$, let $Q' \in \mathbb{P}_{F'_n}$, let $\mathcal{P} := \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(Q' \cap F_n)) \in W(\Gamma_{\mathcal{F}}, n)$ and let $\mathcal{P}' := \sigma^{-1}_{\Gamma_{\mathcal{F}'}}(\mathrm{Path}_{\mathcal{F}'}(Q')) \in W(\Gamma_{\mathcal{F}'}, n)$.

First, we notice the equalities

$$\begin{aligned}\mathcal{P} &= \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(\pi_{\mathrm{Pyr}(\mathcal{F}')/\mathrm{Pyr}(\mathcal{F})}(Q'))) = \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\pi_{W(\mathcal{F}')/W(\mathcal{F})}(\mathrm{Path}_{\mathcal{F}'}(Q'))) \\ &= \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\sigma^{-1}_{\Gamma_{\mathcal{F}'}}(\mathrm{Path}_{\mathcal{F}'}(Q'))) = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') \end{aligned} \tag{158}$$

where the equalities hold by the following reasonings: The first equality holds by the choice of $\mathcal{P} := \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(Q' \cap F_n))$ and the definition of the map $\pi_{\mathrm{Pyr}(\mathcal{F}')/\mathrm{Pyr}(\mathcal{F})}$ via $\pi_{\mathrm{Pyr}(\mathcal{F}')/\mathrm{Pyr}(\mathcal{F})}(Q') = Q' \cap F_n$ in Lemma 76(iii). The second equality holds by the identity in Lemma 76(iv). The third equality holds by the identity in Definition/Lemma 105(ii). The last equality holds by the choice of $\mathcal{P}' := \sigma^{-1}_{\Gamma_{\mathcal{F}'}}(\mathrm{Path}_{\mathcal{F}'}(Q'))$.

Second, combining the equality in (158) and the identity in Lemma 109 yields that

$$\mathcal{P} \in W(\Gamma, n) \text{ if and only if } \mathcal{P}' \in W(\Gamma', n). \tag{159}$$

Now, by the definition of $\mathbb{P}_{F'_n}[\Gamma']$ in Definition 85, by the choice of $\mathcal{P}' = \sigma^{-1}_{\Gamma_{\mathcal{F}'}}(\mathrm{Path}_{\mathcal{F}'}(Q'))$ and by the equivalence in (159), we obtain that

$$Q' \in \mathbb{P}_{F'_n}[\Gamma'] \text{ if and only if } \mathcal{P} \in W(\Gamma, n).$$

But, by the choice of $\mathcal{P} := \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(Q' \cap F_n))$ and by the definition $\mathbb{P}_{F_n}[\Gamma]$, we also conclude that

$$\mathcal{P} \in W(\Gamma, n) \text{ is equivalent to } Q' \cap F_n \in \mathbb{P}_{F_n}[\Gamma] \text{ and, thus, to } Q' \in \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma]).$$

Therefore, combining these equivalences provides the first desired identity $\mathbb{P}_{F'_n}[\Gamma'] = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma])$.

Finally, the second desired identity follows from the computation

$$\mathbb{P}^{(1)}_{F'_n}[\Gamma'] = \mathbb{P}_{F'_n}[\Gamma'] \cap \mathbb{P}^{(1)}_{F'_n} = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma]) \cap \mathbb{P}^{(1)}_{F'_n} = \mathbb{P}^{(1)}_{F'_n}(\mathbb{P}_{F_n}[\Gamma])$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $\mathbb{P}^{(1)}_{F'_n}[\Gamma'] = \mathbb{P}_{F'_n}[\Gamma'] \cap \mathbb{P}^{(1)}_{F'_n}$ in Definition 85. The second equality holds by the first desired identity $\mathbb{P}_{F'_n}[\Gamma'] = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma])$. The last equality holds by the definition of $\mathbb{P}^{(1)}_{F'_n}(\mathbb{P}_{F_n}[\Gamma]) = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}[\Gamma]) \cap \mathbb{P}^{(1)}_{F'_n}$ in (5). □

Lemma 114. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let $\mathcal{F}' = k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k, let $\overline{\mathcal{F}} = \overline{k} \cdot (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$; resp. $\Gamma_{\overline{\mathcal{F}}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$; resp. $\overline{\mathcal{F}}$), let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ and define $\Gamma' := k' \cdot \Gamma$ and $\overline{\Gamma} := \overline{k} \cdot \Gamma$. Moreover, let $n \in \mathbb{N}_0$. Then the following hold:*

(i) The maps

$$\pi_1 : \mathbb{P}_{F'_n}[\Gamma'] \to \mathbb{P}_{F_n}[\Gamma] \text{ via } Q' \mapsto Q' \cap F_n$$

and

$$\pi_2 : \mathbb{P}_{F'_n}(V(\Gamma')) \to \mathbb{P}_{F_n}(V(\Gamma)) \text{ via } Q' \mapsto Q' \cap F_n$$

are well defined surjections such that

$$\deg(Q) = \sum_{Q' \in \pi_i^{-1}(Q)} \deg(Q') \tag{160}$$

for all $i = 1, 2$ and all Q in the codomain of π_i.

(ii) We have the estimates

$$N[\overline{F}_n, \overline{\Gamma}] \geq N[F_n, \Gamma] \text{ and } N(\overline{F}_n, V(\overline{\Gamma})) \geq N(F_n, V(\Gamma))$$

for all $n \in \mathbb{N}_0$.

(iii) If all places in $\mathbb{P}_{F_n}[\Gamma]$ (resp. $\mathbb{P}_{F_n}(V(\Gamma))$) are rational, then π_1 (resp. π_2) is even a bijection and we have the identity $N[F'_n, \Gamma'] = N[F_n, \Gamma]$ (resp. $N(F'_n, V(\Gamma')) = N(F_n, V(\Gamma))$) for all $n \in \mathbb{N}_0$.

(iv) If all places in $\mathbb{P}_{F_n}[\Gamma]$ are rational and Γ is also a forward complete subgraph of $\Gamma_{\mathcal{F}}$, then we have the identities

$$N(F'_n, V(\Gamma')) = N[F'_n, \Gamma'] = N[F_n, \Gamma] = N(F_n, V(\Gamma))$$

for all $n \in \mathbb{N}_0$.

Proof. For (i): First of all, we notice that, for all $i = 1, 2$, the surjectivity of the map π_i follows if we can show that it is well defined and satisfies the desired identity in (160): Indeed, the degree is a positive natural number and, thus, for any Q in the codomain of π_i, the preimage $\pi_i^{-1}(Q)$ cannot be empty then.

Now, we first deal with π_1: Consider the map $\pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}$ in Lemma 76(iii). Then we compute

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(\pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}(Q'))) &= \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\pi_{W(\mathcal{F}')/W(\mathcal{F})}(\mathrm{Path}_{\mathcal{F}'}(Q'))) \\ &= \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\sigma_{\Gamma_{\mathcal{F}'}}^{-1}(\mathrm{Path}_{\mathcal{F}'}(Q')))\end{aligned} \tag{161}$$

for all $Q \in P_{F_n}$ where the first equality holds by the identity

$$\mathrm{Path}_{\mathcal{F}} \circ \pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})} = \pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \mathrm{Path}_{\mathcal{F}'}$$

in Lemma 76(iv) and the second equality holds by the identity

$$\pi_{W(\mathcal{F}')/W(\mathcal{F})} \circ \sigma_{\Gamma_{\mathcal{F}'}} = \sigma_{\Gamma_{\mathcal{F}}} \circ \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$$

in Lemma 105(ii).

On the one hand, for all $Q' \in \mathbb{P}_{F_n'}[\Gamma']$, we have $\sigma_{\Gamma_{\mathcal{F}'}}^{-1}(\mathrm{Path}_{\mathcal{F}'}(Q')) \in W(\Gamma')$ by the definition of $\mathbb{P}_{F_n'}[\Gamma']$ in Definition 85. Combining this fact and the definition of $\Gamma' = k' \cdot \Gamma = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(\Gamma)$ in Definition 107 yields $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\sigma_{\Gamma_{\mathcal{F}'}}^{-1}(\mathrm{Path}_{\mathcal{F}'}(Q'))) \in W(\Gamma)$. Moreover, by the equality in (161) and the equalities $\pi_1(Q') = Q \cap F_m = Q \cap F_{0,m} = \pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}(Q)$, we therefore conclude $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(\pi_1(Q'))) \in W(\Gamma)$. But this again implies $\pi_1(Q') \in \mathbb{P}_{F_n}[\Gamma]$ and, thus, the map π_1 is indeed well defined.

On the other hand, for all $Q \in \mathbb{P}_{F_n}[\Gamma]$, there is some $Q' \in \mathbb{P}_{F_n'}$ with

$$Q = \pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}(Q') = Q' \cap F_n$$

by the surjectivity of $\pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}$ in Lemma 76(iii). Moreover, for all such Q', we even have $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(\pi_{\mathrm{Pyr}(\mathcal{F}')/\,\mathrm{Pyr}(\mathcal{F})}(Q'))) \in W(\Gamma)$ by the choice of $Q \in \mathbb{P}_{F_n}[\Gamma]$. Then, by the equality in (161), we obtain $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\sigma_{\Gamma_{\mathcal{F}'}}^{-1}(\mathrm{Path}_{\mathcal{F}'}(Q'))) \in W(\Gamma)$. Furthermore, by this and the definition of $\Gamma' = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(\Gamma)$, we conclude $\sigma_{\Gamma_{\mathcal{F}'}}^{-1}(\mathrm{Path}_{\mathcal{F}'}(Q')) \in W(\Gamma')$ and, thus, $Q' \in \mathbb{P}_{F_n'}[\Gamma']$.

Therefore, for all $Q \in \mathbb{P}_{F_n}[\Gamma]$, we have the equality $\pi_1^{-1}(Q) = \mathbb{P}_{F_n}(Q)$. Combining this equality and [Sti08, p. 114, Theorem 3.6.3(c)] finally provides the desired identity in (160).

Next, we deal with π_2: First, we notice the equality

$$\mathbb{P}_{F_n'}(V(\Gamma')) = \mathbb{P}_{F_n'}(\mathbb{P}_{F_n}(V(\Gamma))) \tag{162}$$

because of the definition of $\Gamma' = k' \cdot \Gamma$ in Definition 107. But this equality in (162) implies $\pi_2(Q') = Q' \cap F_n \in \mathbb{P}_{F_n}(V(\Gamma))$ for all $Q' \in \mathbb{P}_{F_n'}(V(\Gamma'))$ and the equality $\pi_2^{-1}(Q) = \mathbb{P}_{F_n}(Q)$ for all $Q \in \mathbb{P}_{F_n}(V(\Gamma))$. Consequently, similar to the above reasoning, we conclude that π_2 is well defined and satisfies the desired identity in (160). Hence, (i) follows.

For (ii): On the one hand, we already obtain the first desired estimate in (ii) by the equalities and estimates

$$N[\overline{F}_n, \overline{\Gamma}] = \#\mathbb{P}^{(1)}_{\overline{F}_n}[\overline{\Gamma}] = \#\mathbb{P}_{\overline{F}_n}[\overline{\Gamma}] \geq \#\mathbb{P}_{F_n}[\Gamma] \geq \#\mathbb{P}^{(1)}_{F_n}[\Gamma] = N[F_n, \Gamma]$$

where the equalities and estimates hold by the following reasonings: The first and last equalities hold by the definition of $N[\cdot, \cdot] = \#\mathbb{P}^{(1)}_{\cdot}[\cdot]$ in Definition 85. The second equality holds because $\overline{F}_n$ has an algebraically closed full constant field $\overline{k}$ and, thus, all its places are rational. The first estimate holds because $\pi_1 : \mathbb{P}_{\overline{F}_n}[\overline{\Gamma}] \to \mathbb{P}_{F_n}[\Gamma]$ in Lemma 114(i) is a surjection. The second estimate holds because of the definition of $\mathbb{P}^{(1)}_{F_n}[\Gamma] = \mathbb{P}_{F_n}[\Gamma] \cap \mathbb{P}^{(1)}_{F_n}$ in Definition 85.

On the other hand, we also obtain the second desired estimate in (ii) by the equalities and estimates

$$\begin{aligned} N(\overline{F}_n, V(\overline{\Gamma})) = \#\mathbb{P}^{(1)}_{\overline{F}_n}(V(\overline{\Gamma})) = \#\mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma})) &\geq \#\mathbb{P}_{F_n}(V(\Gamma)) \\ &\geq \#\mathbb{P}^{(1)}_{F_n}(V(\Gamma)) = N(F_n, V(\overline{\Gamma})) \end{aligned}$$

where the equalities and estimates hold by analogous reasonings.

For (iii): Suppose that all places in $\mathbb{P}_{F_n}[\Gamma]$ (resp. $\mathbb{P}_{F_n}(V(\Gamma))$) are rational. Then the identity in (160) supplies that all fibers of π_1 (resp. π_2) must be singletons and, thus, π_1 (resp. π_2) are indeed bijections.

Moreover, this identity in (160) even supplies that all places in $\mathbb{P}_{F'_n}[\Gamma']$ (resp. $\mathbb{P}_{F'_n}(V(\Gamma'))$) are rational. Consequently, π_1 (resp. π_2) is actually a bijection from $\mathbb{P}^{(1)}_{F'_n}[\Gamma']$ to $\mathbb{P}^{(1)}_{F_n}[\Gamma]$ (resp. from $\mathbb{P}^{(1)}_{F'_n}(V(\Gamma'))$ to $\mathbb{P}^{(1)}_{F_n}(V(\Gamma))$). Hence, we obtain the desired equality in (iii) by the definition of $N[\cdot]$ in Definition 85 (resp. of $N(\cdot)$ in (5)).

For (iv): Suppose that all places in $\mathbb{P}_{F_n}[\Gamma]$ are rational and that Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$. Then we already obtain the last the desired identity by the equalities

$$N[F_n, \Gamma] = \#\mathbb{P}^{(1)}_{F_n}[\Gamma] = \#\mathbb{P}_{F_n}[\Gamma] = \#\mathbb{P}_{F_n}(V(\Gamma)) = \#\mathbb{P}^{(1)}_{F_n}(V(\Gamma)) = N(F_n, (V(\Gamma)) \quad (163)$$

where the equalities hold by the following reasonings: The first (resp. last) equality holds by the definition of $N[F_n, \Gamma]$ in Definition 85 (resp. of $N(F_n, (V(\Gamma))$ in (5)). The second equality holds by the assumption that all places in $\mathbb{P}_{F_n}[\Gamma]$ are rational. The third equality holds by the assumption that Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and by the 'moreover'-part in Lemma 86. The fourth equality holds because the second and third equalities already imply that all places in $\mathbb{P}_{F_n}(V(\Gamma))$ are rational.

Finally, the first two desired identities immediately follow from the combination of the identity in (163) and Lemma 114(iii). Hence, (iv) follows. □

5.1.3 Induced Automorphisms on Constant Field Extensions of Subgraphs

Purpose of this subsection. In this subsection, we will prove Lemma 117 which will be important in the proof of Lemma 120(ii).

More concretely, let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the constant field k and let $\mathcal{F}' = k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be a constant field extension of $\mathcal{F}$. Then Lemma 117 supplies that every automorphism of k-algebras on k' induces an automorphism on the tower graph $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$ which respects the fibers of the CFE-projection morphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ and also restricts to an automorphism on the constant field extension $k' \cdot \Gamma$ of every subgraph Γ of $\Gamma_{\mathcal{F}}$.

Induced Automorphisms on tower graphs. More generally, not only the automorphisms of k-algebras in k' will induce automorphisms on the tower graphs. In the following Definition/Lemma 115, we will only need an automorphism τ of rings on F_1 which restricts to automorphisms on k and F_0 and satisfies the identity $\sigma \circ \tau = \tau \circ \sigma$ on F_0.

Here, the first conditions will ensure that τ induces bijections on the sets of places in F_1 and F_0, i.e. on the edge and vertex sets of $\Gamma_{\mathcal{F}}$. The last condition, which is the identity $\sigma \circ \tau = \tau \circ \sigma$ on F_0, will ensure that τ respects the embedding of F_0 in F_1 via σ. If τ also satisfies this condition, then the pair of induced bijections on the edge and vertex sets of $\Gamma_{\mathcal{F}}$ will even be an automorphism of directed graph on $\Gamma_{\mathcal{F}}$.

Definition/Lemma 115. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let $\Gamma_{\mathcal{F}}$ be its tower graph and let τ be an automorphism of rings on F_1 which restricts to automorphisms on k and F_0 and satisfies the identity $\sigma \circ \tau = \tau \circ \sigma$ on F_0.*

Then τ induces an automorphism τ' of directed graphs on $\Gamma_{\mathcal{F}}$ via $\tau'(P) := \tau(P)$ for all $P \in V(\Gamma_{\mathcal{F}})$ and $\tau'(Q) := \tau(Q)$ for all $Q \in E(\Gamma_{\mathcal{F}})$.

Proof. First of all, we notice that τ maps valuation rings of F_1 and F_0 again to valuation rings of F_1 and F_0, respectively, by the definition of valuation rings in function fields in [HJ90, p. 509, Theorem 8.4.5] and by the assumption that τ is an automorphism of rings on F_1 which restricts to automorphisms on k and F_0. Consequently, combining this, the definition of places as the maximal ideals of these valuation rings in [HJ90, p. 509, Theorem 8.4.5] and the definitions of $V(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_0}$ and $E(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_1}$ in Definition 74, yields that $\tau' : V(\Gamma_{\mathcal{F}}) \to V(\Gamma_{\mathcal{F}})$ and $\tau' : E(\Gamma_{\mathcal{F}}) \to E(\Gamma_{\mathcal{F}})$ are indeed well defined bijections.

Now, let $P_0 \xrightarrow{Q} P_1$ be an edge in $\Gamma_{\mathcal{F}}$. Then we compute

$$\begin{aligned}\tau(P_\varepsilon) &= \sigma^{-\varepsilon}(\tau(\sigma^\varepsilon(P_\varepsilon))) = \sigma^{-\varepsilon}(\tau(Q \cap \sigma^\varepsilon(F_0))) = \sigma^{-\varepsilon}(\tau(Q) \cap \tau(\sigma^\varepsilon(F_0))) \\ &= \sigma^{-\varepsilon}(\tau(Q) \cap \sigma^\varepsilon(F_0)) = \sigma^{-\varepsilon}(\tau(Q)) \cap F_0 \end{aligned} \tag{164}$$

for all $\varepsilon = 0, 1$ where the equalities hold by the following reasonings: The first equality holds because σ is bijective and because τ satisfies the identity $\sigma \circ \tau = \tau \circ \sigma$ on F_0. The second equality holds by the definition of the edge $P_0 \xrightarrow{Q} P_1$ in $\Gamma_{\mathcal{F}}$ in Definition 74. The third equality holds because τ is bijective. The fourth equality holds because we derive the equalities $\tau(\sigma^\varepsilon(F_0)) = \sigma^\varepsilon(\tau(F_0)) = \sigma^\varepsilon(F_0)$ from the assumption that τ satisfies the identity $\sigma \circ \tau = \tau \circ \sigma$ on F_0 and from the assumption that τ restricts to an automorphism on F_0.

Finally, the equalities in (164) yield that $\tau(P_0) \xrightarrow{\tau(Q)} \tau(P_1)$ is an edge in $\Gamma_{\mathcal{F}}$. Hence, τ' is indeed an automorphism of directed graphs on $\Gamma_{\mathcal{F}}$ by Definition 65. □

Construction of an automorphism on the tower graph from an automorphism on the extension field. In the following Lemma 116, we will first extend the automorphism τ on k' to an automorphism $\hat{\tau}$ on $k' \cdot F_1 = F_1'$ which satsfies the requirements of Definition/Lemma 115. Then this will yield the desired automorphism $\hat{\tau}'$ on $\Gamma_{\mathcal{F}'}$ which is induced by τ in Lemma 117.

Lemma 116. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let $(F_\nu')_\nu := k' \cdot \mathcal{F}$ be a constant field extension of $\mathcal{F}$ for some algebraic extension k'/k and let τ be an automorphism of k-algebras on k'.*

Then there is some unique extension $\hat{\tau}$ of τ to an automorphism of F_1-algebras on F_1'. Moreover, $\hat{\tau}$ restricts to an automorphism of F_0-algebras on F_0' and satisfies the identity $\hat{\tau} \circ \sigma = \sigma \circ \hat{\tau}$ on F_0'.

Proof. First, let $\overline{k}$ be the algebraic closure of k which is contained in the domain of σ in Definition 5(ii). Then $\overline{k}$ contains k' by the definition of constant field extensions of recursive towers in Definition 21 and, moreover, the extension $\overline{k}/k$ is Galois since it is always assumed that k is a perfect field.

Second, as F_1 has full constant field k, we have the equality $F_1 \cap \overline{k} = k$ and, thus, [Coh91, p. 188, Theorem 5.5] supplies that the extensions F_1/k and $\overline{k}/k$ are linearly disjoint. But then the extensions F_1/k and k'/k must also be linearly disjoint. Consequently, the proof of [Coh91, p. 188, Proposition 5.4] yields that the compositum $F_1' = k' \cdot F_1$ is a tensor product of the k-algebras F_1 and k' (with the inclusion maps). Thus, by the universal property, we obtain some unique automorphism $\hat{\tau}$ which restricts to id_{F_1} on F_1 and τ on k'. Hence, the 'main'-part follows.

Moreover, as $\hat{\tau}$ also restricts to the identity map on $F_0 \subseteq F_1$, we get the equalities $\hat{\tau}(F_0') = \hat{\tau}(k' \cdot F_0) = k' \cdot F_0 = F_0'$ and, consequently, $\hat{\tau}$ restricts to an automorphism of F_0-algebras on F_0'.

Finally, let $x = \sum_{i=1}^{r} a_i b_i \in F_0' = k' \cdot F_0$ with $a_i \in k'$ and $b_i \in F_0$ for all $i = 1, \ldots, r$. Then we deduce the desired identity $\hat{\tau} \circ \sigma = \sigma \circ \hat{\tau}$ on F_0' by the computation

$$\sigma(\hat{\tau}(x)) = \sum_{i=1}^{r} \sigma(\tau(a_i))\sigma(\mathrm{id}_{F_1}(b_i)) = \sum_{i=1}^{r} \tau(\sigma(a_i))\,\mathrm{id}_{F_1}(\sigma(b_i)) = \hat{\tau}(\sigma(x))$$

where the first and last equalities hold because $\hat{\tau}$ and σ are morphisms of rings and because $\hat{\tau}$ restricts to τ on k' and to id_{F_1} on F_1 and the second equality holds because σ is a morphism of $\overline{k}$-algebras by its definition in Definition 5(ii) and, thus, we have the equality $\sigma(\tau(a_i)) = \tau(\sigma(a_i))$. Hence, the 'moreover'-part also follows. ☐

Lemma 117. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0), let $(F_\nu')_\nu := k' \cdot \mathcal{F}$ be a constant field extension of $\mathcal{F}$ for some algebraic extension k'/k, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$), let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ and let $\Gamma' := k' \cdot \Gamma$.*

Moreover, let τ be an automorphism of k-algebras on k', let $\hat{\tau}$ be the extensions of τ to an automorphism of F_1-algebras on F_1' in Lemma 116 and let $\hat{\tau}'$ be the automorphism of directed graphs on $\Gamma_{\mathcal{F}'}$ which is induced by $\hat{\tau}$ as in Definition/Lemma 115.

Then $\hat{\tau}'$ satisfies the identity

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} \circ \hat{\tau}' = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$$

and restricts to an automorphism on Γ'.

Proof. First, for all $P_0' \in V(\Gamma_{\mathcal{F}'})$, all $P_1' \in E(\Gamma_{\mathcal{F}'})$ and all $\varepsilon = 0, 1$, we compute

$$\begin{aligned} \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\hat{\tau}'(P_\varepsilon')) &= \hat{\tau}(P_\varepsilon') \cap F_\varepsilon = \hat{\tau}(P_\varepsilon') \cap \hat{\tau}(F_\varepsilon) = \hat{\tau}(P_\varepsilon' \cap F_\varepsilon) \\ &= P_\varepsilon' \cap F_\varepsilon = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_\varepsilon') \end{aligned} \tag{165}$$

where the first and last equalities hold by the definitions of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ in Definition/Lemma 105 and of $\hat{\tau}'$ in Definition/Lemma 115, the second and fourth equalities hold because $\hat{\tau}$ fixes F_ε by Lemma 116 and the third equality holds because $\hat{\tau}$ is bijective. Hence, the equalities in (165) supply the desired identity

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} \circ \hat{\tau}' = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}. \tag{166}$$

Moreover, we also compute

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\hat{\tau}'(V(\Gamma'))) = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma))) = V(\Gamma) \tag{167}$$

and

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\hat{\tau}'(E(\Gamma'))) = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(E(\Gamma))) = E(\Gamma) \tag{168}$$

where the first equalities hold by the identity in (166) and by the equalities $V(\Gamma') = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma))$ and $E(\Gamma') = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(E(\Gamma))$ in the definition of $\Gamma' = k \cdot \Gamma = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma)$ in Definition/Lemma 107 and the second equalities hold because $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ is an epimorphism by Definition/Lemma 105(i). In particular, the equalities in (167) and (168) provide the inclusions $\hat{\tau}'(V(\Gamma')) \subseteq \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma)) = V(\Gamma')$ and $\hat{\tau}'(E(\Gamma')) \subseteq \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(E(\Gamma)) = E(\Gamma')$. In particular, these inclusions imply that $\hat{\tau}'$ must restricts to a morphism $\Gamma' \to \Gamma'$

Finally, we notice that since the same reasoning can be applied to $(\hat{\tau}^{-1})'$, this morphism also restricts to a morphism $\Gamma' \to \Gamma'$. But $(\hat{\tau}^{-1})'$ is clearly the inverse of $\hat{\tau}'$ and, hence, $\hat{\tau}'$ must even restrict to an automorphism on Γ' which is the last desired statement. ☐

Example 118. *As the $\mathbb{F}_9$-constant field extension of the first weakly connected component Γ in Figure B.2 is the second weakly connected component Γ' in Figure B.1, Lemma 117 provides that the Frobenius automorphism $\tau : \mathbb{F}_9 \to \mathbb{F}_9$, $\alpha \to \alpha^3$ induces an automorphism $\hat{\tau}'$ on Γ' which respects the fibers of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$.*

Indeed, $\hat{\tau}'$ fixes all vertices (resp. edges) except for the two at the bottom and swaps these two bottom vertices (resp. edges).

5.1.4 Path Lifting and Properties Invariant Under Constant Field Extension

Purpose of this subsection. There are many similarities between constant field extensions of subgraphs and covers of topological spaces where the paths in the tower graphs correspond to the paths in the topological spaces. One of these similarities is the following Path Lifting Lemma 119 for constant field extensions of subgraphs.

Moreover, in Lemma 120, we will prove that many properties of subgraphs are in some sense invariant under constant field extensions. These properties will be crucial in the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Path Lifting Lemma.

Lemma 119 (Path Lifting Lemma)**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0) and let $(F'_\nu)_\nu = \mathcal{F}' := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k.*

Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower of $\mathcal{F}$ (resp. $\mathcal{F}'$), let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ and let $\Gamma' := k' \cdot \Gamma$. Finally, let $\mathcal{P}$ be any undirected (resp. directed) path in Γ of length n, let P be the initial vertex of $\mathcal{P}$ and let $P' \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P)$.

*Then there is some undirected (resp. directed) path $\mathcal{P}'$ in Γ' of length n which starts at the vertex P' and satisfies the identity $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \mathcal{P}$. We call $\mathcal{P}'$ a $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$**-lift** of $\mathcal{P}$.*

Proof. Let us write v_0 (resp. v_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}}$ and v'_0 (resp. v'_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}'}$.

We will find the desired $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'$ of $\mathcal{P}$ by induction over the length $n \in \mathbb{N}_0$ of $\mathcal{P}$: For $n = 0$, the path $\mathcal{P}$ is just the vertex P. Thus, the vertex P' is already the desired $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'$ of $\mathcal{P}$.

Now, suppose $n \geq 1$, let Q be the n-th and last edge of $\mathcal{P}$, let $P_i := v_i(Q)$ for all $i = 0, 1$ and let $\mathcal{P}_0$ be the $(0, n-1)$-subpath of $\mathcal{P}$. Then there is some index $\varepsilon \in \{0, 1\}$ such that $\mathcal{P}_0$ and Q are connected via the vertices P_ε. Moreover, we may apply the induction hypothesis to $\mathcal{P}_0$ as it has length $n - 1$ and, hence, obtain a $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'_0$ of $\mathcal{P}_0$. Consequently, the terminal vertex P'_ε of $\mathcal{P}'_0$ must be a $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-preimage of P_ε and it satisfies the equalities

$$\begin{aligned}\sigma^\varepsilon(P'_\varepsilon) \cap \sigma^\varepsilon(F_0) = \sigma^\varepsilon(P'_\varepsilon \cap F_0) = \sigma^\varepsilon(P_\varepsilon) = \sigma^\varepsilon(v_\varepsilon(Q)) = \sigma^\varepsilon(\sigma^{-\varepsilon}(Q) \cap F_0) \\ = Q \cap \sigma^\varepsilon(F_0) \qquad (169)\end{aligned}$$

where the equalities hold by the following reasonings: The first and last equalities hold because σ is bijective. The second equality holds because $P'_\varepsilon \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_\varepsilon) = \mathbb{P}_{F'_0}(P_\varepsilon)$. The third equality holds by the choice of P_ε. The fourth equality holds by the definition of $\Gamma_{\mathcal{F}}$ in Definition 74.

Consequently, because of the equality in (169) and because of the equalities $k'\cdot\sigma^{\varepsilon}(F_0) = \sigma^{\varepsilon}(k'\cdot F_0) = \sigma^{\varepsilon}(F_0')$, we can apply Lemma 26 to the extension $F_1/\sigma^{\varepsilon}(F_0)$ (see Figure 5.2) and obtain some place

$$Q' \in \mathbb{P}_{F_1'}((Q, \sigma^{\varepsilon}(P_{\varepsilon}'))) \subseteq E(\Gamma_{\mathcal{F}'}). \tag{170}$$

In particular, we obtain the following equalities: First, we obtain the equalities

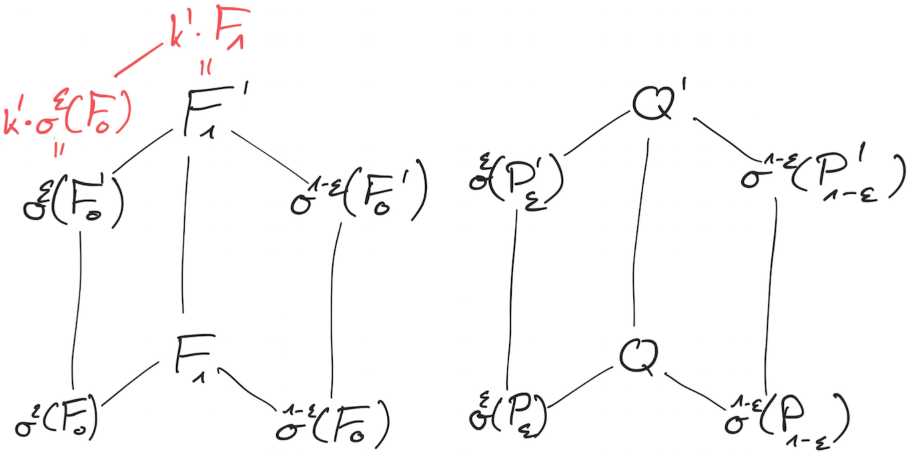


Figure 5.2: Extensions of function fields and places in a proof

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q') = Q' \cap F_1 = Q \tag{171}$$

where the first equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ in Definition/Lemma 105(i) and the second equality holds by the choice of Q' in (170). Second, we obtain the equalities

$$v_{\varepsilon}'(Q') = \sigma^{-\varepsilon}(Q') \cap F_0' = \sigma^{-\varepsilon}(Q' \cap \sigma^{\varepsilon}(F_0')) = \sigma^{-\varepsilon}(\sigma^{\varepsilon}(P_{\varepsilon}')) = P_{\varepsilon}' \tag{172}$$

where the first equality holds by the definition of $\Gamma_{\mathcal{F}'}$, the second equality holds since σ is a bijection, the third equality holds by the choice of Q' in (170) and the last equality is clear. Third, we obain the equalities

$$\begin{aligned}\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(v_{1-\varepsilon}'(Q')) &= v_{1-\varepsilon}'(Q') \cap F_0 = \left(\sigma^{-(1-\varepsilon)}(Q') \cap F_0'\right) \cap F_0 \\ &= \left(\sigma^{-(1-\varepsilon)}(Q') \cap \sigma^{-(1-\varepsilon)}(F_1)\right) \cap F_0 = \sigma^{-(1-\varepsilon)}(Q' \cap F_1) \cap F_0 \\ &= \sigma^{-(1-\varepsilon)}(Q) \cap F_0 = v_{1-\varepsilon}(Q) = P_{1-\varepsilon} \end{aligned} \tag{173}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$. The second equality holds by the definition of $\Gamma_{\mathcal{F}'}$. The third equality holds by the inclusions $F_0' \supseteq F_0$ and $\sigma^{-(1-\varepsilon)}(F_1) = \sigma^{-(1-\varepsilon)}(F_0\cdot\sigma(F_0)) = \sigma^{\varepsilon-1}(F_0)\cdot\sigma^{\varepsilon}(F_0) \supseteq F_0$ where the last inclusion follows from the choice of $\varepsilon \in \{0,1\}$. The fourth equality holds because σ is bijective. The fifth equality holds by the second equality in (171). The second to last equality holds by the definition of $\Gamma_{\mathcal{F}}$. The last equality holds by the choice of $P_{1-\varepsilon}$.

Consequently, for $P_{1-\varepsilon}' := v_{1-\varepsilon}'(Q')$, combining the equalities in (171), (172) and (173) yields the edge

$$Q' \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q) \subseteq \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(E(\Gamma)) = E(\Gamma')$$

satisfying $P_i' = v_i'(Q') \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_i)$ for all $i = 1, 2$. for all $i = 1, 2,$.

Finally, by this conclusion and by the choice of P'_ε as the terminal vertex of the $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'_0$ of $\mathcal{P}_0$, we can compose $\mathcal{P}'_0$ and Q' at the vertex P'_ε and obtain the desired $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'$, i.e $\mathcal{P}'$ is a path in Γ' of length n which starts at P' and satisfies the identity $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \mathcal{P}$.

In particular, if $\mathcal{P}$ is directed, then ε is equal to zero and, hence, in this case, $\mathcal{P}'$ is also a directed path. ☐

Subgraphs properties invariant under constant field extensions. In Lemma 120, we will prove that many properties of subgraphs are invariant under constant field extensions. These properties will be crucial in the proof of the almost complete answer to Conjecture 1(iii).

Lemma 120. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0) and let $(F'_\nu)_\nu = \mathcal{F}' := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k.*

Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower of $\mathcal{F}$ (resp. $\mathcal{F}'$), let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ and let $\Gamma' := k' \cdot \Gamma$. Then the following hold:

(i) Γ is finite if and only if Γ' is finite.

(ii) Suppose that Γ is weakly connected. Then Γ' is a disjoint union of its finitely many weakly connected components $\Gamma'_1, \dots, \Gamma'_r$ and the morphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_i \to \Gamma$ for all $i = 1, \dots, r$.

Moreover, the following are equivalent:

- *(a) Γ is strongly connected.*
- *(b) Γ'_i is strongly connected for some $i = 1, \dots, r$.*
- *(c) Γ'_i is strongly connected for all $i = 1, \dots, r$.*

(iii) Γ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$ if and only if Γ' is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}'}$.

Moreover, if Γ is weakly connected and $\Gamma'_1, \dots, \Gamma'_r$ are the weakly connected components of Γ' in Lemma 120(ii), then the following are equivalent:

- *(a) Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$.*
- *(b) Γ'_i is a weakly connected component of $\Gamma_{\mathcal{F}'}$ for some $i = 1, \dots, r$*
- *(c) Γ'_i is a weakly connected component of $\Gamma_{\mathcal{F}'}$ for all $i = 1, \dots, r$*

(iv) A path $\mathcal{P}' \in W(\Gamma_{\mathcal{F}'})$ is tame if and only if $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') \in W(\Gamma_{\mathcal{F}})$ is tame. In particular, all paths in Γ' are tame if and only if all paths in Γ are tame.

(v) A path $\mathcal{P}' \in W(\Gamma_{\mathcal{F}'})$ has balanced ramification indices if and only if $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') \in W(\Gamma_{\mathcal{F}})$ has balanced ramification indices. In particular, all circles in Γ' have balanced ramification indices if and only if all circles in Γ have balanced ramification indices.

Moreover, if Γ is weakly connected and $\Gamma'_1, \dots, \Gamma'_r$ are the weakly connected components of Γ' in Lemma 120(ii), then the following are equivalent:

- *(a) All circles in Γ have balanced ramification indices.*
- *(b) All circles in Γ'_i have balanced ramification indices for some $i = 1, \dots, r$*
- *(c) All circles in Γ'_i have balanced ramification indices for all $i = 1, \dots, r$*

Proof. For (i): We estimate

$$\#V(\Gamma) \leq \#\mathbb{P}_{F_0'}(V(\Gamma)) \leq \sum_{P \in V(\Gamma)} \deg(P)$$

and

$$\#E(\Gamma) \leq \#\mathbb{P}_{F_1'}(E(\Gamma)) \leq \sum_{Q \in E(\Gamma)} \deg(Q)$$

where the first estimates hold by the definition $\mathbb{P}_{\cdot}(\cdot)$ in (5) and the second estimates hold because $F_i' = k' \cdot F_i$ is a constant field extension of F_i for all $i = 0, 1$ and because of [Sti08, p. 114, Theorem 3.6.3].

Then combining these estimates and the fact that the definition of $\Gamma' = k' \cdot \Gamma$ implies the equalities $V(\Gamma') = \mathbb{P}_{F_0'}(V(\Gamma))$ and $E(\Gamma') = \mathbb{P}_{F_1'}(E(\Gamma))$ already yields the desired equivalence.

For the 'main'-part in (ii): First of all, we notice that the desired statements hold trivially if Γ is empty since $\Gamma' = k' \cdot \Gamma = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma)$ is also empty in this case and, consequently, has no weakly connected components. Therefore, in the following, suppose that Γ is non-empty. Then $\Gamma' = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma)$ is also non-empty since $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ is an epimorphism by Definition/Lemma 105(i).

Second, applying Lemma 68(iii) to Γ' (as a subgraph of itself) yields that Γ' is a disjoint union of all its weakly connected components Γ_0' and since Lemma 109 implies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma' \to \Gamma$, it especially restricts to a morphism $\Gamma_0' \to \Gamma$ for all these components Γ_0'.

Now, we will show that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ even restricts to an epimorphism of $\Gamma_0' \to \Gamma$: Let $P \in V(\Gamma)$. As Γ_0' is non-empty by the definition of weakly connected components in Definition 66(v), there is some vertex $P_{0,0}' \in V(\Gamma_0') \subseteq V(\Gamma') = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma))$. Then $P_{0,0} := \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P_{0,0}')$ is a vertex in Γ and since Γ is weakly connected, there is some undirected path $\mathcal{P}$ from $P_{0,0}$ to P. Consequently, Lemma 119 provides some $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'$ of $\mathcal{P}$ which starts at $P_{0,0}'$, i.e. $\mathcal{P}'$ is some undirected path in Γ' which starts $P_{0,0}'$ and satisfies $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \mathcal{P}$ (see Figure 5.3).

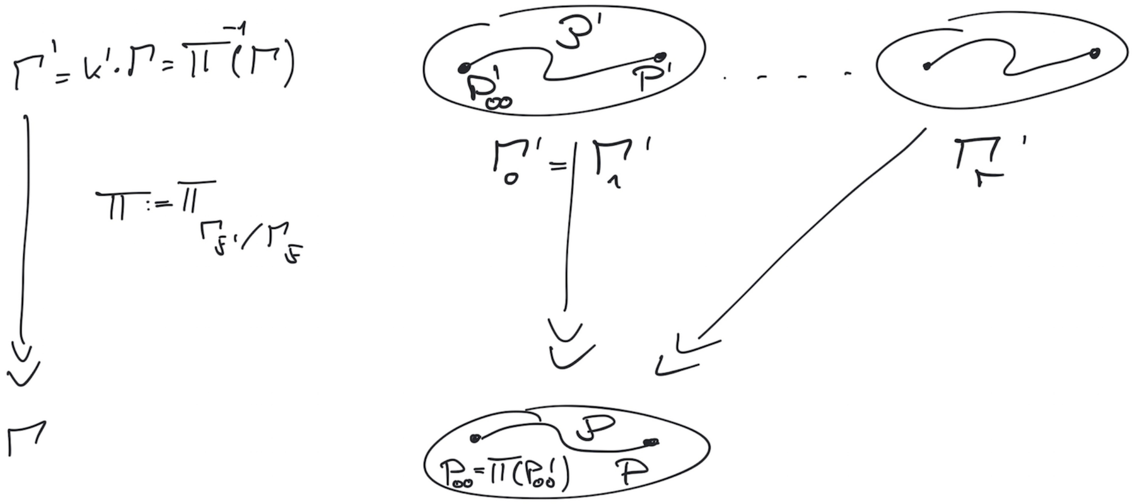


Figure 5.3: Decomposition of constant field extensions of subgraphs into their weakly connected components

But, because Γ_0' is a weakly connected component of Γ' and because the initial vertex $P_{0,0}'$ of $\mathcal{P}'$ is contained in Γ_0', the whole path $\mathcal{P}'$ must be contained in Γ_0'. In particular, combining this conclusion and the identity $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \mathcal{P}$ yields that the terminal vertex

$$P' \text{ of } \mathcal{P}' \text{ is contained in } V(\Gamma'_0) \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P). \tag{174}$$

Therefore, this conclusion in (174) implies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ indeed restricts to a surjection $V(\Gamma'_0) \to V(\Gamma)$.

Next, let $Q \in E(\Gamma)$, Moreover, define $P := v_{\text{init}}(Q) = Q \cap F_0 \in V(\Gamma)$ and choose $P' \in \mathbb{P}_{F'_0}(P) \cap V(\Gamma'_0)$ as in (174). Then Lemma 26 provides some place

$$\begin{aligned} Q' \in \mathbb{P}_{F'_1}((Q, P')) &\subseteq \mathbb{P}_{F'_1}(E(\Gamma)) \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q) \cap \mathbb{P}_{F'_1}(P') \\ &= E_+(\Gamma', P') \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q) \subseteq E(\Gamma'_0) \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q) \end{aligned} \tag{175}$$

where inclusions and equality holds by the following reasonings: The first inclusion holds because of the definition of $\mathbb{P}_{F'_1}(\cdot)$ in (5), because $Q \in E(\Gamma)$ and because of the definition of $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ in Definition/Lemma 105. The equality holds because of the definition of $\Gamma' = k' \cdot \Gamma$ in Definition/Lemma 107 and because the definition of $\Gamma_{\mathcal{F}'}$ in Definition 74 provides that any edge in $Q' \in E(\Gamma') \cap \mathbb{P}_{F'_1}(P)$ has the initial vertex $Q' \cap F'_0 = P$. The second inclusion holds because Γ'_0 is a weakly connected component and, thus, a forward complete subgraph of Γ' and because of the definition of forward complete subgraphs in Definition 66(iii). Therefore, (175) implies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ also restricts to a surjection $E(\Gamma'_0) \to E(\Gamma)$.

Hence, combining both surjections yields that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_0 \to \Gamma$ for all weakly connected components Γ'_0 of Γ'.

Finally, since the preimage $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) = \mathbb{P}_{F'_0}(P)$ contains at most $\deg(P)$ many places for any $P \in V(\Gamma)$, since any two distinct weakly connected components Γ'_0 of Γ' are disjoint and since $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_0 \to \Gamma$, we also conclude that there can only be finitely many weakly connected components Γ'_0 of Γ'.

All together, the 'main'-part of (ii) follows.

For the 'moreover'-part in (ii): We will show the implications from (c) to (b), from (b) to (a) and from (a) to (c): Here, the first implication from (c) to (b) holds trivially and the second implication from (b) to (a) holds because $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_i \to \Gamma$ and because of Lemma 70(v).

Next, we will prove the last implication from (a) to (c) for $k' = \overline{k}$ where $\overline{k}$ is the algebraic closure of k which is contained in the domain of σ by its definition in Definition 5(ii). For this case, let us replace r with s and all apostrophes with overbars, e.g. $\overline{\Gamma}_i := \Gamma'_i$ for all $i = 1, \ldots, s$.

Then, from this, we derive the implication from (a) to (c) for more general k' in the following way: Suppose that Γ is strongly connected. First, by the assumption that the implication from (a) to (c) holds for $\overline{k}$, we obtain that the

$$\text{weakly connected components } \overline{\Gamma}_1, \ldots, \overline{\Gamma}_s \text{ of } \overline{\Gamma} \text{ are all strongly connected.} \tag{176}$$

Next, we notice the equalities

$$\pi^{-1}_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}'}}(\Gamma') = \overline{k} \cdot \Gamma' = \overline{k} \cdot \Gamma = \overline{\Gamma}$$

where the first equality holds by the definition of constant field extensions of subgraphs in Definition 107, the second equality holds by Lemma 110 and the third equality holds by the definition of Γ' in the assumption.

For all $j = 1, \ldots, r$, we then conclude that $\overline{k} \cdot \Gamma'_j = \pi^{-1}_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}'}}(\Gamma'_j)$ is a forward and backward complete subgraph of $\pi^{-1}_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}'}}(\Gamma') = \overline{\Gamma}$ because Γ'_j is a weakly connected component and, thus, a forward and backward complete subgraph of Γ', and because of Lemma

70(iv). But, by Lemma 68(iii), this means that $\overline{k} \cdot \Gamma'_j$ must be a disjoint union of some of the weakly connected components $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_s$ of $\overline{\Gamma}$, say $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_m$.

Hence, this must be the disjoint union of $\overline{k} \cdot \Gamma'_j$ in the 'main'-part of (ii) and since all $\overline{\Gamma}_i$ are strongly connected by (176), we may apply the implication from (b) to (a) for $(\mathcal{F}, \Gamma_{\mathcal{F}}, \Gamma, k', \Gamma', (\Gamma'_1, \ldots, \Gamma'_r))$ chosen as $(\mathcal{F}', \Gamma_{\mathcal{F}'}, \Gamma'_j, \overline{k}, \overline{k} \cdot \Gamma'_j, (\overline{\Gamma}_1, \ldots, \overline{\Gamma}_m))$. This provides the desired statement, namely that Γ'_j is also strongly connected for all $j = 1, \ldots, r$.

Now, for proving the implication from (a) to (c) in the case $k' = \overline{k}$, let $l \in \{1, \ldots, s\}$ and notice that, by Lemma 57, it is enough to show that

$$\text{any } \overline{Q} \in E(\overline{\Gamma}_l) \text{ has some path } \mathcal{P}_{\overline{Q}} \text{ in } \overline{\Gamma}_l \text{ which goes from } v_{\text{term}}(\overline{Q}) \text{ to } v_{\text{init}}(\overline{Q}). \tag{177}$$

Therefore, let $\overline{Q} \in E(\overline{\Gamma}_l)$ and define $\overline{P}_1 := v_{\text{init}}(\overline{Q})$ and $P := \pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\overline{P}_1) \in V(\Gamma)$. Then, as Γ is strongly connected, there is some directed path $\mathcal{P}$ in Γ from $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(v_{\text{term}}(\overline{Q}))$ to P (see Figure 5.4). Moreover, Lemma 119 provides some $\pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}$-lift of this path $\mathcal{P}$, i.e. a directed path

$$\overline{\mathcal{P}}_1 \text{ in } \overline{\Gamma} \text{ which goes from } v_{\text{term}}(\overline{Q}) \text{ to some place } \overline{P}_2 \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) = \mathbb{P}_{\overline{F}_0}(P). \tag{178}$$

Consequently, the composition of $\overline{Q}$ (considered as a path of length one) and $\overline{\mathcal{P}}_1$ is a

$$\text{path } \overline{\mathcal{P}} \text{ from } \overline{P}_1 \text{ to } \overline{P}_2. \tag{179}$$

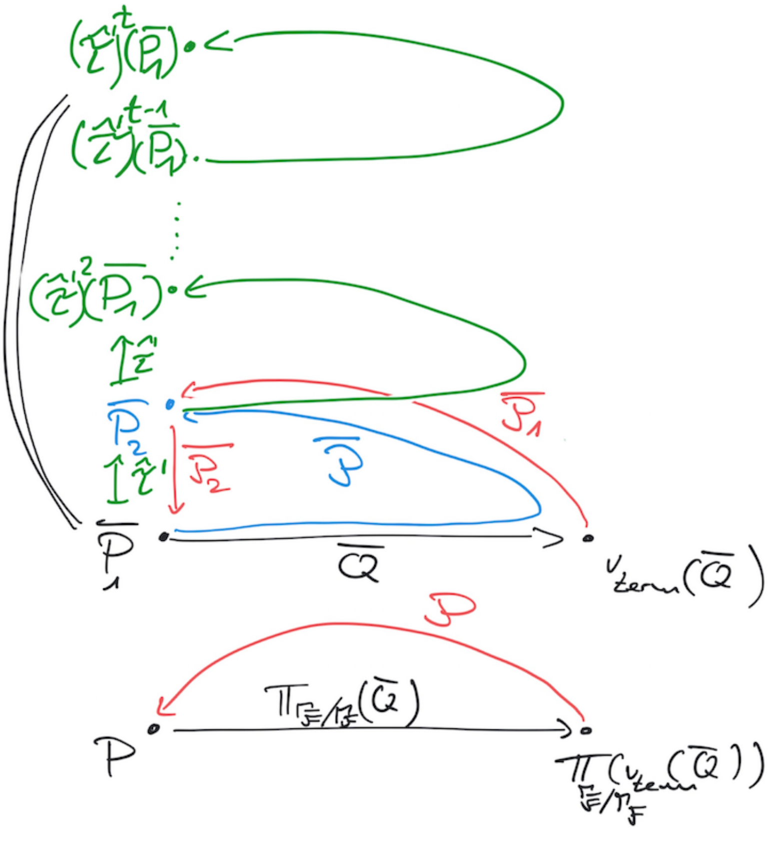


Figure 5.4: Lifting paths in a proof

Now, Lemma 25 supplies some automorphism τ of $\overline{k}$-algebras and some $t \in \mathbb{N}$ such that

$$\tau(\overline{P}_1) = \overline{P}_2 \text{ and } \tau^t(\overline{P}_1) = \overline{P}_1. \tag{180}$$

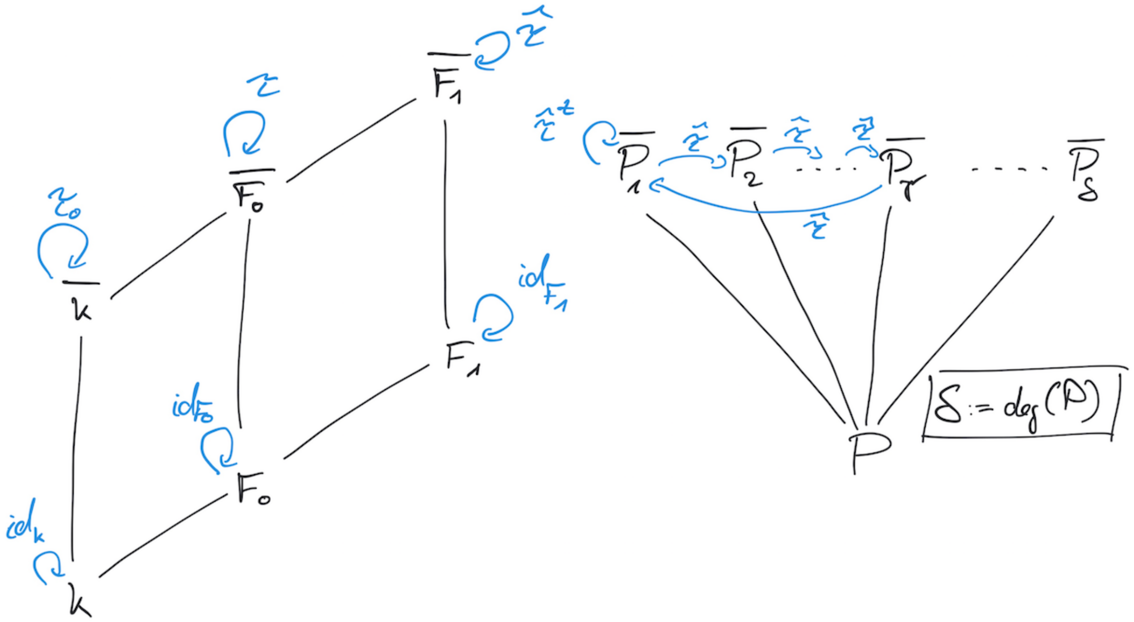


Figure 5.5: Diagrams of extensions of function fields and places and the action of automorphisms in a proof

Let τ_0 be the restriction of τ to an automorphism of k-algebras on $\overline{k}$ (see Figure 5.5). On the one hand, because of the definition $\overline{F}_0 = \overline{k} \cdot F_0$, we conclude that τ is the unique extension of τ_0 to an automorphism of F_0-algebras on $\overline{F}_0$. On the other hand, Lemma 116 also supplies a unique extension $\hat{\tau}$ of τ to an automorphism of F_1-algebras on $\overline{F}_1$. But, as the restriction of $\hat{\tau}$ to an automorphism of F_0-algebras on $\overline{F}_0$ is still an extension of τ_0, it must be equal to τ. Consequently, by this conclusion and the equalities in (180), we obtain

$$\hat{\tau}(\overline{P}_1) = \tau(\overline{P}_1) = \overline{P}_2 \text{ and } \hat{\tau}^t(\overline{P}_1) = \tau^t(\overline{P}_1) = \overline{P}_1. \tag{181}$$

Second to last, by Lemma 117, $\hat{\tau}$ induces an automorphism $\hat{\tau}'$ of directed graphs on $\overline{\Gamma}$ via $\hat{\tau}'(\overline{P}) := \hat{\tau}(\overline{P})$ and $\hat{\tau}'(\overline{Q}_0) := \hat{\tau}(\overline{Q}_0)$ for all $\overline{P} \in V(\overline{\Gamma})$ and all $\overline{Q}_0 \in E(\overline{\Gamma})$. Then, from this definition of $\hat{\tau}'$, from the first equality in (181) and from (179), we derive that $(\hat{\tau}')^i(\overline{\mathcal{P}})$ is a path in $\overline{\Gamma}$ from $(\hat{\tau}')^i(\overline{P}_1)$ to $(\hat{\tau}')^i(\overline{P}_2) = (\hat{\tau}')^{i+1}(\overline{P}_1)$ and, hence, that $(\hat{\tau}')^i(\overline{\mathcal{P}})$ and $(\hat{\tau}')^{i+1}(\overline{\mathcal{P}})$ are composable paths for all $i \in \mathbb{N}_0$.

Consequently, $\overline{\mathcal{P}}_2 := \prod_{i=1}^{t-1}(\hat{\tau}')^i(\overline{\mathcal{P}})$ is a path in $\overline{\Gamma}$ from $\hat{\tau}'(\overline{P}_1) = \overline{P}_2$ to $(\hat{\tau}')^{t-1}(\overline{P}_2) = (\hat{\tau}')^t(\overline{P}_1) = \overline{P}_1$ where the equalities hold by (181). Thus, combining this conclusion and the definition of $\overline{\mathcal{P}}_1$ in (178) provides that $\overline{\mathcal{P}}_1\overline{\mathcal{P}}_2$ is a path in $\overline{\Gamma}$ from $v_{\mathrm{term}}(\overline{Q})$ to $\overline{P}_1 = v_{\mathrm{init}}(\overline{Q})$.

Finally, since $\overline{\Gamma}_l$ is a weakly connected component of $\overline{\Gamma}$ and since the initial vertex $v_{\mathrm{term}}(\overline{Q})$ of $\overline{P}_1 = v_{\mathrm{init}}(\overline{Q})$ is contained in Γ'_l, this path must be completely contained in $\overline{\Gamma}_l$ and, hence, be one of the desired paths $\mathcal{P}_{\overline{Q}}$ in (177).

For the 'main'-part in (iii): The desired equivalences immediately follow because of the definition of $\Gamma' = k' \cdot \Gamma = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma)$ in Definition 107, because Lemma 105(i) supplies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} : \Gamma_{\mathcal{F}'} \to \Gamma_{\mathcal{F}}$ is an epimorphism and because we can thus apply the 'moreover'-part in Lemma 70(iv).

For the 'moreover'-part in (iii): First, the implication from (c) to (b) holds trivially. Second, for the implication from (a) to (c), suppose that Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$. Then, by the definition of weakly connected components in Definition 66(v), Γ is especially a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$. Consequently, the 'main'-part in (iii) supplies that Γ' is also a forward and backward complete subgraph of $\Gamma_{\mathcal{F}'}$. But, since Γ' is the disjoint union of its weakly connected components $\Gamma'_1, \ldots, \Gamma'_r$ and since it immediately follows from Definition 66(iii) that the property of being a forward (resp. backward) complete subgraph is transitive, we also conclude that $\Gamma'_1, \ldots, \Gamma'_r$ are weakly connected components of $\Gamma_{\mathcal{F}'}$.

Third and finally, for the implication from (b) to (a), let $l \in \{1, \ldots, r\}$ be the index such that Γ'_l is a weakly connected component of $\Gamma_{\mathcal{F}'}$. Consequently, by the definition of weakly connected components in Definition 66(v),

$$\Gamma'_l \text{ is a forward and backward complete subgraph of } \Gamma_{\mathcal{F}'}. \tag{182}$$

Now, let Q be an edge in $\Gamma_{\mathcal{F}}$ such that its initial (resp. terminal) vertex P is contained in Γ and set $i := 0$ (resp. $i := 1$). Then we have the equalities

$$Q \cap \sigma^i(F_0) = \sigma^i(\sigma^{-i}(Q) \cap F_0) = \sigma^i(P) \tag{183}$$

and since $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_l \to \Gamma$ by Lemma 120(ii), there is some

$$P' \in V(\Gamma'_l) \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) = V(\Gamma'_l) \cap \mathbb{P}_{F'_0}(P). \tag{184}$$

Consequently, by (183) and (184), Lemma 26 provides some place

$$Q' \in \mathbb{P}_{F'_1}((Q, \sigma^i(P'))) \subseteq E(\Gamma_{\mathcal{F}'}) \cap \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q). \tag{185}$$

In particular, the initial (resp. terminal) vertex of Q' equals

$$\sigma^{-i}(Q') \cap F'_0 = \sigma^{-i}(Q' \cap \sigma^i(F'_0)) = \sigma^{-i}(\sigma^i(P')) = P'. \tag{186}$$

But combining (184), (185), (186) and (182) yields that Q' is even an edge in $E(\Gamma'_l)$. Thus, because of this conclusion, because $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma'_l \to \Gamma$ and because of (185), we conclude that $Q = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q')$ is an edge in $E(\Gamma)$. Therefore, we established that Γ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$.

Finally, since Γ'_l is a weakly connected component of Γ', it is non-empty and, thus, its $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-image Γ is also non-empty. Hence, by the definition of weakly connected components as non-empty weakly connected forward and backward complete subgraphs in Definition 66(v), we conclude the desired statement, namely that Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$.

For (iv) and (v): Let $[P'_{i,j}]_{j-i\leq 1} := \mathcal{P}' \in W(\Gamma_{\mathcal{F}'}, n)$ for some $n \in \mathbb{N}_0$ and $[P_{i,j}]_{j-i\leq 1} := \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}) = [\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P'_{i,j})]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n)$. Then Definition/Lemma 105(i) provides the equality

$$e(P'_{i-1,i}|\sigma^{\varepsilon}(P'_{i-1+\varepsilon.i-1+\varepsilon})) = e(P_{i-1,i}|\sigma^{\varepsilon}(P_{i-1+\varepsilon.i-1+\varepsilon})) \tag{187}$$

for all $i = 1, \ldots, n$ and all $\varepsilon = 0, 1$. Thus, by these equalities in (187) and the definition of tameness (resp. balanced ramification indices) of paths in Definition/Lemma 81(ii) (resp. Definition/Lemma 82(ii)), the desired equivalence in the 'main'-part of (iv) (resp. (v)) immediately follows.

Moreover, the desired equivalence in the 'in particular'-part of (iv) also immediately follows from the first equivalence in (iv) and since $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma' \to \Gamma$ by Lemma 109.

The only remaining parts are the equivalences in the 'in particular'- and 'moreover'-part in (v):

For the 'in particular if'-part in (v): Suppose that

$$\text{all circles in } \Gamma \text{ have balanced ramification indices} \tag{188}$$

and that $\mathcal{P}'$ from above is a circle in Γ'. As $\mathcal{P}'$ is a circle, we have the equality

$$P'_{0,0} = P'_{n,n}. \tag{189}$$

Now, by Lemma 109, we obtain that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = [\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P'_{i,j})]_{j-i\leq 1} = [P_{i,j}]_{j-i\leq 1}$ is an element in $W(\Gamma, n)$ and, moreover, the equality in (189) even yields the equality $P_{0,0} = P_{n,n}$. This means that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}')$ is a closed path and, therefore, combining the assertion in (188) and Lemma 83 provides that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}')$ has balanced ramification indices. Hence, by the equivalence in the 'main'-part in (v), $\mathcal{P}'$ also must have balanced ramification indices.

For the 'in particular only if'-part in (v): Suppose that

$$\text{all circles in } \Gamma' \text{ have balanced ramification indices} \tag{190}$$

and that $\mathcal{P}$ is a circle in Γ which starts and ends at the vertex P. Then by [Sti08, p. 114, Theorem 3.6.3(c)], the set $\mathbb{P}_{F'_0}(P) = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) \subseteq \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma)) = V(\Gamma')$ contains finitely many places, say

$$\{P'_1, \dots, P'_\delta\} := \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) \tag{191}$$

with $\delta = \#\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P)$. Moreover, for all $i = 1, \dots, \delta$, the Path Lifting Lemma 119 supplies some $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift $\mathcal{P}'_i$ of $\mathcal{P}$ in Γ' which starts at P'_i. In particular, since $\mathcal{P}$ is a circle, its lift $\mathcal{P}'_i$ must also stop at some vertex $P'_{s(l)} \in \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P)$.

This defines a map

$$s : \{1, \dots, \delta\} \to \{1, \dots, \delta\} \text{ such that } \mathcal{P}'_i \text{ goes from } P'_i \text{ to } P'_{s(i)} \tag{192}$$

and, therefore, the paths $\mathcal{P}'_i$ and $\mathcal{P}'_{s(i)}$ are composable for all $i = 1, \dots, \delta$. Furthermore, since the set $\{s^\mu(1) : \mu \in \mathbb{N}_0\} \subseteq \{1, \dots, \delta\}$ is finite, there must be repetitions and, thus, we obtain natural numbers $\mu < \nu$ such that $s^\mu(1) = s^\nu(1)$. Consequently, the composition

$$\mathcal{P}' := \prod_{i=\mu}^{\nu-1} \mathcal{P}'_{s^i(1)} \tag{193}$$

is a closed path in Γ'.

Next, we notice that, by the assertion in (190) and by Lemma 83, the closed path $\mathcal{P}'$ must have balanced ramification indices. Then combining this conclusion, the fact that $\mathcal{P}'_{s(i)}$ is a $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$-lift of $\mathcal{P}$ and the equivalence in the 'main'-part in (v) yields that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}') = \prod_{i=\mu}^{\nu-1} \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P}_{s(i)}) = \prod_{i=\mu}^{\nu-1} \mathcal{P}$ has balanced ramification indices. But, by the definition of having balanced ramification indices in Definition 82(ii) and for $[P_{i,j}]_{j-i\leq 1} := \mathcal{P} \in W(\Gamma, n)$, this means that we have the equality

$$\left(\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1})\right)^{\nu-\mu} = \left(\prod_{i=1}^{n} e(P_{i-1,i}|\sigma(P_{i,i}))\right)^{\nu-\mu}$$

Finally, if we take the $(\nu - \mu)$-th root of both sides of this equality, we conclude the desired statement, namely that $\mathcal{P}$ has balanced ramification indices.

For the equivalences in the 'moreover'-part in (v): First, the implication from (c) to (b) holds trivially. Second, for the implication from (a) to (c), suppose that all circles in Γ have balanced ramification indices. Then the 'in particular'-part in (v) also supplies that all circles in Γ' have balanced ramification indices. But, since Γ' is the disjoint union of $\Gamma'_1, \ldots, \Gamma'_r$, we conclude that all circles in $\Gamma'_1, \ldots, \Gamma'_r$ have balanced ramification indices.

Third and finally, for the implication from (b) to (a), suppose that

$$\text{all circles in } \Gamma'_l \text{ have balanced ramification indices} \tag{194}$$

for some $l \in \{1, \ldots, r\}$. Let $\mathcal{P} = [P_{i,j}]_{j-i \leq 1}$ be a circle of length n in Γ. As $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_l \to \Gamma$, one of the vertices in the set $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P) = \{P'_1, \ldots, P'_\delta\}$ in (191) must be contained in $V(\Gamma'_l)$, say P'_1.

Then combining this conclusion, the choice of Γ'_l as a weakly connected component of Γ' and the conclusion in (192) supplies that the paths $\mathcal{P}'_{s^i(1)}$ are completely contained in Γ'_l for all $i \in \mathbb{N}_0$. In particular, this also provides that the closed path $\mathcal{P}'$ in (193) is also completely contained in Γ'_l. Consequently, because of (194), the path $\mathcal{P}'$ also has balanced ramification indices in this case.

Hence, the remaining reasoning in the proof of the 'in particular only if'-part in (v) can be again applied to conclude the desired statement, namely that $\mathcal{P}$ has balanced ramification indices. □

Weakly connected components and the CFE-projection morphism.

Lemma 121. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k which is defined by the pair (σ, F_0) and let $(F'_\nu)_\nu = \mathcal{F}' := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}'}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}'$) and let Γ' be a subgraph of $\Gamma'_{\mathcal{F}}$.*

If Γ' is a weakly connected component of $\Gamma_{\mathcal{F}'}$, then $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is a weakly connected component of $\Gamma_{\mathcal{F}}$.

Proof. Suppose that Γ' is a weakly connected component of $\Gamma_{\mathcal{F}'}$. By Lemma 70(ii) and by the definition of constant field extensions of subgraphs in Definition 107, Γ' is a subgraph of $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')) = k' \cdot \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$.

In particular, since Γ' is a weakly connected component of $\Gamma_{\mathcal{F}'}$, it must especially be one of the weakly connected components of $k' \cdot \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$. But, then the implication from (b) to (a) in the 'moreover'-part in Lemma 120(iii) implies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is indeed a weakly connected component of $\Gamma_{\mathcal{F}}$. □

5.1.5 Constant Field Ext. of Rational, Splitting and Ramification Subgraphs

Purpose of this subsection. Let $\mathcal{F}' = k' \cdot \mathcal{F}$ be a constant field extensions of a recursive tower $\mathcal{F}$. In this subsection, we will connect the constant field extensions $k' \cdot \Gamma$ of the rational, splitting and ramification subgraphs Γ of $\Gamma_{\mathcal{F}}$ with the rational, splitting and ramification subgraphs Γ' of $\Gamma_{\mathcal{F}'}$.

This connection will be crucial for the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Constant field extension of the rational subgraph.

Lemma 122. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a field k, let $\mathcal{F}' = (F'_\nu)_\nu := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k and let $\Gamma^{\mathrm{rat}}_{\mathcal{F}}$ (resp. $\Gamma^{\mathrm{rat}}_{\mathcal{F}'}$) be the rational subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$).*

Then $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ is a subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ and $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an isomorphism $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$.

Proof. First, we notice that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ by Lemma 109. Second, we notice that, for all $P \in V(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) = \mathbb{P}_{F_0}^{(1)}$ and all $Q \in V(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) = \mathbb{P}_{F_1}^{(1)}$, the preimages $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(P) = \mathbb{P}_{F_0'}(P)$ and $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(Q) = \mathbb{P}_{F_1'}(Q)$ are singletons and only consist of rational places by [Sti08, p. 114, Theorem 3.6.3(c)]. But this already implies both desired statements, namely that $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}} = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(\Gamma)$ is a subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ and that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ even restricts to an isomorphism $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$. ☐

Constant field extension of the splitting subgraph.

Lemma 123. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a field k of balanced degree, let $\mathcal{F}' = (F_\nu')_\nu := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension field k' of k and let $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ (resp. $\Gamma_{\mathcal{F}'}^{\mathrm{split}}$) be the splitting subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$).*

Then $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}}$ is a subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{split}}$ and $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an isomorphism $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}} \to \Gamma_{\mathcal{F}}^{\mathrm{split}}$.

Proof. First, by the definition of $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ in Definition 88(ii), we notice that $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ is a subgraph of the rational subgraph $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ of $\Gamma_{\mathcal{F}}$. In particular, this yields that

$$k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}} \text{ is a subgraph of } k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}}. \tag{195}$$

Second, we notice that Lemma 122 provides that $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}}$ is a subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$ and that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an isomorphism $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$. Then these two statements, the conclusion in (195) and Lemma 109 supply that

$$k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}} \text{ is a subgraph of } \Gamma_{\mathcal{F}'}^{\mathrm{rat}} \tag{196}$$

and that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an isomorphism $\phi : k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}} \to \Gamma_{\mathcal{F}}^{\mathrm{split}}$ where the latter conclusion is the second desired statement.

Moreover, because ϕ is an isomorphism, because $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ is d-regular by its definition and because of (196), we also deduce that $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}} = \phi^{-1}(\Gamma_{\mathcal{F}}^{\mathrm{split}})$ is a d-regular subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$. Hence, by the definition of $\Gamma_{\mathcal{F}'}^{\mathrm{split}}$ as the largest d-regular subgraph of $\Gamma_{\mathcal{F}'}$ in Definition 88(ii), we conclude that $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{split}}$ is a subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{split}}$ which is the first desired statement. ☐

Constant field extension of the ramification subgraph.

Lemma 124. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a field k, let $\mathcal{F}' = (F_\nu')_\nu := k' \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k and let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ (resp. $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$) be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$).*

Then we have the identity $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} = k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} \to \Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Proof. For the desired identity $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} = k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$, we first notice that the definition of $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}} = \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(\Gamma)$ in Definition 107 provides the equality

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(E(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = E(k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}). \tag{197}$$

Second, Definition/Lemma 105(i) supplies that the ramification indices of the edges in $\Gamma'_{\mathcal{F}}$ are invariant under the action of the epimorphism $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}:\Gamma_{\mathcal{F}'}\to\Gamma_{\mathcal{F}}$. In particular, this implies that

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} \text{ restricts to a surjection from the set of the ramified (resp. unramified) edges in } E(\Gamma_{\mathcal{F}'}) \text{ to the set of the ramified (resp. unramified) edges in } E(\Gamma_{\mathcal{F}}). \tag{198}$$

Combining the 'ramified'-part of the conclusions in (198), the fact that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ contains all ramified edges in $\Gamma_{\mathcal{F}}$ and the first equality in (197) yields that $k'\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ contains all ramified edges in $\Gamma_{\mathcal{F}'}$.

Then the combination of this conclusion, of the fact that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(iii) and of Lemma 70(iv) implies that $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})=k'\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}'}$ which contains all ramified edges in $\Gamma_{\mathcal{F}'}$. But, by the definition of $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ as the smallest subgraph of $\Gamma_{\mathcal{F}}$ with these properties, we even conclude that $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $k'\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Therefore, by Definition 66(iv), the complementary subgraph

$$\Gamma' := (k'\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}})\backslash\Gamma_{\mathcal{F}'}^{\mathrm{ram}} \text{ is a well defined forward and backward complete subgraph of } \Gamma_{\mathcal{F}'} \text{ which contains none of the ramified edges in } \Gamma_{\mathcal{F}'}^{\mathrm{ram}}. \tag{199}$$

In the following, we will show that Γ' is empty and, for that, we first notice that the image graph

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma') \text{ is also a forward and backward complete subgraph of } \Gamma_{\mathcal{F}}^{\mathrm{ram}} \tag{200}$$

by the following reasoning: First of all, by the choice of

$$\Gamma' = (k'\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}})\backslash\Gamma_{\mathcal{F}'}^{\mathrm{ram}} = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})\backslash\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$$

in (199), it immediately follows that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is a subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Second, let $i:=0$ (resp. $i=1$), let v'_i be the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}'}$, let v_i be the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}}$, and let $Q\in E(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ with $P_i:=v_i(Q)\in V(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma'))=\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(V(\Gamma'))$. Then there is some $P'_i\in V(\Gamma')\subseteq\mathbb{P}_{F'_0}$ such that

$$P'_i\cap F_0=\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(P'_i)=P_i \tag{201}$$

and we also have the equalities

$$Q\cap\sigma^i(F_0)=\sigma^i(\sigma^{-i}(Q)\cap F_0)=\sigma^i(v_i(Q))=\sigma^i(P_i). \tag{202}$$

Consequently, by these equalities in (201) and (202), Lemma 26 implies that there is some $Q'\in\mathbb{P}_{F'_1}((Q,\sigma^i(P'_i)))\subseteq\mathbb{P}_{F'_1}=E(\Gamma_{\mathcal{F}'})$. In particular, we derive that Q' is an edge in $\Gamma_{\mathcal{F}'}$ with the initial (resp. terminal) vertex

$$v'_i(Q')=\sigma^{-i}(Q')\cap F'_0=\sigma^{-i}(Q'\cap\sigma^i(F'_0))=\sigma^{-i}(\sigma^i(P'_i))=P'_i\in V(\Gamma').$$

But, because of this conclusion and because Γ' is forward (resp. backward) complete in (199), we obtain that Q' is even an edge in Γ' and, thus, that $Q=\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q')$ is an edge in $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$. Hence, we indeed conclude that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ which is the desired statement in (200).

Next, the conclusion in (200) supplies that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}\backslash\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is a well defined forward and backward complete subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Then combining the 'unramified'-part in (198) and the fact that Γ' contains no ramified edges by (199) provides that $\Gamma_{\mathcal{F}}^{\mathrm{ram}} \backslash \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ still contains all the ramified edges in $\Gamma_{\mathcal{F}}$. But, as $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is the smallest forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ which satisfies these properties, we deduce that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ and, consequently, $\Gamma' = (k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \backslash \Gamma_{\mathcal{F}'}$ must be empty. Hence, we obtain the desired equality $k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}} = \Gamma_{\mathcal{F}'}$.

Finally, the second desired statement, namely that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} \to \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ immediately follows from the identity $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} = k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and Lemma 109. ☐

Example 125. *Let us again consider the recursive MW-towers $\mathcal{F} = \mathcal{F}_{MW,2}$ over $\mathbb{F}_3$ and $\mathcal{F}' = \mathbb{F}_9 \cdot \mathcal{F} = \mathcal{F}'_{MW,2} = (F'_\nu)_\nu$ over $\mathbb{F}_9$ in Example 77(i) which are defined by the polynomial $f_{MW,2} = Y^2 + XY + 2X^2 + 1$. The second weakly connected component in Figure B.1 is the ramification subgraph $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ and the first weakly connected component in Figure B.2 is the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Moreover, in accordance to Lemma 124, we also see that $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ is the $\mathbb{F}_9$-constant field extension of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.*

5.2 Tower Graphs of Truncations of Recursive Towers

Purpose of this section. In this section, we will connect the tower graph of a recursive tower $\mathcal{F}$ with the tower graph of its truncation $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F})$ and introduce our second new concept, namely truncations of subgraphs. This will also be crucial for the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Structure of this section. We will structure this section very similar to Section 5.1: First, we will connect the tower graph of a recursive tower $\mathcal{F}$ to the tower graph of its truncation $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F})$ via the Trun-projection morphism $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}} : \Gamma_{\mathcal{F}_{\geq m}} \to \Gamma_{\mathcal{F}}$.

In Subsection 5.2.1, we will define truncations of subgraphs and prove some first properties.

In Subsection 5.2.2, we will again list properties of subgraphs which are in some sense invariant under truncations. Here, we will also implicitly lift paths for truncations of subgraphs.

In Subsection 5.2.3, we will relate the rational, splitting and ramification subgraphs of $\mathcal{F}$ to the rational, splitting and ramification subgraphs of $\mathcal{F}_{\geq m}$.

The Trun-projection morphism. In the following Definition/Lemma 126, we will connect the tower graph $\Gamma_{\mathcal{F}_{\geq m}}$ of the level m truncation $\mathcal{F}_{\geq m}$ of a recursive tower $\mathcal{F}$ and the tower graph $\Gamma_{\mathcal{F}}$ via the Trun-projection morphism $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}} : \Gamma_{\mathcal{F}_{\geq m}} \to \Gamma_{\mathcal{F}}$.

Definition/Lemma 126. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$), let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ and define*

$$\pi_V : V(\Gamma_{\mathcal{F}_{\geq m}}) = \mathbb{P}_{F_m} \to V(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_0} \text{ via } P \mapsto P \cap F_0$$

and

$$\pi_E : E(\Gamma_{\mathcal{F}_{\geq m}}) = \mathbb{P}_{F_{m+1}} \to E(\Gamma_{\mathcal{F}}) = \mathbb{P}_{F_1} \text{ via } Q \mapsto Q \cap F_1.$$

Then

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}} := (\pi_V, \pi_E) : \Gamma_{\mathcal{F}_{\geq m}} \to \Gamma_{\mathcal{F}}$$

*is a well defined epimorphism of directed graphs. We call $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ the **Trun-projection morphism** of $\mathcal{F}$ for m.*

Proof. Let σ be the tower map of $\mathcal{F}$ and let us write v_0' (resp. v_1') for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}_{\geq m}}$ and v_0 (resp. v_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}}$ in Definition 55(i). Let $Q \in E(\Gamma_{\mathcal{F}_{\geq m}}) = \mathbb{P}_{F_{m+1}}[\Gamma]$. Then, for all $i = 0, 1$, we compute

$$\begin{aligned} v_i(\pi_E(Q)) = v_i(Q \cap F_1) = \sigma^{-i}(Q \cap F_1) \cap F_0 = \sigma^{-i}(Q) \cap \sigma^{-i}(F_1) \cap F_0 \\ = (\sigma^{-i}(Q) \cap F_m) \cap F_0 = v_i'(Q) \cap F_0 = \pi_V(v_i'(Q)) \end{aligned} \tag{203}$$

where the first (resp. last) equality holds by the definition of π_E (resp. π_V), the second (resp. fourth) equality holds by the definition of the initial and terminal vertex maps on $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) in Definition 74, the third equality holds because σ is a bijection and the fourth equality holds because of the inclusions $F_m \supseteq F_0$ and $\sigma^{-i}(F_1) = \sigma^{-i}(F_0 \cdot \sigma(F_0)) = \sigma^{-i}(F_0) \cdot \sigma^{1-i}(F_0) \supseteq F_0$.

Consequently, the equality in (203) yields that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ is indeed a well defined morphism of directed graphs by its definition in Definition 65. Moreover, [Sti08, p. 71, Proposition 3.1.7(b)] supplies that π_V and π_E are indeed surjections. □

Places lying over image graphs of the Trun-projection morphism. In the following Lemma 127, the assumption that all vertices in Γ' have positive out-degree will be essential. Moreover, without going into details, we just remark that the desired inclusion $\mathbb{P}_{F_n}[\Gamma'] \subseteq \mathbb{P}_{F_n}[\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')]$ can indeed be proper without this assumption.

Lemma 127. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let Γ' be a subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ such that all vertices in Γ' have positive out-degree. Then we have the inclusion*

$$\mathbb{P}_{F_n}[\Gamma'] \subseteq \mathbb{P}_{F_n}[\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')]$$

for all $n \geq m$.

Proof. Let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Let $Q' \in \mathbb{P}_{F_n}[\Gamma']$ for some $n \geq m$. Then we have $\mathcal{P}' := \sigma^{-1}_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(Q')) \in W(\Gamma', n-m)$ by the definition of $\mathbb{P}_{F_n}[\Gamma']$ in Definition 85.

Moreover, since all vertices in Γ' have positive out-degree, there is a path $\mathcal{P}'' \in W(\Gamma', m)$ which starts at the terminal vertex of $\mathcal{P}'$ (see Figure 5.6). Consequently, we can define

$$[P'_{i,j}]_{j-i\leq 1} := \mathcal{P}' \cdot \mathcal{P}'' \in W(\Gamma', n) \tag{204}$$

and, by Lemma 78, obtain the equalities

$$\begin{aligned} \sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}' \cdot \mathcal{P}'') &= \sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}') \cdot \sigma^{n-m}(\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}'')) \\ &= \mathrm{Path}_{\mathcal{F}_{\geq m}}(Q') \cdot \sigma^{n-m}(\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}'')) \in W(\mathcal{F}_{\geq m}, n). \end{aligned}$$

Thus, Lemma 19 supplies some place

$$Q'' \in \mathbb{P}_{F_{n+m}}(\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}' \cdot \mathcal{P}'')) \cap \mathbb{P}_{F_{n+m}}(Q') \subseteq \mathbb{P}_{F_{n+m}}[\Gamma'] \tag{205}$$

Finally, we obtain $Q' \in \mathbb{P}_{F_n}[\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')]$ and, hence, the desired inclusion $\mathbb{P}_{F_n}[\Gamma'] \subseteq \mathbb{P}_{F_n}[\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')]$ by the computation

$$\sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(Q')) = [\sigma^{-i}(Q' \cap F_{i,j})]_{j-i\leq 1} = [\sigma^{-i}((Q'' \cap F_{0,n}) \cap F_{i,j})]_{j-i\leq 1}$$

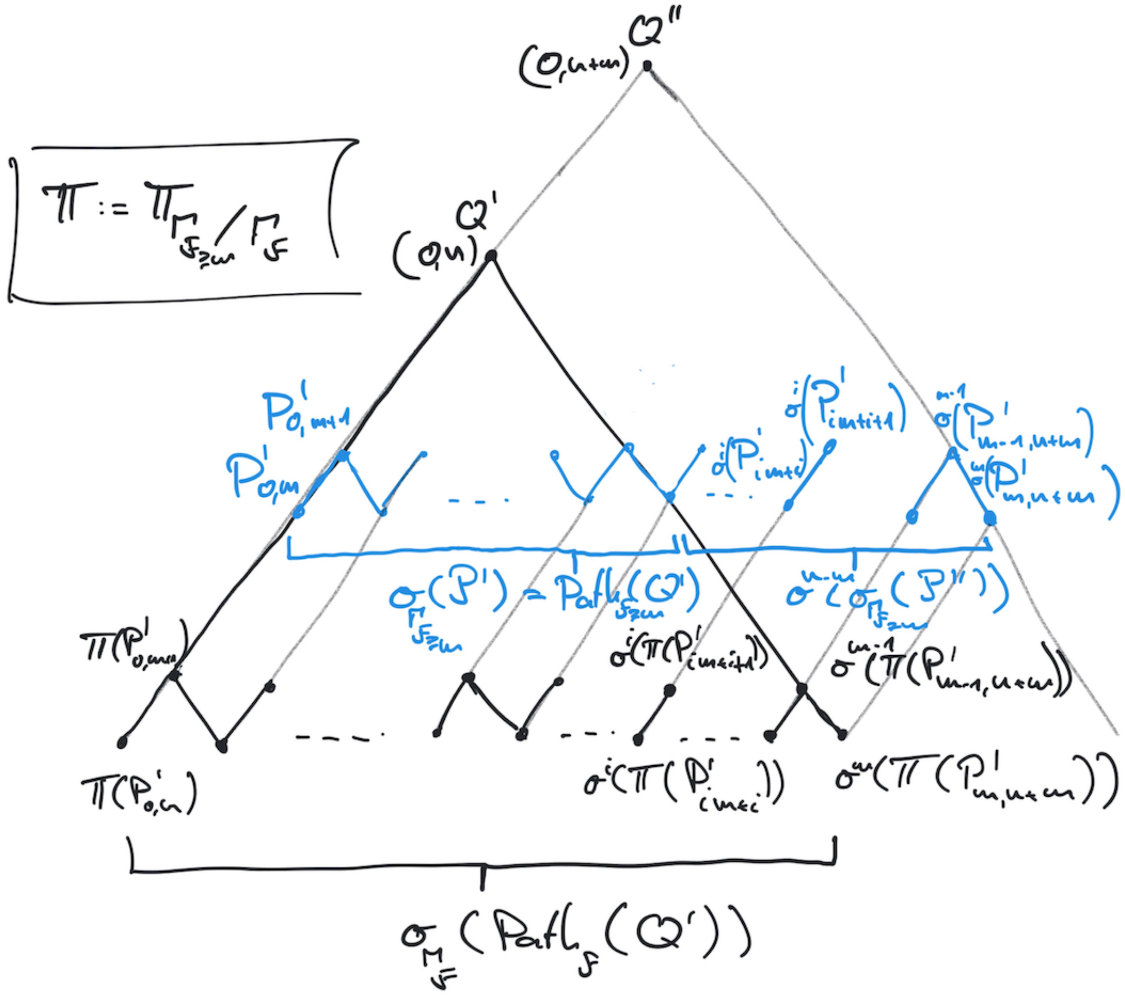


Figure 5.6: Inclusions of places over paths in a proof

$$= [\sigma^{-i}((Q'' \cap F_{i,m+j}) \cap F_{i,j})]_{j-i\leq 1} = [\sigma^{-i}((\sigma^i(P'_{i,m+j}) \cap F_{i,j})]_{j-i\leq 1}$$
$$= [P'_{i,m+j} \cap F_{j-i}]_{j-i\leq 1} = [\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(P'_{i,m+j})]_{j-i\leq 1} \in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'), n)$$

where the equalities and containment-statement holds by the following reasonings: The first equality hold by the definition of $\mathrm{Path}_{\mathcal{F}}$ in Definition/Lemma 17(i) and of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76. The second equality holds by the choice of Q'' in (205). The third equality holds since Lemma 10(i) provides the inclusions $F_{0,n} \supseteq F_{i,j}$ and $F_{i,m+j} \supseteq F_{i,j}$. The fourth equality holds because the choice of Q'' in (205) and the choice of $P'_{i,j}$ in (204) imply the equalities $(\sigma^i(P'_{i,j}))_{j-i\leq 1} = \sigma_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathcal{P}' \cdot \mathcal{P}'') = \mathrm{Path}_{\mathcal{F}_{\geq m}}(Q'') = (Q'' \cap F_{i,m+j})_{j-i\leq 1} \in W(\mathcal{F}_{\geq m}, n)$. The fifth equality holds since σ is a bijection and since Lemma 10(ii) and Lemma 10(i) imply the equalities $\sigma^{-i}(F_{i,j}) = F_{0,j-i} = F_{j-i}$. The last equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$. The containment-statement holds since the $P'_{i,j}$ are contained in Γ' by their choices in (204). □

5.2.1 Truncations of Subgraphs

Purpose of this subsection. In this subsection, we will define truncations of subgraphs of tower graphs and prove some first properties.

Later in the Subsection 5.2.3, we will use truncations of subgraphs to connect the rational, splitting and ramification subgraphs of $\mathcal{F}$ with the rational, splitting and ramification subgraphs of truncation $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F})$. This will play some role in the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Moreover, in Example 129, we will also give the already announced counterexample to Conjecture 1(i), which was proposed in [BGS04, p. 7, Conjecture 1].

Truncations of subgraphs.

Definition/Lemma 128. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\mathcal{F}_{\geq m}$ be the level m truncation $\mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let Γ be a subgraph of $\Gamma_{\mathcal{F}}$. Then vertex and edge sets*

$$V(\mathrm{Trun}_{\geq m}(\Gamma)) := \mathbb{P}_{F_m}[\Gamma] \text{ and } E(\mathrm{Trun}_{\geq m}(\Gamma)) := \mathbb{P}_{F_{m+1}}[\Gamma]$$

define a subgraph $\mathrm{Trun}_{\geq m}(\Gamma)$ of $\Gamma_{\mathcal{F}_{\geq m}}$ (see Figure 5.7). We call $\mathrm{Trun}_{\geq m}(\Gamma)$ the ***level*** *m* ***truncation*** *of Γ.*

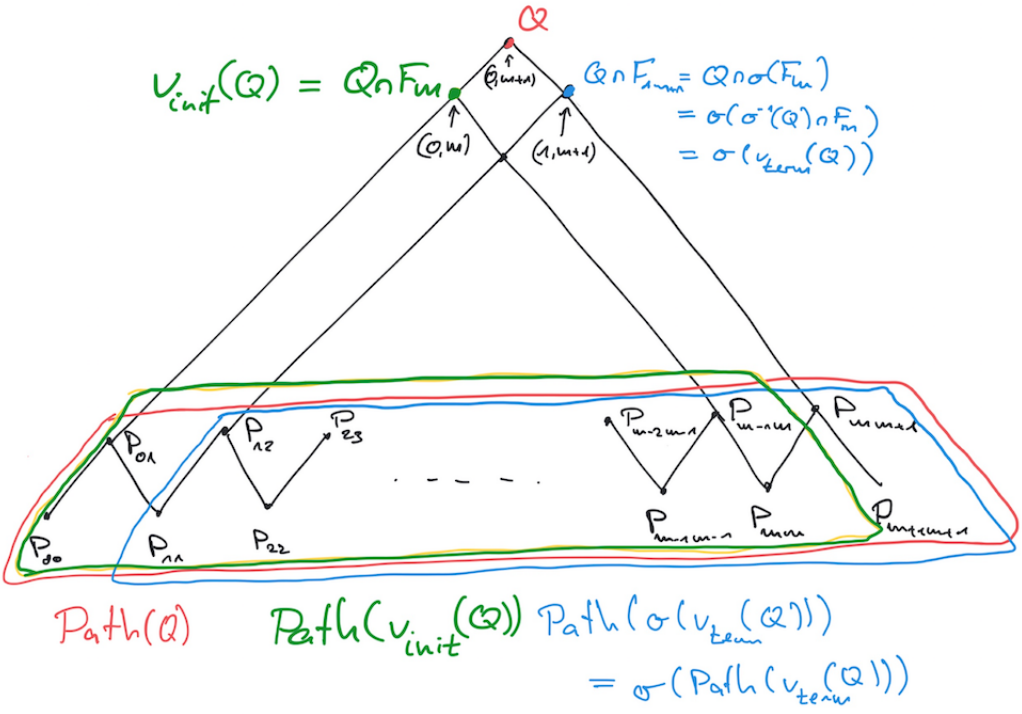


Figure 5.7: Truncations of subgraphs

Proof. Let σ be the tower map of $\mathcal{F}$ and let us write v_0 (resp. v_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}_{\geq m}}$.

Now, let $Q \in E(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_{m+1}}[\Gamma]$ and define $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$. For all $\varepsilon = 0, 1$, we then compute

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(v_\varepsilon(Q))) &= \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(\sigma^{-\varepsilon}(Q) \cap F_m)) = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(\sigma^{-\varepsilon}(P_{\varepsilon,m+\varepsilon})))\\ &= \sigma_{\Gamma_{\mathcal{F}}}^{-1}((\sigma^{-\varepsilon}(P_{i+\varepsilon,j+\varepsilon}))_{j-i\leq 1}) = [\sigma^{-\varepsilon-i}(P_{i+\varepsilon,j+\varepsilon})]_{j-i\leq 1} \in W(\Gamma, m) \qquad (206)\end{aligned}$$

where the equalities and containment-statement hold by the following reasonings: The first equality holds by the definition of the initial and terminal vertex map on $\Gamma_{\mathcal{F}_{\geq m}}$ in Definition 74. For the second equality, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Then we obtain the second equality by the identities

$$\sigma^{-\varepsilon}(Q) \cap F_m = \sigma^{-\varepsilon}(Q \cap \sigma^{\varepsilon}(F_{0,m})) = \sigma^{-\varepsilon}(Q \cap F_{\varepsilon,m+\varepsilon}) = \sigma^{-\varepsilon}(P_{\varepsilon,m+\varepsilon})$$

where the first identity holds because σ is a bijection and because of the identity in Lemma 10(i), the second identity holds because Lemma 10(ii) supplies the equality $\sigma^{\varepsilon}(F_{0,m}) = F_{\varepsilon,m+\varepsilon}$ and the last identity holds by the definitions of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q) = (Q \cap F_{i,j})_{i,j}$ in Definition 11. The third equality holds because Lemma 76(i) first provides $\mathrm{Path}(P_{\varepsilon,m+\varepsilon}) = (P_{i,j})_{i,j}$ where $i, j \in \mathbb{N}_0$ run over all $\varepsilon \leq i \leq j \leq m + \varepsilon$ and Lemma 20(ii) then provides the equality $\mathrm{Path}(\sigma^{-\varepsilon}(P_{\varepsilon,m+\varepsilon})) = (\sigma^{\varepsilon}(P_{i+\varepsilon,j+\varepsilon}))_{j-i\leq 1}$. The last equality holds by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76. The containment-statement holds because $Q \in E(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_{m+1}}[\Gamma]$ implies $[\sigma^{-i}(P_{i,j})]_{j-i\leq 1} = \sigma^{-1}(\mathrm{Path}(Q)) \in W(\Gamma, m+1)$ by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ and by the definition of $\mathbb{P}_{F_{m+1}}[\Gamma]$ in Definition 85.

Finally, combining the definition of $\mathbb{P}_{F_m}[\Gamma]$ and (206) yields

$$v_{\varepsilon}(Q) \in \mathbb{P}_{F_m}[\Gamma] = V(\mathrm{Trun}_{\geq m}(\Gamma))$$

for all $\varepsilon = 0, 1$. Hence, $\mathrm{Trun}_{\geq m}(\Gamma)$ is indeed a well defined subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ by Definition 66(i). ☐

Example 129 (Counterexample for Conjecture 1(i))**.** *Consider the tame recursive tower* $\mathcal{F}_{\geq 1} = \mathcal{F}_{GS,2} = (F_{1+\nu})_{\nu}$ *in [Sti08, p. 261, Proposition 7.3.3] for* $l = 3$ *and* $e = 2$ *over* $\mathbb{F}_q$ *with* $q = l^e = 9$ *which is defined by the polynomial* $f_{GS,2} = Y^m + (X+1)^m - 1$ *with* $m = \frac{q-1}{l-1} = 4$. *See also Examples 8(ii) and Figure 4.1.*

In Figure B.17, the first weakly connected component is the splitting subgraph and the second is the ramification subgraph. In particular, the ramified loop at the bottom vertex in the ramification subgraph supplies that we can apply the Reduction Lemma 30(iii) to $\mathcal{F}_{\geq 1}$ *and add a zeroth level* F_0 *to* $\mathcal{F}_{\geq 1}$. *Let* $\mathcal{F} = (F_{\nu})_{\nu}$ *be the corresponding recursive tower which satisfies* $\mathrm{Trun}_{\geq 1}(\mathcal{F}) = \mathcal{F}_{\geq 1}$ *and let* σ *be its tower map. Notice that* $\mathcal{F}$ *is also a polynomial-recursive tower as* F_0 *is a rational function field.*

In Figure B.18, the degree one subgraph of $\Gamma_{\mathcal{F}}$ *is depicted which is also already the ramification subgraph* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *of* $\Gamma_{\mathcal{F}}$.

(i) On the one hand, because no other rational vertices in $\Gamma_{\mathcal{F}}$ *except for the ones in its degree one subgraph* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *have rational out-going edges and because the paths in* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *and Abhyankar's Lemma provide that all vertices in* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *are ramified in* F_n/F_0 *for some* $n \in \mathbb{N}$, *we conclude that the splitting locus* $\mathrm{Split}(\mathcal{F}/F_0)$ *is empty.*

On the other hand, [Sti08, p. 261, Proposition 7.3.3] provides the estimate $2/7 = 2/(q-2) = \lambda(\mathcal{F}_{\geq 1}) = \lambda(\mathcal{F})$ *and, therefore, the splitting rate* $\nu(\mathcal{F})$ *must also be positive. Hence,* $\mathcal{F}$ *is a counterexample for the original Conjecture 1 which is Conjecture 1(i).*

(ii) More concretely, by using the not yet proven Lemma 141, Lemma 142 and Lemma 144, it can easily be deduced that the first (resp. second) weakly connected component in Figure B.17 is the level one truncation of the first (resp. second) weakly connected component in Figure B.18.

Truncations and the Trun-projection morphism. In general, $\mathrm{Trun}_{\geq m}(\Gamma)$ is only a subgraph of the preimage graph ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma)$. But if Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$, then the following Lemma 130(ii) supplies that these graphs are even equal.

Lemma 130. *Let* $\mathcal{F} = (F_{\nu})_{\nu}$ *be a recursive tower, let* $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_{\nu}$ *be the level* m *truncation of* $\mathcal{F}$, *let* $\Gamma_{\mathcal{F}}$ *(resp.* $\Gamma_{\mathcal{F}_{\geq m}}$*) be the tower graph of* $\mathcal{F}$ *(resp.* $\mathcal{F}_{\geq m}$*) and let* Γ *be a subgraph of* $\Gamma_{\mathcal{F}}$. *Then the following hold:*

(i) The morphism $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ *restricts to a morphism* $\mathrm{Trun}_{\geq m}(\Gamma) \to \Gamma$.

Moreover, if all vertices in Γ have positive out-degrees, this restriction is even an epimorphism and we have the identity

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\mathrm{Trun}_{\geq m}(\Gamma)) = \Gamma.$$

(ii) If Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$, then we have the identity

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma) = \mathrm{Trun}_{\geq m}(\Gamma).$$

Proof. For the 'main'-part in (i): Let Q' be a place in $V(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_m}[\Gamma]$ (resp. a place in $E(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_{m+1}}[\Gamma]$). Then, for the pyramid $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ of $\mathcal{F}$, we derive

$$[\sigma^{-i}(Q' \cap F_{i,j})\ ]_{j-i\leq 1} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q')) \in W(\Gamma) \tag{207}$$

where the equality holds by the definitions of Path in Definition/Lemma 17(i) and of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and the containment-statement holds by the definition of $\mathbb{P}.[\Gamma]$ in Definition 85. In particular, for $\varepsilon := 0$ (resp. $\varepsilon := 1$), we deduce

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(Q') = Q' \cap F_{\varepsilon} = Q' \cap F_{0,\varepsilon} \in V(\Gamma) \text{ (resp. } \in E(\Gamma))$$

where the first equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$, the second equality holds by the identity in Lemma 10(i) and the containment-statement holds by (207). Hence, $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ indeed restricts to a morphism $\mathrm{Trun}_{\geq m}(\Gamma) \to \Gamma$.

For the 'moreover'-part' in (i): For the first desired statement, let $Q \in E(\Gamma)$. Then, by the assumption that all vertices have positive out-degrees, there is some path $\mathcal{P} \in W(\Gamma, m+1)$ which starts at the edge Q, i.e.

$$\mathrm{Edge}_1(\mathcal{P}) = Q \tag{208}$$

Now, Lemma 17(i) supplies some place $Q' \in \mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \subseteq \mathbb{P}_{F_{m+1}}[\Gamma]$. In particular, we obtain

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(Q') = Q' \cap F_1 = \mathrm{Edge}_1(\mathcal{P}) = Q \tag{209}$$

where the first equality holds by the definition of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$, the second equality holds because the choice of Q' provides the identity $\mathrm{Path}(Q') = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$ and because of the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and the third equality holds by the equality in (208).

Next, the assumption that all vertices in Γ have positive out-degrees also provides that any $P \in V(\Gamma)$ is the initial vertex for some edge $Q \in E(\Gamma)$. Consequently, combining (209) and the definition of morphism of directed graphs in Definition 65 yields that the initial vertex P' of Q' is mapped to P via $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$.

Hence, by this conclusion, by (209) and by the 'moreover'-part, we indeed deduce that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\mathrm{Trun}_{\geq m}(\Gamma) \to \Gamma$ which is the first desired statement.

Finally, the desired identity $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\mathrm{Trun}_{\geq m}(\Gamma)) = \Gamma$ immediately follows from the first desired statement in the 'moreover'-part and the definition of the image graph in Definition/Lemma 69(i)

For (ii): Finally, the 'moreover'-part immediately follows from the equalities

$$V(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma)) = \pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(V(\Gamma)) = \mathbb{P}_{F_m}(V(\Gamma)) = \mathbb{P}_{F_m}[\Gamma] = V(\mathrm{Trun}_{\geq m}(\Gamma))$$

and

$$E(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma)) = \pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(E(\Gamma)) = \mathbb{P}_{F_{m+1}}(E(\Gamma)) = \mathbb{P}_{F_{m+1}}[\Gamma] = E(\mathrm{Trun}_{\geq m}(\Gamma))$$

where the equalities hold by the following reasonings: The first equalities hold by the definition of the preimage graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma)$ in Definition/Lemma 69(ii). The second equalities hold by the definition of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$. The third equalities hold by the assumption that Γ is a forward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ and by the 'moreover'-part in Lemma 86. The last equality holds by the definition of the level m truncation $\mathrm{Trun}_{\geq m}(\Gamma)$ in Definition/Lemma 128. ☐

Intersections and disjoint unions of truncations. The 'moreover'-part of the following Lemma 210 is only true for disjoint subgraphs. The problem with a non-disjoint union $\Gamma_1 \cup \Gamma_2$ of subgraphs is that the set of path $W(\Gamma_1 \cup \Gamma_2)$ might be larger than just the union $W(\Gamma_1) \cup W(\Gamma_2)$. Thus, we only obtain that $\mathrm{Trun}_{\geq m}(\Gamma_1) \cup \mathrm{Trun}_{\geq m}(\Gamma_2)$ is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_1 \cup \Gamma_2)$.

Lemma 131. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let M be a set of subgraphs of $\Gamma_{\mathcal{F}}$. Then we have the identity*

$$\mathrm{Trun}_{\geq m}(\bigcap_{\Gamma \in M} \Gamma) = \bigcap_{\Gamma \in M} \mathrm{Trun}_{\geq m}(\Gamma).$$

Moreover, if all subgraphs Γ in M are pairwise disjoint, then their truncations $\mathrm{Trun}_{\geq m}(\Gamma)$ are also pairwise disjoint and we have also have the identity

$$\mathrm{Trun}_{\geq m}(\coprod_{\Gamma \in M} \Gamma) = \coprod_{\Gamma \in M} \mathrm{Trun}_{\geq m}(\Gamma).$$

Proof. We can interpret the vertices as the paths of length zero and the edges as paths of length one, i.e. $V(\cdot) = W(\cdot, 0)$ and $E(\cdot, 1)$. Let us also set $\Phi := \bigcap$ in any case (resp. $\Phi := \coprod$ if all subgraphs Γ in M are pairwise disjoint).

For all $i = 0, 1$, we then obtain the equalities

$$\begin{aligned} W\left(\underset{\Gamma \in M}{\Phi} \mathrm{Trun}_{\geq m}(\Gamma), i\right) &= \underset{\Gamma \in M}{\Phi} V(\mathrm{Trun}_{\geq m}(\Gamma)) = \underset{\Gamma \in M}{\Phi} \mathbb{P}_{F_m}[\Gamma] \\ &= \underset{\Gamma \in M}{\Phi} \mathbb{P}_{F_m}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m))) = \mathbb{P}_{F_m}\left(\sigma_{\Gamma_{\mathcal{F}}}\left(\underset{\Gamma \in M}{\Phi} W(\Gamma, m)\right)\right) \\ &= \mathbb{P}_{F_m}\left(\sigma_{\Gamma_{\mathcal{F}}}\left(W\left(\underset{\Gamma \in M}{\Phi} \Gamma, m\right)\right)\right) = \mathbb{P}_{F_m}\left[\underset{\Gamma \in M}{\Phi} \Gamma\right] \\ &= W\left(\mathrm{Trun}_{\geq m}\left(\underset{\Gamma \in M}{\Phi} \Gamma\right), i\right) \end{aligned} \tag{210}$$

where the equalities holds by the following reasonings: The first equality holds by the definition of the intersection (resp. union) of subgraphs in Definition 66(ii). The second and last equalities hold by the definition of truncations of subgraphs of $\Gamma_{\mathcal{F}}$ in Definition/Lemma 128. The third and second to last equalities hold by the definition of $\mathbb{P}_{F_m}[\cdot]$ in Definition 85. The fourth equality holds by the definition $\mathbb{P}_{F_m}(\cdot)$ in (5) and by the bijectivity of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76. The fifth equality holds because the set of paths on the intersection (disjoint union) graph is clearly the intersection (disjoint union) of the sets of paths on the involved graphs.

These equalities in (210) supply the desired identities in the 'main'- and 'moreover'-part. ☐

Transitivity rule for truncations. Different than for the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')$ of a subgraph Γ' of $\Gamma_{\mathcal{F}'}$ in Lemma 127, the following Lemma 132 will ensure that the places which lie over a subgraph Γ of $\Gamma_{\mathcal{F}}$ are the same the places which lie over its truncation $\mathrm{Trun}_{\geq m}(\Gamma)$.

As an immediate consequence, we will obtain the transitivity rule for truncations of subgraphs in Lemma 133.

Lemma 132. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower. Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. For all $m, n \in \mathbb{N}_0$ with $n \geq m$, we then have the identity*

$$\mathbb{P}_{F_n}[\Gamma] = \mathbb{P}_{F_n}[\mathrm{Trun}_{\geq m}(\Gamma)].$$

Proof. Let σ be the tower map of $\mathcal{F}$ and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Moreover, let $Q \in \mathbb{P}_{F_n}$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$.

First, suppose that $Q \in \mathbb{P}_{F_n}[\Gamma]$. By the definition of $\mathbb{P}_{F_n}[\Gamma]$ in Definition 85, we then have

$$(\sigma^{-i}(P_{i,j}))_{j-i\leq 1} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(Q)) \in W(\Gamma, n) \tag{211}$$

Next, for all $0 \leq r \leq s \leq n-m$ with $s - r \leq 1$, we compute

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}^{-1}\left(\mathrm{Path}_{\mathcal{F}}\left(\sigma^{-r}(P_{r,m+s})\right)\right) = \sigma_{\Gamma_{\mathcal{F}}}^{-1}\left(\sigma^{-r}\left(\mathrm{Path}_{\mathcal{F}}(P_{r,m+s})\right)\right) &= \sigma_{\Gamma_{\mathcal{F}}}^{-1}\left(\sigma^{-r}\left((P_{i,j})_{\substack{r\leq i\leq j\leq m+s\\ j-i\leq 1}}\right)\right)\\ &= \sigma_{\Gamma_{\mathcal{F}}}^{-1}\left((\sigma^{-r}(P_{r+i,r+j}))_{\substack{0\leq i\leq j\leq m+s-r\\ j-i\leq 1}}\right)\\ &= (\sigma^{-(r+i)}(P_{r+i,r+j}))_{i,j} \in W(\Gamma, m)\end{aligned} \tag{212}$$

where the equalities and containment-statement hold by the following reasonings: The first equality holds by Definition/Lemma 20(ii). The second equality holds by 'on the other hand'-part in Lemma 17(ii). The third equality holds by Definition/Lemma 20(i). The last equality holds by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76. The containment-statement holds by (211).

Finally, we obtain $Q \in \mathbb{P}_{F_n}[\mathrm{Trun}_{\geq m}(\Gamma)]$ by

$$\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}^{-1}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(Q)) = (\sigma^{-i}(P_{i,m+j}))_{j-i\leq 1} \in W(\mathrm{Trun}_{\geq m}(\Gamma), n-m) \tag{213}$$

where the equality holds by the definitions of $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$ in Definition 11, of $\mathrm{Path}_{\mathcal{F}_{\geq m}}$ and of $\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}$ and the containment-statement holds because (212) implies that $\sigma^{-i}(P_{i,m+j})$ is contained in $\mathrm{Trun}_{\geq m}(\Gamma)$.

Second, suppose that $Q \in \mathbb{P}_{F_n}[\mathrm{Trun}_{\geq m}(\Gamma)]$. Then proof basically goes in the opposite direction: By assertion, we start with

$$\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}^{-1}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(Q)) \in W(\mathrm{Trun}_{\geq m}(\Gamma), n-m)$$

and, thus, conclude $(\sigma^{-i}(P_{i,m+j}))_{j-i\leq 1} \in W(\mathrm{Trun}_{\geq m}(\Gamma), n-m)$ by the equality in (213). This again implies that the path on the left side of the equalities in (212) is contained in $W(\Gamma, m)$. Hence, the right side $(\sigma^{-(r+i)}(P_{r+i,r+j}))_{i,j}$ is also contained for all $r = 0, \ldots, n-m$. Finally, combining this and the equality in (211) yields $Q \in \mathbb{P}_{F_n}[\Gamma]$.

Hence, we established the desired identity. □

Lemma 133. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower. Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. For all $m, n \in \mathbb{N}_0$, we then have the identity*

$$\mathrm{Trun}_{\geq n}(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathrm{Trun}_{\geq m+n}(\Gamma).$$

Proof. By the definition of truncations of subgraphs in Definition 128, we have to show the identities $\mathbb{P}_{F_{m+n+i}}[\Gamma] = \mathbb{P}_{F_{m+n+i}}[\mathrm{Trun}_{\geq m}(\Gamma)]$ for all $i = 0, 1$. But, these equalities hold by Lemma 132. □

Truncations and subgraphs of the rational subgraph without ramified edges.

Lemma 134. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$) be the rational subgraph of the tower graph $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$). Finally let Γ be a subgraph of $\Gamma_{\mathcal{F}}$ such that all vertices in Γ have positive out-degree and such that Γ contains no ramified edges.*

Then Γ is a subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ if and only if $\mathrm{Trun}_{\geq m}(\Gamma)$ is a subgraph $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$.

Proof. By the definition of the rational subgraph as the subgraph with all the rational vertices and edges in Definition 88(i), it is enough to show that

$$\begin{aligned}&\text{the vertices and edges in } \Gamma \text{ are all rational if and only if}\\ &\text{the vertices and edges in } \mathrm{Trun}_{\geq m}(\Gamma) \text{ are all rational.}\end{aligned} \tag{214}$$

For the 'if'-part in (214): Suppose that the vertices and edges in $\mathrm{Trun}_{\geq m}(\Gamma)$ are all rational. Then, by the definition of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ in Definition/Lemma 126, the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\mathrm{Trun}_{\geq m}(\Gamma))$ also only contains rational vertices and edges. But, due to the assertion that all vertices in Γ have positive out-degree and due to the identity in the 'moreover'-part in Lemma 130(i), the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\mathrm{Trun}_{\geq m}(\Gamma))$ is equal to Γ. Hence, the 'if'-part in (214) follows.

For the 'only if'-part in (214): Suppose that the vertices and edges in Γ are all rational, let $Q \in V(\mathrm{Trun}_{\geq m}(\Gamma))$ (resp. $Q \in E(\mathrm{Trun}_{\geq m}(\Gamma))$) and set $\varepsilon := 0$ (resp. $\varepsilon := 1$). By the definition of truncations of subgraphs in Definition/Lemma 128, we have $Q \in \mathbb{P}_{F_{m+\varepsilon}}[\Gamma]$, i.e. $\mathcal{P} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma, m+\varepsilon)$ due to the definition of $\mathbb{P}_{F_{m+\varepsilon}}[\Gamma]$ in Definition 85.

Now, let $(P_{i,j})_{j-i\leq 1} := \mathrm{Path}(Q) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) \in W(\mathcal{F}, 0, m+\varepsilon)$. On the one hand, because Γ contains no ramified edges, because of the definition of unramified edges in Definition 88(iii) and by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76, all the extensions $P_{i-1,i}/P_{i-1,i-1}$ and $P_{i-1,i}/P_{i,i}$ are unramified. On the other hand, because of the assertion that all vertices and edges in Γ are rational, Lemma 80 supplies that $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$ is rational.

Combining these two conclusions yields that we can apply Lemma 17(iv) to $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$. Consequently, we obtain that Q is the unique place which lies over $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) = \mathrm{Path}(Q)$ and that Q must especially be rational. Hence, we also established the 'only if'-part in (214). □

Truncations and ramification. On the one hand, in the following Lemma 135, we will show that if a subgraph Γ contains no ramified edges, then the same holds for its truncation $\mathrm{Trun}_{\geq m}(\Gamma)$. On the other hand, Lemma 136 will provide that this is no equivalence in general. More concretely, it will come out that if $\mathrm{Trun}_{\geq m}(\Gamma)$ contains no ramified edges, then we can only conclude that all circles in Γ have balanced ramification indices.

Lemma 135. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let Γ be a subgraph of $\Gamma_{\mathcal{F}}$. Finally, let Q be an edge in the level m truncation $\mathrm{Trun}_{\geq m}(\Gamma)$ of Γ and let $\mathcal{P} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma, m+1)$.*

If the edge $Q \in E(\mathrm{Trun}_{\geq m}(\Gamma))$ is ramified, then the initial or terminal edge $P \in E(\Gamma)$ of $\mathcal{P}$ is also ramified.

In particular, if $\mathrm{Trun}_{\geq m}(\Gamma)$ contains ramified edges, then Γ also contains ramified edges.

Proof. Let $(F_{i,j})_{i,j} = \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$ and $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ be the pyramid of Q. By the definitions of Path in Definition and of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition, we get the equality

$[\sigma^{-i}(P_{i,j})]_{i,j} = \mathcal{P} \in W(\Gamma, m+1)$. Thus,

$$P_{0,1} \text{ is the initial and } \sigma^{-m}(P_{m,m+1}) \text{ the terminal edge of } \mathcal{P}. \tag{215}$$

Moreover, by Lemma 10(i), by Lemma 10(ii) and by the definition of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ in Definition 11, we also have the equalities $Q = P_{0,m+1}$, $Q \cap F_m = Q \cap F_{0,m} = P_{0,m}$ and $Q \cap \sigma(F_m) = Q \cap F_{1,m} = P_{1,m}$. Therefore, the definition of ramified edges in Definition 88(iii) provides that $P_{0,m+1}/P_{0,m}$ is ramified in $F_{0,m+1}/F_{0,m}$ or $P_{0,m+1}/P_{1,m+1}$ is ramified in $F_{0,m+1}/F_{1,m+1}$.

In the first case, applying Key Lemma 36(iv) to the diamond $(P_{0,m+1}, P_{0,m}, P_{m,m+1}, P_{m,m})$ and the ramified extension $P_{0,m+1}/P_{0,m}$ yields that $P_{m,m+1}/P_{m,m}$ is also ramified in $F_{m,m+1}/F_{m,m}$. Moreover, combining this, the invariance of ramification indices under the action of isomorphisms in (11), Lemma 10(i) and Lemma 10(ii) implies that $\sigma^{-m}(P_{m,m+1})/\sigma^{-m}(P_{m,m})$ is ramified in F_1/F_0. Hence, by this and by (215), the terminal edge $\sigma^{-m}(P_{m,m+1})$ of $\mathcal{P}$ is indeed ramified in the first case.

In the second case, applying Key Lemma 36(iv) to the diamond $(P_{0,m+1}, P_{0,1}, P_{1,m+1}, P_{1,1})$ and the ramified extension $P_{0,m+1}/P_{1,m+1}$ yields that $P_{0,1}/P_{1,1}$ is also ramified in $F_{0,1}/F_{1,1}$. Moreover, combining this, Lemma 10(i) and Lemma 10(ii) implies that $P_{0,1}/P_{1,1}$ is ramified in $F_1/\sigma(F_0)$. Hence, by this and by (215), the initial edge $P_{0,1}$ of $\mathcal{P}$ is indeed ramified in the second case. □

Lemma 136. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$.*

If $\mathrm{Trun}_{\geq n}(\Gamma)$ contains no ramified edges for some $n \in \mathbb{N}$, then all circles in Γ have balanced ramification indices.

Proof. We show this by contraposition. Thus, suppose that there is a circle $\mathcal{C} = [P'_{i,j}]_{j-i\leq 1}$ in Γ which has unbalanced ramification indices.

Then we have

$$\Delta(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C})) = \frac{\prod_{i=1}^{l} e(P'_{i-1,i}|P'_{i-1,i-1})}{\prod_{i=1}^{l} e(P'_{i-1,i}|\sigma(P'_{i,i}))} \neq 1 \tag{216}$$

where the first equality holds by the equality in (92) and the inequality holds by the definition of unbalanced ramification indices in Definition/Lemma 82(ii). In particular, because circles of length zero are just vertices and therefore have balanced ramification indices, $\mathcal{C}$ must have positive length l.

Now, for all $r \in \mathbb{N}_0$, the surjectivity of the map $\mathrm{Path}_{\mathcal{F}}$ in Definition/Lemma 17(i) supplies some place

$$Q_r \in \mathbb{P}_{F_{lr}}(\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=1}^{r} \mathcal{C})) \subseteq \mathbb{P}_{F_{lr}}[\Gamma] = \mathbb{P}_{F_{lr}}[\mathrm{Trun}_{\geq m}(\Gamma)] \tag{217}$$

where the inclusion holds because $\mathcal{C}$ is a circle in Γ and the equality holds by Lemma 132. Moreover, let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q_r)$. Then we have

$$[\sigma^{-i}(P_{i,m+j})]_{j-i\leq 1} = \sigma^{-1}_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(P_{0,lr})) \in W(\mathrm{Trun}_{\geq m}(\Gamma), lr - m) \tag{218}$$

for all $r \in \mathbb{N}_0$ with $lr \geq m$ where the equality holds by the equality $Q_r = P_{0,lr}$, by the definitions of $\mathrm{Path}_{\mathcal{F}_{\geq m}}$ in Definition/Lemma 17(i) and of $\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}$ in Definition/Lemma 76 and the containment-statement holds by the definition of $\mathbb{P}.[\cdot]$ in Definition 85.

Next, we consider the quotient

$$\frac{e(P_{0,lr}|P_{0,0})}{e(P_{0,lr}|P_{lr,lr})} = \Delta(\mathrm{Path}(Q_r)) = \Delta(\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=1}^{r} \mathcal{C})) = \Delta(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}))^r \tag{219}$$

for all $r \in \mathbb{N}_0$ where the first equality holds by the first equality in (92), the second equality holds by the choice of $Q_r \in \mathbb{P}_{F_{lr}}(\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=1}^{r} \mathcal{C}))$, the third equality follows immediately from the equality in the complete row in (92) and the last equality holds by (216).

But, since $0 < \Delta(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C})) \neq 1$ by (216), the sequence of quotients in (219) either tends to zero or to infinity as $r \to \infty$. This then implies that there must be an index s with $sl \geq m+1$ such that $P_{0,s}/P_{0,m}$ or $P_{0,s}/P_{ls-m,ls}$ is ramified. In particular, by the multiplicative transitivity rule for ramification indices in (7), $P_{0,t+1}/P_{0,t}$ or $P_{t,ls}/P_{t+1,ls}$ is ramified for some $t = m, \ldots, ls-1$.

In the first case, we apply Key Lemma 36(iv) to the extension $P_{0,t+1}/P_{0,t}$ in the diamond $(P_{0,t+1}, P_{0,t}, P_{t-m,t+1}, P_{t-m,t})$ and obtain that $P_{t-m,t+1}/P_{t-m,t}$ is ramified. Thus, by (218), $\sigma^{-(t-m)}(P_{t-m,t+1})$ is a ramified edge in $\mathrm{Trun}_{\geq m}(\Gamma)$.

In the second case, we apply Key Lemma 36(iv) to the extension $P_{t,ls}/P_{t+1,ls}$ in the diamond $(P_{t,ls}, P_{t,t+1+m}, P_{t+1,ls}, P_{t+1,t+1+m})$ and obtain by (218) that $\sigma^{-t}(P_{t,t+1+m})$ is a ramified edge in $\mathrm{Trun}_{\geq m}(\Gamma)$.

Hence, we established that $\mathrm{Trun}_{\geq m}(\Gamma)$ contains a ramified edge in any case. ☐

Truncations and constant field extensions commute. In Lemma 29, we already showed that truncations and constant field extensions of towers $\mathcal{F}$ commute. Lemma 137 will establish the same for subgraphs of $\Gamma_{\mathcal{F}}$.

Lemma 137. *Let $\mathcal{F} =$ be a recursive tower over the field k which is defined by the pair (σ, F_0). Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Finally, let k' be an algebraic extension field of k which is contained in the domain of σ and let $m \in \mathbb{N}_0$. Then we have the identity*

$$\mathrm{Trun}_{\geq m}(k' \cdot \Gamma) = k' \cdot \mathrm{Trun}_{\geq m}(\Gamma).$$

Proof. In order to handle the desired identities for the sets of vertices and edges simultaneously, we notice that the vertex sets can be identified with the sets of paths of length zero, i.e. $V(\cdot) = W(\cdot, 0)$, and the edge sets are the sets of paths of length one, i.e. $E(\cdot) = W(\cdot, 1)$.

Then we already obtain the desired identity $\mathrm{Trun}_{\geq m}(k' \cdot \Gamma) = k' \cdot \mathrm{Trun}_{\geq m}(\Gamma)$ by the equalities

$$W(\mathrm{Trun}_{\geq m}(k' \cdot \Gamma), \varepsilon) = \mathbb{P}_{k' \cdot F_{m+\varepsilon}}[k' \cdot \Gamma] = \mathbb{P}_{k' \cdot F_{m+\varepsilon}}(\mathbb{P}_{F_{m+\varepsilon}}[\Gamma]) = W(k' \cdot \mathrm{Trun}_{\geq m}(\Gamma), \varepsilon)$$

for all $\varepsilon = 0, 1$ where the first and last equalities hold by the definitions of truncation of subgraphs in Definition/Lemma 128 and constant field extensions of subgraphs in Definition 107 and the second equality holds by Lemma 113. ☐

5.2.2 Properties Invariant Under Truncations

Purpose of this subsection. In Lemma 138, we will prove that many properties of subgraphs are invariant under truncations. These invariances will be crucial in the proof of the almost complete answer to Conjecture 1(iii) in Corollary 184.

Properties invariant under truncations.

Lemma 138. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$ for some $m \in \mathbb{N}_0$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let Γ be a subgraph of $\Gamma_{\mathcal{F}}$. Then the following hold:*

(i) If Γ is finite, then $\mathrm{Trun}_{\geq m}(\Gamma)$ is finite.

Moreover, if all vertices in Γ have positive out-degree (resp. positive in-degree), the this is even an equivalence.

(ii) Suppose that $\mathcal{F}$ has balanced degree d and that Γ is d-regular. Then $\mathrm{Trun}_{\geq m}(\Gamma)$ *is also d-regular.*

(iii) If Γ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$, then $\mathrm{Trun}_{\geq m}(\Gamma)$ *is also a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$.*

Moreover, if all vertices in Γ have positive in-degree (resp. out-degree), then this is even an equivalence.

(iv) Suppose that all vertices in Γ have positive in- and out-degree. Then Γ is weakly (resp. strongly) connected if and only if $\mathrm{Trun}_{\geq m}(\Gamma)$ *is weakly (resp. strongly) connected.*

(v) Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$ if and only if all vertices in Γ have positive in- and out-degree and $\mathrm{Trun}_{\geq m}(\Gamma)$ *is a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$.*

Remark 139. *We remark that the items (iii) and (v) (resp. (iv)) in Lemma 138 are wrong without the assumption that all vertices have positive in- and (resp. or) out-degree.*

Proof of Lemma 138. For (i): First, we notice that if Γ is finite, then $\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m))$ and $\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m+1))$ are also finite. Moreover, if all vertices in Γ have positive out-degree (resp. positive in-degree), then this is even an equivalence.

Second, due to Definition 85, we have the equalities $\mathbb{P}_{F_m}[\Gamma] = \mathbb{P}_{F_m}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m)))$ and $\mathbb{P}_{F_{m+1}}[\Gamma] = \mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m+1)))$ and, thus, $\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m))$ and $\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, m+1))$ are finite if and only if $\mathbb{P}_{F_m}[\Gamma] = V(\mathrm{Trun}_{\geq m}(\Gamma))$ and $\mathbb{P}_{F_{m+1}}[\Gamma] = E(\mathrm{Trun}_{\geq m}(\Gamma))$ are finite.

Both conclusion together yield the desired implications in (i).

For (ii): Let $P' \in V(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_m}[\Gamma]$ where the equality holds by the definition of $\mathrm{Trun}_{\geq m}(\Gamma)$ in Definition/Lemma 128. By the definition of $\mathbb{P}_{F_m}[\Gamma]$ in Definition 85, the path $\mathcal{P} := \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(P'))$ is contained in Γ.

First, we notice that since Γ is d-regular, there are d distinct outgoing (resp. ingoing) edges $Q_1, \ldots, Q_d$ in Γ at the terminal (resp. initial) vertex of $\mathcal{P}$ by Definition 55(vii) (see Figure 5.8). Let $\mathcal{P}_i \in W(\Gamma, m+1)$ be the corresponding pairwise distinct compositions of

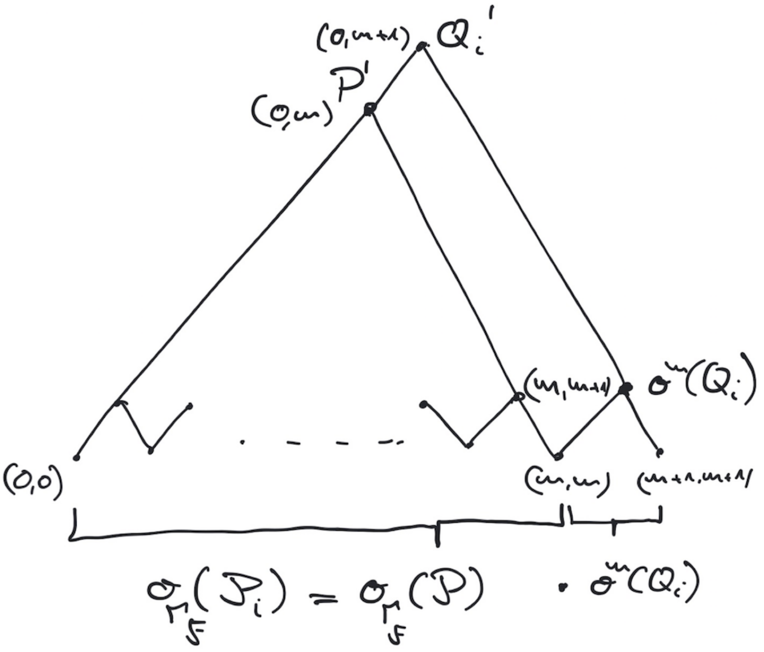


Figure 5.8: d-regularity in a proof

$\mathcal{P}$ and Q_i (resp. Q_i and $\mathcal{P}$) for all $i = 1, \ldots, d$. Then we obtain the equalities

$$\begin{aligned} &\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_i) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P} \cdot Q_i) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(Q_i)) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) \cdot \sigma^m(Q_i) \\ \text{(resp.} \quad &\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_i) = \sigma_{\Gamma_{\mathcal{F}}}(Q_i \cdot \mathcal{P}) = \sigma_{\Gamma_{\mathcal{F}}}(Q_i) \cdot \sigma(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = Q_i \cdot \sigma(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \quad) \end{aligned} \tag{220}$$

for all $i = 1, \ldots, d$ where the first equality holds by the choice of $\mathcal{P}_i$, the second equality holds by Lemma 78 and the third equality holds because $Q_i \in E(\Gamma) = W(\Gamma, 1)$ and because of the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76.

Second, let $\varepsilon := 0$ (resp. $\varepsilon := 1$). By the equalities in (220), Lemma 19 supplies that there are places

$$Q'_i \in \mathbb{P}_{m+1}(\sigma^{\varepsilon}(P')) \cap \mathbb{P}_{m+1}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_i)) \subseteq P_{F_{m+1}}[\Gamma] = E(\mathrm{Trun}_{\geq m}(\Gamma)) \tag{221}$$

for all $i = 1, \ldots, d$ where the inclusion holds since $\mathcal{P}_i$ is a path in Γ by its choice and the equality holds by the definition of $\mathrm{Trun}_{\geq m}(\Gamma)$.

Moreover, since the paths $\mathcal{P}_i$ are pairwise distinct, the places

$$Q'_i \text{ are also pairwise distinct} \tag{222}$$

for all $i = 1, \ldots, d$.

Next, let v'_{ε} be the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}_{\geq m}}$. Then we compute

$$v'_{\varepsilon}(Q'_i) = \sigma^{-\varepsilon}(Q'_i) \cap F_m = \sigma^{-\varepsilon}(Q'_i \cap \sigma^{\varepsilon}(F_m)) = \sigma^{-\varepsilon}(\sigma^{\varepsilon}(P')) = P' \tag{223}$$

for all $i = 1, \ldots, d$ where the first equality holds by the definition $\Gamma_{\mathcal{F}_{\geq m}}$ in Definition 74, the second equality holds because σ is a bijection, the third equality holds by the choice of $Q'_i \in \mathbb{P}_{m+1}(\sigma^{\varepsilon}(P'))$ in (221) and the last equality holds trivially.

Finally, on the one hand, combining (221), (222) and (223) yields that the out-degree (resp. in-degree) of P' is at least d. On the other hand, the 'in-particular'-part in Lemma 87 also provides that d is an upper bound for the out-degree (resp. in-degree) of P'.

This means that Γ is d-regular and, hence, (ii) follows.

For (iii): Let us write v'_0 (resp. v'_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}_{\geq m}}$ and v_0 (resp. v_1) for the initial (resp. terminal) vertex map on $\Gamma_{\mathcal{F}}$ and let $\varepsilon := 0$ (resp. $\varepsilon := 1$).

On the one hand, suppose that Γ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$. Let $Q' \in E(\Gamma_{\mathcal{F}_{\geq m}}) = \mathbb{P}_{F_{m+1}}$ with $v'_{\varepsilon}(Q') \in V(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_m}[\Gamma]$ and define $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q')$. For all $\varepsilon = 0, 1$, we then obtain

$$[\sigma^{-\varepsilon-i}(P_{i+\varepsilon,j+\varepsilon})]_{j-i\leq 1} = \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}(v'_{\varepsilon}(Q'))) \in W(\Gamma, m) \tag{224}$$

where the equality holds by the same reasoning as the equality in (206) and the containment-statement holds because of the assumption $v'_{\varepsilon}(Q') \in \mathbb{P}_{F_m}[\Gamma]$ and because of the definition of $\mathbb{P}_{F_m}[\Gamma]$ in Definition 85. Consequently, we conclude

$$\sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}(Q')) = [\sigma^{-i}(P_{i,j})]_{i,j} \in W(\Gamma, m+1) \tag{225}$$

where the equality holds by the definitions of $\sigma_{\Gamma_{\mathcal{F}}}$ and of Path and the containment-statement holds by combining (224) and the assumption that Γ is forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$. Finally, (225) supplies $Q' \in \mathbb{P}_{F_{m+1}}[\Gamma] = E(\mathrm{Trun}_{\geq m}(\Gamma))$ and, hence, $\mathrm{Trun}_{\geq m}(\Gamma)$ is indeed a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$.

On the other hand, suppose that all vertices in Γ have positive in-degree (resp. out-degree) and that $\mathrm{Trun}_{\geq m}(\Gamma)$ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$. Let $Q \in E(\Gamma_{\mathcal{F}})$ with $v_{\varepsilon}(Q) \in V(\Gamma)$. Because of the assumption that all vertices in Γ have positive in-degree (resp. out-degree), there is some path $\mathcal{P} \in W(\Gamma_{\mathcal{F}}, m+1)$ which stops (resp. starts) at Q and is else completely contained in Γ, i.e.

$$\begin{aligned} &\mathrm{Edge}_{m+1}(\mathcal{P}) = Q \text{ and } \mathrm{Edge}_i(\mathcal{P}) \in E(\Gamma) \text{ for all } i = 1, \dots, m \\ &(\text{resp. } \mathrm{Edge}_1(\mathcal{P}) = Q \text{ and } \mathrm{Edge}_i(\mathcal{P}) \in E(\Gamma) \text{ for all } i = 2, \dots, m+1). \end{aligned} \tag{226}$$

Then Lemma 17(i) provides a place $Q' \in \mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \subseteq E(\Gamma_{\mathcal{F}_{\geq m}})$.

Let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q')$. Then we have $\mathrm{Path}(Q') = (P_{i,j})_{j-i\leq 1}$. Moreover, because of the equality $\mathcal{P} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q')) = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, m+1)$, we also have the equalities $\mathrm{Edge}_i(\mathcal{P}) = \sigma^{-(i-1)}(P_{i-1,i})$ for all $i = 1, \dots, m+1$. Therefore, combining these equalities, the choice of $\mathcal{P}$ in (226) and the equality in (224) implies $v'_\varepsilon(Q') \in \mathbb{P}_{F_m}[\Gamma] = V(\mathrm{Trun}_{\geq m}(\Gamma))$.

But since $\mathrm{Trun}_{\geq m}(\Gamma)$ is a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$, we derive $Q' \in E(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_{m+1}}[\Gamma]$ from its definition in Definition 66(iii). In particular, the definition of $\mathbb{P}_{F_{m+1}}[\Gamma]$ in Definition 85 supplies that $Q = \mathrm{Edge}_{m+1}(\mathcal{P}) = \mathrm{Edge}_{m+1}(\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q')))$ (resp. $Q = \mathrm{Edge}_1(\mathcal{P}) = \mathrm{Edge}_1(\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q')))$) is contained in $E(\Gamma)$. Thus, Γ is indeed a forward (resp. backward) complete subgraph of $\Gamma_{\mathcal{F}}$ and, hence, (iii) follows.

For (iv): The 'if'-parts immediately follow from the identity $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\mathrm{Trun}_{\geq m}(\Gamma)) = \Gamma$ in Lemma 130(i) and from then applying Lemma 70(v).

Now, for the 'only if'-parts, suppose that Γ is weakly (resp. strongly) connected. Let P'_0 and P'_1 be vertices in $V(\mathrm{Trun}_{\geq m}(\Gamma))$. We will find an undirected (resp. directed) path $\mathcal{Q}$ from P'_0 to P'_1 in $\mathrm{Trun}_{\geq m}(\Gamma)$. This will then provide that $\mathrm{Trun}_{\geq m}(\Gamma)$ is also weakly (resp. strongly) connected.

For that, first consider the paths

$$\mathcal{P}_i := \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(P'_i)) \in W(\Gamma_{\mathcal{F}}, m) \tag{227}$$

for all $i = 0, 1$. These are both paths in Γ because we have $P'_i \in V(\mathrm{Trun}_{\geq m}(\Gamma)) = \mathbb{P}_{F_m}[\Gamma]$ for all $i = 0, 1$. Moreover, because Γ is weakly (resp. strongly) connected, there is also some undirected (resp. directed) path $\mathcal{P}$ in Γ from $v_{\mathrm{term}}(\mathcal{P}_0)$ to $v_{\mathrm{init}}(\mathcal{P}_1)$.

In particular, if we take the longest possible directed subpaths in the undirected path $\mathcal{P}_0 \cdot \mathcal{P} \cdot \mathcal{P}_1$, we obtain an odd natural number r and a sequence $(\mathcal{P}_{1,1}, \dots, \mathcal{P}_{r,r})$ of directed paths in Γ of non-zero lengths (except for the trivial case $m = 0$ and $P'_0 = P'_1$) such that

$$\mathcal{P}_{1,1} = \mathcal{P}_0 \cdot \mathcal{P}'_1 \text{ and } \mathcal{P}_{r,r} = \mathcal{P}'_r\mathcal{P}_1 \text{ for some paths } \mathcal{P}'_1 \text{ and } \mathcal{P}'_r, \tag{228}$$

$$v_{\mathrm{term}}(\mathcal{P}_{i,i}) = v_{\mathrm{term}}(\mathcal{P}_{i+1,i+1}) \text{ for all odd } i \in \{1, \dots, r-1\}, \tag{229}$$

$$v_{\mathrm{init}}(\mathcal{P}_{i,i}) = v_{\mathrm{init}}(\mathcal{P}_{i+1,i+1}) \text{ for all even } i \in \{1, \dots, r-1\} \tag{230}$$

(see Figure 5.9). Notice that we have $r = 1$ if and only if $\mathcal{P}$ is a directed path. In this case, we also have the equality $\mathcal{P}_{1,1} = \mathcal{P}_0\mathcal{P}\mathcal{P}_1$.

Second, we define $\mathcal{P}_{0,1} := v_{\mathrm{init}}(\mathcal{P}_{1,1})$ and $\mathcal{P}_{r,r+1} := v_{\mathrm{term}}(\mathcal{P}_{r,r})$. Moreover, because all vertices in Γ have positive out- and in-degree and because of the equalities in (229) and (230), we may choose paths $\mathcal{P}_{i,i+1}$ in Γ of length m such that $v_{\mathrm{term}}(\mathcal{P}_{i,i}) = v_{\mathrm{init}}(\mathcal{P}_{i,i+1}) = v_{\mathrm{term}}(\mathcal{P}_{i+1,i+1})$ for all odd $i \in \{1, \dots, r-1\}$ and $v_{\mathrm{init}}(\mathcal{P}_{i,i}) = v_{\mathrm{term}}(\mathcal{P}_{i,i+1}) = v_{\mathrm{init}}(\mathcal{P}_{i+1,i+1})$ for all even $i \in \{1, \dots, r-1\}$. Consequently, the compositions $\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1}$ for all odd $i \in \{1, \dots, r\}$ and $\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i}$ for all even $i \in \{1, \dots, r\}$ are well defined paths in Γ.

Moreover, for all $i \in \{1, \dots, r\}$, let l_i be the length of $\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1}$ if i is odd and of $\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i}$ if i is even. Then we get the estimate

$$l_i \geq 2m \tag{231}$$

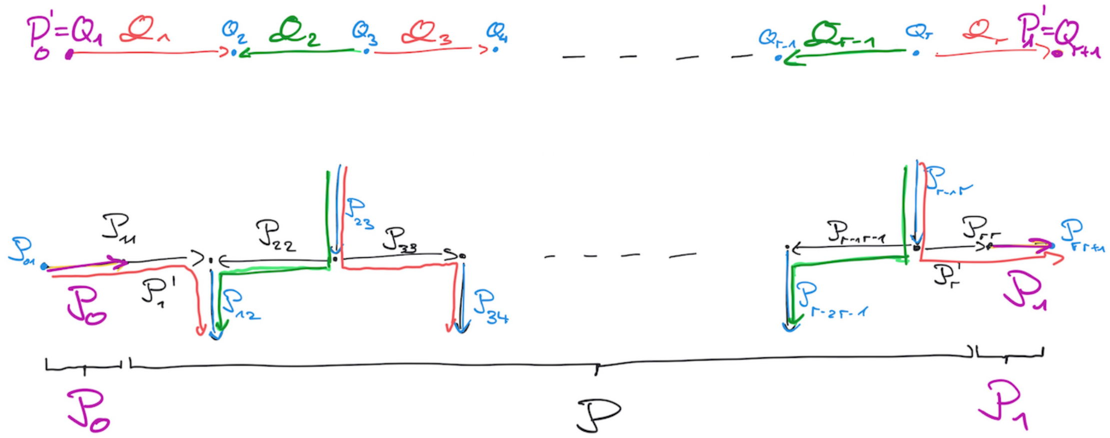


Figure 5.9: Lifting undirected paths from a subgraph to its truncation in a proof

for all $i = 1, \dots, r-1$ by the following reasoning: If $r = 1$, then we have $\mathcal{P}_{1,1} = \mathcal{P}_0\mathcal{P}\mathcal{P}_1$ and, in this case, the estimate in (231) follows from the facts that $\mathcal{P}_0$ and $\mathcal{P}_1$ are paths of length m. Otherwise, if $r \geq 2$, the estimate in (231) follows since the lengths of $\mathcal{P}_{1,1}$ and $\mathcal{P}_{r,r}$ are at least m by (228) and since the length of $\mathcal{P}_{i,i+1}$ is at least m for all $i = 1, \dots, r-1$ by its choice.

Third, for $i = 1 = r$, we obtain the equalities

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{0,1}\mathcal{P}_{1,1}\mathcal{P}_{1,2}) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_0\mathcal{P}\mathcal{P}_1) &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_0)\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))\cdot\sigma^{l_1-m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_1))\\ &= \mathrm{Path}_{\mathcal{F}}(P_0')\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))\cdot\sigma^{l_1-m}(\mathrm{Path}_{\mathcal{F}}(P_1'))\\ &= \mathrm{Path}_{\mathcal{F}}(P_0')\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))\cdot\mathrm{Path}_{\mathcal{F}}(\sigma^{l_1-m}(P_1'))\end{aligned}\tag{232}$$

where the first equality holds by the choices of the involved paths, the second equality holds by Lemma 78, the third equality holds by the choice of $\mathcal{P}_\varepsilon = \sigma^{-1}_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(P_\varepsilon'))$ for all $\varepsilon = 0, 1$ in (227) and the last equality holds by Definition/Lemma 20(ii).

Fourth, by the surjectivity of the map Path in Definition/Lemma 17(i), we get that the set $\mathbb{P}_{F_m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i}))$ is non-empty for all $i = 2, \dots, r$. Thus, for all $i = 2, \dots, r$, there are places

$$Q_i \in \mathbb{P}_{F_m}(\sigma(\mathcal{P}_{i-1,i})) \subseteq \mathbb{P}_{F_m}[\Gamma] = V(\mathrm{Trun}_{\geq m}(\Gamma)).\tag{233}$$

By this choice of Q_i and by reasonings which are similar to the reasonings of the equalities in (232), we make the following four computations: For $i = 1 < r$, we compute

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{0,1}\mathcal{P}_{1,1}\mathcal{P}_{1,2}) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{0,1}\mathcal{P}_0\mathcal{P}_1'\mathcal{P}_{1,2}) &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_0)\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_1'))\cdot\sigma^{l_1-m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{1,2}))\\ &= \mathrm{Path}_{\mathcal{F}}(P_0')\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_1'))\cdot\sigma^{l_1-m}(\mathrm{Path}_{\mathcal{F}}(Q_2))\\ &= \mathrm{Path}_{\mathcal{F}}(P_0')\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_1'))\cdot\mathrm{Path}_{\mathcal{F}}(\sigma^{l_1-m}(Q_2)).\end{aligned}\tag{234}$$

For $1 < i = r$, we compute

$$\begin{aligned}\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{r-1,r}\mathcal{P}_{r,r}\mathcal{P}_{r,r+1}) &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{r-1,r}\mathcal{P}_r'\mathcal{P}_1\mathcal{P}_{r,r+1})\\ &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{r-1,r})\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_r'))\cdot\sigma^{l_r-m}(\sigma(\mathcal{P}_1'))\\ &= \sigma_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(Q_r))\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_r'))\cdot\sigma^{l_r-m}(\mathrm{Path}_{\mathcal{F}}(P_1'))\\ &= \sigma_{\Gamma_{\mathcal{F}}}(\mathrm{Path}_{\mathcal{F}}(Q_r))\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_r'))\cdot\mathrm{Path}_{\mathcal{F}}(\sigma^{l_r-m}(P_1')).\end{aligned}\tag{235}$$

For all odd $i \in \{2, \dots, r-1\}$, we compute

$$\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1}) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i})\cdot\sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i}))\cdot\sigma^{l_i-m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}))$$

$$
\begin{aligned}
&= \mathrm{Path}_{\mathcal{F}}(Q_i) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i})) \cdot \sigma^{l_i-m}(\mathrm{Path}_{\mathcal{F}}(Q_{i+1})) \\
&= \mathrm{Path}_{\mathcal{F}}(Q_i) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i})) \cdot \mathrm{Path}_{\mathcal{F}}(\sigma^{l_i-m}(Q_{i+1}). \qquad (236)
\end{aligned}
$$

For all even $i \in \{2, \dots, r-1\}$, we compute

$$
\begin{aligned}
\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i}) &= \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i})) \cdot \sigma^{l_i-m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i})) \\
&= \mathrm{Path}_{\mathcal{F}}(Q_{i+1}) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i})) \cdot \sigma^{l_i-m}(\mathrm{Path}_{\mathcal{F}}(Q_i)) \\
&= \mathrm{Path}_{\mathcal{F}}(Q_{i+1}) \cdot \sigma^m(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i})) \cdot \mathrm{Path}_{\mathcal{F}}(\sigma^{l_i-m}(Q_i)). \qquad (237)
\end{aligned}
$$

Fifth, we will choose the Q_i in (233) more specifically for all $i = 2, \dots, r$. For that, let $Q_1 := P'_0$ and then we claim that the following iteration in (241), (242) and (243) is well defined for all $i = 1, \dots, r$ and that it produces a sequence $(\mathcal{Q}_1, \dots, \mathcal{Q}_r)$ of directed paths in $\mathrm{Trun}_{\geq m}(\Gamma)$ such that

$$
\begin{aligned}
&v_{\mathrm{init}}(\mathcal{Q}_1) = P'_0 \text{ and } v_{\mathrm{term}}(\mathcal{Q}_r) = P'_1, &(238)\\
&v_{\mathrm{term}}(\mathcal{Q}_i) = v_{\mathrm{term}}(\mathcal{Q}_{i+1}) \text{ for all odd } i \in \{1, \dots, r-1\}, &(239)\\
&v_{\mathrm{init}}(\mathcal{Q}_i) = v_{\mathrm{init}}(\mathcal{Q}_{i+1}) \text{ for all even } i \in \{1, \dots, r-1\}. &(240)
\end{aligned}
$$

Hence, this sequence $(\mathcal{Q}_1, \dots, \mathcal{Q}_r)$ finally provides the desired undirected path $\mathcal{Q}$ from P'_0 to P'_1 in $\mathrm{Trun}_{\geq m}(\Gamma)$. Moreover, if $\mathcal{P}$ is a directed path, we have $r = 1$ and, therefore, $\mathcal{Q} = \mathcal{Q}_1$ is then even a directed path.

As already announced, we consider the following iteration where i runs over all $1, \dots, r$ in ascending order:

$$
R_i \in \begin{cases}
\mathbb{P}_{F_{l_i}}(Q_i) \cap \mathbb{P}_{F_{l_i}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1})) & \text{if } i \neq r \text{ is odd} \\
\mathbb{P}_{F_{l_i}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i})) \cap \mathbb{P}_{F_{l_i}}(\sigma^{l_i-m}(Q_i)) & \text{if } i \text{ is even} \\
\mathbb{P}_{F_{l_r}}((Q_r, \sigma^{l_r-m}(P'_1))) \cap \mathbb{P}_{F_{l_r}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{r-1,r}\mathcal{P}_{r,r}\mathcal{P}_{r,r+1})) & \text{if } i = r
\end{cases} \qquad (241)
$$

$$
\mathcal{Q}_i := \sigma^{-1}_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(R_i)) \qquad (242)
$$

$$
Q_{i+1} := \begin{cases}
v_{\mathrm{term}}(\mathcal{Q}_i) & \text{if } i \text{ is odd} \\
v_{\mathrm{init}}(\mathcal{Q}_i) & \text{if } i \text{ is even}
\end{cases} \qquad (243)
$$

In the following, we will show that the sets in (241) are non-empty in all of the three cases, that $\mathcal{Q}_i$ is a well defined path of length $l_i - m$ in $\mathrm{Trun}_{\geq m}(\Gamma)$, that $Q_{i+1} \in \mathbb{P}_{F_m}(\sigma(\mathcal{P}_{i,i+1}))$ if $i < r$ and that the equalities in (238), (239) and (240) hold.

For showing that the sets (241) are non-empty, notice that we have $Q_1 = P'_0$ and that, for all $i = 2, \dots, r$, we may assume $Q_i \in \mathbb{P}_{F_m}(\sigma(\mathcal{P}_{i-1,i}))$ from the $i-1$-th step. There will be no circular reasoning since proving $Q_i \in \mathbb{P}_{F_m}(\sigma(\mathcal{P}_{i-1,i}))$ will only need the definitions in the $i-1$-th step.

The set in the first case in (241) is non-empty because of the equalities in (234) and (236) and because of Lemma 19.

The set in the second case in (241) is non-empty because of the equality in (237) and because of Lemma 19.

The set in the third case in (241) is non-empty because of the equalities in (232) and (235) and because of Lemma 19.

The path $\mathcal{Q}_i \in W(\Gamma_{\mathcal{F}_{\geq m}}, l_i - m)$ in (242) is well defined because the estimate in (231) implies that $\mathrm{Path}_{\mathcal{F}_{\geq m}}$ can indeed be applied to $R_i \in \mathbb{P}_{F_{l_i}}$. Furthermore, let $(F_{k,l})_{k,l} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$. Then we compute

$$
\begin{aligned}
\mathcal{Q}_i = \sigma^{-1}_{\Gamma_{\mathcal{F}_{\geq m}}}(\mathrm{Path}_{\mathcal{F}_{\geq m}}(R_i)) &= \sigma^{-1}_{\Gamma_{\mathcal{F}_{\geq m}}}((R_i \cap F_{k,m+l})_{l-k\leq 1}) \\
&= [\sigma^{-k}(R_i \cap F_{k,m+l})]_{l-k\leq 1} \in W(\Gamma_{\mathcal{F}_{\geq m}}, l_i - m) \qquad (244)
\end{aligned}
$$

where the first equality and the containment-statement hold by the choice of $\mathcal{Q}_i$ in (241), the second equality holds because Lemma 28 provides the equality $\mathrm{Pyr}(\mathcal{F}_{\geq m}) = (F_{k,m+l})_{k,l}$ and because of the definition of $\mathrm{Path}_{\mathcal{F}_{\geq m}}$ in Definition/Lemma 17(i) and the third equality holds by the definition of $\sigma_{\Gamma_{\mathcal{F}_{\geq m}}}$ in Definition/Lemma 76.

Next, for all $0 \leq k \leq l \leq l_i - m$ with $l - k \leq 1$, by Lemma 76(i), we obtain $\mathrm{Path}_{\mathcal{F}}(R_i \cap F_{k,m+l}) = (R_i \cap F_{s,t})_{t-s \leq 1} \in W(\mathcal{F}, k, m+l)$. Therefore, by the choice of R_i in (243), this implies that

$$\mathrm{Path}_{\mathcal{F}}(R_i \cap F_{k,m+l}) \text{ is the } (k, m+l)\text{-subpath of } \begin{cases} \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1}) & \text{if } i \text{ is odd} \\ \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i}) & \text{if } i \text{ is even} \end{cases}. \tag{245}$$

In particular, since $\mathcal{P}_{i-1,i}\mathcal{P}_{i,i}\mathcal{P}_{i,i+1}$ and $\mathcal{P}_{i,i+1}\mathcal{P}_{i,i}\mathcal{P}_{i-1,i}$ (in their respective cases) are contained in Γ by their choices, we derive that

$$\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\sigma^{-k}(\mathrm{Path}_{\mathcal{F}}(R_i \cap F_{k,m+l}))) = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(\sigma^{-k}(R_i \cap F_{k,m+l})))$$

is also contained in Γ where the equality holds by Definition/Lemma 20(ii). Consequently, by the definition of $\mathbb{P}_{m+l-k}[\Gamma]$ in Definition 85, we even conclude that the place $\sigma^{-k}(R_i \cap F_{k,m+l})$ is contained in $\mathbb{P}_{m+l-k}[\Gamma]$. But, by the definition of $\mathrm{Trun}_{\geq m}(\Gamma)$ in Definition/Lemma 128, this again provides that $\sigma^{-k}(R_i \cap F_{k,m+l})$ is contained in $\mathrm{Trun}_{\geq m}(\Gamma)$. Hence, by this and (244), we obtain the desired containment-statement $\mathcal{Q}_i \in W(\mathrm{Trun}_{\geq m}(\Gamma), l_i - m)$.

For showing $Q_{i+1} \in \mathbb{P}_{F_m}(\sigma(\mathcal{P}_{i,i+1}))$ for all $i = 1, \ldots, r-1$, we distinguish two cases: If i is odd, we compute

$$\begin{aligned} \mathrm{Path}_{\mathcal{F}}(Q_{i+1}) &= \mathrm{Path}_{\mathcal{F}}(\sigma^{-(l_i-m)}(R_i \cap F_{l_i-m,l_i})) = \sigma^{-(l_i-m)}(\mathrm{Path}_{\mathcal{F}}(R_i \cap F_{l_i-m,l_i})) \\ &= \sigma^{-(l_i-m)}(\sigma^{l_i-m}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}))) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}) \end{aligned} \tag{246}$$

where the equalities hold by the following reasonings: The first equality holds because Q_{i+1} is the terminal vertex of the path $\mathcal{Q}_i$ in (242) and because of the equality in (244). The second equality holds by Definition/Lemma 20(ii). The third equality holds by combining (245) with the first two equalities in (234) if $1 = i < r$ and with the first equality in (236) if $2 \leq i$. The fourth equality holds by Definition/Lemma 20(iii).

If i is even, we compute

$$\mathrm{Path}_{\mathcal{F}}(Q_{i+1}) = \mathrm{Path}_{\mathcal{F}}(R_i \cap F_{0,m}) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_{i,i+1}) \tag{247}$$

where the equalities hold by the following reasonings: The first equality holds because Q_{i+1} is the initial vertex of the path $\mathcal{Q}_i$ in (242) and because of the equality in (244). The second equality holds by combining (245) and the first equality in (237).

The first equality in (238) holds by the equalities

$$v_{\mathrm{init}}(\mathcal{Q}_1) = R_1 \cap F_{0,m} = Q_1 = P_0'$$

where the first equality holds by the equality in (244) and the second and third equalities holds by the choice of R_i in the first case in (241) and by the definition of the place $Q_1 = P_0'$ in $F_m = F_{0,m}$.

The second equality in (238) holds by the equalities

$$v_{\mathrm{term}}(\mathcal{Q}_r) = \sigma^{-(l_r-m)}(R_r \cap F_{l_r-m,l_r}) = P_1'$$

where the first equality holds by the equality in (244) and the second equality holds because of the choice of R_i in the third case in (241) and because $\sigma^{l_r-m}(P_1')$ is a place in $\sigma^{l_r-m}(F_m) = F_{l_r-m,l_r}$.

Second to last, the equality in (239) follows from the equalities

$$\begin{aligned} v_{\mathrm{term}}(\mathcal{Q}_{i+1}) &= \sigma^{-(l_{i+1}-m)}(R_{i+1} \cap F_{l_{i+1}-m,l_{i+1}}) = \sigma^{-(l_{i+1}-m)}(\sigma^{l_{i+1}-m}(Q_{i+1})) \\ &= Q_{i+1} = v_{\mathrm{term}}(\mathcal{Q}_i) \end{aligned}$$

where the first equality holds by the equality in (244), the second equality holds because of the choice of R_{i+1} in the second case in (241) and because $\sigma^{l_{i+1}-m}(Q_{i+1})$ is a place in $\sigma^{l_{i+1}-m}(F_m) = F_{l_{i+1}-m,l_{i+1}}$, the third equality is clear and the last equality holds by the definition of Q_{i+1} in (243).

Finally, the equality in (240) follows from the equalities

$$v_{\mathrm{init}}(\mathcal{Q}_{i+1}) = R_{i+1} \cap F_{0,m} = Q_{i+1} = v_{\mathrm{init}}(\mathcal{Q}_i)$$

where the first equality holds by the equality in (244), the second equality holds because of the choice of R_{i+1} in the first and third cases in (241) and because Q_{i+1} is a place in $F_m = F_{0,m}$ and the last equality holds by the definition of Q_{i+1} in (243).

All together, (iv) follows.

For (v): First, we notice that if Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$, then it is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by the definition of weakly connected components in Definition 66(v). In particular, by Lemma 87, this implies that all vertices in Γ have positive in- and out-degree. Thus, for both desired implications, we may assume that all vertices in Γ have positive in- and out-degree. Then Lemma 130(i) supplies that

$$\Gamma \text{ is non-empty if and only if } \mathrm{Trun}_{\geq m}(\Gamma) \text{ is non-empty.} \tag{248}$$

Hence, the desired equivalence in the 'moreover'-part of (v) immediately follows from this conclusion in (248) and from the equivalences in Lemma 138(iii) and Lemma 138(iv). □

Weakly connected components and the Trun-projection morphism.

Lemma 140. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$. Moreover, let $\Gamma_{\mathcal{F}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$) be the tower graph of $\mathcal{F}$ (resp. $\mathcal{F}_{\geq m}$) and let Γ' be a subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$.*

If Γ' is a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$, then $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')$ is a weakly connected component of $\Gamma_{\mathcal{F}}$ such that $\mathrm{Trun}_{\geq m}(\Gamma) = \Gamma'$.

Proof. Suppose that Γ' is a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$. First, we notice that, by Lemma 70(ii),

$$\Gamma' \text{ is a subgraph of } {\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')). \tag{249}$$

Second, we notice that, by Lemma 70(v), the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')$ is also weakly connected. Thus, there is some weakly connected component Γ of $\Gamma_{\mathcal{F}}$ which contains $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')$. In particular, this also implies that ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'))$ is a subgraph of ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma)$. Consequently, by this and by (249), we obtain that

$$\Gamma' \text{ is a subgraph of } {\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma). \tag{250}$$

Third, since Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$, Lemma 130(ii) provides the equality $\mathrm{Trun}_{\geq m}(\Gamma) = {\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma)$ and, moreover, Lemma 138(v) then even provides that $\mathrm{Trun}_{\geq m}(\Gamma) = {\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma)$ is a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$.

Finally, combining this, (250), the assertion that Γ' is also a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$ and the fact that weakly connected components are either equal or disjoint by Lemma 68(ii) yields the equalities $\Gamma' = {\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma) = \mathrm{Trun}_{\geq m}(\Gamma)$. In particular, because Definition/Lemma 126 supplies that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ is an epimorphism and because of the 'moreover'-part in Lemma 70(iii), we conclude that

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma') = \pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}({\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma)) = \Gamma$$

is indeed a weakly connected component of $\Gamma_{\mathcal{F}}$ such that $\Gamma' = \mathrm{Trun}_{\geq m}(\Gamma)$. ☐

5.2.3 Truncations of Rational, Splitting and Ramification Subgraphs.

Purpose of this subsection. Let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F})$ be a truncation of a recursive tower $\mathcal{F}$. In this subsection, we will connect the truncations $\mathrm{Trun}_{\geq m}(\Gamma)$ of the rational, splitting and ramification subgraphs Γ of $\Gamma_{\mathcal{F}}$ with the rational, splitting and ramification subgraphs Γ' of $\Gamma_{\mathcal{F}_{\geq m}}$.

Truncation of the rational subgraph.

Lemma 141. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$ and let $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$) be the rational subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$ of $\mathcal{F}_{\geq m}$).*

Then $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$ is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$ and $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$.

Proof. For all $P' \in V(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}) \cup E(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}) = \mathbb{P}_{F_m}^{(1)} \cup \mathbb{P}_{F_{m+1}}^{(1)}$, Lemma 80 supplies that the path $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}_{\mathcal{F}}(P'))$ is rational and, thus, contained in $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ by the definition of the rational subgraph in Definition 88(i). Consequently, by the definitions of $\mathbb{P}_{.}[\Gamma_{\mathcal{F}}^{\mathrm{rat}}]$ in Definition 85 and of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$ in Definition/Lemma 128, we obtain the inclusions and equalities

$$V(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}) \subseteq \mathbb{P}_{F_m}[\Gamma_{\mathcal{F}}^{\mathrm{rat}}] = V(\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}}))$$

and

$$E(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}) = \mathbb{P}_{F_{m+1}}[\Gamma_{\mathcal{F}}^{\mathrm{rat}}] = E(\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}}))$$

Therefore, we conclude the first desired statement, namely that $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$ is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$.

The second desired statement, namely that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}} \to \Gamma_{\mathcal{F}}^{\mathrm{rat}}$, immediately follows from the first desired statement and from Definition/Lemma 130(i). ☐

Truncation of the splitting subgraph.

Lemma 142. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d which is defined by the pair (σ, F_0), let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$ and let $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$) be the splitting subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$ of $\mathcal{F}_{\geq m}$). Let $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ be the rational subgraph and $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph of $\Gamma_{\mathcal{F}}$.*

Then $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}})$ is a d-regular forward and backward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$. Moreover, if $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$ is finite, then $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$ is even the disjoint union of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}})$ and some subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$ and $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}} \to \Gamma_{\mathcal{F}}^{\mathrm{split}} \cup (\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$.

Remark 143. *Notice that Theorem 155 will sharpen the 'moreover'-part of Lemma 142. It will imply that $\Gamma^{\text{split}}_{\mathcal{F}_{\geq m}}$ is even either empty or a weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}$. But the latter again provides that $\Gamma^{\text{split}}_{\mathcal{F}_{\geq m}}$ is either equal to* $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})$ *or to some subgraph of* $\text{Trun}_{\geq m}(\Gamma^{\text{ram}}_{\mathcal{F}} \cap \Gamma^{\text{rat}}_{\mathcal{F}})$.

Proof of Lemma 142. For the 'main'-part: First, we notice that, by the definition of $\Gamma^{\text{split}}_{\mathcal{F}}$ in Definition 88(ii), it is a d-regular forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$. Therefore, Lemma 138(ii) supplies that

$$\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}}) \text{ is also a } d\text{-regular subgraph of } \Gamma_{\mathcal{F}_{\geq m}}. \tag{251}$$

Second, we obtain the equalities and inclusions

$$V(\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})) = \mathbb{P}_{F_m}[\Gamma^{\text{split}}_{\mathcal{F}}] \subseteq \mathbb{P}_{F_m}(V(\Gamma^{\text{split}}_{\mathcal{F}})) \tag{252}$$

and

$$E(\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})) = \mathbb{P}_{F_{m+1}}[\Gamma^{\text{split}}_{\mathcal{F}}] \subseteq \mathbb{P}_{F_{m+1}}(E(\Gamma^{\text{split}}_{\mathcal{F}})) \subseteq \mathbb{P}_{F_{m+1}}(V(\Gamma^{\text{split}}_{\mathcal{F}})) \tag{253}$$

where the equalities hold by the definition of $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})$ in Definition/Lemma 128, the first inclusions hold by Lemma 86 and the last inclusion holds since the initial vertex $Q \cap F_0$ of any edge $Q \in E(\Gamma^{\text{split}}_{\mathcal{F}})$ is again contained in $\Gamma^{\text{split}}_{\mathcal{F}}$.

Now, combining the inclusions in (252) and (253) and the fact that Lemma 92(i) implies that all places in $V(\Gamma^{\text{split}}_{\mathcal{F}}) \subseteq \mathbb{P}_{F_0}$ are rational and split completely in F_m/F_0 and F_{m+1}/F_0 yields that $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})$ is a subgraph of the rational subgraph $\Gamma^{\text{rat}}_{\mathcal{F}_{\geq m}}$ of $\Gamma_{\mathcal{F}_{\geq m}}$.

Finally, by this conclusion, by the conclusion in (251) and by the definition of the $\Gamma^{\text{split}}_{\mathcal{F}_{\geq m}}$ as the largest d-regular subgraph of $\Gamma^{\text{rat}}_{\mathcal{F}_{\geq m}}$, we conclude the desired statement in the 'main'-part.

For the first desired statement in the 'moreover'-part: Since $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}})$ is a d-regular forward and backward complete subgraph of the finite d-regular forward and backward subgraph $\Gamma^{\text{split}}_{\mathcal{F}_{\geq m}}$ of $\Gamma^{\text{rat}}_{\mathcal{F}_{\geq m}}$, the complementary graph

$$\Gamma' := \Gamma^{\text{split}}_{\mathcal{F}_{\geq m}} \backslash \text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}}) \tag{254}$$

is also a well defined finite d-regular forward and backward complete subgraph of $\Gamma^{\text{rat}}_{\mathcal{F}_{\geq m}}$. In particular, by Lemma 68(iii), this means that

$$\Gamma' \text{ is a disjoint union of its } d\text{-regular weakly connected components } \Gamma'_1, \dots, \Gamma'_r. \tag{255}$$

Now, let $k \in \{1, \dots, r\}$ and consider the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$. Because of Lemma 70(v), the image graph $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is weakly connected. Furthermore, since Γ'_k is a subgraph of $\Gamma^{\text{rat}}_{\mathcal{F}_{\geq m}}$, Lemma 141 even provides that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is a weakly connected subgraph of $\Gamma^{\text{rat}}_{\mathcal{F}}$.

Next, since $\Gamma^{\text{split}}_{\mathcal{F}}$ is also a forward complete subgraph of $\Gamma_{\mathcal{F}}$, Lemma 130(ii) implies the equality $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}}) = \pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma^{\text{split}}_{\mathcal{F}})$. But, because Γ'_k is disjoint from $\text{Trun}_{\geq m}(\Gamma^{\text{split}}_{\mathcal{F}}) = \pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}{}^{-1}(\Gamma^{\text{split}}_{\mathcal{F}})$ by its choice in (255) and (254), we also conclude that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is disjoint from $\Gamma^{\text{split}}_{\mathcal{F}}$. In particular, this means that

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k) \text{ is a weakly connected subgraph of } \Gamma^{\text{rat}}_{\mathcal{F}} \text{ which cannot be } d\text{-regular.} \tag{256}$$

Finally, we will show that

$$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k) \text{ and } \Gamma_{\mathcal{F}}^{\mathrm{ram}} \text{ are not disjoint.} \tag{257}$$

Then, because $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is weakly connected and because $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(iii), we deduce that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is a subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Consequently, we conclude that the preimage graph ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k))$ is a subgraph of ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) = \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ where the equality holds because $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and because of Lemma 130(ii). But, since Γ'_k is also a subgraph of ${\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}}^{-1}(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k))$ by Lemma 70(ii), we obtain that Γ'_k is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$.

Moreover, combining this conclusion, the fact that Γ'_k is a subgraph of $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$ by its choice and the fact that $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}$ is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}})$ by Lemma 141 provides that Γ'_k is even a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{rat}}) = \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$ where the equality holds by Lemma 210.

Finally, because $k \in \{1, \ldots, r\}$ was arbitrary and because of (255), we finally derive that Γ' is also a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$. Hence, by the choice of $\Gamma' = \Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}} \backslash \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}})$, we deduce the first desired statement in the 'moreover'-part, namely that $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$ is a disjoint union of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}})$ and some subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$.

For proving the claim in (257): Assume the contrary, i.e. $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ and $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ are disjoint. By this assumption and by (256), we even obtain that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is a subgraph of $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$. Consequently, we deduce the equalities

$$\begin{aligned} N(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) &= \#\left(\mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cap \mathbb{P}_{F_n}^{(1)}\right) = \#\left(\mathrm{Path}_{\mathcal{F}}^{-1}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cap \mathbb{P}_{F_n}^{(1)}\right) \\ &= \#\left((\sigma_{\Gamma_{\mathcal{F}}}^{-1} \circ \mathrm{Path}_{\mathcal{F}})^{-1}(\mathcal{P}) \cap \mathbb{P}_{F_n}^{(1)}\right) = 1 \end{aligned} \tag{258}$$

for all $n \in \mathbb{N}_0$ and all paths $\mathcal{P} \in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k), n)$ where the first equality holds by the definition of $N(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ in (5), the second equality holds by the definition of $\mathrm{Path}_{\mathcal{F}}$ in Definition/Lemma 17(i), the third equality holds because $\sigma_{\Gamma_{\mathcal{F}}}$ is a bijection by Definition/Lemma 76 and the last equality holds by Lemma 91(ii).

Next, let $A \in \mathbb{N}_0^{s \times s}$ be the standard adjacency matrix of $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ for some enumeration $v = (P_1, \ldots, P_s)$ of its vertices, i.e.

$$A = \left(\#E(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k), P_i, P_j)\right)_{i,j} = \left(\sum_{Q \in E(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k), P_i, P_j)} w_0(Q)\right)_{i,j}$$

for the weight function $w_0 : E(\Gamma) \to \mathbb{Z}$, $Q \mapsto 1$. Then we notice that the j-th column sum in A is the in-degree of P_j and the i-th row sum in A is the out-degree of P_i for all $i, j \in \{1, \ldots, s\}$. In particular, we can choose an enumeration v such that A is a upper block triangular matrix with irreducible matrices $A_1, \ldots, A_s$ on the diagonal blocks.

But combining that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)$ is not d-regular by (256) and that d is an upper bound for the in- and out-degrees of the vertices in $\Gamma_{\mathcal{F}}$ by the 'in particular'-part in Lemma 87, we obtain that, for all $i = 1, \ldots, s$, all column and row sums in A_i have the upper bound d and that there is some column or row sum which is less than d.

Consequently, because A_i is irreducible by the combination of (256) and Lemma 63(ii), we can apply Lemma 64(ii) and get the estimates

$$\rho(A) = \max_{i=1,\ldots,s} \rho(A_i) < d \tag{259}$$

for the spectral radii $\rho(A)$ of A and $\rho(A_i)$ of A_i.

Moreover, by Lemma 59, we notice the equalities

$$\begin{aligned}(a_{i,j}^{(n)})_{i,j} := A^n &= \left(\sum_{\mathcal{P}\in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n,P_i,P_j)} w_0(\mathcal{P})\right)_{i,j} \\ &= \Big(\#W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n,P_i,P_j)\Big)_{i,j} \end{aligned} \tag{260}$$

Now, let J be a canonical Jordan form of A for some transformation matrix $T \in \mathrm{GL}(\mathbb{C},s)$ satisfying $A = TJT^{-1}$. Then, for the vector $v := \begin{pmatrix}1 & \dots & 1\end{pmatrix} \in \mathbb{N}^{1\times s}$, we conclude the equalities

$$\begin{aligned}\#W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n) = \sum_{i,j\in\{1,\dots,s\}} a_{i,j}^{(n)} &= vA^nv^t = (vT)J^n(T^{-1}v^t) \\ &= \mathcal{O}((\rho(A)+\varepsilon)^n)\end{aligned} \tag{261}$$

for all $\varepsilon > 0$ as $n \to \infty$ where the first equality holds by summing up the entries in (260), the second equality holds by the definition of $v = \begin{pmatrix}1 & \dots & 1\end{pmatrix}$, the third equality holds by the choices of J and T satisfying the equality $A = TJT^{-1}$ and the last equality holds by Lemma 61.

Finally, we compute

$$\begin{aligned}\#V(\Gamma'_k)d^{n-m} = N(F_n,V(\Gamma'_k)) = N[F_n,\Gamma'_k] &\leq N[F_n,\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)] \\ &= \sum_{\mathcal{P}\in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n)} N(F_n,\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \\ &= \#W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n) = \mathcal{O}((\rho(A)+\varepsilon)^n)\end{aligned} \tag{262}$$

for all $n \geq m$ and all $\varepsilon > 0$ as $n \to \infty$ where the equalities and estimate holds by the following reasonings: The first equality holds since Γ'_k is a subgraph of $\Gamma^{\mathrm{split}}_{\mathcal{F}_{\geq m}}$ and, therefore, Lemma 92(i) supplies the inclusion $V(\Gamma'_k) \subseteq \mathsf{Split}(\mathcal{F}_{\geq m}/F_m)$. The second equality holds because Γ'_k is a forward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ by (255) and because of the 'in particular'-part of Lemma 86. The estimate holds because the definition of $N[\cdot]$ in Definition 85 and Lemma 127 provide the equalities and estimate

$$N[F_n,\Gamma'_k] = \#\left(\mathbb{P}_{F_n}[\Gamma'] \cap \mathbb{P}^{(1)}_{F_n}\right) \leq \#\left(\mathbb{P}_{F_n}[\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma')] \cap \mathbb{P}^{(1)}_{F_n}\right) = N[F_n,\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)].$$

The third equality holds because the definition of $N[\cdot]$ implies the equalities

$$\begin{aligned}N[F_n,\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k)] &= \#\left(\mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n)) \cap \mathbb{P}^{(1)}_{F_n}\right) \\ &= \#\left(\coprod_{\mathcal{P}\in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n)} \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cap \mathbb{P}^{(1)}_{F_n}\right) \\ &= \sum_{\mathcal{P}\in W(\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}(\Gamma'_k),n)} N(F_n,\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))\end{aligned}$$

The fourth equality holds by the equality in (258). The last equality holds by (261).

But, the estimate in (262) contradicts the estimate in (259) for small $\varepsilon > 0$. Hence, the desired statement in (257) and, by that, the first desired statement in the 'moreover'-part follow.

For the second desired statement in the 'moreover'-part: We first notice that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to morphisms

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \to \Gamma_{\mathcal{F}}^{\mathrm{split}} \text{ and } \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}) \to \Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$$

by Lemma 130(i) and, consequently, to a morphism

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \cup \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}) \to \Gamma_{\mathcal{F}}^{\mathrm{split}} \cup (\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}).$$

Hence, combining this conclusion and the first desired statement in the 'moreover'-part then yields the second desired statement, namely that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}} \to \Gamma_{\mathcal{F}}^{\mathrm{split}} \cup (\Gamma_{\mathcal{F}}^{\mathrm{ram}} \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}})$. ☐

Truncation of the ramification subgraph.

Lemma 144. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\mathcal{F}_{\geq m} = \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ be the level m truncation of $\mathcal{F}$ and let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$) be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ (resp. $\Gamma_{\mathcal{F}_{\geq m}}$ of $\mathcal{F}_{\geq m}$).*

Then $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$, $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$ is a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ and $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restrict so a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}} \to \Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Proof. The first desired statement, namely that $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$, immediately follows from the fact that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(iii) and from Lemma 138(iii).

Next, we claim that

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \text{ contains all the ramified edges in } \Gamma_{\mathcal{F}_{\geq m}}. \tag{263}$$

As $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ and as $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$ is the intersection of all forward and backward complete subgraph of $\Gamma_{\mathcal{F}_{\geq m}}$ containing these ramified edges by its definition, the claim in (263) then provides that $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$ must be a subgraph of $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$. Hence, the second desired statement follows if the claim is true.

Now, for proving the claim in (263), let $Q \in E(\Gamma_{\mathcal{F}_{\geq m}}) = \mathbb{P}_{F_{m+1}}$ be a ramified edge in $\Gamma_{\mathcal{F}_{\geq m}}$. This means by its definition in Definition 88(iii) that $Q/Q \cap F_m$ is ramified in F_{m+1}/F_m or $Q/Q \cap \sigma(F_m)$ is ramified in $F_{m+1}/\sigma(F_m)$.

We will go through both cases: On the one hand, if $Q/Q \cap F_m$ is ramified in F_{m+1}/F_m, then

$$Q/Q \cap F_0 \text{ is also ramified in } F_{m+1}/F_0 \tag{264}$$

by the multiplicative transitivity rule for ramification indices in (7). Therefore, we obtain

$$v_{\mathrm{init}}(\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q))) = (Q \cap F_1) \cap F_0 = Q \cap F_0 \in \mathrm{Ram}(\mathcal{F}/F_0) \subseteq V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \tag{265}$$

where the equalities, the containment-statement and the inclusion hold by the following reasonings: The first equality holds by the definitions of Path in Definition/Lemma 17(i), of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and of the initial vertex map v_{init} for $\Gamma_{\mathcal{F}}$ in Definition 55(iii). The second equality holds as F_1/F_0 is an extension of function fields. The containment-statement holds by the conclusion in (264) and by the definition of the ramification locus $\mathrm{Ram}(\mathcal{F}/F_0)$ in Definition 3(ii). The inclusion holds by Proposition 92(ii).

Then combining that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and (265) supplies that the path $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma_{\mathcal{F}}, m+1)$ is completely contained in $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and, thus, that Q is contained in $\mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma_{\mathcal{F}}, m+1))) = \mathbb{P}_{F_{m+1}}[\Gamma_{\mathcal{F}}^{\mathrm{ram}}] = \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$.

On the other hand, if $Q/Q \cap \sigma(F_m)$ is ramified in $F_{m+1}/\sigma(F_m)$, then applying Key Lemma 36(iv) to the diamond $(Q, Q \cap F_1, Q \cap \sigma(F_m), Q \cap \sigma(F_0))$ of places yields that $Q \cap F_1/Q \cap \sigma(F_0)$ is also ramified in $F_1/\sigma(F_0)$. But this again supplies that $Q \cap F_1 = \mathrm{Edge}(\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)))$ is a ramified edge in $\Gamma_{\mathcal{F}}$ and, thus, contained in $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Consequently, combining this conclusion and the fact that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ supplies that the path $\sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q) \in W(\Gamma_{\mathcal{F}}, m+1)$ is completely contained in $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and, thus, that Q is contained in $\mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma_{\mathcal{F}}, m+1))) = \mathbb{P}_{F_{m+1}}[\Gamma_{\mathcal{F}}^{\mathrm{ram}}] = \mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$. Hence, in any case, the claim in (263) follows.

Finally, the last desired statement, namely that $\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$ restrict so a morphism $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}} \to \Gamma_{\mathcal{F}}^{\mathrm{ram}}$, immediately follows from the second desired statement and from Lemma 130(i). $\square$

Example 145. *Let us again consider the recursive GS-towers* $\mathcal{F}_{\geq 1}$ *and* $\mathcal{F}$ *from Example 129. In Example 129(ii), we mentioned that* $\mathrm{Trun}_{\geq 1}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ *is the disjoint union of* $\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{split}}$ *and* $\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{ram}}$.

5.3 At Most One Finite Balanced Weakly Connected Component.

Summary of the results of this section. In [Bee04, p. 238, Theorem 5.5] and in [HP12, p. 27, Theorem 23], it was shown that most of the Beelen-graphs and all of the HP-graphs have at most one finite d-regular weakly connected component, respectively, where d is the balanced degree of the given recursive tower. As the first major result of this thesis, in Theorem 155 of this section, we will show that the tower graph not only has at most one finite d-regular weakly connected component but even at most one finite balanced weakly connected component.

On the one hand, by Corollary 156, this will especially imply that the Beelen-graph also has at most one finite balanced weakly connected component. On the other hand, in Theorem 154, we will present a simplified proof of [HP12, p. 27, Theorem 23] which will also work on the more general definition of correspondences in Definition 98.

This part is joint work with Florian Heß. More concretely, the first Subsection 5.3.1 is the contribution of Florian Heß and the last two Subsections 5.3.2 and 5.3.3 are the contributions of the author.

Recapitulation of the proof of [HP12, p. 27, Theorem 23]. Let X be a smooth projective geometrically integral curve over an algebraic extension field k of a finite field and let (Y, π_1, π_2) be a correspondence on X of type (d, d) such that the sequence $(C_\nu)_\nu$ in Definition 99 satisfies the requirements for the singular-recursive tower of (Y, π_1, π_2). Then the proof of [HP12, p. 27, Theorem 23] can be divided into the following three steps:

First, in [HP12, p. 25, Proposition 20], it was shown that the number $\#D_n$ of points in the set $D_n := \{Q \in C_n : \pi_{n,0,0}(Q) = \pi_{n,0,n}(Q)\}$ has the upper bound

$$\#D_n \leq 2d^n - \sum_{i=1}^{k} \lambda_i^n \tag{266}$$

for all $n \in \mathbb{N}_0$ and some $\lambda_i \in \mathbb{C}$ with $|\lambda_i| \leq d$ for all $i = 1, \ldots, k$.

Second, via calculations using Jacobian matrices in [HP12, p. 15, Corollary 9], it was concluded that the estimate in (266) can be replaces by the proper estimate

$$\#D_{mn} < 2d^{mn} - \sum_{i=1}^{k} \lambda_i^{mn}. \tag{267}$$

for some $n \in \mathbb{N}$ and all $m \in \mathbb{N}$.

Third and finally, from considering the eigenvalues of a standard adjacency matrix of a finite d-regular weakly connected component Γ of the HP-graph, it was derived that the number of points in $\{Q \in D_{mn} : \pi_{mn,0,0}(Q) \in V(\Gamma)\} \subseteq D_{mn}$ is equal to $d^{mn} + \sum_{j=1}^{s} \tilde{\lambda}_j^{mn}$ with some $\tilde{\lambda}_j \in \mathbb{C}$ with $|\lambda_j| \leq d$ for all $j = 1, \ldots, s$. Then applying [HP12, p. 26, Lemma 21] yielded some $m \in \mathbb{N}$ such that all λ_i^{mn} and $\tilde{\lambda}_j^{mn}$ have positive real parts. Therefore, it came out that the estimate in (267) can only hold if the HP-graph has at most one finite d-regular weakly connected component Γ.

Structure of this section In the first Subsection 5.3.1, we will first show in Theorem 153 that the estimate in (267) even holds for more general constructions of towers of curves. In particular, this will include the tower $(\tilde{C}_\nu)_\nu$ of normalizations $\tilde{C}_n$ of the curves C_n in the singular-recursive tower $(C_\nu)_\nu$.

Here, we will replace the lengthy calculations using Jacobian matrices from [HP12, p. 15, Corollary 9] with a much shorter reasoning using uniformizing elements and, by that, obtain a simplified proof.

Finally, for the tower $(\tilde{C}_\nu)_\nu$ of normalizations, in Theorem 153, we will even improve the estimate in (267) to the estimate

$$\sum_{Q \in D_{mn}} \min_{i=0,mn} e_{\pi_{mn,0,i}}(Q) < 2d^{mn} - \sum_{i=1}^{k} \lambda_i^{mn}. \tag{268}$$

In Subsection 5.3.2, we will next use Theorem 153 to show in Theorem 154 that the HP-graphs of the more general correspondences in Definition 98 also have at most one finite d-regular weakly connected component.

In Subsection 5.3.2, we will then also replace the standard adjacency matrices in the third step of the proof of [HP12, p. 27, Theorem 23] with $w_{\mathbf{P},\mathbf{1}}$-adjacency matrices (see Definition 157). Consequently, considering the eigenvalues of these matrices instead and combining them with Proposition 39 and the estimate in (268) will provide the first major result of this thesis, which is Theorem 155. There it will come out that the tower graph not only has at most one finite d-regular weakly connected component but even at most one finite balanced weakly connected component.

5.3.1 Counting Points in Pyramids of Curves

This subsection is joint work with Florian Heß.

Notation. We will rewrite our notation for recursive towers of function fields using curves. This facilitates the use of the diagonal and techniques from [HP12].

Notation 146. *We are given $d > 1$, projective and integral curves X_i over the algebraically closed field K for all $i \geq 0$ with X_0 regular, and finite surjective morphisms*

$$\pi_{i,j,e} : X_i \to X_j$$

of degree d^{i-j} for each e with $0 \leq e \leq i - j$ satisfiying

$$\pi_{i,i,0} = \mathrm{id}, \quad \pi_{j,k,f} \circ \pi_{i,j,e} = \pi_{i,k,e+f}.$$

We assume that, for all i, e, d with $i \geq e + d$, the diamonds with $\pi_{i,i-e,0}$, $\pi_{i,i-d,d}$, $\pi_{i-e,i-e-d,d}$ and $\pi_{i-d,i-e-d,0}$ represent X_i as a cover of degree one of the fibre product of X_{i-e} and X_{i-d} over X_{i-e-d}.

Degree of the Diagonal. Let $n \geq 1$. By the universal property of products there is a uniquely determined morphism $\psi_n : X_n \to X_0 \times X_0$ factoring $\pi_{n,0,0}$ and $\pi_{n,0,n}$ by the two projections $X_0 \times X_0 \to X_0$. Let Δ denote the diagonal of $X_0 \times X_0$. We define

$$\Delta_n = \psi_n^*(\Delta).$$

Proposition 147. *We have that Δ_n is an effective Cartier divisor of X_n with*

$$\mathrm{Supp}(\Delta_n) = \{Q \in X_n \,|\, \pi_{n,0,0}(Q) = \pi_{n,0,n}(Q)\}$$

and there are $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ with $|\lambda_i| \leq d$ such that

$$\deg(\Delta_n) = 2d^n - \sum_{i=1}^{k} \lambda_i^n$$

for all $n \geq 1$.

Proof. By assumption, X_0 is regular over K and K is perfect, so X_0 is also smooth over K. Smoothness is preserved under base extension and composition, so $X_0 \times X_0$ is smooth over K, hence regular and in particular integral and normal.

Now X_n is projective over K and integral by assumption. The reduced scheme-theoretic image $X'_n = \psi_n(X_n)$ therefore defines a closed integral subscheme of $X_0 \times X_0$, which projects onto X_0. Since X_n and X_0 have dimension one, X'_n has also dimension one and then codimension one in $X_0 \times X_0$ because $X_0 \times X_0$ has dimension two. As $X_0 \times X_0$ is projective over K, X'_n is also projective over K. Finally, X'_n is an effective Cartier divisor of $X_0 \times X_0$ because $X_0 \times X_0$ is integral and normal.

Similarly, Δ is isomorphic to X_0 under the projections, is hence projective over K and regular, and then constitutes an effective Cartier divisor of $X_0 \times X_0$. Now X'_n and Δ do not have irreducible components in common. For otherwise, since they are both irreducible, we would have $X'_n = \Delta$. But this would imply $\pi_{n,0,0} = \pi_{n,0,n}$ which is not possible, since by assumption $\pi_{2n,n,0}$ has degree $d^n > 1$ and is the base extension of $\pi_{n,0,0}$ by $\pi_{n,0,n}$.

Write $i_n : X'_n \to X_0 \times X_0$ for the closed embedding of X'_n into $X_0 \times X_0$. We restrict ψ_n to the surjective morphism $\psi'_n : X_n \to X'_n$. Then $\psi_n = i_n \circ \psi'_n$ and

$$\Delta_n = \psi_n^*(\Delta) = \psi_n'^*(i_n^*(\Delta)). \tag{269}$$

We compute $\deg(\Delta_n)$ via the intersection number $X'_n \cdot \Delta$. As in [HP12, p. 25, Proposition 20] we see that there are $k \in \mathbb{N}_0$, $\lambda_i \in \mathbb{C}$ with $|\lambda_i| \leq d$ such that for all n

$$X'_n \cdot \Delta = 2d^n - \sum_{i=1}^{k} \lambda_i^n. \tag{270}$$

By [Liu02, p. 377, Lemma 9.1.4] the restriction $\Delta|_{X'_n} = i_n^*(\Delta)$ of Δ to X'_n defines an effective Cartier on X'_n such that

$$\deg(i_n^*(\Delta)) = X'_n \cdot \Delta. \tag{271}$$

Using [Liu02, p. 263, Proposition 7.1.38] we see that $\psi_n'^*(i_n^*(\Delta))$ is an effective Cartier divisor on X_n satisfying

$$\deg(\psi_n'^*(i_n^*(\Delta))) = \deg(i_n^*(\Delta)) \tag{272}$$

The assertion on the support of Δ_n is clear from the respective property of Δ, and the combination of (269), (270), (271) and (272) yields the assertion on the degree. ☐

Multiplicities in the Diagonal. We will now investigate the multiplicities of points in the diagonal Δ_n. The local ring of a scheme X at x is denoted by $\mathcal{O}_{X,x}$ and its maximal ideal by $\mathfrak{m}_{X,x}$. If D is an effective Cartier divisor on X, we write $D_{X,x}$ for the principal ideal of $\mathcal{O}_{X,x}$ defined by D.

Lemma 148. *Let $Q \in \operatorname{Supp}(\Delta_n)$. Then $(\Delta_n)_{X_n,Q}$ is is generated by the element $\pi^*_{n,0,n}(z_0) - \pi^*_{n,0,0}(z_0)$, where z_0 is a local uniformizer of $Q_0 = \pi_{n,0,0}(Q) = \pi_{n,0,n}(Q)$.*

Proof. The local ring of $X_0 \times X_0$ at (Q_0, Q_0) satisfies $\mathcal{O}_{X_0\times X_0,(Q_0,Q_0)} \cong \mathcal{O}_{X_0,Q_0} \otimes_K \mathcal{O}_{X_0,Q_0}$. The ideal J of $\mathcal{O}_{X_0,Q_0} \otimes_K \mathcal{O}_{X_0,Q_0}$ corresponding to Δ under this isomorphism is generated by elements of the form $x \otimes 1 - 1 \otimes x$ with $x \in \mathcal{O}_{X_0,Q_0}$, and the maximal ideal $\mathfrak{n}$ of $\mathcal{O}_{X_0,Q_0} \otimes_K \mathcal{O}_{X_0,Q_0}$ contains $\mathfrak{m}_{X_0,Q_0} \otimes 1 + 1 \otimes \mathfrak{m}_{X_0,Q_0}$. Since K is algebraically closed we can write $x = a + bz_0$ with $a \in K$ and $b \in \mathcal{O}_{X_0,Q_0}$. Then

$$\begin{aligned} x \otimes 1 - 1 \otimes x &= bz_0 \otimes 1 - 1 \otimes bz_0 = (b \otimes 1)(z_0 \otimes 1) - (1 \otimes b)(1 \otimes z_0) \\ &= (1 \otimes b)(z_0 \otimes 1 - 1 \otimes z_0) - (z_0 \otimes 1)(1 \otimes b - b \otimes 1) \\ &\in (\mathcal{O}_{X_0,Q_0} \otimes_K \mathcal{O}_{X_0,Q_0})(z_0 \otimes 1 - 1 \otimes z_0) + \mathfrak{n}J \end{aligned}$$

This means that $J/\mathfrak{n}J$ is generated by the residue class of $z_0 \otimes 1 - 1 \otimes z_0$. Since $\mathcal{O}_{X_0,Q_0} \otimes_K \mathcal{O}_{X_0,Q_0}$ is noetherian, J is finitely generated and by the Lemma of Nakayama it is generated by $z_0 \otimes 1 - 1 \otimes z_0$.

Finally, tracing the definition of Δ_n, we see that the element $\pi^*_{n,0,n}(z_0) - \pi^*_{n,0,0}(z_0)$ generates $(\Delta_n)_{X_n,Q}$. ☐

We recall some definitions and basic facts, see [Liu02, p. 260, p. 265, p. 275]. Let $Q \in \operatorname{Supp}(X_n)$ and D be an effective Cartier divisor on X_n. The multiplicity of D at Q is

$$\begin{aligned} \operatorname{mult}_Q(D) &= \operatorname{length}_{\mathcal{O}_{X_n,Q}}(\mathcal{O}_{D,Q}) = \operatorname{length}_{\mathcal{O}_{X_n,Q}}(\mathcal{O}_{X_n,Q}/D_{X_n,Q}) \\ &= \dim_K(\mathcal{O}_{X_n,Q}/D_{X_n,Q}). \end{aligned}$$

The ramification index of $\pi_{n,m,i}$ at Q is

$$\begin{aligned} e_{\pi_{n,m,i}}(Q) &= \operatorname{length}_{\mathcal{O}_{X_n,Q}}(\mathcal{O}_{X_n,Q}/\pi^*_{n,m,i}(\mathfrak{m}_{X_m,\pi_{n,m,i}(Q)})\mathcal{O}_{X_n,Q}) \\ &= \dim_K(\mathcal{O}_{X_n,Q}/\pi^*_{n,m,i}(\mathfrak{m}_{X_m,\pi_{n,m,i}(Q)})\mathcal{O}_{X_n,Q}). \end{aligned}$$

Then

$$\deg(D) = \sum_{Q\in X_n} \operatorname{mult}_Q(D),$$

and if Q is regular then

$$D_{X_n,Q} = \mathfrak{m}_{X_n,Q}^{\operatorname{mult}_Q(D)} \text{ and } \pi^*_{n,m,i}(\mathfrak{m}_{X_m,\pi_{n,m,i}(Q)})\mathcal{O}_{X_n,Q} = \mathfrak{m}_{X_n,Q}^{e_{\pi_{n,m,i}}(Q)},$$

whereby $\operatorname{mult}_Q(D)$ and $e_{\pi_{n,m,i}}(Q)$ are also uniquely determined.

Proposition 149. *Every regular $Q \in \operatorname{Supp}(\Delta_n)$ satisfies the estimate $\operatorname{mult}_Q(\Delta_n) \geq \min(e_{\pi_{n,0,0}}(Q), e_{\pi_{n,0,n}}(Q))$, with equality if $e_{\pi_{n,0,0}}(Q) \neq e_{\pi_{n,0,n}}(Q)$.*

Proof. By Lemma 148 and using its notation, the ideal $(\Delta_n)_{X_n,Q}$ is generated by the element $\pi^*_{n,0,n}(z_0) - \pi^*_{n,0,0}(z_0)$. Since z_0 is a local uniformizer of $\mathcal{O}_{X_0,Q_0}$ and Q is regular, we get

$$\begin{aligned} \pi^*_{n,0,0}(z_0)\mathcal{O}_{X_n,Q} &= \pi^*_{n,0,0}(\mathfrak{m}_{X_0,Q_0})\mathcal{O}_{X_n,Q} = \mathfrak{m}_{X_n,Q}^{e_{\pi_{n,0,0}}}, \\ \pi^*_{n,0,n}(z_0)\mathcal{O}_{X_n,Q} &= \pi^*_{n,0,n}(\mathfrak{m}_{X_0,Q_0})\mathcal{O}_{X_n,Q} = \mathfrak{m}_{X_n,Q}^{e_{\pi_{n,0,n}}} \end{aligned}$$

and

$$(\pi^*_{n,0,n}(z_0) - \pi^*_{n,0,0}(z_0))\mathcal{O}_{X_n,Q} \subseteq \mathfrak{m}_{X_n,Q}^{\min(e_{\pi_{n,0,0}}, e_{\pi_{n,0,n}})},$$

with equality if the ramification indices are different. ☐

Lemma 150. *Let $P \in \operatorname{Supp}(\Delta_1)$. Then there are $Q_n \in X_n$ such that $Q_1 = P$ and $\pi_{n,m,i}(Q_n) = Q_m$ for all n, m, i with $0 \leq m \leq n$ and $0 \leq i \leq n-m$. We have $Q_n \in \operatorname{Supp}(\Delta_n)$ for all $n \geq 1$.*

Proof. We define $Q_1 = P$, $Q_0 = \pi_{1,0,0}(P) = \pi_{1,0,1}(P)$. By the correspondence of points and places, Definition/Lemma 17(i) implies that there are Q_n which satisfy $Q_{n-1} = \pi_{n,n-1,0}(Q_n) = \pi_{n,n-1,1}(Q_n)$ if $n \geq 1$. Since $\pi_{n,m,i}$ is a composition of suitable $\pi_{j,j-1,e}$ this yields the first assertion.

The last assertion holds by observing that $\pi_{n,0,0}(Q_n) = \pi_{n,0,n}(Q_n)$ by the first assertion, and this implies $Q_n \in \operatorname{Supp}(\Delta_n)$ by Proposition 147. ☐

We fix some additional notation: Given P and Q_n as in Lemma 150 write R_n for the local ring of X_n at Q_n and $\mathfrak{m}_n$ for the maximal ideal of R_n. Let z_0 denote a local uniformiser of R_0.

The lower bound for the multiplicities in Proposition 149 can be increased, as we will see now.

Lemma 151. *Assume that P and Q_n can be chosen regular for all $n \geq 1$. If $e_{\pi_{1,0,0}}(P) = e_{\pi_{1,0,1}}(P)$, then there is $u \in K^\times$ such that*

$$\pi^*_{n,0,i}(z_0) \equiv u^i \pi^*_{n,0,0}(z_0) \bmod \mathfrak{m}_n^{e_{\pi_{n,0,0}}(Q_n)+1}$$

for all $n, i \geq 1$. In particular, $e_{\pi_{n,0,i}}(Q_n) = e_{\pi_{n,0,0}}(Q_n)$.

Proof. By the assumptions, we obtain $\pi^*_{1,0,1}(z_0)\pi^*_{1,0,0}(z_0)^{-1} \in R_1^\times$ and thus some $u \in K^\times$ such that $\pi^*_{1,0,1}(z_0)\pi^*_{1,0,0}(z_0)^{-1} \equiv u \bmod \mathfrak{m}_1$. Now $\pi^*_{n,1,j}(\pi^*_{1,0,0}(z_0)) = \pi^*_{n,1,j-1}(\pi^*_{1,0,1}(z_0))$, so

$$\begin{aligned}
\pi^*_{n,0,i}(z_0)\pi^*_{n,0,0}(z_0)^{-1} &= \pi^*_{n,1,i-1}(\pi^*_{1,0,1}(z_0))\pi^*_{n,1,0}(\pi^*_{1,0,0}(z_0))^{-1} \\
&= \pi^*_{n,1,i-1}(\pi^*_{1,0,1}(z_0))\pi^*_{n,1,0}(\pi^*_{1,0,0}(z_0))^{-1} \\
&\quad \cdot \left(\prod_{j=1}^{i-1} \pi^*_{n,1,j}(\pi^*_{1,0,0}(z_0))^{-1}\pi^*_{n,1,j-1}(\pi^*_{1,0,1}(z_0))\right) \\
&= \prod_{j=0}^{i-1} \pi^*_{n,1,j}(\pi^*_{1,0,1}(z_0)\pi^*_{1,0,0}(z_0)^{-1})
\end{aligned}$$

This shows $\pi^*_{n,0,i}(z_0)\pi^*_{n,0,0}(z_0)^{-1} \in R_n^\times$ and $\pi^*_{n,0,i}(z_0)\pi^*_{n,0,0}(z_0)^{-1} \equiv u^i \bmod \mathfrak{m}_n$. We also have $R_n\pi^*_{n,0,0}(z_0) = \mathfrak{m}_n^{e_{\pi_{n,0,0}}(Q_n)}$, so multiplication of this congruence by $\pi^*_{n,0,0}(z_0)$ yields the first assertion.

The second assertion follows from the first assertion because $R_n u^i \pi^*_{n,0,0}(z_0) = \mathfrak{m}_n^{e_{\pi_{n,0,0}}}$. ☐

Proposition 152. *Assume that K is the algebraic closure of a finite field and that P and Q_n can be chosen regular for all $n \geq 1$. If $e_{\pi_{1,0,0}}(P) = e_{\pi_{1,0,1}}(P)$, then there is $n \geq 1$ such that $e_{\pi_{nm,0,0}}(Q_{nm}) = e_{\pi_{nm,0,nm}}(Q_{nm})$ and $\operatorname{mult}_{Q_{nm}}(\Delta_{mn}) \geq e_{\pi_{nm,0,0}}(Q_{nm}) + 1$ for every $m \geq 1$.*

Proof. We abbreviate $z_i = \pi^*_{i,0,0}(z_0)$ and $e_{n,m,i} = e_{\pi_{n,m,i}}(Q_n)$. By Lemma 148 the ideal $(\Delta_i)_{X_i,Q_i}$ is generated by the element $\pi^*_{i,0,i}(z_0) - z_i$ for all $i \geq 1$.

By Lemma 151 and the regularity assumption there is $u \in K^\times$ such that

$$\pi^*_{i,0,i}(z_0) \equiv u^i z_i \bmod \mathfrak{m}_i^{e_{i,0,0}+1}$$

for all $i \geq 1$. Since u lies in a finite field there is $n \geq 1$ with $u^n = 1$. Then

$$\pi^*_{nm,0,nm}(z_0) - z_{nm} \equiv u^{nm} z_{nm} - z_{nm} \equiv z_{nm} - z_{nm} \equiv 0 \bmod \mathfrak{m}_{nm}^{e_{nm,0,0}+1}$$

for all $m \geq 1$. Thus $e_{nm,0,0} = e_{nm,0,nm}$ and Q_{nm} has multiplicity at least $e_{nm,0,0} + 1$ in Δ_{nm}. ☐

Bound. With the results on the degree of the diagonal and its multiplicites we obtain the following bound for the number of points counted with weights, which lie over closed paths of the tower graph.

Theorem 153. *Assume that K is the algebraic closure of a finite field and that there are $r \geq 1$, $P \in \operatorname{Supp}(\Delta_r)$ regular and $Q_n \in X_n$ regular with $Q_r = P$ and $\pi_{n,m,i}(Q_n) = Q_m$ for all n, m, i with $0 \leq m \leq n$, $r|n$, $r|m$ and $1 \leq i \leq n-m$, $r|i$.*

If $\pi_{r,0,0}$ and $\pi_{r,0,r}$ have the same ramification index at P, then there are $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ with $|\lambda_i| \leq d$ and $n \geq 1$ with $r \,|\, n$ such that

$$\sum_{\substack{Q \in \operatorname{Supp}(\Delta_{nm}),\\ Q \text{ regular}}} \min(e_{\pi_{nm,0,0}}(Q), e_{\pi_{nm,0,nm}}(Q)) \;<\; 2d^{nm} - \sum_{i=1}^{k} \lambda_i^{nm}$$

for all $m \geq 1$.

Proof. We apply the results of the previous section to only those X_l, where l is a multiple of r and l is replaced by l/r. To simplify the notation we may thus assume $r = 1$ in the statement of the theorem.

By Proposition 147 and Proposition 149, we get

$$\sum_{\substack{Q \in \operatorname{Supp}(\Delta_{nm}),\\ Q \text{ regular}}} \min(e_{\pi_{nm,0,0}}(Q), e_{\pi_{nm,0,nm}}(Q)) \leq \deg(\Delta_{nm}) = 2d^{nm} - \sum_{i=1}^{k} \lambda_i^{nm}$$

for all $m, n \geq 1$. By the assumption, there is $n \geq 1$ with $r|n$ such that for every $m \geq 1$ there is at least one $Q = Q_{nm}$ in the sum as in Proposition 152. Thus the $\leq$ in the estimate may be changed into $<$ for such n, m. □

As an application, a balanced closed path of length r leads to $P \in \operatorname{Supp}(\Delta_r)$. In the case of [HP12], the path is in the split component and thus P and all Q_n are regular by [HP12, p. 12, Proposition 6]. In the truly balanced case we assume that the X_i are regular, so that P and the Q_n are also regular.

5.3.2 At Most One Finite d-Regular Weakly Connected Component

Summary of the results of this subsection. In Theorem 154 of this subsection, we will finish the simplified proof of [HP12, p. 27, Theorem 23]. The latter proved that the HP-graph contains at most one finite d-regular weakly connected component. This simplified proof will also work on the more general correspondences in Definition 98.

HP-graphs have at most one finite d-regular weakly connected component. The main simplification step of the proof in [HP12, p. 27, Theorem 23] was already accomplished in the proof of Theorem 153. The following proof of Theorem 154 will basically be the proof of [HP12, p. 27, Theorem 23]. Only, instead of applying [HP12, p. 15, Corollary 9] and [HP12, p. 25, Proposition 20], we will apply Theorem 153.

Theorem 154. *Let X be a smooth projective geometrically integral curve over an algebraic extension field k of a finite field and let (Y, π_1, π_2) be a correspondence on X of type (d, d) such that the sequence $(C_\nu)_\nu$ in Definition 99 satisfies the requirements for the singular-recursive tower of (Y, π_1, π_2).*

Then the geometric graph $\mathcal{G}_\infty$ of (Y, π_1, π_2) has at most one finite d-regular weakly connected component.

Proof. Let $\Gamma_1, \dots, \Gamma_r$ be the finite d-regular weakly connected components of $\mathcal{G}_\infty$. We will show $r \leq 1$.

First, we notice that, by the definition of the singular-recursive tower $(C_\nu)_\nu$ with the morphisms $\pi_{i,j,e} : C_i \to C_j$ in Definition 99 and by [HP12, p. 12, Proposition 6], it satisfies the requirements of Theorem 153. Thus, we obtain

$$\#\{P \in C_{mn} : \pi_{mn,0,0}(P) = \pi_{mn,0,mn}(P)\} = \#\operatorname{Supp}(\Delta_{nm}) < 2d^{nm} - \sum_{j=1}^{k} \lambda_j^{mn} \tag{273}$$

for all $m \geq 1$, some $n \geq 1$ and some $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ with $|\lambda_i| \leq d$ for all $i = 1, \dots, k$.

From this point on, the proof is basically the same as proof in [HP12]: Second, by the definitions of the singular-recursive tower $(C_\nu)_\nu$ via the fiber products in Definition 99 and the geometric graph $\mathcal{G}_\infty$ of (Y, π_1, π_2) in Definition 101(i), we also get that the

$$\text{number of closed paths in } \coprod_{i=1}^{r} \Gamma_i \text{ is a lower bound for } \#\operatorname{Supp}(\Delta_{nm}). \tag{274}$$

Third, let A_i be a standard-adjacency matrix of Γ_i and let $\lambda_{i,1}, \dots, \lambda_{i,s_i}$ be the eigenvalues of A_i counted with multiplicities and sorted by their absolute values in descending order for all $i = 1, \dots, r$. Because Γ_i is d-regular, the row sums in A_i are constantly d. Thus, A_i has the eigenvalue $\lambda_{i,1} = d$ with the eigenvector which only consists ones and, by [HJ90, p. 492, Theorem 8.1.22], all other eigenvalues $\lambda_{i,j}$ satisfy $|\lambda_{i,j}| \leq d$ with $j = 2, \dots, s_i$. Because Lemma 59 implies that the trace of A_i^{nm} counts the closed paths in Γ_i, because of (274) and because of the estimate in (273), we obtain the estimates

$$rd^{nm} + \sum_{i=1}^{r} \sum_{j=2}^{s_i} \lambda_{i,j}^{mn} \leq \#\operatorname{Supp}(\Delta_{nm}) < 2d^{nm} - \sum_{j=1}^{k} \lambda_j^{mn} \tag{275}$$

Finally, [HP12, p. 26, Lemma 21] supplies some $m \geq 1$ such that all λ_i^{mn} and $\lambda_{i,j}^{mn}$ in (275) have nonnegative real parts. But this yields the desired estimate $r \leq 1$. □

5.3.3 At Most One Finite Balanced Weakly Connected Component

Summary of the results of this subsection. In this subsection, we will prove the first major result of this thesis, which is Theorem 155. There it will come out that the tower graph not only has at most one finite d-regular weakly connected component but even at most one finite balanced weakly connected component.

Significance of having at most one finite balanced weakly connected component. First, in [Bee04, p. 238, Corollary 5.6] and in [HP12, p. 27, Theorem 24], the fact that most of the Beelen-graphs and all HP-graphs have at most one finite d-regular weakly connected component is used to prove that the limit of a good recursive tower cannot increase after a finite constant field extension if some further technical conditions are satisifed.

By combining Theorem 155 and the main result of this thesis, we will even be able to show in Theorem 188 that these technical conditions can always be dropped. Consequently, it comes out that the limit of a good recursive tower cannot increase after a finite constant field extension. This will be the third major result of this thesis. Moreover, there we will actually need the full statement of Theorem 155, i.e. that the tower graph has at most one finite balanced weakly connected component and not only at most one finite d-regular weakly connected component.

Second, knowing that the tower graph has at most one finite balanced weakly connected component will also come in handy for determining the limits of recursive towers

$\mathcal{F}$ for which we cannot simply print their tower graphs $\Gamma_\mathcal{F}$ (as they are too large or $\mathcal{F}$ is parametrized over $\mathbb{F}_q$ by infinitely many q). More concretely, for most recursive towers $\mathcal{F}$ in the literature, we are in the comfortable situation that we already know that $\mathcal{F}$ has a non-empty splitting subgraph $\Gamma_\mathcal{F}^{\text{split}}$ and that there is also another finite forward and backward complete subgraph Γ of $\Gamma_\mathcal{F}$ which contains the ramification subgraph $\Gamma_\mathcal{F}^{\text{ram}}$. Now, if we add that $\Gamma_\mathcal{F}$ has at most one finite balanced weakly connected component, we obtain that Γ can have no finite weakly connected component without ramified edges since this would be another finite balanced weakly connected component. Thus, Γ must already be equal to $\Gamma_\mathcal{F}^{\text{ram}}$ and all its finite weakly connected components must also contain circles with unbalanced ramification indices.

For a tame recursive tower $\mathcal{F}$, this will already be enough to determine the precise limit $\lambda(\mathcal{F})$ of $\mathcal{F}$ by applying Corollary 195. That will be the first part of the fourth major result of this thesis. For a wild recursive tower $\mathcal{F}$, this will only be enough to determine the splitting rate $\nu(\mathcal{F})$ of $\mathcal{F}$. However, if $\mathcal{F}$ is also α-weakly ramified (see Definition 199), then the second part of the fourth major result in Corollary 200 will also enable us to determine the asymptotic genus $\gamma(\mathcal{F})$ and limit $\lambda(\mathcal{F})$ of $\mathcal{F}$.

Third, just as an insight on recursive towers, it is remarkable that the definition of a recursive tower already implies that the solutions of the defining polynomial $f(X,Y)$ behave in a way such that the tower graph can have at most one finite balanced weakly connected component.

At most one finite balanced weakly connected component. In the proof of Theorem 154, we only used Theorem 153 via the weaker estimate

$$\#\operatorname{Supp}(\Delta_{mn}) < 2d^{mn} - \sum_{i=1}^{k} \lambda_i^{mn}. \tag{276}$$

More concretely, by considering the eigenvalues of a standard adjacency matrix of a finite d-regular weakly connected component Γ of the HP-graph, we showed that the number of points in $\{Q \in \operatorname{Supp}(\Delta_{mn}) : \pi_{mn,0,0}(Q) \in V(\Gamma)\}$ is equal to $d^{mn} + \sum_{j=1}^{s} \tilde{\lambda}_j^{mn}$ with some $\tilde{\lambda}_j \in \mathbb{C}$ with $|\tilde{\lambda}_j| \le d$ for all $j = 1, \dots, s$. Then we chose $m \in \mathbb{N}$ such that all λ_i^{mn} and $\tilde{\lambda}_j^{mn}$ have positive real parts. Consequently, it came out that the estimate in (276) can only hold if the HP-graph has at most one finite d-regular weakly connected component Γ.

Now, for the first major result of this thesis in Theorem 155, we will have to utilize the stronger estimate in Theorem 153 for the tower $(\tilde{C}_\nu)_\nu$ of normalizations $\tilde{C}_n$ of the curves C_n in the singular-recursive tower $(C_\nu)_\nu$. This is the estimate

$$\sum_{Q \in \operatorname{Supp}(\Delta_{mn})} \min_{i=0,mn} e_{\pi_{mn,0,i}}(Q) < 2d^{mn} - \sum_{i=1}^{k} \lambda_i^{mn}. \tag{277}$$

For that, we will also have to bring the ramification indices into the picture. This will be accomplished via Proposition 39. More concretely, by considering the eigenvalues of a $w_{\mathbf{P},\mathbf{1}}$-adjacency matrix (see Definition 157) of a finite balanced weakly connected component Γ of the tower graph and applying Proposition 39 suitably, we will show

$$\sum_{Q \in \operatorname{Supp}(\Delta_{mn}) \cap \psi(\mathbb{P}_{F_{mn}}[\Gamma])} \min_{i=0,mn} e_{\pi_{mn,0,i}}(Q) = d^{mn} + \sum_{j=1}^{s} \tilde{\lambda}_j^{mn}$$

with some $\tilde{\lambda}_j \in \mathbb{C}$ with $|\tilde{\lambda}_j| \le d$ for all $j = 1, \dots, s$ where $\psi(\mathbb{P}_{F_{mn}}[\Gamma])$ is the image of $\mathbb{P}_{F_{mn}}[\Gamma]$ after translating the recursive tower $\mathcal{F}$ of function fields into the recursive tower $(\tilde{C}_\nu)_\nu$ of smooth curves.

Therefore, analogously to the reasoning before, the estimate in (277) can only hold if the tower graph has at most one finite balanced weakly connected component Γ.

Also note that we will already use definitions and lemmata from Subsection 6.1 in the proof of Theorem 155. However, these definitions and lemmata can be used as black boxes and there will be no circular reasonings.

Theorem 155. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d over an algebraic extension field of a finite field. Then the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ has at most one finite weakly connected component which only contains circles with balanced ramification indices.*

Proof. For the geometric tower $\overline{k} \cdot \mathcal{F}$ of $\mathcal{F}$, we notice that, by Lemma 120(i) and by the 'moreover'-parts of Lemma 120(iii) and Lemma 120(v), the $\overline{k}$-constant field extension $\overline{k} \cdot \Gamma$ of a finite balanced weakly connected component Γ of $\Gamma_{\mathcal{F}}$ is the disjoint union of finite balanced weakly connected components of $\Gamma_{\overline{\mathcal{F}}}$. Because of that and because constant field extensions of disjoint subgraphs are again disjoint by Lemma 111, it is enough to show that $\Gamma_{\overline{\mathcal{F}}}$ has at most one finite weakly connected component which only contains circles with balanced ramification indices. Hence, w.l.o.g. we may assume that the constant field k of $\mathcal{F}$ is algebraically closed.

Now, let $\Gamma_1, \ldots, \Gamma_r$ be the finite weakly connected components of $\Gamma_{\mathcal{F}}$ which only contain circles with balanced ramification indices. We will show $r \leq 1$.

First, let $A_1, \ldots, A_r$ be $w_{\mathbf{P},\mathbf{1}}$-adjacency matrices of $\Gamma_1, \ldots, \Gamma_r$, respectively. From combining the 'moreover'-part in Lemma 164 for $x_{\mathbf{P}} = \mathbf{P}$, Lemma 165(ii) and Lemma 165(iii), we derive that all row sums of A_i are equal to d for all $i = 1, \ldots, r$. Consequently, the vector with only ones is an eigenvector to the eigenvalue d and, for the eigenvalues $\lambda_{i,1}, \ldots, \lambda_{i,s_i}$ of A_i counted with multiplicities and sorted by their absolute values in descending order for all $i = 1, \ldots, r$, [HJ90, p. 492, Theorem 8.1.22] thus supplies

$$\lambda_{i,1} = d \text{ and } |\lambda_{i,j}| \leq d \text{ for all } j = 2, \ldots, s_i. \tag{278}$$

Second, let $W^\circ(\Gamma, n)$ denote the set of closed paths in the directed graph Γ. Then we obtain the equalities and estimate

$$\begin{aligned}
r \cdot d^n + \sum_{i=1}^{r} \sum_{j=2}^{s_i} \lambda_{i,j}^n &= \sum_{i=1}^{r} \operatorname{Tr}(A_i^n) = \sum_{i=1}^{r} \sum_{\mathcal{C} \in W^\circ(\Gamma_i, n)} w_{\mathbf{P},\mathbf{1}}(\mathcal{C}) \\
&= \sum_{i=1}^{r} \sum_{\mathcal{C} = [P_{\mu,\nu}]_{\nu-\mu \leq 1} \in W^\circ(\Gamma_i, n)} \prod_{j=1}^{r} e(P_{j-1,j} | P_{j-1,j-1}) \\
&= \sum_{i=1}^{r} \sum_{\mathcal{C} \in W^\circ(\Gamma_i, n)} \sum_{Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}))} e(Q | Q \cap F_0) \\
&= \sum_{i=1}^{r} \sum_{Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(W^\circ(\Gamma_i, n)))} \min(e(Q|Q \cap F_0), e(Q|\sigma^n(Q) \cap F_0)) \\
&\leq \sum_{Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(W^\circ(\Gamma_{\mathcal{F}}, n)))} \min_{i=0,n} e(Q|Q \cap \sigma^i(F_0)) \\
&= \sum_{\substack{Q \in \mathbb{P}_{F_n} \\ Q \cap F_0 = \sigma^{-n}(Q \cap \sigma^n(F_0))}} \min_{i=0,n} e(Q|Q \cap \sigma^i(F_0))
\end{aligned} \tag{279}$$

where the equalities and estimates hold by the following reasonings. The first equality holds because of the choice of $\lambda_{i,1}, \ldots, \lambda_{i,s_i}$ as the eigenvalues of A_i, because of (278) and because it is well known that the trace of the n-th power of a matrix A is the sum of the n-th powers of the eigenvalues of A.

The second equality holds because the trace $\mathrm{Tr}(A_i^n)$ is the sum of the entries on the diagonal of A_i^n and because Lemma 59 implies that the l-th value on the diagonal is the sum of $w_{\mathbf{P},\mathbf{1}}$-values of all closed paths at the l-th vertex in Γ_i.

The third equality holds by the definition of

$$w_{\mathbf{P},\mathbf{1}}(\mathcal{C}) = \prod_{i=1}^{r} w_{\mathbf{P},\mathbf{1}}(P_{i-1,i}) = \prod_{i=1}^{r} e(P_{i-1,i}|P_{i-1,i-1})$$

in Definition 157. The fourth equality holds by Proposition 39.

The fifth equality holds because Lemma 83 implies that not only all circles but even all closed paths in Γ_i have balanced ramification indices and because the implication from (i) to (iv) in Lemma 166 also provides the equalities

$$e(Q|Q \cap F_0) = e(Q|\sigma^n(Q) \cap F_0)) = \min(e(Q|Q \cap F_0), e(Q|\sigma^n(Q) \cap F_0))$$

for all $Q \in \mathbb{P}_{F_n}(\mathcal{C})$ and all closed paths $\mathcal{C}$ in Γ_i.

The estimate follows as the right side runs over all closed paths in $\Gamma_{\mathcal{F}}$ and not only the closed paths in $\Gamma_1, \ldots, \Gamma_r$.

The last equality holds because $\mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(W^\circ(\Gamma_{\mathcal{F}}, n)))$ is the set of all $Q \in \mathbb{P}_{F_n}$ such that $[\sigma^{-i}(Q \cap \sigma^i(F_{j-i}))]_{j-i \leq 1} = [\sigma^{-i}(Q \cap F_{i,j})]_{j-i \leq 1} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) \in W^\circ(\Gamma_{\mathcal{F}}, n)$ and because the definition of $W^\circ(\Gamma_{\mathcal{F}}, n)$ as the set of closed paths provides $Q \cap F_0 = \sigma^{-n}(Q \cap \sigma^n(F_0))$.

Third, as in Proposition 102(iv), let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the finite subcategories of k-algebras consisting of the k-algebras and morphisms which are depicted in the commutative diagrams in Figure 4.9 and Figure 4.10, respectively. Then, by Proposition 102(i) and Proposition 102(iv), there are geometrically integral curves X and Y over k such that X is smooth, (Y, π_1, π_2) is a correspondence on X of type (d, d), the sequence $(C_\nu)_\nu$ in Definition 99 satisfies the requirements for the singular-recursive tower of (Y, π_1, π_2) and there are isomorphisms $\psi_{j-i} \circ \sigma^{-i} : F_{i,j} \to K(C_{j-i})$ for all $0 \leq i \leq j$ which induce an isomorphism $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ of categories. Then we obtain the equality

$$\sum_{\substack{Q \in \mathbb{P}_{F_n} \\ Q \cap F_0 = \sigma^{-n}(Q \cap \sigma^n(F_0))}} \min_{i=0,n} e(Q|Q \cap \sigma^i(F_0)) = \sum_{\substack{Q \in \mathbb{P}_{K(C_n)} \\ (\pi_{n,0,0}^*)^{-1}(Q) = (\pi_{n,0,n}^*)^{-1}(Q)}} \min_{i=0,n} e(Q|Q \cap \pi_{n,0,i}^*(K(C_0))) \tag{280}$$

by the following reasoning: The fact that Ψ is a functor implies that $\psi_0 \circ \sigma^{-i} : F_{i,i} \to K(C_0)$ maps $Q \cap \sigma^i(F_0)$ to $(\pi_{n,0,i}^*)^{-1}(\psi_n(Q))$ for all $i = 0, n$ and we get the equality

$$\psi_0(\sigma^{-i}(Q \cap \sigma^i(F_0))) = (\psi_0 \circ \sigma^{-i})(Q \cap \sigma^i(F_0)) = (\pi_{n,0,i}^*)^{-1}(\psi_n(Q)).$$

Hence, the desired equality in (280) follows because ψ_n maps the places $Q \in \mathbb{P}_{F_n}$ with $Q \cap F_0 = \sigma^{-n}(Q \cap \sigma^n(F_0))$ bijectively to the places $Q \in \mathbb{P}_{K(C_n)}$ with $(\pi_{n,0,0}^*)^{-1}(Q) = (\pi_{n,0,n}^*)^{-1}(Q)$.

Fourth, we consider any circle in Γ, say of length r. As the circle has balanced ramification indices by assumption, Lemma 166 then supplies some $P \in \mathrm{Supp}(\Delta_r)$ such that $\pi_{r,0,0}$ and $\pi_{r,0,r}$ have the same ramification indices. Thus, if we take the normalizations $\tilde{C}_n$ of C_n for all $n \in \mathbb{N}_0$, then $(\tilde{C}_\nu)_\nu$ satisfies the requirements of Theorem 153, which are formulated in Notation 146, with the canonical extension morphisms $\tilde{\pi}_{i,j,e} : \tilde{C}_i \to \tilde{C}_j$ of the morphisms $\pi_{i,j,e} : C_i \to C_j$. Moreover, the normalization morphisms $\tilde{C}_n \to C_n$ also induce an isomorphisms $\tilde{\Psi} : \mathcal{D}_2 \to \mathcal{D}_3$ of categories where $\mathcal{D}_3$ is the same finite subcategory of k-algebras as $\mathcal{D}_2$ but with $\tilde{C}_n$ and $\tilde{\pi}_{i,j,e}$ instead of C_n and $\pi_{i,j,e}$. Hence, we derive the equalities

$$\sum_{\substack{Q \in \mathbb{P}_{F_n} \\ Q \cap F_0 = \sigma^{-n}(Q \cap \sigma^n(F_0))}} \min_{i=0,n} e(Q|Q \cap \sigma^i(F_0)) = \sum_{\substack{Q \in \mathbb{P}_{K(\tilde{C}_n)} \\ (\tilde{\pi}_{n,0,0}^*)^{-1}(Q) = (\tilde{\pi}_{n,0,n}^*)^{-1}(Q)}} \min_{i=0,n} e(Q|Q \cap \tilde{\pi}_{n,0,i}^*(K(\tilde{C}_0)))$$

$$= \sum_{Q \in \mathrm{Supp}(\Delta_n)} \min_{i=0,n} e_{\pi_{n,0,i}}(Q) \tag{281}$$

where the equalities hold by the following reasonings: The first equality holds by the equality in (280) and by the isomorphism $\tilde{\Psi} : \mathcal{D}_2 \to \mathcal{D}_3$ from above. The second equality holds by the one to one correspondence of the rational places in $K(\tilde{C}_n)$ and the k-rational points in $\tilde{C}_n$ and by the identity for the divisor Δ_n in Proposition 147.

Fifth, combining the estimate in (279), the equality in (281) and applying Theorem 153 to $(\tilde{C}_\nu)_\nu$ yields the estimates

$$r \cdot d^{mn} + \sum_{i=1}^{r} \sum_{j=2}^{s_i} \lambda_{i,j}^{mn} \leq \sum_{Q \in \mathrm{Supp}(\Delta_n)} \min_{i=0,n} e_{\pi_{n,0,i}}(Q) < 2d^{mn} - \sum_{i=1}^{k} \lambda_i^{mn} \tag{282}$$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ such that $|\lambda_i| \leq d$ for all $i = 1, \ldots, k$, some $n \in \mathbb{N}$ and all $m \in \mathbb{N}$.

Finally, [HP12, p. 26, Lemma 21] supplies some $m \geq 1$ such that all λ_i^{mn} and $\lambda_{i,j}^{mn}$ in (281) have nonnegative real parts. But this yields the desired estimate $r \leq 1$. ☐

Corollary 156. *Let $\mathcal{F}$ be a recursive tower of balanced degree d over an algebraic extension field k of a finite field. Then the following hold:*

(i) The tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ has at most one finite d-regular weakly connected component.

(ii) Suppose that $\mathcal{F}$ is a polynomial-recursive tower which is defined by the polynomial f. Then the Beelen-graph $\Gamma(f, k)$ has at most one finite weakly connected component which only contains circles with balanced ramification indices.

In particular, $\Gamma(f, k)$ has at most one finite d-regular weakly connected component.

Proof. For (i): By the fundamental equality for the tower graph in Lemma 87, a finite d-regular weakly connected component Γ of $\Gamma_{\mathcal{F}}$ cannot have ramified edges. In particular, this means that Γ only contains circles which balanced ramification indices. Hence, as Theorem 155 provides that there is at most one finite balanced weakly connected component, the desired statement in (i) also follows, namely that $\Gamma_{\mathcal{F}}$ has at most one finite d-regular weakly connected component.

For (ii): Since Proposition 95 provides that the Beelen-graph $\Gamma(f, k)$ is isomorphic to the rational subgraph $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ with an isomorphism which does not change the ramification indices, the desired statements in (ii) immediately follow from the corresponding statements for $\Gamma_{\mathcal{F}}$ in Theorem 155 and Corollary 156(i). ☐

6 An Almost Complete Answer to Conjecture 1(iii)

Summary of the results of this chapter. This is the main chapter of this thesis. Here we will prove three of the five major results. In particular, this will include the main result, namely the almost complete answer to Conjecture 1(iii), which is Corollary 184 and has the following outcome:

Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d over a finite field. If every finite weakly connected component of the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ contains circles with unbalanced ramification indices, then $\nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) = \#\,\mathrm{Split}(\mathcal{F}/F_0)$ holds, i.e. $\mathcal{F}$ satisfies Conjecture 1(ii).

This criterion is very mild and the author is only aware of the wild CNT-tower in Examples 8(v) having no truncation $\mathcal{F}_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ to which Corollary 184 is applicable. By the equalities $\nu(\mathcal{F}/F_m) = \nu(\mathcal{F}_{\geq m})$ and $\mathrm{Split}(\mathcal{F}/F_m) = \mathrm{Split}(\mathcal{F}_{\geq m}/F_m)$, we will obtain that all these recursive towers satisfy Conjecture 1(iv) and, thus, also the weaker Conjecture 1(iii). Hence, we will call Corollary 184 our almost complete answer to Conjecture 1(iii).

Second, we will also establish the third major result of this thesis, which is Theorem 188. In [Bee04, p. 238, Corollary 5.6] and in [HP12, p. 27, Theorem 24], it was shown that the limit of a good recursive tower cannot increase after a finite constant field extension whenever some technical conditions are satisfied. In Theorem 188, we will show that these technical conditions can even be dropped. This means that the limit of a good recursive towers can never increase after a finite constant field extension.

Third, we will also prove the fourth major results of this thesis, which are Corollary 195 and Corollary 200. These corollaries will be sharp versions of the criterion in Theorem 4 and will enable us to determine the precise limits of all α-weakly ramified recursive towers (see Definition 199) with finite ramification indices which only have unbalanced weakly connected components. Again, the only recursive tower known to the author which does not satisfy those conditions is the CNT-tower.

Finally, we will also demonstrate the significance of our results by improving several known and important results from the literature (see Section 6.4). For instance, it will follow that the lower bounds for the limits of the BBGS-towers, which are established in [BBGS15, p. 3] (see also Examples 8(vi)), are already the precise limits. Because the BBGS-towers provide the best known lower bounds for Ihara's constant $A(q)$ for non-prime q, we will deduce that this lower bound cannot be improved further by the BBGS-towers.

Also, we will finish the search for good quadratic polynomial-recursive towers over $\mathbb{F}_2$, which was started in [ST15]. More concretely, we will show that the limits of the four ST-towers Examples 8(iv) are vanishing. In [ST15, p. 667, Theorem 1.4] and [ST15, p. 680, Theorem 2.14], it was shown that these four are the remaining candidates for potentially good quadratic polynomial-recursive towers over $\mathbb{F}_2$.

Main idea and structure for the proof of our almost complete answer to Conjecture 1(iii). The almost complete answer to Conjecture 1(iii), which is Corollary 184, is one of the two essential cases in Corollary 183 and, thus, it will be an immediate consequence of Corollary 183. Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author, Corollary 183 characterizes all recursive towers $\mathcal{F} = (F_\nu)_\nu$ which satisfy $\nu(\mathcal{F}) = \mathrm{Split}(\mathcal{F}/F_0)$, i.e. which satisfy Conjecture 1(ii). Here, Corollary 184 and Corollary 185 cover the two essential cases in Corollary 183 in a less technical and thus more accessible manner. More concretely, Corollary 184 covers the recursive towers that satisfy Conjecture 1(ii) and Corollary 185 covers the tame recursive towers which do not satisfy Conjecture 1(ii). In the following, let us present the main ideas and structures of their proofs:

Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower with balanced degree d over a finite field k which is defined by the pair (σ, F_0) and let $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu = \overline{k} \cdot \mathcal{F}$ be the geometric tower of $\mathcal{F}$. First, Proposition 180 will generalize Theorem 96 (which was proven in [BGS04, p. 15, Theorem 4.10]) from polynomial-recursive to pair-recursive towers. More concretely, Proposition 180 will provide that the splitting rate $\nu(\mathcal{F})$ satisfies the estimates $\# V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \leq \# \,\mathrm{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F}) = \# V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \rho(\mathcal{F})$ where

$$\rho(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{d^n}.$$

Second, from Proposition 176, we will derive the estimate

$$\lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{d^n} \leq \sum_{\Gamma} \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\Gamma))}{d^n} \tag{283}$$

where the sum runs over all finite weakly connected components Γ of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Up to this point, nothing fundamentally new will have happened since we will have hardly done anything more than generalizing the proof of Theorem 96, which was already proven in [BGS04, p. 15, Theorem 4.10]. However, this will change from the next third step on.

Third, we will find useful upper bounds for all quotients $N(\overline{F}_n, V(\Gamma))/d^n$ in the sum in (283). Essentially, finding these useful upper bounds is the largest problem which is solved in this thesis. Indeed, almost everything that we showed so far and that we will show in this chapter up to the Main Theorem 177 and in Section 7.1 of the next chapter will go into the proof of its solution. The basic idea of the solution in Main Theorem 177 is the following:

We will introduce more structure to the tower graphs via suitable weight functions $w_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}} : E(\Gamma_{\overline{\mathcal{F}}}) \to \mathbb{R}_{>0}$ on $\Gamma_{\mathcal{F}}$ which satisfy the estimates

$$\tilde{N}(\overline{F}_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \leq w_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}(\mathcal{P}) \tag{284}$$

for all $\mathcal{P} \in W(\Gamma_{\overline{\mathcal{F}}}, n)$. Then, for all $\varepsilon > 0$ and as $n \to \infty$, we will have the asymptotic behavior

$$\frac{N(\overline{F}_n, V(\Gamma))}{d^n} \leq \frac{\tilde{N}(\overline{F}_n, V(\Gamma))}{d^n} = \sum_{\mathcal{P} \in W(\Gamma, n)} \frac{\tilde{N}(\overline{F}_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))}{d^n} = \mathcal{O}((\frac{\rho(A)}{d} + \varepsilon)^n)$$

by Lemma 59 where $\rho(A)$ denotes the spectral radius of any $w_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}$-adjacency matrix A of Γ.

More specifically, we will choose $w_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}$ in a way such that, on the one hand, the estimate $\rho(A) \leq d$ is satisfied and, on the other hand, the equality

$$\rho(A) = d \text{ holds if and only if all circles in } \Gamma \text{ have balanced ramification indices.} \tag{285}$$

This will provide that $\rho(\mathcal{F}) = 0$ and $\#\operatorname{Split}(\mathcal{F}/F_0) = \nu(\mathcal{F})$ if every Γ has a circle with unbalanced ramification indices. Up to constant field extensions of subgraphs, this is the statement of Corollary 184.

For Corollary 185, we will show that $\rho(k' \cdot \mathcal{F}) > 0$ and $\nu(k' \cdot \mathcal{F}) > \#\operatorname{Split}(k' \cdot \mathcal{F}/k' \cdot F_0)$ hold for some finite extension k'/k if there is some Γ which only contains circles with balanced ramification indices. First, we will derive that the summand $\lim_{n\to\infty} N(\overline{F}_n, V(\Gamma))$ in (283) is positive in this case. Finally, the difficulty will become to conclude that all the corresponding places in $\mathbb{P}_{\overline{F}_n}(V(\Gamma))$ already lie over rational places in $F'_n = k' \cdot F_n$ for some finite extension k'/k.

Structure of this chapter. In Section 6.1, we will introduce the desired weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ on the tower graphs which will enable the proof of the almost complete answer to Conjecture 1(iii).

In Section 6.2, we will exhibit Main Theorem 7 from which all results of this chapter will follow. For that, we will need three interim results which we will already formulate and use in this chapter but only prove in the next Chapter 7.

In Section 6.3, we will derive three of the five major results of this thesis.

In Section 6.4, as an application of the results of the previous sections, we will determine the precise limits of several recursive towers from the literature. By doing so, we will improve some important results from the literature.

6.1 Introducing Weights on Tower Graphs

Purpose of this section. In this section, we will introduce the desired weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ on the tower graphs which will enable the proof of the almost complete answer to Conjecture 1(iii).

Moreover, these weights $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}(\mathcal{P})$ will be parametrized as Laurent polynomials in variables x_p for all $p \in \mathbb{P}$. This parametrization will be crucial and, thus, we will formalize it in this section.

Structure of this section. In Subsection 6.1.1, we will introduce the desired weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ on the tower graphs.

In Subsection 6.1.2, we will formalize the parametrization of the weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ as Laurent polynomials in variables x_p for all $p \in \mathbb{P}$.

6.1.1 Weights With Complex Numbers

Purpose of this subsection. In this subsection, we will define the crucial weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ on the tower graphs $\Gamma_\mathcal{F}$ in Definition 157 and then prove that these weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$ satisfy the crucial estimate in (284).

The weight functions $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}$. In the following Definition 157, we will define the weight function $w_{x_\mathbf{P},\hat{x}_\mathbf{P}} : E(\Gamma_\mathcal{F}) \to \mathbb{C}$ for all $x_\mathbf{P} = (x_p)_p, \hat{x}_\mathbf{P} = (\hat{x}_p)_p \in \mathbb{C}^\mathbb{P}$ which will be crucial in the proof of our almost complete answer to Conjecture 1(iii).

Here, for any edge Q in $\Gamma_\mathcal{F}$, we should think of its weight $w_{x_\mathbf{P},\hat{x}_\mathbf{P}}(Q)$ as first replacing the primes p in the prime decomposition of $e(Q|Q \cap F_0)$ with x_p and of $e(Q|Q \cap \sigma(F_0))$ with $\hat{x}_p$ and then taking their product.

Definition 157. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree which is defined by the pair (σ, F_0), let $\Gamma_\mathcal{F}$ be its tower graph and let $x_\mathbf{P} = (x_p)_p$ and $\hat{x}_\mathbf{P} = (\hat{x}_p)_p$ be sequences* *in* $\mathbb{C}^{\mathbb{P}}$*. Then we define the weight function*

$$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}} : E(\Gamma_{\mathcal{F}}) \to \mathbb{C} \text{ via } Q \mapsto x_{\mathbf{P}}^{v_{\mathbf{P}}(e(Q|Q\cap F_0))} \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))}$$

on $\Gamma_{\mathcal{F}}$*.*

Examples 158. *In the following and for all* $x_{\mathbf{P}} = (x_p)_p, \hat{x}_{\mathbf{P}} = (\hat{x}_p)_p \in \mathbb{C}^{\mathbb{P}}$*, we list the* $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$*-adjacency matrices* A *of some subgraphs* Γ *in Chapter B and their eigenvalues:*

(i) For the second weakly connected component Γ *in Figure B.1 and the enumeration of the vertices in* Γ *from top to bottom, we have*

$$A = \begin{pmatrix} 0 & \hat{x}_2 & 0 & 0 & 0 \\ \hat{x}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_2 \end{pmatrix}$$

and the eigenvalues $\hat{x}_2, 1, x_2$*.*

(ii) For the second weakly connected component Γ *in Figure B.4 and the enumeration of the vertices in* Γ *from top to bottom, we have*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and the eigenvalues $2, 0, \lambda, \overline{\lambda}$ *with* $\lambda = (1 + i\sqrt{3})/2$*.*

(iii) For the first weakly connected component Γ *in Figure B.7 and the enumeration of the vertices in* Γ *from top to bottom, we have*

$$A = \begin{pmatrix} \hat{x}_2 & 1 \\ 0 & x_2 \end{pmatrix}$$

and the eigenvalues $\hat{x}_2, x_2$*.*

(iv) For the second weakly connected component Γ *in Figure B.7 and the enumeration of the vertices in* Γ *from top to bottom, we have*

$$A = \begin{pmatrix} 1 & \hat{x}_2 \\ x_2 & 0 \end{pmatrix}$$

and the eigenvalues $\lambda_\pm$ *with* $\lambda_\pm = (1 \pm \sqrt{1 + 4\hat{x}_2 x_2})/2$*. For instance, for* $\hat{x}_2 x_2 = 2$*, we have* $\lambda_+ = 2$ *and* $\lambda_- = -1$*.*

(v) For the first weakly connected component Γ *in Figure B.18, we have*

$$A = \begin{pmatrix} \hat{x}_4 x_4 \end{pmatrix}$$

and the eigenvalue $\hat{x}_4 x_4$*.*

The crucial estimate for the proof of our almost complete answer to Conjecture 1(iii). In the paragraph on page 202, we pointed out that the choice of the weight functions $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$ should satisfy the estimate in (284). This estimate will be established in the following Lemma 159(ii) for all $x_{\mathbf{P}} = (x_p)_p, \hat{x}_{\mathbf{P}} = (\hat{x}_p)_p \in \mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$ and $x_p, \hat{x}_p \in [1, p]$ for all $p \in \mathbb{P}$.

Lemma 159. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\Gamma_{\mathcal{F}}$ be its tower graph and let $x_{\mathbf{P}} = (x_p)_p$ and $\hat{x}_{\mathbf{P}} = (\hat{x}_p)_p$ be sequences in $\mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$ for all $p \in \mathbb{P}$. Then the following hold:*

(i) We have the identity

$$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))}\hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}$$

for all paths $\mathcal{P} \in W(\Gamma_{\mathcal{F}}, n)$, $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ and $(P_{i,j})_{i,j} := \operatorname{Pyr}(Q)$.

(ii) Let Γ be a finite subgraph of $\Gamma_{\mathcal{F}}$, let A be the $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$-adjacency matrix for some enumeration of the vertices of Γ and suppose that $\hat{x}_p$ and x_p are real numbers in the interval $[1, p]$ for all $p \in \mathbb{P}$. Then we have the estimate

$$\sum_{\mathcal{P} \in W(\Gamma, n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \leq N(A^n).$$

Examples 160. *Let $x_{\mathbf{P}} = (x_p)_p, \hat{x}_{\mathbf{P}} = (\hat{x}_p)_p \in \mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$ and $\hat{x}_p, x_p \in [1, p]$ for all $p \in \mathbb{P}$.*

(i) The paths $\mathcal{P}$ and $\mathcal{P}'$ in Examples 77(i) which are depicted in the figures 4.2 and 4.3 satisfy $\tilde{N}(F_4, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \tilde{N}(F'_4, \sigma_{\Gamma_{\mathcal{F}'}}(\mathcal{P})) = 2 = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}')$.

(ii) The path $\mathcal{P}$ in Examples 77(ii) which is depicted in Figure 4.4 satisfies the equalities $\tilde{N}(F_3, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = 2 = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$.

Proof of Lemma 159. For (i): We show the desired identity in (i) by induction over $n \in \mathbb{N}_0$. For $n = 0$, we have the equalities $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = 1 = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ where the first equality holds by the definition of $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ in Definition 50 and the second equality holds by the definition of weight functions on paths of length zero in Definition 58. As the equalities $Q = P_{0,0} = P_{n,n}$ hold for $n = 0$, the desired identity follows.

Now, let $n \geq 1$ and $(F_{i,j})_{i,j} := \operatorname{Pyr}(\mathcal{F})$. By $(P_{i,j})_{i,j} = \operatorname{Pyr}(Q)$ and by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76, we have the equality $\mathcal{P} = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1}$. Furthermore, let $\mathcal{P}' := [\sigma^{-i}(P_{i,j})]_{j-i\leq 1} \in W(\Gamma, n-1)$ be the truncation of $\mathcal{P}$ to a path of length $n-1$ in Γ. Then the induction hypothesis yields the equality

$$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}') = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}'))x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{0,0}))}\hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{n-1,n-1}))}. \tag{286}$$

Moreover, for all $\varepsilon \in \{0, 1\}$, we notice the equalities

$$\begin{aligned} \sigma^{-(n-1)}(P_{n-1,n}) \cap \sigma^{\varepsilon}(F_0) &= \sigma^{-(n-1)}(P_{n-1,n} \cap \sigma^{n-1+\varepsilon}(F_{0,0})) \\ &= \sigma^{-(n-1)}(P_{n-1+\varepsilon,n-1+\varepsilon}) \end{aligned} \tag{287}$$

where the equalities hold by the following reasonings: The first equality holds because σ is an isomorphism and because of the identity $F_0 = F_{0,0}$ in Lemma 10(i). The second equality holds because Lemma 10(ii) provides the equality $\sigma^{n-1+\varepsilon}(F_{0,0}) = F_{n-1+\varepsilon,n-1+\varepsilon}$ and because of the equalities

$$P_{n-1,n} \cap F_{n-1+\varepsilon,n-1+\varepsilon} = (Q \cap F_{n-1,n}) \cap F_{n-1+\varepsilon,n-1+\varepsilon} = Q \cap F_{n-1+\varepsilon,n-1+\varepsilon}$$

$$= P_{n-1+\varepsilon,n-1+\varepsilon}$$

where the first and last equalities hold by the definition of $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$ in Definition 11 and the second equality holds since Lemma 10(i) provides the inclusion $F_{n-1+\varepsilon,n-1+\varepsilon} \subseteq F_{n-1,n})$.

Then combining the equality in (287) and the invariance of ramification indices under the action of isomorphisms in (11) yields the identity

$$e(\sigma^{-(n-1)}(P_{n-1,n})|\sigma^{-(n-1)}(P_{n-1,n}) \cap \sigma^{\varepsilon}(F_0)) = e(P_{n-1,n}|P_{n-1+\varepsilon,n-1+\varepsilon}) \tag{288}$$

for all $\varepsilon \in \{0,1\}$. Thus, by the definition of $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$ in Definition 157, by the equality in (288) and by the identity $e = \tilde{e}$ on the extensions in $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$ in Definition 41(i), we deduce the equality

$$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\sigma^{-(n-1)}(P_{n-1,n})) = x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{n-1,n}|P_{n-1,n-1}))} \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{n-1,n}|P_{n,n}))}. \tag{289}$$

Finally, we obtain the desired equality by the equalities

$$\begin{aligned}
\frac{w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})}{\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))} &= \frac{w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}')}{\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}'))} \cdot w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\sigma^{-(n-1)}(P_{n-1,n})) \cdot \frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})} \\
&= x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{0,0}))} \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{n-1,n-1}))} \cdot x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{n-1,n}|P_{n-1,n-1}))} \\
&\quad \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{n-1,n}|P_{n,n}))} \cdot \frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})} \\
&= x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{0,0})\tilde{e}(P_{n-1,n}|P_{n-1,n-1})\frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})})} \\
&\quad \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n-1}|P_{n-1,n-1})\tilde{e}(P_{n-1,n}|P_{n,n})\frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})})} \\
&= x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n}|P_{0,0}))} \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(P_{0,n}|P_{n,n}))}
\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds since the definition of weight functions on paths in Definition 58 implies the equality $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}') \cdot w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\sigma^{-(n-1)}(P_{n-1,n}))$ and since the definition of $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ in Definition 50 implies the equalities

$$\begin{aligned}
\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) &= \frac{\prod_{i=0}^{n-1} \tilde{e}(P_{i,i+1}|P_{i,i})}{\tilde{e}(P_{0,n}|P_{0,0})} = \frac{\prod_{i=0}^{n-2} \tilde{e}(P_{i,i+1}|P_{i,i})}{\tilde{e}(P_{0,n-1}|P_{0,0})} \cdot \frac{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})}{\tilde{e}(P_{0,n}|P_{0,n-1})} \\
&= \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}')) \cdot \frac{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})}{\tilde{e}(P_{0,n}|P_{0,n-1})}.
\end{aligned}$$

The second equality holds by the identities in (286) and (289). The third equality holds because the assumption $\hat{x}_p x_p = p$ for all $p \in \mathbb{P}$ implies the equality $a = \mathbf{P}^{v_{\mathbf{P}}(a)} = x_{\mathbf{P}}^{v_{\mathbf{P}}(a)} \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(a)}$ for all $a \in \mathbb{Q}$ and because of elementary arithmetics. The last equality holds because Lemma 13 and the multiplicative transitivity rule for $\tilde{e}$ in Lemma 44(ii) yield the equalities

$$\tilde{e}(P_{0,n-1}|P_{0,0})\tilde{e}(P_{n-1,n}|P_{n-1,n-1})\frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})} = \tilde{e}(P_{0,n}|P_{0,0})$$

and

$$\begin{aligned}
&\tilde{e}(P_{0,n-1}|P_{n-1,n-1})\tilde{e}(P_{n-1,n}|P_{n,n})\frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})} \\
&\quad = \tilde{e}(P_{0,n-1}|P_{n-1,n})\tilde{e}(P_{n-1,n}|P_{n-1,n-1})\tilde{e}(P_{n-1,n}|P_{n,n})\frac{\tilde{e}(P_{0,n}|P_{0,n-1})}{\tilde{e}(P_{n-1,n}|P_{n-1,n-1})}
\end{aligned}$$

$$= \tilde{e}(P_{0,n}|P_{n,n}).$$

For (ii): Let $(a_{i,j}^{(n)})_{i,j} := A^n$. Then we immediately obtain the desired estimate in (ii) by the equalities and estimate

$$\sum_{\mathcal{P}\in W(\Gamma,n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \leq \sum_{\mathcal{P}\in W(\Gamma,n)} w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) = \sum_{i,j\in\{1,\dots,m\}} a_{i,j}^{(n)} = N(A^n)$$

where the estimate holds by the identity in the Lemma 159(i) and by the assumption $x_p, \hat{x}_p \in [1, p]$, the first equality holds by summing up the entries of the matrix in the identity in Lemma 59 and the last equality holds by the definition of N in Definition 60. □

6.1.2 Weights With Laurent Polynomials over Complex Numbers

Purpose of this subsection. In Lemma 159, we saw that $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ can be estimated by the weight $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ for all $x_{\mathbf{P}} = (x_p)_p, \hat{x}_{\mathbf{P}} = (\hat{x}_p)_p \in \mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$ and $x_p, \hat{x}_p \in [1, p]$. Moreover, these weights $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ are basically parametrized as Laurent polynomials in the variables x_p for all $p \in \mathbb{P}$. This parametrization will be crucial and, thus, we will formalize it in this subsection.

Structure of this subsection. First, in Definition 161(i), we will introduce the ring $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ of Laurent polynomials in the variables y_p for all $p \in \mathbb{P}$ and the evaluation morphism $\mathrm{Eval}_{x_{\mathbf{P}}} : \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \to \mathbb{C}$ which evaluates y_p at x_p.

Second, in Definition 162, we will introduce the weight function $w'_{\mathbf{1},\mathbf{P}} : E(\Gamma_{\mathcal{F}}) \to \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$.

Third, in Lemma 164 and for all paths $\mathcal{P}$ in $\Gamma_{\mathcal{F}}$, we will formalize the parametrization of the weights $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ as Laurent polynomials in the variables x_p for all $p \in \mathbb{P}$, i.e. we will connect the weight functions $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$ and $w'_{\mathbf{1},\mathbf{P}}$ via the identity $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}} = \mathrm{Eval}_{x_{\mathbf{P}}} \circ w'_{\mathbf{1},\mathbf{P}}$.

Fourth, in Lemma 165, we will prove some first simple properties of $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices.

Fifth, in Lemma 164, we will characterize the paths $\mathcal{P}$ in $\Gamma_{\mathcal{F}}$ with balanced ramification indices in terms of their weights $w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})$.

Definition 161. *(i) Let R be the polynomial ring over $\mathbb{C}$ in the variables y_p for all $p \in \mathbb{P}$ and let K be its field of fractions. Then we will write*

$$y_{\mathbf{P}} := (y_p)_p \in R^{\mathbb{P}}, \quad \mathbb{C}[y_{\mathbf{P}}] := R, \quad \mathbb{C}(y_{\mathbf{P}}) := K.$$

Moreover, we will also write

$$y_{\mathbf{P}}^{-1} := (y_p^{-1})_p, \quad \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] := \mathbb{C}[\{y_p, y_p^{-1} : p \in \mathbb{P}\}]$$

and $\mathbb{R}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ (resp. $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$) for the set of elements in $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ with coefficients in $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$).

Similarly to usual polynomials (resp. Laurent polynomials; resp. rational functions), we will also often write $h = h(y_{\mathbf{P}})$ for all h in $\mathbb{C}[y_{\mathbf{P}}]$ (resp. $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$; resp $\mathbb{C}(y_{\mathbf{P}})$).

(ii) Let $x_{\mathbf{P}} = (x_p)_p \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$. Then the evaluation morphism

$$\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \to \mathbb{C} \text{ via } \sum_{\beta} c_\beta y_{\mathbf{P}}^{\beta} \mapsto \sum_{\beta} c_\beta x_{\mathbf{P}}^{\beta}$$

of $\mathbb{C}$-algebras is well defined and we will denote it and its extension to the morphism

$$\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m} \to \mathbb{C}^{m\times m} \text{ via } (a_{i,j})_{i,j} \mapsto (\mathrm{Eval}_{x_{\mathbf{P}}}(a_{i,j}))_{i,j}$$

of $\mathbb{C}$-algebras by $\mathrm{Eval}_{x_{\mathbf{P}}}$.

Moreover, we will also often write $h(x_{\mathbf{P}}) := \mathrm{Eval}_{x_{\mathbf{P}}}(h)$ for all $h \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ and $A(x_{\mathbf{P}}) := \mathrm{Eval}_{x_{\mathbf{P}}}(A)$ for all $A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$.

Weights in Laurent polynomials. In the following Definition 162, we will define the already announced weight function $w'_{\mathbf{1},\mathbf{P}}$. Moreover, in Chapter 8, this weight function $w'_{\mathbf{1},\mathbf{P}}$ will also be crucial for finding genus formulas.

Definition 162. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\Gamma_{\mathcal{F}}$ be its tower graph and let $x_{\mathbf{P}} = (x_p)_p$ and $\hat{x}_{\mathbf{P}} = (\hat{x}_p)_p$ be sequences in $\mathbb{C}^{\mathbb{P}}$. Then we define the weight function*

$$w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}} : E(\Gamma_{\mathcal{F}}) \to \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \text{ via } Q \mapsto (x_{\mathbf{P}} * y_{\mathbf{P}})^{v_{\mathbf{P}}(e(Q|Q\cap F_0))} \cdot (\hat{x}_{\mathbf{P}} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))}$$

on $\Gamma_{\mathcal{F}}$ where $*$ denotes the componentwise multiplication of sequences in (2).

Examples 163. *In the following, we consider the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices A of the subgraphs Γ in Examples 158 for the same enumerations.*

(i) For the second weakly connected component Γ in Figure B.1 and the enumeration of the vertices in Γ from top to bottom, we have

$$A = \begin{pmatrix} 0 & 2y_2^{-1} & 0 & 0 & 0 \\ 2y_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 \\ 0 & 0 & 0 & 0 & y_2 \end{pmatrix}.$$

(ii) For the second weakly connected component Γ in Figure B.4 and the enumeration of the vertices in Γ from top to bottom, we have

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

(iii) For the first weakly connected component Γ in Figure B.7 and the enumeration of the vertices in Γ from top to bottom, we have

$$A = \begin{pmatrix} 2y_2^{-1} & 1 \\ 0 & y_2 \end{pmatrix}.$$

(iv) For the second weakly connected component Γ in Figure B.7 and the enumeration of the vertices in Γ from top to bottom, we have

$$A = \begin{pmatrix} 1 & 2y_2^{-1} \\ y_2 & 0 \end{pmatrix}.$$

(v) For the first weakly connected component Γ in Figure B.18, we have

$$A = \begin{pmatrix} 4y_2^{-1} \cdot y_2 \end{pmatrix} = \begin{pmatrix} 4 \end{pmatrix}.$$

Formalizing the parametrization. In the following Lemma 164, we will formalize the parametrization of the weights $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ as Laurent polynomials over the complex numbers via the weight function $w'_{\mathbf{1},\mathbf{P}}$ and the evaluation morphism $\mathrm{Eval}_{x_{\mathbf{P}}}$.

Lemma 164. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower, let $\Gamma_{\mathcal{F}}$ be its tower graph and let $x_{\mathbf{P}} = (x_p)_p$ and $\hat{x}_{\mathbf{P}} = (\hat{x}_p)_p$ be sequences in $\mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$ for all $p \in \mathbb{P}$. Then we have the identity*

$$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}} = \mathrm{Eval}_{x_{\mathbf{P}}} \circ w'_{\mathbf{1},\mathbf{P}}.$$

Moreover, let Γ be a finite subgraph of $\Gamma_{\mathcal{F}}$, let $v = (P_1, \ldots, P_m)$ be any enumeration of the vertices in Γ and let A (resp. B) be the $w'_{\mathbf{1},\mathbf{P}}$ (resp. $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$)-adjacency matrix of Γ for this enumeration v. Then we also have the identity

$$B = A(x_{\mathbf{P}}).$$

Proof. Let (σ, F_0) be the pair by which the recursive tower $\mathcal{F}$ is defined. First, we notice that the assumption $\hat{x}_p x_p = p$ for all $p \in \mathbb{P}$ implies the equality

$$\hat{x}_{\mathbf{P}} = (p x_p^{-1})_p = (\mathrm{Eval}_{x_{\mathbf{P}}}(p y_p^{-1}))_p. \tag{290}$$

Then we already obtain the desired identity in the 'main'-part by the equalities

$$\begin{aligned}(\mathrm{Eval}_{x_{\mathbf{P}}} \circ w'_{\mathbf{1},\mathbf{P}})(Q) &= \mathrm{Eval}_{x_{\mathbf{P}}}((\mathbf{1} * y_{\mathbf{P}})^{v_{\mathbf{P}}(e(Q|Q\cap F_0))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))}) \\ &= x_{\mathbf{P}}^{v_{\mathbf{P}}(e(Q|Q\cap F_0))} \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))} = w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(Q)\end{aligned}$$

for all $Q \in E(\Gamma_{\mathcal{F}})$ where the first equality holds by the definition of $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162, the second equality holds by the definition of the evaluation morphism $\mathrm{Eval}_{x_{\mathbf{P}}}$ of $\mathbb{C}$-algebras in Definition 161(ii) and by (290) and the third equality holds by the definition of $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$ in Definition 157.

Finally, the identity in the 'moreover'-part follows from the equalities

$$\begin{aligned}B &= \left(\sum_{Q\in E(\Gamma,P_i,P_j)} w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(Q)\right)_{i,j} = \left(\sum_{Q\in E(\Gamma,P_i,P_j)} (\mathrm{Eval}_{x_{\mathbf{P}}} \circ w'_{\mathbf{1},\mathbf{P}})(Q)\right)_{i,j} \\ &= \mathrm{Eval}_{x_{\mathbf{P}}}(A) = A(x_{\mathbf{P}})\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of B as the $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$-adjacency matrix of Γ for the enumeration v and by the definition of w-adjacency matrices in Definition 58. The second equality holds by the identity in the 'main'-part. The third equality holds by the definition of $\mathrm{Eval}_{x_{\mathbf{P}}}$ on matrices in Definition and by the definition of A as the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ for the enumeration v. The last equality holds by the definition of $A(x_{\mathbf{P}}) = \mathrm{Eval}_{x_{\mathbf{P}}}(A)$ in Definition 161(ii). □

First simple properties of $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices. The following properties in Lemma 165 will be used at various places. Here, the properties in (ii) and (iii) are consequences of the fundamental equality for the vertices in the tower graph in Lemma 87. In Examples 163, we immediately see that all our examples satisfy the first two properties in Lemma 165.

Lemma 165. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d over the field k which is defined by the pair (σ, F_0), let Γ be a finite subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and let A be the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ for some enumeration $v = (P_1, \ldots, P_m)$ of the vertices in Γ. Then the following hold:*

(i) *The entries of A are contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$.*

(ii) *All column (resp. row) sums in $A(\mathbf{1})$ (resp. $A(\mathbf{P})$) are at most d.*

(iii) *Suppose that k is algebraically closed. Then Γ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ if and only if all column sums in $A(\mathbf{1})$ and all row sums $A(\mathbf{P})$ are equal to d.*

Proof. For (i): By the definition of $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162 and by the definition of the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix A in Definition 58, we already obtain the desired statement that the entries of A are in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$.

For (ii): First, we notice that Lemma 164 provides that

$$\text{$A(\mathbf{1})$ (resp. $A(\mathbf{P})$) is the $w_{\mathbf{1},\mathbf{P}}$ (resp. $w_{\mathbf{P},\mathbf{1}}$)-adjacency matrix of Γ for the enumeration v.} \tag{291}$$

Second, we also notice the equalities

$$w_{\mathbf{1},\mathbf{P}}(Q) = \mathbf{1}^{v_{\mathbf{P}}(e(Q|Q\cap F_0))}\mathbf{P}^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))} = e(Q|Q\cap\sigma(F_0)) = e(Q|\sigma(P_j)) \tag{292}$$

and

$$w_{\mathbf{P},\mathbf{1}}(Q) = \mathbf{P}^{v_{\mathbf{P}}(e(Q|Q\cap F_0))}\mathbf{1}^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))} = e(Q|Q\cap F_0) = e(Q|P_i) \tag{293}$$

for all $Q \in E(\Gamma, P_i, P_j)$ where the first equalities hold by the definition $w_{\mathbf{1},\mathbf{P}}$ and $w_{\mathbf{P},\mathbf{1}}$ in Definition 157, the second equalities hold by the definition of the multi-index notation in (1) and the third equalities hold since the definition of $Q \in E(\Gamma, P_i, P_j) \subseteq E(\Gamma_{\mathcal{F}}, P_i, P_j)$ in Definition 74 implies the equalities $P_j = \sigma^{-1}(Q) \cap F_0$ and $P_i = Q \cap F_0$.

Consequently, we obtain the equalities

$$A(\mathbf{1}) = \left(\sum_{Q\in E(\Gamma,P_i,P_j)} w_{\mathbf{1},\mathbf{P}}(Q)\right)_{i,j} = \left(\sum_{Q\in E(\Gamma,P_i,P_j)} e(Q|\sigma(P_j))\right)_{i,j} \tag{294}$$

and

$$A(\mathbf{P}) = \left(\sum_{Q\in E(\Gamma,P_i,P_j)} w_{\mathbf{P},\mathbf{1}}(Q)\right)_{i,j} = \left(\sum_{Q\in E(\Gamma,P_i,P_j)} e(Q|P_i)\right)_{i,j} \tag{295}$$

where the first equalities hold by (291) and by the definition of w-adjacency matrices in Definition 58 and the second equalities hold by the equalities in (292) and (293). Hence, by the identities in (294) and (295), considering the j-th column sum in $A(\mathbf{1})$ yields

$$\begin{aligned}\sum_{i=1}^{m}\sum_{Q\in E(\Gamma,P_i,P_j)} e(Q|\sigma(P_j)) &= \sum_{Q\in E_-(\Gamma,P_j)} e(Q|\sigma(P_j)) \\ &\leq \sum_{Q\in E_-(\Gamma_{\mathcal{F}},P_j)} e(Q|\sigma(P_j)) \leq d\end{aligned} \tag{296}$$

and considering the i-th row sum in $A(\mathbf{P})$ yields

$$\sum_{j=1}^{m}\sum_{Q\in E(\Gamma,P_i,P_j)} e(Q|P_i) = \sum_{Q\in E_+(\Gamma,P_i)} e(Q|P_i)$$

$$\leq \sum_{Q \in E_+(\Gamma_{\mathcal{F}}, P_i)} e(Q|P_i) \leq d \tag{297}$$

where the equalities hold because $v = (P_1, \ldots, P_m)$ is an enumeration of the vertices in Γ and because of the definitions of $E_-(\Gamma, P_j)$ and $E_+(\Gamma, P_i)$ in Definition 55(vii), the first estimates hold since Γ is a subgraph of $\Gamma_{\mathcal{F}}$ and the second estimates hold by the identities in Lemma 87. Hence, (ii) follows from the estimates in (296) and (297).

For (iii): First, we notice that the assumption that k is algebraically closed provides that all places in F_1 and F_0 are rational. Thus, the second estimates in (296) and (297) are even equalities in this case.

Consequently, the only remaining estimates in (296) and (297) yield that all column sums in $A(\mathbf{1})$ and all row sums $A(\mathbf{P})$ are equal to d if and only if the equalities $E_-(\Gamma, P_j) = E_-(\Gamma_{\mathcal{F}}, P_j)$ and $E_+(\Gamma, P_i) = E_+(\Gamma_{\mathcal{F}}, P_i)$ hold for all $i, j \in \{1, \ldots, m\}$. But, since these equalities are precisely the definitions of Γ being a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ in Definition 66(iii), the desired equivalence in (iii) follows. ☐

Paths of balanced Ramification Indices in terms of weights. The following Lemma 166 characterizes paths of balanced ramification indices in several ways, e.g. in terms of its $w'_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}$-value and in terms of Abhyankar Ramification indices.

Lemma 166. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0), let $\Gamma_{\mathcal{F}}$ be its tower graph, let $\mathcal{P} \in W(\Gamma_{\mathcal{F}}, n)$ for some $n \in \mathbb{N}_0$ and let $(P_{i,j})_{j-i \leq 1} := \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$. Then the following are equivalent:*

(i) $\mathcal{P}$ has balanced ramification indices.

(ii) $w_{\mathbf{1},\mathbf{P}}(\mathcal{P}) = w_{\mathbf{P},\mathbf{1}}(\mathcal{P})$.

(iii) $\tilde{e}(Q|P_{0,0}) = \tilde{e}(Q|P_{n,n})$ for all $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$.

(iv) $e(Q|P_{0,0}) = e(Q|P_{n,n})$ for all $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$.

(v) $\deg_{y_p}(w'_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}(\mathcal{P})) = 0$ for all $x_{\mathbf{P}}, \hat{x}_{\mathbf{P}} \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$ and all $p \in \mathbb{P}$.

(vi) $w'_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}(\mathcal{P}) \in \mathbb{C}\backslash\{0\}$ for all $x_{\mathbf{P}}, \hat{x}_{\mathbf{P}} \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$.

Proof. For the equivalence of (i) and (ii): The equivalence of (i) and (ii) immediately follows from the assertion that $\mathcal{P}$ has balanced ramification indices, from its definition in Definition/Lemma 82(i) and from the equalities

$$\begin{aligned} w_{\mathbf{1},\mathbf{P}}(\mathcal{P}) &= \prod_{i=1}^{n} w_{\mathbf{1},\mathbf{P}}(\sigma^{-(i-1)}(P_{i-1,i})) \\ &= \prod_{i=1}^{n} \mathbf{1}^{v_{\mathbf{P}}(e(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap F_0))} \cdot \mathbf{P}^{v_{\mathbf{P}}(e(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma(F_0)))} \\ &= \prod_{i=1}^{n} e(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma(F_0)) = \prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i}) \end{aligned} \tag{298}$$

and

$$\begin{aligned} w_{\mathbf{P},\mathbf{1}}(\mathcal{P}) &= \prod_{i=1}^{n} w_{\mathbf{P},\mathbf{1}}(\sigma^{-(i-1)}(P_{i-1,i})) \\ &= \prod_{i=1}^{n} \mathbf{P}^{v_{\mathbf{P}}(e(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap F_0))} \cdot \mathbf{1}^{v_{\mathbf{P}}(e(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma(F_0)))} \end{aligned}$$

$$= \prod_{i=1}^{n} e(\sigma^{-(i-1)}(P_{i-1,i}) | \sigma^{-(i-1)}(P_{i-1,i}) \cap F_0)) = \prod_{i=1}^{n} e(P_{i-1,i} | P_{i-1,i-1}) \tag{299}$$

where the equalities hold by the following reasonings: The first equalities hold because the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and the definition of $(P_{i,j})_{j-i\leq 1} = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})$ imply the equalities $\mathcal{P} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1} \in W(\Gamma_{\mathcal{F}}, n)$ and because of the definition of weight functions on paths in Definition 58(i). The second equalities hold by the definitions of $w_{\mathbf{P},\mathbf{1}}$ and $w_{\mathbf{1},\mathbf{P}}$ in Definition 157. The third equalities hold by the definition of the multi-index notation in (1). For the last equalities, we first compute the identities

$$\begin{aligned} \sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma^{\varepsilon}(F_0) &= \sigma^{-(i-1)}(P_{i-1,i} \cap F_{i-1+\varepsilon,i-1+\varepsilon}) \\ &= \sigma^{-(i-1)}(P_{i-1+\varepsilon,i-1+\varepsilon,i-1+\varepsilon,i-1+\varepsilon}) \end{aligned}$$

for all $\varepsilon = 0, 1$ where the first equality holds since Lemma 10(i) and Lemma 10(iii) provide the equalities

$$\sigma^{\varepsilon}(F_0) = \sigma^{-(i-1)}(\sigma^{i-1+\varepsilon}(F_{0,0})) = \sigma^{-(i-1)}(F_{i-1+\varepsilon,i-1+\varepsilon})$$

and the second equality holds since the definition of $(P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n)$ in Definition 16(i) provides the equality $P_{i-1,i} \cap F_{i-1+\varepsilon,i-1+\varepsilon} = P_{i-1+\varepsilon,i-1+\varepsilon}$. Then the last equalities in (298) and (299) follow from the combination of these identities and the invariance of the ramification indices under the action of isomorphisms in (11).

For the equivalence of (ii), (iii) and (iv): Let $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$. Then we immediately derive the desired equivalences from the equalities

$$\frac{w_{\mathbf{P},\mathbf{1}}(\mathcal{P})}{w_{\mathbf{1},\mathbf{P}}(\mathcal{P})} = \frac{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1})}{\prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i})} = \Delta(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \frac{e(Q|P_{0,0})}{e(Q|P_{n,n})} = \frac{\tilde{e}(Q|P_{0,0})}{\tilde{e}(Q|P_{n,n})}$$

where the first equality holds by the equalities in (298) and (299), the second and third equalities hold by the equalities in (92) and the last equality holds by Theorem 47.

For the equivalence of (v) and (vi): This equivalence holds trivially.

For the implication from (vi) to (ii): Lemma 164 implies the equalities

$$\mathrm{Eval}_{\mathbf{1}} \circ w'_{\mathbf{1},\mathbf{P}} = w_{\mathbf{1},\mathbf{P}} \text{ and } \mathrm{Eval}_{\mathbf{P}} \circ w'_{\mathbf{1},\mathbf{P}} = w_{\mathbf{P},\mathbf{1}}. \tag{300}$$

Consequently, we obtain the desired equality in (ii) by the equalities

$$w_{\mathbf{1},\mathbf{P}}(\mathcal{P}) = \mathrm{Eval}_{\mathbf{1}}(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) = w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}) = \mathrm{Eval}_{\mathbf{P}}(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) = w_{\mathbf{P},\mathbf{1}}(\mathcal{P})$$

where the first and last equalities hold by the equalities in (300) and the second and third equalities hold because of the assumption $w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}) \in \mathbb{C}$ in (vi) and because $\mathrm{Eval}_{x_{\mathbf{P}}}$ is a morphism of $\mathbb{C}$-algebras by its definition in Definition 161(ii).

For the implication from (i) to (v): First, we compute

$$\begin{aligned} w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) &= \prod_{i=1}^{n} w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\sigma^{-(i-1)}(P_{i-1,i})) \\ &= \prod_{i=1}^{n} (x_{\mathbf{P}} * y_{\mathbf{P}})^{v_{\mathbf{P}}(e(P_{i-1,i}|P_{i-1,i-1}))} \cdot (\hat{x}_{\mathbf{P}} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(e(P_{i-1,i}|P_{i,i}))} \end{aligned}$$

$$= \prod_{p \in \mathbb{P}} (x_p y_p)^{v_p(\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1}))} \cdot (\hat{x}_p y_p^{-1})^{v_p(\prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i}))} \tag{301}$$

where the equalities hold by the following reasonings: The first (resp. second) equality holds by just replacing $w_{\mathbf{1},\mathbf{P}}$ with $w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$ in the reasoning for the first equality (resp. second and fourth equalities) in (298) and the third equality holds by the definitions of $*$ in (2), of the multi-index notation in (1) and of $v_{\mathbf{P}}$ in (3) and by elementary arithmetics.

Then combining the identity in (301), the assumption in (i), i.e. the equality

$$\prod_{i=1}^{n} e(P_{i-1,i}|P_{i-1,i-1}) = \prod_{i=1}^{n} e(P_{i-1,i}|P_{i,i}),$$

and the assertion $x_p, \hat{x}_p \neq 0$ already yields the desired statement in (v), namely that the y_p-degree of $w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P})$ must vanish for all $p \in \mathbb{P}$. ☐

6.2 Estimates for the Number of Places over Subgraphs

Summary of the results of this section. The largest problem solved in this thesis is the characterization of the finite weakly connected components Γ of the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of a recursive tower $\mathcal{F} = (F_\nu)_\nu$ for which $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ vanishes. Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author, the Main Theorem 177 yields that this limit vanishes if and only if Γ contains a circle with unbalanced ramification indices.

Main Theorem 177 is by far the deepest result of this thesis. More concretely, up to the Sections 2.5, 2.6, 4.3, 4.4, 5.3, and up to some further minor exceptions, everything that we have done so far and everything that we will do in the next Chapter 7 will go into the proof of the Main Theorem 177.

Moreover, for the proof of Main Theorem 177 in Subsection 6.2.2, we will also need the following three interim results in Subsection 6.2.1.

Structure of this section. In Subsection 6.2.1, we will formulate the three interim results which will form the core of the proof of Main Theorem 177.

In Subsection 6.2.2, we will prove the Main Theorem 177 of this thesis.

6.2.1 Three Interim Results

Purpose of this subsection. In this subsection, we will formulate the three interim results which will form the core of the proof of Main Theorem 177. For simplicity, we will shift their proofs to the next Chapter 7 will be dedicated entirely to their proofs.

The first interim result. The first interim result is formulated in Theorem 168(iii) and the cases (b) and (c) will provide the desired equivalence in (285).

Before we can formulate Theorem 168, we will need the following Definition 167.

Definition 167. *Let us denote the restriction of the principal branch of the complex logarithm in [Fre09, p. 29, Theorem I.2.11] to a function* $\mathbb{C}\backslash]-\infty, 0] \to \mathbb{C}$ *by* Log. *Then* Log *is a holomorphic function by [Fre09, p. 55, Theorem I.5.8.] (notice that the notions of holomorphic functions and analytic functions are used synonymously in [Fre09, p. 53]). Moreover, we define*

$$\log_2(\mathbf{P}) := (\log_2(p))_p \in \mathbb{R}_{>0}^{\mathbb{P}}, \quad z^\alpha := \mathrm{Exp}(\alpha\,\mathrm{Log}(z)), \quad z^\beta := (z^{\beta_p})_p \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$$

for all $z \in \mathbb{C}\backslash]-\infty, 0]$*, all* $\alpha \in \mathbb{R}$ *and all* $\beta = (\beta_p) \in \mathbb{R}^{\mathbb{P}}$*. Also notice that* z^α *is the usual* α*-th power of* $z \in \mathbb{C}\backslash]-\infty, 0]$ *for all* $\alpha \in \mathbb{Z}$*.*

Theorem 168 (First interim result)**.** *Let Γ be a finite weakly connected directed graph, let $w : E(\Gamma) \to \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ a weight function on Γ such that its image is contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \backslash \{0\}$, let $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m \times m}$ be the w-adjacency matrix of Γ for some enumeration of the m vertices in Γ and let $\chi_A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}][t]$ be the characteristic polynomial of A. Moreover, suppose that all column sums of $A(\mathbf{1})$ and all row sums of $A(\mathbf{P})$ are at most $d \in \mathbb{R}_{>0}$. Then the following hold:*

(i) A is connected.

(ii) If Γ is not strongly connected, then A is reducible and the estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ holds for all $x \in]1, 2[$.

(iii) There are the following three possible cases:

(a) If there is a column sum in $A(\mathbf{1})$ or a row sum in $A(\mathbf{P})$ which is less than d, then we have the estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$.

(b) If all column sums of $A(\mathbf{1})$ and all row sums of $A(\mathbf{P})$ are constantly d and there is a circle $\mathcal{C}$ in Γ such that $w(\mathcal{C}) \notin \mathbb{R}_{\geq 0}$, then we have the estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$.

(c) If all column sums of $A(\mathbf{1})$ and all row sums of $A(\mathbf{P})$ are constantly d and we have $w(\mathcal{C}) \in \mathbb{R}_{>0}$ for all circles $\mathcal{C}$ in Γ, then we have the identity $\rho(A(x^{\log_2(\mathbf{P})})) = d$ for all $x \in [1, 2]$.

In the last case (c), Γ is also strongly connected and A is irreducible.

We will prove Theorem 168 in Section 7.1 of the next chapter (see Subsection 7.1.6).

Examples 169. *Let us consider the subgraphs in Examples 163 with regards to the three cases in Theorem 168(iii). As all these subgraphs are already weakly connected components of their respective tower graphs, Lemma 165(iii) already supplies that only the two cases (b) and (c) occur.*

Moreover, by Lemma 164, we obtain the eigenvalues of $A(x^{\log_2(\mathbf{P})})$ if we choose $(x_p)_p = x_{\mathbf{P}} = x^{\log_2(\mathbf{P})} = (x^{\log_2(p)})_p$ and $(\hat{x}_p)_p = \hat{x}_{\mathbf{P}} = (px^{-\log_2(p)})_p$ in Examples 158. From that, we then deduce that the subgraphs in (i) (see the second weakly connected component in Figure B.1) and (iii) (see the first weakly connected component in figure B.7) satisfy (b) and that the subgraphs in (ii) (see the second weakly connected component in Figure B.4), (iv) (see the second weakly connected component in Figure B.18) and (v) (see the first weakly connected component in figure B.18) satisfy (c).

The second and third interim result. The second and third interim results are the following Corollary 170 and Corollary 171. In the case that $\mathcal{F}$ is tame and its ramification subgraph has some finite weakly connected component which only contains circles with balanced ramification indices, the second interim result in Corollary 170 will ensure that one of the limits in the sum in (283) does not vanish. Moreover, the third interim result in Corollary 171 will ensure that the corresponding places in $\mathbb{P}_{\overline{F}_n}(V(\Gamma))$ lie over places in F_n will have bounded degrees as $n \to \infty$.

This will then yield Corollary 185 which is the second essential case of Corollary 183.

Corollary 170 (Second interim result)**.** *Let $\mathcal{F}$ be a recursive tower and Γ be a finite strongly connected subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. If all circles in Γ have balanced ramification indices, then the set*

$$\{\tilde{e}(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$$

of Abhyankar ramification indices is finite.

In particular, if all circles in Γ *have balanced ramification indices and all paths in* Γ *are tame, then the set*

$$\{e(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$$

of ramification indices is finite.

We will prove Corollary 170 in Section 7.2 of the next chapter (see Subsection 7.2.3).

Corollary 171 (Third interim result)**.** *Let* $\mathcal{F}$ *be a recursive tower over a finite field and let* Γ *be a finite strongly connected subgraph of the tower graph* $\Gamma_{\mathcal{F}}$ *of* $\mathcal{F}$ *such that all circles in* Γ *have balanced ramification indices and all paths in* Γ *are tame. Then the set* $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ *is finite.*

We will prove Corollary 171 in Section 7.3 of the next chapter (see Subsection 7.3.3).

Examples 172. *With regards to the second and third interim results in Corollary 170 and Corollary 171, let us consider the three subgraphs in Examples 163 which only have circles with balanced ramification indices, namely the subgraphs in (ii) (see the second weakly connected component in Figure B.4), (iv) (see the second weakly connected component in Figure B.18) and (v) (see the first weakly connected component in figure B.18).*

Here, all recursive towers $\mathcal{F}$ *are tame and, thus, by Lemma 44(iii), the usual and Abhyankar ramification indices are equal. Moreover, all subgraphs stop ramifying from the second level on, i.e.* $Q/Q \cap F_1$ *is unramified for all* $Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma)) = \mathbb{P}_{\mathcal{F}}[\Gamma]$ *and, thus, the sets* $\{e(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ *of ramification indices in Corollary 170 only consist of* 1 *and* d *where* d *is the degree of* $\mathcal{F}$*.*

In particular, we also obtain that the sets $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ *of degrees in Corollary 171 also only consist of* 1 *and* d*.*

6.2.2 Main Theorem

In this section, we will prove the Main Theorem 177 of this thesis. We will derive all other results of this chapter from this theorem. Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author, the Main Theorem 177 characterizes the finite weakly connected components Γ of $\Gamma_{\mathcal{F}}$ which contribute to the splitting rate $\nu(\mathcal{F})$, i.e. for which $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ does not vanish. Here, this limit vanishes if and only if Γ contains a circle with unbalanced ramification indices.

The predecessor of Main Theorem 177 is Proposition 175. It will handle the case of algebraically closed constant fields. Then Proposition 176 and Main Theorem 177 will transfer the insights from Proposition 175 to more general constant fields.

Balanced weakly connected components are strongly connected. The requirement that a weakly connected component Γ of $\Gamma_{\mathcal{F}}$ only contains circles with balanced ramification indices is quite restrictive. In particular, the following Lemma 173 yields that Γ must already be strongly connected.

Lemma 173. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a recursive tower of balanced degree* d*, let* Γ *be a finite weakly connected component of the tower graph* $\Gamma_{\mathcal{F}}$ *of* $\mathcal{F}$ *and let* A *be the* $w'_{\mathbf{1},\mathbf{P}}$*-adjacency matrix of* Γ *for some enumeration* v *of the vertices in* Γ*.*

If Γ *only has circles with balanced ramification indices, then* Γ *is strongly connected and* A *is irreducible.*

Proof. For an algebraically closed constant field k of $\mathcal{F}$: First, by the fact that the definition of $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162 implies that all $w'_{\mathbf{1},\mathbf{P}}$-values of the edges are contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$, we conclude that Theorem 168 can be applied to Γ and $w'_{\mathbf{1},\mathbf{P}}$.

Second, by the assumption that Γ only has circles with balanced ramification indices and by the implication from (i) to (vi) in Lemma 166, we obtain $w'_{\mathbf{1},\mathbf{P}}(\mathcal{C}) \in \mathbb{R}_{>0}$ for all circles in Γ.

Third, since Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$, Lemma 165(iii) supplies that all column sums of $A(\mathbf{1})$ and all row sums of $A(\mathbf{P})$ are equal to d.

Hence, combining these three conclusions yields that we are in the case (c) of Theorem 168(iii). Consequently, the desired statements follows, namely that Γ is strongly connected and A is irreducible.

For an arbitrary constant field k of $\mathcal{F}$: Let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F}$ be the geometric tower of $\mathcal{F}$. Then Lemma 120(i) and the 'moreover'-part in Lemma 120(iii) provide that $\overline{k} \cdot \Gamma$ is a union of finite weakly connected components $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ of $\Gamma_{\overline{\mathcal{F}}}$. Furthermore, by the 'moreover'-part in Lemma 120(v), we also obtain that every $\overline{\Gamma}_i$ only contains circles with balanced ramification indices. Consequently, the first part of this proof for algebraically closed constant fields supplies that all $\overline{\Gamma}_i$ are strongly connected. Finally, the desired statement, namely that Γ is strongly connected, follows from the 'moreover'-part in Lemma 120(ii), □

Examples 174. *With regards to Lemma 173 (which can also be shown for non algebraically closed constant fields), let us consider the three subgraphs in Examples 163 which only have circles with balanced ramification indices, namely the subgraphs in (ii) (see the second weakly connected component in Figure B.4), (iv) (see the second weakly connected component in Figure B.18) and (v) (see the first weakly connected component in figure B.18) satisfy (c). These finite weakly connected components of their respective tower graphs are all strongly connected.*

The predecessor of our Main Theorem. The following Proposition 175 is the connecting piece between the first interim result in Theorem 168 and Main Theorem 177. For an algebraically closed constant field of $\mathcal{F} = (F_\nu)_\nu$ and for every finite weakly connected component Γ of $\Gamma_{\mathcal{F}}$ which contains circles with unbalanced ramification indices, the 'moreover'-part of Proposition 175 provides that $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ vanishes.

Proposition 175. *Let $\mathcal{F} = (F_\nu)_\nu$ be recursive tower of balanced degree d over an algebraically closed field. Moreover, let Γ be a finite weakly connected subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and let A be the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ for some enumeration v of the vertices in Γ. If we are in any of the two cases*

(i) Γ is not a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$

(ii) Γ has a circle with unbalanced ramification indices

then we have

$$\rho(A(x^{\log_2(\mathbf{P})})) < d \text{ and } N[F_n, \Gamma] = \mathcal{O}((\rho(A(x^{\log_2(\mathbf{P})})) + \varepsilon)^n)$$

as $n \to \infty$ for all $x \in]1,2[$ and all $\varepsilon > 0$.

Moreover, if Γ is also a weakly connected component of $\Gamma_{\mathcal{F}}$ which has a circle of unbalanced ramification indices, then we have the identity

$$\lim_{n\to\infty} \frac{N(F_n, V(\Gamma))}{d^n} = 0.$$

Proof. Let $x \in]1,2[$, $x_{\mathbf{P}} := x^{\log_2(\mathbf{P})}$, $\hat{x}_{\mathbf{P}} := \mathbf{P} \cdot x^{-\log_2(\mathbf{P})}$ and B_x be the $w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$-adjacency matrix of Γ for the enumeration v.

First, we will show the desired estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$: For that, we notice that, by Lemma 165(ii), the column (resp. row) sums of $A(\mathbf{1})$ (resp. $A(\mathbf{P})$) are at most d. Consequently, combining this and the fact that all $w'_{\mathbf{1},\mathbf{P}}$-values are contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$ by its definition in Definition 162 yields that we can apply Theorem 168.

In the first case (i), i.e. if Γ is not a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$, Lemma 165(iii) even provides that there is a column sum of $A(\mathbf{1})$ or a row sum of $A(\mathbf{P})$ which is less than d. Hence, the desired estimate follows from Theorem 168(iii)(a).

In the second case (ii), i.e. if Γ has a circle $\mathcal{C}$ with unbalanced ramification indices, the equivalence of the items (i) and (v) in Lemma 166 provides that the y_q-degree of $w'_{\mathbf{1},\mathbf{P}}(\mathcal{C})$ does not vanish for some $q \in \mathbb{P}$. Moreover, we may also assume that we are not in the first case and, thus, Lemma 165(iii) supplies that all column sums of $A(\mathbf{1})$ and all row sums of $A(\mathbf{P})$ are equal to d. Hence, the desired estimate follows from Theorem 168(iii)(b).

Next, we will show the desired identity $N[F_n, \Gamma] = \mathcal{O}((\rho(A(x^{\log_2(\mathbf{P})})) + \varepsilon)^n)$ for all $\varepsilon > 0$ as $n \to \infty$: For that, we first notice that the 'moreover'-part in Lemma 164 implies the equalities

$$B_x = A(x_{\mathbf{P}}) = A(x^{\log_2(\mathbf{P})}). \tag{302}$$

Consequently, we already obtain the desired identity for all $\varepsilon > 0$ by the equalities and estimates

$$\begin{aligned} N[F_n, \Gamma] = \sum_{\mathcal{P} \in W(\Gamma, n)} N(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) &\leq \sum_{\mathcal{P} \in W(\Gamma, n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \\ &\leq N(B_x^n) = N(A(x^{\log_2(\mathbf{P})})^n) = \mathcal{O}((\rho(A(x^{\log_2(\mathbf{P})})) + \varepsilon)^n) \end{aligned} \tag{303}$$

as $n \to \infty$ where the equalities and estimates hold by the following reasonings: The first equality holds by the definitions of $N[F_n, \Gamma] = \#\mathbb{P}^{(1)}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, n)))$ in Definition 85 and of $N(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \#\mathbb{P}^{(1)}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ in (5).

The first estimate holds by Corollary 51. For the second estimate, we first notice that $x^{\log_2(p)}$ is monotonically increasing for $x \in [1, 2]$ and satisfies the equalities $1^{\log_2(p)} = 1$ and $p^{\log_2(p)} = p$ for all $p \in \mathbb{P}$. Thus, for all $p \in \mathbb{P}$, we get the estimates $1 \leq x^{\log_2(p)} \leq p$ and $p^{-1} \leq x^{-\log_2(p)} \leq 1$ Consequently, the real numbers $x^{\log_2(p)}$ and $px^{-\log_2(p)}$ are contained in the interval $[1, p]$. Then the second estimate follows from this conclusion, from the choice of B_x as the $w_{x_{\mathbf{P}}, \hat{x}_{\mathbf{P}}}$-adjacency matrix of Γ for the enumeration v and from Lemma 159(ii).

The second equality holds by the equality in (302). The last equality holds by Definition 60 and by Lemma 61.

The desired identity in the 'moreover'-part immediately follows from the equality and estimates

$$0 \leq \lim_{n \to \infty} \frac{N(F_n, V(\Gamma))}{d^n} = \lim_{n \to \infty} \frac{N[F_n, \Gamma]}{d^n} \leq 0 \tag{304}$$

where the equality and estimates hold by the following reasonings: The first estimate is clear as the numerator and denominator are natural numbers. The second equality follows from the combination of the fact that weakly connected components are especially forward complete subgraphs by their definition in Definition 66(v) and of the 'moreover'-part in Lemma 86. The last estimate holds by the 'main'-part. □

Finite subgraphs which are no weakly connected components. In Proposition 175, the constant field k of the recursive tower $\mathcal{F} = (F_\nu)_\nu$ is algebraically closed. Now, we will switch to more general constant fields k which do not need to be algebraically closed.

Here the following Proposition 176 handles the finite weakly connected subgraphs Γ of $\Gamma_{\mathcal{F}}$ which are not already weakly connected components. More concretely, Proposition 176 provides that $\lim_{\nu \to \infty} N[F_\nu, \Gamma]/d^\nu$ vanishes.

Proposition 176. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of balanced degree d and let $(\overline{F}_\nu)_\nu := \overline{F}_q \cdot \mathcal{F}$ be the geometric tower of $\mathcal{F}$. Moreover, let Γ be a finite weakly connected subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and let $\overline{\Gamma} := \overline{\mathbb{F}}_q \cdot \Gamma$.*

If Γ is not a weakly connected component of $\Gamma_{\mathcal{F}}$, then we have the identities

$$\lim_{n\to\infty} \frac{N[F_n, \Gamma]}{d^n} = \lim_{n\to\infty} \frac{N[\overline{F}_n, \overline{\Gamma}]}{d^n} = 0.$$

Proof. Suppose that Γ is not a weakly connected component of $\Gamma_{\mathcal{F}}$. On the one hand, as Γ is weakly connected, it must be the case that Γ is not a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by the definition of weakly connected components in Definition 66(v). Let $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ be the weakly connected components of $\overline{\Gamma}$ in Lemma 120(ii). Then the 'moreover'-part in Lemma 120(iii) implies that, for all $i = 1, \ldots, r$,

$$\overline{\Gamma}_i \text{ is weakly connected but not a for- and backward complete subgraph of } \Gamma_{\overline{\mathcal{F}}}. \tag{305}$$

On the other hand, we deduce the equalities and estimate

$$N[F_n, \Gamma] = \#\mathbb{P}^{(1)}_{F_n}[\Gamma] \leq \#\mathbb{P}_{\overline{F}_n}(\mathbb{P}^{(1)}_{F_n}[\Gamma]) = \#\mathbb{P}^{(1)}_{\overline{F}_n}(\mathbb{P}^{(1)}_{F_n}[\Gamma]) \leq \#\mathbb{P}^{(1)}_{\overline{F}_n}[\overline{\Gamma}] = N[\overline{F}_n, \overline{\Gamma}] \tag{306}$$

for all $n \in \mathbb{N}_0$ where the equalities and estimates hold by the following reasonings: The first (resp. last) equality holds by the definition of $N[F_n, \Gamma]$ (resp. $N[\overline{F}_n, \overline{\Gamma}]$) in Definition 85. The first estimate holds because any place in $\mathbb{P}^{(1)}_{F_n}[\Gamma]$ has at least one place in $\overline{F}_n$ which lies above it. The second equality holds because $\overline{F}_n$ has an algebraically closed full constant field and, thus, only rational places. The second estimate holds by the obvious inclusion $\mathbb{P}^{(1)}_{\overline{F}_n}(\mathbb{P}^{(1)}_{F_n}[\Gamma]) \subseteq \mathbb{P}^{(1)}_{\overline{F}_n}(\mathbb{P}_{F_n}[\Gamma])$ and by the second identity $\mathbb{P}^{(1)}_{\overline{F}_n}(\mathbb{P}_{F_n}[\Gamma]) = \mathbb{P}^{(1)}_{\overline{F}_n}[\overline{\Gamma}]$ in Lemma 113.

Then we estimate

$$0 \leq \lim_{n\to\infty} \frac{N[F_n, \Gamma]}{d^n} \leq \lim_{n\to\infty} \frac{N[\overline{F}_n, \overline{\Gamma}]}{d^n} = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{N[\overline{F}_n, \overline{\Gamma}_i]}{d^n} = 0 \tag{307}$$

where the equalities and estimates hold by the following reasonings: The first estimate holds because $N[F_n, \Gamma] = \#\mathbb{P}_{F_n}[\Gamma] \in \mathbb{N}_0$ is a cardinality of a set by its definition in Definition 85. The second estimate holds by the estimate in (306). The first equality holds because $\overline{\Gamma}$ is a disjoint union of $\overline{\Gamma}_1, \ldots, \Gamma_r$ and because of the definition of $N[\overline{F}_n, \cdot] = \#\mathbb{P}_{F_n}[\cdot] = \#\mathbb{P}_{F_n}(\sigma_{\Gamma_{\overline{\mathcal{F}}}}(W(\cdot, n)))$ in Definition 85. The second equality holds by combining the conclusion in (305) and Lemma 120(i) and applying the estimate and identity in the 'main'-part of Lemma 175(i) to all $\overline{\Gamma}_i$.

Hence, the estimates in (307) must all be equalities which again yields the desired identities. ☐

Main Theorem. Finally, we come to the Main Theorem 177 of this thesis. Up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example in the literature known to the author, Main Theorem 177 characterizes the finite weakly connected components Γ of $\Gamma_{\mathcal{F}}$ which contribute to the splitting rate $\nu(\mathcal{F})$, i.e. for which the limit $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ does not vanish.

Main Theorem 177 is by far the deepest result of this thesis. More concretely, up to the Sections 2.5, 2.6, 4.3, 4.4, 5.3, and up to some further minor exceptions, everything that we have done so far and everything that we will do in the next Chapter 7 will go into the proof of Main Theorem 177.

Theorem 177 (Main Theorem)**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the finite field $\mathbb{F}_q$ of balanced degree d, let $(\overline{F}_\nu)_\nu := \overline{\mathbb{F}}_q \cdot \mathcal{F}$ be the geometric tower of $\mathcal{F}$ and let $(F'_\nu)_\nu := \mathbb{F} \cdot \mathcal{F}$*

be the constant field extension of $\mathcal{F}$ for some intermediate field $\mathbb{F}$ of the extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$. Moreover, let Γ be a finite weakly connected component of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$, let $\overline{\Gamma} := \overline{\mathbb{F}}_q \cdot \Gamma$ and let $\Gamma' := \mathbb{F} \cdot \Gamma$. Finally, define the composite field

$$k := \prod_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma))} \mathbb{F}_{q^{\deg(Q)}}.$$

which is equal to the finite field $\mathbb{F}_{q^l}$ for the natural number

$$l := \operatorname*{lcm}_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma))} \deg(Q).$$

if the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma))\}$ is finite. Then the following hold:

(i) If Γ has a circle with unbalanced ramification indices, then we have the identities

$$\lim_{n\to\infty} \frac{N(F_n', V(\Gamma'))}{d^n} = \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} = 0.$$

(ii) If $\mathbb{F}$ is not an extension field of k, then we have the identity

$$\lim_{n\to\infty} \frac{N(F_n', V(\Gamma'))}{d^n} = 0.$$

(iii) If Γ only has circles with balanced ramification indices and only has tame paths, then the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma))\}$ is finite and, hence, l is a natural number such that

$$0 < \lim_{n\to\infty} \frac{N(F_n', V(\Gamma'))}{d^n} = \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} \qquad \text{if } \mathbb{F}/\mathbb{F}_{q^l}$$

and

$$0 = \lim_{n\to\infty} \frac{N(F_n', V(\Gamma'))}{d^n} < \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} \qquad \text{else.}$$

More precisely, if $\mathbb{F}/\mathbb{F}_{q^l}$, then we even have the identity

$$N(F_n', V(\Gamma')) = N(\overline{F}_n, V(\overline{\Gamma}))$$

for all $n \in \mathbb{N}_0$.

Proof. For (i): Suppose that Γ has a circle with unbalanced ramification indices. By Lemma 120(i) and the 'moreover'-part in Lemma 120(iii), we obtain that $\overline{\Gamma}$ is a disjoint union of finitely many finite weakly connected components $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ of $\Gamma_{\overline{\mathcal{F}}}$. Then the 'moreover'-part of Lemma 120(v) supply that

$$\text{every component } \overline{\Gamma}_i \text{ has circles with unbalanced ramification indices.} \tag{308}$$

Consequently, we obtain the estimates and equalities

$$0 \le \lim_{n\to\infty} \frac{N(F_n', V(\Gamma'))}{d^n} \le \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}_i))}{d^n} = 0 \tag{309}$$

where the estimates and equalities hold by the following reasonings: The first estimate holds because $N(F_n', V(\Gamma')) = \#\mathbb{P}_{F_n'}(V(\Gamma'))$ is the cardinality of a set by its definition in (5). The second estimate holds because $\overline{\Gamma} = \overline{\mathbb{F}}_q \cdot \Gamma'$ is also the $\overline{\mathbb{F}}_q$-constant field extension of Γ' by Lemma 110 and because of Lemma 114(ii). The first equality holds as $\overline{\Gamma}$ is a disjoint union of the $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ and as the definition of $N(\overline{F}_n, \cdot)$ in (5) provides the equality $N(\overline{F}_n, V(\coprod_{i=1}^r \overline{\Gamma}_i)) = \sum_{i=1}^r N(\overline{F}_n, V(\overline{\Gamma}_i))$. The second equality follows from (308) and the 'moreover'-part of Proposition 175.

Hence, all estimates in (309) are actually equalities. In particular, this yields the desired identities in (i).

For (ii): Suppose that $\mathbb{F}$ is not an extension field of k. First, we notice that the items (i), (ii) and (iii) in Lemma 120 supply that Γ' is a disjoint union of finitely many finite weakly connected components $\Gamma'_1, \ldots, \Gamma'_s$ of $\Gamma_{\mathcal{F}'}$. In particular, because of that and because of the definition of $N(F'_n, \cdot)$ in (5), we conclude the equality

$$N(F'_n, V(\Gamma')) = \sum_{i=1}^{s} N(F'_n, V(\Gamma'_i)) \tag{310}$$

In the following, let $j \in \{1, \ldots, s\}$. Then we will show the identity

$$\lim_{n \to \infty} \frac{N(F'_n, V(\Gamma'_j))}{d^n} = 0. \tag{311}$$

Then combining this identity and the equality in (310) provides the desired identity in (ii).

For proving this identity in (311), we first notice that, by the definition of k in the assumptions, by the definition of $\mathbb{P}_{\mathcal{F}}(V(\Gamma)) = \coprod_{n \in \mathbb{N}_0} \mathbb{P}_{F_n}(V(\Gamma))$ in Definition 2(ii) and by the assertion that $\mathbb{F}$ is not an extension field of k, there must be some $m \in \mathbb{N}_0$ and some place $Q_0 \in \mathbb{P}_{F_m}(V(\Gamma))$ such that the residue field $\mathbb{F}_{q^{\deg(Q_0)}}$ of Q_0 is not contained in $\mathbb{F}$. Consequently, as the residue field of any place $Q' \in \mathbb{P}_{F'_m}(Q_0) = \mathbb{P}_{\mathbb{F} \cdot F_m}(Q_0)$ is the compositum $\mathbb{F}_{q^{\deg(Q_0)}} \cdot \mathbb{F}$ by [Sti08, p. 114, Theorem 3.6.3(g)], it is a proper extension field of $\mathbb{F}$. Hence, we deduce that

$$\text{no place in } \mathbb{P}_{F'_m}(Q_0) \text{ is rational.} \tag{312}$$

Moreover, because of the way we chose Γ'_j in the beginning, Lemma 120(ii) supplies that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to an epimorphism $\Gamma'_j \to \Gamma$. Thus, for $P := Q_0 \cap F_0$, there is some place $P' \in \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}^{-1}(P) \cap V(\Gamma'_j) = \mathbb{P}_{F'_0}(P) \cap V(\Gamma'_j)$ and, therefore, Lemma 26 even provides some place $Q' \in \mathbb{P}_{F'_m}((Q_0, P')) \subseteq \mathbb{P}_{F'_m}(Q_0) \cap \mathbb{P}_{F'_m}(V(\Gamma'_j))$. From (312), we moreover derive that this place

$$Q' \text{ is not rational.} \tag{313}$$

Next, consider the level m truncation

$$\mathcal{F}'_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F}') = (\mathcal{F}'_{m+\nu})_{\nu}$$

of $\mathcal{F}'$. As Γ'_j is a weakly connected component of $\Gamma_{\mathcal{F}'}$,

$$\Gamma'_j \text{ is especially a forward complete subgraph of } \Gamma_{\mathcal{F}'}. \tag{314}$$

From this and from Definition/Lemma 130(ii), we therefore derive the equalities

$$G' := \mathrm{Trun}_{\geq m}(\Gamma'_j) = \pi_{\Gamma_{\mathcal{F}'_{\geq m}}/\Gamma_{\mathcal{F}'}}{}^{-1}(\Gamma'_j) \tag{315}$$

for the level m truncation G' of Γ'_j. Then

$$G' \text{ is a finite weakly connected component of } \Gamma_{\mathcal{F}'_{\geq m}} \tag{316}$$

because Γ'_j is a finite weakly connected component of $\Gamma_{\mathcal{F}'}$ and because of the items (i) and (v) in Lemma 138. Moreover, we compute

$$\mathbb{P}_{F'_m}(V(\Gamma'_j)) = \pi_{\Gamma_{\mathcal{F}'_{\geq m}}/\Gamma_{\mathcal{F}'}}{}^{-1}(V(\Gamma'_j)) = V(G') \tag{317}$$

where the first equality holds because of the definitions of $\mathbb{P}_{F'_m}(V(\Gamma'_j))$ in (5) and of $\pi_{\Gamma_{\mathcal{F}'_{\geq m}}/\Gamma_{\mathcal{F}'}}$ in Definition/Lemma 126 and the second equality holds because of the equality in (315) and because of the definition of preimage subgraphs in Definition/Lemma 69(ii). Therefore, combining this equality in (317), the conclusion in (313) and the definition of the rational subgraph $\Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}}$ of $\Gamma_{\mathcal{F}'_{\geq m}}$ in Definition 88(i) yields that the place

$$Q' \in \mathbb{P}_{F'_m}(V(\Gamma'_j)) \text{ is contained in } V(G') \text{ but not in } V(\Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}}). \tag{318}$$

Then, because G' is finite by (316), we obtain that

$$\begin{aligned} &G' \cap \Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}} \text{ is the disjoint union of its finite} \\ &\text{weakly connected components } H'_1, \dots, H'_t. \end{aligned} \tag{319}$$

Furthermore, (318) even supplies that $G' \cap \Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}}$ and all H'_ν are proper subgraphs of the weakly connected graph G'. Hence, Lemma 68(i) provides that none of the H'_ν is a weakly connected component of $\Gamma_{\mathcal{F}'_{\geq m}}$. Thus,

$$\begin{aligned} &\text{all subgraphs } H'_\nu \text{ are finite weakly connected} \\ &\text{but not weakly connected components of } \Gamma_{\mathcal{F}'_{\geq m}}. \end{aligned} \tag{320}$$

Finally, we obtain the estimate and equalities

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \frac{N(F'_n, V(\Gamma'_j))}{d^n} &= \lim_{\substack{n\to\infty \\ n\geq m}} \frac{N(F'_n, \mathbb{P}_{F'_m}(V(\Gamma'_j)))}{d^n} \\ = \lim_{\substack{n\to\infty \\ n\geq m}} \frac{N(F'_n, V(G'))}{d^n} &= \lim_{\substack{n\to\infty \\ n\geq m}} \frac{N[F'_n, G']}{d^n} = \lim_{\substack{n\to\infty \\ n\geq m}} \frac{N[F'_n, G' \cap \Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}}]}{d^n} \\ = \frac{1}{d^m} \sum_{\nu=1}^{t} \lim_{\substack{n\to\infty \\ n\geq m}} \frac{N[F'_n, H'_\nu]}{d^{m-n}} &= 0 \end{aligned} \tag{321}$$

where the estimate and equalities hold by the following reasonings:

The estimate holds since $N(F'_n, V(\Gamma'_j)) = \#\mathbb{P}_{F'_n}(V(\Gamma'_j))$ is the cardinality of a set by its definition in (5).

The first equality holds because excluding finitely many elements of the sequence does not change its limit, because $F'_n/F'_m/F'_0$ are extensions of function fields for all $n \geq m$ and because of the definition of $\mathbb{P}_{F'_m}(V(\Gamma'_j))$ in (5).

The second equality holds by the equality in (317). The third equality holds because the conclusion in (316) and Definition 66(v) first imply that G' is a forward complete subgraph of $\Gamma_{\mathcal{F}'_{\geq m}}$, because then applying the 'moreover'-part in Lemma 86 yields the equality $\mathbb{P}_{F'_n}(V(G')) = \mathbb{P}_{F'_n}[G']$ and, finally, because of the definitions of $N(F'_n, V(G')) = \#(\mathbb{P}_{F'_n}(V(G')) \cap \mathbb{P}^{(1)}_{F'_n})$ in (5) and of $N[F'_n, G'] = \#(\mathbb{P}_{F'_n}[G'] \cap \mathbb{P}^{(1)}_{F'_n})$ in Definition 85.

The fourth equality holds because of the Definition of $N[F'_n, \cdot] = \mathbb{P}^{(1)}_{F'_n}[\cdot]$, because any rational place Q must have a rational path $\sigma^{-1}_{\Gamma_{\mathcal{F}}}(\text{Path}_{\mathcal{F}_{\geq m}}(Q))$ in G' by Lemma 80 and because $\Gamma^{\text{rat}}_{\mathcal{F}'_{\geq m}}$ precisely consists of all rational vertices and edges in $\Gamma_{\mathcal{F}'_{\geq m}}$ and, thus, contains all rational paths in $\Gamma_{\mathcal{F}'_{\geq m}}$.

The fifth equality holds by the definition of $N[F_n', \cdot]$, by the choice of the H_ν in (319) and by elementary arithmetics. The last equality holds by the conclusion in (320) and by Lemma 176.

Hence, the one and only estimate in (321) is also an equality and, more specifically, it is even the desired identity in (311).

For the estimate $0 < \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n}$ in (iii): Suppose that Γ only has circles with balanced ramification indices and only has tame paths. First, we notice that

$$\text{all paths in } \overline{\Gamma} \text{ are tame and all circles in } \overline{\Gamma} \text{ have balanced ramification indices} \tag{322}$$

due to the 'in particular'-parts in Lemma 120(iv) and Lemma 120(v). Second, the 'moreover'-part of Lemma 120(iii) provides that

$$\overline{\Gamma} \text{ is a disjoint union of weakly connected components } \overline{\Gamma}_1, \ldots, \overline{\Gamma}_r \text{ of } \Gamma_{\overline{\mathcal{F}}}. \tag{323}$$

Of course, these components $\overline{\Gamma}_i$ are also the weakly connected components of $\overline{\Gamma}$. Thus, we conclude the equalities

$$\mathbb{P}_{\overline{\mathcal{F}}}[\overline{\Gamma}_i] = \coprod_{n\in\mathbb{N}_0} \mathbb{P}_{\overline{F}_n}[\overline{\Gamma}_i] = \coprod_{n\in\mathbb{N}_0} \mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}_i)) \tag{324}$$

for all $i = 1, \ldots, r$ where equalities hold by the following reasonings: The first equality holds by the definition of $\mathbb{P}_{\mathcal{F}}[\overline{\Gamma}_i]$ in Definition 85. For the second equality, we notice that because $\overline{\Gamma}_i$ is a weakly connected component of $\Gamma_{\overline{\mathcal{F}}}$, it is especially a forward complete subgraph of $\Gamma_{\overline{\mathcal{F}}}$ by the definition of weakly connected components in Definition 66(v). Therefore, the identity $\mathbb{P}_{\overline{F}_n}[\overline{\Gamma}_i] = \mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}_i))$ follows from the 'moreover'-part in Lemma 86 for all $n \in \mathbb{N}_0$. Consequently, the second equality in (324) follows.

Next, we notice that Lemma 120(i) supplies that $\overline{\Gamma}_i$ is even finite. Consequently, combining this fact, the fact that (322) must also hold for all the weakly connected components $\overline{\Gamma}_i$ of $\overline{\Gamma}$, the equality in (324) and Corollary 170 provides that

$$\{e(Q|Q \cap \overline{F}_0) : Q \in \mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}_i)) \text{ for some } n \in \mathbb{N}_0\} \text{ is bounded by some } B_i \in \mathbb{N} \tag{325}$$

for all $i = 1, \ldots, r$.

Second to last, define $B := \max_{i=1,\ldots,r} B_i$. Then we derive the equalities and estimate

$$\begin{aligned} \#V(\overline{\Gamma}) \cdot d^n &= \sum_{P\in V(\overline{\Gamma})} d^n = \sum_{Q\in\mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}))} e(Q|Q\cap\overline{F}_0) \leq \sum_{Q\in\mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}))} B \\ &= N(\overline{F}_n, V(\overline{\Gamma})) \cdot B \end{aligned} \tag{326}$$

where the equalities and estimate hold by the following reasonings: The first equality holds as all summands are constantly d^n. The second equality holds by applying the fundamental equality in (8) to all $P \in V(\overline{\Gamma}) \subset \mathbb{P}_{\overline{F}_0}$ in the extension $\overline{F}_n/\overline{F}_0$ of degree d^n and by the fact that all places in $\overline{F}_n$ are rational since the geometric tower $\overline{\mathcal{F}} = (\overline{F}_r)_r$ has the algebraically closed full constant field $\overline{\mathbb{F}}_q$. The estimate holds because of (325), because of the definition of $B = \max_{i=1,\ldots,r} B_i$ and because the fact that $\overline{\Gamma}$ is the disjoint union of $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_r$ implies the equality $\mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma})) = \coprod_{i=1}^r \mathbb{P}_{\overline{F}_n}(V(\overline{\Gamma}_i))$. The last equality holds because all summands are equal to B, because of the definition of $N(\overline{F}_n, V(\overline{\Gamma})) = \mathbb{P}^{(1)}_{\overline{F}_n}(V(\overline{\Gamma}))$ in (5) and because all places in $\overline{F}_n$ are rational.

Finally, we obtain the desired estimate by the estimates

$$0 < \frac{\#V(\overline{\Gamma})}{B} \leq \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} \tag{327}$$

where the first estimate holds because Γ is a weakly connected component and, thus, non-empty by its definition in Definition 66(v) and because, by that and by the definition of $\overline{\Gamma} = \overline{\mathbb{F}}_q \cdot \Gamma = \pi^{-1}_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}(\Gamma)$ in Definition 107, $\overline{\Gamma}$ must also be non-empty. The second estimate in (327) holds by the estimate in (326).

For the distinction of cases in (iii): First of all, we derive the equalities $\mathbb{P}_{F_n}[\Gamma] = \mathbb{P}_{F_n}(V(\Gamma))$ for all $n \in \mathbb{N}_0$ from the assertion that Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$ and from the 'moreover'-part of Lemma 86. Moreover, since Γ only has circles with balanced ramification indices, Lemma 173 supplies that Γ is even strongly connected. Thus, combining this, Corollary 171 and the definitions of $\mathbb{P}_{\mathcal{F}}[\Gamma]$ in Definition 85 and of $\mathbb{P}_{\mathcal{F}}(V(\Gamma))$ in Definition 2(ii) already yields that the set

$$\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\} = \{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma))\}$$

is finite and, hence, that l is natural number. Moreover, [Sti08, p. 190, Lemma 5.1.9(d)] even supplies that all places in $\mathbb{P}_{\mathbb{F}_{q^l}\cdot F_n}(\mathbb{P}_{F_n}(V(\Gamma)))$ are rational.

Now, we go through both cases: On the one hand, suppose that $\mathbb{F}$ is an extension field of $\mathbb{F}_{q^l}$. Then $F'_n = \mathbb{F} \cdot (\mathbb{F}_{q^l} \cdot F_n)$ is the $\mathbb{F}$-constant field extension of $\mathbb{F}_{q^l} \cdot F_n$. Thus, [Sti08, p. 114, Theorem 3.6.3(c)] provides that all places in

$$\mathbb{P}_{F'_n}(\mathbb{P}_{\mathbb{F}_{q^l}\cdot F_n}(\mathbb{P}_{F_n}(V(\Gamma)))) = \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}(V(\Gamma))) = \mathbb{P}_{F'_n}(\mathbb{P}_{F'_0}(V(\Gamma))) = \mathbb{P}_{F'_n}(V(\Gamma')) \qquad (328)$$

are also rational where the first equality holds because $F'_n/(\mathbb{F}_{q^l} \cdot F_n)/F_n$ are extensions of function fields, the second equality holds because $F'_n/F_n/F_0$ and $F'_n/F'_0/F_0$ are extensions of function fields and the third equality holds because of the identity $\mathbb{P}_{F'_0}(V(\Gamma)) = V(\Gamma')$ in Definition 107.

Consequently, by this conclusion in (328), we may apply Lemma 114(iii) and, thereby, obtain the desired identity $N(\overline{F}_n, V(\overline{\Gamma})) = N(F'_n, V(\Gamma'))$ in the 'more precisely'-part. Then combining this identity and the estimate in (327) finally yields the desired identity and estimate in the first case, namely

$$0 < \lim_{n\to\infty} \frac{N(F'_n, V(\Gamma'))}{d^n} = \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\Gamma}))}{d^n} \qquad \text{if } \mathbb{F}/\mathbb{F}_{q^l}.$$

On the other hand, suppose that $\mathbb{F}$ is no extension field of $\mathbb{F}_{q^l}$. Then the desired identity follows from Lemma 177(ii) and the desired estimate was already proven in (327). □

Examples 178. *With regards to the Main Theorem 177, let us consider the five subgraphs in Examples 163.*

First, the subgraphs Γ in (i) (see the second weakly connected component Γ in Figure B.1) and in (iii) (see the first weakly connected component Γ in Figure B.7) both contain circles with unbalanced ramification indices. Therefore, the Main Theorem 177(i) provides that the limits $\lim_{\nu\to\infty} N(\mathbb{F} \cdot F_0, V(\mathbb{F} \cdot \Gamma))/d^\nu$ vanish for all constant field extensions $\mathbb{F} \cdot \mathcal{F}$ of $\mathcal{F}$.

Second, the subgraphs Γ in (ii) (see the second weakly connected component in Figure B.4), (iv) (see the second weakly connected component in Figure B.18) and (v) (see the first weakly connected component in figure B.18) only contain circles with unbalanced ramification indices. Therefore, the Main Theorem 177(iii) provides that the limits $\lim_{\nu\to\infty} N(\mathbb{F}_{q^l} \cdot F_0, V(\mathbb{F}_{q^l} \cdot \Gamma))/d^\nu$ do not vanish for some finite constant field extensions $\mathbb{F}_{q^l} \cdot \mathcal{F}$ of $\mathcal{F}$. More concretely, we have $l = 1$ in (ii) and $l = 2$ in (iv) and (v).

6.3 Consequences of the Main Theorem

Summary of the results of this section. In this section, we will prove three of the five major results of this thesis.

In Subsection 6.3.1, we will prove the almost complete answer to Conjecture 1(iii), which is Corollary 184. This is the main result of this thesis. Moreover, in Example 181, we will show that the CNT-tower in Examples 8(v) is a counterexample to Conjecture 1(iv).

In Theorem 188 of Subsection 6.3.3, we will deduce that limit of a good recursive tower can never increase after a finite constant field extension.

Finally, in Subsection 6.3.3, we will formulate sharp versions of the criterion in Theorem 4, which are Corollary 195 and Corollary 200. This will enable us to determine the precise limits of all recursive towers with a finite ramification subgraph which only has unbalanced weakly connected components. The only recursive tower known to the author for which these corollaries are not applicable is the wild CNT-tower in Examples 8(v).

6.3.1 The Almost Complete answer to Conjecture 1(iii)

Summary of the results of this subsection. In Corollary 184 of this subsection, we will finally give our almost complete answer to Conjecture 1(iii), which is the main result of this thesis. This Corollary 184 is one of the two essential cases in Corollary 183 and thus it will be an immediate consequence of Corollary 183.

Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example known to the author, Corollary 183 will give the following characterization of recursive towers over finite fields of balanced degree: The recursive tower $\mathcal{F}$ satisfies Conjecture 1(ii), i.e. $\nu(\mathcal{F}) = \text{Split}(\mathcal{F}/F_0)$, if and only if every finite weakly connected component Γ of the ramification subgraph $\Gamma_{\mathcal{F}}^{\text{ram}}$ contains circles with unbalanced ramification indices.

Corollary 184 will cover the 'if'-part of Corollary 183. As far as the author is aware, up to the CNT-tower, all recursive towers $\mathcal{F}$ in the literature satisfy the criterion in the 'if'-part for some truncation $\text{Trun}_{\geq m}(\mathcal{F})$ and thus satisfy Conjecture 1(iv), i.e. $\nu(\mathcal{F}/F_m) = \text{Split}(\mathcal{F}/F_m)$ for some $m \in \mathbb{N}_0$. Therefore, we will call Corollary 184 our almost complete answer to the weaker Conjecture 1(iii).

Moreover, in Example 181, we will also show that the CNT-tower in Examples 8(v) is a counterexample to Conjecture 1(iv).

Structure of this subsection First, we will give a more general definition of $\rho(\mathcal{F})$ for pair-recursive towers in Definition 179 and then prove in Proposition 180 that the splitting rate $\nu(\mathcal{F})$ satisfies $\#V(\Gamma_{\mathcal{F}}^{\text{split}}) \leq \nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\text{split}}) + \rho(\mathcal{F})$.

Second, in Example 181, we will show that the CNT-tower is a counterexample to Conjecture 1(iv).

Third, in Theorem 182, we will determine the value $\rho(k' \cdot \mathcal{F})$ for all finite constant field extensions $k' \cdot \mathcal{F}$ of $\mathcal{F}$.

Fourth, as a corollary of Theorem 182, we will prove Corollary 183, which characterizes recursive towers with respect to Conjecture 1(ii).

Fifth, as the first essential case of Corollary 183, we will provide the almost complete answer to Conjecture 1(iii), which is Corollary 184. Moreover, we will also provide the second essential case of Corollary 183, which is Corollary 185.

Sixth, we will consider this almost complete answer to Conjecture 1(iii) for some of the examples of recursive towers in Examples 8 and then give a justification for the term 'almost complete' answer to Conjecture 1(iii).

Seventh, we will discuss some observations which suggest that, although true for most recursive towers, Conjecture 1(iii) could be wrong for some very specific recursive towers. Moreover, we will elaborate on a strategy to possibly produce counterexamples to Conjecture 1(iii).

An upper bound for the splitting rate. In the following Definition 179, we will give a more general definition of $\rho(\mathcal{F})$ for pair-recursive towers. In [BGS04, p. 15, Definition 4.9], which is Definition 94(ii), the value $\rho(\mathcal{F})$ was already defined for polynomial-recursive towers. Moreover, from the third identity in the 'moreover'-part of Proposition 95, it will follow that the new definition of $\rho(\mathcal{F})$ for pair-recursive towers indeed agrees with the old for polynomial-recursive towers.

Then, in Proposition 180, we will prove that the splitting rate $\nu(\mathcal{F})$ satisfies the estimates and equality $\#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \leq \#\operatorname{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \rho(\mathcal{F})$. This will ensure that the only places $P \in \mathbb{P}_{F_0} \backslash \operatorname{Split}(\mathcal{F}/F_0)$ which can contribute to the splitting rate, i.e. the places P for which $\lim_{\nu\to\infty} N(F_\nu, P)/d^\nu > 0$ holds, are the vertices of the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Definition 179. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d. In accordance to the third identity in the 'moreover'-part in Proposition 95, we more generally define*

$$\rho(\mathcal{F}) := \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{d^n}.$$

Proposition 180. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field of balanced degree d, let $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ be the splitting subgraph and $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Then we have the estimates and identity*

$$\#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \leq \#\operatorname{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \rho(\mathcal{F}) \leq \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \#V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}).$$

Moreover, we have $\#\operatorname{Split}(\mathcal{F}/F_0) = \nu(\mathcal{F})$ if and only if $\#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) = \nu(\mathcal{F})$.

Proof. For the 'main'-part: The first desired estimate $\#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \leq \#\operatorname{Split}(\mathcal{F}/F_0)$ holds by the inclusion $V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \subseteq \operatorname{Split}(\mathcal{F}/F_0)$ in Lemma 92(i).

The second desired estimate $\#\operatorname{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F})$ holds by the equalities and estimates

$$\#\operatorname{Split}(\mathcal{F}/F_0) = \frac{\#\operatorname{Split}(\mathcal{F}/F_0)\cdot d^n}{d^n} = \frac{N(F_n, \operatorname{Split}(\mathcal{F}/F_0))}{d^n} \leq \frac{N(F_n)}{d^n} \to \nu(\mathcal{F})$$

as $n \to \infty$ where the first equality and the estimate are clear, the second equality holds by the definitions of $\operatorname{Split}(\mathcal{F}/F_0)$ in Definition 3(i) and of $N(F_n, \operatorname{Split}(\mathcal{F}/F_0)) = \#\mathbb{P}_{F_n}^{(1)}(\operatorname{Split}(\mathcal{F}/F_0))$ in (5) and, finally, we have $\nu(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_n)}{d^n}$ by its definition in Definition 2(iii).

For the desired identity $\nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \rho(\mathcal{F})$, we consider the rational subgraph $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ of $\Gamma_{\mathcal{F}}$, which is finite since $\mathcal{F}$ is defined over a finite field. First, we notice that, by the definition of $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ as the largest d-regular subgraph of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ in Definition 88(ii) and by the equalities in Lemma 87, $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ cannot contain any of the ramified edges in $\Gamma_{\mathcal{F}}$. Moreover, because of this conclusion, because $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ is also a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ and because of the definition of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ as the smallest forward and backward complete subgraph which contains all the ramified edges of $\Gamma_{\mathcal{F}}$, we deduce that $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ and $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ must be disjoint.

Consequently, the finite subgraph $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ is the disjoint union of $\Gamma_{\mathcal{F}}^{\mathrm{split}}$, $\Gamma_{\mathcal{F}}^{\mathrm{rat}} \cap \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and its remaining finitely many finite weakly connected components $\Gamma_1, \ldots, \Gamma_r$. In particular, these components cannot be d-regular as they are disjoint from $\Gamma_{\mathcal{F}}^{\mathrm{split}}$. Hence, because of this and because they are finite subgraphs of $(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}) \cap \Gamma_{\mathcal{F}}^{\mathrm{rat}}$, Lemma 91(i) supplies that none of the $\Gamma_1, \ldots, \Gamma_r$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and, especially, that

$$\text{none of the } \Gamma_1, \ldots, \Gamma_r \text{ is a weakly connected component of } \Gamma_{\mathcal{F}}. \tag{329}$$

Then we finally derive the desired identity $\nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\text{split}}) + \rho(\mathcal{F})$ from the equalities

$$\begin{aligned}\nu(\mathcal{F}) &= \lim_{n\to\infty} \frac{N(F_n)}{d^n} = \lim_{n\to\infty} \frac{N[F_n, \Gamma_{\mathcal{F}}^{\text{rat}}]}{d^n} \\ &= \lim_{n\to\infty} \frac{N[F_n, \Gamma_{\mathcal{F}}^{\text{split}}]}{d^n} + \lim_{n\to\infty} \frac{N[F_n, \Gamma_{\mathcal{F}}^{\text{rat}} \cap \Gamma_{\mathcal{F}}^{\text{ram}}]}{d^n} + \sum_{i=1}^{r} \lim_{n\to\infty} \frac{N[F_n, \Gamma_i]}{d^n} \\ &= \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\text{split}}))}{d^n} + \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\text{ram}}))}{d^n} = \#V(\Gamma_{\mathcal{F}}^{\text{split}}) + \rho(\mathcal{F})\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of the splitting rate in Definition 2(iii). The second equality holds because of the definitions of $N(F_n) = \#\mathbb{P}_{F_n}^{(1)}$ in (4) and of $N[F_n, \Gamma_{\mathcal{F}}^{\text{rat}}] = \#\mathbb{P}_{F_n}^{(1)}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma_{\mathcal{F}}^{\text{rat}}, n)))$ in Definition 85, because any rational place must lie over a rational path by Lemma 80 and because $\Gamma_{\mathcal{F}}^{\text{rat}}$ already contains all these rational paths by its definition in Definition 88(i). The third equality holds because $\Gamma_{\mathcal{F}}^{\text{rat}}$ is the disjoint union of $\Gamma_{\mathcal{F}}^{\text{split}}$, $\Gamma_{\mathcal{F}}^{\text{rat}} \cap \Gamma_{\mathcal{F}}^{\text{ram}}$ and $\Gamma_1, \ldots, \Gamma_r$.

For the fourth equality, we first conclude the identity $N[F_n, \Gamma_{\mathcal{F}}^{\text{split}}] = N(F_n, V(\Gamma_{\mathcal{F}}^{\text{split}}))$ by the fact that $\Gamma_{\mathcal{F}}^{\text{split}}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and by the 'moreover'-part in Lemma 86. Second, we conclude the identities

$$N[F_n, \Gamma_{\mathcal{F}}^{\text{rat}} \cap \Gamma_{\mathcal{F}}^{\text{ram}}] = N[F_n, \Gamma_{\mathcal{F}}^{\text{ram}}] = N(F_n, V(\Gamma_{\mathcal{F}}^{\text{ram}}))$$

where the first identity holds because any rational place in $\mathbb{P}_{F_n}[\Gamma_{\mathcal{F}}^{\text{ram}}]$ lies over a rational path in $\Gamma_{\mathcal{F}}$ and the second equality holds because $\Gamma_{\mathcal{F}}^{\text{ram}}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}}$ and because of the 'moreover'-part in Lemma 86. Third, combining (329) and Lemma 176 supplies the identity

$$\lim_{n\to\infty} \frac{N[F_n, \Gamma_i]}{d^n} = 0.$$

Combining these three identities yields the desired fourth equality .

The last equality holds because the inclusion $V(\Gamma_{\mathcal{F}}^{\text{split}}) \subseteq \text{Split}(\mathcal{F}/F_0)$ in Lemma 92 implies the equality $N(F_n, V(\Gamma_{\mathcal{F}}^{\text{split}})) = \#V(\Gamma_{\mathcal{F}}^{\text{split}}) \cdot d^n$ and because of the definition of $\rho(\mathcal{F})$ in Definition 179.

Finally, the last desired estimate $\#V(\Gamma_{\mathcal{F}}^{\text{split}}) + \rho(\mathcal{F}) \leq \#V(\Gamma_{\mathcal{F}'}^{\text{split}}) + \#V(\Gamma_{\mathcal{F}'}^{\text{ram}})$ follows from the equalities and estimate

$$\rho(\mathcal{F}) = \lim_{n\to\infty} \frac{N(F_n, V(\Gamma_{\mathcal{F}}^{\text{ram}}))}{d^n} \leq \lim_{n\to\infty} \frac{\#V(\Gamma_{\mathcal{F}}^{\text{ram}}) \cdot d^n}{d^n} = \#V(\Gamma_{\mathcal{F}}^{\text{ram}})$$

where the first equality holds by the definition of $\rho(\mathcal{F})$, the estimate holds because of the definition of $N(F_n, V(\Gamma_{\mathcal{F}}^{\text{ram}})) = \#\mathbb{P}_{F_n}^{(1)}(V(\Gamma_{\mathcal{F}}^{\text{ram}}))$ any place in F_0 has at most d^n places in F_n which lie above it and the last equality is clear.

For the 'moreover'-part: On the one hand, the 'if'-part immediately follows from the estimates $\#V(\Gamma_{\mathcal{F}}^{\text{split}}) \leq \# \text{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F})$ in the 'main'-part.

On the other hand, for the 'only if'-part, suppose $\# \text{Split}(\mathcal{F}/F_0) = \nu(\mathcal{F})$. Because $\lim_{n\to\infty} N(F_n, P)/d^n = 1$ holds for all $P \in \text{Split}(\mathcal{F}/F_0)$, we have the equalities

$$\nu(\mathcal{F}) = \sum_{P \in \mathbb{P}_{F_0}^{(1)}} \lim_{n\to\infty} \frac{N(F_n, P)}{d^n} = \# \text{Split}(\mathcal{F}/F_0) + \sum_{P \in \mathbb{P}_{F_n}^{(1)} \setminus \text{Split}(\mathcal{F}/F_0)} \lim_{n\to\infty} \frac{N(F_0, P)}{d^n}.$$

Hence, combining this equality and the assertion $\# \text{Split}(\mathcal{F}/F_0) = \nu(\mathcal{F})$ yields that

$$\text{all places } P \in \mathbb{P}_{F_0} \text{ which satisfy } \lim_{n\to\infty} \frac{N(F_n, P)}{d^n} > 0$$

$$\text{are already contained in } \operatorname{Split}(\mathcal{F}/F_0). \tag{330}$$

Now, let $P \in \operatorname{Split}(\mathcal{F}/F_0)$ and let Γ' be the subgraph of $\Gamma_{\mathcal{F}}$ which consists of all vertices and edges of the paths which start at P. Since $P \in \operatorname{Split}(\mathcal{F}/F_0)$ splits completely on every level F_n/F_0, the vertices and edges of every path which start at P are rational. Consequently, as the constant field of $\mathcal{F}$ is finite and as there are only finitely many rational vertices and edges in $\Gamma_{\mathcal{F}}$, the subgraph

$$\Gamma' \text{ is finite and only contains rational vertices and edges.} \tag{331}$$

Next, we notice that any place $Q \in \mathbb{P}_{F_m}(P)$ with some $m \in \mathbb{N}_0$ must also be rational and split completely in every extension F_n/F_m with $n \geq m$. Thus, we obtain the estimate $\lim_{n\to\infty} N(F_n, Q)/d^n > 0$ for all $Q \in \mathbb{P}_{F_m}(P)$. Moreover, let $P' \in V(\Gamma')$. By the choice of Γ', there is a path $\mathcal{P}$ from P to P', say of length m. Then Lemma 17(i) supplies some place $Q \in \mathbb{P}_{F_m}(\mathcal{P}) \subseteq \mathbb{P}_{F_m}((P, P'))$. But this again implies $\lim_{n\to\infty} N(F_n, P')/d^n \geq \lim_{n\to\infty} N(F_n, Q)/d^n > 0$. Hence, by (330), we conclude that P' is contained in $\operatorname{Split}(\mathcal{F}/F_0)$ and, thus, that

$$\text{any vertex in } \Gamma' \text{ has } d \text{ outgoing edges.} \tag{332}$$

Combining this conclusion in (332) and the finiteness of Γ' in (331) yields the equality $\#E(\Gamma') = d \cdot \#V(\Gamma')$. Then the 'in particular'-part of Lemma 87 and the fact that Γ' only contains rational vertices and edges in (331) provide that, for all $P' \in \Gamma'$, there must also be d rational ingoing edges in Γ'. But this means that Γ' is a d-regular subgraph of $\Gamma_{\mathcal{F}}^{\text{rat}}$ and, thus, by the definition $\Gamma_{\mathcal{F}}^{\text{split}}$ in Definition 88(ii), even a subgraph of $\Gamma_{\mathcal{F}}^{\text{split}}$. Hence, we established the desired statement, namely that P is contained in $V(\Gamma_{\mathcal{F}}^{\text{split}})$ and $\# \operatorname{Split}(\mathcal{F}/F_0) = \#V(\Gamma_{\mathcal{F}}^{\text{split}})$. □

The CNT-tower - A counterexample to Conjecture 1(iv). In the following Example 181, we will show that the CNT-tower in Examples 8(v) does not satisfy Conjecture 1(iii).

Example 181 (Counterexample to Conjecture 1(iv))**.** *We will prove that the CNT-tower* $\mathcal{F}_{CNT} =: \mathcal{F}$ *is a counterexample to Conjecture 1(iv), i.e. that* $\nu(\mathcal{F}_{\geq m}) > \# \operatorname{Split}(\mathcal{F}_{\geq m}/F_m)$ *holds for all truncations* $\mathcal{F}_{\geq m} := \operatorname{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$ *of* $\mathcal{F} = (F_\nu)_\nu = (\mathbb{F}_4(x_0, \ldots, x_\nu))_\nu$ *with* $m \in \mathbb{N}_0$*. For that, let* σ *be the tower map of* $\mathcal{F}$ *in Definition 5(ii).*

First, we consider the degree one subgraph of $\mathcal{F}$ *in Figure B.27, which is also the ramification subgraph* $\Gamma_{\mathcal{F}}^{\text{ram}}$ *of* $\mathcal{F}$*. Here we already added a zeroth level to the CNT-tower by the Reduction Lemma 30(ii). Next we list all possible paths* $\mathcal{P}$ *of length three in* $\Gamma_{\mathcal{F}}^{\text{ram}}$ *with their ramification indices. Let* P_β *be the place in* F_0 *which is generated by* $x_0 - \beta$ *for all* $\beta \in \mathbb{F}_4$ *and let* P_∞ *be the place at infinity in* F_0*. Then we see that the only paths* $\mathcal{P}$ *of length* 3 *in* $\Gamma_{\mathcal{F}}^{\text{ram}}$ *which can have a place* $Q \in \mathbb{P}_{F_3}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ *such that* $Q/Q \cap F_2$ *is ramified are the two circles* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *with the vertex sequence* $(P_0, P_\infty, P_1, P_0)$*. By checking the genera of* F_2 *and* F_3 *via Magma [BCP97], we obtain* $g(F_2) = 1$ *and* $g(F_3) = 5$*. Consequently, the Hurwitz-Genus-Formula in (9) supplies that there must be ramified places in* F_3/F_2*. But this means that*

$$\begin{aligned}&\textit{there is some } i = 1, 2 \ (\textit{say } i = 1) \textit{ and some place } Q_3 \in \mathbb{P}_{F_3}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_i)) \\ &\textit{such that } Q_3/Q_3 \cap F_2 \textit{ is ramified.}\end{aligned} \tag{333}$$

Second, let us consider the following path $\mathcal{P}'_{m+1}$ *in* $\Gamma_{\mathcal{F}}^{\text{ram}}$ *of length* $m+1$ *for all* $m \in \mathbb{N}_0$ *with the vertex set*

$$(P_0, P_0, \ldots, P_0, P_\infty, P_1, P_0)$$

where the subpath consisting of the last three edges is $\mathcal{P}_1$. *For* $m \leq 1$, *we immediately conclude by applying Abhyankar's Lemma to the elementary extensions in the pyramid of any place* $Q \in \mathbb{P}_{F_{m+1}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}'_{m+1}))$ *that* $Q/Q \cap F_m$ *is ramified in* F_{m+1}/F_m. *For* $m \geq 2$, *we first notice that* $\sigma^{m-2}(Q_3)$ *lies over* $\sigma^{m-2}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}_1))$ *and that we can therefore apply Lemma 19. By that, we obtain a place* $Q \in \mathbb{P}_{F_{m+1}}((\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}'_{m+1}), \sigma^{m-2}(Q_3)))$. *On the one hand, we notice that* $Q \cap F_{m-2}/Q \cap \sigma^{m-2}(F_0)$ *is unramified because the loop at* P_0 *in* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *is unramified. On the other hand, by (333), we deduce that* $Q \cap \sigma^{m-2}(F_3) = \sigma^{m-2}(Q_3)$ *is ramified in* $\sigma^{m-2}(F_3)/\sigma^{m-2}(F_2)$. *Consequently, again iteratively applying Abhyankar's Lemma to the extensions in the pyramid of* Q *yields that* $Q/Q \cap F_m$ *is ramified in* F_{m+1}/F_m. *Next, we notice that all places which lie over* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *must be rational since otherwise Main Theorem 177(ii) supplies the first equality in*

$$0 = \lim_{\nu \to \infty} \frac{N(F_\nu, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{2^\nu} = \nu(\mathcal{F}) > 0$$

where the second equality holds because $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *is also the degree one subgraph of* $\Gamma_{\mathcal{F}}$ *and the estimate follows from [CNT18, p. 19, Corollary 4.13]. Consequently,* Q *is a rational ramified edge of* $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ *and, hence, we established that*

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \text{ contains rational ramified edges for all } m \in \mathbb{N}_0. \tag{334}$$

Third, by the fact that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *is a finite weakly connected component of* $\Gamma_{\mathcal{F}}$, *by Lemma 138(i) and by Lemma 138(v), we obtain that*

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}}) \text{ is also a finite weakly connected component of } \Gamma_{\mathcal{F}_{\geq m}}. \tag{335}$$

Moreover, since $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *is the degree one subgraph of* $\Gamma_{\mathcal{F}}$, *we deduce that all rational places in* F_m *lie over* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *and that* $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$ *is therefore a forward and backward complete subgraph of* $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$. *Thus, combining this, (334) and (335) yields that* $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}}$ *must be empty. But, because of* $\nu(\mathcal{F}_{\geq m}) = d^m \cdot \nu(\mathcal{F}) > 0 = \#V(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{split}})$, *applying the contraposition of the equivalence in the 'moreover'-part of Proposition 180 to* $\mathcal{F}_{\geq m}$ *supplies the desired estimate* $\#\nu(\mathcal{F}_{\geq m}) > \mathrm{Split}(\mathcal{F}_{\geq m}/F_m)$ *for all* $m \in \mathbb{N}_0$.

Determining the ρ-value of a recursive tower. Proposition 180 provided the estimates $\#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \leq \#\,\mathrm{Split}(\mathcal{F}/F_0) \leq \nu(\mathcal{F}) = \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) + \rho(\mathcal{F})$. In the following Theorem 182, we will determine the value $\rho(\mathcal{F}')$ for all finite constant field extensions $\mathcal{F}'$ of $\mathcal{F}$.

Also note for the assumptions in Theorem 182 that since every weakly connected component of the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ contains some ramified edges and since there are only finitely many ramified edges in the tower graph, the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ can only have finitely many weakly connected components.

Even more so, Theorem 155 supplies that there is at most one finite weakly connected component Γ_i of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ with balanced ramification indices. Hence, in the assumptions of Theorem 182, we could even assume $r \leq 1$. This means that Γ is either empty or a finite weakly connected weakly connected component of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Theorem 182. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a recursive tower over a finite field* $\mathbb{F}_q$ *of balanced degree* d, *let* $\overline{\mathcal{F}} = \overline{\mathbb{F}}_q \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ *be the geometric tower of* $\mathcal{F}$, *let* $\mathcal{F}' := \mathbb{F} \cdot \mathcal{F} = (F'_\nu)_\nu$ *be the constant field extension of* $\mathcal{F}$ *for some be an intermediate field* $\mathbb{F}$ *of the extension* $\overline{\mathbb{F}}_q/\mathbb{F}_q$.

Moreover, let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *(resp.* $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$*) be the ramification subgraph of the tower graph* $\Gamma_{\mathcal{F}}$ *of* $\mathcal{F}$ *(resp.* $\Gamma_{\mathcal{F}'}$ *of* $\mathcal{F}'$*), let* Γ *be the disjoint union of all the finitely many finite weakly connected components* $\Gamma_1, \dots, \Gamma_r$ *of* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *which only have circles with balanced ramification indices.*

Finally, define the composite field

$$k_i := \prod_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))} \mathbb{F}_{q^{\deg(Q)}}.$$

for all $i = 1, \dots, r$ which is equal to the finite field $\mathbb{F}_{q^{l_i}}$ for the natural number

$$l_i := \operatorname*{lcm}_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))} \deg(Q).$$

if the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))\}$ is finite. Then the following hold:

(i) The constant field extension $\mathbb{F} \cdot \Gamma$ is also the disjoint union of all the finitely many finite weakly connected components of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ which only contain circles with balanced ramification indices.

(ii) We have the estimate

$$\rho(\mathcal{F}') \geq \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma))}{d^n} = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma_i))}{d^n}.$$

Moreover, if $\mathbb{F}$ or $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is finite, then the estimate is even an identity.

(iii) If $\mathbb{F}$ is not an extension field of k_i, then we have the identity

$$\lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma_i))}{d^n} = 0$$

(iv) Suppose that Γ_i only has tame paths. Then the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))\}$ is finite and, hence, l_i is a natural number such that

$$0 < \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma_i))}{d^n} = \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\mathbb{F}}_q \cdot \Gamma_i))}{d^n} \qquad \text{if } \mathbb{F}/\mathbb{F}_{q^{l_i}}$$

and

$$0 = \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma_i))}{d^n} < \lim_{n\to\infty} \frac{N(\overline{F}_n, V(\overline{\mathbb{F}}_q \cdot \Gamma_i))}{d^n} \qquad \text{else.}$$

In particular, we have the estimate $\rho(\mathcal{F}') > 0$ if $\mathbb{F}/\mathbb{F}_{q^{l_i}}$.

Proof. For (i): First, we will show that all finite weakly connected components Γ' of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ which only contain circles with balanced ramification indices are already subgraphs of $\mathbb{F} \cdot \Gamma$: Let Γ' be such a finite weakly component of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$. Then we consider its image graph

$$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma') \text{ which is a finite weakly connected connected component of } \Gamma^{\mathrm{ram}}_{\mathcal{F}} \tag{336}$$

because of of Lemma 121 and because $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$ restricts to a morphism $\Gamma^{\mathrm{ram}}_{\mathcal{F}'} \to \Gamma^{\mathrm{ram}}_{\mathcal{F}}$ by Lemma 124. Moreover, its preimage graph $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma'))$ is again a subgraph of $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma^{\mathrm{ram}}_{\mathcal{F}}) = \mathbb{F} \cdot \Gamma^{\mathrm{ram}}_{\mathcal{F}} = \Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ where the second equality holds again by Lemma 124. In particular, because Γ' is a subgraph of $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma'))$ by Lemma 70(ii) and because it is even a weakly connected component of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$,

$$\begin{gathered}\Gamma' \text{ must be one of the weakly connected components} \\ \text{of } \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')) \text{ in Lemma 120(ii).}\end{gathered} \tag{337}$$

Consequently, the combination of (336), of (337), of the assertions that Γ' is finite and only contains circles with balanced ramification indices and of the 'moreover'-part in Lemma 120(v) yields that $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ is also a finite weakly connected component of $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ which only has circles with balanced ramification indices. Hence, $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')$ must be one of the Γ_i.

Finally, because Γ' is a subgraph of $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma')) = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma_i)$ and because Γ_i is a subgraph of Γ, we conclude the desired statement, namely that Γ' is also a subgraph of $\pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma) = \mathbb{F} \cdot \Gamma$.

Next, we will show that $\mathbb{F} \cdot \Gamma$ is a disjoint union of finitely many finite weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}'}$ which only contain circles with balanced ramification indices: Let $j \in \{1, \ldots, r\}$. Because of the definition of the assertion that Γ_j is a finite weakly connected component of $\Gamma^{\text{ram}}_{\mathcal{F}}$ in Definition 66(v), because $\Gamma^{\text{ram}}_{\mathcal{F}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ and because the property of being a forward (resp. backward) complete subgraph is clearly transitive, we notice that Γ_j is also a finite weakly connected component of $\Gamma_{\mathcal{F}}$. Therefore, the combination of this conclusion, of the assertion that Γ_j only contains circles with balanced ramification indices and of the items (i), (iii) and (v) in Lemma 120 provides that $\mathbb{F} \cdot \Gamma_j$ is a disjoint union of finitely many finite weakly connected components of $\Gamma_{\mathcal{F}'}$ which only contain circles with balanced ramification indices.

Moreover, these components are also weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}'}$ because $\mathbb{F} \cdot \Gamma_j$ is a subgraph of $\mathbb{F} \cdot \Gamma^{\text{ram}}_{\mathcal{F}} = \Gamma^{\text{ram}}_{\mathcal{F}'}$ where the equality holds by Lemma 124.

Finally, because of this last conclusion and because Lemma 111 supplies the equalities

$$\mathbb{F} \cdot \Gamma = \pi^{-1}_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\Gamma) = \mathbb{F} \cdot \coprod_{i=1}^{r} \Gamma_i = \coprod_{i=1}^{r} \mathbb{F} \cdot \Gamma_i,$$

we conclude that $\mathbb{F} \cdot \Gamma$ is indeed a disjoint union of finitely many finite weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}'}$ which only contain circles with balanced ramification indices.

For the 'main'-part in (ii): We immediately obtain the 'main'-part in (ii) by the equalities and estimates

$$\begin{aligned} \rho(\mathcal{F}') &= \lim_{n\to\infty} \frac{N(F'_n, V(\Gamma^{\text{ram}}_{\mathcal{F}'}))}{d^n} \geq \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma))}{d^n} \\ &= \sum_{i=1}^{r} \lim_{n\to\infty} \frac{N(F'_n, V(\mathbb{F} \cdot \Gamma_i))}{d^n} \end{aligned} \tag{338}$$

where the first equality holds by the definition of $\rho(\mathcal{F}')$ in Definition 179, the estimate holds because of the assertion that Γ is a subgraph $\Gamma^{\text{ram}}_{\mathcal{F}}$ and because $\mathbb{F} \cdot \Gamma$ is a subgraph of $\mathbb{F} \cdot \Gamma^{\text{ram}}_{\mathcal{F}} = \Gamma^{\text{ram}}_{\mathcal{F}'}$ by Lemma 182(i) and the last equality holds because $\mathbb{F} \cdot \Gamma$ is the disjoint union of the $\mathbb{F} \cdot \Gamma_i$ and because of the definition of $N(F'_n, \cdot)$ in (5).

For the 'moreover'-part in (ii): Suppose that $\mathbb{F}$ or $\Gamma^{\text{ram}}_{\mathcal{F}}$ is finite. First of all, we notice that because $\mathbb{F} \cdot \Gamma$ is a union of weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}'}$ by Lemma 182(i) and because unions of forward and backward complete subgraphs are again forward and backward complete by the last remark in Definition 66(iii), we get that

$$\mathbb{F} \cdot \Gamma \text{ is a forward and backward complete subgraph of } \Gamma^{\text{ram}}_{\mathcal{F}'}. \tag{339}$$

Now, let $\Gamma^{\text{rat}}_{\mathcal{F}'}$ be the rational subgraph of $\Gamma_{\mathcal{F}'}$ which is finite if $\mathbb{F}$ is finite. Thus, in any case, the intersection subgraph $\Gamma^{\text{ram}}_{\mathcal{F}'} \cap \Gamma^{\text{rat}}_{\mathcal{F}'}$ is finite. Moreover, let $\Gamma'_1, \ldots, \Gamma'_t$ be the finitely many finite weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}'} \cap \Gamma^{\text{rat}}_{\mathcal{F}'}$. Since $\mathbb{F} \cdot \Gamma$ is a forward and backward complete subgraph of $\Gamma^{\text{ram}}_{\mathcal{F}'}$ by (339), it immediately follows from the definitions of forward and backward complete subgraphs in Definition 66(iii) and of intersection subgraphs in Definition 66(ii) that $\mathbb{F} \cdot \Gamma \cap \Gamma^{\text{rat}}_{\mathcal{F}'}$ is also a forward and backward complete subgraph of $\Gamma^{\text{ram}}_{\mathcal{F}'} \cap \Gamma^{\text{rat}}_{\mathcal{F}'}$.

Therefore, the complementary subgraph $(\Gamma^{\text{ram}}_{\mathcal{F}'} \cap \Gamma^{\text{rat}}_{\mathcal{F}'}) \backslash (\mathbb{F} \cdot \Gamma \cap \Gamma^{\text{rat}}_{\mathcal{F}'})$ is a well defined forward and backward complete subgraph of $\Gamma^{\text{ram}}_{\mathcal{F}'} \cap \Gamma^{\text{rat}}_{\mathcal{F}'}$ by Definition 66(iv). In particular, Lemma 68(iii) implies that $(\Gamma^{\mathrm{ram}}_{\mathcal{F}'} \cap \Gamma^{\mathrm{rat}}_{\mathcal{F}'})\backslash(\mathbb{F}\cdot\Gamma\cap\Gamma^{\mathrm{rat}}_{\mathcal{F}'})$ is a disjoint union of the weakly connected components of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'} \cap \Gamma^{\mathrm{rat}}_{\mathcal{F}'}$ which are subgraphs of $(\Gamma^{\mathrm{ram}}_{\mathcal{F}'} \cap \Gamma^{\mathrm{rat}}_{\mathcal{F}'})\backslash(\mathbb{F}\cdot\Gamma\cap\Gamma^{\mathrm{rat}}_{\mathcal{F}'})$ and, hence, disjoint from $\mathbb{F}\cdot\Gamma\cap\Gamma^{\mathrm{rat}}_{\mathcal{F}'}$.

Let $\Gamma'_{s+1},\dots,\Gamma'_t$ with $s\leq t$ be these components. Then, for all $j=s+1,\dots,t$, there are two possible cases: If Γ'_j is a already a finite weakly connected component of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$, then it cannot only have circles with balanced ramification indices because it otherwise would be a subgraph of $\mathbb{F}\cdot\Gamma$ by Lemma 182(i). Else, if Γ'_j is not a weakly connected component of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$, then it is also not a weakly connected component of $\Gamma_{\mathcal{F}'}$. Thus, in any case and for all $i=s+1,\dots,t$, we derive the equality

$$\lim_{n\to\infty}\frac{N[F'_n,\Gamma'_i]}{d^n}=0 \tag{340}$$

from the 'moreover'-part in Lemma 86 and Lemma 177(i) in the first case and from Lemma 176 in the second case.

Second to last, we obtain the equalities and estimates

$$\begin{aligned}\lim_{n\to\infty}\frac{N(F'_n,V(\Gamma^{\mathrm{ram}}_{\mathcal{F}'}))}{d^n}&=\lim_{n\to\infty}\frac{N[F'_n,\Gamma^{\mathrm{ram}}_{\mathcal{F}'}]}{d^n}=\lim_{n\to\infty}\frac{N[F'_n,\Gamma^{\mathrm{rat}}_{\mathcal{F}'}\cap\Gamma^{\mathrm{ram}}_{\mathcal{F}'}]}{d^n}=\sum_{j=1}^{t}\lim_{n\to\infty}\frac{N[F'_n,\Gamma'_i]}{d^n}\\&=\sum_{j=1}^{s}\lim_{n\to\infty}\frac{N[F'_n,\Gamma'_i]}{d^n}=\lim_{n\to\infty}\frac{N[F'_n,\coprod_{j=1}^{s}\Gamma'_i]}{d^n}\leq\lim_{n\to\infty}\frac{N[F'_n,\mathbb{F}\cdot\Gamma])}{d^n}\\&=\lim_{n\to\infty}\frac{N(F'_n,V(\mathbb{F}\cdot\Gamma))}{d^n}\leq\lim_{n\to\infty}\frac{N(F'_n,V(\Gamma^{\mathrm{ram}}_{\mathcal{F}'}))}{d^n}\end{aligned} \tag{341}$$

where the equalities and estimates hold by the following reasonings: The first equality holds because $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}'}$ by its definition in Definition 88(iii) and because of the 'moreover'-part in Lemma 86.

The second equality holds because of the definition of $N[F'_n,\cdot]=\#\mathbb{P}^{(1)}_{F'_n}(\sigma_{\Gamma_{\mathcal{F}'}}(W(\cdot,n)))$ in Definition 85, because Lemma 80 implies that any rational place in F'_n must lie above a rational path in $\Gamma_{\mathcal{F}}$ and because $\Gamma^{\mathrm{rat}}_{\mathcal{F}'}$ contains all these rational paths in $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(i).

The third (resp. fifth) equality again holds because of the definition of $N[F'_n,\cdot]=\#\mathbb{P}^{(1)}_{F'_n}(\sigma_{\Gamma_{\mathcal{F}'}}(W(\cdot,n)))$ and because $\Gamma^{\mathrm{rat}}_{\mathcal{F}'}\cap\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ is the disjoint union of the Γ'_i by their choice.

The fourth equality holds by the equality in (340). The first estimate holds because the $\Gamma'_1,\dots\Gamma'_s$ are chosen to be disjoint subgraphs of $\mathbb{F}\cdot\Gamma$.

For the sixth equality, we first notice that because of the conclusion in (339), because $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ is a forward complete subgraph of $\Gamma_{\mathcal{F}'}$ by its definition in Definition 88(iii) and because being a forward complete subgraph is transitive, $\mathbb{F}\cdot\Gamma$ is even a forward complete subgraph of $\Gamma_{\mathcal{F}'}$. Hence, we derive the sixth equality from this conclusion and from the 'moreover'-part in Lemma 86. The last estimate holds because $\mathbb{F}\cdot\Gamma$ is a subgraph of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ by (339).

Consequently, both estimates in (341) must actually be equalities. But the last estimate in (341) is exactly the only estimate in (338) and, hence, we obain the desired identity in the 'moreover'-part in (ii).

For (iii) and (iv): Because $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is a forward and backward complete subgraph of $\Gamma_{\mathcal{F}}$ by its definition in Definition 88(iii) and because the property of being a forward (resp. backward) complete subgraph is clearly transitive, we conclude that Γ_i is also a finite weakly connected component of $\Gamma_{\mathcal{F}}$ which only has circles with balanced ramification indices. Thus, Theorem 177(ii) provides the desired identity in (iii) and Lemma 177(iii) provides the 'main'-part in (iv).

Moreover, the 'in particular'-part immediately follows from the 'main'-part and from Lemma 182(ii). □

Characterization of recursive towers with respect to Conjecture 1(ii). Up to finite constant field extensions and up to some very specific wild recursive towers for which the CNT-tower in Examples 8(v) is the only example known to the author, the following Corollary 183 characterizes the recursive towers over finite fields of balanced degree with respect to Conjecture 1(ii): A recursive tower $\mathcal{F}$ satisfies Conjecture 1(ii), i.e. $\nu(\mathcal{F}) = \operatorname{Split}(\mathcal{F}/F_0)$, if and only if every finite weakly connected component Γ of the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ contains circles with unbalanced ramification indices.

Moreover, Corollary 183 also provides more precise information about the finite constant field extensions.

Also note that Theorem 155 supplies that the ramification subgraph of a recursive tower over a finite field has at most one finite balanced weakly connected component. Hence, in the assumptions of Corollary 183, we could even assume $s \leq r \leq 1$.

Corollary 183. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field $\mathbb{F}_q$ of balanced degree d and let $\mathcal{F}' := \mathbb{F}_{q^l} \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some finite extension $\mathbb{F}_{q^l}/\mathbb{F}_q$.*

Let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$, let $r \in \mathbb{N}_0$ be the number of all the finitely many finite weakly connected components $\Gamma_1, \ldots, \Gamma_r$ of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ which only have circles with balanced ramification indices. Moreover, after maybe reordering $\Gamma_1, \ldots, \Gamma_r$, suppose that $\Gamma_1, \ldots, \Gamma_s$ are exactly the Γ_i which only have tame paths.

Finally, define the composite field

$$k_i := \prod_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))} \mathbb{F}_{q^{\deg(Q)}}.$$

for all $i = 1, \ldots, r$ which is equal to the finite field $\mathbb{F}_{q^{l_i}}$ for the natural number

$$l_i := \operatorname*{lcm}_{Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))} \deg(Q).$$

if the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))\}$ is finite.

Then Theorem 182(iv) implies that, for all $i = 1, \ldots, s$, the sets $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}(V(\Gamma_i))\}$ are finite and, hence, l_i are a natural numbers. Furthermore, the following hold:

(i) If $r = 0$, i.e. if there is no finite weakly connected component of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ which only has circles with balanced ramification indices, then we have the identities

$$\rho(\mathcal{F}') = 0 \text{ and } \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) = \# \operatorname{Split}(\mathcal{F}'/F_0') = \nu(\mathcal{F}')$$

(ii) If, for all $i = 1, \ldots, r$, the finite field $\mathbb{F}_{q^l}$ is not an extension field of k_i, then we have the identities

$$\rho(\mathcal{F}') = 0 \text{ and } \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) = \# \operatorname{Split}(\mathcal{F}'/F_0') = \nu(\mathcal{F}')$$

(iii) If, for some $i = 1, \ldots, s$, the finite field $\mathbb{F}_{q^l}$ is an extension field of $k_i = \mathbb{F}_{q^{l_i}}$, i.e. l_i divides l, then we have the estimates

$$\rho(\mathcal{F}') > 0 \text{ and } \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) \leq \# \operatorname{Split}(\mathcal{F}/F_0) < \nu(\mathcal{F}') \leq \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) + \#V(\Gamma_{\mathcal{F}'}^{\mathrm{ram}})$$

Proof of Corollary 183. For (i) and (ii): The first desired identiy $\rho(\mathcal{F}') = 0$ in (i) (resp. (ii)) immediately follows from the 'moreover'-part in Theorem 182(ii) (resp. and from Theorem 182(iv)).

The second and third desired identities follow because the already proven first desired identity $\rho(\mathcal{F}') = 0$ supplies that the first two estimates in Proposition 180 must actually be our desired identities.

For (iii): Suppose that, for some $i = 1, \ldots, s$, the finite field $\mathbb{F}_{q^l}$ is an extension field of k_i. Then the first desired estimate $\rho(\mathcal{F}') > 0$ in (iii) immediately follows from the 'in particular'-part in Theorem 182(iv).

The second desired estimate $\#V(\Gamma^{\text{split}}_{\mathcal{F}'}) < \nu(\mathcal{F}')$ in (iii) follows from the first desired estimate $\rho(\mathcal{F}') > 0$ and from the identity $\nu(\mathcal{F}') = \#V(\Gamma^{\text{split}}_{\mathcal{F}'}) + \rho(\mathcal{F}')$ in Proposition 180. Furthermore, the 'moreover'-part in Proposition 180 also supplies the estimates $\#V(\Gamma^{\text{split}}_{\mathcal{F}'}) \leq \#\operatorname{Split}(\mathcal{F}/F_0) < \nu(\mathcal{F}')$.

Finally, the last desired $\nu(\mathcal{F}') \leq \#V(\Gamma^{\text{split}}_{\mathcal{F}'}) + \#V(\Gamma^{\text{ram}}_{\mathcal{F}'})$ in (iii) also holds by Proposition 180. ☐

The almost complete answer to Conjecture 1(iii). Finally, in the following Corollary 184, we will present our almost complete answer to Conjecture 1(iii). Here, Corollary 184 will be the first essential case of Corollary 183 and will provide the following statement for recursive towers $\mathcal{F}$ over finite fields of balanced degree: If every finite weakly connected component of the the ramification subgraph of $\mathcal{F}$ contains circles with unbalanced ramification indices, then $\mathcal{F}$ satisfies Conjecture 1(ii), i.e. $\nu(\mathcal{F}) = \operatorname{Split}(\mathcal{F}/F_0)$.

The author is only aware of the wild CNT-tower in Examples 8(v) which does not satisfy the 'if'-condition for some truncation $\operatorname{Trun}_{\geq m}(\mathcal{F})$. In fact, in Example 181 , we even showed that the CNT-tower is a counterexample to Conjecture 1(iv), i.e. does not satisfy Conjecture 1(ii) for all its truncations.

All other recursive towers $\mathcal{F}$ known to the author satisfy the 'if'-condition for some truncation $\operatorname{Trun}_{\geq m}(\mathcal{F})$ and thus also Conjecture 1(iv), which is $\nu(\mathcal{F}/F_m) = \operatorname{Split}(\mathcal{F}/F_m)$ for some $m \in \mathbb{N}_0$. As Conjecture 1(iv) implies the weaker Conjecture 1(iii), we will call Corollary 184 our almost complete answer to Conjecture 1(iii).

Then, in Corollary 185, we will also present the second essential case of Corollary 183 for tame recursive towers $\mathcal{F}$: If the ramification subgraph of $\mathcal{F}$ has some weakly connected component which only contains circles with balanced ramification indices, then $\mathcal{F}$ does not satisfy Conjecture 1(ii), i.e. $\nu(\mathcal{F}) > \operatorname{Split}(\mathcal{F}/F_0)$.

Corollary 184 (Almost Complete Answer to Conjecture 1(iii))**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field $\mathbb{F}_q$ of balanced degree d and let $\mathcal{F}' := \mathbb{F}_{q^l} \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some finite extension $\mathbb{F}_{q^l}/\mathbb{F}_q$.*

Moreover, let $\Gamma^{\text{ram}}_{\mathcal{F}}$ be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ and suppose that all finite weakly connected components of $\Gamma^{\text{ram}}_{\mathcal{F}}$ have circles with unbalanced ramification indices. Then we have the identities

$$\rho(\mathcal{F}') = 0 \text{ and } \nu(\mathcal{F}') = \#\operatorname{Split}(\mathcal{F}'/F'_0) = \#V(\Gamma^{\text{split}}_{\mathcal{F}'}).$$

In particular, this holds for $l = 1$, i.e. $\mathcal{F}' = \mathcal{F}$.

Proof. This is just a reformulation of Corollary 183(i). ☐

Corollary 185. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower over a finite field $\mathbb{F}_q$ of balanced degree d and let $\mathcal{F}' := \mathbb{F}_{q^l} \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some finite extension $\mathbb{F}_{q^l}/\mathbb{F}_q$.*

Moreover, let $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ be the ramification subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$, suppose that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ has some finite weakly connected component which only has circles with balanced ramification indices. Then there is some natural number l_1 such that we have the estimates

$$\rho(\mathcal{F}') > 0 \text{ and } \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) \leq \#\operatorname{Split}(\mathcal{F}/F_0) < \nu(\mathcal{F}') \leq \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}) + \#V(\Gamma_{\mathcal{F}'}^{\mathrm{ram}})$$

if l_1 divides l.

Proof. By the assertion that $\mathcal{F}$ is tame, we especially have that all paths in $\Gamma_{\mathcal{F}}$ are tame. Moreover, by this and by the assertion that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ has some finite weakly connected component which only has circles with balanced ramification indices, we obtain $r = s \geq 1$ in Corollary 183.

Finally, we can choose l_1 as the natural number l_1 in this corollary and derive the desired statement from Corollary 183(iii). ☐

Examples 186. *Up to the CNT-tower in Examples 8(v), the author is not aware of any recursive towers in the literature, to which Corollary 184 and Corollary 185 cannot be applied for some truncation. In particular, this holds for all examples in this thesis (see Examples 8, Figure 4.1 and Chapter B).*

For instance, Corollary 184 supplies $\rho(\mathcal{F}_{MW,2}) = 0$ (see Figure B.1) and, thus, the MW-tower $\mathcal{F}_{MW,2}$ satisfies Conjecture 1(iv).

On the other hand, Corollary 185 supplies $\rho(\mathbb{F}_{9^l} \cdot \mathcal{F}_{MW,11}) > 0$ for some $l \in \mathbb{N}$ (see Figure B.7). More concretely, we can choose $l = 1$.

Furthermore, it can easily be shown that every finite weakly connected component of the ramification subgraphs of $\operatorname{Trun}_{\geq 1}(\mathcal{F}_{MW,11})$ contains circles with unbalanced ramification indices. Thus, Corollary 184 provides that $\operatorname{Trun}_{\geq 1}(\mathcal{F}_{MW,11})$ satisfies Conjecture 1(ii) and, because of that, $\mathcal{F}_{MW,11}$ satisfies Conjecture 1(iv).

Justification of the term 'almost complete' answer to Conjecture 1(iii) and the last two missing pieces for a complete answer. In the following, we will give a justification for the term 'almost complete' answer to Conjecture 1(iii):

First, the only recursive tower known to the author for which Corollary 184 is not applicable is the wild CNT-tower in Examples 8(v). Thus, in that sense, our almost complete answer works on most recursive towers in the literature and provides that Conjecture 1(iv) is true for all these examples.

Second, although Conjecture 1(ii) is already disproven (see Example 129), most of the recursive towers in the literature satisfy this conjecture. Moreover, as we pointed out earlier, up to finite constant field extensions and up to some very specific wild recursive towers, we can interpret Corollary 183 as an almost complete characterization of the recursive towers which satisfy Conjecture 1(ii). Only some wild recursive towers are not covered by this characterization, namely the ones which have finite weakly connected components Γ of the ramification subgraphs such that all circles in Γ have balanced ramification indices.

However, these missing wild recursive towers are rare special cases. This is also partly supported by the observation that the CNT-tower in Examples 8(v) is apparently the only example of such a wild recursive tower in the literature. Here we only write 'partly supported' because there seems to be a bias in the search algorithms and constructions which disfavor the missing wild recursive towers. The only exceptions of which the author is aware are [BGS06] and [ST15]. In [BGS06], the authors classified all recursive towers of so called Kummer-type and Artin-Schreier-type and, in [ST15], the authors classified all potentially good recursive towers of degree two over $\mathbb{F}_2$ without any further restrictions.

Consequently, these rare wild recursive towers are the first missing piece of our only almost complete answer to Conjecture 1(iii). The second missing piece is a final answer to the question whether Conjecture 1(iii) is true for all recursive towers or only to most of them as we established via the main result in Corollary 184.

Is Conjecture 1(iii) true? In Corollary 184, we gave an almost complete answer to Conjecture 1(iii). This Conjecture 1(iii) was proposed in [Sti10, p. 5, Problem 1] (2010) for the first time and, in [Bee22, p. 10, p. 24] (2022), it was confirmed that it is still an open conjecture. All other conjectures in Conjecture 1(i)-(iv) are disproven.

Now, although the almost complete answer in Corollary 184 yields that Conjecture 1(iii) is true for most recursive towers, we will cast some doubts on the validity of Conjecture 1(iii) in the following.

First of all, as we mentioned in the paragraph above, there seems to be a bias in the computational searches and constructions of good recursive towers in the literature and this bias automatically rejects all possible counterexamples to Conjecture 1(iii). Indeed, by the first estimate in Lemma 180, the easiest way to ensure a positive splitting rate $\nu(\mathcal{F}) \geq \#\operatorname{Split}(\mathcal{F}/F_0)$ is to have a non-empty splitting locus $\operatorname{Split}(\mathcal{F}/F_0)$. Therefore, all computational searches (e.g. in [MW05], [Wul02], [Lö07]) are looking for non-empty splitting loci and all constructions (e.g. [BR20], [BBGS15] [Sti08]) are containing non-empty splitting loci.

However, there are two exceptions of this possible bias known to the author, namely [BGS06] and [ST15]. On the one hand, in [BGS06], the authors classified all recursive towers of Kummer-type and Artin-Schreier-type and from there the CNT-tower $\mathcal{F}_{CNT,s}$ also arose. We showed in Example 181 that all truncations of this CNT-tower have a ramification subgraph with a finite balanced weakly connected component and that it therefore is a counterexample to Conjecture 1(iv).

On the other hand, in [ST15], the authors classified all potentially good recursive towers of degree two over $\mathbb{F}_2$. More concretely, in [ST15, p. 680, Theorem 2.14], they are finally left with only four remaining potentially good recursive towers, namely the four ST-towers in Examples 8(iv). Despite the fact that we will show that the splitting rates of these towers vanish in Corollary 202, the ramification subgraph of $\mathcal{F}_{ST,4}$ depicted in Figure B.24 is still remarkably different from all other examples in the literature. It is the only example of a ramification subgraph with a finite weakly connected component Γ which is also strongly connected but has circles with unbalanced ramification indices.

Consequently, these two examples $\mathcal{F}_{CNT,s}$ and $\mathcal{F}_{ST,4}$ could indicate that there are many more different types of ramification subgraphs which do not occur in the literature because of some possible bias in the search algorithms and constructions.

A sufficient criterion to disprove Conjecture 1(iii). In the following Lemma 187, we will formulate a sufficient criterion for a tame recursive tower to be a counterexample to Conjecture 1(iii).

Note that Theorem 155 implies that the tower graph of a recursive tower over a finite field has at most one finite balanced weakly connected component. Hence, in Lemma 187, the existence of a finite weakly connected component of $\Gamma_{\mathcal{F}}$ which satisfies the second property (ii) already implies that all other finite weakly connected components must satisfy the first property (i).

Lemma 187. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower of balanced degree d over a finite field k. Suppose that every finite weakly connected component Γ of $\Gamma_{\mathcal{F}}$ satisfies one of the following two properties:*

(i) Γ has circles with unbalanced ramification indices.

(ii) All circles in Γ have balanced ramification indices and, for all vertices P in Γ and for all $m \in \mathbb{N}_0$, there is some prime q and a path $\mathcal{P} = [P_{i,j}]_{j-i\leq 1}$ of length $n \geq m+1$ in Γ which starts at $P = P_{0,0}$ such that $v_q(e(P_{n-1,n}|P_{n-1,n-1})) \geq 1$ and $v_q(e(P_{i-1,i}|\sigma(P_{i,i}))) = 0$ for all $i = 1, \dots, n$.

If there is at least one finite weakly connected component Γ *of* $\Gamma_{\mathcal{F}}$ *which satisfies the second property (ii), then there is some finite constant field extension* $\mathcal{F}' = k' \cdot \mathcal{F} = (F'_\nu)_\nu$ *which disproves Conjecture 1(iii), i.e. which satisfies* $\nu(\mathcal{F}'/F'_m) > 0$ *and* $\mathrm{Split}(\mathcal{F}'/F'_m) = \emptyset$ *for all* $m \in \mathbb{N}_0$*.*

Proof. Suppose that Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$ which satisfies the second property in (ii). Then Theorem 177(iii) provides some finite extension k'/k such that $\nu(\mathcal{F}') > 0$. On the one hand, by Lemma 120(iii) and Lemma 121, we notice that the finite weakly connected components of $\Gamma_{\mathcal{F}'}$ are the weakly connected components of the k'-constant field extensions of the finite weakly connected components of $\Gamma_{\mathcal{F}}$. On the other hand, if Γ is a finite weakly connected component as in (i) (resp. in (ii)), then, by Lemma 120(v) (resp. the Path Lifting Lemma 119), any finite weakly connected component of $k' \cdot \Gamma$ also satisfies (i) (resp. (ii)). Thus, w.l.o.g. we may assume $k = k'$ and $\mathcal{F} = \mathcal{F}'$.

Next, we have to show that the splitting locus $\mathrm{Split}(\mathcal{F}/F_m)$ is empty for all $m \in \mathbb{N}_0$. For that, let $m \in \mathbb{N}_0$ and let $\mathcal{F}_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F}) = (F_{m+\nu})_\nu$. Moreover, let $\Gamma_1, \ldots, \Gamma_s$ be the finite weakly connected components of $\Gamma_{\mathcal{F}}$ which are also subgraphs of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$, suppose that $\Gamma_1, \ldots, \Gamma_r$ satisfy (i), suppose that$\Gamma_{r+1}, \ldots, \Gamma_s$ satisfy (ii) and let $\Gamma := \Gamma_{\mathcal{F}}^{\mathrm{rat}} \setminus \coprod_{i=1}^s \Gamma_i$. Then we have

$$\begin{aligned} \mathrm{Split}(\mathcal{F}/F_m) = \mathrm{Split}(\mathcal{F}_{\geq m}/F_m) &\subseteq V(\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{rat}}) \subseteq \mathbb{P}_{F_m}(V(\Gamma_{\mathcal{F}}^{\mathrm{rat}})) \\ &\subseteq \mathbb{P}_{F_m}(V(\Gamma)) \sqcup \coprod_{i=1}^{s} \mathbb{P}_{F_m}(V(\Gamma_i)). \end{aligned} \tag{342}$$

First, we conclude

$$\begin{aligned} \#(\mathrm{Split}(\mathcal{F}/F_m) \cap \mathbb{P}_{F_m}(V(\Gamma))) &= \sum_{P \in \mathrm{Split}(\mathcal{F}/F_m) \cap \mathbb{P}_{F_m}(V(\Gamma))} \lim_{\nu \to \infty} \frac{d^{\nu-m}}{d^{\nu-m}} \\ &\leq \sum_{P \in \mathbb{P}^{(1)}_{F_m}(V(\Gamma))} \lim_{\nu \to \infty} \frac{N(F_\nu, P)}{d^{\nu-m}} = \lim_{\nu \to \infty} \frac{N(F_\nu, V(\Gamma))}{d^{\nu-m}} \\ &= \lim_{\nu \to \infty} \frac{N[F_n, \Gamma]}{d^{\nu-m}} = 0 \end{aligned} \tag{343}$$

where the equalities and estimates hold by the following reasonings: The first equality is clear. The first estimate holds since there lie d^ν rational places in $\mathbb{P}_{F_\nu}$ over $P \in \mathrm{Split}(\mathcal{F}/F_m)$ and because we added further nonnegative summands. The second equality holds by the definition of $N(F_\nu, \cdot)$ in (5). The second to last equality holds because, by the definition of Γ, it already contains all rational path which start in Γ and, thus, for all $Q \in \mathbb{P}^{(1)}_{F_\nu}(V(\Gamma))$, we have $Q \in \mathbb{P}^{(1)}_{F_\nu}[\Gamma]$. For the last equality, we first notice that, by the choice of Γ and Γ_i for all $i = 1, \ldots, s$, all the weakly connected components G of Γ are finite but none of them is a weakly connected component of $\Gamma_{\mathcal{F}}$. Thus, the first equality follows from applying Proposition 176 to all these weakly connected components G of Γ.

Second, for all $i = 1, \ldots, r$, we obtain

$$\begin{aligned} \#(\mathrm{Split}(\mathcal{F}/F_m) \cap \mathbb{P}_{F_m}(V(\Gamma_i))) &= \#(\mathrm{Split}(\mathcal{F}/F_m) \cap V(\mathrm{Trun}_{\geq m}(\Gamma_i))) \\ &\leq \lim_{\nu \to \infty} \frac{N(F_\nu, V(\mathrm{Trun}_{\geq m}(\Gamma_i)))}{d^{\nu-m}} = 0 \end{aligned} \tag{344}$$

where the equalities and estimate hold by the following reasonings: The first equality holds because combining that Γ_i is a weakly connected component of $\Gamma_{\mathcal{F}}$ and Lemma 130(ii) yields the equality $\mathbb{P}_{F_m}(V(\Gamma_i)) = V(\mathrm{Trun}_{\geq m}(\Gamma_i))$. The estimate holds by the same reasonings as the first two equalities and first estimate in (343). The last equality holds because, by the choice of Γ_i for all $i = 1, \ldots, r$, it contains circles with unbalanced ramification indices and, thus, we can apply Lemma 177(i).

Third, let $i \in \{r+1, \ldots, s\}$, let $P' \in \mathbb{P}_{F_m}(V(\Gamma_i)) = V(\mathrm{Trun}_{\geq m}(\Gamma))$, let $P := P' \cap F_0$ and, for $P \in V(\Gamma)$, choose q, $n \geq m+1$ and $\mathcal{P}$ as in (ii). Then Lemma 19 supplies some place $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cap \mathbb{P}_{F_n}(P')$. But, by the choice of q and $\mathcal{P}$, we see that iteratively applying Abhyankar's Lemma in (10) to the extensions of places in $\mathrm{Pyr}(Q)$ yields $v_q(e(Q|Q \cap F_{n-1})) \geq 1$. In particular, this means that Q/P' is ramified in F_n/F_m and, thus, $P' \notin \mathrm{Split}(\mathcal{F}/F_m)$.

All together, by this and by the equalities in (342), (343) and (344), we established that the splitting locus $\mathrm{Split}(\mathcal{F}/F_m)$ is indeed empty. ☐

A finite weakly connected component which satisfies the sufficient criterion. With Lemma 187, we now have a sufficient criterion to disprove Conjecture 1(iii). Thus, our next task is to construct or find a tame recursive tower which satisfies the two requirements in Lemma 187, i.e. every finite weakly connected component of the tower graph has to satisfy (i) or (ii) and at least one of these components has to satisfy (ii).

Let us analyze these two requirements in more depth and, for that, let $\Gamma_1, \ldots, \Gamma_r$ be the finite weakly connected components of the tame recursive tower $\mathcal{F}$. In the generic case Γ_i has circles with unbalanced ramification indices, i.e. Γ satisfies (i). Basically, the first requirement ensures that the splitting locus is empty and the second requirement ensures that the splitting rate is still positive. So, our final question becomes: Is there a tame recursive tower $\mathcal{F}$ of balanced degree d which has a finite weakly connected component Γ satisfying (ii)?

For that, let us consider the following $\{1, \ldots d\}^2$-weighted directed graph Γ_0 in Figure 6.1 where, if not labeled otherwise, the weights are equal to $(1,1)$. Here, $e_1, e_2 \in \{2, \ldots d\}$

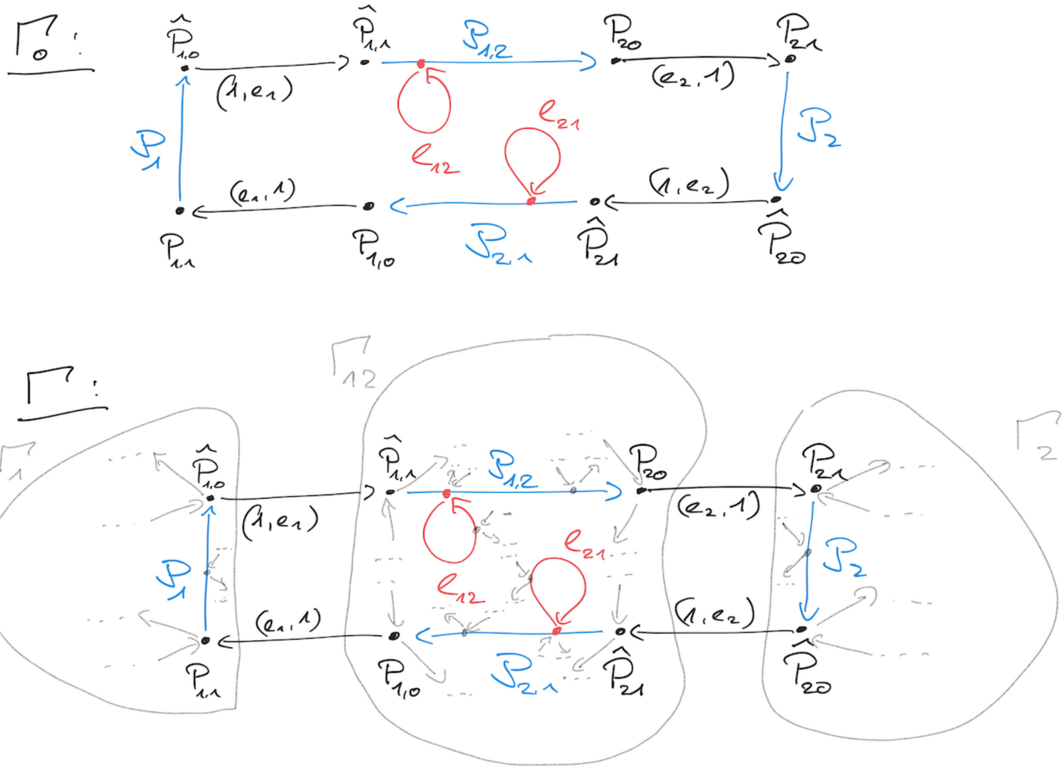


Figure 6.1: A directed graph for a counterexample to Conjecture 1(iii)

are coprime, $\mathcal{P}_i$ is a path from $P_{i,1}$ to $\hat{P}_{i,0}$ without repetitions for all $i = \{1,2\}$, $\mathcal{P}_{i,j}$ is a path from $\hat{P}_{i,1}$ to $P_{j,0}$ without repetitions for all $\{i,j\} = \{1,2\}$ and $\mathcal{C}_{i,j}$ is a circle of positive length which starts at one of the vertices in $\mathcal{P}_{i,j}$ for all $\{i,j\} = \{1,2\}$.

If we interpret the weights as the ramification indices in F_1/F_0 and $F_1/\sigma(F_0)$, respectively, this directed graph satisfies the condition in (ii).

Furthermore, any finite weakly connected component Γ of a recursive tower $\mathcal{F}$ of balanced degree d also has to satisfy the identities in Lemma 87. But there is no problem with just filling up Γ_0 with additional unramified edges to the second $\{1,\dots d\}^2$-weighted directed graph Γ in Figure 6.1 until the identities in Lemma 87 are satisfied: Here, Γ_i contains the path from $P_{i,1}$, $\Gamma_{1,2}$ contains the paths $\mathcal{P}_{i,j}$ and the circles $\mathcal{C}_{i,j}$ for all $\{i,j\} = \{1,2\}$ and Γ is a weakly connected component of $\Gamma_{\mathcal{F}}$.

Notice, that there is even a lot of flexibility in this 'filling up'-process, e.g. Γ can be arbitrarily large and Γ_1, Γ_2 and $\Gamma_{1,2}$ are far from being unique.

Consequently, if we can find a tame recursive tower of degree d such that Γ is a finite weakly connected component of $\Gamma_{\mathcal{F}}$ and all other finite weakly connected components of $\Gamma_{\mathcal{F}}$ also satisfy (i) or (ii) in Lemma 187, then $\mathcal{F}$ is a counterexample to Conjecture 1(iii).

Finally, we have to construct/find a tame recursive tower which has a subgraph Γ as in Figure 6.1. But we stop here. However, in [HP16], the authors proposed constructing recursive towers from prescribed subgraphs Γ of the tower graph and, without going into details, such an approach looks promising: Start with a parametrized defining polynomial f, reduce the parameters via coordinate transformations, deduce polynomial equations for the parametrized coefficients of f from the edges in Γ, solve the polynomial equations via Gröbner-basis methods.

6.3.2 Positive Limits Are Stable under Finite Constant Field Extensions

Summary of the results of this subsection. In [Bee04, p. 238, Corollary 5.6] and in [HP12, p. 27, Theorem 24], it was shown that the limit of a good recursive tower cannot increase after a finite constant field extension if some technical conditions are satisfied.

As the third major result of this thesis, we will show in Theorem 188 that these technical conditions can even be dropped. Consequently, the limit of a good recursive tower can never increase after a finite constant field extension.

From this result, we will derive in Corollary 207 that the CNT-towers $\mathcal{F}_{CNT,s}$ in Examples 8(v) with even s have the precise limit $\lambda(\mathcal{F}_{CNT,s}) = 1$. This is an immediate consequence of Theorem 188 and [CNT18, p. 19, Corollary 4.13, Corollary 4.14]. There $\lambda(\mathcal{F}_{CNT,2}) = 1$ and $\lambda(\mathcal{F}_{CNT,s}) \geq 1$ were shown for all even s.

Limits of good recursive towers are stable under finite constant field extensions. As the asymptotic genus $\gamma(\mathcal{F})$ is stable under constant field extensions anyways, the only critical value for the limit $\lambda(\mathcal{F}) = \nu(\mathcal{F})/\gamma(\mathcal{F})$ is the splitting rate $\nu(\mathcal{F})$. The following Theorem 188 will basically be a combination of the first two major results. First, the main result will provide that the splitting rate can only increase if there is a finite balanced weakly connected component which only has rational places lying above it. Then the first major result will supply that there is at most one such finite balanced weakly connected component. Hence, it will come out that the limit can only increase once after a finite constant field extension, namely from zero to non-zero. But as good recursive towers already have positive splitting rates, this will ensure that their limits cannot increase after finite constant field extensions at all.

Theorem 188. *Let $\mathcal{F}$ be a good recursive tower over a finite field k of balanced degree and let $\mathcal{F}' = k' \cdot \mathcal{F}$ be a finite constant field extensions of $\mathcal{F}$. Then we have*

$$\nu(\mathcal{F}') = \nu(\mathcal{F}), \qquad \gamma(\mathcal{F}') = \gamma(\mathcal{F}), \qquad \lambda(\mathcal{F}') = \lambda(\mathcal{F}).$$

Proof. Let $(F_\nu)_\nu := \mathcal{F}$ and let $(F'_\nu)_\nu := \mathcal{F}'$. First, the second desired identity $\gamma(\mathcal{F}') = \gamma(\mathcal{F})$ immediately follows from the equalities

$$\gamma(\mathcal{F}) = \lim_{\nu\to\infty} \frac{g(F_\nu)}{[F_\nu : F_0]} = \lim_{\nu\to\infty} \frac{g(F'_\nu)}{[F'_\nu : F'_0]} = \gamma(\mathcal{F}')$$

where the first and last equalities hold by the definition of $\gamma(\cdot)$ in Definition 2(iii) and the second equality holds by Lemma 21(i).

Second, the third desired identity $\lambda(\mathcal{F}') = \lambda(\mathcal{F})$ immediately follows from the first two desired identities and the definition of $\lambda(\cdot) = \nu(\cdot)/\gamma(\cdot)$ in Definition 2(iii).

Third, we will show the first desired identity $\nu(\mathcal{F}') = \nu(\mathcal{F})$. For that, let $\Gamma_1, \ldots, \Gamma_r$ be the finite weakly connected components of the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ which only contain circles with balanced ramification indices and let Γ_0 be the splitting subgraph $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ of $\Gamma_{\mathcal{F}}$. Then we have the estimate and equalities

$$\begin{aligned} 0 < \nu(\mathcal{F}) = \#V(\Gamma^{\mathrm{split}}_{\mathcal{F}}) + \rho(\mathcal{F}) &= \#V(\Gamma^{\mathrm{split}}_{\mathcal{F}}) + \sum_{i=1}^{r} \lim_{\nu\to\infty} \frac{N(F_\nu, V(\Gamma_i))}{d^\nu} \\ &= \sum_{i=0}^{r} \lim_{\nu\to\infty} \frac{N(F_\nu, V(\Gamma_i))}{d^\nu} \end{aligned} \tag{345}$$

where the estimate holds because of the assumption that $\mathcal{F}$ is a good tower, the first equality holds by Proposition 180, the second equality holds because k is a finite field and because we can therefore apply the 'moreover'-part of Theorem 182(ii) for $\mathbb{F} = k$ and the last equality holds by the choice of the $\Gamma_0 = \Gamma^{\mathrm{split}}_{\mathcal{F}}$.

Consequently, by (345), there must be at least one Γ in $\{\Gamma_0, \ldots, \Gamma_r\}$ such that

$$0 < \lim_{\nu\to\infty} \frac{N(F_\nu, V(\Gamma))}{d^\nu}.$$

Moreover, Theorem 155 provides that there is at most one finite weakly connected component of $\Gamma_{\mathcal{F}}$ which only contains circles with balanced ramification indices. But this means

$$\{\Gamma\} = \{\Gamma_0, \ldots, \Gamma_r\}. \tag{346}$$

Next, from the estimate in (346) and from applying the contraposition of Theorem 182(iii) for $\mathbb{F} = k$, we conclude that all places in $\mathbb{P}_{F_n}(V(\Gamma))$ must be rational for all $n \in \mathbb{N}_0$. In particular, since Γ is a weakly connected component and since the 'moreover'-part of Lemma 86 therefore implies the equality $\mathbb{P}_{F_n}(V(\Gamma)) = \mathbb{P}_{F_n}[\Gamma]$, all places in $\mathbb{P}_{F_n}[\Gamma]$ are rational. Thus, Lemma 114(iv) provides the equality

$$N(F_n, V(\Gamma)) = N(F'_n, V(k' \cdot \Gamma)) \tag{347}$$

for all $n \in \mathbb{N}_0$.

Finally, Theorem 182(i) supplies that $k' \cdot \Gamma$ is the disjoint union of all finite weakly connected components of $\Gamma^{\mathrm{split}}_{\mathcal{F}}$ which only contain circles with balanced ramification indices. But applying Theorem 155 to $\mathcal{F}'$ yields that $k' \cdot \Gamma$ must already be the only finite weakly connected component of $\Gamma_{\mathcal{F}'}$ which only contains circles with balanced ramification indices. Hence, we obtain the desired equality $\nu(\mathcal{F}) = \nu(\mathcal{F}')$ by the equalities

$$\nu(\mathcal{F}) = \lim_{\nu\to\infty} \frac{N(F_\nu, V(\Gamma))}{d^\nu} = \lim_{\nu\to\infty} \frac{N(F'_\nu, V(k' \cdot \Gamma))}{d^\nu} = \nu(\mathcal{F}')$$

where the first equality holds by combining (345) and (346), the second equality holds by the equality in (347) and the last equality holds by applying the same reasoning from the first equality to $\mathcal{F}'$ and its only finite balanced weakly connected component $k' \cdot \Gamma$. $\square$

6.3.3 Determining Precise Limits

Summary of the results of this subsection. In this subsection, we will prove the fourth major result of this thesis, which is Corollary 196 and Corollary 200. These will basically be sharp versions of the criterion in Theorem 4 for recursive towers, i.e. they will even yield precise limits for recursive towers and not only lower bounds. Here, Corollary 196 handles tame recursive towers and Corollary 200 handles wild recursive towers.

Moreover, these corollaries will work on all α-weakly ramified (see Definition 199) recursive towers $\mathcal{F}$ for which the finite ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ only has unbalanced weakly connected components. The author is only aware of one recursive tower to which the corollaries are not applicable for some truncation, namely the CNT-tower in Examples 8(v). In particular, all limits in Figure 4.1 except for f_{CNT} can be obtained from applying the corollaries. In the next Section 6.4.1, we will demonstrate how Corollary 196 and Corollary 200 can be applied to determine precise limits for some important recursive towers in the literature.

In this subsection, we will also make preparation for Chapter 8. There we will compute genus formulas for tame recursive towers.

The significance of determining precise limits. First, Corollary 195 and Corollary 200 settle that recursive towers which have no finite balanced weakly connected component are asymptotically bad. For instance, because of that, it will come out in Corollary 202 that the four ST-towers are bad and that, consequently, there are no good polynomial-recursive towers of degree two over $\mathbb{F}_2$. More generally, for classifying good recursive towers with certain parameters, being able to decide whether a tower is good or bad is crucial.

Second, if we know the precise limit of a tower $\mathcal{F}$ over $\mathbb{F}_q$ and not only a lower bound, we can be sure whether the tower is optimal or improves a lower bound for Ihara's constant $A(q)$. For instance, in Corollary 203, Corollary 204 and Corollary 205, it will come out that the lower bounds which were established in [BBGS15, p. 4, Theorem 1.2] for the limits of the BBGS-towers, in [BGS05, p. 161, Main Theorem] for the limits of the BezGS-towers and in [BR20, p. 2] for the limits of the BR-towers are already their precise limits.

For the BBGS-towers and BezGS, this will confirm that the lower bounds for Ihara's constant $A(q)$ which were established in [BBGS15, p. 3, Theorem 1.1] and [BGS05, p. 174, Corollary 3.4] cannot be improved further via these towers. This is of particular interest, since the BezGS-towers provide the best known lower bounds for Ihara's constant $A(q)$ for all cubic q and the BBGS-towers for all non prime q.

For the BR-towers, this will confirm that the BR-towers are not improving any lower bounds for Ihara's constant $A(q)$.

Main idea for the sharp versions. The limit $\lambda(\mathcal{F}) = \nu(\mathcal{F})/\gamma(\mathcal{F})$ consists of the splitting rate $\nu(\mathcal{F})$ and the asymptotic genus $\gamma(\mathcal{F})$. On the one hand, with our almost complete answer to Conjecture 1(iii) in Corollary 184, we already characterized the recursive towers for which $\nu(\mathcal{F}) = \#\operatorname{Split}(\mathcal{F}/F_0)$ holds. On the other hand, for determining the precise value of $\gamma(\mathcal{F})$, the main idea will be to show that the estimate in Theorem 4(ii) is even an equality for the same recursive towers for which we already concluded the equality $\nu(\mathcal{F}) = \#\operatorname{Split}(\mathcal{F}/F_0)$.

Structure of this subsection. First, in Definition 189, we will formalize the error which is introduced in the proof of the estimates for $g(F_n)$ and $\gamma(\mathcal{F})$ in Theorem 4(ii) and prove some useful properties. Then, in Proposition 192, we will write down the precise formulas for $g(F_n)$ and $\gamma(\mathcal{F})$.

Second, in Corollary 195, we will prove the sharp version of Theorem 4 for tame recursive towers.

Third, in Corollary 200, we will prove the sharp version of Theorem 4 for wild recursive towers.

Formalizing the error. In Proposition 192, it will come out that the terms $\mathcal{D}_\alpha(F_n, A)/2$ and $\delta_\alpha(\mathcal{F}, A)/2$ from the following Definition 189 are the errors of the estimates for $g(F_n)$ and $\gamma(\mathcal{F})$ in the proof of Theorem 4(ii), respectively.

Definition 189. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower. Then we define the map*

$$\mathbb{1}_{F_0} : \mathbb{P}_{F_0} \to \mathbb{R} \text{ via } P \mapsto 1.$$

More generally, let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map and let A be a finite subset of $\mathbb{P}_{F_0}$. Then we define

$$\mathcal{D}_\alpha(F_n, A) := \sum_{Q \in \mathbb{P}_{F_n}(A)} (\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)) \cdot \deg(Q)$$

for all $n \in \mathbb{N}_0$ and if the sequence $(\frac{\mathcal{D}_\alpha(F_\nu, A)}{[F_\nu : F_0]})_\nu$ converges in $\mathbb{R}$, we also define

$$\delta_\alpha(\mathcal{F}, A) := \lim_{n\to\infty} \frac{\mathcal{D}_\alpha(F_n, A)}{[F_n : F_0]}.$$

Lemma 190. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower, let A_1 and A_2 be finite disjoint subsets of $\mathbb{P}_{F_0}$ and let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map. Then we have the identity*

$$\mathcal{D}_\alpha(F_n, A_1 \sqcup A_2) = \mathcal{D}_\alpha(F_n, A_1) + \mathcal{D}_\alpha(F_n, A_2)$$

for all $n \in \mathbb{N}_0$ and if the sequences $(\frac{\mathcal{D}_\alpha(F_\nu, A_i)}{[F_\nu : F_0]})_\nu$ converge in $\mathbb{R}$ for all $i = 1, 2$, then we also have the identity

$$\delta_\alpha(\mathcal{F}, A_1 \sqcup A_2) = \delta_\alpha(\mathcal{F}, A_1) + \delta_\alpha(\mathcal{F}, A_2).$$

Proof. The first desired identity immediately follows because of the definition $\mathcal{D}_\alpha(F_n, \cdot)$ in Definition 189 and because the definition $\mathbb{P}_{F_0}(\cdot)$ in (5) implies the equality $\mathbb{P}_{F_0}(A_1 \sqcup A_2) = \mathbb{P}_{F_0}(A_1) \sqcup \mathbb{P}_{F_0}(A_2)$.

The second desired identity follows from the definition $\delta_\alpha(\mathcal{F}, \cdot)$ in Definition 189 and from the first already proven desired identity. □

The errors are invariant under constant field extensions. In the following Definition/Lemma 191, we will show that the errors $\mathcal{D}_\alpha(F_n, A)/2$ and $\delta_\alpha(\mathcal{F}, A)/2$ are invariant under constant field extensions.

Definition/Lemma 191. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the field k of degree d and let $\mathcal{F}' = k' \cdot \mathcal{F} = (F_\nu)_\nu$ be the constant field extension of $\mathcal{F}$ for some algebraic extension k'/k. Moreover, let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map such that the sequence $(\frac{\mathcal{D}_\alpha(F_\nu, A)}{[F_\nu : F_0]})_\nu$ converges in $\mathbb{R}$. Then we define the extension map*

$$k' \cdot \alpha = \alpha' : \mathbb{P}_{F_0'} \to \mathbb{R} \text{ via } P' \mapsto \alpha(P' \cap F_0).$$

and have the identities

$$k' \cdot \mathbb{1}_{F_0} = \mathbb{1}_{F_0'}, \quad \mathcal{D}_\alpha(F_n, A) = \mathcal{D}_{\alpha'}(F_n', \mathbb{P}_{F_0'}(A)), \quad \delta_\alpha(\mathcal{F}, A) = \delta_{\alpha'}(\mathcal{F}', \mathbb{P}_{F_0'}(A))$$

for all finite subsets A of $\mathbb{P}_{F_0}$ and all $n \in \mathbb{N}_0$.

Proof. The first desired identity $k' \cdot \mathbb{1}_{F_0} = \mathbb{1}_{F'_0}$ immediately follows from the definition of $k' \cdot \mathbb{1}_{F_0}$.

The second desired identity $\mathcal{D}_\alpha(F_n, A) = \mathcal{D}_{\alpha'}(F'_n, \mathbb{P}_{F'_0}(A))$ follows for all $n \in \mathbb{N}_0$ from the equalities

$$\begin{aligned}\mathcal{D}_\alpha(F_n, A) &= \sum_{Q \in \mathbb{P}_{F_n}(A)} (\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)) \cdot \deg(Q) \\ &= \sum_{Q \in \mathbb{P}_{F_n}(A)} (\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)) \cdot \sum_{Q' \in \mathbb{P}_{F'_n}(Q)} \deg(Q') \\ &= \sum_{Q' \in \mathbb{P}_{F'_n}(\mathbb{P}_{F_n}(A))} (\alpha'(Q' \cap F'_0) \cdot e(Q'|Q' \cap F'_0) - d(Q'|Q' \cap F'_0)) \cdot \deg(Q') \\ &= \sum_{Q' \in \mathbb{P}_{F'_n}(\mathbb{P}_{F'_0}(A))} (\alpha'(Q' \cap F'_0) \cdot e(Q'|Q' \cap F'_0) - d(Q'|Q' \cap F'_0)) \cdot \deg(Q') \\ &= \mathcal{D}_{\alpha'}(F'_n, \mathbb{P}_{F'_0}(A))\end{aligned}$$

where the equalities holds by the following reasonings: The first and last equalities hold by the definitions of $\mathcal{D}_\alpha(F_n, A)$ and $\mathcal{D}_{\alpha'}(F'_n, P_{F'_0}(A))$ in Definition 189. The second equality holds because [Sti08, p. 114, Theorem 3.6.3(c)] provides the equality $\sum_{Q' \in \mathbb{P}_{F'_n}(Q)} \deg(Q') = \deg(Q)$. The third equality holds by the definition of $\alpha' = k' \cdot \alpha$, by the independence of the ramification indices and different exponents under constant field extensions. The fourth equality holds because $F'_n/F_n/F_0$ and $F'_n/F'_0/F_0$ are extensions of function fields.

Finally, the last desired identity $\delta_\alpha(\mathcal{F}, A) = \delta_{\alpha'}(\mathcal{F}', \mathbb{P}_{F'_0}(A))$ immediately follows from the definitions of $\delta_\alpha(\mathcal{F}, A)$ and $\delta_{\alpha'}(\mathcal{F}', \mathbb{P}_{F'_0}(A))$ in Definition 189 and from the second already proven desired identity. □

Precise formulas for the genera. The following Proposition 192 provides precise formulas for $g(F_n)$ and $\gamma(\mathcal{F})$, i.e. it includes the errors $\mathcal{D}_\alpha(F_n, A)/2$ and $\delta_\alpha(\mathcal{F}, A)/2$ from the estimates in the proof of Theorem 4(ii).

Proposition 192. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower. Moreover, let A be a finite subset of $\mathbb{P}_{F_0}$ which contains* $\operatorname{Ram}(\mathcal{F}/F_0)$ *and let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map. Then we have the identity*

$$g(F_n) = \frac{1}{2}\left(2 + [F_n : F_0]\left((2g(F_0) - 2) + \sum_{P \in A} \alpha(P) \cdot \deg(P)\right) - \mathcal{D}_\alpha(F_n, A)\right)$$

Moreover, if the sequence $(\frac{\mathcal{D}_\alpha(F_\nu, A)}{[F_\nu : F_0]})_\nu$ converges in $\mathbb{R}$, then we have the identity

$$\gamma(\mathcal{F}) = g(F_0) - 1 + \frac{\sum_{P \in A} \alpha(P) \cdot \deg(P)}{2} - \frac{\delta_\alpha(\mathcal{F}, A)}{2}.$$

In particular, this is the case if the estimates

$$\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) \geq d(Q|Q \cap F_0)$$

hold for all $Q \in \mathbb{P}_{\mathcal{F}}(A)$.

Proof. We will modify the proof of [Sti08, p. 249, Theorem 7.2.10(b)]: Let $g_n := g(F_n)$ and $d_n := [F_n : F_0]$ for all $n \in \mathbb{N}_0$. Then we compute the equalities

$$2g_n - 2 = d_n(2g_0 - 2) + \sum_{\substack{P \in A \\ Q \in \mathbb{P}_{F_n}(P)}} d(Q|P) \cdot \deg(P)$$

$$
\begin{aligned}
&= d_n(2g_0-2) + \sum_{\substack{P\in A\\ Q\in\mathbb{P}_{F_n}(P)}} (\alpha(P)\cdot e(Q|P) - (\alpha(P)\cdot e(Q|P) - d(Q|P)))\cdot\deg(Q)\\
&= d_n(2g_0-2) + \left(\sum_{P\in A}\alpha(P)\cdot\deg(P)\sum_{Q\in\mathbb{P}_{F_n}(P)} e(Q|P)\cdot f(Q|P)\right) - \mathcal{D}_\alpha(F_n,A)\\
&= d_n\left((2g_0-2)+\sum_{P\in A}\alpha(P)\cdot\deg(P)\right) - \mathcal{D}_\alpha(F_n,A) \qquad (348)
\end{aligned}
$$

for all $n\in\mathbb{N}_0$ where the equalities hold by the following reasonings: The first equality holds because of the Hurwitz Genus Formula in (9) for F_n/F_0, because of the assertion $\mathrm{Ram}(\mathcal{F}/F_0)\subseteq A$ in Lemma 92(ii) and because [Sti08, p. 106, Corollary 3.5.5(a)] implies the equality $d(Q|P)=0$ for all $P\in\mathbb{P}_{F_0}\backslash\mathrm{Ram}(\mathcal{F}/F_0)$ and $Q\in\mathbb{P}_{F_n}(P)$.

The second equality is clear. The third equality holds by the identity $f(Q|P)\deg(P)=\deg(Q)$, by the definition of $\mathcal{D}_\alpha(F_n/F_0)$ in Definition 189 and by elementary arithmetics.

The last equality follows from the fundamental equality $\sum_{Q\in\mathbb{P}_{F_n}(P)} e(Q|P)f(Q|P) = [F_n:F_0]=d_n$ in (8).

Now, the 'main'-part follows if we add 2 and divide by 2 in the equality in (348). For the 'moreover'-part, we also divide by d_n and apply $\lim_{n\to\infty}$ and then the assertion that $(\frac{\mathcal{D}_\alpha(F_\nu,A)}{[F_\nu:F_0]})_\nu$ converges in $\mathbb{R}$ yields the desired identity in the 'moreover'-part.

For the 'in particular'-part, we notice that the sequence $(\frac{g(F_\nu)}{d_\nu})_\nu$ converges in $\mathbb{R}_{>0}$ because $\mathrm{Ram}(\mathcal{F}/F_0)$ is the subset of the finite set A and because of Theorem 4(ii). Hence, by this conclusion and the identity in (348), we also obtain the desired statement in the in particular'-part, namely that $(\frac{\mathcal{D}_\alpha(F_\nu,A)}{[F_\nu:F_0]})_\nu$ converges in $\mathbb{R}$. □

Sharp Criterion for Precise Limits of Tame Recursive Towers.

Corollary 195 of this subsubsection will be our sharp version of Theorem 4 for tame recursive towers.

The errors for tame recursive towers. For tame recursive towers, we will choose $\alpha=\mathbb{1}_{F_0}$ and then the following Lemma 193(i) will supply that the errors $\mathcal{D}_{\mathbb{1}_{F_0}}(F_n,V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ and $\delta_{\mathbb{1}_{F_0}}(\mathcal{F},V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ can be expressed in terms of $N(\overline{F}_n,V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))$.

Moreover, Lemma 193(ii) will even supply that the error $\delta_{\mathbb{1}_{F_0}}(\mathcal{F},V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ in the estimate for $\gamma(\mathcal{F})$ (see Proposition 192) vanishes if $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is finite and only has unbalanced weakly connected components Γ.

Lemma 193. *Let $\mathcal{F}=(F_\nu)_\nu$ be a tame recursive tower over the field k of degree d and let $\overline{\mathcal{F}}=\overline{k}\cdot\mathcal{F}=(\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$. Then the following hold:*

(i) For all finite subgraphs Γ of $\Gamma_{\mathcal{F}}$, we have the identities

$$\mathcal{D}_{\mathbb{1}_{F_0}}(F_n,V(\Gamma)) = \mathcal{D}_{\mathbb{1}_{\overline{F}_0}}(\overline{F}_n,V(\overline{k}\cdot\Gamma)) = N(\overline{F}_n,V(\overline{k}\cdot\Gamma))$$

for all $n\in\mathbb{N}_0$. In particular, the sequence $(\frac{\mathcal{D}_{\mathbb{1}_{F_0}}(F_\nu,A)}{d^\nu})_\nu$ converges in $\mathbb{R}$ and we have the identities

$$\delta_{\mathbb{1}_{F_0}}(\mathcal{F},V(\Gamma)) = \delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}},V(\overline{k}\cdot\Gamma)) = \lim_{n\to\infty}\frac{N(\overline{F}_n,V(\overline{k}\cdot\Gamma))}{d^n}.$$

(ii) If $\mathcal{F}$ has balanced degree d and Γ is a finite weakly connected component of $\Gamma_{\mathcal{F}}$ which has a circle with unbalanced ramification indices, then we have the identities

$$\delta_{\mathbb{1}_{F_0}}(\mathcal{F},V(\Gamma)) = \delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}},V(\overline{k}\cdot\Gamma)) = 0.$$

Proof. For (i): The third and fourth desired identities immediately follow from the first two desired identities and from the definitions of $\delta_{\mathbb{1}_{F_0}}(\mathcal{F}, V(\Gamma))$ and $\delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}}, V(\overline{k}\cdot\Gamma))$ in Definition 189.

The first desired identity $\mathcal{D}_{\mathbb{1}_{F_0}}(F_n, V(\Gamma)) = \mathcal{D}_{\mathbb{1}_{\overline{F}_0}}(\overline{F}_n, V(\overline{k}\cdot\Gamma))$ follows for all $n \in \mathbb{N}_0$ from the equality $V(\overline{k}\cdot\Gamma) = \mathbb{P}_{\overline{F}_0}(V(\Gamma))$ in Definition 107 and from the first and second identities in Definition/Lemma 191.

Finally, the second desired identity $\mathcal{D}_{\mathbb{1}_{\overline{F}_0}}(\overline{F}_n, V(\overline{k}\cdot\Gamma)) = N(\overline{F}_n, V(\overline{k}\cdot\Gamma))$ follows for all $n \in \mathbb{N}_0$ from the equalities

$$\begin{aligned}\mathcal{D}_{\mathbb{1}_{\overline{F}_0}}(\overline{F}_n, V(\overline{k}\cdot\Gamma)) &= \sum_{\overline{Q}\in\mathbb{P}_{\overline{F}_n}(V(\overline{k}\cdot\Gamma))} (e(\overline{Q}|Q\cap\overline{F}_0) - d(\overline{Q}|Q\cap\overline{F}_0))\cdot\deg(\overline{Q}) \\ &= \sum_{\overline{Q}\in\mathbb{P}^{(1)}_{\overline{F}_n}(V(\overline{k}\cdot\Gamma))} e(\overline{Q}|Q\cap\overline{F}_0) - (e(\overline{Q}|Q\cap\overline{F}_0) - 1) = N(\overline{F}_n, V(\overline{k}\cdot\Gamma))\end{aligned}$$

where the equalities holds by the following reasonings: The first equality holds by the definitions of $\mathcal{D}_{\mathbb{1}_{\overline{F}_0}}(\overline{F}_n, V(\overline{k}\cdot\Gamma))$ and $\mathbb{1}_{\overline{F}_0}$ in Definition 189. The second equality holds because combining the assertion that $\mathcal{F}$ is tame and [Sti08, p. 106, Corollary 3.5.5] supplies the equality $d(\overline{Q}|Q\cap\overline{F}_0) = e(\overline{Q}|Q\cap\overline{F}_0) - 1$ and because $\overline{F}_0$ has the algebraically closed full constant field $\overline{k}$ and, hence only rational places. The third equality holds because all summand are equal to one and because of the definition of $N(\overline{F}_n, V(\overline{k}\cdot\Gamma)) = \#\mathbb{P}^{(1)}_{\overline{F}_0}(V(\overline{k}\cdot\Gamma))$ in (5)

For (ii): The first desired identity $\delta_{\mathbb{1}_{F_0}}(\mathcal{F}, V(\Gamma)) = \delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}}, V(\overline{k}\cdot\Gamma))$ holds by the first identity in the 'in particular'-part in Lemma 193(i).

Moreover, the second desired identity $\delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}}, V(\overline{k}\cdot\Gamma)) = 0$ follows from the equalities

$$\delta_{\mathbb{1}_{\overline{F}_0}}(\overline{\mathcal{F}}, V(\overline{k}\cdot\Gamma)) = \lim_{n\to\infty}\frac{N(\overline{F}_n, V(\overline{k}\cdot\Gamma))}{d^n} = 0 \tag{349}$$

where the first equality holds by the second identity in the 'in particular'-part in Lemma 193(i) and the second equality holds by Lemma 177(i). □

Precise formulas for the genera in tame recursive towers. The following Theorem 194 provides precise formulas for $g(F_n)$ and $\gamma(\mathcal{F})$ for tame recursive towers.

Moreover, the identity for $g(F_n)$ will also be our starting point in Chapter 8. There we will find ways to compute explicit formulas for $g(F_n)$ by calculating the only unknown value $N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}))$ on the right side of this identity.

Theorem 194. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower over a field k of degree d and let $\overline{\mathcal{F}} = \overline{k}\cdot\mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$. Suppose that the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is finite and let Γ be the disjoint union of all the weakly connected components $\Gamma_1, \dots, \Gamma_r$ of $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ which only have circles with balanced ramification indices. Then we have the identity*

$$g(F_n) = \frac{1}{2}\left(2 + d^n\left((2g(F_0) - 2) + \#V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})\right) - N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}))\right).$$

Moreover, if k is a finite field and $\mathcal{F}$ has balanced degree, then we also have the identities

$$\delta_{\mathbb{1}_{F_0}}(\mathcal{F}, V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})) = \rho(\overline{\mathcal{F}}) = \lim_{n\to\infty}\frac{N(\overline{F}_n, V(\overline{\mathbb{F}}_q\cdot\Gamma))}{d^n} = \sum_{i=1}^{r}\lim_{n\to\infty}\frac{N(\overline{F}_n, V(\overline{\mathbb{F}}_q\cdot\Gamma_i))}{d^n}$$

and

$$\gamma(\mathcal{F}) = g(F_0) - 1 + \frac{\sum_{P\in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})}\deg(P)}{2} - \frac{\rho(\overline{\mathcal{F}})}{2}.$$

Proof. We already obtain the desired identity in the 'main'-part by the following equalities

$$
\begin{aligned}
g(F_n) &= \frac{1}{2}\left(2 + d^n\left((2g(F_0)-2) + \sum_{P\in V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})} \mathbb{1}_{F_0}(P)\cdot \deg(P)\right) - \mathcal{D}_{\mathbb{1}_{F_0}}(F_n, A)\right)\\
&= \frac{1}{2}\left(2 + d^n\left((2g(F_0)-2) + \sum_{P\in V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})} \deg(P)\right) - N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))\right)\\
&= \frac{1}{2}\left(2 + d^n\left((2g(F_0)-2) + \#V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})\right) - N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))\right) \qquad (350)
\end{aligned}
$$

where the equalities hold by the following reasonings: The first equality holds by applying the 'main'-part in Proposition 192 to $\mathcal{F}$, $A = V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})$ and $\alpha = \mathbb{1}_{F_0}$. The second equality holds because Lemma 193(i) and the equality $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} = \overline{k}\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ in Lemma 124 provide the equalities $\mathcal{D}_{\mathbb{1}_{F_0}}(F_n, A) = N(\overline{F}_n, V(\overline{k}\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))$ and because we have $\mathbb{1}_{F_0}(P) = 1$ for all $P \in \mathbb{P}_{F_0}$ by the definition of $\mathbb{1}_{F_0}$ in Definition 189. The third equality holds because the equality $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} = \overline{k}\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and Definition 107 supply the equality $V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}) = \mathbb{P}_{\overline{F}_n}(V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ and because the map π_2 in Lemma 114(i) then supplies the equality $\sum_{P\in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \deg(P) = \#V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})$.

For the 'moreover'-part, suppose that k is finite and that $\mathcal{F}$ has balanced degree d. The second desired identity $\delta_{\mathbb{1}_{F_0}}(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = \rho(\overline{\mathcal{F}})$ then follows from the identities in the 'in particular'-part in Lemma 193(i), from the equality $\overline{k}\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}} = \Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ in Lemma 124 and from the definition $\rho(\overline{\mathcal{F}})$ in Definition 179.

The third and fourth desired identities follow from the fact that $\overline{k}\cdot\Gamma_{\mathcal{F}}^{\mathrm{ram}} = \Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ is also finite by Lemma 120(i) and from the 'moreover'-part in Theorem 182(ii)

Finally, the last desired identity for $\gamma(\mathcal{F})$ follows from the already proven first desired identity $\delta_{\mathbb{1}_{F_0}}(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = \rho(\overline{\mathcal{F}})$ and from the identity in the 'moreover'-part of Proposition 192. □

Sharp criterion for precise limits of tame recursive towers. The following Corollary 195 and Corollary 196 of Theorem 194 will evaluate the formula for $\gamma(\mathcal{F})$ in Theorem 194 for the only two possible cases:

On the one hand, in Corollary 195, the error $\rho(\overline{\mathcal{F}})$ vanishes if $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is finite and each of its weakly connected components contains circles with unbalanced ramification indices. Hence, in this case, the estimate in Theorem 4 is already an equality.

Furthermore, as this is the generic case and as this corollary can be applied to all tame recursive towers in the literature known to the author (or at least to some of their truncations), we will call Corollary 195 the sharp criterion for precise limits of tame recursive towers.

On the other hand, in Corollary 196, we will also cover the complementary case, i.e. if the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is still finite but has some weakly connected component which only contains circles with balanced ramification indices.

Corollary 195 (Sharp Criterion for Precise Limits of Tame Recursive Towers)**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower over the finite field $\mathbb{F}_q$ of balanced degree d and let $\mathcal{F}' := \mathbb{F}_{q^l}\cdot\mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some finite extension $\mathbb{F}_{q^l}/\mathbb{F}_q$.*

Moreover, let $\Gamma_{\mathcal{F}'}^{\mathrm{split}}$ be the splitting subgraph of the tower graph $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$ and suppose that the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is finite and that all its weakly connected components have circles with unbalanced ramification indices.

Then we have the identities

$$\nu(\mathcal{F}') = \#\,\mathrm{Split}(\mathcal{F}'/F_0') = \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}}), \qquad \gamma(\mathcal{F}') = g(F_0) - 1 + \frac{\sum_{P\in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \deg(P)}{2}$$

and

$$\lambda(\mathcal{F}') = \frac{2 \cdot \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}})}{2 \cdot g(F_0) - 2 + \sum_{P \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \deg(P)}.$$

In particular, this holds for $l = 1$, i.e. $\mathcal{F}' = \mathcal{F}$.

Proof. The first two desired identities $\nu(\mathcal{F}') = \#\,\mathrm{Split}(\mathcal{F}'/F_0') = \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}})$ immediately follow from Corollary 184.

The third desired identity follows from first applying the 'moreover'-part in Theorem 182(ii) and the Main Theorem 177(i) to obtain the equality $\rho(\overline{\mathcal{F}}) = 0$ for the geometric tower $\overline{\mathcal{F}}$ of $\mathcal{F}$, from then applying Theorem 194 to obtain the desired identity for $\mathcal{F}$ and from the equality $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$.

The last desired identity follows from the first two already proven identities and from the equality $\lambda(\mathcal{F}') = \frac{\nu(\mathcal{F}')}{\gamma(\mathcal{F}')}$ in Definition 2(iii). □

Corollary 196. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower over the finite field $\mathbb{F}_q$ of balanced degree, let $\overline{\mathcal{F}} = \overline{\mathbb{F}}_q \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$ and let $\mathcal{F}' := \mathbb{F}_{q^l} \cdot \mathcal{F}$ be the constant field extension of $\mathcal{F}$ for some finite intermediate field $\mathbb{F}_{q^l}$ of $\overline{\mathbb{F}}_q/\mathbb{F}_q$.*

Moreover, suppose that the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is finite and that it has some weakly connected component which only containes circles with balanced ramification indices.

Then we have the identity and estimate

$$\left(g(F_0) - 1 + \frac{\sum_{P \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \deg(P)}{2} \right) - \gamma(\mathcal{F}') = \rho(\overline{\mathcal{F}}) > 0$$

Proof. The desired identity immediately follows from applying Theorem 194 to first obtain the desired identity for $\mathcal{F}$ and from then applying the equality $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$.

The desired estimate $\rho(\overline{\mathcal{F}}) > 0$ immediately follows from the 'in particular'-part in Theorem 182(iv). □

Sharp Criterion for Precise Limits of Wild Recursive Towers.

Corollary 200 of this subsubsection will be our sharp version of Theorem 4 for wild recursive towers.

The errors for wild recursive towers. For wild recursive towers, the situation becomes more difficult because the different exponents of the extensions Q/P can now be larger than $e(Q|P) - 1$ and, thus, we cannot any longer simply choose $\alpha = \mathbb{1}_{F_0}$ for $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ and expect that the error $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ vanishes.

However, by its definition in Definition 189 and by Definition/Lemma 191, the error $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ is still a weighted sum which runs over the places in $\mathbb{P}_{\overline{F}_n}(V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))$. For wild recursive towers, the basic idea will become to estimate the weights by a common value b_n for every $n \in \mathbb{N}_0$ and then to obtain the upper bound

$$\frac{\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{2} \leq \lim_{n \to \infty} \frac{b_n \cdot N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))}{2 \cdot d^n}. \tag{351}$$

Consequently, if we can choose α and b_n such that the product $b_n \cdot N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))$ is growing slower than d^n, then the error again vanishes. For our final sharp criterion in Corollary 200, this will exactly be the case. There we will make use of the following two observations:

On the one hand, if the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of a recursive tower $\mathcal{F}$ is finite and only has weakly connected components which contain circles with unbalanced ramification indices, then Proposition 175 supplies some $\rho < d$ such that the equality $N(\overline{F}_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = \mathcal{O}(\rho^n)$ holds as $n \to \infty$. Most wild recursive tower $\mathcal{F}$ in the literature satisfy this property for the ramification subgraph and the author is only aware of one exception, which is the CNT-tower in Examples 8(v).

On the other hand, in Definition 199, we will define α-weakly ramified recursive towers. It will come out that b_n can even be chosen as a constant in n if $\mathcal{F}$ is α-weakly ramified. All tame towers, the wild GS-tower $\mathcal{F}_{GS,3}$, the wild BezGS-towers and the wild BBGS-towers will be α-weakly ramified. In fact, the author is not aware of any recursive towers which are not α-weakly ramified.

Finally, combining these two observations yields that the error $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ in (351) indeed vanishes.

Two predecessors for the sharp criterion for precise limits of wild recursive towers. If we apply the following Theorem 197 to $A = V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$, then this captures that, in (351), the error $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))/2$ vanishes if $b_n \cdot N(\overline{F}_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ grows slower than d^n as $n \to \infty$.

Moreover, Corollary 198 will apply Proposition 175 to provide the desired $\rho \in [1, d[$ in the requirements of Theorem 197 for wild recursive towers.

Theorem 197. *Let $\mathcal{F} = (F_\nu)_\nu$ be a tower over the field k and let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$ Moreover, A be a finite subset of $\mathbb{P}_{F_0}$ and let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map.*

If there are real numbers $\rho \in [1, d[$ and $\beta \in [1, \frac{d}{\rho}[$ such that, for all $\varepsilon > 0$,

$$N(\overline{F}_n, \mathbb{P}_{\overline{F}_0}(A)) = \mathcal{O}((\rho + \varepsilon)^n)$$

and

$$\max_{Q \in \mathbb{P}_{F_n}(A)} |\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)| = \mathcal{O}((\beta + \varepsilon)^n)$$

as $n \to \infty$, then we have the identities

$$\mathcal{D}_\alpha(F_n, A) = \mathcal{O}((\rho \cdot \beta + (\rho + \beta)\varepsilon + \varepsilon^2)^n) \quad \text{and} \quad \delta_\alpha(\mathcal{F}, A) = 0$$

as $n \to \infty$.

Proof. test Suppose that there are such real numbers ρ and β and corresponding $\varepsilon > 0$. Then the first desired identity already follows from the estimates and equalities

$$\begin{aligned} |\mathcal{D}_\alpha(F_n, A)| &\leq \sum_{Q \in \mathbb{P}_{F_n}(A)} |\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)| \cdot \deg(Q) \\ &\leq \sum_{Q \in \mathbb{P}_{F_n}(A)} \deg(Q) \cdot \max_{Q \in \mathbb{P}_{F_n}(A)} |\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)| \\ &= N(\overline{F}_n, \mathbb{P}_{\overline{F}_0}(A)) \cdot \max_{Q \in \mathbb{P}_{F_n}(A)} |\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)| \\ &= \mathcal{O}((\rho + \varepsilon)^n) \cdot \mathcal{O}((\beta + \varepsilon)^n) = \mathcal{O}((\rho \cdot \beta + (\rho + \beta)\varepsilon + \varepsilon^2)^n) \end{aligned} \tag{352}$$

as $n \to \infty$ where the estimates and equalities hold by the following reasonings: The first estimate holds by the definition of $\mathcal{D}_\alpha(F_n, A) = \sum_{Q \in \mathbb{P}_{F_n}(A)} (\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0)) \cdot \deg(Q)$ in Definition 189 and by the triangle inequality. The second estimate is clear. The first equality holds by the identities

$$\sum_{Q \in \mathbb{P}_{F_n}(A)} \deg(Q) = \sum_{Q \in \mathbb{P}_{F_n}(A)} \sum_{\overline{Q} \in \mathbb{P}_{\overline{F}_n}} 1 = \#\mathbb{P}_{\overline{F}_n}(\mathbb{P}_{F_n}(A)) = \#\mathbb{P}_{\overline{F}_n}(\mathbb{P}_{\overline{F}_0}(A)) = N(\overline{F}_n, \mathbb{P}_{\overline{F}_0}(A))$$

where the first identity holds by [Sti08, p. 119, Lemma 3.6.5], the second identity holds by the definition of $\mathbb{P}_{\overline{F}_n}(\mathbb{P}_{F_n}(A))$ in (5), the third identity holds because $\overline{F}_n/F_n/F_0$ and $\overline{F}_n/\overline{F}_0/F_0$ are extensions of function fields and the last identity holds because the fact that $\overline{\mathbb{F}}_q$ is algebraically closed implies that all places in $\mathbb{P}_{\overline{F}_n}$ are rational and because of the definition of $N(\overline{F}_n, \mathbb{P}_{\overline{F}_0}(A)) = \mathbb{P}^{(1)}_{\overline{F}_n}(\mathbb{P}_{\overline{F}_0}(A))$ in (5).

The second equality in (352) holds by the choices of ρ and β in the assumptions. The last equality in (352) is a well known rule for the $\mathcal{O}$-notation.

The second desired identity $\delta_\alpha(\mathcal{F}, A) = 0$ follows from the definition of $\delta_\alpha(\mathcal{F}, A) = \lim_{n\to\infty} \mathcal{D}_\alpha(F_n, A)/d^n$ in Definition 189 and from the equality in (352). ☐

Corollary 198. *Let $\mathcal{F} = (F_\nu)_\nu$ be a possibly wild recursive tower over the finite field $\mathbb{F}_q$ of balanced degree d, let $\overline{\mathcal{F}} = \overline{\mathbb{F}}_q \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$ and let $\mathcal{F}' = \mathbb{F}_{q^l} \cdot \mathcal{F} = (F'_\nu)_\nu$ be the constant field extension of $\mathcal{F}$ for some finite intermediate field F_{q^l} of the extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$.*

Moreover, let $\Gamma^{\mathrm{split}}_{\mathcal{F}'}$ be the splitting subgraph of the tower graph $\Gamma_{\mathcal{F}'}$ of $\mathcal{F}'$ and suppose that the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is finite and that all its weakly connected components $\Gamma_1, \ldots, \Gamma_r$ have circles with unbalanced ramification indices. Finally, let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map.

On the one hand, we then have real numbers $\rho_i \in [1, d[$ for all $i = 1, \ldots, r$ such that

$$N(\overline{F}_n, V(\overline{\mathbb{F}}_q \cdot \Gamma_i)) = \mathcal{O}((\rho_i + \varepsilon)^n)$$

as $n \to \infty$ for all $\varepsilon > 0$

On the other hand, if there are also real numbers $\beta_i \in [1, \frac{d}{\rho_i}[$ for all $i = 1, \ldots, r$ such that

$$\max_{Q\in\mathbb{P}_{F_n}(V(\Gamma_i))} |\alpha(Q\cap F_0)\cdot e(Q|Q\cap F_0) - d(Q|Q\cap F_0)| = \mathcal{O}((\beta_i+\varepsilon)^n)$$

as $n \to \infty$ for all $\varepsilon > 0$, then we have the identities

$$\nu(\mathcal{F}') = \#\operatorname{Split}(\mathcal{F}'/F'_0) = \#V(\Gamma^{\mathrm{split}}_{\mathcal{F}'}), \qquad \gamma(\mathcal{F}') = g(F_0) - 1 + \frac{\sum_{P\in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})} \alpha(P)\deg(P)}{2}$$

and

$$\lambda(\mathcal{F}') = \frac{2\cdot\#V(\Gamma^{\mathrm{split}}_{\mathcal{F}'})}{2\cdot g(F_0) - 2 + \sum_{P\in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})} \alpha(P)\deg(P)}.$$

In particular, this then also holds for $l = 1$, i.e. $\mathcal{F}' = \mathcal{F}$.

Proof. For the 'on the one hand'-part: Let $j \in \{1, \ldots, r\}$. First, we notice that, by Lemma 120(i) and by the 'moreover'-parts in Lemma 120(iii) and and Lemma 120(v), the constant field extension $\overline{\mathbb{F}}_q \cdot \Gamma_j$ is a disjoint union of finitely many finite weakly connected components $\overline{\Gamma}_{j,1}, \ldots, \overline{\Gamma}_{j,s_j}$ of the tower graph $\Gamma_{\overline{\mathcal{F}}}$ of $\overline{\mathcal{F}}$ which all have circles with unbalanced ramification indices.

Then we obtain the desired real number $\rho_j \in [1, d[$ in the 'on the one hand'-part by the equalities

$$N(\overline{F}_n, V(\overline{\mathbb{F}}_q \cdot \Gamma_j)) = \sum_{i=1}^{s_j} N(\overline{F}_n, V(\overline{\Gamma}_{j,i})) = \sum_{i=1}^{s_j} N[\overline{F}_n, \overline{\Gamma}_{j,i}] = \mathcal{O}((\rho_j+\varepsilon)^n)$$

as $n \to \infty$ for all $\varepsilon > 0$ where the equalities holds by the following reasonings: The first equality holds by the definition $N(\overline{F}_n, \cdot)$ in (5) and because $\overline{\mathbb{F}}_q \cdot \Gamma_j$ is a disjoint union of the $\overline{\Gamma}_{j,1}, \ldots, \overline{\Gamma}_{j,s_j}$. The second equality holds because all $\overline{\Gamma}_{j,i}$ are weakly connected components and, thus especially, forward complete subgraphs of $\Gamma_{\overline{\mathcal{F}}}$ and because of the 'moreover'-part in Lemma 86. For the last equality, we first notice that all the $\overline{\Gamma}_{j,i}$ are finite weakly connected subgraphs of $\Gamma_{\overline{\mathcal{F}}}$ which have circles with unbalanced ramification indices and, thus, Lemma 175(ii) provides real numbers $\rho_{j,i} \in [1, d[$ such that $N[\overline{F}_n, \overline{\Gamma}_{j,i}] = \mathcal{O}((\rho_{j,i} + \varepsilon)^n)$. The last equality then follows for $\rho_j := \max_{i=1,\ldots s_j} \rho_{j,i}$.

For the 'on the other hand'-part: Suppose that there are also real number $\beta_i \in [1, \frac{d}{\rho_i}[$ as in the assumptions for all $i = 1, \ldots, r$.

Now, the first two desired identities $\nu(\mathcal{F}') = \#\operatorname{Split}(\mathcal{F}'/F_0') = \# V(\Gamma^{\mathrm{split}}_{\mathcal{F}'})$ in the 'on the other hand'-part immediately follow from Corollary 184.

The third desired identity in the 'on the other hand'-part follows from equalities

$$\begin{aligned}\gamma(\mathcal{F}') = \gamma(\mathcal{F}) = \mathrm{g}(F_0) - 1 + \frac{\sum_{P \in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})} \alpha(P) \cdot \deg(P)}{2} - \frac{\delta_\alpha(\mathcal{F}, V(\Gamma^{\mathrm{ram}}_{\mathcal{F}}))}{2} \\ = \mathrm{g}(F_0) - 1 + \frac{\sum_{P \in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})} \alpha(P) \cdot \deg(P)}{2} - \sum_{i=1}^{r} \frac{\delta_\alpha(\mathcal{F}, V(\Gamma_i))}{2} \\ = \mathrm{g}(F_0) - 1 + \frac{\sum_{P \in V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})} \alpha(P) \cdot \deg(P)}{2}\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds as the genus of a function field is invariant under constant field extensions by [Sti08, p. 114, Theorem 3.6.3(b)]. The second equality holds by Proposition 192. The third equality holds because $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is the disjoint union of its weakly connected components $\Gamma_1, \ldots, \Gamma_r$ and because of Lemma 190. The last equality holds because of the choices of the ρ_i and β_i in the assertions and because, therefore, Theorem 197 supplies $\delta_\alpha(\mathcal{F}, V(\Gamma_i)) = 0$.

The last desired identity in the 'on the other hand'-part immediately follows from the first two already proven identities and from the equality $\lambda(\mathcal{F}') = \frac{\nu(\mathcal{F}')}{\gamma(\mathcal{F}')}$ in Definition 2(iii). ☐

Sharp criterion for precise limits of wild tower. Motivated by the definition of weakly ramified recursive towers in [Sti08, p. 267, Definition 7.4.12], we will define α-weakly ramified recursive towers for maps $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ in the following Definition 199. The author is not aware of any recursive towers $\mathcal{F}$ which are not α-weakly ramified for some map $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$. For instance, we will show in the proof of Corollary 203 that the important BBGS-towers in Examples 8(vi) are α-weakly ramified.

For any wild recursive tower $\mathcal{F}$ which is α-weakly ramified and which has a finite ramification subgraph with only unbalanced weakly connected components, the following Corollary 198 will provide the precise limit $\lambda(\mathcal{F})$ of $\mathcal{F}$. The only wild recursive tower known to the author to which Corollary 198 cannot be applied is the CNT-tower in Examples 8(v).

In particular, because this corollary is easier to apply than Corollary 198, we will call Corollary 200 the sharp criterion for precise limits of wild recursive towers.

Definition 199. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over the finite field and let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map. Then $\mathcal{F}$ is called α-**bounded** if the estimate*

$$\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) \geq d(Q|Q \cap F_0)$$

*holds for all $Q \in \mathbb{P}_{\mathcal{F}}$. Moreover, $\mathcal{F}$ is called α-**weakly ramified** if the set*

$$\{\alpha(Q \cap F_0) \cdot e(Q|Q \cap F_0) - d(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}\}$$

is finite.

Corollary 200 (Sharp Criterion for Precise Limits of Wild Recursive Towers)**.** *Let $\mathcal{F} = (F_\nu)_\nu$ be a possibly wild recursive tower over the finite field $\mathbb{F}_q$ of balanced degree d and let $\mathcal{F}' = \mathbb{F}_{q^l} \cdot \mathcal{F} = (F'_\nu)_\nu$ be the constant field extension of $\mathcal{F}$ for some finite extension field $\mathbb{F}_{q^l}/\mathbb{F}_q$.*

Moreover, suppose that the ramification subgraph $\Gamma_{\mathcal{F}}^{\text{ram}}$ of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is finite and that all its weakly connected components have circles with unbalanced ramification indices. Finally, let $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ be a map such that $\mathcal{F}$ is α-weakly ramified.

Then we have the identities

$$\nu(\mathcal{F}') = \#\,\mathrm{Split}(\mathcal{F}'/F'_0) = \#V(\Gamma_{\mathcal{F}'}^{\text{split}}), \qquad \gamma(\mathcal{F}') = g(F_0) - 1 + \frac{\sum_{P \in V(\Gamma_{\mathcal{F}}^{\text{ram}})} \alpha(P)\deg(P)}{2}$$

and

$$\lambda(\mathcal{F}') = \frac{2 \cdot \#V(\Gamma_{\mathcal{F}'}^{\text{split}})}{2 \cdot g(F_0) - 2 + \sum_{P \in V(\Gamma_{\mathcal{F}}^{\text{ram}})} \alpha(P)\deg(P)}.$$

In particular, this also holds for $l = 1$, i.e. $\mathcal{F}' = \mathcal{F}$.

Proof. The assumption that $\mathcal{F}$ is α-weakly ramified immediately implies that the maximum in Corollary 198 is even bounded as $n \to \infty$. Hence, we can apply Corollary 198 to $\mathcal{F}$ for any $\rho_i \in [1, d[$. ☐

Remark 201. *In Corollary 200, we assumed that $\mathcal{F}$ has constant degree. However, if we drop this assumption and add the assumption that the splitting subgraph $\Gamma_{\mathcal{F}}^{\text{split}}$ is non-empty, then the formula for the limit $\lambda(\mathcal{F})$ of $\mathcal{F}$ in Corollary 200 still holds.*

Sketch of the Proof. Since this remark will not be used in this thesis, we will only sketch the proof.

First, we notice that there must be a level $m \in \mathbb{N}_0$ from which on the degree of $\mathcal{F}$ is constant. Thus, the level m truncation $\mathcal{F}_{\geq m} := \mathrm{Trun}_{\geq m}(\mathcal{F})$ is a recursive tower in the sense of Definition 5(ii). Moreover, we notice that all places in $A := \mathbb{P}_{F_m}(V(\Gamma_{\mathcal{F}}^{\text{split}}))$ are rational and split completely in $\mathcal{F}_{\geq m}$. Now, let us consider these rational places in A as vertices in $\Gamma_{\mathcal{F}_{\geq m}}$. because of the assumption that $\Gamma_{\mathcal{F}}^{\text{split}}$ is non-empty, it is even is a weakly connected component of $\Gamma_{\mathcal{F}}$. Consequently, the edges in $\Gamma_{\mathcal{F}_{\geq m}}$ cannot connect vertices in A with vertices outside of A. Thus, $\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\text{split}})$ is a non-empty finite balanced weakly connected component of $\Gamma_{\mathcal{F}_{\geq m}}^{\text{split}}$. But as splitting subgraphs are always balanced weakly connected components, applying Theorem 155 yields the equality

$$\mathrm{Trun}_{\geq m}(\Gamma_{\mathcal{F}}^{\text{split}}) = \Gamma_{\mathcal{F}_{\geq m}}^{\text{split}} \tag{353}$$

of non-empty graphs. Furthermore, the assumption on the finiteness of $\Gamma_{\mathcal{F}}^{\text{ram}}$ and Theorem 155 even imply that

all weakly connected components of $\Gamma_{\mathcal{F}_{\geq m}}^{\text{ram}}$ must be finite and unbalanced. (354)

In particular, the map $\alpha : \mathbb{P}_{F_0} \to \mathbb{R}$ from Corollary 200 can be used to define a map $\alpha_m : \mathbb{P}_{F_m} \to \mathbb{R}$ such that Corollary 200 can be applied to $\mathcal{F}_{\geq m}$. By doing so, we obtain the equalities

$$\lambda(\mathcal{F}) = \lambda(\mathcal{F}_{\geq m}) = \frac{\#V(\Gamma_{\mathcal{F}_{\geq m}}^{\text{split}})}{\gamma(\mathcal{F}_{\geq m})} = \frac{[F_m : F_0] \cdot \#V(\Gamma_{\mathcal{F}}^{\text{split}})}{[F_m : F_0] \cdot \gamma(\mathcal{F})} = \frac{\#V(\Gamma_{\mathcal{F}}^{\text{split}})}{\gamma(\mathcal{F})} \tag{355}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of limits in Definition/Lemma 2(iii) and the definition of truncations in Definition 27.

The second equality holds by Corollary 200. The third equality holds by the equality in (353) and by the definition of $\gamma(\mathcal{F}_{\geq m}) = \lim_{m \leq n \to \infty} g(F_n)/[F_n : F_m] = [F_m : F_0] \cdot \lambda(\mathcal{F})$ in Definition/Lemma 2(iii). The last equality holds by canceling the factors.

Second, notice that we have not applied the formula for $\gamma(\mathcal{F}_{\geq m})$ from Corollary 200 in (354). Instead, we will directly compute $\gamma(\mathcal{F})$ in this case. For that, we first notice that the proof of the genus formula in Proposition 192 does not need the assumption that $\mathcal{F}$ has constant degree. Hence, we also have the equality

$$\gamma(\mathcal{F}) = g(F_0) - 1 + \frac{\sum_{P \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \alpha(P) \cdot \deg(P)}{2} - \frac{\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{2}.$$

Finally, we have to show that $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ vanishes, since combining this and the equality in (355) yields the same limit as Corollary 200 which was the desired statement. However, we will only sketch the remaining steps: When we go through the proof of Corollary 200 and the proofs of the theorems used in this proof, we see that $\delta_\alpha(\mathcal{F}, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ vanishes if the number of places which lie over $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ in the geometric tower $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu$ are negligible in relation to the degree $[F_n : F_0]$ as $n \to \infty$, i.e.

$$\lim_{n \to \infty} \frac{N(\overline{F}_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{[F_n : F_0]} = 0.$$

Now, assume that the latter is not the case. Then the number of places in $\overline{\mathcal{F}}$ which lie over $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$ is also not negligible in relation of the degree as $n \to \infty$. But we already concluded that $\Gamma_{\mathcal{F}_{\geq m}}^{\mathrm{ram}}$ has no finite balanced weakly connected in (354) and, consequently, obtain a contradiction with Corollary 184. □

6.4 Further Improvements of Known Results

Summary of the results of this section. In this section, we will use Theorem 188 and the sharp criteria for precise limits of recursive towers in Corollary 195 and Corollary 200 to determine the precise limits for several important recursive towers in the literature. In most cases, this will also have further implications.

6.4.1 ST-Towers - Quadratic Polynomial-Recursive Towers over $\mathbb{F}_2$

As we already discussed in Examples 8(iv), in [ST15, p. 667, Theorem 1.4], the authors searched for all good polynomial-recursive towers $\mathcal{F}$ of degree two over $\mathbb{F}_2$. Up to isomorphisms, the authors showed in [ST15, p. 667, Theorem 1.4] and [ST15, p. 680, Theorem 2.14] that there are only the four ST-polynomials

$$\begin{aligned} f_{ST,1} &:= Y^2X + Y + X^2 + 1, & f_{ST,3} &:= X^2Y^2 + XY^2 + Y + X, \\ f_{ST,2} &:= X^2 + XY^2 + X + Y, & f_{ST,4} &:= X^2Y^2 + XY^2 + Y + X^2 + 1 \end{aligned} \tag{356}$$

which can potentially define a good recursive tower of degree two over $\mathbb{F}_2$.

The improvement for [ST15]. In the following Corollary 202 and for all $i = 1, \ldots, 4$, we will apply Corollary 184 to the ST-towers $\mathcal{F}_{ST,i}$ which are defined by the ST-polynomials $f_{ST,i}$ and show that the splitting rate of $\mathcal{F}_{ST,i}$ is zero. In particular, this implies that there are no good polynomial-recursive towers of degree two over $\mathbb{F}_2$.

Corollary 202. *For all $i = 1, \ldots, 4$, we have the identities $\lambda(\mathcal{F}_{ST,i}) = 0$. In particular, this means that there are no good polynomial-recursive towers of degree two over $\mathbb{F}_2$.*

Proof. In the figures B.21, B.22, B.23, B.24, the rational subgraphs of the ST-towers $\mathcal{F}_{ST,i}$ are depicted. Here, we see that the splitting subgraph of $\mathcal{F}_{ST,i}$ is empty and that its rational and ramification subgraphs are equal for all $i = 1, \ldots, 4$.

In these figures, we can also see that each of the ramification subgraphs only consist of exactly one finite weakly connected component and it contains circles with unbalanced ramification indices. Consequently, we can apply Corollary 184 to $\mathcal{F}_i$ for all $i = 1, \ldots, 4$. As the splitting subgraphs are also empty, we then obtain that sll splitting rates $\nu(\mathcal{F}_{ST,i})$ vanish. In particular, this also implies that the limit $\lambda(\mathcal{F}_{ST,i}) = \nu(\mathcal{F}_{ST,i})/\gamma(\mathcal{F}_{ST,i})$ vanishes for all $i = 1, \ldots, 4$. Hence, we established the 'main'-part.

Finally, the 'in particular'-part follows from the vanishing of the limits of all $\mathcal{F}_{ST,i}$ and from [ST15, p. 667, Theorem 1.4] and [ST15, p. 680, Theorem 2.14] (see (356)). □

6.4.2 BBGS-Towers: Ihara's Constant $A(q)$ for Non Prime q

As we already discussed in Examples 8(vi), in [BBGS15, p. 3], the authors introduced the BBGS-towers $\mathcal{F}_{BBGS,q,i,j}$ over $\mathbb{F}_{q^m}$ for all prime powers q and all $m := i + j$ with $i, j \in \mathbb{N}$ and $\gcd(i, j) = 1$. Then, in [BBGS15, p. 4, Theorem 1.2], they proved the lower bound

$$\lambda(\mathcal{F}_{BBGS,q,i,j}) \geq 2 \cdot \left(\frac{1}{q^j - 1} + \frac{1}{q^i - 1} \right)^{-1} \tag{357}$$

for the limit of $\mathcal{F}_{BBGS,q,i,j}$. By that, in [BBGS15, p. 3, Theorem 1.1], the authors provided the lower bound

$$A(p^m) \geq 2 \cdot \left(\frac{1}{p^{\lfloor m/2 \rfloor} - 1} + \frac{1}{p^{\lceil m/2 \rceil} - 1} \right)^{-1} \tag{358}$$

for all $p \in \mathbb{P}$ and all $m \geq 2$ which is currently the largest known lower bound for $A(p^m)$ with $m \geq 2$.

In Figure B.25, we depicted the rational subgraph $\Gamma_{\mathcal{F}}^{\text{rat}}$ of $\mathcal{F} = \mathcal{F}_{BBGS,q,i,j}$ for $q = 3$ and $i = j = 1$. Here, the first weakly connected component is the ramification subgraph $\Gamma_{\mathcal{F}}^{\text{ram}}$ and the second weakly connected component is the splitting subgraph $\Gamma_{\mathcal{F}}^{\text{split}}$.

The Improvement for [BBGS15]. In the following Corollary 203, we will be able to apply Corollary 200 to the BBGS-towers $\mathcal{F}_{BBGS,q,i,j}$ and show that the lower bound in (357) is already the precise limit $\lambda(\mathcal{F}_{BBGS,q,i,j})$. By that, we will ensure that the BBGS-towers do not actually provide a larger lower bound for $A(p^m)$ than the one in (358).

Corollary 203. *The BBGS-tower $\mathcal{F} = \mathcal{F}_{BBGS,q,i,j}$ in (19) has the limit*

$$\lambda(\mathcal{F}) = 2 \cdot \left(\frac{1}{q^j - 1} + \frac{1}{q^i - 1} \right)^{-1}$$

for all at least quadratic prime powers q and all $i, j \in \mathbb{N}$ with $\gcd(i, j) = 1$.

In particular, for all $p \in \mathbb{P}$ and all $m \geq 2$, the BBGS-towers do not provide a larger lower bound for $A(p^m)$ than already established in [BBGS15, p. 3, Theorem 1.1], namely

$$A(p^m) \geq 2 \cdot \left(\frac{1}{p^{\lfloor m/2 \rfloor} - 1} + \frac{1}{p^{\lceil m/2 \rceil} - 1} \right)^{-1}.$$

Proof. Let $\mathcal{F} = (F_\nu)_\nu = (\mathbb{F}_{q^m}(x_0, \ldots, x_\nu))_\nu$ where the sequence of elements x_n comes from the definition of a polynomial recursive tower in Definition 5(i). Then we first notice that [BBGS15, p. 7, Proposition 2.5] and [BBGS15, p. 8, Proposition 2.8] imply that

$$\text{the vertex set of } \Gamma_{\mathcal{F}}^{\text{split}} \text{ consists of the } q^m - 1 \text{ places } (x_0 = \beta) \text{ with } \beta \in \mathbb{F}_{q^m}^*, \tag{359}$$

that $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a weakly connected component of $\Gamma_{\mathcal{F}}$ with the only two vertices $P_0 := (x_0 = 0)$ and $P_\infty := (x_0 = \infty)$. Moreover, there it was is also deduced that any edge with initial vertex P_0 is a loop which is ramified in F_1/F_0 but not in $F_1/\sigma(F_0)$. In particular, this means that

$$\begin{aligned}&\Gamma_{\mathcal{F}}^{\mathrm{ram}} \text{ is a finite weakly connected component of } \Gamma_{\mathcal{F}}\\ &\text{containing circles with unbalanced ramification indices.}\end{aligned} \tag{360}$$

Second, let

$$b_\infty := \frac{q^m - 1}{q^j - 1} + 1 \text{ and } b_0 := \frac{q^m - 1}{q^i - 1} + 1 \tag{361}$$

Then it was shown in [BBGS15, p. 14] that P_i is b_i-bounded in $\mathcal{F}$ for all $i = 0, \infty$, i.e. we have the estimates

$$d(P'|P_i) \leq b_i \cdot (e(P'|P_i) - 1) \tag{362}$$

for all $n \in \mathbb{N}_0$, all $P' \in \mathbb{P}_{F_n}(P_i)$ and all $i = 0, \infty$. By that, the authors in [BBGS15] ensured that the asymptotic genus $\gamma(\mathcal{F})$ is finite.

Even better for us, the authors actually computed the errors $b_i \cdot (e(P'|P_i) - 1) - d(P'|P_i)$ of these estimates: On the one hand, in [BBGS15, p. 14], it was concluded (even if it is not explicitly formulated) that the estimate in (362) is actually an equality for $i = \infty$, i.e. we have the equality

$$d(P'|P_\infty) = b_\infty \cdot e(P'|P_\infty) - b_\infty \tag{363}$$

for all $n \in \mathbb{N}_0$ and all $P' \in \mathbb{P}_{F_n}(P_\infty)$.

On the other hand, in [BBGS15, p. 21] and [BBGS15, p. 22], the authors computed

$$d(P'|P_0) = b_0 \cdot e(P'|P_0) - \frac{q^m - 1}{q - 1} \tag{364}$$

for all $n \in \mathbb{N}_0$ and all $P' \in \mathbb{P}_{F_n}(P_0)$.

Combining the conclusions in (359), (360), (361), (363) and (364) supplies that $\mathcal{F}$ is α-weakly ramified for

$$\alpha : \mathbb{P}_{F_0} \to \mathbb{R} \text{ via } \alpha(P) := 0,\ \alpha(P_0) = b_0,\ \alpha(P_\infty) = b_\infty$$

Thus, we can apply the 'in particular'-part in Corollary 200 to $\mathcal{F}$ and obtain the desired identity

$$\lambda(\mathcal{F}) = \frac{2 \cdot \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}})}{2g(F_0) - 2 + \sum\limits_{P \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \alpha(P) \deg(P)} = \frac{2 \cdot (q^m - 1)}{b_0 + b_\infty - 2} = 2 \cdot \left(\frac{1}{q^j - 1} + \frac{1}{q^i - 1} \right)^{-1}.$$

Note that, we could only apply Corollary 200 directly because, in [CCH21, p. 3, Main Theorem] and [CCH21, p. 10, Lemma 2.4] , it was proven that the BBGS towers have constant degree. Alternatively, we could also have applied Remark 201 which sketched the proof of how Corollary 200 can be extended to recursive towers with non-constant degree and non-empty splitting locus.

Finally, the 'in particular'-part immediately follows from applying the 'main'-part to $q = p^m$, $i = \lfloor m/2 \rfloor$ and $j = \lceil m/2 \rceil$. □

6.4.3 BezGS-Towers: Ihara's Constant $A(q)$ for Cubic q

The Improvement for [BGS05]. In complete analogy to Corollary 203, we will also show that the lower bounds for the BezGS-towers which were established in [BGS05, p. 161, Main Theorem] are already the precise limits. Together with the BBGS-towers, the BezGS-towers provide the best known lower bound for Ihara's constant $A(l^3)$ with prime powers l.

Corollary 204. *The BezGS-tower* $\mathcal{F} = \mathcal{F}_{BezGS,l}$ *has the limit*

$$\lambda(\mathcal{F}) = \frac{2(l^2-1)}{l+2}$$

for all $q = l^3$ *with a prime power* l.

In particular, for all prime powers l*, the BezGS-towers do not provide a larger lower bound for* $A(l^3)$ *than already established in [BGS05, p. 161, Main Theorem], which is*

$$A(l^3) \geq \frac{2(l^2-1)}{l+2}.$$

Proof. This proof is completely analogous to the proof of Corollary 203. Thus, we only list the conclusions and provide the references from which the conclusions can be derived.

First, from [BS07, p. 157, Theorem 3], it can be concluded that the splitting subgraph $\Gamma_{\mathcal{F}}^{\text{split}}$ is non-empty and that its vertex set $V(\Gamma_{\mathcal{F}}^{\text{split}})$ contains exactly $l(l+1)$ many vertices.

Second, from [BS07, p. 160-, Lemma 6] [BS07, p. 162, Figure 7], we can deduce that the ramification subgraph is finite and its vertex set $V(\Gamma_{\mathcal{F}}^{\text{ram}})$ is equal to the set $V(\mathcal{F}/F_0)$ in the notation from there. Moreover, it satisfies $\#V(\Gamma_{\mathcal{F}}) = l+3$ and $P_0, P_\infty \in V(\Gamma_{\mathcal{F}}^{\text{ram}})$ where P_i denote the places at i for all $i = 0, \infty$. In particular, it also comes out that $\Gamma_{\mathcal{F}}^{\text{ram}}$ is already a finite weakly connected component of $\Gamma_{\mathcal{F}}$ which contains circles with unbalanced ramification indices. Also see Figure B.20 for the splitting subgraph and ramification subgraph of the GV-tower, which is the BezGS-tower for $l = 2$.

Third, from [BS07, p. 160-168, Figure 5, 6, 7, Proof of Corollary 15], we can also derive that $\mathcal{F}$ is α-weakly ramified for the map

$$\alpha : \mathbb{P}_{F_0} \to \mathbb{R}, \quad P \mapsto \begin{cases} 2 & P = P_\infty \\ \frac{l}{l-1} & P \in V(\Gamma_{\mathcal{F}}^{\text{ram}}) \backslash \{P_\infty\} \\ 0 & P \in P \in \mathbb{P}_{F_0} \backslash V(\Gamma_{\mathcal{F}}^{\text{ram}}) \end{cases}.$$

Finally, by these first three conclusions, we can apply Corollary 200 and obtain the desired identity

$$\lambda(\mathcal{F}) = \frac{2 \cdot \#V(\Gamma_{\mathcal{F}}^{\text{split}})}{2 \cdot g(F_0) - 2 + \sum_{P \in V(\Gamma_{\mathcal{F}}^{\text{ram}})} \alpha(P) \cdot \deg(P)} = \frac{2l(l+1)}{\frac{l}{l-1}(l+2)} = \frac{2(l^2-1)}{l+2}.$$

□

6.4.4 BR-Towers - Good Recursive Towers over Prime Fields

In this subsection, we will show that Corollary 195 can be applied to the **BR-towers** which are the recursive tower $\mathcal{T}$ in [BR20, p. 4, Theorem 2.3]. This will then provide that the estimate

$$\lambda(\mathcal{T}) \geq \frac{2}{q-2} \tag{365}$$

in [BR20, p. 4, Theorem 2.3] is even an identity for all $q = p^s \geq 5$ with $p \in \mathbb{P}$ and $s \in \mathbb{N}$. Here, for $q \in \mathbb{P}$ with $q \notin \{2, 3\}$, these BR-towers $\mathcal{T}$ are the first recursive towers over prime fields.

Translating $\mathcal{T}$ into a recursive tower $\mathcal{F}$ of function fields. The tower $\mathcal{T}$ is not a recursive tower of function fields but a recursive tower of integral curves over $k := \mathbb{F}_q$ for all $q = p^s \geq 5$ with $p \in \mathbb{P}$ and $s \in \mathbb{N}$. Thus, we will first translate this recursive tower $\mathcal{T}$ of curves into a recursive tower of function fields $\mathcal{F}$. For that, let us consider the construction of $\mathcal{T}$ in [BR20]:

In the third to last and second to last paragraphs of the introduction in [BR20, p. 2], the authors use two suitable finite morphisms $g, h : C_1 := \mathbb{P}^1 \to C_0 := \mathbb{P}^1$ to construct $\mathcal{T} = (C_\nu)_{\nu \geq 1}$ over $k = \mathbb{F}_q$ for all $q = p^s \geq 5$ with $p \in \mathbb{P}$ and $s \in \mathbb{N}$. Notice that the indices in [BR20] and here differ by one, e.g. C_{-1} in [BR20] is denoted by C_0 here, and that (f, g) in [BR20] are denoted by (h, g) here.

First, by taking the function fields of the involved curves, this construction translates into a construction of recursive towers of function fields $\mathcal{F}_{\geq 1} = (F_{1+\nu})_\nu = (k(x_1, \ldots, x_\nu))_\nu$ which is defined by the equation $g(Y) = h(X)$. Let σ be the tower map of $\mathcal{F}_{\geq 1}$ in Lemma 7.

Second, for simplicity and convenience, let us add a zeroth level F_0 to $\mathcal{F}_{\geq 1}$ in the following way: We notice that, by the construction of g and h in the second to last paragraph of the introduction in [BR20, p. 2], there is a place in $F_1 = k(x_1)$ which totally ramifies in one of the extensions $k(x_1)/k(g(x_1))$ and $k(x_1)/k(h(x_1))$ but is unramified in the other. Then applying the Reduction Lemma 30(iii) supplies that we can add the zeroth level $F_0 := k(h(x_1))$ to $\mathcal{F}_{\geq 1}$. Let $\mathcal{F} := (F_\nu)_\nu$ be the corresponding recursive tower and also notice that we have the equalities $\sigma(h(x_1)) = h(x_2) = g(x_1)$.

For $q = 5$, the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ and ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of $\mathcal{F}$ are depicted in Figure B.28.

The improvement for [BR20]. Now, we will be able to apply Corollary 195 to $\mathcal{F}$ and, by that, prove that the estimate in [BR20, p. 4, Theorem 2.3] and (365) is even an identity.

Corollary 205. *The estimate in [BR20, p. 4, Theorem 2.3] is even an identity, i.e. we have the identities* $\lambda(\mathcal{T}) = \lambda(\mathcal{F}) = \frac{2}{q-2}$.

Proof. In order to apply Corollary 195 to $\mathcal{F}$, we first need to identify the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ and ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of $\mathcal{F}$. By the construction in the third to last and second to last paragraphs of the introduction in [BR20, p. 2], they are of the following form:

On the one hand, the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ consists of a single vertex which is the place at infinity of $F_0 = k(h(x_1))$ and of the $q + 1$ edges which are all the rational places of the rational function field $F_1 = k(x_1)$.

On the other hand, by the construction of g and h in the third to last and second to last paragraphs of the introduction in [BR20, p. 2], the minimal subgraph Γ which contains all of the degree two edges is a forward and backward complete subgraph. Note that since $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ contains all degree one edges, we obtain that the edges of Γ are exactly the places of degree two in F_1. In particular, the degree two subgraph of $\Gamma_{\mathcal{F}}$ must be the disjoint union of $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ and Γ.

Moreover, the fact that F_1 has genus zero supplies that all ramified places have at most degree two. But since the edges of degree one in $\Gamma_{\mathcal{F}}$ are exactly the edges of $\Gamma_{\mathcal{F}}^{\mathrm{split}}$, all ramified edges have degree two. But the construction of g and h also implies that all ramified edges are even totally ramified. Combining these conclusions yields that there are exactly two ramified edges in $\Gamma_{\mathcal{F}}$: Both of degree two, one totally ramified in F_1/F_0 and one totally ramified in $F_1/\sigma(F_0)$ and, thus, both contained in Γ. Hence, the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a forward and backward complete subgraph of Γ and the difference graph $\Gamma \backslash \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a union of finite weakly connected components of $\Gamma_{\mathcal{F}}$ which contain no ramified edges and, thus, are balanced. But as $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ is already a finite balanced weakly connected component of $\Gamma_{\mathcal{F}}$, Theorem 155 provides that the difference graph must be empty and, by that, $\Gamma = \Gamma_{\mathcal{F}}^{\mathrm{ram}}$. Even more, we conclude that all weakly connected component of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ must be unbalanced.

One a side note, we note that, by the identity in Lemma 87 and by summing up the out-degrees of the vertices, any finite weakly connected component of $\Gamma_{\mathcal{F}}$ contains an edge which is ramified in the F_1/F_0 if and only if it also contains an edge which is ramified in $F_1/\sigma(F_0)$. This implies that both ramified edges are contained in the same weakly connected component of Γ. Consequently, this component must already be the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of $\mathcal{F}$. However, this fact will not be relevant in the following.

Finally, we obtain the desired identities by the equalities

$$\lambda(\mathcal{T}) = \lambda(\mathcal{F}) = \frac{2 \cdot \#V(\Gamma_{\mathcal{F}'}^{\mathrm{split}})}{2 \cdot g(F_0) - 2 + \sum\limits_{P \in V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})} \deg(P)} = \frac{2}{\#h(\mathbb{P}^1(\mathbb{F}_{q^2}) \backslash \mathbb{P}^1(\mathbb{F}_q)) - 2} = \frac{2}{q-2}$$

where the equalities hold by the following reasonings: The first equality holds by the construction of $\mathcal{F}$. The second equality holds because $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ does not contain finite balanced weakly connected components and because we can therefore apply the 'in particular'-part in Corollary 195. The third equality holds by the equality $\Gamma = \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ from above, by the choice of Γ with $E(\Gamma) = \mathbb{P}_{F_1}^{(2)}$ and by the construction of h in the third to last and second to last paragraphs of the introduction in [BR20, p. 2]. The fourth equality holds because $\mathbb{P}^1(\mathbb{F}_{q^2}) \backslash \mathbb{P}^1(\mathbb{F}_q)$ has $(q^2+1)-(q+1) = q^2-q$ many elements, because h is totally ramified at two of these elements and splits completely at all other $q^2-q-2 = (q-2)(q+1)$ elements and because this implies the equalities $\#h(\mathbb{P}^1(\mathbb{F}_{q^2}) \backslash \mathbb{P}^1(\mathbb{F}_q)) = 2 + (q^2-q-2)(q+1)^{-1} = 2 + (q-2) = q$. □

6.4.5 A Recipe to Determine Limits of Tame Recursive Towers

The sharp criterion for precise limits of tame recursive towers in Corollary 195 provides the following recipe to determine limits of tame recursive towers $\mathcal{F}$ which, to the best knowledge of the author, works on all tame recursive towers in the literature (e.g [MW05], [Sti08], [BR20]):

First, we have to determine whether there is some level $l \in \mathbb{N}_0$ such that we can apply Corollary 195 to $\mathcal{F}_{\geq l} := \mathrm{Trun}_{\geq l}(\mathcal{F})$, i.e. whether there is an l such that $\Gamma_{\mathcal{F}_{\geq l}}^{\mathrm{ram}}$ is finite and each of its weakly connected components has circles with unbalanced ramification indices. For apparently all tame recursive towers in the literature, this is indeed the case. Second, we have to identify the splitting subgraph $\Gamma_{\mathcal{F}_{\geq l}}^{\mathrm{split}}$ and the ramification subgraph $\Gamma_{\mathcal{F}_{\geq l}}^{\mathrm{ram}}$ of $\mathcal{F}_{\geq l}$. Third and finally, Corollary 195 yields the desired limit of $\mathcal{F}_{\geq l}$ and, by that, also the limit of $\mathcal{F}$. Let us demonstrate this recipe on the following Example 206.

Example 206. *Let us consider the tame recursive MW-tower $\mathcal{F}_{MW,12} = \mathcal{F} = (F_\nu)_\nu$ over $\mathbb{F}_{25}$ in [MW05, p. 213, f_{12}] which is defined by the polynomial $f_{MW,12} = X^2Y^2 + (X^2 + 3X + 3)Y + 4$. Let σ be a tower map of $\mathcal{F}$ from Lemma 7(iii). In Figure B.9, the degree one subgraph of $\mathcal{F}$ is depicted. Let $\Gamma_1, \ldots, \Gamma_4$ be the depicted weakly connected components of $\Gamma_{\mathcal{F}}^{\mathrm{rat}}$ indexed from left to right.*

By the identities in Lemma 87, Γ_1 and Γ_4 are even finite weakly connected components of $\Gamma_{\mathcal{F}}$ and, moreover, because Γ_4 is $(d =)2$-regular, it is the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$ of $\Gamma_{\mathcal{F}}$.

Now, since $f_{MW,12}$ has degree two in both variables, Riemann's Inequality in [Sti08, p. 148, Corollary 3.11.4] provides the estimate $g(F_1) \leq 1$. Then combining this estimate and the Hurwitz Genus Formula in (9) yields that there are at most four places in F_1 which are ramified in F_1/F_0 (resp. $F_1/\sigma(F_0)$). Consequently, there are at most eight ramified edges in the tower graph of $\mathcal{F}$. More precisely, as we can see in figure B.9, there are *exactly eight ramified edges contained in the depicted degree one subgraph. In particular, this also implies*

$$g(F_1) = 1. \tag{366}$$

Thus, we conclude that the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *of* $\Gamma_{\mathcal{F}}$ *is the disjoint union of* Γ_1 *and the smallest forward and backward complete subgraph* Γ *of* $\Gamma_{\mathcal{F}}$ *which contains* Γ_2 *and* Γ_3*. In particular, we notice that all circles in* Γ *must have balanced ramification indices and, thus, we cannot directly apply Corollary 195 to* $\mathcal{F}$*.*

However, if we consider the paths of length two in Γ_2 *and* Γ_3 *and apply Abhyankar's Lemma in (10), we notice that no place* $Q \in \mathbb{P}_{F_2}(V(\Gamma))$ *is ramified in* F_2/F_1 *and* $F_2/\sigma(F_1)$*. Consequently, on the one hand, Lemma 144 supplies that the ramification subgraph* $\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{ram}}$ *of* $\mathcal{F}_{\geq 1} = \mathrm{Trun}_{\geq 1}(\mathcal{F})$ *is equal to the finite weakly connected component*

$$\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{ram}} = \mathrm{Trun}_{\geq 1}(\Gamma_1) \tag{367}$$

of the tower graph $\Gamma_{\mathcal{F}_{\geq 1}}$ *of* $\mathcal{F}_{\geq 1}$*. On the other hand, since* $\mathbb{F}_{25}$ *is finite, the splitting subgraph* $\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{split}}$ *of* $\mathcal{F}_{\geq 1}$ *is also finite. In particular, combining Lemma 142 and Theorem 155 yields the equality*

$$\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{split}} = \mathrm{Trun}_{\geq 1}(\Gamma_4). \tag{368}$$

Thus, we can apply Corollary 195 to $\mathcal{F}_{\geq 1}$ *and obtain the limit*

$$\begin{aligned}\lambda(\mathcal{F}) &= \lim_{\nu \to \infty} \frac{N(F_\nu)}{g(F_\nu)} = \lim_{1 \leq \nu \to \infty} \frac{N(F_\nu)}{g(F_\nu)} = \lambda(\mathcal{F}_{\geq 1}) = \frac{2 \cdot \# V(\Gamma_{\mathcal{F}_{\geq 1}})}{2 \cdot g(F_1) - 2 + \sum\limits_{P \in V(\Gamma_{\mathcal{F}_{\geq 1}}^{\mathrm{ram}})} \deg(P)} \\ &= \frac{2 \cdot \# E(\Gamma_4)}{\sum\limits_{P \in E(\Gamma_1)} \deg(P)} = 4 = \sqrt{25} - 1\end{aligned} \tag{369}$$

where the equalities hold by the following reasonings: The first three equalities hold because of the definition of limits of towers in Definition 2(iii) and because finitely many elements in a sequence do not matter for its limit. The fourth equality holds by applying Corollary 195 to $\mathcal{F}_{\geq 1}$*. The fifth equality holds because of the equality* $g(F_1) = 1$ *in (366) and because of the equalities in (367) and (368) and because the vertices in* $\Gamma_{\mathcal{F}_{\geq 1}}$ *are the edges in* $\Gamma_{\mathcal{F}}$*.*

Thus, this tower $\mathcal{F}$ *is not only asymptotically good, as it was already shown in [MW05, p. 213,* f_{12}*] with the lower bound* $3 \leq \lambda(\mathcal{F})$*, but it is even optimal.*

Note that since the tower $\mathcal{F}_{\geq 1}$ *is already optimal, it would already have been sufficient to apply Theorem 4(iii) instead of Corollary 195.*

A list of limits for some recursive tame towers. As already pointed out, the recipe from above works on all tame recursive towers in the literature which are known to the author. Only sometimes, there are slight modifications necessary in the arguments inside of the three steps of the recipe. For instance, if we have a tower $\mathcal{F}$ over $\mathbb{F}_q$ which is parametrized by infinitely many q, it is of course not possible to just print all tower graphs. But then we can go on as in the proof of Theorem 205 where we determined the precise limit of the tame BR-tower for all $q \notin \{2, 3\}$. Correspondingly, one can show by this strategy that the lower bound $2/(\#\Lambda - 2)$ for the limit of a tame recursive tower $\mathcal{F}$ which satisfies the conditions of [Sti08, p. 256, Theorem 7.3.1] is already equal to its limit. In particular, this implies that the lower bounds $2/(l - 1)$ and $2/(q - 1)$ for the respective limits of the tame recursive towers $\mathcal{F}_{GS,1}$ in [Sti08, p. 262, Proposition 7.3.2] and $\mathcal{F}_{GS,2}$ in [Sti08, p. 261, Proposition 7.3.3] (see Examples 8(ii)) are already equal to their limits.

Finally, in Figure 4.1, we already included the precise limits $\lambda(\mathcal{F})$ for all recursive towers in Examples 8. For all these examples of tame recursive towers, the precise limits can be determined by the above recipe.

In particular, for $f_{MW,i}$ with $i = 8, 12, 14, 15, 16, 20, 21$, the authors of [MW05, p. 213] explicitly raised the question whether their computed lower bounds b were already the precise limits.

6.4.6 CNT-tower.

In [CNT18, p. 19, Corollary 4.13, Corollary 4.14], it was established that the CNT-tower $\mathcal{F}_{CNT,2}$ in Examples 8(v) has limit $\lambda(\mathcal{F}_{CNT,2}) = 1$ and that the other CNT-towers $\mathcal{F}_{CNT,s}$ satisfy $\lambda(\mathcal{F}_{CNT,s}) = 0$ for odd s and $\lambda(\mathcal{F}_{CNT,s}) \geq 1$ for even s.

But since we established in the third major result, which is Theorem 188, that limits of good towers are stable under finite constant field extensions, we conclude that the equality $\lambda(\mathcal{F}_{CNT,s}) = 1$ must hold for all even s.

Corollary 207. *We have* $\lambda(\mathcal{F}_{CNT,s}) = 1$ *for all even* s *and* $\lambda(\mathcal{F}_{CNT,s}) = 0$ *for all odd* s.

Proof. This follows immediately from [CNT18, p. 19, Corollary 4.13, Corollary 4.14] and Theorem 188. □

7 Proofs of the Three Interim Results

Summary of the results of this chapter. In this chapter, we will prove the three interim results from Subsection 6.2.1, which are Theorem 168, Corollary 170 and Corollary 171. These three interim results form the core of the proof of the Main Theorem 177.

Furthermore, in this chapter, we will prove Theorem 218 as a part of the proof of the first interim result. Here, Theorem 218 is a further result of this thesis on itself and provides a new interpretation of the characteristic polynomials of adjacency matrices in terms of the circles in the directed graph.

Finally, we will also prove Theorem 225 as a part of the proof of the third interim result. This Theorem 225 will provide even more than just what is needed to prove the third interim result and, therefore, should also be seen as a further result of this thesis.

More concretely, for any recursive tower $\mathcal{F}$ over a finite field and any place $Q \in \mathbb{P}_{\mathcal{F}}$, Theorem 225 will provide upper bounds C_Q for the degree $\deg(Q)$ of Q which can be expressed entirely in terms of the degree d of $\mathcal{F}$, of the ramification indices of the extensions in $\mathrm{Pyr}(Q)$ and the degrees of the places in $\mathrm{Path}(Q)$.

Structure of this chapter. In Section 7.1, we will prove the first interim result, which is Theorem 168. In Section 7.2, we will prove the second interim result, which is Corollary 170. In Section 7.3, we will prove the third interim result, which is Corollary 171.

7.1 First Interim Result - Key Lemma II and Applications

Summary of the results of this section. In this section, we will prove the first interim result in Theorem 168.

Main idea of the proof. The main idea of the proof of the first interim result in Theorem 168 is captured in Key Lemma 211: We will deduce that the function of spectral radii $\rho(A(x^{\log_2(\mathbf{P})}))$ in x on the interval $[1,2]$ can even be extended to a holomorphic function in a neighborhood of $[1,2]$. Then the maximum modulus principle will provide the two cases $\rho(A(x^{\log_2(\mathbf{P})})) = d$ and $\rho(A(x^{\log_2(\mathbf{P})})) < d$ on $]1,2[$.

Finally, we will also have to characterize these two cases in terms of the weights $w(\mathcal{C})$ of the circles $\mathcal{C}$ in Γ.

Structure of this section. In Subsection 7.1.1, we will define all notions which will appear in Key Lemma 211 and make some preparations for the proof of Key Lemma 211.

In Subsection 7.1.2, we will formulate and prove Key Lemma 211 which is one of the keys to the proof of our Main Theorem 177.

In Subsection 7.1.3, we will use Key Lemma 211 to characterize the two possible cases for the function of spectral radii $\rho(A(x^{\log_2(\mathbf{P})})) : [1,2] \to \mathbb{R}_{\geq 0}$ of an irreducible matrix A with entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ (see Lemma 214).

In Subsection 7.1.4, we will extend the insights of Subsection 7.1.3 to all connected matrices A with entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ (see Lemma 216).

In Subsection 7.1.5, for all weight functions $w : E(\Gamma) \to \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ such that the image of w is contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$ and all w-adjacency matrices A of Γ, we will translate the characterization of the two possible cases for the function of spectral radii $\rho(A(x^{\log_2(\mathbf{P})})) : [1, 2] \to \mathbb{R}_{\geq 0}$ from Subsection 7.1.4 into terms of circles in the directed graph Γ (see Corollary 219).

In Subsection 7.1.6, we will then finally prove our first interim result, which is Theorem 168.

7.1.1 Some Prerequisites

Purpose of this subsection. In this Subsection, we will define all notions which will appear in Key Lemma 211 and make some preparations for its proof.

A group algebra over the complex numbers. For the proof of the first interim result in Theorem 168, we will interpret the entries in $A(x^{\log_2(\mathbf{P})})$ as functions $[1, 2] \to \mathbb{C}$ in x and even as holomorphic functions in open neighborhoods of $[1, 2]$.

Here, analogously to polynomial functions and formal polynomials, we will replace the input variable x of these functions $h(x) : [1, 2] \to \mathbb{C}$ with a formal symbol y and distinguish between the functions $h(x)$ and the formal elements $h(y)$ in y. As for polynomials, this formality will be helpful to provide clarity.

Consequently, in the following Definition 208, we will introduce the group algebra $\mathcal{B}$ over $\mathbb{C}$ which contains these elements $h(y)$ and, in Definition/Lemma 209, we will relate the functions $h(x)$ and the elements $h(y)$ via the evaluation morphism $\sigma_x : \mathcal{B} \to \mathbb{C}$

Definition 208. *Let $\mathcal{B}$ be the group $\mathbb{C}$-algebra of the additive group of $\mathbb{R}$. Similar to the notation for polynomials, we will also write y^α for the elements $\alpha \in \mathbb{R}$ and, thus, have the equalities*

$$\mathcal{B} = \{\sum_{\alpha \in I} c_\alpha y^\alpha : c_\alpha \in \mathbb{C} \text{ for all } \alpha \in I \text{ and all finite subsets } I \subset \mathbb{R}\}$$

and

$$\sum_{\alpha \in I} c_\alpha y^\alpha \cdot \sum_{\beta \in J} d_\beta y^\beta = \sum_{\alpha \in I, \beta \in J} c_\alpha d_\beta y^{\alpha+\beta}$$

for all $\sum_{\alpha \in I} c_\alpha y^\alpha, \sum_{\beta \in J} d_\beta y^\beta \in \mathcal{B}$. Moreover, we define the subset

$$\mathcal{B}_{\geq 0} := \{\sum_{\alpha \in I} c_\alpha y^\alpha : c_\alpha \in \mathbb{R}_{\geq 0} \text{ for all } \alpha \in I \text{ and all finite subsets } I \subset \mathbb{R}\}$$

of $\mathcal{B}$ which is an additive and multiplicative submonoid. For all $\beta = (\beta_p)_p \in \mathbb{R}^{\mathbb{P}}$, we also define

$$y^\beta := (y^{\beta_p})_p \in \mathcal{B}^{\mathbb{P}}$$

and the subalgebra

$$\mathbb{C}[y^\beta] := \mathbb{C}[\{y^{\beta_p} : p \in \mathbb{P}\}] \subset \mathcal{B}$$

and denote the fraction field of $\mathbb{C}[y^\beta]$ by $\mathbb{C}(y^\beta)$.

Finally, for all polynomials $h = h(y_{\mathbf{P}}) = \sum_{\alpha \in I} c_\alpha y_{\mathbf{P}}^\alpha \in \mathbb{C}[y_{\mathbf{P}}]$, we will also denote the element $\sum_{\alpha \in I} c_\alpha y^\beta$ by $h(y^\beta)$. Consequently, all elements in $\mathbb{C}(y^\beta)$ are of the form $\frac{h(y^\beta)}{g(y^\beta)}$ for some $h, g \in \mathbb{C}[y_{\mathbf{P}}]$ with $g(y^\beta) \neq 0$.

Definition/Lemma 209. *Let U be the open subset $\mathbb{C}\backslash]-\infty, 0] \times \mathbb{C}$ of $\mathbb{C}^2$. We will call a function $f : U \to \mathbb{C}$* ***holomorphic*** *if it is holomorphic in the sense of [Kau11, p. 2, Definition 1.1], i.e. f is continuous and partially holomorphic, where f is called* ***partially holomorphic*** *if the functions $\mathbb{C}\backslash]-\infty, 0] \to \mathbb{C}$, $z \mapsto f(z, s_0)$ and $\mathbb{C} \to \mathbb{C}$, $s \mapsto f(z_0, s)$ are holomorphic for all $(z_0, s_0) \in U$.*

We will denote the $\mathbb{C}$-algebra of all these holomorphic functions $U \to \mathbb{C}$ by $\mathcal{O}(U)$. Then the map

$$\pi_{\mathcal{O}(U)} : \mathcal{B}[t] \to \mathcal{O}(U) \text{ via } \pi_{\mathcal{O}(U)}(\sum_{\alpha,i} c_{\alpha,i} y^{\alpha} t^{i})(z, s) := \sum_{\alpha,i} c_{\alpha,i} z^{\alpha} s^{i}$$

is a well defined morphism of $\mathbb{C}$-algebras where we defined $z^{\alpha} = \operatorname{Exp}(\alpha \operatorname{Log}(z))$ in Definition 167. Moreover, the map

$$\sigma_z : \mathcal{B} \to \mathbb{C} \text{ via } \sum_{\alpha \in I} c_{\alpha} y^{\alpha} \mapsto \sum_{\alpha \in I} c_{\alpha} z^{\alpha}$$

and its extension

$$\sigma_z : \mathcal{B}^{m \times m} \to \mathbb{C}^{m \times m} \text{ via } (a_{i,j})_{i,j} \mapsto (\sigma_z(a_{i,j}))_{i,j}$$

are also a well defined morphism of $\mathbb{C}$-algebras for all $z \in \mathbb{C}\backslash]-\infty, 0]$. Similar to the evaluation maps on polynomials, we will also use the notation

$$f(z, s) := \pi_{\mathcal{O}(U)}(f)(z, s), \quad g(z) := \sigma_z(g), \quad A(z) := \sigma_z(A)$$

for all $f \in \mathcal{B}[t]$, $g \in \mathcal{B}$, $A \in \mathcal{B}^{m \times m}$ and $(z, s) \in U$.

Proof. On the one hand, since the principal branch $\operatorname{Log} : \mathbb{C}\backslash]-\infty, 0] \to \mathbb{C}$ of the complex logarithm and the complex exponential function $\operatorname{Exp} : \mathbb{C} \to \mathbb{C}$ are holomorphic, the composition

$$\operatorname{Exp}(\alpha \operatorname{Log}(\cdot)) : \mathbb{C}\backslash]-\infty, 0] \to \mathbb{C} \text{ via } z \mapsto z^{\alpha} = \pi_{\mathcal{O}(U)}(y^{\alpha})(z, s_0)$$

is also holomorphic for all $\alpha \in \mathbb{R}$ and $s_0 \in \mathbb{C}$. On the other hand, since

$$\mathbb{C} \to \mathbb{C} \text{ via } s \mapsto z_0^{\alpha} = \pi_{\mathcal{O}(U)}(y^{\alpha})(z_0, s)$$

is constant and, thus, also holomorphic for all $\alpha \in \mathbb{R}$ and $z_0 \in \mathbb{C}\backslash]-\infty, 0]$, we conclude that $\pi_{\mathcal{O}(U)}(y^{\alpha}) : U \to \mathbb{C}$ is a continuous partially holomorphic function and, hence, holomorphic in the sense of [Kau11, p. 2, Definition 1.1]. Analogously, we obtain that $\pi_{\mathcal{O}(U)}(t) : U \to \mathbb{C}$ is also homomorphic.

Combining these two conclusions, the identity

$$\pi_{\mathcal{O}(U)}(\sum_{\alpha,i} c_{\alpha,i} y^{\alpha} t^{i}) = \sum_{\alpha,i} c_{\alpha,i} \cdot \pi_{\mathcal{O}(U)}(y^{\alpha}) \cdot \pi_{\mathcal{O}(U)}(t)^{i}$$

for all $\sum_{\alpha,i} c_{\alpha,i} y^{\alpha} t^{i} \in \mathcal{B}[t]$ and the fact that $\mathcal{O}(U)$ is a $\mathbb{C}$-algebra by [Kau11, p. 2, Definition 1.1] yields that $\pi_{\mathcal{O}(U)}$ is a well defined morphism of $\mathbb{C}$-algebras.

Finally, it is obvious that σ_z is a well defined morphism of $\mathbb{C}$-algebras. □

Lemma 210. *We have the estimate $|h(z)| \leq h(|z|)$ for all $h \in \mathcal{B}_{\geq 0}$ and $z \in \mathbb{C}\backslash]-\infty, 0]$.*

Proof. Choose $\varphi \in \mathbb{R}$ such that

$$z = |z| \operatorname{Exp}(i\varphi) \tag{370}$$

is a presentation of $z \in \mathbb{C}\backslash]-\infty, 0]$ in polar form. Then we compute

$$
\begin{aligned}
|z^\alpha| &= |\operatorname{Exp}(\alpha \operatorname{Log}(z))| = |\operatorname{Exp}(\alpha \operatorname{Log}(|z| \operatorname{Exp}(i\varphi)))| = |\operatorname{Exp}(\alpha \operatorname{Log}(|z|) + i\varphi)| \\
&= |\operatorname{Exp}(\alpha \operatorname{Log}(|z|))| = |z|^\alpha
\end{aligned} \tag{371}
$$

where the equalities hold by the following reasonings: The first and last equalities hold by the definition of $w^\alpha = \operatorname{Exp}(\alpha \operatorname{Log}(w))$ for all $w \in \mathbb{C}\backslash]-\infty, 0]$ in Definition 167. The second equality holds by the equality in (370). The third equality holds by the equalities $\operatorname{Log}(|z| \operatorname{Exp}(i\varphi)) = \operatorname{Log}(|z|) + \operatorname{Log}(\operatorname{Exp}(i\varphi)) = \operatorname{Log}(|z|) + i\varphi$. The fourth equality holds because of the equality $\operatorname{Exp}(\alpha \operatorname{Log}(|z|) + i\varphi) = \operatorname{Exp}(\alpha \operatorname{Log}(|z|)) \operatorname{Exp}(i\varphi)$ and because $\varphi \in \mathbb{R}$ implies the equality $|\operatorname{Exp}(i\varphi)| = 1$.

Next, for all $h \in \mathcal{B}_{\geq 0}$, we have the presentation $h = \sum_{\alpha \in I} c_\alpha y^\alpha$ for some finite subset I of $\mathbb{R}$ and $c_\alpha \in \mathbb{R}_{\geq 0}$ for all $\alpha \in I$ by the definition of $\mathcal{B}_{\geq 0}$ in Definition 208. Thus, for all $z \in \mathbb{C}\backslash]-\infty, 0]$, we obtain the desired estimate by the equalities and estimate

$$
|h(z)| \leq \sum_{\alpha \in I} c_\alpha |z^\alpha| = \sum_{\alpha \in I} c_\alpha |z|^\alpha = h(|z|)
$$

where the estimate holds by the triangle inequality and by $c_\alpha \in \mathbb{R}_{\geq 0}$ for all $\alpha \in I$, the first equality holds by the estimate in (371) and the second equality holds by the definitions of $h = \sum_{\alpha \in I} c_\alpha y^\alpha$ and $h(|z|) = \sigma_{|z|}(h)$ in Lemma 209. □

7.1.2 Key Lemma II

Summary of the results of this subsection. In the following Key Lemma 211, we will estimate the spectral radius $\rho(A(x))$ of any irreducible quadratic matrix A with entries in $\mathcal{B}_{\geq 0}$ and for all $x \in \mathbb{R}_{>0}$. As the name already suggests, this lemma is one of the keys to the proof of Main Theorem 177. It enables us to show the estimate for the spectral radius in Proposition 175 which is the most important ingredient for the proof of Main Theorem 177.

Main ideas of the proof of Key Lemma II. For (i) and (ii) in Key Lemma 211, we will first use Perron-Frobenius theory and the implicit function theorem to show that the spectral radius $\rho(A(x))$ is already an algebraically simple eigenvalue $\lambda(x)$ of $A(x)$ for all $x \in \mathbb{R}_{>0}$ and that this function $\lambda : \mathbb{R}_{>0} \to \mathbb{C}$ can be extended to a holomorphic function in a neighborhood of $\mathbb{R}_{>0}$. Then, for (iii) and (iv), we will apply the maximum modulus principle and more Perron Frobenius theory to conclude that, for any closed interval $I = [a, b] \subset \mathbb{R}_{>0}$, the function of spectral radii $\rho(A(x))$ is either constant or attains its maximum at one of the boundary points but not at the interior points.

Lemma 211 (Key Lemma II)**.** *Let $A = (a_{i,j})_{i,j} \in \mathcal{B}_{\geq 0}^{m \times m}$ be an irreducible matrix and $\rho(A(z))$ be the spectral radius of $A(z) \in \mathbb{C}^{m \times m}$ for all $z \in \mathbb{C}\backslash]-\infty, 0]$. Then we have the following:*

(i) The spectral radius $\lambda(x) := \rho(A(x)) \in \mathbb{R}_{\geq 0}$ is an algebraically simple eigenvalue of $A(x) \in \mathbb{R}_{\geq 0}^{m \times m}$ for all $x \in \mathbb{R}_{>0}$ and defines a map $\lambda : \mathbb{R}_{>0} \to \mathbb{C}$.

(ii) The map $\lambda : \mathbb{R}_{>0} \to \mathbb{C}$ can be extended to a holomorphic function on a domain $G \subseteq \mathbb{C}\backslash]-\infty, 0]$ which contains $\mathbb{R}_{>0}$. Let us also denote this extension by λ. Then $\lambda(z)$ is an eigenvalue of $A(z)$ for all $z \in G$.

(iii) For all closed intervals $I = [a, b] \subset \mathbb{R}_{>0}$ with $a \leq b$, we have the identity

$$
\max_{x \in I} \rho(A(x)) = \max(\rho(A(a)), \rho(A(b))) =: d
$$

(iv) and either $\rho(A(x)) = d$ or $\rho(A(x)) < d$ for all $x \in]a, b[$.

Proof. For (i): By the assumption, the (i, j)-th entry $a_{i,j}$ of A is an element in $\mathcal{B}_{\geq 0}$. Thus, for all $x \in \mathbb{R}_{>0}$ and all $i, j \in \{1, \ldots, m\}$, we conclude that, $a_{i,j}(x) \in \mathbb{R}_{>0}$ if $a_{i,j} \neq 0$. Combining this and the fact that the irreducibility of a matrix only depends on the position of its non-zero entries by its definition in Definition 62, we deduce that $A(x) \in \mathbb{R}^{m \times m}_{\geq 0}$ is an irreducible matrix for all $x \in \mathbb{R}_{>0}$. Therefore, [HJ90, p.508] supplies that the spectral radius $\lambda(x) = \rho(A(x))$ is an algebraically simple eigenvalue of $A(x)$ for all $x \in \mathbb{R}_{>0}$. Hence, (i) follows.

For (ii): Let χ_A be the characteristic polynomial of A. Then $\pi_{\mathcal{O}(U)}(\chi_A) : U \to \mathbb{C}$, $(z, s) \mapsto \chi_A(z, s)$ is a holomorphic function by the definition of $\pi_{\mathcal{O}(U)} : \mathcal{B}[t] \to \mathcal{O}(U)$ in Lemma 209 on page 263. Then the algebraic simplicity of the eigenvalue $\lambda(x)$ in Lemma 211(i) implies

$$(\frac{d}{ds}\pi_{\mathcal{O}(U)}(\chi_A))(x, \lambda(x)) = (\frac{d}{dt}\chi_A(x, t))(\lambda(x)) \neq 0 \tag{372}$$

for all $x \in \mathbb{R}_{>0}$. By the inequality in (372) and the implicit mapping theorem in [Kau11, p. 28, Theorem 8.6], there is an open ball B_x in $\mathbb{C}\backslash]-\infty, 0]$ around any $x \in \mathbb{R}_{>0}$ such that $\lambda : \mathbb{R}_{>0} \to \mathbb{C}$ has a holomorphic extension λ_x on B_x and we have the equalities

$$0 = \pi_{\mathcal{O}(U)}(\chi_A)(z, \lambda_x(z)) = \chi_A(z, t)(\lambda_x(z))$$

for all $z \in B_x$. Thus, $\lambda_x(z)$ is an eigenvalue of $A(z)$ for all $z \in B_x$.

Now, since the intersection of two such balls is empty or contains a non-empty open interval of $\mathbb{R}_{>0}$ on which the extensions are identical with λ, they are compatible by the identity theorem for holomorphic functions in [Fre09, p.125, corollary III.3.2] (notice that the notions of holomorphic functions and analytic functions are used synonymously in [Fre09, p. 53]).

Next, let G be the union of these open balls B_x. Then G is open and connected and, thus, a domain. Consequently, we obtain that λ can be extended to a holomorphic function on this domain $G \subseteq \mathbb{C}\backslash]-\infty, 0]$ which contains $\mathbb{R}_{>0}$ (see Figure 7.1 on page 265) and $\lambda(z)$ is an eigenvalue of $A(z)$ for all $z \in G$. Hence, (ii) follows.

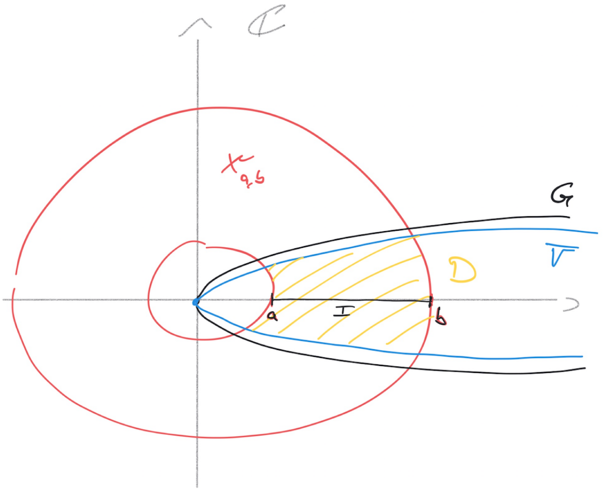


Figure 7.1: The domain and annulus in a proof

For (iii) and (iv): First, for the (i, j)-th entry $a_{i,j} \in \mathcal{B}_{\geq 0}$ of A, we derive the estimate $|a_{i,j}(z)| \leq a_{i,j}(|z|)$ from Lemma 210. Therefore, if we define $|B| := (|b_{i,j}|)_{i,j}$ for all

$B = (b_{i,j})_{i,j} \in \mathbb{C}^{m\times m}$, we get the entrywise estimate

$$|A(z)| \leq A(|z|) \tag{373}$$

for all $z \in \mathbb{C}\backslash]-\infty, 0]$. Now, we estimate

$$|\lambda(z)| \leq \rho(A(z)) \leq \rho(A(|z|)) = \lambda(|z|) \tag{374}$$

for all $z \in G$ where the estimates and equality holds by the following reasonings: The first estimate holds by the definition of the spectral radius $\rho(A(z))$ of $A(z)$ and by the fact that Lemma 211(ii) provides that $\lambda(z)$ is an eigenvalue of $A(z)$. The second estimate holds by the estimate in (373) and by applying [HJ90, p.491, Theorem 8.11.18]. The equality holds by the definition of λ in Lemma 211(ii).

Next, we may take a connected open subset V of G (see Figure 7.1) such that the inclusions $\mathbb{R}_{>0} \subseteq \overline{V} \subseteq G \cup \{0\}$ hold for the closure $\overline{V}$ of V and such that the intersection of V and the annulus $X_{a,b} := \{z \in \mathbb{C} : a \leq |z| \leq b\}$ is still connected. Let D be the interior of $\overline{V} \cap X_{a,b}$. Then D is a bounded domain which contains the open interval $]a, b[$ but neither a or b. Moreover, the closure $\overline{D}$ of D is contained in $\overline{V} \cap X_{a,b} \subseteq G$ due to $a > 0$. Thus, λ restricts to a continuous function on $\overline{D}$ which is also holomorphic on D.

On the one hand, by the maximum principle in [Fis03, p. 91, Satz 6.4], either λ is constant on $\overline{D}$ and the desired identity in (iii) and the first case in (iv) hold or there is an element $z' \in \partial D$ such that $|\lambda(z')| = \max_{z\in\overline{D}} |\lambda(z)|$ and

$$|\lambda(z')| > |\lambda(z)| \tag{375}$$

for all $z \in D$. On the other hand, the estimate in (374) also yields the estimate

$$|\lambda(z')| \leq |\lambda(|z'|)|. \tag{376}$$

Combining (375) and (376) provides $|z'| \notin D$. But this and $z' \in \overline{D} \subseteq X_{a,b}$ imply $|z'| \in [a, b]\backslash D = \{a, b\}$ and, thus, the equality

$$\max_{z\in\overline{D}} |\lambda(z)| = \max(\lambda(a), \lambda(b)). \tag{377}$$

Second to last, we estimate

$$d \leq \max_{x\in I} \rho(A(x)) = \max_{x\in I} \lambda(x) \leq \max_{z\in\overline{D}} |\lambda(z)| = \max(\lambda(a), \lambda(b)) = d \tag{378}$$

where the first estimate holds since $I = [a, b]$ contains the elements a and b, the first and last equalities hold by the definition of λ in Lemma 211(i), the second estimate holds by the inclusion $I \subseteq \overline{D}$ and the second equality holds by the identity in (377). Consequently, all the estimates in (378) must be equalities and, hence, the desired equality $\max_{x\in I} \rho(A(x)) = d$ in (iii) follows.

Finally, the estimate in (375) especially implies $\rho(A(x)) = \lambda(x) < \max_{z\in\overline{D}} |\lambda(z)| = d$ for all $x \in]a, b[\subset D$ and, hence, (iv) also follows. □

7.1.3 Application to Irreducible Matrices in Several Variables

Summary of the results of this subsection. The matrix in Key Lemma 211 has entries in $\mathcal{B}_{\geq 0}$, i.e. only in one variable y. But, in the first interim result in Theorem 168, the matrix $A = A(y_{\mathbf{P}})$ has entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$, i.e. in several independent variables y_p. In this subsection, we will make the transition from $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ to $\mathbb{R}_{\geq 0}[y^{\log_2(\mathbf{P})}]$ via the evaluation isomorphism $L : \mathbb{C}(y_{\mathbf{P}}) \to \mathbb{C}(y^{\log_2(\mathbf{P})})$ in Definition/Lemma 212 which replaces y_p with $y^{\log_2(p)}$ for all $p \in \mathbb{P}$. By Lemma 213, this isomorphism L will then enable us to transfer conclusions for the characteristic polynomial $\chi_{L(A)}$ of $L(A) = A(y^{\log_2(\mathbf{P})})$ to the characteristic polynomial χ_A of $A = A(y_{\mathbf{P}})$.

Consequently, in Lemma 214, we will characterize the two possible cases for the function of spectral radii $\rho(A(x^{\log_2(\mathbf{P})})) : [1,2] \to \mathbb{R}_{\geq 0}$ of an irreducible matrix A with entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$.

Definition/Lemma 212. *The map*

$$L : \mathbb{C}(y_{\mathbf{P}}) \to \mathbb{C}(y^{\log_2(\mathbf{P})}) \text{ via } \frac{h(y_{\mathbf{P}})}{g(y_{\mathbf{P}})} \mapsto \frac{h(y^{\log_2(\mathbf{P})})}{g(y^{\log_2(\mathbf{P})})}$$

and its extension

$$L : \mathbb{C}(y_{\mathbf{P}})^{m\times m} \to \mathbb{C}(y^{\log_2(\mathbf{P})})^{m\times m} \text{ via } (a_{i,j})_{i,j} \mapsto (L(a_{i,j}))_{i,j}.$$

are well defined isomorphisms of $\mathbb{C}$-algebras.

Proof. It is clear that the second desired statement, namely that $L : \mathbb{C}(y_{\mathbf{P}})^{m\times m} \to \mathbb{C}(y^{\log_2(\mathbf{P})})^{m\times m}$ is a well defined isomorphism of $\mathbb{C}$-algebras, immediately follows from the first desired statement, i.e. from $L : \mathbb{C}(y_{\mathbf{P}}) \to \mathbb{C}(y^{\log_2(\mathbf{P})})$ being a well defined isomorphism of $\mathbb{C}$-algebras.

Now, for the first desired statement, it is obvious that $L_{|\mathbb{C}[y_{\mathbf{P}}]}$ is a well defined morphism of $\mathbb{C}$-algebras as $\mathbb{C}[y_{\mathbf{P}}]$ is the polynomial ring in the variables y_p for all $p \in \mathbb{P}$.

We first show the identity $\ker L_{|\mathbb{C}[y_{\mathbf{P}}]} = \{0\}$ which then provides that $L_{|\mathbb{C}[y_{\mathbf{P}}]}$ is a monomorphism of $\mathbb{C}$-algebras: For that, let $y_{\mathbf{P}}^{\beta}$ be a monomial in $\mathbb{C}[y_{\mathbf{P}}]$, i.e. $\beta = (\beta_p)_p \in (\mathbb{N}_0^{\mathbf{P}})'$. Then we obtain the equality

$$L(y_{\mathbf{P}}^{\beta}) = y^{\sum_p \log_2(p)\beta_p} = y^{\log_2(\mathbf{P}^{\beta})} \tag{379}$$

where the first equality holds by the definition of L and by the multi-index notation in (1) and the second equality holds by the definition of $\log_2(p) = \log_2(p)$ for all $p \in \mathbb{P}$ in Definition 167 and by well known rules for logarithms.

Next, let $h \in \ker L_{|\mathbb{C}[y_{\mathbf{P}}]}$. Then we have the equality $h = \sum_{\beta\in I} c_\beta y_{\mathbf{P}}^{\beta}$ for some finite subset I of $(\mathbb{N}_0^{\mathbb{P}})'$ and $c_\beta \in \mathbb{C}$ for all $\beta \in I$. Thus, we also compute

$$0 = L(h) = \sum_{\beta\in I} c_\beta y^{\log_2(\mathbf{P}^{\beta})} \in \mathcal{B} \tag{380}$$

where the first equality holds by the choice of $h \in \ker L_{|\mathbb{C}[y_{\mathbf{P}}]}$, the second equality holds by the presentation of $h = \sum_{\beta\in I} c_\beta y_{\mathbf{P}}^{\beta}$, by the definition of L and by the equality in (379).

Now, since $\log_2$ is injective and since any natural number uniquely factorizes into prime numbers, we obtain that these elements $y^{\log_2(\mathbf{P}^{\beta})}$ in (380) are pairwise distinct for all $\beta \in I$ and, thus, are $\mathbb{C}$-linearly independent elements in $\mathcal{B}$ by Definition 208. Hence, the equalities $c_\beta = 0$ follow for all $\beta \in I$. Therefore, the desired equalities $h = 0$ and $\ker L_{|\mathbb{C}[y_{\mathbf{P}}]} = \{0\}$ follow and, consequently, $L_{|\mathbb{C}[y_{\mathbf{P}}]}$ is indeed a monomorphism.

Furthermore, we conclude that L must also be a well defined monomorphism of $\mathbb{C}$-algebras since it is the canonical extension of the monomorphism $L_{|\mathbb{C}[y_{\mathbf{P}}]}$ via the universal property for the field $\mathbb{C}(y_{\mathbf{P}})$ of fractions of $\mathbb{C}[y_{\mathbf{P}}]$.

Finally, the surjectivity of L immediately follows from the definition of the field $\mathbb{C}(y^{\log_2(\mathbf{P})})$ of fractions of the $\mathbb{C}$-subalgebra $\mathbb{C}[y^{\log_2(\mathbf{P})}] = \mathbb{C}[\{y^{\alpha_p} : p \in \mathbb{P}\})$ of $\mathcal{B}$ in Definition 208 which is clearly the image of L. ☐

Lemma 213. *Let $\chi_A \in \mathbb{C}(y_{\mathbf{P}})[t]$ and $\chi_{L(A)} \in \mathbb{C}(y^{\log_2(\mathbf{P})})[t]$ be the characteristic polynomials of the matrices $A \in \mathbb{C}(y_{\mathbf{P}})^{m\times m}$ and $L(A) \in \mathbb{C}(y^{\log_2(\mathbf{P})})^{m\times m}$, respectively. Then $\chi_A \in \mathbb{C}[t]$ if and only if $\chi_{L(A)} \in \mathbb{C}[t]$.*

Proof. Let $A = (a_{i,j})_{i,j}$ and $\chi_A = \sum_{i=0}^{r} c_i t^i$ with $c_i \in \mathbb{C}(y_{\mathbf{P}})$ for all $i = 0, \ldots, r$. Combining the definition of $L(A) = (L(a_{i,j}))_{i,j}$ in Definition/Lemma 212, the Leibniz formula for determinants and the fact that L is a morphism of rings by again Definition/Lemma 212 yields that the i-th coefficient of $\chi_{L(A)} = \det(tI - L(A))$ is the L-image $L(c_i)$ of the i-th coefficient c_i of $\chi_A = \det(tI - A)$. But since L is even an isomorphism of $\mathbb{C}$-algebras, we conclude that $L(c_i) \in \mathbb{C}$ if and only if $c_i \in \mathbb{C}$. Hence, the desired equivalence follows. □

Lemma 214. *Let $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m \times m}$ be an irreducible matrix, let d be the maximum $\max(\rho(A(\mathbf{1})), \rho(A(\mathbf{P}))) \in \mathbb{R}_{>0}$ and let $\chi_A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}][t]$ be the characteristic polynomial of A. Then there are the following two possible cases:*

(i) $\chi_A \notin \mathbb{C}[t]$ *and* $\rho(A(x^{\log_2(\mathbf{P})})) < d$ *for all* $x \in]1, 2[$.

(ii) $\chi_A \in \mathbb{C}[t]$ *and* $\rho(A(x^{\log_2(\mathbf{P})})) = d$ *for all* $x \in [1, 2]$.

Proof. First, we notice that the matrix

$$L(A(y_{\mathbf{P}})) \in \mathcal{B}_{\geq 0}^{m \times m} \text{ is irreducible} \tag{381}$$

because of the definition of $L : \mathbb{C}(y_{\mathbf{P}})^{m \times m} \to \mathbb{C}(y^{\log_2(\mathbf{P})})^{m \times m}$ in Definition/Lemma 212 via the isomorphism $L : \mathbb{C}(y_{\mathbf{P}}) \to \mathbb{C}(y^{\log_2(\mathbf{P})})$ and because the irreducibility of a matrix only depends on the positions of the non-zero-entries by its definition on Definition 62.

Second, we also notice that we have the identity $\mathrm{Eval}_{x^{\log_2(\mathbf{P})}} = \sigma_x \circ L$ for all $x \in [1, 2]$ by the definitions of these maps in Definition 161(ii), Definition/Lemma 209 and Definition 212. Therefore, we have the equalities

$$A(x^{\log_2(\mathbf{P})}) = \mathrm{Eval}_{x^{\log_2(\mathbf{P})}}(A) = \sigma_x(L(A)) = L(A)(x). \tag{382}$$

Now, because $\rho(A(1^{\log_2(\mathbf{P})})) = \rho(A(\mathbf{1})) \leq d$ and $\rho(A(2^{\log_2(\mathbf{P})})) = \rho(A(\mathbf{P})) \leq d$, combining (382), (381), Key Lemma 211(iii) and Key Lemma 211(iv) yields that the function of the spectral radius $\rho(A(x^{\log_2(\mathbf{P})}))$ in the argument $x \in [1, 2]$ reaches its maximum at one of the boundary points and that we either have the estimate

$$\rho(A(x^{\log_2(\mathbf{P})})) < d \tag{383}$$

or the equality

$$\rho(A(x^{\log_2(\mathbf{P})})) = d \tag{384}$$

for all $x \in]1, 2[$.

We will distinguish two cases: First, suppose the estimate in (383) holds. Since $\max_{x=1,2} \rho(A(x^{\log_2(\mathbf{P})})) = d$ holds by assumption and since $\rho(A(\cdot))$ is continuous on $[1, 2]$ by Key Lemma 211(ii), it cannot be constant on $[1, 2]$. Hence, we must have $\chi_A \notin \mathbb{C}[t]$ and arrive at the first desired case (i).

Second, suppose that the equality in (384) holds. We will show that $\chi_A \in \mathbb{C}[t]$ holds and, hence, arrive at the second desired case (ii): For all $x \in \mathbb{R}_{>0}$, let $\lambda_{x,1}, \ldots, \lambda_{x,m}$ be the eigenvalues of $A(x^{\log_2(\mathbf{P})})$ (counted with multiplicities). Then, on the one hand, we conclude

$$|\lambda_{x,k}| \leq \rho(A(x^{\log_2(\mathbf{P})})) = d \tag{385}$$

for all $k = 1, \ldots, m$ and all $x \in \mathbb{R}_{>0}$ where the estimate holds by the definition of the spectral radius and the equality holds because of the equality in (384), because Lemma 211(ii) implies that the partially constant function $\rho(A(\cdot)) : \mathbb{R}_{>0} \to \mathbb{C}$ can be extended to a holomorphic function on a domain which contains $\mathbb{R}_{>0}$ and because the identity theorem in [Fre09, p. 125, Corollary 3.3.2.1] then implies that $\rho(A(\cdot))$ must be constantly d on all of $\mathbb{R}_{>0}$. This again yields

$$|\chi_{A(x^{\log_2(\mathbf{P})})}(s)| = \prod_{k=1}^{m} |s - \lambda_{x,k}| \leq \prod_{k=1}^{m} |s| + |\lambda_{x,k}| \leq (s+d)^m \tag{386}$$

for all $x \in \mathbb{R}_{>0}$ and all $s \in \mathbb{R}_{>0}$ where the equality holds since $\lambda_{x,1}, \ldots, \lambda_{x,m}$ are the eigenvalues of $A(x^{\log_2(\mathbf{P})})$ (counted with multiplicities) and, thus, $\chi_{A(x^{\log_2(\mathbf{P})})}$ is of the form $\prod_{k=1}^{m} t - \lambda_{x,k}$, the first estimate holds by the triangular inequality and the second estimate holds by the estimate in (385).

One the other hand, because the matrix A has entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ and because of the Leibniz formula for determinants, we have a presentation $\chi_A = \sum_{\beta \in X} h_\beta(t) y_{\mathbf{P}}^{\beta}$ for some finite subset X of $(\mathbb{Z}^{\mathbb{P}})'$ and polynomials $h_\beta(t) \in \mathbb{R}[t]\backslash\{0\}$ for all $\beta \in X$. Then we obtain the equalities

$$\chi_{L(A)} = \sum_{\beta} h_\beta(t)(y^{\log_2(\mathbf{P})})^{\beta} = \sum_{\beta} h_\beta(t) y^{\log_2(\mathbf{P}^{\beta})} \tag{387}$$

where the equalities hold by the following reasonings: The first two equalities hold by combining the definitions of the morphisms $L : \mathbb{C}(y_{\mathbf{P}})^{m\times m} \to \mathbb{C}(y^{\log_2(\mathbf{P})})^{m\times m}$, of $L : \mathbb{C}(y_{\mathbf{P}}) \to \mathbb{C}(y^{\log_2(\mathbf{P})})$ in Definition/Lemma 212, the fact that the latter is a morphism of $\mathbb{C}$-algebras with $L(y_p) = y^{\log_2(p)}$ and the above presentation of χ_A. The second equality holds by the definitions of $y^{\log_2(\mathbf{P})} = (y^{\log_2(p)})_p$ in Definition 208, of $\log_2(p) = \log_2(p)$ for all $p \in \mathbb{P}$ in Definition 167 and of the multi-index notation for powers in (1) and by applying well known rules for logarithms.

Since $\log_2$ is injective and since any natural number has a unique decomposition into prime numbers, $\log_2(\mathbf{P}^{\beta})$ is an injective map in the argument $\beta \in (\mathbb{Z}^{\mathbb{P}})'$ and we moreover have $\log_2(\mathbf{P}^{\beta}) = 0$ if only if $\beta = 0$. Therefore, combing this fact and the identity in (387) provides that

$$\chi_{L(A)} \in \mathbb{C}[t] \text{ if and only if } X = \{0\}. \tag{388}$$

Now, assume that there is a non-zero index $\beta \in X$. Then there is also a non-zero index $\gamma \in X$ such that $\mathbf{P}^{\gamma} = \max\{\mathbf{P}^{\beta} : \beta \in X\} > 1$ or $\mathbf{P}^{\gamma} = \min\{\mathbf{P}^{\beta} : \beta \in X\} < 1$. Moreover, since $h_\gamma(t) \neq 0$, there is an element $s \in \mathbb{R}_{>0}$ such that $h_\gamma(s) \neq 0$. Then, by the injectivity of $\log_2(\mathbf{P}^{\beta})$ as a function in β and by the identity in (387), we obtain a small $\epsilon > 0$ such that

$$|\chi_{A(x^{\log_2(\mathbf{P})})}(s)| = |\chi_{L(A)(x)}(s)| = |h_\gamma(s) x^{\log_2(\mathbf{P}^{\gamma})} + \mathcal{O}(x^{\log_2(\mathbf{P}^{\gamma})-\epsilon})| \to \infty$$

as $x \in \mathbb{R}_{>0}$ tends to ∞ in the first case $\mathbf{P}^{\gamma} > 1$ and

$$|\chi_{A(x^{\log_2(\mathbf{P})})}(s)| = |\chi_{L(A)(x)}(s)| = |h_\gamma(s) x^{\log_2(\mathbf{P}^{\gamma})} + \mathcal{O}(x^{\log_2(\mathbf{P}^{\gamma})+\epsilon})| \to \infty$$

as $x \in \mathbb{R}_{>0}$ tends to 0 in the second case $\mathbf{P}^{\gamma} < 1$. But, both cases contradict that (386) implies that $\chi_{A(x^{\log_2(\mathbf{P})})}(s)$ is bounded as a function in $x \in \mathbb{R}_{>0}$. Therefore, we must have $X = \{0\}$. Combining this equality and the equivalence in (388) provides $\chi_{L(A)} \in \mathbb{C}[t]$. Finally, we derive the desired statement $\chi_A \in \mathbb{C}[t]$ from Lemma 213. □

7.1.4 Application to Reducible Matrices in Several Variables

Summary of the results of this subsection. In this subsection, we will extend the insights from Lemma 214 of the last Subsection 7.1.3 to connected matrices.

For that, we will first prove the following Lemma 215 which will enable us to apply Lemma 214 to the irreducible submatrices $A_1, \ldots, A_r$ of PAP^t. Then we will extend Lemma 214 to Lemma 216.

Lemma 215. *Let $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$ be connected.*

(i) Then there is some permutation matrix $P \in \{0,1\}^{m\times m}$ such that PAP^t is an upper block triangular matrix with quadratic irreducible matrices $A_1, \dots, A_r$ on the block diagonal.

Moreover, let $B_1, \dots, B_{r-1}, C_2, \dots C_r$ be the matrices with entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ such that

$$PAP^t = \begin{pmatrix} A_1 & * & \dots & * & \dots & \dots & * \\ 0 & \ddots & \ddots & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & * & & & \vdots \\ 0 & \dots & 0 & A_k & * & \dots & * \\ \vdots & & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 & \dots & 0 & A_r \end{pmatrix} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \\ \vdots \\ Z_r \end{pmatrix} = \begin{pmatrix} S_1 & \dots & S_k & \dots & S_r \end{pmatrix}$$

with

$$Z_1 := \begin{pmatrix} A_1 & B_1 \end{pmatrix} \text{ and } Z_k := \begin{pmatrix} 0 & A_k & B_k \end{pmatrix} \text{ and } Z_r := \begin{pmatrix} 0 & A_r \end{pmatrix}$$

and

$$S_1 := \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \text{ and } S_k := \begin{pmatrix} C_k \\ A_k \\ 0 \end{pmatrix} \text{ and } S_r := \begin{pmatrix} C_r \\ A_r \end{pmatrix}$$

for all $k = 2, \dots, r-1$. If $r \geq 2$, then we also have the inequalities

$$B_1 \neq 0 \text{ and } (B_k \neq 0 \text{ or } C_k \neq 0) \text{ and } C_r \neq 0 \tag{389}$$

for all $k = 2, \dots, r-1$.

(ii) Suppose that all column sums in $A(\mathbf{1}) \in \mathbb{R}_{\geq 0}^{m\times m}$ and all row sums in $A(\mathbf{P}) \in \mathbb{R}_{\geq 0}^{m\times m}$ are at most $d \in \mathbb{R}_{>0}$ Then the following hold for the matrix PAP^t in Lemma 215(i):

(a) All column (resp. row) sums in $A_k(\mathbf{1})$ (resp. $A_k(\mathbf{P})$) are at most d for all $k = 1, \dots, r$

(b) If $r \geq 2$, then there is a column sum in $A_k(\mathbf{1})$ or a row sum in $A_k(\mathbf{P})$ which is less than d for all $k = 1, \dots, r$.

(c) If $r \geq 2$, then the estimates

$$\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) \leq d \text{ and } \min(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d$$

hold for all $k = 1, \dots, r$.

Proof. For (i): If A is irreducible, we can choose P as the identity matrix and set $r := 1$.

Otherwise, by first applying the definition of reducible matrices in Definition 62(ii) to A and then iteratively to all reducible block matrices which iteratively appear in the corresponding upper block triangular matrices on the block diagonals, we obtain a permutation matrix $P \in \{0,1\}^{m\times m}$, a natural number $r \geq 2$, quadratic irreducible matrices $A_1, \dots, A_r$ and matrices $B_1, \dots, B_{r-1}, C_2, \dots C_r$ with entries in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ such that PAP^t is of the desired upper block triangular form in the 'main'-part.

Finally, the desired inequalities in (389) follow by the following reasoning: If the contrary were true, then there would be a permutation matrix $T \in \{0,1\}^{m\times m}$ such that $T(PAP^t)T^t = (TP)A(TP)^t$ is a block diagonal matrix with at least two blocks. But this is impossible because a product TP of permutation matrices is again a permutation matrix and because of the definition of A being connected in Definition 62.

For (ii): First of all, for all $x_{\mathbf{P}} \in \{\mathbf{1}, \mathbf{P}\}$, we notice the equalities

$$(PAP^t)(x_{\mathbf{P}}) = P(x_{\mathbf{P}})A(x_{\mathbf{P}})P(x_{\mathbf{P}})^t = PA(x_{\mathbf{P}})P^t$$

where the first equality holds since $\mathrm{Eval}_{x_{\mathbf{P}}}$ is a morphism of $\mathbb{C}$-algebras in Definition 161(ii) and the second equality holds by $P \in \{0,1\}^{m\times m}$. Consequently, for all $x_{\mathbf{P}} \in \{\mathbf{1}, \mathbf{P}\}$, as multiplying P from left to $A(x_{\mathbf{P}})$ only permutes the rows of $A(x_{\mathbf{P}})$ and as then multiplying P^t from the right to $PA(x_{\mathbf{P}})$ only permutes the columns of $PA(x_{\mathbf{P}})$, we conclude that d is still an upper bound for all column sums in $PA(\mathbf{1})P^t$ and for all row sums in $PA(\mathbf{P})P^t$. Therefore, w.l.o.g we may assume that A is already of the upper block triangular form in Lemma 215(i).

Now, due to $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$, we get that $A(\mathbf{1}) = \mathrm{Eval}_{\mathbf{1}}(A)$ and $A(\mathbf{P}) = \mathrm{Eval}_{\mathbf{P}}(A)$ only have non-negative real entries. For $(a_{i,j})_{i,j} := A$, then combining this fact, the assumption that all the column sums in $A(\mathbf{1})$ and the assumption that all the row sums in $A(\mathbf{P})$ are at most d implies the first desired item (a).

Next, suppose $r \geq 2$. Then we notice that $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$ even provides that the entry $a_{i,j}$ vanishes if and only if $a_{i,j}(\mathbf{1})$ (resp. $a_{i,j}(\mathbf{P})$) vanishes. Therefore, combining these equivalences, the assumption that all the column sums in $A(\mathbf{1})$ are at most d, the assumption that all the row sums in $A(\mathbf{P})$ are at most d and the inequalities in (389) yields that there must be a column sum in $A_k(\mathbf{1})$ or row sum in $A_k(\mathbf{P})$ which is less than d for all $k = 1, \ldots, r$. Hence, (b) follows.

Second to last, we derive the estimate

$$\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) \leq d$$

for all $k = 1, \ldots, r$ from the estimates in (a) and from then applying [HJ90, p.492, Theorem 8.1.22].

Finally, we also derive the last desired estimate

$$\min(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d$$

for all $k = 1, \ldots, r$ from the irreducibility of A_k, from the first two items (a) and (b) and from Lemma 64(ii). Hence, (c) follows. □

Lemma 216. *Let $A \in \mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$ be a connected matrix such that the row sums of $A(\mathbf{P})$ and the column sums of $A(\mathbf{1})$ are equal to some $d \in \mathbb{R}_{>0}$ and let $\chi_A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}][t]$ be the characteristic polynomial of A. Then there are the following two possible cases:*

(i) $\chi_A \notin \mathbb{C}[t]$ and $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$.

(ii) $\chi_A \in \mathbb{C}[t]$ and $\rho(A(x^{\log_2(\mathbf{P})})) = d$ for all $x \in [1, 2]$.

In case (ii), the matrix A is necessarily irreducible.

Proof. By [HJ90, p.492, Theorem 8.1.22], we obtain

$$\rho(A(\mathbf{1})) = \rho(A(\mathbf{P})) = d. \tag{390}$$

Thus, if A is irreducible, then the desired statement immediately follows from the items (i) and (ii) in Lemma 214.

Now, let A be reducible. We have to show that the case (i) holds. For that, we consider the matrix PAP^t in Lemma 215(i) which is in upper block triangular form with quadratic irreducible matrices $A_1, \ldots, A_r$ on the block diagonal. Since A is reducible, we have $r \geq 2$ and, thus, Lemma 215(ii)(c) provides the estimates

$$\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) \leq d \text{ and } \min(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d \tag{391}$$

for all $k = 1, \ldots, r$. Then we conclude the desired estimate by the equality and estimate

$$\rho(A(x^{\log_2(\mathbf{P})})) = \max_{k=1,\ldots,r} \rho(A_k(x^{\log_2(\mathbf{P})})) < d$$

for all $x \in]1, 2[$ where the equality and estimate hold by the following reasonings: The equality holds because $A(x^{\log_2(\mathbf{P})})$ and

$$P(x^{\log_2(\mathbf{P})})A(x^{\log_2(\mathbf{P})})P(x^{\log_2(\mathbf{P})})^t = PA(x^{\log_2(\mathbf{P})})P^t = PA(x^{\log_2(\mathbf{P})})P^{-1}$$

have the same eigenvalues. For the estimate, we distinguish two cases:

If $\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d$, then both cases in Lemma 214 provide the estimate $\rho(A_k(x^{\log_2(\mathbf{P})})) < d$. Else, if $\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) = d$, then combining the equalities $1^{\log_2(\mathbf{P})} = \mathbf{1}$ and $2^{\log_2(\mathbf{P})} = \mathbf{P}$, the second estimate in (391) and Lemma 214(i) also provides the estimate $\rho(A_k(x^{\log_2(\mathbf{P})})) < \ d$.

Moreover, since $\rho(A(1^{\log_2(\mathbf{P})})) = \rho(A(\mathbf{1})) = d$ holds by the equalities in (390), the function of the spectral radius $\rho(A(x^{\log_2(\mathbf{P})}))$ cannot be constant on $[1, 2]$ and, hence, the other desired statement $\chi_A \notin \mathbb{C}[t]$ follows. □

7.1.5 Finite Weighted Directed Graphs and its Characteristic Polynomial

Summary of the results of this subsection. The last missing piece for the proof of Theorem 168 in the next Subsection 7.1.6 is Corollary 219 of this subsection. For all weight functions $w : E(\Gamma) \to \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ such that the image of w is contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$ and all w-adjacency matrices A of Γ, Corollary 219 translates the two possible cases for the characteristic polynomial χ_A in Lemma 216 into terms of the weights $w(\mathcal{C})$ of the circles $\mathcal{C}$ in Γ.

In order to prove this Corollary 219, we will first need the following Definition 217 and Theorem 218. Here, Theorem 218 is a further result of this thesis and provides a new interpretation of the characteristic polynomials of adjacency matrices in terms of the circles in the directed graph.

Definition 217. *Let $m \in \mathbb{N}$, $X \subseteq X_0 := \{1, \ldots, m\}$ and R be a commutative ring.*

(i) We write $S(X)$ for the symmetric group on X, i.e. the set of all bijections $\sigma : X \to X$.

(ii) Let $\sigma \in S(X)$. Then we define $O(\sigma, i)$ as the orbit $\{\sigma^k(i) : k \in \mathbb{N}_0\}$ of $i \in X$ under the action of σ, i.e. the action of $\langle\sigma\rangle$ where $\langle\sigma\rangle$ is the subgroup of $S(X)$ which is generated by σ, the set

$$F(\sigma) := \{i \in X : \sigma(i) = i\}$$

of elements which are fixed under the action of σ, the set

$$G(\sigma) := X \backslash F(\sigma)$$

of elements which are not fixed under action of σ and the set

$$R(\sigma) := \{\min O(\sigma, i) : i \in X\}$$

of representatives of the quotient set $\langle\sigma\rangle\backslash X$ containing only the minimal elements of all orbits.

(iii) For all $\sigma \in S(X)$, we define the canonical extension

$$\hat{\sigma} \in S(X_0) \text{ of } \sigma \text{ via } i \mapsto \begin{cases} \sigma(i) & \text{if } i \in X \\ i & \text{else} \end{cases},$$

the sign-*value* $\operatorname{sign}(\sigma) := \operatorname{sign}(\hat{\sigma})$ *of σ and we call σ a k-cycle of X if $\hat{\sigma}$ is a k-cycle in $S(X_0)$ for all $k \in \mathbb{N}_0$.*

(iv) Let $A = (a_{i,j})_{i,j} \in R^{m\times m}$. Then we define $a_{\sigma,i} := \prod_{l=1}^{\#O(\sigma,i)} a_{\sigma^{l-1}(i),\sigma^l(i)}$ for all $i \in X$ and $a_\sigma := \prod_{i\in R(\sigma)} a_{\sigma,i}$.

(v) Let Γ be a finite directed graph, $v = (P_1, \dots, P_m)$ be some enumeration of its vertices, let $w : E(\Gamma) \to R$ be a weight function on Γ, let $\sigma \in S(X)$, $i \in X$ and $k := \#O(\sigma, i)$. Then we define $C(\sigma, i)$ to be set of all cycles $\mathcal{C}$ in Γ for which $(P_i, P_{\sigma(i)}, \dots, P_{\sigma^{k-1}(i)}, P_{\sigma^k(i)}) = (P_i, P_{\sigma(i)}, \dots, P_{\sigma^{k-1}(i)}, P_i)$ is the sequence of vertices of $\mathcal{C}$.

Finite Weighted Directed Graphs and its Characteristic Polynomial The following Theorem 218 is a new interpretation of the characteristic polynomials of adjacency matrices in terms of the circles in the directed graph.

Theorem 218. *Let $X_0 := \{1, \dots, m\}$, let R be a commutative ring, let Γ be a finite directed graph with weight function $w : E(\Gamma) \to R$, let $A = (a_{i,j})_{i,j} \in R^{m\times m}$ be the w-adjacency matrix of Γ for some enumeration $v = (P_1, \dots, P_m)$ of its vertices and let $\chi_A = \sum_{k=0}^m c_k t^k \in R[t]$ with $c_k \in R$ for all $k = 0, \dots, m$ be the characteristic polynomial of A. Then the following hold:*

(i) For all subsets $X \subseteq X_0$ and $\sigma \in S(X)$, we have the identity

$$a_\sigma = \prod_{i\in R(\sigma)} \sum_{\mathcal{C}\in C(\sigma,i)} w(\mathcal{C}).$$

(ii) We have the identity

$$c_k = (-1)^{m+k} \sum_{X,\sigma} \operatorname{sign}(\sigma) a_\sigma$$

where the sum runs over all $X \subseteq X_0$ with $\#X = m - k$ and all $\sigma \in S(X)$.

Proof. For (i): Let $X \subseteq X_0$, $\sigma \in S(X)$ and, moreover, let us shortly write $E(\Gamma, i, j) := E(\Gamma, P_i, P_j)$. Then we first compute

$$a_{\sigma,i} = \prod_{l=1}^{\#O(\sigma,i)} a_{\sigma^{l-1}(i),\sigma^l(i)} = \prod_{l=1}^{\#O(\sigma,i)} \sum_{Q_l\in E(\Gamma,\sigma^{l-1}(i),\sigma^l(i))} w(Q_l) = \sum_{\mathcal{C}\in C(\sigma,i)} w(\mathcal{C}) \tag{392}$$

for all $i \in R(\sigma)$ where the equalities hold by the following reasonings: The first equality holds by the definition of $a_{\sigma,i}$ in Definition 217(iv). The second equality holds by the definition of the w-adjacency matrix $A = (a_{i,j})_{i,j} := (\sum_{Q\in E(\Gamma,i,j)} w(Q))_{i,j}$ of Γ in Definition 58. For the third equality, we expand the product on the left side of this third equality: On the one hand, for $k := \#O(\sigma, i)$, the summands in the expanded product are of the form $\prod_{l=1}^k w(Q_l)$ with $Q_l \in E(\Gamma, \sigma^{l-1}(i), \sigma^l(i))$. But, this implies that $\mathcal{C} : P_i \xrightarrow{Q_1} P_{\sigma(i)} \xrightarrow{Q_2} \dots \xrightarrow{Q_k} P_{\sigma^k(i)} = P_i$ is a cycle of length k. Consequently, we obtain that $\mathcal{C}$ is an element in $C(\sigma, i)$ by its definition in Definition 217(v). Moreover, we also obtain the equality $\prod_{l=1}^k w(Q_l) = w(\mathcal{C})$ by the definition of $w(\mathcal{C})$ in Definition 58. On the other hand, because the edges Q_l run over all edges $P_{\sigma^{l-1}(i)} \to P_{\sigma^l(i)}$ and because of the definition of $C(\sigma, i)$ in Definition 217(v), all circles in $C(\sigma, i)$ are uniquely covered in this way.

Finally, combining the definition of $a_\sigma = \prod_{i \in R(\sigma)} a_{\sigma,i}$ in Definition 217(iv) and the equality in (392) yields the desired identity in (i).

For (ii): Let I_m be the identity matrix in $R^{m\times m}$. First, we obtain the equalities

$$\begin{aligned}\chi_A &= \det(tI_m - A) = (-1)^m \det(A - tI_m)\\ &= (-1)^m \sum_{\sigma\in S(X_0)} \operatorname{sign}(\sigma) \prod_{i\in G(\sigma)} a_{i,\sigma(i)} \prod_{i\in F(\sigma)} (a_{i,\sigma(i)} - t).\end{aligned} \tag{393}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of characteristic polynomials. The second equality holds by the multi-linearity of the determinant. The third equality holds by applying the Leibniz formula for determinants and the definitions of $F(\sigma)$ and $G(\sigma)$ in Definition 217(ii). Now, on the one hand, we have the equalities

$$\prod_{i\in G(\sigma)} a_{i,\sigma(i)} = \prod_{i\in G(\sigma)\cap R(\sigma)} \prod_{l=1}^{\#O(\sigma,i)} a_{\sigma^{l-1}(i),\sigma^l(i)} = \prod_{i\in G(\sigma)\cap R(\sigma)} a_{\sigma,i} \tag{394}$$

for all $\sigma \in S(X_0)$ where the first equality holds by the decomposition of $G(\sigma)$ into a disjoint union of the σ-orbits $O(\sigma, i)$ for all $i \in G(\sigma) \cap R(\sigma)$ and the second equality holds by the definition of $a_{\sigma,i} = \prod_{l=1}^{\#O(\sigma,i)} a_{\sigma^{l-1}(i),\sigma^l(i)}$ in Definition 217(iv). On the other hand, we have

$$\begin{aligned}\prod_{i\in F(\sigma)} (a_{i,\sigma(i)} - t) &= \sum_{k=0}^{m} \sum_{\substack{J\subseteq F(\sigma)\\ \#J+k=\#F(\sigma)}} (-t)^k \prod_{i\in J} a_{i,\sigma(i)}\\ &= \sum_{k=0}^{m} \sum_{\substack{J\subseteq F(\sigma)\\ \#(J\sqcup G(\sigma))=m-k}} (-t)^k \prod_{i\in J} a_{i,\sigma(i)}\\ &= \sum_{k=0}^{m} (-t)^k \sum_{\substack{J\subseteq F(\sigma)\\ \#(J\sqcup G(\sigma))=m-k}} \prod_{i\in J} a_{\sigma,i}\end{aligned} \tag{395}$$

for all $\sigma \in S(X_0)$ where the first equality holds as any summand of the expanded product $\prod_{i\in F(\sigma)}(a_{i,\sigma(i)} - t)$ is of the form $(-t)^k \prod_{i\in J} a_{i,\sigma(i)}$ for some subset $J \subseteq F(\sigma)$ with $\#J + k = \#F(\sigma)$, the second equality holds by the equalities $\#(J \sqcup G(\sigma)) + k = \#J + k + \#G(\sigma) = \#F(\sigma) + \#G(\sigma) = \#X_0 = m$ and the third equality holds by elementary arithmetics and by the fact that the equalities $a_{\sigma,i} = \prod_{l=1}^{\#O(\sigma,i)} a_{\sigma^{l-1}(i),\sigma^l(i)} = \prod_{l=1}^{1} a_{\sigma^{l-1}(i),\sigma^l(i)} = a_{i,\sigma(i)}$ hold for all $i \in J \subseteq F(\sigma)$. Then we conclude the equalities

$$\begin{aligned}\chi_A &= (-1)^m \sum_{\sigma\in S(X_0)} \operatorname{sign}(\sigma) \left(\prod_{i\in G(\sigma)\cap R(\sigma)} a_{\sigma,i}\right) \left(\sum_{k=0}^{m} (-t)^k \sum_{\substack{J\subseteq F(\sigma)\\ \#(J\sqcup G(\sigma))=m-k}} \prod_{i\in J} a_{\sigma,i}\right)\\ &= \sum_{k=0}^{m} t^k (-1)^{m+k} \sum_{\sigma\in S(X_0)} \sum_{\substack{J\subseteq F(\sigma)\\ \#(J\sqcup G(\sigma))=m-k}} \operatorname{sign}(\sigma) \left(\prod_{i\in G(\sigma)\cap R(\sigma)} a_{\sigma,i}\right) \left(\prod_{i\in J} a_{\sigma,i}\right)\\ &= \sum_{k=0}^{m} t^k (-1)^{m+k} \sum_{\substack{\sigma\in S(X_0)\\ J\subseteq F(\sigma)\\ \#(J\sqcup G(\sigma))=m-k}} \operatorname{sign}(\sigma) \prod_{i\in (G(\sigma)\sqcup J)\cap R(\sigma)} a_{\sigma,i}\end{aligned} \tag{396}$$

where the equalities hold by the following reasonings: The first equality holds by applying the two identities in (394) and (395) to the corresponding factors in the identity in (393). The second equality holds by elementary arithmetics. The third equality holds because the inclusions $J \subseteq F(\sigma) \subseteq R(\sigma)$ hold by the definitions of $F(\sigma)$ and $R(\sigma)$ in Definition 217(ii) and because combining these inclusions and one of de Morgan' rules then yields the equality $(G(\sigma) \sqcup J) \cap R(\sigma) = (G(\sigma) \cap R(\sigma)) \sqcup J$.

Now, combining the identity in (396) and the identity $\chi_A = \sum_{k=0}^{m} c_k t^k$ provides the equality

$$c_k = (-1)^{m+k} \sum_{\substack{\sigma \in S(X_0) \\ J \subseteq F(\sigma) \\ \#(J \sqcup G(\sigma)) = m-k}} \operatorname{sign}(\sigma) \prod_{i \in (G(\sigma) \sqcup J) \cap R(\sigma)} a_{\sigma,i} \tag{397}$$

for all $k = 0, \ldots, m$.

In (397), the sum runs over all elements in the set

$$T_{m-k} = \{(\sigma, J) \in S(X_0) \times \{J : J \subseteq X_0\} : J \subseteq F(\sigma) \text{ and } \#(G(\sigma) \sqcup J) = m - k\}.$$

We will show that the following maps ϕ and ψ are well defined and inverse to each other:

$$\phi : T_{m-k} \to \coprod_{\substack{X \subseteq X_0 \\ \#X = m-k}} S(X) \text{ via } (\sigma, J) \mapsto \sigma'$$

where σ' is the canonical restriction of σ to an element in $S(G(\sigma) \sqcup J)$ and

$$\psi : \coprod_{\substack{X \subseteq X_0 \\ \#X = m-k}} S(X) \to T_{m-k} \text{ via } \sigma \mapsto (\hat{\sigma}, F(\sigma))$$

where $\hat{\sigma} \in S(X_0)$ is the canonical extension of σ in Definition 217(iii).

First, let $(\sigma, J) \in T_{m-k}$. Then we have $X := G(\sigma) \sqcup J \subseteq X_0$ with $\#X = m - k$ by the definition of T_{m-k}. Moreover, as σ is a bijection and fixes all elements in $X_0 \backslash X \subseteq F(\sigma)$, we deduce the equality $\sigma(X) = X$ and, hence, σ canonically restricts to an element in $\sigma' \in S(X)$. Thus, ϕ is well defined. Furthermore, we also obtain the equalities

$$\widehat{(\sigma')} = \sigma \text{ and } F(\sigma') = J. \tag{398}$$

by the following reasonings: The first equality holds since the canonical extension and restriction are clearly inverse to each other. The second equality holds since σ' fixes any element in $F(\sigma) \cap X = J$ and does not fix any element in $G(\sigma) = X \backslash J$.

Next, let $X \subseteq X_0$ with $\#X = m - k$ and $\sigma \in S(X)$. Then $\hat{\sigma} \in S(X_0)$ and $J := F(\sigma) \subseteq F(\hat{\sigma}) \subseteq X_0$. Moreover, we clearly have the equality $G(\sigma) = G(\hat{\sigma})$ and, consequently, we conclude the equalities $\#(G(\hat{\sigma}) \sqcup J) = \#(G(\sigma) \sqcup F(\sigma)) = \#X = m - k$. Hence, ψ is well defined. Furthermore, we also obtain the identity

$$(\hat{\sigma})' = \sigma \tag{399}$$

since the canonical restriction and extension are obviously inverse to each other. Thus, (398) yields $\psi \circ \phi = \mathrm{id}$ and (399) yields $\phi \circ \psi = \mathrm{id}$. Especially, this means that ϕ is bijective.

On the one hand, for all $(\sigma, J) \in T_{m-k}$, we get the equalities

$$\operatorname{sign}(\sigma') = \operatorname{sign}(\widehat{(\sigma')}) = \operatorname{sign}(\sigma) \tag{400}$$

where the first equality holds by the definition of sign in Definition 217(iii) and the second equality holds by the identity $\widehat{(\sigma')} = \sigma$ in (398). On the other hand, for all $(\sigma, J) \in T_{m-k}$ and for $X := G(\sigma) \sqcup J$, we also derive the equalities

$$\begin{aligned} R(\sigma') &= (R(\sigma') \cup X_0 \backslash X) \cap X = (\{\min O(\sigma, i) : i \in X\} \cup X_0 \backslash X) \cap X \\ &= (\{\min O(\sigma, i) : i \in X\} \cup \{\min O(\sigma, i) : i \in X_0 \backslash X\}) \cap X \\ &= \{\min O(\sigma, i) : i \in X_0\} \cap X = R(\sigma) \cap X \end{aligned} \tag{401}$$

where the equalities hold by the following reasonings: The first equality holds by applying one of de Morgan's rules, by the equality $X_0 \backslash X \cap X = \emptyset$ and by the inclusion $R(\sigma') \subseteq X$. The second equality hold by the definition of R in Definition 217(ii) and by the fact that $\sigma(i) = \sigma'(i)$ holds for all $i \in X$ and, thus, implies the equality $O(\sigma'.i) = O(\sigma, i)$ for all $i \in X$. The third equality holds as the inclusion $X_0 \backslash X \subseteq F(\sigma)$ implies the equality $O(\sigma, i) = \{i\}$ for all $i \in X_0 \backslash X$. The fourth equality holds by the equality $X_0 \backslash X \cup X = X_0$. The last equality holds by the definition of R in Definition 217(ii).

Altogether, we conclude the desired identity in (ii) by the equalities

$$c_k = (-1)^{m+k} \sum_{\substack{X \subseteq X_0 \\ \#X = m-k \\ \sigma \in S(X)}} \operatorname{sign}(\sigma) \prod_{i \in R(\sigma)} a_{\sigma,i} = (-1)^{m+k} \sum_{\substack{X \subseteq X_0 \\ \#X = m-k \\ \sigma \in S(X)}} \operatorname{sign}(\sigma) a_\sigma$$

where the first equality holds by combining the identity in (397), the bijectivity of ϕ and the identities in (400) and (401) and the second equality holds by the definition of $a_\sigma = \prod_{i \in R(\sigma)} a_{\sigma,i}$ in Definition 217(iv). □

Corollary 219. *Let Γ be a finite directed graph, let $w : E(\Gamma) \to \mathbb{C}[y_\mathbf{P}, y_\mathbf{P}^{-1}]$ be a weight function on Γ such that the image of w is contained in $\mathbb{R}_{\geq 0}[y_\mathbf{P}, y_\mathbf{P}^{-1}] \backslash \{0\}$. Moreover, let $A \in \mathbb{R}_{\geq 0}[y_\mathbf{P}, y_\mathbf{P}^{-1}]^{m \times m}$ be the w-adjacency matrix of Γ for some enumeration $v = (P_1, \dots, P_m)$ of the vertices in Γ and let $\chi_A \in \mathbb{C}[y_\mathbf{P}, y_\mathbf{P}^{-1}][t]$ be the characteristic polynomial of A.*

Then $\chi_A \in \mathbb{C}[t]$ if and only if $w(\mathcal{C}) \in \mathbb{R}_{>0}$ for all circles $\mathcal{C}$.

Proof. By the combination of the items (i) and (ii) in Theorem 218, the coefficients $c_k \in \mathbb{R}[y_\mathbf{P}, y_\mathbf{P}^{-1}]$ of $\chi_A = \sum_{k=0}^m c_k t^k$ are of the form

$$c_k = \sum_{X,\sigma} (-1)^{m+k} \operatorname{sign}(\sigma) \prod_{i \in R(\sigma)} \sum_{\mathcal{C} \in C(\sigma,i)} w(\mathcal{C}) \tag{402}$$

for all $k = 0, \dots, m$ where the first sum runs over all $X \subseteq X_0$ with $\#X = m - k$ and all $\sigma \in S(X)$.

First, assume that the $w(\mathcal{C}) \in \mathbb{R}_{>0}$ for all circles $\mathcal{C}$ in Γ. Then this is especially true for all circles $\mathcal{C} \in C(\sigma, i)$ on the right side of the identity in (402). Consequently, the coefficient c_k of χ_A is an element in $\mathbb{R}$ and, hence, we arrive at the desired statement $\chi_A = \sum_{k=0}^m c_k t^k \in \mathbb{C}[t]$.

Second, assume the contrary, i.e that there is a circle in Γ such that its w-value has non-zero y_q-degree for some $q \in \mathbb{P}$. Then there is also a shortest circle with this property, say of length l. Let $\mathcal{C}_0 : P_{i_1} \to P_{i_2} \to \cdots \to P_{i_{l+1}} = P_{i_1}$ be such a shortest circle. Since $\mathcal{C}_0$ is a circle, there are no repeating vertices in $\mathcal{C}_0$ except for $P_{i_{l+1}} = P_{i_1}$. Therefore, the l-cycle $(i_1 \; i_2 \; \dots \; i_l) \in S(X_0)$ is well defined and we may define $\sigma_0 \in S(X)$ as the canonical restriction of this l-cycle on the subset $X := \{i_1, \dots, i_l\} \subseteq X_0$ which has l elements. Then we immediately conclude

$$\mathcal{C}_0 \in C(\sigma_0, i_1) \tag{403}$$

by the definitions of $\mathcal{C}_0$ and σ_0 from above and of $C(\sigma_0, i_1)$ in Definition 217(v). We also conclude the equality $O(\sigma_0, i_1) = X$ by the definitions of X and σ_0. Moreover, as $\mathcal{C}_0$ is a circle, we may choose $\mathcal{C}_0$ such that the first index i_1 is also the minimal element in X and, thus, even obtain the identity

$$R(\sigma_0) = \{i_1\} \tag{404}$$

by the definition of R in Definition 217(ii). Hence, combining (402), (403), (404) and the assumption that the codomain of w does not contain zero yields that

$$\begin{aligned}&(-1)^l \operatorname{sign}(\sigma_0) w(\mathcal{C}_0) \text{ is a non-zero summand of } c_{m-l} \text{ in } (402)\\ &\text{and has non-vanishing } y_q\text{-degree.}\end{aligned} \tag{405}$$

Next, we notice that any subset $X \subseteq X_0$ with $\#X = l$ and any $\sigma \in S(X)$ with $\#R(\sigma) \geq 2$ supplies that the quotient set $\langle\sigma\rangle\backslash X$ contains $\#R(\sigma) \geq 2$ orbits $O(\sigma, i)$ and, thus, that all these orbits contain less than l elements. Therefore, for all such σ and $i \in R(\sigma)$, we conclude that the set $C(\sigma, i)$ only contains circles $\mathcal{C}$ with a length which is less than l. But, by the choice of l, this means that

$$\text{the } y_q\text{-degree of } w(\mathcal{C}) \text{ vanishes} \tag{406}$$

for all $X \subseteq X_0$ with $\#X = l$, all $\sigma \in S(X)$ with $R(\sigma) \geq 2$, all $i \in R(\sigma)$ and all $\mathcal{C} \in C(\sigma, i)$.

Let us finally check that all coefficients of the summands of c_{m-l} (as polynomials in y_q) with $\#R(\sigma) = 1$ have the same sign: For all $X \subseteq X_0$ with $\#X = l$, all $\sigma \in S(X)$ with $\#R(\sigma) = 1$, we have the equality $\#O(\sigma, i) = l$ and, thus, σ is a restriction of an l-cycle in $S(X_0)$. This then implies the identity

$$\operatorname{sign}(\sigma) = (-1)^{l-1} \tag{407}$$

by the definition of $\operatorname{sign}(\sigma)$ in Definition 217(iii). Moreover, as $w(\mathcal{C})$ is an element in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$,

$$\text{the non-zero coefficients of } w(\mathcal{C}) \text{ are positive real numbers} \tag{408}$$

for all $X \subseteq X_0$ with $\#X = l$, all $\sigma \in S(X)$ with $R(\sigma) = 1$, all $i \in R(\sigma)$ and all $\mathcal{C} \in C(\sigma, i)$.

Finally, combining (405), (406), (407) and (408) provides that there is no way that the non-zero summand $(-1)^l \operatorname{sign}(\sigma_0) w(\mathcal{C}_0) = -w(\mathcal{C}_0)$ of c_{m-l} in (402) can be canceled out. Hence, c_{m-l} must be non-zero and have non-vanishing y_q-degree. This again implies the desired statement $\chi_A = \sum_{k=0}^{m} c_k t^k \notin \mathbb{C}[t]$. □

7.1.6 Proof of the First Interim Result

Finally, we are prepared to prove the first interim result in Theorem 168.

Proof of Theorem 168. First of all, we notice that since the sum of two elements in the image $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$ of w never vanishes, we have the inequality

$$\sum_{Q \in E(\Gamma, v, v')} w(Q) \neq 0 \tag{409}$$

for all $v, v' \in V(\Gamma)$ with $E(\Gamma, v, v') \neq \emptyset$.

For (i): Because of (409), we may apply Lemma 63(i) to A and, by that, already obtain (i).

For (ii): On the one hand, because of (409), we may apply Lemma 63(ii) to A and, by that, already obtain the first desired statement in (ii), namely that A is reducible.

On the other hand, consider the upper block triangular matrix PAP^t in Lemma 215(i) with quadratic irreducible matrices $A_1, \ldots, A_r$ on the block diagonal. Then we compute

$$\rho(A(x^{\log_2(\mathbf{P})})) = \rho((PAP^t)(x^{\log_2(\mathbf{P})})) = \max_{k=1,\ldots,r} \rho(A_k(x^{\log_2(\mathbf{P})})). \tag{410}$$

for all $x \in]1, 2[$ where the first equality holds since it is well known that the matrices $A(x^{\log_2(\mathbf{P})})$ and

$$\begin{aligned}(PAP^t)(x^{\log_2(\mathbf{P})}) &= P(x^{\log_2(\mathbf{P})})A(x^{\log_2(\mathbf{P})})P(x^{\log_2(\mathbf{P})})^t = PA(x^{\log_2(\mathbf{P})})P^t \\ &= PA(x^{\log_2(\mathbf{P})})P^{-1}\end{aligned}$$

have the same eigenvalues and the second equality holds by the upper block triangular form of $(PAP^t)(x^{\log_2(\mathbf{P})})$ with the matrices $A_k(x^{\log_2(\mathbf{P})})$ on its block diagonal.

Moreover, since A is reducible, we have $r \geq 2$ and, thus, we can apply Lemma 215(ii)(c). From doing so, we derive the estimates

$$\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) \leq d \text{ and } \min(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d \tag{411}$$

for all $k = 1, \ldots, r$. Now, we distinguish two cases: If $\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) < d$, then both cases in Lemma 214 provide the estimate $\rho(A_k(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$. Else, if $\max(\rho(A_k(\mathbf{1})), \rho(A_k(\mathbf{P}))) = d$, then combining the equalities $1^{\log_2(\mathbf{P})} = \mathbf{1}$ and $2^{\log_2(\mathbf{P})} = \mathbf{P}$, the second estimate in (411) and Lemma 214(i) also provides the estimate $\rho(A_k(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$. We have established that $\rho(A_k(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$ and all $k = 1, \ldots, r$.

Hence, by these estimates and by the equality in (410), we finally deduce the desired estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$.

For (iii)(a): On the one hand, if Γ is not strongly connected, the desired estimate in (iii)(a) immediately follows from Lemma 168(ii).

On the other hand, suppose that Γ is strongly connected and that one of the column sums of $A(\mathbf{1})$ or one of the row sums of $A(\mathbf{P})$ is less than d. Combining the assumption that Γ is strongly connected, (409) and Lemma 63(ii) provides that A is irreducible.

Let $(a_{i,j})_{i,j} := A$. Because the image of w is contained in $\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]\backslash\{0\}$, we conclude that $a_{i,j}(\mathbf{1})$ (resp. $a_{i,j}(\mathbf{P})$) vanishes if and only if $a_{i,j}$ vanishes. But, since the irreducibility of a matrix only depends on the position of its non-zero entries, we conclude that $A(\mathbf{1})$ and $A(\mathbf{P})$ are irreducible real matrices with non-negative entries. Consequently, we derive the estimate $\max(\rho(A(\mathbf{1})), \rho(A(\mathbf{P}))) \leq d$ from the assertion that all the row sums of $A(\mathbf{1})$ and the column sums of $A(\mathbf{P})$ are at most d and from [HJ90, p. 492, Theorem 8.1.22].

Furthermore, we even derive the estimate $\min(\rho(A(\mathbf{1})), \rho(A(\mathbf{P})) < d$ from the assertion that the sums are not only at most d, but there is even a sum which is less than d, and from Lemma 64(ii). Now, if $\max(\rho(A(\mathbf{1})), \rho(A(\mathbf{P})) < d$, both cases in Lemma 214 yield the desired estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$. Else, if $\max(\rho(A(\mathbf{1})), \rho(A(\mathbf{P})) = d$, then, because of the estimate $\min(\rho(A(\mathbf{1})), \rho(A(\mathbf{P})) < d$, we can apply Lemma 214(i) and also obtain the desired estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1, 2[$. Hence, in any case, (iii)(a) follows.

For (iii)(b): Suppose that all row sums of $A(\mathbf{1})$ and the column sums of $A(\mathbf{P})$ are constantly d and that there is a circle $\mathcal{C}$ in Γ such that $w(\mathcal{C}) \notin \mathbb{R}_{\geq 0}$. Then Corollary 219 yields $\chi_A \notin \mathbb{C}[t]$. Consequently, combining this, the assumption that all sums are equal to d and Lemma 168(i) provides that we can apply Lemma 216(i) to A. By doing so, we obtain the desired estimate $\rho(A(x^{\log_2(\mathbf{P})})) < d$ for all $x \in]1,2[$. Hence, (iii)(b) follows.

For (iii)(c): Suppose that all row sums of $A(\mathbf{1})$ and the column sums of $A(\mathbf{P})$ are constantly d and that all circles $\mathcal{C}$ in Γ satisfy $w(\mathcal{C}) \in \mathbb{R}_{>0}$. Then Corollary 219 yields $\chi_A \in \mathbb{C}[t]$. Consequently, combining this, the assumption that all sums are equal to d and Lemma 168(i) provides that we can apply Lemma 216(ii). By doing so, we obtain the desired identity $\rho(A(x^{\log_2(\mathbf{P})})) = d$ for all $x \in [1,2]$ in (iii)(c) and that A is irreducible. Finally, by (409), Lemma 63(ii) supplies that Γ is strongly connected and, hence, we also established the two remaining desired statements in (iii). ☐

7.2 Second Interim Result - Balanced Ramification Indices

Summary of the results of this section. In this section, we will prove the second interim result in Corollary 170 which will be a corollary of Theorem 221.

Main idea of the proof. Without giving a formal definition, we notice that any closed path $\mathcal{P}$ in a directed graph Γ has a cactus-like form via some circles in Γ as on the left side in Figure 7.2. Now we will insert closed paths from Γ into $\mathcal{P}$ until the resulting closed path $\mathcal{P}'$ has a bouquet-like form as on the right side in Figure 7.2.

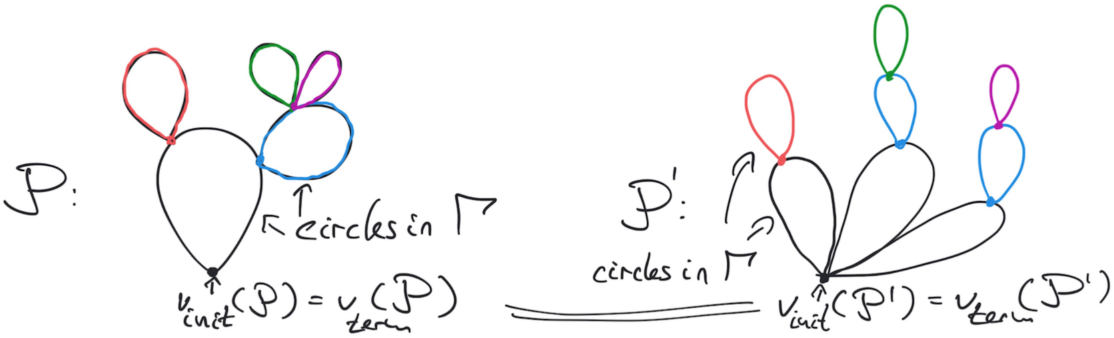


Figure 7.2: Cactus- and bouquet-like forms of closed paths

Then, for any insertion step, say from $\mathcal{P}_1$ to $\mathcal{P}_2$, we will show in Lemma 220(iii) that the Abhyankar ramification indices $\tilde{e}(R_1|R_1 \cap F_0)$ for all $R_1 \in \mathbb{P}_{\mathcal{F}}[\mathcal{P}_1]$ are divisors of the Abhyankar ramification indices $\tilde{e}(R_2|R_2 \cap F_0)$ for all $R_2 \in \mathbb{P}_{\mathcal{F}}[\mathcal{P}_2]$. Consequently, we will obtain that $\tilde{e}(Q|Q \cap F_0)$ is a divisor $\tilde{e}(Q'|Q' \cap F_0)$ for all $Q \in \mathbb{P}_{\mathcal{F}}[\mathcal{P}]$ and all $Q' \in \mathbb{P}_{\mathcal{F}}[\mathcal{P}']$.

Finally, from Lemma 220(ii) and Lemma 220(iv), we will derive that the Abhyankar ramification index $\tilde{e}(Q'|Q' \cap F_0)$ of every place $Q' \in \mathbb{P}_{\mathcal{F}}[\mathcal{P}']$ which lies over this closed path $\mathcal{P}'$ in bouquet-like form is bounded from above by a single number only depending on Γ.

An unusual strategy. Also note that although a usual strategy to conclude that one ramification index $e(Q|P)$ divides another ramification index $e(Q'|P)$ is to show that Q'/Q is an extension of places, this will not be our strategy here. Indeed, because we will insert paths, such a relation Q'/Q cannot be expected at all.

However, we are not interested in the usual ramification indices $e(Q|P)$ anyways but in the Abhyankar ramification indices $\tilde{e}(Q|P)$. Here, in order to compute the Abhyankar ramification index $\tilde{e}(Q|P)$, we have to iteratively apply the $\tilde{e}$-version of Abhyankar's Lemma 44(i) to the elementary extensions in the pyramid $\mathrm{Pyr}(Q)$. Thus, $\tilde{e}(Q|P)$ only depends on the positions and values of the ramification indices in the path $\mathrm{Path}(Q)$ and we therefore have to show that this 'Abhyankar's Lemma'-iteration provides a multiple after we insert a closed path.

Structure of this section. In Subsection 7.2.1, we will prove the auxiliary Lemma 220 which, for instance, proves that the aforementioned insertion steps indeed provide multiples.

In Subsection 7.2.2, we will prove Theorem 221 which supplies the concrete upper bound for Corollary 170.

In Subsection 7.2.3, we will then finally prove our second interim result, which is Corollary 170. This will be an immediate consequence of Theorem 221.

7.2.1 Properties of Paths and Circles with Balanced Ramification Indices

Purpose of this subsection. In this subsection, we will make further preparations for the proof of Theorem 221 by proving the auxiliary Lemma 220. For instance, in Lemma 220(iii), we will show that the aforementioned insertion steps from the paragraph indeed provide multiples.

Lemma 220. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0) and let $\Gamma_\mathcal{F}$ be its tower graph. Moreover, for all $i = 1, \dots, r$, let $\mathcal{C}_i$ be closed paths in $\Gamma_\mathcal{F}$ of length n_i and balanced ramification indices which all start at the same initial vertex and let $Q_i \in \mathbb{P}_{F_{n_i}}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}_i))$. Then the following hold:*

(i) Let $\mathcal{P}$ and $\mathcal{P}'$ be paths and $\mathcal{C}$ be a closed path in $\Gamma_\mathcal{F}$ such that the terminal vertex of $\mathcal{P}$ is the initial vertex of $\mathcal{P}'$ and $\mathcal{C}$ (see the middle part of Figure 7.3).

On the one hand, if $\mathcal{P}$ and $\mathcal{P}'$ have balanced ramification indices, then their composition $\mathcal{P}\mathcal{P}'$ also has balanced ramification indices.

On the other hand, if $\mathcal{P}\mathcal{P}'$ and $\mathcal{C}$ have balanced ramification indices, then $\mathcal{P}\mathcal{C}\mathcal{P}'$ also has balanced ramification indices.

(ii) Let $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}))$ for $n := \sum_{i=1}^r n_i$ and $\mathcal{C} := \prod_{i=1}^r \mathcal{C}_i$ (see the left part of Figure 7.3). Then the closed path $\mathcal{C}$ also has balanced ramification indices and we have the identity

$$\tilde{e}(Q|Q \cap F_0) = \operatorname*{lcm}_{i=1,\dots,r} \tilde{e}(Q_i|Q_i \cap F_0).$$

(iii) Let $\mathcal{P}$ and $\mathcal{P}'$ be paths and $\mathcal{C}$ be a closed path in $\Gamma_\mathcal{F}$ such that $\mathcal{C}$ has balanced ramification indices and the terminal vertex of $\mathcal{P}$ is the initial vertex of $\mathcal{C}$ and $\mathcal{P}'$ (see the middle part of Figure 7.3), let $\mathcal{P}_1 := \mathcal{P}\mathcal{P}'$ be of length l_1, let $\mathcal{P}_2 := \mathcal{P}\mathcal{C}\mathcal{P}'$ be of length l_2 and let $R_i \in \mathbb{P}_{F_{l_i}}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_i))$ for all $i = 1, 2$. Then $\tilde{e}(R_1|R_1 \cap F_0)$ divides $\tilde{e}(R_2|R_2 \cap F_0)$.

(iv) Let $\mathcal{P}_i$ be paths in $\Gamma_\mathcal{F}$ such that the terminal vertex of $\mathcal{P}_i$ is the initial vertex of $\mathcal{P}_{i+1}$ for all $i = 1, \dots, 2r-1$, let $\mathcal{P} := \prod_{i=1}^{2r} \mathcal{P}_i$ be a path of length l and let $R \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}))$. If $\mathcal{C}_i = \mathcal{P}_i\mathcal{P}_{2r+1-i}$ for all $i = 1, \dots, r$ (see the right part of Figure 7.3), then $\mathcal{P}$ has balanced ramification indices and $\tilde{e}(R|R \cap F_0)$ divides $\prod_{i=1}^r \tilde{e}(Q_i|Q_i \cap F_0)$.

Proof. For (i): Both statements immediately follow from the equivalence of the items (i) and (vi) in Lemma 166 and from the fact that the multiplicativity of weight functions on paths in Definition 58 imply the equalities

$$w'_{x_\mathbf{P}, \hat{x}_\mathbf{P}}(\mathcal{P}\mathcal{P}') = w'_{x_\mathbf{P}, \hat{x}_\mathbf{P}}(\mathcal{P}) \cdot w'_{x_\mathbf{P}, \hat{x}_\mathbf{P}}(\mathcal{P}')$$

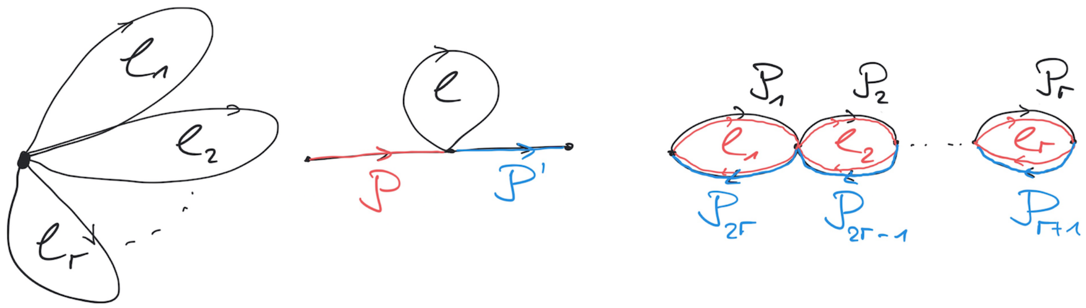


Figure 7.3: Figures in an auxiliary Lemma

and

$$w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}\mathcal{C}\mathcal{P}) = w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) \cdot w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{C}) \cdot w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}') = w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}\mathcal{P}') \cdot w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{C})$$

for all $x_{\mathbf{P}}, \hat{x}_{\mathbf{P}} \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$.

For (ii): First, we notice that the 'on the one hand'-part in Lemma 220(i) immediately implies the first desired statement in (ii), namely that $\mathcal{C} = \prod_{i=1}^r \mathcal{C}_i$ also has balanced ramification indices.

Second, let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ (see Figure 7.4). Then we have the equalities

$$(P_{i,j})_{j-i\leq 1} = \mathrm{Path}(Q) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}) = \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}_1) \cdot \sigma^{n_1}(\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=2}^{r} \mathcal{C}_i)) \tag{412}$$

where the first equality hold by the definition of Path in Definition/Lemma and by the choice of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$, the second equality holds by the choice of $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}))$ and the third equality holds by the equality $\mathcal{C} = \sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=1}^r \mathcal{C}_i)$ and by Lemma 78. In particular, the equality in (412) provides $\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=2}^r \mathcal{C}_i) = (\sigma^{-n_1}(P_{i+n_1,j+n_1}))_{j-i\leq 1} \in W(\mathcal{F}, 0, n-n_1)$ and, thus,

$$\sigma^{-n_1}(P_{n_1,n}) \in \mathbb{P}_{F_{n-n_1}}(\sigma_{\Gamma_{\mathcal{F}}}(\prod_{i=2}^{r} \mathcal{C}_i)). \tag{413}$$

We will show the desired identity in (ii) by induction over $r \in \mathbb{N}$. For $r = 1$, there is nothing to show. For $r \geq 2$, we already obtain the desired equality by the equalities

$$\begin{aligned}\tilde{e}(Q|Q\cap F_0) = \tilde{e}(P_{0,n}|P_{0,n_1})\tilde{e}(P_{0,n_1}|P_{0,0}) &= \frac{\tilde{e}(P_{n_1,n}|P_{n_1,n_1})\tilde{e}(P_{0,n_1}|P_{0,0})}{\gcd(\tilde{e}(P_{n_1,n}|P_{n_1,n_1}), \tilde{e}(P_{0,n_1}|P_{n_1,n_1}))} \\ &= \frac{\tilde{e}(P_{n_1,n}|P_{n_1,n_1})\tilde{e}(Q_1|Q_1\cap F_0)}{\gcd(\tilde{e}(P_{n_1,n}|P_{n_1,n_1}), \tilde{e}(Q_1|Q_1\cap F_0))} = \mathrm{lcm}(\tilde{e}(Q_1|Q_1\cap F_0), \tilde{e}(P_{n_1,n}|P_{n_1,n_1})) \\ &= \mathrm{lcm}(\tilde{e}(Q_1|Q_1\cap F_0), \tilde{e}(\sigma^{-n_1}(P_{n_1,n})|\sigma^{-n_1}(P_{n_1,n})\cap F_0)) = \underset{i=1,\ldots,r}{\mathrm{lcm}}\, \tilde{e}(Q_i|Q_i\cap F_0)\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ in Definition 11 and by the multiplicative transitivity rule for the Abhyankar ramification indices in Lemma 44(ii).

The second equality holds by applying the $\tilde{e}$-version of Abhyankar's Lemma in Lemma 44(i) to $\tilde{e}(P_{0,n}|P_{0,n_1})$.

For the third equality, we first notice that the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 implies the equality $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}_1) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n_1)$ and, thus, P_{0,n_1} and Q_1 are both places in $\mathbb{P}_{F_{n_1}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}_1))$. Then combining that $\mathcal{C}_1$ has balanced ramification indices and

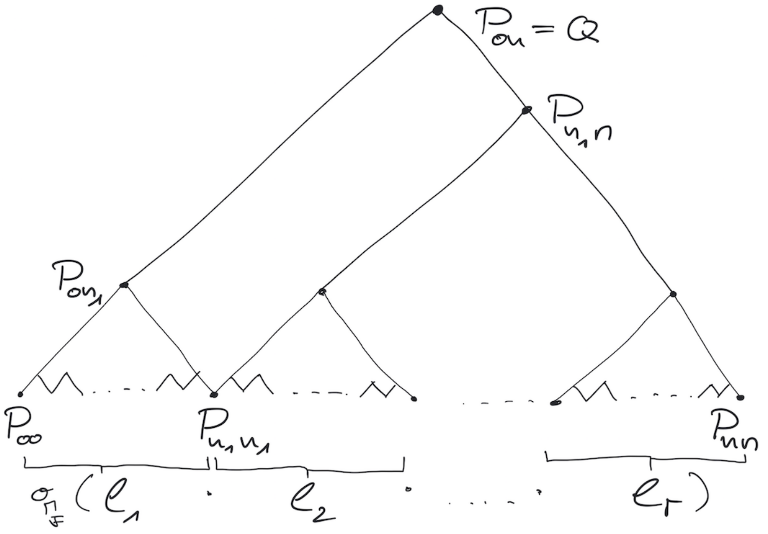


Figure 7.4: Composition of closed paths in a proof

the equivalence of the items (i) and (iii) in Lemma 166 yields the equality $\tilde{e}(P_{0,n_1}|P_{n_1,n_1}) = \tilde{e}(P_{0,n_1}|P_{0,0})$. Thus, the third equality follows from this equality, from the fact that the Abhyankar ramification indices only depend on the path $\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}_1)$ by their definition in Definition 41 and from the fact that the equalities $\mathrm{Path}(Q_1) = \sigma_{\Gamma_\mathcal{F}}(\mathcal{C}_1) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, n_1)$ provide the equality $P_{0,0} = Q_1 \cap F_0$ by Definition 11.

The fourth equality is a well known rule for the gcd and lcm of a natural numbers. The fifth equality holds since Lemma 46 supplies the equality $\tilde{e}(P_{n_1,n}|P_{n_1,n_1}) = \tilde{e}(\sigma^{-n_1}(P_{n_1,n})|\sigma^{-n_1}(P_{n_1,n_1}))$ and since $\sigma^{-n_1}(P_{n_1,n_1})$ is a place in F_0 which lies under $\sigma^{-n_1}(P_{n_1,n})$.

The last equality holds by $\sigma^{-n_1}(P_{n_1,n}) \in \mathbb{P}_{F_{n-n_1}}(\sigma_{\Gamma_\mathcal{F}}(\prod_{i=2}^{r}\mathcal{C}_i))$ in (413) and by then applying the induction hypothesis to $\prod_{i=2}^{r}\mathcal{C}_i$.

For (iii): Let $(P'_{i,j})_{i,j} := \mathrm{Pyr}(R_1)$, let $(P_{i,j})_{i,j} := \mathrm{Pyr}(R_2)$, let k be the length of $\mathcal{P}$, let $k' := l_1 - k$ be the length of $\mathcal{P}'$ and let n be the length of $\mathcal{C}$ (see Figure 7.5).

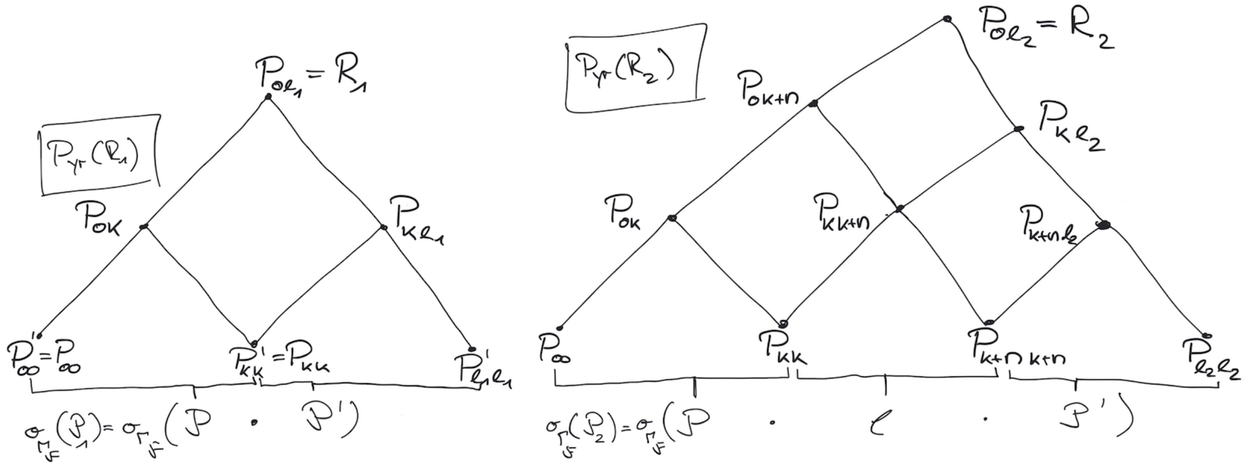


Figure 7.5: Inserting closed paths in a proof

We notice that the choices $(P'_{i,j}) = \mathrm{Pyr}(R_1)$, $(P_{i,j})_{i,j} = \mathrm{Pyr}(R_2)$, the assumptions $\mathcal{P}_1 = \mathcal{P}\mathcal{P}'$, $\mathcal{P}_2 = \mathcal{P}\mathcal{C}\mathcal{P}'$ and $R_i \in \mathbb{P}_{F_{l_i}}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}_i))$ for all $i = 1, 2$ and the definition of $\sigma_{\Gamma_\mathcal{F}}$ in

Definition/Lemma 76 provide the equalities

$$\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}) = (P'_{i,j})_{j-i\leq 1} = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, k), \tag{414}$$

$$\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}) = (\sigma^{-k}(P_{k+i,k+j}))_{j-i\leq 1} \in W(\mathcal{F}, n), \tag{415}$$

$$\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}') = (\sigma^{-k}(P'_{k+i,k+j}))_{j-i\leq 1} = (\sigma^{-(k+n)}(P_{k+n+i,k+n+j}))_{j-i\leq 1} \in W(\mathcal{F}, k'). \tag{416}$$

First, the equalities in (414) yields that $P'_{0,k}$ and $P_{0,k}$ are both places in $\mathbb{P}_{F_k}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$. Moreover, we obtain the equalities

$$\tilde{e}(P'_{0,k}|P'_{0,0}) = \tilde{e}(P_{0,k}|P_{0,0}) \text{ and } \tilde{e}(P'_{0,k}|P'_{k,k}) = \tilde{e}(P_{0,k}|P_{k,k}) \tag{417}$$

because the Abhyankar ramification indices only depend on the path by their definition in Definition 41.

Second, the equality in (415) yields

$$\sigma^{-k}(P_{k,k+n}) \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C})). \tag{418}$$

Furthermore, we obtain the equalities

$$\begin{aligned}\tilde{e}(P_{k,k+n}|P_{k+n,k+n}) &= \tilde{e}(\sigma^{-k}(P_{k,k+n})|\sigma^{-k}(P_{k+n,k+n})) = \tilde{e}(\sigma^{-k}(P_{k,k+n})|\sigma^{-k}(P_{k,k}))\\ &= \tilde{e}(P_{k,k+n}|P_{k,k})\end{aligned} \tag{419}$$

where the first and last equality hold by the invariance of the Abhyankar ramification indices under the action of σ in Lemma 46 and the second equality holds by combining the assumption that $\mathcal{C}$ has balanced ramification indices, the equivalence of the items (i) and (iii) in Lemma 166 and (418).

Third, the equalities in (416) yield that $\sigma^{-k}(P'_{k,l_1})$ and $\sigma^{-(k+n)}(P_{k+n,l_2})$ are both places in $\mathbb{P}_{F_{k'}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}'))$. Moreover, we obtain the equalities

$$\begin{aligned}\tilde{e}(P'_{k,l_1}|P'_{k,k}) &= \tilde{e}(\sigma^{-k}(P'_{k,l_1})|\sigma^{-k}(P'_{k,k})) = \tilde{e}(\sigma^{-(k+n)}(P_{k+n,l_2})|\sigma^{-(k+n)}(P_{k+n,k+n}))\\ &= \tilde{e}(P_{k+n,l_2}|P_{k+n,k+n})\end{aligned} \tag{420}$$

where the first and last equalities hold by the invariance of the Abhyankar ramification indices under the action of σ in Lemma 46 and the second equality holds because the Abhyankar ramification indices only depend on the path in their definition in Definition 41.

Next, we make the following three computations: First, we compute

$$\begin{aligned}\tilde{e}(R_1|R_1\cap F_0) = \tilde{e}(P'_{0,l_1}|P'_{0,k})\tilde{e}(P'_{0,k}|P'_{0,0}) &= \frac{\tilde{e}(P'_{0,k}|P'_{0,0})\tilde{e}(P'_{k,l_1}|P'_{k,k})}{\gcd(\tilde{e}(P'_{0,k}|P'_{k,k}), \tilde{e}(P'_{k,l_1}|P'_{k,k}))}\\ &= \frac{\tilde{e}(P'_{0,k}|P'_{0,0}) \operatorname{lcm}(\tilde{e}(P'_{0,k}|P'_{k,k}), \tilde{e}(P'_{k,l_1}|P'_{k,k}))}{\tilde{e}(P'_{0,k}|P'_{k,k})}\\ &= \frac{\tilde{e}(P_{0,k}|P_{0,0}) \operatorname{lcm}(\tilde{e}(P_{0,k}|P_{k,k}), \tilde{e}(P_{k+n,l_2}|P_{k+n,k+n}))}{\tilde{e}(P_{0,k}|P_{k,k})}\end{aligned} \tag{421}$$

where the equalities hold by the following reasonings: The first equality holds because the definition of $\text{Pyr}(R_1) = (P'_{i,j})_{i,j}$ in Definition 11 implies the equalities $R_1 = P'_{0,l_1}$ and $R_1 \cap F_0 = R_1 \cap F_{0,0} = P'_{0,0}$ and because of the multiplicative transitivity rule for $\tilde{e}$ in Lemma 44(ii). The second equality holds by applying the $\tilde{e}$-version of Abhyankar's Lemma in Lemma 44(i) to $\tilde{e}(P'_{0,l_1}|P'_{0,k})$. The third equality holds by the well known formula $a \cdot b = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$ for all $a, b \in \mathbb{N}$. The last equality holds by the equalities in (417) and (420).

Second, we compute

$$\begin{aligned}\tilde{e}(P_{k,l_2}|P_{k,k}) &= \tilde{e}(P_{k,l_2}|P_{k,k+n})\tilde{e}(P_{k,k+n}|P_{k,k}) \\ &= \frac{\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n})\tilde{e}(P_{k,k+n}|P_{k,k})}{\gcd(\tilde{e}(P_{k,k+n}|P_{k+n,k+n}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n}))} \\ &= \frac{\tilde{e}(P_{k,k+n}|P_{k,k})\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n})}{\gcd(\tilde{e}(P_{k,k+n}|P_{k,k}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n}))} \\ &= \operatorname{lcm}(\tilde{e}(P_{k,k+n}|P_{k,k}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n})) \end{aligned} \tag{422}$$

where the first and second equalities follow analogously to the first two equalities in (421), the third equality holds by swapping the two factors in the numerator and by applying the equality in (419) to the first argument in the gcd and the last equality holds by the well known formula $ab = \operatorname{lcm}(a,b)\gcd(a,b)$ for all $a,b \in \mathbb{N}$.

Third, we compute

$$\begin{aligned}\tilde{e}(R_2|R_2 \cap F_0) &= \frac{\tilde{e}(P_{0,k}|P_{0,0})\operatorname{lcm}(\tilde{e}(P_{0,k}|P_{k,k}),\tilde{e}(P_{k,l_2}|P_{k,k}))}{\tilde{e}(P_{0,k}|P_{k,k})} \\ &= \frac{\tilde{e}(P_{0,k}|P_{0,0})\operatorname{lcm}(\tilde{e}(P_{0,k}|P_{k,k}),\tilde{e}(P_{k,k+n}|P_{k,k}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n}))}{\tilde{e}(P_{0,k}|P_{k,k})} \end{aligned} \tag{423}$$

where the first equality follows analogously to the first three equalities in (421) (with $\mathcal{P}'$ replaced by $\mathcal{C} \cdot \mathcal{P}'$) and the second equality holds by the equality in (422) and by the well known rule $\operatorname{lcm}(a, \operatorname{lcm}(b,c)) = \operatorname{lcm}(a,b,c)$ for all $a,b,c \in \mathbb{N}$.

Finally, combining the equalities in (421) and (423) supplies the desired statement in (iii), namely that $\tilde{e}(R_1|R_1 \cap F_0)$ divides $\tilde{e}(R_2|R_2 \cap F_0)$.

For (iv): We will show the statement in (iv) by induction over $r \in \mathbb{N}$. For $r = 1$, we have $\mathcal{P} = \mathcal{C}_1$ and the places Q_1 and R are both contained in $\mathbb{P}_{F_{n_1}}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}_1))$. Hence, as Abhyankar ramification indices only depend on the path $\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{C}_1)$ in Definition (417), we even obtain the equality $\tilde{e}(R|R \cap F_0) = \tilde{e}(Q_1|Q_1 \cap F_0)$ in this case.

Next, let $r \geq 2$ and define $\mathcal{C} := \prod_{i=2}^{2r-1} \mathcal{P}_i$. First, we notice that $\mathcal{C}$ has balanced ramification indices by iteratively applying the 'on the other hand'-part in Lemma 220(i) and, hence, we are in the situation of the proof of Lemma 220(iii) for $\mathcal{P} = \mathcal{P}_1\mathcal{C}\mathcal{P}_{2r}$ where we choose $(\mathcal{P}, \mathcal{P}', \mathcal{C}, \mathcal{P}_1, \mathcal{P}_2, R_1, R_2)$ in this lemma as $(\mathcal{P}_1, \mathcal{P}_{2r}, \mathcal{C}, \mathcal{C}_1, \mathcal{P}, Q_1, R)$ with the notation from here.

Consequently, for

$$\gamma := \operatorname{lcm}(\tilde{e}(P_{0,k}|P_{k,k}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n})),$$

we obtain the equalities

$$\begin{aligned}\tilde{e}(R|R \cap F_0) &= \frac{\tilde{e}(P_{0,k}|P_{0,0})\operatorname{lcm}(\tilde{e}(P_{0,k}|P_{k,k}),\tilde{e}(P_{k,k+n}|P_{k,k}),\tilde{e}(P_{k+n,l_2}|P_{k+n,k+n}))}{\tilde{e}(P_{0,k}|P_{k,k})} \\ &= \frac{\tilde{e}(P_{0,k}|P_{0,0}) \cdot \gamma}{\tilde{e}(P_{0,k}|P_{k,k})} \cdot \frac{\tilde{e}(P_{k,k+n}|P_{k,k})}{\gcd(\gamma,\tilde{e}(P_{k,k+n}|P_{k,k}))} \\ &= \tilde{e}(Q_1|Q_1 \cap F_0) \cdot \frac{\tilde{e}(P_{k,k+n}|P_{k,k})}{\gcd(\gamma,\tilde{e}(P_{k,k+n}|P_{k,k}))} \end{aligned} \tag{424}$$

where the equalities hold by the following reasonings: The first equality holds by the equality $R = R_2$ and the equality in (423). The second equality holds by first applying the well known rule $\operatorname{lcm}(a,b,c) = \operatorname{lcm}(\operatorname{lcm}(a,c),b)$ and then applying the well known rule $\operatorname{lcm}(d,b) = \frac{db}{\gcd(d,b)}$ for $d := \operatorname{lcm}(a,c)$ and by the definition of γ. The third equality holds by the equality in (421) and the equality $R_1 = Q_1$.

Second to last, we deduce the equalities

$$\tilde{e}(P_{k,k+n}|P_{k,k}) = \tilde{e}(\sigma^{-k}(P_{k,k+n})|\sigma^{-k}(P_{k,k})) = \tilde{e}(\sigma^{-k}(P_{k,k+n})|\sigma^{-k}(P_{k,k+n}) \cap F_0) \tag{425}$$

where the first equality holds by the last equality in (419), the second equality holds since combining (418), (415), the definition of $\mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}))$ in (5) and the definition of Path in Definition/Lemma 17(i) yields the equalities

$$(\sigma^{-k}(P_{k+i,k+j}))_{j-i\leq 1} = \sigma_{\Gamma_\mathcal{F}}(\mathcal{C}) = \mathrm{Path}(\sigma^{-k}(P_{k,k+n})) = (\sigma^{-k}(P_{k,k+n} \cap F_{i,j}))_{j-i\leq 1}$$

and since this equality at the $(0,0)$-th position provides the equalities

$$\sigma^{-k}(P_{k,k}) = \sigma^{-k}(P_{k,k+n}) \cap F_{0,0} = \sigma^{-k}(P_{k,k+n}) \cap F_0.$$

Moreover, we can apply the induction hypothesis to $\mathcal{C} = \prod_{i=2}^{2r-1} \mathcal{P}_i$ and $\sigma^{-k}(P_{k,k+n}) \in \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}))$. Because of the equality in (425), we then even derive that $\tilde{e}(P_{k,k+n}|P_{k,k})$ divides $\prod_{i=2}^{r} \tilde{e}(Q_i|Q_i \cap F_0)$.

Finally, combining this conclusion and the equality in (424) provides the desired statement in (iv), namely that $\tilde{e}(R|R \cap F_0)$ divides $\prod_{i=1}^{r} \tilde{e}(Q_i|Q_i \cap F_0)$. □

7.2.2 Subgraphs Only Containing Circles with Bal. Ramification Indices

Summary of the results of this subsection. In this subsection, we will prove the following Theorem 221 which establishes the desired upper bound ${B_\Gamma}^{\#V(\Gamma)}$ for the Abhyankar indices $\tilde{e}(Q|Q \cap F_0)$ for all $Q \in \mathbb{P}_\mathcal{F}[\Gamma]$.

Note that since the set of circles in a finite subgraph Γ is finite, the set of places $\mathbb{P}_\mathcal{F}^\circ[\Gamma]$ which lie over these circles is also finite.

Theorem 221. *Let $\mathcal{F}$ be a recursive tower, let Γ be a non-empty finite strongly connected subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$ such that all circles in Γ have balanced degree, let $Q \in \mathbb{P}_\mathcal{F}[\Gamma]$ and define*

$$B_\Gamma := \operatorname*{lcm}_{Q' \in \mathbb{P}_\mathcal{F}^\circ[\Gamma]} \tilde{e}(Q'|Q' \cap F_0).$$

Then $\tilde{e}(Q|Q \cap F_0)$ divides ${B_\Gamma}^{\#V(\Gamma)}$.

Remark 222. *By using the dual recursive tower $\hat{\mathcal{F}}$, we can also show that $\tilde{e}(Q|Q \cap \sigma^n(F_0))$ divides ${B_\Gamma}^{\#V(\Gamma)}$.*

Sketch of the proof of Theorem 221. First, let $\mathcal{P} := \sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}(Q))$. Then we will deduce that it is enough to show Theorem 221 for closed paths. More, concretely, we will show this by induction over the length n of $\mathcal{P}$ where the statement is trivial for $n = 0$.

Second, for $n \geq 1$, we will decompose the black closed path $\mathcal{P}$ as in Figure 7.6: There, $\mathcal{P}$ is the composition $\mathcal{P}'\mathcal{C}\mathcal{P}'''$ of the red path $\mathcal{P}'$ which has no repetitions, the blue circle $\mathcal{C}$ which is the first circle of positive length in $\mathcal{P}$ and the remaining yellow path $\mathcal{P}'''$.

Third, the assertion that Γ is strongly connected will supply the green path $\mathcal{P}''$ in Figure 7.6 which is the shortest path in Γ from the terminal vertex of $\mathcal{P}'$ and $\mathcal{C}$, namely P, to the initial vertex of $\mathcal{P}$ and $\mathcal{P}'$, namely P_0. Consequently, we will conclude that $\mathcal{P}'\mathcal{C}\mathcal{P}''$ must be of the form $\prod_{i=1}^{2r} \mathcal{P}_i$ (purple in Figure 7.6) where $r \leq \#V(\Gamma)$ and $\mathcal{P}_i\mathcal{P}_{2r+-i}$ is a circle for all $i = 1, \ldots r$.

Fourth, by this choice and by Lemma 220(iv), we will then obtain that $e(Q'|Q' \cap F_0)$ divides ${B_\Gamma}^{\#V(\Gamma)}$ for all places Q' which lie over the purple path $\mathcal{P}'\mathcal{C}\mathcal{P}'' = \prod_{i=1}^{2r} \mathcal{P}_i$. Furthermore, since $\mathcal{P}'\mathcal{P}'''$ is a shorter closed path than $\mathcal{P} = \mathcal{P}'\mathcal{C}\mathcal{P}'''$, the induction hypothesis will provide the same 'dividing'-statement for all places Q' which lie over $\mathcal{P}'\mathcal{P}'''$. Hence, Lemma 220(ii) will even imply this 'dividing'-statement for all places Q' which lie over their composition $(\mathcal{P}'\mathcal{C}\mathcal{P}'')(\mathcal{P}'\mathcal{P}''')$.

Last, as $\mathcal{P}''\mathcal{P}'$ is also a closed path and, therefore, has balanced ramification indices by Lemma 83, applying Lemma 220(iii) to $\mathcal{P} = \mathcal{P}'\mathcal{C}\mathcal{P}'''$ will finally provide the desired statement, namely that $e(Q|Q \cap F_0)$ divides ${B_\Gamma}^{\# V(\Gamma)}$.

Proof of Theorem 221. Let $\mathcal{P} := \sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}(Q))$ be the path of Q and let $n \in \mathbb{N}_0$ be its length. Then, by the definition of $\mathbb{P}_\mathcal{F}[\Gamma]$ in Definition 85, the path $\mathcal{P}$ is contained in Γ.

First, we notice that since Γ is strongly connected, there is a path $\tilde{\mathcal{P}}$ from the terminal vertex to the initial vertex of $\mathcal{P}$ and, thus, $\mathcal{P}\tilde{\mathcal{P}}$ is a closed path, say of length l. Moreover, as $\tilde{e}(Q|Q \cap F_0)$ divides $\tilde{e}(Q'|Q' \cap F_0)$ for all $Q' \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}\tilde{\mathcal{P}}))$ with $Q'|Q$, it is enough to show the desired statement that $\tilde{e}(Q|Q \cap F_0)$ divides ${B_\Gamma}^{\# V(\Gamma)}$ under the additional assumption that $\mathcal{P}$ is a closed path.

We will show this by induction over the length $n \in \mathbb{N}_0$ of $\mathcal{P}$. For $n = 0$, the closed path $\mathcal{P}$ is only a vertex and we have $\tilde{e}(Q|Q \cap F_0) = \tilde{e}(Q|Q) = 1$ and, thus, the desired statement follows trivially in this case.

Now, suppose $n \geq 1$. We will first make some preparations: As $\mathcal{P}$ is a closed path, there are repeating vertices in $\mathcal{P}$. Let P be the first vertex which appears in $\mathcal{P}$ a second time. Then there are paths (see Figure 7.6)

$$\begin{aligned}&\mathcal{P}' \text{ and } \mathcal{P}''' \text{ and a circle } \mathcal{C} \text{ of positive length with initial vertex } P \text{ such that}\\ &\mathcal{P} = \mathcal{P}'\mathcal{C}\mathcal{P}''' \text{ and only } P \text{ appears more than once in } \mathcal{P}'\mathcal{C}.\end{aligned} \tag{426}$$

Let P_0 be the initial vertex of $\mathcal{P}$. As Γ is strongly connected, there is a path $\mathcal{P}''$ in Γ from P to P_0 without repeating vertices.

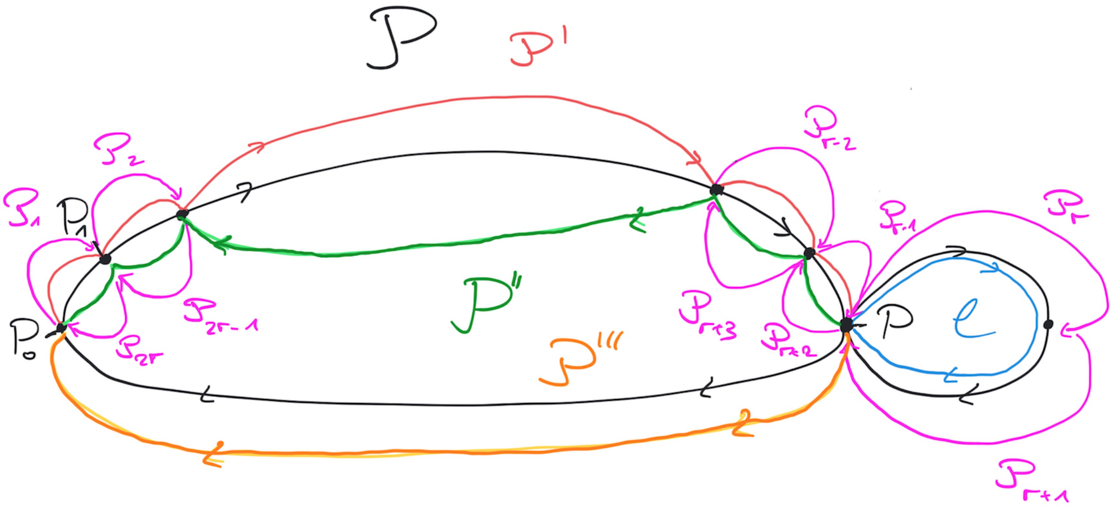


Figure 7.6: Auxiliary paths in a proof

Although $\mathcal{P}'$ and $\mathcal{P}''$ have no repeating vertices, they have common vertices since $\mathcal{P}'$ goes from P_0 to P and $\mathcal{P}''$ goes from P to P_0. Next, we will show by induction over the length m of $\mathcal{P}'$ that the composition $\mathcal{P}'\mathcal{C}\mathcal{P}''$ is of the form

$$\mathcal{P}'\mathcal{C}\mathcal{P}'' = \prod_{i=1}^{2r} \mathcal{P}_i \tag{427}$$

such that

$$\mathcal{C}_i := \mathcal{P}_i\mathcal{P}_{2r+1-i} \text{ is a circle of positive length and } \mathcal{C}_r = \mathcal{C} \tag{428}$$

for all $i = 1, \dots, r$: First, we choose arbitrary subpaths $\mathcal{P}_r$ and $\mathcal{P}_{r+1}$ of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{P}_r\mathcal{P}_{r+1}$. This is a valid choice because $\mathcal{C}$ has positive length by its choice in (426).

Now, for $m = 0$, we have $P_0 = P$ and, consequently, $\mathcal{P}'$ and $\mathcal{P}''$ both just consist of this vertex $P_0 = P$ by their choices. Hence, we can choose $r = 1$ and are done with this case.

Next, suppose $m \geq 1$ and, therefore, $P \neq P_0$. Moreover, let P_1 be the first vertex in $\mathcal{P}'$ after P_0 which also appears in $\mathcal{P}''$, define $\mathcal{P}_1$ as the subpath of $\mathcal{P}'$ which goes from P_0 to P_1 and define $\mathcal{P}_{2r}$ as the subpath of $\mathcal{P}''$ which goes from P_1 to P_0. By the choice of P_1 as the first index $\mathcal{P}'$ after P_0 which also appears in $\mathcal{P}''$, the paths $\mathcal{P}_1$ and $\mathcal{P}_{2r}$ can have no other common vertices than P_0 and P_1. Consequently, by this conclusion, by the assertion that $\mathcal{P}'$ and $\mathcal{P}''$ have no repeating vertices and by the choice of $P_1 \neq P_0$, we conclude that $\mathcal{C}_1 = \mathcal{P}_1\mathcal{P}_{\ 2r}$ is indeed a circle of positive length.

Finally, let $\mathcal{P}' = \mathcal{P}_1\mathcal{P}_1'$ and $\mathcal{P}'' = \mathcal{P}_{2r}''\mathcal{P}_{2r}$. Then applying the induction hypothesis to the shorter closed path $\mathcal{P}_1'\mathcal{C}\mathcal{P}_{2r}''$ yields the desired remaining paths $\mathcal{P}_2, \dots, \mathcal{P}_{2r-1}$ in (427) and (428).

Furthermore, because of the equality $\mathcal{P}' = \prod_{i=1}^{r-1} \mathcal{P}_i$ and because $\mathcal{P}'$ has no repeating vertices, we also derive the estimate

$$r \leq \#V(\Gamma) \tag{429}$$

and are done with our preparations.

By the following reasoning, these preparations first provide that

$$\tilde{e}(Q'|Q' \cap F_0) \text{ divides } B_\Gamma{}^{\#V(\Gamma)} \tag{430}$$

for all $Q' \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}'\mathcal{C}\mathcal{P}''))$ where l be the length of $\mathcal{P}'\mathcal{C}\mathcal{P}''$: First, the conclusions in (428) and (427) and the assumption that all circles have balanced ramification indices ensure that we can apply Lemma 220(iv). By that, we conclude that $\tilde{e}(Q'|Q' \cap F_0)$ divides the product $\prod_{i=1}^{r} \tilde{e}(Q_i|Q_i \cap F_0)$ for all $Q_i \in \mathbb{P}_{F_{l_i}}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{C}_i))$ where l_i be the length of $\mathcal{C}_i$ for all $i = 1, \dots, r$. But, because $\mathcal{C}_i$ is a circle by (428), because of the definition of B_Γ and because of the estimate in (429), we even derive that this product $\prod_{i=1}^{r} \tilde{e}(Q_i|Q_i \cap F_0)$ divides $B_\Gamma{}^{\#V(\Gamma)}$. Both conclusions together then yield (430).

Second, we notice that since $\mathcal{C}$ has non-zero length by its choice in (426), the length of the closed path $\mathcal{P}'\mathcal{P}'''$ is at most $n-1$. Thus, applying the induction hypothesis to $\mathcal{P}'\mathcal{P}'''$ yields that

$$\tilde{e}(Q'|Q' \cap F_0) \text{ divides } B_\Gamma{}^{\#V(\Gamma)} \tag{431}$$

for all $Q' \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}'\mathcal{P}'''))$ where l be the length of $\mathcal{P}'\mathcal{P}'''$. Consequently, combining the fact that Lemma 83 implies that the closed paths $\mathcal{P}'\mathcal{C}\mathcal{P}''$ and $\mathcal{P}'\mathcal{P}'''$ have balanced ramification indices, Lemma 220(ii) and the conclusions in (430) and (431) supplies that

$$\tilde{e}(Q'|Q' \cap F_0) \text{ divides } B_\Gamma{}^{\#V(\Gamma)} \tag{432}$$

for all $Q' \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}((\mathcal{P}'\mathcal{C}\mathcal{P}'')(\mathcal{P}'\mathcal{P}''')))$ where l be the length of $(\mathcal{P}'\mathcal{C}\mathcal{P}'')(\mathcal{P}'\mathcal{P}''')$.

Third, we notice that $\mathcal{P}''\mathcal{P}'$ is also a closed path with balanced ramification indices by Lemma 83. Therefore, Lemma 220(iii) provides that

$$\tilde{e}(Q|Q \cap F_0) \text{ divides } \tilde{e}(Q'|Q' \cap F_0) \tag{433}$$

for all $Q' \in \mathbb{P}_{F_l}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}'\mathcal{C}(\mathcal{P}''\mathcal{P}')\mathcal{P}'''))$ where l be the length of $\mathcal{P}'\mathcal{C}(\mathcal{P}''\mathcal{P}')\mathcal{P}'''$.

Finally, combining (433) and (432) yields the desired statement that $\tilde{e}(Q|Q \cap F_0)$ divides $B_\Gamma{}^{\#V(\Gamma)}$. □

7.2.3 Proof of the Second Interim Result

Finally, we are prepared to prove the second interim result in Corollary 170.

Proof of Corollary 170. If all circles in Γ have balanced ramification indices, then Theorem 221 already implies the desired statement, namely that the set $\{\tilde{e}(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ is finite, since all it elements are divisors of the natural number ${B_\Gamma}^{\#V(\Gamma)}$.

Moreover, if all paths in Γ are tame, the combination of Definition 81(i) and Lemma 44(iii) implies the equality $e(Q|Q \cap F_0) = \tilde{e}(Q|Q \cap F_0)$ for all $Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]$. But, this again provides the equality $\{\tilde{e}(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\} = \{e(Q|Q \cap F_0) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ and, hence, the 'in particular'-part also follows. ☐

7.3 Third Interim Result - Degree Bounds for Places

Summary of the results of this section. In this section, we will prove the third interim result in Corollary 171. This Corollary 171 will be a corollary of Theorem 228. Theorem 228 will again be a consequence of Theorem 225. This Theorem 225 should be seen as another result of this thesis, because it will provide upper bounds C_Q for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{\mathcal{F}}$ where $\mathcal{F}$ only needs to be a recursive tower over a finite field. These upper bounds C_Q will be expressed entirely in terms of the degree d of $\mathcal{F}$, of the ramification indices of the extensions in $\mathrm{Pyr}(Q)$ and the degrees of the places in $\mathrm{Path}(Q)$.

For any tame recursive tower $\mathcal{F}$ over a finite field and finite strongly connected subgraph Γ of $\Gamma_{\mathcal{F}}$ which has at least one circle of positive degree but only contains circles with balanced ramification indices, the upper bound C_Γ for $\delta_\Gamma := \{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ in Theorem 228 will be $l_\Gamma \cdot (d!)^{{B_\Gamma}^{\#V(\Gamma)}}$ where $l_\Gamma := \mathrm{lcm}_{Q \in E(\Gamma)} \deg(Q)$ is defined in Definition 226 and $B_\Gamma := \mathrm{lcm}_{Q \in \mathbb{P}^\circ_{\mathcal{F}}[\Gamma]} \tilde{e}(Q|Q \cap F_0)$ is defined in Theorem 221. Thus, C_Γ will only depend on Γ. Although this upper bound C_Γ will be sufficient for the purpose of proving the third interim result in Corollary 171, it will usually have a large error $C_\Gamma - \max_{Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]} \deg(Q)$. However, for any concretely given place $Q \in \mathbb{P}_{\mathcal{F}}$, the upper bounds C_Q for $\deg(Q)$ in Theorem 225 should rather be used.

Quality of the final upper bound. For the final upper bound C_Γ of the set $\delta_\Gamma = \{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ in Theorem 228, we will just focus on finding an upper bound C_Γ which only depends on Γ. Moreover, we will also care more about C_Γ having a simple presentation than about minimizing the error $C_\Gamma - \max_{Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]} \deg(Q)$. This will be sufficient for the purpose of proving the third interim result in Corollary 171.

However, this does not mean that all upper bounds C_Q for $\deg(Q)$ which will appear in this section will have awfully large errors $C_Q - \deg(Q)$. More concretely, we will split the problem of finding the upper bound C_Γ in Theorem 228 into two parts:

In the first Subsection 7.3.1, which will finally yield Theorem 225, we will find upper bounds C_Q for $\deg(Q)$ for all $Q \in \mathbb{P}_{\mathcal{F}}$ which can be expressed entirely in terms of the degree d of $\mathcal{F}$, of the ramification indices of the extensions in $\mathrm{Pyr}(Q)$ and the degrees of the places in $\mathrm{Path}(Q)$. Here, we will have several upper bounds C_Q for $\deg(Q)$ and the rule of thumb will be that the worser upper bounds C_Q will have simpler presentations.

In the second Subsection 7.3.2, we will then prefer to only use the worst of the upper bounds C_Q from Theorem 225 to construct the final upper bound C_Γ for δ_Γ. Although we could also use all the better upper bounds C_Q from Theorem 225 to construct better upper bounds C'_Γ for δ_Γ, the resulting presentations of these better upper bounds C'_Γ would be unproportionally more complicated and technical.

Structure of this section. In Subsection 7.3.1, we will prove Theorem 225 which provides the upper bounds C_Q for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{\mathcal{F}}$ where $\mathcal{F}$ is a recursive tower over a finite field.

In Subsection 7.3.2, we will prove Theorem 228 which provides the upper bound C_Γ for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]$ where $\mathcal{F}$ is a tame recursive tower over a finite field and Γ is a finite strongly connected subgraph of $\Gamma_{\mathcal{F}}$ which has at least one circle of positive degree but only contains circles with balanced ramification indices.

In Subsection 7.3.3, we will then finally prove our third interim result, which is Corollary 171.

7.3.1 Bounds Only Depending on Ramification Indices in the Pyramid

Summary of the results of this subsection. In this subsection, we will prove another result of this thesis, namely Theorem 225. There we will find several upper bounds C_Q for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{\mathcal{F}}$ where $\mathcal{F} = (F_\nu)_\nu$ is a recursive tower over a finite field. More concretely, the upper bounds C_Q will be expressed solely in terms of the degree d of $\mathcal{F}$, of the ramification indices of the extensions in $\mathrm{Pyr}(Q)$ and the degrees of the places in $\mathrm{Path}(Q)$

Difficulty of coming up with the final upper bounds. In the end, after we have properly defined the final upper bounds C_Q for $\deg(Q)$ in Definition 223 and Theorem 225, it will be, although sometimes technical, mostly straight forward to prove that all C_Q are upper bounds for $\deg(Q)$ via induction over the level n of the place $Q \in \mathbb{P}_{F_n}$. However, as it is often the case with proofs using induction, the main challenge is to come up with the right formula, estimate or statement in the first place. This is also the case here. The main difficulty was to first come up with the final degree bounds C_Q in Theorem 225.

Although we will not present it here, there is a way of deducing the degree bounds C_Q in a more intuitive way. We will skip this deduction since it is quite long and since it does not improve the results of Theorem 225. However, skipping this deduction will also have the following further drawback. Alone from the induction, it will be impossible to assess the quality of the degree bounds C_Q in Theorem 225. Consequently, as these assessments are also not vital for the proof of the third interim result, which is our primary goal of this section, we will skip them too.

Main idea of the proof. Although we will not present the above mentioned deduction of the final upper bounds C_Q of $\deg(Q)$ in Theorem 225, we will at least provide its main idea:

All upper bounds C_Q in Theorem 225 will be of the form $C_Q = l_Q \cdot \prod_{p \in \mathbb{P}, p \leq d} p^{\beta_{Q,p}}$ for some suitable $\beta_{Q,p}$ and $l_Q := \mathrm{lcm}_{k=1,\ldots,n} \deg(P_{k-1,k})$ (see the last definition in Definition 223) where $(P_{i,j})_{j-j} := \mathrm{Path}(Q)$. For the first factor l_Q, we will reduce the situation to rational paths $\mathrm{Path}(Q)$ by extending the constant field $\mathbb{F}_q$ of $\mathcal{F}$ to $\mathbb{F}_{q^{l_Q}}$. This will already provide the first factor l_Q in $C_Q = l_Q \cdot \prod_{p \leq d} p^{\beta_{Q,p}}$.

For the second factor $\prod_{p \leq d} p^{\beta_{Q,p}}$ in $C_Q = l_Q \cdot \prod_{p \leq d} p^{\beta_{Q,p}}$, the key ingredient will be the estimate

$$\deg(Q') \leq \mathrm{lcm}(\deg(P_1), \deg(P_2)) \cdot \frac{e(P_1|P) \cdot e(P_2|P)}{e(Q'|P)} \tag{434}$$

in Key Lemma 36(iii) where (Q', P_1, P_2, P) forms a diamond of places in a diamond (E, F_1, F_2, F) of function fields with $E = F_1 \cdot F_2$.

The main idea here will be to somehow apply this estimate iteratively for all elementary diamonds in $\mathrm{Pyr}(Q)$ from the bottom level, i.e. $\mathrm{Path}(Q)$, to the top level, i.e. Q. Doing this is non-trivial because estimating the factor $\mathrm{lcm}(\deg(P_1), \deg(P_2))$ on the higher levels needs more than just estimates for $\deg(P_i)$, $i = 1, 2$. We will need an upper bound for $\deg(P_i)$ which will even be a multiple of $\deg(P_i)$.

But, for recursive towers $\mathcal{F}$ which are not locally Galois, we cannot rule out any prime factor $p \leq d$ to appear in $\deg(Q)$. This will explain that the second factor of $C_Q = l_Q \cdot \prod_{p \leq d} p^{\beta_{Q,p}}$ is a product of prime powers $p^{\beta_{Q,p}}$ which run over all primes $p \leq d$.

Moreover, this will also explain why we can provide further upper bounds C_Q for locally Galois recursive towers $\mathcal{F}$ in Theorem 225. For locally Galois recursive towers $\mathcal{F}$, in this product $\prod_{p \leq d} p^{\beta_{Q,p}}$, we will only have to take prime divisors $p \leq d$ into account which divide d.

Finally, the main challenge becomes to find the exponents $\beta_{Q,p}$. Notice that since we extended the constant field to $\mathbb{F}_{q^{l_Q}}$ in the beginning, these exponents $\beta_{Q,p}$ will be upper bounds for $v_p(\deg(Q))$. Deducing these upper bounds $\beta_{Q,p}$ for $v_p(\deg(Q))$ would have been part of the above mentioned deduction. There the pyramidal configuration of the elementary diamonds in $\mathrm{Pyr}(Q)$ would have played the key role.

However, in the following Definition 223, we will just present the final definitions of the upper bounds $\beta_{Q,p}$ for $v_p(\deg(Q))$ as $e_{Q,p}$ and $b(Q, A)$ for all $A \in \mathcal{M}$ and leave out any deductions.

The upper bounds for the exponents. In the following Definition 223, we will define the upper bounds $\beta_{Q,p}$ for $v_p(\deg(Q))$ as $e_{Q,p}$ and $b(Q, A)$ for all $A \in \mathcal{M}$. All these values will be defined via maxima which run over the diamonds in $(Q, P_{0,k}, P_{k,n}, P_{k,k})$ in the pyramid $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$ of Q (see Figure 7.7) and this is also how we should think about them.

First, $\#\,\mathrm{Count}(Q, k)$ is the ramification index $e(P_{k,n}|P_{k,k})$. Second, $\#\,\mathrm{Primes}(Q, k)$ is the sum of prime exponents in the prime decomposition of $e(P_{k,n}|P_{k,k})$. Third, $\#\mathrm{Ram}(Q, k)$ is the number of elementary extensions $P_{k,j+1}/P_{k,j}$ for $j = k, \ldots, n-1$ inside of the extension $P_{k,n}/P_{k,k}$ which are ramified. Fourth, $\#\,\mathrm{Kill}(Q, k)$ is the number of elementary extensions $P_{k,j+1}/P_{k,j}$ for $j = k, \ldots, n-1$ inside of the extension $P_{k,n}/P_{k,k}$ for which ramification indices get killed up to the extensions $P_{0,j+1}/P_{0,j}$. Fifth, up to flooring, $e_{Q,p}$ is the maximum of the $\log_p$-values of the quotients in (434) which runs of all the diamonds $(Q, P_{0,k}, P_{k,n}, P_{k,k})$.

Here, the already mentioned rule of thumb for the quality and simplicity of the upper bounds can be observed. From one to five, on the one hand, the values get smaller but, on the other hand, their descriptions get more complicated.

Definition 223. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a finite field, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be its pyramid, let $Q \in \mathbb{P}_{F_n}$ for some $n \in \mathbb{N}$ with $n \geq 2$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$. Then we define (also see Figure 7.7)*

- $\mathrm{Kill}(Q, k) := \{j \in \{k, \ldots, n-1\} : e(P_{k,j+1}|P_{k,j}) > e(P_{0,j+1}|P_{0,j})\}$,
- $\mathrm{Ram}(Q, k) := \{j \in \{k, \ldots, n-1\} : e(P_{k,j+1}|P_{k,j}) \geq 2\}$,
- $\mathrm{Primes}(Q, k) := \{(p, i) \in \mathbb{P} \times \mathbb{N} : i \leq v_p(e(P_{k,n}|P_{k,k}))\}$,
- $\mathrm{Count}(Q, k) := \{1, \ldots, e(P_{k,n}|P_{k,k})\}$,
- $\mathrm{Kill}\hat{}(Q, k) := \{i \in \{1, \ldots, k\} : e(P_{i-1,k}|P_{i,k}) > e(P_{i-1,n}|P_{i,n})\}$,
- $\mathrm{Ram}\hat{}(Q, k) := \{i \in \{1, \ldots, k\} : e(P_{i-1,k}|P_{i,k}) \geq 2\}$,
- $\mathrm{Primes}\hat{}(Q, k) := \{(p, i) \in \mathbb{P} \times \mathbb{N} : i \leq v_p(e(P_{0,k}|P_{k,k}))\}$
- $\mathrm{Count}\hat{}(Q, k) := \{1, \ldots, e(P_{0,k}|P_{k,k})\}$,

for all $k = 1, \dots, n-1$ *and*

$$b(Q, A) := \max_{k=1,\dots,n-1} \#A(Q, k)$$

for all $A \in \mathcal{M} := \{\mathrm{Ram}, \mathrm{Kill}, \mathrm{Primes}, \mathrm{Count}, \mathrm{Ram}\hat{}, \mathrm{Kill}\hat{}, \mathrm{Primes}\hat{}, \mathrm{Count}\hat{}\}$*. Moreover, we also define*

$$e_{Q,p} := \left\lfloor \log_p \left(\max_{k=1,\dots,n-1} \frac{e(P_{0,k}|P_{k,k}) \cdot e(P_{k,n}|P_{k,k})}{e(Q|P_{k,k})} \right) \right\rfloor$$

for all $p \in \mathbb{P}$ *and*

$$c_Q := \prod_{p \in \mathbb{P},\, p \le d} p^{e_{Q,p}}, \quad \rho(d) := \prod_{p \in \mathbb{P}} p^{\lfloor \log_p d \rfloor}, \quad l_Q := \operatorname*{lcm}_{k=1,\dots,n} \deg(P_{k-1,k}).$$

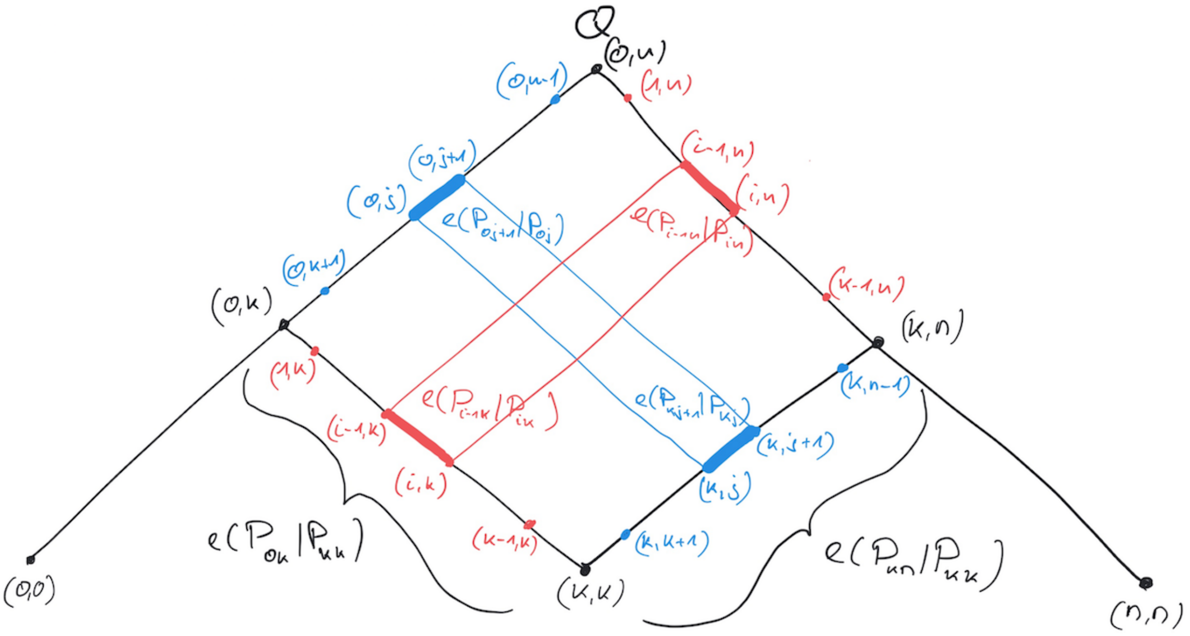


Figure 7.7: Pyramid of places with ramification indices

An auxiliary Lemma. The following Lemma 224(iv) is an auxiliary lemma for the subsequent proof of Theorem 225.

On the one hand, the estimates in Lemma 224(iv) will imply that, in Theorem 225, it is enough to only prove that $\deg(Q)$ divides $l_Q \cdot c_Q$ in any case and $l_Q \cdot d^{b(Q,A)}$ for $A \in \{\mathrm{Kill}, \mathrm{Kill}\hat{}\}$ if $\mathcal{F}$ is locally Galois. On the other hand, Lemma 224(v) will make it possible to prove Theorem 225 by induction over $n \in \mathbb{N}$ with $n \ge 2$.

Lemma 224. *Let* $\mathcal{F} = (F_\nu)_\nu$ *be a recursive tower of degree* d *over a finite field, let* $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ *be its pyramid, let* $Q \in \mathbb{P}_{F_n}$ *for some* $n \in \mathbb{N}$ *with* $n \ge 2$ *and let* $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ *be the pyramid of* Q*. Then the following hold:*

- *(i) We have the inclusions* $\mathrm{Kill}(Q, k) \subseteq \mathrm{Ram}(Q, k)$ *and* $\mathrm{Kill}\hat{}(Q, k) \subseteq \mathrm{Ram}\hat{}(Q, k)$ *for all* $k = 1, \dots, n-1$.
- *(ii) We have the identities*

$$\#\,\mathrm{Primes}(Q, k) = \sum_{p \in \mathbb{P}} v_p(e(P_{k,n}|P_{k,k})) \text{ and } \#\,\mathrm{Count}(Q, k) = e(P_{k,n}|P_{k,k})$$

and

$$\#\,\mathrm{Primes}\hat{}(Q,k) = \sum_{p\in\mathbb{P}} v_p(e(P_{0,k}|P_{k,k})) \text{ and } \#\,\mathrm{Count}\hat{}(Q,k) = e(P_{0,k}|P_{k,k})$$

for all $k = 1, \ldots, n-1$*.*

(iii) For all $A \in \{\mathrm{Kill}, \mathrm{Kill}\hat{}\}$ *and all* $k = 1, \ldots, n-1$*, we have the estimates*

$$\frac{e(P_{0,k}|P_{k,k}) \cdot e(P_{k,n}|P_{k,k})}{e(Q|P_{k,k})} \leq d^{\#A(Q,k)}$$

and

$$\#\,\mathrm{Kill}(Q,k) \leq \#\mathrm{Ram}(Q,k) \leq \#\,\mathrm{Primes}(Q,k) \leq \#\,\mathrm{Count}(Q,k)$$

and

$$\#\,\mathrm{Kill}\hat{}(Q,k) \leq \#\mathrm{Ram}\hat{}(Q,k) \leq \#\,\mathrm{Primes}\hat{}(Q,k) \leq \#\,\mathrm{Count}\hat{}(Q,k).$$

(iv) For all $A \in \{\mathrm{Kill}, \mathrm{Kill}\hat{}\}$*, we have the estimate*

$$e_{Q,p} \leq \lfloor \log_p d \rfloor \cdot b(Q,A).$$

Moreover, we have the estimates

$$b(Q,\mathrm{Kill}) \leq b(Q,\mathrm{Ram}) \leq b(Q,\mathrm{Primes}) \leq b(Q,\mathrm{Count})$$

and

$$b(Q,\mathrm{Kill}\hat{}) \leq b(Q,\mathrm{Ram}\hat{}) \leq b(Q,\mathrm{Primes}\hat{}) \leq b(Q,\mathrm{Count}\hat{}).$$

(v) Let $\varepsilon \in \{0,1\}$ *and suppose that* $\mathrm{Path}(Q) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F},0,n)$ *is rational. Then the path* $\mathrm{Path}(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})) = (\sigma^{-\varepsilon}(P_{i+\varepsilon,j+\varepsilon}))_{j-i\leq 1} \in W(\mathcal{F},0,n-1)$ *of the place* $\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}) \in \mathbb{P}_{F_{n-1}}$ *is also rational.*

  (a) Suppose $n \geq 3$ *and let* $A_0 := \mathrm{Kill}$ *and* $A_1 := \mathrm{Kill}\hat{}$*. Then* $A_\delta(\cdot,k)$ *and* $b(\cdot,A_\delta)$ *can be applied to* $\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})$ *for all* $\delta = 0,1$ *and* $k = 1,\ldots,n-2$ *and we have the inclusions*

  $$\varepsilon + A_\delta(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),k) \subseteq A_\delta(Q,k+\varepsilon)$$

  and the estimate

  $$b(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),A_\delta) \leq b(Q,A_\delta).$$

  (b) Suppose $n \geq 3$*. Then we have the identity*

  $$e_{\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),p} = \left\lfloor \log_p \left(\max_{k=1+\varepsilon,\ldots,n-2+\varepsilon} \frac{e(P_{\varepsilon,k}|P_{k,k}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k})}{e(P_{\varepsilon,n-1+\varepsilon}|P_{k,k})} \right) \right\rfloor$$

  for all $p \in \mathbb{P}$*.*

Proof. For (i): The desired inclusions immediately follow from the definitions of Kill, Kill$\hat{}$, Ram and Ram$\hat{}$ in Definition 223 and from the fact that ramification indices are positive natural numbers and, thus, are at least one.

For (ii): The desired identities follow immediately from the definitions of Primes, Primes$\hat{}$, Count and Count$\hat{}$ in Definition 223.

For (iii): We conclude the first desired estimate by the computation

$$\frac{e(P_{0,k}|P_{k,k}) \cdot e(P_{k,n}|P_{k,k})}{e(Q|P_{k,k})} = \begin{cases} \prod_{i=1}^{k} \frac{e(P_{i-1,k}|P_{i,k})}{e(P_{i-1,n}|P_{i,n})} & \text{if } A = \mathrm{Kill}\hat{} \\ \prod_{j=k}^{n-1} \frac{e(P_{k,j+1}|P_{k,j})}{e(P_{0,j+1}|P_{0,j})} & \text{if } A = \mathrm{Kill} \end{cases} \leq d^{\# A(Q,k)}$$

for all $k = 1, \ldots, n-1$ where the equality and estimate hold by the following reasonings:

The equality holds because the multiplicative transitivity rule for ramification indices in (7) first implies the identities

$$e(Q|P_{k,k}) = e(Q|P_{k,n}) \cdot e(P_{k,n}|P_{k,k}) = e(P_{0,n}|P_{k,n}) \cdot e(P_{k,n}|P_{k,k})$$

and

$$e(Q|P_{k,k}) = e(Q|P_{0,k}) \cdot e(P_{0,k}|P_{k,k}) = e(P_{0,n}|P_{0,k}) \cdot e(P_{0,k}|P_{k,k})$$

and then implies the identities

$$\frac{e(P_{0,k}|P_{k,k})}{e(P_{0,n}|P_{k,n})} = \prod_{i=1}^{k} \frac{e(P_{i-1,k}|P_{i,k})}{e(P_{i-1,n}|P_{i,n})}$$

and

$$\frac{e(P_{k,n}|P_{k,k})}{e(P_{0,n}|P_{0,k})} = \prod_{j=k}^{n-1} \frac{e(P_{k,j+1}|P_{k,j})}{e(P_{0,j+1}|P_{0,j})}.$$

The first case in the estimate holds, on the one hand, because the combination of the definition of $\mathrm{Kill}\hat{(}Q,k)$ in Definition 223 and of Key Lemma 36(iv) provides the equality $\frac{e(P_{i-1,k}|P_{i,k})}{e(P_{i-1,n}|P_{i,n})} = 1$ for all $i \in \{1, \ldots, k\} \setminus \mathrm{Kill}\hat{(}Q,k)$ and, on the other hand, because the fundamental equality in (8) provides the estimate $\frac{e(P_{i-1,k}|P_{i,k})}{e(P_{i-1,n}|P_{i,n})} \leq d$ for all $i \in \mathrm{Kill}\hat{(}Q,k)$.

The second case in the estimate holds analogously if we just use the definition of $\mathrm{Kill}(Q,k)$ in Definition 223.

Next, the estimates $\#\,\mathrm{Kill}(Q,k) \leq \#\mathrm{Ram}(Q,k)$ and $\#\,\mathrm{Kill}\hat{(}Q,k) \leq \#\mathrm{Ram}\hat{(}Q,k)$ immediately follow from the inclusions $\mathrm{Kill}(Q,k) \subseteq \mathrm{Ram}(Q,k)$ and $\mathrm{Kill}\hat{(}Q,k) \subseteq \mathrm{Ram}\hat{(}Q,k)$ in Lemma 224(i) for all $k = 1, \ldots, n-1$.

For the estimates $\#\mathrm{Ram}(Q,k) \leq \#\,\mathrm{Primes}(Q,k)$ and $\#\mathrm{Ram}\hat{(}Q,k) \leq \#\,\mathrm{Primes}\hat{(}Q,k)$, we first notice the factorizations

$$e(P_{k,n}|P_{k,k}) = \prod_{j=k}^{n-1} e(P_{k,j+1}|P_{k,j}) = \prod_{j \in \mathrm{Ram}(Q,k)} e(P_{k,j+1}|P_{k,j}) \tag{435}$$

and

$$e(P_{k,n}|P_{k,k}) = \prod_{p \in \mathbb{P}} p^{v_p(e(P_{k,n}|P_{k,k}))} = \prod_{(p,i) \in \mathrm{Primes}(Q,k)} p \tag{436}$$

for all $k = 1, \ldots, n-1$ by the definition of Ram and Primes in Definition 223. Now, combining that the factorization in (436) of $e(P_{k,n}|P_{k,k})$ only consists of primes and that the factorization in (435) only consists of natural numbers ≥ 2 yields that the factorization in (436) must run over more factors than the factorization in (435). Hence, the desired estimate $\#\mathrm{Ram}(Q,k) \leq \#\,\mathrm{Primes}(Q,k)$ follows for all $k = 1, \ldots, n-1$. Moreover, the other desired estimate $\#\mathrm{Ram}\hat{(}Q,k) \leq \#\,\mathrm{Primes}\hat{(}Q,k)$ follows from using the sets $\mathrm{Ram}\hat{(}Q,k)$ and $\mathrm{Primes}\hat{(}Q,k)$ and from factorizing $e(P_{0,k}|P_{k,k})$ in the two analogous ways.

Finally, the last desired estimates $\#\,\mathrm{Primes}(Q,k) \leq \#\,\mathrm{Count}(Q,k)$ and $\#\,\mathrm{Primes}\hat{(}Q,k)$ $\leq \#\,\mathrm{Count}\hat{(}Q,k)$ follows from the identities in Lemma 224(ii) and the estimates

$$x \geq \log_2(x) = \log_2(\prod_{p \in \mathbb{P}} p^{v_p(x)}) \geq \log_2(2^{\sum_{p \in \mathbb{P}} v_p(x)}) = \sum_{p \in \mathbb{P}} v_p(x)$$

for all $x \in \mathbb{N}$ where all estimates and equalities are obvious, except for the first estimate. For the first estimate, we first notice that it holds for $x = 1, 2$. Then, because of the well known equality $\log_2 = \log_2(e)\log_e$, we compute

$$\log_2'(2) = \log_2(e)\log_e'(2) = \log_2(e)/2 < \log_2(4)/2 = 1.$$

Thus, this estimate ensures that the derivative of $\log_2$ is less than the derivative of $\mathrm{id}_{\mathbb{R}_{>0}}$ at 2. Finally, combining this fact and the well known fact that $\log_2 = \log_2(e)\log_e$ is a concave function $\mathbb{R}_{>0} \to \mathbb{R}$ yields the desired estimate $x \geq \log_2(x)$ for all $x \in \mathbb{N}$.

For (iv): We already obtain the first desired estimate by the estimates

$$\begin{aligned} e_{Q,p} &= \lfloor \log_p(\max_{k=1,\dots,n-1} \frac{e(P_{0,k}|P_{k,k}) \cdot e(P_{k,n}|P_{k,k})}{e(Q|P_{k,k})})\rfloor \leq \lfloor \log_p(\max_{k=1,\dots,n-1} d^{\#A(Q,k)})\rfloor \\ &= \lfloor \log_p(d^{b(Q,A)})\rfloor = \lfloor \log_p d\rfloor \cdot b(Q,A) \end{aligned}$$

where the equalities and estimate hold by the following reasonings: The first equality is just the definition of $e_{Q,p}$ in Definition 223. The estimate holds because the first estimate in Lemma 224(iii) and because of all the involved maps $\lfloor\cdot\rfloor$, $\log_p$ and $\max_{k=1,\dots,n-1}(\cdot)$ are monotonically increasing. The second second equality holds because the power map $\mathbb{N}_0 \to \mathbb{N}$, $x \mapsto d^x$ is monotonically increasing and because of the definition of $b(Q,A)$ in Definition 223. The last equality holds because of the well known rule $\log_p(x)\cdot y = \log_p(x^y)$ for all $x, y \in \mathbb{N}$ and because the definition of $b(Q,A)$ in Definition 223 implies $b(Q,A) \in \mathbb{N}_0$ and, thus, the equality $\lfloor \log_p d\rfloor \cdot b(Q,A) = \lfloor \log_p(d)\cdot b(Q,A)\rfloor$.

The other desired estimates immediately follow from the definition of $b(Q,A)$ in Definition 223 and the estimates in Lemma 224(iii).

For (v): First, we notice that Lemma 10(i) and Lemma 10(ii) provide the identities $\sigma^{-\varepsilon}(F_{\varepsilon,n-1+\varepsilon}) = F_{0,n-1} = F_{n-1}$. This implies that $\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})$ is indeed a place $\sigma^{-\varepsilon}(F_{\varepsilon,n-1+\varepsilon}) = F_{n-1}$.

Second, we compute

$$\begin{aligned} \mathrm{Pyr}(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})) &= \sigma^{-\varepsilon}(\mathrm{Pyr}(P_{\varepsilon,n-1+\varepsilon})) = \sigma^{-\varepsilon}((P_{i,j})_{\varepsilon\leq i\leq j\leq n-1+\varepsilon}) \\ &= (\sigma^{-\varepsilon}(P_{i+\varepsilon,j+\varepsilon}))_{0\leq i\leq j\leq n-1} \end{aligned} \tag{437}$$

where the first and last equalities hold by Definition/Lemma 15(i) and the second equality holds by the Lemma 13.

Finally, because of the assertion that the path $\mathrm{Path}(Q) = (P_{i,j})_{j-i\leq 1} \in W(\mathcal{F}, 0, n)$ is rational, because of the definition of rational paths in Definition 16(iv), because of the equality in (437) and because of the 'on the one hand'-part in Lemma 17(ii) we conclude the desired statement in (v), namely that the path $\mathrm{Path}(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})) = (\sigma^{-\varepsilon}(P_{i+\varepsilon,j+\varepsilon}))_{j-i\leq 1} \in W(\mathcal{F}, 0, n-1)$ is also rational.

For (v)(a): The first already proven statement $\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}) \in \mathbb{P}_{F_{n-1}}$ provides that $A_\delta(\cdot, k)$ and $b(\cdot, A_\delta)$ can be applied to $\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon})$ for all $\delta = 0, 1$ and all $k = 1, \dots, n-2$. These are the first two desired statements in (v)(a).

On the one hand, we derive the desired inclusion $A_\delta(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), k) \subseteq \varepsilon + A_\delta(Q, k+\varepsilon)$ in (v)(a) for $\delta = 1-\varepsilon$ and all $k = 1, \dots, n-2$ from the equalities and inclusions

$$\begin{aligned} A_1(P_{0,n-1}, k) &= \{i \in \{1, \dots, k\} : e(P_{i-1,k}|P_{i,k}) > e(P_{i-1,n-1}|P_{i,n-1})\} \\ &\subseteq \{i \in \{1, \dots, k\} : e(P_{i-1,k}|P_{i,k}) > e(P_{i-1,n}|P_{i,n})\} \\ &= A_1(Q, k) \end{aligned}$$

and

$$\begin{aligned} 1 + A_0(\sigma^{-1}(P_{1,n}), k) &= 1 + \{j \in \{k, \ldots, n-2\} : e(P_{k+1,j+2}|P_{k+1,j+1}) > e(P_{1,j+2}|P_{1,j+1})\} \\ &= \{j \in \{k+1, \ldots, n-1\} : e(P_{k+1,j+1}|P_{k+1,j}) > e(P_{1,j+1}|P_{1,j})\} \\ &\subseteq \{j \in \{k+1, \ldots, n-1\} : e(P_{k+1,j+1}|P_{k+1,j}) > e(P_{0,j+1}|P_{0,j})\} \\ &= A_0(Q, k+1) \end{aligned}$$

where the equalities and inclusions hold by the following reasonings: The first equalities hold by the equality in (437), the definition of $A_{1-\varepsilon}(\sigma^{-\varepsilon}(P_{1,n}), k)$ in Definition 223 for all $k = 1, \ldots, n-2$ and the invariance of the ramification indices under the action of isomorphisms in (11). The inclusions hold because Key Lemma 36(iv) provides the estimates $e(P_{i-1,n-1}|P_{i,n-1}) \geq e(P_{i-1,n}|P_{i,n})$ and $e(P_{1,j+1}|P_{1,j}) \geq e(P_{0,j+1}|P_{0,j})$. The last equalities are the definitions of $A_{1-\varepsilon}(Q, k+\varepsilon)$ in Definition 223 for all $k+\varepsilon = 1+\varepsilon, \ldots, n-2+\varepsilon$. The remaining second equality in the second chain holds because the set on the left side consists of the indices $j+1$ in $\{k+1, \ldots, n-1\}$ which satisfy $e(P_{k+1,(j+1)+1}|P_{k+1,j+1}) > e(P_{1,(j+1)+1}|P_{1,j+1})$ and because then substituting $j+1$ with j yields the set on the right side.

On the other hand, we also derive the other desired inclusion $\varepsilon + A_\delta(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), k) \subseteq A_\delta(Q, k+\varepsilon)$ in (v)(a) for $\delta = \varepsilon$ and all $k = 1, \ldots, n-2$ from the equalities and inclusions

$$\begin{aligned} A_0(P_{0,n-1}, k) &= \{j \in \{k, \ldots, n-2\} : e(P_{k,j+1}|P_{k,j}) > e(P_{0,j+1}|P_{0,j})\} \\ &\subseteq \{j \in \{k, \ldots, n-1\} : e(P_{k,j+1}|P_{k,j}) > e(P_{0,j+1}|P_{0,j})\} \\ &= A_0(Q, k) \end{aligned}$$

and

$$\begin{aligned} 1 + A_1(\sigma^{-1}(P_{1,n}), k) &= 1 + \{i \in \{1, \ldots, k\} : e(P_{i,k+1}|P_{i+1,k+1}) > e(P_{i,n}|P_{i+1,n})\} \\ &= \{i \in \{2, \ldots, k+1\} : e(P_{i-1,k+1}|P_{i,k+1}) > e(P_{i-1,n}|P_{i,n})\} \\ &\subseteq \{i \in \{1, \ldots, k+1\} : e(P_{i-1,k+1}|P_{i,k+1}) > e(P_{i-1,n}|P_{i,n})\} \\ &= A_1(Q, k+1) \end{aligned}$$

where the equalities and inclusions hold by the following analogous reasonings: The first equalities hold by the equality in (437), the definition of $A_\varepsilon(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), k)$ in Definition 223 for all $k = 1, \ldots, n-2$ and the invariance of the ramification indices under the action of isomorphisms in (11). The inclusions are obvious The last equalities hold by the definition of $A_\delta(Q, k+\varepsilon)$ in Definition 223 for all $k+\varepsilon = 1+\varepsilon, \ldots, n-2+\varepsilon$. The last equalities are the definitions of $A_{1-\varepsilon}(Q, k+\varepsilon)$ in Definition 223 for all $k+\varepsilon = 1+\varepsilon, \ldots, n-2+\varepsilon$. The remaining second equality in the second chain holds because the set on the left side consists of the indices $i+1$ in $\{2, \ldots, k+1\}$ which satisfy $e(P_{(i+1)-1,k+1}|P_{i+1,k+1}) > e(P_{(i+1)-1,n}|P_{i+1,n})$ and because then substituting $i+1$ with i yields the set on the right side.

Hence, we established the desired inclusion

$$A_\delta(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), k) \subseteq \varepsilon + A_\delta(Q, k+\varepsilon) \tag{438}$$

in (v)(a) for all $\delta = 0, 1$.

Finally, the desired estimate $b(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), A_\delta) \leq b(Q, A_\delta)$ in (v)(a) follows from the estimates and equalities

$$\begin{aligned} b(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), A_\delta) = \max_{k=1,\ldots,n-2} \# A_\delta(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), k) \quad &\leq \max_{k=1,\ldots,n-2} \# A_\delta(Q, k+\varepsilon) \\ \leq \max_{k=1,\ldots,n-1} \# A_\delta(Q, k) = b(Q, A_\delta) \end{aligned}$$

for all $\delta = 0, 1$ where the equalities hold by the definition of $b(\cdot, A_\delta)$ in Definition 223, the first estimate by the inclusion in (438) and the second estimate holds because the assertion $\varepsilon \in \{0, 1\}$ implies that the maximum on the right side runs over one more element than the maximum on the left side.

For (v)(b): We already obtain the desired equality in (v)(b) by the equalities

$$\begin{aligned} e_{\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),p} &= \left\lfloor \log_p \left(\max_{k=1,\dots,n-2} \frac{e(P_{\varepsilon,k+\varepsilon}|P_{k+\varepsilon,k+\varepsilon}) \cdot e(P_{k+\varepsilon,n-1+\varepsilon}|P_{k+\varepsilon,k+\varepsilon})}{e(P_{\varepsilon,n-1+\varepsilon}|P_{k+\varepsilon,k+\varepsilon})} \right) \right\rfloor \\ &= \left\lfloor \log_p \left(\max_{k=1+\varepsilon,\dots,n-2+\varepsilon} \frac{e(P_{\varepsilon,k}|P_{k,k}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k})}{e(P_{\varepsilon,n-1+\varepsilon}|P_{k,k})} \right) \right\rfloor \end{aligned}$$

for all $p \in \mathbb{P}$ where the first equality holds by the equality in (437), by the definition of $e_{\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),p}$ in Definition 223 and by the invariance of the ramification indices under the action of isomorphisms in (11) and the second equality holds since we only changed the indexing via the bijection $\{1, \dots, n-2\} \to \{1+\varepsilon, \dots, n-2+\varepsilon\}$, $k \mapsto k + \varepsilon$ ☐

Theorem 225 with the degree bounds C_Q. The following Theorem provides the desired upper bounds C_Q for the degree $\deg(Q)$ of any place $Q \in \mathbb{P}_{\mathcal{F}}$ which can be expressed solely in terms of the degree d of $\mathcal{F}$, of the ramification indices of extensions in $\mathrm{Pyr}(Q)$ and the degrees of the places in $\mathrm{Path}(Q)$.

Theorem 225. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a finite field, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be its pyramid, let $Q \in \mathbb{P}_{F_n}$ for some $n \in \mathbb{N}$ with $n \geq 2$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ be the pyramid of Q.*

Then $\deg(Q)$ divides $l_Q \cdot \beta$ for all $\beta \in \{c_Q, \rho(d)^{b(Q,A)}, (d!)^{b(Q,A)}\}$ and any $A \in \mathcal{M}$. If $\mathcal{F}$ is locally Galois, then $\deg(Q)$ also divides $l_Q \cdot d^{b(Q,A)}$ for all $A \in \mathcal{M}$.

Sketch of the proof. Before, we come to the actual proof of Theorem 225, we want to sketch it and provide some understanding of what is basically going on.

We will divide the proof into two parts: First, we will prove all desired statements under the auxiliary assertion that $\mathrm{Path}(Q)$ is rational. Then, in the second part, we will derive the general statements from reducing them to the more special statements in the first part.

For the first part, we will conlcude that it is enough to show the estimates

$$v_p(\deg(Q)) \leq e_{Q,p} \tag{439}$$

for all $p \in \mathbb{P}$ in any case and

$$v_p(\deg(Q)) \leq v_p(d) \cdot b(Q, A) \tag{440}$$

for all $p \in \mathbb{P}$ and all $A \in \{\mathrm{Kill}, \mathrm{Kill}^\wedge\}$ if $\mathcal{F}$ is locally Galois. Then we will prove the estimates in (439) and (440) by induction over $n \in \mathbb{N}$ with $n \geq 2$.

Although, in the end, we will have to distinguish the six cases which are given by the six tuples in the set

$$\{ \text{'}n = 2\text{'}, \text{'}n \geq 3\text{'} \} \times \{ \text{'}\mathcal{F} \text{ arbitrary'}, \text{'}\mathcal{F} \text{ loc. Galois, } v = 0\text{'}, \text{'}\mathcal{F} \text{ loc. Galois, } v \geq 1\text{'} \}$$

where $v := \min_{i=0,1} v_p(f(P_{0,n}|P_{i,n-1+i}))$, the structure will always be the following: We will estimate each of the two summands of the sum $v_p(\deg(Q)) = v_p(f(P_{0,n}|P_{\gamma,n-1+\gamma})) + v_p(\deg(P_{\gamma,n-1+\gamma}))$ for some suitable $\gamma \in \{0, 1\}$ and then conclude that the resulted estimate has the desired upper bound in (439) resp. (440).

Let us illustrate this for the case $n \geq 3$, $\mathcal{F}$ locally Galois, $v \geq 1$ and $A = \mathrm{Kill}$: On the one hand, we have the estimate $v_p(f(P_{0,n}|P_{0,n-1})) \leq v_p(d)$ because of the fundamental equality in (8) for Galois extensions and, on the other hand, we also derive the estimate $v_p(\deg(P_{0,n-1})) \leq v_p(d) \cdot b(P_{0,n-1}, \mathrm{Kill})$ from applying the induction hypothesis to $P_{0,n-1} \in \mathbb{P}_{F_{n-1}}$. Next, we will also derive the estimate $b(P_{0,n-1}, \mathrm{Kill}) + 1 \leq b(Q, \mathrm{Kill})$ from the assumption $v \geq 1$ and finally bring everything together

$$\begin{aligned} v_p(\deg(Q)) = v_p(f(P_{0,n}|P_{0,n-1})) + v_p(\deg(P_{0,n-1})) &\leq v_p(d) + v_p(d) \cdot b(P_{0,n-1}, \mathrm{Kill}) \\ &\leq v_p(d) \cdot b(Q, \mathrm{Kill}). \end{aligned}$$

For the second part, we will drop the auxiliary assertion that $\mathrm{Path}(Q)$ is rational. Here, we will consider the constant field extension $\mathcal{F}' := \mathbb{F}_{q^{l_Q}} \cdot \mathcal{F} = (F'_\nu)_\nu$ of $\mathcal{F}$ and check that $\mathrm{Path}(Q')$ is rational for all $Q' \in \mathbb{P}_{F'_n}(Q)$. Consequently, we will conclude that the corresponding statements of the first part can be applied to Q'. Finally, combining the so obtained multiples for $\deg(Q')$ and the fact that $\mathcal{F}' = \mathbb{F}_{q^{l_Q}} \cdot \mathcal{F}$ emerges from a constant field extension via $\mathbb{F}_{q^{l_Q}}$ will eventually provide all the desired statements in Theorem 225.

Proof of Theorem 225. We first show all statements in Theorem 225 under the

$$\text{auxiliary assertion that } \mathrm{Path}(Q) = (P_{i,j})_{j-i \leq 1} \in W(\mathcal{F}, 0, n) \text{ is rational.} \tag{441}$$

In particular, by this assertion, the places $P_{k-1,k}$ are rational for all $k = 1, \ldots, n$ and, hence, we then have the equality $l_Q = 1$. Moreover, we also notice that since $\rho(d) := \prod_{p \in \mathbb{P}} p^{\lfloor \log_p d \rfloor}$ is a product of the maximal prime powers $\leq d$, it is a divisor of $d!$. Combining these two conclusions, the estimates in Lemma 224(iv) and the identities $\deg(Q) = \prod_{p \in \mathbb{P}} p^{v_p(\deg(Q))}$, $c_Q = \prod_{p \in \mathbb{P},\, p \leq d} p^{e_{Q,p}}$, $\rho(d)^{b(Q,A)} = \prod_{p \in \mathbb{P},\, p \leq d} p^{\lfloor \log_p d \rfloor \cdot b(Q,A)}$ and $d^{b(Q,A)} = \prod_{p \in \mathbb{P}} p^{v_p(d) \cdot b(Q,A)}$ for all $A \in \mathcal{M}$ yields that it is enough to show the estimates

$$v_p(\deg(Q)) \leq e_{Q,p} \tag{442}$$

for all $p \in \mathbb{P}$ in any case and

$$v_p(\deg(Q)) \leq v_p(d) \cdot b(Q, A) \tag{443}$$

for all $p \in \mathbb{P}$ and all $A \in \{\mathrm{Kill}, \mathrm{Kill}\hat{}\}$ if $\mathcal{F}$ is locally Galois.

In the following, let (σ, F_0) be the pair by which $\mathcal{F}$ is recursively defined in Definition 5(ii). We will show the desired estimates in (442) and (443) by induction over $n \in \mathbb{N}$ with $n \geq 2$: First, we notice that the well known rule $\deg(P_{0,n})) = f(P_{0,n}|P_{i,n-1+i}) \deg(P_{i,n-1+i})$ implies the equality equality $v_q(\deg(P_{0,n})) = v_q(f(P_{0,n}|P_{i,n-1+i}) + v_q(\deg(P_{i,n-1+i}))$ for all $i = 0, 1$ and all $q \in \mathbb{P}$. But, this again implies the equality

$$v_q(\deg(P_{0,n})) = \min_{i=0,1} v_q(f(P_{0,n}|P_{i,n-1+i}) + \max_{i=0,1} v_q(\deg(P_{i,n-1+i})) \tag{444}$$

for all $q \in \mathbb{P}$. Consequently, we compute

$$\begin{aligned} \operatorname*{lcm}_{i=0,1} \deg(P_{i,n-1+i}) \cdot \operatorname*{gcd}_{i=0,1} f(P_{0,n}|P_{i,n-1+i}) &= \deg(P_{0,n}) \\ &\leq \operatorname*{lcm}_{i=0,1} \deg(P_{i,n-1+i}) \cdot \frac{e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})}{e(P_{0,n}|P_{1,n-1})} \end{aligned} \tag{445}$$

(see Figure 7.8) where the first equality follows from the equality in (444) and from the well known equalities $\mathrm{lcm}(a, b) = \prod_{q \in \mathbb{P}} q^{v_q(\max(a,b))}$ and $\gcd(a, b) = \prod_{q \in \mathbb{P}} q^{v_q(\min(a,b))}$ for all $a, b \in \mathbb{N}$ and the estimate follows from applying the first estimate in Key Lemma 36(iii) to the diamond $(P_{0,n}, P_{0,n-1}, P_{1,n}, P_{1,n-1})$ of places.

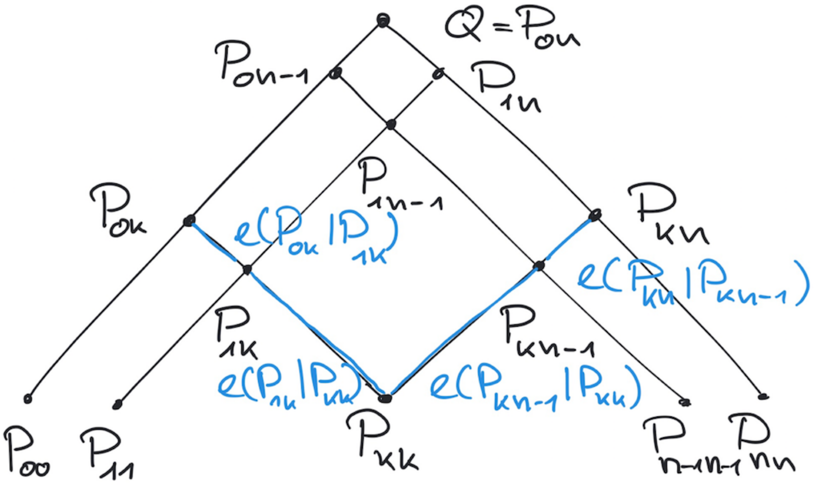


Figure 7.8: Pyramid of places with ramification indices in a proof

Now, let $\varepsilon \in \{0,1\}$ be an index such that

$$v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) = \min_{i=0,1} v_p(f(P_{0,n}|P_{i,n-1+i})). \tag{446}$$

Then we derive the equality and estimates

$$\begin{aligned} v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) &= v_p\left(\gcd_{i=0,1} f(P_{0,n}|P_{i,n-1+i})\right) \le \left\lfloor \log_p\left(\gcd_{i=0,1} f(P_{0,n}|P_{i,n-1+i})\right)\right\rfloor \\ &\le \left\lfloor \log_p\left(\frac{e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})}{e(P_{0,n}|P_{1,n-1})}\right)\right\rfloor =: \alpha_{Q,p} \end{aligned} \tag{447}$$

where the equality holds by the choice of ε in (456) and by the well known equality $\gcd(a,b) = \prod_{q\in\mathbb{P}} q^{v_q(\min(a,b))}$ for all $a,b \in \mathbb{N}$, the first estimate holds because $v_p(a)$ is a natural number which is at most $\log_p(a)$ for all $a \in \mathbb{N}$ and the second estimate holds because of the estimate in (445) and because $\lfloor\cdot\rfloor$ and $\log_p$ are monotonically increasing functions.

Consequently, for $n \ge 3$, we get the equalities and estimates

$$\begin{aligned} \alpha_{Q,p} + e_{\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),p} &= \left\lfloor \log_p\left(\frac{e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})}{e(P_{0,n}|P_{1,n-1})}\right)\right\rfloor \\ &\quad + \left\lfloor \log_p\left(\max_{k=1+\varepsilon,\dots,n-2+\varepsilon} \frac{e(P_{\varepsilon,k}|P_{k,k}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k})}{e(P_{\varepsilon,n-1+\varepsilon}|P_{k,k})}\right)\right\rfloor \\ &\le \left\lfloor \log_p\left(\max_{k=1+\varepsilon,\dots,n-2+\varepsilon}\left(\frac{e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})}{e(P_{0,n}|P_{1,n-1})}\right.\right.\right. \\ &\qquad \left.\left.\left. \cdot \frac{e(P_{\varepsilon,k}|P_{k,k}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k})}{e(P_{\varepsilon,n-1+\varepsilon}|P_{k,k})}\right)\right)\right\rfloor \\ &= \left\lfloor \log_p\left(\max_{k=1+\varepsilon,\dots,n-2+\varepsilon} \frac{e(P_{1-\varepsilon,n-\varepsilon}|P_{1,n-1}) \cdot e(P_{\varepsilon,k}|P_{k,k}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k})}{e(Q|P_{k,k})}\right)\right\rfloor \\ &\le \left\lfloor \log_p\left(\max_{k=1+\varepsilon,\dots,n-2+\varepsilon} \frac{e(P_{0,k}|P_{k,k}) \cdot e(P_{k,n}|P_{k,k})}{e(Q|P_{k,k})}\right)\right\rfloor \\ &\le e_{Q,p} \end{aligned} \tag{448}$$

where the equalities and estimates hold by the following reasonings:

The first equality holds by the choice of $\alpha_{Q,p}$ in (447) and by the 'moreover'-part in Lemma 224(v)(b) which is applicable because $n \geq 3$ and because of the auxiliary assertion in (441).

The first estimate holds by the well known rules $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$ for all $x, y \in \mathbb{R}_{\geq 0}$ and $\log_p(x) + \log_p(y) = \log_p(xy)$ for all $x, y \in \mathbb{R}_{>0}$.

The second equality holds because Lemma 13 and the multiplicative transitivity rule for ramification indices in (7) supply the equalities $e(P_{0,n}|P_{1,n-1}) = e(P_{0,n}|P_{\varepsilon,n-1+\varepsilon}) \cdot e(P_{\varepsilon,n-1+\varepsilon}|P_{1,n-1})$ and $e(P_{0,n}|P_{\varepsilon,n-1+\varepsilon}) \cdot e(P_{\varepsilon,n-1+\varepsilon}|P_{k,k}) = e(P_{0,n}|P_{k,k}) = e(Q|P_{k,k})$ and because we can represent the numerator in the first quotient as the product $e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1}) = e(P_{\varepsilon,n-1+\varepsilon}|P_{1,n-1}) \cdot e(P_{1-\varepsilon,n-\varepsilon}|P_{1,n-1})$.

The second estimate holds for $\varepsilon = 0$ because Key Lemma 36(iv) provides the estimate $e(P_{1-\varepsilon,n-\varepsilon}|P_{1,n-1}) \leq e(P_{k,n}|P_{k,n-1+\varepsilon})$, because $\lfloor \cdot \rfloor$, $\log_p$ and max are monotonically increasing and because of the equality $e(P_{k,n}|P_{k,n-1+\varepsilon}) \cdot e(P_{k,n-1+\varepsilon}|P_{k,k}) = e(P_{k,n}|P_{k,k})$.

The second estimate holds for $\varepsilon = 1$ because Key Lemma 36(iv) provides the estimate $e(P_{1-\varepsilon,n-\varepsilon}|P_{1,n-1}) \leq e(P_{0,k}|P_{\varepsilon,k})$, because $\lfloor \cdot \rfloor$, $\log_p$ and max are monotonically increasing and because of the equality $e(P_{0,k}|P_{\varepsilon,k}) \cdot e(P_{\varepsilon,k}|P_{k,k}) = e(P_{0,k}|P_{k,k})$.

The last estimate holds because the maximum in the definition of $e_{Q,p}$ in Definition 223 runs over $k = 1, \ldots, n-1$ and, thus over one more element as the maximum on the left side.

Next, if

$$v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) \geq 1 \tag{449}$$

then we derive the estimates

$$e(P_{0,n}|P_{0,n-1}) < e(P_{1,n}|P_{1,n-1}) \leq e(P_{k,n}|P_{k,n-1}) \tag{450}$$

and

$$e(P_{0,n}|P_{1,n}) < e(P_{0,n-1}|P_{1,n-1}) \leq e(P_{0,k}|P_{1,k}) \tag{451}$$

for all $k = 1, \ldots, n-2$ where the first estimates hold as, otherwise for all $\delta = 0, 1$, we derive the impossible estimate

$$\begin{aligned} 0 = \left\lfloor \log_p(1) \right\rfloor &\geq \left\lfloor \log_p \left(\frac{e(P_{\delta,n-1+\delta}|P_{1,n-1})}{e(P_{0,n}|P_{1-\delta,n-\delta})} \right) \right\rfloor \\ &= \left\lfloor \log_p \left(\frac{e(P_{\delta,n-1+\delta}|P_{1,n-1}) \cdot e(P_{1-\delta,n-\delta}|P_{1,n-1})}{e(P_{0,n}|P_{1-\delta,n-\delta}) \cdot e(P_{1-\delta,n-\delta}|P_{1,n-1})} \right) \right\rfloor \\ &= \left\lfloor \log_p \left(\frac{e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})}{e(P_{0,n}|P_{1,n-1})} \right) \right\rfloor \\ &= \alpha_{Q,p} \geq v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) \geq 1 \end{aligned}$$

where the equalities and estimates hold by the following reasonings: The first and second equalities are clear. The first estimate holds because of the assumption $e(P_{0,n}|P_{1-\delta,n-\delta}) \geq e(P_{\delta,n-1+\delta}|P_{1,n-1})$ of this contradiction and because $\log_p$ and $\lfloor \cdot \rfloor$ are monotonically increasing functions. The third equality holds because the multiplicative transitivity rule for ramification indices in (7) supplies the equality $e(P_{0,n}|P_{1-\delta,n-\delta}) \cdot e(P_{1-\delta,n-\delta}|P_{1,n-1}) = e(P_{0,n}|P_{1,n-1})$ and because the product is the numerator of the quotient on the left side can be represented as $e(P_{\delta,n-1+\delta}|P_{1,n-1}) \cdot e(P_{1-\delta,n-\delta}|P_{1,n-1}) = e(P_{0,n-1}|P_{1,n-1}) \cdot e(P_{1,n}|P_{1,n-1})$. The fourth equality and second estimate holds by the equalities and estimates in (447). The last estimate holds by the assertion in (457).

Consequently, if $v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) \geq 1$, then the estimates in (450) and (451) and the definitions of Kill and $\mathrm{Kill}\hat{}$ in Definition 223 yield

$$n-1 \in \mathrm{Kill}(Q,k) \text{ and } 1 \in \mathrm{Kill}\hat{}(Q,k) \tag{452}$$

for all $k = 1, \ldots, n-2$. On the other hand, these definitions also imply $\mathrm{Kill}(P_{0,n-1},k) \subseteq \{k, \ldots, n-2\}$ and $\mathrm{Kill}\hat{}(\sigma^{-1}(P_{1,n}),k) \subseteq \{1, \ldots, k\}$ and, thus,

$$n-1 \notin \mathrm{Kill}(P_{0,n-1},k) \text{ and } 1 \notin 1 + \mathrm{Kill}\hat{}(\sigma^{-1}(P_{1,n}),k) \tag{453}$$

for all $k = 1, \ldots, n-2$. Let $A_0 := \mathrm{Kill}$, $A_1 := \mathrm{Kill}\hat{}$. Combining (452), (453) and the inclusion $\delta + A_\delta(\sigma^{-\delta}(P_{\delta,n-1+\delta}),k) \subseteq A_\delta(Q,k+\delta)$ in Lemma 224(v)(a) yields the estimate

$$\#A_\delta(\sigma^{-\delta}(P_{\delta,n-1+\delta}),k) + 1 \leq \#A_\delta(Q,k+\delta) \tag{454}$$

for all $\delta = 0,1$ and $k = 1, \ldots, n-2$. Hence, if $v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) \geq 1$, we obtain

$$\begin{aligned} b(\sigma^{-\delta}(P_{\delta,n-1+\delta}),A_\delta) + 1 &= \max_{k=1,\ldots,n-2} \#A_\delta(\sigma^{-\delta}(P_{\delta,n-1+\delta}),k) + 1 \\ &\leq \max_{k=1,\ldots,n-2} \#A_\delta(Q,k+\delta) \\ &\leq \max_{k=1,\ldots,n-1} \#A_\delta(Q,k) = b(Q,A_\delta) \end{aligned} \tag{455}$$

where the equalities hold by the definitions of $b(\sigma^{-\delta}(P_{\delta,n-1+\delta}),A_\delta)$ and $b(Q,A_\delta)$ in Definition 223, the first estimate holds by the estimate in (454) and the second estimate holds as the maximum on the right side runs over one more element than the maximum on the left side.

Second to last, we obtain the equality and estimate

$$\begin{aligned} v_p(\deg(P_{\gamma,n-1+\gamma})) &= v_p(\deg(\sigma^{-\gamma}(P_{\gamma,n-1+\gamma}))) \\ &\leq \begin{cases} 0 & \text{if } n = 2 \\ e_{\sigma^{-\gamma}(P_{\gamma,n-1+\gamma}),p} & \text{if } n \geq 3 \\ v_p(d) \cdot b(\sigma^{-\gamma}(P_{\gamma,n-1+\gamma}),A_\delta) & \text{if } n \geq 3 \text{ and } \mathcal{F} \text{ is locally Galois} \end{cases} \end{aligned} \tag{456}$$

for all $\gamma = 0,1$ and all $\delta = 0,1$ where the equality and estimate hold by the following reasonings: The equality holds by the invariance of the degree of places under the action of isomorphisms in (11). The first case of the estimate holds because the auxiliary assertion in (441) implies that $P_{\gamma,n-1+\gamma} = P_{\gamma,1+\gamma}$ is rational for $n = 2$. The second and third cases of the estimate hold because $\mathrm{Path}(\sigma^{-\gamma}(P_{\gamma,n-1+\gamma}))$ is also rational by Lemma 224(v) and because we can therefore apply the induction hypothesis to $\sigma^{-\gamma}(P_{\gamma,n-1+\gamma}) \in \mathbb{P}_{F_{n-1}}$.

Finally, we bring everything together to derive the desired estimates in (442) and (443): First, the desired estimate in (442) follows from the equality and estimates

$$\begin{aligned} v_p(\deg(Q)) &= v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) + v_p(\deg(P_{\varepsilon,n-1+\varepsilon})) \\ &\leq \begin{cases} \alpha_{Q,p} & \text{if } n = 2 \\ \alpha_{Q,p} + e_{\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}),p} & \text{if } n \geq 3 \end{cases} \\ &\leq e_{Q,p} \end{aligned} \tag{457}$$

where the equality and estimates hold by the following reasonings: The first equality holds because of the well known equality $\deg(Q) = f(P_{0,n}|P_{\gamma,n-1+\gamma}) \cdot \deg(P_{\gamma,n-1+\gamma})$ for all $\gamma = 0,1$ and because $v_p : \mathbb{N} \to \mathbb{N}_0$ is a morphism of monoids. The first estimate holds by combining the estimates in (447) and the first two cases in (456). The first case of the second estimate holds because the definitions of $\alpha_{Q,p}$ in (447) and of $e_{Q,p}$ in Definition 223 even imply the equality $\alpha_{Q,p} = e_{Q,p}$ for $n = 2$. The second case of the second estimate holds by the estimate in (448) for all $n \geq 3$.

Second, if $\mathcal{F}$ is locally Galois and $v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) = 0$ holds, then the desired estimate in (443) follows from the equality and estimates

$$\begin{aligned} v_p(\deg(Q)) &= v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) + v_p(\deg(P_{\varepsilon,n-1+\varepsilon})) \\ &\leq \begin{cases} 0 & \text{if } n = 2 \\ v_p(d) \cdot b(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), A_\delta) & \text{if } n \geq 3 \end{cases} \\ &\leq v_p(d) \cdot b(Q, A_\delta) \end{aligned}$$

for all $\delta = 0, 1$ where the equality and estimates hold by the following reasonings: The equality holds by the same reasoning as the first equality in (457). The first estimate holds by combining the assertion $v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) = 0$ and the first and third cases of the estimate in (456). The first case of the second estimate holds because the definition of $b(Q, A_\delta)$ in Definition 223 implies that it is nonnegative. The second case of the second estimate holds by the estimate $b(\sigma^{-\varepsilon}(P_{\varepsilon,n-1+\varepsilon}), A_\delta) \leq b(Q, A_\delta)$ in Lemma 224(v)(a).

Third, if $\mathcal{F}$ is locally Galois and $v_p(f(P_{0,n}|P_{\varepsilon,n-1+\varepsilon})) \geq 1$ holds, then the desired estimate in (443) follows from the equality and estimates

$$\begin{aligned} v_p(\deg(Q)) &= v_p(f(P_{0,n}|P_{\delta,n-1+\delta})) + v_p(\deg(P_{\delta,n-1+\delta})) \\ &\leq \begin{cases} v_p(d) & \text{if } n = 2 \\ v_p(d) + v_p(d) \cdot b(\sigma^{-\delta}(P_{\delta,n-1+\delta}), A_\delta) & \text{if } n \geq 3 \end{cases} \\ &\leq v_p(d) \cdot b(Q, A_\delta) \end{aligned}$$

for all $\delta = 0, 1$ where the equality and estimates hold by the following reasonings: The equality holds by the same reasoning as the first equality in (457). For the first estimate, we first notice that the assertion that $\mathcal{F}$ is locally Galois and Lemma 33 imply that $F_{0,n}/F_{\delta,n-1+\delta}$ is Galois. Thus, the fundamental equality for Galois extensions in (8) supplies that $f(P_{0,n}|P_{\delta,n-1+\delta})$ divides d and the estimate $v_p(f(P_{0,n}|P_{\delta,n-1+\delta})) \leq d$. Then the first estimate follows from combining this estimate and the first and third cases of the estimate in (456). The last estimate holds because the estimate in (455) and the fact that $b(\sigma^{-\delta}(P_{\delta,n-1+\delta}), A_\delta)$ is nonnegative imply the estimates $1 \leq b(\sigma^{-\delta}(P_{\delta,n-1+\delta}), A_\delta) + 1 \leq b(Q, A_\delta)$.

All together, we established that Theorem 225 follows for all $n \in \mathbb{N}$ with $n \geq 2$ under the auxiliary assertion that $\mathrm{Path}(Q)$ is rational.

Finally, we drop this auxiliary assertion: Let $\mathbb{F}_q$ be the finite field over which $\mathcal{F}$ is defined. Then we consider the constant field extension $\mathcal{F}' := \mathbb{F}_{q^{l_Q}} \cdot \mathcal{F} = (F'_\nu)_\nu$ and its pyramid $(F'_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F}')$. In particular, Definition/Lemma 21 provides the equality

$$F'_{i,j} = \mathbb{F}_{q^{l_Q}} \cdot F_{i,j} \tag{458}$$

for all $i, j \in \mathbb{N}_0$ with $i \leq j$ and Lemma 34 provides that

$$\mathcal{F}' \text{ is locally Galois if } \mathcal{F} \text{ is locally Galois.} \tag{459}$$

Now, let $Q' \in \mathbb{P}_{F'_n}(Q)$. Then [Sti08, p.190, Lemma 5.1.9(d)] supplies that

$$\deg(Q) = \gcd(l_Q, \deg(Q)) \cdot \deg(Q') \text{ is a divisor of } l_Q \cdot \deg(Q'). \tag{460}$$

Furthermore, for the path $(P'_{i,j})_{j-i\leq 1} := \mathrm{Path}_{\mathcal{F}'}(Q') \in W(\mathcal{F}', 0, n)$ of Q', we obtain the equalities

$$(P'_{i,j} \cap F_{i,j})_{j-i\leq 1} = \pi_{W(\mathcal{F}')/W(\mathcal{F})}((P'_{i,j})_{j-i\leq 1}) = \pi_{W(\mathcal{F}')/W(\mathcal{F})}(\mathrm{Path}_{\mathcal{F}'}(Q'))$$

$$= \mathrm{Path}_{\mathcal{F}}(\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}}(Q')) = \mathrm{Path}_{\mathcal{F}}(Q) = (P_{i,j})_{j-i\leq 1} \tag{461}$$

where the first equality holds by the definition of $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$ in Lemma 76(iv), the second equality holds by the choice of $(P'_{i,j})_{j-i\leq 1} = \mathrm{Path}_{\mathcal{F}'}(Q')$, the third equality holds by the identity in Lemma 76(iv), the fourth equality holds because of the definition of $\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}}(Q') = Q' \cap F_{0,n} = Q' \cap F_n = Q$ in Lemma 76(iii) and the last equality holds by the choice of $(P_{i,j})_{j-i\leq 1} = \mathrm{Path}_{\mathcal{F}}(Q)$.

Consequently, by the equality in (461), we conclude that $P'_{i,j}/P_{i,j}$ is an extension of places in $F'_{i,j}/F_{i,j}$ for all $0 \leq i \leq j \leq n$. Furthermore, we also notice that since any of the places $P_{i,i}$ with $i = 0, \ldots, n$ is contained in some of the places $P_{i-1,i}$ with $i = 1, \ldots, n$, the degree of of the places in $(P_{i,j})_{j-i\leq 1} = \mathrm{Path}_{\mathcal{F}}(Q)$ divides $l_Q = \mathrm{lcm}_{k=1,\ldots,n} \deg(P_{k-1,k})$. Therefore, combining this conclusion, the equality in (459) and [Sti08, p.190, Lemma 5.1.9(d)] yields that the path $(P'_{i,j})_{j-i\leq 1} = \mathrm{Path}_{\mathcal{F}'}(Q')$ of Q' is rational.

Hence, by this last conclusion, by (459) and for all of the desired statements for Q, the first part of this proof provides the corresponding statements for Q'. But, combining this, (460) and the fact that the invariance of ramification indices under constant field extensions in (12) implies the equality $e(P'_{i,j}|P'_{k,l}) = e(P_{i,j}|P_{k,l})$ for all $0 \leq i \leq k \leq l \leq j \leq n$ finally also supplies all the desired statements for Q. □

7.3.2 Degree Bounds Which Only Depend on Subgraphs

Summary of the results of this subsection. In this subsection, we will bring the tower graph $\Gamma_{\mathcal{F}}$ into the picture and prove Theorem 228. For every tame recursive towers $\mathcal{F}$ and every finite strongly connected subgraphs Γ of $\Gamma_{\mathcal{F}}$ which has a circle with positive length but only contains circles with balanced ramification indices, this theorem provides an upper bound C_Γ for the set $\delta_\Gamma = \{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}\}$. This upper bound C_Γ will be $l_\Gamma \cdot {B_\Gamma}^{\#V(\Gamma)}$ and, thus, only depend on Γ. The third interim result in Corollary 171 will be an immediate consequence of this Theorem 228.

We will first prove Proposition 227. Then Theorem 228 will follow from combining this proposition and Theorem 225.

Degree bounds for places lying over subgraphs. Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a finite field and Γ be a finite subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Then, for $Q \in \mathbb{P}_{F_n}[\Gamma]$ with $n \geq 2$, Theorem 225 supplies the multiples $l_Q \cdot \rho(d)^{b(Q,\mathrm{Count})}$ and $l_Q \cdot (d!)^{b(Q,\mathrm{Count})}$ of $\deg(Q)$ in any case and $l_Q \cdot d^{b(Q,\mathrm{Count})}$ if $\mathcal{F}$ is locally Galois. Then the following Proposition 227(i) will provide the larger multiples $l_\Gamma \cdot \rho(d)^{B_{\Gamma,n}}$ and $l_\Gamma \cdot (d!)^{B_{\Gamma,n}}$ of $\deg(Q)$ in any case and $l_\Gamma \cdot d^{B_{\Gamma,n}}$ if $\mathcal{F}$ is locally Galois. The point here will be that these larger multiples only depend on Γ and n but not on Q anymore.

Furthermore, in Proposition 227(ii), we will prove that if $\mathcal{F}$ is tame, Γ is strongly connected and only contains circles with balanced ramification indices, then we can even get rid of the dependency on n, i.e. we will obtain the estimate $B_{\Gamma,n} \leq {B_\Gamma}^{\#V(\Gamma)}$ where ${B_\Gamma}^{\#V(\Gamma)}$ only depends on Γ. This is the last missing piece to prove the desired Theorem 228.

Definition 226. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a finite field which is defined by the pair (σ, F_0), let Γ be a finite subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ with at least one circle of positive length. Then we define*

$$B_{\Gamma,n} := \max_{Q \in \coprod_{i=0}^{n} \mathbb{P}_{F_i}[\Gamma]} e(Q|Q \cap F_0) \quad \text{and} \quad l_\Gamma := \operatorname*{lcm}_{Q \in E(\Gamma)} \deg(Q)$$

for all $n \in \mathbb{N}_0$.

Note that $B_{\Gamma,n}$ is a natural number because the assertion that Γ is finite with at least one circle of positive length implies that it contains at least one but finitely many paths of every given length $i \in \mathbb{N}_0$.

Proposition 227. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a finite field and let Γ be a finite subgraph of the tower graph $\Gamma_\mathcal{F}$ of $\mathcal{F}$ with at least one circle of positive length. Then the following hold:*

(i) Let $Q \in \mathbb{P}_n[\Gamma]$ for some $n \geq 2$. Then l_Q divides l_Γ and we have the estimate

$$b(Q, \mathrm{Count}) \leq B_{\Gamma,n}.$$

(ii) If Γ is strongly connected, all circles in Γ have balanced ramification indices and all paths in Γ are tame, then $B_{\Gamma,n}$ divides ${B_\Gamma}^{\#V(\Gamma)}$ for all $n \in \mathbb{N}_0$.

Proof. For (i): Let σ be the tower map of $\mathcal{F}$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ be the pyramid of Q. Then we derive

$$\left[\sigma^{-i}(P_{i,j})\right]_{j-i\leq 1} = \sigma_{\Gamma_\mathcal{F}}^{-1}((P_{i,j})_{j-i\leq 1}) = \sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}_\mathcal{F}(Q)) \in W(\Gamma, n) \tag{462}$$

where the first equality follows from the definition of $\sigma_{\Gamma_\mathcal{F}}$ in Definition/Lemma 76, the second equality follows from the 'on the one hand'-part in Lemma 17(ii) and the containment-statement follows from the definition of $\mathbb{P}_n[\Gamma]$ in Definition 85 and the assertion $Q \in \mathbb{P}_n[\Gamma]$.

Now, the definition of l_Q in Definition 223 and the invariance of the degree of places under the action of isomorphisms in (11) provide the equality

$$l_Q = \operatorname*{lcm}_{k=1,\ldots,n} \deg(P_{k-1,k}) = \operatorname*{lcm}_{k=1,\ldots,n} \deg(\sigma^{-k}(P_{k-1,k}))$$

But since the equality in (462) provides $\sigma^{-k}(P_{k-1,k}) \in E(\Gamma)$, we obtain that the lcm in l_Q runs over a subset of the lcm in the definition of $l_\Gamma = \mathrm{lcm}_{P\in E(\Gamma)} \deg(P)$ in Definition . In particular, this yields the first desired statement in (i), namely that l_Q divides l_Γ.

For the desired estimate $b(Q, \mathrm{Count}) \leq b_{\Gamma,n}$ in (i), we first conclude the equalities

$$\begin{aligned}\sigma_{\Gamma_\mathcal{F}}^{-1}\left(\mathrm{Path}\left(\sigma^{-k}(P_{k,n})\right)\right) &= \sigma_{\Gamma_\mathcal{F}}^{-1}\left(\sigma^{-k}(\mathrm{Path}\,(P_{k,n}))\right) = \sigma_{\Gamma_\mathcal{F}}^{-1}\left(\sigma^{-k}\left((P_{i,j})_{\substack{k\leq i\leq j\leq n\\ j-i\leq 1}}\right)\right)\\ &= \sigma_{\Gamma_\mathcal{F}}^{-1}\left(\sigma^{-k}(P_{i+k,j+n})_{\substack{0\leq i\leq j\leq n-k\\ j-i\leq 1}}\right) = \left[\sigma^{-(i+k)}(P_{i+k,j+n})\right]_{j-i\leq 1} \in W(\Gamma, n-k)\end{aligned} \tag{463}$$

for all $k = 1, \ldots, n-1$ where the equalities and containment-statement hold by the following reasonings: The first equality holds by Definition/Lemma 20(ii). The second equality holds by Definition/Lemma 17(ii). The third equality holds by Definition/Lemma 20(i). The last equality holds by the definition of $\sigma_{\Gamma_\mathcal{F}}$. The containment-statement holds because this path is a subpath of $\sigma_{\Gamma_\mathcal{F}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma)$ by (462).

Finally, we obtain the desired estimate in (i) by the equalities and estimate

$$\begin{aligned}b(Q, \mathrm{Count}) &= \max_{k=1,\ldots,n-1} e(P_{k,n}|P_{k,k}) = \max_{k=1,\ldots,n-1} e(\sigma^{-k}(P_{k,n})|\sigma^{-k}(P_{k,k}))\\ &= \max_{k=1,\ldots,n-1} e(\sigma^{-k}(P_{k,n})|\sigma^{-k}(P_{k,n}) \cap F_0)\\ &\leq \max_{Q'\in\coprod_{i=0}^{n}\mathbb{P}_{F_i}[\Gamma]} e(Q'|Q' \cap F_0) = B_{\Gamma,n}\end{aligned}$$

where the equalities and estimate hold by the following reasonings: The first and last equalities hold by the definitions of $b(Q, \mathrm{Count})$ in Definition 223 and of $B_{\Gamma,n}$ in Definition

226. The second equality holds by the invariance of the ramification indices under the action of isomorphisms in (11). The third equality holds since, for $(F_{i,j})_{i,j} := \operatorname{Pyr}(\mathcal{F})$, $\sigma^{-k}(P_{k,k})$ is a place in $\sigma^{-k}(F_{k,k}) = F_{0,0} = F_0$ which lies under $\sigma^{-k}(P_{k,n})$. The estimate holds because (463) supplies $\sigma^{-k}(P_{k,n}) \in \mathbb{P}_{F_{n-k}}[\Gamma]$.

For (ii): By the definitions of $B_{\Gamma,n}$ in Definition 226 and of B_Γ in Lemma 221, we have the equalities

$$B_{\Gamma,n} = \max_{Q \in \coprod_{i=0}^{n} \mathbb{P}_{F_i}[\Gamma]} e(Q|Q \cap F_0) \quad \text{and} \quad B_\Gamma = \operatorname*{lcm}_{Q \in \mathbb{P}^{\circ}_{\mathcal{F}}[\Gamma]} \tilde{e}(Q|Q \cap F_0). \tag{464}$$

Then we notice that because of the assertion that all paths in Γ are tame, Lemma 44(iii) supplies the equality $e = \tilde{e}$ for all the extensions of places in the definitions of $B_{\Gamma,n}$ and B_Γ in (464). Consequently, applying Lemma 221 yields that all ramification indices in the maximum in $B_{\Gamma,n}$ divide ${B_\Gamma}^{\#V(\Gamma)}$. Hence, $B_{\Gamma,n}$ indeed divides ${B_\Gamma}^{\#V(\Gamma)}$ and (ii) follows. ☐

Theorem 228 with the degree bound C_Γ which only depends on the subgraph Γ. In the following Theorem 228, we finally provide the desired upper bounds C_Γ of $\delta_\Gamma = \{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}\}$ which only depends on the subgraph Γ.

Theorem 228. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower of degree d over a finite field, let Γ be a finite strongly connected subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ with at least one edge such that all circles in Γ have balanced ramification indices and all paths in Γ are tame and let $Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]$.*

Then $\deg(Q)$ divides $l_\Gamma \cdot \rho(d)^{{B_\Gamma}^{\#V(\Gamma)}}$ and $l_\Gamma \cdot (d!)^{{B_\Gamma}^{\#V(\Gamma)}}$. If $\mathcal{F}$ is locally Galois, then $\deg(Q)$ also divides $l_\Gamma \cdot d^{{B_\Gamma}^{\#V(\Gamma)}}$.

Proof. Let $n \in \mathbb{N}_0$ such that $Q \in \mathbb{P}_{F_n}[\Gamma]$. Then we first notice that, because of the assertion that Γ is strongly connected with at least one edge, it must also have circles of positive lengths. Next, we distinguish four cases:

For $n \geq 2$, all statements are direct combinations of statements in Theorem 225 and Proposition 227.

For $n = 1$, we have $Q \in \mathbb{P}_{F_1}[\Gamma] = \mathbb{P}_{F_1}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, 1))) = \mathbb{P}_{F_1}(E(\Gamma)) = E(\Gamma)$ and, thus, $\deg(Q)$ is already a divisor of $l_\Gamma = \operatorname{lcm}_{Q' \in E(\Gamma)} \deg(Q')$.

For $n = 0$, we have $Q \in \mathbb{P}_{F_0}[\Gamma] = \mathbb{P}_{F_0}(\sigma_{\Gamma_{\mathcal{F}}}(W(\Gamma, 0))) = \mathbb{P}_{F_0}(V(\Gamma)) = V(\Gamma)$. Because Γ is strongly connected and because of the assertion that Γ contains at least one edge, there must also be an edge Q' in Γ with $v_{\text{init}}(Q') = Q$. Consequently, $\deg(Q)$ is a divisor of $\deg(Q') = f(Q'|Q) \cdot \deg(Q)$ and, thus, of l_Γ. ☐

7.3.3 Proof of the Third Interim Result

Finally, we are prepared to prove the third interim result in Corollary 171.

Proof of Corollary 171. If Γ is empty or only consists of a single vertex, then $\mathbb{P}_{\mathcal{F}}[\Gamma]$ is also empty or consists of a single place, respectively. Hence, the finiteness of the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ follows trivially in these cases.

If Γ contains more than one vertex, then it must also contain at least one edge because it is strongly connected. Thus, in all the remaining cases, Γ contains at least one circle of positive length. But then the finiteness of the set $\{\deg(Q) : Q \in \mathbb{P}_{\mathcal{F}}[\Gamma]\}$ immediately follows from Theorem 228. ☐

8 Computing Genus Formulas

Summary of the results of this chapter. In this chapter, we will prove Corollary 246, which is a further result of this thesis and enables us to compute genus formulas for all recursive towers which have finite separating power ramification subgraphs (see Definition 229(iii) and Definition 239). Moreover, in (A4), we will even provide an approach on how to apply Corollary 246 to effectively compute genus formulas. This approach (A4) will also have a first naive implementatios in Subsection 8.3.2. The author is not aware of any tame recursive tower in the literature on which this implementation does not work. Consequently, in Examples 250, we will list genus formulas for some representative tame recursive towers from the literature.

Significance of being able to compute genus formulas. In some situations, it is useful to have explicit formulas for the genera in towers of functions fields:

First, good towers $\mathcal{F} = (F_\nu)_\nu$ of function fields over finite fields can be used to construct error correcting Goppa Codes with good parameters (see the proof of [Sti08, p. 300, Proposition 8.4.6]). In this construction, more precise approximations of the values $N(F_n)$ and $g(F_n)$ also yield more precise approximations for the parameters of the corresponding Goppa Codes constructed from F_n. Thus, being able to compute an explicit formula for $g(F_n)$ is advantageous. Also note that, although we will not present details, the methods of this chapter can also be used to compute the upper bound $N[\overline{F}_n, \overline{k} \cdot \Gamma_{\mathcal{F}}^{\text{rat}}]$ for $N(F_n)$ where $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ is the geometric tower of $\mathcal{F}$.

Second, in the generic case, computing the genus $g(F_n)$ via a computer algebra system like Magma [BCP97] is already unfeasible for $n \geq 6$. Thus, to compute the genera $g(F_n)$ for higher levels F_n, there is hardly any other chance than to already compute a formula for $g(F_n)$ for all n.

Third, it is crucial to have an actual genus formula in some situations. For instance, in [HP16, p. 12, Proposition 12], the authors computed the genus formula for the tame recursive HP-tower $\mathcal{F}_{HP,q}$, which we also already introduced in Examples 8(vii). In Examples 250(i), the implementation of Subsection 8.3.2 will also compute this formula.

The challenges of computing precise values for $N(F_n)$ and $g(F_n)$. In Chapter 6, we estimated $N(F_n)$ and $g(F_n)$ asymptotically in such a way that we could determine the precise limits $\nu(\mathcal{F}) = \lim_{\nu\to\infty} \frac{N(F_\nu)}{d^\nu}$ and $\gamma(\mathcal{F}) = \lim_{\nu\to\infty} \frac{g(F_\nu)}{d^\nu}$ in most cases, e.g. for tame $\mathcal{F}$ where every weakly connected component of the finite ramification subgraph $\Gamma_{\mathcal{F}}^{\text{ram}}$ has circles with unbalanced ramification indices. This was only possible because we could show that the quotient with the error of our estimate for $N(F_n)$ (resp. $g(F_n)$) and d^n vanishes as $n \to \infty$.

In this chapter, we will try to actually compute these errors. For that, we will briefly recapitulate the estimates from the last chapters for the case that $\mathcal{F}$ is tame and every weakly connected component of the finite ramification subgraph $\Gamma_{\mathcal{F}}^{\text{ram}}$ has circles with unbalanced ramification indices: First, we considered the equalities

$$N(F_n) = N(F_n, V(\Gamma_{\mathcal{F}}^{\text{split}})) + N(F_n, V(\Gamma_{\mathcal{F}}^{\text{ram}})) + N(F_n, V(\Gamma_{\mathcal{F}} \backslash (\Gamma_{\mathcal{F}}^{\text{split}} \sqcup \Gamma_{\mathcal{F}}^{\text{ram}})))$$

$$= \#V(\Gamma_{\mathcal{F}}^{\mathrm{split}}) \cdot d^n + N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) + \#W(\Gamma_{\mathcal{F}}^{\mathrm{rat}} \backslash (\Gamma_{\mathcal{F}}^{\mathrm{split}} \sqcup \Gamma_{\mathcal{F}}^{\mathrm{ram}}), n)$$

and

$$g(F_n) = \frac{1}{2}\left(2 + d^n\left((2g(F_0) - 2) + \#V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})\right) - N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))\right) \tag{465}$$

and, second, we proved

$$0 = \lim_{\nu\to\infty} \frac{\#W(\Gamma_{\mathcal{F}}^{\mathrm{rat}} \backslash (\Gamma_{\mathcal{F}}^{\mathrm{split}} \sqcup \Gamma_{\mathcal{F}}^{\mathrm{ram}}), \nu)}{d^\nu} = \lim_{\nu\to\infty} \frac{N(F_\nu, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))}{d^\nu} = \lim_{\nu\to\infty} \frac{N(\overline{F}_\nu, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))}{d^\nu}.$$

Thus, the errors which we want to compute are $\#W(\Gamma_{\mathcal{F}}^{\mathrm{rat}} \backslash (\Gamma_{\mathcal{F}}^{\mathrm{split}} \sqcup \Gamma_{\mathcal{F}}^{\mathrm{ram}}), n)$, $N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ and $N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}))$.

But, although it is simple to compute $\#W(\Gamma_{\mathcal{F}}^{\mathrm{rat}} \backslash (\Gamma_{\mathcal{F}}^{\mathrm{split}} \sqcup \Gamma_{\mathcal{F}}^{\mathrm{ram}}), n)$, it is basically impossible to compute $N(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))$ without more concrete information on the defining equations of the recursive tower $\mathcal{F}$: Indeed, this comes down to the question of predicting the degree of the place Q in a diamond (Q, P_1, P_2, P) of places for non-coprime $e(P_1|P)$ and $e(P_2|P)$, i.e. where killing of ramification indices appears. Here, Key Lemma 36(iii) at least supplies the degree bounds

$$\mathrm{lcm}(\deg(P_1), \deg(P_2)) \leq \deg(Q) \leq \mathrm{lcm}(\deg(P_1), \deg(P_2)) \cdot \gcd(e(P_1|P), e(P_2|P)).$$

Unfortunately, the degree of Q can attain any value inside of this range depending on the concrete defining equations of the involved function fields. Consequently, we cannot hope to find a general approach to compute $N(F_n)$.

Nevertheless, as we already mentioned in the last paragraph, the methods of this chapter can also be used to compute the upper bound $N[\overline{F}_n, \overline{k} \cdot \Gamma_{\overline{\mathcal{F}}}^{\mathrm{rat}}]$ for $N(F_n)$ where $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ denotes the geometric tower of $\mathcal{F}$.

Genus Formulas. On the other hand, this last problem disappears if we switch to the geometric tower $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu$, i.e. if we want to compute the error

$$N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})) = \sum_{\mathcal{P} \in W(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}, n)} N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P})) \tag{466}$$

of our estimate for $g(F_n)$ in (465). Here, Corollary 51 provides an approach to compute

$$N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P})) = \tilde{N}(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P})) = \frac{\prod_{i=1}^n \tilde{e}(P_{i-1,i}|P_{i-1,i-1})}{\tilde{e}(Q|P_{0,0})} = \frac{\prod_{i=1}^n e(P_{i-1,i}|P_{i-1,i-1})}{e(Q|P_{0,0})}$$

for all $\mathcal{P} = [P_{i,j}]_{j-i\leq 1} \in W(\Gamma_{\overline{\mathcal{F}}}, n)$ and all $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P}))$. Hence, we obtain the number of places $N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P}))$ which lie over the path $\mathcal{P}$ from iteratively applying Abhyankar's Lemma to the extensions in the pyramid $\mathrm{Pyr}(Q)$ of Q.

However, although iteratively applying Abhyankar's Lemma is effective for computing $e(Q|Q \cap F_0) = e(Q|P_{0,0})$ for any explicitly given path $\mathcal{P} = [P_{i,j}]_{j-i\leq 1}$, we still have to face two non-trivial challenges (TwoCh): First, we basically need to compute $N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P}))$ for all $\mathcal{P} \in W(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}, n)$ at the same time (e.g. via a suitable parametrization of the paths $\mathcal{P}$) and, second, we need to find a useful presentation of the sum of these values $N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P}))$ in (466).

In this chapter, we will overcome these two challenges for tame recursive towers $\mathcal{F}$ with finite power separating ramification subgraphs $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ (see Definition 229(iii) and Definition 239). More concretely, we will develop an approach for computing genus formulas for these tame recursive towers $\mathcal{F} = (F_\nu)_\nu$. In Corollary 246, which is the main result of this chapter, we will provide some finite subset $\Lambda \subset \overline{\mathbb{Q}}$ and some polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ for all $\lambda \in \Lambda$ such that

$$g(F_n) = \frac{1}{2}\left(2 + d^n\left(g(F_0) - 2 + \#V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})\right) - \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n\right) \tag{467}$$

holds for all $n \geq c(\mathcal{F})$ with some lower bound $c(\mathcal{F}) \in \mathbb{N}_0$.

All the results of this chapter will be applicable to all tame recursive towers in the literature known to the author (e.g. [MW05], [Sti08], [BR20]). Thus, in the end, we will have an effective approach (A4) to compute genus formulas and even a first naive implementation in Subsection 8.3.2 which automatizes this approach and also works on all the tame recursive towers in the literature known to the author.

Note that, for some tame recursive towers in the literature (e.g. [MW05, p. 212, f_{11}]) the results are only applicable to some level l truncation of the tower with $l \leq 2$.

Structure of this chapter. In the first Section 8.1, we will introduce separating subgraphs and elaborate on how we can overcome our two challenges in (TwoCh) for separating subgraphs.

The second Section 8.2 will form the core of this chapter. Here we will first demonstrate how the insights of the first Section 8.1 can be effectively applied to compute genus formulas for some example. Second, we will introduce the notion of e-power subgraphs Γ (see Definition 239). Third, we will prove the main result of this chapter, namely Corollary 246: For all tame recursive towers $\mathcal{F}$ satisfying that every weakly connected component Γ_i of the finite ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is a separating power subgraph, Corollary 246 will provide the explicit finite sets Λ, upper bounds for the degrees of the polynomials $f_\lambda(n)$ and the lower bound $c(\mathcal{F})$ in the genus formula in (467). Finally, we will also develop our final approach (A4) on how to effectively apply Corollary 246 to compute genus formulas.

In the third Section 8.3, we will present a first naive implementation of the approach (A4) to compute genus formulas automatically (see Subsection 8.3.2). This implementation will work on all tame recursive towers in the literature known to the author. Correspondingly, at the end in Examples 250, we will also list genus formulas for some representative tame recursive towers from the literature.

8.1 An Approach to Compute Genus Formulas

Purpose of this section. In this section, we will introduce separating subgraphs and elaborate on how we can overcome our two challenges in (TwoCh) for separating subgraphs.

Structure of this section. In the first Subsection 8.1.1, for all primes q, we will first define q-separating subgraphs Γ in Definition 229 and prove some characterizations of q-separating subgraphs in Lemma 231. Here, the characterization in Lemma 231(iv) will carry the key feature of q-separating subgraphs, namely that, for all $Q \in \mathbb{P}_{F_n}[\Gamma]$, at least one of the numbers $v_q(e(Q|Q \cap F_0))$ and $v_q(e(Q|Q \cap \sigma^n(F_0)))$ vanishes.

In the second Subsection 8.1.2, we will then define the map N' in Definition 233. Then this map N' together with the weight function $w'_{\mathbf{1},\mathbf{P}}$ on Γ in Definition 162 and the above key feature of q-separating subgraphs will supply Key Lemma III of this chapter, which is Key Lemma 234. Finally, this Key Lemma 234 will enable us to overcome our two challenges in (TwoCh).

8.1.1 Separating Subgraphs

Purpose of this subsection. In this subsection, we will introduce separating subgraphs and characterize them in Lemma 231. Here, the characterization in Lemma 231(iv) will be the key feature of separating subgraphs for overcoming our two challenges in (TwoCh) for tame recursive towers with finite separating ramification subgraphs.

Motivating the definition of separating subgraphs. For the definition of so called q-separating subgraphs, we first consider the ramification subgraphs $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ which are depicted in the figures B.28, B.2, B.4, B.19, B.16. There we observe for all these examples that the ramified edges of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ distribute in the following way: The ramifiaction subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ consists of two disjoint subgraphs Γ_1 and Γ_2 which are only connected via edges from Γ_1 to Γ_2 such that all the initial vertices of the edges Q which are ramified in $F_1/\sigma(F_0)$ are contained in Γ_1 and all the terminal vertices of the edges Q which are ramified in F_1/F_0 are contained in Γ_2. Since the connecting edges from Γ_1 and Γ_2 are also unramified in at least one of the extensions F_1/F_0 and $F_1/\sigma(F_0)$, we notice that these ramification subgraphs satisfy the property that there is no path in $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ which starts at an edge Q_1 being ramified in F_1/F_0 and ends at an edge Q_2 being ramified in $F_1/\sigma(F_0)$.

q-separating subgraphs. If we now specialize this last property from above for the q-powers in the ramification indices, we already arrive at the definition of q-separating subgraphs in Definition 229. In the following, we will shortly say that an extension Q/P of places in an extension F/E of function fields is q-ramified if $v_q(e(Q|P)) > 0$. Otherwise, we will say that Q/P is q-unramified.

Definition 229. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Finally, let $q \in \mathbb{P}$.*

*(i) We call a pair (Q_1, Q_2) of edges in $\Gamma_{\mathcal{F}}$ **q-critical** if $v_q(e(Q_1|Q_1 \cap F_0)) \geq 1$ and $v_q(e(Q_2|Q_2 \cap \sigma(F_0))) \geq 1$.*

*(ii) We call Γ **q-separating** if there is no path from Q_1 to Q_2 for any q-critical pair (Q_1, Q_2) of edges in Γ.*

*(iii) We call Γ **$(q_1, \ldots, q_r)$-separating (resp. separating)** if Γ is p-separating for all $p \in \{q_1, \ldots, q_r\} \subset \mathbb{P}$ (resp. $p \in \mathbb{P}$).*

Distribution of ramification indices in paths of separating subgraphs. The following Lemma is an immediate consequence of the definition of q-separating subgraphs and basically states that the sequence of edges $(P_{0,1}, \ldots, P_{n-1,n})$ of any path $\mathcal{P}$ in a q-separating subgraph Γ can be split into a first part $(P_{0,1}, \ldots, P_{r-1,r})$ of edges which are q-unramified in F_1/F_0 and a second part $(P_{r-1,r}, \ldots, P_{n-1,n})$ of edges which are q-unramified in $F_1/\sigma(F_0)$.

Lemma 230. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let $q \in \mathbb{P}$ and let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Finally, let $Q \in \mathbb{P}_{F_n}[\Gamma]$ for some $n \in \mathbb{N}$ and let $(P_{i,j})_{j-i\leq 1} := \mathrm{Path}(Q) \in W(\mathcal{F}, 0, n)$.*

If Γ is q-separating, then there is some $r \in \{1, \ldots, n\}$ such that $v_q(\tilde{e}(P_{i-1,i}|P_{i-1,i-1})) = 0$ for all $i = 1, \ldots, r$ and $v_q(\tilde{e}(P_{i-1,i}|P_{i,i})) = 0$ for all $i = r, \ldots, n$.

Proof. We show this by contraposition: Suppose that there is no such index $r \in \{1, \ldots, n\}$. First, by this assertion, we conclude that there are some indices i_1 and i_2 in $\{1, \ldots, n\}$ with $i_1 < i_2$ such that

$$q \text{ divides } \tilde{e}(P_{i_1-1,i_1}|P_{i_1-1,i_1-1}) \text{ and } \tilde{e}(P_{i_2-1,i_2}|P_{i_2,i_2}). \tag{468}$$

Second, for all $\varepsilon = 0, 1$ and all $i = 1, \ldots, n$, we compute

$$\begin{aligned}\tilde{e}(P_{i-1,i}|P_{i-1+\varepsilon,i-1+\varepsilon}) &= \tilde{e}(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1+\varepsilon,i-1+\varepsilon})) \\ &= \tilde{e}(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i} \cap F_{i-1+\varepsilon,i-1+\varepsilon})) \\ &= \tilde{e}(\sigma^{-(i-1)}(P_{i-1,i})|\sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma^{\varepsilon}(F_0)) \end{aligned} \tag{469}$$

where the equalities hold by the following reasonings: The first equality holds by the invariance of the action of σ on the Abhyankar ramification indices in Lemma 46. The second equality holds because the definition of $(P_{i,j})_{j-i\leq 1} = \mathrm{Path}(Q)$ in Definition 17(i) and the inclusion $F_{i-1+\varepsilon,i-1+\varepsilon} \subseteq F_{i-1,i}$ in Lemma 10(i) imply the equalities

$$P_{i-1+\varepsilon,i-1+\varepsilon} = Q \cap F_{i-1+\varepsilon,i-1+\varepsilon} = (Q \cap F_{i-1,i}) \cap F_{i-1+\varepsilon,i-1+\varepsilon} = P_{i-1,i} \cap F_{i-1+\varepsilon,i-1+\varepsilon}.$$

The last equality holds because the fact that σ is a bijection implies the equality

$$\sigma^{-(i-1)}(P_{i-1,i} \cap F_{i-1+\varepsilon,i-1+\varepsilon}) = \sigma^{-(i-1)}(P_{i-1,i}) \cap \sigma^{-(i-1)}(F_{i-1+\varepsilon,i-1+\varepsilon})$$

and because Lemma 10(ii) and Lemma 10(i) imply the equalities

$$\sigma^{-(i-1)}(F_{i-1+\varepsilon,i-1+\varepsilon}) = \sigma^{\varepsilon}(F_{0,0}) = \sigma^{\varepsilon}(F_0).$$

Third, we notice

$$[\sigma^{-i}(P_{i,j})]_{j-i\leq 1} = \sigma_{\Gamma_{\mathcal{F}}}^{-1}(\mathrm{Path}(Q)) \in W(\Gamma, n) \tag{470}$$

where the equality holds by the definition of $\sigma_{\Gamma_{\mathcal{F}}}$ in Definition/Lemma 76 and by the choice of $(P_{i,j})_{j-i\leq 1} = \mathrm{Path}(Q)$ in the assumptions and the containment-statement holds because of the definition of $Q \in \mathbb{P}_{F_n}[\Gamma]$ in Definition 85.

Finally, from (468), (469) and (470) and from the definition of q-critical pairs of edges in Definition 229(i), we derive that

$$(Q_1, Q_2) := \left(\sigma^{-(i_1-1)}(P_{i_1-1,i_1}), \sigma^{-(i_2-1)}(P_{i_2-1,i_2})\right) \text{ is a } q\text{-critical pair of edges in } \Gamma_{\mathcal{F}}.$$

But (470) even supplies that $[\sigma^{-(i_1-1+i)}(P_{i_1-1+i,i_1-1+j})]_{j-i\leq 1} \in W(\Gamma, i_2 - i_1 + 1)$ is a path in Γ from Q_1 to Q_2. Hence, by the definition of q-separating subgraphs in Definition 229(ii), we deduce the desired statement, namely that Γ is not q-separating. □

Characterizations of q-separating subgraphs. In the following Lemma 231, we will prove three useful characterizations of q-separating subgraphs. Here the characterization in (iii) captures our motivating observation for the definition of q-separating subgraphs Γ from the beginning of this Section 8.1.1, namely that Γ consists of suitable disjoint subgraphs Γ_1 and Γ_2 with connecting edges only going from Γ_1 to Γ_2.

The characterization in (iv) carries the key feature of q-separating subgraphs for overcoming our two challenges in (TwoCh) for tame recursive towers, namely that any place $Q \in \mathbb{P}_{F_n}[\Gamma]$ is q-unramified in at least one of the extensions F_n/F_0 or $F_n/\sigma^n(F_0)$.

Lemma 231. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Finally, let $q \in \mathbb{P}$. Then the following statements are equivalent:*

- *(i) Γ is q-separating.*
- *(ii) For all $n \in \mathbb{N}$ and all paths $[P_{i,j}]_{j-i\leq 1} \in W(\Gamma, n)$, there is some index r such that $v_q(e(P_{i-1,i}|P_{i-1,i-1})) = 0$ for all $i = 1, \ldots, r$ and $v_q(e(P_{i-1,i}|\sigma(P_{i,i}))) = 0$ for all $i = r, \ldots, n$.*

(iii) Then Γ consists of disjoint subgraphs Γ_1 and Γ_2 which are possibly connected via some edges from Γ_1 to Γ_2 and moreover satisfy the following properties:

All these connecting edges Q satisfy $v_q(e(Q|Q \cap \sigma^i(F_0))) = 0$ for some $i = 0, 1$, the initial vertex of any edge Q with $v_q(e(Q|Q \cap \sigma(F_0))) \geq 1$ is contained in Γ_1 and the terminal vertex of any edge Q with $v_q(e(Q|Q \cap F_0)) \geq 1$ is contained in Γ_2.

(iv) For all $n \in \mathbb{N}$ and all $Q \in \mathbb{P}_{F_n}[\Gamma]$, we have $v_q(\tilde{e}(Q|Q \cap F_0)) = 0$ or $v_q(\tilde{e}(Q|Q \cap \sigma^n(F_0))) = 0$.

Proof. For the implication from (i) to (ii): Suppose that Γ is q-separating and let $n \in \mathbb{N}$ and $\mathcal{P} := [P_{i,j}]_{j-i \leq 1} \in W(\Gamma, n)$. Then Lemma 17(i) provides some $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}))$. By the definitions of $\sigma_{\Gamma_\mathcal{F}}$ in Definition/Lemma 76 and of Path in Definition/Lemma 17(i), this yields $(\sigma^i(P_{i,j}))_{j-i \leq 1} = \sigma_{\Gamma_\mathcal{F}}(\mathcal{P}) = \mathrm{Path}(Q)$.

Consequently, by Lemma 230, we deduce that there is some $r \in \{1, \ldots, n\}$ such that $v_q(\tilde{e}(\sigma^{i-1}(P_{i-1,i})|\sigma^{i-1}(P_{i-1,i-1}))) = 0$ for all $i = 1, \ldots, r$ and $v_q(\tilde{e}(\sigma^{i-1}(P_{i-1,i})|\sigma^i(P_{i,i}))) = 0$ for all $i = r, \ldots, n$.

But, because we have $e = \tilde{e}$ on the extensions in paths in $\mathcal{F}$ by Definition 41(i) and because of the invariance of ramification indices under the action of isomorphisms in (11), the desired statement in (ii) follows.

For the implication from (ii) to (iii): Suppose that, for all paths $[P_{i,j}]_{j-i \leq 1} \in W(\Gamma, n)$ with $n \in \mathbb{N}$, there is some index r such that $v_q(e(P_{i-1,i}|P_{i-1,i-1})) = 0$ for all $i = 1, \ldots, r$ and $v_q(e(P_{i-1,i}|\sigma(P_{i,i}))) = 0$ for all $i = r, \ldots, n$.

Define Γ_2' as the union of all strongly connected components of Γ which contain the terminal vertices of edges Q with $v_q(e(Q|Q \cap F_0)) \geq 1$. Moreover, choose Γ_2 as the smallest forward complete subgraph of Γ which contains Γ_2' and let Γ_1 be the subgraph of Γ with vertex set $V(\Gamma_1) := V(\Gamma) \backslash V(\Gamma_2)$ and edge set $E(\Gamma_1) := \coprod_{P_1, P_2 \in V(\Gamma_1)} E(\Gamma, P_1, P_2)$.

Then we notice that Γ_1 and Γ_2 are indeed disjoint by the definition of Γ_1. Furthermore, since Γ_2 is forward complete, there cannot be a an edge from Γ_1 and Γ_2.

Next, since any connecting edge Q from Γ_1 and Γ_2 is especially a path of length one from $Q \cap F_0$ to $\sigma^{-1}(Q) \cap F_0$ by the definition of tower graphs in Definition 74, the assertion from the beginning and the invariance of ramification indices under the action of isomorphisms in (11) imply the equalities

$$0 = v_q(e(\sigma^{-i}(Q)|\sigma^i(\sigma^{-i}(Q) \cap F_0))) = v_q(e(Q|Q \cap \sigma^i(F_0)))$$

for some $i = 0, 1$.

Finally, let Q be an edge in Γ with initial vertex P in Γ_2. By the definition of Γ_2 there is a path from some vertex P_1 to P which is the terminal vertex of some edge Q_1 with $v_q(e(Q_1|Q_1 \cap \sigma(F_0))) \geq 1$. But the assertion from the beginning then implies the estimate $v_q(e(Q|Q \cap F_0)) = 0$. Hence, we concluded that the initial vertex P in Γ of any edge Q in Γ with $v_q(e(Q|Q \cap F_0)) \geq 1$ must be contained in Γ_1. All together, we established (iii).

For the implication from (iii) to (i): Suppose that Γ is of the form in (iii) and let (Q_1, Q_2) be a q-critical pair of edges in Γ.

Because of the definition of q-critical pairs in Definition 229(i) and because of the properties of Γ_1 and Γ_2, the terminal vertex of Q_1 must be contained in Γ_2 and the initial vertex of Q_2 in Γ_1.

Since the connecting edges Q from Γ_1 to Γ_2 satisfy $v_q(e(Q|Q \cap \sigma^i(F_0))) = 0$ for some $i = 0, 1$, we also obtain that the edges Q_1 and Q_2 cannot be the same connecting edge Q. Finally, since there are no connecting edges from Γ_2 to Γ_1, we conclude the desired statement in (i), namely that Γ is q-separating by 229(ii).

For the implication from (i) to (iv): Suppose that Γ is q-separating, let $(F_{i,j})_{i,j} := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$, let $n \in \mathbb{N}$, let $Q \in \mathbb{P}_{F_n}[\Gamma]$, and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ by the pyramid of Q.

Then we have the equality $\mathrm{Path}(Q) = (P_{i,j})_{j-i\leq 1}$ by Lemma 17(ii). Consequently, Lemma 230 supplies some $r \in \{1, \dots, n\}$ such that

$$v_q(\tilde{e}(P_{i-1,i}|P_{i-1,i-1})) = 0 = v_q(\tilde{e}(P_{j-1,i}|P_{j,j})) \tag{471}$$

for all $i = 1, \dots, r$ and all $j = r, \dots, n$.

Next, we obtain the estimates and equalities

$$0 \leq v_q(\tilde{e}(P_{0,r}|P_{0,0})) = \sum_{i=1}^{r} v_q(\tilde{e}(P_{0,i}|P_{0,i-1})) \leq \sum_{i=1}^{r} v_q(\tilde{e}(P_{i-1,i}|P_{i-1,i-1})) = 0 \tag{472}$$

where the estimates and equalities hold by the following reasonings: The first estimate holds since the Abhyankar ramification indices are positive natural numbers by Definition 41. The first equality holds because of the multiplicative transitivity rule for the Abhyankar ramification indices in Lemma 44(ii) and because $v_q : \mathbb{N} \to \mathbb{N}_0$ is a morphism of monoids. The second estimate holds because applying the $\tilde{e}$-version of Abhyankar's Lemma 44(i) to the diamond $(P_{0,i}, P_{0,i-1}, P_{i-1,i}, P_{i-1,i-1})$ yields the equality

$$\tilde{e}(P_{0,i}|P_{0,i-1}) = \frac{\tilde{e}(P_{i-1,i}|P_{i-1,i-1})}{\gcd(\tilde{e}(P_{0,i-1}|P_{i-1,i-1}), \tilde{e}(P_{i-1,i}|P_{i-1,i-1}))}.$$

The second equality hold by the first equality in (471).

Analogously, we also obtain the estimates and equalities

$$0 \leq v_q(\tilde{e}(P_{r,n}|P_{n,n})) = \sum_{i=r+1}^{n} v_q(\tilde{e}(P_{i-1,n}|P_{i,n})) \leq \sum_{i=r+1}^{n} v_q(\tilde{e}(P_{i-1,i}|P_{i,i})) = 0. \tag{473}$$

Second to last, let us define

$$m := \min(v_q(\tilde{e}(P_{0,r}|P_{r,r})), v_q(\tilde{e}(P_{r,n}|P_{r,r}))). \tag{474}$$

Then we get the equalities

$$\begin{aligned} v_q(\tilde{e}(Q|Q \cap F_0)) &= v_q(\tilde{e}(P_{0,n}|P_{0,0})) = v_q(\tilde{e}(P_{0,n}|P_{0,r})) + v_q(\tilde{e}(P_{0,r}|P_{0,0})) \\ &= v_q(\tilde{e}(P_{0,n}|P_{0,r})) = v_q(\tilde{e}(P_{r,n}|P_{r,r})) - m \end{aligned} \tag{475}$$

where the equalities hold by the following reasonings: The first equality holds because the assertion $Q \in \mathbb{P}_{F_n}$, the definition of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ in Definition 11 and Lemma 10(i) provide the equalities

$$Q = Q \cap F_n = Q \cap F_{0,n} = P_{0,n} \quad \text{and} \quad Q \cap F_0 = Q \cap F_{0,0} = P_{0,0}.$$

The second equality holds because of the multiplicative transitivity rule for Abhyankar ramification indices in (7) and because v_q is a morphism of monoids. The third equality holds since the estimates in (472) must even be equalities. The last equality holds because applying the $\tilde{e}$-version of Abhyankar's Lemma 44(i) to the diamond $(P_{0,n}, P_{0,r}, P_{r,n}, P_{r,r})$ yields the equality

$$\tilde{e}(P_{0,n}|P_{0,r}) = \frac{\tilde{e}(P_{r,n}|P_{r,r})}{\gcd(\tilde{e}(P_{0,r}|P_{r,r}), \tilde{e}(P_{r,n}|P_{r,r}))},$$

because of the well known equality $v_q(\gcd(a, b)) = \min(v_q(a), v_q(b))$ for all $a, b \in \mathbb{N}$ and because of the definition of m in (474).

Analogously to the equalities in (475), we get the equalities

$$\begin{aligned} v_q(\tilde{e}(Q|Q\cap\sigma^n(F_0))) &= v_q(\tilde{e}(P_{0,n}|P_{n,n})) = v_q(\tilde{e}(P_{0,n}|P_{r,n})) + v_q(\tilde{e}(P_{r,n}|P_{n,n})) \\ &= v_q(\tilde{e}(P_{0,n}|P_{r,n})) = v_q(\tilde{e}(P_{0,r}|P_{r,r})) - m \end{aligned} \tag{476}$$

Finally, because of the definition of m in (474) and because of the equalities in (475) and (476), we conclude the desired statement, namely that at least one of the values $v_q(\tilde{e}(Q|Q\cap F_0))$ and $v_q(\tilde{e}(Q|Q\cap\sigma^n(F_0)))$ vanishes.

For the implication from (iv) to (i): We will show this implication by contraposition: Suppose that Γ is not q-separating. By the definition of q-separating subgraphs in Definition 229(ii), there is a q-critical pair (Q_1, Q_2) of edges in Γ and a path $\mathcal{P}$ in Γ, say of length $n \in \mathbb{N}$, from Q_1 to Q_2.

Now, we can choose some $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P}))$ by Definition/Lemma 17(i). Let $(P_{i,j}) := \mathrm{Path}(Q)$ be the path of Q. By the definition of $\sigma_{\Gamma_\mathcal{F}}$ in Definition/Lemma 76, we then have the equality $\mathcal{P} = [\sigma^{-i}(P_{i,j})]_{j-i\leq 1}$ and, thus,

$$Q_1 = P_{0,1} \text{ and } Q_2 = \sigma^{-(n-1)}(P_{n-1,n}). \tag{477}$$

Moreover, we also compute

$$\begin{aligned} Q_1 \cap F_0 = P_{0,1}\cap F_0 &= P_{0,1} \cap F_{0,0} = (Q\cap F_{0,1})\cap F_{0,0} \\ &= Q \cap F_{0,0} = P_{0,0} \end{aligned} \tag{478}$$

and

$$\begin{aligned} Q_2 \cap \sigma(F_0) &= \sigma^{-(n-1)}(P_{n-1,n}) \cap \sigma(F_0) = \sigma^{-(n-1)}(P_{n-1,n}\cap F_{n,n}) \\ &= \sigma^{-(n-1)}((Q\cap F_{n-1,n})\cap F_{n,n}) = \sigma^{-(n-1)}((Q\cap F_{n-1,n})\cap F_{n,n}) \\ &= \sigma^{-(n-1)}(Q\cap F_{n,n}) = \sigma^{-(n-1)}(P_{n,n}) \end{aligned} \tag{479}$$

where the equalities hold by the following reasonings: The first equalities hold by the equalities in (477). The second equalities hold because σ is bijective and because Lemma 10(i) and Lemma 10(ii) imply the equalities $F_0 = F_{0,0}$ and $\sigma^{-n}(F_{0,0}) = F_{n,n}$. The third and last equalities hold by the definition of $\mathrm{Path}(Q) = (P_{i,j})_{j-i\leq 1}$ in Definition/Lemma 17(i). The fourth equalities hold because Lemma 10(i) implies the equalities $F_{0,1} \supseteq F_{0,0}$ and $F_{n-1,n} \supseteq F_{n,n}$.

Consequently, we obtain the equalities and estimates

$$\begin{aligned} v_q(\tilde{e}(Q|Q\cap F_0)) &= v_q(\tilde{e}(Q|P_{0,1})) + v_q(\tilde{e}(P_{0,1}|P_{0,0})) \geq v_q(\tilde{e}(P_{0,1}|P_{0,0})) \\ &= v_q(\tilde{e}(Q_1|Q_1\cap F_0)) \geq 1 \end{aligned} \tag{480}$$

where the equalities and estimates hold by the following reasonings: The first equality holds because the choice of $\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$ supplies the equalities $Q\cap F_{0,\varepsilon} = P_{0,\varepsilon}$ for all $\varepsilon = 0, 1$, because of the multiplicative transitivity rule for Abhyankar ramification indices in Lemma 44(ii) and because v_q is a morphism of monoids. The first estimate holds because Abhyankar ramification indices are positive natural numbers by Definition 41. The second equality hold by the equalities in (477) and (478). The second estimate holds by the definition that (Q_1, Q_2) is a q-critical pair of edges in Definition 229(i).

Analogously, we also obtain the equalities and estimates

$$\begin{aligned} v_q(\tilde{e}(Q|Q\cap\sigma^n(F_0))) &= v_q(\tilde{e}(Q|P_{n-1,n})) + v_q(\tilde{e}(P_{n-1,n}|P_{n,n})) \geq v_q(\tilde{e}(P_{n-1,n}|P_{n,n})) \\ &= v_q(\tilde{e}(Q_2|Q_2\cap\sigma(F_0))) \geq 1 \end{aligned} \tag{481}$$

where we also need the invariance of the Abhyankar ramification indices under the action of σ for the second equality.

Finally, the estimates in (480) and (481) provide that $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_\mathcal{F}}(\mathcal{P})) \subseteq \mathbb{P}_{F_n}[\Gamma]$ is the desired place in the contraposition. $\square$

8.1.2 Key Lemma III

Summary of the results of this subsection. In this subsection, we will overcome our two challenges in (TwoCh) for computing genus formulas for tame recursive towers having finite separating ramification subgraphs.

For that and via Lemma 232, we will first connect the map $\tilde{N}$ in Definition 50 and the weight function $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162. Then combining this connection and the key feature of separating subgraphs in Lemma 231(iv) will yield the Key Lemma III of this chapter, which is Key Lemma 234. More concretely, this Key Lemma 234 will enable us to reduce the computation of the desired value

$$N(\overline{F}_n, V(\Gamma)) = \tilde{N}(\overline{F}_n, V(\Gamma))$$

for finite separating weakly connected components of $\Gamma_{\mathcal{F}}$ to the following to steps: First compute the n-th power A^n of any $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix $A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m\times m}$ of the separating finite subgraph Γ. Then apply the map N' to the sum of the entries in A^n. More concretely, the map N' will evaluate the y_q-principal part of any Laurent polynomial $h \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ at q and its y_q-non-principal part at 1 (see Definition 233).

The connection of the map $\tilde{N}$ and the weight function $w'_{\mathbf{1},\mathbf{P}}$. In the following Lemma 232, we will extend Lemma 159 and obtain a connection of the map $\tilde{N}$ in Definition 50 and the weight function $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162.

More suggestively, we should think of the identity in Lemma 232 as follows: Computing the desired value $N(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P})) = \tilde{N}(\overline{F}_n, \sigma_{\Gamma_{\overline{\mathcal{F}}}}(\mathcal{P}))$ by iteratively applying Abhyankar's Lemma to the extensions in $\mathrm{Pyr}(Q)$ translates into computing the product $w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}) = \prod_{i=1}^{n} w'_{\mathbf{1},\mathbf{P}}(P_{i-1,i})$, but with a small error. This small error is then captured via the additional factor $y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}$.

Lemma 232. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let $\Gamma_{\mathcal{F}}$ be the tower graph of $\mathcal{F}$. Finally, let $n \in \mathbb{N}_0$, let $\mathcal{P} \in W(\Gamma_{\mathcal{F}}, n)$, let $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$. Then we have the identity*

$$w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}) = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cdot y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}.$$

Proof. For all $x_{\mathbf{P}}$ and $\hat{x}_{\mathbf{P}}$ in $\mathbb{C}^{\mathbb{P}}$ with $\hat{x}_p x_p = p$, we compute

$$\begin{aligned}\mathrm{Eval}_{x_{\mathbf{P}}}(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) &= w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}(\mathcal{P}) = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cdot x_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot \hat{x}_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))} \\ &= \mathrm{Eval}_{x_{\mathbf{P}}}\left(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cdot y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}\right) \quad (482)\end{aligned}$$

where the first equality holds by Lemma 164, the second equality holds by Lemma 159(i) and the last equality holds by the definition of the evaluation morphism $\mathrm{Eval}_{x_{\mathbf{P}}}$ in Definition 161(ii).

But, since $w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})$ and $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cdot y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}$ are two Laurent polynomials in $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ which have the same values at all points $x_{\mathbf{P}} \in (\mathbb{C}\backslash\{0\})^{\mathbb{P}}$ by the equalities in (482), they must already be equal. ☐

Removing the error for separating subgraphs by applying N'. Now the key feature of separating subgraphs from Lemma 231(iv) comes into play and provides that the error in Lemma 232 only consists of one of the two factors $y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))}$ and $(\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}$. Thus, we can remove this error by evaluating the y_q-principal part of $w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})$ at q and its y_q-non-principal part at 1.

In the following Definition 233, we will define the corresponding map N' on the set $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ of Laurent polynomials and then obtain the Key Lemma 234 of this chapter.

Definition 233. *Consider the $\mathbb{C}$-algebra $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ in Definition 161(i) and let $q \in \mathbb{P}$. Then we define*

$$\mathrm{N}'(y_q^{\alpha}) := \mathrm{Eval}_{\mathbf{1}}(y_q^{\alpha}) = 1, \quad \mathrm{N}'(y_q^{-\alpha}) := \mathrm{Eval}_{\mathbf{P}}(y_q^{-\alpha}) = q^{\alpha}, \quad \mathrm{N}'(y_{\mathbf{P}}^{\beta}) := \prod_{p \in \mathbb{P}} \mathrm{N}'(y_p^{\beta_p})$$

for all $\alpha \in \mathbb{N}_0$ and all $\beta = (\beta_p)_p \in (\mathbb{Z}^{\mathbb{P}})'$ and the morphism

$$\mathrm{N}' : \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \to \mathbb{C} \text{ via } \sum_{\beta} c_{\beta} \cdot y_{\mathbf{P}}^{\beta} \mapsto \sum_{\beta} c_{\beta} \cdot \mathrm{N}'(y_{\mathbf{P}}^{\beta})$$

of $\mathbb{C}$-vector spaces.

Lemma 234 (Key Lemma III)**.** *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$.*

(i) Let $q \in \mathbb{P}$ and let $\mathcal{P}$ be a path in Γ of some length $n \in \mathbb{N}_0$. If Γ is q-separating, then we have the identity

$$v_q(\mathrm{N}'(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}))) = v_q(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))).$$

In particular, if Γ is separating, then we even have the identity

$$\mathrm{N}'(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})).$$

(ii) Suppose that Γ is finite and let $A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m \times m}$ be the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ for some enumeration of the m vertices in Γ, let $v = (1 \ldots 1) \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{1 \times m}$ and let $n \in \mathbb{N}_0$. If Γ is separating, then we have the identity

$$\sum_{\mathcal{P} \in W(\Gamma, n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \mathrm{N}'(v \cdot A^n \cdot v^t).$$

Proof. For the 'main'-part in (i): Suppose that Γ is q-separating, let $(F_{i,j}) := \mathrm{Pyr}(\mathcal{F})$ be the pyramid of $\mathcal{F}$, choose $Q \in \mathbb{P}_{F_n}(\sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ by Lemma 17(i) and let $(P_{i,j})_{i,j} := \mathrm{Pyr}(Q)$ be the pyramid of Q.

First, for all $\varepsilon = 0, 1$, we have the equalities

$$Q \cap \sigma^{\varepsilon n}(F_0) = Q \cap F_{\varepsilon n, \varepsilon n} = P_{\varepsilon n, \varepsilon, n} \tag{483}$$

where the first equality holds because Lemma 10(i) provides the equality $F_0 = F_{0,0}$ and Lemma 10(ii) then provides the equality $\sigma^{\varepsilon n}(F_{0,0}) = F_{\varepsilon n, \varepsilon n}$ and the second equality holds by the definition of $(P_{i,j})_{i,j} = \mathrm{Pyr}(Q)$ in Definition 11.

Then we already obtain the desired identity in the 'main'-part in (i) by the equalities

$$\begin{aligned} v_q(\mathrm{N}'(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P}))) &= v_q\left(\mathrm{N}'\left(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \cdot y_{\mathbf{P}}^{v_{\mathbf{P}}(\tilde{e}(Q|P_{0,0}))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(\tilde{e}(Q|P_{n,n}))}\right)\right) \\ &= v_q(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))) + \sum_{p \in \mathbb{P}} v_q\left(\mathrm{N}'\left(y_p^{v_p(\tilde{e}(Q|P_{0,0}))} \cdot (p y_p^{-1})^{v_p(\tilde{e}(Q|P_{n,n}))}\right)\right) \\ &= v_q(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))) + v_q\left(\mathrm{N}'\left(y_q^{v_q(\tilde{e}(Q|P_{0,0}))} \cdot (q y_q^{-1})^{v_q(\tilde{e}(Q|P_{n,n}))}\right)\right) \\ &= v_q(\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))) \end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by Lemma 232. The second equality holds because $\tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) \in \mathbb{C}$, because of the definition of the morphism N' of $\mathbb{C}$-vector spaces in Definition 233 and because $v_q : \mathbb{Z} \to \mathbb{N}_0$ is a morphism of monoids. The third equality holds because the definition of N' implies that the summands vanish for all $p \in \mathbb{P}$ which are distinct from q. The last equality holds because combining the assertion that Γ is q-separating, the implication from (i) to (iv) in Lemma 231 and the equalities in (483) first yields that one of the factors in the product $y_q^{v_q(\tilde{e}(Q|P_{0,0}))} \cdot (qy_q^{-1})^{v_q(\tilde{e}(Q|P_{n,n}))}$ is already equal to one and because the definition of N' in Definition 233 then yields that both N'-values of these factors are equal to one.

For the 'in particular'-part in (i): Suppose that Γ is separating. By the definition of separating subgraphs in Definition 229(iii), this means that Γ is p-separating for all $p \in \mathbb{P}$. Therefore, the equality in the 'main'-part of (i) holds for all $q = p \in \mathbb{P}$. Hence, from this, the desired identity $\mathrm{N}'(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) = \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P}))$ 'in particular'-part already follows.

For (ii): Suppose that Γ is separating and let $(a_{i,j}^{(n)})_{i,j} := A^n$. Then we already obtain the desired identity in (ii) by the equalities

$$\begin{aligned}\sum_{\mathcal{P} \in W(\Gamma,n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) &= \sum_{\mathcal{P} \in W(\Gamma,n)} \mathrm{N}'(w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})) = \mathrm{N}'\left(\sum_{\mathcal{P} \in W(\Gamma,n)} w'_{\mathbf{1},\mathbf{P}}(\mathcal{P})\right) \\ &= \mathrm{N}'\left(\sum_{i,j} a_{i,j}^{(n)}\right) = \mathrm{N}'(v \cdot A^n \cdot v^t)\end{aligned}$$

where the equalities hold by the following reasonings: The first equality holds by the 'in particular'-part in Lemma 234(i). The second equality holds because N' is a morphism of vector spaces by the definition of N' in Definition 233. The third equality holds if we sum up the entries of the matrix in the identity in Lemma 59. The last equality holds because the sum $\sum_{i,j} a_{i,j}^{(n)}$ of the entries $a_{i,j}^{(n)}$ of A^n is equal to the product $v \cdot A^n \cdot v^t$. □

8.2 Computing Genus Formulas in the Tame and Separating Case

Purpose of this section. In the last Section 8.1, Key Lemma 234 enabled us to overcome our two challenges in (TwoCh) for computing the desired value of $N(\overline{F}_n, V(\Gamma))$ for finite separating subgraphs Γ. There we exchanged these two challenges for the following two fairly accessible steps (TwoSt): First compute the n-th power A^n of a $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix $A \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]^{m \times m}$ of Γ. Then apply the map N' to the sum of the entries in A^n. Consequently, in the following, we will focus on finding ways to execute these two steps effectively. This will peak in our main result of this section and chapter, which is Corollary 246.

Structure of this section. In the first Subsection 8.2.1, we will consider a motivating example and derive the main idea on how to execute the two steps in (TwoSt).

In the second Subsection 8.2.2, we will concentrate on power subgraphs and make preparations for the next Subsection 8.2.3.

In the third Subsection 8.2.3, we will finally prove the desired Corollary 246. This will involve the only elaborate proof of this chapter, which is the proof of Theorem 243.

In the fourth and last Subsection 8.2.4, we will formulate the final approach (A4) for effectively computing genus formulas via applying Corollary 246. Our objective here will be to formulate this final approach (A4) in such a way that we will be able to automatize (A4) and to provide a first naive implementation of (A4) in Subsection 8.3.2.

8.2.1 A Motivating Example

Purpose of this subsection. In this subsection, we will consider a motivating example and derive the main idea on how to execute the two steps in (TwoSt).

Computing genus formulas. For any tame recursive tower $\mathcal{F} = (F_\nu)_\nu$ of degree d over some field k with finite ramification subgraph, Theorem 194 and Corollary 51 provide the identities

$$\begin{aligned} g(F_n) &= \frac{1}{2}\left(2 + d^n\left(g(F_0) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}})\right) - N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}))\right) \\ &= \frac{1}{2}\left(2 + d^n\left(g(F_0) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}})\right) - \tilde{N}(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}))\right) \end{aligned} \tag{484}$$

for all $n \in \mathbb{N}_0$. If $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is separating, this identity in (484), Key Lemma 234(ii) and Lemma 237 can be combined to provide an approach for computing a formula for $g(F_n)$ for all large n. For instance, under the additional assumption that the ramification indices in F_1/F_0 and $F_1/\sigma(F_0)$ are powers of the same natural number, Theorem 246 will even enable us to predict the terms λ^n appearing in the final formula for $g(F_n)$ in (467).

A motivating example. In the following Example 235, in order to demonstrate the basic ideas of the approach and of the proof of Theorem 246, we will compute the genus formula for some explicitly given tame recursive tower $\mathcal{F}$. Moreover, as a sanity check, we will choose the HP-tower $\mathcal{F}_{HP,q} = \mathcal{F} = (F_\nu)_\nu$ from [HP16, p. 12, Proposition 12] (see also 8(vii)) since there is a genus formula already known for $\mathcal{F}$.

Example 235. *Let q be a power of some $p \in \mathbb{P}\backslash\{2,3\}$ and let $\mathcal{F}_{\geq 1} = (F_\nu)_{\nu\geq 1} = (\mathbb{F}_q(x_1, \dots, x_\nu))_{\nu \geq 1}$ be the recursive HP-tower over $\mathbb{F}_q$ from Example 8(vii) which is defined by the polynomial $f := f_{HP,q} = Y^2(3X-1)-(X^2+X)$. In [HP16, p. 12, Proposition 12], it is shown that the genus sequence of $\mathcal{F}$ is equal to*

$$(g(F_n))_{n\geq 1} = (2^n + 1 - (2 + n \bmod 2) \cdot 2^{\lfloor \frac{n}{2} \rfloor})_{n \geq 1} = (0, 1, 3, 9, 21, 49, 105, \dots). \tag{485}$$

Moreover, let σ be the tower map of $\mathcal{F}_{\geq 1}$ in Lemma 7, i.e. $\sigma(x_n) = x_{n+1}$ for all $n \in \mathbb{N}$.

First, we make the following simplification step which is not essential and only serves to keep this example as simple as possible: Let us add a zeroth level to $\mathcal{F}$ to simplify the ramification subgraph. For that, define $z_0 := x_1^2$ and $F_0 := \mathbb{F}_q(z_0)$. Then it follows from Reduction Lemma 30(ii) that $\mathcal{F} = (F_\nu)_\nu$ is a recursive tower of balanced degree 2 *which is defined by the tuple (σ, F_0) and satisfies* $\mathrm{Trun}_{\geq 1}(\mathcal{F}) = \mathcal{F}_{\geq 1}$.

Second, consider the ramification subgraph of $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ of $\mathcal{F}$ which, as can be seen in Figure B.26 for $q = 5$, is a disjoint union of two finite weakly connected components Γ and Γ' of $\Gamma_{\mathcal{F}}$. Moreover, we obtain the equalities

$$\begin{aligned} \tilde{N}(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}})) = \tilde{N}(F_n, V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})) &= \tilde{N}(F_n, V(\Gamma)) + \tilde{N}(F_n, V(\Gamma')) \\ &= 2 \cdot \tilde{N}(F_n, V(\Gamma)) \end{aligned} \tag{486}$$

for all $n \in \mathbb{N}_0$ where the equalities hold by the following reasonings: For the first equality, we first notice that all vertices and edges in $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ are rational and, thus, combining Lemma 124 and Lemma 122 yield that $\pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}}}$ even restricts to an isomorphism $\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}} \to \Gamma^{\mathrm{ram}}_{\mathcal{F}}$. Moreover, by Lemma 105(i), the ramification indices of the edges are stable under this isomorphism. Consequently, the first equality follows from the fact that $\tilde{N}(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}))$ (resp. $\tilde{N}(F_n, V(\Gamma^{\mathrm{ram}}_{\mathcal{F}}))$) only depends on the ramification indices in the paths of $\Gamma_{\overline{\mathcal{F}}}$ (resp. $\Gamma_{\mathcal{F}}$) which start at vertices in $V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}) \subset \mathbb{P}_{\overline{F}_0}$ (resp. $V(\Gamma^{\mathrm{ram}}_{\mathcal{F}}) \subset \mathbb{P}_{F_0}$) by Definition 50.

The second equality holds by the definition of $\tilde{N}(F_n, \cdot)$ *in Definition 50 and since* $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ *is the disjoint union of the finite weakly connected components* Γ *and* Γ' *of* $\Gamma_{\mathcal{F}}$*. For the last equality, we notice that* Γ *and* Γ' *are clearly isomorphic and that the ramification indices of the edges are stable under this isomorphism. Consequently, the last equality follows with the same reasoning as the first equality.*

Third, we notice that Γ *is q-separating (even separating) where* Γ_∞ *and* Γ_0 *in Lemma 231(iii) both exactly consist of a single vertex, say* P_∞ *and* P_0*, respectively, and a single edge and there is exactly one connecting edge from* Γ_∞ *to* Γ_0*. For* $x := y_2$ *and* $y := 2x^{-1}$*, the* $w'_{\mathbf{1},\mathbf{P}}$*-adjacency matrix of* Γ *for the enumeration* (P_∞, P_0) *is then of the form*

$$A = \begin{pmatrix} y & 1 \\ 0 & x \end{pmatrix} \in \mathbb{C}[x, x^{-1}]^{2\times 2}$$

Then A *can be diagonalized with the eigenvalues* x *and* y*. Let* $D \in \mathbb{C}(x)^{2\times 2}$ *be a corresponding diagonal matrix with some transformation matrix* $T \in \mathrm{GL}(2, \mathbb{C}(x))$*, i.e.* $A = T^{-1}DT$ *and let* $v = \begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathbb{C}^{1\times 2}$*. A quick computation yields the equalities*

$$a_n := v \cdot A^n \cdot v^t = v \cdot (T^{-1}D^nT) \cdot v^t = \frac{(x^2 + x - 2)x^n}{x^2 - 2} + \frac{(x^2 - x - 2)y^n}{x^2 - 2} \tag{487}$$

$$= a_{n,0} + a_{n,\infty} \in \mathbb{C}[x, x^{-1}] \tag{488}$$

where $a_{n,0}$ *denotes the first and* $a_{n,\infty}$ *denotes the second summand in the sum in (487). Thus, the definition of* $\tilde{N}(F_n, V(\Gamma))$ *in Definition 50, Lemma 234(ii) and the 'in particular'-part in Lemma 237 provide the equalities*

$$\tilde{N}(F_n, V(\Gamma)) = \sum_{\mathcal{P} \in W(\Gamma, n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \mathrm{N}'(v \cdot A^n \cdot v^t)$$

$$= \mathrm{Eval}_2(\mathrm{Princ}(a_n)) + \mathrm{Eval}_1(\mathrm{NonPrinc}(a_n)) \tag{489}$$

for all $n \in \mathbb{N}_0$*.*

Fourth, let $S_0 := \mathbb{C}[x]$ *and* $S_\infty := \mathbb{C}[x^{-1}]_{(x^{-1})} \cap \mathbb{C}[x^{-1}]_{(x^{-1}-e^{-1})}$ *where* $\mathbb{C}[x^{-1}]_{(x^{-1}-a)}$ *denotes the localization of* $\mathbb{C}[x^{-1}]$ *at the maximal ideal* $(x^{-1} - a)\mathbb{C}[x^{-1}]$ *for all* $a \in \mathbb{C}$*. Then we can extend the morphism* $\mathrm{Eval}_2 : \mathbb{C}[x^{-1}] \to \mathbb{C}$ *to the morphism*

$$\mathrm{Eval}_2 : S_\infty \to \mathbb{C} \text{ via } h \mapsto h(2).$$

of $\mathbb{C}$*-algebras. Moreover, since* $S_0 \cap S_\infty = \mathbb{C}$*, we can even extend any* $\mathbb{C}$*-basis of* $\mathbb{C}$ *to a* $\mathbb{C}$*-basis of* S_0 *and of* S_∞*. The union of these bases is a basis of* $S_0 + S_\infty$ *which can again be extended to a* $\mathbb{C}$*-basis of* $\mathbb{C}(x)$*. Let* U *be the* $\mathbb{C}$*-span of the basis vectors which where added in the last step. By that, any element* $z \in \mathbb{C}(x)$ *has a presentation of the form* $z = z_0 + z_\infty + u$ *with* $z_0 \in S_0$*,* $z_\infty \in S_\infty$*,* $u \in U$ *and, for any other presentation* $z = z'_0 + z'_\infty + u'$*, we have* $u' = u$*,* $z'_0 = z_0 + a$*,* $z'_\infty = z_\infty - a$ *for some* $a \in \mathbb{C}$*. Hence, we can also extend the restriction* $\mathrm{N}'_{|\mathbb{C}[x,x^{-1}]}$ *of the morphism* N' *in Definition 233 to the morphism*

$$\mathrm{N}' : \mathbb{C}(x) \to \mathbb{C} \text{ via } z = z_0 + z_\infty + u \mapsto \mathrm{Eval}_2(z_0) + \mathrm{Eval}_1(z_\infty) \tag{490}$$

of $\mathbb{C}$*-vector spaces.*

Fifth, on the one hand, since $a_{n,\infty} \in S_\infty$ *in (488), we obtain the equality*

$$\mathrm{N}'(a_{n,\infty}) = \mathrm{Eval}_2\left(\frac{(x^2 - x - 2)y^n}{x^2 - 2}\right) = 0. \tag{491}$$

On the other hand, for the first summand $a_{n,0}$ in (488), let $\gamma := x^2+x-2$ be the first factor in the numerator of $a_{n,0}$, let $\varepsilon_{\sqrt{2}} := (2\sqrt{2})^{-1} \cdot (x+\sqrt{2}), \varepsilon_{-\sqrt{2}} := -(2\sqrt{2})^{-1} \cdot (x-\sqrt{2}) \in S_0$. Then we have $\varepsilon_{\pm\sqrt{2}}(\pm\sqrt{2}) = 1$ and $\varepsilon_{\pm\sqrt{2}}(\mp\sqrt{2}) = 0$ and compute

$$\begin{aligned} a_{n,0} = \frac{(x^2+x-2)x^n}{x^2-2} &= \frac{(x^2+x-2)x^n - \left(\varepsilon_{\sqrt{2}} \cdot \mathrm{Eval}_{\sqrt{2}}(\gamma \cdot x^n) + \varepsilon_{-\sqrt{2}} \cdot \mathrm{Eval}_{-\sqrt{2}}(\gamma \cdot x^n)\right)}{x^2-2} \\ &\qquad + \frac{\varepsilon_{\sqrt{2}} \cdot \mathrm{Eval}_{\sqrt{2}}(\gamma \cdot x^n) + \varepsilon_{-\sqrt{2}} \cdot \mathrm{Eval}_{-\sqrt{2}}(\gamma \cdot x^n)}{x^2-2} \qquad (492) \\ &= a_{n,0,0} + a_{n,0,1} \qquad (493) \end{aligned}$$

where $a_{n,0,0}$ denotes the first and $a_{n,0,1}$ denotes the second summand in the sum in (492).

Sixth, since the numerator of $a_{n,0,0}$ has zeroes at $\sqrt{2}$ and $-\sqrt{2}$ by its construction, it is divisible by $(x-\sqrt{2})(x+\sqrt{2}) = x^2-2$ and, hence, $a_{n,0,0}$ is an element in S_0. Consequently, we obtain

$$\mathrm{N}'(a_{n,0,0}) = \mathrm{Eval}_1(a_{n,0,0}) = \frac{1+\sqrt{2}}{2} \cdot \sqrt{2}^n - \frac{1-\sqrt{2}}{2} \cdot (-\sqrt{2})^n. \qquad (494)$$

Moreover, $a_{n,0,1}$ is clearly an element in S_∞ and, thus,

$$\mathrm{N}'(a_{n,0,1}) = \mathrm{Eval}_2(a_{n,0,1}) = \frac{2+\sqrt{2}}{4} \cdot \sqrt{2}^n - \frac{2-\sqrt{2}}{4} \cdot (-\sqrt{2})^n. \qquad (495)$$

Second to last, combining (489), (488), (492) and (495) yields

$$\begin{aligned} \tilde{N}(\overline{F}_n, V(\Gamma)) &= \mathrm{N}'(a_n) = \mathrm{N}'(a_{n,0,0} + a_{n,0,1} + a_{n,\infty}) \\ &= \mathrm{N}'(a_{n,0,0}) + \mathrm{N}'(a_{n,0,1}) + \mathrm{N}'(a_{n,\infty}) \\ &= \frac{(4+3\sqrt{2})}{4} \cdot \sqrt{2}^n + \frac{(4-3\sqrt{2})}{4} \cdot (-\sqrt{2})^n \\ &= (2 + n \bmod 2) \cdot 2^{\lfloor \frac{n}{2} \rfloor} \qquad (496) \end{aligned}$$

Finally, combining (484), (486) and (496) yields the same formula of $g(F_n)$ as in (485) but with another expression for $(2 + n \bmod 2) \cdot 2^{\lfloor \frac{n}{2} \rfloor}$:

$$\begin{aligned} g(F_n) &= \frac{1}{2}\left(2 + 2^n\left(-2 + \#V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})\right) - \tilde{N}(\overline{F}_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}}))\right) \\ &= 2^n + 1 - \tilde{N}(\overline{F}_n, V(\Gamma)) \\ &= 2^n + 1 - \frac{(4+3\sqrt{2})}{4} \cdot \sqrt{2}^n - \frac{(4-3\sqrt{2})}{4} \cdot (-\sqrt{2})^n \\ &= 2^n + 1 - (2 + n \bmod 2) \cdot 2^{\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

8.2.2 Some Last Preparations

Purpose of this subsection. In this subsection, we will make some last preparations for the proofs of Theorem 243 and Corollary 246.

Decomposing the map N' into its q-parts. The crucial map N' for our approach (TwoSt) can be decomposed into maps N'_q for all primes q where each map N'_q will only handle the q-part of the map N'.

Definition 236. *Let $e \in \mathbb{N}$. Then we define*

$$R_e := \mathbb{C}[\{y_p, y_p^{-1} : p \in \mathbb{P} \text{ with } v_p(e) = 0\}] \subset \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}] \quad \text{and} \quad x_e := y_{\mathbf{P}}^{v_{\mathbf{P}}(e)}$$

and K_e as the field of fractions of R_e inside of some fixed field of fractions K_1 of $R_1 = \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$. Notice that $R_e[x_e, x_e^{-1}]$ (resp. $K_e[x_e, x_e^{-1}]$) is a ring of Laurent polynomials in the variable x_e over R_e (resp. K_e).

*Next, let $h = \sum_{i=r}^{s} h_i \cdot x_e^i \in K_e[x_e, x_e^{-1}]$ where $h_i \in K_e$ for all $i = r, \ldots, s$. Then we define the e-**principal part** of h as*

$$\mathrm{Princ}_e(h) := \sum_{i=r}^{-1} h_i \cdot x_e^i$$

*and the e-**non-principal part** of h as*

$$\mathrm{NonPrinc}_e(h) := h - \mathrm{Princ}_e(h) = \sum_{i=0}^{s} h_i \cdot x_e^i.$$

Finally, for all $a \in \mathbb{K}_e \backslash \{0\}$, we denote the evaluation morphism on $K_e[x_e, x_e^{-1}]$ which evaluates x_e at a by

$$\mathrm{Eval}_a : K_e[x_e, x_e^{-1}] \to K_e \text{ via } h \mapsto h(a)$$

and define the morphism

$$\mathrm{N}'_e : K_e[x_e, x_e^{-1}] \to K_e \text{ via } h \mapsto \mathrm{Eval}_e(\mathrm{Princ}_e(h)) + \mathrm{Eval}_1(\mathrm{NonPrinc}_e(h))$$

of K_e-vector spaces.

Notice that Princ_q and $\mathrm{NonPrinc}_q$ is well defined for all $h \in \mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$, that we have $R_e = R_{e'}$ and $K_e = K_{e'}$ if the prime decompositions of e and e' have the same support and that $\mathrm{Princ}_e : K_e[x_e, x_e^{-1}] \to K_e[x_e^{-1}]$ and $\mathrm{NonPrinc}_e : K_e[x_e, x_e^{-1}] \to K_e[x_e]$ are morphisms of K_e-vector spaces.

Lemma 237. *Let $q_1, \ldots, q_s \in \mathbb{P}$ be pairwise distinct, let $R := \mathbb{C}[\{y_p, y_p^{-1} : p \in \{q_1, \ldots, q_s\}]$ and let $h \in R$. Then we have the identity*

$$\mathrm{N}'(h) = (\mathrm{N}'_{q_1} \circ \cdots \circ \mathrm{N}'_{q_s})(h)$$

Moreover, let $e \in \mathbb{N}$ such that the support of the prime decomposition of e is equal to $\{q_1, \ldots, q_r\}$ with $r \leq s$. If $h \in R \cap R_e[x_e, x_e^{-1}]$, then we have the identity

$$\mathrm{N}'(h) = (\mathrm{N}'_{q_{r+1}} \circ \cdots \circ \mathrm{N}'_{q_s})(\mathrm{N}'_e(h)).$$

In particular, if $h \in \mathbb{C}[x_e, x_e^{-1}]$, then we even have the identities

$$\mathrm{N}'(h) = \mathrm{N}'_e(h) = \mathrm{Eval}_e(\mathrm{Princ}_e(h)) + \mathrm{Eval}_1(\mathrm{NonPrinc}_e(h)).$$

Proof. All identities immediately follow from the definitions of the maps N' in Definition 233 and of N'_{q_i} and N'_e in Definition 236. □

Separating enumerations. An immediate consequence of the form of separating subgraphs in Lemma 231(iii) will be the existence of so called separating enumerations of the vertices in the subgraph. These separating enumerations will be defined in the following Definition 238.

Definition 238. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Let Γ be a finite subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Let $q \in \mathbb{P}$ and let $q_1, \ldots, q_r \in \mathbb{P}$ be pairwise distinct*

*We call an enumeration $v = (P_1, \ldots, P_m)$ of the vertices in Γ q-**separating** if there is an index m_∞ such that the initial vertices of all edges Q in Γ with $v_q(e(Q|Q \cap \sigma(F_0))) \geq 1$ are contained in $\{P_1, \ldots, P_{m_\infty}\}$ and the terminal vertices of all edges Q in Γ with $v_q(e(Q|Q \cap F_0)) \geq 1$ are contained in $\{P_{m_\infty+1}, \ldots, P_m\}$. Here, we call m_∞ the q-**separating index** of v.*

*We even call v $(q_1, \ldots, q_r)$-**separating (resp. separating)** if it is p-separating for all $p \in \{q_1, \ldots, q_r\}$ (resp. $p \in \mathbb{P}$) and the p-separating index m_∞ can be chosen as the same index for all p.*

Power subgraphs. As we already pointed out, all tame recursive towers $\mathcal{F}$ in the literature known to the author have finite separating ramification subgraph. Moreover, all these tame recursive towers $\mathcal{F} = (F_\nu)_\nu$ also satisfy a second property which is advantageous for computing genus formulas: All ramification indices in F_1/F_0 and $F_1/\sigma(F_0)$ are powers of the same natural number e.

The desired Theorem 243 will only require the slightly weaker property that the all weakly connected components of the ramifiaction subgraph are power subgraphs.bgraph Γ_i of the tower graph are powers of the same natural number e_i.

Definition 239. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Let Γ be a subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. Let $q_1, \ldots, q_r \in \mathbb{P}$ be pairwise distinct and let $e = \prod_{i=1}^r q_i^{e_r} \in \mathbb{N}$ for some $e_1, \ldots, e_r \in \mathbb{N}$.*

*We say that the $(q_1, \ldots, q_r)$-**parts of the ramification indices in** Γ **are powers of** $e \in \mathbb{N}$ if, for all $Q \in E(\Gamma)$ and all $i = 0, 1$, the ramification indices $e(Q|Q \cap \sigma^i(F_0))$ are of the form $e^{w_{Q,i}} \cdot a_{Q,i}$ with $w_{Q,i} \in \mathbb{N}_0$ and $a_{Q,i} \in \mathbb{N}$ such that $v_{q_\nu}(a_{Q,i}) = 0$ for all $\nu = 1, \ldots, r$.*

*We even say that Γ is an e-**power subgraph** if the ramification indices $e(Q|Q \cap \sigma^i(F_0))$ are powers of e for all $Q \in E(\Gamma)$ and all $i = 0, 1$.*

Notice that the q-parts of the ramification indices in a subgraph Γ are always powers of q for all $q \in \mathbb{P}$.

The $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices for finite separating power subgraphs. Next, we will bring both definitions from above together and obtain that the corresponding $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of such a subgraph for a separating enumeration is of the form in the 'in particular'-part of the following Lemma 240.

Lemma 240. *Let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Let Γ be a finite subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ with m vertices. Let $q_1, \ldots, q_s \in \mathbb{P}$ be pairwise distinct with $s \in \mathbb{N}_0$ minimal such that the supports of the prime decompositions of all ramification indices in Γ (i.e. $e(Q|Q \cap \sigma^i(F_0))$ for all $Q \in E(\Gamma)$ and all $i = 0, 1$) are contained in $\{q_1, \ldots, q_s\}$.*

If the $(q_1, \ldots, q_r)$-parts of the ramification indices in Γ are powers of e and Γ is q_1-separating, then there is some $(q_1, \ldots, q_r)$-separating enumeration $v = (P_1, \ldots, P_m)$ with separating index m_∞ and the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix A of Γ for this enumeration v is of the form

$$A = \begin{pmatrix} A_\infty & B \\ 0 & A_0 \end{pmatrix} \in R_e[x_e, x_e^{-1}]^{m \times m} \subset K_e[x_e, x_e^{-1}]^{m \times m} \tag{497}$$

with $A_\infty \in K_e[x_e^{-1}]^{m_\infty \times m_\infty}$ and $A_0 \in K_e[x_e]^{m_0 \times m_0}$ where $m_0 := m - m_\infty$.

In particular, if Γ *is a separating* e*-power subgraph, then there is even some separating enumeration* v *of the vertices in* Γ *and the corresponding* $w'_{\mathbf{1},\mathbf{P}}$*-adjacency matrix* A *is of the form in (497) with* $A \in \overline{\mathbb{Q}}[x_e, x_e^{-1}]^{m\times m}$ *where* $\overline{\mathbb{Q}}$ *denotes the algebraic closure of* $\mathbb{Q}$ *in* $\mathbb{C}$*.*

Proof. For the 'main'-part: Suppose that the $(q_1,\dots,q_r)$-parts of the ramification indices in Γ are powers of e and Γ is q-separating with $q := q_1$. Then the implication from (i) to (iii) in Lemma 231 supplies disjoint subgraphs Γ_∞ and Γ_0 such that Γ consists of Γ_∞ and Γ_0 with possibly connecting edges from Γ_∞ to Γ_0.

Now, choose m_∞ as the number of vertices in Γ_∞ and let $(P_1,\dots,P_{m_\infty})$ and $(P_{m_\infty+1},\dots,P_m)$ be enumerations of the vertices in Γ_∞ and Γ_0, respectively. By Lemma 231(iii), the enumeration $v := (P_1,\dots,P_{m_\infty},P_{m_\infty+1},\dots,P_m)$ of the vertices in Γ is a q-separating with separating index m_∞.

We claim that v is even one of the desired $(q_1,\dots,q_r)$-separating enumerations in the 'main'-part: For that, let us use the notation $e(Q|Q\cap\sigma^i(F_0)) = e^{w_{Q,i}}a_{Q,i}$ in Definition 239. First, we compute

$$v_{q_\nu}(e(Q|Q\cap\sigma^i(F_0))) = v_{q_\nu}(e^{w_{Q,i}}a_{Q,i}) = v_{q_\nu}(e^{w_{Q,i}}) = e_\nu \cdot w_{Q,i} \tag{498}$$

for all $\nu = 1,\dots,r$ and all $i = 0,1$ where the first equality holds by the definitions of $e^{w_{Q,i}}$ and $a_{Q,i}$ in Definition 239, the second equality holds since $v_q : \mathbb{N} \to \mathbb{N}_0$ is a morphism of monoids and since $v_{q_v}(a_{Q,i}) = 0$ by its choice in Definition 239 and the last equality holds by the definition of $e = \prod_{\mu=1}^r q_\mu^{e_r}$ in the assumptions.

Then for all $\nu = 1,\dots,r$, all edges Q in Γ with $v_{q_\nu}(e(Q|Q\cap\sigma^i(F_0))) \geq 1$, the equality in (498) and the assertion $e_\nu \in \mathbb{N}$ imply the estimate $w_{Q,i} \geq 1$. Hence, for all $\nu = 1,\dots,r$ and all $i = 0,1$, we obtain that $v_{q_\nu}(e(Q|Q\cap\sigma^i(F_0))) \geq 1$ if and only if $v_q(e(Q|Q\cap\sigma^i(F_0))) \geq 1$. Consequently, this ensures that v is also an $(q_1,\dots,q_r)$-separating enumeration with $(q_1,\dots,q_r)$-separating index m_∞ by Definition 238.

Second, let $(a_{i,j})_{i,j} := A$. Then we obtain that the entries $a_{i,j}$ are indeed elements in $R_e[x_e, x_e^{-1}] \subset K_e[x_e, x_e^{-1}]$ for all $i,j \in \{1,\dots,m\}$ by the equalities

$$\begin{aligned} a_{i,j} &= \sum_{Q\in E(\Gamma,P_i,P_j)} w'_{\mathbf{1},\mathbf{P}}(Q) = \sum_{Q\in E(\Gamma,P_i,P_j)} y_{\mathbf{P}}^{v_{\mathbf{P}}(e(Q|Q\cap F_0))} \cdot (\mathbf{P} * y_{\mathbf{P}}^{-1})^{v_{\mathbf{P}}(e(Q|Q\cap\sigma(F_0)))} \\ &= \sum_{Q\in E(\Gamma,P_i,P_j)} \prod_{p\in\{q_1\dots,q_s\}} y_p^{v_p(e(Q|Q\cap F_0))} \cdot \prod_{p\in\{q_1\dots,q_s\}} (py_p^{-1})^{v_p(e(Q|Q\cap\sigma(F_0)))} \\ &= \sum_{Q\in E(\Gamma,P_i,P_j)} \prod_{p\in\{q_{r+1}\dots,q_s\}} y_p^{v_p(e(Q|Q\cap F_0))} \cdot \prod_{p\in\{q_{r+1}\dots,q_r\}} (py_p^{-1})^{v_p(e(Q|Q\cap\sigma(F_0)))} \\ &\qquad \cdot x_e^{w_{Q,0}} \cdot (ex_e^{-1})^{w_{Q,1}} \end{aligned} \tag{499}$$

where the equalities hold by the following reasonings: The first equality holds by the definition of weighted adjacency matrices in Definition 58(ii). The second equality holds by the definition of the weight function $w'_{\mathbf{1},\mathbf{P}}$ in Definition 162. The third equality holds by the assertion that the supports of the prime decompositions of all ramification indices in Γ are contained in $\{q_1,\dots,q_s\}$ and since the minimality of s implies that the primes $q_1,\dots,q_s$ are pairwise distinct. The fourth equality holds by the equalities in (498) and by the definition of $x_e = y_{\mathbf{P}}^{v_{\mathbf{P}}(e)} = \prod_{\nu=1}^r y_p^{v_{q_\nu}(e)} = \prod_{\nu=1}^r y_p^{e_\nu}$ in Definition 236.

Second to last, combining the assertion that the terminal (resp. initial) vertex P_j (resp. P_i) of any edge Q with $v_q(e(Q|Q\cap F_0)) \geq 1$ (resp. $v_q(e(Q|Q\cap\sigma(F_0))) \geq 1$) is contained in Γ_0, i.e. $j > m_\infty$, (resp. Γ_∞, i.e. $i \leq m_\infty$,) and the equality in (498) yields

$$v_q(e(Q|Q\cap F_0)) = 0 = w_{Q',0} \quad (\text{resp. } v_q(e(Q|Q\cap\sigma(F_0))) = 0 = w_{Q',1}) \tag{500}$$

for all $i,j \in \{1,\dots,m_\infty\}$ (resp. $i,j \in \{m_\infty+1,\dots,m\}$) and all $Q' \in E(\Gamma,P_i,P_j)$.

Finally, the equalities in (499) and (500) together supply the last desired statements, namely that the entries of A_∞, i.e. $a_{i,j}$ with $i, j \in \{1, \dots, m_\infty\}$, are contained in $K_e[x_e^{-1}]$ and that the entries of A_0, i.e. $a_{i,j}$ with $i, j \in \{m_\infty + 1, \dots, m\}$, are contained in $K_e[x_e]$.

For the 'in particular-part: Suppose that Γ is a separating e-power subgraph.

By Definition 238 and by Definition 229(iii), the assertion especially implies that the $(q_1, \dots, q_s)$-parts of the ramification indices in Γ are powers of e and Γ is q_1-separating. Hence, the 'main'-part provides some $(q_1, \dots, q_s)$-separating enumeration v of the vertices in Γ.

But, because of the assertion that the supports of the prime decompositions of all ramification indices in Γ are contained in $\{q_1, \dots, q_s\}$ and because of the definition of separating enumerations in Definition 238, this enumeration v is the desired separating enumeration of the vertices in Γ.

Moreover, the presentation of the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ for v in (497) is indeed of the desired form with $A \in R_e[x_e, x_e^{-1}]^{m\times m}$. But, since the support of all ramification indices in Γ are contained in the $\{q_1, \dots, q_s\}$, we have $r = s$ in the equality in (499). Hence, the last desired statement follows, namely that $A \in \overline{\mathbb{Q}}[x_e, x_e^{-1}]^{m\times m}$. ☐

8.2.3 A Genus Formula

Summary of the results of this subsection. In this subsection, we will come to the core of this chapter, namely the proofs of Theorem 243 and its Corollary 246, which is the main result of this chapter. This Corollary 246 will provide the desired genus formulas for all tame recursive towers with a finite separating power ramification subgraph.

Extending the map N'_e. In the 'in particular part of Lemma 237, we concluded that N' and N'_e are equal on $\mathbb{C}[x_e, x_e^{-1}]$, i.e. N' evaluates the principal part of $h \in \mathbb{C}[x_e, x_e^{-1}]$ at e and its non-principal part at 1.

In the following Definition 241, we will extend the definition of N'_e to any Laurent polynomial ring $K[x, x^{-1}]$ and prove Theorem 243 for this more general definition of N'_e.

Definition 241. *Let K be a field and $K[x, x^{-1}]$ be a ring of Laurent polynomials in the variable x. Moreover, let $e \in K\backslash\{0\}$, let $\mathrm{Eval}_1 : K[x, x^{-1}] \to K$ be the evaluation morphism $h(x) \mapsto h(1)$ and let $\mathrm{Eval}_e : K[x, x^{-1}] \to K$ be the evaluation morphism $h(x) \mapsto h(e)$ of K-algebras. Then we extend the definition of N'_e in Definition 236 to K and e, i.e.*

$$\mathrm{N}'_e : K[x, x^{-1}] \to K \textit{ via } h \mapsto \mathrm{Eval}_1(\mathrm{NonPrinc}(h)) + \mathrm{Eval}_e(\mathrm{Princ}(h))$$

where $\mathrm{Princ}(h) \in x^{-1}K[x^{-1}]$ denotes the principal part of h and $\mathrm{NonPrinc}(h) = h - \mathrm{Princ}(h) \in K[x]$ denotes the non-principal part of h.

We also extend this morphism N'_e to the morphism

$$\mathrm{N}''_e : K[x, x^{-1}]^{m\times m} \to K \textit{ via } (a_{i,j}) \mapsto \mathrm{N}'_e(\sum_{i,j} a_{i,j}) = \sum_{i,j} \mathrm{N}'_e(a_{i,j})$$

of K-vector spaces for all $m \in \mathbb{N}$.

A crucial but technical definition. The following Definition 242 defines the set $\Lambda(A)$, numbers $b(A, \lambda)$ for all $\lambda \in \Lambda(A)$ and the number $c(A)$ in Theorem 243. This set $\lambda(A)$ and the number $c(A)$ will be crucial for the choices of $\Lambda \subset \overline{\mathbb{Q}}$ and $c(\mathcal{F})$ in the desired identity

$$g(F_n) = \frac{1}{2}\left(2 + d^n\left(g(F_0) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\mathcal{F}})\right) - \sum_{\lambda\in\Lambda} f_\lambda(n)\cdot\lambda^n\right)$$

for all $n \geq c(\mathcal{F})$ in Corollary 246. Moreover, the numbers $b(A, \lambda)$ will provide degree bounds for the involved polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$. Unfortunately, the definitions in Definition 242 are quite technical and yet quite unmotivated from the 'First, we define'-part on.

Therefore and for now, it will be sufficient for us to go through this Definition 242 cursorily and only to the point that we are convinced that the sum in the desired identity in Theorem 243 is well defined. In the proof of Theorem 243, all definitions of Definition 242 will occur naturally one by one.

Definition 242. *Let $e \in \mathbb{N}$, let K be an algebraically closed field of characteristic zero, let $K[x, x^{-1}]$ be the ring of Laurent polynomials over K in the variable x and let*

$$A = \begin{pmatrix} A_\infty & B \\ 0 & A_0 \end{pmatrix} \in K[x, x^{-1}]^{m \times m}$$

with $A_0 \in K[x]^{m_0 \times m_0}$ and $A_\infty \in K[x^{-1}]^{m_\infty \times m_\infty}$. Now, let us fix the following notation:

- *Let $x_1, \ldots, x_{r_0}$ be the eigenvalues of A_0 and $x_{r_0+1}, \ldots, x_r$ of A_∞.*
- *Let $\chi_{A_0}(t) = \prod_{\nu=1}^{r_0}(t - x_\nu)^{\rho_\nu} \in K[x][t]$ and $\chi_{A_\infty}(t) = \prod_{\nu=r_0+1}^{r}(t - x_\nu)^{\rho_\nu} \in K[x^{-1}][t]$ be the characteristic polynomials of A_0 and A_∞, respectively. Then $\chi_A(t) = \chi_{A_0}(t) \cdot \chi_{A_\infty}(t) \in K[x, x^{-1}][t]$ is the characteristic polynomial of A.*
- *Let s_i be the maximal size of any Jordan blocks of A for the eigenvalue x_i for all $i = 1, \ldots, r$, and let $s := \sum_{i=1}^{r} s_i \leq m$.*
- *Let $\chi_0(t) = \prod_{\nu=1}^{r_0}(t - x_\nu)^{s_\nu} \in K[x, x_1, \ldots, x_{r_0}][t]$, let $\chi_\infty(t) = \prod_{\nu=r_0+1}^{r}(t - x_\nu)^{s_\nu} \in K[x^{-1}, x_{r_0+1}, \ldots, x_r][t]$ and let $\chi(t) := \chi_0(t) \cdot \chi_\infty(t) = \prod_{\nu=1}^{r}(t - x_\nu)^{s_\nu} \in K[x, x^{-1}, x_1, \ldots, x_r][t]$.*
- *Let $(a_{i,j}^{(n)})_{i,j} := A^n$.*
- *Let $E := K(x)$, $y := ex^{-1}$ and $F := E(x, x_1, \ldots, x_n)$. Then F/E is a finite extension of function fields with the algebraically closed full constant fields K.*
- *Let R_0 be the integral closure of $K[x]$ in F and let R_∞ be the integral closure of $K[x^{-1}]$ in F.*

First, we define

$$\begin{aligned} &\omega_0', \omega_\infty' \in \mathbb{N}_0 \text{ as the minimal with the properties} \\ &x^{\omega_0'}\chi_\infty \in R_0[t] \text{ and } x^{-\omega_\infty'}\chi_0 \in R_\infty[t], \end{aligned} \tag{501}$$

and

$$\omega_0'' := \max\{\deg_{x^{-1}}(a_{i,j}^{(\nu)}) : \nu = 0, \ldots, s-1,\ i, j \in \{1, \ldots, m\}\} \tag{502}$$

and

$$\omega_\infty'' := \max\{\deg_x(a_{i,j}^{(\nu)}) : \nu = 0, \ldots, s-1,\ i, j \in \{1, \ldots, m\}\} \tag{503}$$

and

$$\omega_i := s_i \cdot \omega_0' + \omega_0'' \text{ and } \omega_j := s_j \cdot \omega_\infty' + \omega_\infty'' \tag{504}$$

for all $i = 1, \ldots, r_0$ and $j = r_0 + 1, \ldots, r$.

Next, we define

$$\begin{aligned}\chi_i(t) &:= \frac{\chi(t)}{(t-x_i)^{s_i}} = \frac{\chi_0(t)\cdot\chi_\infty(t)}{(t-x_i)^{s_i}} = \prod_{\nu=1,\nu\neq i}^{r}(t-x_\nu)^{s_\nu} \\ &= \begin{cases}\prod_{\nu=1,\nu\neq i}^{r_0}(t-x_\nu)^{s_\nu}\cdot\chi_\infty(t) & \textit{if } i\le r_0\\ \chi_0(t)\cdot\prod_{\nu=r_0+1,\nu\neq i}^{r}(t-x_\nu)^{s_\nu} & \textit{if } i> r_0\end{cases}\end{aligned} \tag{505}$$

for all $i=1,\dots,r$

Moreover, we define

$$\begin{aligned} &M_{0,i} := \operatorname{Supp}(\operatorname{div}_0((x-1)x^{\omega_i}\chi_i(x_i)^{s_i})), && M_{0,j} := M_{0,\infty} := \operatorname{Supp}(\operatorname{div}_0(y(y-1)))\\ &\Lambda_{0,i} := \{x_i(P) : P\in M_{0,i}\}, && \Lambda_{0,j} := \{x_j(Q) : Q\in M_{0,j}\},\\ &b_{0,i}(P) := v_P((x-1)x^{\omega_i}\chi_i(x_i)^{s_i})-1, && b_{0,j}(Q) := v_Q((y-1)y^{\omega_j}\chi_j(x_j)^{s_j})-1,\\ &b_{0,i}(\lambda) := \max_{P'\in M_{0,i}, x_i(P')=\lambda} b_{0,i}(P'), && b_{0,j}(\mu) := \max_{Q'\in M_{0,j}, x_j(Q')=\mu} b_{0,j}(Q')\end{aligned} \tag{506}$$

for all $i=1,\dots,r_0$*, all* $j=r_0+1,\dots,r$*, all* $P\in M_{0,i}$*, all* $Q\in M_{0,j}$*, all* $\lambda\in\Lambda_{0,i}$ *and all* $\mu\in\Lambda_{0,j}$ *and*

$$\begin{aligned} &M_{\infty,i} := M_{\infty,0} := \operatorname{Supp}(\operatorname{div}_0(x(x-1))), && M_{\infty,j} := \operatorname{Supp}(\operatorname{div}_0((y-1)y^{\omega_j}\chi_j(x_j)^{s_j})),\\ &\Lambda_{\infty,i} := \{x_i(P) : P\in M_{\infty,i}\}, && \Lambda_{\infty,j} := \{x_j(Q) : Q\in M_{\infty,j}\},\\ &b_{\infty,i}(P) := v_P((x-1)x^{\omega_i}\chi_i(x_i)^{s_i})-1, && b_{\infty,j}(Q) := v_Q((y-1)y^{\omega_j}\chi_j(x_j)^{s_j})-1,\\ &b_{\infty,i}(\lambda) := \max_{P'\in M_{\infty,i}, x_i(P')=\lambda} b_{\infty,i}(P'), && b_{\infty,j}(\mu) := \max_{Q'\in M_{\infty,j}, x_j(Q')=\mu} b_{\infty,j}(Q')\end{aligned} \tag{507}$$

for all $i=1,\dots,r_0$*, all* $j=r_0+1,\dots,r$*, all* $P\in M_{\infty,i}$*, all* $Q\in M_{\infty,j}$*, all* $\lambda\in\Lambda_{\infty,i}$ *and all* $\mu\in\Lambda_{\infty,j}$*.*

Finally, we define

$$\begin{aligned} &\Lambda_0 := \bigcup_{i=1}^{r}\Lambda_{0,i},\quad \Lambda_\infty := \bigcup_{i=1}^{r}\Lambda_{\infty,i},\quad \Lambda(A) := \Lambda_0\cap\Lambda_\infty,\\ &b_0(\lambda_0) := \max\{s_i-1+b_{0,i}(\lambda_0) : i=1,\dots,r \textit{ with } \lambda_0\in\Lambda_{0,i}\},\\ &b_\infty(\lambda_\infty) := \max\{s_i-1+b_{\infty,i}(\lambda_\infty) : i=1,\dots,r \textit{ with } \lambda_\infty\in\Lambda_{\infty,i}\},\\ &b(A,\lambda) := \min\{b_0(\lambda), b_\infty(\lambda)\}\end{aligned} \tag{508}$$

for all $\lambda_0\in\Lambda_0$*, all* $\lambda_\infty\in\Lambda_\infty$ *and all* $\lambda\in\Lambda(A)$ *and we also define*

$$c(A) := \max\{s_i-1+b_{j,i}(\lambda) : i=1,\dots,r \textit{ and } j=0,\infty \textit{ and } \lambda\in\Lambda_{j,i}\cap\Lambda(A)\}. \tag{509}$$

The main result of this chapter. As we already mentioned, the main result of this chapter is Corollary 246. It is a corollary of the following Theorem 243. Essentially, this theorem provides a formula for the two steps in (TwoSt).

Theorem 243. *Let everything be as in Definition 242. Then we have the equality*

$$\mathrm{N}''_e(A^n) = \sum_{\lambda\in\Lambda(A)} f_\lambda(n)\cdot\lambda^n$$

for all $n\ge c(A)$ *and some polynomials* $f_\lambda(n)\in K[n]$ *with* $\deg(f(n))\le b(A,\lambda)$ *for all* λ *in the finite subset* $\Lambda(A)\subset K$*.*

In the following Corollary 244, we can apply Theorem 243 to the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices in the 'in particular'-part in Lemma 240.

Corollary 244. *Let $e \in \mathbb{N}$ and let $\mathcal{F}$ be a recursive tower which is defined by the pair (σ, F_0). Moreover, let Γ be a finite separating e-power subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. The 'in particular'-part in Lemma 240 supplies a separating enumeration v of the vertices in Γ such that the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix $A \in \overline{\mathbb{Q}}[x_e, x_e^{-1}]$ of Γ for v is of the form in Definition 242.*

Then we have the identity

$$\sum_{\mathcal{P} \in W(\Gamma,n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \sum_{\lambda \in \Lambda(A)} f_\lambda(n) \cdot \lambda^n \tag{510}$$

for all $n \geq c(A)$ and some polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ with $\deg(f(n)) \leq b(A, \lambda)$ for all $\lambda \in \Lambda(A) \subset \overline{\mathbb{Q}}$.

Proof. Let $v \in \mathbb{C}^{1\times m}$ be the vector with only ones and consider the maps N'_e and N''_e in Definition 241 for $K := \overline{\mathbb{Q}}$ and $x := x_e = y_{\mathbf{P}}^{v_{\mathbf{P}}(e)}$ where the last equality holds by the definition of x_e in Definition 236. Then we obtain the equalities

$$\sum_{\mathcal{P} \in W(\Gamma,n)} \tilde{N}(F_n, \sigma_{\Gamma_{\mathcal{F}}}(\mathcal{P})) = \mathrm{N}'(vA^n v^t) = \mathrm{N}'_e(vA^n v^t) = \mathrm{N}''_e(A^n) \tag{511}$$

where the equalities hold by the following reasonings: The first equality holds by Lemma 234(ii). The second equality holds because $vA^n v^t$ is a Laurent polynomial in $\overline{\mathbb{Q}}[x_e, x_e^{-1}]$ and because the 'in particular-part of Lemma 237 implies that N' and N'_e are equal on $\overline{\mathbb{Q}}[x_e, x_e^{-1}]$. The third equality holds by definition of N''_e in Definition 241.

Finally, applying Proposition 243 to A yields that $\mathrm{N}''_e(A^n)$ is of the desired form in (511). Hence, combining this and the equality in (510) yields the desired statement. ☐

The main result of this chapter: Corollary 246. In Corollary 246, we will apply Corollary 244 to the weakly connected components of the ramification subgraph of the geometric tower $\overline{\mathcal{F}}$ of $\mathcal{F}$ and finally obtain the desired genus formula in (467). Consequently, this genus formula will hold for all $n \geq c(\mathcal{F})$ where, in the following Definition 245, $c(\mathcal{F})$ will be defined as the maximum of all lower bounds $c(A)$ which will appear from applying Corollary 244.

Definition 245. *Let $\mathcal{F}$ be a recursive tower, let $\overline{\mathcal{F}}$ be the geometric tower of $\mathcal{F}$, and let $\Gamma_1, \ldots, \Gamma_r$ be the finite weakly connected components of $\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}$. Moreover, let A_i be a $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ_i for all $i = 1, \ldots, r$. and let $c(A_i) + 1$ be the size of the largest Jordan-block of A_i for all $i = 1, \ldots, r$. Then we define*

$$c(\mathcal{F}) := \max_{i=1,\ldots r} c(A_i).$$

Corollary 246 Computing Genus Formulas**.** *Let $e_i \in \mathbb{N}$ for all $i = 1 \ldots, r$, let $\mathcal{F}$ be a tame recursive tower over a field k which is defined by the pair (σ, F_0) and let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$. Moreover, suppose that $\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}$ is finite with only separating weakly connected components $\Gamma_1, \ldots, \Gamma_r$ such that the ramification indices in Γ_i are powers of e_i for all $i = 1 \ldots, r$.*

Then we have the identities

$$N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}})) = \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n$$

and

$$g(F_n) = \frac{1}{2}\left(2 + d^n \left(g(F_0) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}})\right) - \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n\right)$$

for all $n \geq c(\mathcal{F})$, some finite subset $\Lambda \subset \overline{\mathbb{Q}}$ and some polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ for all $\lambda \in \Lambda$.

Proof. Let A_i be the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of Γ_i from Corollary 244 for all $i = 1, \dots, r$. First, we obtain the equalities

$$\tilde{N}(F_n, V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})) = \sum_{i=1}^{r} \tilde{N}(F_n, V(\Gamma_i)) = \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n \tag{512}$$

for all $n \geq \max_{i=1,\dots,r} c(A_i)$, some finite subset $\Lambda \subset \overline{\mathbb{Q}}$ and some polynomials $f_\lambda(n) \in \overline{\mathbb{Q}}[n]$ for all $\lambda \in \Lambda$ where the equalities hold by the following reasonings: The first equality holds because of the definition of $\tilde{N}(F_n, \cdot)$ in Definition 50 and because $\Gamma_1, \dots, \Gamma_r$ are the weakly connected components of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ and, thus, any path in Γ is contained in exactly one of these components. The second equality holds by applying Corollary 244 to these separating weakly components components $\Gamma_1, \dots, \Gamma_r$.

Second, we notice that $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} = \overline{k} \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ is also finite by Lemma 124 and by Lemma 120(i). Then the 'moreover'-part in Lemma 120(iii) supplies that $\overline{k} \cdot \Gamma_i$ is a disjoint union of weakly connected components $\overline{\Gamma}_{i,1}, \dots, \overline{\Gamma}_{i,s_i}$ of $\Gamma_{\overline{\mathcal{F}}}$ for all $i = 1, \dots, r$. Moreover, by Lemma 105(i), by Lemma 109 and by Definition 239, we obtain that, for all $i = 1, \dots, r$, the weakly connected components $\overline{\Gamma}_{i,1}, \dots, \overline{\Gamma}_{i,s_i}$ of $\Gamma_{\overline{\mathcal{F}}}$ are even separating $e_i \in \mathbb{N}$-power subgraphs for all $j = 1, \dots, s_i$. But $\overline{\Gamma}_{1,1}, \dots, \overline{\Gamma}_{1,s_1}, \dots, \overline{\Gamma}_{r,1}, \dots, \overline{\Gamma}_{r,s_r}$ are exactly the weakly connected components of $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}} = \overline{k} \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$.

Let $A_{i,j}$ be the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix of $\Gamma_{i,j}$ from Corollary 244 for all $i = 1, \dots, r$ and all $j = 1, \dots, s_i$. Then we deduce the first desired equality by the equalities

$$N(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})) = \tilde{N}(\overline{F}_n, V(\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}})) = \sum_{\lambda \in \Lambda} f_\lambda(n) \cdot \lambda^n \tag{513}$$

for all $n \geq \max_{i,j} c(A_{i,j}) = c(\mathcal{F})$ where the first equality holds because $\mathcal{F}$ is tame and because we can therefore apply the last identity in Corollary 51 and the second equality holds because the above reasoning ensures that we can apply (512) to $\overline{\mathcal{F}}$ and because of the definition of $c(\mathcal{F})$ in Definition 245.

Finally, the second desired identity for $g(F_n)$ in the 'moreover'-part immediately follows from the first identity in Lemma 194 and from the equality in (513). □

The proof of Theorem 243. Finally, we will prove Theorem 243. This is the only elaborate proof in this chapter. However, the proof is basically a generalization of the ideas in Example 235.

Proof of Theorem 243. First, let $v \in F^{1 \times m}$ be the vector with only ones. Then Lemma 61 provides the presentation

$$a_n := v \cdot A^n \cdot v^t = v \cdot TJ^nT \cdot v^t = \sum_{i=1}^{r} \sum_{j=0}^{s_i - 1} c_{i,j} \binom{n}{j} x_i^{n-j} \tag{514}$$

of the sum of the entries in A^n for some suitable $c_{i,j} \in F$ and all $n \in \mathbb{N}_0$ where the binomial coefficient $\binom{n}{j}$ is zero if $n < j$.

Second, we want to solve for the coefficients $c_{i,j}$ in (514) and, for that, consider the confluent Vandermonde matrix

$$V := \begin{pmatrix} V(x_1, s_1) & \dots & V(x_r, s_r) \end{pmatrix} \in K[x_1, \dots, x_r]^{s \times s}$$

where

$$V(x_i, s_i) := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ x_i & 1 & 0 & \dots & 0 \\ x_i^2 & x_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_i^{s_i-1} & \binom{s_i-1}{1}x_i^{s_i-2} & \binom{s_i-1}{2}x_i^{s_i-3} & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_i^{s-1} & \binom{s-1}{1}x_i^{s-2} & \binom{s-1}{2}x_i{}^{s-3} & \dots & \binom{s-1}{s_i-1}x_i^{s-1-(s_i-1)} \end{pmatrix} \in K[x_i]^{s\times s_i}$$

for all $i = 1, \dots, r$. Let

$$a := \begin{pmatrix} a_0 & \dots & a_{s-1} \end{pmatrix}, c := \begin{pmatrix} c_{1,0} & \dots & c_{1,s_1-1} & \dots & c_{r,0} & \dots & c_{r,s_r-1} \end{pmatrix} \in K[x_1, \dots, x_r]^{1\times s}.$$

Then we have the equality

$$a^t = V \cdot c^t \tag{515}$$

Now, [MZ21, p. 4, Theorem 2.4] provides that V is invertible and a presentation for the $c_{i,j}$. Before we write down this presentation, let us first introduce an adaption of the notation in [MZ21, p. 2]: For all $i = 1, \dots, r$ and $j = 0, \dots, s_i$, we define

$$\chi_{i,j}(t) := (t - x_i)^j \prod_{\nu=1,\nu\neq i}^{r} (t - x_\nu)^{s_\nu}. \tag{516}$$

Notice that we have $\chi_i(t) = \chi_{i,0}(t)$ by the definition of $\chi_i(t)$ in (505). Moreover, we define

$$\begin{aligned} L_{i,j}(t) &:= \chi_i(t) \cdot (t - x_i)^j \sum_{\mu=0}^{s_i-1-j} \frac{1}{\mu!} \left(\chi_i(t)^{-1}\right)^{(\mu)}(x_i) \cdot (t - x_i)^\mu \\ &= \sum_{\mu=0}^{s_i-1-j} \frac{1}{\mu!} \left(\chi_i(t)^{-1}\right)^{(\mu)}(x_i) \cdot \chi_{i,j+\mu}(t) \end{aligned} \tag{517}$$

where $\left(\chi_i(t)^{-1}\right)^{(\mu)}$ denotes the μ-th derivative of $\chi_i(t)^{-1}$ in the variable t. Next, we consider the Faa di Bruno's Formula for the μ-th derivative of the composition $f \circ g$ of two functions f and g in [Fla01, p. 1], i.e.

$$(f \circ g)(t)^{(\mu)} = \sum_{(m_1,\dots,m_\mu)\in X_\mu} \frac{\mu!}{\prod_{\nu=1}^{\mu} m_\nu! \cdot \nu!^{m_\nu}} \cdot f^{(w(m_1,\dots,m_\mu))}(g(t)) \cdot \prod_{\nu=1}^{\mu} \left(g(t)^{(\nu)}\right)^{m_\nu} \tag{518}$$

where $w(m_1, \dots, m_\mu) := \sum_{\nu=1}^{\mu} m_\nu$ and $X_\mu := \{(m_1, \dots, m_\mu) : \sum_{\nu=1}^{\mu} \nu \cdot m_\nu = \mu\}$. Now, we want to apply Faa di Bruno's Formula to $f = t^{-1}$ and $g = \chi_i(t)$ and, for that, notice the equality

$$(t^{-1})^{(w(m_1,\dots,m_\mu))} = (-1)^{w(m_1,\dots,m_\mu)} \cdot w(m_1, \dots, m_\mu)! \cdot t^{-1-w(m_1,\dots,m_\mu)}.$$

Hence, for all $i = 1, \dots, r$ and all $\mu = 0, \dots, s_i - 1$ and for

$$c(m_1, \dots, m_\nu) := \frac{\mu! \cdot (-1)^{w(m_1,\dots,m_\mu)} \cdot w(m_1, \dots, m_\mu)!}{\prod_{\nu=1}^{\mu} m_\nu! \cdot \nu!^{m_\nu}},$$

we derive the equality

$$\left(\chi_i(t)^{-1}\right)^{(\mu)} = \sum_{(m_1,\dots,m_\mu)\in X_\mu} c(m_1, \dots, m_\nu) \cdot \chi_i(t)^{-1-w(m_1,\dots,m_\mu)} \cdot \prod_{\nu=1}^{\mu} \left(\chi_i(t)^{(\nu)}\right)^{m_\nu}$$

$$= \frac{\sum\limits_{(m_1,\ldots,m_\mu)\in X_\mu} c(m_1,\ldots,m_\nu)\cdot\chi_i(t)^{\mu-w(m_1,\ldots,m_\mu)}\cdot\prod\limits_{\nu=1}^{\mu}\left(\chi_i(t)^{(\nu)}\right)^{m_\nu}}{\chi_i(t)^{\mu+1}} \tag{519}$$

$$= \frac{\mathcal{L}(\chi_i(t),\mu)}{\chi_i(t)^{\mu+1}} \tag{520}$$

where $\mathcal{L}(\chi_i(t),\mu)$ denotes the numerator in (519). Since $\mu = \max\{w(m_1,\ldots,m_\mu) : (m_1,\ldots,m_\mu)\in X(\mu)\}$, the numerator $\mathcal{L}(\chi_i(t),\mu)$ is a polynomial in t.

Consequently, [MZ21, p. 6, Corollary 2.5] provides that, for all $i = 1,\ldots,r$ and $j = 1,\ldots,s_i-1$, the elements $c_{i,j}$ in the solution c of the equation $a^t = Vc^t$ in (515) are of the form

$$\begin{aligned}
c_{i,j} &= \sum_{\nu=0}^{s-1}\frac{a_\nu}{\nu!}L_{i,j}^{(\nu)}(0) = \sum_{\nu=0}^{s-1}\sum_{\mu=0}^{s_i-1-j}\frac{a_\nu}{\nu!}\cdot\left(\frac{d}{dt}\right)^{\nu}\left(\frac{1}{\mu!}\left(\chi_i(t)^{-1}\right)^{(\mu)}(x_i)\cdot\chi_{i,j+\mu}(t)\right)(0)\\
&= \sum_{\nu=0}^{s-1}\sum_{\mu=0}^{s_i-1-j}\frac{a_\nu}{\nu!}\frac{1}{\mu!}\left(\chi_i(t)^{-1}\right)^{(\mu)}(x_i)\cdot\chi_{i,j+\mu}(t)^{(\nu)}(0)\\
&= \sum_{\nu=0}^{s-1}\sum_{\mu=0}^{s_i-1-j}\frac{a_\nu}{\nu!}\frac{1}{\mu!}\frac{\mathcal{L}(\chi_i(t),\mu)(x_i)}{\chi_i(x_i)^{\mu+1}}\cdot\chi_{i,j+\mu}(t)^{(\nu)}(0)\\
&= \frac{\sum_{\nu=0}^{s-1}\sum_{\mu=0}^{s_i-1-j}\frac{1}{\nu!\cdot\mu!}\cdot\mathcal{L}(\chi_i(t),\mu)(x_i)\cdot\chi_i(x_i)^{s_i-(\mu+1)}\cdot\chi_{i,j+\mu}(t)^{(\nu)}(0)\cdot a_\nu}{\chi_i(x_i)^{s_i}} && (521)\\
&= \frac{\gamma'_{i,j}}{\chi_i(x_i)^{s_i}} && (522)
\end{aligned}$$

where $\gamma'_{i,j}$ denotes the numerator in (521) and the equalities hold by the following reasonings: The first equality holds by [MZ21, p. 6, Corollary 2.5] and by our adaption of the notation from there in 517. The second equality holds by the equality in (517). The third equality holds since only $\chi_{i,j+\mu}(t)$ depends on t in the derivative. The fourth equality holds by the equality in (520). The fifth equality holds by extending the numerator and denominators with $\chi_i(x_i)^{s_i-(\mu+1)}$ and by elementary arithmetics. The last equality holds by the choice of $\gamma'_{i,j}$.

Combining the equalities in (514) and (522) yields the equality

$$a_n = \sum_{i=1}^{r}\frac{\sum\limits_{j=0}^{s_i-1}\gamma'_{i,j}\binom{n}{j}x_i^{n-j}}{\chi_i(x_i)^{s_i}}. \tag{523}$$

Third, on the one hand, let us also write $S_0 = R_0$ for the integral closure of $K[x]$ in F. On the other hand, let $S_\infty \supset R_\infty$ be the integral closure of $K[x^{-1}]_{(x^{-1})}\cap K[x^{-1}]_{(x^{-1}-e^{-1})} = K[y]_{(y)}\cap K[y]_{(y-1)}$ in F where $K[z]_{(z-a)}$ denotes the localization of $K[z]$ at the maximal ideal $(z-a)K[z]$ for all $z\in\{x^{-1},y\}$ and all $a\in K$.

Since $x_1,\ldots,x_{r_0}$ are zeroes of the characteristic polynomial $\chi_{A_0}(t)\in K[x][t]$ of $A_0\in K[x]^{m_0\times m_0}$ and since $x_{r_0+1},\ldots,x_r$ are zeroes of the characteristic polynomial $\chi_{A_\infty}(t)\in K[x^{-1}][t]$ of $A_\infty\in K[x^{-1}]^{m_\infty\times m_\infty}$, we obtain

$$x_1,\ldots,x_{r_0}\in S_0 \text{ and } x_{r_0+1},\ldots,x_r\in S_\infty. \tag{524}$$

By [Sti08, p. 79, Theorem 3.2.6(b)] and [Sti08, p. 77, Definition 3.2.2], we then have the equalities

$$S_0 = \{z\in F : v_Q(z)\geq 0 \text{ for all } Q\in\mathbb{P}_F\backslash\operatorname{Supp}(\operatorname{div}_\infty(x))\} \tag{525}$$

and

$$S_\infty = \{z \in F : v_Q(z) \geq 0 \text{ for all } Q \in \operatorname{Supp}(\operatorname{div}_0(y)) \cup \operatorname{Supp}(\operatorname{div}_0(y-1))\} \tag{526}$$

Moreover, we also have the equality

$$S_0 \cap S_\infty = \{z \in F : v_Q(z) \geq 0 \text{ for all } Q \in \mathbb{P}_F\} = K \tag{527}$$

where the first equality holds by the equalities $\operatorname{div}_\infty(y) = \operatorname{div}_\infty(ex^{-1}) = \operatorname{div}_0(x)$ and by the equalities in (525) and (526) and the second equality holds because of [Sti08, p. 8, Corollary 1.1.20] and because K ss an algebraically closed field and, thus, must already be the full constant field of F.

By the equality in (527), we can extend any K-basis of K to some basis of the K-vector space S_0 and also of S_∞. Then the union of these two bases is a basis of $S_0 + S_\infty$. Finally, we can even extend this last basis of $S_0 + S_\infty$ to a K-basis of all of F. This means that any element in $z \in F$ has a presentation $z = z_0 + z_\infty + u$ with $z_0 \in S_0$, $z_\infty \in S_\infty$ and $u \in U$ where any other such presentation of $z = z_0' + z_\infty' + u$ satisfies $u = u'$, $z_0' = z_0 + a$ and $z_\infty' = z_\infty - a$ for some constant $a \in K$. In particular, for $z \in K[x, x^{-1}]$, we can choose $z_0 = \operatorname{NonPrinc}(z)$, $z_\infty = \operatorname{Princ}(z)$ and $u = 0$.

Next, let

$$\begin{aligned}&\sigma_1 : S_0 \to K \text{ be any extension morphism of } K\text{-algebras of the evaluation}\\ &\text{morphism } \operatorname{Eval}_1 : R_0 \to K,\ h(x) \mapsto h(1)\end{aligned} \tag{528}$$

and let

$$\begin{aligned}&\sigma_e : S_\infty \to K \text{ be any extension morphism of } K\text{-algebras of the evaluation}\\ &\text{morphism } \operatorname{Eval}_e : R_\infty \to K,\ h(x) \mapsto h(e).\end{aligned} \tag{529}$$

Because distinct presentations $z = z_0 + z_\infty + u = z_0' + z_\infty + u$ satisfy $z_0' = z_0 + a$ and $z_\infty' = z_\infty - a$ for some constant $a \in K$, we can even extend the morphism

$$\mathrm{N}_e' : K[x, x^{-1}] \to K \text{ via } z \to \operatorname{Eval}_1(\operatorname{NonPrinc}(z)) + \operatorname{Eval}_e(\operatorname{NonPrinc}(z))$$

in Definition 241 to the morphism

$$\mathrm{N}_e' : F \to K \text{ via } z = z_0 + z_\infty + u \mapsto \sigma_1(z_0) + \sigma_e(z_\infty) \tag{530}$$

of K-vector spaces.

Fourth, we will show that the exponents ω_i in (504) are chosen such that

$$\gamma_{i,j} := x^{\omega_i}\gamma_{i,j}' \text{ and } h_i := x^{\omega_i}\chi_i(x_i)^{s_i} \text{ are elements in } S_0 \tag{531}$$

for all $i = 1, \ldots, r_0$ and $j = 0, \ldots, s_i - 1$ and

$$\gamma_{i,j} := x^{-\omega_i}\gamma_{i,j}' \text{ and } h_i := x^{-\omega_i}\chi_i(x_i)^{s_i} \text{ are elements in } S_\infty \tag{532}$$

for all $i = r_0 + 1, \ldots, r$ and all $j = 0, \ldots, s_i - 1$.

For that, we first notice the equality

$$x^{\omega_0'}\chi_i(t)^{(\mu)} = \left(x^{\omega_0'} \cdot \chi_\infty(t) \cdot \prod_{\nu=1,\nu\neq i}^{r_0} (t - x_\nu)^{s_\nu}\right)^{(\mu)} \in S_0[t] \tag{533}$$

for all $i = 1, \ldots, r_0$ and all $\mu = 0 \ldots, s_i - 1$ where the equality holds because $x^{\omega_0'}$ is constant in t and because of the equality $\chi_i(t) = \prod_{\nu=r_0+1,\nu\neq i}^{r} (t - x_\nu)^{s_\nu} \cdot \chi_\infty(t)$ in (505) and the containment-statement holds because of the definition of ω_0' in (501) as the minimal exponent such that $x^{\omega_0'} \cdot \chi_\infty(t) \in R_0[t] = S_0[t]$, because we have $x_\nu \in S_0$ for all $\nu = 1, \ldots, r_0$ by (524) and because the derivative of a polynomial in $S_0[t]$ still has coefficients in the K-algebra S_0.

In particular, for all $i = 1, \ldots, r_0$, if we evaluate the s_i-th power of this equality in (533) for $\mu = 0$ at x_i, we already obtain the second desired containments-statement in (531) because of the definition of $\omega_i := s_i \cdot \omega_0' + \omega_0''$ in (504) and because we have $x_i \in S_0$ by (524).

Next, we conclude

$$\begin{aligned}
&\left(x^{\omega_0'}\right)^{\mu} \cdot \chi_i(t)^{\mu - w(m_1,\ldots,m_\mu)} \cdot \prod_{\nu=1}^{\mu} \left(\chi_i(t)^{(\nu)}\right)^{m_\nu} \\
&\qquad = \left(x^{\omega_0'}\right)^{\mu - w(m_1,\ldots,m_\mu)} \cdot \chi_i(t)^{\mu - w(m_1,\ldots,m_\mu)} \cdot \left(x^{\omega_0'}\right)^{\sum_{\nu=1}^{\mu} m_\nu} \cdot \prod_{\nu=1}^{\mu} \left(\chi_i(t)^{(\nu)}\right)^{m_\nu} \\
&\qquad = \left(x^{\omega_0'} \cdot \chi_i(t)\right)^{\mu - w(m_1,\ldots,m_\mu)} \cdot \prod_{\nu=1}^{\mu} \left(x^{\omega_0'} \cdot \chi_i(t)^{(\nu)}\right)^{m_\nu} \in S_0[t] \qquad (534)
\end{aligned}$$

for all $i = 1, \ldots, r_0$ and all $\mu = 0 \ldots, s_i - 1$ where the first equality by the definition of $w(m_1, \ldots, m_\mu) = \sum_{\nu=1}^{\mu} m_\nu$ and by elementary arithmetics, the second equality holds by elementary arithmetics and the containment-statement holds by (533). In particular, combining (533) and the definition of $\mathcal{L}(\chi_i(t), \mu)(x_i)$ in (520) yields

$$\left(x^{\omega_0'}\right)^{\mu} \cdot \mathcal{L}(\chi_i(t), \mu)(x_i) \in S_0 \qquad (535)$$

for all $i = 1, \ldots, r_0$ and all $\mu = 0 \ldots, s_i - 1$. Thus, we derive

$$\begin{aligned}
&\left(x^{\omega_0'}\right)^{s_i} \cdot \mathcal{L}(\chi_i(t), \mu)(x_i) \cdot \chi_i(x_i)^{s_i - (\mu+1)} \cdot \chi_{i,j+\mu}(t)^{(\nu)}(0) \\
&\quad = \left(x^{\omega_0'}\right)^{\mu} \cdot \mathcal{L}(\chi_i(t), \mu)(x_i) \cdot \left(x^{\omega_0'} \cdot \chi_i(x_i)\right)^{s_i - (\mu+1)} \cdot \left(x^{\omega_0'} \cdot \chi_{i,j+\mu}(t)\right)^{(\nu)}(0) \in S_0 \quad (536)
\end{aligned}$$

for all $i = 1, \ldots, r_0$, all $\mu = 0 \ldots, s_i - 1$ and all $\nu = 0, \ldots, s - 1$ where the equality holds by the presentation of s_i as the sum $s_i = \mu + (s_i - (\mu + 1)) + 1$ and by elementary arithmetics and the containment-statement holds by (535) and by (533).

Finally, because of (536), because of the definition of $\gamma_{i,j}'$ in (522), because the definition of ω_0'' in (502) supplies $x^{\omega_0''} a_\nu = \sum_{i,j} x^{\omega_0''} a_{i,j}^{(\nu)} \in S_0$ and because of the definition of $\omega_i := s_i \cdot \omega_0' + \omega_0''$ in (504), we get the desired containment-statements in (531).

Now, the desired containment-statements in (532) follow analogously (just use the presentation $\chi_i(t) = \chi_0(t) \cdot \prod_{\nu=r_0+1, \nu \neq i}^{r} (t - x_\nu)^{s_\nu}$ in (505)).

On the one hand, let us define

$$\begin{aligned}
a_{n,0} &:= \sum_{i=1}^{r_0} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}' \binom{n}{j} x_i^{n-j}}{\chi_i(x_i)^{s_i}} = \sum_{i=1}^{r_0} \frac{x^{\omega_i}}{x^{\omega_i}} \cdot \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}' \binom{n}{j} x_i^{n-j}}{\chi_i(x_i)^{s_i}} \\
&= \sum_{i=1}^{r_0} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} x_i^{n-j}}{h_i} \qquad (537)
\end{aligned}$$

where, by (531), the numerators $\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} x_i^{n-j}$ and the denominators h_i are elements in S_0 for all $i = 1, \ldots, r_0$. On the other hand, we also define

$$a_{n,\infty} := \sum_{i=r_0+1}^{r} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}' \binom{n}{j} x_i^{n-j}}{\chi_i(x_i)^{s_i}} = \sum_{i=r_0+1}^{r} \frac{x^{-\omega_i}}{x^{-\omega_i}} \cdot \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}' \binom{n}{j} x_i^{n-j}}{\chi_i(x_i)^{s_i}}$$

$$= \sum_{i=r_0+1}^{r} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} x_i^{n-j}}{h_i} \tag{538}$$

where, by (532), the numerators $\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} x_i^{n-j}$ and the denominators h_i are elements in S_∞ for all $i = r_0+1, \dots, r$. By (523), we then get the equality

$$a_n = a_{n,0} + a_{n,\infty}. \tag{539}$$

Fifth, for all polynomials $h \in F[t]$, all $\lambda \in K$ and all $e \in \mathbb{N}_0$, we denote the e-th Taylor polynomial of h at λ as

$$T_{\lambda,e}(h) := \sum_{k=0}^{e} \frac{h^{(k)}(\lambda)}{k!} (t-\lambda)^k. \tag{540}$$

It is well known that

$$T_{\lambda,e}(h) = h \tag{541}$$

if $\deg(h) \le e$. More specifically, we define

$$T_{Q,e}(x_i^{n-j}) := T_{x_i(Q),e}(t^{n-j})(x_i) \tag{542}$$

for all $e \in \mathbb{N}_0$, all $i = 1, \dots, r$, all $j = 0 \dots, s_i-1$ and all $Q \in \mathbb{P}_L$ with $Q \notin \operatorname{Supp}(\operatorname{div}_\infty(x)) \supseteq \operatorname{Supp}(\operatorname{div}_\infty(x_i))$ if $i \le r_0$ and $Q \notin \operatorname{Supp}(\operatorname{div}_0(x)) \supseteq \operatorname{Supp}(\operatorname{div}_\infty(x_i))$ if $i > r_0$ where the inclusions hold since x_i is a zero of $\chi_{A_0}(t) \in K[x][t]$ if $i \le r_0$ and of $\chi_{A_\infty}(t) \in K[x^{-1}][t]$. Then we have

$$\begin{aligned} v_Q\left(x_i(Q)^{n-j} - T_{Q,e}(x_i^{n-j})\right) &= v_Q\left(T_{Q,n-j}(x_i^{n-j}) - T_{Q,e}(x_i^{n-j})\right) \\ &= v_Q\left(\sum_{k=e+1}^{n-j} \frac{t^{(k)}(x_i(Q))}{k!}(x_i - x_i(Q))^k\right) \ge e+1 \end{aligned} \tag{543}$$

for all $e \in \mathbb{N}_0$, all $i = 1, \dots, r$, all $j = 0 \dots, s_i-1$ and all $Q \in \mathbb{P}_L$ with $Q \notin \operatorname{Supp}(\operatorname{div}_\infty(x)) \supseteq \operatorname{Supp}(\operatorname{div}_\infty(x_i))$ if $i \le r_0$ and $Q \notin \operatorname{Supp}(\operatorname{div}_0(x)) \supseteq \operatorname{Supp}(\operatorname{div}_\infty(x_i))$ if $i > r_0$ where the first equality holds by the definition of $T_{Q,e}(x_i^{n-j})$ in (542) and by the equality in (541), the second equality holds by the equality in (540) and the estimate holds since the argument is a K-linear combination of $(x_i - x_i(Q))^{e+1}, \dots, (x_i - x_i(Q))^{n-j}$.

Sixth, let $h_0 := \prod_{i=1}^{r_0} h_i$. By the Strong Approximation Theorem in [Sti08, p. 33, Theorem 1.6.5], we can choose elements $\varepsilon_Q \in F$ for all $Q \in \operatorname{Supp}(\operatorname{div}_0(h_0(x-1)))$ such that

$$\begin{aligned} &v_Q(\varepsilon_Q - 1) = v_Q(h_0(x-1)) \ge v_Q(h_i(x-1)) \ge 0, \quad v_R(\varepsilon_Q) \ge 0, \\ &v_P(\varepsilon_Q) = v_P(h_0(x-1)) \ge v_P(h_i(x-1)) \ge 0 \end{aligned} \tag{544}$$

for all $i = 1, \dots, r_0$, all $P \in \operatorname{Supp}(\operatorname{div}_0(h_0(x-1))) \backslash \{Q\}$ and all $R \in \mathbb{P}_F \backslash \operatorname{Supp}(\operatorname{div}_\infty(x))$ where the estimates $v_Q(h_0(x-1)) \ge v_Q(h_i(x-1)) \ge 0$ and $v_P(h_0(x-1)) \ge v_P(h_i(x-1)) \ge 0$ hold by the definition $h_0 := \prod_{i=1}^{r_0} h_i$, by (531) and by (525). Thus, by (525), we even obtain $\varepsilon_Q \in S_0$. Next, we consider the equalities

$$a_{n,0} = \sum_{i=1}^{r_0} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} x_i^{n-j}}{h_i} - \sum_{i=1}^{r_0} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \sum_{Q \in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})}{h_i}$$

$$
\begin{aligned}
&\qquad\qquad + \sum_{i=1}^{r_0} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}\binom{n}{j} \cdot \sum\limits_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})}{h_i} \\
&= \sum_{i=1}^{r_0} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}\binom{n}{j} \cdot \left(x_i^{n-j} - \sum\limits_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right)}{h_i} \\
&\qquad\qquad + \sum_{i=1}^{r_0} \frac{\sum\limits_{j=0}^{s_i-1} \gamma_{i,j}\binom{n}{j} \cdot \sum\limits_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})}{h_i} && (545) \\
&= a_{n,0,0} + a_{n,0,1} && (546)
\end{aligned}
$$

where $a_{n,0,0}$ denotes the first summand and $a_{n,0,1}$ the second summand in (545) and the equalities hold by the following reasonings: The first equality holds because the first summand is already $a_{n,0}$ by (537) and because the second and the third summands are canceling each other out. The second equality holds by elementary arithmetics. The last equality holds by the choices of $a_{n,0,0}$ and $a_{n,0,1}$.

Seventh, for all $i = 1, \ldots, r_0$ and all the places $P \in M_{0,i} = \operatorname{Supp}(\operatorname{div}_0((x-1)x^{\omega_i}\chi_i)) = \operatorname{Supp}(\operatorname{div}_0((x-1)h_i))$ in (506), we estimate

$$
\begin{aligned}
& v_P\left(\sum_{j=0}^{s_i-1} \gamma_{i,j}\binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right)\right) \\
&\geq \min_{j=0,\ldots,s_i-1} v_P\left(\gamma_{i,j}\binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right)\right) \\
&\geq \min_{j=0,\ldots,s_i-1} v_P\left(x_i^{n-j} - \sum_{Q\in M_{0,i}} \varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right) \\
&\geq \min_{\substack{j=0,\ldots,s_i-1\\ Q\in M_{0,i}\setminus\{P\}}} \left\{ v_P\left(x_i^{n-j} - \varepsilon_P \cdot T_{P,b_{0,i}(P)}(x_i^{n-j}) \right), v_P\left(\varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right) \right\} \\
&\geq \min_{j=0,\ldots,s_i-1} \left\{ v_P\left(x_i^{n-j} - T_{P,b_{0,i}(P)}(x_i^{n-j}) + (1-\varepsilon_P) \cdot T_{P,b_{0,i}(P)}(x_i^{n-j}) \right), \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. v_P(h_i(x-1)) \right\} \\
&\geq \min_{j=0,\ldots,s_i-1} \left\{ v_P\left(x_i^{n-j} - T_{P,b_{0,i}(P)}(x_i^{n-j}) \right), v_P\left((1-\varepsilon_P) \cdot T_{P,b_{0,i}(P)}(x_i^{n-j}) \right), \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. v_P(h_i(x-1)) \right\} \\
&\geq v_P(h_i(x-1)) && (547)
\end{aligned}
$$

where the estimates hold by the following reasonings: The first, third and fifth estimates hold by the well known rule $v_P(a+b) \geq \min\{v_P(a), v_P(b)\}$. The second estimate holds because of the well known rule $v_P(a\cdot b) = v_P(a) + v_P(b)$ and because $\gamma_{i,j}\binom{n}{j}$ is contained in S_0 by (531) and therefore has no pole at $P \in \mathbb{P}_F \setminus \operatorname{Supp}(\operatorname{div}_\infty(x)) \supseteq M_{0,i} = \operatorname{Supp}(\operatorname{div}_0((x-1)h_i))$ where the inclusions hold since $h_i(x-1) \in S_0$. The fourth estimate holds because of elementary arithmetics, because of the equality

$$
v_P\left(\varepsilon_Q \cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right) = v_P(\varepsilon_Q) + v_P\left(T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \right),
$$

because of choice of ε_Q in (544) and because $T_{Q,b_{0,i}(Q)}(x_i^{n-j}) \in S_0$ by its definition in (542). The last estimate holds by the choice of ε_P in (544), by the definition of $b_{0,i}(P) = v_P((x-1)x^{\omega_i}\chi_i) - 1$ in (506) for all $P \in M_{0,i}$ and by the estimate in (543).

Consequently, we obtain

$$\begin{aligned}&v_P\left(\frac{\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\left(x_i^{n-j}-\sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})\right)}{h_i}\right)\\&=v_P\left(\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\left(x_i^{n-j}-\sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})\right)\right)-v_P(h_i)\\&\geq v_P(x-1)\end{aligned}\tag{548}$$

for all $i=1,\ldots,r_0$ and all $P\in\mathbb{P}_F\backslash\operatorname{Supp}(\operatorname{div}_\infty(x))$ where the equality is a well known rule and the estimate holds by the following reasonings: On the one hand, the estimate holds for $P\in\mathbb{P}_F\backslash(\operatorname{Supp}(\operatorname{div}_0(h_i(x-1)))\sqcup\operatorname{Supp}(\operatorname{div}_\infty(x)))$ because the v_P-argument of the first summand is an element in S_0 and because $v_P(h_i)=0=v_P(x-1)$ for these P. On the other hand, the estimate holds for $P\in\operatorname{Supp}(\operatorname{div}_0(h_i(x-1)))$ because of the estimate in (547).

Hence, by (525), we concluded that all summands of $a_{n,0,0}$ in (546) are contained in S_0 and are even S_0-multiples of $(x-1)$. Therefore, the same holds for the whole sum, i.e.

$$a_{n,0,0}=\sum_{i=1}^{r_0}\frac{\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\left(x_i^{n-j}-\sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})\right)}{h_i}\in(x-1)S_0\tag{549}$$

In particular, we obtain the equalities

$$\mathrm{N}'_e(a_{n,0,0})=\sigma_1(a_{n,0,0})=0\tag{550}$$

where the first equality holds because of the definition of the extension of N'_e on F in (530) and because $a_{n,0,0}$ is an element in S_0 by (549) and the second equality holds because $a_{n,0,0}$ is even an element in $(x-1)S_0$ and because the σ_1 is an extension morphism of K-algebras of the evaluation morphism Eval_1 by (528).

Eighth, let us consider the second summand in (546), i.e.

$$\begin{aligned}a_{n,0,1}&=\sum_{i=1}^{r_0}\frac{\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})}{h_i}\\&=\sum_{i=1}^{r_0}\sum_{j=0}^{s_i-1}\sum_{Q\in M_{0,i}}\binom{n}{j}\cdot\frac{\gamma_{i,j}\cdot\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})}{h_i}\\&=\sum_{i=1}^{r_0}\sum_{j=0}^{s_i-1}\sum_{Q\in M_{0,i}}\sum_{k=0}^{b_{0,i}(Q)}\binom{n}{j}\cdot\binom{n-j}{k}\cdot x_i(Q)^{n-j-k}\cdot\frac{\gamma_{i,j}\cdot\varepsilon_Q\cdot(x_i-x_i(Q))^k}{h_i}\end{aligned}\tag{551}$$

where the first equality holds by the choice of $a_{n,0,1}$ in (546), the second equality holds by elementary arithmetics and resorting the sums and the third equality holds by the definition of $T_{Q,b_{0,i}(Q)}(x_i^{n-j})$ in (542) and (540) and because of the equalities

$$\frac{(t^{n-j})^{(k)}}{k!}=\frac{(n-j)!\cdot t^{n-j-k}}{(n-j-k)!\cdot k!}=\binom{n-j}{k}\cdot t^{n-j-k}.$$

Consequently, since N'_e is a morphism K-vector spaces, the equality in (551) yields

$$\mathrm{N}'_e(a_{n,0,1}) = \sum_{i=1}^{r_0} \sum_{j=0}^{s_i-1} \sum_{Q\in M_{0,i}} \sum_{k=0}^{b_{0,i}(Q)} \binom{n}{j}\binom{n-j}{k} x_i(Q)^{n-j-k} \cdot \mathrm{N}'_e\left(\frac{\gamma_{i,j}\cdot\varepsilon_Q\cdot(x_i - x_i(Q))^k}{h_i}\right). \tag{552}$$

Ninth, we will deal with the second summand $a_{n,\infty}$ of the sum $a_n = a_{n,0} + a_{n,\infty}$ in (539). Here, we basically do the steps six to eight but for S_∞ instead of S_0 and, therefore, let us just list the outcome: For all $i = r_0+1,\ldots,r$ and all $Q \in M_{0,i} = M_{0,\infty} = \mathrm{Supp}(\mathrm{div}_0(y(y-1)))$ in (506), choose elements $\varepsilon_Q \in S_\infty$

$$v_Q(\varepsilon_Q - 1) = v_Q(h_0(y-1)) \geq v_Q(h_i(y-1)) \geq 0,$$
$$v_P(\varepsilon_Q) = v_P(h_0(y-1)) \geq v_P(h_i(y-1)) \geq 0$$

for all $P \in M_{0,\infty}\backslash\{Q\}$. Next, let

$$a_{n,\infty,0} := \sum_{i=r_0+1}^{r} \frac{\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\left(x_i^{n-j} - \sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}(Q)}(x_i^{n-j})\right)}{h_i}$$

and

$$a_{n,\infty,1} := \sum_{i=r_0+1}^{r} \frac{\sum_{j=0}^{s_i-1}\gamma_{i,j}\binom{n}{j}\cdot\sum_{Q\in M_{0,i}}\varepsilon_Q\cdot T_{Q,b_{0,i}}(x_i^{n-j})}{h_i}.$$

Then we get the equality

$$a_{n,\infty} = a_{n,\infty,0} + a_{n,\infty,1} \tag{553}$$

On the one hand, we analogously obtain $a_{n,\infty,0} \in (y-1)S_\infty = (\frac{e}{x}-1)S_\infty$ and, hence,

$$\mathrm{N}'_e(a_{n,\infty,0}) = \sigma_e(a_{n,\infty,0}) = 0. \tag{554}$$

On the other hand, we also analogously obtain

$$\mathrm{N}'_e(a_{n,\infty,1}) = \sum_{i=r_0+1}^{r} \sum_{j=0}^{s_i-1} \sum_{Q\in M_{0,i}} \sum_{k=0}^{b_{0,i}(Q)} \binom{n}{j}\binom{n-j}{k} x_i(Q)^{n-j-k} \mathrm{N}'_e\left(\frac{\gamma_{i,j}\cdot\varepsilon_Q\cdot(x_i - x_i(Q))^k}{h_i}\right). \tag{555}$$

Tenth, we collect everything which we have established so far. For that, let

$$\kappa_{i,j,Q,k} := \mathrm{N}'_e\left(\frac{\gamma_{i,j}\cdot\varepsilon_Q\cdot(x_i - x_i(Q))^k}{h_i}\right) \in K \tag{556}$$

for all $i = 1,\ldots,r$, all $j = 0,\ldots,s_i - 1$, all $Q \in M_{0,i}$ and all $k = 0,\ldots,b_{0,i}(Q)$.

Then combining the definition of $\mathrm{N}''_e(A^n)$ in Definition 241, (514), (539), (546), (550), (552), (553), (554) and (555) yields

$$\begin{aligned}\mathrm{N}''_e(A^n) &= \mathrm{N}'_e(a_n) = \mathrm{N}'_e(a_{n,0,1}) + \mathrm{N}'_e(a_{n,\infty,1})\\ &= \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \sum_{Q\in M_{0,i}} \sum_{k=0}^{b_{0,i}(Q)} \binom{n}{j}\cdot\binom{n-j}{k}\cdot x_i(Q)^{n-j-k}\cdot\kappa_{i,j,Q,k}.\end{aligned} \tag{557}$$

Furthermore, let us resort the sum in (557) for the appearing $\lambda = x_i(Q)$ and since we are not interested in the constants in K, let us also just write const for the appearing constants. Then we deduce the equalities

$$\begin{aligned} \mathrm{N}_e''(A^n) &= \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \sum_{\lambda\in\Lambda_{0,i}} \sum_{k=0}^{b_{0,i}(\lambda)} \binom{n}{j} \cdot \binom{n-j}{k} \cdot \lambda^n \cdot \mathrm{const}_{i,j,\lambda,k} \\ &= \sum_{\lambda\in\Lambda_0} \sum_{k=0}^{b_0(\lambda)} \mathrm{const}_{\lambda,k} \cdot n^k \cdot \lambda^n \end{aligned} \tag{558}$$

for all $n \geq n_0 := \max\{s_i - 1 + b_{0,i}(\lambda) : i = 1,\ldots,r \text{ and } \lambda \in \Lambda_{0,i}\}$ where the equalities hold by the following reasonings: The first equality holds because of the equality in (557), because $\Lambda_{0,i}$ precisely consists of the elements $x_i(Q)$ with $Q \in M_{0,i}$ and because $b_{0,i}(\lambda)$ is defined as the maximum of all $b_{0,i}(Q)$ with $x_i(Q) = \lambda$ in (506). The second equality holds because Λ_0 is defined as the union of the $\Lambda_{0,i}$, because the product $\binom{n}{j} \cdot \binom{n-j}{k}$ is a polynomial in n for $n \geq n_0$ with degree $j + k \leq s_i - 1 + b_{0,i}(\lambda)$ for all $i = 1,\ldots,r$, all $j = 0,\ldots,s_i - 1$, all $\lambda \in \Lambda_{0,i}$ and all $k = 0,\ldots,b_{0,i}(\lambda)$ and because $b_0(\lambda)$ is defined as the maximum of these upper bounds $s_i - 1 + b_{0,i}(\lambda)$ in (508).

But, (558) is now of the form

$$\mathrm{N}_e''(A^n) = \sum_{\lambda\in\Lambda_0} f_\lambda(n) \cdot \lambda^n \tag{559}$$

for all $n \geq n_0$ and some polynomials $f_\lambda(n) \in K[n]$ with

$$\deg(f_\lambda(n)) \leq b_0(\lambda) \tag{560}$$

for all $\lambda \in \Lambda_0$.

Eleventh, in the steps three to ten, we can interchange the roles of x and y and obtain another presentation of $\mathrm{N}_e''(A^n)$ which is similar to the one in (560). Let us just briefly go through the steps:

In the third step, we set S_∞ to the integral closure of $K[y] = K[x^{-1}]$ and S_0 to the integral closure of $K[x]_{(x)} \cap K[x]_{(x-1)}$ in F. Then we have $x_1,\ldots,x_{r_0} \in S_0$ and $x_{r_0+1},\ldots,x_r \in S_\infty$ and the equalities

$$S_\infty = \{z \in F : v_Q(z) \geq 0 \text{ for all } Q \in \mathbb{P}_F \setminus \mathrm{Supp}(\mathrm{div}_\infty(y))\}$$

and

$$S_0 = \{z \in F : v_Q(z) \geq 0 \text{ for all } Q \in \mathrm{Supp}(\mathrm{div}_0(x)) \cup \mathrm{Supp}(\mathrm{div}_0(x-1))\}.$$

Moreover, the equality in (527) still holds and since we have $K[x] \subseteq S_0$ and $K[x^{-1}] \subseteq S_\infty$, we can again extend $\mathrm{N}_e' : K[x,x^{-1}] \to K$ to another morphism

$$\mathrm{N}_e' : F \to K \text{ via } z = z_0 + z_\infty + u \mapsto \sigma_1(z_0) + \sigma_e(z_\infty)$$

of K-vector spaces where $\sigma_1 : S_0 \to K$ is an extension morphism of K-algebras of Eval_1 and $\sigma_e : S_\infty \to K$ of Eval_e.

The fourth and fifth step do not need any modifications. For the sixth step, we define $h_\infty := \prod_{i=r_0+1}^{r} h_i$ and choose elements $\varepsilon_Q \in F$ for all $Q \in \mathrm{Supp}(\mathrm{div}_0(h_\infty(y-1)))$ such that

$$\begin{aligned} &v_Q(\varepsilon_Q - 1) = v_Q(h_\infty(y-1)) \geq v_Q(h_i(y-1)) \geq 0, \quad v_R(\varepsilon_Q) \geq 0, \\ &v_P(\varepsilon_Q) = v_P(h_\infty(x-1)) \geq v_P(h_i(x-1)) \geq 0 \end{aligned}$$

for all $i = r_0+1, \dots, r$, all $P \in \mathrm{Supp}(\mathrm{div}_0(h_\infty(y-1))) \setminus \{Q\}$ and all $R \in \mathbb{P}_F \setminus \mathrm{Supp}(\mathrm{div}_\infty(y))$. Then we obtain the equality $a_{n,\infty} = a_{n,\infty,0} + a_{n,\infty,1}$ for

$$a_{n,\infty,0} := \sum_{i=r_0+1}^{r} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}(Q)}(x_i^{n-j}) \right)}{h_i}$$

and

$$a_{n,\infty,1} := \sum_{i=r_0+1}^{r} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}(Q)}(x_i^{n-j})}{h_i}.$$

In the seventh step, we analogously estimate

$$v_P \left(\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}(Q)}(x_i^{n-j}) \right) \right) \geq v_P(h_i(y-1))$$

for all $i = r_0 + 1, \dots, r$ and all $P \in M_{\infty,i} = \mathrm{Supp}(\mathrm{div}_0((y-1)y^{\omega_i}\chi_i)) = \mathrm{Supp}(\mathrm{div}_0((y-1)h_i))$ in (507) and then we estimate

$$v_P \left(\frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}(Q)}(x_i^{n-j}) \right)}{h_i} \right) \geq v_P(y-1)$$

for all $i = r_0+1, \dots, r$ and all $P \in \mathbb{P}_F \setminus \mathrm{Supp}(\mathrm{div}_\infty(y))$. Consequently, this yields $a_{n,\infty,0} \in (y-1)S_\infty$ and $\mathrm{N}'_e(a_{n,\infty,0}) = \sigma_e(a_{n,\infty,0}) = 0$.

The eight step work completely analogous and, here, we get the equality

$$\mathrm{N}'_e(a_{n,\infty,1}) = \sum_{i=r_0+1}^{r} \sum_{j=0}^{s_i-1} \sum_{Q \in M_{\infty,i}} \sum_{k=0}^{b_{\infty,i}(Q)} \binom{n}{j} \binom{n-j}{k} x_i(Q)^{n-j-k} \mathrm{N}'_e \left(\frac{\gamma_{i,j} \varepsilon_Q (x_i - x_i(Q))^k}{h_i} \right).$$

In the ninth step, for all $i = 1, \dots, r_0$ and all $Q \in M_{\infty,i} = M_{\infty,0} = \mathrm{Supp}(\mathrm{div}_0(x(x-1)))$ in (507), we choose elements $\varepsilon_Q \in S_0$

$$\begin{aligned} v_Q(\varepsilon_Q - 1) &= v_Q(h_0(x-1)) \geq v_Q(h_i(x-1)) \geq 0, \\ v_P(\varepsilon_Q) &= v_P(h_0(x-1)) \geq v_P(h_i(x-1)) \geq 0 \end{aligned}$$

for all $P \in M_{\infty,0} \setminus \{Q\}$. Next, we define

$$a_{n,0,0} := \sum_{i=1}^{r_0} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \left(x_i^{n-j} - \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}(Q)}(x_i^{n-j}) \right)}{h_i}$$

and

$$a_{n,0,1} := \sum_{i=1}^{r_0} \frac{\sum_{j=0}^{s_i-1} \gamma_{i,j} \binom{n}{j} \cdot \sum_{Q \in M_{\infty,i}} \varepsilon_Q \cdot T_{Q,b_{\infty,i}}(x_i^{n-j})}{h_i}.$$

Then we get the equality $a_{n,0} = a_{n,0,0} + a_{n,0,1}$, conclude $a_{n,0,0} \in (x-1)S_0$ and $\mathrm{N}'_e(a_{n,\infty,0}) = \sigma_e(a_{n,\infty,0}) = 0$ and

$$\mathrm{N}'_e(a_{n,0,1}) = \sum_{i=1}^{r_0} \sum_{j=0}^{s_i-1} \sum_{Q \in M_{\infty,i}} \sum_{k=0}^{b_{0,i}(Q)} \binom{n}{j} \binom{n-j}{k} x_i(Q)^{n-j-k} \mathrm{N}'_e \left(\frac{\gamma_{i,j} \cdot \varepsilon_Q \cdot (x_i - x_i(Q))^k}{h_i} \right).$$

Second to last, with the sets and numbers in (507) and (508), the tenth step provides a presentation

$$\mathrm{N}''_e(A^n) = \sum_{\lambda \in \Lambda_\infty} f_\lambda(n) \cdot \lambda^n \tag{561}$$

for all $n \geq n_\infty := \max\{s_i - 1 + b_{\infty,i}(\lambda) : i = 1, \ldots, r \text{ and } \lambda \in \Lambda_{\infty,i}\}$ and some polynomials $f_\lambda(n) \in K[n]$ with

$$\deg(f_\lambda(n)) \leq b_\infty(\lambda) \tag{562}$$

for all $\lambda \in \Lambda_\infty$.

Finally, combining the presentations in (559) and (561) and the degree bounds in (560) and (562) supplies the desired presentation in Theorem 243 because of the definitions of $\Lambda(A) := \Lambda_0 \cap \Lambda_\infty$ and $b(A, \lambda) := \min\{b_0(\lambda), b_\infty(\lambda)\}$(508) and because the family of elements $n^k \lambda^n$ for all $n \geq \max(n_0, n_\infty)$ are K-linearly independent.

Moreover, since the summands in the final formula vanish for all $\lambda \notin \lambda(A)$, we can even replace $\Lambda_{j,i}$ with $\Lambda_{j,i} \cap \Lambda(A)$ in the definitions of n_0 after (558) and n_∞ after (561). Doing so yields that the desired final formula even holds for all $n \geq max(n_0, n_\infty) = c(A)$, which is the last missing piece of Theorem 243. ☐

8.2.4 An Effective Approach to Compute Genus Formulas

Purpose of this subsection. In this subsection, we will first elaborate in Remark 247 that there is reason to suspect that the set $\Lambda(A)$ in Theorem 243 can actually be chosen smaller.

Second, we will discuss possible approaches for effectively applying Corollary 246 to compute genus formulas. By that, we will come up with our preferred final approach (A4). This approach will then be automated in a first naive implementation of (A4) in Subsection 8.3.2.

Third and finally, we will also discuss a first way to generalize Corollary 246.

Remark 247. *The proof of Theorem 243 is basically an adhoc approach to apply the ideas in Example 235 for restricting the candidates for λ^n in the desired formula*

$$\mathrm{N}'_e(vA^n v^t) = \sum_\lambda f_\lambda(n) \lambda^n \tag{563}$$

to a finite set $\Lambda(A)$ and for providing degree bounds $b(A, \lambda)$ for all $\lambda \in \Lambda(A)$. But, except for the fact that $\Lambda(A)$ is defined as the the intersection $\Lambda_i \cap \Lambda_\infty$, the proof completely neglects possible cancellations of the terms with λ^n.

Therefore, it is not surprising that, for all considered examples, the appearing λ in the final formula were even contained in a smaller subset, namely

$$\begin{aligned} \Lambda' := & \bigcup_{i=1}^{r_0} \left\{ x_i(Q) : Q \in \mathrm{Supp}(\mathrm{div}_0(\prod_{\nu=r_0+1}^{r} (x_i - x_\nu))) \right\} \\ & \cup \bigcup_{i=r_0+1}^{r} \left\{ x_i(Q) : Q \in \mathrm{Supp}(\mathrm{div}_0(\prod_{\nu=1}^{r_0} (x_i - x_\nu))) \right\} \subseteq \Lambda(A). \end{aligned} \tag{564}$$

Moreover, the following further observations also indicate that this could hold more generally:

First, it can be explicitly calculated that the terms with $x_i(Q)^n$ and $x_k(Q)^n$ are canceling each other out for all distinct $i, k \in \{1, \dots, r_0\}$ if $x_i(Q) = x_k(Q) \notin \{x_\nu(Q) : \nu \in \{1, \dots, r_0\} \backslash \{i, k\}\}$ (resp. $i, k \in \{r_0 + 1, \dots, r\}$ if $x_i(Q) = x_k(Q) \notin \{x_\nu(Q) : \nu \in \{r_0 + 1, \dots, r\} \backslash \{i, k\}\}$). Otherwise, e.g. if $x_i(Q) = x_k(Q) = x_l(Q)$ for pairwise distinct $i, k, l \in \{1, \dots, r_0\}$, it becomes inevitable to extract more information from the numerators $\gamma_{i,j}$, $\gamma_{k,j}$ and $\gamma_{l,j}$ in the identity in (522) to maybe then also obtain the desired cancellations of these terms.

Second, as we already pointed out, $\Lambda(A)$ is defined as the intersection of the two sets Λ_0 and Λ_∞ in (508). This identity already excludes all the $x_i(Q)$ which do not appear in both sets Λ_0 and Λ_∞. In particular, notice that this does not effect Λ' as Λ' is a subset of of both sets by their definitions.

Third, in some simpler cases, it can also be calculated that the terms with $x_i(Q)^n$ do no appear in the formula for all $Q \in \mathrm{Supp}(\mathrm{div}(x))$ and all $i = 1, \dots, r$.

Finally, we could not find another reason besides the mentioned examples for why the terms with $x_i(Q)^n$ for $Q \in \mathrm{Supp}(\mathrm{div}_0(x-1))$ and $i = 1, \dots, r_0$ (resp. $Q \in \mathrm{Supp}(\mathrm{div}_0(x-e))$ and $i = r_0 + 1, \dots, r$) should not appear in the formula in (563).

Possible ways to compute a genus formula. Let us discuss some possible ways to compute the formula

$$\mathrm{N}''_e(A^n) = \sum_{\lambda \in \Lambda(A)} f_\lambda(n) \cdot \lambda^n \tag{565}$$

for all $n \geq c(A)$ in Theorem 243.

(A1) First, a naive approach is to just use the constructive proof of Theorem 243, i.e. to just compute the finite subset $\Lambda(A) \subset \overline{\mathbb{Q}}$, the degree bounds $b(A, \lambda)$ for all $\lambda \in \Lambda(A)$ and the values of $\mathrm{N}''_e(A^n)$ from $c(A)$ up to $n_0 = c(A) + \sum_{\lambda \in \Lambda(A)}(b(A, \lambda) + 1)$. This will then provide a linear equation system (LES) with a confluent Vandermonde matrix. Hence the coefficients of the polynomials $f_\lambda(n)$ are provided by the unique solution of this LES.

However, as we discussed in Remark 247, the set $\Lambda(A)$ seems to be unnecessarily large. This can especially become problematic if also the degree bounds $b(A, \lambda)$ are large. Although most of the bounds $b(A, x_i(Q))$ appear to be reasonably small, there are exceptions: For all $i = 1, \dots, r_0$ and all $Q \in \mathrm{Supp}(\mathrm{div}_0(x))$ (resp. all $i = r_0 + 1, \dots, r$ and all $Q \in \mathrm{Supp}(\mathrm{div}_\infty(x))$), the bounds $b(A, x_i(Q))$ are at least ω''_0 in (502) (resp. ω''_∞ in (503)) which is increasing with s.

(A2) Second, to avoid these unnecessarily large degree bounds, we can just concretely compute the bothersome numerators $\gamma'_{i,j}$ in (521) for the concrete matrix A and specify the problematic degree bounds by that.

(A3) Third, we can use the construction which is provided in the proof of Theorem 243. Here the only problematic step becomes to compute the concrete values of

$$\kappa_{i,j,Q,k} = \mathrm{N}'_e\left(\frac{\gamma_{i,j} \cdot \varepsilon_Q \cdot (x_i - x_i(Q))^k}{h_i}\right)$$

in (556). In the proof, we could neglect these constants but now we have to find a presentation of the argument in the form $z_0 + z_\infty + u \in F = S_0 + S_\infty + U$ in (530).

(A4) Fourth, finally and preferably, we can consider the sub LES in the first naive approach with the subset $\Lambda' \subseteq \Lambda(A)$ in (564) and check whether the solution of this sub LES is also a solution of the original LES from $c(A)$ up to n_0. If $\Lambda(A)$ and the problematic degree bounds are indeed inflated as we suspected in Remark 247, then the solution of the sub LES will also be the solution for the LES. Otherwise we have a counterexample for our suspicions and can dynamically add elements from $\Lambda(A)$ to Λ', increase the degree bounds, again solve the new larger sub LES and again check the solution for the original LES from $c(A)$ up to $n_0 = c(A) + \sum_{\lambda \in \Lambda(A)}(b(A,\lambda) + 1)$. Eventually, this will supply the desired solution of the LES and thereby the coefficients of the polynomials $f_\lambda(n)$ in (565).

Generality of Corollary 246. We want to point out that, although the assumptions in Corollary 246 are quite restrictive in theory, all tame recursive towers $\mathcal{F}$ in the literature known to the author have some truncation to which Corollary 246 can be applied. This includes all tame recursive towers in [MW05], [Sti08] and [BR20]. In fact, all these tame recursive towers have a finite ramification separating power subgraph.

In fact, it even comes out that, for all the finite weakly connected components $\overline{\Gamma}$ of $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ and its corresponding $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix A, we have $F = \overline{\mathbb{Q}}(x, x_1, \dots, x_r) = E = \overline{\mathbb{Q}}(x)$ in Definition 242.

The next more general case. As we pointed out above, there is no urge to improve the generality of Theorem 243 yet. It already overpowers all existing examples. In particular, there could even be a reason why (good) recursive towers must all be of this simple form which we described in the last paragraph. This would make any generalizing attempt redundant. However, the author is not convinced that such a reason must exist and rather suspects that the ways by which the recursive towers in the current literature were found or constructed are biased.

Out of this suspicion, let us at least briefly discuss how a genus formula can be computed for the following next more general case of Corollary 246: Let $\mathcal{F} = (F_\nu)_\nu$ be a tame recursive tower over a field k and let $\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu = \overline{k} \cdot \mathcal{F}$ be the geometric tower of $\mathcal{F}$. Suppose that all weakly connected components $\Gamma_1, \dots, \Gamma_r$ of the finite ramification subgraph $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ are separating. Hence we only dropped the assumption that Γ_i has to be a power subgraph and will still be able to compute a genus formula. The only drawback will be that we basically have to apply the proof of Theorem 243 iteratively and it therefore becomes cumbersome (but apparently not impossible) to provide a set $\Lambda(A)$ and degree bounds $b(A, \lambda)$ as in Theorem 243.

Now, let Γ be any of the weakly connected components of $\Gamma_{\overline{\mathcal{F}}}^{\mathrm{ram}}$ and let $q_1, \dots, q_s \in \mathbb{P}$ be pairwise distinct with $s \in \mathbb{N}_0$ minimal such that the supports of the prime decompositions of all ramification indices in Γ are contained in $\{q_1, \dots, q_s\}$.

Since Γ is separating and after maybe changing the enumeration of the primes $q_1, \dots, q_s$, we can partition $\{q_1, \dots, q_s\}$ into subsets

$$M_1 := \{q_1, \dots q_{r_1}\},\ M_2 := \{q_{r_1+1}, \dots, q_{r_2}\}, \dots,\ M_t := \{q_{r_{t-1}+1}, \dots, q_{r_t}\}$$

such that the M_j-parts of the ramification indices in Γ are powers of some fixed natural number e_j where the support of the prime decomposition of e_j is equal to M_j. Notice that this is at least always possible by choosing $t = s$, $M_j = \{q_j\}$ and $e_j = q_j$ for all $j = 1, \dots, s$.

By Lemma 240, any of these sets M_j comes with an M_j-separating enumeration of the vertices in Γ. Moreover, it is not difficult to see that we can even find one enumeration v of the vertices in Γ which is M_j-separating for all $j = 1, \dots, t$ but with possibly differing separating indices $m_{j,\infty}$. Consequently, Lemma 240 implies that, for all $j = 1, \dots, t$, the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix A for v is of the form

$$A = \begin{pmatrix} A_{j,\infty} & B_j \\ 0 & A_{j,0} \end{pmatrix} \in R_{e_j}[x_{e_j}, x_{e_j}^{-1}]^{m \times m} \subset K_{e_j}[x_{e_j}, x_{e_j}^{-1}]^{m \times m}$$

with $A_{j,\infty} \in K_{e_j}[x_{e_j}^{-1}]^{m_{j,\infty} \times m_{j,\infty}}$ and $A_{j,0} \in K_{e_j}[x_{e_j}]^{m_{j,0} \times m_{j,0}}$ where $m_{j,0} := m - m_{j,\infty}$.

Then the eigenvalues $x_1, \dots, x_\rho$ of A can be enumerated in a way such that, for all $j = 1, \dots, t$, there are indices $\rho_{j,0}$ satisfying that $x_1, \dots, x_{\rho_{j,0}}$ are the eigenvalues of $A_{j,0}$ and $x_{\rho_{j,0}+1}, \dots, x_\rho$ are the eigenvalues of $A_{j,\infty}$.

Finally, we can first compute the presentation of $vA^n v^t$ in the form

$$a_n := v \cdot A^n \cdot v^t = \sum_{i=1}^{r} \sum_{j=0}^{s_i - 1} c_{i,j} \binom{n}{j} x_i^{n-j}$$

for all $n \geq c(A)$ as in (514) and then iteratively compute the N'_{e_j}-value of $(\mathrm{N}'_{e_{j-1}} \circ \dots \circ \mathrm{N}'_{e_1})(a_n)$ in the same way as in the proof of Theorem 243. Of course, again as in (A4), some steps in the proof can be replaced by solving linear equation systems for suitable confluent Vandermonde matrices.

8.3 A First Naive Implementation to Compute Genus Formulas

Summary of the results of this section In this section, we will provide a first naive implementation of the approach in (A4) (see Subsection 8.3.2). This implementation will work on all tame recursive towers in the literature which are known to the author, including all tame recursive towers in [MW05], [Sti08], [BR20]. Consequently, in Examples 250, we will also list genus formulas for some representative tame recursive towers from the literature.

8.3.1 Three Final Challenges for an Implementation.

Purpose of this subsection. We will have to face three final challenges before we can provide the first naive implementation of our approach in (A4) in Subsection 8.3.2. In combination with Corollary 246, this will enable us to compute genus formulas for all tame recursive towers $\mathcal{F}$ in the literature which are known to the author, including all tame recursive towers in [MW05], [Sti08], [BR20].

The first challenge will be that, for some tame recursive towers $\mathcal{F} = (F_\nu)_\nu$ in the literature, Corollary 246 is only applicable for some level l truncation $\mathrm{Trun}_{\geq l}(\mathcal{F}) = (F_{l+\nu})_\nu$ of $\mathcal{F}$.

The second challenge will be that the proof of Corollary 246 applies Theorem 243 to the infinite constant field extension $\overline{k} \cdot \mathcal{F}$ of $\mathcal{F}$ where $\overline{k}$ is an algebraic closure of the constant field k of $\mathcal{F}$.

The third challenge will be to compute the upper bound $n_0 := c(A) + \sum_{\lambda \in \Lambda(A)} (b(A, \lambda) + 1)$ in (A4) which is the level up to which we need to check our computed formula.

A solution for the first two challenges. The following Lemma 248 will solve the first two challenges by translating the genus formula in Corollary 248 into a formula which is expressed in terms of the N''_e-values of the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices $A_{i,j}$ of the weakly connected components $\Gamma'_{i,j}$ of the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}'_{\geq l}}$ of $\mathcal{F}'_{\geq l} = \mathrm{Trun}_{\geq l}(k' \cdot \mathcal{F})$. Here k'/k is a finite extension which can easily be determined for all tame recursive towers in the literature which the author considered.

Lemma 248. *Let $\mathcal{F} = (F_\nu)_\nu$ be a recursive tower over a field k which is defined by the pair (σ, F_0). Let $\Gamma_1, \ldots, \Gamma_s$ be the weakly connected components of the finite ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$. On the one hand, for all $i = 1, \ldots, \rho$, suppose that Γ_i is a separating e_i-power subgraph for some $e_i \in \mathbb{N}$. On the other hand, suppose $\Gamma_{\rho+1}, \ldots, \Gamma_s$ only have circles with balanced ramification indices and that there is some $l \in \mathbb{N}_0$ such that the level l truncation $\mathrm{Trun}_{\geq l}(\Gamma_i)$ has no ramified edges for all $i = \rho + 1, \ldots, s$*

Moreover, let $\mathcal{F}' := k' \cdot \mathcal{F} = (F'_\nu)_\nu$ be the constant field extension of $\mathcal{F}$ for some algebraic extension field k' of k which contains all residue fields of the edges $Q \in \bigcup_{i=1}^{\rho} E(\Gamma_i) \subseteq \mathbb{P}_{F_1}$, and let $\Gamma'_{i,1}, \ldots, \Gamma'_{i,r_i}$ be the weakly connected components of $k' \cdot \Gamma_i$ for all $i = 1, \ldots, \rho$. Then the following hold:

(i) The $\Gamma'_{i,j}$ are exactly the weakly connected components of $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ which have circles with unbalanced ramification indices and the level l truncations of all other weakly connected components of $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$ contain no ramified edges.

Moreover, for all $i = 1, \ldots, \rho$ and all $j = 1, \ldots, r_i$, $\Gamma'_{i,j}$ is even a finite separating e_i-power subgraph of $\Gamma_{\mathcal{F}'}^{\mathrm{rat}}$.

In particular, for all $i = 1, \ldots, \rho$ and all $j = 1, \ldots, r_i$, $\Gamma'_{i,j}$, we can choose some e_i-separating enumeration $v_{i,j}$ of the $m_{i,j}$ vertices in $\Gamma'_{i,j}$ such that the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrix $A_{i,j}$ of $\Gamma'_{i,j}$ for $v_{i,j}$ is of the form

$$A_{i,j} = \begin{pmatrix} A_{i,j,\infty} & B_{i,j} \\ 0 & A_{i,j,0} \end{pmatrix} \in \overline{\mathbb{Q}}[x_{e_i}, x_{e_i}^{-1}]^{m_{i,j} \times m_{i,j}}$$

with $A_{i,j,0} \in \overline{\mathbb{Q}}[x_{e_i}]^{m_{i,i,0} \times m_{i,j,0}}$ and $A_{i,j,\infty} \in \overline{\mathbb{Q}}[x_{e_i}^{-1}]^{m_{i,j,\infty} \times m_{i,j,\infty}}$ with $m_{i,j,0} + m_{i,j,\infty} = m_{i,j}$.

(ii) For all $n \geq l$, we have the identity

$$g(F_n) = \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \mathrm{N}''_{e_i}(A_{i,j}^{l})\right) d^{n-l} - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \mathrm{N}''_{e_i}(A_{i,j}^{n})\right).$$

Proof. For the 'main'-part in (i): First, we notice that since Γ_i is a weakly connected component of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ by assertion, Lemma 124 and Lemma 120(iii) supply that any weakly connected component $\Gamma'_{i,j}$ of $k' \cdot \Gamma_i$ is also a weakly connected component of $\Gamma_{\mathcal{F}'}^{\mathrm{ram}}$.

On the one hand, the assertion that Γ_i is a separating weakly connected component of $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ implies that Γ_i must also have circles with unbalanced ramification indices for all $i = 1, \ldots, \rho$. By this and by Lemma 120(v), we conclude that the weakly connected components $\Gamma'_{i,j}$ of $k' \cdot \Gamma_i$ indeed also have circles with unbalanced ramification indices for all $j = 1, \ldots, r_i$.

On the other hand, any other weakly connected component Γ' of $\Gamma_{\mathcal{F}'}^{\mathrm{ram}} = k' \cdot \Gamma_{\mathcal{F}}^{\mathrm{ram}}$ must be a weakly connected component of $k' \cdot \Gamma_i$ for some $i = \rho + 1, \ldots, s$. But here Lemma 120(v) and the assertion that Γ_i only has circles with unbalanced ramification indices also supplies that Γ' also only has circles with balanced ramification indices.

Finally, for all $i = \rho+1, \ldots, s$, the assertion that $\mathrm{Trun}_{\geq l}(\Gamma_i)$ contains no ramified edges implies that the same holds for $k' \cdot \mathrm{Trun}_{\geq l}(\Gamma_i) = \mathrm{Trun}_{\geq l}(k' \cdot \Gamma_i)$ and, thus, also for the level l truncations of the weakly connected components of $k' \cdot \Gamma_i$.

Hence, we established the 'main'-part in (i).

For the 'moreover'- and 'in particular'-parts in (i): First, we notice that Lemma 120(i) provides that $k' \cdot \Gamma_i$ and, thus, $\Gamma'_{i,j}$ is finite.

Second, we notice that the assertion that k' contains all residue fields of the edges in Γ_i implies that $k' \cdot \Gamma_i$ only contains rational vertices and edges. Consequently, any weakly connected component $\Gamma'_{i,j}$ of $k' \cdot \Gamma_i$ is also a subgraph of $\Gamma^{\mathrm{rat}}_{\mathcal{F}'}$ by the definition of the rational subgraph in Definition 88(i).

Third, Lemma 109 and Lemma 105(i) supply that

$$\begin{aligned}&\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}} \text{ restricts to an epimorphism } k' \cdot \Gamma_i \to \Gamma_i \text{ such that}\\ &\text{the ramification indices are invariant under the action of } \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}.\end{aligned} \tag{566}$$

Hence, the assertion that all ramification indices of Γ_i are powers of e_i implies that the same holds for all weakly connected components of $\Gamma_{i,j}$ of $k' \cdot \Gamma_i$.

Fourth, from (566) and from the definition of separating subgraphs in Definition 229, it also immediately follows that any path $\mathcal{P}$ which goes from Q'_1 to Q'_2 for some critical pair (Q'_1, Q'_2) of edges in $k' \cdot \Gamma_i$ also provides a path $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(\mathcal{P})$ from $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q'_1)$ to $\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q'_2)$ for some critical pair $(\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q'_1), \pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}(Q'_2))$ of edges in Γ_i. Therefore, $k' \cdot \Gamma_i$ and all its weakly connected components $\Gamma_{i,j}$ must also be separating. Hence, we established the 'moreover'-part in (i).

Finally, the 'in particular'-part follows from combining the 'moreover'-part and the 'in particular-part in Lemma 240.

For (ii): Let $\overline{\mathcal{F}} = \overline{k} \cdot \mathcal{F} = (\overline{F}_\nu)_\nu$ be the geometric tower of $\mathcal{F}$, let $\overline{\mathcal{F}}_{\geq l} = \mathrm{Trun}_{\geq l}(\overline{\mathcal{F}}) = (\overline{F}_{l+\nu})_\nu$ be its level l truncation and let $\overline{\Gamma}_{i,j} := \overline{k} \cdot \Gamma'_{i,j}$ for all $i = 1, \ldots, \rho$ and all $j = 1, \ldots, r_i$.

First, by the 'moreover'-part in Lemma 248(i), we have that $\Gamma'_{i,j}$ is a subgraph of $\Gamma^{\mathrm{rat}}_{\mathcal{F}'}$. Then Lemma 109 and Lemma 122 together imply that

$$\pi_{\Gamma_{\overline{\mathcal{F}}}/\Gamma_{\mathcal{F}'}} \text{ restricts to an isomorphism } \overline{\Gamma}_{i,j} \to \Gamma'_{i,j}. \tag{567}$$

Consequently, because of this and because $\tilde{N}$-value only depends on the ramification indices in the paths by Definition 50, we even obtain the equality

$$\tilde{N}(\overline{F}_n, V(\overline{\Gamma}_{i,j})) = \tilde{N}(F'_n, V(\Gamma'_{i,j})) \tag{568}$$

for all $n \in \mathbb{N}_0$.

Second, because the 'main'-part of Lemma 248(i) provides that $\Gamma'_{i,j}$ contains circles with unbalanced ramification indices and because of Lemma 136, we deduce that $\mathrm{Trun}_{\geq l}(\Gamma'_{i,j})$ still contains ramified edges.

Moreover, the 'main'-part of Lemma 248(i) also provides that the level l truncations of the other weakly connected components of $\Gamma^{\mathrm{ram}}_{\mathcal{F}'}$ have no ramified edges.

Hence, combining these two conclusions, Lemma 138(v) and Lemma 144 yields that

$$\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j}) \text{ are the weakly connected components of } \Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}_{\geq l}} \tag{569}$$

for all $i = 1, \ldots, \rho$ and all $j = 1, \ldots, r_i$.

Third and finally, the desired identity in (ii) now follows from the equalities

$$\begin{aligned} g(F_n) &= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}_{\geq l}})\right) d^{n-l} - N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}_{\geq l}}))\right)\\ &= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \#V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}_{\geq l}})\right) d^{n-l} - \tilde{N}(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}_{\geq l}}))\right)\\ &= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \#V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j}))\right) d^{n-l}\right.\\ &\qquad\qquad\left. - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(\overline{F}_n, V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j})))\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(\overline{F}_l, V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j})))\right) d^{n-l}\right.\\
&\qquad\qquad\qquad \left. - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(\overline{F}_n, V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j})))\right)\\
&= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(\overline{F}_l, V(\overline{\Gamma}_{i,j}))\right) d^{n-l} - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(\overline{F}_n, V(\overline{\Gamma}_{i,j}))\right)\\
&= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(F'_l, V(\Gamma'_{i,j}))\right) d^{n-l} - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \tilde{N}(F'_n, V(\Gamma'_{i,j}))\right)\\
&= \frac{1}{2}\left(2 + \left(g(F_l) - 2 + \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \mathrm{N}''_{e_i}(A^l_{i,j})\right) d^{n-l} - \sum_{i=1}^{\rho}\sum_{j=1}^{r_i} \mathrm{N}''_{e_i}(A^n_{i,j})\right)
\end{aligned}$$

for all $n \geq l$ where the equalities hold by the following reasonings:

The first equality holds because the ramification subgraph $\Gamma^{\mathrm{ram}}_{\mathcal{F}_{\geq l}}$ of $\mathcal{F}_{\geq l} := \mathrm{Trun}_{\geq l}(\mathcal{F})$ is a subgraph of $\mathrm{Trun}_{\geq l}(\Gamma^{\mathrm{ram}}_{\mathcal{F}})$ by Lemma 144, because combining the assertion that $\Gamma^{\mathrm{ram}}_{\mathcal{F}}$ is finite and Lemma 138(i) supplies that $\mathrm{Trun}_{\geq l}(\Gamma^{\mathrm{ram}}_{\mathcal{F}})$ is also finite and because we can therefore apply Theorem 194 to $\mathcal{F}_{\geq l} = \mathrm{Trun}_{\geq l}(\mathcal{F}) = (F_{l+\nu})_\nu$.

The second equality holds because the tameness of $\mathcal{F}$ also implies that $\overline{\mathcal{F}}_{\geq l}$ is tame and because we can therefore apply the identity in the 'in particular'-part in Corollary 51. The third equality holds by (569) and by the definition of $\tilde{N}$ in Definition 50.

The fourth equality holds because the assertion $V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j})) \subset \mathbb{P}_{\overline{F}_l}$ and the definition of $\tilde{N}$ imply the equality $\tilde{N}(\overline{F}_l, V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j}))) = \#V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j}))$.

The fifth equality follows again from the definition of $\tilde{N}$ and from the identities

$$V(\mathrm{Trun}_{\geq l}(\overline{\Gamma}_{i,j})) = V(\pi^{-1}_{\Gamma_{\overline{\mathcal{F}}_{\geq l}}/\Gamma_{\overline{\mathcal{F}}}}(\overline{\Gamma}_{i,j})) = \mathbb{P}_{\overline{F}_l}(V(\overline{\Gamma}_{i,j}))$$

where the first identity holds because $\overline{\Gamma}_{i,j}$ is a weakly connected component of $\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}$ by (569), because $\Gamma^{\mathrm{ram}}_{\overline{\mathcal{F}}}$ is also a weakly connected component of $\Gamma_{\overline{\mathcal{F}}}$, because this property of being a weakly connected component is transitive and because of Lemma 130(ii) and the second identity holds by the definitions of $\pi_{\Gamma_{\overline{\mathcal{F}}_{\geq l}}/\Gamma_{\overline{\mathcal{F}}}}$ in Definition/Lemma 126 and of preimage graphs in Definition 69(ii).

The second to last equality holds because the isomorphism in (567) respects ramification indices by Lemma 105(i) and because the $\tilde{N}(\cdot, V_0)$-value only depends on the ramification indices in the paths which start in V_0.

The last equality holds by applying Lemma 234(ii), the 'in particular'-part in Lemma 237, by the definitions of the matrices $A_{i,j}$ as the $w'_{\mathbf{1},\mathbf{P}}$-adjacency matrices of the $\Gamma'_{i,j}$ in the 'in particular'-part in Lemma 248(i) and by the definition of N''_{e_i} in Definition 241. $\square$

A solution for the third challenge. All tame recursive towers $\mathcal{F}$ in the literature known to the author are of a very simple form, which includes all tame recursive towers in [MW05], [Sti08], [BR20]. They all satisfy the assumptions in Lemma 249. As a consequence of this simple form, Lemma 249 provides a quite convenient upper bound for $n_0 = c(A) + \sum_{\lambda \in \Lambda(A)}(b(A, \lambda) + 1)$.

Lemma 249. *Let everything be as in Definition 242 and suppose* $F = K(x, x_1, \ldots, x_r) = E = K(x)$. *Then we immediately obtain* $x_1, \ldots, x_{r_0} \in R_0 = K[x]$ *and* $x_{r_0+1}, \ldots, x_r \in R_\infty = K[x^{-1}]$. *Moreover, suppose* $\deg_x(x_i) \leq 1$ *for all* $i = 1, \ldots, r_0$ *and* $\deg_{x^{-1}}(x_i) \leq 1$

for all $i = r_0 + 1, \dots, r$ *and let* $b := \max\{\deg_x(a_{i,j}), \deg_{x^{-1}}(a_{i,j}) : i, j \in \{1, \dots, m\}\}$. *Then we have the estimates*

$$\sum_{\lambda \in \Lambda(A)} (b(A, \lambda) + 1) \leq m + (b + 2)m^2 \quad \text{and} \quad c(A) \leq (b + 3)m.$$

Proof. On the one hand, since we have $\deg_{x^{-1}}(x_i) \leq 1$ for all $i = r_0 + 1, \dots, r$ by assertion, the x^{-1}-degrees of the coefficients of $\chi_\infty(t) = \prod_{\nu=r_0+1}^{r}(t - x_\nu)^{s_\nu}$ are at most $\sum_{\nu=r_0+1}^{r} s_\nu \leq m_\infty$. Consequently, we derive $\omega_0' \leq m_\infty$ from the definition of ω_0' as the minimal natural number such that $x^{\omega_0'}\chi_\infty(t) \in R_0[t] = K[x][t]$ in (501). Moreover, combining this, the fact that $\chi_\infty(t)$ is a polynomial in $R_\infty[t] = K[x^{-1}][t]$ by its definition in Definition 242 and the assertion that $\deg_x(x_i) \leq 1$ for all $i = 1, \dots, r_0$ yields the estimate

$$\deg_x(x^{\omega_0'}\chi_\infty(x_i)) \leq 2m_\infty \tag{570}$$

for all $i = 1, \dots, r_0$. On the other hand, we also estimate

$$\deg_x\Big(\prod_{\nu=1, \nu \neq i}^{r_0} (x_i - x_\nu)^{s_\nu}\Big) \leq \sum_{\nu=1}^{r_0} s_\nu \leq m_0 \tag{571}$$

where the first estimate holds by $x_\mu \in R_0 = K[x]$ and the assumption $\deg_x(x_\mu) \leq 1$ for all $\mu = 1, \dots, r_0$ and the second estimate holds since $x_1, \dots, x_{r_0}$ are eigenvalues of A_0 and since s_ν is the size of the largest Jordan block of A for the eigenvalue x_ν and, thus, also of A_0.

Putting both estimates together supplies

$$\begin{aligned}\deg_x(x^{\omega_0'}\chi_i(x_i)) &= \deg_x\Big(\prod_{\nu=1, \nu \neq i}^{r_0} (x_i - x_\nu)^{s_\nu}\Big) + \deg_x(x^{\omega_0'}\chi_\infty(x_i)) \\ &\leq m_0 + 2m_\infty \leq 2m \end{aligned} \tag{572}$$

for all $i = 1, \dots, r_0$ where the equality and estimates hold by the following reasonings: The equality holds because of the definition of $\chi_i(t) = \prod_{\nu=1, \nu \neq i}^{r_0}(x_i - x_\nu)^{s_\nu}\chi_\infty(t)$ in (505), because $\prod_{\nu=1, \nu \neq i}^{r_0}(x_i - x_\nu)^{s_\nu}$ and $x^{\omega_0'}\chi_\infty(x_i)$ are elements in $K[x]$ and because we can therefore apply the well known rule $\deg(f \cdot g) = \deg(f) + \deg(g)$ for all $f, g \in K[x]$. The first estimate holds by the estimates in (570) and (571). The last estimate holds by the equality $m_0 + m_\infty = m$ in Definition 242.

Analogously, we estimate

$$\begin{aligned}\deg_x(x^{-\omega_\infty'}\chi_i(x_i)) &= \deg_x\Big(\prod_{\nu=r_0+1, \nu \neq i}^{r} (x_i - x_\nu)^{s_\nu}\Big) + \deg_x(x^{-\omega_\infty'}\chi_0(x_i)) \\ &\leq m_\infty + 2m_0 \leq 2m \end{aligned} \tag{573}$$

Next, we already obtain the desired estimate $\sum_{\lambda \in \Lambda(A)}(b(A, \lambda) + 1) \leq m + (b + 2)m^2$ by the equalities and estimates

$$\begin{aligned}\sum_{\lambda \in \Lambda(A)} (b(A, \lambda) + 1) \leq \sum_{\lambda \in \Lambda_0} (b_0(\lambda) + 1) &= \sum_{\lambda \in \Lambda_0} \max\{s_i + b_{0,i}(\lambda) : i = 1, \dots, r \text{ with } \lambda \in \Lambda_{0,i}\} \\ &\leq \sum_{i=1}^{r} s_i + \sum_{\lambda \in \Lambda_{0,i}} b_{0,i}(\lambda) \leq m + \sum_{i=1}^{r} \sum_{P \in M_{0,i}} b_{0,i}(P) \\ &= m + \sum_{i=1}^{r_0} \sum_{P \in M_{0,i}} (v_P((x - 1)x^{\omega_i}\chi_i(x_i)^{s_i}) - 1)\end{aligned}$$

$$\begin{aligned}
&\qquad + \sum_{i=1+r_0}^{r} \sum_{P\in M_{0,i}} (v_P((y-1)y^{\omega_i}\chi_i(x_i)^{s_i}) - 1) \\
&\le m + \sum_{i=1}^{r_0} \sum_{P\in M_{0,i}} v_P(x^{\omega_i}\chi_i(x_i)^{s_i}) + \sum_{i=1+r_0}^{r} \sum_{P\in M_{0,i}} v_P(x^{-\omega_i}\chi_i(x_i)^{s_i}) \\
&= m + \sum_{i=1}^{r_0} \sum_{P\in M_{0,i}} v_P(x^{\omega_0''}) + v_P(x^{s_i\omega_0'}\chi_i(x_i)^{s_i}) \\
&\qquad + \sum_{i=1+r_0}^{r} \sum_{P\in M_{0,i}} v_P(x^{-\omega_\infty''}) + v_P(x^{-s_i\omega_\infty'}\chi_i(x_i)^{s_i}) \\
&\le m + \sum_{i=1}^{r_0} \deg_x(x^{\omega_0''}) + \deg_x(x^{s_i\omega_0'}\chi_i(x_i)^{s_i}) \\
&\qquad + \sum_{i=1+r_0}^{r} \deg_{x^{-1}}(x^{\omega_\infty''}) + \deg_{x^{-1}}(x^{-s_i\omega_\infty'}\chi_i(x_i)^{s_i}) \\
&\le m + bm^2 + \sum_{i=1}^{r_0} \deg_x(x^{s_i\omega_0'}\chi_i(x_i)^{s_i}) + \sum_{i=1+r_0}^{r} \deg_{x^{-1}}(x^{-s_i\omega_\infty'}\chi_i(x_i)^{s_i}) \\
&= m + bm^2 + \sum_{i=1}^{r_0} s_i \deg_x(x^{\omega_0'}\chi_i(x_i)) + \sum_{i=1+r_0}^{r} s_i \deg_{x^{-1}}(x^{-\omega_\infty'}\chi_i(x_i)) \\
&\le m + bm^2 + \sum_{i=1}^{r_0} s_i \cdot 2m + \sum_{i=1+r_0}^{r} s_i \cdot 2m \\
&\le m + (b+2)m^2
\end{aligned}$$

where the equalities and estimates hold by the following reasonings:

The first estimate holds by the definition of $b(A,\lambda)$ in (508) as the minimum of $b_0(\lambda)$ and $b_\infty(\lambda)$.

The first equality holds because, in (508), $b_0(\lambda)$ is defined as the maximum of all $s_i - 1 + b_{0,i}(\lambda)$ with $i = 1, \ldots, r$ and $\lambda \in \Lambda_{0,i}$ and because the ones are canceling each other out.

The second estimate holds because, in (508), Λ_0 is defined as the union of all $\Lambda_{0,i}$ for all $i = 1, \ldots, r$ and because, therefore, the double sum on the right side of the estimate contains all summands of the left side of the estimate.

The third estimate holds because we have $\sum_{i=1} s_i = s \le m$ in Definition 242 and because, in (506), $b_{0,i}(\lambda)$ is defined as the maximum of all $b_{0,i}(P)$ with $P \in M_{0,i}$ and $x_i(P) = \lambda$.

The second equality holds because of the definition of $b_{0,i}(P)$ in (506). The fourth estimate holds by the equalities $v_P((x-1)x^{\omega_i}\chi_i(x_i)^{s_i}) = v_P(x-1) + v_P(x^{\omega_i}\chi_i(x_i)^{s_i})$ and $v_P((y-1)y^{\omega_i}\chi_i(x_i)^{s_i}) = v_P(y-1) + v_P(y^{\omega_i}\chi_i(x_i)^{s_i})$, by the estimates $v_P(x-1) \le 1$ and $v_P(y-1) \le 1$ and by the equality $y = ex^{-1}$.

The third equality holds because of the definitions of $\omega_i = s_i\omega_0' + \omega_0''$ for all $i = 1, \ldots, r_0$ and of $\omega_i = s_i\omega_\infty' + \omega_\infty''$ for all $i = r_0 + 1, \ldots, r$ in (504) and because of the well known rule $v_P(f \cdot g) = v_P(f) + v_P(g)$ for all $f, g \in F\backslash\{0\} = K(x)\backslash\{0\}$ (the elements are either obviously non-zero or because of the definition of $\chi_i(t) = \prod_{\nu=1,\nu\neq i}(t - x_\nu)^{s_\nu}$ in (505)).

The fifth estimate holds because the corresponding elements are contained in $K[x]$ and $K[x^{-1}]$, respectively, by the definitions of ω_0' and ω_∞' in (501) and of $\chi_i(t)$ in (505).

For the sixth estimate, we first notice the equality and estimates

$$\deg_x(x^{\omega_0''}) = \omega_0'' \le bs \le bm \tag{574}$$

where the equality is clear, the first estimate holds because the definition of b in the assumptions implies that the x-degree of any entry of A^n is bounded by bn and because of the definition of ω_0'' in (502) and the last estimate holds because of the estimate $s \leq m$. Consequently, we derive the estimates

$$\sum_{i=1}^{r_0} \deg_x(x^{\omega_0''}) \leq \sum_{i=1}^{r_0} bm \leq bmr_0.$$

Analogously, we also derive the estimates

$$\sum_{i=r_0+1}^{r} \deg_{x^{-1}}(x^{\omega_\infty''}) \leq \sum_{i=r_0+1}^{r} bm \leq bm(r-r_0).$$

Combining these two estimates and

$$bmr_0 + bm(r-r_0) = bmr \leq bm^2$$

then yield the sixth estimate.

The fourth equality holds by the well known rule $\deg(f^s) = s\deg(f)$ for all $f \in K[t]\backslash$. The second to last estimate holds by the estimates in (572) and (573). The last estimate holds because $\sum_{i=1}^{r} s_i = s \leq m$.

Finally, going through the definition of $c(A)$ in (509) and only applying a subset of the estimates from above also yields the second desired estimate $c(A) \leq m + (b+2)m = (b+3)m$. ☐

8.3.2 The Implementation

Summary of the results of this subsection. Finally, we will come to the already announced first naive implementation of the approach in (A4) which works on all tame recursive towers $\mathcal{F}$ in the literature which are known to the author, including all tame recursive towers in [MW05], [Sti08], [BR20]. Consequently, in Examples 250, we will also list genus formulas for some representative tame recursive towers from the literature.

The implementation. More concretely, we will demonstrate the implementation on the tame recursive tower $\mathcal{F} = (F_\nu)_\nu$ in [MW05, p. 212, f_{11}] which is defined by the polynomial $f = f_{11} = Y^2 + (X^2+1)Y + X^2 + X + 1$ over $\mathbb{F}_3$.

The degree two subgraph of the tower graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$ is depicted in Figure B.8. Here, the first two weakly connected components Γ and Γ' form the ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$ of $\Gamma_{\mathcal{F}}$.

First, we need to check from which level l on the approach in (A4) is applicable: On the one hand, Γ is separating, only contains rational vertices and edges and all ramification indices in Γ are powers of two. On the other hand, Γ' has only circles with balanced ramification indices and if we consider the paths of length two in Γ' and apply Abhyankar's Lemma, we also see that $\mathrm{Trun}_{\geq 1}(\Gamma')$ contains no ramified edges. Consequently, we can apply Lemma 248 to $\mathcal{F}$ for $l = 1$ and $k' = k = \mathbb{F}_3$ and thereby obtain the equality

$$\begin{aligned} g(F_n) &= \frac{1}{2} \cdot \Big(2 + \big(\#V(\Gamma^{\mathrm{ram}}_{\mathcal{F}_{\geq l}}) - 2 + 2g(F_l)\big)\, d^{n-l} - N(\overline{F}_n, V(\Gamma^{\mathrm{ram}}_{\mathcal{F}_{\geq l}}))\Big) \\ &= \frac{1}{2} \cdot \Big(2 + \big(r \cdot \mathrm{N}_e''(A^l) - 2 + 2g\big)\, d^{n-l} - r \cdot \mathrm{N}_e''(A^n)\Big) \end{aligned} \tag{575}$$

for all $n \geq \max(1, c(A))$ where $d := 2$, $e := 2$, $g := g(F_l) = 1$, $r := 1$ and

$$A := \begin{pmatrix} 2x^{-1} & 1 \\ 0 & x \end{pmatrix} \in \overline{\mathbb{Q}}[x, x^{-1}]^{m \times m}. \tag{576}$$

for $m := 2$. Furthermore, we notice that $x_1 = x$ is the only eigenvalue of A_0 and $x_2 = 2x^{-1}$ is the only eigenvalue of A_∞ and, thus, Lemma 249 is applicable for $b := 1$.

We have all the necessary input data and can start the computation of the genus formula in Magma [BCP97] via the following first naive implementation of the approach in (A4): First, we begin with the constant field $\mathbb{Q}$:

```
> QQ := Rationals( );
> K0 := QQ;
> K0t< t > := PolynomialRing( K0 );
> K0x< x > := RationalFunctionField( K0 );
> K0xu< u > := PolynomialRing( K0x );
```

More generally, K_0 must be chosen such that the eigenvalues of A are contained in $K_0(x)$, i.e. such that the characteristic polynomial χ_A of A decomposes into linear factors over K_0. There are some examples where higher roots of unity must be adjoint to $\mathbb{Q}$. Of course, this can also be automated if necessary.

Second, we initialize the input data and the matrix A and compute its eigenvalues.

```
> // Input Data [* d, e, m, r, b, g, l, Aseq *]
> FormulaData := [* 2, 2, 2, 1, 1, 1, 1, [  2/x, 1, 0, x ] *];
> d := FormulaData[ 1 ];
> e := FormulaData[ 2 ];
> m := FormulaData[ 3 ];
> r := FormulaData[ 4 ];
> b := FormulaData[ 5 ];
> g := FormulaData[ 6 ];
> l := FormulaData[ 7 ];
> Aseq := FormulaData[ 8 ];
>
> // Initialize the matrix A and compute its eigenvalues
> A := Matrix( K0x, m, m, Aseq );
> A;
[2/x   1]
[  0   x]
> f := CharacteristicPolynomial( A );
> f;
u^2 + (-x^2 - 2)/x*u + 2
> ff := Factorization( f );
> ff;
[
<u - 2/x, 1>,
<u - x, 1>
]
> ff_zeroes := < - Evaluate( f[ 1 ],  0 ) : f in ff >;  // Eigenvalues of A
> ff_zeroes;
<2/x, x>
```

Third, we extend the constant field K_0 further to some K which also contains all the zeroes of the differences of the eigenvalues x_i.

```
> // Extend the constant field to some suitable field K
> df := Numerator( &*[ xi - xj : xi in ff_zeroes, xj in ff_zeroes | \
xi ne xj ] );
```

```
> df;
-t^4 + 4*t^2 - 4
> K< w > := SplittingField( df );
> MinimalPolynomial( w );
t^2 - 2
> Kt< t > := PolynomialRing( K );
> Kx< x > := RationalFunctionField( K );
```

Fourth, we compute the set Λ of evaluations $x_i(Q)$ such that Q is a zero of $x_i - x_j$ for some $j \neq i$. Out of convenience, instead of Λ' in (A4), we compute this superset $\Lambda \supseteq \Lambda'$. However, in this example, we already have $\Lambda = \Lambda'$.

```
> // Compute the set Lambda of evalutions x_i( Q )
> // where Q is a zero of x_i - x_j for some j ne i
> fi_seq := [ [* ff_zeroes[ i ], [ ff_zeroes[ i ] - ff_zeroes[ j ] : j in /
[ 1 .. #ff ] | i ne j ] *] : i in [ 1 .. #ff ] ];
> fi_seq;
[ [*
2/x,
[
(-x^2 + 2)/x
]
*], [*
x,
[
(x^2 - 2)/x
]
*] ]
> Lambda := { };
> for xifi in fi_seq do
for>   xi := xifi[ 1 ];
for>   fi := xifi[ 2 ];
for>   zz := &cat[ Roots( Kt ! Numerator( fij ) ) : fij in fi ];
for>   Lambda join:= {Evaluate( xi, z[ 1 ] ) : z in zz };
for> end for;
> Lambda;
{
-w,
w
}
```

Fifth, we do not bother to compute the upper bounds $b(A, \lambda)$ of the degrees of the polynomials $f_\lambda(n)$ in (565). In all examples for which the author computed a genus formula, the polynomials $f_\lambda(n)$ were constant. Moreover, we do also not bother to compute $c(A)$. We even try to compute a genus formula for all $n \geq 1$. Of course, in case that the verification step at the end fails, these two steps must be refined.

Next, with these additional hypotheses, we solve the corresponding linear equation system in (A4) and obtain candidates for the desired formulas for $\mathrm{N}''_e(A^n)$ and, by the equality in (575), also for $g(F_n)$.

```
> // Compute the coefficients in the polynomial under the
> // additional assumption that the polynomials have degree zero
```

```
> s := #Lambda;
> an_seq := [ My_Get_NValue( &+Eltseq( A^n ), e ) : n in [ 0 .. s - 1 ] ];
> a := Matrix( K, s, 1, an_seq );
> a;
[2]
[3]
> Lambdass := [ [ x, 1 ] : x in Lambda ];
> V := My_Get_ConfluntVandermondeMatrix( Lambdass, s );
> V;
[ 1  1]
[-w  w]
> c := V^-1 * a;
> c;
[1/4*(-3*w + 4)]
[ 1/4*(3*w + 4)]
>
> // Put everything together for the desired N- and g-formulas
> Lambda := [ d ] cat [ Lambdass[ i ][ 1 ] : i in [ 1 .. #Lambdass ] ];
> R := PolynomialRing( K, #Lambda );
> AssignNames( ~R, [ "(" cat Sprint( Lambda[ i ] ) cat ")^n\n" : i i\
n [ 1 .. #Lambda ] ] );
> NFormula := &+[ c[ i - 1, 1 ] * R.i : i in [ 2 .. #Lambda ] ];
> NFormula;
1/4*(-3*w + 4)*(-w)^n
 + 1/4*(3*w + 4)*(w)^n
> lambdal := [ Lambda[ i ]^l :  i in [ 1 .. #Lambda ] ];
> lambdal;
[
2,
-w,
w
]
> gFormula := ( 2 + ( r*Evaluate( NFormula, lambdal ) - 2 + 2*g ) * \
R.1 / d^l - r * NFormula ) / 2;
> gFormula;
3/4*(2)^n
 + 1/8*(3*w - 4)*(-w)^n
 + 1/8*(-3*w - 4)*(w)^n
 + 1
```

Finally, by (A4) and Lemma 249, we only have to check our computed candidate for the formula for $\mathrm{N}''_e(A^n)$ up to the level $(b+4)m+(b+2)m^2 \geq n_0 = c(A)+\sum_{\lambda\in\Lambda(A)}(b(A,\lambda)+1) = \sum_{\lambda\in\Lambda(A)}(b(A,\lambda)+1)$.

```
> // Check the N-formula up to n_end := ( b + 4 ) * m + ( b + 2 ) * m^2;
> n_end := ( b + 4 ) * m + ( b + 2 ) * m^2;
> n_end;
22
> Nval_seq := < Evaluate( NFormula, [ Lambda[ i ]^n :  i in [ 1 .. #La \
mbda ] ] ) :  n in [ 0 .. n_end ] >;
> a := < My_Get_NValue( &+Eltseq( A^n ), e ) : n in [ 0 .. n_end ] >;
> Nval_seq eq a;
```

```
true
```

Hence, we established the genus formula

$$g(F_n) = \frac{3}{4} \cdot 2^n + \frac{3\sqrt{2}-4}{8} \cdot (-\sqrt{2})^n + \frac{-3\sqrt{2}-4}{8} \cdot \sqrt{2}^n + 1$$

for all $n \geq 1$ and the first elements of the sequence are equal to the following numbers.

```
> // Compute first n_end many genera in the genus sequence
> n_end := 20;
> gn_seq := < Evaluate( gFormula, [ Lambda[ i ]^n :  i in [ 1 .. #La\
mbda ] ] ) : n in [ 1 .. n_end ] >;
> gn_seq;
<1, 2, 4, 9, 19, 41, 85, 177, 361, 737, 1489, 3009, 6049, 12161, 24385,
48897, 97921, 196097, 392449, 785409>
```

Finally, separately computing the genera of the function fields in Magma confirms the first three elements in the above genus formula. However, it already takes over ten minutes to compute the next element $g(F_4)$ of the genus sequence.

```
> // Check the first few genera in the computed genus formula
> ZZ := Integers( );
> ZZxy< x, y > := PolynomialRing( ZZ, 2 );
> fq := < 3, x^2*y + x^2 + x + y^2 + y + 1  >;
> q := fq[ 1 ];
> k := GF( q );
> kxy< x, y > := PolynomialRing( k, 2 );
> f := kxy ! fq[ 2 ];
> F0< x0 > := RationalFunctionField( k );
> F0t< t > := PolynomialRing( F0 );
> FF := [* F0 *];
> n_end := 3;
> Fn_prev< xn_prev > := F0;
> for n in [ 1 .. n_end ] do
for>   Fn_prevt< t > := PolynomialRing( Fn_prev );
for>   Fn< xn_prev > := FunctionField( Evaluate( f, [ xn_prev, t ] )\
 );
for>   FF cat:= [* Fn *];
for>   Fn_prev := Fn;
for> end for;
> < Genus( Fn ) : Fn in FF >;
<0, 1, 2, 4>
```

A list of genus formulas. In the following Examples 250, we will finish this section with a list of genus formulas for some tame recursive towers. All genus formulas were computed with the above implementation and only the input data had to be adjusted.

Here, the tower in (i) is the tame recursive tower in Example 235 and the tower in (ii) is the tame recursive tower from the above illustration of the implementation.

Finally, we want to remark that we could have also included many more examples, e.g from [MW05]. But the final genus formulas of these examples are only minor alternations of the formulas in Examples 250 and thus they do not provide more insights.

Examples 250. *Any member of the following list consists of the following data: First, we provide a polynomial $f \in \mathbb{F}_q[X, Y]$ which already exist in the literature and is known to define a tame recursive tower $\mathcal{F} = (F_\nu)_\nu$.*

Second, except for the tower in (i) on which we already elaborated extensively in Example 235, we also provide a reference to a figure with the minimal subgraph $\Gamma_{i,c}$ of the tower graph of $\mathbb{F}_{q^i} \cdot \mathcal{F}$ which contains all edges Q with $\deg(Q) \leq c$. All these subgraphs $\Gamma_{i,j}$ contain the corresponding ramification subgraphs.

Third, we provide the genus formula $g(F_n)$ for all $n \geq l$ which was computed with the above implementation.

(i) For $q \in \mathbb{P}\backslash\{2, 3\}$, $f = Y^2(3Y - 1) - (X^2 + X)$ in [HP16, p. 12, Proposition 12] (see also Example 235) and for all $n \geq 1$, we obtain

$$g(F_{n-1}) = 2^n - \frac{4 + 3\sqrt{2}}{4} \cdot \sqrt{2}^n - \frac{4 - 3\sqrt{2}}{4} \cdot (-\sqrt{2})^n + 1.$$

(ii) For $q = 3$, $f = Y^2 + (X^2 + 1)Y + X^2 + X + 1$ in [MW05, p. 212, f_{11}] with $\Gamma_{1,2}$ in Figure B.8 and for all $n \geq 1$, we obtain

$$g(F_n) = \frac{3}{4} \cdot 2^n + \frac{3\sqrt{2} - 4}{8} \cdot (-\sqrt{2})^n + \frac{-3\sqrt{2} - 4}{8} \cdot \sqrt{2}^n + 1.$$

(iii) For $q = 5$, $f = (X^6 + X + 2)(Y^5 - Y) - (X^5 - X)(Y^6 + Y^5 + 2Y + 3)$ in [BR20, p. 4] with $\Gamma_{1,2}$ in Figure B.28 and for all $n \geq 0$, we obtain

$$g(F_n) = 9 \cdot 6^n + 15 \cdot 2^n + (-\frac{25}{2} + 5\sqrt{6}) \cdot (-\sqrt{6})^n + (-\frac{25}{2} - 5\sqrt{6}) \cdot \sqrt{6}^n + 1.$$

(iv) For $q = 3$, $f = Y^2 + X^2Y + 1$ in [MW05, p. 212, f_4] with $\Gamma_{1,2}$ in Figure B.4 and for all $n \geq 0$, we obtain

$$g(F_n) = 2 \cdot 2^n + (\sqrt{2} - \frac{3}{2}) \cdot (-\sqrt{2})^n + (-\sqrt{2} - \frac{3}{2}) \cdot \sqrt{2}^n + 1.$$

(v) For $q = 3$, $f = 2X^2 + XY + Y^2 + 1$ in [MW05, p. 212, f_2] with $\Gamma_{2,1}$ in Figure B.1 and for all $n \geq 0$, we obtain

$$g(F_n) = \frac{3}{2} \cdot 2^n + (\sqrt{2} - \frac{3}{2}) \cdot (-\sqrt{2})^n + (-\sqrt{2} - \frac{3}{2}) \cdot \sqrt{2}^n + 1 + \frac{1}{2} \cdot 0^n$$

where $0^0 = 1$ and $0^n = 0$ for all $n \geq 1$.

(vi) For $q = 7$, $f = Y^6 + (X + 1)^6 - 1$ in [Sti08, p. 262, Proposition 7.3.2] with $\Gamma_{1,2}$ in Figure B.16 and for all $n \geq 0$, we obtain

$$g(F_n) = \frac{5}{2} \cdot 6^n + \frac{2\sqrt{6} - 7}{4} \cdot (-\sqrt{6})^n + \frac{-2\sqrt{6} - 7}{4} \cdot \sqrt{6}^n + 1.$$

9 Prospects

In this final chapter, we will give suggestions for possible future directions which are closely related to this thesis.

Finding a counterexample to Conjecture 1(iii). In Conjecture 1(iii), it was suspected that any recursive tower $\mathcal{F} = (F_\nu)_\nu$ over a finite field with positive splitting rate $\nu(\mathcal{F})$ must have a rational place $P \in \mathbb{P}_{F_m}$ on some level $m \in \mathbb{N}_0$ which splits on all further levels $n \geq m$, i.e. in F_n/F_m. This conjecture was first introduced in [BGS04, p. 7, Conjecture 1] via its stronger version in Conjecture 1(i) and then it was confirmed to be still open in [Sti10, p. 5, Problem 1] and [Bee22, p. 10, 24]. It is the weakest of the Conjectures 1(i)-(iv) and also the only one which is not disproven yet.

In Corollary 184, we showed that Conjecture 1(iii) is true for all recursive towers $\mathcal{F}$ such that the ramification subgraph of some truncation of $\mathcal{F}$ only has finite unbalanced weakly connected components. Note that, in correspondence to the formulation of the main result in the summary in Section **??**, the latter property is equivalent to the property that the tower graph of $\mathcal{F}$ only has finite balanced weakly connected components which stop ramifying from some level on. This condition is very mild and there is only one recursive tower known to the author which does not fulfill this condition, namely the CNT-tower in Examples 8(v).

Nonetheless, at the end of Subsection 6.3, we laid out a strategy to find a counterexample to Conjecture 1(iii) (see Lemma 187 and Figure 6.1). The single missing step in the strategy is to find a recursive tower to the directed graph which is depicted in Figure 6.1.

A **suggestion** for future work is therefore to find a recursive tower $\mathcal{F}$ such that its tower graph $\Gamma_{\mathcal{F}}$ has a subgraph as in Figure 6.1. Then $\mathcal{F}$ would be a counterexample to Conjecture 1(iii). We will discuss in the next paragraph how to find recursive towers with prescribed subgraphs.

Finding recursive towers using directed graphs. With any recursive tower, we can associate the tower graph (or the Beelen-graph or the HP-graph). For all tame recursive towers $\mathcal{F}$ with finite ramification subgraph $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$, Corollary 195 and Corollary 196 provide up to finite constant field extensions that $\mathcal{F}$ is good if and only if $\Gamma_{\mathcal{F}}$ has a finite balanced weakly connected component Γ. For wild towers with finite ramification subgraphs, the 'only if'-part of this equivalence is still true. Moreover, the necessary finite constant field extension can be estimated. For instance, if Γ only contains rational vertices and edges and is d-regular, i.e. Γ is equal to the splitting subgraph $\Gamma_{\mathcal{F}}^{\mathrm{split}}$, then there is no constant field extension necessary and the precise limit $\lambda(\mathcal{F})$ of $\mathcal{F}$ can even be read from $V(\Gamma)$, $V(\Gamma_{\mathcal{F}}^{\mathrm{ram}})$ and α via the formulas in Corollary 195 and Corollary 200.

Because of that, it is natural to try to reverse the direction by finding recursive towers to prescribed subgraphs of the tower graph. This was also proposed in [HP16] for HP-graphs. First, we can specify a directed graph Γ with weights $w(Q) \in \{(e, \hat{e}) : 1 \leq e, \hat{e} \leq d\}$ on its edges Q which at least does not violate the most obvious necessary conditions for being a subgraph of a tower graph. Then we can try to find a recursive tower $\mathcal{F}$ such that

Γ is a subgraph of its tower graph $\Gamma_{\mathcal{F}}$ and the weights $w(Q)$ of the edges Q are equal to the ramification indices $(e(Q|Q \cap F_0), e(Q|Q \cap \sigma(F_0))$ with the tower map σ of $\mathcal{F}$.

There are several reasons for why this is a promising approach to find new good recursive towers or to characterize all good recursive towers having certain subgraphs or to solve further problems related to recursive towers:

First, to this point, the BR-towers $\mathcal{F}_{BR,q}$ over $\mathbb{F}_q$ are the only good recursive towers over $\mathbb{F}_q$ with prime $q \neq 2, 3$ and it took 25 years to find them. Yet, their splitting subgraph and ramification subgraph are fairly simple and small. For $q = 5$, these subgraphs are depicted in Figure B.28 and only consist of four vertices. A systematic search for all good recursive towers $\mathcal{F} = (F_\nu)_\nu$ with rational F_1 for all small potential subgraphs would have been sufficient to at least find $\mathcal{F}_{BR,5}$.

Second, as pointed out in the paragraph before, finding a tame recursive tower to the potential subgraph in Figure 6.1 would provide a counterexample to Conjecture 1(iii).

Third, we can classify all recursive towers with a prescribed subgraph. Let us demonstrate such an attempt for a subgraph Γ of the BR-towers and try to classify all recursive towers $\mathcal{F} = (F_\nu)_\nu$ with rational F_1.

Note that the rationality of F_1 is not as restrictive as it might seem at first. Many recursive towers in the literature, including the BR-towers, can be reduced to this situation by the Reduction Lemma 30. Because $F_1 = \mathbb{F}_q(z)$ is rational, we have $\mathbb{F}_q(g(z))$ and $\mathbb{F}_q(h(z))$ for two rational function $g(z), h(z) \in \mathbb{F}_q(z)$ of degree d. Moreover, we can try to first find all recursive towers over the algebraic closure $\overline{\mathbb{F}}_q$ and then check which of the discovered candidates already define recursive towers over $\mathbb{F}_q$.

Let us fix the subgraph Γ as the disjoint union of one d-regular subgraph with one edge (as the splitting subgraph in Figure B.28), two loops which are totally ramified in F_1/F_0 and not ramified in $F_1/\sigma(F_0)$ and two further edges which are totally ramified in $F_1/\sigma(F_0)$ and not ramified in F_1/F_0 (as it is the case in the $\overline{\mathbb{F}}_5$-constant field extension of ramification subgraph in Figure B.28). Then we just mention without proof that, up to coordinate transformations, the following can be concluded for $q = p^n$ with p prime and $\gcd(p, d) = 1$: The tower graph $\Gamma_{\mathcal{F}}$ has the subgraph Γ if and only if

$$g(z) = z^d \quad \text{and} \quad h(z) = -\frac{1}{a^d} \frac{(\frac{z-a}{z-a^{-p^s}})^d - a^{2d}}{(\frac{z-a}{z-a^{-p^s}})^d - 1}$$

for some $s \in \mathbb{N}_0$ such that $d = 1 + p^s$ and some a in $\mathbb{F}_{p^{2s}}$ if $s \neq 0$ and in $\overline{\mathbb{F}}_p$ if $s = 0$ which is not a root of $f = t(t^d - 1)(t^d + 1)$.

Moreover, for $s \neq 0$, i.e. $d \neq 2$, it can be shown that this already defines a good recursive tower over $\mathbb{F}_{p^{2s}}$. Finally, considering all possible coordinate transformations provides that the BR-towers $\mathcal{F}_{BR,p}$ are the only good recursive towers over $\mathbb{F}_p$ with this subgraph Γ and with $\gcd(d, p) = 1$, $d > 2$ and F_1 rational.

Fourth, it would be interesting to see if there are tame recursive towers with more complicated tower graphs. As far as the author knows, all tame recursive towers in the literature have simple separating ramification subgraphs.

A **suggestion** for future work is therefore to find and/or classify recursive towers $\mathcal{F}$ with certain subgraphs Γ of $\Gamma_{\mathcal{F}}$. For this purpose, it would also be useful to have an implementation in some computer algebra system like Magma [BCP97]. For a given directed graph with weights, this implementation could automatically solve for the coefficients in the defining polynomial of the polynomial-recursive tower, for d and for the possible prime powers q.

Furthermore, instead of only finite d-regular weakly connected components, we should also take the more general finite balanced weakly connected subgraphs into consideration.

Handling the remaining wild recursive towers in the Main Theorem 177. Let $\mathcal{F}$ be a recursive tower over the finite field k of balanced degree d and let Γ be a finite weakly connected component of Γ. In the Main Theorem 177, up to finite constant field extensions and up to some very specific wild recursive towers, the limit $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ vanishes if and only if Γ contains circles with unbalanced ramification indices. More concretely, the only recursive towers to which Main Theorem 177 is not applicable are wild recursive towers $\mathcal{F}$ with a finite balanced weakly connected component in the tower graph which contain wild paths. The single example of such a wild recursive tower known to the author is the CNT-tower $\mathcal{F}_{CNT,s}$ in Examples 8(v). Consequently, although the ramification subgraph of the CNT-tower $\mathcal{F}_{CNT,s}$ is a finite balanced weakly connected component, we cannot simply conclude from Main Theorem 177(iii) that the splitting rate of $\mathcal{F}_{CNT,s}$ is positive for some s.

Let us investigate where the proof goes wrong if there is some wild path. For finite balanced weakly connected components Γ which only contains tame paths, the proof can be divided into three steps:

First, because of Abhyankar's Lemma, roughly speaking, there is so much killing of ramification indices going on in the pyramids of places over Γ that Corollary 170 provides that the ramification indices $e(Q|Q \cap F_0)$ are bounded by a single number for all places Q in $\mathcal{F}$ which lie over Γ.

Second, because of this, Corollary 171 even implies that not only the ramification indices are bounded by a single number but also the degrees of the places Q over Γ.

Third, as the degrees of the places over Γ are bounded by a single number, they are all rational after a suitable finite constant field extension. But then the number of rational places in $\mathcal{F}$ over Γ is the same as the number of rational places in the geometric tower $\overline{k}\cdot\mathcal{F}$ of $\mathcal{F}$. Thus, we can even extend the constant field k of $\mathcal{F}$ to the algebraically closed field $\overline{k}$ and count the places over the $\overline{k}$-constant field extension of Γ. Therefore, without loss of generality, let us assume that k is algebraically closed. Consequently, the fundamental equality in (8) supplies the equality

$$\sum_{Q\in\mathbb{P}_{F_n}(V(\Gamma))} e(Q|Q \cap F_0) = \#V(\Gamma) \cdot d^n.$$

Finally, since the ramification indices are bounded by a single number due to the first step, the number $N(F_\nu, V(\Gamma))$ of summands has to satisfy the desired estimate

$$\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu > 0.$$

Hence, the only issue for finite balanced weakly connected components Γ which contain wild paths is that, in the first step, Abhyankar's Lemma is not applicable to every extension in the pyramids of places over Γ. Therefore, a priori, we do not know whether the ramification indices $e(Q|Q \cap F_0)$ are bounded by a single number for all places Q in $\mathcal{F}$ which lie over Γ. But the second and third step also work in the wild case if the ramification indices are bounded by a single number.

A **suggestion** for future work is therefore to find criteria which help deciding whether the ramification indices over a finite balanced weakly connected component Γ with wild paths are bounded by a single number and, thus, whether the $\lim_{\nu\to\infty} N(F_\nu, V(\Gamma))/d^\nu$ is positive.

Appendices

A Algebraic Geometry

In this chapter, we will prove the algebraic geometric statements which are used in Subsection 4.4.2.

Lemma 251. *Let $\varphi : D \to C$ be a finite morphism of projective integral algebraic curves over a field k.*

Then φ maps the generic point of D to the generic point of C and closed points of D to closed points of C.

Proof. One the one hand, by [Liu02, p.277, Lemma 3.10(iii)], we obtain that the generic point of D maps to the generic point of C. On the other hand, finite morphisms are proper and, therefore, closed. Thus, φ maps closed points to closed points. □

Let us denote the function field of an integral curve C over some field k by $K(C)$ and the pullback of a morphism $\varphi : D \to C$ of integral curves over k by $\varphi^* : K(C) \hookrightarrow K(D)$ be the pullback of φ.

Lemma 252. *Let $\varphi : D \to C$ be a finite morphism of projective integral curves over the perfect field k, suppose that C is regular. Moreover, let $\varphi^* : K(C) \to K(D)$ be the pullback of φ. Finally, let Q be a place in $K(D)$ and let q be the unique closed point in D such that Q dominates the maximal ideal m_q of its local ring $\mathcal{O}_{D,q}$.*

Then φ is etale at q if and only if D is regular at q and Q is unramified in the extension $K(D)/\varphi^*(K(C))$.

Proof. First, we notice because of Lemma 251 that the image $p := \varphi(q)$ is a closed point on X. Let m_p be the maximal ideal of the local ring $\mathcal{O}_{C,p}$ of p.

On the one hand, because C is an integral curve over k and because p is a closed point, we have the inclusions $k \subsetneq \mathcal{O}_{C,p} \subsetneq K(C)$ and the local ring $\mathcal{O}_{C,p}$ has $K(C)$ as its fraction field. On the other hand, we also obtain that the local ring $\mathcal{O}_{C,p}$ is also Noetherian and has Krull-dimension one. Moreover, because C is regular at p, the local ring $\mathcal{O}_{C,p}$ is also regular. But, regular Noetherian local rings with Krull-dimension one are discrete valuation rings. Hence, combining these two conclusions provides that m_p is a place in $K(C)$ and that $\varphi^*(m_p)$ is a place in $\varphi^*(K(C))$.

Moreover, by the choice of $p = \varphi(q)$, the place $\varphi^*(m_p)$ is also contained in the maximal ideal m_q of the local ring $\mathcal{O}_{D,q}$ of q. In particular, this means that $Q/\varphi^*(m_p)$ is an extension of places in $K(D)/\varphi^*(K(C))$.

For the 'if'-part: Suppose that φ is etale at q. On the one hand, Lemma 251 supplies that q is a closed point in D. Moreover, as projective varieties are locally Noetherian, [Liu02, p.140, Corollary 3.24] implies the first desired statement, namely that D is regular at q.

Now, because q is a regular closed point on D and because of the same reasoning as at the beginning, we conclude that m_q is a place in $K(D)$ and, therefore, must already be equal to the place Q.

On the other hand, let $\mathcal{O}_Q$ be the valuation ring of Q. Then the etaleness of φ at q also implies the second equality $Q = m_q = \varphi^*(m_p)\mathcal{O}_{D,q} = \varphi^*(m_p)\mathcal{O}_Q$ by [Liu02, p.139, Definition 3.17]. Therefore, the extension $Q/\varphi^*(m_p)$ of places in $K(D)/\varphi^*(K(C))$ is unramified and, hence, the 'if'-part follows.

For the 'only if'-part: Suppose that D is regular at q and that the extension $Q/\varphi^*(m_p)$ is unramified in the extension $K(D)/\varphi^*(K(C))$. Then, because q is a regular closed point on D, the same reasoning from the beginning can also be applied to q instead of p. By that, we conclude that m_q is a also place in $K(D)$ and, therefore, must already be equal to the place Q.

Now, on the one hand, the assertion that $Q/\varphi^*(m_p)$ is unramified supplies that φ is unramified at q. On the other hand, [Liu02, p.137, Corollary 3.10] provides that φ is flat at q since finite morphisms are non-constant by [Liu02, p.277, Lemma 3.10]. Then, by definition of etaleness in [Liu02, p.139, Definition 3.17], these two properties already include that φ is etale at q. Hence, the 'only if'-part also follows. ☐

Lemma 253. *Let $\psi : W \to Y$ and $\phi : Y \to X$ be finite morphisms of schemes and let w, y and x be points of W, Y and X, respectively, such that $\phi(y) = x$, and $\psi(w) = y$. Moreover, let*

$$\rho_W : W' := W \times_X \mathcal{O}_{X,x}, \quad \rho_Y : Y' := Y \times_X \mathcal{O}_{X,x} \to Y, \quad \rho_X : X' := X \times_X \mathcal{O}_{X,x} \to X$$

be the canonical projection morphisms. Finally, let $\eta_x \in X'$ with $\rho_X(\eta_x) = x$, let $\eta_y \in Y'$ with $\rho_Y(\eta_y) = y$ and $(\phi \times_X \mathrm{id})(\eta_y) = \eta_x$ and let $\eta_w \in W'$ with $\rho_W(\eta_w) = w$ and $(\psi \times_X \mathrm{id})(\eta_w) = \eta_y$.

Then the induced morphisms $\rho_W^* : \mathcal{O}_{W.w} \to \mathcal{O}_{W',\eta_w}$, $\rho_Y^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{Y',\eta_y}$ and $\rho_X^* : \mathcal{O}_{X,x} \to \mathcal{O}_{X',\eta_x}$ are isomorphisms of rings which make the second diagram in Figure A.1 commutative.

$$\begin{tikzcd}
W' := W \times_X \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\rho_W"] \arrow[d, "\psi \times_X id"] & W \arrow[d, "\psi"'] \\
Y' := Y \times_X \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\rho_Y"] \arrow[d, "\phi \times_X id"] & Y \arrow[d, "\phi"'] \\
X' := X \times_X \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r, "\rho_X"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}_{W',\eta_w} & \mathcal{O}_{W,w} \arrow[l, "\sim", "\rho_W^*"'] \\
\mathcal{O}_{Y',\eta_Y} \arrow[u, "(\psi \times_X id)^*"] & \mathcal{O}_{Y,y} \arrow[l, "\rho_Y^*", "\sim"'] \arrow[u, "\psi^*"'] \\
\mathcal{O}_{X',\eta_x} \arrow[u, "(\phi \times_X id)^*"] & \mathcal{O}_{X,x} \arrow[l, "\rho_X^*", "\sim"'] \arrow[u, "\phi^*"']
\end{tikzcd}$$

Figure A.1: First commutative diagram for local rings

Proof. The commutativity of the second diagram in Figure A.1 immediately follows from the fact that it is induced by the morphisms of schemes in the first commutative diagram in this figure. Thus, it is enough to show that ρ_Y^* is an isomorphism.

Now, for any scheme Z, any open subscheme U of Z and any point $z \in U$, the inclusion morphisms $U \to X$ induces an isomorphism of the local rings of z in Z and in U. Therefore, we can assume Y and X to be affine schemes $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(A)$. In particular, this means that x and y are prime ideals of A and B, respectively.

Moreover, we have $Y' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_x) \cong \mathrm{Spec}(B \otimes_A A_x) \cong \mathrm{Spec}(B_x)$. Hence, Y' is also an affine scheme and these isomorphisms map η_Y to the extension yB_x of y in the localization B_x of B at the prime ideal x. Moreover, ρ_Y^* corresponds to the canonical morphism $B_y \to (B_x)_{yB_x}$. But, it is well known that this is even an isomorphism. □

Let $\phi : Y \to X$ be a morphism of schemes and let $x \in X$. Then we denote the fiber $Y \times_X \kappa(x)$ of ϕ by Y_x where $\kappa(x)$ denotes the reside field of x.

Lemma 254. *Let X be a regular integral curve over a field k with generic point x, let $\psi : W \to Y$ and $\phi : Y \to X$ be finite morphisms of varieties over k and let $f := \phi \circ \psi$ be flat.*

Then W is a curve over k, $f^{-1}(x)$ consists of all the generic points of W and W_x is affine.

Moreover, there are isomorphisms

$$\prod_{w \in f^{-1}(x)} \mathcal{O}_{W,w} \to \mathcal{O}_{W_x}(W_x), \qquad \prod_{y \in \phi^{-1}(x)} \mathcal{O}_{Y,y} \to \mathcal{O}_{Y_x}(Y_x), \qquad \mathcal{O}_{X,x} \to \mathcal{O}_{X_x}(X_x)$$

such that the diagram in Figure A.2 is commutative where the vertical morphisms are canonical.

$$\begin{tikzcd}
\prod_{w \in f^{-1}(x)} \mathcal{O}_{W,w} \arrow[r, "\sim"] & \mathcal{O}_{W_x}(W_x) \\
\prod_{y \in \phi^{-1}(x)} \mathcal{O}_{Y,y} \arrow[r, "\sim"] \arrow[u] & \mathcal{O}_{Y_x}(Y_x) \arrow[u] \\
\mathcal{O}_{X,x} \arrow[r, "\sim"] \arrow[u] & \mathcal{O}_{X_x}(X_x) \arrow[u]
\end{tikzcd}$$

Figure A.2: Second commutative diagram for local rings

Proof. For the 'main'-part: First, since compositions of finite morphisms are finite by [GW10, p. 574], the composition $f = \phi \circ \psi$ is finite, Therefore, [GW10, p. 326, Remark 12.15(2)] provides that all fibers of f only contain finitely many points. Combining this and [Liu02, p. 82, Proposition 2.1.16] especially means that

the fiber scheme $W_{x'}$ only contains finitely many points for all $x' \in X$
and therefore has dimension zero. (577)

Second, we notice that X is a Dedekind scheme because it is a regular curve over k and because of the equivalence at the end of [Liu02, p. 128, Example 2.9]. Consequently, combining this, the assumption that f is flat, (577) and the 'moreover'-part in [Liu02, p. 137, Theorem 4.3.12] supplies that the local rings $\mathcal{O}_{W,w}$ and $\mathcal{O}_{X,x'}$ have the same dimension for all $w \in f^{-1}(x')$ and all $x' \in X$. In particular, this implies

$$\dim(W) \leq \dim(X) = 1. \tag{578}$$

Also, as the generic points of W can be characterized as the points w in W such that $\mathcal{O}_{W,w}$ is zero-dimensional, we derive the second desired statement, namely that $f^{-1}(x)$ precisely consists of all the generic points in W.

Next, let $W_1, \dots, W_r$ be the irreducible components of W, let w_i be the generic point of W_i in W and let f_i be the restriction of f to a finite morphism $W_i \to X$ for all $i = 1, \dots, r$.

Since $f_i(w_i) = f(w_i) = x$, [GW10, p. 126, Proposition 5.22(3)] then implies the estimate $\dim(W_i) \geq \dim(X)$. In particular, by this and by (578), we also obtain

$$1 \geq \dim(W) \geq \dim(W_i) \geq \dim(X) = 1.$$

Hence, these estimates must be equalities and we established that all irreducible components W_i of W have dimension one. By the definition of curves over k in Definition 97, we therefore conclude the first desired statement, namely that W is a curve over k.

For the 'moreover'-part: First, we notice that the canonical morphisms $\psi_x : W_x \to Y_x$ and $\phi_x : Y_x \to X_x$ induce morphisms of rings which make the diagram in Figure A.3 commutative for all $\xi \in W_x$, $\psi_x(\xi) = \eta$ and $\phi_x(\eta) = \eta_x$. Next, we take the canonical product

$$\begin{tikzcd}
\mathcal{O}_{W_x,\xi} & \mathcal{O}_{W_x}(W_x) \arrow[l] \\
\mathcal{O}_{Y_x,\eta} \arrow[u] & \mathcal{O}_{Y_x}(Y_x) \arrow[l] \arrow[u] \\
\mathcal{O}_{X_x,\eta_x} \arrow[u] & \mathcal{O}_{X_x}(X_x) \arrow[l] \arrow[u]
\end{tikzcd}$$

Figure A.3: Third commutative diagram for local rings

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y_x,\eta} \to \prod_{\xi \in \psi_x^{-1}(\eta)} \mathcal{O}_{W_x,\xi} \tag{579}$$

for all $\eta \in (\phi \times \mathrm{id})^{-1}(\eta_x) = (\phi \times \mathrm{id})^{-1}(X_x) = Y_x$ where the first equality holds since X_x only consists of the point η_x. If we then again take the canonical product of these morphisms in (579) over all $\eta \in Y_x$, we morphisms

$$\mathcal{O}_{X_x,\eta_x} \to \prod_{\eta \in Y_x} \mathcal{O}_{Y_x,\eta} \to \prod_{\xi \in W_x} \mathcal{O}_{W_x,\xi}$$

which make the two right squares in the diagram in Figure A.4 commutative.

Second, we notice that the reasonings in the proof of the 'main'-part also apply to $\phi : Y \to X$ (just choose $\psi = \mathrm{id}$) and $\mathrm{id}_X : X \to X$ (just choose $\psi = \mathrm{id} = \phi$). Thus, combining (577) and [GW10, p. 125, Proposition 5.20(ii)] yields that the horizontal maps in the left two squares in the diagram in Figure A.4 are even isomorphisms.

Finally, the canonical products of the isomorphisms in Lemma 253 provide the isomorphisms in the left half of the commutative diagram in Figure A.4 since the local rings of generic points are already their residue fields. Hence, the composition of the isomorphisms on the rows in this diagram are the desired isomorphisms which make the diagram in Figure A.2 commutative. □

$$\begin{tikzcd}
\prod\limits_{w\in f^{-1}(x)} \mathcal{O}_{W,w} \arrow[r, "\sim"] & \prod\limits_{\xi\in W_x} \mathcal{O}_{W_x,\xi} & \mathcal{O}_{W_x}(W_x) \arrow[l, "\sim"'] \\
\prod\limits_{y\in \phi^{-1}(x)} \mathcal{O}_{Y,y} \arrow[r, "\sim"] \arrow[u] & \prod\limits_{\eta\in Y_x} \mathcal{O}_{Y_x,\eta} \arrow[u] & \mathcal{O}_{Y_x}(Y_x) \arrow[l, "\sim"'] \arrow[u] \\
\mathcal{O}_{X,x} \arrow[r, "\sim"] \arrow[u] & \mathcal{O}_{X_x,\eta_x} \arrow[u] & \mathcal{O}_{X_x}(X_x) \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

Figure A.4: Fourth commutative diagram for local rings

We denote the function fields of integral curves C by $K(C)$ and the pullback of finite morphisms $\psi : D \to C$ of integral curves by $\psi^* : K(D) \to K(C)$ which makes $K(C)$ to an $K(D)$-algebra.

Lemma 255. *Let X be an regular integral curve over a field k and let $\phi_X : Y \to X$ and $\phi_Z : Z \to X$ be finite morphisms of integral curves over k. Moreover, let $Y \times_X Z$ be the fiber product of k-schemes with the canonical projection morphisms $\rho_Y : Y \times_X Z \to Y$ and $\rho_Z : Y \times_X Z \to Z$ Then the following hold:*

(i) $Y \times_X Z$ is a curve over k and ρ_Y and ρ_Z are finite.

(ii) $Y \times_X Z$ is integral if and only if the tensor product $K(Y) \otimes_{K(X)} K(Z)$ of k-algebras is a field.

(iii) If $Y \times_X Z$ is an integral curve over k, then the function field $K(Y \times_X Z)$ is a tensor product $K(Y) \otimes_{K(X)} K(Z)$ of k-algebras with the structural morphisms ρ_Y^* and ρ_Z^*.

Proof. For (i): Let us shortly write $W := Y \times_X Z$. Then we first notice that [GW10, p. 574] already supplies the two last desired statements in (i), namely that ρ_Y and ρ_Z are finite. We even obtain that the composition

$$f := \phi_Y \circ \rho_Y = \phi_Z \circ \rho_Z : W \to X$$

is a finite morphism. Moreover, combining the equivalence at the end of [Liu02, p. 128, Example 2.9], [Liu02, p. 137, Corollary 3.10] and [Liu02, p. 136, Proposition 4.3.3(d)] yields that f is also flat.

Next, we also notice that [Liu02, p. 88, Example 2.2.3] and [Liu02, p. 88, Proposition 2.2.4(b),(c)] provide that W is a variety over k. Hence, Lemma 254 also supplies that W is a curve over k.

For (ii): Before, we come to the proof of the equivalence, we need to make some observations: First, because Y, Z and X are integral, [Liu02, p. 65, Proposition 2.4.18] supplies that any of these curves has exactly one generic point, say y, z and x, respectively, and that the corresponding local rings $\mathcal{O}_{Y,y}$, $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{X,x}$ are the function fields $K(Y)$, $K(Z)$ and $K(X)$, respectively. In particular, this means that the local rings $\mathcal{O}_{Y,y}$, $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{X,x}$ are already the residue fields of x, y and z, respectively.

Second, we apply the 'moreover'-part of Lemma 254 separately to the morphisms $\rho_Y : W \to Y$ and $\phi_Y : Y \to X$ and to the morphisms $\rho_Z : W \to Z$ and $\phi_Z : Z \to X$ and obtain the corresponding commutative diagram for Y and Z. Moreover, since Y and Z are integral and thus only have the generic point y and z, respectively, the products in the middle of the commutative diagrams are just $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Z,z}$. In particular, we notice that these two commutative diagrams and the universal property of tensor products of the $\mathcal{O}_{X,x}$-algebras provide morphisms such that the first diagram in Figure A.5 is commutative.

We claim that the morphism

$$\rho : K(Y) \otimes_{K(X)} K(Z) = \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} \to \prod_{w \in f^{-1}(x)} \mathcal{O}_{W,w} \tag{580}$$

in this diagram is even an isomorphism. To prove this claim, we will show that the

$$\begin{tikzcd}
& \prod\limits_{w \in f^{-1}(x)} \mathcal{O}_{W,w} \arrow[rrr, "\sim"] & & & \mathcal{O}_{W_x}(W_x) & \\
& \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} \arrow[u, "\rho", "\wr"'] \arrow[rrr, "\sim"] & & & \mathcal{O}_{Y_x}(Y_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z_x}(Z_x) \arrow[u, "\wr", "\rho'"'] & \\
\mathcal{O}_{Y,y} \arrow[ur] \arrow[rrr, bend right, "\sim"] & & \mathcal{O}_{Z,z} \arrow[ul] \arrow[rrr, bend left, "\sim"] & \mathcal{O}_{Y_x}(Y_x) \arrow[ur] & & \mathcal{O}_{Z_x}(Z_x) \arrow[ul] \\
& \mathcal{O}_{X,x} \arrow[ul] \arrow[ur] \arrow[rrr, equal] & & & \mathcal{O}_{X,x} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& W_x \arrow[d, "g"', "\wr"] & \\
& Y_x \times_{\mathrm{Spec}(\mathcal{O}_{X,x})} Z_x \arrow[dl] \arrow[dr] & \\
Y_x \arrow[dr] & & Z_x \arrow[dl] \\
& \mathrm{Spec}(\mathcal{O}_{X,x}) &
\end{tikzcd}$$

Figure A.5: Fifth commutative diagram for local rings

morphism

$$\rho' : \mathcal{O}_{Y_x}(Y_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z_x}(Z_x) \to \mathcal{O}_{W_x}(W_x)$$

in this diagram is an isomorphism. Since the horizontal morphisms in the corresponding square of this diagram are also isomorphisms and since the square is commutative, we then indeed obtain that ρ is an isomorphism.

Now, in order to show that ρ' is an isomorphism, we notice that since W_x, Y_x and Z_x are affine schemes, the right half of the first diagram in Figure A.5 is induced by the second commutative diagram in this figure. Therefore, we conclude that ρ' is an isomorphism if $g : W_x \to Y_x \times_{\mathrm{Spec}(\mathcal{O}_{X,x})} Z_x$ is an isomorphism.

By the universal property of fiber products, the morphism g is the unique morphism which makes the second diagram in Figure A.5 commutative. But the canonical isomorphism

$$\begin{aligned} W_x &= (Y \times_X Z) \times_X \mathrm{Spec}(\mathcal{O}_{X,x}) \cong (Y \times_X \mathrm{Spec}(\mathcal{O}_{X,x})) \times_{\mathrm{Spec}(\mathcal{O}_{X,x})} (Z \times_X \mathrm{Spec}(\mathcal{O}_{X,x})) \\ &= Y_x \times_{\mathrm{Spec}(\mathcal{O}_{X,x})} Z_x \end{aligned}$$

in [Bos13, p. 302, Proposition 13(ii)] also makes the second diagram in Figure A.5 commutative. Therefore, g must be this canonical isomorphism and we are ready with our observation.

Next, we come to the proof of the desired equivalence in (ii): On the one hand, we conclude that

$$\begin{aligned} &W = Y \times_X Z \text{ is integral if and only if } W \text{ has exactly one generic point} \\ &\text{and the local ring of this generic point is a field.} \end{aligned} \tag{581}$$

where the implications hold by the following reasonings: The 'only if'-part immediately follows from [Liu02, p. 65, Proposition 2.4.18(a)]. For the 'if'-part, we first notice [Liu02, p. 137, Corollary 3.9] implies that ρ_Y and ρ_Z are flat since Y and Z are integral and X is regular. Consequently, by applying [Liu02, p. 136, Proposition 4.3.3], we also obtain that $f : W \to X$ is flat and can apply [Liu02, p. 137, Proposition 4.3.8]. Then the 'if'-part follows because the generic fiber of W_x is affine by Lemma 254, because this implies that W_x is integral if $\mathcal{O}_{W_x}(W_x)$ is an integral domain and because of the isomorphism in the top row in the first diagram in Figure A.5.

On the other hand, the isomorphism ρ in (580) supplies that the tensor product

$$K(Y) \otimes_{K(X)} K(Z) \text{ is a field if and only if } W \text{ has exactly one generic point and that the local ring of this generic point is a field.} \tag{582}$$

Finally, combining the equivalences in (581) and (582) then yields the desired equivalence in (ii).

For (iii): Suppose that W is integral. Then (580) even supplies that ρ is an isomorphism $K(Y) \otimes_{K(X)} K(Z) \to \mathcal{O}_{W,w}$ where w is the unique generic point w of W. But this local ring is exactly the function field $K(Y \times_X Z)$ of $W = Y \times_X Z$. Moreover, the morphisms $K(Y) \to K(Y \times_X Z)$ and $K(Z) \to K(Y \times_X Z)$ are also the pullbacks ρ_Y^* and ρ_Z^*. Hence, (iii) also follows. □

B Figures of Tower Graphs

This chapter contains figures displaying the degree one or degree two subgraphs of the tower graphs for all recursive towers which we use as examples (see Examples 8 and Figure 4.1). Note that the displayed data in the figures are explained thoroughly in Examples 75.

B.1 From [MW05]

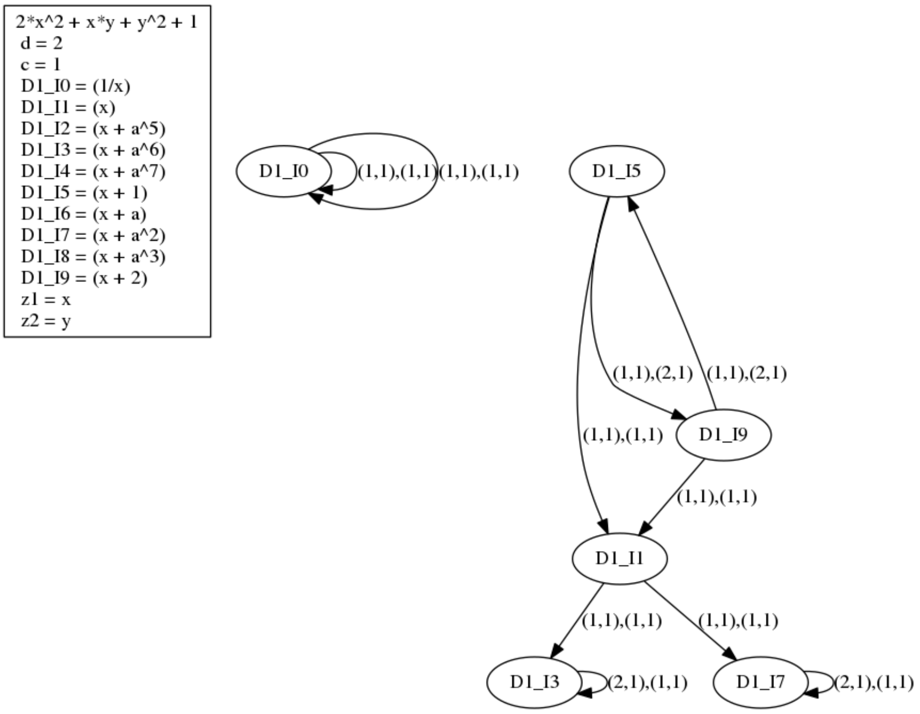


Figure B.1: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,2} = (F_\nu)_\nu$ in [MW05, p. 212, f_2] over $\mathbb{F}_9$ which is defined by the polynomial $f_{MW,2} = Y^2 + XY + 2X^2 + 1$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,2}) = 2/3$ and $g(F_1) = 0$. Here the first weakly connected component is the splitting subgraph and the second is the ramification subgraph.

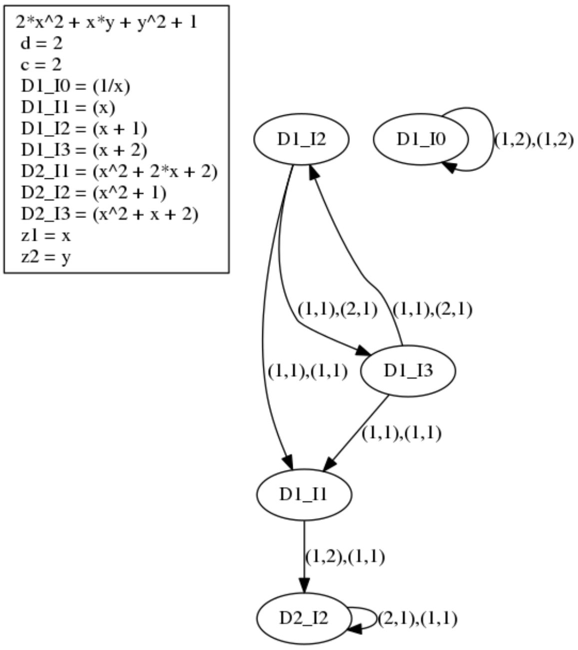


Figure B.2: Degree two subgraph of the tower graph of the tame recursive tower $\mathcal{F}'_{MW,2} = (F_\nu)_\nu$ in [MW05, p. 212, f_2] but over $\mathbb{F}_3$ instead of $\mathbb{F}_9$. See Examples 8(i). This tower is defined by the polynomial $f_{MW,2} = Y^2 + XY + 2X^2 + 1$ and satisfies $\lambda(\mathcal{F}_{MW,2}) = 0$ and $g(F_1) = 0$. Here the first weakly connected component is the splitting subgraph and the second is the ramification subgraph.

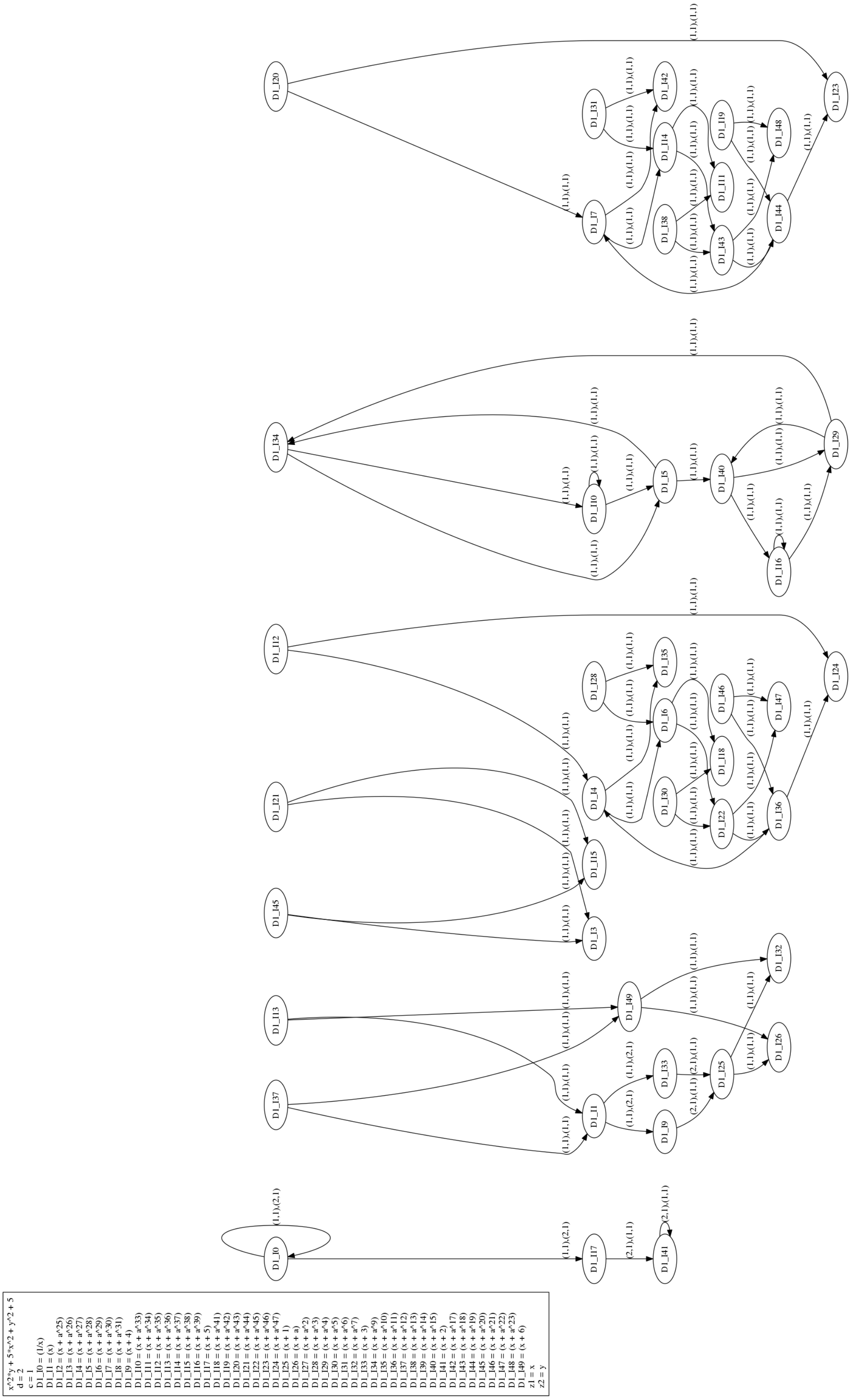


Figure B.3: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,3} = (F_\nu)_\nu$ in [MW05, p. 214, f_3] over $\mathbb{F}_{49}$ which is defined by the polynomial $f_{MW,3} = Y^2 + X^2Y + 5X^2 + 5$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,3}) = 6$ and $g(F_1) = 1$. Here the first weakly connected component is a weakly connected component of the ramification subgraph and the fifth is the splitting subgraph.

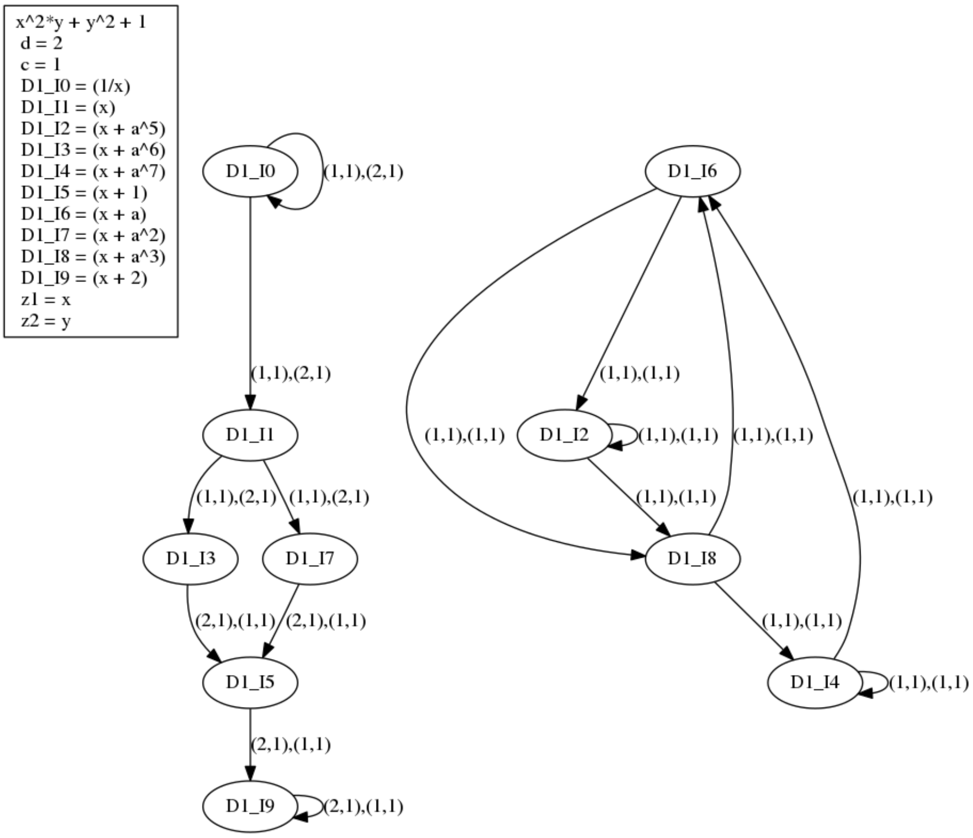


Figure B.4: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,4} = (F_\nu)_\nu$ in [MW05, p. 212, f_4] over $\mathbb{F}_9$ which is defined by the polynomial $f_{MW,4} = Y^2 + X^2Y + 1$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,4}) = 2$ and $g(F_1) = 1$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

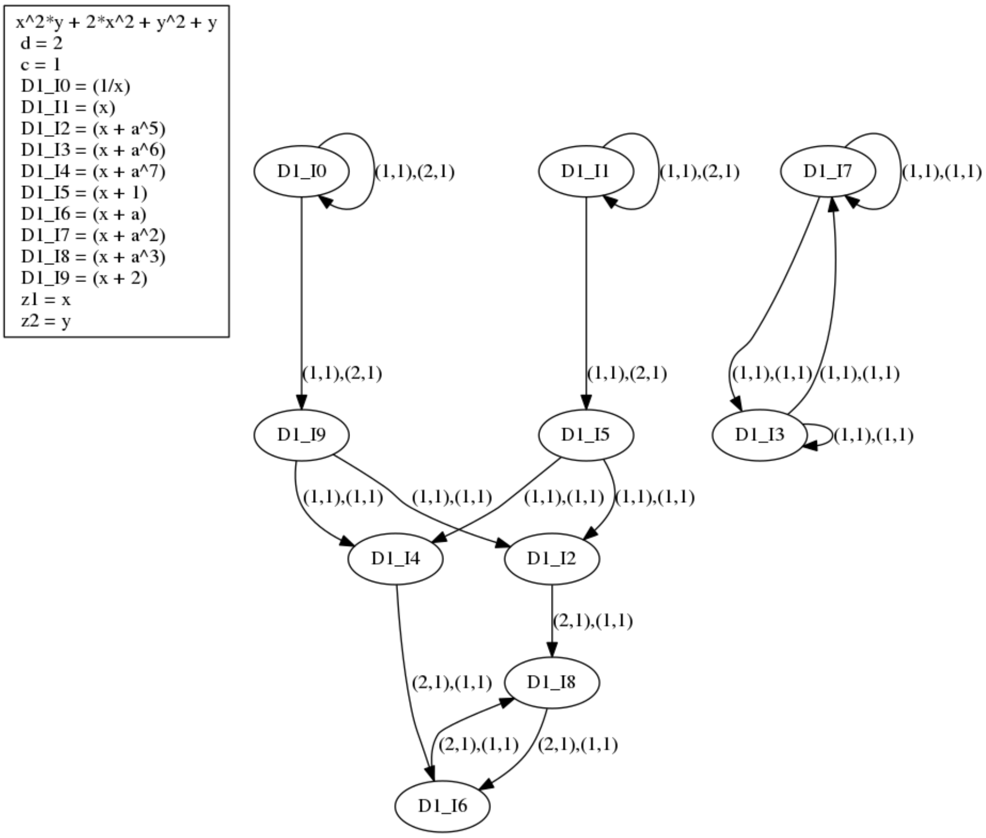


Figure B.5: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,6} = (F_\nu)_\nu$ in [MW05, p. 212, f_6] over $\mathbb{F}_9$ which is defined by the polynomial $f_{MW,6} = Y^2 + (X^2+1)Y + 2X^2$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,6}) = 2/3$ and $g(F_1) = 1$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

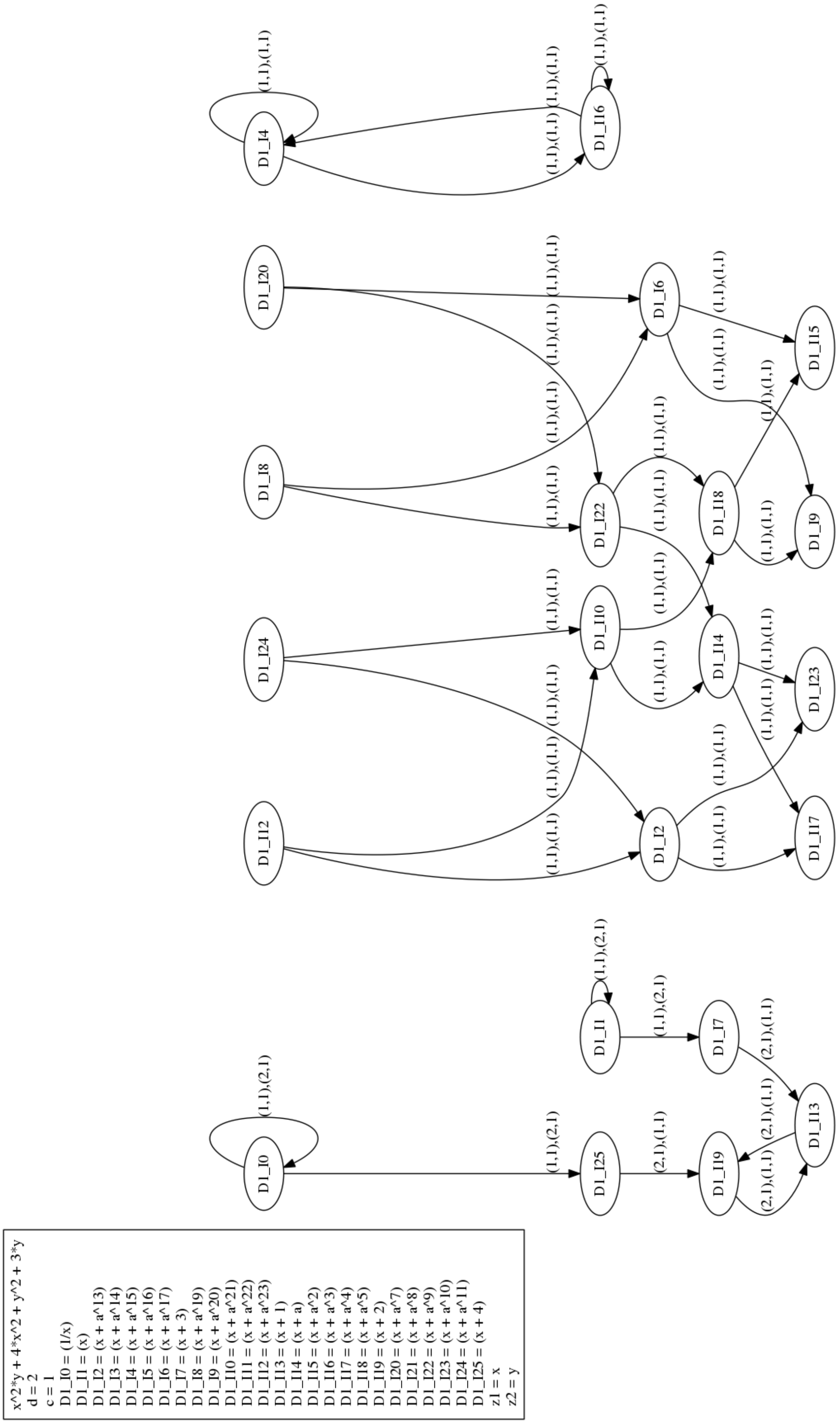


Figure B.6: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,8} = (F_\nu)_\nu$ in [MW05, p. 213, f_8] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,8} = Y^2 + (X^2 + 3)Y + 4X^2$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,8}) = 1$ and $g(F_1) = 1$. Here the first weakly connected component is the ramification subgraph and the third is the splitting subgraph.

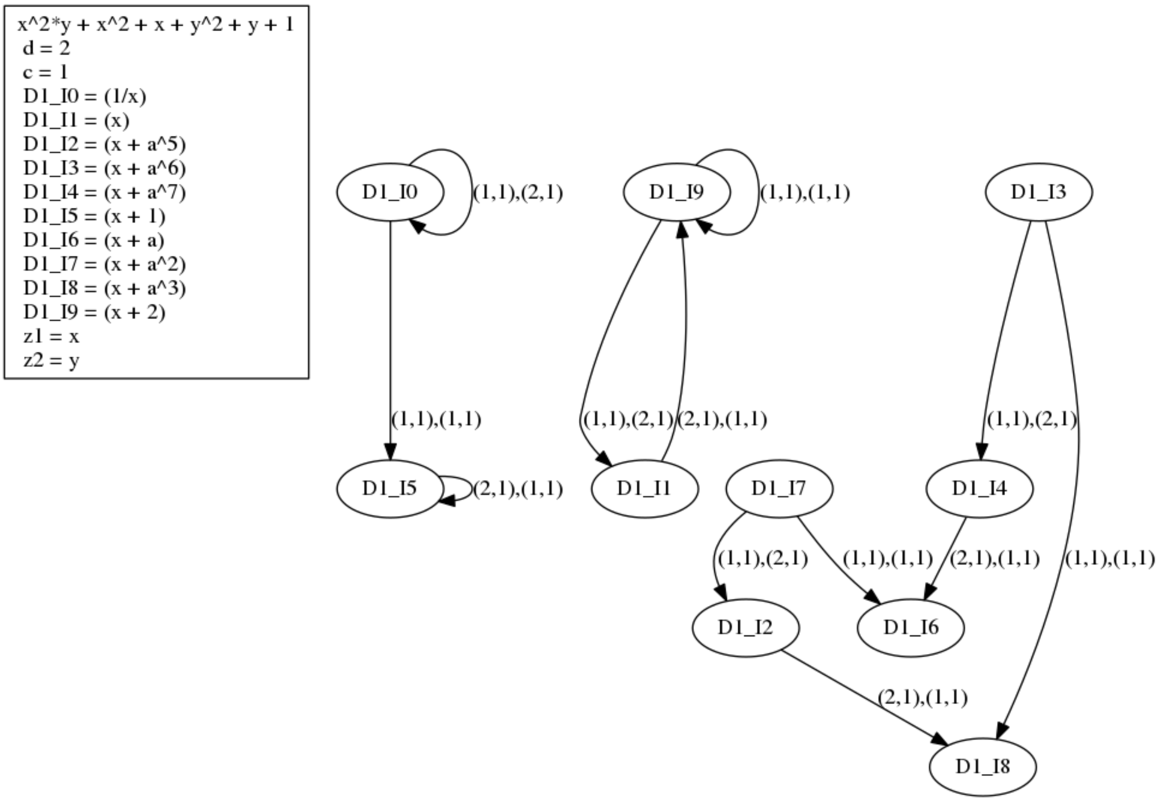


Figure B.7: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,11} = (F_\nu)_\nu$ in [MW05, p. 212, f_{11}] over $\mathbb{F}_9$ which is defined by the polynomial $f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,11}) = 2$ and $g(F_1) = 1$. Here the first two weakly connected components are weakly connected components of the ramification subgraph.

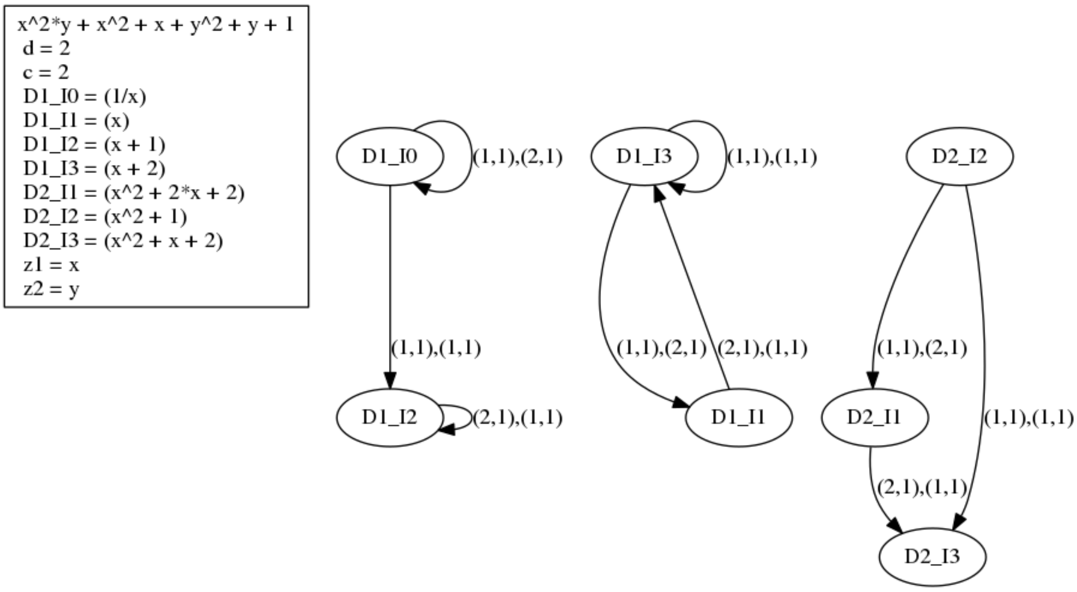


Figure B.8: Degree two subgraph of the tower graph of the tame recursive tower $\mathcal{F}'_{MW,11} = (F_\nu)_\nu$ in [MW05, p. 212, f_{11}] but over $\mathbb{F}_3$ instead of $\mathbb{F}_9$. This tower is defined by the polynomial $f_{MW,11} = Y^2 + (X^2+1)Y + X^2 + X + 1$ and satisfies $\lambda(\mathcal{F}'_{MW,11}) = 0$ and $g(F_1) = 1$. See Examples 8(i). Here the first two weakly connected components are weakly connected components of the ramification subgraph.

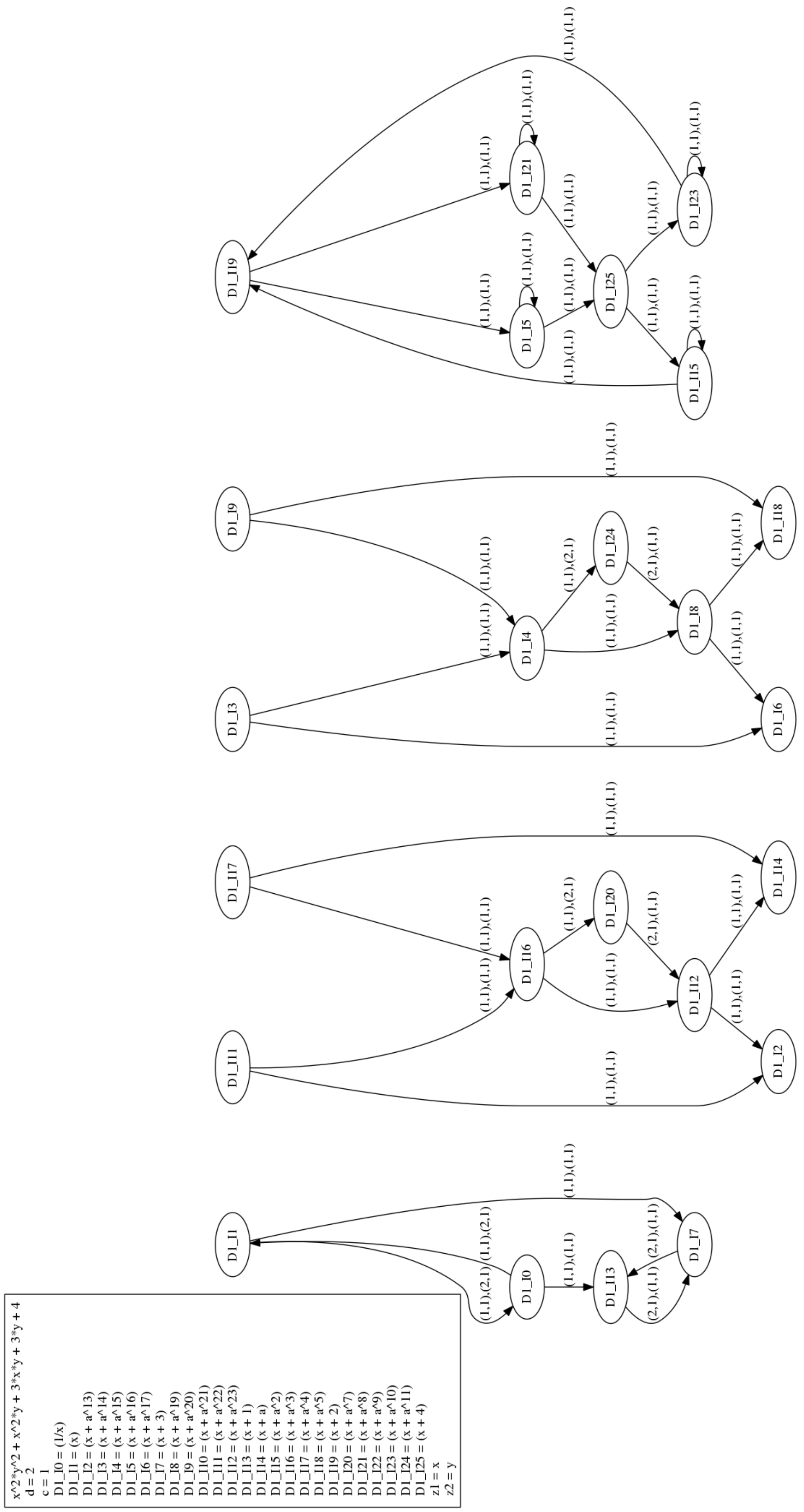


Figure B.9: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,12} = (F_\nu)_\nu$ in [MW05, p. 213, f_{12}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,12} = X^2Y^2 + (X^2 + 3X + 3)Y + 4$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,12}) = 4$ and $g(F_1) = 1$. Here the first weakly connected components is a weakly connected component of the ramification subgraph and the fourth is the splitting subgraph.

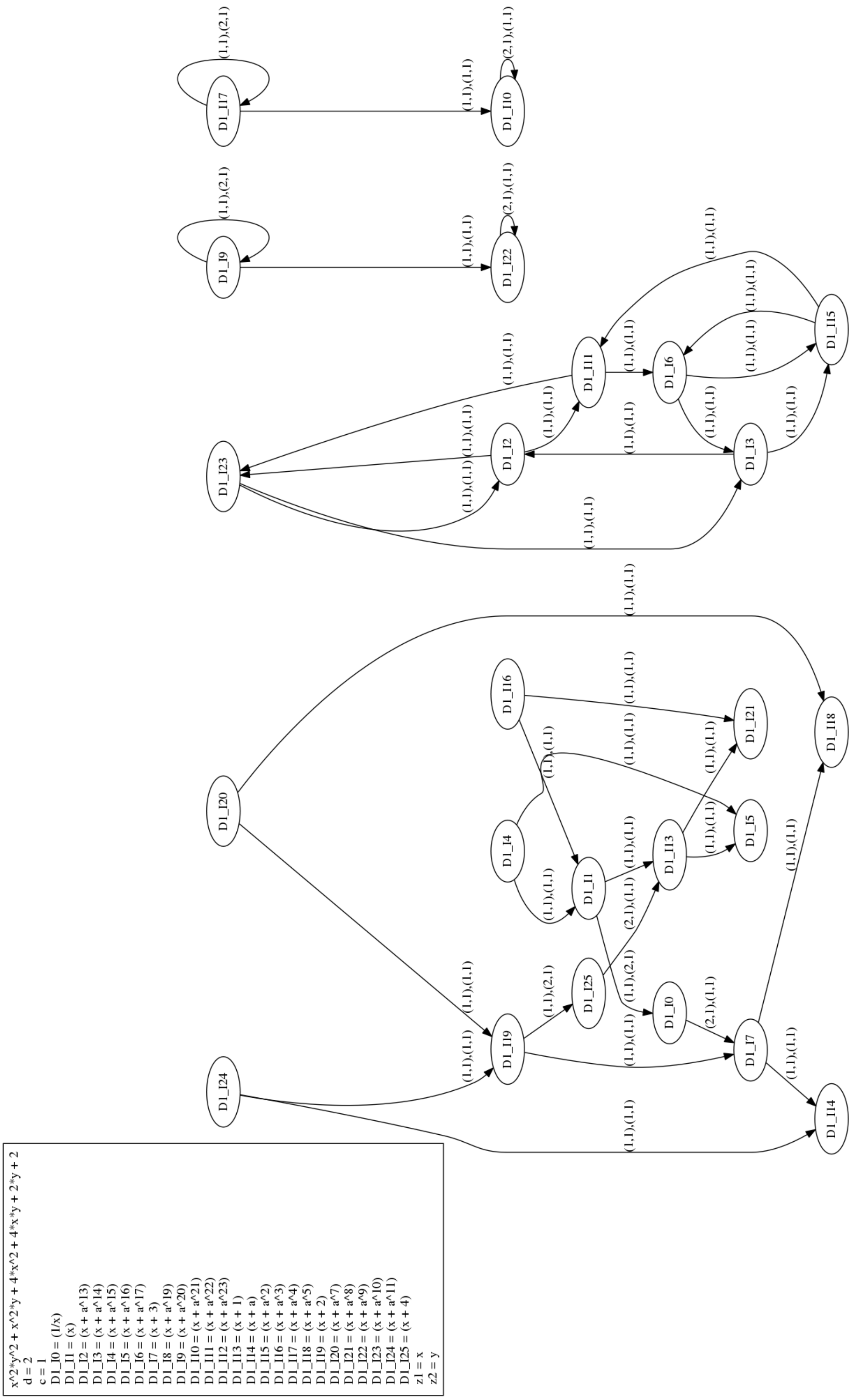


Figure B.10: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,14} = (F_\nu)_\nu$ in [MW05, p. 213, f_{14}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,14} = X^2Y^2 + (X^2 + 4X + 2)Y + 4X^2 + 2$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,14}) = 4$ and $g(F_1) = 1$. Here the second weakly connected component is the splitting subgraph and the last two are weakly connected components of the ramification subgraph.

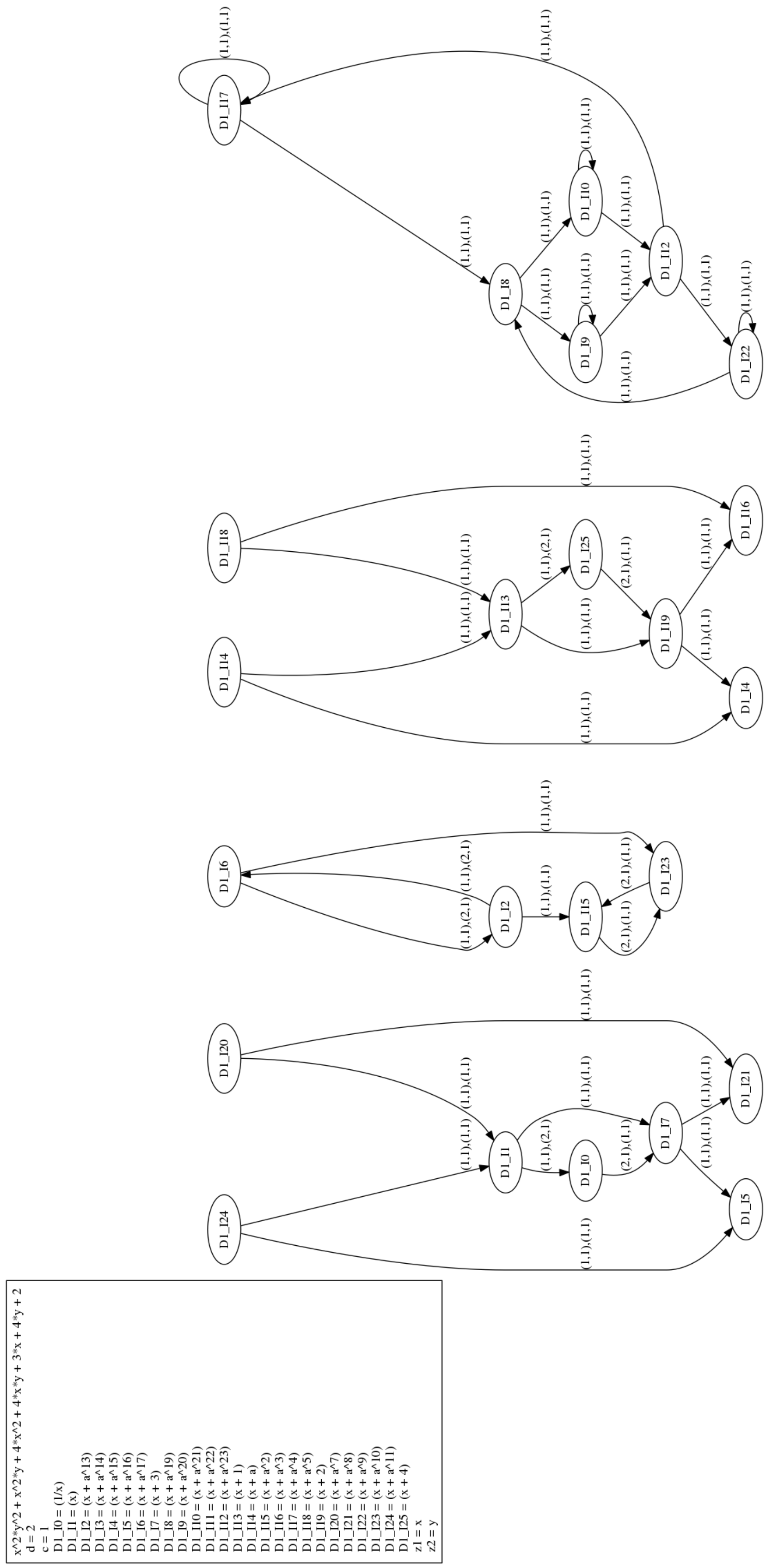


Figure B.11: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,15} = (F_\nu)_\nu$ in [MW05, p. 213, f_{15}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,15} = X^2Y^2 + (X^2 + 4X + 4)Y + 4X^2 + 3X + 2$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,15}) = 4$ and $g(F_1) = 1$. Here the second weakly connected component is a weakly connected components of the ramification subgraph and the fourth is the splitting subgraph.

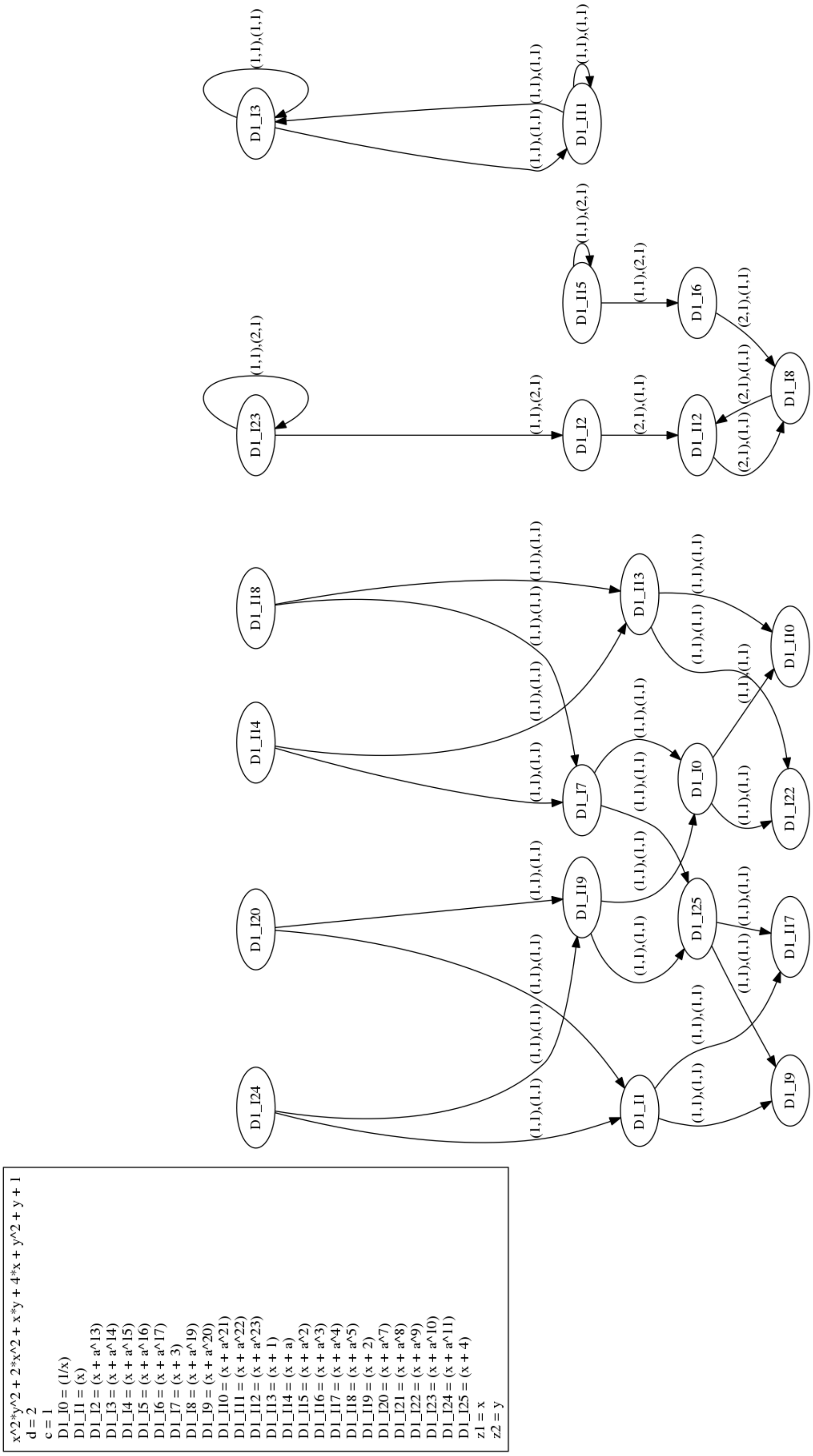


Figure B.12: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,16} = (F_\nu)_\nu$ in [MW05, p. 213, f_{16}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,16} = (X^2+1)Y^2 + (X+1)Y + 2X^2 + 4X + 1$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,16}) = 1$ and $g(F_1) = 1$. Here the second weakly connected component is the ramification subgraph and the third is the splitting subgraph.

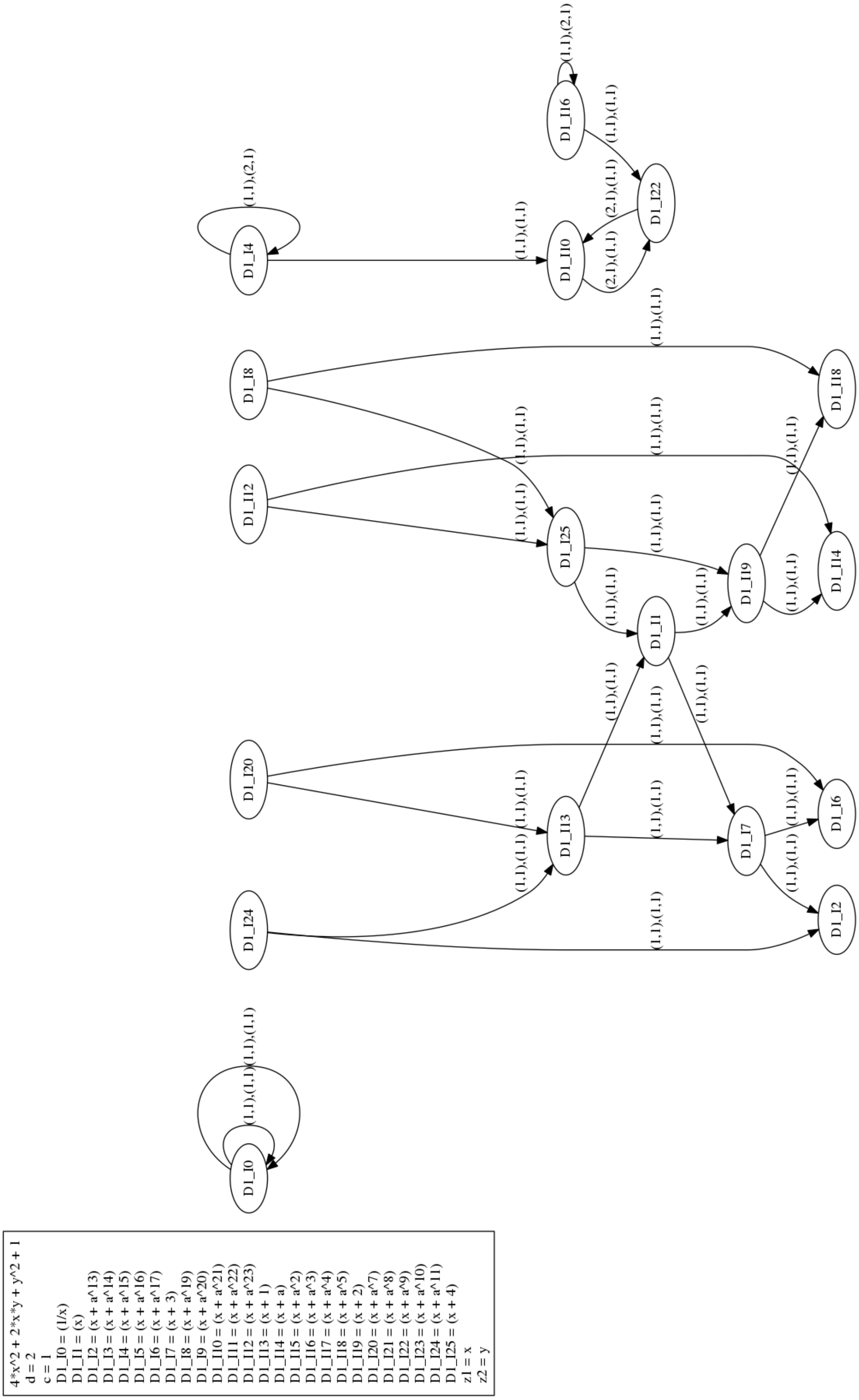


Figure B.13: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,20} = (F_\nu)_\nu$ in [MW05, p. 213, f_{20}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,20} = Y^2 + 2XY + 4X^2 + 1$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,20}) = 1$ and $g(F_1) = 0$. Here the first weakly connected component is the splitting subgraph and the third is the ramification subgraph.

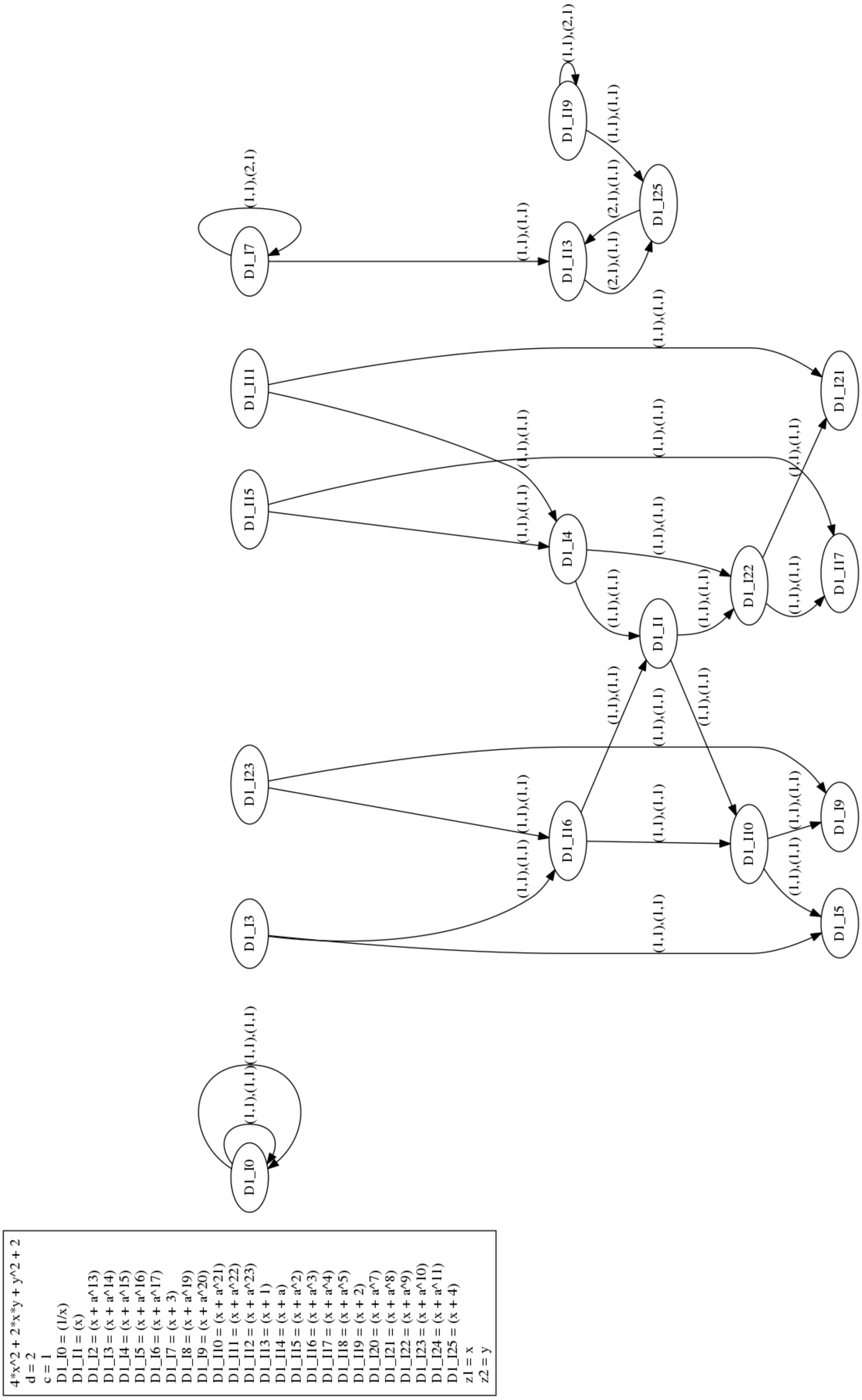


Figure B.14: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,21} = (F_\nu)_\nu$ in [MW05, p. 213, f_{21}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,21} = Y^2 + 2XY + 4X^2 + 2$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,21}) = 1$ and $g(F_1) = 0$. Here the first weakly connected component is the splitting subgraph and the third is the ramification subgraph.

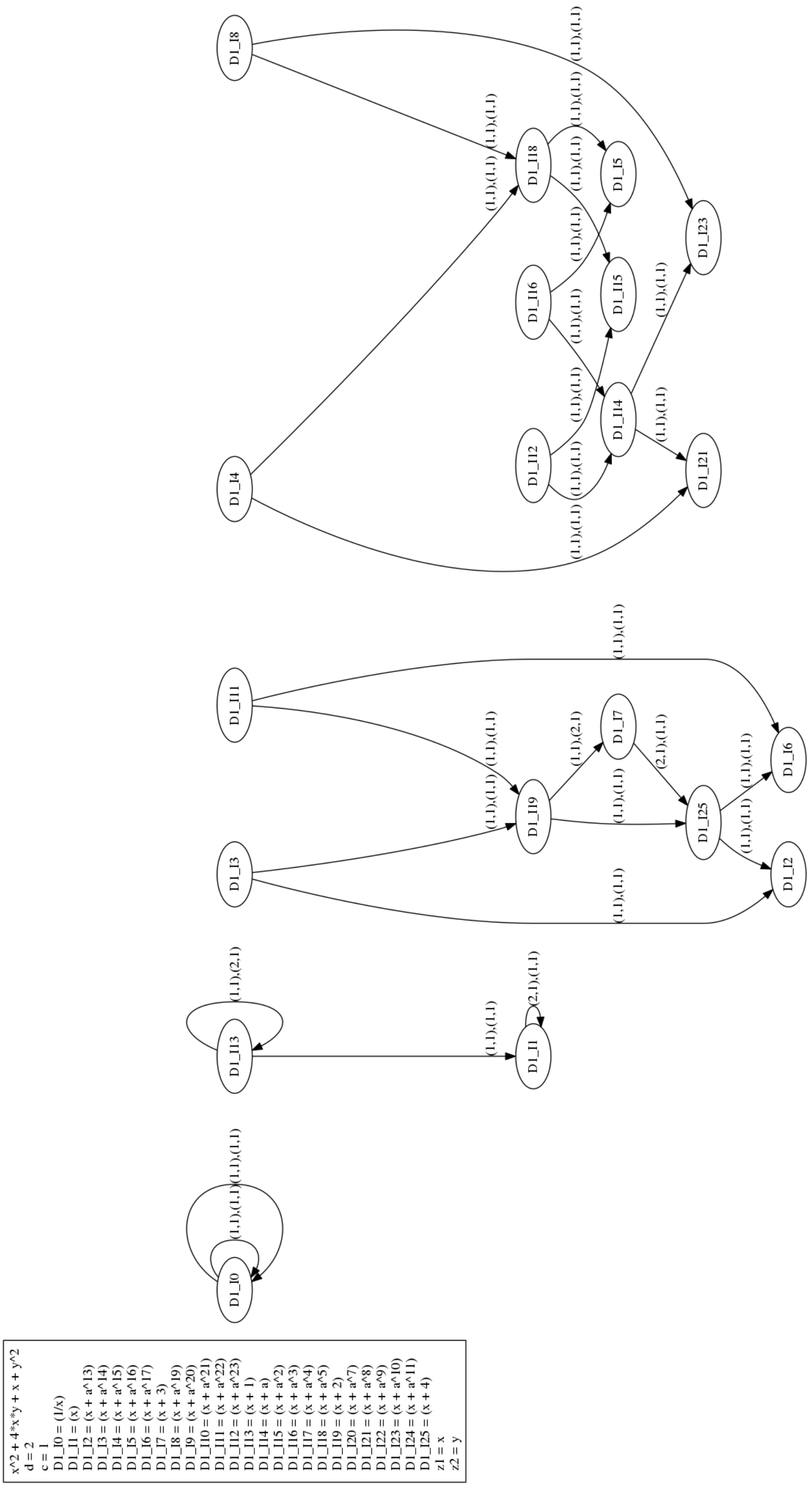
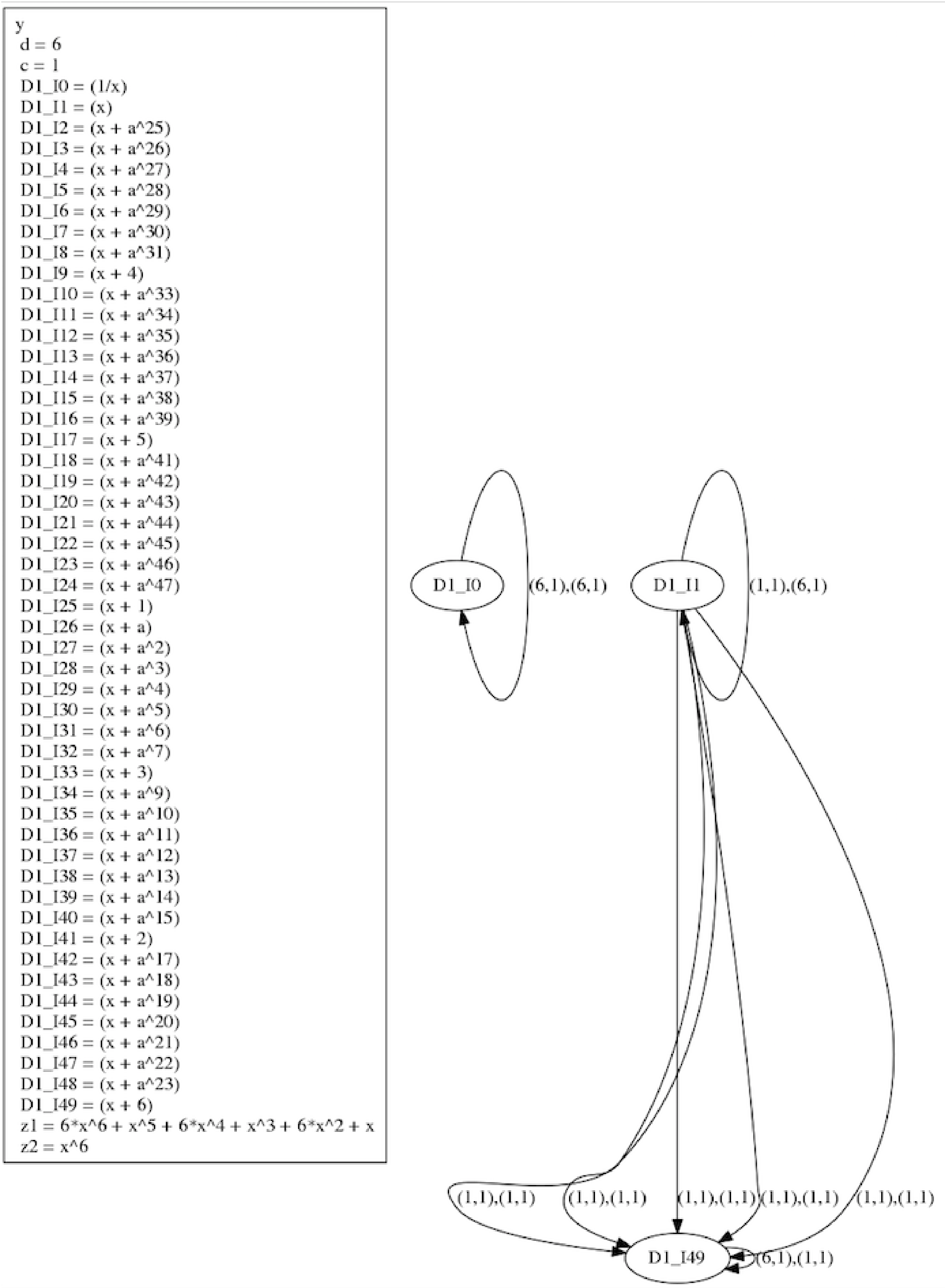


Figure B.15: Degree one subgraph of the tame recursive tower $\mathcal{F}_{MW,22} = (F_\nu)_\nu$ in [MW05, p. 213, f_{22}] over $\mathbb{F}_{25}$ which is defined by the polynomial $f_{MW,22} = Y^2 + 4XY + X^2 + X$. See Examples 8(i). This tower satisfies $\lambda(\mathcal{F}_{MW,22}) = 4$ and $g(F_1) = 0$. Here the first weakly connected component is the splitting subgraph and the second is a weakly connected component of the ramification subgraph.

B.2 From [Sti08]



```
y
d = 6
c = 1
D1_I0 = (1/x)
D1_I1 = (x)
D1_I2 = (x + a^25)
D1_I3 = (x + a^26)
D1_I4 = (x + a^27)
D1_I5 = (x + a^28)
D1_I6 = (x + a^29)
D1_I7 = (x + a^30)
D1_I8 = (x + a^31)
D1_I9 = (x + 4)
D1_I10 = (x + a^33)
D1_I11 = (x + a^34)
D1_I12 = (x + a^35)
D1_I13 = (x + a^36)
D1_I14 = (x + a^37)
D1_I15 = (x + a^38)
D1_I16 = (x + a^39)
D1_I17 = (x + 5)
D1_I18 = (x + a^41)
D1_I19 = (x + a^42)
D1_I20 = (x + a^43)
D1_I21 = (x + a^44)
D1_I22 = (x + a^45)
D1_I23 = (x + a^46)
D1_I24 = (x + a^47)
D1_I25 = (x + 1)
D1_I26 = (x + a)
D1_I27 = (x + a^2)
D1_I28 = (x + a^3)
D1_I29 = (x + a^4)
D1_I30 = (x + a^5)
D1_I31 = (x + a^6)
D1_I32 = (x + a^7)
D1_I33 = (x + 3)
D1_I34 = (x + a^9)
D1_I35 = (x + a^10)
D1_I36 = (x + a^11)
D1_I37 = (x + a^12)
D1_I38 = (x + a^13)
D1_I39 = (x + a^14)
D1_I40 = (x + a^15)
D1_I41 = (x + 2)
D1_I42 = (x + a^17)
D1_I43 = (x + a^18)
D1_I44 = (x + a^19)
D1_I45 = (x + a^20)
D1_I46 = (x + a^21)
D1_I47 = (x + a^22)
D1_I48 = (x + a^23)
D1_I49 = (x + 6)
z1 = 6*x^6 + x^5 + 6*x^4 + x^3 + 6*x^2 + x
z2 = x^6
```

Figure B.16: The degree two subgraph of the tame recursive tower $\mathcal{F}_{GS,1} = (F_\nu)_\nu$ in [Sti08, p. 260, Proposition 7.3.2] for $l = 7$ over $\mathbb{F}_{l^2}$ which is defined by the polynomial $f_{GS,1} = Y^{l-1} + (X+1)^{l-1} - 1$ from the first level on. See Examples 8(ii). We added a zeroth level F_0 to $\mathcal{F}_{GS,1}$ via the Reduction Lemma 30(iii). This tower satisfies $\lambda(\mathcal{F}_{GS,1}) = 2/(l-1)$ and $g(F_1) = 0$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

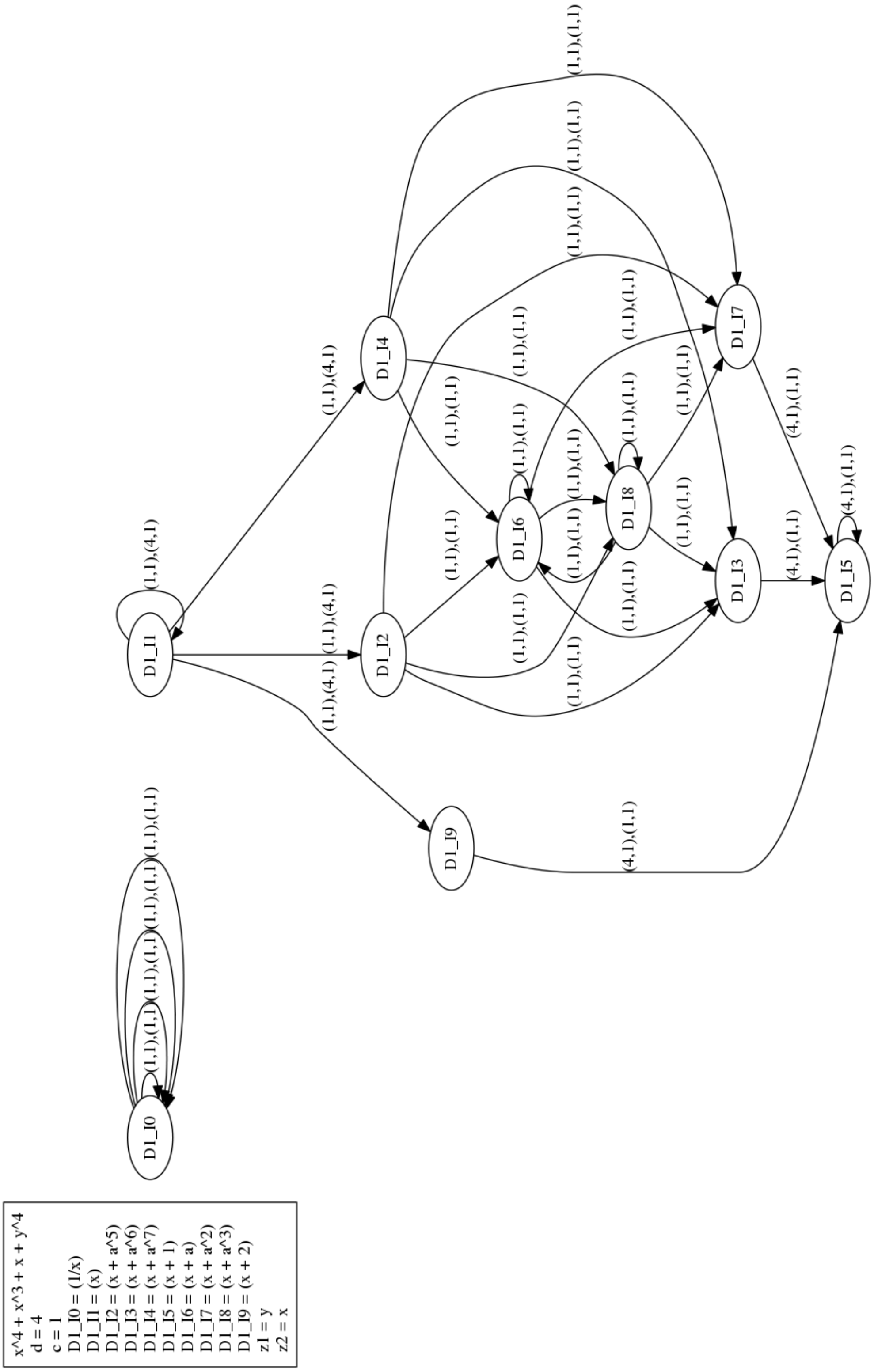


Figure B.17: Degree one subgraph of the tame recursive tower $\mathcal{F}_{GS,2} = (F_\nu)_\nu$ in [Sti08, p. 261, Proposition 7.3.3] for $l = 3$ and $e = 2$ over $\mathbb{F}_q$ with $q = l^e$ which is defined by the polynomial $f_{GS,2} = Y^m + (X+1)^m - 1$ with $m = \frac{q-1}{l-1}$. See Examples 8(ii). This tower satisfies $\lambda(\mathcal{F}_{GS,2}) = 2/(q-1)$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

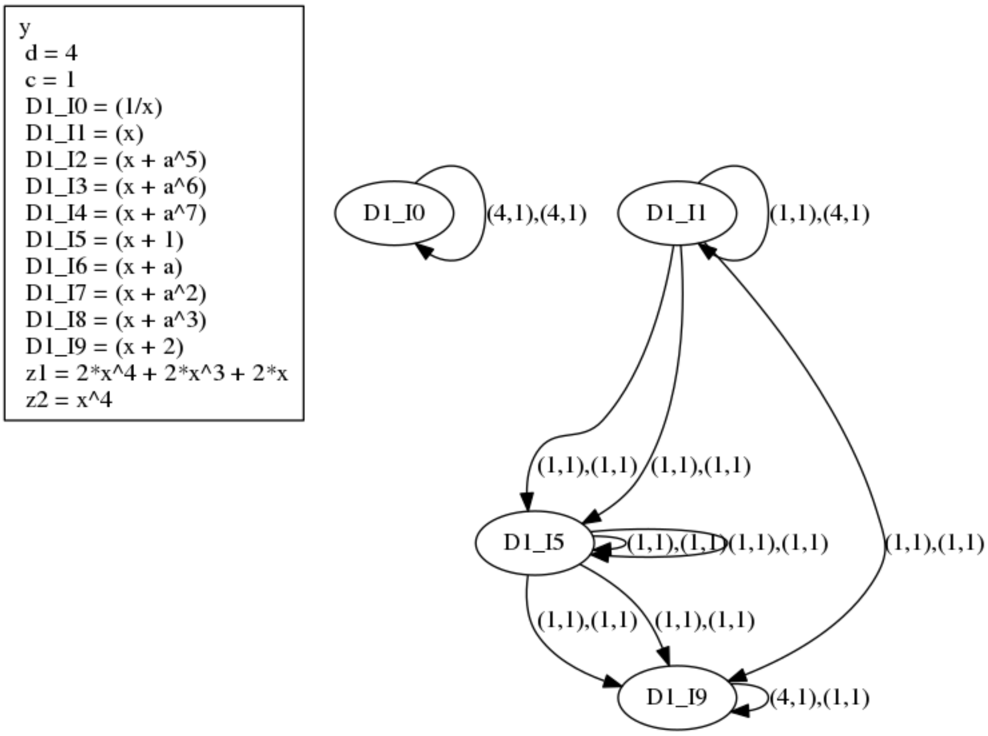


Figure B.18: Degree one subgraph of the tame recursive tower $\mathcal{F}'_{GS,2} = (F_\nu)_\nu$ in [Sti08, p. 261, Proposition 7.3.3] for $l = 3$ and $e = 2$ over $\mathbb{F}_q$ with $q = l^e$ which is defined by the polynomial $f_{GS,2} = Y^m + (X+1)^m - 1$ with $m = \frac{q-1}{l-1}$ from the first level on. See Examples 8(ii). We added a zeroth level F_0 to $\mathcal{F}'_{GS,2}$ via the Reduction Lemma 30(iii). This tower satisfies $\lambda(\mathcal{F}'_{GS,2}) = 2/(q-1)$ and $g(F_1) = 0$. Here the two weakly connected components form the ramification subgraph.

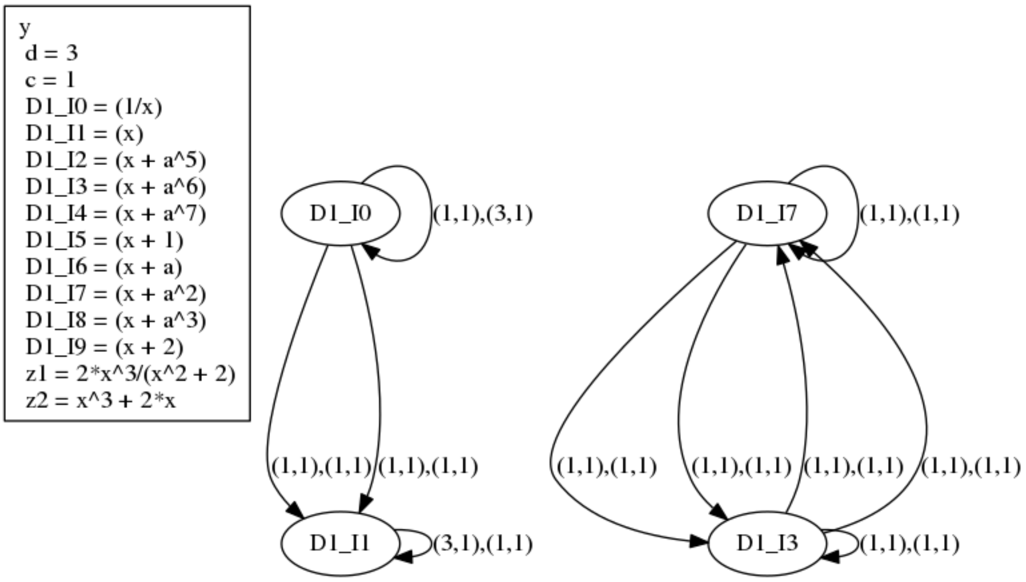


Figure B.19: Degree one subgraph of the wild recursive GS-tower $\mathcal{F}_{GS,3} = (F_\nu)_\nu$ in [Sti08, p. 262, Definition 7.4.1] for $l = 3$ over $\mathbb{F}_{l^2}$ which is defined by the polynomial $f_{GS,3} = (Y^l - Y)(1 - X^{l-1}) - X^l$ from the first level on. See Examples 8(ii). We added a zeroth level F_0 to $\mathcal{F}_{GS,3}$ via the Reduction Lemma 30(iii). This tower satisfies $\lambda(\mathcal{F}_{GS,3}) = l - 1$ and $g(F_1) = 0$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

B.3 From [vdGvdV02]

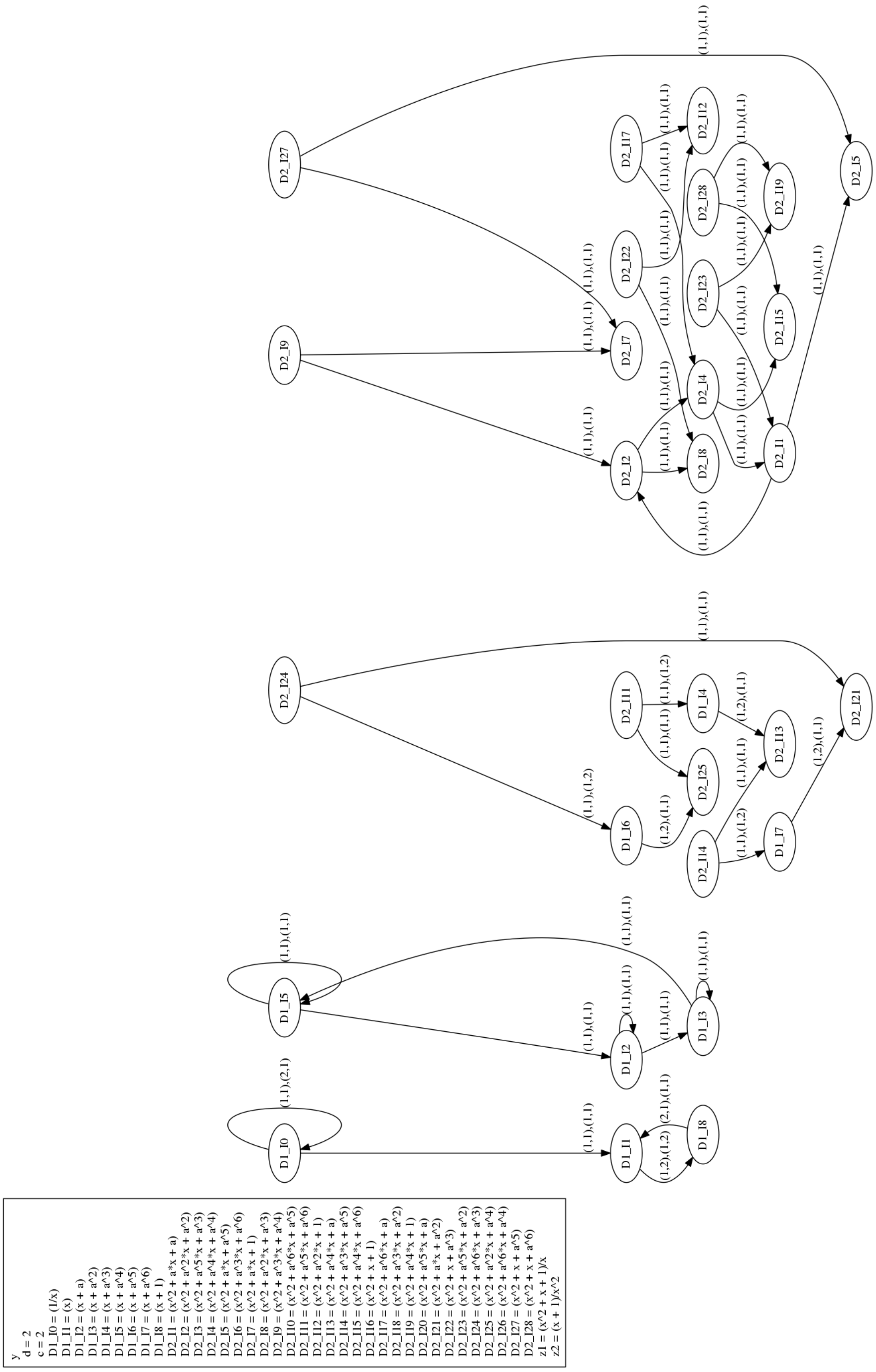


Figure B.20: Degree one subgraph of the wild recursive GV-tower $\mathcal{F}_{GV} = (F_\nu)_\nu$ in [vdGvdV02, p. 292] over $\mathbb{F}_8$ which is defined by the polynomial $f_{GV} = (Y^2 + Y)X + X^2 + X + 1$. See Examples 8(iii). We added a zeroth level F_0 to $\mathcal{F}_{GV}$ via the Reduction Lemma 30(iii).
This tower satisfies $\lambda(\mathcal{F}_{GV}) = 3/2$ and $g(F_1) = 0$. Here the first weakly connected component is the ramification subgraph and the second is the splitting subgraph.

B.4 From [ST15]

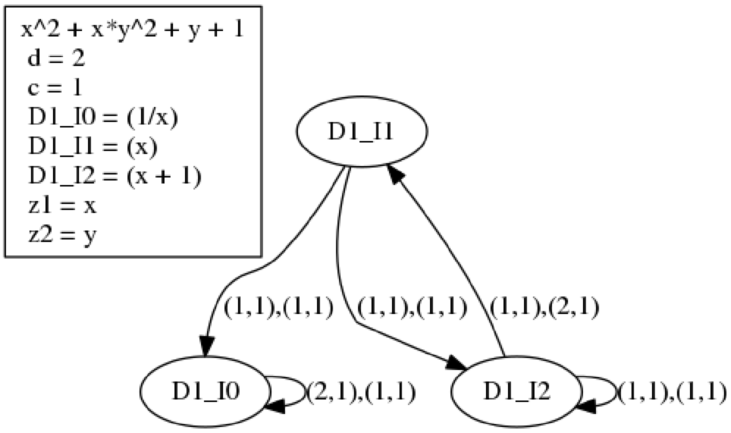


Figure B.21: Degree one subgraph of the wild recursive tower $\mathcal{F}_{ST,1} = (F_\nu)_\nu$ in [ST15, p. 680, Theorem 2.14(1)] $\mathbb{F}_2$ which is defined by the polynomial $f_{ST,1} = Y^2X + Y + X^2 + 1$. See Examples 8(iv). This tower satisfies $\lambda(\mathcal{F}_{ST,1}) = 0$ and $g(F_1) = 1$. Here the weakly connected component is the ramification subgraph.

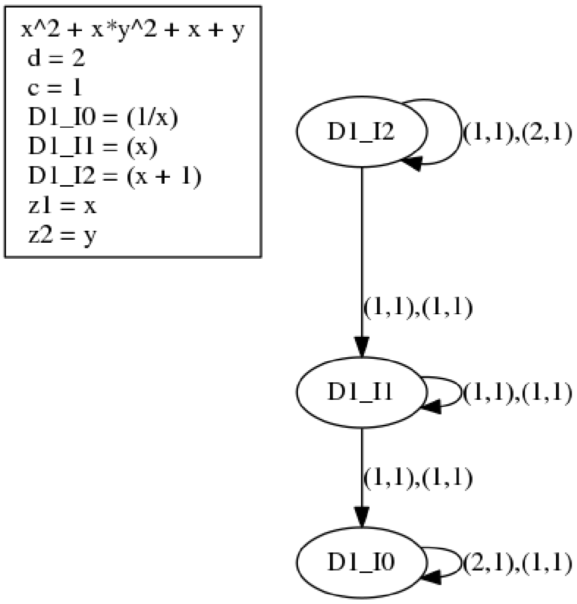


Figure B.22: Degree one subgraph of the wild recursive tower $\mathcal{F}_{ST,3} = (F_\nu)_\nu$ in [ST15, p. 680, Theorem 2.14(2)] $\mathbb{F}_2$ which is defined by the polynomial $f_{ST,2} = X^2 + XY^2 + X + Y$. See Examples 8(iv). This tower satisfies $\lambda(\mathcal{F}_{ST,2}) = 0$ and $g(F_1) = 1$. Here the weakly connected component is the ramification subgraph.

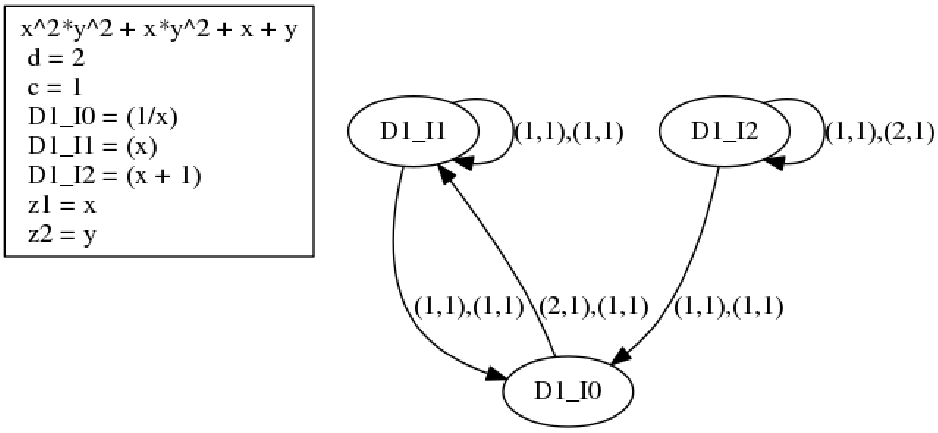


Figure B.23: Degree one subgraph of the wild recursive tower $\mathcal{F}_{ST,3} = (F_\nu)_\nu$ in [ST15, p. 680, Theorem 2.14(3)] $\mathbb{F}_2$ which is defined by the polynomial $f_{ST,3} = X^2Y^2 + XY^2 + Y + X$. See Examples 8(iv). This tower satisfies $\lambda(\mathcal{F}_{ST,3}) = 0$ and $g(F_1) = 1$. Here the weakly connected component is the ramification subgraph.

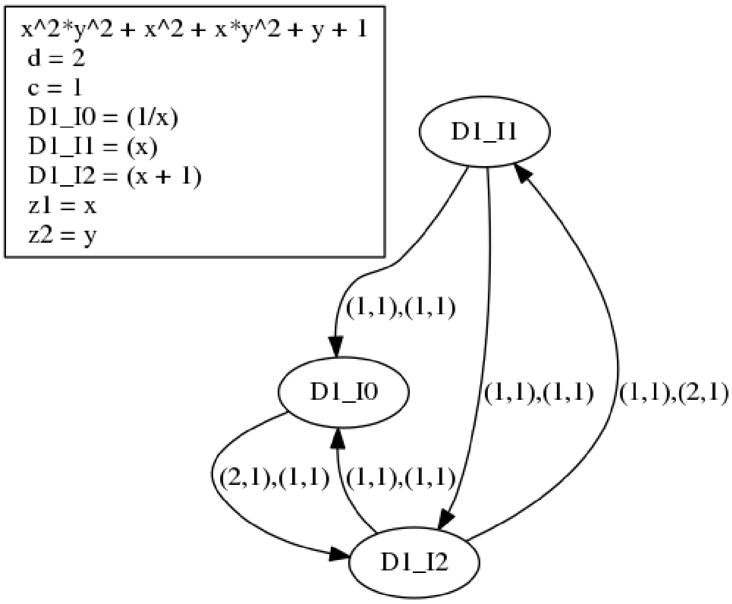


Figure B.24: Degree one subgraph of the wild recursive tower $\mathcal{F}_{ST,4}F = (F_\nu)_\nu$ in [ST15, p. 680, Theorem 2.14(4)] $\mathbb{F}_2$ which is defined by the polynomial $f_{ST,4} = X^2Y^2 + XY^2 + Y + X^2 + 1$. See Examples 8(iv). This tower satisfies $\lambda(\mathcal{F}_{ST,4}) = 0$ and $g(F_1) = 1$. Here the weakly connected component is the ramification subgraph.

B.5 From [BBGS15]

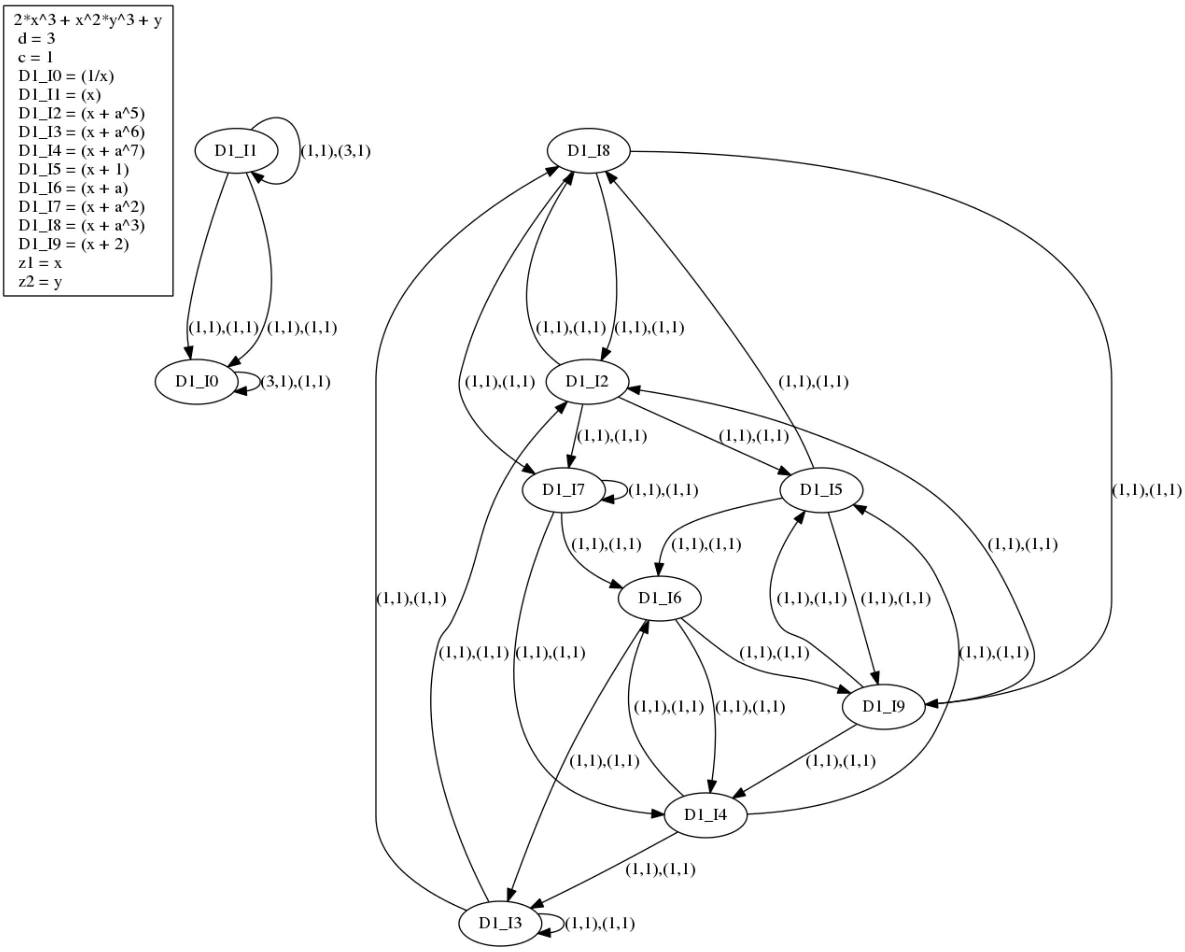


Figure B.25: Degree one subgraph of the wild recursive BBGS-tower $\mathcal{F}_{BBGS,q,i,j} = (F_\nu)_\nu$ in [BBGS15, p. 4, Equation (9)] for $q = 3$, $i = j = 1$ over $\mathbb{F}_{q^m}$ with $m = i + j$ which is defined by the polynomial $f_{BBGS,3,1,1} = \varepsilon_{q,i,j} \cdot (\mathrm{Tr}_j(Y/X^{q^i}) + \mathrm{Tr}_i(Y^{q^j}/X) - 1)$. See Examples 8(vi). This tower satisfies $\lambda(\mathcal{F}_{BBGS,3,1,1}) = 2 \cdot ((q^j - 1)^{-1} + (q^i - 1)^{-1})^{-1}$. Here the first weakly connected component is the ramification subgraph and the second is the ramification subgraph.

B.6 From [HP16]

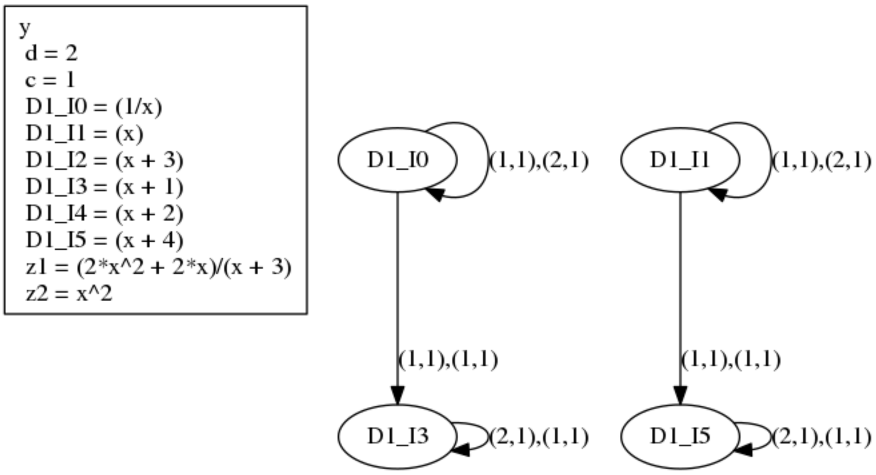


Figure B.26: Degree one subgraph of the tame recursive tower $\mathcal{F}_{HP,5} = (F_\nu)_\nu$ in [HP16, p. 12, Proposition 12] $\mathbb{F}_5$ which is defined by the polynomial $f_{HP,5} = Y^2(3X - 1) - (X^2 + X)$ from level one on. See Examples 8(vii). We added a zeroth level F_0 via the Reduction Lemma 30(iii). This tower satisfies $\lambda(\mathcal{F}_{HP,5}) = 0$ and $g(F_1) = 0$. Here the weakly connected components are the two weakly connected components of the ramification subgraph.

B.7 From [CNT18]

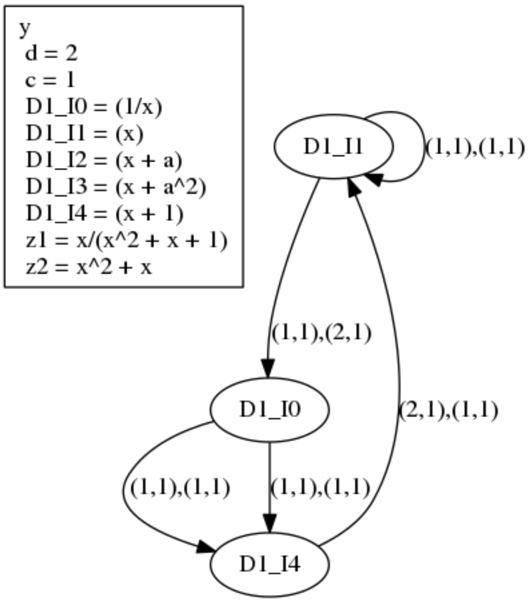


Figure B.27: Degree one subgraph of the wild recursive CNT-tower $\mathcal{F}_{CNT,2} = (F_\nu)_\nu$ in [CNT18, p. 19, Corollary 4.13] over $\mathbb{F}_4$ which is defined by the polynomial $f_{CNT} = (Y^2 + Y)(X^2 + X + 1) + X$. See Examples 8(v). We added a zeroth level F_0 to $\mathcal{F}_{CNT,2}$ via the Reduction Lemma 30(iii).
This tower satisfies $\lambda(\mathcal{F}_{CNT,2}) = 1$ and $g(F_1) = 0$. Here the degree one subgraph is equal to the ramification subgraph.

B.8 From [BR20]

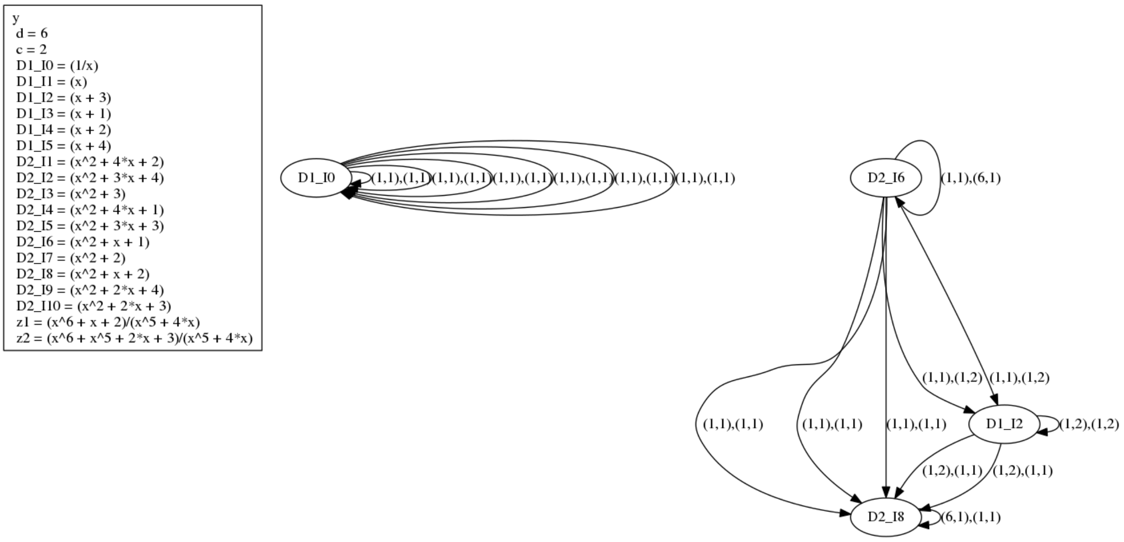


Figure B.28: Degree two subgraph of the tame recursive BR-tower $\mathcal{F}_{BR,5} = (F_\nu)_\nu$ in [BR20, p. 4] for $q = 5$ over $\mathbb{F}_q$. From level one on, this tower is defined by the polynomial $f_{BR,5} = (X^6 + X + 2)(Y^5 - Y) - (X^5 - X)(Y^6 + Y^5 + 2Y + 3)$ and satisfies $\lambda(\mathcal{F}_{BR,5}) = 2/(q-1)$. See Examples 8(viii). We added a zeroth level F_0 to $\mathcal{F}_{BR,5}$ via the Reduction Lemma 30(iii). Here the first weakly connected component is the splitting subgraph and the second weakly connected component is the ramification subgraph.

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List of Figures

List of Symbols

What follows is an overview of symbols used in this thesis which are non-canonical, often with references to the corresponding definitions. Also note that although the symbols in the following list are mostly used in the described way, there are some instances where this is not the case.

(σ, F_0)	Pair which defines recursive tower (see Definition 5(ii))
(E, F_1, F_2, F)	Diamond of function fields (see (6))
(Q, P_1, P_2, P)	Diamond of places (see (6))
$(R^{\mathbb{N}})'$, $(R^{\mathbb{N}_0})'$, $(R^{\mathbb{P}})'$	Set of all sequences with elements in the the ring R which are almost all zero
$(x = \alpha)$, $(x = \infty)$	The place in the rational function field $k(x)$ which is generated by the element $x - \alpha$ and the place at infinity, respectively
$<, \lneqq, \leq$	The first two symbols denote the proper estimate and the third symbol also allows equality
$[a, b]$, $]a, b[$	Open and closed interval in $\mathbb{R} \cup \{\infty\}$ from a to b, respectively
α	Edge map on a directed graph (see Definition 55(i)) or a map as in Theorem 4(ii) and in Definition 189
$\alpha^e = \prod_{p \in \mathbb{P}} \alpha_p^{e_p}$	Multi-index notation (see (1))
$\log_2(\mathbf{P}) = (\log_2(p))_{p \in \mathbb{P}}$	See Definition 167
$\mathcal{O}(U)$, $\pi_{\mathcal{O}(U)}$	See Definition 209
$\mathbb{C}[y^\beta]$, $\mathbb{C}(y^\beta)$	See Definition 208
$\mathcal{C}$	A closed path or circle in a directed graph (see Definition 55(iv))
$\mathcal{F} = (F_\nu)_\nu$	A tower of function fields (see Definition 2(i)) or, more specifically, a recursive tower of function fields (see Definition 5)
$\mathcal{F}' = (F'_\nu)_\nu = k' \cdot \mathcal{F}$	k'-constant field extension of the tower $\mathcal{F}$ (see Definition 21(i))
$\mathcal{F}_{BBGS,q,i,j}$, $f_{BBGS,q,i,j}$	BBGS-towers and -polynomials, respectively (see 8(vi))
$\mathcal{F}_{BezGS}$, $f_{BezGS,l}$	BezGS-towers and -polynomials, respectively (see 8(iii))
$\mathcal{F}_{BR,q}$, $f_{BR,q}$	BR-towers and -polynomials, respectively (see 8(viii))
$\mathcal{F}_{CNT}$, f_{CNT}	CNT-tower and -polynomial, respectively (see 8(v))
$\mathcal{F}_{GS,i}$, $f_{GS,i}$	GS-towers and -polynomials, respectively (see 8(ii))

$\mathcal{F}_{GV}$, f_{GV}	GV-tower and -polynomial, respectively (see 8(iii))
$\mathcal{F}_{HP,q}$, $f_{HP,q}$	HP-towers and -polynomials, respectively (see 8(vii))
$\mathcal{F}_{MW,i}$, $f_{MW,i}$	MW-towers and -polynomials, respectively (see 8(i))
$\mathcal{F}_{ST,i}$, $f_{ST,i}$	ST-towers and -polynomials, respectively (see 8(iv))
$\mathcal{G}_\infty$, $\mathcal{G}_{\text{sing}}$	Geometric and singular graph (see Definition 101)
$\mathcal{M}$	See Definition 223
$\mathcal{P} \cdot \mathcal{P}'$	Composition of the paths $\mathcal{P}$ and $\mathcal{P}$ in a recursive tower (see Definition 16(iii)) or a directed graph (see Definition 55(v))
$\mathcal{P}, [P_{i,j}]_{j-i\leq 1}, (P_{i,j})_{j-i\leq 1}$	A path in a directed graph (see Definition 55(ii) and Notation 56) or a path in the pyramid of a recursive tower (see Definition 17(i))
$\coprod$, $\sqcup$	Disjoint unions
Count, Countˆ	See Definition 223
c_Q	See Definition 223
$\mathcal{D}_\alpha$, δ_α	See Definition189
$\deg_-(P)$, $\deg_+(P)$	In- and out-degree of the vertex P in a directed graph, respectively (see Definition 55(vii))
$\Delta(\mathcal{P})$	See Remark 49
$e_{Q,p}$	See Definition 223
Eval_a	Evaluation morphism on Laurent polynomials (see Definition 236)
$\mathrm{Eval}_{x_{\mathbf{P}}}$	Evaluation morphism on Laurent polynomials in $\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$ (see Definition 161(ii))
$\mathrm{Ext}(\mathrm{Pyr}(Q))$, $\mathrm{ElemExt}(\mathrm{Pyr}(Q))$	Sets of extensions and elementary extensions in pyramid $\mathrm{Pyr}(Q)$ of places, respectivly (see Definition 14(ii))
$\mathbb{F}_{q^l}$, $\mathbb{F}$	Finite field with q^l elements and an algebraic extension field of a finite field, respectively
$\mathbf{F}$, $\overline{\mathbf{F}}$	Union $\mathbf{F}$ of all function fields in the recursive tower $\mathcal{F}$ and its algebraic closure $\overline{\mathbf{F}}$ (see Definition 5(ii))
$c(\mathcal{F})$	See Definition 245
$c(A)$	See Definition 242
$\mathcal{B}$, $\mathcal{B}_{\geq 0}$	See Definition 208
y, y^α, $y^\beta = (y^{\beta_p})_p$	See Definition 208
Γ	Directed graph (see Definition 55(i))
$\Gamma \backslash \Gamma'$	Difference subgraph (see Definition 66(iv))
Γ_c	Degree c subgraph of a tower graph (see Lemma 61)

γ_X	See (136)
$\mathrm{Graph}(\mathrm{Pyr}(Q))$	Pyramidal graph of the pyramid $\mathrm{Pyr}(Q)$ of places for some place Q in a recursive tower (see Definition 72(i))
$\hat{\mathcal{F}}$	Dual of the recursive tower $\mathcal{F}$ (see Definition/Lemma 35)
$\hat{\sigma}$	See Definition 217(iii)
Kill, Kill$\hat{\ }$	See Definition 223
$\lambda(\mathcal{F}), \nu(\mathcal{F}), \gamma(\mathcal{F})$	Limit, splitting rate and genus of the tower $\mathcal{F}$ of function fields, respectively (see Definition 2(iii))
$\mathbb{C}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$, y_p	Ring of Laurent polynomials in the variables y_p for all $p \in \mathbb{P}$ (see Definition 161(i))
$\mathbb{R}_{\geq 0}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$, $\mathbb{R}[y_{\mathbf{P}}, y_{\mathbf{P}}^{-1}]$	See Definition 161(i)
$\mathrm{Length}(\mathcal{P})$	Length of the path $\mathcal{P}$ in a directed graph (see Definition 55(ii))
$\lfloor \alpha \rfloor$, $\lceil \alpha \rceil$	The real number α rounded down and rounded up, respectively
$L(h)$, $L(A)$	See Definition 212
$\mathbb{N}$, $\mathbb{N}_0$	Set of positive and nonnegative integers, respectively
N'	See Definition 233
N'_e, N'_q	See Definition 236
N''_e	See Definition 241
$\tilde{N}$	See Definition 50
$\mathbb{1}_{F_0}$	Map $\mathbb{1}_{F_0} : \mathbb{P}_{F_0} \to \mathbb{R}$ via $P \mapsto 1$ (see Definition 189)
$\mathbf{1}$	Sequence $(1, 1, \dots)$ consisting of only ones which is indexed by $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{P}$
$\overline{\mathcal{F}} = (\overline{F}_\nu)_\nu = \overline{k} \cdot \mathcal{F}$	Geometric tower of the tower $\mathcal{F}$ (see Definition 21(i))
$\overline{k}$	An algebraic closure of k; in most instances, the algebraic closure of k inside of the domain of σ (see Definition 5(ii))
$\overline{X}$, $\overline{\varphi}$	Fiber product $X \times_k \overline{k}$ for a curve over k and an algebraic closure of k and the canonical morphism $\overline{\varphi} = \varphi \times_k \overline{k}$ for the morphism φ of curves over k
$\mathbb{P}$	Set of primes in $\mathbb{N}$
$\mathbb{P}^{\circ}_{F_n}[\Gamma]$	Set of places in F_n which lie over a circle in Γ (see Definition 85)
$\mathbb{P}_F$, $\mathbb{P}_F^{(r)}$	Set of places in the function field F and its subset of places P with $\deg(P) = r$ (see (4))
$\mathbb{P}_{\mathcal{F}}$	Union of all sets of places in F_n for all $n \in \mathbb{N}_0$ (see Definition 2(ii))
$\mathbb{P}_{\mathcal{F}}(A)$	Union of all sets $\mathbb{P}_{F_n}(A)$ for all $n \in \mathbb{N}_0$ (see Definition 2(ii))
$\mathbb{P}_{\mathcal{F}}[\Gamma]$	Union of all sets $\mathbb{P}_{F_n}[\Gamma]$ for all $n \in \mathbb{N}_0$ (see Definition 85)
$\mathbb{P}^{\circ}_{\mathcal{F}}[\Gamma]$	Union of all sets $\mathbb{P}^{\circ}_{F_n}[\Gamma]$ for all $n \in \mathbb{N}_0$ (see Definition 85)

$\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}$	Set of all places which are contained in one of the function fields $F_{i,j}$ of the pyramid $(F_{i,j})_{i,j} = \mathrm{Pyr}(\mathcal{F})$ (see Definition 9(iv))
$\mathbb{P}_F(\mathcal{P})$, $\mathbb{P}_F^{(r)}(\mathcal{P})$	Set of places in F which lie over all places in $\mathcal{P}$ and its subset of places P with $\deg(P) = r$(see Definition (5)))
$\mathbb{P}_F(A)$, $\mathbb{P}_F^{(r)}(A)$	Set of places in F which lie some of some $\mathcal{P} \in A$ and its subset of places P with $\deg(P) = r$ (see Definition (5))
$\mathbb{P}_{F_n}((Q_1, \dots, Q_r))$	Set of places in F_n which lie over all places $Q_1, \dots, Q_r$(see Definition (5))
$\mathbb{P}_{F_n}[\Gamma]$, $\mathbb{P}_{F_n}^{(1)}[\Gamma]$	Set of places in F_n which lie over a path in Γ and its subset of rational places (see Definition 85)
$\mathrm{Path}(Q) = (P_{i,j})_{j-i \leq 1}$	Path of the place Q in the pyramid of a recursive tower (see Definition/Lemma 17)
$\phi(\Gamma)$, $\phi^{-1}(\Gamma)$	Image and preimage graph, respectively, of the subgraph Γ for the morphism ϕ of directed graphs (see Definition 69)
$\pi_{\mathbb{P}_{\mathrm{Pyr}(\mathcal{F}')}/\mathbb{P}_{\mathrm{Pyr}(\mathcal{F})}}$, $\pi_{W(\mathcal{F}')/W(\mathcal{F})}$	Projection maps for constant field extensions $\mathcal{F}'/\mathcal{F}$ of recursive towers (see Lemma 76)
$\rho(d)$	See Definition 223
P	Sequence $(p)_p = (2, 3, 5, \dots)$ of all primes p
Primes, Primesˆ	See Definition 223
$\mathrm{Princ}(h)$, $\mathrm{NonPrinc}(h)$	Principal and non-principal part of the Laurent polynomial h, respectively (see Definition 236)
$\mathrm{Pyr}(\mathcal{F}) = (F_{i,j})_{i,j}$	Pyramid of the recursive tower $\mathcal{F}$ (see Definition 9(i))
$\mathrm{Pyr}(Q) = (P_{i,j})_{i,j}$	Pyramid of the place Q in the pyramid of a recursive tower (see Definition 11)
Ram, Ramˆ	See Definition 223
$\mathrm{Ram}(\mathcal{F}/F_m)$, $\mathrm{Split}(\mathcal{F}/F_m)$	Ramification and splitting loci of $\mathcal{F}$ over F_m, respectively (see Definition 3(ii))
$\rho(\mathcal{F})$	See Definition 179 for pair-recursive towers $\mathcal{F}$ and Definition 94(ii) for polynomial-recursive towers $\mathcal{F}$
$\rho(A)$	Spectral radius of the quadratic complex matrix A (see Examples 75)
σ_z	See Definition 209
$\mathrm{sign}(\sigma)$	Either the usual signum-map if σ is a permutation on the set $\{1, \dots, m\}$ or its extension in Definition 217(iii)
$\mathrm{Sp}(A)$, $\mathrm{Sp}(\phi)$	Spectrum of eigenvalues of the matrix A and the linear map ϕ
$\subset, \subsetneq, \subseteq$	The first two symbols denote the proper inclusion of sets and the third symbol also allows the sets to be equal
$\sigma_{\Gamma_{\mathcal{F}}}$	Tower graph map of a recursive tower $\mathcal{F}$ (see Definition/Lemma 76)

$\pi_{\Gamma_{\mathcal{F}'}/\Gamma_{\mathcal{F}}}$	CFE-projection morphism $\Gamma_{\mathcal{F}'} \to \Gamma_{\mathcal{F}}$ (see Definition/Lemma 105(i))
$\Gamma_{\mathcal{F}}$	Tower graph of the recursive tower $\mathcal{F}$ (see Definition 74)
$\Gamma_{\mathcal{F}}^{\mathrm{rat}}$, $\Gamma_{\mathcal{F}}^{\mathrm{split}}$, $\Gamma_{\mathcal{F}}^{\mathrm{ram}}$	Rational, splitting and ramification subgraph, respectively (see Definition 88)
$\pi_{\Gamma_{\mathcal{F}_{\geq m}}/\Gamma_{\mathcal{F}}}$	Trun-projection morphism (see Definition/Lemma 126)
$\tilde{g}(F_n)$	See Remark 53
σ	Tower map of a recursive tower (see Definition 5(ii))
$\sigma^k(\mathrm{Pyr}(Q))$, $\sigma^k(\mathrm{Path}(Q))$, $\sigma^k(\mathcal{P})$	Action of σ^k on the pyramid $\mathrm{Pyr}(Q)$, the path $\mathrm{Path}(Q)$ and the path $\mathcal{P}$, respectively (see Definition/Lemma 15(i), Definition/Lemma 20(i), respectively)
Tr	Trace map on quadratic matrices or endomorphisms of finite vector spaces
Tr_a	More specific trace map in the definition of the BBGS-towers (see Examples 8(vi))
$\mathrm{Trun}_{\geq m}(\mathcal{F})$	Level m truncation of tower $\mathcal{F}$ (see Definition/Lemma 27)
$\mathrm{Trun}_{\geq m}(\Gamma)$	Level m truncation of the subgraph Γ of the tower graph of a recursive tower (see Definition/Lemma 128)
$\tau_{\mathcal{F}}$	See Lemma 91(ii)
φ^c	Induced map on the closed points of the finite morphism φ of curves
$\mathrm{Vertex}_i(\mathcal{P})$, $\mathrm{Edge}_i(\mathcal{P})$	i-th vertex and edge of the path $\mathcal{P}$ in a directed path (see Definition 55(ii))
$v_{\mathrm{init}}(Q)$, $v_{\mathrm{init}}(\mathcal{P})$, $v_{\mathrm{term}}(Q)$, $v_{\mathrm{term}}(\mathcal{P})$	Initial and terminal vertex of the edge Q or path $\mathcal{P}$ in a directed graph, respectively (see Definition 55(i) and Definition 55(ii))
$v_{\mathbf{P}}(a)$	Sequence $(v_p(a))_p \in (\mathbb{Z}^{\mathbb{P}})' \subset \mathbb{Z}^{\mathbb{P}}$ for all $a \in \mathbb{Q}\backslash\{0\}$ (see (3))
$w'_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$	The weight function on the tower graph of a recursive tower in Definition 162
$w_{x_{\mathbf{P}},\hat{x}_{\mathbf{P}}}$	The weight function on the tower graph of a recursive tower in Definition 157
$y_{\mathbf{P}} = (y_p)_p$	See Definition 161(i)
$a * b$	Componentwise product of the sequences a and b (see (2))
$A(x_{\mathbf{P}}) = \mathrm{Eval}_{x_{\mathbf{P}}}(A)$	See Definition 161(ii)
$A(q)$	Ihara's constant (see Definition 2(iv))
$a_{\sigma,i}$, a_σ	See Definition 217(iv)
$C(\sigma, i)$	See Definition 217(v)
d	Degree of a tower of function fields (see Definition 2(ii))
$E(\Gamma)$	Set of edges of the directed graph Γ (see Definition 55)
$E_-(\Gamma, P)$, $E_+(\Gamma, P)$	Set of in- and outgoing edges at the vertex P in the directed graph Γ, respectively (see Definition 55(vii))

F/E	Extension of function fields (see Definition 1(ii))
$F(\sigma)$	See Definition 217(ii)
F_n	Function fields of the sequence of a tower of function fields (see Definition 2(i))
$G(\sigma)$	See Definition 217(ii)
$g(F)$, $g(C)$	Genus of the function field F and the curve C, respectively
k	Either an index or a perfect field; in most instances, even the constant field of a function field (see Definition 1(i))
$K \cdot L$	Composite field of K and L in some common extension field
k'	An algebraic extension field of k
$k' \cdot \alpha$	See Definition/Lemma191
$k' \cdot \Gamma$	Constant field extension of the subgraph Γ of a tower graph of a recursive tower $\mathcal{F}$ via an algebraic extension field k' of the constant field of $\mathcal{F}$ (see Definition 107)
$K(X)$	Function field of the integral curve X
l_Γ	See Definition 226
l_Q	See Definition 223
m_∞	Separating index (see Definition 238)
$N(A)$	Sum of the entries of the complex quadratic matrix A (see Definition 60)
$N(F)$	Number of rational places in the function field F (see (4))
$N(F, \mathcal{P})$	Cardinality of $\mathbb{P}_F^{(1)}(\mathcal{P})$ (see (5))
$N(F, A)$	Cardinality of $\mathbb{P}_F^{(1)}(A)$ (see (5))
$N[F_n, \Gamma]$	Cardinality of $\mathbb{P}_F^{(1)}[\Gamma]$ (see Definition 85)
$O(\sigma, i)$	See Definition 217(ii)
P	Either a place in a function field or an edge in a directed graph
Q	Either a place in a function field or an edge in a directed graph
Q/P	Extension Q/P of places in some extension of function fields
$R(\sigma)$	See Definition 217(ii)
$R^{\mathbb{N}}$, $R^{\mathbb{N}_0}$, $R^{\mathbb{P}}$	Set of all sequences with elements in the the ring R for the index set $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{P}$, respectively
R^M	Set of maps from M to R
R_e, K_e, x_e	See Definition 236
$S(X)$	Symmetric group on the set X (see Definition 217(i))

$V(\Gamma)$	Set of vertices of the directed graph Γ (see Definition 55)
$v_p(a)$	Exponent of p in the prime decomposition of $a \in \mathbb{Z}\backslash\{0\}$ and $v_p(b) - v_p(c)$ if $a = \frac{b}{c} \in \mathbb{Q}\backslash\{0\}$ with $b, c \in \mathbb{Z}$
$W(\mathcal{F}, m, n)$, $W(\mathcal{F}, n)$, $W(\mathcal{F})$	Sets of paths in the pyramid of $\mathcal{F}$ (see Definition 16(i))
$W(\Gamma)$	Set of paths in the directed graph Γ (see Definition 55(iii))
$W(\Gamma, n)$	Set of paths in the directed graph Γ of length n (see Definition 55(iii))
$W(\Gamma, n, P)$	Set of paths in the directed graph Γ of length n which start at P (see Definition 55(iii))
$W(\Gamma, n, P_0, P_1)$	Set of paths in the directed graph Γ of length n which start at P_0 and end at P_1 (see Definition 55(iii))
$w(Q)$, $w(\mathcal{P})$	Weight of an edge and path in a directed graph (see Definition 58(i))
$W_{\text{rat}}(\mathcal{F}, m, n)$, $W_{\text{rat}}(\mathcal{F}, n)$, $W_{\text{rat}}(\mathcal{F})$	Sets of rational paths in the pyramid of $\mathcal{F}$ (see Definition 16(iv))
$W_{\text{rat}}(\Gamma)$	Set of rational paths in the directed graph Γ (see Definition 79)
X^c, $X(k)$	Closed and k-rational points of the curve X, respectively
z^α, z^β	See Definition 167

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Declaration

Hereby, I declare, that the presented dissertation with the title

Recursive Towers over Finite Fields

is my own work and, that I have not used other sources than the sources stated in the text or in the bibliography. The dissertation has, neither as a whole, nor in part, been submitted for assessment in a doctoral procedure at another university. I confirm that I am aware of the guidelines of good scientific practice of the Carl von Ossietzky University Oldenburg and that I observed them. Furthermore, I declare that I have not availed myself of any commercial placement or consulting services in connection with my doctoral procedure.