CARL VON OSSIETZKY
UNIVERSITÄT
OLDENBURG

# On the prescribed mean curvature flow of graphical hypersurfaces in certain globally hyperbolic Lorentzian manifolds with non-compact Cauchy hypersurfaces

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von Herren **Giuseppe Gentile**

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Gutachter: **Prof. Dr. Boris Vertman**
Zweitgutachter: **Prof. Dr. Daniel Grieser**

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# Declaration

Hereby I declare that the proposed thesis, titled "**On the prescribed mean curvature flow of graphical hypersurfaces in certain globally hyperbolic Lorentzian manifolds with non-compact Cauchy hypersurfaces**", is my own work and that I did not use any other sources then those explicitly cited in the text and in the bibliography. Furthermore, I want to point out that this thesis has not been submitted to other universities for assessment in a doctoral degree procedure either in its entirety, or in its parts. Also I attest that I am aware of and have observed the guidelines of good scientific practice of the Carl von Ossietzky Universität Oldenburg. Finally I assert that I have not made use of any sort of commercial recruitment or consulting service.

Giuseppe Gentile

# Abstract

In this thesis we firstly prove parabolic Schauder estimates for the Laplace-Beltrami operator on a manifold $\overline{M}$ with fibered boundary equipped with a $\Phi$-metric $\widetilde{g}$. This setting generalizes the asymptotically conical (scattering) spaces and includes special cases of magnetic and gravitational monopoles. As a consequence, we establish existence and regularity of solutions for some quasilinear parabolic equations on $(\overline{M}, \widetilde{g})$. This is the crucial groundwork for the analysis of many geometric flows. In particular, we focus on the prescribed mean curvature flow of graphs over a space-like slice of a generalised Robertson-Walker space-time having $\Phi$-manifolds as space-like slices. We prove that the flow exists for short time and that it preserves the space-likeness condition. Our discussion generalizes previous work by Ecker, Huisken, Gerhardt and others with respect to a crucial aspect: we consider a class of non-compact Cauchy hypersurface. Moreover, we specialize the aforementioned works by considering globally hyperbolic Lorentzian space-times equipped with a specific class of warped product metrics.



# Zusammenfassung

In dieser Dissertation zeigen wir zuerst parabolische Schauder Abschätzungen für den Laplace-Beltrami Operator auf* einer Mannigfaltigkeit $\overline{M}$, dessen gefaserter Rand mit einer $\Phi$-Metrik $\widetilde{g}$ ausgestattet ist. Dies generalisiert asymptotisch konische Mannigfaltigkeiten (scattering) und schließt Spezialfälle der magnetischen und gravitationalen Monopole ein. Als Konsequenz zeigen wir die Existenz und die Regularität quasi-linearer parabolischer Gleichungen auf $(\overline{M}, \widetilde{g})$. Das ist entscheidend für die Analysis des mittlerer Krümmungsflusses. Insbesondere studieren wir den vorgeschriebenen mittleren Krümmungsfluss des Graphen einer Funktion, die über eine raumartige Faser einer generalisierten Robertson-Walker Raumzeit definiert ist, wobei die raumartigen Fasern $\Phi$-Mannigfaltigkeiten sind. Wir zeigen, dass der Fluss für kurze Zeit existiert und raumartig bleibt. Diese Arbeit verallgemeinert frühere Arbeiten von Ecker, Huisken und Gerhardt hinsichtlich eines wichtigen Aspekts: wir analysieren den Fluss für nicht-kompakte Cauchy-Hyperflächen. Im Kontext dieser Arbeiten schränken wir uns auf den Spezialfall einer Lorentz-Raumzeit mit globaler, hyperbolischer, verzerrter Produktmetrik ein.

# Contents

# List of Figures

# Chapter 1

# Introduction

In 1916 the human species has been gifted one of the most successful physical theories, the "General theory of Relativity" by Albert Einstein [Ein23]. Einstein's theory tells us that the universe we live in can be modelled as a 4-dimensional Lorentzian manifold. In particular the "non-flatness" nature of the universe is due to the famous Einstein's field equations

$$\mathrm{Ric}_{\alpha\beta} - \frac{1}{2}\,\mathrm{scal}\,\overline{g}_{\alpha\beta} + \Lambda\overline{g}_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}. \tag{1.0.1}$$

The left hand side of the equation represents the geometry of the manifold, i.e. the universe, and the right hand side represents the matter and energy content of the universe.
Constant mean curvature space-like hypersurfaces play an important role in the analysis of (1.0.1). Indeed, among the mathematicians and physicists who devoted their carrier to the analysis of (1.0.1), we would like to recall the amazing work of Choquet-Bruhat. In [Cho52] Choquet-Bruhat proves that an initial data set $(\overline{M}, g, k)$, consisting of a 3-dimensional Riemannian manifold $(\overline{M}, g)$ equipped with a $(0, 2)$-tensor $k$, admits a development into a Lorentzian manifold $(N, \overline{g})$ satisfying (1.0.1). Furthermore $(N, \overline{g})$ has $(\overline{M}, g)$ as a space-like Cauchy hypersurface whose second fundamental form is given by the $(0, 2)$-tensor $k$. The initial data set $(\overline{M}, g, k)$ can not be chosen arbitrarily, it has to satisfy certain conditions, usually referred to as Einstein constraint equations. Solving such equations can be made much easier by assuming constant mean curvature initial data.
Moreover, space-like hypersurfaces of constant mean curvature have been used for the first proof of the positive mass theorem by Schoen and Yau [ShYa79].
Finally, as stated in [Ren96], the existence of a foliation consisting of space-like hypersurfaces of constant mean curvature is closely related to the famous cosmic censorship conjecture.
The crucial role of constant mean curvature space-like hypersurfaces in general relativity is therefore undeniable. Thus, one question naturally arises.

**Question 1**
*Are there space-like hypersurfaces of prescribed mean curvature in a given Lorentzian manifold* $(N, \overline{g})$*?*

The above can be formally expressed as follows.

**Question 2**
*Let $(N, \overline{g})$ be an ambient Lorentzian manifold. Does an embedding $F : \overline{M} \to N$ from a smooth manifold $\overline{M}$ such that:*

*i) denoted by* H *the mean curvature of $\overline{M}$ as a submanifold of $N$,*

$$\mathrm{H} = \mathcal{H} \tag{1.0.2}$$

*where $\mathcal{H} : \overline{M} \to \mathbb{R}$ is a prescribing function.*

*ii) $F(\overline{M})$ is space-like.*

The aim of this work is to get some step closer to find an answer to Question 2 for a class of Lorentzian manifolds with non-compact Cauchy hypersurfaces.

## 1.1 How to answer to Question 2

Before diving directly into the Lorentzian analysis of the Question 2, we want to point out that it can be discussed in the Riemannian setting as well. It is clear that in this case condition *ii*) is automatically satisfied. For convenience let us begin by discussing Question 2 for the special case $\mathcal{H} = 0$ and assuming $(N, \overline{g})$ being the 3-dimensional Euclidean space $\mathbb{R}^3$. In particular we are asking if there exist surfaces (2-dimensional submanifolds) in $\mathbb{R}^3$ so that their mean curvature is 0. In the Riemannian setting (hyper)surfaces with vanishing mean curvature are also called minimal (hyper)surfaces since they are minimizers for the area functional. It is well known, see e.g. [Pér17], that minimal surfaces in $\mathbb{R}^3$ are locally expressed as graphs of a function $u$ satisfying the second order quasi-linear elliptic PDE

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|}}\right) = 0. \tag{1.1.1}$$

The first noticeable solution of (1.1.1) is the catenoid, formally described by Euler in 1744. Thus, it is clear that in $\mathbb{R}^3$ an answer to Question 2 is provided by finding solutions to (1.1.1).

**Remark 1.1.1**
*It is important to notice that the left-hand side of equation (1.1.1) is nothing but the mean curvature of a graphical hypersurface in $\mathbb{R}^3$ equipped with the Euclidean metric.*

For non-graphical hypersurfaces in a generic ambient Riemannian manifold, equation (1.0.2), even for the case $\mathcal{H} = 0$, does not lead to an equation "as easy as" (1.1.1).
In the Lorentzian setting, on the other hand, assuming the manifold $(N, \overline{g})$ to be globally hyperbolic (cf. [BeSa05, Theorem 1.1]), one has that space-like hypersurfaces can be written as graphs (cf. [BeSa03, Proposition 3.9]). Therefore one can prove the existence of solutions to (1.1.1) to give a positive answer to Question 2. The first two example in this direction are given by [Ger83] and [Bar84]. Less

than a decade later, in [EcHu91] Ecker and Huisken provide a parabolic method to answer to Question 2.
Let $F : \overline{M} \times [0,T] \to N$ be a family of embeddings. We say that $F$ is a prescribed mean curvature flow if it satisfies the Cauchy problem

$$\partial_s F = -(\mathrm{H} - \mathcal{H})\mu, \quad F(s=0) = F_0; \tag{1.1.2}$$

where H and $\mu$ represent the mean curvature and the unit normal to $\overline{M}$ at time $s$ and $\mathcal{H} : \overline{M} \to \mathbb{R}$ is a fixed prescribing function and $F_0$ is some initial embedding. If a solution to (1.1.2) exists for $T = \infty$ and it converges, it is not difficult to see that it has to converge to a stationary solution of (1.1.2), i.e. an embedding $F^*$ so that $\mathrm{H}(F^*) = \mathcal{H}$. Thus, a positive answer to Question 2 is obtained if a prescribed mean curvature flow exists for long time and it converges.

## 1.2 Mean curvature flow

Equation (1.1.2) is a slight generalisation of the classical mean curvature flow (MCF) (obtained by setting $\mathcal{H} = 0$). The mean curvature flow described the evolution of hypersurfaces in the direction of the unit normal with velocity given by the mean curvature H.

In the Riemannian setting one sees that the (MCF) is an area decreasing flow. This means that, if the ambient manifold is compact, a mean curvature flow is forced to collapse hence ending in finite time (see [Smo12, Proposition 3.10]). By having finite time existence, in the Riemannian setting one can not expect convergence of the flow. Although we are interested in the (prescribed) mean curvature flow in the Lorentzian setting, let us mention some useful reading about the topic in the Riemannian setting, [Hui84] [Hui90], [Eck04], [Whi05], [Man11], [CoMi12] and [Smo12], to cite just a few.

Contrarily, in the Lorentzian setting, the MCF is area increasing, thus one may expect long time existence and study its possible convergence. There is therefore hope to answer to Question 2. Hence it is interesting to study the long time behaviour of the prescribed mean curvature flow, described in equation (1.1.2), in the Lorentzian setting.

## 1.3 State of the art and placement of the thesis

The first analysis on the prescribed mean curvature flow in the Lorentzian setting, as stated above, can be traced back to [EcHu91]. Ecker and Huisken prove long time existence and convergence of the flow on a spatially-compact globally hyperbolic Lorentzian manifold subject to certain geometric restriction, e.g.

1. Time-like convergence condition: For every time-like vector field $X$ over the ambient Lorentzian manifold $(N, \overline{g})$, the Ricci tensor $\mathrm{Ric}^N(X,X) \geq 0$.

2. A structure condition for the mean curvature (see [Bar84, §3]).

In [GER00] Gerhardt proves the same results as in [ECHU91] but relaxing conditions 1. and 2. above at the expense of some regularity condition. We want to point out that both [ECHU91] and [GER00] take advantage of the ambient manifold being spatially compact. Some notably examples where the spatial compactness condition is relaxed are [ECOT97] and [KPLOT21]. In [ECOT97] the author deals with Minkowski spaces and highly relies on the ambient manifold being $\mathbb{R}^n$. Similarly in [KPLOT21] the authors deal with manifolds which are euclidean outside a compact region, therefore relying on the ambient manifolds being "almost" $\mathbb{R}^n$.

In this work we set-up the ground work for an answer to Question 2 in a class of Lorentzian manifolds (called generalised Robertson-Walker space-times) having a non-compact Cauchy hypersurface. In particular, due to the abundance of the geometry at infinity of non-compact manifolds, we focus our attention to $\Phi$-manifolds. This generalises all the existing works cited above.

## 1.4 Structure of the thesis

The thesis will be divided in 3 parts. The first part is devoted to the analytical background necessary for the analysis of heat-type equations in the setting of $\Phi$-manifolds. The second part can be thought as a compendium for the geometry of generalised Robertson-Walker space-times and its graphical submanifolds. In part three we proceed with the analysis of the prescribed (graphical) mean curvature flow in generalised Robertson-Walker space-times having a $\Phi$-manifold as space-like slice. Let us describe each part in more details.

### 1.4.1 Part I

We relax the spatial compactness condition in [ECHU91] by requiring the space-like slice to be a $\Phi$-manifold. $\Phi$-manifolds are a class of stochastically complete manifolds (cf. Proposition 3.5). Stochastically complete manifolds have the advantage to satisfy the so called Omori-Yau maximum principle. Since a parabolic maximum principle will be crucial in the analysis of (1.1.2), chapter 2 will be devoted to a parabolic maximum principle for stochastically complete manifolds.
In chapter 3 we present briefly, but carefully, $\Phi$-manifolds. In particular we describe the singular nature of the $\Phi$-metric (cf. equation (3.1.1)) and introduce a class of differential operators encapsulating the aforementioned singular behaviour of the metric. Finally we present a notion of Hölder regular functions with respect to such a singularity.
From equation (1.1.1), Remark 1.1.1 and the overall discussion in §1.1, one can conclude that the prescribed mean curvature flow for graphical submanifolds in a Lorentzian ambient manifold, will be some slight variation of the heat-equation. Therefore denoted by $(\overline{M}, \widetilde{g})$ a $\Phi$-manifold, in Chapter 4 we analyse in details the mapping property of the heat kernel operator $\mathbf{H}$, i.e. the inverse of the heat operator $(\partial_s + \widetilde{\Delta})$, on $(\overline{M}, \widetilde{g})$. Here $\widetilde{\Delta}$ denotes the Laplace-Beltrami operator associated to $\widetilde{g}$. Finally, in chapter 5 we begin by analysing linear Cauchy problems of the form

$$(i)\ (\partial_s + a\widetilde{\Delta})u = \ell, \;\; u(\_,0) = 0; \;\; (ii)\ (\partial_s + a\widetilde{\Delta})u = 0, \;\; u(\_,0) = u_0; \qquad (1.4.1)$$

for suitable functions $a, \ell$ and $u_0$. By employing microlocal techniques we prove that parametrix, i.e. approximate inverse, for the equations above exist (cf. Proposition 5.2.1). In conclusion we use the above mentioned result to prove short time existence of solutions to non-linear Cauchy problems of the form

$$(\partial_s + a\widetilde{\Delta})u = F(u); \tag{1.4.2}$$

with $F$ satisfying the assumptions in §5.3.
All the material covered in Part I will be employed later for the proof of short-time existence of prescribed mean curvature flows in generalised Robertson-Walker space-times having Φ-manifolds as space-like slice.

### 1.4.2 Part II

As mentioned above Part II can be thought as a geometry handbook. In Chapter 6 we furnish the reader with a quick overview on Lorentzian geometry, highlighting the differences arising from classical Riemannian geometry.
In order to make the reader comfortable with the notations as well as with the geometric quantities involved in the flow, in Chapter 7 we rigorously define all the needed intrinsic and extrinsic geometric quantities.
Finally, in Chapter 8 we introduce generalised Robertson-Walker space-times and explicitly compute the geometric quantities, defined in Chapter 7, for graphical hypersurfaces in generalised Robertson-Walker space-times.

### 1.4.3 Part III

Part III is the conclusion of the work done in Part I and Part II. Namely, in Chapter 9 we derive an expression for (1.1.2) in the context of graphical submanifolds. This will lead us to an expression formally equivalent to the left hand side of (1.1.1). Indeed the equation we will be analysing is

$$\begin{aligned} \partial_s u &= -\Delta_g u + \frac{f'(u)}{f(u)}\left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}, \\ u(\_,0) &= u_0. \end{aligned} \tag{1.4.3}$$

The above equation is clearly non linear in $u$ thus in the same chapter we also provide a linearisation. Notice that, in view of the notation presented in Question 2, $\Delta_g$ is the Laplace-Beltrami operator associated to the pull-back metric $g = F^*\overline{g}$ on $\overline{M}$ while $\widetilde{\nabla}$ denotes the gradient with respect to the Φ-metric $\widetilde{g}$.
In Chapter 10 we proceed with the analysis by proving that solutions to (1.4.3) do exist for short time. As mentioned earlier, the proof of this result will heavily rely on the results in Part I. We conclude the chapter by noticing that, as long as it exists, the flow stays stochastically complete.
Our aim is to get a little bit closer to an answer to Question 2. Thus, as stated in §1.1, one would like to have long time existence and convergence of solutions to (1.4.3). Although long time existence and convergence are far from the aim of this work, we will prove the most difficult step towards long time existence, that is

proving prescribed (graphical) mean curvature flows to stay space-like for as long as they exist. As usual in geometric analysis, namely the analysis of geometric flows, this is achieved by means of evolution equations for certain geometric quantities, which will be derived in Chapter 11, and an application of the parabolic maximum principle, obtained in §2.4.
We conclude this thesis with the claimed result. Indeed in Chapter 12 we prove that, under some assumptions comparable to the ones in [EcHu91] cited above, a prescribed mean curvature flow stays space-like for as long as it exists.

# Part I

# Singular analysis

# Chapter 2

# Maximum principle and stochastic completeness

One of the main goals of this work is to prove the existence of solutions of heat-type equations on $\Phi$-manifolds. As it will be clear in §3.5, $\Phi$-manifolds, which will be introduced in §3.1, represent an example of a much wider family of manifolds called stochastically complete manifolds.
A well known tool for the analysis of parabolic PDE's, as heat-type equations, is the (parabolic) maximum principle. Although quite classical in the context of compact manifolds, the maximum principle may not hold for generic non-compact manifolds. For non-compact manifolds one usually deals with a weaker version of the maximum principle, known as the Omori-Yau maximum principle. Contrarily to compact manifolds, the Omori-Yau maximum principle holds for non-compact manifolds satisfying certain analytic-geometric conditions. In this chapter we will therefore focus on one of these conditions, i.e. stochastic completeness. We present the notion of stochastic completeness in §2.1, as well as the Omori-Yau maximum principle in §2.2, for a generic Riemannian manifolds $(X, g_X)$. Finally, by making use of the Omori-Yau maximum principle, in §2.4 we furnish the reader with a version of the parabolic maximum principle suitable for our analysis.
We want to point out that the maximum principle will not only be used in §5.1 for the parametrix construction but in the study of the behaviour of prescribed mean curvature flows (i.e. in gaining a priori estimates) as well.

## 2.1 Stochastically complete manifolds

First, we want to make clear, once and for all, our convention for the Laplace-Beltrami operator.

**Definition 2.1**
*Let $(X, g_X)$ be an $m$-dimensional Riemannian manifold. For a function $u \in C^2(X)$ the Laplace-Beltrami operator is defined to be*

$$\Delta_X u = -\operatorname{div} \nabla u \tag{2.1.1}$$

*where* div *and* $\nabla$ *are the divergence and the gradient taken with respect to the metric tensor $g_X$ respectively.*

*In particular, given coordinates* $(x^i)_{i=1,\dots,m}$*, the local coordinate expression for the Laplacian is given by*

$$\Delta_X u = -g_X^{ij}\left(u_{ij} - \Gamma_{ij}^k u_k\right); \tag{2.1.2}$$

*where* $\Gamma_{ij}^k$ *denotes the Christoffel symbols of the metric tensor* $g_X$*.*

Next we present the notion of stochastic completeness.

**Definition 2.2**
*Let* $(X, g_X)$ *be a Riemannian manifold. It is stochastically complete if the heat kernel of the (positive) Laplace-Beltrami operator* $\Delta_X$*, associated to* $g_X$ *satisfies*

$$\int_X H(s, p, \widetilde{p})\, \mathrm{dvol}_{g_x}(\widetilde{p}) = 1. \tag{2.1.3}$$

Stochastic completeness can be equivalently characterized by a volume growth condition, due to Grigor'yan [Gri86], cf. also Theorem 2.11 in [AMR16].

**Theorem 2.1**
*Let* $(X, g_X)$ *be a complete Riemannian manifold. Consider for some reference point* $p \in X$ *the geodesic ball* $B_R(p)$ *of radius* $R$ *around* $p$*. If the function*

$$\frac{R}{\log\left(\mathrm{Vol}\left(B_R(p)\right)\right)} \notin L^1(1, \infty) \tag{2.1.4}$$

*then* $(X, g_X)$ *is stochastically complete.*

## 2.2 Omori-Yau maximum principle

The Omori-Yau maximum principle for the Laplacian, defined in e.g. [AMR16, Definition 2.1] states that for any function $u \in C^2(X)$ with bounded supremum there is a sequence $\{p_k\}_k \subset X$ satisfying

$$u(p_k) > \sup_X u - \frac{1}{k}, \quad |\nabla u(p_k)| \le \frac{1}{k}, \quad -\Delta_X u(p_k) < \frac{1}{k}. \tag{2.2.1}$$

Similarly, provided $u$ has bounded infimum, there exists a sequence $\{p_k'\}_k \subset M$ such that

$$u(p_k') < \inf_X u + \frac{1}{k}, \quad |\nabla u(p_k')| \le \frac{1}{k}, \quad -\Delta_X u(p_k') > \frac{1}{k}. \tag{2.2.2}$$

By [AMR16, Theorem 2.3], the Omori-Yau maximum principle for the Laplacian holds on any $(X, g_X)$ with Ricci curvature bounded from below. We shall refer to this principle as the *strong* Omori-Yau maximum principle in order to distinguish it from another version of the principle on stochastically complete manifolds.

**Remark 2.2.1**
*We want to point out a difference with [AMR16] in the different sign convention for the Laplace-Beltrami operator.*

According to [AMR16, Theorem 2.8 (i) and (iii)], a similar version of the Omori-Yau maximum principle holds for stochastically complete manifolds. More precisely, for any $(X, g_X)$ satisfying e.g. the volume growth condition in Theorem 2.1, and any function $u \in C^2(X)$, there is a sequence $\{p_k\}_k \subset X$ such that

$$u(p_k) > \sup_X u - \frac{1}{k} \quad \text{and} \quad -\Delta_X u(p_k) < \frac{1}{k}. \tag{2.2.3}$$

Similarly, there exists a sequence $\{p'_k\}_k \subset M$ such that

$$u(p'_k) < \inf_X u + \frac{1}{k} \quad \text{and} \quad -\Delta_X u(p'_k) > \frac{1}{k}. \tag{2.2.4}$$

## 2.3 Classical Hölder spaces

In order to discuss the maximum principle, appropriate functional spaces have to be introduced. Although here we are going to present here a classical definition of Hölder spaces, in §3.3 we will give a slightly different definition of object, called with a slight abuse of notation Hölder spaces, which will encode the singular behaviour of $\Phi$-metrics.

**Definition 2.3**
*Let $\alpha$ be a number in $(0,1)$. We define the semi-norm*

$$[u]_\alpha := \sup_{X_T^2} \left\{ \frac{|u(p,t) - u(p',t')|}{d(p,p')^\alpha + |s-s'|^{\alpha/2}} \right\} < \infty, \tag{2.3.1}$$

*where the supremum is over $X_T^2$ with $X_T = X \times [0,T]$; the distance $d$ is induced by the metric $g_X$. The Hölder space $C^\alpha(X \times [0,T])$, for $\alpha \in (0,1)$, is defined as the space of continuous functions $u \in C^0(X \times [0,T])$ with bounded $\alpha$-norm*

$$\|u\|_\alpha := \|u\|_\infty + [u]_\alpha. \tag{2.3.2}$$

Once equipped with the $\alpha$-norm (2.3.2), the resulting normed vector space $C^\alpha(X \times [0,T])$ is a Banach space. Similarly one defines higher order Hölder spaces.

**Definition 2.4**
*Let $(X, g_X)$ be a Riemannian manifold. Consider $k, l_1$ and $l_2$ to be non negative integers. We say that a function $u$ lies in $C^{k,\alpha}(X \times [0,T])$ if $(P \circ \partial_s^{l_2})u$ lies in $C^\alpha(\overline{M} \times [0,T])$, for $P \in \mathrm{Diff}^{l_1}(X)$, $0 \le l_1 + 2l_2 \le k$; where with $\mathrm{Diff}^{l_1}(X)$ denotes the space of differential operators of order $l_1$ over $\overline{M}$. In particular, this is equivalent to require that the $(k,\alpha)$-norm, defined by*

$$\|u\|_{k,\alpha} = \|u\|_\alpha + \sum_{l_1+2l_2 \le k} \sum_{P \in \mathrm{Diff}^{l_1}(\overline{M})} \|(P \circ \partial_s^{l_2})u\|_\alpha. \tag{2.3.3}$$

*is bounded.*

**Remark 2.3.1**
*By definition we have the chain of inclusions $C^{l,\alpha}(X \times [0,T]) \subset C^{k,\alpha}(X \times [0,T])$ for every $0 \le k \le l$.*

## 2.4 Parabolic maximum principle

Based on the Omori-Yau maximum principle in §2.2, [CHV21] proved the following enveloping theorem.

**Proposition 2.4.1** (Proposition 3.1 in [CHV21])
*Let $(X, g_X)$ be a stochastically complete manifold and consider $u \in C^{2,\alpha}(X \times [0,T])$. Then*

$$u_{\sup}(s) := \sup_X u(\cdot, s), \quad u_{\inf}(s) := \inf_X u(\cdot, s)$$

*are locally Lipschitz hence differentiable almost everywhere in $(0,T)$. Moreover, at those differentiable times $s \in (0,T)$ we find, in the notation of (2.2.3) and (2.2.4),*

$$\begin{aligned} \frac{\partial}{\partial s} u_{\sup}(s) &\leq \lim_{\varepsilon \to 0^+} \left( \limsup_{k\to\infty} \frac{\partial u}{\partial s} \left(p_k(s+\varepsilon), s+\varepsilon\right) \right), \\ \frac{\partial}{\partial s} u_{\inf}(s) &\geq \lim_{\varepsilon \to 0^+} \left( \liminf_{k\to\infty} \frac{\partial u}{\partial s} \left(p'_k(s+\varepsilon), s+\varepsilon\right) \right). \end{aligned} \tag{2.4.1}$$

**Remark 2.4.1**
*We want to point out that, although [CHV21, Proposition 3.1] has been proved for $\Phi$-manifolds; the same proof holds for stochastically complete manifolds as well. Indeed, in [CHV21], the authors make use of $\Phi$-metric in order to use inequality (2.2.3) which, as stated in §2.2 holds for stochastically complete manifolds.*

We are now in the position to prove the claimed maximum principle. We want to point out that the upcoming results, although similar, deal with two completely different situations. In the first one one deals with a **fixed** stochastically complete metric; this will be enough to prove the (short-time) existence of solution for heat-type equations on $\Phi$-manifolds (see chapter 5.3). In the second one instead one allows the metric to be $s$-dependent. In particular the first two will be used in the proof of existence of solutions of heat-type equations, while the third one will be used in the analysis of prescribed mean curvature flows.

**Theorem 2.2**
*Let $(X, g_X)$ be a $m$-dimensional stochastically complete manifold. Furthermore, let $a \geq \delta > 0$ be a bounded function on $X$. If $u \in C^{2,\alpha}(X \times [0,T])$ is a solution of the Cauchy problem*

$$(\partial_s + a\Delta_X)u = 0, \; u|_{s=0} = 0; \tag{2.4.2}$$

*then $u = 0$.*

***Proof:***
Combining the first inequality in Proposition 2.4.1 and (2.2.3), it follows that

$$\frac{\partial}{\partial s} u_{\sup}(s) \leq \lim_{\varepsilon\to 0} \left( \limsup_{k\to\infty} \frac{a(p_k(s+\varepsilon), s+\varepsilon)}{k} \right) \leq 0.$$

Analogously, by combining the second inequality in Proposition 2.4.1 and (2.2.4), we get

$$\frac{\partial}{\partial s} u_{\inf}(s) \geq \lim_{\varepsilon\to 0} \left( \liminf_{k\to\infty} \frac{-a(p_k(s+\varepsilon), s+\varepsilon)}{k} \right) \geq 0.$$

This means that the infimum of the function $u$ over $X$ is non-decreasing in time, while the supremum of the function $u$ over $X$ is non-increasing in time; since $u = 0$ at time $s = 0$, follows directly that $u = 0$ on $M \times [0, T]$. □

As mentioned at the beginning of this chapter, a maximum principle will be further used to gain a priori estimates for the analysis of prescribed mean curvature flows. For this further analysis the maximum principle presented above will not be enough. We will indeed have to consider a family of stochastically complete metrics. Therefore a parabolic maximum principle is needed.

**Theorem 2.3**
*Let $(X, g_X(s))$ be a family of stochastically complete manifolds with $s \in [0, T]$. Denote the corresponding family of Laplace-Beltrami operators by $\Delta_s$. Consider solutions $u^{\pm} \in C^{2,\alpha}_{\Phi}(X \times [0, T])$, solving the differential inequalities*

$$\left( \frac{\partial}{\partial s} + \Delta_s \right) u^+ \leq 0, \quad \left( \frac{\partial}{\partial s} + \Delta_s \right) u^- \geq 0. \tag{2.4.3}$$

*Then $u^+_{\sup}(s) \leq u^+_{\sup}(0)$ and $u^-_{\inf}(s) \geq u^-_{\inf}(0)$ for every $s \in [0, T]$.*

***Proof:***
Note first by (2.2.3) and (2.2.4)

$$\frac{\partial}{\partial s} u^+ \big( p_k(s), s \big) \leq \frac{1}{k}, \quad \frac{\partial}{\partial s} u^- \big( p'_k(s), s \big) \geq -\frac{1}{k}.$$

Then in view of Proposition 2.4.1 we find almost everywhere

$$\frac{\partial}{\partial s} u^+_{\sup}(t) \leq 0, \quad \frac{\partial}{\partial s} u^-_{\inf}(t) \geq 0.$$

The claim now follows. □

# Chapter 3

# $\Phi$-manifolds

As mentioned in the introduction, our aim is to study the prescribed mean curvature flow of certain hypersurfaces of (a specific class of) Lorentzian manifolds, arising as graphs over non-compact manifolds. Contrarily to compact manifolds, non-compact manifolds come with much more interesting geometry; due to the varied behaviour at infinity. Therefore, in order to perform some analysis on non-compact manifolds, one usually focuses to specific geometries.
In this work we decided to restrict our attention to a specific class of geometry at infinity usually referred to as $\Phi$-manifolds. Therefore we devote this chapter to introduce the reader to the concept of $\Phi$-manifolds. This chapter is far from being a complete compendium on $\Phi$-manifolds; thus we refer the reader to the pioneering work of Mazzeo and Melrose [MaMe98] for more details.

## 3.1 Manifolds with fibered boundary

The first step toward the definition of $\Phi$-manifolds is the definition of manifolds with fibred boundary.

**Definition 3.1**
*Let $\overline{M} = M \cup \partial\overline{M}$ be a compact smooth manifold with boundary. We say that $\overline{M}$ is a manifold with fibred boundary if the boundary $\partial\overline{M}$ of $\overline{M}$ is the total space of a fibration. That is*

$$\partial\overline{M} \xrightarrow{\phi} Y$$

*with typical fibre $Z$ such that both $Y$ and $Z$ are closed manifolds*

Having the notion of manifolds with fibred boundary at our disposal, it is not hard to give the notion of $\Phi$-metric.

**Definition 3.2**
*Let $\overline{M}$ be a manifold with fibred boundary with $\partial\overline{M}$ being the total set of a fibration with base $Y$ and typical fibre $Z$. Assume, further, $g_Y$ to be a Riemannian metric on $Y$ and $g_Z$ to be a symmetric bilinear form on $\partial\overline{M}$ restricting to Riemannian metrics on each fiber. We call a Riemannian metric $\widetilde{g}$ over $\overline{M}$ a $\Phi$-metric if, when restricted to the open collar neighbourhood $(0,1) \times \partial M$, $\widetilde{g}$ admits locally the expression*

$$\widetilde{g} = \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z + h =: \widehat{g} + h, \tag{3.1.1}$$

*where $\widehat{g}$ is called the exact fibred boundary metric and $h$ is a perturbation (gathering all cross-terms in $\widetilde{g}$) such that $|h|_{\widehat{g}} = O(x)$ as $x \to 0$.*

From this point on, we will denote by $b$ the dimension of $Y$ and by $f$ the dimension of $Z$. Let $x$ be a choice of a boundary defining function for $\partial M$. That is, $x$ is a non-negative function on $\overline{M}$ lying in $C^\infty(\overline{M})$, so that $\partial\overline{M} = \{p \in \overline{M} \,|\, x(p) = 0\}$ and $\mathrm{d}\,x \neq 0$ on $\partial\overline{M}$. Since $\overline{M}$ is compact, there exists a collar neighbourhood $U$ of $\partial\overline{M}$ in $\overline{M}$ such that $U \simeq [0,1) \times \partial\overline{M}$. It is therefore possible to write every point in $U$ as a pair $(x, w)$, with $x \in [0,1)$ and $w \in \partial\overline{M}$. Since $\partial\overline{M}$ is the total space of a fibration over the base space $Y$ with typical fibre $Z$, there is an open cover $\{V_i\}$ of $Y$ such that $\phi^{-1}(V_i) \simeq V_i \times Z$ for every $i$. Thus, in such open subsets, every point can be written as a pair $(\widehat{y}, \widehat{z})$. It is therefore locally possible to write every $p \in U$ as the triple $p = (x, \widehat{y}, \widehat{z})$. In conclusion, by means of the above identification, every point $p$ in $U$ has coordinates $p = (x, y^1, \dots, y^b, z^1, \dots, z^f)$ where $(y^1, \dots, y^b)$ and $(z^1, \dots, z^f)$ are coordinates for $\widehat{y} \in Y$ and $\widehat{z} \in Z$ respectively.

**Remark 3.1.1**
*Due to the abundance of indices, some times we will use $y$ and $z$ to denote either the whole coordinate $(y^1, \dots, y^b)$ and $(z^1, \dots, z^f)$ respectively or a generic coordinate element, i.e. $y = y^k$ for some $k$ and $z = z^j$ for some $j$.*

**Example 1**
*A classical example for Φ-manifolds is the space $\mathbb{R}^m$ equipped with the Euclidean metric expressed in polar coordinates.*
*Indeed, by denoting with $g_{S^{m-1}}$ the standard round metric of the $m-1$-dimensional sphere, the Euclidean metric on $\mathbb{R}^m$ in polar coordinates can be expressed as*

$$\widetilde{g} = \mathrm{d}\,r^2 + r^2 g_{S^{m-1}}.$$

*Performing the change of coordinates $x = 1/r$ we obtain an expression which agrees with the one in (3.1.1); namely*

$$\widetilde{g} = \frac{\mathrm{d}\,x^2}{x^4} + \frac{g_{S^{m-1}}}{x^2}.$$

*Note that, in this example, the fibre $Z$ consist of only a point, i.e. $Z = \{pt\}$.*

**Remark 3.1.2**
*The example above displays how, although defined as compact manifolds with boundary, Φ-manifolds model certain class of non-compact manifolds.*
*Indeed, the change of coordinates presented above represents a choice of placement of the "singular region". In fact, by setting $x = 1/r$ we "moved" the singular region $\{r = \infty\}$ of $\mathbb{R}^m$ to the origin $\{x = 0\}$.*

## 3.2 Φ-vector fields and Φ-one forms

In the context of Φ-manifolds a set of "well behaved" vector fields can be defined. Here by "well behaved" we mean that they take care of the singular nature of the metric tensor. We define the space of Φ-vector fields to be

$$\mathcal{V}_\Phi(\overline{M}) = \left\{ V \in \mathcal{V}(M) \,\middle|\, \begin{array}{c} Vx \in x^2 C^\infty(M) \text{ and} \\ V_p \in T_p\phi^{-1}(\phi(p)) \text{ for every } p \in \partial M \end{array} \right\}.$$

In local coordinates $(x, y, z)$ (cf. Remark 3.1.1) near $\partial\overline{M}$,

$$\mathcal{V}_\Phi(\overline{M}) := \operatorname{span}_{C^\infty(M)} \left\{ x^2\partial_x, x\partial_{y_1}, ..., x\partial_{y_b}, \partial_{z_1}, ..., \partial_{z_f} \right\}.$$

The Φ-tangent bundle ${}^\Phi T\overline{M}$ is, by definition, a vector bundle over $\overline{M}$ whose sections are given by $\mathcal{V}_\Phi(\overline{M})$.

**Remark 3.2.1**
*It is worth to point out that, the inner product, i.e. the metric paring, of any two Φ-vector fields is bounded.*

Analogously to the classical differential geometry, one can also consider the dual bundle ${}^\Phi T^*\overline{M}$. This is the bundle whose sections are the Φ-one forms on $\overline{M}$, which are differential forms on $\overline{M}$ generated by the family

$$\left\{ \frac{\mathrm{d}\,x}{x^2}, \frac{\mathrm{d}\,y_1}{x}, ..., \frac{\mathrm{d}\,y_b}{x}, \mathrm{d}\,z_1, ..., \mathrm{d}\,z_f \right\}.$$

Since $\mathcal{V}_\Phi(\overline{M})$ is a Lie algebra and a $C^\infty(\overline{M})$ module (see [MaMe98]), one can consider the algebra $\operatorname{Diff}^*_\Phi(\overline{M})$. In particular one defines higher order Φ-differential operators as operators on $C^\infty(\overline{M})$ which can be written as a $C^\infty(\overline{M})$ linear combination of elements of $\mathcal{V}_\Phi(\overline{M})$. Hence, we define the space of Φ-differential operators of order k, denoted by $\operatorname{Diff}^k_\Phi(\overline{M})$ or alternatively by $\mathcal{V}^k_\Phi(\overline{M})$, as the space of linear operators $P : C^\infty(\overline{M}) \to C^\infty(\overline{M})$ which can be locally expressed by

$$P = \sum_{|\alpha|+|\beta|+q\leq k} P_{\alpha,\beta,q}(x, y, z)(x^2\partial_x)^q(x\partial_y)^\beta\partial_z^\alpha,$$

where $\alpha$ and $\beta$ are multi-indices, $\partial_y = \partial_{y_1}, \ldots, \partial_{y_b}$, $\partial_z = \partial_{z_1}, \ldots, \partial_{z_f}$ and $P_{\alpha,\beta,q}$ is a smooth function.

## 3.3 Hölder continuity on Φ-manifolds

The analysis of PDEs relies upon the choice of suitable function spaces. For our purposes, Hölder spaces and slight variations (weighted Hölder spaces) are needed. Although a classical definition of Hölder spaces has already be given (cf. §2.3), we devote this section to the definition of a space of functions (called again Hölder spaces) encoding the singular behaviour of Φ-metrics.

We denote by $C^k_\Phi(M \times [0, T])$ the set of functions that, together with their Φ-derivatives up to order $k$, are continuous on $\overline{M} \times [0, T]$. We want to point out that time derivatives will be considered as second order derivatives. For $\alpha \in (0, 1)$, we define the $\alpha$-norm as the map $\|\_\|_\alpha : C^0_\Phi(M \times [0, T]) \to [0, \infty)$ given by

$$\|u\|_\alpha = \|u\|_\infty + \sup \left\{ \frac{|u(p, t) - u(p', t')|}{d(p, p')^\alpha + |t - t'|^{\alpha/2}} \right\} =: \|u\|_\infty + [u]_\alpha. \tag{3.3.1}$$

Sometimes we will refer to the term $[u]_\alpha$ as the ($\alpha$ or Hölder)-seminorms.

The distance between $p$ and $p'$, appearing on the denominator of (3.3.1), is defined in terms of $x^4 g_\Phi$ and it is, locally near the boundary, equivalent to

$$d(p,p') = \sqrt{|x-x'|^2 + (x+x')^2\|y-y'\| + (x+x')^4\|z-z'\|^2}.$$

We define the $\alpha$-Hölder continuous functions as the space of functions continuous up to the boundary $\partial\overline{M}$ and whose $\alpha$-norm is bounded. That is

$$C^\alpha_\Phi(M\times[0,T]) := \{u \in C^0_\Phi(M\times[0,T]) \,|\, \|u\|_\alpha < \infty\}.$$

Once endowed with the $\alpha$-norm (3.3.1), this functional space turns into a Banach space. Higher order Hölder regularity is defined as follows. For $k, l_1$ and $l_2$ being non-negative integers, the $(k,\alpha)$-Hölder space is given by

$$C^{k,\alpha}_\Phi(M\times[0,T]) = \left\{ u \in C^k_\Phi(M\times[0,T]) \,\middle|\, \begin{array}{l} (\mathcal{V}^{l_1}_\Phi \circ \partial^{l_2}_s)u \in C^\alpha_\Phi(M\times[0,T]), \\ \text{for } l_1 + 2l_2 \leq k \end{array} \right\} \tag{3.3.2}$$

From [BaVe14, Proposition 3.1] follows that the $(k,\alpha)$-Hölder space $C^{k,\alpha}_\Phi(M\times[0,T])$, when equipped with the norm

$$\|u\|_{k,\alpha} = \|u\|_\alpha + \sum_{l_1+2l_2\leq k} \sum_{X\in\mathcal{V}^{l_1}_\Phi} \|(X\circ\partial^{l_2}_s)u\|_\alpha$$

is also a Banach space.

**Remark 3.3.1**
*For every $0 \leq k^1 \leq k^2$ and for every $\alpha \in (0,1)$, one has*

$$C^{k^2,\alpha}_\Phi(M\times[0,T]) \subset C^{k^1,\alpha}_\Phi(M\times[0,T]).$$

*In particular, this means that, for every $k \geq 0$, $C^{k,\alpha}_\Phi(M\times[0,T]) \subset C^\alpha_\Phi(M\times[0,T])$.*

Finally, for $\gamma$ a real number, one can define the weighted Hölder spaces as follow:

$$x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) = \{x^\gamma u \,|\, u \in C^{k,\alpha}_\Phi(M\times[0,T])\} \tag{3.3.3}$$

On $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, consider then the modified norm

$$\|x^\gamma u\|_{k,\alpha,\gamma} := \|u\|_{k,\alpha}.$$

Whenever $\gamma \neq 0$, the above definition turns the multiplication by $x^\gamma$ into an isometry between $C^{k,\alpha}_\Phi(M\times[0,T])$ and $x^\gamma C^{k,\alpha}_\Phi(M\times[0,T])$, naturally implying that the weighted Hölder spaces are also Banach spaces.

The above gives a notion of $\alpha$-Hölder functions $u = u(p,s)$ defined over $\overline{M}\times[0,T]$. Sometimes we will be dealing with functions not depending on $s$. Thus a notion of $\alpha$-Hölder functions on $\overline{M}$ is also needed. This is achieved just by suppressing the $s$-terms in the above definitions. In particular (3.3.1) takes now the form: for $u : \overline{M} \to \mathbb{R}$,

$$\|u\|_\alpha = \|u\|_\infty + \sup_{p,p'\in\overline{M}} \left\{ \frac{u(p)-u(p')}{\mathrm{d}(p,p')^\alpha} \right\} =: \|u\|_\infty + [u]_\alpha. \tag{3.3.4}$$

The $(0,\alpha)$-space remains formally unaltered,

$$C^{\alpha}_{\Phi}(M) =: \left\{ u \in C^{0}_{\Phi}(M) \,\middle|\, \|u\|_{\alpha} < \infty \right\},$$

where, similarly to the previous case, $C^{k}_{\Phi}(M)$ denotes functions that, together with their $\Phi$-derivatives up to order $k$, are continuous on $\overline{M}$.
Concerning (3.3.2), the $s$-derivative vanishes completely from the definition resulting in

$$C^{k,\alpha}_{\Phi}(M) = \left\{ u \in C^{k}_{\Phi}(M) \,\middle|\, Xu \in C^{\alpha}_{\Phi}(M) \, \forall \, X \in \mathcal{V}^{l}_{\phi} \text{ for } l \leq k \right\}. \tag{3.3.5}$$

Furthermore, again just by removing the $s$-derivative from the equation, we define the $(k,\alpha)$-norm on $C^{k,\alpha}_{\Phi}(M)$ by

$$\|u\|_{k,\alpha} = \|u\|_{\alpha} + \sum_{l=1}^{k} \sum_{X \in \mathcal{V}^{l}_{\Phi}} \|Xu\|_{\alpha}. \tag{3.3.6}$$

As for the more general case one has that the $(k,\alpha)$-Hölder space $C^{k,\alpha}_{\Phi}(M)$ is a Banach space, once equipped with the $(k,\alpha)$-norm defined above.
Finally one can define weighted $(k,\alpha)$-Hölder spaces exactly as it has already been done.

## 3.4 Properties of Φ-Hölder spaces

Here we will present some useful properties regarding $(k,\alpha)$-Hölder functions over $\overline{M}$. Let $(\overline{M}, \widetilde{g})$ be a $\Phi$-manifold and consider $k \in \mathbb{N}$ and $\alpha \in (0,1)$.

**Lemma 3.4.1**
*Let $a$ and $b$ be functions in $C^{k,\alpha}_{\Phi}(M)$. The product $a \cdot b = ab$ is also in $C^{k,\alpha}_{\Phi}(M)$.*

***Proof:***
For simplicity we will prove the result only for $k = 0$. The general case follows along the same lines.
The continuity up to the boundary condition is clearly satisfied. It is also trivial to notice that $\|ab\|_{\infty} < \infty$. In order to prove $ab \in C^{k,\alpha}_{\Phi}(M)$ it is necessary to estimate $[ab]_{\alpha}$. Let $p, p'$ be points in $\overline{M}$.

$$\begin{aligned} |a(p)b(p) - a(p')b(p')| &\leq |a(p)||b(p) - b(p')| + |b(p')||a(p) - a(p')| \\ &\leq \|a\|_{\infty}|b(p) - b(p')| + \|b\|_{\infty}|a(p) - a(p')|. \end{aligned}$$

Thus, it follows

$$[ab]_{\alpha} \leq \|a\|_{\infty}[b]_{\alpha} + \|b\|_{\infty}[a]_{\alpha} < \infty.$$

The general result holds by applying Leibniz rule. □

The same proof as for the previous lemma can be used to prove the following.

**Proposition 3.4.1**
*Let $a$ be a function in $C^{k,\alpha}_{\Phi}(M)$ and $b$ a function in $C^{k,\alpha}_{\Phi}(M \times [0,T])$. Then $a \cdot b = ab \in C^{k,\alpha}_{\Phi}(M \times [0,T])$. Here $ab$ is defined by, for every $(p,s) \in \overline{M} \times [0,T]$, $ab(p,s) := a(p)b(p,s)$.*

**Lemma 3.4.2**
*Let $a$ be a function in $C^{k,\alpha}_{\Phi}(M)$ and assume there exists some constant $C > 0$ so that $a \geq C > 0$. Then $1/a$ lies on $C^{k,\alpha}_{\Phi}(M)$ as well.*

***Proof:***
As for the previous result, we will present a proof only for $k = 0$. For higher $k$ the same idea can be employed.
Notice that, by assumption, $\|1/a\|_{\infty} < \infty$. Let now $p, p'$ be points in $\overline{M}$. An application of the mean value theorem leads to

$$\left|\frac{1}{a(p)} - \frac{1}{a(p')}\right| = \left|\frac{1}{\xi}\right| |a(p) - a(p')|;$$

with $\xi \in (a(p), a(p'))$ where, without loss of generality, we have assumed $a(p) \leq a(p')$. Note that $\xi > a(p) \geq C$ hence we can conclude

$$\left[\frac{1}{a}\right]_{\alpha} \leq \frac{1}{C^2}[a]_{\alpha} < \infty.$$

□

**Corollary 3.4.1**
*Let $a$ be a function in $C^{k,\alpha}_{\Phi}(M)$. If there exists $C > 0$ so that $0 < C \leq a$ then $\sqrt{a}$ is in $C^{k,\alpha}_{\Phi}(M)$.*

***Proof:***
The proof for $k = 0$ follows exactly the proof of lemma 3.4.2 hence we will not repeat it. We will present instead to proof for $k = 1$. For $X \in \mathcal{V}_{\Phi}(M)$ one has

$$X\sqrt{a} = \frac{1}{2\sqrt{a}} Xa.$$

From the result for $k = 0$ we know that $\sqrt{(a)} \in C^{\alpha}_{\Phi}(M)$. In particular it follows from lemma 3.4.2 that $1/\sqrt{a}$ is also a function in $C^{\alpha}_{\Phi}(M)$. The result now follows by applying lemma 3.4.1. □

Next we want to see for which $\gamma > 0$ the space $x^{\gamma} C^{k,\alpha}_{\Phi}(M \times [0,T])$ is a subspace of $C^{k,\alpha}_{\Phi}(M \times [0,T])$. To this end, in view of Proposition 3.4.1, it is enough to see under which assumption on $\gamma$ the function $x^{\gamma} \in C^{k,\alpha}_{\Phi}(M)$.

**Lemma 3.4.3**
*Let $p \geq q > 0$. There exists some constant $c > 0$ so that the function $F(x) = x^p - x^q$ is decreasing for $x \in (0, c]$.*

**Proposition 3.4.2**
*For $\gamma \geq 1$ the function $x^{\gamma} : \overline{M} \to \mathbb{R}$ lies in $C^{\alpha}_{\Phi}(M)$.*

***Proof:***
First of all notice that boundedness of $x^{\gamma}$ is trivial. Let us now analyse the term $[x^{\gamma}]_{\alpha}$. For $p, p' \in \overline{M}$ we write $x(p) = x$ and $x'(p) = x'$. If $x = 0$ or $x' = 0$ one has

$$[x^{\gamma}]_{\alpha} \leq \frac{\mathrm{d}(p,p')^{\gamma}}{\mathrm{d}(p,p')^{\alpha}} < \infty;$$

where the last inequality follows from $\gamma \geq 1 \geq \alpha$. We can thus assume $x \neq 0$, $x' \neq 0$ and, without loss of generality, $x \geq x'$. By employing the previous lemma we will compare $x^\gamma$ with $x^{[\gamma]}$ where $[\gamma]$ denotes the integer part of $\gamma$. Let us denote by $A$ the maximum of $x$ and consider $c$ to be the constant for $\gamma \geq [\gamma] > 0$ as in the lemma 3.4.3. It is now clear that

$$\frac{c}{A}x \leq c \text{ and } \frac{c}{A}x \geq \frac{c}{A}x'.$$

One can therefore apply lemma 3.4.3 leading to

$$\left(\frac{c}{A}\right)^\gamma (x^\gamma - x'^\gamma) \leq \left(\frac{c}{A}\right)^{[\gamma]} (x^{[\gamma]} - x'^{[\gamma]}).$$

Thus one finds

$$x^\gamma - x'^\gamma \leq \left(\frac{A}{c}\right)^{\gamma-[\gamma]} (x^{[\gamma]} - x'^{[\gamma]}) \leq c_1(x^{[\gamma]} - x'^{[\gamma]});$$

where $c_1$ is a constant depending on $A$ and $\gamma$. The result now follows by noticing that $[\gamma]$ is an integer hence

$$x^\gamma - x'^\gamma \leq c_2(x - x') \leq c_2 \,\mathrm{d}(p, p')$$

for some constant $c_2$. □

**Remark 3.4.1**
*The above result can be extended to $x^\gamma \in C^{k,\alpha}_\Phi(M)$ for every $k$. Indeed for $X \in \mathcal{V}_\Phi(M)$, $Xx^\gamma$ is either $0$ or $x^{\gamma+1}$.*

The above result completely describes what happens for $\gamma \geq 1$. Let now analyse the case with $\gamma \in (0, 1)$.

**Lemma 3.4.4**
*For every $x_1, x_2 \in \mathbb{C}$ and for every $\gamma \in (0, 1)$ one has*

$$|x_1 + x_2|^\gamma \leq |x_1|^\gamma + |x_2|^\gamma. \tag{3.4.1}$$

***Proof:***
By making use of the triangle inequality one sees that

$$|x_1 + x_2|^\gamma \leq (|x_1| + |x_2|)^\gamma.$$

Thus (3.4.1) follows if

$$(|x_1| + |x_2|)^\gamma \leq |x_1|^\gamma + |x_2|^\gamma. \tag{3.4.2}$$

It is therefore enough to prove (3.4.2) for $x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq 0$. Moreover, for $x_1 = 0$ or $x_2 = 0$ the result is trivial; so we can assume $x_1, x_2 > 0$ and, without loss of generality $x_1 \geq x_2$. By setting $\widetilde{x} = x_1/x_2$ one sees that proving (3.4.2) is equivalent to prove

$$(\widetilde{x} + 1)^\gamma \leq \widetilde{x}^\gamma + 1 \tag{3.4.3}$$

for $\widetilde{x} \geq 1$. Let us define the function $F(\widetilde{x}) := \widetilde{x}^\gamma + 1 - (\widetilde{x} + 1)^\gamma$. Since $\gamma \in (0, 1)$, $F(1) \geq 0$. The result follows by noticing that $F$ is increasing for $\widetilde{x} \geq 1$. □

A straight forward application of the above lemma leads to the following result.

**Corollary 3.4.2**
*For every $x_1, x_2 \in \mathbb{C}$ and for every $\gamma \in (0,1)$*

$$|x_1|^\gamma - |x_2|^\gamma \leq |x_1 - x_2|^\gamma. \tag{3.4.4}$$

We are now in the position to prove the following.

**Proposition 3.4.3**
*For every $\gamma \in (0,1)$ such that $\gamma \geq \alpha$ the function $x^\gamma \in C^{k,\alpha}_\Phi(M)$.*

***Proof:***
In view of remark 3.4.1 it is enough to prove the result for $k = 0$. As for proposition 3.4.2 we only need to estimate $[x^\gamma]_\alpha$. For $p, p' \in \overline{M}$ one sees that, on account of corollary 3.4.2,

$$|x^\gamma - x'^\gamma| \leq |x - x'|^\gamma \leq \mathrm{d}(p,p')^\gamma.$$

Thus

$$[x^\gamma]_\alpha \leq \frac{\mathrm{d}(p,p')^\gamma}{\mathrm{d}(p,p')^\alpha}$$

which is bounded for $\gamma \geq \alpha$. □

We conclude this section by summing up, in the next corollary, what has been obtained in propositions 3.4.2 and 3.4.3.

**Corollary 3.4.3**
*For $\gamma \geq 1$, $x^\gamma C^{k,\alpha}_\Phi(M) \subset C^{k,\alpha}_\Phi(M)$. For $\gamma \in (0,1)$, $x^\gamma C^{k,\alpha}_\Phi(M) \subset C^{k,\alpha}_\Phi(M)$ if $\gamma \geq \alpha$.*

## 3.5 Stochastic completeness for Φ- and generalised Φ-manifolds

As stated in chapter 2, Φ-manifolds are an example of stochastically complete manifolds. To see this, we show that condition (2.1.4) holds for Φ-manifolds.

We begin by pointing out that Φ-manifolds, defined as compact manifolds with boundary, can be understood as open Riemannian manifolds (cf. remark 3.1.2). Such an interpretation is achieved by expressing $\overline{M}$ as the union of a compact region $K$ with an open subset $U$, with $U$ equipped with the Riemannian metric locally given by the expression (3.1.1). This is obtained by considering $U \simeq (0,1) \times \partial\overline{M}$ and identifying $K = \{p \in \overline{M} \,|\, x(p) \geq 1\}$.
Let $(\overline{M}, \widetilde{g})$ be a Φ-manifold. By performing the change of coordinates $r = x^{-1}$ on $\overline{M}$, one can rewrite the expression for $\widetilde{g}$ as

$$\widetilde{g} = \mathrm{d}\, r^2 + r^2 \phi^* g_Y + g_Z + h. \tag{3.5.1}$$

With such a change of coordinates we note that, since both $Y$ and $Z$ are compact, the distance between two points towards the boundary ($\partial\overline{M} = \{r = \infty\}$) is proportional to $r$. This can be checked by noticing that the distance from the boundary is given by the term $\mathrm{d}\, r^2$; therefore the distance in this direction is proportional to the Euclidean distance given in polar coordinates.

In view of Theorem 2.1, let $p \in K$ (therefore away from the singular region) be fixed and consider $B(p, R)$ to be the open disc centred at $p$ of radius $R$. For any positive number $S$ let us consider the truncated compact subset $\overline{M}_S = \{q \in \overline{M} \,|\, r(q) \leq S\}$. As mentioned above, the distance on $(\overline{M}, \widetilde{g})$ is proportional to $r$; therefore, for $R > 0$ big enough, one has $B(p, R) \subset \overline{M}_R$. Thus

$$\frac{R}{\log \mathrm{Vol}\, B(p, R)} \sim \frac{R}{\log \mathrm{Vol}\, \overline{M}_R} \quad \text{as} \quad R \to \infty,$$

meaning that the two functions agree up to bounded functions. In particular, the latter is integrable if and only if the former is. The expression for the Φ-metric implies $\mathrm{dvol}_\Phi(p) = h_0 r(p)^b \,\mathrm{d}\, r \,\mathrm{d}\, y \,\mathrm{d}\, z$, with $h_0$ being a bounded smooth function. Thus, as $R$ goes to $\infty$, $\mathrm{Vol}\, \overline{M}_R \sim R^{b+1} \leq e^{CR^2}$, for some positive constant $C$, that is

$$\frac{\cdot}{\log \mathrm{Vol}\, \overline{M}.} \notin L^1(1, \infty), \tag{3.5.2}$$

implying, in particular, $(\overline{M}, \widetilde{g})$ to be stochastically complete.

The argument above shows that a key role for stochastic completeness (in this setting) is the distance being proportional to $r$. Thus the same will be true even if we allow the distance to be proportional to a bounded functions times $r$. For future references we give a name to these "new" objects.

**Definition 3.3**
*Let $\overline{M}$ be a compact manifold with fibred boundary as in Definition 3.1. We say that $\overline{M}$ is a generalized Φ-manifold if the interior $M$ of $\overline{M}$ is equipped with a metric $g \in C^0(\overline{M}, \mathrm{Sym}^2\, {}^\Phi T^*M)$. As for Φ-manifolds one has that, locally near the boundary, $g$ is of the form*

$$g = a(x, y, z)\frac{\mathrm{d}\, x^2}{x^4} + b(x, y, z)\frac{\Phi^* g_Y}{x^2} + c(x, y, z) g_Z + h \tag{3.5.3}$$

*with $a, b, c \in C^{0,\alpha}_\Phi(\overline{M})$ (cf. §3.3).*

It is clear that generalised Φ-manifolds are a slight generalisation of Φ-manifolds; indeed if $a = b = c = 1$ then the expression in (3.5.3) is exactly the same as (3.1.1).

**Remark 3.5.1**
*The condition $g \in C^0(\overline{M}, \mathrm{Sym}^2\, {}^\Phi T^*M)$ in Definition 3.3 is equivalent to require that the inner product of any two Φ-vector fields (cf. §3.2) is bounded.*

In particular, with the same argument as for Φ-manifolds, we conclude the following.

**Proposition 3.5.1**
*Generalized Φ-manifolds are stochastically complete.*

# Chapter 4

# Mapping properties of the heat kernel

Chapter 5 will be devoted to proving one of the main results of this work. That being the existence of solutions for short time to heat-type Cauchy problems. This will be achieved by constructing an inverse for heat type operators $P = \partial_s + a\widetilde{\Delta}$, with $\widetilde{\Delta}$ denoting the Laplace-Beltrami operator on a Φ-manifold $(\overline{M}, \widetilde{g})$ and $a$ being some suitable function.
As it will be clear later, the construction of an inverse for operators of heat type, is intimately related to the inverse of the heat operator $\partial_s + \widetilde{\Delta}$; the so called heat kernel operator **H**. Recall that **H** is nothing but a convolution with the fundamental solution $H$ of the heat-equation (cf. §4.2). As usual we refer to such a function $H$ as the heat kernel.

In the context of manifolds with fibred boundaries, following the construction of the heat kernel **H** carried over in [TaVe21], the asymptotic behaviour of the heat kernel $H$ near its singularity is "conveniently studied" in the heat space $M^2_h$ which we will briefly present in §4.1. We want to point out that the construction of the heat-double space $M^2_h$ presented here is "ready made"; meaning that we do not go into too much details. For a better explanation on the reason why we use blow-up spaces we refer the reader to [TaVe21, §4 and §5]. In there one sees that the space $M^2_{h,2}$ (in our notation) arises from other spaces, namely (in their notation) $M^2_\Phi$ and $M^2_b$. The space $M^2_\Phi$ is classical in the analysis of Φ-pseudodifferential operators (see e.g. [GTV21, §4.2], [MaMe98]) while the space $M^2_b$ is classical in the analysis of b-pseudodifferential operator (see e.g [GTV21, §3.2], moreover we recommend the reader the amazing introduction to b-calculus in [Gri01]). Next, in §4.2, we give an explicit formulation of the asymptotics of the heat kernel based on a result due to [TaVe21]. Finally, through §4.3, §4.4, §4.5 and §4.6 we prove mapping properties for the heat operator (cf. Theorem 4.2 and Theorem 4.3).

## 4.1 Review of the heat space

As in the previous chapter $(\overline{M}, \widetilde{g})$ will denote a Φ-manifold. We begin by presenting the three iterated blow-ups giving rise to the heat space $M^2_h$. The blow-ups are necessary to study the asymptotic behaviour of the heat kernel near its singular-

ity (i.e. space diagonal, $s = 0$) near the boundary $\partial M$. We refer the reader to [TaVe21] for a more detailed construction.
Furthermore using the same notation as in §3.1, $x : \overline{M} \to \mathbb{R}$ will denote the boundary defining function for $\partial M$ and we will use the short cut notation $(x, y, z)$ $(y = y^1, \ldots, y^b$ and $z = z^1, \ldots, z^f)$ for local coordinates on the collar neighbourhood of the boundary $\partial M$.

Before proceeding with the construction, we want to briefly recall the notion of p-submanifolds.

**Definition 4.1**
*Let $\overline{M}$ denote an m-dimensional manifolds with boundary and consider $N \subset \overline{M}$ to be an n-dimensional submanifold. We say that $N$ is a p-submanifold if it can be locally expresse as $(x^{n+1} = 0, \ldots, x^m = 0)$ where $(x^i)_i$ are local coordinates on $M$.*

### 4.1.1 First blow up and projective coordinates

In here as well as in the upcoming subsections, §4.1.2 and §4.1.3, we will briefly sketch the blow-up procedure allowing for the construction of the heat space $M^2_h$. As in [TaVe21, §5], we begin with the "intermediate heat blow up space". This is obtained by multiplying the Φ-double space , meticulously constructed in [TaVe21, §3] or [GTV21, §4.2] , with the $s$-axis $[0, \infty)$.
Consider first the submanifold $S_1 = (\partial M)^2 \times [0, \infty)_s$ of $\overline{M}^2 \times [0, \infty)_s$. Notice that, since $\partial M$ is a p-submanifold of $\overline{M}$, $S_1$ is a p-submanifold of $\overline{M}^2 \times [0, \infty)$. By blowing up $S_1$ in $\overline{M}^2 \times [0, \infty)_s$ we get the pair

$$M^2_{h,1} := [\overline{M}^2 \times [0, \infty)_s; S_1], \quad \beta_1 : M^2_{h,1} \to \overline{M}^2 \times [0, \infty)_t.$$

The object $M^2_{h,1}$ is a "new" manifold obtained by cutting out the codimension 2 submanifold $S^1 = \{x = 0; \widetilde{x} = 0\}$ of $\overline{M}^2 \times [0, \infty)_s$ (displayed below as an edge). After the cutting, we glue the reminder with its spherical normal bundle (under appropriate identification) giving rise to a new boundary hypersurface (which is the conormal bundle of $S_1$ in $M^2 \times [0, \infty)_s$). We want to point out that such a construction is more convoluted than it seems. The identification discussed above is in the sense of [Mel93, §7.1]. We refer the reader once again to [Gri01] for a very nice introduction to blow-ups.

The new manifold $M^2_{h,1}$ comes equipped with a blow-down map $\beta_1 : M^2_{h,1} \to \overline{M}^2 \times [0, \infty)$. The blow-down map is completely described by appropriate projective coordinates. Before presenting the projective coordinates, we furnish the reader with a picture describing the blow up process from $\overline{M}^2 \times [0, \infty)$ (left) to $M^2_{h,1}$ (right). Moreover, in the picture one can see the names of each boundary hypersurface as well as the projective coordinates in regime 1 and 2 respectively, which will be defined below. We recall that the boundary hypersurfaces on $\overline{M}^2 \times [0, T)$ (left in figure 4.1) are given by the following boundary defining functions,

$$\mathrm{lf} = \{\widetilde{x} = 0\}; \quad \mathrm{rf} = \{x = 0\}; \quad \mathrm{tb} = \{s = 0\}.$$

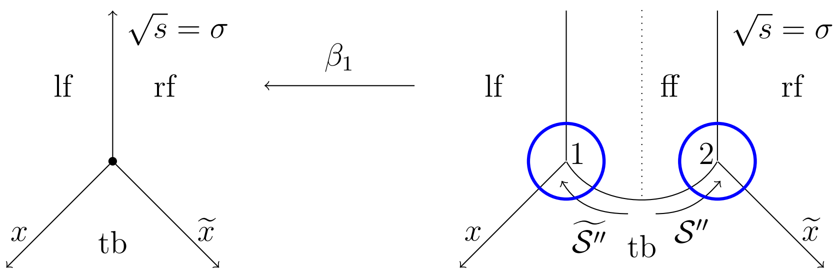


Figure 4.1: First blow-up $M^2_{h,1}$

Following the steps described in [Gri01], one can describe the projective coordinates for $M^2_{h,1}$ by considering two regimes:

• **Regime near the intersection of lf, ff and tb:** In Figure 4.1 this is denoted by "regime 1". This regime is identified with the region where $\widetilde{x} \ll x$ implying, in particular, that the function $\widetilde{\mathcal{S}''} = \widetilde{x}x^{-1}$ is bounded. Therefore, by writing $\sqrt{s} =: \sigma$, the projective coordinates for the lower-left corner are

$$\left(x, y, z, \frac{\widetilde{x}}{x}, \widetilde{y}, \widetilde{z}, \sqrt{s}\right) = (\sigma, x, y, z, \widetilde{\mathcal{S}''}, \widetilde{y}, \widetilde{z}). \tag{4.1.1}$$

Hence, on Regime 1 one has $\rho_{\rm ff} = x$, $\rho_{\rm lf} = \widetilde{s}$ and $\rho_{\rm tb} = \sigma$, where we write $\rho_\star$ for a defining function of a boundary hypersurface $\star$.

• **Regime near the intersection of rf, ff and tb:** "Regime 2" in Figure 4.1. As for the description of regime 1 above but with the role of $x$ and $\widetilde{x}$ interchanged, this regime can be identified with $x \ll \widetilde{x}$, resulting in the function $\mathcal{S}'' := x\widetilde{x}^{-1}$ being bounded. Hence, for $\sigma$ as in regime 1, the projective coordinates for the right-hand corner are

$$\left(\sqrt{s}, \frac{x}{\widetilde{x}}, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z}\right) = (\sigma, \mathcal{S}'', y, z, \widetilde{x}, \widetilde{y}, \widetilde{z}). \tag{4.1.2}$$

The boundary defining functions for the regime 2 are given by $\rho_{\rm ff} = \widetilde{x}$, $\rho_{\rm rf} = \mathcal{S}''$ and $\rho_{\rm tb} = \sigma$.

**Remark 4.1.1**
*The projective coordinates defined above for Regimes 1 and 2 are valid in "larger" regions. In fact, one can define both $\mathcal{S}''$ and $\widetilde{\mathcal{S}''}$ as long as one stays away from $\{\widetilde{x} = 0\}$ and $\{x = 0\}$ respectively. This perspective will be useful for computing the parabolic Schauder estimates throughout §4.4 to §4.6.*

We can finally give a precise expression for the blow-down map $\beta_1$. We will focus only to regime 1 since for regime 2 can be described similarly. When restricted to the lower-left corner, the blow-down map takes the expression

$$(\beta_1)\big|_1(\sigma, x, y, z, \widetilde{\mathcal{S}''}, \widetilde{y}, \widetilde{z}) = (\sigma, x, y, z, x\widetilde{\mathcal{S}''}, \widetilde{y}, \widetilde{z}).$$

### 4.1.2 Second blow-up

The second blow-up consists in blowing up the temporal fibre diagonal (the dashed line in Figure 4.1). That is, we want to blow-up the submanifold $S_2$ of $M^2_{h,1}$

given by
$$S_2 := \left\{ \frac{\widetilde{x}}{x} - 1 = 0 \;\text{ and }\; y = \widetilde{y} \right\}.$$
For the technical reasoning behind this blow-up we refer the read to [GTV21, §4.2] and the references therein. As for the first blow-up, the "new" manifold can be pictured by replacing $S_2$ by its spherical inward pointing normal bundle (see Figure 4.2). As for the first blow-up, the procedure is more involved and we refer the reader to [Mel93, §7.1]. The "new" manifold is defined by the pair
$$M^2_{h,2} := [M^2_{h,1}; S_2], \quad \beta_2 : M^2_{h,1} \to M^2_{h,1}.$$
Similarly to the first blown-up space, this second blow-up gives rise to a manifold with boundary. In particular this "new manifold" has one additional boundary hypersurface compared to the first blow-up space $M^2_{h,1}$. Such a boundary hypersurface will be denoted by fd and it is given by fd $= \{\widetilde{s} - 1 = 0 \;\text{ and }\; y = \widetilde{y}\}$. Furthermore, one considers the iterated blow-down map as the composition $\beta_1 \circ \beta_2 : M^2_{h,2} \to \overline{M}^2 \times [0, \infty)_\infty$. As for the first blow-up, the next picture shows the blown up space as well as the projective coordinates.

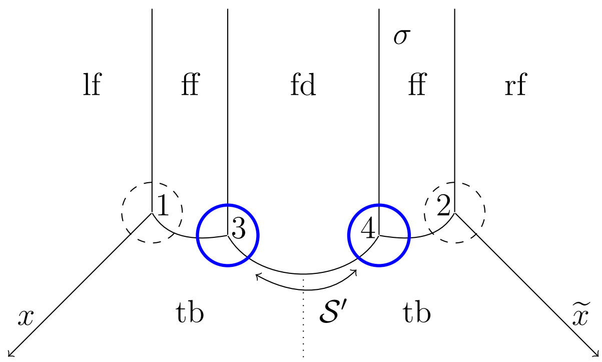


Figure 4.2: Second blow-up $M^2_{h,2}$

Following again the steps described in [Gri01], it is possible to define the projective coordinates on fd by taking
$$\left( \sigma, x, y, z, \frac{\widetilde{\mathcal{S}''} - 1}{x}, \frac{\widetilde{y} - y}{x}, \widetilde{z} - z \right) =: (\sigma, x, y, z, \mathcal{S}', \mathcal{U}', \mathcal{Z}') \tag{4.1.3}$$
away from $x = 0$ (which corresponds to "regime 3" in Figure 4.2). Similarly, one can consider the projective coordinates on ff away from $\widetilde{x} = 0$ (corresponding to "regime 4" in Figure 4.2) as
$$\left( \sigma, \widetilde{x}, \widetilde{y}, \widetilde{z}, \frac{\mathcal{S}'' - 1}{\widetilde{x}}, \frac{y - \widetilde{y}}{\widetilde{x}}, z - \widetilde{z} \right) =: \left( \sigma, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\mathcal{S}}', \widetilde{\mathcal{U}}', \widetilde{\mathcal{Z}}' \right).$$

**Remark 4.1.2**

*Despite the projective coordinates given above for Regimes 3 and 4, one can actually use just one of the coordinates above to work on both Regimes, since one can understand that approaching* ff *from* fd *means that* $(\mathcal{S}', \mathcal{U}', \mathcal{Z}') \to \infty$ *(and similarly for* $\widetilde{\mathcal{S}}', \widetilde{\mathcal{U}}', \widetilde{\mathcal{Z}}')$*). Hence, one can say that on both Regimes 3 and 4,* $\rho_{\text{tb}} = \sigma$*,* $\rho_{\text{fd}} = x$ *and one approaches* ff *if* $\|(\mathcal{S}', \mathcal{U}', \mathcal{Z}')\| \to \infty$.

### 4.1.3 Third blow-up

We are now ready to present our third and last blow-up. This blow-up arises from the classical singularity of the heat kernel on the spacial diagonal (corresponding to the dotted line in Figure 4.2). Therefore, the heat space $M_h^2$ is built by replacing (in the sense of [Mel93, §7.1]) diag($M$) (at $t=0$) by its spherical normal bundle on $M_{h,2}^2$ (see Figure 4.3). More precisely

$$M_h^2 := \left[M_{h,2}^2; (\beta_1 \circ \beta_2)^{-1}(\mathrm{diag}(M) \times \{s=0\})\right], \ \beta : M_h^2 \to \overline{M}^2 \times [0,\infty)_s$$

with $\beta$ being the iterated blow-down map. As we have already seen for the second blown-up space, the third blow-up gives rise to an additional boundary hypersurface when compared to $M_{h,2}^2$. Such an hypersurface will be denoted by td. In total the heat space $M_h^2$ has six boundary hypersurfaces which we collect in the set $\mathcal{M}_1(M_h^2) = \{\mathrm{lf}, \mathrm{rf}, \mathrm{tb}, \mathrm{ff}, \mathrm{fd}, \mathrm{td}\}$. For a visual understanding we propose a picture for the third blown up space as well. In order to keep the pictures as clear as possible, in Figure 4.3 we suppressed the projective coordinates in regime 5 which will be displayed in Figure 4.4.

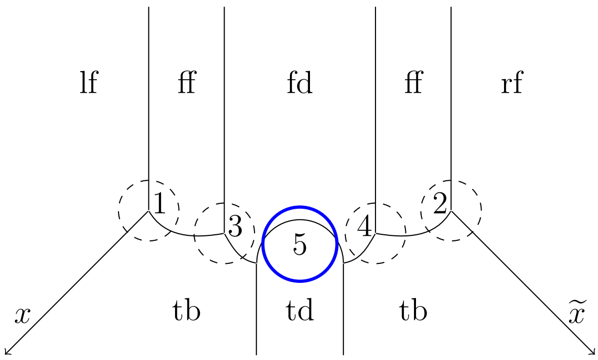


Figure 4.3: Third blow-up

The projective coordinates near the intersection of fd and td (which is represented by "regime 5" in Figure 4.3) are given by

$$\left(\sigma, x, y, z, \frac{\mathcal{S}'}{\sigma}, \frac{\mathcal{U}'}{\sigma}, \frac{\mathcal{Z}'}{\sigma}\right) =: (\sigma, x, y, z, \mathcal{S}, \mathcal{U}, \mathcal{Z}). \tag{4.1.4}$$

Using the same notations as in the previous blow-ups, we have the boundary defining functions $\rho_{\mathrm{fd}} = x$, $\rho_{\mathrm{td}} = \sigma$ and $(|\mathcal{S}|, \|\mathcal{U}\|, \|\mathcal{Z}\|) \to \infty$ corresponds to tb.

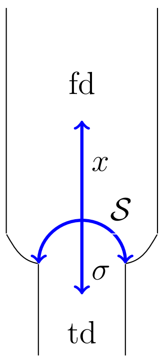


Figure 4.4: Projective coordinates in regime 5

**Remark 4.1.3**
*In the interior of* td*, away from* fd*, we can also use projective coordinates* $(\sigma, (\theta - \widetilde{\theta})/\sigma, \widetilde{\theta})$*, where* $\theta, \widetilde{\theta}$ *are two copies of any local coordinates on* $M$*.*

## 4.2 Asymptoptics of the heat kernel

As mentioned at the beginning of this chapter, we will denote by $H$ the fundamental solution of the heat operator $\partial_s + \widetilde{\Delta}$ and by $\mathbf{H}$ the operator acting via convolution with the fundamental solution $H$. We will mostly refer to the former as the heat kernel while to the latter as the heat kernel operator. Here by "acting via convolution with the heat kernel" we mean that, given a function $u \in C^{k,\alpha}_{\Phi}(M \times [0,T])$, for some integer $k$ and some $\alpha \in (0,1)$,

$$\mathbf{H}u(p,s) = \int_0^s \int_M H(s - \widetilde{s}, p, \widetilde{p}) u(\widetilde{p}, \widetilde{s})\, \mathrm{dvol}_{\Phi}(\widetilde{p})\, \mathrm{d}\,\widetilde{s}; \tag{4.2.1}$$

where $\mathrm{dvol}_{\Phi}(\widetilde{p})$ means that we are considering the volume form with respect to the coordinate on the copy of $\overline{M}$ with coordinates $(\widetilde{x}, \widetilde{y}, \widetilde{z})$.
The aim of this section is to describe the asymptotic behaviour of the heat kernel $H$ on the heat space $M^2_h$. To this end we begin by recalling an important result obtained by [TaVe21]. This result is of crucial importance since it gives the asymptotics of $\beta^* H$ on $M^2_h$. In particular, in view of the construction of $M^2_h$ in §4.1, the result due to Talebi and Vertman gives explicit behaviour for $\beta^* H$ approaching the boundary hypersurfaces of $M^2_h$ which we collected in the set $\mathcal{M}_1(M^2_h)$ (see above).

For every boundary face in $M^2_h$ (displayed in Figure 4.3) we will denote by $\rho$ with a subscript the boundary defining function of the face denoted in the subscript.

**Theorem 4.1** (Theorem 6.1 in [TaVe21])
*Let* $(\overline{M}, \widetilde{g})$ *be an* $m$*-dimensional complete manifold with fibered boundary endowed with a* $\Phi$*-metric. Denote by* $H$ *the heat kernel associated to the unique self-adjoint extension of the corresponding Laplace-Beltrami operator. The lift* $\beta^* H$ *is a polyhomogeneous function on* $M^2_h$ *with asymptotic behavior described by*

$$\beta^* H \sim \rho_{\mathrm{lf}}^{\infty} \rho_{\mathrm{ff}}^{\infty} \rho_{\mathrm{rf}}^{\infty} \rho_{\mathrm{tb}}^{\infty} \rho_{\mathrm{fd}}^{0} \rho_{\mathrm{td}}^{-m} G_0 \tag{4.2.2}$$

*with* $G_0$ *being a bounded function. In particular, the above means that* $\beta^* H$ *is of leading order* $-m$ *on* td*, smooth on* fd *and vanishes to infinite order on* lf*,* ff*,* rf *and* td*.*

In order to make the computations in §4.3, 4.4, 4.3.6 and 4.6 more readable we will compute here the asymptotic behaviour of the lift of certain quantities which will be needed in the aforementioned sections.

**Remark 4.2.1**
*In the following formulae we will be describing only the worst case scenario, i.e. the most singular behaviour. Bounds for better behaved terms in the asymptotic expansion follow from the most singular one and the error estimate from the definition of polyhomogeneous conormal functions.*

**•Lifts in the intersection of lf, ff and tb:** In this regime the asymptotic behaviors of both $\beta^* H$ and $\beta^* \operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}$ are appropriately described by the projective coordinates

$$x, \;\; y, \;\; z, \;\; \widetilde{\mathcal{S}''} = \frac{\widetilde{x}}{x}, \;\; \widetilde{y}, \;\; \widetilde{z}, \;\; \sigma = \sqrt{s}.$$

Recall that, with respect to these coordinates $\rho_{\mathrm{lf}} = \widetilde{\mathcal{S}''}$, $\rho_{\mathrm{ff}} = x$ and $\rho_{\mathrm{tb}} = \sigma$. From Theorem 4.1 and by computing directly the pull-back of the volume form, one has

$$\begin{aligned} &\beta^*(XH) \sim \sigma^{-1}(x\widetilde{\mathcal{S}''}\sigma)^\infty G_0 = (x\widetilde{s}\sigma)^\infty G_0 \quad \text{for } X \in \{\mathrm{id}, \mathcal{V}_\phi, \mathcal{V}_\phi^2\} \\ &\beta^*(\operatorname{dvol}_\Phi \operatorname{d}\widetilde{\mathcal{S}''}) \sim 2(\widetilde{\mathcal{S}''}x)^{-2-b}\sigma x h \operatorname{d}\widetilde{\mathcal{S}''} \operatorname{d}\widetilde{y} \operatorname{d}\widetilde{z} \operatorname{d}\sigma. \end{aligned} \tag{4.2.3}$$

**•Lifts in the intersection of** rf**,** ff **and** tb**:** The asymptotic behaviours of both $\beta^* H$ and $\beta^* \operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}$ are suitably described by the projective coordinates

$$\mathcal{S}'' = \frac{x}{\widetilde{x}}, \;\; y, \;\; z, \;\; \widetilde{x}, \;\; \widetilde{z}, \;\; \sigma = \sqrt{s - \widetilde{s}}.$$

The boundary defining function with respect to these coordinates are $\rho_{\mathrm{rf}} = \mathcal{S}''$, $\rho_{\mathrm{ff}} = \widetilde{x}$ and $\rho_{\mathrm{tb}} = \sigma$. As above we can conclude that

$$\begin{aligned} &\beta^*(XH) \sim \sigma^{-1}(\widetilde{x}\mathcal{S}''\sigma)^\infty G_0 = (\widetilde{x}\mathcal{S}''\sigma)^\infty G_0 \quad \text{for } X \in \{\mathrm{id}, \mathcal{V}_\phi, \mathcal{V}_\phi^2\} \\ &\beta^*(\operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}) \sim 2\widetilde{x}^{-2-b}\sigma h \operatorname{d}\widetilde{x} \operatorname{d}\widetilde{y} \operatorname{d}\widetilde{z} \operatorname{d}\sigma. \end{aligned} \tag{4.2.4}$$

**•Lift in the intersection of** ff**,** fd **and** tb**:** In this regime the asymptotic behaviours of both $\beta^* H$ and $\beta^* \operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}$ are fittingly described using projective coordinates

$$x, \;\; y, \;\; z, \;\; \mathcal{S}' = \frac{\widetilde{x} - x}{x^2}, \;\; \mathcal{U}' = \frac{y - \widetilde{y}}{x}, \;\; \mathcal{Z}' = z - \widetilde{z}, \;\; \sigma = \sqrt{s - \widetilde{s}}.$$

In the next section we will encounter some extra quantity in this regime hence it is useful to collect it here. As in the previous cases one has

$$\begin{aligned} &\beta^*(XH) = \sigma^{-1}(\sigma)^\infty G_0 \sim \sigma^\infty G_0 \quad \text{for } X \in \{\mathrm{id}, \mathcal{V}_\phi, \mathcal{V}_\phi^2\}. \\ &\beta^*(\operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}) \sim 2(1 + \mathcal{S}'x)^{-2-b}\sigma h \operatorname{d}\mathcal{S}' \operatorname{d}\mathcal{U}' \operatorname{d}\mathcal{Z}' \operatorname{d}\sigma. \\ &\beta^*(\partial_i XH) \sim x^{-2}\sigma^\infty G_0 \quad \text{with } i = x, y, z. \end{aligned} \tag{4.2.5}$$

Note that, in the above, $G_0$ vanishes to infinite order as $\|(\mathcal{S}', \mathcal{U}', \mathcal{Z}')\|$ goes to $\infty$.

**•Lifts in the intersection of** fd **and** td**:** The appropriate projective coordinate for a suitable description of the asymptotic behaviour of both $\beta^* H$ and $\beta^* \operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}$ in this regime are

$$x, \;\; y, \;\; z, \;\; \mathcal{S} = \frac{\widetilde{x} - x}{\sigma x^2}, \;\; \mathcal{U} = \frac{y - \widetilde{y}}{\sigma x}, \;\; \mathcal{Z} = \frac{z - \widetilde{z}}{\sigma}, \;\; \sigma = \sqrt{s - \widetilde{s}}.$$

Similarly to the regime where ff intersects fd and tb one finds

$$\begin{aligned} &\beta^*(XH) \sim (\sigma)^{-m-2} G_0 \quad \text{for } X \in \{\mathrm{id}, \mathcal{V}_\phi, \mathcal{V}_\phi^2\}. \\ &\beta^*(\operatorname{dvol}_\Phi \operatorname{d}\widetilde{s}) \sim 2(1 + \mathcal{S}\sigma x)^{-2-b}\sigma^{m+1} h \operatorname{d}\mathcal{S} \operatorname{d}\mathcal{U} \operatorname{d}\mathcal{Z} \operatorname{d}\sigma. \\ &\beta^*(\partial_i XH) \sim x^{-2}\sigma^{-m-3} G_0 \quad \text{with } i = x, y, z. \end{aligned} \tag{4.2.6}$$

with $G_0$ vanishing to infinite order as $(\mathcal{S}, \mathcal{U}, \mathcal{Z})$ goes to $\infty$.

## 4.3 Mapping properties

As often mentioned, the main goal of Part I is to show existence, for short time, of solutions to heat type equations

$$(\partial_s + a\widetilde{\Delta})u = \ell, \quad u|_{s=0} = 0 \tag{4.3.1}$$

for suitable functions $a$ and $\ell$. This will be proved in Chapter 5. As we will see there, in order to prove existence of solutions to (4.3.1) one may consider approximate inverse (parametrix) of the parabolic operator $P := \partial_s + a\widetilde{\Delta}$ (also referred to as a heat type operator). The construction of parametrix for heat type operators, carried over in §5.1, relies heavily on the heat kernel operator $\mathbf{H}$ whose kernel's asymptotics have been discussed in §4.2.
Parametrices are, by definition, approximate inverse, thus affected by some error terms. Such error terms must be small, so it is convenient to construct them between suitable Hölder spaces. The Hölder spaces under consideration are clearly Hölder spaces where the heat kernel $\mathbf{H}$ is well behaved, i.e. bounded. It is therefore important to know between which Hölder spaces $\mathbf{H}$ acts as a bounded operator.
In this section we present a result in this spirit. Due to highly involved computations, the proof of such a result will be carried over in §4.4, 4.3.6 and 4.6. ed Hölder spaces.

**Theorem 4.2**
*Let $(\overline{M}, \widetilde{g})$ be an $m$-dimensional manifold with fibered boundary equipped with a $\Phi$-metric. The heat kernel operator $\mathbf{H}$ as a map between the Hölder spaces*

$$\mathbf{H} : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^{k+2,\alpha}_\Phi(M \times [0,T]), \tag{4.3.2}$$

*acting by convolution, as in (4.2.1), is bounded.*

***Proof:***
We will prove the statement for $k = 0$, that is

$$\mathbf{H} : x^\gamma C^{\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^{2,\alpha}_\Phi(M \times [0,T]).$$

The more general case can be proved similarly with additional integration by part argument near td and by employing the vanishing order of the heat kernel near the boundary $\partial M^2_h$ (similar argument has been employed in [BaVe14]). Furthermore, note that, for $\overline{u} \in x^\gamma C^\alpha_\Phi(M \times [0,T])$, there exists some $u \in C^\alpha_\Phi(M \times [0,T])$ so that $\overline{u} = x^\gamma u$. In particular, it follows that $H\overline{u}$ lies in $x^\gamma C^{2,\alpha}_\Phi(M \times [0,T])$ if and only if $x^{-\gamma} H x^\gamma u$ lies in $C^{2,\alpha}_\Phi(M \times [0,T])$. This is equivalent to prove that

$$\mathbf{H}_\gamma := X^{-\gamma} \circ \mathbf{H} \circ X^\gamma : C^\alpha_\Phi(M \times [0,T]) \to C^{2,\alpha}_\Phi(M \times [0,T]), \tag{4.3.3}$$

is bounded, with $X^\gamma$ being the "multiplication by $x^\gamma$" operator. Moreover, from (3.3.2), it follows that proving (4.3.2) is equivalent to prove that the operator $G$, defined by $\mathbf{G} = X\mathbf{H}_\gamma$ with $X \in \{\mathrm{id}, \mathcal{V}_\Phi, \mathcal{V}^2_\Phi\}$, is a bounded operator mapping

$$\mathbf{G} : C^\alpha_\Phi(M \times [0,T]) \to C^\alpha_\Phi(M \times [0,T]).$$

Therefore, given a function $u$ in $C^\alpha_\Phi(M \times [0,T])$, the goal is to prove

$$\|\mathbf{G}u\|_\alpha \leq c\|u\|_\alpha \tag{4.3.4}$$

for some uniform constant $c > 0$. This will be obtained directly by estimating $\|\mathbf{G}u\|_\alpha$. From the definition of the $\alpha$-norm in (3.3.1) we find

$$\|\mathbf{G}u\|_\alpha = [\mathbf{G}u]_\alpha + \|\mathbf{G}u\|_\infty.$$

One can see that

$$[\mathbf{G}u]_\alpha \leq \sup_{\substack{p,p' \in M \\ p \neq p'}} \frac{|\mathbf{G}u(p,s) - \mathbf{G}u(p',s)|}{\mathrm{d}(p,p')^\alpha} + \sup_{\substack{s,s' \geq 0 \\ s \neq s'}} \frac{|\mathbf{G}u(p,s) - \mathbf{G}u(p,s')|}{|s-s'|^{\alpha/2}},$$

leading to

$$\|\mathbf{G}u\|_\alpha \leq \sup_{\substack{p,p' \in M \\ p \neq p'}} \frac{|\mathbf{G}u(p,s) - \mathbf{G}u(p',s)|}{\mathrm{d}(p,p')^\alpha} + \sup_{\substack{s,s' \geq 0 \\ t \neq s'}} \frac{|\mathbf{G}u(p,s) - \mathbf{G}u(p,s')|}{|s-s'|^{\alpha/2}} + \|Gu\|_\infty.$$

Thus (4.3.4) is satisfied if the following are satisfied

$$|\mathbf{G}u(p,s) - \mathbf{G}u(p',s)| \leq c\|u\|_\alpha d(p,p')^\alpha, \tag{4.3.5}$$
$$|\mathbf{G}u(p,s) - \mathbf{G}u(p,s')| \leq c\|u\|_\alpha |s-s'|^{\alpha/2}, \tag{4.3.6}$$
$$|\mathbf{G}u(p,s)| \leq c\|u\|_\alpha. \tag{4.3.7}$$

We will therefore proceed in three steps:

i) Uniform estimates of Hölder differences in space (4.3.5),

ii) Uniform estimates of Hölder differences in time (4.3.6),

iii) Uniform estimates of the supremum norm (4.3.7).

As mentioned above, these three steps will be treated separately in sections 4.4, 4.5 and 4.6 respectively. □

**Remark 4.3.1**
*As for the heat kernel operator* $\mathbf{H}$*, we will denote by* $G$ *the integral kernel of the operator* $\mathbf{G}$*.*

From Theorem 4.2 other mapping properties can be derived.

**Theorem 4.3**
*Let* $(\overline{M}, \widetilde{g})$ *be an* $m$ *dimensional manifold with fibered boundary equipped with a* $\Phi$*-metric. The heat kernel* $\mathbf{H}$ *as a map between the Hölder spaces*

$$\mathbf{H} : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to \sqrt{s} x^\gamma C^{k+1,\alpha}_\Phi(M \times [0,T]),$$
$$\mathbf{H} : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to s^{\alpha/2} x^\gamma C^{k+2}_\Phi(M \times [0,T]),$$

*acting via convolution, as in* (4.2.1)*,are bounded.*

***Proof:***
We will present only the argument for the first mapping property as the second follows along the same lines.

The same argument as in the previous result leads to an equivalent formulation of the statement. That is, one has to prove that the operator

$$s^{-1/2}x^{-\gamma}\mathbf{H}x^{\gamma} : C^{\alpha}_{\Phi}(M \times [0,T]) \to C^{1,\alpha}_{\Phi}(M \times [0,T])$$

is bounded. As in the previous theorem one deduces that the above is equivalent to prove that the operator $\mathbf{G}_s$, defined by $\mathbf{G}_s u = X(s^{-1/2}x^{-\gamma}\mathbf{H}x^{\gamma})u$ with $X \in \{id, \mathcal{V}_{\phi}\}$, mapping

$$\mathbf{G}_s : C^{\alpha}_{\Phi}(M \times [0,T]) \to C^{\alpha}_{\Phi}(M \times [0,T])$$

is bounded. One has

$$(\mathbf{G}_s u)(p,s) = \int_0^t \int_M X((s-\widetilde{s})^{-1/2}H_{\gamma}(s-\widetilde{s},p,\widetilde{p}))u(\widetilde{p},\widetilde{s})\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\,\widetilde{s};$$

where $H_{\gamma}$ is defined in (4.3.3). The estimates in sections 4.4, 4.5 and 4.6 will already cover the case $X \in \{\{\mathrm{id}\}, \mathcal{V}_{\Phi}\}$. Moreover it is crucial to note that we are no longer considering elements in $\mathcal{V}^2_{\Phi}(M)$, which will lead to an extra $\sigma$ term. On the other hand, the term $(s-\widetilde{s})^{-1/2}$ inside the integrand lifts to an extra $\sigma^{-1}$ in every region of $M^2_h$. This means that the presence of the term $(s-\widetilde{s})^{-1/2}$ is proportionally compensated by the absence of second order Φ-differential operators (i.e. elements in $\mathcal{V}^2_{\Phi}$). Thus, in attempting to get these estimates following the same computations as in the upcoming sections, the integrands obtained will have the exact same asymptotics. □

## 4.4 Estimates of Hölder differences in space

The aim of this section is to prove the inequality in (4.3.5). Consider $p, p'$ to be some fixed points in $M$ and define the sets

$$M^+ = \left\{\widetilde{p} \in M \,\middle|\, d(p,\widetilde{p}) \le 3d(p,p')\right\}, \quad M^- = \left\{\widetilde{p} \in M \,\middle|\, d(p,\widetilde{p}) \ge 3d(p,p')\right\}.$$

Let us denote by $X$ any element which is either the identity, a Φ-derivative or a second order Φ-differential operator. For any $u$ function in $C^{\alpha}_{\Phi}(M \times [0,T])$ one has

$$Gu(s,p) - Gu(s,p') = I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^s \int_{M^+} [G(s-\widetilde{s},p,\widetilde{p}) - G(s-\widetilde{s},p',\widetilde{p})]\,[u(\widetilde{s},\widetilde{p}) - u(\widetilde{s},p)]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\,\widetilde{s};$$
$$I_2 = \int_0^s \int_{M^-} [G(s-\widetilde{s},p,\widetilde{p}) - G(s-\widetilde{s},p',\widetilde{p})]\,[u(\widetilde{s},\widetilde{p}) - u(\widetilde{s},p)]\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\,\widetilde{s};$$
$$I_3 = \int_0^s \int_{M} [G(s-\widetilde{s},p,\widetilde{p}) - G(s-\widetilde{s},p',\widetilde{p})]\,u(\widetilde{s},p)\,\mathrm{dvol}_{\Phi}(\widetilde{p})\,\mathrm{d}\,\widetilde{s}.$$

Hence, it is clear that (4.3.5) is satisfied if $|I_j| \le c\|u\|_{\alpha}\,\mathrm{d}(p,p')^{\alpha}$ for $j = 1,2,3$.

Since the heat kernel $H$, hence also $G$, is smooth in the interior of $M^2_h$, the claimed estimates need only to be provided near the boundary hypersurfaces of $M^2_h$.

Moreover, as stated in Theorem 4.1, $\beta^* H$, hence the lift of $G$ as well, vanish to infinite order away from $\mathrm{fd} \cup \mathrm{td}$ (i.e. in regimes 1,2,3 and 4), resulting in trivial estimates. We will therefore focus solely on the estimates of $I_j$ near $\mathrm{fd} \cup \mathrm{td}$ for every $j = 1, 2, 3$. In estimating the integrals we will assume, without loss of generality, $G$ to be compactly supported in the regime of interest. In conclusion, in order to simplify the notation, we will identify the integration regions $M$, $M^+$ and $M^-$ with their lifts; furthermore we will denote by $y$ either one or all of the base coordinates $y_1, \ldots, y_f$ and by $z$ either one or all of the fiber coordinates $z_1, \ldots, z_b$ and similarly for $y'$ and $z'$.

### 4.4.1 Estimates for $I_2$

First of all notice that, in this setting, $\widetilde{p}$ is ranging in $M^-$. We begin by noticing the following useful fact.

**Lemma 4.4.1**
*Let $p''$ be a point in $M$ such that $d(p', p'') \leq d(p, p')$. For every point $\widetilde{p}$ in $M^-$ one has*

$$\frac{1}{3} d(p, \widetilde{p}) \leq d(p'', \widetilde{p}).$$

***Proof:***
Using the triangle inequality, the assumption on $p''$ and the fact that $\widetilde{p}$ lies in $M^-$, one has

$$\begin{aligned} d(p, \widetilde{p}) &\leq d(p, p') + d(p', \widetilde{p}) \leq d(p, p') + d(p', p'') + d(p'', \widetilde{p}) \\ &\leq d(p, p') + d(p, p') + d(p'', \widetilde{p}) = 2d(p, p') + d(p'', \widetilde{p}) \\ &\leq \frac{2}{3} d(p, \widetilde{p}) + d(p'', \widetilde{p}). \end{aligned}$$

The result follows by cancellation. □

In order to estimate $I_2$ we employ the Mean Value Theorem leading to

$$\begin{aligned} I_2 =& |x - x'| \int_0^s \int_{M^-} \partial_\xi G\big|_{(s-\widetilde{s}, \xi, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z})} \left[ u(\widetilde{s}, \widetilde{x}, \widetilde{y}, \widetilde{z}) - u(\widetilde{s}, x, y, z) \right] \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{s} \\ &+ \|y - y'\| \int_0^s \int_{M^-} \partial_\eta G\big|_{(s-\widetilde{s}, x, \eta, z, \widetilde{x}, \widetilde{y}, \widetilde{z})} \left[ u(\widetilde{s}, \widetilde{x}, \widetilde{y}, \widetilde{z}) - u(\widetilde{s}, x, y, z) \right] \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{s} \\ &+ \|z - z'\| \int_0^s \int_{M^-} \partial_\zeta G\big|_{(s-\widetilde{s}, x, y, \zeta, \widetilde{x}, \widetilde{y}, \widetilde{z})} \left[ u(\widetilde{s}, \widetilde{x}, \widetilde{y}, \widetilde{z}) - u(\widetilde{s}, x, y, z) \right] \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\widetilde{s}. \end{aligned}$$

Let $p''$ denote one, and all, the points in $M$ arising from the application of the Mean Value Theorem. That is $p''$ has coordinates in the set $\{(\xi, y, z), (x', \eta, z), (x', y', \zeta)\}$. Clearly $p''$ satisfies either $d(p, p'') \leq d(p, p')$ or $d(p', p'') \leq d(p, p')$. If the latter holds, Lemma 4.4.1 gives $d(p, \widetilde{p}) \leq 3d(p'', \widetilde{p})$. Similarly, arguing by means of the triangle inequality, one sees that the same estimate holds if the former case is satisfied. In conclusion, for every $\widetilde{p} \in M^-$, and for $p''$ as above,

$$d(p, \widetilde{p}) \leq 3d(p'', \widetilde{p}).$$

Moreover, since $u$ lies in $C^\alpha_\Phi(M \times [0,T])$, one finds

$$\begin{aligned} I_2 \leq & c\|u\|_\alpha |x-x'| \int_0^s \int_{M^-} \partial_\xi G(s-\widetilde{s}, p'', \widetilde{p}) d\,(p'', \widetilde{p})^\alpha \operatorname{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\, \widetilde{s} \\ & + c\|u\|_\alpha \|y-y'\| \int_0^s \int_{M^-} \partial_\eta G(s-\widetilde{s}, p'', \widetilde{p}) d\,(p'', \widetilde{p})^\alpha \operatorname{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\, \widetilde{s} \\ & + c\|u\|_\alpha \|z-z'\| \int_0^s \int_{M^-} \partial_\zeta G(s-\widetilde{s}, p'', \widetilde{p}) d\,(p'', \widetilde{p})^\alpha \operatorname{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\, \widetilde{s}. \end{aligned}$$

In the above, with abuse of notation, we denoted by $p''$ any of the occurrences of the point arising from the Mean value theorem. In particular in the first integral $p''$ has coordinates $(\xi, y, z)$, in the second $(x', \eta, z)$ and in the third $(x', y', \zeta)$. For readability reasons we will denote the summands in the estimate above respectively with $I_{2,1}$, $I_{2,2}$ and $I_{2,3}$. Clearly estimates for $I_{2,1}$, $I_{2,2}$ and $I_{2,3}$ can be obtained similarly. Therefore we present explicit computation only for $I_{2,1}$.

The formulae in (4.2.6) give us the asymptotic behaviour of $\partial_\xi X H_\gamma$ in this regime. Using projective coordinates $(\sigma, x, y, z, \mathcal{S}', \mathcal{U}', \mathcal{Z}')$ given by

$$\mathcal{S}' = \frac{\widetilde{x} - \xi}{\xi^2}, \ \mathcal{U}' = \frac{\widetilde{y} - y}{\xi}, \ \mathcal{Z}' = \widetilde{z} - z \ \text{ and } \ \sigma = \sqrt{t - \widetilde{s}};$$

one has

$$|I_{2,1}| \leq c\|u\|_\alpha |x-x'| \int_0^{\sqrt{s}} \int_{M^-} \sigma^{-m-2} \xi^{-2} G_0 \beta^*(d(p'', \widetilde{p})^\alpha) \, \mathrm{d}\, \mathcal{S}' \, \mathrm{d}\, \mathcal{U}' \, \mathrm{d}\, \mathcal{Z}' \, \mathrm{d}\, \sigma \quad (4.4.1)$$

with $G_0$ being bounded.

Let us now analyse the distance $\beta^*(d(p'', \widetilde{p})^\alpha)$. Recall that, in regime 5, $\xi \sim \widetilde{x}$ implying, in particular, $\widetilde{\mathcal{S}''} = \widetilde{x}/\xi \sim 1$ and thus giving

$$\begin{aligned} d((\xi, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})) &= \sqrt{|\xi - \tilde{x}|^2 + (\xi + \tilde{x})^2 \|y - \tilde{y}\|^2 + (\xi + \tilde{x})^4 \|z - \tilde{z}\|^2} \\ &\sim \xi^2 \sqrt{|\mathcal{S}'|^2 + |\mathcal{U}'|^2 + |\mathcal{Z}'|^2} \\ &=: \xi^2 r(\mathcal{S}', \mathcal{U}', \mathcal{Z}'). \end{aligned}$$

Notice that the function $r$, is nothing but the radial distance in polar coordinates from the origin. From the above we conclude the existence of some constant $c$ such that

$$\beta^* \left( d\left( (\xi, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}) \right)^\alpha \right) \leq c(\xi^2 r)^\alpha. \quad (4.4.2)$$

By using $r$ as the radial coordinate in $\mathbb{R}^m$ we can perform a change of coordinates in (4.4.1) leading to

$$|I_{2,1}| \leq c\|u\|_\alpha |x-x'| \int_0^{\sqrt{s}} \int_{M^-} \sigma^{-m-2} \xi^{-2+2\alpha} r^{m-1+\alpha} G_0 \, \mathrm{d}\, r \, \mathrm{d}\,(\text{angle}) \, \mathrm{d}\, \sigma.$$

Finally, setting $\varsigma = r^{-1}\sigma$, it follows that the asymptotic behaviour of $\varsigma^{-1}$ is (cf. (4.1.4))

$$\varsigma^{-1} \sim \sqrt{|\mathcal{S}|^2 + |\mathcal{U}|^2 + |\mathcal{Z}|^2}.$$

This implies that integrating $G_0$ against any negative power of $\varsigma$ leads to a bounded term. Moreover, for $r$ defined as above, $M^- \subset \{\xi^{-2} d(p, p') \leq cr\}$ for some constant

$c > 0$. Thus, once the angular variables are being integrated out, it follows, by performing yet another change of coordinates given by $\sigma \mapsto \varsigma$,

$$
\begin{aligned}
|I_2'| &\leq c\|u\|_\alpha |x - x'| \int_{\xi^{-2}d(p,p')}^{\infty} r^{-2+\alpha} \xi^{-2+2\alpha} \,\mathrm{d}\, r \\
&= c\|u\|_\alpha |x - x'| \xi^{-2+2\alpha} (\xi^{-2} d(p,p'))^{-1+\alpha} \\
&\leq c\|u\|_\alpha d(p,p')^\alpha,
\end{aligned}
$$

as claimed.

### 4.4.2 Estimates of $I_1$

As for the estimates of $I_2$ we see that

$$
\begin{aligned}
I_1 = &\int_0^s \int_{M^+} G(s - \widetilde{s}, p, \widetilde{p})[u(\widetilde{p}, \widetilde{s}) - u(p, \widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\,\widetilde{s} \\
&- \int_0^s \int_{M^+} G(s - \widetilde{s}, p', \widetilde{p})[u(\widetilde{p}, \widetilde{s}) - u(p', \widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\,\widetilde{s} \\
&+ \int_0^s \int_{M^+} G(s - \widetilde{s}, p', \widetilde{p})[u(p, \widetilde{s}) - u(p', \widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\,\widetilde{s} \\
=: &I_{1,1} - I_{1,2} + I_{1,3}.
\end{aligned}
$$

Clearly the estimates for $I_{1,1}$ and $I_{1,2}$ will be similar, thus we present the full computations only for $I_{1,1}$. Moreover, as for $I_2$, estimates away from $\mathrm{fd} \cup \mathrm{td}$ are trivial thus we will focus on the estimates near regime 5.

#### Estimate of $I_{1,1}$

In regime 5, the asymptotics of $G$ are given by the expression in (4.2.6); in particular $G \sim \sigma^{-m-2} G_0$, with $G_0$ vanishing to infinite order when $(|\mathcal{S}|, \|\mathcal{U}\|, \|\mathcal{Z}\|) \to \infty$. Let us choose projective coordinates given by $(\sigma, x, y, z, \mathcal{S}', \mathcal{U}', \mathcal{Z}')$, with

$$
\mathcal{S}' = \frac{\widetilde{x} - x}{x^2}, \;\; \mathcal{U}' = \frac{\widetilde{y} - y}{x}, \;\; \mathcal{Z}' = \widetilde{z} - z \;\text{ and }\; \sigma = \sqrt{s - \widetilde{s}}.
$$

Recall that, in these projective coordinates, the lift of the volume form is expressed as in (4.2.5); resulting in

$$
|I_{1,1}| \leq \|u\|_\alpha \int_0^{\sqrt{s}} \int_{M^+} \sigma^{-m-1} G_0 \beta^* d((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z}))^\alpha \,\mathrm{d}\,\mathcal{S}' \,\mathrm{d}\,\mathcal{U}' \,\mathrm{d}\,\mathcal{Z}' \,\mathrm{d}\,\sigma.
$$

Furthermore, in regime 5, $x \sim \widetilde{x}$. Thus, as already done for the estimates for $I_2$, set $r(\mathcal{S}', \mathcal{U}', \mathcal{Z}') := \sqrt{|\mathcal{S}'|^2 + \|\mathcal{U}'\|^2 + \|\mathcal{Z}'\|^2}$. This implies

$$
\begin{aligned}
\beta^* d((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})) &= \sqrt{|x - \tilde{x}|^2 + (x + \tilde{x})^2 \|y - \tilde{y}\|^2 + (x + \tilde{x})^4 \|z - \tilde{z}\|^2} \\
&\sim x^2 \sqrt{|\mathcal{S}'|^2 + \|\mathcal{U}'\|^2 + \|\mathcal{Z}'\|^2} \\
&= c x^2 r(\mathcal{S}', \mathcal{U}', \mathcal{Z}').
\end{aligned}
$$

It follows, $M^+ = \{r \leq cx^{-2}d(p,p')\}$ for some constant $c > 0$.
Let us now denote $\varsigma = \sigma/r$. Arguing as in the estimates for $I_2$, $r$ can be thought as the radial distance in $\mathbb{R}^m$ with coordinates given by $(\mathcal{S}', \mathcal{U}', \mathcal{Z}')$. We can therefore consider polar coordinates and perform a change of coordinates in the integral above. Integrating out once again the angular coordinates, we find

$$|I_{1,1}| \leq c\|u\|_\alpha x^{2\alpha} \int_{I(\varsigma)} \int_0^{x^{-2}d(p,p')} \varsigma^{-m-1} r^{-1+\alpha} G_0 \,\mathrm{d}\, r \,\mathrm{d}\, \sigma.$$

The estimate now follows by noticing $\varsigma^{-m-1}G_0$ to be bounded (due to the decay properties of $G_0$).

**Estimate of $I_{1,3}$**

As mentioned earlier, the estimates for $I_{1,3}$ will be slightly different from the one for $I_{1,1}$ and they rely on an integration by parts.
First of all we consider projective coordinates $(\sigma, x, y, z, \mathcal{S}, \mathcal{U}, \mathcal{Z})$ with

$$\mathcal{S} = \frac{\widetilde{x} - x}{\sigma x^2}, \;\; \mathcal{U} = \frac{y - \widetilde{y}}{\sigma x}, \;\; \mathcal{Z} = \frac{z - \widetilde{z}}{\sigma}.$$

Notice that the "worst case scenario" for $I_{1,3}$ is given by $G = \sigma^{-m-2}(X_1 X_2 G_0)$ with both $X_1, X_2 \in \{\partial_\mathcal{S}, \partial_\mathcal{U}, \partial_\mathcal{Z}\}$. For the sake of simplicity, since the general case is similar, we assume $X_1 = \partial_\mathcal{S}$.
on the other hand, for fixed $(\sigma, \mathcal{U}, \mathcal{Z})$ one has $M^+ = \{|\mathcal{S}| \leq r(\sigma, \mathcal{U}, \mathcal{Z})\}$, where this expression for $M^+$ comes from the fact that $r$ is taken originally in terms of $(\mathcal{S}', \mathcal{U}', \mathcal{Z}') = (\sigma\mathcal{S}, \sigma\mathcal{U}, \sigma\mathcal{Z})$. Hence, since

$$\beta^*(\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\, \widetilde{s}) = h(x + x^2\sigma\mathcal{S}, y + x\sigma\mathcal{U}, z + \sigma\mathcal{Z})\sigma^{m+1} \,\mathrm{d}\, \varsigma \,\mathrm{d}\, \eta \,\mathrm{d}\, \zeta \,\mathrm{d}\, \sigma,$$

for $h$ being smooth function. Moreover, let us denote by $\delta u$ the term $\delta u := [u(p, \widetilde{s}) - u(p', \widetilde{s})]$. Clearly $\delta u$ does not depend on $\widetilde{p}$; thus $I_{1,3}$ can be written as

$$\begin{aligned} I_{1,3} &= \int_0^{\sqrt{s}} \delta u \int_{M^+} \sigma^{-1}(\partial_\mathcal{S} X_2 G_0) h \,\mathrm{d}\, \mathcal{S} \,\mathrm{d}\mathcal{U} \,\mathrm{d}\, \mathcal{Z} \,\mathrm{d}\, \sigma \\ &= \int_0^{\sqrt{s}} \delta u \int_{\partial M^+} \sigma^{-1}(X_2 G_0)\big|_{|\mathcal{S}|=r} h \,\mathrm{d}\, \mathcal{S} \,\mathrm{d}\mathcal{U} \,\mathrm{d}\, \mathcal{Z} \,\mathrm{d}\, \sigma \\ &\quad - \int_0^{\sqrt{s}} \delta u \int_{M^+} \sigma^{-1}(X_2 G_0) \partial_\mathcal{S} h \,\mathrm{d}\, \mathcal{S} \,\mathrm{d}\mathcal{U} \,\mathrm{d}\, \mathcal{Z} \,\mathrm{d}\, \sigma \\ &=: I_{1,3}^1 - I_{1,3}^2. \end{aligned}$$

We begin by noticing that the derivative of the smooth function $h$ with respect to $\mathcal{S}$ can be written as $\partial_\mathcal{S} h = x^2\sigma h'$. The $\sigma$ appearing in this derivative clearly cancels with the $\sigma^{-1}$. Thus, since $x(X_2 G_0)$ is bounded, the whole integral over $M^+$ in $I_{1,3}^2$ is bounded. The estimate follows by estimating $\delta u$ against the $\alpha$-norm of $u$ multiplied by the distance $d(p,p')^\alpha$.

Let us now focus on the integral $I_{1,3}^1$. For simplicity we will denote $X_2 G_0$ just by $G_0'$. We begin by performing a change of coordinates in $I_{1,3}^1$. In particular we choose the projective coordinates $(\sigma, x, y, z, \mathcal{S}', \mathcal{U}', \mathcal{Z}')$ with

$$\mathcal{S}' = \frac{\widetilde{x} - x}{x^2}, \quad \mathcal{U}' = \frac{\widetilde{y} - y}{x}, \quad \mathcal{Z}' = \widetilde{z} - z.$$

This accounts into a change of coordinates of the form $\mathcal{S}' = \sigma\mathcal{S}$, $\mathcal{U}' = \sigma\mathcal{U}$ and $\mathcal{Z}' = \sigma\mathcal{Z}$, leading to

$$|I^1_{1,3}| \leq \|u\|_\alpha \int_0^{\sqrt{s}} \int_{\partial M^+} \sigma^{-m} (G'_0)\big|_{|\mathcal{S}|=r} \beta^* d((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z}))^\alpha \,\mathrm{d}\, U \,\mathrm{d}\, Z \,\mathrm{d}\, \sigma.$$

We proceed exactly as for the estimates of $I_{1,1}$. This means, we consider polar coordinates in $\mathbb{R}^{m-1}$ (with coordinates $(\mathcal{U}', \mathcal{Z}')$) and denote by $R$ the radial component. Next we set $\varsigma = \sigma/R$. Furthermore, from the expression of $M^+$ above one has $\{|\mathcal{S}| = r\} = \partial M^+ = \{\widetilde{p} \,|\, d(p,\widetilde{p}) = 3d(p,p')\}$. In particular, for $\widetilde{p}$ being a point lying in the boundary of $\partial M^*$ of $M^+$,

$$2d(p,p') \leq d(p,\widetilde{p}) \leq 4d(p,p').$$

Integrating out the angular component results in

$$\begin{aligned}|I^1_{1,3}| &\leq \|u\|_\alpha \int_0^\infty \int_0^{4d(p,p')} R^{-1+\alpha} \varsigma^{-m} \left( \sqrt{\frac{|\mathcal{S}'|^2 + \|\mathcal{U}'\|^2 + \|\mathcal{Z}'\|^2}{\|\mathcal{U}'\|^2 + \|\mathcal{Z}'\|^2}} \right)^\alpha (G'_0)\big|_{|\mathcal{S}|=r} \,\mathrm{d}\, R \,\mathrm{d}\, \sigma \\ &\leq c\|u\|_\alpha d(p,p')^\alpha.\end{aligned}$$

**Remark 4.4.1**
*For $X = \mathrm{id}$ the asymptotics of the integrand will have an improvement by $\sigma^2$. This means in particular, that integration by part will no longer be necessary.*

### 4.4.3 Estimates of $I_3$

As usual let us assume $p = (x,y,z)$ and $p' = (x',y',z')$. By a simple adding and subtracting the same quantity we can express $I_3$ as the sum of three integrals $I_{3,1}$, $I_{3,2}$ and $I_{3,3}$ as displayed below.

$$\begin{aligned}I_3 = &\int_0^s \int_M [G(s-\widetilde{s}, p, \widetilde{p}) - G(s-\widetilde{s}, (x',y,z), \widetilde{p})] u(\widetilde{s}, p) \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\, \widetilde{s} \\ &+ \int_0^s \int_M [G(s-\widetilde{s}, (x',y,z), \widetilde{p}) - G(s-\widetilde{s}, (x',y',z), \widetilde{p})] u(\widetilde{s}, p) \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\, \widetilde{s} \\ &+ \int_0^s \int_M [G(s-\widetilde{s}, (x',y',z), \widetilde{p}) - G(s-\widetilde{s}, p', \widetilde{p})] u(\widetilde{s}, p) \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\, \widetilde{s} \\ =: &I_{3,1} + I_{3,2} + I_{3,3}.\end{aligned}$$

Our first aim is to show that the expression above can actually be reduced to $I_3 = I_{3,1}$. To see this we show that both $I_{3,2}$ and $I_{3,3}$ are vanishing. Due to similarity we will prove only $I_{3,2} = 0$.

Recall that, as discussed in §3.5 (cf. Proposition 3.5.1), $\Phi$-manifolds are stochastically. Thus, by recalling the expression for the integral kernel $G$, one has that $I_{3,2}$

can be written as

$$\begin{aligned} I_{3,2} &= \int_0^s x^\gamma u(p,\widetilde{s}) \left( \int_M [X(x^{-\gamma}H)(s-\widetilde{s},(x',y,z),\widetilde{p}) \right. \\ &\qquad \left. - X(x^{-\gamma}H)(s-\widetilde{s},(x',y',z),\widetilde{p})] \,\mathrm{dvol}_\Phi(\widetilde{p}) \right) \mathrm{d}\,\widetilde{s} \\ &= \int_0^s ((x')^{-\gamma} - (x')^{-\gamma}) x^\gamma u(p,\widetilde{s}) \,\mathrm{d}\,\widetilde{s} = 0. \end{aligned}$$

We can now estimate $I_3$. As for the previous cases, estimates away from $\mathrm{fd} \cup \mathrm{td}$ are trivial thus will not be presented here.

An application of the Mean Value Theorem results in

$$I_3 = |x - x'| \int_0^s \int_M \partial_\xi G\big|_{(s-\widetilde{s},\xi,y,z,\widetilde{x},\widetilde{y},\widetilde{z})} u(x,y,z,\widetilde{s}) \,\mathrm{dvol}_\Phi(\widetilde{p}) \,\mathrm{d}\,\widetilde{s} \tag{4.4.3}$$

In projective coordinate $(\sigma,\xi,y,z,\mathcal{S},\mathcal{U},\mathcal{Z})$, with

$$\mathcal{S} = \frac{\widetilde{x} - \xi}{\sigma\xi^2}, \quad \mathcal{U} = \frac{y - \widetilde{y}}{\sigma\xi}, \quad \mathcal{Z} = \frac{z - \widetilde{z}}{\sigma},$$

one finds that the lifted vector field $\beta^*(\partial_\xi)$ admits an expression of the form

$$\beta^*(\partial_\xi) = \partial_\xi - [2\xi^{-1}\mathcal{S} + \xi^{-2}\sigma^{-1}]\partial_{\mathcal{S}} - \xi^{-1}\mathcal{U}\partial_{\mathcal{U}}.$$

In particular we conclude $\beta^*(\partial_\xi G) \sim \xi^{-2}\sigma^{-1}\partial_{\mathcal{S}} G_0'$. Here the asymptotics of $G_0'$ are similar to those of $G_0$ in $I_{1,3}$ near td. Furthermore, since the function $u(p,\widetilde{s})$ is constant with respect to the spatial integration (i.e. with respect to $\widetilde{p}$), integration by parts gives

$$\begin{aligned} &\int_0^s \int_M \xi^{-2}\sigma^{-1}\partial_{\mathcal{S}} G_0' u(p,s-\sigma^2) h \,\mathrm{d}\,\mathcal{S} \,\mathrm{d}\,\mathcal{U} \,\mathrm{d}\,\mathcal{Z} \,\mathrm{d}\,\sigma \\ &\quad = \int_0^s \int_{\partial M} \xi^{-2}\sigma^{-1} G_0'\big|_{|\mathcal{S}|=\infty} u(p,s-\sigma^2) h \,\mathrm{d}\,\mathcal{S} \,\mathrm{d}\,\mathcal{U} \,\mathrm{d}\,\mathcal{Z} \,\mathrm{d}\,\sigma \\ &\qquad - \int_0^t \int_M \xi^{-2}\sigma^{-1} G_0' u(p,s-\sigma^2) \partial_{\mathcal{S}} h \,\mathrm{d}\,\mathcal{S} \,\mathrm{d}\,\mathcal{U} \,\mathrm{d}\,\mathcal{Z} \,\mathrm{d}\,\sigma. \end{aligned}$$

Due to the decay properties of $H_\gamma$ near $\partial\overline{M}$ (cf. (4.2.6)), the integral along the boundary vanishes.

Similarly to the estimates of $I_{1,3}^2$ we find $\partial_{\mathcal{S}} h = \xi^2\sigma h'$. In particular $\sigma\xi^2$ and $\sigma^{-1}\xi^{-2}$ cancel out leading to

$$\begin{aligned} &\int_0^s \int_M \xi^{-2}\sigma^{-1}\partial_{\mathcal{S}} G_0' u(p,s-\sigma^2) h \,\mathrm{d}\,\mathcal{S} \,\mathrm{d}\,\mathcal{U} \,\mathrm{d}\,\mathcal{Z} \,\mathrm{d}\,\sigma \\ &\qquad = - \int_0^s \int_M G_0' u(p,s-\sigma^2) h' \,\mathrm{d}\,\mathcal{S} \,\mathrm{d}\,\mathcal{U} \,\mathrm{d}\,\mathcal{Z} \,\mathrm{d}\,\sigma. \end{aligned}$$

The estimate now follows by proceeding as for the estimates of $I_{1,3}^2$. It is important to point out that, contrarily to $I_{1,3}^2$, the boundary term is vanishing.

With this last inequality we conclude the proof of Hölder differences in space.

## 4.5 Estimates for Hölder differences in time

In this section we will prove the estimates stated in (4.3.6). Without loss of generality we can consider $s < s'$. Indeed, in order to gain the estimates for $s' < s$ it will be enough to repeat all the upcoming estimates interchanging the role of $s$ and $s'$.
We begin by proving the estimates under the initial further assumption $2s' - s \geqslant 0$ (i.e., $s' < s \leq 2s'$). At the end of the section we will explain how to proceed for the other case, i.e. $2s' - s < 0$; as it will clarified, the estimates are obtained similarly to the one which we are going to present.

Let $T_-$, $T_+$ and $T'_+$ denote the intervals

$$T_- := [0, 2s' - s], \;\; T_+ := [2s' - s, t] \;\text{ and }\; T'_+ := [2s' - s, s'].$$

As it has already been done for the estimates of Hölder differences in space (cf. §4.4) we denote by $\mathbf{G}$ the operator $\mathbf{G} = X\mathbf{H}_\gamma$ for $X \in \{\{\mathrm{id}\}, \mathcal{V}_\Phi, \mathcal{V}_\Phi^2\}$. Using the same argument Bahuaud and Vertman used in [BaVe14, §3.2] we deduce

$$\begin{aligned}
\mathbf{G}u(p,s) - \mathbf{G}u(p,s') =& |s - s'| \int_{T_-} \int_M \partial_\theta G\big|_{(s-\widetilde{s},p,\widetilde{p})} [u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\
&+ \int_{T_+} \int_M G(s - \widetilde{s}, p, \widetilde{p}) [u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\
&- \int_{T'_+} \int_M G(s' - \widetilde{s}, p, \widetilde{p}) [u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})] \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\
&+ \int_0^s \int_M G(s - \widetilde{s}, p, \widetilde{p}) u(p,\widetilde{s}) \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\
&- \int_0^{s'} \int_M G(s' - \widetilde{s}, p, \widetilde{p}) u(p,\widetilde{s}) \,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\
=:& L_1 + L_2 - L_3 + L_4 - L_5
\end{aligned}$$

First of all notice that space-variable $p$ is constant in the integration. Second, from Proposition 3.5.1, $(\overline{M}, \widetilde{g})$ is stochastically complete, i.e.

$$\int_M H(s, p, \widetilde{p}) \,\mathrm{dvol}_\Phi(\widetilde{p}) = 1.$$

These two key observations joined together result into the following

$$L_4 - L_5 = \int_0^s u(p,\widetilde{s}) \,\mathrm{d}\,\widetilde{s} - \int_0^{s'} u(p,\widetilde{s}) \,\mathrm{d}\,\widetilde{s} \leq C\|u\|_\infty |s - s'|^{\alpha/2}.$$

It is therefore clear that, in order to prove the inequality claimed in (4.3.6), it is only necessary to estimate $L_1, L_2$ and $L_3$. However, due to similarities between the terms $L_2$ and $L_3$, we will only present one of them. In conclusion, in what follows we will present the estimates, at each regime, of the terms $L_1$ and $L_2$. Moreover, as for the Hölder differences in space, estimates away from $\mathrm{fd} \cup \mathrm{td}$ are trivial thus they will be omitted.

### 4.5.1 Estimates for $L_1$

In projective coordinates $(\sigma, x, y, z, \mathcal{S}, \mathcal{U}, \mathcal{Z})$ with

$$\mathcal{S} = \frac{\widetilde{x} - x}{\sigma x^2}, \quad \mathcal{U} = \frac{y - \widetilde{y}}{\sigma x}, \quad \mathcal{Z} = \frac{z - \widetilde{z}}{\sigma},$$

the asymptotics of $H_\gamma$ in regime 5 are given by (4.2.6). In particular it follows $\beta^* \partial_\theta G \sim \sigma^{-m-4} G'_0$, with $G'_0$ being polyhomogeneous and vanishing to infinite order when $(|\mathcal{S}|, \|\mathcal{U}\|, \|\mathcal{Z}\|) \to \infty$. Also, the volume form has asymptotics of the form

$$\beta^*(\mathrm{dvol}_\Phi(\widetilde{p})\, \mathrm{d}\,\widetilde{s}) \sim \sigma^{m+1} h \,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma, \tag{4.5.1}$$

with $h$ a smooth function of $\widetilde{p} = (x + x^2 \mathcal{S}\sigma, y + x\mathcal{U}\sigma, z + \sigma\mathcal{Z})$ (cf. (4.2.6)). Further, recall that in regime 5 $x \sim \widetilde{x}$; thus

$$d(p, \widetilde{p}) \leq c\sigma\rho_{\mathrm{fd}} \sqrt{|\mathcal{S}|^2 + \|\mathcal{U}\|^2 + \|\mathcal{Z}\|^2} =: c\sigma x r(\mathcal{S}, \mathcal{U}, \mathcal{Z}). \tag{4.5.2}$$

Notice that, due to its expression, $r$ is bounded as long as its entries are bounded; therefore $G'_0 r^\alpha$ is bounded everywhere.
Moreover, for every $\widetilde{s} \in T_-$, $|\theta - \widetilde{s}| \geq |s - s'|$. In conclusion the integral $L_1$ can be estimated as follows

$$\begin{aligned}
|L_1| &\leq \|u\|_\alpha |s - s'| \int_{T_-} \int_M |\sigma^{-3} G'_0 d(p, \widetilde{p})^\alpha| \,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma \\
&\leq C\|u\|_\alpha |s - s'| \int_{\sqrt{s-s'}}^\infty \int_M |\sigma^{-3+\alpha} x^\alpha G'_0 r^\alpha| \,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma \\
&\leq C\|u\|_\alpha |s - s'|^{\alpha/2};
\end{aligned}$$

thus completing the estimates for $L_1$.

### 4.5.2 Estimates for $L_2$

Proceeding exactly as for $L_1$, in the same projective coordinates $(\sigma, x, y, z, \mathcal{S}, \mathcal{U}, \mathcal{Z})$, we recall, from (4.2.6), that the asymptotics of $\beta^* G$ are given by $\beta^*(G) \sim \sigma^{-m-2} G'_0$, with $G'_0$ being polyhomogeneous and vanishing to infinite order whenever $(|\mathcal{S}|, \|\mathcal{U}\|, \|\mathcal{Z}\|) \to \infty$. Furthermore, in these coordinates, the lift of the volume form has the expression as in (4.5.1). Therefore, by expressing $L_2$ in projective coordinates one finds

$$\begin{aligned}
|L_2| &\leq \|u\|_\alpha \int_{T_+} \int_M |\sigma^{-1} G'_0 d(p, \widetilde{p})^\alpha| \,\mathrm{d}\,\varsigma\,\mathrm{d}\,\eta\,\mathrm{d}\,\zeta\,\mathrm{d}\,\sigma \\
&\leq c\|u\|_\alpha \int_{T_+} \int_M |\sigma^{-1+\alpha} G'_0 r^\alpha| \,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma \\
&\leq c\|u\|_\alpha |s - s'|^{\alpha/2},
\end{aligned}$$

concluding the estimates for the $L_2$-term.
This completes the estimates for time difference with derivatives under the assumption that $2s' - s \geqslant 0$.

Let us now remove the initial further assumption, meaning that we assume $2s' - s < 0$. By assumption $s > s' \geq 0$, thus by making use of $2s' < 3s$, hence one concludes $s' < s < 2|s - s'|$. Therefore the Hölder differences in time, under the assumption $2s' - s < 0$ can be expressed as

$$\begin{aligned}\mathbf{G}u(p,s) - \mathbf{G}u(p,s') &= \int_0^s \int_M G(s-\widetilde{s},p,\widetilde{p})[u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\ &\quad - \int_0^{s'} \int_M G(s'-\widetilde{s},p,\widetilde{p})[u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s}.\end{aligned}$$

The integrals above can be treated similarly to $L_2$; the proof of the estimate claimed in (4.3.6) is therefore complete.

## 4.6 Estimates for the supremum norm

We finally reached the last step for the proof of Theorem 4.2; that is we prove the estimates for the supremum of $\mathbf{G}u$ claimed in (4.3.7).
For $(p,s)$ in $\overline{M} \times [0,T]$, $\mathbf{G}u(p,s)$ we write $\mathbf{G}u(p,s)$ as

$$\begin{aligned}\mathbf{G}u(p,s) &= \int_0^s \int_M G(s-\widetilde{s},p,\widetilde{p})u(\widetilde{p},\widetilde{s})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\ &= \int_0^s \int_M G(s-\widetilde{s},p,\widetilde{p})[u(\widetilde{p},\widetilde{s}) - u(p,\widetilde{s})]\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\ &\quad + \int_0^s \int_M G(s-\widetilde{s},p,\widetilde{p})u(p,\widetilde{s})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s} \\ &= J_1 + J_2.\end{aligned}$$

As for the previous cases, the estimates away from $\mathrm{fd} \cup \mathrm{td}$ are trivial hence they will be omitted.

We begin by estimating $J_1$.
In projective coordinates $(\sigma, x, y, z, \mathcal{S}, \mathcal{U}, \mathcal{Z})$ given by

$$\mathcal{S} = \frac{\widetilde{x} - x}{\sigma x^2}, \quad \mathcal{U} = \frac{y - \widetilde{y}}{\sigma x}, \quad \mathcal{Z} = \frac{z - \widetilde{z}}{\sigma};$$

one has in view of (4.2.6)

$$\beta^*(G(s-\widetilde{s},p,\widetilde{p})\,\mathrm{dvol}_\Phi(\widetilde{p})\,\mathrm{d}\,\widetilde{s}) = \sigma^{-1} G_0'\,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma$$

with $G_0$ vanishing to infinite order as $\|(\mathcal{S}, \mathcal{U}, \mathcal{Z})\| \to \infty$.
Further, recall that in regime 5 $x \sim \widetilde{x}$; thus

$$d(p,\widetilde{p}) \sim \sigma x r(\mathcal{S}, \mathcal{U}, \mathcal{Z})$$

where $r = \sqrt{|\mathcal{S}|^2 + \|\mathcal{U}\|^2 + \|\mathcal{Z}\|^2}$ bounded for as long as its entries are bounded. Therefore $J_1$ can be estimated as

$$\begin{aligned}|J_1| &\le C\|u\|_\alpha \int_0^{\sqrt{s}} \int_M \sigma^{-1} G_0' d(p,\widetilde{p})^\alpha\,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma \\ &= C\|u\|_\alpha \int_0^{\sqrt{s}} \int_M |\sigma^{-1+\alpha} G_0' r(\mathcal{S}, \mathcal{U}, \mathcal{Z})^\alpha|\,\mathrm{d}\,\mathcal{S}\,\mathrm{d}\,\mathcal{U}\,\mathrm{d}\,\mathcal{Z}\,\mathrm{d}\,\sigma \\ &\le C\|u\|_\alpha s^{\alpha/2};\end{aligned}$$

thus proving the claimed inequality.

The estimate in (4.3.7) now follows if $J_2$ satisfies a similar estimate to the one for $|J_1|$ above. To see this one may argue exactly as we have done for the estimates of $L_4$ in §4.5.

# Chapter 5

# Solutions to heat type equations

For a given $\Phi$-manifold $(\overline{M}, \widetilde{g})$, the heat kernel operator $\mathbf{H}$, whose mapping properties have been widely analysed through section §4.3 to §4.6, represents an inverse of the heat operator $(\partial_s + \widetilde{\Delta})$; where $\widetilde{\Delta}$ denotes the unique self-adjoint extension of the Laplace-Beltrami operator associated to the $\Phi$-metric $\widetilde{g}$. This means that, given some function $\ell \in x^{\gamma} C^{k,\alpha}_{\Phi}(M \times [0,T])$, $u = \mathbf{H}(\ell)$ is a solution of the Cauchy problem

$$\left(\partial_s + \widetilde{\Delta}\right) u = \ell, \quad u|_{s=0} = 0.$$

Here the aim is to construct a right inverse of heat type operators

$$P := \partial_s + a\widetilde{\Delta}, \tag{5.0.1}$$

where $a$ is a function on $\overline{M} \times [0,T]$ satisfying the assumptions of Theorem 5.1. Such an inverse will allow us to conclude the existence of solutions to non-linear heat type Cauchy problems of the form

$$(\partial_s + a\widetilde{\Delta})u = F(u), \quad u|_{s=0} = 0, \tag{5.0.2}$$

where $F$ is an operator satisfying certain conditions described below.

Our first goal is, therefore, to construct a right inverse of $P$. This will be achieved by constructing first an approximate inverse, i.e. a parametrix, for $P$. The structure of $P$ makes it reasonable to ground the construction of parametrix on the heat kernel operator $\mathbf{H}$.
The parametrix construction will be divided in two parts, a boundary and an interior parametrix. A combination of those will then give rise to a parametrix for heat type operators. A boundary parametrix will be constructed in §5.1.1. Our construction follows along the same steps of the boundary parametrix in [BaVe19]. It is a technical construction since it requires a careful analysis near the boundary. The construction of an interior parametrix, along with a parametrix for heat type operators, will instead take place in §5.1.2. The interior parametrix will follow as a consequence of the standard analysis of parabolic PDE's on compact manifolds.

Once a right parametrix for $P$ has been constructed we conclude the following result.

**Theorem 5.1**
*Let $\beta$ be in $(0,1)$ and consider a positive function $a$ in $C^{k,\beta}_{\Phi}(M\times[0,T])$ to so that it is bounded from below away from $0$. There exist two operators $\mathbf{Q}$ and $\mathbf{E}$ so that, for every $\alpha\in(0,1)$, $\alpha<\beta$ and for every $\gamma\in\mathbb{R}$,*

$$\begin{aligned}\mathbf{Q}&: x^{\gamma}C^{\alpha}_{\Phi}(M\times[0,T])\to x^{\gamma}C^{2,\alpha}_{\Phi}(M\times[0,T]),\\ \mathbf{E}&: x^{\gamma}C^{2,\alpha}_{\Phi}(M)\to x^{\gamma}C^{2,\alpha}_{\Phi}(M\times[0,T]),\end{aligned}$$

*are both bounded. Furthermore, for $\ell\in x^{\gamma}C^{\alpha}(M\times[0,T])$ and $u_0\in C^{2,\alpha}(M)$, $\mathbf{Q}\ell$ and $\mathbf{E}u_0$ are solutions of the Cauchy problems*

$$\textbf{\textit{(i)}}Pu=\ell;\ u|_{s=0}=0\ \text{ and }\ \textbf{\textit{(ii)}}Pu=0;\ u|_{s=0}=u_0 \tag{5.0.3}$$

*respectively for an heat type operator $P$ as in (5.0.1).*

In particular the above result gives the existence of solutions to non-homogeneous linear heat type Cauchy problems with null initial conditions (Cauchy problem (i) above) and to homogeneous linear heat type Cauchy problems with non vanishing initial condition (Cauchy problem (ii) above). The proof of Theorem 5.1 will be the core of §5.2. Finally, in §5.3 we move to the analysis of non linear heat type problems as in (5.0.2).

## 5.1 Parametrix for heat type operators

### 5.1.1 Boundary parametrix

As in [BaVe19], the boundary parametrix will be constructed by localizing the problem in appropriate coordinate patches by making use of two partitions of unity. Thus, we will firstly construct a localised parametrix, then by summing over the partition of unity, we get an approximate inverse of $P$ near the boundary. The next Lemma explains the reason why the choice of partitions of unity, localised near the boundary, are useful for the purposes described at the beginning of this section.

**Lemma 5.1.1**
*Let $(\overline{M},\widetilde{g})$ be a $\Phi$-manifold and consider two functions $\varphi,\psi\in C^{\infty}(M)$ to be compactly supported. Assume, furthermore, that $\varphi$ and $\psi$ lie in $C^{\alpha}_{\Phi}(M)$ (cf. §3.3) and that $\psi$ is supported away from the boundary $\partial\overline{M}$ of $\overline{M}$. Let $\mathbf{H}$ be the operator, described in §4.3, i.e. the heat kernel operator of the unique self-adjoint extension of the Laplace-Beltrami operator associated to the $\Phi$-metric $\widetilde{g}$. Denote by $R^0$ the operator defined by $R^0=\psi\mathbf{H}\varphi$ where $\psi$ and $\varphi$ act by multiplication. That is $R^0u=\psi\mathbf{H}(\varphi u)$. For every non negative integer $k$, and for every $\alpha\in(0,1)$ and $\gamma\in\mathbb{R}$, The operator $R^0$ acting between the weighted Hölder spaces*

$$R^0: x^{\gamma}C^{k,\alpha}_{\Phi}(M\times[0,T])\to\sqrt{t}x^{\gamma}\,C^{k+1,\alpha}_{\Phi}(M\times[0,T])$$

*has operator norm $\|R^0\|_{\mathrm{op}}$ satisfying*

$$\|R^0\|_{\mathrm{op}}\xrightarrow{T\to 0}0.$$

***Proof:***
With the same argument employed in the proof of Theorem 4.2, as well as Theorem 4.3, it is enough to prove the result for $k = 0$. It is important to point out that the operator $R^0$ acts as a convolution, i.e. for $u \in x^\gamma C^\alpha(M \times [0,T])$,

$$R^0 u(p,t) = \int_0^t \int_M \psi(p) H(t - \widetilde{t}, p, \widetilde{p}) \varphi(\widetilde{p}) u(\widetilde{p}, \widetilde{t}) \, \mathrm{dvol}_\Phi(\widetilde{p}) \, \mathrm{d}\,\widetilde{t};$$

with $H$ being the heat kernel whose asymptotics have been discussed in §4.2. For simplicity we will denote the kernel of the operator $R^0$ just by $\psi H \varphi$.
Now, since $\psi$ is supported away from the boundary $\partial\overline{M}$ of $\overline{M}$, the lift of $\psi H \varphi$ to the heat space $\overline{M}^2_h$ is (compactly) supported away from ff, fd, lf and rf (see Figure 4.3). Therefore, by looking at theorem 4.1, we conclude that the asymptotic behaviour of $\psi H \varphi$ is given by the asymptotic of the operator $H$ near td (i.e. regime 5); that is

$$\beta^*(\psi H \varphi) \sim \sigma^{-m} G_1$$

where $G_1$ is a bounded function vanishing to infinite order as $|(\mathcal{S}, \mathcal{U}, \mathcal{Z})| \to \infty$.
It is important to notice that the proof of Theorem 4.2 does not hold for the operator $R^0$ for it being non stochastically complete (see (2.1.3)). But, arguing with the exact same estimates, we conclude that Theorem 4.3 holds for the operator $R^0$ as well. In conclusion

$$\|R^0\|_{\mathrm{op}} = \sup_{\|u\|_\alpha = 1} \|R^0 u\|_{1,\alpha} = \sup_{\|u\|_\alpha = 1} \|R^0 u\|_\alpha + \sup_{\substack{\|u\|_\alpha = 1 \\ X \in \mathcal{V}_\Phi}} \|X(R^0 u)\|_\alpha \leq c\sqrt{s}.$$

The above estimate implies the result since, for $T \to 0$, $\sqrt{s} \to 0$. □

We can now construct the specific partition of unity.

### Partitions of unity

Using the same notation as in Chapter 3 let us fix some $\mathfrak{R} > 0$ and, consequently, the subset of $\overline{M}$ given by $U_\mathfrak{R} = \left\{p \in \overline{M} \,|\, x(p) \leq \mathfrak{R}\right\}$. Furthermore, for $d > 0$ let us define the family of half-cubes

$$B(d) = [0, d) \times (-d, d)^b \times (-d, d)^f \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^b \times \mathbb{R}^f,$$

where $b$ and $f$ denote the dimension of the closed manifolds $Y$ and $Z$ respectively. Since $\overline{M}$ is a compact manifolds with boundary, for every $\overline{p} \in \partial\overline{M}$, there exists some coordinate char $A$ around $\overline{p}$ and a diffeomorphism $\phi : B(1) \to A$. Moreover, due to compactness of $\partial\overline{M}$, we can consider finitely many charts $\{\overline{p}_i, \phi_i : B(1) \to A_i\}$ where the $\overline{p_i}$'s are points on the boundary $\partial\overline{M}$. By choosing $\mathfrak{R}$ small enough, the finite family $(A_i)_i$ will cover the whole collar neighbourhood $U_\mathfrak{R}$. Such a covering can be extended to a covering of the whole manifold $\overline{M}$ by considering an additional open set $A_0 = \{p \in \overline{M} \,|\, x(p) > \mathfrak{R}/2\} \big( = \overline{M} \setminus U_{\mathfrak{R}/2} \big)$.

We will now define bump functions supported on the finite family of open neighbourhoods of the points $\overline{p_i} \in \partial\overline{M}$. We begin by setting $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}$ to be a compactly supported function so that, $\sigma(x) \leq 1$ with $\sigma(x) = 1$ for $x \in [0, 1/2]$ and $\sigma(x) = 0$ for $x > 1$. By an application of the Mean Value theorem, it is easy to see that $\sigma$ lies in $C^{k,\alpha}(\mathbb{R}_{\geq 0})$ for every $k \geq 0$ and for every $\alpha \in (0, 1]$.

**Remark 5.1.1**
*The Hölder space $C^{k,\alpha}(\mathbb{R}_{\geq 0})$ above denotes the classical Hölder space; i.e. in this case the Hölder differences are given by*

$$[\sigma]_\alpha = \sup_{x,x' \in \mathbb{R}_{\geq 0}} \frac{|\sigma(x) - \sigma(x')|}{|x - x'|^\alpha}.$$

**Lemma 5.1.2**
*Let $\varepsilon$ be a number in $(0,1)$. Denote by $\widehat{\sigma} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ the function defined by*

$$\widehat{\sigma}(x) = \sigma\left(\frac{x}{\varepsilon}\right).$$

*For every $\alpha \in (0,1]$ there exists some constant $C > 0$ so that*

$$[\widehat{\sigma}]_\alpha \leq C\varepsilon^{-\alpha}.$$

*In particular, $\widehat{\sigma} \in C^{k,\alpha}(\mathbb{R}_{\geq 0})$ for every $k \geq 0$.*

***Proof:***
From direct computations one sees that

$$|\widehat{\sigma}(x) - \widehat{\sigma}(x')| = \left|\sigma\left(\frac{x}{\varepsilon}\right) - \sigma\left(\frac{x'}{\varepsilon}\right)\right| \leq C\left|\frac{x}{\varepsilon} - \frac{x'}{\varepsilon}\right|^\alpha = C\varepsilon^{-\alpha}|x - x'|^\alpha,$$

for some constant $C > 0$. Notice that the inequality above follows from the assumption on $\sigma$. □

Before proceeding with the definition of the bump functions, we need a further intermediate result. As it has been already stated in Remark 3.1.1, we will use the short hand notation $y$ and $z$ for $(y^1, \ldots, y^b)$ and $(z^1, \ldots, z^f)$ respectively.

**Lemma 5.1.3**
*Let $(\overline{M}, \widetilde{g})$ be a $\Phi$-manifold. For every $q \in [1,\infty]$ the following distances are equivalent*

$$\mathrm{d}_{q,\Phi}(p,p') = \left(|x - x'|^q + (x + x')^q\|y - y'\|^q + (x + x')^2 q\|z - z'\|^q\right)^{1/q} \quad \textit{for } q < \infty;$$
$$\mathrm{d}_{\infty,\Phi}(p,p') = \max\{|x - x'|, (x + x')\|y - y'\|, (x + x')^2\|z - z'\|\} \quad \textit{for } q = \infty.$$

*Here by equivalent we mean that for every $q, q' \in [1,\infty]$ there exists constants $c, C > 0$ so that for every $p, p' \in \overline{M}$, $c\,\mathrm{d}_{q',\Phi}(p,p') \leq \mathrm{d}_{q,\Phi}(p,p') \leq C\,\mathrm{d}_{q',\Phi}(p,p')$.*

***Proof:***
Notice that, it is enough to prove that for a given $q \in [1,\infty)$ there exists some constant $c, C > 0$ so that

$$c\,\mathrm{d}_{\infty,\Phi}(p,p') \leq \mathrm{d}_{q,\Phi}(p,p') \leq C\,\mathrm{d}_{\infty,\Phi}(p,p')$$

for every $p, p' \in \overline{M}$. Indeed one can use the transitive property to gain the other inequalities.
Let $q \in [1,\infty)$. For given $p, p' \in \overline{M}$ it is obvious that $\mathrm{d}_{\infty,\Phi}(p,p') \leq \mathrm{d}_{q,\Phi}(p,p')$. The other inequality is trivial too.

$$\mathrm{d}_{q,\Phi}(p,p') \leq \left(\mathrm{d}_{\infty,\Phi}(p,p')^q + \mathrm{d}_{\infty,\Phi}(p,p')^q + \mathrm{d}_{\infty,\Phi}(p,p')^q\right)^{1/q} = 3^{1/q}\,\mathrm{d}_{\infty,\Phi}(p,p').$$

□

**Remark 5.1.2**
*Due to the equivalence above we will not distinguish between the various $\Phi$-distances.*

We are now in the position to define the appropriate bump functions. Let $\overline{p} \in \partial\overline{M}$ be fixed. From the definition of the open covering, defined above, there exists some $\phi_i : B(1) \to A_i$ so that $\phi_i(\overline{p}) = (0, \overline{y}, \overline{z})$ for some $\overline{y} \in (-1,1)^b$ and $\overline{z} \in (-1,1)^f$.

**Proposition 5.1.1**
*Let $\varepsilon \in (0,1)$ be fixed. Consider the functions $\widehat{\psi}_{i,\overline{p}}, \widehat{\varphi}_{i,\overline{p}} : A_i \to \mathbb{R}$ defined by, for $p \in A_i$ so that $\phi_i^{-1}(p) = (x,y,z)$,*

$$\widehat{\varphi}_{i,\overline{p}}(p) = \sigma\left(\frac{x}{\varepsilon}\right)\sigma(x\|y-\overline{y}\|)\sigma(\varepsilon x^2\|z-\overline{z}\|); \tag{5.1.1}$$

$$\widehat{\psi}_{i,\overline{p}}(p) = \sigma\left(\frac{x}{2\varepsilon}\right)\sigma\left(\frac{x\|y-\overline{y}\|}{2}\right)\sigma\left(\frac{\varepsilon x^2\|z-\overline{z}\|}{2}\right). \tag{5.1.2}$$

*Then $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ satisfy:*

*I. $\widehat{\psi}_{i,\overline{p}} \equiv 1$ on the support of $\widehat{\varphi}_{i,\overline{p}}$.*

*II. There exist constants (all of which will be denoted by $C$) so that $[\widehat{\psi}_{i,\overline{p}}]_\alpha \le C\varepsilon^{-\alpha}$, $[\widehat{\varphi}_{i,\overline{p}}]_\alpha \le C\varepsilon^{-\alpha}$.*

*III. $\widehat{\varphi}_{i,\overline{p}}, \widehat{\psi}_{i,\overline{p}} \in C^\alpha_\Phi(M)$ (see §3.3 for the definition of Hölder spaces on $\Phi$-manifolds).*

*IV. There exists some constant $\overline{C} > 0$ (depending solely on the dimension of $Y$ and $Z$) so that* diam $(\mathrm{supp}(\widehat{\varphi}_{i,\overline{p}})) \le \overline{C}\varepsilon$. *Here by* diam *we mean the diameter, that is*

$$\mathrm{diam}\,(\mathrm{supp}(\widehat{\varphi}_{i,\overline{p}})) = \max_{p,p' \in \mathrm{supp}\left(\widehat{\varphi}_{i,\overline{p}}\right)} \mathrm{d}_\Phi(p,p').$$

*Note that, in the spirit of Remark 5.1.2, in the above we did not specify whit respect to which $\Phi$-distance on $\overline{M}$ is the diameter considered.*

***Proof:***
Property I follows directly from the definition of $\sigma$.

Clearly, the fact that $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ lie in $C^\alpha_\Phi(M)$ is a direct consequence of II, due to $\widehat{\psi}_{i,\overline{p}}$ and $\widehat{\varphi}_{i,\overline{p}}$ being bounded. Let us therefore prove II. Since $\widehat{\psi}_{i,\overline{p}}$ is just a rescaling of $\widehat{\varphi}_{i,\overline{p}}$, it is enough to prove II for the function $\widehat{\varphi}_{i,\overline{p}}$.

Let $p, p' \in A_i$ and assume $\phi_i^{-1}(p) = (x,y,z)$ while $\phi_i^{-1}(p') = (x',y',z')$. We have the following chain of inequalities.

$$\begin{aligned}
\widehat{\varphi}_{i,\overline{p}}(p)-\widehat{\varphi}_{i,\overline{p}}(p') = &\ \sigma\left(\frac{x}{\varepsilon}\right)\sigma(x\|y-\overline{y}\|)\sigma(\varepsilon x^2\|z-\overline{z}\|) \\
&- \sigma\left(\frac{x'}{\varepsilon}\right)\sigma(x'\|y'-\overline{y}\|)\sigma(\varepsilon x'^2\|z'-\overline{z}\|) \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\left(\sigma(x\|y-\overline{y}\|)-\sigma(x'\|y'-\overline{y}\|)\right) \\
&+ C\left(\sigma(\varepsilon x^2\|z-\overline{z}\|)-\sigma(\varepsilon x'^2\|z'-\overline{z}\|)\right) \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\left((x-x')\|y-\overline{y}\|+x'\|y-\overline{y}\|-x'\|y'-\overline{y}\|\right)^\alpha \\
&+ C\varepsilon^\alpha\left(x^2\|z-\overline{z}\|-x'^2\|z-\overline{z}\|+x'^2\|z-\overline{z}\|-x'^2\|z'-\overline{z}\|\right)^\alpha \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\left(|x-x'|\|y-\overline{y}\|\right)^\alpha + C\left(x'\|y-y'\|\right)^\alpha \\
&+ C\varepsilon^\alpha\left(|x^2-x'^2|\|z-\overline{z}\|\right)^\alpha + C\varepsilon^\alpha\left(x'^2\|z-z'\|\right)^\alpha \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\varepsilon^{-\alpha}\left(|x-x'|\|y-\overline{y}\|\right)^\alpha + C\varepsilon^{-\alpha}\left(x'\|y-y'\|\right)^\alpha \\
&+ C\varepsilon^{-\alpha}\left(|x-x'|(x+x')\|z-\overline{z}\|\right)^\alpha + C\varepsilon^{-\alpha}\left(x'^2\|z-z'\|\right)^\alpha \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\varepsilon^{-\alpha}\left(x'\|y-y'\|\right)^\alpha + C\varepsilon^{-\alpha}\left(x'^2\|z-z'\|\right)^\alpha \\
\leq & C\varepsilon^{-\alpha}|x-x'|^\alpha + C\varepsilon^{-\alpha}\left(x'\|y-y'\|+x\|y-y'\|\right)^\alpha \\
&+ C\varepsilon^{-\alpha}\left(x'^2\|z-z'\|+(2xx'+x^2)\|z-z'\|\right)^\alpha \\
\leq & C\varepsilon^{-\alpha}\left(|x-x'|^\alpha+(x+x')^\alpha\|y-y'\|^\alpha+(x+x')^{2\alpha}\|z-z'\|^\alpha\right) \\
\leq & C\varepsilon^{-\alpha}\,\mathrm{d}_{\infty,\Phi}(p,p')^\alpha \leq C\,\mathrm{d}_{2,\Phi}(p,p')^\alpha.
\end{aligned} \tag{5.1.3}$$

It is important to mention that the $C$'s in the above estimate represent (perhaps different) uniform constants. Furthermore, the first and second inequalities are obtained by making use of the $\alpha$-Hölder regularity of $\sigma$ and, especially, it being bounded. The third inequality is obtained by making use of the reverse triangle inequality and sublinearity of $x^\alpha$ (with $\alpha \in (0,1)$). The fourth inequality follows by noticing that $\varepsilon, \alpha \in (0,1)$ thus $\varepsilon^\alpha \leq 1 \leq \varepsilon^{-\alpha}$. The fifth inequality is a direct consequence of $\|y-\overline{y}\|$ as well as $|x+x'|$ and $\|z-\overline{z}\|$ being bounded. The sixth inequality is a consequence of the function $x^\alpha$ being increasing. The seventh is just a rewriting of the above one. Finally the first to last inequality is just the definition of the "$\infty, \Phi$-distance defined in Lemma 5.1.3 from which follows the last inequality. So far we have seen $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ to lie in $C^\alpha_\Phi(M)$. This result can be extended to $C^{2,\alpha}_\Phi(M)$ just by noticing $\sigma$ to be constant near $\overline{p}$.

Finally, let us prove IV. Consider $p, p' \in \operatorname{supp}(\widehat{\varphi}_{i,\overline{p}})$ with $\phi_i^{-1}(p) = (x,y,z)$ and $\phi_i^{-1}(p') = (x',y',z')$. By definition of $\sigma$ the following hold:

$$x \leq \varepsilon;\quad x\|y-\overline{y}\| \leq 1;\quad \varepsilon x^2\|z-\overline{z}\| \leq 1 \tag{5.1.4}$$

$$x' \leq \varepsilon;\quad x'\|y'-\overline{y}\| \leq 1;\quad \varepsilon x'^2\|z'-\overline{z}\| \leq 1. \tag{5.1.5}$$

In particular, $x, x' \in (0,\varepsilon)$. Thus, computing $\mathrm{d}_{1,\Phi}(p,p')$ we get

$$\begin{aligned}
\mathrm{d}_{1,\Phi}(p,p') =& |x-x'| + (x+x')\|y-y'\| + (x+x')^2\|z-z'\| \\
&\leq \varepsilon + 4\sqrt{b}\varepsilon + 4\sqrt{f}\varepsilon^2 \leq \overline{C}\varepsilon
\end{aligned}$$

with $\overline{C} = \max\{1, 4\sqrt{b}, 4\sqrt{f}\}$. Notice that the values $2\sqrt{b}$ and $2\sqrt{f}$ come from the Euclidean length of the diagonal of the cubes $(-1,1)^b$ and $(-1,1)^f$ respectively. □

**Remark 5.1.3**
*We want to point out that the function $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ defined in Proposition 5.1.1 are defined on the open sets $A_i$. Due to the nature of the function $\sigma$ they can be extended to the whole manifold $\overline{M}$ by setting them to be 0 outside their support. With a slight abuse of notation we will denote these extensions again with $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ respectively.*

**Corollary 5.1.1**
*In the same assumptions as in Proposition 5.1.1, there exists constants (all of which will be denoted by $C$) such that*

$$[X(\widehat{\varphi}_{i,\overline{p}})]_\alpha \leq C\varepsilon^{1-\alpha};\ [Y(\widehat{\varphi}_{i,\overline{p}})]_\alpha \leq C\varepsilon^{2-\alpha} \ \textit{ for } X \in \mathcal{V}_\Phi(\overline{M}) \textit{ and } Y \in \mathcal{V}^2_\Phi(\overline{M}); \quad (5.1.6)$$

*with similar estimates holding for $\widehat{\psi}_{i,\overline{p}}$ as well. In particular $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ lie in $C^{2,\alpha}_\Phi(M)$.*

***Proof:***
As for II in Proposition 5.1.1, the proof relies on estimates similar to (5.1.3). Although similar, these estimates are much longer and tedious, therefore they will not be presented here. Nonetheless we will explain how and why the argument holds. Regarding the why, notice that (first order) $\Phi$-derivatives lie in the algebra $\text{span}\{x^2\partial/\partial_x, x\partial/\partial_y, \partial/\partial_z\}$. Now, by looking at the expression of $\widehat{\varphi}_{i,\overline{p}}$ in (5.1.1), one has

$$x^2\frac{\partial}{\partial_x}\widehat{\varphi}_{i,\overline{p}} = \frac{x^2}{\varepsilon}\widehat{\Phi};\ \ x\frac{\partial}{\partial_y}\widehat{\varphi}_{i,\overline{p}} = x^2\widehat{\Phi};\ \ \frac{\partial}{\partial_z}\widehat{\varphi}_{i,\overline{p}} = \varepsilon x^2\widehat{\Phi}$$

for some functions $\widehat{\Phi}$. Similar results hold for second order $\Phi$-derivatives as well. In this latter case one sees that the worst case scenario (i.e. the one where $\varepsilon$ has the highest negative power) is given by

$$x^2\frac{\partial}{\partial_x}\left(x^2\frac{\partial}{\partial_x}\widehat{\varphi}_{i,\overline{p}}\right) = \frac{x^4}{\varepsilon^2}\widehat{\Phi} + \frac{x^3}{\varepsilon}\widehat{\Phi} + \text{ more regular terms in } \varepsilon.$$

In the above $\widehat{\Phi}$ denotes different functions.
If $x$ is small, i.e. $x \leq \varepsilon$, the Hölder condition for $\sigma$ (which will hold for the functions $\widehat{\Phi}$) leads indeed to the claimed inequalities.

It is therefore clear that ensuring $x \leq \varepsilon$ leads to the claimed inequalities. To this end we begin by noticing that it is enough to compute the $\alpha$-seminorms only for $p$ and $p'$ in $\text{supp}\,(\widehat{\varphi}_{i,\overline{p}})$. Indeed, for $p, p'$ both not lying in the support, $[X(\widehat{\varphi}_{i,\overline{p}})]_\alpha = 0$ for $X \in \mathcal{V}_\Phi(\overline{M})$ and $X \in \mathcal{V}^2_\Phi(\overline{M})$. Assume now $p \in \text{supp}(\widehat{\varphi}_{i,\overline{p}})$ while $p' \notin \text{supp}(\widehat{\varphi}_{i,\overline{p}})$. Then, by looking at (5.1.5), at least one of the following is satisfied,

$$x' > \varepsilon;\ \ x'\|y' - \overline{y}\| > 1;\ \ \varepsilon x'^2\|z' - \overline{z}\| > 1.$$

Assume, without loss of generality, the condition $x' > \varepsilon$ to be satisfied. Then one can subtract from $X(\widehat{\varphi}_{i,\overline{p}})(p)$ the function $X(\widehat{\varphi}_{i,\overline{p}})(p'')$ where $p''$ is a point whose coordinates are given by $(x', y, z)$. The estimate now follows trivially.
Thus, under the assumption $p, p' \in \text{supp}(\widehat{\varphi}_{i,\overline{p}})$, (5.1.4) and (5.1.5) imply that $x$ and $x'$ are both bounded from above by $\varepsilon$. The inequality will therefore follow. $\square$

**Remark 5.1.4**
*It is worth to point out that the functions $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ have much better regularity than $C^2$. Indeed, one can employ the same proof as in the above and conclude $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ to lie in $C^{k,\alpha}_{\Phi}(M)$ for every $k > 0$.*

The functions $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ will allow us to construct the claimed partitions of unity. Recall that, for a partition of unity, only a finite number of functions may be non-vanishing in a neighbourhood. Although we have a fine family of open sets $(A_i)_i$, the functions $\widehat{\varphi}_{i,\overline{p}}$, $\widehat{\psi}_{i,\overline{p}}$ are defined for every point on the boundary $\partial\overline{M}$ of $\overline{M}$. This makes virtually impossible to have only finitely many non-vanishing functions in neighbourhoods of points in a collar neighbourhood of the boundary. Hence, the final step for the construction of partitions of unity is to reduce the amount of points $\overline{p} \in \partial\overline{M}$ by means of which we defined the bump functions $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$. To this end let us consider the following set. For $\vartheta \in (0,1)$ consider

$$E_{i,\vartheta} = A_i \cap \left\{ \phi_i(0, \vartheta\Lambda) \,|\, \Lambda \in \mathbb{Z}^{b+f} \right\}.$$

Recall that $\phi_i : B(1) \to A_i$ is a diffeomorphism, in particular bijective, thus the set $E_{i,\vartheta}$ consists of finitely many boundary points in $A_i$. This especially means that the family of functions $(\widehat{\varphi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$, as well as for the family $(\widehat{\psi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$, are finite.

**Remark 5.1.5**
*By definition of $\sigma$ we can conclude that there exists an open neighbourhood of the boundary $\partial\overline{M}$ of $\overline{M}$, contained in the collar neighbourhood $U_{\mathfrak{R}}$, so that every point $q$ in such a neighbourhood lies in the support of at most finitely many of the functions $(\widehat{\varphi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$ and $(\widehat{\psi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$.*

The only thing left to get partitions of unity (on an open neighbourhood of the boundary) is to let the families $(\widehat{\varphi}_{i,\overline{p}})$ and $(\widehat{\psi}_{i,\overline{p}})$ to sum up to 1. To this end we only need to "normalize". That is we define, for some $\vartheta \in (0,1)$ and for every $\overline{p} \in E_{i,\vartheta}$, the functions $\varphi_{i,\overline{p}}$ and $\psi_{i,\overline{p}}$ as follows. If $p \in \overline{M}$ does not lie in the support of any of the functions $(\widehat{\varphi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$, respectively $(\widehat{\psi}_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$, we set $\varphi_{i,\overline{p}}(p) = 0$ and $\psi_{i,\overline{p}}(p) = 0$. Otherwise we set

$$\varphi_{i,\overline{p}}(p) := \frac{\widehat{\varphi}_{i,\overline{p}}(p)}{\sum_j \sum_{\overline{p}\in E_{j,\vartheta}} \widehat{\varphi}_{j,\overline{p}}(p)} \quad \text{and} \quad \psi_{i,\overline{p}}(p) := \frac{\widehat{\psi}_{i,\overline{p}}(p)}{\sum_j \sum_{\overline{p}\in E_{j,\vartheta}} \widehat{\psi}_{j,\overline{p}}(p)}. \tag{5.1.7}$$

It is now clear that the family $(\varphi_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$ and $(\psi_{i,\overline{p}})_{i,\overline{p}\in E_{i,\vartheta}}$ are partition of unity on open neighbourhoods of $\partial\overline{M}$. Furthermore, since (5.1.7) holds only for points contained in the support of some of the functions $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$; it follows that properties I to IV in Proposition 5.1.1 hold for the families $(\varphi_{i,\overline{p}})_{i,\overline{p}}$ and $(\psi_{i,\overline{p}})$.

**Remark 5.1.6**
*Notice that the functions $\widehat{\varphi}_{i,\overline{p}}$ and $\widehat{\psi}_{i,\overline{p}}$ are defined in terms of some $\varepsilon \in (0,1)$. Thus the families $(\varphi_{i,\overline{p}})_{i,\overline{p}}$ and $(\psi_{i,\overline{p}})_{i,\overline{p}}$ are partitions of unity for any choice of $\varepsilon$.*

Finally, the function

$$\phi := \sum_i \sum_{\overline{p}\in E_{i,\vartheta}} \varphi_{i,\overline{p}} \tag{5.1.8}$$

is constantly equal to 1 on an open neighbourhood of $\partial\overline{M}$ and satisfies properties I to IV in Proposition 5.1.1 as well.

### Boundary Parametrix

The partitions of unity constructed in §5.1.1 allow us to construct a boundary parametrix for heat type operators $P$ (cf. (5.0.1)).

Let $\gamma \in \mathbb{R}$ and $\alpha \in (0,1)$ be fixed and consider $\ell \in x^\gamma C^\alpha_\Phi(M \times [0,T])$. A parametrix for an heat type operator $P$ is an operator $\mathbf{Q} : x^\gamma C^\alpha_\Phi(M \times [0,T]) \to x^\gamma C^{2,\alpha}_\Phi(M \times [0,T])$ so that $u = \mathbf{Q}\ell$ is a solution for the parabolic Cauchy problem

$$Pu = (\partial_s + a\widetilde{\Delta})u = \ell; \quad u|_{s=0} = 0. \tag{5.1.9}$$

Our first step towards the construction of such an operator $\mathbf{Q}$ is the construction of an operator $\mathcal{Q}_B : x^\gamma C^\alpha_\Phi(M\times[0,T]) \to x^\gamma C^{2,\alpha}_\Phi(M\times[0,T])$ giving rise to approximate solutions of (5.1.9) near the boundary. (The notion of approximate solutions is in the spirit of Lemma 5.1.4 below). Thus the first order of business is to localise (5.1.9) near the boundary. Let us therefore fix some $\overline{p} \in \partial\overline{M}$. As pointed out in Remark 5.1.5 every point on the boundary lies in the support of at most finitely many of the functions defined in (5.1.7). Thus, without loss of generality, we can assume $\overline{p}$ to lie in some $E_{i,\vartheta}$ for some $i$ and some $\vartheta \in (0,1)$, which, from now on, will be considered to be fixed. Next we freeze the coefficient $a$ of the Laplace-Beltrami operator at $s = 0$. In particular we focus our attention to the parabolic Cauchy problem

$$P(\overline{p},0)\overline{u}_{\overline{p}} := (\partial_s + a(\overline{p},0)\widetilde{\Delta})\overline{u}_{\overline{p}} = \varphi_{i,\overline{p}}\ell, \quad \overline{u}_{\overline{p}}|_{t=0} = 0. \tag{5.1.10}$$

Note that the Cauchy problem (5.1.10) is formally different from the Cauchy problem in (5.1.9), not only due to the localisation but especially because the coefficient $a$ of the Laplace-Beltrami operator is now a constant.
By assuming $a$ to be positive and bounded from below away from zero, it is clear that, upon rescaling, the heat kernel operator of $a(\overline{p},0)\widetilde{\Delta}$, denoted by $\mathbf{H}_{\gamma,\overline{p}}$ is the same as the one analysed by [TaVe21] whose mapping properties have been extensively discussed in §4.3. It follows that a solution for (5.1.10) is given by $\mathbf{H}_{\gamma,\overline{p}}(\varphi_{i,\overline{p}}\ell)$. In particular, by defining

$$u_{\overline{p}} = \mathcal{Q}_{\gamma,i,\overline{p}}(\ell) := \psi_{i,\overline{p}}\mathbf{H}_{\gamma,\overline{p}}(\varphi_{i,\overline{p}}\ell), \tag{5.1.11}$$

we have the following.

**Lemma 5.1.4**
*Let $\alpha < \beta \leq 1$ and assume $a \in C^\beta_\Phi(M \times [0,T])$ to be positive and bounded from below away from zero. Then, for every $\ell \in C^\alpha_\Phi(M\times[0,T])$, the function $u_{\overline{p}} = \mathcal{Q}_{\gamma,i,\overline{p}}$, defined in (5.1.11), satisfies*

$$Pu_{\overline{p}} := (\partial_s + a\widetilde{\Delta})u_{\overline{p}} = \varphi_{i,\overline{p}}\ell + R^1_{i,\overline{p}}\ell + R^2_{i,\overline{p}}\ell \tag{5.1.12}$$

*where*

*a) $R^1_{i,\overline{p}} : x^\gamma C^\alpha_\Phi(M\times[0,T]) \to x^\gamma C^\alpha_\Phi(M\times[0,T])$ is a bounded operator. Moreover, there exists some constant $C > 0$ so that*

$$\|R^1_{i,\overline{p}}\ell\|_\alpha \leq C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right)\|\ell\|_\alpha.$$

*b)* $R^2_{i,\overline{p}} : x^\gamma C^\alpha_\Phi(M \times [0,T]) \to x^\gamma C^\alpha_\Phi(M \times [0,T])$ *is a bounded operator and its operator norm goes to* $0$ *as* $T \to 0^+$ *i.e.*

$$\lim_{T\to 0^+} \|R^2_{i,\overline{p}}\|_{\mathrm{op}} = 0.$$

***Proof:***
In order to avoid the plethora of indices we will suppress all the indices on $\varphi, \psi$ and the error terms $R^0$ and $R^1$. Following the same computations as in [BaVe14, Lemma 4.3] one gets

$$\begin{aligned} Pu_{\overline{p}} =& \psi\partial_s \mathbf{H}_{\overline{p}}(\varphi\ell) + [a\widetilde{\Delta}, \psi]\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) + \psi a\widetilde{\Delta}\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ =& \psi(\partial_s + a\widetilde{\Delta})\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) + [a\widetilde{\Delta}, \psi]\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ =& \psi\,(\partial_s + a(\overline{p},0))\,\widetilde{\Delta})\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) + \psi\left(a - a(\overline{p},0)\widetilde{\Delta}\right)(\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ & + [a\widetilde{\Delta}, \psi]\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ =& \psi\varphi\ell + \psi\,(a - a(\overline{p},0))\,\widetilde{\Delta}\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) + [a\widetilde{\Delta}, \psi]\,(\mathbf{H}_{\overline{p}}(\varphi\ell)) \\ =& :\psi\varphi\ell + R^1\ell + R^2\ell = \varphi\ell + R^1\ell + R^2\ell; \end{aligned} \tag{5.1.13}$$

where $[a\widetilde{\Delta}, \psi]$ denotes the commutator between the differential operators $a\widetilde{\Delta}$ and the operator multiplication by $\psi$. Note that the fourth equality in (5.1.13) follows from $\mathbf{H}_{\overline{p}}(\varphi f)$ being a solution of the localized Cauchy problem. Moreover, the last equality is a consequence of property I in proposition 5.1.1.
We will estimate the norms of $R^1$ and $R^2$ with $\gamma = 0$; the case for generic $\gamma$ is slightly more involved but it follows along the same lines. Furthermore, the estimates will be performed on $\mathrm{supp}(\psi)$ since the $\alpha$-norm is not effected by such a change.

Let us begin by estimating the $\alpha$-norm of the operator $R^1$ applied to the function $\ell$.

$$\begin{aligned} \|R^1\ell\|_\alpha =& \|R^1\ell\|_\infty + [R^1\ell]_\alpha \\ \leq& \|\psi\|_\infty \|a - a(\overline{p},0)\|_\infty \|\widetilde{\Delta}\mathbf{H}(\varphi\ell)\|_\infty \\ & + [\psi]_\alpha \|a - a(\overline{p},0)\|_\infty \|\widetilde{\Delta}\mathbf{H}(\varphi\ell)\|_\infty \\ & + \|\psi\|_\infty [a - a(\overline{p},0)]_\alpha \|\widetilde{\Delta}\mathbf{H}(\varphi\ell)\|_\infty \\ & + \|\psi\|_\infty \|a - a(\overline{p},0)\|_\infty [\widetilde{\Delta}\mathbf{H}(\varphi\ell)]_\alpha. \end{aligned} \tag{5.1.14}$$

We will estimate each term in (5.1.14) separately. In what follows, unless otherwise specified, we will denote all the uniform constants by $C$.
We begin by estimating the first term in (5.1.14). By assumption $a \in C^\beta_\Phi(M \times [0,T])$ with $\beta > \alpha$. Thus one deduces

$$\|a - a(\overline{p},0)\|_\infty \leq C(\varepsilon^\beta + T^{\beta/2}); \tag{5.1.15}$$

for some constant $C > 0$. From Theorem 4.3 one has boundedness of the operator $\widetilde{\Delta}\mathbf{H} : C^\alpha_\Phi(M \times [0,T]) \to s^{\alpha/2} C^0_\Phi(M \times [0,T])$; thus resulting in the estimate

$$\|\widetilde{\Delta}\,(\mathbf{H}(\varphi\ell))\,\|_\infty \leq C s^{\alpha/2} \|\varphi\ell\|_\infty \leq C T^{\alpha/2} \|\ell\|_\infty \leq C T^{\alpha/2} \|\ell\|_\alpha. \tag{5.1.16}$$

In the above the $C$'s denote different uniform constants; moreover the last inequality follows from the definition of the $\alpha$-norm. Hence the first term in (5.1.14) can be estimated by

$$\|\psi\|_\infty \|a - a(\overline{p}, 0)\|_\infty \|\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)\|_\infty \leq C(\varepsilon^\beta + T^{\beta/2})T^{\alpha/2}\|\ell\|_\alpha. \tag{5.1.17}$$

For the second term in (5.1.14) we use property II in Proposition 5.1.1 paired with (5.1.15) and (5.1.16), resulting in

$$[\psi]_\alpha \|a - a(\overline{p}, 0)\|_\infty \|\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)\|_\infty \leq C\varepsilon^{-\alpha}T^{\alpha/2}(\varepsilon^\beta + T^{\beta/2}). \tag{5.1.18}$$

The third term in (5.1.14) can be estimated by noticing the following. Recall that we are estimating on the supp($\psi$); thus, from property IV in Proposition 5.1.1, for very $p, p'$ lying in the support of $\psi$, $\mathrm{d}_\Phi(p, p') \leq \overline{C}\varepsilon$. By choosing $\varepsilon$ small enough, e.g. $\varepsilon < 1/\overline{C}$, and $T < 1$, which will be consistent for our future applications (cf. Proposition 5.1.2), we have

$$\mathrm{d}_\Phi(p, p')^\beta + |s - s'|^{\beta/2} \leq \mathrm{d}_\Phi(p, p')^\alpha + |s - s'|^{\alpha/2}.$$

This implies, in particular, by recalling that $a \in C^\beta_\Phi(M \times [0, T])$ thus $[a]_\beta \leq C$ for some constant $C > 0$, $[a - a(\overline{p}, 0)]_\alpha = [a]_\alpha \leq C$. Therefore we find

$$\|\psi\|_\infty [a - a(\overline{p}, 0)]_\alpha \|\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)\|_\infty \leq CT^{\alpha/2}\|\ell\|_\alpha. \tag{5.1.19}$$

Finally, in order to estimate the fourth, and last, term in (5.1.14) we use the mapping property discussed in Theorem 4.2 to deduce $\widetilde{\Delta}\mathbf{H} : C^\alpha_\Phi(M \times [0, T]) \to C^\alpha_\Phi(M \times [0, T])$ to be bounded. Thus

$$[\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)]_\alpha \leq \|\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)\|_\alpha \leq C\|\varphi\ell\|_\alpha \leq C\varepsilon^{-\alpha}\|\ell\|_\alpha;$$

which, in turn, implies

$$\|\psi\|_\infty \|a - a(\overline{p}, 0)\|_\infty [\widetilde{\Delta}\left(\mathbf{H}(\varphi\ell)\right)]_\alpha \leq C(\varepsilon^\beta + T^{\beta/2})\varepsilon^{-\alpha}\|\ell\|_\alpha. \tag{5.1.20}$$

Joining (5.1.17)-(5.1.20) together, in view of (5.1.14), we conclude

$$\begin{aligned}\|R^1\ell\|_\alpha \leq& C\left(T^{\alpha/2}(\varepsilon^\beta + T^{\beta/2}) + \varepsilon^{-\alpha}T^{\alpha/2}(\varepsilon^\beta + T^{\beta/2}) + T^{\alpha/2} + \varepsilon^{-\alpha}(\varepsilon^\beta + T^{\beta/2})\right)\|\ell\|_\alpha \\ \leq& C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right)\|\ell\|_\alpha;\end{aligned}$$

where the $C$'s denote different uniform constants. We want to point out that the estimate above holds due to $\varepsilon \leq 1/\overline{C} < 1$ and $0 < \alpha < \beta \leq 1$. The above estimate proves the first part of the statement.

For the second part we argue as follows. By making use of the product rule one sees that for every function $w$ twice differentiable,

$$[a\widetilde{\Delta}, \psi]w = a\widetilde{\Delta}(\psi) \cdot w - 2a\widetilde{g}(\widetilde{\nabla}\psi, \widetilde{\nabla}w).$$

Note that, our choice of $\psi$ implies that all of its derivatives are vanishing near the boundary $\partial\overline{M}$. Thus, by choosing $w = \mathbf{H}_{\overline{p}}(\varphi\ell)$, we are in the assumption of Lemma 5.1.1; hence $R^2 : C^\alpha_\Phi(M \times [0, T]) \to C^\alpha_\Phi(M \times [0, T])$ is a bounded operator with operator norm converging to 0 as $T \to 0^+$. ☐

**Remark 5.1.7**
*We want to point out the main difference between the result presented here and the analogous result for edge manifolds [BaVe14, Lemma 4.3]. In [BaVe14] the authors use the Mean Value Theorem to estimate the supremum norm of the coefficient a of the Laplace-Beltrami operator. This leads to terms which can be estimate against the incomplete edge distance. In particular they reach an estimate of the form*

$$\|a - a(\overline{p}, 0)\|_\infty \leq C(\varepsilon + T^{\alpha/2})$$

*for some positive constant $C$ (cf. [BaVe14, page 21]).*
*In our case an application of the Mean Value Theorem does not lead to something comparable with any of the $\Phi$-distances $\mathrm{d}_\Phi$. Therefore we could assume less regularity, from a differentiability point of view. But, the assumption $a \in C^\alpha_\Phi(M \times [0, T])$ is not enough to guarantee the existence of a boundary parametrix (see Proposition 5.1.2). Indeed one can see that, by assuming $a \in C^\alpha_\Phi(M \times [0, T])$, the estimates performed in the proof of Lemma 5.1.4 lead to*

$$\|R^1\ell\|_\alpha \leq C(T^{\alpha/2}\varepsilon^{-\alpha} + 1)$$

*which, in turn, can not be made less than one thus making it impossible for $R^1$ to have small operator norm.*

By means of the operators $\mathcal{Q}_{\gamma,i,\overline{p}}$ we define the operator

$$\mathcal{Q}_B = \sum_i \sum_{\overline{p} \in E_{i,\vartheta}} \mathcal{Q}_{\gamma,i,\overline{p}},$$

so that, for a given function $\ell$ in $x^\gamma C^\alpha_\Phi(M \times [0, T])$, one has

$$\mathcal{Q}_B\ell = \sum_i \sum_{\overline{p} \in E_{i,\vartheta}} \psi_{i,\overline{p}} \mathbf{H}_{\gamma,\overline{p}}(\varphi_{i,\overline{p}}\ell). \tag{5.1.21}$$

**Proposition 5.1.2**
*For every $0 < \delta < 1$ there exist $\varepsilon$ and $T$ positive and small enough so that*

$$\begin{aligned} \mathcal{Q}_B &: x^\gamma C^\alpha_\Phi(M \times [0, T]) \to x^\gamma C^{2,\alpha}_\Phi(M \times [0, T]), \\ \mathcal{Q}_B &: x^\gamma C^\alpha_\Phi(M \times [0, T]) \to x^\gamma \sqrt{s} C^{1,\alpha}_\Phi(M \times [0, T]) \end{aligned} \tag{5.1.22}$$

*are bounded operators. Moreover, in terms of the function $\phi$ defined in (5.1.8) one has, for every $\ell \in x^\gamma C^\alpha_\Phi(M \times [0, T])$,*

$$(\partial_s + a\widetilde{\Delta})(\mathcal{Q}_B\ell) = \phi\ell + R^1\ell + R^2\ell$$

*with $\|R^1\|_{\mathrm{op}} \leq \delta$ and $\|R^2\|_{\mathrm{op}}$ converging to $0$ as $T$ goes to $0$.*

***Proof:***
The mapping properties in (5.1.22) are a straightforward consequence of the mapping properties of the heat kernel operator $\mathbf{H}$ (cf. Theorem 4.2 and Theorem 4.3) and by noticing that multiplication by $\psi_{i,\overline{p}}$ as well as multiplication by $\varphi_{i,\overline{p}}$ are bounded operators preserving the regularity.

For the second part of the statement we begin by explicitly computing $(\partial_t + a\widetilde{\Delta})(\mathcal{Q}_B\ell)$. Since the sum defining $\mathcal{Q}_B$ in (5.1.21) is finite, by Lemma 5.1.4 we conclude

$$\begin{aligned}(\partial_s + a\widetilde{\Delta})\mathcal{Q}_B\ell =& \sum_i \sum_{\overline{p}\in E_{i,\vartheta}} (\partial_s + a\widetilde{\Delta})\left(\psi_{i,\overline{p}}\mathbf{H}_{\gamma,\overline{p}}(\varphi_{i,\overline{p}}\ell)\right) \\ =& \phi\ell + \sum_i \sum_{\overline{p}\in E_{i,\vartheta}} R^1_{i,\overline{p}}\ell + \sum_i \sum_{\overline{p}\in E_{i,\vartheta}} R^2_{i,\overline{p}}\ell.\end{aligned}$$

For simplicity let us denote $R^j\ell = \sum_i \sum_{\overline{p}\in E_{i,\vartheta}} R^j_{i,\overline{p}}\ell$ for $j = 1, 2$. Lemma 5.1.4 gives

$$\|R^1_{i,\overline{p}}\ell\|_\alpha \leq C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right)\|\ell\|_\alpha.$$

Hence, by letting $\|\ell\|_\alpha \leq 1$ we find that the operator norm of $R^1$ is bounded by

$$\|R^1\|_{\mathrm{op}} \leq C\left(T^{(\alpha+\beta)/2}\varepsilon^{-\alpha} + T^{\alpha/2}\varepsilon^{\beta-\alpha} + \varepsilon^{\beta-\alpha}\right).$$

For a give $0 < \delta < 1$ we can choose $0 < T < 1$ and $\varepsilon < \min\{1, 1/\overline{C}\}$ small enough so that

$$T^{(\alpha+\beta)/2}\varepsilon^{-\alpha},\ T^{\alpha/2}\varepsilon^{\beta-\alpha},\ \varepsilon^{\beta-\alpha} < \frac{\delta}{3C};$$

and $x = \varepsilon$ is a smooth hypersurface. This might be accomplished, for instance, by choosing

$$T^{\alpha/2} \leq \frac{\delta}{3C}\varepsilon^\alpha;\quad \varepsilon^{\beta-\alpha} \leq \frac{\delta}{3C}.$$

In concerns of the operator norm of $R^2$ one argues directly by making use of Lemma 5.1.4. □

### 5.1.2 Construction of the Parametrix

In §5.1.1 we constructed an approximate boundary parametric for an heat type operator $P$. Here we will first construct an approximate parametrix $\mathcal{Q}_I$ for $P$ in the interior $M$ of $\overline{M}$. Next we will see that a combination of $\mathcal{Q}_B$, as in (5.1.21), and $\mathcal{Q}_I$, defined below in (5.1.24), will lead to an approximate parametrix $\mathcal{Q}$ for $P$ on the whole $\overline{M}$. As it is usual in operator theory, we will then get rid of the error, arising from $\mathcal{Q}$ being an approximate parametrix, via von Neumann series resulting in the claimed parametrix $\mathbf{Q}$ for $P$. Finally we will prove short-time existence for non-homogeneous heat type equations on $\Phi$-manifolds, i.e. Theorem 5.1.

Let $0 < \delta < 1$ be fixed and consider $\varepsilon$ and $T$ as in Proposition 5.1.2. From $\varepsilon$ being fixed, it follows that an $\varepsilon$-neighbourhood of $\partial\overline{M}$ is also fixed and the function $\phi$ (defined in (5.1.8)) is identically 1 on this neighbourhood. The idea now is to cut off a neighbourhood of $\partial\overline{M}$ from $\overline{M}$. Let $M_\varepsilon := \{p \in \overline{M} \,|\, x(p) \geq \varepsilon/2\}$. Clearly $M_\varepsilon$ is a compact manifold with boundary; we can therefore consider its double space $\widehat{M}$. Recall that the double space consists of two copies of $M_\varepsilon$ glued along the boundary and, for compact manifolds with boundary, it is a compact manifold without boundary. Note that the double space construction does not lead to a smooth metric on $\widehat{M}$. In order to smooth it up we consider a smoothing of such a metric so that the metric on $\widehat{M}$ and the one on $M$ coincide on $M_{2\varepsilon}$. Moreover, in dealing with $\widehat{M}$ we are working away from the boundary $\partial\overline{M}$ of $\overline{M}$; thus the $\alpha$-Hölder spaces are exactly the classical ones.

We can extend the function $(1-\phi)$ to a function, still denoted by $(1-\phi)$, on $\widehat{M}$ by setting it to be 0 on the second copy of $M_\varepsilon$. Hence $(1-\phi)$ defines, in particular, a smooth cut off function over $M_\varepsilon$ in $\widehat{M}$. Similarly, let $\overline{P}$ denote the uniform parabolic extension of $P|_{M_\varepsilon}$ to $\widehat{M}$. From classical parabolic PDE theory, it is well known that there exists a parametrix $\overline{Q}_I$ for the heat operator $\overline{P}$ so that the maps

$$\begin{aligned} \overline{Q}_I &: C^{k,\alpha}(\overline{Y}\times[0,T]) \to C^{k+2,\alpha}(\overline{Y}\times[0,T]) \\ \overline{Q}_I &: C^{k,\alpha}(\overline{Y}\times[0,T]) \to \sqrt{s}C^{k+1,\alpha}(\overline{Y}\times[0,T]) \end{aligned} \tag{5.1.23}$$

are bounded. The idea is to use such a parametrix $\overline{Q}_I$ and the boundary parametrix constructed above to construct a parametrix $\mathcal{Q}$ for the Cauchy problem (5.1.9).

Note that, for a given function $\widehat{u}\in C^{k,\alpha}(\widehat{M}\times[0,T])$, $\overline{Q}_I\widehat{u}\in\sqrt{s}C^{k+1,\alpha}(\widehat{M}\times[0,T])$. In order to turn $\overline{Q}_I\widehat{u}$ into a function in $C^{k,\alpha}_\Phi(M\times[0,T])$, let us consider a cut off function $\overline{\Psi}$ on $\widehat{M}$ so that $\overline{\Psi}=1$ on $\operatorname{supp}(1-\phi)$. We can now define the operator

$$\mathcal{Q}_I := \overline{\Psi}\overline{Q}_I(1-\phi) \tag{5.1.24}$$

where $\overline{\Psi}$ and $1-\phi$ act by multiplication. As pointed out in the proof of Proposition 5.1.2, multiplication by $\overline{\Psi}$ and $(1-\phi)$ preserve the regularity and are bounded operators. Therefore the operator $\mathcal{Q}_I$

$$\begin{aligned} \mathcal{Q}_I : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) &\xrightarrow{(1-\phi)} C^{k,\alpha}(\widehat{M}\times[0,T]) \xrightarrow{\overline{Q}_I} \\ &\xrightarrow{\overline{Q}_I} \sqrt{s}C^{k+1,\alpha}(\widehat{M}\times[0,T]) \xrightarrow{M(\overline{\Psi})} \sqrt{s}C^{k+1,\alpha}(M_\varepsilon\times[0,T]) \end{aligned}$$

acts continuously. Moreover, since we are working away from the boundary of $\overline{M}$, the spaces $C^{k+1,\alpha}(M_\varepsilon\times[0,T])$ can be identified with the space $x^\gamma C^{k+1,\alpha}_\Phi(M_\varepsilon\times[0,T])$. We can hence conclude that the operator $\mathcal{Q}_I$ mapping

$$\mathcal{Q}_I : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \to x^\gamma\sqrt{s}C^{k+1,\alpha}_\Phi(M\times[0,T])$$

is bounded. We can therefore construct an approximate parametrix $\mathcal{Q}$ for the operator $P$ by setting

$$\mathcal{Q}\ell = \mathcal{Q}_B\ell + \mathcal{Q}_I\ell.$$

In particular, in view of the construction above and Proposition 5.1.2, one sees that

$$\begin{aligned} \mathcal{Q} &: x^\gamma C^{\alpha}_\Phi(M\times[0,T]) \to x^\gamma C^{2,\alpha}_\Phi(M\times[0,T]) \\ \mathcal{Q} &: x^\gamma C^{\alpha}_\Phi(M\times[0,T]) \to x^\gamma\sqrt{s}C^{1,\alpha}_\Phi(M\times[0,T]) \end{aligned} \tag{5.1.25}$$

are bounded.

## 5.2 Linear heat type Cauchy problems

Proposition 5.1.2 shows exactly the approximate nature of the constructed solution for the heat type equation

$$(\partial_s + a\widetilde{\Delta})u = \ell.$$

Nevertheless, if the error terms have small enough norm, as Proposition 5.1.2 also suggests, then one can iterate away the error by means of von Neumann series. With this in mind, a preliminary version of Theorem 5.1 can be proved.

**Proposition 5.2.1**
*Let $0 < \alpha < \beta \leq 1$ and consider $a \in C^{\beta}_{\Phi}(M \times [0,T])$ to be positive and bounded from below away from zero. For $T_0 > 0$ sufficiently small there exists an operator $\mathbf{Q}$ so that the maps*

$$\mathbf{Q} : x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T_0]) \to x^{\gamma} C^{2,\alpha}_{\Phi}(M \times [0,T_0]),$$
$$\mathbf{Q} : x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T_0]) \to x^{\gamma} \sqrt{s} C^{1,\alpha}_{\Phi}(M \times [0,T_0])$$

*are bounded. Moreover, for every function $\ell$ in $x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T])$, $\mathbf{Q}\ell$ is a solution of the inhomogeneous Cauchy problem*

$$(\partial_s + a\widetilde{\Delta})u = \ell, \quad u|_{s=0} = 0. \tag{5.2.1}$$

***Proof:***
Let $\ell$ be a function in $x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T])$ and $\delta \in (0,1)$ small enough, e.g. $\delta < 1/3$. By Proposition 5.1.2 and the construction above one computes

$$(\partial_s + a\widetilde{\Delta})(\mathcal{Q}\ell) = \phi\ell + R^1\ell + R^2\ell + (1-\phi)\ell + R^3\ell;$$

where $R^1$ and $R^2$ are the operators arising from Proposition 5.1.2 while $R^3$ is given by

$$R^3\ell = [a\widetilde{\Delta}, \overline{\psi}]\left(\overline{Q}_I\left((1-\phi)\ell\right)\right).$$

Clearly $R^3 : x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T]) \to x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T])$ is bounded. Furthermore, the operator norm of $R^3$ can be estimated in the same way as it has been done for $R^2$ in Lemma 5.1.4. In particular it follows that, both $\|R^2\|_{\text{op}}$ and $\|R^3\|_{\text{op}}$ converge to 0 as $T$ goes to 0 while $\|R^1\|_{\text{op}} < \delta$. In terms of the chosen $\delta$, for $T_0$ sufficiently small, $R := R^1 + R^2 + R^3$ is such that

$$\|R\|_{\text{op}} \leq \|R^1\|_{\text{op}} + \|R^2\|_{\text{op}} + \|R^3\|_{\text{op}} < 1.$$

It is now clear that $\mathrm{id} + R$ is invertible, with inverse obtained via the von Neumann series of $R$. The claimed right parametrix of $P$ will then be

$$\mathbf{Q} = \mathcal{Q}(\mathrm{id} + R)^{-1}.$$

□

**Corollary 5.2.1**
*Let $a \in C^{\beta}_{\Phi}(M \times [0,T])$ be positive and bounded from below away from zero. For $T_0$ sufficiently small (depending on $(\beta - \alpha)$) there exists an operator* $\mathbf{E}$

$$\mathbf{E} : x^{\gamma} C^{2,\alpha}_{\Phi}(M) \to x^{\gamma} C^{2,\alpha}_{\Phi}(M \times [0,T_0])$$

*so that* $\mathbf{E}$ *is bounded. Moreover, for every $u_0$ in $x^{\gamma} C^{2,\alpha}_{\Phi}(M)$, $u = \mathbf{E} u_0$ is a solution of the homogeneous Cauchy problem*

$$(\partial_s + a\widetilde{\Delta})u = 0, \quad u|_{s=0} = u_0. \tag{5.2.2}$$

***Proof:***
Since $u_0 \in C^{2,\alpha}_{\Phi}(M)$, $a\widetilde{\Delta} u_0$ lies in $C^{\alpha}_{\Phi}(M \times [0,T])$. Using the right parametrix for the inhomogeneous Cauchy problem constructed in Proposition 5.2.1, set

$$\mathbf{E} u_0 = u_0 - \mathbf{Q}(a\widetilde{\Delta} u_0).$$

An easy computation shows that $\mathbf{E} u_0$ solves the homogeneous Cauchy problem. □

Note that, unlike the statement of Theorem 5.1, the last two results gives us a solution only on an interval $[0, T_0]$ which can be different from the initial interval $[0, T]$.

***Proof of Theorem 5.1:***
Consider a function $\ell \in x^{\gamma} C^{\alpha}_{\Phi}(M \times [0,T])$ and the Cauchy problem

$$(\partial_s + a\widetilde{\Delta})u = \ell; \quad u|_{s=0} = 0,$$

From Proposition 5.2.1, we know that the Cauchy problem above admits a solution $u$ lying in $x^{\gamma} C^{2,\alpha}_{\Phi}(M \times [0,T_0])$. If $T_0 \geq T$ the statement is already proved. Let us therefore assume $T_0 < T$.
Our goal is then to prove that $u$ can be extended to a solution on the entire interval $[0, T]$. Consider $0 < \lambda < T_0$ and the Cauchy problem

$$(\partial_t + a\widetilde{\Delta})v_1 = 0; \quad v_1|_{s=0} = u|_{s=T_0-\lambda}. \tag{5.2.3}$$

The parametrix construction (Corollary 5.2.1) ensures the existence of a solution $v_1$ in $x^{\gamma} C^{2,\alpha}_{\Phi}(M \times [0,T_0])$. A change of variables, given by the translation $s \mapsto s + (T_0 - \lambda)$, allows us to define $v_1$ as a function defined on $[T_0 - \lambda, 2T_0 - \lambda]$.
Let us now consider the inhomogeneous Cauchy problem

$$(\partial_s + a\widetilde{\Delta})u_1 = \ell; \quad u_1|_{s=0} = 0. \tag{5.2.4}$$

As before, such a Cauchy problem admits a solution $u_1$ in $x^{\gamma} C^{2,\alpha}_{\Phi}(M \times [0,T_0])$. Moreover, as it has already been done earlier for $v_1$, we can consider $u_1$ as a function defined on $[T_0 - \lambda, 2T_0 - \lambda]$. In particular, it follows that the Cauchy problem

$$(\partial_t + a\widetilde{\Delta})(u_1 + v_1) = \ell; \quad (u_1 + v_1)|_{s=T_0-\lambda} = u(p, T_0 - \lambda),$$

is satisfied by the function $(u_1 + v_1) \in x^\gamma(M \times [T_0 - \lambda, T_0])$. Furthermore, the same Cauchy problem is satisfied by the function $u$ as well; thus Theorem 2.2 implies $u = u_1 + v_1$ on $[T_0 - \lambda, T_0]$. Hence $u$ can be extended beyond $T_0$ by setting

$$\widetilde{u}(p,s) = \begin{cases} u(p,s), & \text{if } 0 \le s \le T_0 \\ (u_1 + v_1)(p,s), & \text{if } T_0 < s \le 2T_0 - \lambda. \end{cases}$$

Now, if $2T_0 - \lambda \ge T$, the result is proved. If not, repeat the process with $\widetilde{u}$ until $nT_0 - n\lambda \ge T$ (which is possible in a finite number of repetitions since $[0,T]$ is compact). Thus we have an extension of $u$ defined on $M \times [0,T]$.

Note that this extension was obtained employing the parametrix construction, i.e. the maps $\mathbf{Q}$ and $\mathbf{E}$. Such maps are bounded, thus the extended map $\mathbf{Q}$ so that $\ell \mapsto \widetilde{u}$ is also bounded. The proof of Corollary 5.2.1 implies that the operator $\mathbf{E}$ can be extended as well, thus completing the proof. □

Although Theorem 5.1 gives already a satisfactory answer to the existence of solutions to different kind of heat type Cauchy problems, for our future applications, the gained regularity is not enough. In view of Corollary 5.1.1 the statement of Theorem 5.1 can be extended as follows.

**Corollary 5.2.2**

*Let $\beta$ be in $(0,1)$ and consider $a \in C^{2,\beta}_\Phi(M \times [0,T])$ to be bounded from below away from zero. The right inverse $\mathbf{Q}$ of the heat type operator $P = \partial_s + a\widetilde{\Delta}$, analysed in Proposition 5.2.1 and subsequently in Theorem 5.1, is a bounded map when considered as acting between the Banach spaces*

$$\mathbf{Q} : x^\gamma C^{2,\alpha}_\Phi(M \times [0,T]) \to C^{4,\alpha}_\Phi(M \times [0,T]). \tag{5.2.5}$$

***Proof:***

These improved mapping properties for the operator $\mathbf{Q}$ can be proven following the same arguments for the proof of Theorem 5.1. The key point is therefore to gain similar estimates as in Lemma 5.1.4. In this case the estimates are much more convoluted and will not be presented here. We point out that the main reasoning here, as well as in the proof of Lemma 5.1.4, is to use appropriate time decay properties of the operator $\widetilde{\Delta}\mathbf{H}$. In particular in view of Theorem 4.2, Theorem 4.3 and Corollary 5.1.1, denoting by $C$ perhaps different uniform constants, one has (cf. [BaVe19, Proof of Corollary 4.7])

$$\begin{aligned} \|\widetilde{\Delta}\mathbf{H}(\varphi\ell)\|_2 &\le CT^{\alpha/2}\|\varphi\ell\|_{2,\alpha} \le CT^{\alpha/2}\varepsilon^{2-\alpha}\|\ell\|_{2,\alpha}; \\ \|\widetilde{\Delta}\mathbf{H}(\varphi\ell)\|_{1,\alpha} &\le CT^{1/2}\|\varphi\ell\|_{2,\alpha} \le CT^{\alpha/2}\varepsilon^{2-\alpha}\|\ell\|_{2,\alpha}; \\ [\widetilde{\Delta}\mathbf{H}(\varphi\ell)]_{2,\alpha} &\le C\|\varphi\ell\|_{2,\alpha} \le C\varepsilon^{2-\alpha}\|\ell\|_{2,\alpha}; \end{aligned}$$

where in the above we have just suppressed the index $i, \overline{p}$ for readability reasons. □

**Remark 5.2.1**

*In view of Remark 5.1.4 and the reasoning above, it is clear that the statement in Theorem 5.1 can be extend to a mapping property*

$$\mathbf{Q} : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^{k+2,\alpha}_\Phi(M \times [0,T])$$

*for every positive $k$ provided that the function $a$ lies in $C^{k,\beta}_\Phi(M \times [0,T])$ for some $\beta \ge \alpha$.*

## 5.3 Non linear heat type Cauchy problems

In §5.2 we proved the existence of solutions for non-homogeneous Cauchy problems with vanishing initial condition (cf. Theorem 5.1, Corollary 5.2.2 and Remark 5.2.1). In the analysis of the mean curvature flow, one deals with non linear heat type Cauchy problems. It is therefore useful to analyse non-linear heat type Cauchy problems in the setting of $\Phi$-manifolds. Although in our future application we will only need the upcoming result only for $k = 2$, in what follows we present it in the spirit of Remark 5.2.1; that is our set-up is any Banach space $C^{k,\alpha}_{\Phi}(M \times [0,T])$.
For $0 < \alpha < \beta \leq 1$ and $a \in C^{k,\beta}_{\Phi}(M \times [0,T])$, we are interested in Cauchy problems of the form

$$(\partial_s + a\widetilde{\Delta})u = F(u), \quad u|_{s=0} = 0, \tag{5.3.1}$$

with the operator $F$ subject to some restrictions. Those are, in particular:

**Assumptions 1**

1. $F : x^{\gamma}C^{k+2,\alpha}_{\Phi}(M \times [0,T]) \to x^{\gamma}C^{k,\alpha}_{\Phi}(M \times [0,T])$.
2. *$F$ can be written as a sum $F = F_1 + F_2$ with*
   *i)* $F_1 : x^{\gamma}C^{k+2,\alpha}_{\Phi} \to x^{\gamma}C^{k+1,\alpha}_{\Phi}(M \times [0,T])$,
   *ii)* $F_2 : x^{\gamma}C^{k+2,\alpha}_{\Phi} \to x^{\gamma}C^{k,\alpha}_{\Phi}(M \times [0,T])$.
3. *For $u, u' \in x^{\gamma}C^{k+2,\alpha}_{\Phi}(M \times [0,T])$ with $k+2, \alpha, \gamma$-norm bounded from above by some $\eta > 0$, i.e. $\|u\|_{k+2,\alpha,\gamma}, \|u'\|_{k+2,\alpha,\gamma} \leq \eta$, there exists some $C_{\eta} > 0$ such that*
   *i)* $\|F_1(u) - F_1(u')\|_{k+1,\alpha,\gamma} \leq C_{\eta}\|u - u'\|_{k+2,\alpha,\gamma}$, $\|F_1(u)\|_{k+1,\alpha,\gamma} \leq C_{\eta}\|u\|_{k+2,\gamma,\alpha}$,
   *ii)* $\|F_2(u) - F_2(u')\|_{k,\alpha,\gamma} \leq C_{\eta} \max\{\|u\|_{k+2,\alpha,\gamma}, \|u'\|_{k+2,\alpha,\gamma}\}\|u - u'\|_{k+2,\alpha,\gamma}$, $\|F_2(u)\|_{k,\alpha,\gamma} \leq C_{\eta}\|u\|^2_{k+2,\alpha,\gamma}$.

Note that, due to Theorem 5.1 (Corollary 5.2.2 or Remark 5.2.1), if there exists some $u^* \in x^{\gamma}C^{;\alpha}_{\Phi}(M \times [0,T])$ so that

$$\mathbf{Q}\left(F(u^*)\right) = u^*,$$

then $u^*$ is a solution of (5.3.1). We have successfully transformed the problem of finding a solution to a non-linear Cauchy problem of the form (5.3.1) into a fixed point problem. Existence of fixed points of maps on Banach spaces is guaranteed by Banach's fixed point Theorem. Thus the proof of the next Theorem will consist on an application of Banach's fixed point Theorem.

**Theorem 5.2**
*Consider the non-linear Cauchy problem*

$$(\partial_s + a\widetilde{\Delta})u = F(u), \quad u|_{t=0} = 0 \tag{5.3.2}$$

*with $a \in C^{k,\beta}_{\Phi}(M \times [0,T])$ positive and bounded from below away from zero for some $1 \geq \beta > \alpha$. Furthermore, let $F$ satisfy the conditions 1, 2 and 3 in Assumption 1. There exists $u^* \in x^{\gamma}C^{k+2,\alpha}_{\Phi}(M \times [0,T'])$ solution of (5.3.2) for some $T'$ sufficiently small.*

***Proof:***
Let $\eta$ and $T$ be positive numbers to be specified later and set

$$Z_{\eta,T} := \left\{ u \in x^\gamma C_\Phi^{k+2,\alpha}(M \times [0,T]) \middle| \, u(\_,0) = 0, \|u\|_{k+2,\alpha,\gamma} \leq \eta \right\}.$$

$Z_{\eta,T}$ is a closed subset of the Banach space $x^\gamma C_\Phi^{k+2,\alpha}(M \times [0,T])$ hence a complete metric space. For $\mathbf{Q}$ defined as in Theorem 5.1, consider the map $\Psi(u) := (\mathbf{Q} \circ F)(u)$. The assumption of $F$ implies that $\Psi$ maps $x^\gamma C_\Phi^{k,\alpha}(M \times [0,T])$ to itself. As mentioned earlier, our aim is to prove $\Psi$ to be a contraction on $Z_{\eta,T}$, for some $\eta$ and $T$ small enough. Due to linearity of $\mathbf{Q}$, it is enough to prove $\Psi_1 = \mathbf{Q} \circ F_1$ and $\Psi_2 = \mathbf{Q} \circ F_2$ to be contractions on $Z_{\eta,T}$. For simplicity let us denote by $\mathbf{C}$ the number

$$\mathbf{C} := \frac{1}{3\|\mathbf{Q}\|_{\text{op}} C_\eta}.$$

Let us first prove that $\Psi_1$ and $\Psi_2$ map $Z_{\eta,T}$ to itself. We begin with $\Psi_1$.
By requiring $T \leq \mathbf{C}^2$ one has, for every $u \in Z_{\eta,T}$,

$$\|\Psi_1(u)\|_{k+2,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\text{op}} \sqrt{T} \|F_1(u)\|_{k+1,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\text{op}} \sqrt{T} C_\eta \|u\|_{k+2,\alpha,\gamma} \leq \frac{\eta}{3}. \quad (5.3.3)$$

In the above the first estimate follows from the second displayed mapping property of the operator $\mathbf{Q}$ in Proposition 5.2.1 while the second follows from the assumption on $F_1$.
For $\Psi_2$ we argue in a similar manner. Let $u$ be an element in $Z_{\eta,T}$. One has that the chain of inequalities

$$\|\Psi_2(u)\|_{k+2,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\text{op}} \|F_2(u)\|_{k,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\text{op}} C_\mu \|u\|^2_{k+2,\alpha,\gamma} \leq \frac{\eta}{3}, \quad (5.3.4)$$

holds by choosing $\eta \leq \mathbf{C}$. Contrarily to the previous case, here the first estimate follows from the first displayed mapping property of $\mathbf{Q}$ in Proposition 5.2.1.
It is then clear that, by choosing $\eta \leq \mathbf{C}$ and $T \leq \mathbf{C}^2$ then (5.3.3) and 5.3.4 are both satisfied, resulting in $\Psi = \Psi_1 + \Psi_2$ mapping $Z_{\eta,T}$ to itself.

The only thing left to prove is $\Psi$ to be a Lipschitz function with Lipschitz constant less than 1. For $\eta$ and $T$ as above one sees that, arguing in the exact same way as before:

$$\|\Psi_1(u) - \Psi_1(u')\|_{k+2,\alpha,\gamma} \leq \|\mathbf{Q}\|_{\text{op}} \sqrt{T} C_\eta \|u - u'\|_{k+2,\alpha,\gamma} \leq \frac{1}{3} \|u - u'\|_{k+2,\alpha,\gamma}$$

and

$$\begin{aligned} \|\Psi_2(u) - \Psi_2(u')\|_{k+2,\alpha,\gamma} &\leq \|\mathbf{Q}\|_{\text{op}} C_\eta \max\{\|u\|_{k+2,\alpha,\gamma}, \|u'\|_{k+2,\alpha,\gamma}\} \|u - u'\|_{k+2,\alpha,\gamma} \\ &\leq \frac{1}{3} \|u - u'\|_{k+2,\alpha,\gamma}. \end{aligned}$$

The above imply, in particular, $\|\Psi(u) - \Psi(u')\|_{k+2,\alpha,\gamma} \leq 2/3 \|u - u'\|_{k+2,\alpha,\gamma}$. Hence $\Psi$ is a Lipschitz function with Lipschitz constant $2/3 < 1$ thus showing $\Psi$ to be a contraction. □

# Part II

# The geometry of the problem

# Chapter 6

# Lorentzian manifolds

Our analysis concerns upon the evolution of hypersurfaces via the prescribed mean curvature flow in a class of Lorentzian manifolds. To give the reader a feeling for such a setting, in this chapter we will briefly introduce the concept of Lorentzian manifolds and we will present some notions regarding submanifolds, in particular hypersurfaces, in such an ambient space.
We will assume the reader familiar with classical Riemannian geometry, therefore we will mainly point out some specific features of Lorentzian manifolds.
We begin by generalising the concept of inner products on real vector spaces (§6.1) which will easily allow to define semi-Riemannian manifolds (which are a generalisation of Lorentzian manifolds) (§6.2). Next we furnish the reader with an idea on orientability in the contest of Lorentzian manifolds. Finally we extend the causal character, presented in §6.2 for tangent vectors, to submanifolds hence, in particular, to hypersurfaces.

We will be following O'Neill's amazing book on semi-Riemannian geometry [O'Ne83] where the reader can find a much more detailed discussion of every aspect treated in this chapter.

## 6.1 Scalar products on real vector spaces

Let $V$ be a fixed real vector space. We will be considering symmetric bilinear forms defined on $V$.

**Definition 6.1** (Def. 17 in [O'Ne83])
*Let $B$ be a symmetric bilinear form on $V$. We say that $B$ is:*

1. *Positive (negative) definite if $v \neq 0$ implies $B(v,v) > 0$ $(B(v,v) < 0)$.*

2. *Positive (negative) semidefinite if for every $v \in V$, $B(v,v) \geq 0$ $(B(v,v) \leq 0)$.*

3. *Non degenerate if $B(v,w) = 0$ for every $w \in V$ implies $v = 0$.*

For a given symmetric bilinear form $B$ on $V$ we define the index as follows.

**Definition 6.2** (Def. 18 in [O'Ne83])
*Let $B$ be a bilinear form on $V$. We say that the index of $B$ is $\eta$ if $\eta$ is the largest*

*integer such that there exists an $\eta$-dimensional subspace $W$ of $V$ so that $B$ restricted to $W \times W$ is negative definite.*

A special class of bilinear forms are scalar products. To preserve consistency with the upcoming chapters, we will denote scalar products with $\overline{g}$.

**Definition 6.3** (Def. 20 in [O'Ne83])
*A symmetric bilinear form $\overline{g}$ on a real vector space $V$ is a scalar product if it is non degenerate.*

It is clear that the notion of scalar product is a generalisation of the notion of inner product (viz. Euclidean product); indeed a scalar product of index 0 is an inner product. We call scalar product of index different from 0 indefinite. In particular we will refer to scalar products of index 1 as Lorentzian products. A real vector spaces equipped with a Lorentzian vector space will be called Lorentzian vector space.
A special feature of indefinite scalar products is the existence of null vectors. That is, for $\overline{g}$ being an indefinite scalar product on a real vector space $V$ there exists some non zero vector $v \in V$ such that $g(v, v) = 0$.

**Example 2** (Minkowski vector spaces)
*We begin by presenting the two-dimensional Minkowski vector space. Let us consider $\mathbb{R}^2$ with bilinear form defined as follows. For $v = (v_0, v_1)$ and $w = (w_0, w_1)$ we set*

$$\overline{g}(v, w) = -v_0 w_0 + v_1 w_1. \tag{6.1.1}$$

*We notice the following. For $v = (v_0, v_1)$*

*i) If $v_0^2 < v_1^2$, $\overline{g}(v, v) > 0$.*

*ii) If $v_0^2 = v_1^2$, $\overline{g}(v, v) = 0$.*

*iii) If $v_0^2 > v_1^2$, $\overline{g}(v, v) < 0$.*

*The above is easily outlined in the following picture.*

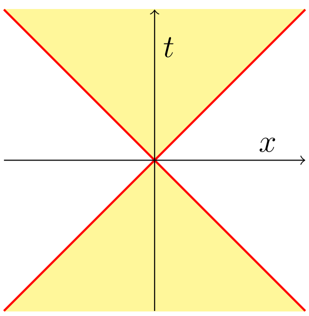


Figure 6.1: Vectors in $(\mathbb{R}^2, \overline{g})$.

*Vectors along the red lines (i.e. the two diagonals at $45^\circ$ degrees) are null vectors. A vector $v$ lying inside the yellow areas satisfy $\overline{g}(v, v) < 0$ while every other vector satisfies $\overline{g}(v, v) > 0$.*

*The scalar product in (6.1.1) can be generalised to dimension $n \geq 2$. Let $v, w$ be vectors in $V$; $v = (v_0, v_1, \ldots, v_{n-1})$ and $w = (w_0, w_1, \ldots, w_{n-1})$. Then*

$$\overline{g}(v,w) = -v_0 w_0 + \sum_{j=1}^{n-1} v_j w_j. \tag{6.1.2}$$

**Convention 1**
*For sake of consistency with other literature, we start the numbering of indices in Lorentzian vector spaces (and, in general, in Lorentzian manifolds) by 0. Thus, when referring to an $n$-dimensional vector space $V$ we mean $n = m + 1$ for some non negative integer $m$.*
*In particular from now on, for $n = m + 1$ being the dimension of a vector space $V$, Greek indices will run in $\{0, \ldots, m\}$ while Latin indices will run in $\{1, \ldots, m\}$.*

Most of the properties of inner products carry over to scalar products. Therefore, as usual, one extends the classical definitions for inner products to scalar products.

**Definition 6.4**
*Let $(V, \overline{g})$ be a vector space equipped with a scalar product. We say that $v$ and $w$ in $V$ are orthogonal if $\overline{g}(v, w) = 0$.*
*Similarly, for $V_1$ and $V_2$ subsets of $V$ we say that $V_1$ and $V_2$ are orthogonal if $\overline{g}(v_1, v_2) = 0$ for every $v_1 \in V_1$ and $v_2 \in V_2$.*

**Remark 6.1.1**
*With example 2 in mind we can easily see that null vectors are orthogonal to themselves.*

Finally we define the orthogonal complement to a subspace.

**Definition 6.5**
*Let $(V, \overline{g})$ be a real vector space equipped with a scalar product. Consider a subspace $W$ of $V$. We denote by $W^\perp$ the set*

$$W^\perp = \{v \in V \,|\, \overline{g}(v, w) = 0 \; \forall \, w \in W\}.$$

**Remark 6.1.2** (Page 49 in [O'Ne83])
*For a given subspace $W$ of a scalar product space $(V, \overline{g})$, in general, we can not refer to $W^\perp$ as the orthogonal complement of $W$; as it is usual for inner product spaces. The reason being that, in general, $W \oplus W^\perp$ may not be the whole $V$. As an example consider the subspace $W = \mathrm{span}\{(1, 1)\}$ of the Minkowski vector space (see example 2). In this case $W^\perp = W$.*

Although the above remark shows up a different behaviour for the "orthogonal complement" in the case of scalar product spaces; we can characterise subspaces which behave nicely, i.e. as expected, once taken the orthogonal complement.

**Definition 6.6**
*Let $(V, \overline{g})$ be a scalar product space. A subspace $W \subset V$ is said to be non degenerate if $\overline{g}\big|_{W \times W}$ is non degenerate.*

**Lemma 6.1.1** (Lemma 2.23 in [O'NE83])
*A subspace $W$ of a scalar product space $(V, \overline{g})$ is non degenerate if and only if $V = W \oplus W^{\perp}$.*

One of the important features of inner product spaces is the existence of orthonormal basis. This property still holds for scalar product spaces.

**Definition 6.7**
*Let $(V, \overline{g})$ be a scalar product space. A vector $v$ in $V$ is said to be unital if $|\overline{g}(v, v)| = 1$.*

**Example 3**
*In the same notation as in example 2 we see that $v = (0, 1)$ is clearly unital, since $\overline{g}(v, v) = 1$. Moreover, the vector $v = (1, 0)$ is unital as well since $\overline{g}(v, v) = -1$.*

**Lemma 6.1.2** (Lemma 2.24 in [O'NE83])
*Let $(V, \overline{g})$ be a non empty scalar product space. Then $V$ admits a basis of orthonormal vectors, i.e. unital vectors which are mutually orthogonal.*

For a given orthonormal basis $(e_\alpha)_{\alpha=0,\dots,m}$ of an $n = m + 1$-dimensional scalar product space $(V, \overline{g})$, the matrix associated to the scalar product in terms of the orthonormal basis is diagonal. Indeed

$$\overline{g}_{\alpha\beta} = \overline{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}\varepsilon_\beta.$$

In the above the Kronecker delta encapsulates the orthogonal nature of the basis, while $\varepsilon_\beta = \overline{g}(e_\beta, e_\beta) = \pm 1$. Usually the collection $(\varepsilon_0, \dots, \varepsilon_m)$ (ordered with the negative signs appearing first) is called "the signature" of $\overline{g}$.

An orthogonal decomposition for vectors can be obtained in scalar product spaces too.

**Proposition 6.1.1** (Lemma 2.25 in [O'NE83])
*Let $(V, \overline{g})$ be an $n = m+1$-dimensional scalar product space and consider $(e_0, \dots, e_m)$ to be an orthonormal basis with $\varepsilon_\alpha = \overline{g}(e_\alpha, e_\alpha)$ for $\alpha \in \{0, \dots, m\}$. For every $v \in V$ there is a unique orthogonal decomposition*

$$v = \sum_{\alpha=0}^{m} \varepsilon_\alpha \overline{g}(v, e_\alpha) e_\alpha. \tag{6.1.3}$$

**Example 4**
*Let $(V, \overline{g})$ be the two-dimensional Minkowski vector space as in example 2. It is easy to see that for $e_0 = (1, 0)$ and $e_1 = (0, 1)$, $\{e_0, e_1\}$ is an orthonormal basis for $V$. In particular for every $v \in V$ we can write*

$$v = -\overline{g}(v, e_0)e_0 + \overline{g}(v, e_1)e_1.$$

Finally we want to recall that the signature and the index of a scalar product are intimately related.

**Lemma 6.1.3** (Lemma 2.26 in [O'NE83])
*Let $(V, \overline{g})$ be an $n = m + 1$-dimensional scalar product space and $(e_0, \dots, e_m)$ an orthonormal basis. The number of negative signs in the signature $(\varepsilon_0, \dots, \varepsilon_m)$ of $\overline{g}$ is the index of $\overline{g}$.*

## 6.2 Semi-Riemannian and Lorentzian manifolds

We begin by recalling and "generalising" a bit the definition of a metric tensor on a manifold.

**Definition 6.8**
*Let $N$ be a smooth manifold. A metric tensor $\overline{g}$ over $N$ is a symmetric non degenerate $(0,2)$-tensor field of constant index.*

In particular, a metric tensor is a smooth map $\overline{g} : N \to \mathrm{Sym}^2(T^*N)$ so that for every $p \in N$, $\overline{g}(p) = \overline{g}_p$ with $\overline{g}_p$ being a scalar product on $T_pN$.

**Definition 6.9**
*A smooth manifold $N$ equipped with a metric tensor $\overline{g}$ is called a semi-Riemannian manifold (or pseudo Riemannian manifold).*
*In particular, if the index is $1$ and $N$ is of dimension $n \geq 2$, $(N, \overline{g})$ will be called a Lorentzian manifold.*

**Remark 6.2.1**
*Semi-Riemannian manifolds generalise the concept of Riemannian manifolds; indeed if the index is $0$ then the resulting semi-Riemannian manifold is a classical Riemannian manifold.*

The convention introduced in Convention 1 carries over in the case of semi-Riemannian manifolds as well.

**Convention 2**
*We denote by $n$ the dimension of a Lorentzian manifold. In particular $n = m + 1$ with $m$ being a non-negative integer. Greek indices will run in $\{0, \ldots, m\}$ while Latin indices will run in $\{1, \ldots, m\}$. Finally, the $0$-th coordinate $x^0$ will be generally denoted by $t$.*

As usual we will use lower indices to denote the component of the metric tensor at each point $p \in N$. In particular for local coordinates $x^0, \ldots, x^m$ on $N$ we write

$$\overline{g}_{\alpha\beta} = \overline{g}\left(\partial_\alpha, \partial_\beta\right) \quad \text{for } \alpha, \beta = 0, \ldots, m.$$

Notice that, in the above, we have used a short cut notation $\partial_\alpha$ for $\partial/\partial x^\alpha$.
Due to non degeneracy, we can consider the inverse of $\overline{g}$. As usual we denote the component of the inverse with upper indices, namely

$$\overline{g}^{\alpha\beta} = (\overline{g}^{-1})_{\alpha\beta}.$$

From Lemma 6.1.2 and Lemma 6.1.3 we deduce the following.

**Corollary 6.2.1**
*Let $(N, \overline{g})$ be an $n = m+1$-dimensional semi-Riemannian manifold. There exists an orthonormal frame $(e_\alpha)_\alpha$ for the metric tensor $\overline{g}$, with $\alpha = 0, \ldots, m$. Furthermore, by considering such a frame to be ordered so that the negative signs appear first, the matrix representation of the metric tensor $\overline{g}$ is given by*

$$\overline{g}_{\alpha\beta} = \delta_{\alpha\beta}\varepsilon_\beta$$

*with $\varepsilon_\beta = \overline{g}(e_\beta, e_\beta) = \pm 1$.*

**Example 5** (Minkowski space-time)
*Let $N = \mathbb{R}^n$ with $n \geq 2$. Since for every $p \in N$, $T_pN = \mathbb{R}^n$, the inner product in example 2 (see equation (6.1.2)) gives rise to a metric tensor. The resulting metric tensor $\overline{g}$ has index 1 hence $(N, \overline{g})$ is a Lorentzian manifold usually referred to as (n-dimensional) Minkowski space-time.*

Next we want to point out that one of the main features of having a Lorentzian metric rather than a Riemannian metric, is the possibility to distinguish vectors by their behaviour under the metric $\overline{g}$. By borrowing some notation from gravitational physics, we distinguish the causal character of tangent vectors as follows.

**Definition 6.10** (Definition 3.3 in [O'Ne83])
*Let $(N, \overline{g})$ be a Lorentzian manifold. Let $p$ be a point in $N$ and consider $v \in T_pN$. The tangent vector $v$ is said to be:*

*i) Space-like if $\overline{g}(v, v) > 0$ or $v = 0$.*

*ii) Null or light-like if $\overline{g}(v, v) = 0$ and $v \neq 0$.*

*iii) Time-like if $\overline{g}(v, v) < 0$.*

*The set of light-like vectors in $T_pN$ is called light cone of $T_pN$.*

**Remark 6.2.2**
*The light cone as well as the set of time-like vectors have two connected components.*

**Example 6** (Causal character in 3D Minkowski space-time)
*Let us consider the 3-dimensional Minkowski space-time; i.e. $\mathbb{R}^3$ equipped with the metric tensor locally given by the matrix*

$$\overline{g}_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*For every $p \in \mathbb{R}^3$, one can sketch the causal character of vectors in $T_p\mathbb{R}^3$ as in the following picture.*

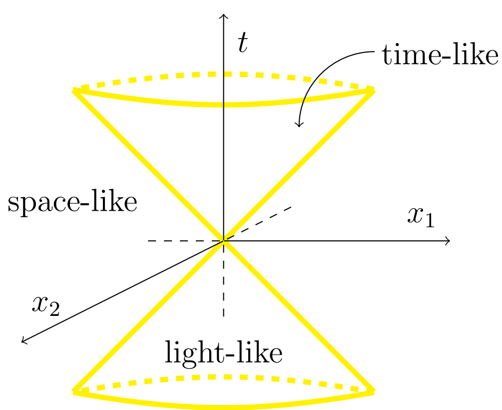


Figure 6.2: Causal characters of vectors in $T_p\mathbb{R}^3$.

*In the above picture vectors lying in the boundary of the two-folded cone (in yellow) are time-like. Instead, vectors falling inside the two-folded cone are time-like while those outside the cone are space-like. It is important to notice that the situation here is just a generalisation of example 2. Indeed, apart from being three dimensional, the result is nothing but example 2 at each point.*

**Remark 6.2.3**
*We would like to point out that, given a Lorentzian manifold $(N, \overline{g})$, for every point $p$ in $N$ the tangent space $T_pN$ looks exactly as the one displayed in Figure 6.2.*

In differential geometry there exists a notion of orientability depending solely on the topological nature of the manifold. Aside from that one, in Lorentzian geometry we can give a notion of time-orientability relying on the Lorentzian structure as well. Indeed, in view of Remark 6.2.3, it is clear that, for every $p$ lying in a Lorentzian manifold $(N, \overline{g})$, every time-like vector in $T_pN$ can fall in either one of the two connected components of the light cone (cf. Figure 6.2). Thus we define time orientability as follows.

**Definition 6.11**
*A Lorentzian manifold $(N, \overline{g})$ is called time orientable if there exists a nowhere vanishing time-like (continuous) vector field $\tau$.*

Time-orientability condition is relevant in general relativity; indeed this is equivalent to say that an "arrow" of time exists. Furthermore this gives also a distinguished notion of past and future. In fact, having a time orientation allows to define future (and past) directed vector fields.

**Definition 6.12**
*Let $(N, \overline{g})$ be a time orientable Lorentzian manifold with nowhere vanishing vector field $\tau$. A time-like vector field $\mu$ is said to be future oriented if*

$$-\overline{g}(\tau, \mu) > 0. \tag{6.2.1}$$

Finally we recall that the product of semi-Riemannian manifolds stays semi-Riemannian.

**Lemma 6.2.1** (Lemma 3.5 in [O'Ne83])
*Let $(N_1, \overline{g}_1)$ and $(N_2, \overline{g}_2)$ be semi-Riemannian manifolds. Denote by $\rho_1$ and $\rho_2$ the projection of $N_1 \times N_2$ onto $N_1$ and $N_2$ respectively and consider $\overline{g} = \rho_1^*(\overline{g}_1) + \rho_2^*(\overline{g}_2)$. Then $(N_1 \times N_2, \overline{g})$ is a semi-Riemannian manifold.*

**Example 7**
*Let $(\overline{M}, \widetilde{g})$ be an $m$-dimensional Riemannian manifold. Considering the following negative definite metric on $\mathbb{R}$ defined by, for every $v, w \in \mathbb{R}$*

$$\langle v, w \rangle = -vw.$$

*Clearly, $(\mathbb{R}, \langle \_, \_ \rangle)$ is a semi-Riemannian manifold. We can therefore consider the product manifold $(N, \overline{g})$. $N$ is an $n = m+1$-dimensional smooth manifolds given by $N = \mathbb{R} \times \mathbb{R}$ and $\overline{g}$ is defined as in lemma 6.2.1. It is easy to see that the* *resulting semi-Riemannian manifold is actually a Lorentzian manifold. Notice that, by denoting with $t$ the coordinate on $\mathbb{R}$ and by $(x^1, \dots, x^m)$ the coordinates on $(\overline{M}, \widetilde{g})$ then $\overline{g}$ can be expressed in local coordinates as*

$$\overline{g} = -\,\mathrm{d}\,t^2 + \widetilde{g}_{ij}\,\mathrm{d}\,x^i\,\mathrm{d}\,x^j \tag{6.2.2}$$

*with the obvious summation over repeated indices. Notice the absence of mixed terms $\mathrm{d}\,t\,\mathrm{d}\,x^k$ for $k = 1, \dots, m$.*

*Minkowski space-times described in example 5 fall exactly in this category of Lorentzian manifolds. To see this it is enough to consider $(\overline{M}, \widetilde{g})$ to be the standard Euclidean space.*

## 6.3 Causal characters of hypersurfaces

Lorentzian manifolds, in particular a specific class of them, represent the ambient space where the evolution under the prescribed mean curvature flow will take place. The objects that will be evolving are space-like hypersurfaces. In this section we will therefore briefly introduce some notions regarding submanifolds, in particular hypersurfaces, of Lorentzian manifolds.
For the rest of the chapter we will always assume $(N, \overline{g})$ to be an $n = m + 1$-dimensional Lorentzian manifold. Further, unless otherwise specified, we will assume submanifolds $\overline{M}$ of $N$ to be embedded, i.e. there exists a smooth map $F : \overline{M} \to N$ so that $F$ and its derivative $DF$ are injective.

From definition 6.9 we deduce that the tangent space at each point of a Lorentzian manifold is a Lorentzian vector space. It is clear that the causal character distinction in definition 6.10 can be carried over to a generic Lorentzian vector space. Furthermore, it can be extended to subspaces of a Lorentzian vector space. We refer to [O'Ne83, §5.6] for more details.

**Definition 6.13**
*Let $(V, \overline{g})$ be a Lorentzian vector space and $W \subset V$ a subspace. The scalar product $\overline{g}$ restricted to $W$ can either be degenerate or have index 1 or 0. We distinguish between these three possibilities as follows.*

*i) $W$ is called space-like if $\overline{g}\big|_{W\times W}$ has index 0, i.e. it is positive definite, that is $\overline{g}$ restricted to $W$ is an inner product.*

*ii) $W$ is said to be time-like if $\overline{g}\big|_{W\times W}$ has index 1, i.e. $W$ equipped with the restricted scalar product is again a Lorentzian vector space.*

*iii) $W$ is degenerate or light-like if $\overline{g}\big|_{W\times W}$ is degenerate.*

From [O'Ne83, Lemma 5.26] and the following remarks we can deduce the following.

**Lemma 6.3.1** (Lemma 5.26 in [O'Ne83])
*Let $(V, \overline{g})$ be a Lorentzian vector space and $W \subset V$ a subspace. $W$ is space-like (time-like) if and only if $W^\perp$ is time-like (space-like).*

The causal character distinction in definition 6.13 can be carried over to submanifolds of a Lorentzian manifold $(N, \overline{g})$.

**Definition 6.14**
*Let $(N, \overline{g})$ be a Lorentzian manifold, $F : \overline{M} \to N$ be a submanifold and denote by $g$ the pull-back metric on $M$, i.e $g = F^*\overline{g}$.*

*i) $(\overline{M}, g)$ is space-like if $g$ is a Riemannian metric.*

*ii) $(\overline{M}, g)$ is time-like if $g$ is a Lorentzian metric.*

*iii) $(\overline{M}, g)$ is light-like if $g$ is degenerate.*

**Remark 6.3.1**
*Notice that the conditions in definition 6.14 are equivalent to require $DF(T_pM)$ to have the same causal character in $T_{F(p)}N$ for every $p \in M$ (cf. [O'Ne83, Page 142]).*

At the beginning of this chapter we used the word hypersurface; therefore here we fix this notation once and for all.

**Definition 6.15**
*Let $(N, \overline{g})$ be an $n = m + 1$-dimensional Lorentzian manifold. We say that $\overline{M}$ is an hypersurface if it is a codimension 1 submanifold.*

Lemma 6.3.1 for hypersurfaces takes the form: $(\overline{M}, g)$ is space-like if and only if the future directed (cf. Definition 6.12) unit normal $\mu$ is time-like. If $(N, \overline{g})$ is a time-orientable Lorentzian manifolds, the space-like condition for hypersurfaces can be codified by means of the gradient function.

**Definition 6.16**
*Let $(\overline{M}, g)$ be a space-like hypersurface of a time orientable Lorentzian manifold $(N, \overline{g})$. Denote by $\mu$ the future directed unit normal to $\overline{M}$ and by $\tau$ a nowhere vanishing time-like vector field (the existence of which is guaranteed by $(N, \overline{g})$ being time orientable. The gradient function $v$ is then defined by*

$$v := -\overline{g}(\tau, \mu). \tag{6.3.1}$$

# Chapter 7

# Intrinsic and extrinsic geometry

The geometry of Lorentzian manifolds is described via the same tensors used to describe the geometry of Riemannian manifolds. Thus the aim of this section is to recall some of those tensors.
We want to furnish the reader with a clear and precise understanding of the objects involved in the analysis of the prescribed mean curvature flow. To this end we will give abstract definitions, namely connection and curvature on vector bundles and pull-back bundles. While presenting such abstract concepts we will give, presented as examples, their "to work on" counterpart.

## 7.1 Connections and curvature

We begin by recalling the notion of connection on vector bundles. In this section we will always consider $\pi : E \to N$ to be a vector bundle over a smooth (Lorentzian when necessary) manifold. Moreover, we will denote by $\Gamma(E)$ the space of sections of the bundle $E$.

**Definition 7.1**
*A connection $D$ on $N$ is a map*

$$D : \Gamma(TN) \otimes \Gamma(E) \to \Gamma(E), \quad D_X S := D(X, S)\, \forall\, X \in \Gamma(TN)\, S \in \Gamma(E),$$

*that is $C^\infty(N)$ linear in $X$ and a derivation in $S$. Here by derivation we mean that, for every $\phi \in C^\infty(N)$, $D_X(\phi S) = X(\phi) \cdot S + \phi \cdot D_X(S)$.*

**Example 8**
*Let $(N, \overline{g})$ be a Lorentzian manifold and $E$ its tangent bundle $E = TN$. Then there is a unique connection, denoted by $\overline{\nabla}$ such that for every $X, Y$ and $Z$ vector fields over $N$,*

$$\begin{aligned} (i) \quad & X\left(\overline{g}(Y, Z)\right) = \overline{g}\left(\overline{\nabla}_X Y, Z\right) + \overline{g}\left(Y, \overline{\nabla}_X Z\right); \\ (ii) \quad & [X, Y] = \overline{\nabla}_X Y - \overline{\nabla}_Y X. \end{aligned} \tag{7.1.1}$$

*In the $(ii)$ above $[X, Y]$ denotes the Lie-bracket of the vector fields $X$ and $Y$.*

**Remark 7.1.1**
*Condition $(i)$ in (7.1.1) can be generalised to any vector bundle $E$ over $N$ equipped*

*with a scalar product. In particular, if a connection on a vector bundle $E$ over $N$ equipped with a scalar product satisfies $(i)$ we will say that the connection is metric. Also we will refer to $(i)$ as "the metric property".*
*We want to point out that example 8 is actually an important result in geometry, usually referred to as fundamental theorem of (semi)-Riemannian geometry. Compare with [O'Ne83, Theorem 3.11].*

Let now $q$ be a fixed point in $N$ and consider $U$ to be an open set containing $q$.

**Definition 7.2**
*A local frame for the vector bundle $E$ is a set of section $(\sigma_1, \dots, \sigma_k)$ of $E$ over $U$ so that, for every $p$ lying in $U$, $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis of $E_p$.*

From now on we will refer to a local frame without specifying the open subset of $N$ over which the (local) section are defined.

**Remark 7.1.2**
*If $S$ is a (local) section for $E$ then $S$ can be written in terms of a local frame $(\sigma_1, \dots, \sigma_k)$ as*

$$S(p) = S^i(p)\sigma_i(p);$$

*with $S^i : U \to \mathbb{R}$.*

For a fixed (local) section $S$ of $E$, the connection $D$ can be thought as a map sending (local) vector fields to (local) sections of $E$. Thus, once a local frame is fixed, $D_X S$ can be expressed in terms of the local frame, for every (local) vector field $X$.

**Definition 7.3**
*Let $(\sigma_1, \dots, \sigma_k)$ be a local frame for $E$. For every (local) vector field $X$ and for every local section $S$ of $E$*

$$(D_X S)_p = (D_X S)^i_p \sigma_i(p).$$

*We define the connection 1-form of the connection $D$ to be*

$$\omega^j_i(X) = (D_X \sigma_i)^j.$$

**Remark 7.1.3**
*Since $\omega^j_i$ is a 1-form over an open subset $U$ of $N$, it can be expressed in local coordinates as*

$$\omega^j_i = \omega^j_{ik} \, \mathrm{d}\, x^k$$

*with $\omega^j_{ik} : U \to \mathbb{R}$.*

In particular one has that, for every (local) section $S = S^k \sigma_k$ of $E$ and for every local vector field $X$

$$D_X S = \left(X(S^k) + S^i \omega^k_i(X)\right) \sigma_k. \tag{7.1.2}$$

**Example 9**
*The connection 1-form for the Levi-Civita connection $\overline{\nabla}$ of a Lorentzian manifold $(N, \overline{g})$ is locally given by the so called Christoffel symbols. Considering $(\partial_0, \dots, \partial_n) =: (\partial_\alpha)_\alpha$ to be a coordinate frame for $TN$. The Christoffel symbols can be expressed*

*entirely in terms of the metric tensor $\overline{g}$ and its derivatives by the formula (compare [O'Ne83, Proposition 3.13 (2)])*

$$\overline{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2}\overline{g}^{\gamma\delta}\left(\overline{g}_{\alpha\delta,\beta} + \overline{g}_{\beta\delta,\alpha} - \overline{g}_{\alpha\beta,\delta}\right). \tag{7.1.3}$$

*In the above $\overline{g}_{\alpha\beta}$ denotes the $(\alpha,\beta)$ component of the metric tensor $\overline{g}$, $\overline{g}^{\alpha\beta}$ its inverse, and we used the comma to denote the partial derivative, i.e.*

$$\overline{g}_{\alpha\delta,\beta} = \frac{\partial}{\partial x^{\beta}}\overline{g}_{\alpha\delta}.$$

*In particular one writes*

$$\overline{\nabla}_{\partial_\alpha}\partial_\beta = \overline{\Gamma}^{\gamma}_{\alpha\beta}\partial_\gamma. \tag{7.1.4}$$

Given a connection on a vector bundle one defines the curvature as follows.

**Definition 7.4**
*Let $\pi : E \to N$ be a vector bundle equipped with a connection $D$. The curvature of $D$ is a map $R^D : \Gamma(TN)\otimes\Gamma(TN)\otimes\Gamma(E) \to \Gamma(E)$ defined by, for every vector fields $X$ and $Y$ and for every section $S$ of $E$*

$$R^D(X,Y)S = D_X D_Y S - D_Y D_X S - D_{[X,Y]}S. \tag{7.1.5}$$

It is easy to see that, by making use of (7.1.2), for a fixed local frame $(\sigma_1,\ldots,\sigma_k)$ of $E$, for every (local) vector fields $X$ and $Y$ and for every local section $S = S^i\sigma_i$ of $E$

$$R^D(X,Y)S = S^i(\mathrm{d}\,\omega_i^k + [\omega,\omega]_i^k)(X,Y)\sigma_k. \tag{7.1.6}$$

In the above $\mathrm{d}\,\omega_i^k$ denotes the exterior derivative of the connection 1-form, while

$$[\omega,\omega]_i^k(X,Y) := \omega_j^k(X)\omega_i^j(Y) - \omega_j^k(Y)\omega_i^j(X).$$

**Example 10**
*Let $\overline{\nabla}$ be the Levi-Civita connection of a Lorentzian manifold $(N,\overline{g})$. Usually one refers to the curvature of the Levi-Civita connection as the Riemann curvature. The Riemann curvature $R^N$ is a $(1,3)$-tensor*

$$R^N : \Gamma(TN)\times\Gamma(TN)\times\Gamma(TN) \to \Gamma(TN).$$

*In terms of any coordinate frame $(\partial_\alpha)_\alpha$ for $TN$, the components*

$$R^N_{\alpha\beta\gamma}{}^{\delta}\partial_\delta = R^N(\partial_\alpha,\partial_\beta)\partial_\gamma. \tag{7.1.7}$$

*can be expressed in terms of Christoffel symbols, in (7.1.3), by*

$$R^N_{\alpha\beta\gamma}{}^{\delta} = \overline{\Gamma}^{\delta}_{\beta\gamma,\alpha} - \overline{\Gamma}^{\delta}_{\alpha\gamma,\beta} + \sum_\eta \overline{\Gamma}^{\eta}_{\beta\gamma}\overline{\Gamma}^{\delta}_{\alpha\eta} - \overline{\Gamma}^{\eta}_{\alpha\gamma}\overline{\Gamma}^{\delta}_{\beta\eta}. \tag{7.1.8}$$

## 7.2 Intrinsic geometry

Examples 8, 9 and 10 give the basic tools to describe the intrinsic geometry of Lorentzian manifolds.
Let then $(N, \overline{g})$ be a Lorentzian manifold and consider $E = TN$ equipped with the Levi-Civita connection $\overline{\nabla}$. The presence of a metric tensor allows us to consider pairing and contractions.

By making use of the metric tensor we can contract indices, from the Riemann curvature $R^N$ defined in example 10, gaining a $(0,4)$-tensor. Such a $(0,4)$-tensor will be denoted by $R^N$ as well and is given by

$$\begin{aligned} R^N &: \Gamma(TN) \times \Gamma(TN) \times \Gamma(TN) \times \Gamma(TN) \to C^\infty(N), \\ &R^N(X, Y, Z, W) := \overline{g}(R^N(X, Y)Z, W), \\ &R^N_{\alpha\beta\gamma\delta} := R^N(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta) = \overline{g}_{\delta\zeta} R^N_{\alpha\beta\gamma}{}^{\zeta}. \end{aligned} \tag{7.2.1}$$

We proceed by presenting, without proving, the basic symmetries of the Riemann $(0,4)$ curvature tensor. For more details see [Lee18, Proposition 7.12].

**Proposition 7.2.1**
*Let $(N, \overline{g})$ be a Lorentzian manifold. For every $X, Y, Z, W \in \Gamma(TN)$ the following equalities hold.*

*(i)* $R^N(X, Y, Z, W) = -R^N(Y, X, Z, W)$.

*(ii)* $R^N(X, Y, Z, W) = -R^N(X, Y, W, Z)$.

*(iii)* $R^N(X, Y, Z, W) + R^N(Y, Z, X, W) + R^N(Z, X, Y, W) = 0$.

*(iv)* $R^N(X, Y, Z, W) = R^N(Z, W, X, Y)$.

By contracting the $(0,4)$-Riemann tensor we gain another tensor called the Ricci tensor.

**Definition 7.5**
*Let $(N, \overline{g})$ be a Lorentzian manifold. The Ricci tensor is defined to be the contraction on the first and the fourth entry of the Riemann curvature tensor; that is*

$$\mathrm{Ric}^N(Y, Z) = \mathrm{trace}(X \mapsto R^N(X, Y)Z).$$

In local coordinates $(x^\alpha)$ on $N$ one has that the component of the Ricci tensor are given by

$$\mathrm{Ric}^N_{\beta\gamma} = R^N_{\alpha\beta\gamma}{}^{\alpha} = \overline{g}^{\alpha\delta} R^N_{\alpha\beta\gamma\delta}. \tag{7.2.2}$$

In particular this means that, given a local orthonormal frame $(e_\alpha)$ for $TN$,

$$\mathrm{Ric}^N(Y, Z) = R^N(e_\alpha, Y, Z, e_\alpha).$$

Finally we recall how to covariantly differentiate a tensor field.

**Definition 7.6**
*Let $(N, \overline{g})$ be a Lorentzian manifold and let denote by $\overline{\nabla}$ the associated Levi-Civita connection. For simplicity we will denote also by $\overline{\nabla}$ the connection induced on the cotangent bundle as well as on the tensor bundle. Let $T$ be an $(s, r)$ tensor, then $\overline{\nabla} T$ is an $(s, r+1)$ tensor defined by, for every $X, X_1, \ldots, X_r \in \Gamma(TN)$ and for every $\alpha_1, \ldots, \alpha_s \in \Gamma(T^*N)$,*

$$\begin{aligned} &\overline{\nabla} T(X, X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s) = \overline{\nabla}_X(T(X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s)) \\ &- T(\overline{\nabla}_X X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s) - \cdots - T(X_1, \ldots, \overline{\nabla}_X X_r, \alpha_1, \ldots, \alpha_s) \\ &- T(X_1, \ldots, X_r, \overline{\nabla}_X \alpha_1, \ldots, \alpha_s) - \cdots - T(X_1, \ldots, X_r, \alpha_1, \ldots, \overline{\nabla}_X \alpha_s). \end{aligned} \tag{7.2.3}$$

## 7.3 Pull-back bundle

In order to make the content of §7.4 more formal and precise, as in §7.1 we will present some abstract notions. In particular, here we will briefly describe how a vector bundle $E$ over $N$ can be pulled-back via a smooth map $F$.
Recall that, contrarily to §6.3, here $F : \overline{M} \to N$, is merely a smooth map.

**Definition 7.7**
*Let $\pi : E \to N$ be a vector bundle, $\overline{M}$ a smooth manifold and $F : \overline{M} \to N$ a smooth map. For every $p$ in $\overline{M}$ we defined*

$$(F^*E)_p = \{(p, s) \in \overline{M} \times E \,|\, \pi(s) = F(p)\}.$$

*We define the pull-back bundle of $E$ via $F$ as the disjoint union of the vector spaces defined above; i.e.*

$$F^*E = \bigcup_{p \in M} (F^*E)_p.$$

**Remark 7.3.1**
*Any local frame $(\sigma_i)_i$ for $E$ gives rise to a local frame for $F^*E$ given by its restriction to the image of $F$; that is $(\sigma_i \circ F)_i$.*

Whenever the vector bundle $\pi : E \to N$ is equipped with a connection $D$, then the connection can also be pulled back to the pull-back bundle $F^*E$. Indeed, let $\omega_i^j$ denote the connection 1-form associated to the connection $D$. For every (local) vector field $X$ over $\overline{M}$, one considers a one form

$${\omega^*}_i^j(X) := (F^* \omega_i^j)(X). \tag{7.3.1}$$

It is a classical exercise to prove that ${\omega^*}_i^j$ gives rise to a connection on $F^*E$. We will denote the resulting connection by $D^{F^*E}$.

Finally we discuss the curvature associated to the pull-back connection $D^{F^*E}$. Let $(\sigma_1, \ldots, \sigma_k)$ be a local frame for the vector bundle $E$. Clearly $(\sigma_1 \circ F, \ldots, \sigma_k \circ F)$ is a local frame for $F^*E$. Let us further denote by $\omega_i^k$ the connection 1-form associated to $D$. Thus, from (7.1.6) we infer, for every local vector fields $X$ and $Y$ over $M$ and for every local section $S$ of $F^*E$

$$R^{D^{F^*E}}(X, Y)S = S^i F^*(\mathrm{d}\,\omega_i^k + [\omega, \omega]_i^k)(X, Y)\sigma_k \circ F.$$

In particular one has

$$D_X^{F^*E} D_Y^{F^*E} S - D_Y^{F^*E} D_X^{F^*E} S - D_{[X,Y]}^{F^*E} S = R^D(DF(X), DF(Y))S. \tag{7.3.2}$$

**Example 11**
*Let $E = TN$ and consider a smooth map $F : \overline{M} \to N$. We can therefore pull-back the Levi-Civita connection and we will denote it by $\nabla^{F^*TN}$.*

*First of all notice that the push-forward $DF$ maps tangent vectors over $\overline{M}$ to sections of the pull-back bundle $F^*TN$. Therefore, an example of sections of the pull-back bundle $F^*TN$ are push forwards of vector fields over $\overline{M}$.*
*Another example of sections of $F^*TN$ is given by the restriction onto the image of $F$ of vector fields over $N$.*

*Let $(\partial_\alpha)_\alpha$ be a coordinate frame for $TN$. As stated in remark 7.3.1, the restriction of $(\partial_\alpha)_\alpha$ to the image of $F$ gives a local frame for $F^*TN$. We will always denote this frame again with $(\partial_\alpha)_\alpha$. By looking at the expression for the connection 1-form (7.3.1) and keeping in mind that a connection is a derivation, cf. definition 7.1, one sees that, for every $X \in \Gamma(T\overline{M})$ and $Y \in \Gamma(F^*TN)$, $Y = Y^\alpha \partial_\alpha$ with $Y^\alpha \in C^\infty(\overline{M})$,*

$$\nabla_X^{F^*TN} Y = X(Y^\alpha)\partial_\alpha + Y^\alpha \overline{\nabla}_{DF(X)} \partial_\alpha = X(Y^\alpha)\partial_\alpha + Y^\alpha X(F^\beta) \overline{\nabla}_{\partial_\beta} \partial_\alpha. \tag{7.3.3}$$

*Since $TN$ is equipped with the inner product given by $\overline{g}$, the pull-back bundle can also be equipped with an inner product by considering the restriction of $\overline{g}$. With a slight abuse of notation, we denote the inner product on $F^*TN$ again with $\overline{g}$. For every $p \in \overline{M}$ and for every $V, W \in (F^*TN)_p$, $V = (p, v)$ and $W = (p, w)$, we write*

$$\overline{g}_p(V, W) := \overline{g}_{F(p)}(v, w). \tag{7.3.4}$$

*As a consequence of computation in local coordinates we find that $\nabla^{F^*TN}$ satisfies the metric property; meaning that, for every $X \in \Gamma(T\overline{M})$ and for every $Y, Z \in \Gamma(F^*TN)$,*

$$X\left(\overline{g}\left(Y, Z\right)\right) = \overline{g}\left(\nabla_X^{F^*TN} Y, Z\right) + \overline{g}\left(Y, \nabla_X^{F^*TN} Z\right). \tag{7.3.5}$$

*Finally, formula (7.3.2) gives a nice commutativity result for the pull-back connection. For simplicity, we will restate (7.3.2) in terms of the pull-back derivative. Let $X, Y$ and $Z$ be vector fields over $\overline{M}$; then*

$$\begin{aligned} \nabla_X^{F^*TN} \nabla_Y^{F^*TN} DF(Z) - \nabla_Y^{F^*TN} \nabla_X^{F^*TN} DF(Z) - \nabla_{[X,Y]}^{F^*TN} DF(Z) \\ = R^N(DF(X), DF(Y)) DF(Z). \end{aligned} \tag{7.3.6}$$

## 7.4 Extrinsic geometry

Let $(N, \overline{g})$ be an $n = m + 1$-dimensional Lorentzian manifold. In this section we will introduce some geometric quantities describing the geometry of hypersurfaces. Although here we will give definitions for the extrinsic geometric quantities for hypersurfaces, we want to point out that these can be easily generalised to submanifolds of higher codimension, see e.g. [Smo12].

Let $\overline{M}$ be an $m$-dimensional smooth manifolds and consider $F : \overline{M} \to N$ to be an immersion so that $F(\overline{M})$ is an hypersurface in $N$, i.e. a codimension one submanifold of $N$. This means, in particular, that $n = m + 1$.
We begin by defining the normal bundle.

**Definition 7.8**
*Given $F : \overline{M} \to (N, \overline{g})$ to be an immersion, we define the normal bundle $T^{\perp}\overline{M}$ to be the disjoint union, ranging over the points of $\overline{M}$, of the normal space*

$$T_p^{\perp}\overline{M} = \left\{ v \in T_{F(p)}N \,\middle|\, \overline{g}(v, D_pF(w)) = 0 \,\forall\, w \in T_p\overline{M} \right\}.$$

The above implies that pointwise

$$T_{F(p)}N = DF(T_p\overline{M}) \oplus T_p^{\perp}\overline{M}.$$

It is worth noticing that $T_{F(p)}N$ can be identified with the fibre at $p$ of the pull-back bundle $F^*TN$ as defined in Definition 7.7.

**Remark 7.4.1**
*Since $F(\overline{M})$ is an hypersurface, at each point the normal space will be spanned by a unit normal vector field $\mu$.*
*In particular, as described in Example 11, for $(\partial_i)_{i=1,\dots,m}$ being a coordinate frame (or in general any local frame) for $T\overline{M}$ we conclude*

$$(\mu, DF(\partial_1), \dots, DF(\partial_m))$$

*is a frame for the pull-back bundle $F^*TN$.*

From Lemma 6.3.1, looking at Definition 6.14, we deduce the following.

**Corollary 7.4.1**
*Let $(N, \overline{g})$ be a Lorentzian manifold. An hypersurface $(\overline{M}, g)$ is space-like (time-like) if and only if the unit normal $\mu$ is time-like (space-like).*

The main object needed for describing the extrinsic geometry of a submanifold is the second fundamental form.

**Definition 7.9**
*Let $F : \overline{M} \to (N, \overline{g})$ be an immersion and $g = F^*\overline{g}$ the metric induced on $\overline{M}$ by pulling-back $\overline{g}$. Denote by $\nabla$ the Levi-Civita connection on $T\overline{M}$ associated with the induced metric $g$. The second fundamental form is defined as the covariant derivative of the tangential map $DF$. Looking at $DF$ as a vector valued 1-form $DF \in \Gamma(T^*M \otimes F^*(TN))$, its covariant derivative can be evaluated as in Definition 7.6 by using the appropriate induced connections. In particular for every $X, Y \in \Gamma(T\overline{M})$*

$$\mathrm{II}(X, Y) = \nabla DF(X, Y) = \nabla_X(DF)(Y) = \nabla_X^{F^*TN}(DF(Y)) - DF(\nabla_X Y). \quad (7.4.1)$$

Next we collect some well known properties of the second fundamental form without proving them. See [Lee18, Proposition 8.1] for more details.

**Proposition 7.4.1**

*(i) The second fundamental form is symmetric and bilinear.*

*(ii) The value of the second fundamental form at $p \in \overline{M}$ depends solely on the value of the vector fields at $p \in \overline{M}$.*

*(iii) The second fundamental form is normal. That is for every $X, Y, Z \in \Gamma(T\overline{M})$*

$$\overline{g}(\mathrm{II}(X,Y), DF(Z)) = 0. \tag{7.4.2}$$

**Remark 7.4.2**
*From (iii) above, equation (7.4.1) implies $\mathrm{II}(X,Y)$ to be the normal component of $\nabla_X^{F^*TN} DF(Y)$ while $DF(\nabla_X Y)$ to be its tangential component. Moreover, for every $X, Y \in \Gamma(T\overline{M})$, $\mathrm{II}(X,Y)$ lies in the $C^\infty(\overline{M})$-span of $\mu$; where $\mu$ denotes the unit normal to $\overline{M}$ as in remark 7.4.1.*

By means of the second fundamental form one can define all the geometric quantities describing the extrinsic geometry of a submanifold.

**Definition 7.10**
*Let $F : \overline{M} \to (N, \overline{g})$ be an immersion and let $\mu$ denote the unit normal to $\overline{M}$. The scalar second fundamental form is a map $\mathrm{h} : \Gamma(T\overline{M}) \otimes \Gamma(T\overline{M}) \to \mathbb{R}$ so that for every vector fields $X$ and $Y$ over $\overline{M}$*

$$\mathrm{h}(X,Y) = \overline{g}(\mathrm{II}(X,Y), \mu). \tag{7.4.3}$$

**Corollary 7.4.2**
*Let $F : \overline{M} \to (N, \overline{g})$ be a space-like hypersurface. For every $X, Y \in \Gamma(T\overline{M})$ one has*

$$\mathrm{II}(X,Y) = -\,\mathrm{h}(X,Y)\mu. \tag{7.4.4}$$

***Proof:***
As stated in remark 7.4.1, $(\mu, DF(\partial_1), \ldots, DF(\partial_m))$ is a local frame for $F^*TN$, with $\mu$ time-like and orthogonal to $DF(\partial_i)$ for every $i$. Thus we can consider the following orthogonal decomposition for the second fundamental form. Let $X, Y$ be two vector fields on $\overline{M}$; then

$$\mathrm{II}(X,Y) = -\overline{g}(\mathrm{II}(X,Y), \mu)\mu + g^{ij}\overline{g}(\mathrm{II}(X,Y), DF(\partial_i))DF(\partial_j)$$

where $g^{ij}$ denotes the inverse of the induced metric on $\overline{M}$. The result follows from $(iii)$ in proposition 7.4.1. □

**Corollary 7.4.3**
*The value of the scalar second fundamental form at a point $p \in \overline{M}$ depends only on the values of the vector fields at the point $p$.*

***Proof:***
This follows directly from proposition 7.4.1 and the fact that the value of the metric pairing at a point only depends on the value of the vector fields at that point. □

**Definition 7.11**
*Let $F : \overline{M} \to (N, \overline{g})$ be a space-like immersion and let $\mu$ be the time-like future oriented unit normal to $\overline{M}$. For every $X \in \Gamma(T\overline{M})$ we define the shape operator $S : \Gamma(T\overline{M}) \to \Gamma(F^*TN)$ as*

$$SX = -\nabla_X^{F^*TN} \mu. \tag{7.4.5}$$

**Corollary 7.4.4**
*For every $X \in \Gamma(TM)$*

$$\overline{g}(SX, \mu) = 0.$$

***Proof:***
Keeping in mind that $\overline{g}(\mu, \mu) = -1$ then for every $X \in \Gamma(TM)$

$$0 = X(\overline{g}(\mu, \mu)) = 2\overline{g}(\nabla_X^{F^*TN} \mu, \mu);$$

where we have used the metric property of the pull-back connection as in (7.3.5). □

**Corollary 7.4.5**
*For every $X, Y \in \Gamma(T\overline{M})$ the following equality holds*

$$\overline{g}(SX, DF(Y)) = \overline{g}(\mathrm{II}(X, Y), \mu) = \mathrm{h}(X, Y). \tag{7.4.6}$$

***Proof:***
Using the metric property of the pull-back connection (cf. (7.3.5)) and the fact that $\mu$ is orthogonal to $DF(T\overline{M})$, one has

$$0 = X(\overline{g}(\mu, DF(Y))) = \overline{g}(\nabla_X^{F^*TN} \mu, DF(Y)) + \overline{g}(\mu, \nabla_X^{F^*TN} DF(Y)).$$

Moreover, by definition $\mathrm{II}(X, Y)$ is the normal component of $\nabla_X^{F^*TN} DF(Y)$ hence the above reads

$$\overline{g}(SX, DF(Y)) = \overline{g}(\mu, \mathrm{II}(X, Y)).$$

□

**Definition 7.12**
*Let $F : \overline{M} \to (N, \overline{g})$ be an immersion, the mean curvature vector $\vec{H}$ is defined as the trace of the second fundamental form*

$$\vec{H} = \mathrm{trace}\,\mathrm{II}\,.$$

**Definition 7.13**
*Let $F : \overline{M} \to (N, \overline{g})$ be an immersion. The mean curvature $\mathrm{H}$ is defined as a function $\mathrm{H} : \overline{M} \to \mathbb{R}$ so that $\mathrm{H} = \mathrm{trace}\,\mathrm{h}$.*

**Corollary 7.4.6**
*Let $F : \overline{M} \to (N, \overline{g})$ be a spacelike hypersurface.*

$$\vec{H} = -\,\mathrm{H}\,\mu.$$

# Chapter 8

# Geometry of graphs in GRWST

The aim of this work is to analyse the evolution, under the (prescribed) mean curvature flow, of space-like (cf. Definition 6.14) graphical hypersurfaces in a generalised Robertson-Walker space-time (defined below in Definition 8.1).
In this chapter we present the local expression of the geometric quantities which will be relevant for the analysis of the (prescribed) mean curvature flow (introduced in §9.2).

In particular, in §8.1 we give a compendium, far from being complete, of the geometry of generalised Robertson-Walker space-times, i.e. we compute the objects defined in §7.2.
The mean curvature flow describes the evolution of submanifolds. Thus in §8.2 we furnish some formulae for the intrinsic geometry of graphical submanifolds in generalised Robertson-Walker space-times.
In §8.4 we describe instead how graphical hypersurfaces are "bent" in relation to the ambient manifold, that is we derive formulae for the extrinsic quantities defined in §7.4.

## 8.1 Geometry of GRWST

The ambient manifold where the (prescribed) mean curvature flow will take place is a generalised Robertson-Walker space-time whose definition we now state explicitly before continuing in studying its intrinsic geometry.

**Definition 8.1**
*Let $(\overline{M}, \widetilde{g})$ be an $m$-dimensional Riemannian manifold. A generalized Robertson-Walker space-time (GRWST) is an $(m+1)$-dimensional Lorentzian manifold $(N, \overline{g})$ satisfying the following conditions:*

1. *there exists a diffeomorphism $\Phi : \mathbb{R} \times \overline{M} \to N$,*
2. *there exists $f \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that $\overline{g}$ is a warped product, i.e.*

$$\Phi^*\overline{g} = -dt^2 + f(t)^2\widetilde{g}. \tag{8.1.1}$$

In what follows we will always identify $(N, \overline{g})$ with $(\mathbb{R} \times \overline{M}, \Phi^*\overline{g})$. From time to time we will refer to a GRWST as above as "a GRWST having $\overline{M}$ as space-like slice". Moreover, from now on we will always work in the assumption of Definition 8.1 without specifying it.

**Remark 8.1.1**
*In view of Definition 6.11, GRWST's are automatically time-oriented, i.e. admit a nowhere vanishing time-like vector field* $\tau = \partial_t$.

For $(N, \overline{g})$ being a warped product manifold, its geometry, i.e. the geometric quantities in §7.2, can be fully described in terms of the warping function $f : \mathbb{R} \to \mathbb{R}^+$ and of the geometry of the space-like slice $(\overline{M}, \widetilde{g})$. To see this we will compute such geometric quantities in coordinates. We begin by fixing the following coordinate frame on $N$.
Consider local coordinates $(x^1, \dots, x^m)$ on $\overline{M}$. As usual $(\partial_1, \dots, \partial_m)$ will denote the the corresponding local frame on $T\overline{M}$. Identifying $N$ with $\mathbb{R} \times \overline{M}$, local coordinates on $N$ are given by $(t, x^1, \dots, x^m)$ (recall the convention introduced in Convention 2) and a local frame for $TN$ is

$$(\partial_t, \partial_1, \dots, \partial_m).$$

The intrinsic geometry of a Lorentzian (Riemannian) manifold is fully described by the Christoffel symbols (cf. Example 9). From straightforward computations, i.e. explicitly computing (7.1.3) for $\overline{g}$ given by (8.1.1), we find the following.

**Lemma 8.1.1**
*The Christoffel symbols of a generalized Robertson-Walker space-time are given by*

$$\begin{aligned} &\overline{\Gamma}^{\alpha}_{00} = 0, && \overline{\Gamma}^{0}_{ij} = f(t)f'(t)\widetilde{g}_{ij}, \\ &\overline{\Gamma}^{0}_{0i} = 0, && \overline{\Gamma}^{k}_{ij} = \widetilde{\Gamma}^{k}_{ij}, \\ &\overline{\Gamma}^{k}_{0i} = \frac{f(t)f'(t)}{f(t)^2}\delta^k_i, && \end{aligned} \tag{8.1.2}$$

*where* $\delta^k_i$ *is the Kronecker Delta and* $\widetilde{\Gamma}^k_{ij}$ *are the Christoffel symbols of* $(\overline{M}, \widetilde{g})$.

By making use of the Christoffel symbols listed above, we can compute the local expression for the Riemannian curvature tensor on $(N, \overline{g})$ (cf. Example 10). In local coordinates $(t, x^1, \dots, x^m)$ as above, we obtain from (8.1.2), (7.1.8) and (7.2.1) the following list of formulae for $R^N$.

**Lemma 8.1.2**
*For every* $i, j, k \in \{1, \dots, m\}$ *we have for the Riemann curvature tensor*

$$\begin{aligned} R^N(\partial_t, \partial_i, \partial_j, \partial_k) &= R^N_{0ijk} = 0, \\ R^N(\partial_t, \partial_i, \partial_j, \partial_t) &= R^N_{0ij0} = -f(t)f''(t)\widetilde{g}_{ij}. \end{aligned} \tag{8.1.3}$$

Finally, we compute the values of the Ricci tensor $\mathrm{Ric}^N$, defined in Definition 7.5.

**Corollary 8.1.1**
*In view of (7.2.2), for every* $i, j = 1, \dots, m$ *we have for the Ricci curvature tensor*

$$\begin{aligned} &\mathrm{Ric}^N(\partial_t, \partial_t) = -m\frac{f''(t)}{f(t)}, \quad \mathrm{Ric}^N(\partial_t, \partial_i) = 0, \\ &\mathrm{Ric}^N(\partial_i, \partial_j) = \widetilde{\mathrm{Ric}}(\partial_i, \partial_j) + f(t)f''(t)\widetilde{g}_{ij} + (m-1)(f'(t))^2\widetilde{g}_{ij}; \end{aligned} \tag{8.1.4}$$

*where* $\widetilde{\mathrm{Ric}}$ *denotes the Ricci tensor of the metric* $\widetilde{g}$ *on* $\overline{M}$.

## 8.2 Intrinsic geometry of graphical hypersurfaces

Let $(N, \overline{g})$ be a GRWST and consider a map $u : \overline{M} \to \mathbb{R}$. A graphical hypersrufaces in $(N, \overline{g})$ is then the set

$$\mathrm{graph}(u) = \{(u(p), p) |\, p \in \overline{M}\} \subset N.$$

Note that, in particular, the graph of $u$ is the image of $\overline{M}$ via the embedding

$$F : \overline{M} \to N \quad F = u \otimes \mathrm{id} \tag{8.2.1}$$

in $N$. From now on we will not distinguish between $\overline{M}$ and its image under the embedding $F$.

**Remark 8.2.1**
*Here we will focus on a "single" embedding $F$. Later on (cf. §9.2) we will allow $u : \overline{M} \times \mathbb{R}_s \to \mathbb{R}$ ($s$ being the coordinates on $\mathbb{R}$), thus leading to the family of embeddings*

$$F : \overline{M} \times \mathbb{R}_s \to N.$$

*The (prescribed) graphical mean curvature flow will be defined in terms of such a family of embeddings.*
*Nonetheless, all the geometric quantities, which will be derived here and in §8.4, will not change, they will only become $s$-dependent as $u$ is.*

The metric tensor determines all the intrinsic geometry of the manifold; thus the first order of business is to derive an expression for the induced metric $g = F^*\overline{g}$, i.e. the metric induced on $\overline{M}$ via the embedding $F$.

**Convention 3**
*To simplify the otherwise heavy notation we will always denote by $f(u)$ the composition $f \circ u$.*

**Proposition 8.2.1**
*Given a point $p$ in $\overline{M}$, the induced metric $g$ on $\overline{M}$ can be described in a local coordinate neighbourhood of $p$ by*

$$g_{ij} = -u_i u_j + f(u)^2 \widetilde{g}_{ij}. \tag{8.2.2}$$

*The inverse of the metric tensor $g$ on $\overline{M}$ near $p$ has expression*

$$g^{ij} = \frac{1}{f(u)^2}\widetilde{g}^{ij} + \frac{1}{f(u)^2}\frac{\widetilde{g}^{jl}u_l\widetilde{g}^{iq}u_q}{f(u)^2 - |\widetilde{\nabla}u|^2_{\widetilde{g}}} \tag{8.2.3}$$

*where $\widetilde{\nabla}$ denotes the gradient of $u$ with respect to the metric $\widetilde{g}$ and $|\widetilde{\nabla}u|^2_{\widetilde{g}} = \widetilde{g}(\widetilde{\nabla}u, \widetilde{\nabla}u)$.*

***Proof:***
The induced metric $g$ on $\overline{M}$ is obtained by pulling back the metric tensor $\overline{g}$ via the embedding $F$. This leads to

$$g_{ij} = \overline{g}(DF(\partial_i), DF(\partial_j)).$$

Since $F = u \times \mathrm{id}$ we find $DF(\partial_i) = u_i\partial_t + \partial_i$. The result follows from the definition of $\overline{g}$ in (8.1.1).

Direct computation shows that (8.2.3) is indeed an inverse for $g$. □

We emphasize that, the geometry induced on $\overline{M}$ by the embedding $F$ is different from the geometry arising from the metric $\widetilde{g}$. Therefore we fix the following convention

**Convention 4**
*The geometric quantities associated to $g = F^*\overline{g}$ are distinguished from those associated to $\widetilde{g}$ as follows. Those associated to the latter will be indicated by a subscript $\sim$ while the one associated to the former will come without any subscript.*

For instance $\nabla$ and $\widetilde{\nabla}$ denote the gradient (or covariant derivative) on $\overline{M}$ with respect to $g$ and $\widetilde{g}$ respectively. We compute

$$\nabla u = \frac{\widetilde{\nabla} u}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}, \quad |\nabla u|^2_g = \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}, \tag{8.2.4}$$

where $|\cdot|_g$ denotes the pointwise norm with respect to $g$, while $|\cdot|_{\widetilde{g}}$ refers to the pointwise norm with respect to $\widetilde{g}$.

We are now ready to compute the Christoffel symbols of $(\overline{M}, g)$. We begin with the following intermediate equation. From straightforward computations involving (8.2.2) it follows

$$g_{jk,i} = \frac{\partial}{\partial x^i} g_{jk} = 2u_i f(u) f'(u) \widetilde{g}_{jk} - u_{ij} u_k - u_j u_{ik} + f(u)^2 \widetilde{g}_{jk,i}. \tag{8.2.5}$$

It is now easy to compute the Christoffel symbols of the induced metric $g$ on $\overline{M}$.

**Proposition 8.2.2**
*The Christoffel symbols of $(\overline{M}, g)$ are given by*

$$\begin{aligned}\Gamma^k_{ij} =& \widetilde{\Gamma}^k_{ij} + \frac{f(u)f'(u)}{f(u)^2}\left(u_i \delta^k_j + u_j \delta^k_i\right) - \frac{f(u)f'(u)}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \widetilde{g}_{ij} \widetilde{g}^{kq} u_q \\ & - \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\left(u_{ij} - \widetilde{\Gamma}^l_{ij} u_l\right) \widetilde{g}^{kq} u_q + 2 \frac{f(u)f'(u)}{f(u)^2 (f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}})} u_i u_j \widetilde{g}^{kq} u_q\end{aligned} \tag{8.2.6}$$

*where $\widetilde{\Gamma}^k_{ij}$ denotes the Christoffel symbols of $(\overline{M}, \widetilde{g})$.*

***Proof:***
Recall that, as pointed out in Example 9, the Christoffel symbols can be defined by the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right).$$

The result follows by substituting (8.2.5) into the above. In particular:

$$\begin{aligned}\Gamma^k_{ij} =& \frac{1}{2} g^{kl}\big(2u_j f(u) f'(u) \widetilde{g}_{il} - u_{ij} u_l - u_i u_{jl} + f(u)^2 \widetilde{g}_{il,j} \\ & + 2u_i f(u) f'(u) \widetilde{g}_{jl} - u_{ij} u_l - u_j u_{il} + f(u)^2 \widetilde{g}_{jl,i} \\ & - 2u_l f(u) f'(u) \widetilde{g}_{ij} + u_{il} u_j + u_i u_{jl} - f(u)^2 \widetilde{g}_{ij,l}\big).\end{aligned}$$

That is

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}\big(2f(u)f'(u)(u_j\widetilde{g}_{il} + u_i\widetilde{g}_{jl} - u_l\widetilde{g}_{ij}) - 2u_{ij}u_l + f(u)^2(\widetilde{g}_{il,j} + \widetilde{g}_{jl,i} - \widetilde{g}_{ij,l})\big).$$

Substituting $g^{kl}$ with the expression in (8.2.3) one has

$$\begin{aligned}\Gamma_{ij}^k =& \frac{f(u)f'(u)}{f(u)^2}\big(u_i\delta_j^k + u_j\delta_i^k - \widetilde{g}^{kq}u_q\widetilde{g}_{ij}\big)\\ &- \frac{1}{f(u)^2}u_{ij}\widetilde{g}^{kq}u_q + \frac{1}{2}\widetilde{g}^{kl}\big(\widetilde{g}_{il,j} + \widetilde{g}_{jl,i} - \widetilde{g}_{ij,l}\big)\\ &+ \frac{f(u)f'(u)}{f(u)^2(f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}})}\big(2u_iu_j\widetilde{g}^{kq}u_q - |\widetilde{\nabla} u|^2_{\widetilde{g}}\widetilde{g}_{ij}\widetilde{g}^{kq}u_q\big)\\ &+ \frac{1}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\frac{1}{2}\widetilde{g}^{lr}\big(\widetilde{g}_{il,j} + \widetilde{g}_{jl,i} - \widetilde{g}_{ij,l}\big)u_r\widetilde{g}^{kq}u_q\\ &- \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2(f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}})}u_{ij}\widetilde{g}^{kq}u_q.\end{aligned}$$

In the second and the fourth lines one can find the formula defining the Christoffel symbols of $(\overline{M}, \widetilde{g})$. Noticing that

$$1 + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} = \frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}},$$

the above simplifies leading to the desired result. □

## 8.3 Space-like graphs in GRWST and the gradient function

In §8.2 we equipped the graphical hypersurface $\overline{M}$ with a metric $g$. The manifold $(\overline{M}, g)$ lies inside the GRWST $(N, \overline{g})$. Since $(N, \overline{g})$ is a Lorentzian manifold, $(\overline{M}, g)$ has a causal character which falls in one of the three possibilities in Definition 6.14. In this work we will focus our attention to space-like hypersurfaces, that is $g = F^*\overline{g}$ to be a Riemannian metric. In view of Lemma 6.3.1 this is equivalent to require, the unit normal $\mu$ to be time-like. This means that

$$|\mu|^2_{\overline{g}} = \overline{g}(\mu, \mu) = -1 < 0. \tag{8.3.1}$$

Due to $(\overline{M}, g)$ being a graphical hypersurface in a GRWST, condition (8.3.1) can be expressed in terms of the warping function $f$ and the function $u : \overline{M} \to \mathbb{R}$. Here we will firstly present such a condition. Further, we will define an object which will play a key role in the analysis of the space-like condition along the (prescribed) mean curvature flow.

Since $F(\overline{M})$ is an hypersurface in $N$, $T_p\overline{M}^\perp$ (see Definition 7.8) is a vector space of dimension one; thus the normal bundle

$$T^\perp\overline{M} = \bigcup_{p\in\overline{M}} T_p\overline{M}^\perp$$

is a line bundle.

**Remark 8.3.1**
*A section $\widetilde{\mu}$ of $T^{\perp}\overline{M}$ is of the form $\widetilde{\mu} = \widetilde{\mu}^{\alpha}\partial_{\alpha}$ with $\widetilde{\mu}^{\alpha} \in C^{\infty}(\overline{M})$.*

**Proposition 8.3.1**
*The element*

$$\widehat{\mu} = \partial_t + \frac{1}{f(u)^2}\widetilde{\nabla}u \tag{8.3.2}$$

*is a section of $T^{\perp}\overline{M}$.*

***Proof:***
Notice that the vectors $DF(\partial_i) = u_i\partial_t + \partial_i$ are tangent to $F(M)$ at $F(p)$. By looking at (8.1.1) it is clear that $\partial_t + 1/f(u)^2\widetilde{\nabla}u$ is orthogonal to $DF(\partial_i)$. □

From (8.3.2) and the condition $\overline{g}(\widehat{\mu}, \widehat{\mu}) < 0$ we conclude the following characterisation for space-like graphs in GRWST's.

**Corollary 8.3.1**
*Let $(N, \overline{g})$ be a GRWST and consider $g$ to be the graphical metric induced on $\overline{M}$ by means of the function $u : \overline{M} \to \mathbb{R}$. The graphical hypersurface $(\overline{M}, g)$ is space-like if and only if*

$$|\widetilde{\nabla}u|^2_{\widetilde{g}} < f(u)^2. \tag{8.3.3}$$

From now on we will assume $(\overline{M}, g)$ to be a space-like graphical hypersurface. Notice that the element described in (8.3.2) is not unital, thus, in order to obtain an unit normal (denoted by $\mu$) we have to normalise it, i.e. divide it by its $\overline{g}$-norm. Furthermore, since $(\overline{M}, g)$ is a space-like hypersurface, we choose, as a convention, the unit normal $\mu$ to be future oriented (cf. Definition 6.12). In particular we conclude that the (future oriented) unit normal $\mu$ can be locally expressed as

$$\mu = \frac{\widehat{\mu}}{|\widehat{\mu}|_{\overline{g}}} = \frac{\partial_t + \frac{1}{f(u)^2}\widetilde{\nabla}u}{\sqrt{1 - \frac{1}{f(u)^2}|\widetilde{\nabla}u|^2_{\widetilde{g}}}} = \frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla}u|^2}}\Big(\partial_t + \frac{1}{f(u)^2}\widetilde{g}^{ij}u_j\partial_i\Big). \tag{8.3.4}$$

Remark 8.1.1 points out that GRWST are time-orientable; thus, in view of the discussion in §6.3 the space-like condition for graphical hypersrufaces in GRWST can be codified in terms of the gradient function $v$ (cf. Definition 6.16). In particular, for $\mu$ as in (8.3.4), we find

$$v := -\overline{g}(\partial_t, \mu) = \frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla}u|^2}}. \tag{8.3.5}$$

From (8.3.5) and (8.2.4) we conclude the following list of properties for the gradient function.

**Proposition 8.3.2**
*The gradient function $v$ in (8.3.5) satisfies the following properties.*

*(i) the gradient function $v$ and the gradient of $u$ (with respect to the induced metric $g = F^{*}\overline{g}$) are related by*

$$|\nabla u|^2_g = \frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla}u|^2_{\widetilde{g}}} - 1 = v^2 - 1. \tag{8.3.6}$$

*(ii) The gradient function $v$ satisfies $v \geq 1$.*

*(iii) The pointwise $g$-norm of $\nabla u$ is bounded from above by*

$$|\nabla u|_g^2 \leq v^2. \tag{8.3.7}$$

*(iv) The following equality holds*

$$v^2 |\widetilde{\nabla} u|_{\widetilde{g}}^2 = f(u)^2 |\nabla u|_g^2. \tag{8.3.8}$$

## 8.4 Extrinsic geometry of space-like graphs

With the term extrinsic geometry we refer to the geometry of the (space-like graphical) hypersurface $(\overline{M}, g)$ in relation to the ambient manifold $(N, \overline{g})$. In particular, since we are dealing with graphical hypersurfaces in a GRWST, the extrinsic geometry of $(\overline{M}, g)$ can be fully described in terms of the function $u : \overline{M} \to \mathbb{R}$ and of the warping function $f$.

The extrinsic geometry of an embedded submanifold is fully described by the second fundamental form II. In terms of the coordinate frame fixed in §8.1 we find the following. The second fundamental form can be thought as a tensor in $\Gamma(F^*TN \otimes T^*M \otimes T^*M)$; hence it has components

$$\mathrm{II} = \mathrm{II}^0_{ij}\, \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j \otimes \partial_t + \mathrm{II}^k_{ij}\, \mathrm{d}\, x^i \otimes \mathrm{d}\, x^j \otimes \partial_k.$$

In particular one has (cf. [Smo12, Equation (10)])

$$\mathrm{II}^\alpha_{ij} = F^\alpha_{ij} + \overline{\Gamma}^\alpha_{\mu\eta} F^\mu_i F^\eta_j - \Gamma^k_{ij} F^\alpha_k, \tag{8.4.1}$$

where $\overline{\Gamma}^\alpha_{\mu\eta}$ denotes the Christoffel symbols of $(N, \overline{g})$ while $\Gamma^k_{ij}$ the Christoffel symbols of $(\overline{M}, g)$. Moreover, $F^\alpha$ denotes the $\alpha$-th component of the embedding $F = u \times \mathrm{id}$ and the lower indices in e.g. $F^\alpha_{ij}$ denote partial derivatives. For more details we refer the reader to [Smo12].

Substituting now (8.1.2), (8.2.6) and the value of the partial derivatives of $F$ in the above one can prove the following proposition.

**Proposition 8.4.1**
*The coordinate expression of the second fundamental form of the embedded manifold $(\overline{M}, g)$ is*

$$\begin{aligned} \mathrm{II}^0_{ij} &= u_{ij} - \Gamma^k_{ij} u_k + f(u) f'(u) \widetilde{g}_{ij} \\ \mathrm{II}^k_{ij} &= -\Gamma^k_{ij} + \frac{f(u) f'(u)}{f(u)^2} (u_i \delta^k_j + u_j \delta^k_i) + \widetilde{\Gamma}^k_{ij} \end{aligned} \tag{8.4.2}$$

*where $\widetilde{\Gamma}^k_{ij}$ are the Christoffel symbols of $(\overline{M}, \widetilde{g})$.*

By means of (8.4.2) one can find an expression for the mean curvature vector. To this end will be useful to notice the following result, which is a straightforward consequence of proposition 8.2.2 and (8.4.2).

**Proposition 8.4.2**
*The mean curvature vector* $\vec{H}$ *of* $\overline{M}$*, defined in Definition 7.12,* $\vec{H} = \vec{H}^0\partial_t + \vec{H}^k\partial_k$ *has components*

$$\begin{aligned}\vec{H}^0 &= \frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} g^{ij}\left(u_{ij} - \widetilde{\Gamma}^k_{ij}u_k - 2\frac{f(u)f'(u)}{f(u)^2}u_iu_j + f(u)f'(u)\widetilde{g}_{ij}\right),\\ \vec{H}^k &= \frac{(\widetilde{\nabla} u)^k}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} g^{ij}\left(u_{ij} - \widetilde{\Gamma}^l_{ij}u_l - 2\frac{f(u)f'(u)}{f(u)^2}u_iu_j + f(u)f'(u)\widetilde{g}_{ij}\right),\end{aligned} \tag{8.4.3}$$

*where* $(\widetilde{\nabla} u)^k$ *denotes the* $k$*-th component of the gradient of* $u$ *with respect to the metric* $\widetilde{g}$*.*

For our purpose it will also be useful to compute the scalar mean curvature. To this end an expression for the scalar second fundamental form will be derived.

**Proposition 8.4.3**
*The local coordinate expression of the scalar second fundamental form* $\mathrm{h} \in \Gamma(T^*M \otimes T^*M)$ *(as in Definition 7.10) is given by*

$$\mathrm{h}_{ij} = -\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}\left(u_{ij} - \widetilde{\Gamma}^k_{ij}u_k - 2\frac{f(u)f'(u)}{f(u)^2}u_iu_j + f(u)f'(u)\widetilde{g}_{ij}\right). \tag{8.4.4}$$

**Remark 8.4.1**
*By combining* (8.4.2) *and* (8.4.4) *we find, written in terms of the gradient function* $v$*,*

$$v\,\mathrm{h}_{ij} = -(u_{ij} - \Gamma^k_{ij}u_k) - f(u)f'(u)\widetilde{g}_{ij}. \tag{8.4.5}$$

As for the mean curvature vector, from computations in coordinates and (8.4.4) one can get the following.

**Proposition 8.4.4**
*The scalar mean curvature* H *(cf. Definition 7.13) can be expressed by*

$$\mathrm{H} = -\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}} g^{ij}\left(u_{ij} - \widetilde{\Gamma}^k_{ij}u_k - 2\frac{f(u)f'(u)}{f(u)^2}u_iu_j + f(u)f'(u)\widetilde{g}_{ij}\right). \tag{8.4.6}$$

The scalar mean curvature H can be expressed in terms of $\Delta_g$, the Laplace operator on $\overline{M}$ taken with respect to the induced metric $g$.

**Proposition 8.4.5**
*The scalar mean curvature* H *can be expressed as*

$$\mathrm{H} = \frac{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}{f(u)}\Delta_g u - \frac{f'(u)\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}{f(u)^2}\left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}})}\right); \tag{8.4.7}$$

*where* $m = \dim(\overline{M})$*.*
*The above can be written in terms of the gradient function* $v$ *as:*

$$v\,\mathrm{H} = \Delta_g u - \frac{f'(u)}{f(u)}(m + v^2 - 1). \tag{8.4.8}$$

***Proof:***
Computing the coordinate expression of the Laplacian $\Delta_g$ one finds

$$\begin{aligned}\Delta_g u &= -g^{ij}(u_{ij} - \Gamma^k_{ij} u_k) = \\ &= -\frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} g^{ij} \left( u_{ij} - \widetilde{\Gamma}^k_{ij} u_k - 2\frac{f'(u)}{f(u)} u_i u_j + \frac{f'(u)|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)} \widetilde{g}_{ij} \right)\end{aligned} \tag{8.4.9}$$

which follows from the coordinate expression of the Christoffel symbols in (8.2.6). The claimed result follows by substituting the appropriate terms in (8.4.6) into the above.

Equation (8.4.8) can be alternatively proved by taking the $\mathrm{trace}_g$ on both sides of Equation (8.4.5). Then one may use (8.2.3) to compute the product $g^{ij}\widetilde{g}_{ij}$ and finally use (8.2.4) and (i) in Proposition 8.3.2 to conclude the result. □

# Part III

# Analysis of the prescribed mean curvature flow

# Chapter 9

# Prescribed mean curvature flow

Given the analysis and geometry background, presented in Part I and Part II respectively, we can set-up and analyse the prescribed mean curvature flow.
As stated in §1 our aim is to lay down the first steps toward the existence of space-like hypersurfaces of prescribed mean curvature in GRWST's having a Φ-manifold as space-like slice. From now on we will assume $(\overline{M}, \widetilde{g})$ to be a fixed Φ-manifold (cf. §3) and $f : \mathbb{R} \to \mathbb{R}$ to be a fixed warped function for the GRWST $(N, \overline{g})$ having $(\overline{M}, \widetilde{g})$ as space-like slice (cf. §8.1).
The pioneering work [EcHu91] suggests to tackle the existence problem above by means of a parabolic PDE, the prescribed mean curvature flow equation. Our first goal is therefore to present such an equation (§9.1). We will see that it can be interpreted, in the case of graphical hypersurfaces, as an heat type parabolic PDE (as the ones discussed in §5). It will be immediately clear that such a parabolic PDE is non linear; thus, in conclusion, it will be linearised in §9.2.

## 9.1 Prescribed mean curvature equation

The prescribed (graphical) mean curvature flow describes the evolution of embedded graphical hypersurfaces in the direction of the unit normal with velocity given by $\mathcal{H} - \mathrm{H}$, where $\mathcal{H} : \overline{M} \to \mathbb{R}$ is a prescribing function and H denotes the mean curvature.
As mentioned in Remark 8.2.1, we consider, for some $T \in \mathbb{R}$, $u : \overline{M} \times [0, T]_s \to \mathbb{R}$ to be a family of function $u(\_, s) : \overline{M} \to \mathbb{R}$ for every $s \in [0, T]$. Consequently, we consider the family of graphical embeddings

$$F : \overline{M} \times [0, T]_s \to N, \quad F(p, s) = (u(p, s), p)\,. \tag{9.1.1}$$

Note that, in the above, the lower index $s$ denotes the coordinate on $[0, T]$. It will be omitted from now on. Using the same notation as in Part II, we say that a family of embedding $F$ (as in (9.1.1)) is a prescribed (graphical) mean curvature flow if it satisfies the initial value problem

$$\partial_s F(s) = -(\mathrm{H} - \mathcal{H})\mu, \quad F(s = 0) = F_0, \tag{9.1.2}$$

where $\mathcal{H} : \overline{M} \to \mathbb{R}$ is the prescribing function, H is the mean curvature of $F(s)(\overline{M}) \subset N$ and $F_0$ is some initial embedding. Under the mean curvature flow, for every point

$p$ in $\overline{M}$ the normal velocity at which $F(p,s)$ moves is given by the mean curvature of $F(s)(\overline{M})$ at $F(s,p)$ minus $\mathcal{H}$.

**Remark 9.1.1**
*If $\mathcal{H} \equiv 0$, the flow is referred to as the (usual) mean curvature flow.*

**Remark 9.1.2**
*Sometimes we will refer to $\big(\overline{M}, g = g(s)\big)$ as a prescribed mean curvature flow. Here, in the notation of Part II, $g(s) = F(s)^*\overline{g}$ and $F = F(s)$ is a prescribed mean curvature flow in the sense above.*

In §8.4 we have seen that the extrinsic geometry of a graphical hypersurface in a GRWST can be completely described in terms of the function $u$ and the warping function $f$. It is then reasonable to rewrite (9.1.2) with respect to $u$ and $f$. In particular, recall that $F = u \times \mathrm{id}$, thus, by taking the $s$-derivative componentwise we find

$$\partial_s u = -\mu^0(\mathrm{H} - \mathcal{H}) = -(\mathrm{H} - \mathcal{H})v \tag{9.1.3}$$

where $\mu^0$ is the $\partial_t$-component of the normal $\mu$. Substituting now the $\partial_t$-component of the unit normal $\mu$ (cf.equation (8.3.4)) we find

$$\partial_s u = -\,\mathrm{H}\,\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}} + \mathcal{H}\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}. \tag{9.1.4}$$

Notice that (9.1.4) is equivalent to (9.1.2). This means that a family of graphical embedding $F = u \times \mathrm{id}$ is a prescribed mean curvature flow if and only if the family of functions $u : \overline{M} \times [0,T] \to \mathbb{R}$ satisfies (9.1.4).

Although Equation (9.1.4) already expresses Equation (9.1.2) in terms of $u$ and $f$, it is not manifestly parabolic. To see that (9.1.4) is indeed an heat type parabolic PDE for the function $u$ we employ Proposition 8.4.5.

**Proposition 9.1.1**
*Let $(\overline{M}, \widetilde{g})$ be an $m$-dimensional $\Phi$-manifold and $(N, \overline{g})$ a GRWST having $(\overline{M}, \widetilde{g})$ as a space-like slice. A family of functions $u : \overline{M} \times ]0,T] \to \mathbb{R}$ give rise to a prescribed graphical mean curvature flow if they satisfy the following Cauchy problem*

$$\begin{cases} \partial_s u = -\Delta_g u + \frac{f'(u)}{f(u)}\left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}} \\ u(\_,0) = u_0 \end{cases} \tag{9.1.5}$$

*where $\Delta_g$ is the $s$-dependent Laplace-Beltrami operator, i.e. the Laplace-Beltrami operator with respect to the induced metric $g = F^*\overline{g}$ on $\overline{M}$.*

**Remark 9.1.3**
*Equation (9.1.5) can be equivalently expressed in terms of the gradient function $v$ by*

$$(\partial_s + \Delta_g)u = \mathcal{H}v + \frac{f'(u)}{f(u)}(m + v^2 - 1). \tag{9.1.6}$$

## 9.2 The linearised equation

The first order of business concerning the analysis of Equation (9.1.5) is to prove that it admits solutions. Before doing so (cf. Chapter 10), we notice that (9.1.5) is non-linear in $u$. Thus, in order to conclude local (in time) existence of solutions for (9.1.5) we need to linearise it. A suitable linearisation can be obtained as follows. We proceed by writing $u = u_0 + w$, meaning that $u(p, s) = u_0(p) + w(p, s)$ for every $(p, s) \in \overline{M} \times [0, T]$, with $u_0 = u(\_, 0)$, i.e. the initial value of equation (9.1.5). Note that, in particular, $u_0$ gives rise to a space-like graph. Since $\partial_s u = \partial_s w$ we will be able to derive, from (9.1.5), an heat type equation for $w$ where the Laplace-Beltrami operator is taken with respect to the initial induced metric $g_0 := F(\_, 0)^* \overline{g}$.

### 9.2.1 Linearisation of lower order terms

We will proceed by presenting an overview on how to find a linearisation of the terms on the right hand side of equation (9.1.5) aside from the Laplacian of $u$. Unless otherwise specified, the upcoming computations have to be considered at a point $p \in \overline{M}$.

Using Taylor series with Lagrange form of the reminder, we can write

$$f(u) = f(u_0) + f'(u_0)w + Q(w); \;\; Q(w) = \frac{1}{2} f''(\xi) w^2 \tag{9.2.1}$$

where, for each $s \in [0, T]$, $\xi(s)$ is a real number lying between $u_0(p)$ and $w(p, s)$. In a similar manner one finds a linearisation of $f'(u)$. In particular this gives

$$f'(u) = f'(u_0) + f''(u_0)w + Q(w); \;\; Q(w) = \frac{1}{2} f'''(\xi) w^2 \tag{9.2.2}$$

where, again, $\xi(s) \in (u_0(p), w(p, s))$. Other terms like $1/f(u)$ and $f(u)^2$ can be linearised in the same way. Indeed, writing $q(t)$ for some rational function, we find

$$q(f(u)) = q(f(u_0)) + f'(u_0) q'(f(u_0)) w + \frac{1}{2} (q \circ f)''(\xi) w^2. \tag{9.2.3}$$

For simplicity we will write $Q(w)$ for the terms which are at least quadratic in $w$ as we have already done in (9.2.1) and (9.2.2).
By linearity of the metric tensor $\widetilde{g}$ and of the gradient with respect to $\widetilde{g}$, $\widetilde{\nabla}$ we also find

$$|\widetilde{\nabla} u|^2_{\widetilde{g}} = |\widetilde{\nabla} u_0|^2_{\widetilde{g}} + 2\widetilde{g}(\widetilde{\nabla} u_0, \widetilde{\nabla} w) + Q(w, \widetilde{\nabla}\, w); \tag{9.2.4}$$

where $Q(w, \widetilde{\nabla}\, w) \left( = \widetilde{g}(\widetilde{\nabla}\, w, \widetilde{\nabla}\, w) \right)$ denotes terms at least quadratic in $w$ or its $\Phi$-derivatives.
It is therefore clear that from (9.2.3) and (9.2.4) we can linearise any product of $f^{(j)}(u)$, $f^k(u)$ and $|\widetilde{\nabla}\, u|^2_{\widetilde{g}}$ for any integers $j$ and $k$.

Furthermore, by making used of the so far obtained linearisations, one finds

$$f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}} = f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}} + 2 f(u_0) f'(u_0) w - 2\widetilde{g}(\widetilde{\nabla} u_0, \widetilde{\nabla} w) + Q(w, \widetilde{\nabla}\, w); \tag{9.2.5}$$

where, as usual, $Q(w, \widetilde{\nabla}\, w)$ denotes terms which are at least quadratic in either $w$ or its $\Phi$-derivatives with coefficients depending on $u_0$ and its first order $\Phi$-derivatives. Set $x_1 = f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}$ and $x_0 = f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}$. Note that $x_0 > 0$ since, by assumption, $u_0$ gives rise to a space-like graph. We can therefore employ Taylor series with Lagrange form of the reminder once again at $x_0$ to linearise $1/\sqrt{f(u)^2 - |\widetilde{\nabla}\, u|^2_{\widetilde{g}}}$ and $1/f(u)^2 - |\widetilde{\nabla}\, u|^2_{\widetilde{g}}$.

**Proposition 9.2.1**
*The lower order terms in the right hand side of equation (9.1.5) can be linearised as follows.*

$$\begin{aligned}
&\frac{f'(u)}{f(u)}\left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}} = \\
=&\frac{f'(u_0)}{f(u_0)}\left(m + \frac{|\widetilde{\nabla} u_0|^2_{\widetilde{g}}}{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}}\right) + \mathcal{H}\frac{f(u_0)}{\sqrt{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}}} \\
&+ L(w, \widetilde{\nabla}\, w) + Q(w, \widetilde{\nabla}\, w)
\end{aligned} \tag{9.2.6}$$

*where*

$$\begin{aligned}
L(w, \widetilde{\nabla}\, w) =& -m\frac{f'(u_0)^2}{f(u_0)^2}w + m\frac{f''(u_0)}{f(u_0)}w - 2\frac{f'(u_0)^2}{\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}\right)^2}w \\
&- \frac{f'(u_0)^2|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}{f(u_0)^2\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}\right)}w + \frac{f''(u_0)|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}{f(u_0)\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}\right)}w \\
&- \mathcal{H}\frac{f'(u_0)|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}{\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}\right)^{3/2}}w + 2\frac{f(u_0)f'(u_0)}{\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}\right)^2}\widetilde{g}\left(\widetilde{\nabla}\, u_0, \widetilde{\nabla}\, w\right) \\
&+ \mathcal{H}\frac{f(u_0)}{\left(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2\right)^{3/2}}\widetilde{g}\left(\widetilde{\nabla}\, u_0, \widetilde{\nabla}\, w\right);
\end{aligned}$$

*and $Q(w, \widetilde{\nabla}\, w)$ denotes terms at most quadratic either in $w$ or its first order $\Phi$-derivatives with coefficients in $u_0$ and its first order $\Phi$-derivative.*

The only thing left is to linearise the Laplace-Beltrami operator of $u$.

### 9.2.2 Linearisation of the Laplace-Beltrami operator

The linearisation of the Laplace-Beltrami operator is slightly more complicated. For every function $h : \overline{M} \to \mathbb{R}$, in local coordinates, the Laplace operator with respect to the induced metric $g = F^*\overline{g}$ can be written as

$$\begin{aligned}
\Delta_g h &= -g^{ij}\left(h_{ij} - \Gamma^k_{ij}h_k\right) \\
&= \frac{1}{f(u)^2}\Delta_{\widetilde{g}} h + \frac{1}{f(u)^2}\widehat{\Delta} h \\
&+ \widetilde{g}(\widetilde{\nabla}\, u, \widetilde{\nabla}\, h)\frac{1}{f(u)^2(f(u)^2 - |\widetilde{\nabla}\, u\,|^2_{\widetilde{g}})}\Delta_{\widetilde{g}} u \\
&+ \widetilde{g}(\widetilde{\nabla}\, u, \widetilde{\nabla}\, h)\frac{1}{f(u)^2(f(u)^2 - |\widetilde{\nabla}\, u\,|^2_{\widetilde{g}})}\widehat{\Delta} u \\
&- (m-1)\widetilde{g}(\widetilde{\nabla}\, u, \widetilde{\nabla}\, h)\frac{f'(u)}{f(u)(f(u)^2 - |\widetilde{\nabla}\, u\,|^2_{\widetilde{g}})} \\
&+ \widetilde{g}(\widetilde{\nabla}\, u, \widetilde{\nabla}\, h)\frac{f(u)f'(u)}{(f(u)^2 - |\widetilde{\nabla}\, u\,|^2_{\widetilde{g}})^2}
\end{aligned} \tag{9.2.7}$$

where $\widehat{\Delta}$ is an operator acting on functions over $\overline{M}$ locally defined by

$$\widehat{\Delta} h = -\frac{\widetilde{g}^{jl}u_l\widetilde{g}^{im}u_m}{f(u)^2 - |\widetilde{\nabla}\, u|^2_{\widetilde{g}}}\left(h_{ij} - \widetilde{\Gamma}^k_{ij}h_k\right). \tag{9.2.8}$$

To get the expression in (9.2.7) one uses the expression of the inverse of the induced metric tensor $g = F^*\overline{g}$ as in (8.2.3), the expression for the Christoffel symbols of $g$ (cf. proposition (8.2.2)) and the fact that

$$1 + \frac{|\widetilde{\nabla}\, u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla}\, u|^2_{\widetilde{g}}} = \frac{f(u)^2}{f(u)^2 - |\widetilde{\nabla}\, u|^2_{\widetilde{g}}}.$$

It is clear that the operator $\Delta_0$, denoting the Laplace-Beltrami operator with respect to the initial induced metric $g_0 = F(\_, 0)^*\overline{g}$ is obtained by substituting every occurrence of $u$ in (9.2.7) with $u_0$. This implies, in particular,

$$\Delta_0 w = \frac{1}{f(u_0)^2}\left(\Delta_{\widetilde{g}} + \widehat{\Delta}_0\right)w + L(w, \widetilde{\nabla}\, w) + Q(w, \widetilde{\nabla}\, w); \tag{9.2.9}$$

where, as usual, $L(w, \widetilde{\nabla}\, w)$ and $Q(w, \widetilde{\nabla}\, w)$ denote expressions depending respectively at most linearly and at least quadratically on the entries in the brackets with coefficients given in terms of $u_0$, $\widetilde{\nabla}\, u_0$ and $\widetilde{\nabla}^2\, u_0$. Moreover, the operator $\widehat{\Delta}_0$ is the operator $\widehat{\Delta}$ defined in (9.2.8) at time $s = 0$; i.e. for every $h : \overline{M} \to \mathbb{R}$

$$\widehat{\Delta}_0 h = -\frac{\widetilde{g}^{jl}u_{0l}\widetilde{g}^{im}u_{0m}}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}\left(h_{ij} - \widetilde{\Gamma}^k_{ij}h_k\right). \tag{9.2.10}$$

**Proposition 9.2.2**
*For $u = u_0 + w$, the PDE appearing in (9.1.5) has the following linearisation.*

$$\left(\partial_s + \frac{f(u_0)^2}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}\Delta_0\right)w = F_1(w, \widetilde{\nabla}\, w) + F_2(w, \widetilde{\nabla}\, w); \tag{9.2.11}$$

*where $F_1$ and $F_2$ denote expressions depending at most linearly and at least quadratically on the entries in the brackets with coefficients given in terms of $u_0$ and its $\Phi$-derivatives up to order two. In particular, the coefficients of the expressions in $F_1$ and $F_2$ are combinations of $f^{(j)}(u_0)$, with $j = 0, 1, 2, 3$, $f(u_0)^k$ and $|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}$.*

***Proof:***
For every function $h : \overline{M} \to \mathbb{R}$, from equation (9.2.7) we find

$$\begin{aligned}\Delta_g h =&\Delta_0 h + \widetilde{g}\left(\widetilde{\nabla}\, u_0, \widetilde{\nabla}\, h\right) \frac{1}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} \frac{1}{f(u_0)^2}\left(\Delta_{\widetilde{g}} + \widehat{\Delta}_0\right) w \\ &+ L^*(w, \widetilde{\nabla}\, w) + Q^*(w, \widetilde{\nabla}\, w) \\ =&\Delta_0 h + \widetilde{g}(\widetilde{\nabla}\, u_0, \widetilde{\nabla}\, h) \frac{1}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} \Delta_0 w \\ &+ L(w, \widetilde{\nabla}\, w) + L^*(w, \widetilde{\nabla}\, w) + Q(w, \widetilde{\nabla}\, w) + Q^*(w, \widetilde{\nabla}\, w).\end{aligned} \tag{9.2.12}$$

In the above the first equality follows by linearising each term in equation (9.2.7) while the second follows from equation (9.2.9).
Notice that the expression in (9.2.6) is contained in the term $L(w, \widetilde{\nabla}\, w)$. Furthermore, we distinguished the at most linear and at least quadratic expressions in $w$ and its $\Phi$-derivatives respectively with $L$, $L^*$ and $Q$, $Q^*$ due to the stared ones depending on $h$ up to its second order derivatives.

Finally, by substituting $h$ with $u = u_0 + w$ in (9.2.12), keeping in mind that the metric tensor $\widetilde{g}$ and its associated gradient $\widetilde{\nabla}$ are linear, we find

$$\begin{aligned}\Delta_g u =&\Delta_0 w + \frac{|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} \Delta_0 w + F_1(w, \widetilde{\nabla}\, w) + F_2(w, \widetilde{\nabla}\, w) \\ =&\frac{f(u_0)^2}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} \Delta_0 w + F_1(w, \widetilde{\nabla}\, w) + F_2(w, \widetilde{\nabla}\, w);\end{aligned} \tag{9.2.13}$$

where the second equality follows from

$$1 + \frac{|\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} = \frac{f(u_0)^2}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}.$$

Formula (9.2.11) now follows by noticing that $u_0$ is not $s$-dependent; thus $\partial_s u = \partial_s w$. □

**Remark 9.2.1**
*It is not hard to see why the coefficients of $F_1(w, \widetilde{\nabla} w)$ and $F_2(w, \widetilde{\nabla} w)$ contain derivative of $f$ evaluated at $u_0$ up to order 3. Indeed, by looking at (9.2.12) it is clear that a linearisation of the Christoffel symbols is needed. Now the Christoffel symbols of a graphical hypersurface in a GRWST equipped with the induced metric are described in Proposition 8.2.2. From (8.2.6), one sees that such a linearisation requires a linearisation of, at most, $f'(u_0)$. A similar argument can be employed when dealing with (9.2.6). Thus, from (9.2.2) we infer that 3 is the highest possible differentiation degree for $f$ appearing as a coefficient for $F_1(w, \widetilde{\nabla} w)$ and $F_2(w, \widetilde{\nabla} w)$.*

It is important to point out one crucial implication of proposition 9.2.2. Namely, short time existence of prescribed graphical mean curvature flows (i.e. short time existence of solutions of (9.1.5)) is guaranteed if there exist some $T > 0$ and a family of functions $w : \overline{M} \times [0, T) \to \mathbb{R}$ satisfying the Cauchy problem

$$\begin{cases}\left(\partial_s + \frac{f(u_0)^2}{f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}} \Delta_0\right) w = F_1(w, \widetilde{\nabla}\, w) + F_2(w, \widetilde{\nabla}\, w) \\ w(\_, 0) = 0\end{cases} \tag{9.2.14}$$

# Chapter 10

# Short time existence of solutions to (9.1.5)

Chapter 9 has been concluded by pointing out a fundamental fact; that is, one can relate short-time existence of prescribed mean curvature flows (i.e. short-time solutions of (9.1.5)) with the existence of solutions for Equation (9.2.14) for some suitable $T > 0$ small enough.
By looking at (9.2.14) we notice that it is an heat type Cauchy problem of the form (5.3.2). Therefore, the analysis provided in Part I will come into play.

## 10.1 Preparation for short time existence

The description of the geometric quantities in §8.1 and of the linearised equation in §9.2 are general. That is they do not depend on the underlying geometry of the space-like slice $(\overline{M}, \widetilde{g})$ but only on the geometry of the ambient manifold $(N, \overline{g})$. As we can see from the structure of equation (9.2.11), it is clear that, in order to discuss short time existence of solutions of (9.1.5), the geometry of $\overline{M}$ equipped with the initial induced metric $g_0 = F(\_, 0)^*\overline{g}$ will play a crucial role.

For $u_0 : \overline{M} \to \mathbb{R}$ being a function so that graph$(u_0)$ is a space-like hypersurface in the GRWST $(N, \overline{g})$, the Cauchy problem (9.2.14) is of the form

$$(\partial_s + a\Delta_0)\, w = F_1(w, \widetilde{\nabla}\, w) + F_2(w, \widetilde{\nabla}\, w); \quad w(\_, 0) = 0, \tag{10.1.1}$$

with $a > 0$. In §5.3, we proved that solutions to (10.1.1) exist for short time in the context of $\Phi$-manifolds, i.e. the Laplace-Beltrami operator appearing in (10.1.1) is the Laplace-Beltrami operator associated to a $\Phi$-metric.

**Remark 10.1.1**
*Once equipped with the initial induced metric $g_{0_{ij}} = -u_{0i}u_{0j} + f(u_0)^2\widetilde{g}_{ij}$ (cf. equation (8.2.2)) $(\overline{M}, g_0)$ is not a $\Phi$-manifold.*

Thus Theorem 5.2 can not be employed to conclude short time existence of solutions to (9.2.11).
Under suitable assumptions on the initial condition $u_0 : \overline{M} \to \mathbb{R}$ of the Cauchy problem (9.1.5) and on the warping function $f : \mathbb{R} \to \mathbb{R}$, the geometry of the initial

induced metric $g_0$ will not be "too far" from the geometry of a $\Phi$-manifold. Here by "not too far" we mean that the initial embedded graphical hypersurface $(\overline{M}, g_0)$ is a generalised $\Phi$-manifold (cf. Definition 3.3).

Dealing with generalised $\Phi$-manifolds implies that the parametrix construction for the heat operator on $\Phi$-manifolds presented in §5.1, can be repeated also in this slightly more general setting. Hence an equivalent result to Theorem 5.2 can be derived leading to the claimed short time existence of solutions to (9.1.5). For convenience of the reader we will quickly recap the steps that led to construct a parametrix for the heat operator in §5.1.

1. Talebi and Vertman in [TaVe21] construct the heat kernel for $\Delta_{\widetilde{g}}$, the Laplacian on $\Phi$-manifolds, by inverting the normal operator for $(\partial_s + \Delta_{\widetilde{g}})$ and then iterating away the error.

2. Making use of the aforementioned construction and Theorem 7.2 in [TaVe21], in §4 we provided mapping properties for the heat kernel in the blown up space.

3. The mapping properties and the fact that $\Phi$-manifolds are stochastically complete allow to construct a parametrix for $(\partial_s + a\Delta_{\widetilde{g}})$ for some positive function $a : \overline{M} \to \mathbb{R}$.

As mentioned above, the geometry of generalised $\Phi$-manifolds is not "too far" from the geometry of a $\Phi$-manifold. Indeed, in view of Proposition 3.5.1 we already know that generalised $\Phi$-manifolds are stochastically complete. Thus, for the steps provided above, it is only necessary to make sure that one can construct the heat kernel for the operator $\Delta_0$.

## 10.2 Short time existence

To prove short time existence of solution to (9.1.5) we need to see under which conditions the initial induced metric $g_0 = F(\_, 0)^*\overline{g}$ on $\overline{M}$ gives rise to a generalised $\Phi$-metric. We also need to see under which conditions the leading asymptotics of the Laplace-Beltrami operator at time $s = 0$, $\Delta_0$, agree with the leading asymptotics of the $\Phi$-Laplacian $\Delta_{\widetilde{g}}$.

The next result will give us the necessary conditions so that the initial induced metric $g_0 = F(\_, 0)^*\overline{g}$ on $\overline{M}$ gives rise to a generalised $\Phi$-metric. In what follows we will denote by $C$ all the uniform constants.

**Proposition 10.2.1**
*Let $f \in C^{k+1}(J)$, whit $J \subseteq \mathbb{R}$ is an interval. Moreover assume $|f(t)| \geq C > 0$ for some constant $C$ and that $\|f^{(j)}\|_\infty < \infty$ for every $j \in \{0, \dots, k+1\}$. Further, let $u_0 \in x^\gamma C^{k,\alpha}_\Phi(M)$ for some $\gamma \geq \alpha > 0$. Assume that $u_0$ is a function such that its graph in the generalized Robertson-Walker spacetime $N = \mathbb{R} \times_f M$ is strongly space-like, i.e. there exists some constant $C > 0$ so that $f(u_0)^2 - |\nabla u_0|^2_{\widetilde{g}} \geq C < 0$. Then*

1. *$f(u_0) \in C^{k,\alpha}_\Phi(M)$.*

2. $1/f(u_0) \in C^{k,\alpha}_{\Phi}(M)$.

3. $\sqrt{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}} \in C^{k-1,\alpha}_{\Phi}(M)$.

4. $1/\sqrt{f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}}} \in C^{k-1,\alpha}_{\Phi}(M)$.

***Proof:***
Notice that (2) and (4) follow from (1), (3) and lemma 3.4.2. Furthermore, (3) follows from Corollary 3.4.1 if one has $f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}} \in C^{k-1,\alpha}_{\Phi}(M)$. In conclusion the result follows once we know that (1) holds and that $|\widetilde{\nabla} u_0|^2_{\widetilde{g}} \in C^{k-1,\alpha}_{\Phi}(M)$.
Let us assume first $u_0 \in C^{k,\alpha}_{\Phi}(M)$. We will prove (1) only for the case $k = 0$ and $k = 1$ since the general case follows the same lines.
Boundedness of $f(u_0)$ follows from boundedness of $f$. The only thing left to estimate is $[f]_\alpha$. Let $p, p' \in \overline{M}$. An application of the Mean Value theorem applied to $f$ leads to, for $\xi \in (u_0(p), u_0(p'))$,

$$|f(u_0(p)) - f(u_0(p'))| = |f'(\xi)||u_0(p) - u_0(p')| \leq \|f'\|_\infty |u_0(p) - u_0(p')|.$$

In particular one has

$$[f(u_0)]_\alpha \leq \|f'\|_\infty [u_0]_\alpha < \infty.$$

Let now consider $X \in \mathcal{V}_\Phi(M)$. Leibniz rule gives

$$X(f(u_0)) = f'(u_0) \cdot X(u_0).$$

Boundedness of the above follows trivially from the assumptions. Hence let us consider $p, p' \in \overline{M}$.

$$\begin{aligned} |X(f(u_0))(p) - &X(f(u_0))(p')| \\ &\leq |f'(u_0)(p)||Xu_0(p) - Xu_0(p')| + |f'(u_0(p)) - f'(u_0(p'))||Xu_0(p')| \\ &\leq \|f'\|_\infty |Xu_0(p) - Xu_0(p')| + |f''(\xi)||u_0(p) - u_0(p')|\|Xu_0\|_\infty; \end{aligned}$$

for $\xi \in (u_0(p), u_0(p'))$ arising from the Mean Value theorem. In conclusion

$$[X(f(u_0))]_\alpha \leq \|f'\|_\infty [Xu_0]_\alpha + \|Xu_0\|_\infty \|f''\|_\infty [u_0]_\alpha < \infty.$$

Let us now move to $|\widetilde{\nabla} u_0|^2_{\widetilde{g}}$. In local coordinates near the boundary, $|\widetilde{\nabla} u_0|^2_{\widetilde{g}}$ can be viewed as the product of two $\Phi$-derivatives. Indeed

$$|\widetilde{\nabla} u_0|^2_{\widetilde{g}} = \widetilde{g}^{ij} u_{0i} u_{0j}.$$

Notice that every upper index gives rise to a positive power of $x$, as can be easily inferred from equation (3.1.1). Moreover, every upper index is paired with a lower index denoting a partial derivative. Thus, for $p$ near the boundary, $|\widetilde{\nabla} u_0|^2_{\widetilde{g}}(p) = Xu_0(p)Yu_0(p)$ for some $X, Y \in \mathcal{V}_\Phi(M)$. In conclusion, near the boundary, $|\widetilde{\nabla} u_0|^2_{\widetilde{g}} = Xu_0 Yu_0$; leading to the desired result.
Let now $u_0 \in x^\gamma C^{k,\alpha}_{\Phi}(M)$. This means, in particular, that $u_0$ can be written as $u_0 = x^\gamma \widetilde{u_0}$ for some $\widetilde{u_0} \in C^{k,\alpha}_{\Phi}(M)$. Hence, for $p, p' \in \overline{M}$ one has estimates of the form

$$\begin{aligned} |x^\gamma u_0(p) - x'^\gamma u_0(p')| &= |x^\gamma \widetilde{u_0}(p) - x^\gamma \widetilde{u_0}(p') + x^\gamma \widetilde{u_0}(p') - x'^\gamma \widetilde{u_0}(p')| \\ &\leq |x^\gamma||\widetilde{u_0}(p) - \widetilde{u_0}(p')| + |x^\gamma - x'^\gamma||\widetilde{u_0}(p)|. \end{aligned}$$

Thus for $u_0 \in x^\gamma C^{k,\alpha}_\Phi(M)$ the result follows from corollary 3.4.3 along the same lines as in the above. □

### 10.2.1 Proof of Short time existence

Let $\alpha \in (0,1)$ and $\gamma \geq \alpha$ be fixed. From now on, we will assume the warping function $f = f(t) : \mathbb{R} \to \mathbb{R}$ and the initial condition of the Cauchy problem (9.1.5) $u_0 : \overline{M} \to \mathbb{R}$ to satisfy the following.

**Assumptions 2**
*Denoting by $c > 0$ uniform constants,*

*(i)* *$f = f(t) : \mathbb{R} \to \mathbb{R}$, lies in $C^2(\mathbb{R})$ and satisfies $\|f^{(j)}\|_\infty \leq c$ for $j = 0, 1, 2$ (here $f^{(j)}$ denotes the $j$-th derivative of $f$ with respect to $t$).*

*(ii)* *$u_0 : \overline{M} \to \mathbb{R}$ is a function in $x^\gamma C^{1,\alpha}_\Phi(M)$ so that $\mathrm{graph}(u_0)$ is a strongly space-like hypersurface of the GRWST $(N, \overline{g})$, i.e. there exists some $c > 0$ so that $f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}} \geq c > 0$.*

We begin by noticing the following trivial fact.

**Proposition 10.2.2**
*Let $l$ be a positive integer. For every $P \in \mathcal{V}^l_\Phi(M)$ (cf. §3.2)*

$$P : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^{k-l,\alpha}_\Phi(M \times [0,T]) \tag{10.2.1}$$

*with $k \geq l$.*
*In particular, for every $X \in \mathcal{V}_\Phi(\overline{M})$ and for every $a \in x^\gamma C^{k,\alpha}(M)$ with $k \geq 1$, there exists some $b \in C^{k-1,\alpha}_\Phi(M)$ such that $Xa = x^\gamma b$.*

**Corollary 10.2.1**
*Let $f : \mathbb{R} \to \mathbb{R}$ and $u_0 : \overline{M} \to \mathbb{R}$ satisfy (i) and (ii) in Assumption 2 respectively. The initial induced metric $g_0 = F(\_, 0)^* \overline{g}$ on $\overline{M}$ is a generalised $\Phi$-metric. Moreover $(\overline{M}, g_0)$ is stochastically complete.*

***Proof:***
Proposition 3.5.1 tells that every generalised $\Phi$-manifold is stochastically complete, thus it is only necessary to prove that $g_0$ is a generalised $\Phi$-metric. In view of remark 3.5.1, this is equivalently achieved by proving that $g_0(X,Y)$ is bounded for every $X, Y \in \mathcal{V}^1_\Phi(M) = \mathcal{V}_\Phi(M)$. Therefore, let $X, Y \in \mathcal{V}_\Phi(\overline{M})$; there exists some $c > 0$ such that

$$|g_0(X,Y)| \leq |X(u_0)Y(u_0)| + |f(u_0)|^2\, |\widetilde{g}(X,Y)| \leq \|u_0\|^2_{\alpha,1} + c' \leq c.$$

The chain of inequalities is an easy consequence of lemma 3.4.1, proposition 10.2.1, proposition 10.2.2 and the fact that $\Phi$-manifolds are generalised $\Phi$-manifolds. □

The only thing left to see now, in order to employ the same parametrix construction as in §5.1, is that the leading asymptotics of the initial Laplace-Beltrami operator $\Delta_0$ agree with those of the $\Phi$-Laplacian $\Delta_{\widetilde{g}}$ (cf. discussion in section §10.1).

Recall that, as displayed in equation (9.2.7), the initial Laplace-Beltrami operator can be written as

$$\Delta_0 = \frac{1}{f(u_0)^2}\left(\Delta_{\widetilde{g}} + \widehat{\Delta}_0\right) + F_1 + F_2 \tag{10.2.2}$$

with $F_1$ and $F_2$ denoting operator of lower order, i.e. order less or equal to 1, and with $\widehat{\Delta}_0$ locally defined by

$$\widehat{\Delta}_0 = -\frac{\widetilde{g}^{jl} u_{0l} \widetilde{g}^{im} u_{0m}}{f(u_0)^2 - |\,\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}}\left(\partial_i\partial_j - \widetilde{\Gamma}^k_{ij}\partial_k\right). \tag{10.2.3}$$

From proposition 10.2.2 we deduce that the $\Phi$-Laplacian $\Delta_{\widetilde{g}}$ satisfies:

$$\Delta_{\widetilde{g}} : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^{k-2,\alpha}_\Phi(M \times [0,T]).$$

If we prove that

$$\widehat{\Delta}_0 : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^\lambda C^{k-2,\alpha}_\Phi(M \times [0,T]), \tag{10.2.4}$$

for some $\lambda > \gamma$, then we conclude that the leading asymptotic of $\Delta_0$ and those of $\Delta_{\widetilde{g}}$ agree.
Now, equations (10.2.2) and (10.2.3), joined with proposition 10.2.1 and lemma 3.4.1, imply that (10.2.4) holds if and only if the claimed mapping properties hold for the numerator, i.e.

$$-\widetilde{g}^{jl} u_{0l} \widetilde{g}^{im} u_{0m}\left(\partial_i\partial_j - \widetilde{\Gamma}^k_{ij}\partial_k\right) : x^\gamma C^{k,\alpha}_\Phi(M\times[0,T]) \to x^\lambda C^{k-2,\alpha}_\Phi(M\times[0,T]) \tag{10.2.5}$$

for some $\lambda > \gamma$.
To prove (10.2.5) we will perform a weight analysis on each term. We begin by estimating the behaviour of the coefficient of the term involving second order derivatives in (10.2.5). These are

$$\widetilde{g}^{jl} u_{0l} \widetilde{g}^{im} u_{0m} \partial_i\partial_j.$$

Notice that every lower index accompanying $u_0$ denotes a classical partial derivative. Moreover every lower index is matched with an upper index denoting the inverse of the $\Phi$-metric $\widetilde{g}$. Thus, by recalling the local expression for a $\Phi$-metric (cf. §3.1) we conclude that every pair of repeated indices lead to a $\Phi$-derivative. In particular, from proposition 10.2.2 we conclude

$$-\widetilde{g}^{jl} u_{0l} \widetilde{g}^{im} u_{0m} \partial_i\partial_j = X(u_0)Y(u_0)P : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^{3\gamma} C^{k-2,\alpha}_\Phi(M \times [0,T]);$$

with $X, Y \in \mathcal{V}_\Phi(M)$ and $P \in \mathcal{V}^2_\Phi(\overline{M})$.
The other term in (10.2.5) is slightly more complicated. We will focus only on one of the factors generating the Christoffel symbols (cf. (7.1.3)). Without loss of generality let us consider

$$\frac{1}{2}\widetilde{g}^{jl} u_{0l} \widetilde{g}^{im} u_{0m} \widetilde{g}^{kp} \widetilde{g}_{ij,p}$$

where the ",$p$" denotes a partial derivative. The repeated indices accompanying $u_0$ can be treated as for the previous case. As before, we can conclude that the repeated index $p$ gives rise to a $\Phi$-derivative of the $(i,j)$-th component of the $\Phi$-metric $\widetilde{g}$. Moreover, such a $\Phi$-derivative is coupled with upper indices $i$ and $j$.

Thus the resulting operation can be modelled via $x^q X(x^{-q}a)$ with $q \in \{0, \dots, 4\}$, $X \in \mathcal{V}_\Phi(\overline{M})$ and $a$ being a function. By the nature of $\Phi$-vector fields (cf. §3.2) we can see the lowest possible outcome for the power of $x$ is 0. Hence

$$-\widetilde{g}^{jl}u_{0l}\widetilde{g}^{im}u_{0m}\widetilde{\Gamma}^k_{ij}\partial_k : x^\gamma C^{k,\alpha}_\Phi(M \times [0,T]) \to x^{3\gamma}C^{k-1,\alpha}_\Phi(M \times [0,T]).$$

We can therefore conclude that the leading asymptotics of the operator $\Delta_0$ are the same as those of $\Delta_{\widetilde{g}}$. This shows that the normal operator for $\partial_s + \Delta_0$ and $\partial_s + \Delta_{\widetilde{g}}$ are the same. In particular this implies that the construction in [TaVe21], the mapping properties in §4 and the construction of parametrix in §5.1 can be carried over to $\Delta_0$.

The above indicate that steps from (1) to (3) in §10.1 hold for $\Delta_0$, thus Theorem 5.2 can be achieved for the operator $\Delta_0$ as well. In conclusion we can prove the claimed existence of solutions, for short time, of the prescribed mean curvature flow.

**Theorem 10.1**
*Let $(\overline{M}, \widetilde{g})$ be a $\Phi$-manifold and $(N, \overline{g})$ a GRWST having $(\overline{M}, \widetilde{g})$ as space-like slice. For c denoting uniform constants, let us additionally assume that the warping function $f = f(t) : \mathbb{R} \to \mathbb{R}$ satisfies a slight variation of (i) in Assumption 2. That is, $f$ lies in $C^3(\mathbb{R})$, $f(t) \geq c > 0$ and that $\|f^{(j)}\|_\infty \leq c < \infty$ for $j \in \{0,1,2,3\}$. Similarly, let us assume some slightly stronger conditions than (ii) in 2 on the initial value $u_0$. In particular, let $u_0 : \overline{M} \to \mathbb{R}$ be in $x^\gamma C^{2,\beta}_\Phi(M)$, with $1 \geq \beta > \alpha$, so that* graph$(u_0)$ *is a strongly space-like hypersurface of $N$, i.e. $f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}} \geq c > 0$. Then, for every $\mathcal{H} \in C^{0,\alpha}_\Phi(M)$ there exists some $T' > 0$ and $u^* \in x^\gamma C^{2,\alpha}_\Phi(M \times [0,T])$ solution of the Cauchy problem (9.1.5).*

***Proof:***
Note that, in view of Proposition 10.2.1, the conditions on the warping function $f$ and on the initial value $u_0$ imply $f(u_0)^2/(f(u_0)^2 - |\widetilde{\nabla}\, u_0|^2_{\widetilde{g}}) \in C^\beta_\Phi(M)$ for $\beta > \alpha$. In this setting $f(u_0)^2/(f(u_0)^2 - |\widetilde{\nabla} u_0|^2_{\widetilde{g}})$ plays the role of the function $a$ in (5.0.2). Now, from Theorem 5.2 it is enough to show that the linear term $F_1 := F_1(w, \widetilde{\nabla}\, w)$ and the quadratic term $F_2 := F_2(w, \widetilde{\nabla}\, w, \widetilde{\nabla}^2\, w)$ in (10.1.1) satisfy the assumptions

i) $F(w, \widetilde{\nabla} w) := F_1(w, \widetilde{\nabla} w) + F_2(w, \widetilde{\nabla} w) : x^\gamma C^{2,\alpha}_\Phi(M \times [0,T]) \to x^\gamma C^\alpha_\Phi(M \times [0,T])$,

ii) $\|F_1(w_1, \widetilde{\nabla}\, w_1) - F_1(w_2, \widetilde{\nabla}\, w_2)\|_{k+1,\alpha,\gamma} \leq C_\mu \|w_1 - w_2\|_{k+2,\alpha}$ with $\|F_1(w, \widetilde{\nabla}\, w)\|_{k+1,\alpha,\gamma} \leq C_\mu$ for every $w_1, w_2 \in x^\gamma C^{k+2,\alpha}_\Phi(M \times [0,T])$,

iii) $\|F_2(w_1) - Q(w_2)\|_{k,\alpha,\gamma} \leq C_\mu \max\{\|w_1\|_{k+2,\alpha,\gamma}, \|w_2\|_{k+2,\alpha,\gamma}\}\|w_1 - w_2\|_{k+2,\alpha,\gamma}$ where $\|F_2(w)\|_{k,\alpha,\gamma} \leq C_\mu$.

The above are clearly satisfied by Proposition 10.2.1, Lemma 3.4.1 and by the structure of the linear and quadratic terms in Proposition 9.2.2. □

**Remark 10.2.1**
*Theorem 10.1 and the local expression of the induced metric (8.2.2) imply that under the prescribed mean curvature flow, at least for short time a generalised $\Phi$-manifolds stays a generalised $\Phi$-manifold.*
*In particular this shows that the flows stays stochastically complete.*

**Remark 10.2.2**
*In view of Theorem 5.2,one could gain more regularity from Theorem 10.1. Indeed, by assuming the warping function* $f \in C^\infty(\mathbb{R})$, $u_0 \in x^\gamma C^{3,\beta}_\Phi(M)$ *and* $\mathcal{H} \in C^1_\Phi(M)$, *then a solution* $u^*$ *lies in* $x^\gamma C^{3,\alpha}_\Phi(M \times [0,T])$. *Note that the requirement* $f$ *being smooth is strong, but it is just required in order to avoid computations; indeed it turns out* $f \in C^5(\mathbb{R})$ *is enough to ensure the solution to be* $C^3$.
*We wanted to point this out since in the next section we will be computing the Laplace-Beltrami operator of the gradient function* $v$ *of a solution* $u$ *to* (9.1.5), *thus resulting in a third order degree differentiation of* $u$.

# Chapter 11

# Evolution equations

Theorem 10.1 states the existence, for short time, of solutions to the parabolic Cauchy problem (9.1.5). In order to prove long time existence (not part of this work) one argues as follows.
Assume that a prescribed (graphical) mean curvature flow exists for a finite maximum time $T_{\max}$. If the relevant geometric properties, i.e. the assumption on $u_0$ in Theorem 10.1, are preserved along the flow, i.e. for every $s \in [0, T_{\max}]$, then the flow can be restarted with $u(T_{\max})$ as an initial condition. This shows that $T_{\max}$ was not a maximal time hence $T_{\max} = \infty$.

It is therefore clear that a method, proving the aforementioned relevant geometric properties to be preserved along the flow, is needed.
Usually in geometric analysis one argues by means of evolution equations, of those geometric quantities, and a parabolic maximum principle. A parabolic maximum principle for stochastically complete manifolds has been already discussed in §2. Thus, for our purposes, we need to compute some evolution equations.

As it has already been suggested in §8.4, the gradient function $v$ will play a crucial role (more details on this in Chapter 12). Therefore in §11.1 we present the evolution equations for the gradient function $v$. Nonetheless, some other evolution equations will be useful for our analysis; namely the evolution equations for the term $\mathrm{H} - \mathcal{H}$ which will be provided in §11.2.

From now on, unless otherwise specified, we will assume $u : \overline{M} \times [0, T]$ to be a solution of the prescribed (graphical) mean curvature flow equation (9.1.5), whose existence is guaranteed by Theorem 10.1.

## 11.1 Evolution equation for the gradient function

Our central aim is to prove that a space-like prescribed graphical mean curvature flow stays uniformly space-like along the flow. To this end we will prove that the gradient function $v$, defined in 6.16, satisfies a partial differential inequality of the form (2.4.3). Such an inequality will follow from the next theorem.

**Theorem 11.1**
*Let $u(s)$ be a solution to the prescribed graphical mean curvature flow (9.1.5) of an*

*$m$-dimensional space-like Cauchy hypersurface. Then the gradient function $v \equiv v(s)$ for the graph of $u(s)$, $s \in [0,T]$ satisfies the following evolution equation*

$$\begin{aligned}(\partial_s + \Delta)v = &- \| \mathrm{h} \|^2 v - \mathrm{Ric}^N(\mu,\mu)v - 2\frac{f'(u)}{f(u)}\,\mathrm{H} + \frac{f'(u)}{f(u)}\mathcal{H} - V(\mathcal{H}) \\ &- \frac{f'(u)}{f(u)}\mathcal{H}v^2 + 2\frac{f'(u)}{f(u)}g(\nabla u, \nabla v) + m\frac{f''(u)}{f(u)}v \\ &- \left(\frac{f'(u)}{f(u)}\right)^2 \|\nabla u\|^2 v - \frac{f''(u)}{f(u)}\|\nabla u\|^2 v - m\left(\frac{f'(u)}{f(u)}\right)^2 v.\end{aligned} \tag{11.1.1}$$

*In the above $V$ is a vector field over $M$ so that*[1]

$$DF(V) = \overline{g}^{ij}\overline{g}(\partial_t, DF(\partial_i))DF(\partial_j).$$

The above theorem is a direct consequence of the following propositions.

**Proposition 11.1.1**
*The gradient function $v$ evolves as*

$$\partial_s v = V(\mathrm{H} - \mathcal{H}) - (\mathrm{H} - \mathcal{H})\frac{f'(u)}{f(u)} + (\mathrm{H} - \mathcal{H})\frac{f'(u)}{f(u)}v^2. \tag{11.1.2}$$

**Proposition 11.1.2**
*The Laplacian of the gradient function $v$ can be expressed as*

$$\begin{aligned}\Delta v = &- \frac{f'(u)}{f(u)}\,\mathrm{H} - \frac{f'(u)}{f(u)}\,\mathrm{H}\,v^2 - V(\mathrm{H}) + 2\frac{f'(u)}{f(u)}g(\nabla u, \nabla v) \\ &- \| \mathrm{h} \|^2 v - \mathrm{Ric}^2(\mu,\mu)v + m\frac{f''(u)}{f(u)}v + \left(\frac{f'(u)}{f(u)}\right)\|\nabla u\|^2 v \\ &- \frac{f''(u)}{f(u)}\|\nabla u\|^2 v - 2\left(\frac{f'(u)}{f(u)}\right)\|\nabla u\|^2 v - m\left(\frac{f'(u)}{f(u)}\right)^2 v.\end{aligned} \tag{11.1.3}$$

We will prove Proposition 11.1.1 and 11.1.2 in §11.1.1 and §11.1.2, respectively.

### 11.1.1 Time derivative of the gradient function

It is important to notice that the unit normal $\mu = \mu(s)$ is a section of an $s$-dependent vector bundle over $\overline{M}$, namely $F^*TN \equiv F(s)^*TN$ with $F(s) \equiv F(\cdot, s) : \overline{M} \to N$ being the graphical embedding given by $F(p,s) = (u(p,s), p)$ for any $p \in \overline{M}$. The pull-back connection on $F^*TN$ is denoted by $\nabla^{F^*TN}$, as introduced in Example 11.

In order to treat the partial derivative $\partial_s$ as a vector field, we consider as in [Smo12, Section 3.2] the pull-back bundle $\mathcal{F}^*TN = F^*TN \times [0,T]$, where $\mathcal{F} = F : \overline{M} \times [0,T] \to N$ is the graphical embedding as above, with the parameter $s \in [0,T]$ now considered as a coordinate on $\overline{M} \times [0,T]$. In such a way an $s$-derivative becomes a covariant derivative in the direction of $\partial_s$ with respect to the pull-back covariant

[1] Recall that $\overline{g}$ defines an inner product on $F^*TN$ by (7.3.4).

derivative $\nabla^{\mathcal{F}^*TN}$. In particular, for every $\sigma \in \Gamma(\mathcal{F}^*TN)$, given in local coordinates by $\sigma = \sigma^\alpha \partial_\alpha$ (recall, $\overline{\nabla}$ is the covariant derivative of $(N, \overline{g})$)

$$\begin{aligned} \partial_s \sigma :=& \nabla^{\mathcal{F}^*TN}_{\partial_s} \sigma = \frac{\partial}{\partial s} \sigma^\alpha \cdot \partial_\alpha + \sigma^\alpha \cdot \overline{\nabla}_{D\mathcal{F}(\partial_s)} \partial_\alpha, \\ \nabla^{\mathcal{F}^*TN}_{\partial_i} \sigma =& \nabla^{F^*TN}_{\partial_i} \sigma. \end{aligned} \tag{11.1.4}$$

**Remark 11.1.1**
*For every $i = 1, \dots, m = \dim \overline{M}$ we obtain for the differential of $\mathcal{F}$*

$$\begin{aligned} D\mathcal{F}(\partial_s) &= -(\mathrm{H} - \mathcal{H})\mu, \\ D\mathcal{F}(\partial_i) &= DF(\partial_i). \end{aligned} \tag{11.1.5}$$

*Due to the symmetry of the second fundamental form associated to $\mathcal{F}$ one has*

$$\nabla^{\mathcal{F}^*TN}_{\partial_s} D\mathcal{F}(\partial_i) = \nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_s) = -\partial_i(\mathrm{H} - \mathcal{H}) \cdot \mu - (\mathrm{H} - \mathcal{H}) \cdot \nabla^{F^*TN}_{\partial_i} \mu. \tag{11.1.6}$$

**Proposition 11.1.3**
*For $u$ being a solution of (9.1.5), the unit normal $\mu$ evolves as*

$$\partial_s \mu := \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu = -DF(\nabla(\mathrm{H} - \mathcal{H})). \tag{11.1.7}$$

***Proof:***
Viewing $\mu$ as a section of the pull-back bundle $\mathcal{F}^*TN$, $\partial_s \mu$ lies in $\Gamma(\mathcal{F}^*TN)$ as well. Thus, taking $(\partial_1, \dots, \partial_m)$ as a local coordinate frame on $T\overline{M}$, we get a local frame for $\mathcal{F}^*(TN)$ given by $(D\mathcal{F}(\partial_s), D\mathcal{F}(\partial_1), \dots, D\mathcal{F}(\partial_m))$. We can therefore express $\partial_s \mu \equiv \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu$ with respect to that frame (recall that $\overline{g}$ defines an inner product on $F^*TN$ by (7.3.4) in Example 11)

$$\begin{aligned} \partial_s \mu = & |D\mathcal{F}(\partial_s)|^{-1}_{\overline{g}} \cdot \overline{g} \left( \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu, D\mathcal{F}(\partial_s) \right) D\mathcal{F}(\partial_s) \\ & + g^{ij} \overline{g} \left( \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu, D\mathcal{F}(\partial_i) \right) D\mathcal{F}(\partial_j). \end{aligned}$$

From (11.1.5) the first term reads

$$\begin{aligned} \overline{g} \left( \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu, D\mathcal{F}(\partial_s) \right) D\mathcal{F}(\partial_s) &= (\mathrm{H} - \mathcal{H})^2 \overline{g} \left( \nabla^{\mathcal{F}^*TN}_{\partial_s} \mu, \mu \right) \mu \\ &= \frac{1}{2} (\mathrm{H} - \mathcal{H})^2 \cdot \partial_s \overline{g} \left( \mu, \mu \right) \cdot \mu = 0, \end{aligned} \tag{11.1.8}$$

where the second equality follows by the metric property of the pull-back connection (cf. Example 11), and the last equality follows by $\mu$ being of unit length. Note that $\overline{g} \left( \mu, DF(\partial_i) \right) = 0$, since $\mu$ is normal. We conclude again by the metric property of the pull-back connection $\nabla^{F^*TN}$, and using (11.1.6) in the second equality

$$\begin{aligned} \partial_s \mu &= -g^{ij} \overline{g} \left( \mu, \nabla^{F^*TN}_{\partial_s} DF(\partial_i) \right) DF(\partial_j) \\ &= -g^{ij} \left( -\partial_i (\mathrm{H} - \mathcal{H}) \right) \overline{g} \left( \mu, \mu \right) DF(\partial_j) + g^{ij} (\mathrm{H} - \mathcal{H}) \overline{g} \left( \mu, \nabla^{F^*TN}_{\partial_i} \mu \right) DF(\partial_j) \\ &= -g^{ij} \partial_i (\mathrm{H} - \mathcal{H}) DF(\partial_j) + \frac{1}{2} g^{ij} (\mathrm{H} - \mathcal{H}) \partial_i \overline{g} \left( \mu, \mu \right) DF(\partial_j), \end{aligned}$$

where we used $\overline{g}(\mu, \mu) = -1$ in the last equation. The second summand vanishes by unitarity of $\mu$, which is a similar argument as in (11.1.8), and thus the statement follows. □

We are now in the position to prove Proposition 11.1.1.

*Proof of Proposition 11.1.1:*
Recall that, by definition $v = -\overline{g}(\mu, \partial_t)$ hence

$$\partial_s v = -\overline{g}\left(\partial_s \mu, \partial_t\right) - \overline{g}\left(\mu, \nabla^{\mathcal{F}^*TN}_{\partial_s} \partial_t\right). \tag{11.1.9}$$

Formula (11.1.7) implies that $\partial_s \mu$ lies in $\Gamma(F^*TN)$ and is tangential to the graph of $u$; that is $\overline{g}\left(\partial_s \mu, \mu\right) = 0$. Now $(\mu, DF(\partial_1), \ldots, DF(\partial_m))$ is a local frame for $F^*TN$, with $\mu$ orthogonal to the other frame elements and time-like. Thus we can write

$$\partial_t = -\overline{g}(\partial_t, \mu)\mu + \partial_t^\top = v\mu + \partial_t^\top, \quad \partial_t^\top := g^{ij}\overline{g}\left(\partial_t, DF(\partial_i)\right) DF(\partial_j). \tag{11.1.10}$$

Defining a local vector field $V \in \Gamma(T\overline{M})$ by $V = g^{ij}\overline{g}\left(\partial_t, DF(\partial_i)\right)\partial_j$, so that $DF(V) = \partial_t^\top$, we conclude from Proposition 11.1.3 (recall $g = F^*\overline{g}$)

$$\begin{aligned} \overline{g}\left(\partial_s \mu, \partial_t\right) &= -\overline{g}\left(DF\left(\nabla(\mathrm{H} - \mathcal{H})\right), DF(V)\right) \\ &= -g\left(\nabla(\mathrm{H} - \mathcal{H}), V\right) = -V(\mathrm{H} - \mathcal{H}). \end{aligned}$$

For the second term in (11.1.9) let us express $\mu$ in the local frame $(\partial_t, \partial_1, \ldots, \partial_m)$

$$\mu = -\overline{g}(\mu, \partial_t)\partial_t + \overline{g}^{ij}\overline{g}(\mu, \partial_i)\partial_j = v\partial_t + vb^j\partial_j \tag{11.1.11}$$

where $b^j : \overline{M} \to \mathbb{R}$, $b^j := \widetilde{g}^{ij}u_i/f(u)^2$, using (8.3.4) in the last equality. In particular one writes

$$\nabla^{\mathcal{F}^*TN}_{\partial_s} \partial_t = -(\mathrm{H} - \mathcal{H})v\overline{\nabla}_{\partial_t}\partial_t - (\mathrm{H} - \mathcal{H})vb^j\overline{\nabla}_{\partial_j}\partial_t.$$

From equation (8.1.2) and applying (11.1.11) one concludes

$$\nabla^{\mathcal{F}^*TN}_{\partial_s} \partial_t = -\frac{f'(u)}{f(u)}(\mathrm{H} - \mathcal{H})\mu + v\frac{f'(u)}{f(u)}(\mathrm{H} - \mathcal{H})\partial_t.$$

The result now follows by substituting the above in (11.1.9). □

### 11.1.2 Laplacian of the gradient function

In order to prove Theorem 11.1 it remains to compute $\Delta v$ (recall $\Delta$ is the Laplacian with respect to $g = F^*\overline{g}$) at a fixed time $s \in [0, T]$. For simplicity we will suppress the parameter $s$. Let us consider a local orthonormal (with respect to $g$) frame $(e_i)_i$ of $T\overline{M}$ over an open neighbourhood $U$, such that $\nabla_{e_i} e_j(p) = 0$ for every $i, j$ at some fixed $p \in \overline{M}$. Then we can write for $\Delta v$ at $p$

$$\Delta v = e_i e_i \overline{g}(\mu, \partial_t) = e_i(\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_t)) + e_i(\overline{g}(\mu, \nabla^{F^*TN}_{e_i}\partial_t)). \tag{11.1.12}$$

The second summand in (11.1.12) will be computed using the next proposition (in (11.1.16)). The first summand is computed below in Lemma 11.1.1.

Let $u$ be a solution of (9.1.5) and $(e_i)_i$ a local parallel orthonormal frame at $p \in \overline{M}$ as above. With respect to a local coordinate frame $(\partial_k)_k$ we can write $e_i = e_i^k \partial_k$ for some smooth coefficients $e_i^k : U \to \mathbb{R}$. Note also that $e_i(u) = -\overline{g}\left(DF(e_i), \partial_t\right)$. We then obtain the following useful formulae at $p \in M$ (recall the definition in (7.4.1) and the fact that we assumed $\nabla_{e_i} e_j(p) = 0$)

$$-\mathrm{h}(e_i, e_j)\mu = \mathrm{II}(e_i, e_j) = \nabla^{F^*TN}_{e_i} DF(e_j), \tag{11.1.13}$$

$$DF(e_i) = -\overline{g}\left(DF(e_i), \partial_t\right)\partial_t + e_i^k\partial_k = e_i(u)\partial_t + e_i^k\partial_k. \tag{11.1.14}$$

**Proposition 11.1.4**
*Let $u$ be a solution of (9.1.5) and $F$ the corresponding family of graphical embeddings. Let $(e_i)_i$ be a local orthonormal frame such that $\nabla_{e_i} e_j(p) = 0$ for every $i, j$ at some fixed $p \in \overline{M}$. Then at $p$ we have*

1. *the covariant derivative of $\partial_t$, as a section of $F^*TN$, can be expressed as*
$$\begin{aligned}\nabla^{F^*TN}_{e_i}\partial_t &= \frac{f'(u)}{f(u)}DF(e_i) + \frac{f'(u)}{f(u)}\overline{g}(DF(e_i), \partial_t)\partial_t \\ &= \frac{f'(u)}{f(u)}DF(e_i) - \frac{f'(u)}{f(u)}e_i(u)\partial_t.\end{aligned} \tag{11.1.15}$$
2. *For $\mu$ being the unit normal to the graph of $u$ one has*
$$\overline{g}\left(\mu, \nabla^{F^*TN}_{e_i}\partial_t\right) = v\frac{f'(u)}{f(u)}e_i(u). \tag{11.1.16}$$
3. *For every $i$ and $j$ ranging between 1 and $m = \dim M$, one finds*
$$e_i\left(\overline{g}\left(\partial_t, DF(e_j)\right)\right) = \frac{f'(u)}{f(u)}\delta_{ij} + \frac{f'(u)}{f(u)}e_i(u)e_j(u) + v\,\mathrm{h}(e_i, e_j); \tag{11.1.17}$$
*where $\delta_{ij}$ denotes the Kronecker delta. In particular, for $i = j$, with the obvious summation convention over repeated indices, one concludes*
$$e_i\left(\overline{g}\left(\partial_t, DF(e_i)\right)\right) = v\,\mathrm{H} + \frac{f'(u)}{f(u)}\left(m + |\nabla u|^2_g\right). \tag{11.1.18}$$

***Proof:***

1. From equation (11.1.14) we see that
$$\nabla^{F^*TN}_{e_i}\partial_t = e_i(u)\overline{\nabla}_{\partial_t}\partial_t + e_i^k\overline{\nabla}_{\partial_k}\partial_t.$$
Equation (11.1.15) now follows by substituting the appropriate values of the covariant derivatives on the right hand side, described by the Christoffel symbols of $(N, \overline{g})$ in (8.1.2), and using (11.1.14) once more.

2. Equation (11.1.16) is a direct consequence of (11.1.15) and the fact that $\mu$ is normal to the graph of $u$, that is $\overline{g}\left(\mu, DF(e_i)\right) = 0$ for every $i = 1, \dots, m$.

3. The metric property of the pull-back connection $\nabla^{F^*TN}$ gives
$$e_i\left(\overline{g}\left(\partial_t, DF(e_j)\right)\right) = \overline{g}\left(\nabla^{F^*TN}_{e_i}\partial_t, DF(e_i)\right) + \overline{g}\left(\partial_t, \nabla^{F^*TN}_{e_i}DF(e_j)\right).$$
From equations (11.1.15) and (11.1.13) we deduce
$$\begin{aligned}e_i\left(\overline{g}\left(\partial_t, DF(e_j)\right)\right) = &\frac{f'(u)}{f(u)}\overline{g}\left(DF(e_i), DF(e_j)\right) \\ &- \frac{f'(u)}{f(u)}e_i(u)\overline{g}\left(\partial_t, DF(e_j)\right) - \mathrm{h}(e_i, e_j)\overline{g}\left(\partial_t, \mu\right).\end{aligned}$$
By assumption, $(e_i)_i$ is a local orthonormal frame, with respect to the metric $g = F^*\overline{g}$, thus $\overline{g}\left(DF(e_i), DF(e_j)\right) = \delta_{ij}$. Moreover, from equation (11.1.14) we compute $-\overline{g}\left(\partial_t, DF(e_j)\right) = e_j(u)$. Finally, the result follows by recalling the definition of the gradient function (cf. Definition 6.16).

□

**Remark 11.1.2**
*Notice that equation (11.1.18) is nothing but (at $p \in \overline{M}$)*

$$\Delta u = -e_i\left(e_i(u)\right) = v\,\mathrm{H} + \frac{f'(u)}{f(u)}\left(m + |\nabla u|_g^2\right) = v\,\mathrm{H} + \frac{f'(u)}{f(u)}(m + v^2 - 1)$$

*where we have used (i) in Proposition 8.3.2. This is exactly the same result as in Proposition 8.4.7.*

**Lemma 11.1.1**
*In the notation of Proposition 11.1.4 we have at $p$*

$$\begin{aligned} e_i(\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_t)) = &- e_i(\overline{g}(\partial_t, DF(e_j))\,\mathrm{h}(e_i, e_j) \\ &- \overline{g}(\partial_t, DF(e_j))e_i(\mathrm{h}(e_i, e_j)). \end{aligned} \tag{11.1.19}$$

***Proof:***
The result follows by the Leibniz rule once we prove that

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_t) = -\overline{g}(\partial_t, DF(e_j))\,\mathrm{h}(e_i, e_j). \tag{11.1.20}$$

To this end, notice that $\partial_t$, as a section of $F^*TN$, decomposes with respect to the orthonormal frame $(\mu, DF(e_1), \ldots, DF(e_m))$ as

$$\partial_t = -\overline{g}(\partial_t, \mu)\mu + \overline{g}(\partial_t, DF(e_j))DF(e_j).$$

Substituting this into the left hand side of (11.1.20) one finds

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_t) = -\overline{g}\left(\partial_t, \mu\right)\overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, \mu\right) + \overline{g}\left(\partial_t, DF(e_j)\right)\overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, DF(e_j)\right).$$

The first summand now vanishes, since $\mu$ is of unit length. Using (11.1.13) we now conclude at $p \in M$

$$\overline{g}(\nabla^{F^*TN}_{e_i}\mu, \partial_t) = -\overline{g}\left(\partial_t, DF(e_j)\right)\mathrm{h}(e_i, e_j).$$

□

**Lemma 11.1.2**
*In the notation of Proposition 11.1.4 we have at $p$* [2]

$$e_i(\mathrm{h}(e_i, e_j)) = e_j(\mathrm{H}) - \mathrm{Ric}^N(DF(e_j), \mu); \tag{11.1.21}$$

*with the obvious summation convention on repeated indices.*

***Proof:***
From equation (7.4.6) in Corollary 7.4.5 we compute, by making use of the metric property of the pull-back connection $\nabla^{F^*TN}$,

$$\begin{aligned} e_i(\mathrm{h}(e_i, e_j)) = &- \overline{g}(\nabla^{F^*TN}_{e_i}DF(e_i), \nabla^{F^*TN}_{e_j}\mu) - \overline{g}(DF(e_i), \nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu) \\ = &- \overline{g}(DF(e_i), \nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu), \end{aligned}$$

[2] $\mathrm{Ric}^N$ applies to $DF(e_j) \in F^*TN$ similar to (7.3.4).

where the second equality follows from the fact that $\overline{g}(\nabla^{F^*TN}_{e_i} DF(e_i), \nabla^{F^*TN}_{e_j}\mu) = 0$ due to (11.1.13) and the fact that $\mu$ is of unit length, i.e. $\overline{g}\,(\mu,\mu) = -1$.

Recall that, at $p$, the curvature form of the pull-back connection is the pull-back of the curvature of the connection, that is

$$\nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu - \nabla^{F^*TN}_{e_j}\nabla^{F^*TN}_{e_i}\mu - \nabla^{F^*TN}_{[e_i,e_j]}\mu = R^N\left(DF(e_i), DF(e_j)\right)\mu.$$

Thus, using $\nabla^{F^*TN}_{[e_i,e_j]}\mu = 0$ due to naturality of the pull-back and the computations being performed at $p$, we obtain (summing over double indices $i$)

$$\begin{aligned}
e_i\left(\mathrm{h}(e_i,e_j)\right) &= -\,\overline{g}\left(\nabla^{F^*TN}_{e_i}\nabla^{F^*TN}_{e_j}\mu, DF(e_i)\right)\\
&= -\,R^N\left(DF(e_i), DF(e_j), \mu, DF(e_i)\right) - \overline{g}\left(\nabla^{F^*TN}_{e_j}\nabla^{F^*TN}_{e_i}\mu, DF(e_i)\right)\\
&= -\,\mathrm{Ric}^N\left(DF(e_j),\mu\right) - e_j\left(\overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, DF(e_i)\right)\right) + \overline{g}\left(\nabla^{F^*TN}_{e_i}\mu, \nabla^{F^*TN}_{e_j}\mu\right)\\
&= -\,\mathrm{Ric}^N\left(DF(e_j),\mu\right) + e_j(\mathrm{H}).
\end{aligned}$$

In the above, the second equality is obtained by making use of (7.2.1). The first term in the third equality is a consequence of $(\mu, DF(e_1), \ldots, DF(e_m))$ being an orthonormal basis of $T_{F(p)}N$ with $\mu$ time-like. The second term is instead a mere application of the metric property of the connection $\nabla^{F^*TN}$. Finally the fourth identity is the result of the formula (7.4.6), Definition 7.13 and of $\mu$ being of unit length. □

We now conclude with the following expression for (11.1.12).

**Proposition 11.1.5**
*Let $u$ be a solution of (9.1.5). Then the Laplacian of the gradient function $v$ can be expressed in terms of a local vector field $V = g^{ij}\overline{g}\left(\partial_t, DF(\partial_i)\right)\partial_j \in \Gamma(T\overline{M})$, such that $DF(V) = \partial_t^\top$, as follows:*

$$\begin{aligned}
\Delta v = &-\frac{f'(u)}{f(u)}\,\mathrm{H} - \frac{f'(u)}{f(u)}\,\mathrm{h}(\nabla u,\nabla u) - v\|\,\mathrm{h}\,\|^2 - V(H) + \mathrm{Ric}^N(DF(V),\mu)\\
&+ \frac{f'(u)}{f(u)}g(\nabla u,\nabla v) + \frac{f''(u)}{f(u)}\|\nabla u\|^2 v - 2\left(\frac{f'(u)}{f(u)}\right)^2\|\nabla u\|^2 v\\
&- \frac{f'(u)}{f(u)}\,\mathrm{H}\,v^2 - m\left(\frac{f'(u)}{f(u)}\right)^2 v.
\end{aligned} \tag{11.1.22}$$

***Proof:***
Plugging (11.1.16), Lemma 11.1.1 and 11.1.2 into (11.1.12) yields the following intermediate expression that holds at $p \in \overline{M}$

$$\begin{aligned}
\Delta v = &-e_i(\overline{g}(\partial_t, DF(e_j))\,\mathrm{h}(e_i,e_j)\\
&-\overline{g}(\partial_t, DF(e_j))\Big(e_j(\mathrm{H}) - \mathrm{Ric}^N(DF(e_j),\mu)\Big) + e_i\Big(v\frac{f'(u)}{f(u)}e_i(u)\Big).
\end{aligned} \tag{11.1.23}$$

Noticing that $V = \overline{g}\,(\partial_t, DF(e_k))\, e_k$ with summation over $k$, we conclude from formula (11.1.17), Lemma 11.1.1 and Lemma 11.1.2

$$\begin{aligned}
\Delta v = &-\Big(\frac{f'(u)}{f(u)}\delta_{ij} + \frac{f'(u)}{f(u)}e_i(u)e_j(u) + v\,\mathrm{h}(e_i,e_j)\Big)\,\mathrm{h}(e_i,e_j)\\
&- V(\mathrm{H}) + \mathrm{Ric}^N\,(DF(V),\mu) + e_i\Big(v\frac{f'(u)}{f(u)}e_i(u)\Big)\\
= &-\frac{f'(u)}{f(u)}\,\mathrm{H} - \frac{f'(u)}{f(u)}\,\mathrm{h}(\nabla u,\nabla u) - v\|\,\mathrm{h}\,\|^2\\
&- V(\mathrm{H}) + \mathrm{Ric}^N\,(DF(V),\mu) + e_i\Big(v\frac{f'(u)}{f(u)}e_i(u)\Big).
\end{aligned} \tag{11.1.24}$$

In order to conclude the statement, it remains to study the last term in (11.1.24). We compute using Remark 11.1.2, arriving at an expression that holds globally

$$\begin{aligned}
e_i\Big(v\frac{f'(u)}{f(u)}e_i(u)\Big) &= e_i(v)\frac{f'(u)}{f(u)}e_i(u) + ve_i\left(\frac{f'(u)}{f(u)}\right)e_i(u) - v\frac{f'(u)}{f(u)}\Delta u\\
&= e_i(v)e_i(u)\frac{f'(u)}{f(u)} + v\frac{f''(u)}{f(u)}e_i(u)e_i(u) - v\left(\frac{f'(u)}{f(u)}\right)^2 e_i(u)e_i(u)\\
&\quad - v^2\frac{f'(u)}{f(u)}\,\mathrm{H} - v\left(\frac{f'(u)}{f(u)}\right)^2 m - v\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\\
&= g\,(\nabla u,\nabla v)\,\frac{f'(u)}{f(u)} + v\frac{f''(u)}{f(u)}|\nabla u|_g^2 - 2v\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\\
&\quad - v^2\frac{f'(u)}{f(u)}\,\mathrm{H} - mv\left(\frac{f'(u)}{f(u)}\right)^2.
\end{aligned}$$

□

Notice that Proposition 11.1.5 is not yet a proof for Proposition 11.1.2. In particular the terms $\mathrm{h}\,(\nabla u,\nabla u)$ and $\mathrm{Ric}^N\,(DF(V),\mu)$ appearing in (11.1.22) need to be simplified.

**Proposition 11.1.6**
*Let $u$ be a solution of (9.1.5). Then*

$$\mathrm{h}(\nabla u,\nabla u) = -g(\nabla u,\nabla v) - \frac{f'(u)}{f(u)}|\nabla u|^2 v. \tag{11.1.25}$$

***Proof:***
Notice that the statement is a direct consequence of a local indentity

$$v_i = -g^{jk}u_j h_{ki} - \frac{f'(u)}{f(u)}vu_i. \tag{11.1.26}$$

Indeed, assuming (11.1.26) to hold locally, we find

$$\begin{aligned}
g(\nabla u,\nabla v) = g^{im}u_m v_i &= -g^{im}u_m g^{jk}u_j\,\mathrm{h}_{ik} - g^{im}u_m\frac{f'(u)}{f(u)}vu_i\\
&= -\,\mathrm{h}(\nabla u,\nabla u) - \frac{f'(u)}{f(u)}|\nabla u|_g^2 v.
\end{aligned}$$

This is precisely the statement after rearrangement. Let us therefore prove (11.1.26). By making use of (8.3.6) one has

$$vv_i = \frac{1}{2}\partial_i v^2 = \frac{1}{2}\partial_i(1+|\nabla u|^2) = g(\nabla_{\partial_i}\nabla u, \nabla u).$$

Furthermore, $\nabla_{\partial_i}\nabla u$ can be expressed locally as

$$\nabla_{\partial_i}\nabla u = \partial_i(g^{jk}u_k)\partial_j + g^{jk}u_k\Gamma^l_{ij}\partial_l.$$

By keeping in mind that $\partial_i g^{jk} = -g^{jl}\Gamma^k_{il} - \Gamma^j_{il}g^{lk}$, one finds

$$g(\nabla_{\partial_i}\nabla u, \nabla u) = g^{jk}u_j(u_{ki} - \Gamma^l_{ik}u_l).$$

The result now follows by substituting (8.4.5) and by noticing that

$$g^{jk}u_j\widetilde{g}_{ki} = \frac{v^2}{f(u)^2}u_i.$$

□

The only thing left to prove Proposition 11.1.2 is a formula for $\mathrm{Ric}^N(DF(V),\mu)$.

**Proposition 11.1.7**
*Let $u$ be a solution for (9.1.5) and $F$ the corresponding family of graphical embeddings. Then we have the following formula for the vector field $V = g^{ij}\overline{g}\left(\partial_t, DF(\partial_i)\right)\partial_j \in \Gamma(T\overline{M})$, with $DF(V) = \partial_t^\top$*

$$\mathrm{Ric}^N(DF(V),\mu) = -\frac{f''(u)}{f(u)}mv - v\,\mathrm{Ric}^N(\mu,\mu). \tag{11.1.27}$$

***Proof:***
By definition of $V$ we have $\partial_t^\top = DF(V) = \partial_t + \overline{g}(\partial_t,\mu)\mu$. Thus

$$\begin{aligned}\mathrm{Ric}^N(DF(V),\mu) &= \mathrm{Ric}^N(\partial_t,\mu) - v\,\mathrm{Ric}^N(\mu,\mu)\\ &= v\,\mathrm{Ric}^N(\partial_t,\partial_t) + vb^i\,\mathrm{Ric}^N(\partial_t,\partial_i) - v\,\mathrm{Ric}^N(\mu,\mu)\\ &= -\,vm\frac{f''(u)}{f(u)} + v\,\mathrm{Ric}^N(\mu,\mu).\end{aligned}$$

The second identity is obtained by considering the orthogonal decomposition of the unit normal $\mu$ with respect to the local frame $(\partial_t, \partial_1, \ldots, \partial_m)$ of $F^*TN$ (cf. formula (11.1.11)); in particular $b^i = \widetilde{g}^{ij}u_j/f(u)^2$. The last identity is a consequence of the values for the Ricci tensor described in Corollary 8.1.1. □

## 11.2 Evolution equation for the mean curvature

As before, let $u(\cdot, s)$ be a solution to the prescribed graphical mean curvature flow (9.1.5) of an $m$-dimensional space-like Cauchy hypersurface. The solution induces a family of embeddings $F(s) : \overline{M} \to N$ with $F(p,s) = (u(,s),p)$ for any $p \in \overline{M}$. The induced metric $g$ is defined by the pullback $g = F(s)^*\overline{g}$. We begin with the following basic evolution equations for the metric tensor.

**Proposition 11.2.1**
*The metric tensor $(g_{ij})$ and its inverse $(g^{ij})$, written in local coordinates, satisfy the following evolution equations along* (9.1.5)

$$\partial_s g_{ij} = 2(\mathrm{H} - \mathcal{H})\, \mathrm{h}_{ij}, \tag{11.2.1}$$
$$\partial_s g^{ij} = -2(\mathrm{H} - \mathcal{H}) g^{ik}\, \mathrm{h}_{kl}\, g^{lj}. \tag{11.2.2}$$

***Proof:***
Notice that (11.2.2) is a direct consequence of (11.2.1). So we only need to prove (11.2.1). Using the same notation as in §11.1.1 we compute (see right below for the explanation of the individual steps)

$$\begin{aligned}
\partial_s g_{ij} &= \partial_s \overline{g}\left(DF(\partial_i), DF(\partial_j)\right) \\
&= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_s} DF(\partial_i), DF(\partial_j)\right) + \overline{g}\left(DF(\partial_i), \nabla^{\mathcal{F}^*TN}_{\partial_s} DF(\partial_j)\right) \\
&= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_s), DF(\partial_j)\right) + \overline{g}\left(DF(\partial_i), \nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_s)\right) \\
&= -\left(\mathrm{H} - \mathcal{H}\right)\left(\overline{g}(\nabla^{F^*TN}_{\partial_i}\mu, DF(\partial_j)) + \overline{g}(DF(\partial_i), \nabla^{F^*TN}_{\partial_j}\mu)\right) = 2\left(\mathrm{H} - \mathcal{H}\right) \mathrm{h}_{ij}\,.
\end{aligned}$$

In the above the first identity follows by definition of the induced metric tensor $g = F^*\overline{g}$. The second is just a consequence of the metric property of the pull-back derivative $\nabla^{\mathcal{F}^*TN}$. The third line comes from the commutativity $[\partial_s, \partial_i] = 0$ of local coordinate fields. Finally the last equality is a consequence of (11.1.6) and the fact that $\mu$ is normal, i.e. $\overline{g}\left(DF(X), \mu\right) = 0$ for any vector field $X$ over $\overline{M}$. □

Next we study evolution of the scalar second fundamental form.

**Proposition 11.2.2**
*The tensor $(h_{ij})$ of the scalar second fundamental form satisfies the following evolution equation (summing over double indices as usual) along* (9.1.5)

$$\begin{aligned}
\partial_s \mathrm{h}_{ij} &= (\mathrm{H} - \mathcal{H})\left(g^{kl}\, \mathrm{h}_{ik}\, \mathrm{h}_{jl} - R^N(\mu, DF(\partial_i), DF(\partial_j), \mu)\right) \\
&\quad + (\mathrm{H} - \mathcal{H})_{ij} - \Gamma^k_{ij}\partial_k\,(\mathrm{H} - \mathcal{H}) \\
&= (\mathrm{H} - \mathcal{H})\left(g^{kl}\, \mathrm{h}_{ik}\, \mathrm{h}_{jl} - R^N(\mu, DF(\partial_i), DF(\partial_j), \mu)\right) + \nabla^2_{ij}\,(\mathrm{H} - \mathcal{H})\,.
\end{aligned} \tag{11.2.3}$$

***Proof:***
We compute with respect to a local coordinate frame $(\partial_i)$

$$\begin{aligned}
\partial_s h_{ij} &= \partial_s\, \mathrm{h}(\partial_i, \partial_j) = \partial_s \overline{g}\left(\nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) \\
&= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_s} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) + \overline{g}\left(\nabla^{F^*TN}_{\partial_i} DF(\partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_s}\mu\right) \\
&= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_s} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) - h_{ij}\overline{g}\left(\mu, \nabla^{\mathcal{F}^*TN}_{\partial_s}\mu\right) + \overline{g}\left(DF(\nabla_{\partial_i}\partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_s}\mu\right) \\
&= \overline{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_s} \nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) + \overline{g}\left(DF(\nabla_{\partial_i}\partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_s}\mu\right).
\end{aligned}$$

In the first line we just used (7.4.6). The second line is obtained by making use of the metric property of the pull-back connection $\nabla^{\mathcal{F}^*TN}$. The third is a consequence of (7.4.1). The last equality follows from the fact that $\mu$ is of unit length, which implies vanishing of the second term in the third line.

We shall now compute these two terms above. By Proposition 11.1.3

$$\bar{g}\left(DF(\nabla_{\partial_i}\partial_j), \nabla^{\mathcal{F}^*TN}_{\partial_s}\mu\right) \equiv \bar{g}\left(DF(\nabla_{\partial_i}\partial_j), \partial_s\mu\right)$$
$$= -g\Big(\nabla_{\partial_i}\partial_j, \nabla\left(\mathrm{H}-\mathcal{H}\right)\Big) = -\Gamma^k_{ij}\partial_k\left(\mathrm{H}-\mathcal{H}\right).$$

This computes the last term. For the first term we proceed as follows. Noting that that $[\partial_i, \partial_s] = 0$, we obtain from the definition of the Riemann curvature tensor

$$\nabla^{\mathcal{F}^*TN}_{\partial_s}\nabla^{\mathcal{F}^*TN}_{\partial_i} D\mathcal{F}(\partial_j) - \nabla^{\mathcal{F}^*TN}_{\partial_i}\nabla^{\mathcal{F}^*TN}_{\partial_s} D\mathcal{F}(\partial_j) = R^N\left(D\mathcal{F}(\partial_s), D\mathcal{F}(\partial_i)\right) D\mathcal{F}(\partial_j).$$

This implies directly

$$\begin{aligned} &\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_s}\nabla^{F^*TN}_{\partial_i} DF(\partial_j), \mu\right) \\ &= R^N\left(D\mathcal{F}(\partial_s), D\mathcal{F}(\partial_i), D\mathcal{F}(\partial_j), \mu\right) + \bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_i}\nabla^{\mathcal{F}^*TN}_{\partial_s} D\mathcal{F}(\partial_j), \mu\right) \\ &= -(\mathrm{H}-\mathcal{H})R^N\left(\mu, D\mathcal{F}(\partial_i), D\mathcal{F}(\partial_j), \mu\right) + \partial_i\left(\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_s), \mu\right)\right) \\ &\quad - \bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_s), \nabla^{\mathcal{F}^*TN}_{\partial_i}\mu\right), \end{aligned} \tag{11.2.4}$$

where the second equality is a consequence of the (11.1.5), (11.1.6) and the metric property of the pull-back connection. Let us now describe the second term at the right hand side of the second equation in (11.2.4). From (11.1.6) we write

$$\begin{aligned} &\partial_i\left(\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_s), \mu\right)\right) \\ &= -\partial_i\Big(\partial_j\left(\mathrm{H}-\mathcal{H}\right)\bar{g}(\mu,\mu)\Big) - \partial_i\left(\left(\mathrm{H}-\mathcal{H}\right)\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j}\mu, \mu\right)\right) \\ &= \partial_i\partial_j\left(\mathrm{H}-\mathcal{H}\right), \end{aligned} \tag{11.2.5}$$

where in the second equation we used the fact that $\bar{g}\left(\mu,\mu\right) = -1$.

To finish computation of (11.2.4), we need $\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_s), \nabla^{\mathcal{F}^*TN}_{\partial_i}\mu\right)$. To express this we will use equation (11.1.6) once more. Before presenting the expression let us notice the following. In view of (11.1.5), $\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu = \nabla^{F^*TN}_{\partial_i}\mu$ is a section of the pull-back bundle $F^*TN$. Hence it can be linearly decomposed in terms of the local frame $(\mu, DF(\partial_1), \ldots, DF(\partial_m))$. In particular, by keeping in mind that $\mu$ is a unit length time-like vector we conclude

$$\nabla^{\mathcal{F}^*TN}_{\partial_i}\mu = g^{jk}\bar{g}\left(\nabla^{F^*TN}_{\partial_i}\mu, DF(\partial_j)\right) DF(\partial_k) = -g^{jk}\,\mathrm{h}_{ij}\,DF(\partial_k)$$

with the obvious summation over the indices $j$ and $k$. Thus we find by (11.1.5)

$$\begin{aligned} &\bar{g}\left(\nabla^{\mathcal{F}^*TN}_{\partial_j} D\mathcal{F}(\partial_s), \nabla^{\mathcal{F}^*TN}_{\partial_i}\mu\right) \\ &= -\partial_j\left(\mathrm{H}-\mathcal{H}\right)\cdot\bar{g}\left(\mu, \nabla^{F^*TN}_{\partial_i}\mu\right) - (\mathrm{H}-\mathcal{H})\bar{g}\left(\nabla^{F^*TN}_{\partial_j}\mu, \nabla^{F^*TN}_{\partial_i}\mu\right) \\ &= -(\mathrm{H}-\mathcal{H})g^{kl}\,\mathrm{h}_{ik}\,\mathrm{h}_{jl}, \end{aligned} \tag{11.2.6}$$

where we used $\bar{g}\left(\mu, \nabla^{F^*TN}_{\partial_i}\mu\right) = 0$ by the metric property of the pull-back connection and the fact that $\mu$ is of unit length. Equation (11.2.7) now follows by substituting (11.2.6) and (11.2.5) in (11.2.4). □

**Corollary 11.2.1**
*The mean curvature evolves along* (9.1.5) *by*

$$
\begin{aligned}
(\partial_s + \Delta)(\mathrm{H} - \mathcal{H}) &= -(\mathrm{H} - \mathcal{H})\Big(\|\,\mathrm{h}\,\|^2 + \mathrm{Ric}^N(\mu,\mu)\Big), \\
(\partial_s + \Delta)(\mathrm{H} - \mathcal{H})^2 &= -2\,(\mathrm{H} - \mathcal{H})^2\,\Big(\|\,\mathrm{h}\,\|^2 + \mathrm{Ric}^N(\mu,\mu)\Big) \\
&\qquad - 2|\nabla\,(\mathrm{H} - \mathcal{H})\,|^2.
\end{aligned}
\tag{11.2.7}
$$

***Proof:***
The second evolution equation is a direct consequence of the first one. For the first equation we compute by Propositions 11.2.1 and 11.2.2

$$
\begin{aligned}
\partial_s\,\mathrm{H} =& \partial_s\left(g^{ij}\,\mathrm{h}_{ij}\right) = \partial_s g^{ij}\cdot\mathrm{h}_{ij} + g^{ij}\cdot\partial_s\,\mathrm{h}_{ij} \\
=& -2\,(\mathrm{H} - \mathcal{H})\,\|\,\mathrm{h}\,\|^2 + g^{ij}\partial_s\,\mathrm{h}_{ij} \\
=& -2\,(\mathrm{H} - \mathcal{H})\,\|\,\mathrm{h}\,\|^2 + (\mathrm{H} - \mathcal{H})\left(\|\,\mathrm{h}\,\|^2 - \mathrm{Ric}^N(\mu,\mu)\right) - \Delta\,(\mathrm{H} - \mathcal{H})\,.
\end{aligned}
$$

□

**Remark 11.2.1**
*We want to point out a difference between the first equation in* (11.2.7) *and the same evolution equation in the proof of [*EcHu91*, Proposition 4.6]. In the latter one sees an extra term* $\overline{g}\left(\overline{\nabla}\mathcal{H},\mu\right)$*. Its presence is due to the function* $\mathcal{H}$ *being defined in [*EcHu91*] on the ambient Lorentzian manifold* $(N,\overline{g})$ *while in our case* $\mathcal{H}$ *is defined on* $(\overline{M},\widetilde{g})$*. In particular, in our case* $\partial_s\mathcal{H}$ *is just vanishing.*

# Chapter 12

# $C^1$-estimates: preserving space-likeness

Everything is now set-up to prove the final result of this work. We claim that, for as long as a prescribed (graphical) mean curvature flow exists, it stays space-like. As mentioned in Chapter 11 this revolves around the evolution equations derived in §11.1 and §11.2 and the parabolic maximum principle in §2.4.

In Chapter 11 we also mentioned that the gradient function would have played a crucial role here; indeed the following holds.

**Proposition 12.0.1**
*If the gradient function $v$ is uniformly bounded along the flow (9.1.5), then the prescribed mean curvature flow (9.1.5) stays space-like.*

***Proof:***
Assume there exists some $K > 1$ so that $v = v(p, s) \leq K$ for every $(p, s) \in M \times [0, T]$. Note that the requirement $K > 1$ follows from Proposition 8.3.2 $(ii)$. Equation (8.3.5) implies

$$f(u) \leq K\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}},$$

where $\widetilde{\nabla} u$ is as before the gradient of $u$ with respect to $\widetilde{g}$. We conclude

$$|\widetilde{\nabla} u|^2_{\widetilde{g}} \leq \left(1 - \frac{1}{K^2}\right) f(u)^2 < f(u)^2.$$

Notice that the above is precisely the condition required for a graph to be space-like as pointed out in Corollary 8.3.1. □

It is therefore clear that our aim is to $v$ to be uniformly bounded along the flow. Unfortunately, in order to do so, we need to add some extra conditions. Namely we will put ourselves in the following setting.

**Assumptions 3**
*Consider the Hölder spaces $C^{k,\alpha}_{\Phi}(M)$ with integer $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, defined with respect to the Riemannian metric $\widetilde{g}$. We impose*

1. **Well behaved warping:** *The warping function $f : \mathbb{R} \to \mathbb{R}$ is smooth and bounded from below away from zero, with uniformly bounded derivatives.*

2. **initial regularity:** *$u_0 \in C^{3,\alpha}_{\Phi}(M)$ and $\mathcal{H} \in C^{\iota,\alpha}_{\Phi}(M)$ with $\iota \geq 2$.*

3. **upper barrier:** *$\mathrm{H}(s=0) - \mathcal{H} \geq \delta > 0$ for some positive $\delta$.*

4. **Time-like convergence:** *$\mathrm{Ric}^N(X,X) > 0$ for any time-like $X \in TN$.*

While the initial regularity assumptions are natural (compare 1. and 2. in Assumption 3 with Remark 10.2.2), the other two assumptions are rather restrictive. Still, they already appear in [EcHu91], cf. also page 606 therein for the time-like convergence assumption. Gerhardt [Ger00] studies mean curvature flow without time-like convergence assumption, however we did not succeed in extending his arguments to the non-compact setting.

The conclusion of this chapter, and of the whole work, is then to prove the following theorem.

**Theorem 12.1**
*Consider the flow (9.1.5) with $\mathcal{H} \in C^{1,\alpha}_{\Phi}(M)$ and solution $u \in C^{2,\alpha}_{\Phi}(M \times [0,T])$. Assume that $u$ is uniformly bounded from above. Then the gradient function $v$ is uniformly bounded along the flow, with the bound depending only on the upper bound of $u$. In particular, the prescribed mean curvature flow stays space-like as long as the flow exists.*

The remainder of the section is concerned with proving Theorem 12.1. Following [Ger00], the proof idea consists on a clever application of the parabolic maximum principle. Before doing that some preparation is needed.

## 12.1 Preliminaries

We begin by showing that $\Phi$-manifolds satisfy the following bounded geometry condition.

**Lemma 12.1.1**
*Let $(\overline{M}, \widetilde{g})$ be a $\Phi$-manifold. Then for every $X \in \Gamma(T\overline{M})$, the Ricci tensor (cf. Definition 7.5) satisfies*

$$|\widetilde{\mathrm{Ric}}(X,X)| \leq C\|X\|^2_{\widetilde{g}}, \tag{12.1.1}$$

*for some positive constant $c$.*

***Proof:***
The proof relies on a tedious analysis of the $\widetilde{g}$-norm of the Ricci tensor. The computations get very messy quickly so we just present an idea on how this should work. Due to the nature of $\Phi$-manifolds we focus on $\widetilde{\mathrm{Ric}}$ only near the boundary. Recall that, in local coordinates,

$$|\widetilde{\mathrm{Ric}}|_{\widetilde{g}} = \widetilde{g}^{ik}\widetilde{g}^{jl}\widetilde{\mathrm{Ric}}_{ij}\widetilde{\mathrm{Ric}}_{kl}. \tag{12.1.2}$$

Now (cf. Equation (7.2.2) and Equation (7.1.8))

$$\widetilde{\mathrm{Ric}}_{ij} = \widetilde{\Gamma}^q_{ij,q} - \widetilde{\Gamma}^q_{qj,i} + \sum_p \widetilde{\Gamma}^p_{ij}\widetilde{\Gamma}^1_{pq} - \widetilde{\Gamma}^p_{qj}\widetilde{\Gamma}^q_{ip}.$$

Recall that repeated upper and lower indices, e.g. $\widetilde{g}^{ik}\widetilde{g}_{lm,i}$ give rise to Φ-derivatives of the term $\widetilde{g}_{lm}$. Thus, by looking at the expression for the Christoffel symbols (Equation (7.1.3) in Example 7.1.4) one sees that the terms arising from (12.1.2) are of one of the two possible form listed below

$$x^q X\left(Y(x^{-q})\right);\quad x^q X(x^{-q})x^p Y(x^{-p})$$

with $X$ and $Y$ being Φ-derivatives. Thus we conclude that the worst possible outcome of (12.1.2) is given by $x^0$ thus implying that it is bounded. □

We will now proceed by presenting some estimates which will be useful for the proof of Theorem 12.1, and hold for any given graphical embedding (not necessarily along the (9.1.5) flow).

**Proposition 12.1.1**
*Consider Assumptions 3. Assume the embeddings $F(\overline{M}) \equiv F(s)(\overline{M})$ are space-like for $s \in [0, T]$. Recall, $g = F^*\overline{g}$ denotes the induced metric on $\overline{M}$, $\mathrm{h}$ the scalar second fundamental and $v$ the gradient function. Then there exists a constant $c > 0$ independent of $u$, such that*

$$|g(\nabla u, \nabla v)| \leq \|\,\mathrm{h}\,\||\nabla u|_g^2 + c|\nabla u|_g^2 v. \tag{12.1.3}$$

***Proof:***
In local coordinates one has

$$g(\nabla u, \nabla v) = g^{ik}u_k v_i.$$

From equation (11.1.26) we infer

$$\begin{aligned} g(\nabla u, \nabla v) &= -g^{ik}u_k g^{jm}u_m\,\mathrm{h}_{ij} - \frac{f'(u)}{f(u)} g^{im}u_m u_i v \\ &= -g^{ik}u_k g^{jm}u_m\,\mathrm{h}_{ij} - \frac{f'(u)}{f(u)}|\nabla u|_g^2. \end{aligned}$$

Recall, by Assumptions 3 there exists some constant $c > 0$ so that $|f'/f|_\infty \leq c$. Thus we conclude

$$|g(\nabla u, \nabla v)| \leq |g^{ik}u_k g^{jm}u_m\,\mathrm{h}_{ij}\,| + \left|\frac{f'(u)}{f(u)}\right||\nabla u|_g^2 v \leq \|\,\mathrm{h}\,\||\nabla u|_g^2 + c_1|\nabla u|_g^2 v,$$

where $\|\,\mathrm{h}\,\|$ denotes the norm of the scalar second fundamental form h with respect to the metric $g$. □

Next we present an estimate for $\mathrm{Ric}^N(\mu, \mu)$.

**Proposition 12.1.2**
*We continue as in Proposition 12.1.1. Then there exists a constant $c > 0$ independent of $u$, such that*

$$|\,\mathrm{Ric}^N(\mu, \mu)| \leq cv^2. \tag{12.1.4}$$

***Proof:***
From the local expression of the unit normal $\mu$ in (8.3.4) we find

$$\operatorname{Ric}^N(\mu,\mu) = v^2 \operatorname{Ric}^N(\partial_t,\partial_t) + \frac{2v^2}{f(u)^2}\operatorname{Ric}^N(\partial_t,\widetilde{\nabla}u) + \frac{v^2}{f(u)^4}\operatorname{Ric}^N(\widetilde{\nabla}u,\widetilde{\nabla}u).$$

Proposition 8.1.1 gives

$$\operatorname{Ric}^N(\mu,\mu) = -mv^2\frac{f''(u)}{f(u)} + \frac{2v^2}{f(u)^2}\operatorname{Ric}^N(\partial_t,\widetilde{\nabla}u) + \frac{v^2}{f(u)^4}\operatorname{Ric}^N(\widetilde{\nabla}u,\widetilde{\nabla}u).$$

The second term vanishes due to Proposition 8.1.1. Again, from Proposition 8.1.1 we infer for the third term

$$\frac{v^2}{f(u)^4}\operatorname{Ric}^N(\widetilde{\nabla}u,\widetilde{\nabla}u) = \frac{v^2}{f(u)^4}\widetilde{\operatorname{Ric}}(\widetilde{\nabla}u,\widetilde{\nabla}u) + \frac{f''(u)}{f(u)^3}v^2|\widetilde{\nabla}u|^2_{\widetilde{g}} + (m-1)\frac{f'(u)^2}{f(u)^4}v^2|\widetilde{\nabla}u|^2_{\widetilde{g}}.$$

We plug this back into the expression for $\operatorname{Ric}^N(\mu,\mu)$ and conclude from *(iv)* in Proposition 8.3.2

$$\operatorname{Ric}^N(\mu,\mu) = mv^2\frac{f''(u)}{f(u)} + \frac{v^2}{f(u)^4}\widetilde{\operatorname{Ric}}(\widetilde{\nabla}u,\widetilde{\nabla}u) + \frac{f''(u)}{f(u)}|\nabla u|^2_g + (m-1)\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|^2_g.$$

By Setting 3 there exist some constants $c_1, c_2, c_3 > 0$ so that

$$|f(t)| \geq c_1; \quad \left|\frac{f'(t)}{f(t)}\right| \leq c_2; \quad \left|\frac{f''(t)}{f(t)}\right| \leq c_3; \quad \forall t \in \mathbb{R}, \tag{12.1.5}$$

By taking the absolute value and keeping in mind Equation 12.1.1 in Lemma 12.1.1 we obtain the following estimate

$$|\operatorname{Ric}^N(\mu,\mu)| \leq mc_3v^2 + \frac{c_4}{f(u)^4}v^2 + c_3|\nabla u|^2_g + (m-1)c_2^2|\nabla u|^2_g.$$

The statement now follows by noticing that $|\nabla u|^2_g \leq v^2$ by Proposition 8.3.2 and since $|f(t)| \geq c_1 > 0$ is bounded uniformly from below away from zero. □

Next we prove that hypersurfaces of $(N,\overline{g})$ arising as graphs of some Hölder regular functions satisfy the mean curvature structure condition, cf. chapter 3 in [Bar84].

**Proposition 12.1.3**
*We continue as in Proposition 12.1.1. Recall,* H *denotes the scalar mean curvature and* $h$ *the scalar second fundamental form. Then for any* $\varepsilon > 0$ *and some uniform constant* $c > 0$ *(independent of* $u$*) we have*

$$|\operatorname{H} + \operatorname{h}(\nabla u,\nabla u)| \leq \varepsilon v\|h\| + c\varepsilon^{-1}v^3. \tag{12.1.6}$$

***Proof:***
At any fixed $(p,s) \in \overline{M}\times[0,T]$ there exists an orthonormal (with respect to $g$) basis $\{e_i\}$ of $h$-eigenvectors, i.e. for the Kronecker delta $\delta_{ij}$

$$h(e_i,e_j) = h_i\delta_{ij}, \quad g(e_i,e_j) = \delta_{ij}.$$

With respect to that basis we compute at $(p, s)$ (writing $(\nabla u)_i := g(\nabla u, e_i)$)

$$\begin{aligned}|\,\mathrm{H}+\mathrm{h}(\nabla u,\nabla u)| &= \left|\sum_{i=1}^m h_i + \sum_{i=1}^m h_i(\nabla u)_i^2\right| \le \sum_{i=1}^m \Big(\frac{1+(\nabla u)_i^2}{\sqrt{\varepsilon v}}\Big)\sqrt{\varepsilon v}|h_i| \\ &\le \sum_{i=1}^m (v\varepsilon)^{-1}\Big(1+(\nabla u)_i^2\Big)^2 + \varepsilon v \sum_{i=1}^m h_i^2 \\ &\le (v\varepsilon)^{-1}\Big(m + 2|\nabla u|_g^2 + |\nabla u|_g^4\Big) + \varepsilon v\|h\|^2.\end{aligned}$$

By (8.3.7) we conclude for some $c > 0$ (independent of $u$ and $(p, s)$)

$$\mathrm{H}+\mathrm{h}(\nabla u,\nabla u) \le c\varepsilon^{-1}v^3 + \varepsilon v\|h\|^2.$$

□

We will need one last estimate.

**Proposition 12.1.4**
*We continue as in Proposition 12.1.1. Consider as above the (local) vector field $V$ on $\overline{M}$, so that $DF(V) = \partial_t^\top$. Then for every function $\mathcal{H} \in C^{1,\alpha}_\Phi(M)$ there exists some uniform constant $c > 0$ (independent of $u$) such that*

$$|V(\mathcal{H})| \le c\|\nabla u\|_g\|\mathcal{H}\|_{1,\alpha}. \tag{12.1.7}$$

***Proof:***
It is easy to see that the condition $DF(V) = \partial_t^\top$ gives $V = -\frac{\widetilde{\nabla} u}{f(u)^2}$.
Therefore in local coordinates, we obtain using (8.2.4) in the last estimate

$$|V(\mathcal{H})| = \left|\frac{1}{f(u)^2}\widetilde{g}^{ij}u_i\mathcal{H}_j\right| \le c|\widetilde{\nabla} u|_{\widetilde{g}}\|\mathcal{H}\|_{1,\alpha} \le c|\nabla u|_g\|\mathcal{H}\|_{1,\alpha},$$

where we used the fact that $f > 0$ is uniformly bounded away from zero. □

We are now ready to prove Theorem 12.1.

## 12.2 Proof of the main theorem

We will use the ideas of the argument of [Ger00] with some adaptations due to non-compact geometry. In the upcoming computations we will systematically suppress the point $(p, s) \in \overline{M} \times [0, T]$ from notation. We consider some constants $\lambda, \rho > 0$, which we will specify later.

Let $\varphi = e^{\rho e^{\lambda u}}$. Assume, without loss of generality that $u > 1$, if it is not the case we can consider $u + C$ for some constant $C > 0$ large enough. An easy computation gives

$$(\partial_s + \Delta)\varphi = -\rho\lambda^2 e^{\lambda u}(1 + \rho e^{\lambda u})\varphi|\nabla u|_g^2 + \rho\lambda e^{\lambda u}\varphi(\partial_s + \Delta)u. \tag{12.2.1}$$

Let us now set $w = \varphi v$. Therefore we find (recall $\mu$ is defined in (8.3.4))

$$\begin{aligned}(\partial_s + \Delta)w &= v(\partial_s + \Delta)\varphi + \varphi(\partial_s + \Delta)v - 2g(\nabla\varphi, \nabla v) \\ &= v(\partial_s + \Delta)\varphi + \varphi(\partial_s + \Delta)v - 2\rho\lambda e^{\lambda u}\varphi g(\nabla u, \nabla v).\end{aligned}$$

Substituting (12.2.1) and (11.1.1) in the above, we obtain

$$(\partial_s + \Delta)w = I_1 + I_2.$$

where $I_1$ and $I_2$ are explicitly given as follows (recall $\mu$ is defined in (8.3.4))

$$\begin{aligned} I_1 := & -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2 \varphi v - \|\,\mathrm{h}\,\|^2\varphi v - V(\mathcal{H})\varphi - 2\frac{f'(u)}{f(u)}\,\mathrm{H}\,\varphi \\ & - 2\left(\rho\lambda e^{\lambda u} - \frac{f'(u)}{f(u)}\right) g(\nabla u, \nabla v)\varphi - \left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2 \varphi v, \\ I_2 := & \,\rho\lambda e^{\lambda u}\varphi v(\partial_s + \Delta)u - \mathrm{Ric}^N(\mu,\mu)\varphi + \frac{f'(u)}{f(u)}\mathcal{H}\varphi - \frac{f'(u)}{f(u)}\mathcal{H}\varphi v^2 \\ & + m\frac{f''(u)}{f(u)}\varphi v - \frac{f''(u)}{f(u)}|\nabla u|_g^2\varphi v - m\left(\frac{f'(u)}{f(u)}\right)^2 \varphi v \end{aligned}$$

First, we estimate $I_2$ from above. By Assumptions 3 there exist some constants $c_1, c_2, c_3 > 0$ such that the warping function $f : \mathbb{R} \to \mathbb{R}^+$ satisfies for any $t \in \mathbb{R}$

$$|f(t)| \geq c_1, \quad \left|\frac{f'(t)}{f(t)}\right| \leq c_2, \quad \left|\frac{f''(t)}{f(t)}\right| \leq c_3.$$

From equation (9.1.6) we now deduce for some $c_4 > 0$ depending on $\|\mathcal{H}\|_\infty$

$$(\partial_s + \Delta)u \leq c_4 v^2.$$

Since $|\nabla u|_g \leq v$ by *(iii)* in Proposition 8.3.2, we arrive by Propositions 12.1.2 and 12.1.4 at the following estimate of $I_2$ (we write $c > 0$ for any uniform positive constant)

$$I_2 \leq c\rho\lambda e^{\lambda u}\varphi v^2 + c|\nabla u|_g^2\varphi v + c|\nabla u|_g\varphi \leq c\rho\lambda e^{\lambda u}\varphi v^3.$$

The estimate of $I_1$ is slightly more involved. Using the formula from Proposition (11.1.6)

$$\mathrm{h}(\nabla u, \nabla u) = -g(\nabla u, \nabla v) - \frac{f'(u)}{f(u)}|\nabla u|^2 v,$$

we can rewrite $I_1$ as follows

$$\begin{aligned} I_1 = & -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2\varphi v - \|\,\mathrm{h}\,\|^2\varphi v - 2\frac{f'(u)}{f(u)}\Big(\mathrm{H} + \mathrm{h}(\nabla u, \nabla u)\Big)\varphi \\ & - 3\left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2\varphi v - m\left(\frac{f'(u)}{f(u)}\right)^2 \varphi v - 2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi. \end{aligned}$$

By Proposition 12.1.3 we find for some uniform constant $c > 0$ (in fact we will not differentiate between positive uniform constants and denote them all by $c$)

$$\begin{aligned} I_1 \leq & -\rho\lambda^2 e^{\lambda u}(1+\rho e^{\lambda u})|\nabla u|_g^2\varphi v \\ & - \left(1 - 2\left|\frac{f'(u)}{f(u)}\right|\varepsilon\right)\|\,\mathrm{h}\,\|^2\varphi v - 3\left(\frac{f'(u)}{f(u)}\right)^2 |\nabla u|_g^2\varphi v \\ & + 2c\left|\frac{f'(u)}{f(u)}\right|\varepsilon^{-1}\varphi v^3 - 2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi. \end{aligned} \tag{12.2.2}$$

We now want to estimate the last term above. By Proposition 12.1.1 we have for some uniform constant $c > 0$

$$
\begin{aligned}
-2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi &\leq 2\rho\lambda e^{\lambda u} |g(\nabla u, \nabla v)|\varphi \\
&\leq 2\rho\lambda e^{\lambda u}\Big(\|h\||\nabla u|_g^2 + c|\nabla u|_g^2 v\Big)\varphi.
\end{aligned}
$$

We estimate this further for any $\varepsilon' > 0$ and using (8.3.7) in the last step

$$
\begin{aligned}
-2\rho\lambda e^{\lambda u} g(\nabla u, \nabla v)\varphi &\leq \frac{2\rho\lambda e^{\lambda u}|\nabla u|_g^2}{\sqrt{2(1-\varepsilon')v}}\sqrt{2(1-\varepsilon')v}\|\,\mathrm{h}\,\|\varphi + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2\varphi v \\
&\leq \frac{\rho^2\lambda^2 e^{2\lambda u}|\nabla u|_g^4}{(1-\varepsilon')v}\varphi + (1-\varepsilon')\|\,\mathrm{h}\,\|^2\varphi v + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2\varphi v \\
&\leq \frac{\rho^2\lambda^2 e^{2\lambda u}}{(1-\varepsilon')}|\nabla u|_g^2\varphi v + (1-\varepsilon')\|\,\mathrm{h}\,\|^2\varphi v + 2c\rho\lambda e^{\lambda u}|\nabla u|_g^2\varphi v.
\end{aligned}
$$

Choosing for any given $\varepsilon' \in (0,1)$ an $\varepsilon > 0$ sufficiently small such that $\varepsilon' > 2\left|\frac{f'(u)}{f(u)}\right|\varepsilon$ and plugging the last estimate into (12.2.2), we arrive at

$$
\begin{aligned}
I_1 \leq\ &-\rho\lambda e^{\lambda u}\Big(\lambda - 2c\Big)|\nabla u|_g^2\varphi v - 3\left(\frac{f'(u)}{f(u)}\right)^2|\nabla u|_g^2\varphi v \\
&+ 2c\left|\frac{f'(u)}{f(u)}\right|\varepsilon^{-1}\varphi v^3 + \frac{\varepsilon'}{(1-\varepsilon')}\rho^2\lambda^2 e^{2\lambda u}|\nabla u|_g^2\varphi v.
\end{aligned} \tag{12.2.3}
$$

Set $\varepsilon' = e^{-\lambda u}$ and $\rho = 1/2$. Choose $\overline{\lambda} > 0$ so that for every $\lambda > \overline{\lambda}$

$$
\frac{\rho}{1 - e^{-\lambda u}} \leq \frac{3}{4}.
$$

Then we can estimate $I_1$ even further by (recall $|\nabla u|_g \leq v$ by Proposition 8.3.2)

$$
I_1 \leq\ -\frac{1}{8}\lambda e^{\lambda u}\Big(\lambda - c\Big)|\nabla u|_g^2\varphi v + c\lambda e^{\lambda u}\varphi v^3.
$$

We want to point out that the above estimates follows by considering $\overline{\lambda}$ to be large enough so that the second and third term in (12.2.3) can be estimated by the second term in the equation above.
Summarizing, we arrive at the following intermediate estimate

$$
(\partial_s + \Delta)w \leq -\frac{1}{8}\lambda e^{\lambda u}\Big(\Big(\lambda - c\Big)|\nabla u|_g^2 - cv^2\Big)w. \tag{12.2.4}
$$

We want to turn this into a differential inequality for the supremum

$$
v_{\sup}(s) = \sup_{p\in M} v(p,s).
$$

Let us assume that there exists some $s_0 \in [0,T]$ such that $v_{\sup}(s_0) > 2$, otherwise the statement is trivial. Since by Proposition 2.4.1, $v_{\sup}(s)$ is locally Lipschitz, $v_{\sup}(s) > 2$ in an open intervall $I = (a,b) \subset [0,T]$ containing $s_0$. We take the minimal possible such $a \geq 0$, such that by continuity of $v_{\sup}(s)$ we have either $a = 0$ or $v_{\sup}(a) = 2$.

Consider $s \in (a, b)$ and a sequence $(p_k(s)) \subset M$ satisfying (2.2.3). Then for $k \in \mathbb{N}$ sufficiently large, $v(p_k(s), s) > 2$ and we establish a differential evolution inequality for $v$ at those points as follows. We consider $v$ and $w$ evaluated at $(p_k(s), s)$ without making it notationally explicit. Since $v \geq 2$, we have $-4 \geq -v^2$ and from $(i)$ in Proposition 8.3.2 we find

$$|\nabla u|_g^2 = v^2 - 1 \geq v^2 - \frac{v^2}{4} = \frac{3}{4}v^2. \tag{12.2.5}$$

Choosing $\lambda > \overline{\lambda}$ sufficiently large (note that these choices do not depend on $u$) the right hand side of (12.2.4), evaluated at $(p_k(s))$ for $k \in \mathbb{N}$ sufficiently large, turns negative and we conclude

$$(\partial_s + \Delta)w(p_k(s), s) \leq 0.$$

This implies by Proposition 2.4.1 for any $s \in (a, b)$ in the limit $k \to \infty$

$$\partial_s w_{\sup}(s) \leq 0.$$

Thus for any $s \in (a, b)$ we conclude $w(\cdot, s) \leq w_{\sup}(s) \leq w_{\sup}(a)$. In particular, we find for any $(p, s) \in M \times (a, b)$ and some constant $c > 0$, depending only on $\mathcal{H}$, $u(s = a)$ and the ambient geometry, that (note that $e^{\rho e^{\lambda u}} > 1$)

$$v(p, s) \leq \exp\Big(\rho e^{\lambda u_{\sup}(a)}\Big) v_{\sup}(a) < c v_{\sup}(a), \tag{12.2.6}$$

where the second estimate holds, provided $u$ is bounded uniformly from above. Now, since we have either $a = 0$ or $v_{\sup}(a) = 2$, we conclude that $v$ is uniformly bounded.

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