Quasihomogeneous Blow-Ups and Pseudodifferential Calculus on $\overline{\mathrm{SL}}(n, \mathbb{R})$

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Kurzfassung

Der erste Teil der vorliegenden Dissertation beschäftigt sich mit *quasihomogenen Blow-ups* einer Untermannigfaltigkeit Y in einer umgebenen Mannigfaltigkeit mit Ecken. Quasihomogene Blow-ups verallgemeinern das Konzept von radiellen Blow-ups. Die Grundidee ist das Zuordnen von *Gewichten* zu Funktionen, die auf Y verschwinden. Diese Idee wird durch eine Filtrierung

$$\mathcal{I}(Y) = \mathcal{F}^{(1)} \subseteq \mathcal{F}^{(2)} \subseteq \mathcal{F}^{(3)} \subseteq \dots \tag{0.1}$$

realisiert, wobei $\mathcal{F}^{(m)}$ ein Raum glatter Funktionen ist, deren *quasihomogene Ordnung* bei Y mindestens m ist. Der Spezialfall $\mathcal{I} \subset \mathcal{I}^2 \subset \mathcal{I}^3 \subset \dots$ entspricht hierbei dem klassischen radiellen Blow-up. Obwohl quasihomogene Blow-ups bereits in verschiedenen Arbeiten verwendet werden (siehe zum Beispiel [7, 9, 12]), wurden in diesen Arbeiten bisher nur Spezialfälle eingeführt. Eine Ausarbeitung allgemeiner quasihomogenen Blow-ups wurde in [22] begonnen, und teilweise aufgeschrieben. In dieser Arbeit liefern wir eine formale Definition allgemeiner quasihomogener Blow-ups und vervollständigen die vorläufige Einführung in [22].

Der zweite Teil dieser Arbeit beschäftigt sich mit der *Hd-Kompaktifizierung* $\overline{\mathrm{SL}}(n, \mathbb{R})$ von $\mathrm{SL}(n, \mathbb{R})$. Diese Kompaktifizierung wurde für allgemeine semisimple Lie Gruppen von Albin, Dimakis, Melrose und Vogan in [1] eingeführt. $\overline{\mathrm{SL}}(n, \mathbb{R})$ ist eine Mannigfaltigkeit mit Ecken, deren Randflächen in einer Bijektion mit den Äquivalenzklassen von parabolischen Untergruppen von $\mathrm{SL}(n, \mathbb{R})$ stehen. Wir konstruieren eine Resolution von $\overline{\mathrm{SL}}(n, \mathbb{R})$, die wir mit X bezeichnen, sodass die rechts-invarianten Differentialoperatoren auf $\mathrm{SL}(n, \mathbb{R})$ zu Operatoren auf X mit simplem Verhalten an den verschieden Randflächen von X geliftet werden.

Wir konstruieren eine Algebra von Pseudodifferentialoperatoren Ψ_X^m auf X, die sowohl die Lifts von rechts-invarianten Differentialoperatoren auf $\mathrm{SL}(n, \mathbb{R})$ enthalten, als auch erste Parametrixen für solche Operatoren. Wir definieren Ψ_X^m mithilfe einer Resolution von X^2, die wir mit X_e^2 bezeichnen und die aus X^2 durch eine Folge von quasihomogenen Blow-ups konstruiert wird. Wir zeigen ein Kompositions-Theorem für Operatoren in Ψ_X^m mithilfe eine Resolution von X^3, die wir mit X_e^3 bezeichnen.

Abstract

In the first part of this thesis we consider the *quasihomogeneous blow-up* of a submanifold Y in a surrounding manifold with corners X. It generalizes the concept of radial blow-up and revolves around the idea of assigning different *weights* to functions vanishing at the submanifold Y. This yields a filtration of the space $\mathcal{I}(Y)$ of smooth functions vanishing on Y, given by subspaces of function vanishing to a certain *quasihomogeneous order*

$$\mathcal{I}(Y) = \mathcal{F}^{(1)} \subseteq \mathcal{F}^{(2)} \subseteq \mathcal{F}^{(3)} \subseteq \dots \tag{0.2}$$

It generalizes the filtration $\mathcal{I} \subset \mathcal{I}^2 \subset \mathcal{I}^3 \subset \dots$. Although quasihomogeneous blow-ups are used in [7, 9, 12], only special cases of this concept are introduced in these works, sufficient for the situations considered in them. An exhaustive treatment of the general case was started and written down partially in [22]. In this thesis, we provide a detailed formal definition of quasihomogeneous blow-ups and thereby, complete the preliminary introduction to this concept in [22].

In the second part of this thesis we consider the *hd-compactification* of $\mathrm{SL}(n, \mathbb{R})$, denoted by $\overline{\mathrm{SL}}(n, \mathbb{R})$, introduced by Albin, Dimakis, Melrose and Vogan in [1] for arbitrary semisimple Lie groups. $\overline{\mathrm{SL}}(n, \mathbb{R})$ is a compact manifold with corners, with boundary faces corresponding to conjugacy classes of parabolic subgroups. We introduce a resolution of $\overline{\mathrm{SL}}(n, \mathbb{R})$, denoted by X, such that the right-invariant differential operators on $\mathrm{SL}(n, \mathbb{R})$ lift to X, and become operators with simple degeneracies at the different boundary faces of X.

We construct an algebra of pseudodifferential operators Ψ^m_X on X that contains the lifts of right-invariant differential operators on $\mathrm{SL}(n, \mathbb{R})$, together with basic parametrices of these operators. It is constructed using a resolution of X^2, denoted X^2_e, by a series of blow-ups. Since the lift of right-invariant operators on X vanish to a variety of different degrees at the boundary faces, quasihomogeneous blow-ups are used. A composition theorem for operators in Ψ^m_X is proven, using a resolution of the triple product space X^3_e.

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Contents

CHAPTER 1

Introduction

The analysis of elliptic operators on noncompact and singular spaces is a large field of research and has been growing steadily in the past decades. A rather geometric approach used in this field is to compactifiy or desingularize the space in question to a manifold with boundary (or corners), on which the operators one wants to analyze have a specific degeneracy at the boundary. It is a general scheme in this field to 'trade' non-compactness or singularities for boundary together with certain degeneracy at this boundary. For a very simple example of such a process, consider the '$+\infty$'-end of the real line $\mathbb{R}$, say the interval $(1, \infty)$. Using the map $f : (1, \infty) \to (0, 1),\ x \mapsto 1/x$, one may take $[0, 1]$ as a compactification. When the old coordinate x approaches ∞, the new coordinate $t = 1/x$ approaches the new boundary $t = 0$. The vector field ∂_x (and similarly any differential operator) then becomes the degenerate vector field $-t^2\partial_t$.
One approach to analyze operators on these compact manifolds with boundaries is by studying algebras of pseudodifferential operators adapted to the degenerate behavior at the boundary. Unlike the case of a compact manifold without boundary, where such a calculus is discussed by Hörmander in [10], there is not one specific natural algebra, but rather several different ones, all arising naturally from the underlying geometry under consideration. Due to the different phenomena that may occur there is little general theory surrounding these different pseudodifferential calculi. Rather, great effort has gone into a large collection of such algebras. The first of these was the b-calculus by Richard Melrose, introduced in [21], also present in [23]. Using similar methods, a lot of which introduced by Melrose, there followed numerous different calculi, for example [16, 15, 26, 7, 20, 17, 14, 13, 9, 25, 6]. Many of the steps involved in building an elliptic theory for these different calculi are based on the same set of principles. One of those is the definition of the algebra itself. In the original pseudodifferential calculus on a compact manifold X without boundary, the algebra of pseudodifferential operators is defined by characterizing their Schwartz kernels as distributions on the double product space X^2. More specifically, they are characterized by their singular behavior near the diagonal $\triangle_X = \{(p, p) \mid p \in X\} \subset X^2$. In the more singular settings mentioned above, this characterization needs to include the behavior at the boundary of X. Even if X has smooth boundary, the double space X^2 is already a manifold with corners and the diagonal $\triangle_X \subset X^2$ meets this boundary in an analytically rather poor way. Analyzing the Schwartz kernels near the intersection of the diagonal with the boundary in X^2 is usually one of the biggest challenges of these approaches. A commonly used geometric approach is to construct a resolution of the space X^2 into a manifold with corners $B(X^2)$ via a sequence of *real blow-ups*, which is a geometric way of constructing a new manifold form an old one by introducing polar coordinates along a

submanifold. The resolved space $B(X^2)$ is equipped with a map back $\beta : B(X^2) \to X^2$ called the *blow-down map*, under which the diagonal lifts and becomes resolved at the boundary of $B(X^2)$. Then an algebra of pseudodifferential operators can be defined as having Schwartz kernels on X^2 that are given push-forwards of distributions on $B(X^2)$ with certain simple degeneracies at the different boundaries of $B(X^2)$.

1.1. Major contributions of this thesis

The concept of manifolds with corners and real blow-up of submanifolds are essential tools in the different calculi mentioned above. In some situations (for example in [7, 9, 12]), a generalization of real blow-up of submanifolds is used, called *quasihomogeneous blow-ups*. It revolves around the concept of assigning weights to functions vanishing at the submanifold in question. Although quasihomogeneous blow-ups are used in [7, 9, 12], only special cases of this concept are introduced in these works, sufficient for the situations considered in them. An exhaustive treatment of the general case was started and written down partially in [22].

The first major contribution of this thesis, is to provide a detailed formal definition of quasihomogeneous blow-ups and thereby, complete the preliminary introduction to this concept in [22].

In Part 2 of this thesis we consider right- (or left-) invariant differential operators on the Special Linear Group $\mathrm{SL}(n,\mathbb{R})$. This is a non-compact smooth manifold. In [1], Albin, Dimakis, Melrose and Vogan introduced the *hd-compactification* for semisimple Lie Groups. For the case of $\mathrm{SL}(n,\mathbb{R})$, we denote this compactification by $\overline{\mathrm{SL}}(n,\mathbb{R})$. It is a compact manifold with corners. We move to a resolution of $\overline{\mathrm{SL}}(n,\mathbb{R})$, denoted by X. The algebra of right-invariant differential operators on $\mathrm{SL}(n,\mathbb{R})$ lifts to X and becomes an algebra of operators with simple degeneracies at the different boundary faces of X.

The second major contribution of this thesis is the construction of an algebra of pseudodifferential operators Ψ^m_X on X that contains the lifts of right-invariant differential operators on $\mathrm{SL}(n,\mathbb{R})$, together with basic parametrices of these operators. It is constructed using a resolution of X^2, denoted X^2_e, by a series of blow-ups. Here, quasihomogeneous blow-ups are used. A composition theorem for operators in Ψ^m_X is proven, using a resolution of the triple product space X^3_e.

The thesis is structured as follows: In Section 1.2 of this introduction we shortly recall the basic definitions surrounding manifolds with corners and real radial blow-up, in an effort to clearly outline the generalization to the quasihomogeneous case. In Section 1.3 we introduce the concept of a quasihomogeneous structure and the corresponding blow-up. In Section 1.4, we introduce the hd-compactification $\overline{\mathrm{SL}}(n,\mathbb{R})$, the resolution space X, and the lifted differential operators on X.

In Chapter 2, we give the definition and basic properties of quasihomogeneous blow-ups. In Chapter 3, we generalize commutativity results from the case of classical radial blow-ups to the case of quasihomogeneous ones. In Chapter 4, marking the beginning of Part 2 of this thesis, we shortly recall some Lie Group theory, that is necessary for the following analysis. In Chapter 5, we introduce the hd-compactification of $\mathrm{SL}(n, \mathbb{R})$. In Chapter 6, we restrict to the case of $\mathrm{SL}(3, \mathbb{R})$ and introduce the resolution space of the hd-compactification, analyze the lift of right- and left invariant operators to this resolved space, and start to develop an elliptic theory for theses operators. In Chapter 7, we generalize the results of Chapter 6 for arbitrary n. Finally, we give a outlook on potential future work in Chapter 8.

1.2. Basic definitions of manifolds and radial blow-up

A smooth real *manifold with boundary* of dimension n is a topological space that is equipped with a smooth structure locally modeled over the half space $\mathbb{R}^n_1 = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ where $\mathbb{R}_+ := [0, \infty)$. Analogously, a *manifold with corners* X is a space with a smooth structure locally modeled over $\mathbb{R}^n_k := \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$. Thus for each point $p \in X$ there is a diffeomorphism mapping an open neighborhood of p into an open domain in $\mathbb{R}^n_k$ (equipped with the restricted topology of $\mathbb{R}^n$). Each point $p \in X$ has a uniquely defined *codimension* k given by the minimal k such that a neighborhood of p may be modeled over $\mathbb{R}^n_k$[1]. The closure of a connected component of points with a fixed codimension is called a *boundary face* of X. The set of all boundary faces is denoted by $M(X)$, those of codimension k by $M_k(X)$. As part of the definition, we require each boundary face to be embedded. It follows from this that each boundary face of a manifold with corners is again a manifold with corners. Boundary faces of codimension 1 are called *boundary hypersurfaces.* Each of the hypersurfaces $H \in M_1(X)$ has a globally defined *boundary defining function* x_H, which is a non-negative smooth function on X such that $H = \{x_H = 0\}$ and dx_H is non vanishing on H. This is of course not unique.

As an example consider the manifold with corners $\mathbb{R}^3_2 = \{(x_1, x_2, y) \mid x_i \geq 0\}$. It has two hypersurfaces $H_1 = \{x_1 = 0\}$, $H_2 = \{x_2 = 0\}$ with boundary defining functions for example $x_1,\ x_2$ respectively. And it has one codimension 2 boundary face $F = H_1 \cap H_2$.

Vector fields: The tangent bundle TX of a manifold with corners may be constructed analogously to a manifold with boundary. Recall however, that tangent vectors do not need to relate to the boundary. For each $p \in X$, the tangent space has a inward pointing part $T^+_p X \subset T_p X$. These no longer form a vector bundle. There is however another natural appearing vector bundle other then TX on a manifold with corners that does relate nicely to the boundary, called the *compressed tangent bundle* and denoted by ${}^bTX$. More precisely it is defined as follows.

Let $\mathcal{V}_b$ be the space of all smooth vector fields on X that are tangent to all boundary faces. In a local model $\mathbb{R}^n_k = \{(x_1, \dots, x_k, y_1, \dots y_{n-k}) \mid x_i \geq 0\}$ they are spanned by $x_i \partial_{x_i},\ \partial_{y_i}$. They form a Lie algebroid and thus are the C^∞-sections of a vector bundle, denoted by

[1] Notice that one could also model a smooth manifold without boundary over $\mathbb{R}^n_k$, simply by staying away from the boundary. A point p having codimension k is equivalent to being able to choose a local model that maps p to $0 \in \mathbb{R}^n_k$.

${}^bTX$. This bundle can also be constructed directly by setting

$$ {}^bT_pX = \mathcal{V}_b(X) \big/ \mathcal{I}(p) \cdot \mathcal{V}_b(X). \tag{1.1} $$

Submanifolds: As with vector fields, general submanifolds may relate well or poorly to the boundary faces of a manifold with corners. There are several different classes for submanifolds defined on manifolds with corners (for an extensive treatment see, for example, [22]). The most relevant class of submanifolds in this thesis are so called *p-submanifolds*. A subset $Y \subset X$ is called a p-submanifold of X if for each $p \in Y$ there are local coordinates (x, x', y, y') $(x, x' \geq 0)$ of X such that locally $Y = \{x' = y' = 0\}$. The 'p' stands for product, since this is equivalent to saying that X takes local product form $X = Y \times X'$ near each point $p \in Y$. Boundary faces are always p-submanifolds. For another example, consider the two subsets $S, T \subset \mathbb{R}^2_2 = \{(x_1, x_2) \mid x_i \geq 0\}$ given by $S = \{x_1 = 1\}$ and $T = \{x_1 = x_2\}$. Then S is a p-submanifold of $\mathbb{R}^2_2$ and T is not, since near the origin no product coordinates can be chosen.

Radial blow-up: The concept of blowing up submanifolds $Y \subset X$ is a systematic and geometric way of introducing polar coordinates along Y. For an exhaustive treatment see [22], we only give a brief introduction here. We start with the local model. Consider again a p-submanifold $Y \subset X$. By definition, we may choose a local model such that

$$ Y = \mathbb{R}^{n'}_{k'} \times \{0\} \subset \mathbb{R}^n_k = X \ (n' < n, k' \leq k), \tag{1.2} $$

with standard coordinates x, x', y, y' such that $Y = \{x = y = 0\}$. Then local polar coordinates are given by x, y, r, ϕ with

$$ r \in (0, \infty), \ \phi \in \mathbb{S}^{n-n'-1}_{k-k'} := \mathbb{S}^{n-n'-1} \cap \mathbb{R}^{n-n'}_{k-k'} \tag{1.3} $$

where $\mathbb{S}^l \subset \mathbb{R}^l$ is the unit sphere. This identifies $X \setminus Y$ locally with $\mathbb{R}^{n'}_{k'} \times (0, \infty) \times \mathbb{S}^{n-n'-1}_{k-k'}$. The radial blow-up of $Y \subset X$ is given by compactifying this local model at $r = 0$. Thus the blow-up of $\mathbb{R}^{n'}_{k'} \times \{0\} \subset \mathbb{R}^n_k$ is a new manifold, denoted by $[\mathbb{R}^n_k; \mathbb{R}^{n'}_{k'} \times \{0\}]$ given by

$$ [\mathbb{R}^n_k; \mathbb{R}^{n'}_{k'} \times \{0\}] = \mathbb{R}^{n'}_{k'} \times [\mathbb{R}^{n-n'}_{k-k'}; \{0\}] = \mathbb{R}^{n'}_{k'} \times [0, \infty) \times \mathbb{S}^{n-n'-1}_{k-k'}. \tag{1.4} $$

This is again a manifold with corners[2].The newly created boundary hypersurface $\{r = 0\} \subset [\mathbb{R}^n_k; \mathbb{R}^{n'}_{k'} \times \{0\}]$ is called the *front face*, denoted by $\mathrm{ff}(\mathbb{R}^{n'}_{k'} \times \{0\})$. The blown-up space is equipped with a blow-down map

$$ \beta : [\mathbb{R}^n_k; \mathbb{R}^{n'}_{k'} \times \{0\}] \to \mathbb{R}^n_k \ , \ (x, y, r, \phi) \mapsto (x, y, r\phi). \tag{1.5} $$

The radius r is the lift of $r = \sqrt{|x|^2 + |y|^2}$ and is a global boundary defining function of $\mathrm{ff}(\mathbb{R}^{n'}_{k'} \times \{0\})$.

For a general submanifold $Y \subset X$, one can define the blow-up of Y on X, denoted by $[X; Y]$ using the local models above. Of course the local polar coordinates depend on the local product coordinates chosen in this construction. A (rather non trivial) fundamental result about blow-up, and ultimately the reason for its usefulness, is the fact the the space $[X; Y]$ is a uniquely defined manifold with corners independent of the choice of

[2]The only thing one needs to check is that $\mathbb{S}^{n-n'-1}_{k-k'} := \mathbb{S}^{n-n'-1} \cap \mathbb{R}^{n-n'}_{k-k'}$ is a manifold with corners, which is straight forward using standard polar coordinates on the sphere.

local coordinates. More precisely, any diffeomorphism on X that is equal to the identity on Y lifts to a diffeomorphism on $[X;Y]$. An alternative way of formulating this (that may give an better intuition of what this uniqueness actually means) is to say that the statement 'a function f is smooth in polar coordinates along Y' is well stated without any further information on what polar coordinates are meant precisely. One consequence of this naturality is that blow-up may be iterated easily, meaning a new p-submanifold in $[X;Y]$ may be chosen and blown up, allowing for the resolution on rather complicated situations.

There is also a global construction of $[X;Y]$ without the use of local coordinates: Given $Y \subset X$, the inward pointing part of its normal bundle is given at each $p \in Y$ by $N_p^+Y = {T_p^+X}/{T_pY}$ The inwards pointing part of the *spherical normal bundle* of Y is defined at each point by

$$S^+N_pY = {(N_p^+Y \setminus \{0\})}/{\mathbb{R}_+}. \tag{1.6}$$

The blown-up space may then be set as

$$[X;Y] := X \setminus Y \sqcup S^+NY. \tag{1.7}$$

The C^∞-structure is then defined using the existence of a normal fibration, which identifies a neighborhood of Y with a neighborhood of the zero section of the inward pointing part of the normal bundle N^+Y.

One of the most fundamental result about vector fields, regarding the question of how to use blow-ups as a means of resolution, is the following:

Theorem 1.1: *Let $Y \subset X$ be a closed p-submanifold. Then the space of vector fields $V \in \mathcal{V}_b(X)$ that are tangent to Y lift under $\beta : [X;Y] \to X$ to become smooth vector fields on $[X;Y]$ that span, over $C^\infty([X;Y])$, the space $\mathcal{V}_b([X;Y])$.*

A proof may be found in [22], Proposition 5.3.1.

1.3. Quasihomogeneous blow-up

In the first part of this thesis we generalize the construction of radial blow-up to quasihomogeneous ones. In special cases, quasihomogeneous blow-ups have already been used extensively, for example in [12] or [7]. An exhaustive treatment of the general case was started and written down partially in [22]. The first major contribution of this thesis, is to provide a detailed formal definition of quasihomogeneous blow-ups and thereby, complete the preliminary introduction to this concept in [22].

Consider again a p-submanifold $Y \subset X$. Denote by $\mathcal{I}^m(Y)$ the ideal of those functions that vanish to order at least $m \in \mathbb{N}$ at Y, which form a filtration

$$\mathcal{I}(Y) \supset \mathcal{I}^2(Y) \supset \mathcal{I}^3(Y) \supset \dots \tag{1.8}$$

It is easy to see by definition of radial blow-up, that each $\mathcal{I}^m(Y)$ lifts to $[X;Y]$ to become elements of the ideal $\mathcal{I}^m(\mathrm{ff}(Y))$ of functions vanishing to order at least m on $\mathrm{ff}(Y)$. In local product coordinates $Y = \{x_i = 0\}$ these spaces are given by

$$\mathcal{I}^m(Y) = \sum_{|\alpha| \geq m} x^\alpha C^\infty(X). \tag{1.9}$$

A *quasihomogeneous structure* at $Y \subset X$ is a more general structure of this type. Intuitively, it is constructed by giving different coordinate functions different *weights*, resulting in a filtration of $\mathcal{I}(Y)$

$$\mathcal{I}(Y) = \mathcal{F}^{(1)} \supset \mathcal{F}^{(2)} \supset \mathcal{F}^{(3)} \supset \dots \tag{1.10}$$

given by ideals $\mathcal{F}^{(m)}$ consisting of those function whose 'weighted order' at Y is at least m (see Definition 2.27). As a simple example, consider the p-submanifold $\{0\} \subset \mathbb{R}^2$ and choose the standard coordinates x_1, x_2. Then giving x_1 the weight 1 and x_2 the weight 3 results in the ideals

$$\mathcal{F}^{(m)} = \sum_{a+3b \geq m} x_1^a x_2^b C^\infty(\mathbb{R}^2). \tag{1.11}$$

Meaning

$$\begin{aligned}
\mathcal{F}^{(1)} &= \operatorname{span}_{C^\infty}\{x_1, x_2\} \\
\mathcal{F}^{(2)} &= \operatorname{span}_{C^\infty}\{x_1^2, x_2\} \\
\mathcal{F}^{(3)} &= \operatorname{span}_{C^\infty}\{x_1^3, x_2\} \\
\mathcal{F}^{(4)} &= \operatorname{span}_{C^\infty}\{x_1^4, x_1x_2, x_2^2\} \\
\mathcal{F}^{(5)} &= \operatorname{span}_{C^\infty}\{x_1^5, x_1^2x_2, x_1x_2^2, x_2^2\} \\
&\dots
\end{aligned} \tag{1.12}$$

Locally, one possibility to define such a structure in general near a point in a submanifold $Y \subset X$ is to chose local product coordinates x_i, x'_j, $Y = \{x_i = 0\}$ (some of the x_i might be in $\mathbb{R}_+$) and associate to each x_i a weight κ_i. We may assume for simplicity that the κ_i are ordered $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_k$. Using multi indecies $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and $\kappa \cdot \alpha := \kappa_1\alpha_1 + \dots + \kappa_k\alpha_k$ we get a (locally defined) quasihomogeneous structure by

$$\mathcal{F}^{(m)} = \sum_{\kappa\alpha \geq m} x^\alpha C^\infty(X). \tag{1.13}$$

Such a filtration of $\mathcal{I}(Y)$ also yields a filtration of the conormal bundle N^*Y

$$N^*Y = S_1 \supset S_2 \supset S_3 \supset \dots \tag{1.14}$$

given by $S_m = \{df_{N^*Y} \mid f \in \mathcal{F}^{(m)}\}$. The largest m for which $S_m \neq 0$ is also the largest weight given to a function. We call such a filtration of the conormal bundle Y a *conormal filtration*. The *quasihomogeneous blow-up* of Y with respect to a quasihomogeneous structure will then again be a new manifold $[X;Y]_{qh}$ consisting of $X \setminus Y$ disjointly united with a front face $\text{ff}(Y)$ together with a blow-down map $\beta : [X;Y]_{qh} \to X$ under which each $\mathcal{F}^{(m)}$ lifts to become smooth and an element of the ideal $\mathcal{I}^m(\text{ff}(Y))$.

The idea of assigning different coordinate functions weights is used in several different areas of mathematics. Besides the use in differential geometry presented here, it is also used in algebraic geometry and many other fields, where it is also called *weighting* or goes

under yet another term. A recent paper by Meinrenken and Loizides ([18], at the present a paper in preparation) gives a comprehensive overview of the use of weighting in the setting of differential geometry. The idea of using a filtration as described above to define a quasihomogeneous structure in the setting of manifolds with corners and real blow-up goes back to Richard Melrose in [22]. There, the fundamental ideas and definitions surrounding such a structure and the blown-up space are already present, although details and proofs are only partially available. The fist part of the thesis aims to contribute a comprehensive treatment of quasihomogeneous blow-ups in the setting of p-submanifolds of manifolds with corners and thus, provide a theoretical foundation for potential future use in the resolution of singular structures.
More precisely, in Chapter 2 we introduce the basic definition of a quasihomogeneous structure at a submanifold $Y \subset X$ and the blown-up space $[X;Y]_{qh}$. In order to introduce this construction, consider again the local model in (1.2), however using simplified notation, given by coordinates x_i, x'_j with $Y = \{x_i = 0\}$ (thus, some of the x_i, x'_i are restricted to being positive). In the case of the radial blow-up $r = \sqrt{x_1^2 + \cdots + x_k^2}$ is a 1-homogeneous function that lifts to become a boundary defining function of the front face and a local model for the blown-up space was $[0,\infty) \times \mathbb{S}^{n-n'-1}_{k-k'} = [0,\infty) \times \{x \in \mathbb{R}^{n-n'-1}_{k-k'} \mid r(x) = 1\}$. In the quasihomogeneous case, where each x_i has an associated weight κ_i, we replace the radial function by one that has 'weighted homogeneity' 1, for example

$$r_\kappa(x) = \left(\sum_{i=1}^{k} x_i^{2(\kappa_k)!/\kappa_i}\right)^{1/2(\kappa_k)!}. \tag{1.15}$$

Since each x_i has 'weighted homogeneity' κ_i, the homogeneity of each summand is $2(\kappa_k)!$, making r_κ a 1-homogeneous function. The use of $(\kappa_k)!$ is somewhat arbitrary. It could be replaced with any number that is a multiple of all κ_i, that appear. We then may use as a model for the blown-up space

$$\underbrace{[0,\infty)}_{\ni r} \times (\mathbb{S}^{n-n'-1}_{k-k'})_{qh}, \tag{1.16}$$

where $(\mathbb{S}^{n-n'-1}_{k-k'})_{qh}$ is the inward pointing part of the non-round sphere

$$(\mathbb{S}^{n-n'-1}_{k-k'})_{qh} = \{x \in \mathbb{R}^{n-n'-1}_{k-k'} \mid r_\kappa(x) = 1\}. \tag{1.17}$$

Similar to the radial case, we will also give a global definition of the blown-up space in Chapter 2. One of the main challenges of the construction is to prove its naturality, meaning its independence of the chosen local coordinates. Of course, the construction of $[X;Y]_{qh}$ depends on the quasihomogeneous structure $\mathcal{F}^{(m)}$. One of the main theorems (Theorem 2.26) of Chapter 2 states that the blown-up space depends only on this filtration. To be more precise, given two quasihomogeneous structures $\mathcal{F}^{(m)}, \tilde{\mathcal{F}}^{(m)}$ at Y, the identity lifts to a diffeomorphism $[X;Y]_{qh} \cong [X;Y]_{\tilde{qh}}$ if and only if $\mathcal{F}^{(m)} = \tilde{\mathcal{F}}^{(m)}$.
Of course, this leaves an even more fundamental question open: what exactly defines a quasihomogeneous structure? One question that might be asked is if the filtration of the normal bundle $N^*Y = S_1 \supset S_2 \supset \dots$ defined above already fully defines such a structure. For an example that shows that this is not true, consider again $0 \in \mathbb{R}^2$ with

weights associated to x_1, x_2 being 1 and 3 respectively. Consider the two coordinate transformations

$$\begin{aligned} &\tilde{x}_1 = x_1,\ \tilde{x}_2 = x_2 + x_1^2 \\ &\tilde{x}_1 = x_1 + x_2^2,\ \tilde{x}_2 = x_2 \end{aligned} \tag{1.18}$$

with associated weights staying 1 and 3 in both cases respectively. A simple calculation shows that both transformations still yield the same filtration of N^*Y, but only the second one yields the same filtration of $\mathcal{I}(Y)$.

In Chapter 2, we give a full characterization of coordinate transformations under which a given quasihomogeneous structure is stable. Furthermore, we give a global, coordinate-free characterization of what filtrations $\mathcal{F}^{(m)}$ actually define a quasihomogeneous structure and show its equivalence to the existence of local coordinates under which it takes the form (1.13).

After these basic definitions and theorems, we continue in Chapter 2 by discussing vector fields and their lifts under quasihomogeneous blow-up. Recall that under radial blow-up, the ideal of all smooth vector fields $V \in \mathcal{V}_b(X)$ that are tangent to Y lift to become smooth vector fields on $[X;Y]$ and span, over C^∞, the ideal $\mathcal{V}_b([X;Y])$. Such a vector field V satisfies $V\mathcal{I}(Y)^m \subset \mathcal{I}(Y)^m$ for all m. In other words, a vector field is tangent to Y if and only if it does not decrease the order of vanishing of a function at Y. Similarly, given a quasihomogeneous structure at Y, one can define the Lie algebra of vector fields $V \in \mathcal{V}_b(X)$ that do not decrease the weighted order of a function, i.e. those that satisfy $V\mathcal{F}^{(m)} \subset \mathcal{F}^{(m)}$, denoted by $\mathcal{V}_\Pi^0$. We show, that these vector fields lift to $[X;Y]_{qh}$ to become smooth elements of $\mathcal{V}_b([X;Y]_{qh})$. Unfortunately, they only span $\mathcal{V}_b([X;Y]_{qh})$ over C^∞ almost everywhere. Denote by q the largest weight appearing in the quasihomogeneous structure at Y. Then $\mathcal{V}_\Pi^0$ is in fact only one element of a filtration

$$\mathcal{V}_b(X) = \mathcal{V}_\Pi^{-q} \supset \mathcal{V}_\Pi^{-q+1} \supset \cdots \supset \mathcal{V}_\Pi^0 \supset \mathcal{V}_\Pi^1 \supset \dots \tag{1.19}$$

where for $m \in \mathbb{Z}$ we set $\mathcal{V}_\Pi^{-m} = \{V \in \mathcal{V}_b(X) \mid V\mathcal{F}^{(m')} \subset \mathcal{F}^{m'-m}\ \forall m'\}$, i.e. the set of all vector fields that decreases the weighted order of a function at Y by at most m. It is easy to see that any vector field can reduce the weighted order of a function by at most q, showing that $\mathcal{V}_\Pi^{-q} = \mathcal{V}_b(X)$. Also, by definition we have $[\mathcal{V}^{-m}, \mathcal{V}^{-m'}] \subset \mathcal{V}^{-m-m'}$. In fact, such a filtration of $\mathcal{V}_b(X)$ may also be used to characterize a quasihomogeneous structure. Filtrations of the space of smooth vector fields satisfying such a commutator condition are often called singular Lie filtrations and have been studied extensively in different contexts of differential analysis (see e.g. [18] [4])

We continue in Chapter 3 with analyzing the question of lifting- and commutativity results in this more general setting. This thesis contributes analogous versions of standard commutativity result for radial blow-up and also states some less frequently used results generalized to this setting. One main difficulty in doing so is that one needs to extend the definition of lifting submanifolds under blow-up to lifting quasihomogeneous structures.

These results mark the end of Part 1 and will be used to resolve a structure associated to left- and right-invariant differential operators in the Lie Group $\mathrm{SL}(n, \mathbb{R})$ in Part 2 of this thesis.

1.4. Pseudodifferential calculus on $\mathrm{SL}(n,\mathbb{R})$

In the second part of this thesis, we consider the real reductive Lie group $\mathrm{SL}(n,\mathbb{R})$. We shortly recall some basics on Lie Groups in Chapter 4. We consider the Lie algebra of right- (or left-) invariant vector fields on $\mathrm{SL}(n,\mathbb{R})$. Their universal enveloping algebra is a ring of right invariant differential operators, denoted by Diff^*_{ri}, for which we aim to develop an elliptic theory. The group $\mathrm{SL}(n,\mathbb{R})$ is, of course, not compact. As described in the beginning of the introduction, we 'trade' non-compactness for a boundary by moving to a compactification. Different compactifications of Lie groups have been studied extensively and a large amount of research has gone into the study of several different options. One of the most famous one being the one by De Concini and Procesi [5], called the *wonderful compactification*. Since the goal of this thesis is a differential analysis of operators acting on the (real) manifold $\mathrm{SL}(n,\mathbb{R})$, we chose the *hd-compactification* introduced by Albin, Dimakis, Melrose and Vogan in [1]. It is viewed by the authors as a real analog of the wonderful compactification and is closely related to it. To be more precise, for a complex semisimple Lie group, the real blow-up of the exceptional divisor of the adjoint group in the wonderful compactification yields a hd-compactification.

1.4.1. Hd-compactification of $\mathrm{SL}(n,\mathbb{R})$. We define a *compactification* of a real non compact manifold M as a compact manifold with corners $\bar{M}$ together with a map $I : M \hookrightarrow \bar{M}$ that is a diffeomorphism onto the interior of $\bar{M}$. This is not unique, and a 'good' compactification always fulfills additional properties. In the case of the manifold M being a group, one can demand that the compactification relates nicely to the group action and the invariant vector fields on G. In this thesis, we consider the specific group $\mathrm{SL}(n,\mathbb{R})$. In this case, the hd-compactification can be constructed explicitly as follows.
First, take the standard Hilbert Schmidt norm on the space of $n \times n$-matrices and denote by

$$\mathbb{SH}(n) = \{A \in \mathrm{Mat}(n \times n) \mid ||A|| = 1\} \tag{1.20}$$

the unit sphere of matrices. Then $A \mapsto \frac{A}{||A||}$ maps each element of $\mathrm{SL}(n,\mathbb{R})$ onto $\mathbb{SH}(n)$. We denote the image of this map by $\mathbb{SI}_+(n) \subset \mathbb{SH}(n)$. It consists of all matrices with norm one that have positive determinant $\mathbb{SI}_+(n) = \{A \in \mathbb{SH}(n) \mid \det A > 0\}$. It is easy to see that this map is a diffeomorphism onto its image $\mathbb{SI}_+(n)$, since the inverse is explicitly given by $A \mapsto \frac{A}{\det(A)^{1/n}}$. Of course, one possibility of compactifying $\mathrm{SL}(n,\mathbb{R})$ would be to simply take the closure of $\mathbb{SI}_+(n)$ as a subset of the compact manifold $\mathbb{SH}(n)$, however, for $n > 2$ we take further steps. The boundary $\partial\,\mathbb{SI}_+(n)$ in $\mathbb{SH}(n)$ is the set of all matrices with norm 1 and determinant 0

$$\partial\,\mathbb{SI}_+(n) = \{A \in \mathbb{SH}(n) \mid \det A = 0\}. \tag{1.21}$$

This space is stratified by the matrices with a fixed rank $k = 1, \dots n-1$. Thus, by setting $S_k = \{A \in \mathbb{SH}(n) \mid \operatorname{rank} A = n-k\}$ we get

$$\partial\,\mathbb{SI}_+(n) = \bigsqcup_{k=1}^{n-1} S_k. \tag{1.22}$$

A schematic visualization of the situation in the case $\mathrm{SL}(3,\mathbb{R})$ is given in the left half of Figure 1.1. For a better intuition one can consider the following matrices in $\mathrm{SL}(3,\mathbb{R})$ as an example:

$$A_1(\varepsilon) = \begin{pmatrix} 1/\varepsilon^2 & & \\ & \varepsilon & \\ & & \varepsilon \end{pmatrix}, \; A_2(\varepsilon) = \begin{pmatrix} 1/\varepsilon & & \\ & 1/\varepsilon & \\ & & \varepsilon^2 \end{pmatrix}. \tag{1.23}$$

Both $A_1(\varepsilon)$ and $A_2(\varepsilon)$ are elements of $\mathrm{SL}(3,\mathbb{R})$ for $\varepsilon > 0$. As $\varepsilon \to 0$, the matrices 'approach infinity'. Their projections onto $\mathbb{S}\mathrm{H}(3,\mathbb{R})$ are given (up to higher order terms in ε) by

$$\bar{A}_1(\varepsilon) = \begin{pmatrix} 1 & & \\ & \varepsilon^2 & \\ & & \varepsilon^2 \end{pmatrix}, \; \bar{A}_2(\varepsilon) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varepsilon^2 \end{pmatrix}. \tag{1.24}$$

As $\varepsilon \to 0$ they tend to the boundary of $\mathbb{S}\mathrm{I}_+(3)$ where they hit it and $\bar{A}_1(0) \in S_1, \ \bar{A}_2(0) \in S_2$.

For a general n, the deepest stratum S_{n-1} is a compact submanifold without boundary of $\mathbb{S}\mathrm{H}(n)$. Thus we may define the (real radial) blow-up $[\mathbb{S}\mathrm{H}(n); S_{n-1}]$. By a general result about group actions (see [2], Theorem 7.5) the next higher stratum S_{n-2} lifts under this blow-up to become a compact p-submanifold of $[\mathbb{S}\mathrm{H}(n); S_{n-1}]$. Iterating this process and blowing up all S_k yields a compact manifold with corners

$$\overline{\mathbb{S}\mathrm{H}}(n) := [\mathbb{S}\mathrm{H}(n); S_{n-1}; \ldots; S_1]. \tag{1.25}$$

The closure of the lift of $\mathbb{S}\mathrm{I}_+(n)$ to this space is one of two connected components of $\overline{\mathbb{S}\mathrm{H}}(n)$ and by definition the hd-compactification of $\mathrm{SL}(n,\mathbb{R})$, denoted by $\overline{SL}(n)$ (see right site of Figure 1.1). We denote the boundary hypersurface generated from S_k by H_k.

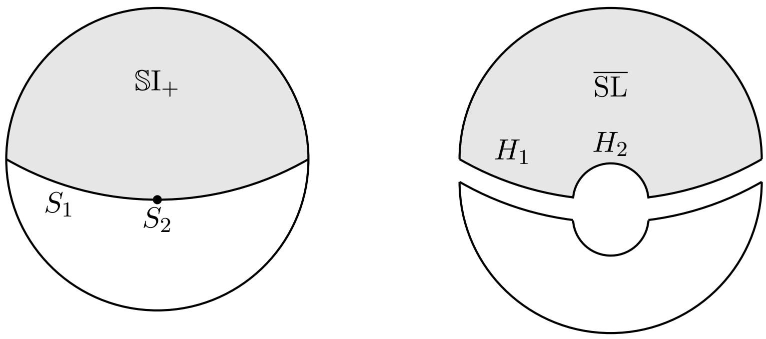

FIGURE 1.1. A schematic visualization of $\mathbb{S}\mathrm{H}(3)$, $\mathbb{S}\mathrm{I}_+(3)$, $\overline{\mathbb{S}\mathrm{H}}(3)$, $\overline{\mathrm{SL}}(3)$ and S_q.

Each boundary hypersurface H_k fibers over two copies of the Grassmannian of type k

$$\begin{array}{ccc} \overline{\mathbb{S}\mathrm{H}}(n-k) \times \overline{\mathbb{S}\mathrm{H}}(k) & \text{———} & H_k \\ & & \downarrow \\ & & \mathrm{Gr}(\mathbb{R}^n, k) \times \mathrm{Gr}(\mathbb{R}^n, k) \end{array} \tag{1.26}$$

One of the two Grassmanians corresponding to the image of an element $A \in S_q$, the other to the (orthogonal complement) of the kernel. The fiber is modeled inductively over

$\mathbb{S}$H of lower dimension. The first factor of the fiber corresponds to the restriction of A to the orthogonal complement of its kernel, and the other one to the remainder, when approaching A.

The different boundary hypersurfaces H_k intersect each other. A phenomenon that one might expect, but is not the case here, is that the different fibrations would be *iterated*, in the sense that whenever two hypersurfaces meet, the two fibrations restricted to the intersection, form a tower in the fibers. This is not the case here, however, there is a similar, but slightly weaker structure, that one might call partially ordered iteration: Let $\bar{q} = \{\bar{q}_1, \dots \bar{q}_l\} \subset \{1, \dots, n-1\}$ be any index subset. Then the intersection $F_{\bar{q}} = H_{\bar{q}_1} \cap \dots \cap H_{\bar{q}_l}$ is non empty and is a connected boundary face of $\overline{\mathrm{SL}}(n)$. In fact, the set of boundary surfaces of $\overline{\mathrm{SL}}(n)$ is indexed by these $\bar{q}$. The fibers of the different fibrations $H_{\bar{q}_i} \to \mathrm{Gr}(\bar{q}_i)^2$ restricted to $F_{\bar{q}}$ intersect each other cleanly and their intersections form smaller fibers of a common larger fibration: Denote by $\mathcal{F}_{\bar{q}}$ the Flag manifold of type $\bar{q}$, meaning the space of all flags $U_1 \subset U_2 \subset \dots U_l$ where $\dim U_i = \bar{q}_i$. Then the boundary face $F_{\bar{q}}$ fiber over two copies of this Flag variety $F_{\bar{q}} \to \mathcal{F}_{\bar{q}} \times \mathcal{F}_{\bar{q}}$. and each of the fibrations of $H_{\bar{q}_i}$ restrict to $F_{\bar{q}}$ to form a tower

$$F_{\bar{q}} \to \mathcal{F}_{\bar{q}} \times \mathcal{F}_{\bar{q}} \to \mathrm{Gr}(\bar{q}_i) \times \mathrm{Gr}(\bar{q}_i), \tag{1.27}$$

where the second map is simply given by projection from the flag onto its dim $\bar{q}_i$-subspace. In fact, given any two boundary faces $F_{\bar{q}}, F_{\bar{q}'}$ such that the first is also a boundary face of the second $F_{\bar{q}} \hookrightarrow F_{\bar{q}'}$ we necessarily have $\bar{q} \supset \bar{q}'$ and a natural projection map between the flag varieties $\mathcal{F}_{\bar{q}} \to \mathcal{F}_{\bar{q}'}$ such that the diagram

$$\begin{tikzcd} F_{\bar{q}} \arrow[r, hook] \arrow[d, "\phi_{\bar{q}}"] & F_{\bar{q}'} \arrow[d, "\phi_{\bar{q}'}"] \\ \mathcal{F}_{\bar{q}} \arrow[r, "\pi_{\bar{q},\bar{q}'}"] & \mathcal{F}_{\bar{q}'} \end{tikzcd} \tag{1.28}$$

commutes. The single boundary face of maximal codimension corresponding to $\bar{q} = \{1, \dots, n-1\}$ is called the Borel face. A point p in the base fixes an Iwasawa decomposition

$$\mathrm{SL}(n) = \mathrm{SO}(n) A_p N_p \tag{1.29}$$

of the group and a corresponding decomposition of the Lie algebra

$$\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p \tag{1.30}$$

which may be used to analyze the lift of the right-invariant vector fields on $\overline{SL}(n, \mathbb{R})$. We will see that $\mathfrak{a}_p$ decomposes into $n-1$ one-dimensional spaces, each of which induces a local boundary defining function of the hypersurface H_i near p. By smoothness in p, this gives rise to a set of globally defined boundary defining functions τ_i for H_i under which all the fibrations $F_{\bar{q}} \to \mathcal{F}_{\bar{q}}^2$ extend into the interior and, restricted to a single fiber, $\tau_i \partial_{\tau_i}$ is an element of the Lie algebra. The root space decomposition of $\mathfrak{n}_p$ induces a filtration on the tangent space of each flag manifold $\mathcal{F}_{\bar{q}}$ indexed over multi indexes

$$
\begin{aligned}
&E^{\alpha} \subset T\mathcal{F}_{\bar{q}} \ , \ \alpha \leq \beta \Rightarrow E^{\alpha} \subset E^{\beta} \ , \\
&E^{\alpha} + E^{\beta} \subset E^{\alpha+\beta} \ , \ [\mathcal{V}_{\alpha}, \mathcal{V}_{\beta}] \subset \mathcal{V}_{\alpha+\beta} \ ,
\end{aligned} \tag{1.31}
$$

where $\mathcal{V}_{\alpha} = C^{\infty}(\mathcal{F}_{\bar{q}}; E^{\alpha})$. The notation E^{α} for one subbundle of $T\mathcal{F}_{\bar{q}}$ for all $\bar{q}$ is justified by the fact that they behave as expected under projection $\mathcal{F}_{\bar{q}} \to \mathcal{F}_{\bar{q}'}$.
This structure induces a *Lie algebroid* of vector fields $\mathcal{V}_e \subset \mathcal{V}_b$ consisting of those vector fields tangent to the boundary and tangent to the fibers of all fibrations $F_{\bar{q}} \longrightarrow \mathcal{F}^2_{\bar{q}} \xrightarrow{\pi_*} \mathcal{F}_{\bar{q}}$ over the left or right flag manifold (depending on considering right- or left-invariant vector fields) and with normal vanishing properties encoded by the bundles E^{α}.
Right-invariant vector fields are elements of this Lie algebroid $\mathcal{V}_e$. Intuitively, $\mathcal{V}_e$ consist of those vector fields whose vanishing behavior at the boundary of the compactification $\overline{\mathrm{SL}}(n, \mathbb{R})$ is modeled by the right-invariant vector fields.

1.4.2. Resolution of the compactification. The main goal of the second part in this thesis is the resolution of $\mathcal{V}_e$. Even though it is a Lie algebroid, the fact that the fibrations of the different hypersurfaces H_i are not iterated in the strong sense complicates the analysis.
The solution pursued in this thesis is to move to a resolution space, denoted by X that is constructed from $\overline{\mathrm{SL}}(n, \mathbb{R})$ via a series of real radial blow-up. To be more precise, it is the *total boundary blow-up* of $\overline{\mathrm{SL}}(n, \mathbb{R})$, meaning that every boundary face $F_{\bar{q}}$ is blown up to generate a boundary hypersurface $H_{\bar{q}}$. The fibration of $F_{\bar{q}}$ lifts to a fibration of $H_{\bar{q}}$. It is an easy consequence of (1.28) that wherever some collection of these newly created hypersurfaces $H_{\bar{q}_1}, \dots, H_{\bar{q}_k}$ meet, their intersection $F = H_{\bar{q}_1} \cap \dots \cap H_{\bar{q}_k}$ is a boundary face of X, the corresponding multi-indexes are totally ordered $\bar{q}_1 \geq \dots \geq \bar{q}_k$, and the fibrations of the different $H_{\bar{q}_i}$ restricted to F form a tower

$$
\begin{tikzcd}
F \arrow[r, "\phi_{\bar{q}_1}"] \arrow[rr, bend right, "\phi_{\bar{q}_2}"'] \arrow[rrr, bend right, "\phi_{\bar{q}_3}"'] \arrow[rrrrr, bend right, "\phi_{\bar{q}_k}"'] & \mathcal{F}_{\bar{q}_1} \arrow[r, "\pi_{\bar{q}_1,\bar{q}_2}"] & \mathcal{F}_{\bar{q}_2} \arrow[r, "\pi_{\bar{q}_2,\bar{q}_3}"] & \mathcal{F}_{\bar{q}_3} \arrow[r] & \dots \arrow[r] & \mathcal{F}_{\bar{q}_k}.
\end{tikzcd} \tag{1.32}
$$

This means that X carries an *iterated fibration structure*. Such structures are the focus of recent work in several different situations, for example [6], [24], [3].
The Lie algebroid $\mathcal{V}_e$ lifts to X and becomes a sub Lie-algebra of $\mathcal{V}_b(X)$. Due to the fact that new fiber coordinates are introduced by the total boundary blow-up, the lift of $\mathcal{V}_e$ to X is no longer a Lie algebroid. However, its C^{∞} span is again a Lie algebroid on X, denoted by $\mathcal{V}_{\mathrm{SL}}$.
Of course, X is also a compactification of $\mathrm{SL}(n, \mathbb{R})$, since the blow down map $X \to \overline{\mathrm{SL}}$ is a diffeomorphism of the interiors, but one should think of it more as a resolution of $\overline{\mathrm{SL}}$ into a manifold carrying an iterated fibration structure.
Lifting both the Lie algebra of right-invariant vector fields $\mathcal{V}_{ri}$ and $\mathcal{V}_e$ to X we get

$$
\mathcal{V}_{ri} \subset \mathcal{V}_e \subset \mathcal{V}_{\mathrm{SL}} \subset \mathcal{V}_b. \tag{1.33}
$$

The universal enveloping algebras of theses spaces give rise to different classes of differential operators on X, namely

$$\mathrm{Diff}_{ri} \subset \mathrm{Diff}_{e} \subset \mathrm{Diff}_{\mathrm{SL}}\,. \tag{1.34}$$

Although operators in Diff_{ri} are the main interest of this study, for large parts of the elliptic theory it is more natural to study elements of $\mathrm{Diff}_{\mathrm{SL}}$, since $\mathcal{V}_{\mathrm{SL}}$ is a Lie algebroid on X and thus consist of the smooth sections of a vector bundle ${}^{\mathrm{SL}}TX$.
Schwartz kernels of operators in $\mathrm{Diff}_{\mathrm{SL}}$ can thus be studied by analyzing the lift of $\mathcal{V}_{\mathrm{SL}}$ from either the left or the right to the double space $(X)^2$. As usual when dealing with manifolds with any kind of boundary, the diagonal $\triangle \subset X^2$, which is of primary interest in the analysis of the Schwartz kernels, hits the boundary rather poorly from an analytical standpoint and analyzing the vector fields $\mathcal{V}_{\mathrm{SL}}$ near the intersection of the diagonal with the boundary in X^2 is the main challenge when developing an elliptic theory for $\mathrm{Diff}_{\mathrm{SL}}$.
One of the main contributions of the second part of the thesis is the construction of a resolution of X^2, denoted by X^2_e under which the diagonal lifts to become a p-submanifold and $\mathcal{V}_i$ lifts from either the left or right to become transversal to the diagonal. The space X^2_e is constructed via a series of blow-ups. Here, quasihomogeneous blow-ups are used in order to resolve the different orders of vanishing appearing from (1.31).
This resolution allows to define an algebra of pseudodifferential operators, whose Schwartz kernels are the push-forward of certain (rather simple) distributions on X^2_e. This algebra contains $\mathrm{Diff}_{\mathrm{SL}}$ together with weak parametrices for these operators. A composition theorem for these pseudodifferential operators is proven, again using a geometric approach by constructing a resolution of the triple space X^3, denoted by X^3_e.
Part 2 of this thesis is structured as follows: In Chapter 4 we briefly recall basic definitions and properties of semisimple Lie Groups. In Chapter 5 we discuss the hd-compactification of $\mathrm{SL}(n,\mathbb{R})$ together with the lift of right-invariant vector fields on it. In Chapter 6 we construct the resolution and the pseudodifferential calculus as described above for the special case of $n = 3$. Most of the challenges for the general case are already present in this case and it serves as a guidance for general n, which is described in Chapter 7.

Part 1

Quasihomogeneous blow-ups of p-submanifolds

CHAPTER 2

Construction and basic properties

We start with a short discussion of the general case of arbitrary p-submanifolds Y of a surrounding manifold with corners X. We then restrict the construction to the special case of blowing up 0 in a vector space, and continue by moving to more and more general cases before finishing this chapter with arbitrary p-submanifolds.

First, we define a *conormal filtration* at a p-submanifold. It is motivated by the idea of giving different codirections in the cotangent bundle weights. For this, we briefly need to recall some basic definitions about manifolds with corners: Let $Y \subset X$ be a boundary p-submanifold. We denote by $\mathrm{Fa}(Y)$ the smallest boundary face of X that contains Y. Furthermore we set $\mathrm{Hu}(Y) = \mathrm{Hu}(\mathrm{Fa}(Y)) \subseteq M_1(X)$ to be the set of all boundary hypersurfaces of X that contain $\mathrm{Fa}(Y)$. Recall that we then have

$$\mathrm{Fa}(Y) \subseteq \bigcap_{H \in \mathrm{Hu}(Y)} H \quad \text{and} \quad N^* \mathrm{Fa}(Y) = \bigoplus_{H \in \mathrm{Hu}(Y)} N^* H. \tag{2.1}$$

Definition 2.1: *Let M be a manifold (with corners) and $Y \subset X$ a p-submanifold. A* ***conormal filtration*** *at p is a fan*

$$N^*_Y X = S_1 \supseteq S_2 \supseteq \dots S_q \supseteq \{0\} = S_{q+1} = \dots \tag{2.2}$$

of decreasing subbundles of $N^*_Y X$. When Y is a boundary submanifold, we have to pose a condition for the S_k to relate nicely to the boundary, namely that there is a sequence of integers κ_H for each $H \in \mathrm{Hu}(Y)$ such that

$$S_k \cap N^*_Y \mathrm{Fa}(Y) = \bigoplus_{\kappa_H \geq k} N^*_Y H \text{ for all } k \tag{2.3}$$

There are some things to note about this definition:

(1) It is to be interpreted as following: A codirection $dx \in S_k$ has the associated weight at least k. Since the subspaces are nested, each codirection dx has a assigned weight, given by the highest index k for which $dx \in S_k$.
(2) Some consecutive S_k might be equal. For example, if there are some codirections with weight 3 but no codirections with weight 2, then $S_2 = S_3$. However, this information is not redundant, since it encodes the weights of the different codirections [1].
(3) It is convenient to require S_q to be of dimension at least 1. Therefore q is the highest homogeneity that actually occurs.

[1] Equivalently, one could define the fan as strictly decreasing in dimension and additionally assign a (strictly increasing) weight to each S_k.

For concrete calculations below, the notion of *matching* coordinates will become important:

Definition 2.2: *Let (S_k) be a conormal filtration at $Y \subset X$. Local product coordinates (x_i, y_j) of Y (where Y is given by $\{x_i = 0\}$) are said to* **match** *(S_k) if there is an increasing series of numbers $1 = N(1) \leq N(2) \leq \dots N(q)$ such that*

$$S_k|_y = \operatorname{span}\{dx_i \mid i = N(k), \dots, n\} \tag{2.4}$$

for $k = 1 \dots q$. Each dx_i then has the associated weight $\kappa_i = j :\Leftrightarrow\ dx_i \in S_j$ but $dx_i \notin S_{j+1}$. The κ_i are increasing and $\kappa_n = q$.

Notice that the multi-index $\kappa \in \mathbb{N}^n$ does not depend on the choice of matching coordinates, but only on (S_k). In fact, it only depends on the dimensions of the (S_k), where the first $(\dim S_1 - \dim S_2)$ of the κ_i are equal to one, the next $(\dim S_2 - \dim S_3)$ are equal to 2 and so forth.
Similarly, any choice of local product coordinates (x_i, y_i) of Y where $Y = \{x_i = 0\}$ together with an associated weight $\kappa_i \in \mathbb{N}$ for each x_i yields locally a conormal filtration given by $S_k = \operatorname{span}\{dx_i|_{N^*Y} \mid \kappa_i \geq k\}$.

Normal fibration: Before we start, we will briefly recall the concept of a normal fibration, since it is of vital importance later on. Let $Y \subset X$ be a p-submanifold of a manifold with corners. Then the normal bundle NY of Y is a vector-bundle over Y with fiber over each $p \in Y$ given by T_pX/T_pY. It has the structure of a manifold with corners. Let 0_{NY} denote the zero-section of NY, which is naturally identified with Y. A *normal fibration* of Y is an identification of a neighborhood of Y (in X) and a neighborhood of 0_{NY} in NY. Formally it is a map $\phi : U_{\subseteq NY} \to V_{\subseteq X}$ (where U is an open neighborhood of 0_{NY} and V is an open neighborhood of Y) such that ϕ is a diffeomorphism and $\phi|_{0_{NY}}$ is equal to the identification of 0_{NY} and Y. Since NY is a vector bundle it can be locally trivialized. This local product structure can be pulled back under ϕ^{-1} to V. The tubular neighborhood theorem states that such a normal fibration always exists.

2.1. Zero in a vector space

We start by considering $\{0\} \subset V$ where V is a n-dimensional real vector space. A conormal filtration at $\{0\}$ is then simply a filtration of the dual space

$$V^* = T_0^*V = S_1 \supseteq S_2 \cdots \supseteq S_q. \tag{2.5}$$

As we will see below, in this special case such a filtration already defines a unique quasi-homogeneous blow-up $[V, 0]_{(S_k)}$ as a manifold with boundary constructed as follows:
We start by constructing the front face of the blown-up space as a set and add the C^∞-structure afterwards. The annihilators of the (S_k) define a filtration of V

$$V = S_{q+1}^\perp \supseteq S_q^\perp \supset \cdots \supset S_2^\perp \supset S_1^\perp = \{0\}. \tag{2.6}$$

Using this, we define the associated graded space of V to be

$$V_S := {}^{V}\!/_{S_q^\perp} \oplus {}^{S_q^\perp}\!/_{S_{q-1}^\perp} \oplus \cdots \oplus {}^{S_3^\perp}\!/_{S_2^\perp} \oplus S_2^\perp, \tag{2.7}$$

which is again a vector space of the same dimension as V and is the direct sum of $q-1$ real vector spaces ${S_{k+1}^{\perp}}/{S_k^{\perp}}$ (some of these might be zero dimensional). V_S and V are not naturally identified, however any choice of linear coordinates on V that *match* the conormal filtration (S_k) as in Definition 2.2 yields such a identification. As an example, consider $V=\mathbb{R}^3$ with standard coordinates x_1, x_2, x_3. Associate the weights $1,1,3$ to these functions respectively. In other words, the conormal filtration is given by $S_1 = V^*,\ S_2 = S_3 = \operatorname{span}\{dx_3\}$. The dual flag is given by $S_3^{\perp} = S_2^{\perp} = \operatorname{span}\{x_1, x_2\},\ S_1^{\perp} = \{0\}$. Thus V_S has the two (not zero dimensional) factors ${V}/{S_3^{\perp}}$ and $S_2^{\perp}$. The linear coordinates x_1, x_2, x_3 give an identification $V \cong V_S$. In general, one should think of the factor ${S_{k+1}^{\perp}}/{S_k^{\perp}}$ as the part of the vector space with weight exactly k (compared to $S_{k+1}^{\perp}$ being the part of weight at most k). This statement is of course only meaningful on V_S, not V.

Recall that on any real vector space W there is a canonically defined 'standard outgoing vector field' given in any linear coordinates x_i by $\sum x_i \partial_{x_i}$. It is a straight forward calculation to check that this is well-defined, independent of the linear coordinates. This standard outgoing vector field of often simply denoted by $x\partial_x$. We now set R_S to be the (well-defined) vector field on V_S that is given on ${S_{k+1}^{\perp}}/{S_k^{\perp}}$ by the k-th multiple of the standard outgoing vector field. Given any linear matching coordinates x_i on V (and thus an identification $V \cong V_S$) with associated weights κ_i it is given by

$$R_S = \sum_{i=1}^{n} \kappa_i x_i \partial_{x_i} \tag{2.8}$$

Again, it is not unique on V, only in V_S. The flow-out of this vector field defines a $\mathbb{R}_{>0}$-group action $V_S \times R_+ \to V_S,\ (x,t) \mapsto \Phi_{R_S}^t(x)$. The quotient of $V_S \setminus \{0\}$ by this action will be our model for the front face:

$$\mathrm{ff} := {(V_S \setminus \{0\})}/{\mathbb{R}_{>0}}\ ,\ \ [V;0]_{(S_k)} := (V \setminus \{0\}) \sqcup \mathrm{ff}. \tag{2.9}$$

For the C^∞-structure choose linear coordinates (x_i) that match (S_k). Under the $\mathbb{R}_{>0}$ action of R_S, the function

$$r_\kappa(x) = \left(\sum_{i=1}^{n} x_i^{2(\kappa_n)!/\kappa_i} \right)^{1/2(\kappa_n)!} \tag{2.10}$$

is homogeneous of degree 1, since each x_i is homogeneous of degree κ_i [2]. This function will serve as a replacement for the standard radius used in the radial blow-up. Since r_κ is homogeneous of degree 1 under the $\mathbb{R}_{>0}$ action of R_S, it defines a diffeomorphism

$$V \setminus \{0\} \ni x \mapsto (r_\kappa(x), \omega) \in (0,\infty) \times \mathbb{S}_\kappa^{n-1} \tag{2.11}$$

where $\mathbb{S}_\kappa^{n-1} := \{x \in V\ :\ r_\kappa(x) = 1\}$ can be identified with ff . The second factor of the map is given by the projection along the flow of R_κ, so

$$\omega_i = \frac{x_i}{r_\kappa(x)^{\kappa_i}}. \tag{2.12}$$

The inverse of this maps extends smoothly to $r=0$ and will define our blow-up:

[2]The use of $(\kappa_n)!$ is somewhat arbitrary. The least common multiple of all κ_i would work as well.

Definition 2.3: *Let (S_k) be a conormal filtration at $0 \in V$. Let (x_i) be linear coordinates that match (S_k). We define the blow-up of* 0 *with respect to (S_k) as in* (2.9) *together with the C^∞-structure and blow-down map given by*

$$\beta : [V; 0]_{(S_k)} = [0, \infty) \times \mathbb{S}^{n-1}_\kappa \ni (r, \omega) \mapsto (r^{\kappa_1}\omega_1, \dots, r^{\kappa_n}\omega_n) \in V. \tag{2.13}$$

Of course we need to check that the C^∞-structure is independent of the choice of linear coordinates. To do this, projective coordinates on the new manifold are very useful.

2.1.1. Projective coordinates. Similar to the radial blow-up, concrete calculations are often the easiest using *projective coordinates*. We start again linear coordinates x_i on V. Let $i \in \{1, \dots n\}$ be a fixed index. Let $\tilde{x}_j = \beta^*(x_j)$ be the pullback of x_j to the blown-up space (2.13). If there is no room for confusion we omit the $\tilde{}$. On the blown-up space $[0, \infty) \times \mathbb{S}^{n-1}_\kappa$ we clearly have $\tilde{x}_j = 0$ at (r, ω) if and only of $r = 0$ or $\omega_j = 0$. On $\{(r, \omega) \in [V; 0]_{(S_K)} \mid r \neq 0,\ \omega_i \neq 0\}$ define the functions $\xi_1, \dots, \xi_n$ by

$$\xi_j = \begin{cases} \tilde{x}_i^{1/\kappa_i} & (j = i) \\ \tilde{x}_j \tilde{x}_i^{-\kappa_j/\kappa_i} & (j \neq i) \end{cases}. \tag{2.14}$$

Proposition 2.4: *The functions (ξ_j) extend smoothly to*

$$D_i = \{(r, \omega) \in [V; 0]_{(S_k)} \mid \omega_i \neq 0\} \tag{2.15}$$

and form a coordinate system on this domain where ξ_i is a local boundary defining function of the front face.

Proof. In the interior, i.e. for $r > 0$ there is nothing to show, since then β is a local diffeomorphism and the ξ_j are smooth with independent differential. Now let us consider the behavior for $r \to 0$. We first consider ξ_i and then ξ_j for $j \neq i$. At an interior point of the domain $(r, \omega) \in D_i^\circ$ (i.e $r > 0,\ \omega_i \neq 0$) we have

$$\omega_i = \frac{\tilde{x}_i}{r^{\kappa_i}} \quad \text{and therefore} \quad \tilde{x}_i = \omega_i r^{\kappa_i} \tag{2.16}$$

as calculated earlier. Therefore, at (r, ω), we have

$$\xi_i = \tilde{x}_i^{1/\kappa_i} = \omega_i^{1/\kappa_i} r. \tag{2.17}$$

This is smooth down to $r = 0$ and a local boundary defining function of the front face. Now let $j \neq i$. Then we have

$$\xi_j = \tilde{x}_j \tilde{x}_i^{-\kappa_j/\kappa_i} = \omega_j r^{\kappa_j} \omega_i^{-\kappa_j/\kappa_i} r^{-\kappa_i} = \omega_j \omega_i^{-\kappa_j/\kappa_i} \tag{2.18}$$

which is independent of r and thus smooth down to $r = 0$. On D_i, these ξ_j form local coordinates. □

It is easy to see that the set of all the domains D_i cover the blown-up space:

$$\bigcup_{i=1}^{n} D_i = [V; \{0\}]_{(S_k)}. \tag{2.19}$$

We are now able to show the naturality of the blow-up construction.

Lemma 2.5: *Let $(x_i), (\tilde{x}_i)$ be two matching linear coordinate systems with respect to (S_k). Then the coordinate change lifts to a diffeomorphism on $[V;0]_{(S_k)}$.*

Proof. We have $\bar{x} = Ax$ for a real invertible matrix. Since the coordinates both match (S_k) we have

$$dx_i \in S_k \Rightarrow d(Ax_i) \in S_k. \tag{2.20}$$

Therefore the k-th row A_k of A has only non zero entries $A_{k,l}$ whenever $\kappa_l \geq \kappa_k$. The lift of this coordinate change in the interior is given by $\bar{\beta}^{-1} \circ A \circ \beta$. Recall that in projective coordinates corresponding to ξ_i we have

$$(\xi_1, \dots, \xi_n) \xrightarrow{\beta} (\xi_1 \xi_i^{\kappa_1}, \dots, \xi_i^{\kappa_i}, \dots, \xi_n \xi_i^{\kappa_n}) \tag{2.21}$$

and in projective coordinates η corresponding to $\bar{x}_j$ we have

$$(\bar{x}_1, \dots, \bar{x}_n) \xrightarrow{\bar{\beta}^{-1}} (\bar{x}_1 \bar{x}_j^{-\kappa_1/\kappa_j}, \dots, \bar{x}_j^{1/\kappa_j}, \dots, \bar{x}_n \bar{x}_j^{\kappa_n/\kappa_j}). \tag{2.22}$$

We therefore get

$$\begin{aligned}
&(\xi_1, \dots \xi_n) \\
\xrightarrow{\beta}&(\xi_1 \xi_i^{\kappa_1}, \dots, \xi_i^{\kappa_i}, \dots, \xi_n \xi_i^{\kappa_n}) \\
\xrightarrow{A}&(A_1(\beta(\xi)), \dots, A_n(\beta(\xi))) \\
\xrightarrow{\bar{\beta}^{-1}}&(A_1(\beta(\xi)) A_j(\beta(\xi))^{-\kappa_1/\kappa_j}, \dots, A_j(\beta(\xi))^{1/\kappa_j}, \dots, A_n(\beta(\xi)) A_j(\beta(\xi))^{-\kappa_n/\kappa_j}).
\end{aligned}$$

Notice that each term of $A_j(\beta(\xi))$ has ξ_j-power of at least κ_j, making the map smooth down to $\xi_i \to 0$. Thus A lifts to become a diffeomorphism on the blown-up space.

□

2.1.2. 0 in $\mathbb{R}^n_k$. The results above immediately generalize to blowing up 0 in $\mathbb{R}^n_k$, if we interpret it as a subspace of the vector space $\mathbb{R}^n$, i.e. only considering linear coordinates on it. A conormal filtration then defines a filtration of the inward pointing tangent space at 0 where each space in (2.7) is replaced by its inward-pointing part:

$$V_S = {T_0^+ \mathbb{R}^n_k}\big/{S_q^{\perp +}} \oplus {S_q^{\perp +}}\big/{S_{q-1}^{\perp +}} \oplus \dots \oplus {S_3^{\perp +}}\big/{S_2^{\perp +}} \oplus S_2^{\perp +}. \tag{2.23}$$

For the C^∞-structure, $\mathbb{S}^{n-1}_\kappa$ is replaced by $\mathbb{S}^{k,n-1}_\kappa := \mathbb{S}^{n-1}_\kappa \cap \mathbb{R}^n_k$. Notice that the extra condition in Definition 2.1 becomes necessary for the identification of $\mathbb{S}^{n-1}_\kappa \cap \mathbb{R}^n_k$ with the new front face ff . The rest is completely analogous. In particular, the definition of the projective coordinates ξ_j and their range D_i are identical.

2.1.3. The function spaces $\mathcal{F}^{(m)}$. Recall that $\mathcal{I}(0) \subset C^\infty(V)$ is the space of smooth function on V (or $\mathbb{R}^n_k$) that vanish at 0. Furthermore, there is a decreasing sequence of subspaces

$$\mathcal{I}(0) \supset \mathcal{I}^2(0) \supset \mathcal{I}^3(0) \dots \tag{2.24}$$

where $\mathcal{I}^m(0)$ is the space of functions vanishing to order at least m at 0. For a given conormal filtration (S_k) there are related subspaces $\mathcal{F}^{(m)}$ of functions vanishing at 0 to 'quasihomogeneous'-order at least m:

Definition 2.6: *Let (S_k) be a conormal filtration at $0 \in V$ with associated weights $\kappa \in \mathbb{N}^n$. We set*

$$\mathcal{F}^{(m)} = \sum_{\kappa\alpha \geq m} x^\alpha C^\infty(V). \tag{2.25}$$

In particular $x_i \in \mathcal{F}^{(\kappa_i)}$ but $x_i \notin \mathcal{F}^{(\kappa_i+1)}$. Of course, $\mathcal{F}^{(m)} \subset C^\infty(V)$ depends on the conormal filtration (S_k), which is not represented in the notation since it rarely leads to confusion.

The fact that this is well-defined is a simple consequence of the fact that any linear coordinate change $\bar{x}_i = \sum_l a_{i,l} x_l$ that fixes (S_k) necessarily satisfies $a_{i,l} = 0$ whenever $\kappa_i < \kappa_k$.

Notice that one can reconstruct the S_k from these spaces since

$$S_k = \{df \mid f \in \mathcal{F}^{(k)}\}. \tag{2.26}$$

The following statements are direct consequences of this definition:

Lemma 2.7: *Let (S_k) be a conormal filtration at $0 \in V$ and $\mathcal{F}^{(m)}$ as above.*

(1) $\mathcal{F}^{(m)}$ is a C^∞ module.
(2) $\mathcal{I}(p) = \mathcal{F}^{(1)} \supseteq \mathcal{F}^{(2)} \supseteq \dots$.
(3) $\mathcal{F}^{(m)} \cdot \mathcal{F}^{(m')} \subseteq \mathcal{F}^{(m+m')}$.
(4) $\mathcal{F}^{(\infty)} = \mathcal{I}^{(\infty)}$ where $\mathcal{F}^{(\infty)} = \bigcap_m \mathcal{F}^{(m)}$ and $\mathcal{I}^{(\infty)} = \bigcap_m \mathcal{I}^{(m)}$.

Proposition 2.8: *Let $\beta : [V;0]_{(S_k)} \to X$ be a quasihomogeneous blow-up. Then the elements $f \in \mathcal{F}^{(m)}$ are precisely those smooth functions, whose lifts $\beta^* f \in C^\infty([V;0]_{(S_k)})$ are elements of $\mathcal{I}^m(\mathit{ff})$.*

Proof. We will use projective coordinates on $[V;0]_{(S_k)}$ corresponding to linear coordinates (x_i) on V matching the conormal filtration. Recall that the projective coordinates (ξ_j) corresponding to one of the coordinate functions x_i are given by

$$\xi_i = x_i^{1/\kappa_i} \text{ and } \xi_j = x_j x_i^{-\kappa_j/\kappa_i} \text{ for } j \neq i \tag{2.27}$$

with the blow-down map locally taking the form

$$x_i = \xi_i^{\kappa_i} \text{ and } x_j = \xi_j \xi_i^{\kappa_j} \text{ for } j \neq i. \tag{2.28}$$

Furthermore, ξ_i is a boundary defining function of ff on its domain. Therefore, we locally have $\mathcal{I}^m(\mathrm{ff}) = \xi_i^m \cdot C^\infty(X)$. Now, take $f \in \mathcal{F}^{(m)}$. Since $\mathcal{F}^{(m)}$ is generated by x^α with $\alpha\kappa \geq m$ we only need to consider those as f. We have

$$\begin{aligned} \beta^*(x^\alpha)(\xi_1, \dots, \xi_n) &= \xi_1^{\alpha_1} \xi_i^{\kappa_1} \cdot \xi_2^{\alpha_2} \xi_i^{\alpha_2\kappa_2} \cdots \xi_i^{\alpha_i\kappa_i} \cdots \xi_n^{\alpha_n} \xi_i^{\alpha_n\kappa_n} \\ &= \xi_i^{\alpha\kappa} \cdot \prod_{j \neq i} \xi_j^{\alpha_j} \in \mathcal{I}^m(\mathrm{ff}). \end{aligned} \tag{2.29}$$

For the other direction, take a smooth function f that satisfies $\beta^* f \in \mathcal{I}^m(\mathrm{ff})$. Writing f in Taylor series $f = \sum_\alpha \lambda_\alpha x^\alpha$, we can see from (2.29) that $\beta^* f \in \mathcal{I}^m(\mathrm{ff})$ implies $\lambda_\alpha = 0$ whenever $\kappa\alpha < m$. $\square$

Proposition 2.9: *Given (S_k) and matching coordinates (x_i) at $0 \in V$ with associated function spaces $\mathcal{F}^{(m)}$, the following sequence is exact for each m:*

$$0 \longrightarrow \mathcal{F}^{(m+1)} + (\mathcal{F}^{(m)} \cap \mathcal{I}^2(0)) \longrightarrow \mathcal{F}^{(m)} \xrightarrow{d|_0} {S_m}/{S_{m+1}} \longrightarrow 0. \tag{2.30}$$

Furthermore

$$\mathcal{F}^{(m)} \cap \mathcal{I}^2(0) = \sum_{\substack{n_1+\cdots+n_k=m \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)}. \tag{2.31}$$

Proof. The first map is injective, since it is an embedding. Now consider the second map. We know that S_m is spanned by those dx_i with $\kappa_i \geq m$. Furthermore $x_i \in \mathcal{F}^{(\kappa_i)}$. Therefore $\mathcal{F}^{(m)} \xrightarrow{d|_0} S_m$ is well-defined and surjective.
Next, take $f = f_1 + f_2 \in \mathcal{F}^{(m+1)} + (\mathcal{F}^{(m)} \cap \mathcal{I}^2(0))$. Since $f_2 \in \mathcal{I}^2(0)$, we have $df_2|_0 = 0$. Furthermore, since $f_1 \in \mathcal{F}^{(m+1)}$, we have $df_1|_0 \in S_{m+1}$ which shows that $df|_0 = [0] \in {S_m}/{S_{m+1}}$.
Now take $f \in \mathcal{F}^{(m)}$ with $df|_0 = [0] \in {S_m}/{S_{m+1}}$, yielding $df|_0 \in S_{m+1}$. Writing f in Taylor series with respect to the coordinates (x_i), we can see that only those x_i-coefficients may be non zero, where $dx_i \in S_{m+1}$. Thus we can divide f into its linear part, which is in $\mathcal{F}^{(m+1)}$, and the rest, which is in $\mathcal{I}^2$. Since $\mathcal{F}^{(m)}$ is a module, this rest is also in $\mathcal{F}^{(m)}$. This proves the exactness.
Moving on to the second statement, first take $f \in \sum_{n_1+\cdots+n_k=m} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)}$. Since $k \geq 2$, the fact that $\mathcal{F}^{(n)} \subseteq \mathcal{I}$ implies $f \in \mathcal{I}^2$ and the fact that $\mathcal{F}^{(n)} \cdot \mathcal{F}^{(n')} \subset \mathcal{F}^{(n+n')}$ implies $f \in \mathcal{F}^{(m)}$. Now take $f \in \mathcal{F}^{(m)} \cap \mathcal{I}^2$. Using Taylor series again, it is clear that the space $\mathcal{F}^{(m)} \cap \mathcal{I}^2$ is spanned by those x^α where $\kappa\alpha \geq m$ and $|\alpha| \geq 2$. We can therefore divides the summands arbitrarily in $x^\alpha = x^{\alpha_1} x^{\alpha_2}$ with $\alpha_1 + \alpha_2 = \alpha$, $|\alpha_i| > 0$ which is clearly an element of $\mathcal{F}^{(\kappa\alpha_1)} \cdot \mathcal{F}^{(\kappa\alpha_2)}$ where $\kappa\alpha_1 + \kappa\alpha_2 = \kappa\alpha \geq m$. $\square$

Remark: The second part of the proposition shows that all the information of the $\mathcal{F}^{(m)}$ is contained in the first $\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(q)}$, since $\mathcal{F}^{(m)} \cap \mathcal{I}^2 = \mathcal{F}^{(m)}$ for $m > q$, which becomes clear in Taylor series.

2.2. Zero section of a vector bundle

We can immediately generalize these definitions to the zero section of a vector bundle or $\mathbb{R}^n_k$-bundle by blowing up each fiber. Let V be an $\mathbb{R}^n_k$-bundle over the base Y (being a manifold with corners). We then identify Y with the zero-section 0_V in V. We denote the fiber over a point $y \in Y$ by V_y. Let (S_k) be a conormal filtration at Y. Then clearly $(S_k)|_y$ is a conormal filtration at 0 in V_y for each $y \in Y$. We then define the blow-up of the zero section Y with respect to (S_k) as

$$[V;Y]_{(S_k)} = \bigsqcup_{y\in Y} [V_y; 0]_{(S_k)|_y} \tag{2.32}$$

with blow-down map defined on each fiber. The C^∞-structure on $[V;Y]_{(S_k)}$ is given by the local product structure. To show that this is well-defined, we need to show that local bundle-isomorphisms $\Phi : V \to U$ lift to a diffeomorphism $\tilde{\Phi} : [V;Y]_{(S_k)} \to [U;Y]_{(S_k)}$. This follows directly from Lemma 2.5 since Φ is linear on the fibers and depends smoothly on local coordinates in the base.

Projective coordinates: Near a point $p \in Y \subset V$ let y_i be local coordinates in Y. Let x_i be linear coordinates in $\mathbb{R}^n_k$. Then a local trivialization of the vector bundle near $p \ni U \backsimeq W_{\subseteq Y} \oplus \mathbb{R}^n_k$ allows the x_i to be pulled back to V to become local fiber coordinates. With the ξ_j defined as before (y_i, ξ_j) become local coordinates on $[V;Y]_{(S_k)}$.

The function spaces $\mathcal{F}^{(m)}$: The function spaces $\mathcal{F}^{(m)}$ may be defined identically as in (2.25), when choosing local coordinates (x_i, y_j) as above. Furthermore Lemma 2.7, Proposition 2.8 and Proposition 2.9 all still hold.

2.3. Point in a manifold

Next, we to consider the case of a point $p \in X$ in a manifold with corners. Let (S_k) be a conormal filtration at p. The construction of the new front face as in (2.9) still works in the same way. Following the construction of radial blow-ups, it would be natural to construct the C^∞-structure by using a normal fibration of p. The normal bundle of a single point is of course just the tangent space T_pX. Let $\phi : U_{\subseteq T_pX} \to V_{\subseteq X}$ be such a normal fibration. The cotangent space of $p \in X$ is naturally identified with the cotangent space of $0 \in T_pX$ (there are the same space after all), so the conormal filtration at p is naturally identified with a conormal filtration at $0 \in T_pX$. Since ϕ is a diffeomorphism, we may define the blow-up of $p \in X$ by blowing up $0 \in T_pX$ (using linear matching coordinates on T_pX). For a single point p, local coordinates (x_i) centered at p always define a normal fibration, since it identifies T_pX with $\operatorname{span}\{\partial_{x_1}, \dots \partial_{x_n}\}$. On the other hand, a normal fibration ϕ together with linear coordinates (x_i) on T_pX also defines local coordinates at p by pulling back the x_i under ϕ^{-1}. Thus, in the case of a single point, choosing a normal fibration coincides with choosing local coordinates (although there is no 1-1 correspondence). Of course, this opens the question of whether or not the C^∞-structure is independent of the choice of normal fibration. We have already seen in the last section that it is independent of linear transformations on T_pX. However two normal fibrations ϕ, ψ may vary in more complicated ways.

> **Example:** Consider $0 \in \mathbb{R}^2$ in standard coordinates x_1, x_2. We define the conormal filtration by
>
> $$S_1 = \operatorname{span}\{\, dx_1,\, dx_2\} \ , \ S_2 = S_3 = \operatorname{span}\{\, dx_2\} \ , \ S_4 = \{0\} = \dots \tag{2.33}$$
>
> Therefore, the weights $\kappa = (\kappa_1, \kappa_2)$ associated with dx_1 and dx_2 are 1 and 3 respectively. By definition, the coordinates x_1, x_2 match the (S_k), therefore we can construct the blow-up and blow-down map $\beta : [\mathbb{R}^2; 0]_{(S_k)} \to \mathbb{R}^2$ using these coordinates. There are two systems of projective coordinates, corresponding to x_1

and x_2 respectively (see figure 2.1). Now consider the coordinate transformation

$$(x_1, x_2) \overset{\Phi}{\mapsto} (x_1, x_2 + x_1^2) = (\bar{x}_1, \bar{x}_2). \tag{2.34}$$

Clearly the new coordinates also match (S_k) since, $d\bar{x}_1 = dx_1$ and $d\bar{x}_2 = dx_2$. Therefore, we may define the blow-up $\bar{\beta} : [\mathbb{R}^2; 0]_{(S_k)} \to \mathbb{R}^2$ together with the analog projective coordinates $(\bar{\xi}_1, \bar{\xi}_2)$, $(\bar{\eta}_1, \bar{\eta}_2)$ (see figure 2.1). We can now write

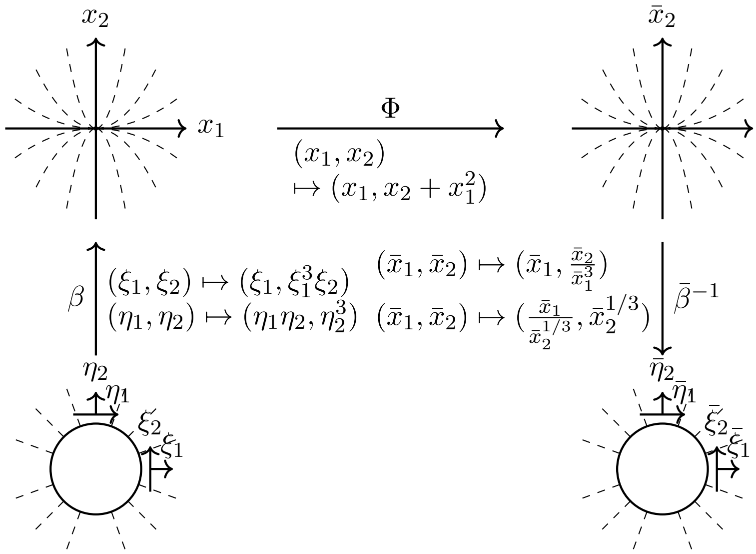

FIGURE 2.1. An example for two quasihomogeneous blow-ups of the same (S_k).

down the lift of Φ in local coordinates (η_1, η_2) and $(\bar{\eta}_1, \bar{\eta}_2)$:

$$\begin{aligned} (\eta_1, \eta_2) &\overset{\beta}{\mapsto} (\eta_1\eta_2, \eta_2^3) \overset{\Phi}{\mapsto} (\eta_1\eta_2, \eta_2^3 + \eta_1^2\eta_2^2) \\ &\overset{\bar{\beta}^{-1}}{\mapsto} \left(\underbrace{\frac{\eta_1\eta_2}{(\eta_2^3 + \eta_1^2\eta_2^2)^{1/3}}}_{=\bar{\eta}_1}, \underbrace{(\eta_2^3 + \eta_1^2\eta_2^2)^{1/3}}_{=\bar{\eta}_2} \right). \end{aligned} \tag{2.35}$$

Notice that the $\bar{\eta}_2$-variable is *not* smooth, when $\eta_2 \to 0$. Therefore, the diffeomorphism Φ does not lift to become a diffeomorphism on the blown-up spaces.

So what happened? Looking at the coordinate change, it seems that we just added a higher order term to the second coordinate. However, since we are performing a quasihomogeneous blow-up, the standard notion of order is not the correct one to consider. Instead, we really should be considering a quasihomogeneous order: Since dx_2 has assigned weight 3, the function x_2 should be considered to have quasihomogeneous weight 3. And since dx_1 has assigned weight 1, the function x_1^2 should be considered to have quasihomogeneous weight 2. Therefore, we actually changed the second coordinate to *lower* order than the coordinate itself. This leads us back the function spaces $\mathcal{F}^{(m)}$:

2.3.1. The function spaces $\mathcal{F}^{(m)}$.

Definition 2.10: *Let* (S_k) *be a conormal filtration at a point* $p \in X$ *and* $\phi : U_{\subseteq T_pX} \to V_{\subseteq X}$ *be a normal fibration. We set*

$$\mathcal{F}_\phi^{(m)} := \{f \in C^\infty(X) \mid f|_V \in (\phi^{-1})^*(\mathcal{F}^{(m)})\} \subset C^\infty(X). \tag{2.36}$$

Lemma 2.11: *Let* (S_k), ϕ, $\mathcal{F}_\phi^{(m)}$ *be as above.*

(1) *If* ϕ *is given by matching coordinates* (x_i) *(i.e. the* (x_i) *are the pullbacks of matching linear coordinates on* T_pX*) then* $\mathcal{F}_\phi^{(m)}$ *again takes the local coordinate form as in* (2.25)

$$\mathcal{F}_\phi^{(m)} = \mathcal{F}_{(x_i)}^{(m)} = \sum_{\kappa\alpha \geq m} x^\alpha C^\infty(X). \tag{2.37}$$

(2) *Lemma 2.7, proposition 2.8 and proposition 2.9 are all true for the spaces* $\mathcal{F}_\phi^{(m)} \subset C^\infty(X)$.

Proof. The first point is clear by definition. The second follows from the fact that ϕ is a diffeomorphism and all the mentioned results are stable under diffeomorphism. □

Lemma 2.7 and Proposition 2.9 actually characterize what kind of sequences may occur:

Proposition 2.12: *Let* $\mathcal{F}^{(m)} \subset C^\infty(X)$ *be a sequence of function spaces for* $m \in \mathbb{N}$ *that satisfy:*

(1) $S_k = \{df \mid f \in \mathcal{F}^{(k)}\}$ *is a conormal filtration at* $p \in X$.
(2) $\mathcal{F}^{(m)}$ *satisfies (1)-(5) in Lemma 2.7.*
(3) *The sequence as in* (2.30) *is well-defined and exact.*

Then there exists a normal fibration $\phi : U_{\subseteq T_pX} \to V_{\subseteq X}$ *that linearizes* $\mathcal{F}^{(m)}$ *in the sense that* $\mathcal{F}^{(m)} = \mathcal{F}_\phi^{(m)}$ *with respect to the conormal filtration* (S_k)*. In other words, there exist coordinates* (x_i) *centered at* p *matching the conormal filtration such that the* $\mathcal{F}^{(m)}$ *takes the form* (2.25).

Proof. Since $S_1 = T_p^*X$ and $S_{q+1} = \{0\}$ we have

$$\dim\left(\bigoplus_m {}^{S_m}\!\big/_{S_{m+1}}\right) = n. \tag{2.38}$$

Therefore we can choose $x_1, \ldots, x_n$ with independent differential at p (and therefore local coordinates) such that

$$S_k = \mathrm{span}_{i=N(k)\ldots n}\{dx_i\} \tag{2.39}$$

with a suitable increasing sequence of integers $N(k)$. These then also define the integers κ_i associated to each x_i as before. Furthermore, since we assume the sequences in (2.30) to be exact, we can choose the x_i in such a way that $x_i \in \mathcal{F}^{(\kappa_i)}$ but $x_i \notin \mathcal{F}^{(\kappa_i+1)}$. The first of which immediately implies

$$\left\{\sum_{\kappa\alpha \geq m} x^\alpha C^\infty(X)\right\} \subseteq \mathcal{F}^{(m)}. \tag{2.40}$$

Knowing this for all m, we can prove the set equality by induction over m. Since $\mathcal{F}^{(1)} = \mathcal{I}$, the equality is clear for $m = 1$. Now assume the set equality holds in (2.40) for all $\tilde{m} < m$ and assume that there exists a $0 \neq f \in \mathcal{F}^{(m)} \setminus \left\{ \sum_{\kappa\alpha \geq m} x^\alpha C^\infty(X) \right\}$. Writing f in Taylor series and using (2.40) we may assume without loss of generality that it takes the form

$$f = \sum_{\kappa\alpha < m} \lambda_\alpha x^\alpha. \tag{2.41}$$

Furthermore, we know that the linear part of f is mapped by d into S_m, which means by choice of the x_i that only those x_i-coefficients are non zero where $\kappa_i \geq m$. We can therefore again by (2.40) assume that f has no order one terms, yielding $d|_p f = 0$. By the exactness of the sequence we then get

$$f \in \mathcal{F}^{(m+1)} + \sum_{\substack{n_1+\dots+n_k=m \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)}. \tag{2.42}$$

However, by induction, any function in the sum on the right has only Taylor coefficients non zero whenever $\kappa\alpha \geq m$, yielding that we in fact have $f \in \mathcal{F}^{(m+1)}$. Since we already know that $d|_p f = 0$, we can use the exact sequence for $m + 1$ to get

$$f \in \mathcal{F}^{(m+2)} + \sum_{\substack{n_1+\dots+n_k=m+1 \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)}. \tag{2.43}$$

However, since we clearly have

$$\sum_{\substack{n_1+\dots+n_k=m+1 \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)} \subseteq \sum_{\substack{n_1+\dots+n_k=m \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)} \tag{2.44}$$

we again already have $f \in \mathcal{F}^{(m+2)}$ and so forth, yielding $f \in \mathcal{F}^k$ for all k. Using the last condition (6) from Lemma 2.7, this yields $f \in \mathcal{I}^\infty$, contradicting (2.41). $\square$

2.3.2. Quasihomogeneous structure. We now have all the ingredients we need to define a full *quasihomogeneous structure.* In fact, we can give two equivalent definitions.

Definition/Theorem 2.13: *Let $p \in M$.*

(1) A series of function spaces $\mathcal{F}^{(m)}$ satisfying the conditions of Proposition 2.12 is call a ***quasihomogeneous structure*** *at $p \in M$.*

(2) For an equivalent characterization, let (S_k) be a conormal filtration at $p \in X$. We say two matching coordinate systems (x_i) and $(\bar{x}_i)$ (or two normal fibrations) are equivalent, if $\mathcal{F}^{(m)}_{(x_i)} = \bar{\mathcal{F}}^{(m)}_{(\bar{x}_i)}$ for all m as in Definition 2.10, written $(x_i) \sim_S (\bar{x}_i)$. This defines an equivalence relation on the set of all matching coordinates. A tuple $\Pi_p = ((S_k), [x]_{\sim_S})$ is also a quasihomogeneous structure at $p \in X$. We call such a equivalence class $[x]$ an jet of quasihomogeneous order (S_k).

We denote a quasihomogeneous structure given in either form by Π.

Proof. The only thing to show is that the two definitions coincide. As shown in proposition 2.12, for each $(\mathcal{F}^{(m)})$ there is a corresponding $((S_k), [x]_{\sim_S})$ and visa versa. $\square$

Remark 2.14: *Notice that the definition of quasihomogeneous jet actually generalizes the definition of a jet, where a classical jet of order r corresponds to the conormal filtration given by $S_k = T_p^*X$ for $k \leq r$ and $S_k = \{0\}$ for $k > r$.*

Before we state the theorems that justify this definition we want to characterize the equivalence of coordinate systems in Taylor series.

Proposition 2.15: *Let (S_k) with associated $\kappa \in \mathbb{N}^n$ be as before. Let $\phi, \bar{\phi}$ be two normal fibrations at p given by two matching coordinate system (x_i) and $(\bar{x}_i)$. Let us write the coordinate transformation in Taylor series*

$$x_i \sim \sum_\alpha \lambda_{i,\alpha} \bar{x}^\alpha. \tag{2.45}$$

Then $(x_i) \sim_S (\bar{x}_i)$ if and only if $\lambda_{i,\alpha} = 0$ whenever $\kappa\alpha < \kappa_i$.

Proof. To start with, assume the coordinate transformation takes the form above. We know that $\mathcal{F}^{(m)}_{(x_i)}$ is spanned by x^β with $\kappa\beta \geq m$. Given such a β, we have

$$\begin{aligned} x^\beta &= \prod_{i=1}^n \left(\sum_\alpha \lambda_{i,\alpha} \bar{x}^\alpha \right)^{\beta_i} \\ &= \text{Sum of terms } \lambda (x^{\alpha_1})^{\beta_1} \cdots (x^{\alpha_n})^{\beta_n} \text{ where } \alpha_i \kappa \geq \kappa_i \ \forall j. \end{aligned} \tag{2.46}$$

Here, each α_i is a multi-index and $\beta = (\beta_j)$ is a single multi-index. We then have

$$\kappa \cdot (\alpha_1\beta_1 + \cdots + \alpha_n\beta_n) = \kappa \cdot \alpha_1\beta_1 + \cdots + \kappa \cdot \alpha_n\beta_n \geq \kappa_1\beta_1 + \cdots + \kappa_n\beta_n = \kappa\beta \geq m \tag{2.47}$$

which implies $x^\beta \in \mathcal{F}^{(m)}_{\bar{x}_i}$ and therefore $\mathcal{F}^{(m)}_{x_i} \subseteq \mathcal{F}^{(m)}_{\bar{x}_i}$.
For the reverse set inequality, it seems that one needs to deal with the Taylor coefficients of the inverse transformation. Fortunately, one does not. Instead we use induction over m and a dimension argument. Clearly $\mathcal{F}^{(1)}_{x_i} = \mathcal{F}^{(1)}_{\bar{x}_i} = \mathcal{I}(p)$. Now assume $\mathcal{F}^{(m)}_{x_i} = \mathcal{F}^{(m)}_{\bar{x}_i}$ for some fixed m. Consider the space $\mathcal{F}^{(m)}_{x_i} / \mathcal{F}^{(m+1)}_{x_i}$. Equation (2.37) implies that this quotient is a finite dimensional vector space spanned over $\mathbb{R}$ by x^α with $\kappa\alpha = m$. Furthermore all these monomials are linear independent in $\mathcal{F}^{(m)}_{x_i} / \mathcal{F}^{(m+1)}_{x_i}$, since a linear combination of them never lies in $\mathcal{F}^{(m+1)}_{x_i}$ (this is again clear form the coordinate representation). Therefore it is a finite-dimensional vector space of dimension $\#\{\alpha \in \mathbb{N}^n \mid \kappa\alpha = m\}$. By the same argument we get

$$\dim \left(\mathcal{F}^{(m)}_{x_i} / \mathcal{F}^{(m+1)}_{x_i} \right) = \dim \left(\mathcal{F}^{(m)}_{\bar{x}_i} / \mathcal{F}^{(m+1)}_{\bar{x}_i} \right). \tag{2.48}$$

Together with $\mathcal{F}^{(m)}_{x_i} = \mathcal{F}^{(m)}_{\bar{x}_i}$ and $\mathcal{F}^{(m+1)}_{x_i} \subseteq \mathcal{F}^{(m+1)}_{\bar{x}_i}$ this implies $\mathcal{F}^{(m+1)}_{x_i} = \mathcal{F}^{(m+1)}_{\bar{x}_i}$.

Now take $(x_i) \sim_S (\bar{x}_i)$, i.e. $\mathcal{F}^{(m)}_{x_i} = \mathcal{F}^{(m)}_{\bar{x}_i}$ for all m. Since $x_i \in \mathcal{F}^{\kappa_i}_{x_i}$ we know that

$$x_i = \sum_\alpha \lambda_{i,\alpha} \bar{x}^\alpha \in \mathcal{F}^{(\kappa_i)}_{\bar{x}_i} \tag{2.49}$$

which implies that $\lambda_{i,\alpha} = 0$ whenever $\kappa\alpha < \kappa_i$, showing the equivalence. $\square$

Remark 2.16: *Proposition 2.15 actually proves a non trivial result about the inversion of power series: Given a smooth invertible function $f : \mathbb{R}^n \to \mathbb{R}^n$ with Taylor series $f_i = \sum_\alpha \lambda_{i,\alpha} x^\alpha$ and a multi-index κ, then the following holds: If $\lambda_{i,\alpha} = 0$ whenever $\kappa\alpha < \kappa_i$, then the Taylor series of f^{-1} satisfies the same.*

We can now define the quasihomogeneous blow-up of a point p as before

Definition 2.17: *Let $p \in X$ and $\mathcal{F}^{(m)}$ a quasihomogeneous structure at p. Let the set ff be defined as before in (2.9) as*

$$
\begin{aligned}
V_S &= {T_p^+ X}\big/{S_q^{\perp+}} \oplus {S_q^{\perp+}}\big/{S_{q-1}^{\perp+}} \oplus \cdots \oplus {S_3^{\perp+}}\big/{S_2^{\perp+}} \oplus S_2^{\perp+} \\
\mathit{ff} &= {(V_S \setminus \{0\})}\big/{\mathbb{R}_{>0}}
\end{aligned}
\tag{2.50}
$$

with the $\mathbb{R}_{>0}$ action defined as before by the vector field R_S given by the k-th multiple of the standard outgoing vector field on ${S_{k+1}^{\perp+}}\big/{S_k}$.
We define the blow-up of Y with respect to $\mathcal{F}^{(m)}$ as

$$
[X;p]_\Pi := \mathit{ff} \sqcup (X \setminus \{p\})
\tag{2.51}
$$

with C^∞ structure and blow-down map constructed using a matching normal fibration $\phi : U_{\subseteq T_p X} \to V_{\subseteq X}$.

Projective coordinates on the blown-up space are defined as before in Section 2.1.1. Before we can prove that the C^∞ structure of $[X;p]_\Pi$ is well-defined, we need to study how vector fields behave under quasihomogeneous blow-ups.

2.3.3. Vector fields.

Definition 2.18: *Let Π be a quasihomogeneous structure at $p \in X$. We say a vector field $V \in C^\infty(X, {}^bTX)$ is of quasihomogeneous order $-m$ with respect to Π, if $V\mathcal{F}^{(m')} \subseteq \mathcal{F}^{(m'-m)}$ for each m' (Where for negative m we extend the filtration by $\mathcal{F}^{(m)} = C^\infty(X)$). We denote the space of these vector fields by $\mathcal{V}_\Pi^{(-m)}$.*

Intuitively, a vector field of order $-m$ reduces the order of a function by at most m. As the notation suggest, the space $\mathcal{V}_\Pi^{(-m)}$ is well-defined, that is it only depends on Π, which is clear since the $\mathcal{F}^{(m)}$ are well-defined.

Lemma 2.19: *$\mathcal{V}_\Pi^{(-m)}$ is a $\mathbb{R}$-vector-space, a $\mathcal{C}^\infty$-module, and the Lie bracket satisfies*

$$
[\mathcal{V}_\Pi^{(-m)}, \mathcal{V}_\Pi^{(-m')}] \subset \mathcal{V}_\Pi^{(-m-m')}.
\tag{2.52}
$$

In particular, $\mathcal{V}_\Pi^{(0)}$ is a Lie-Algebra.

Proof. The first two statements follow directly from the definition, since the $\mathcal{F}^{(m)}$ are vector spaces and C^∞-modules. The last part also follows directly from the definition. For $V \in \mathcal{V}_\Pi^{(-m)}$, $W \in \mathcal{V}_\Pi^{(-m')}$ and $f \in \mathcal{F}^{(n)}$ we have $W(Vf) - V(Wf) \in \mathcal{F}^{(n-m-m')}$. $\square$

Lemma 2.20: *Let $\Pi = ((S_k), [x]_{\sim_S})$ with associated κ. Then*

$$
\mathcal{V}_\Pi^{(-m)} = \sum_{\kappa\alpha \geq \kappa_j - m} C^\infty(X)\; x^\alpha \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} \mathcal{F}^{(\kappa_j - m)} \frac{\partial}{\partial x_j}.
\tag{2.53}
$$

PROOF. First, we check that the spanning vector fields are actually elements of $\mathcal{V}_\Pi^{(-m)}$. So take α with $\kappa_\alpha \geq \kappa_j - m$ and set $V = x^\alpha \partial/\partial x_j$. We know that $\mathcal{F}^{(m')}$ is spanned by x^β with $\kappa\beta \geq m'$. If $\beta_j = 0$ then $Vx^\beta = 0 \in \mathcal{F}^{(m'-m)}$. Otherwise $Vx^\beta = x^\alpha x^{\beta - e_j}$ where $e_j = (0, \dots 1, \dots 0)$ is the j-th unit index. We have $\kappa(\alpha + \beta - e_j) = \kappa\alpha + \kappa\beta - \kappa_j \geq \kappa_j - m + \kappa\beta + \kappa_j = \kappa\beta - m \geq m' - m$ and therefore $Vx^\beta \in \mathcal{F}^{(m'-m)}$.
Next take any $V \in \mathcal{V}_\Pi^{(-m)}$ with

$$V = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}. \tag{2.54}$$

Since $x_j \in \mathcal{F}^{(\kappa_j)}$ and $Vx_j = a_j$, we know that $a_j \in \mathcal{F}^{(\kappa_j - m)}$. Therefore, when writing a_j in Taylor series, it only has coefficients x^α with $\kappa\alpha \geq \kappa_j - m$, proving the statement. $\square$

Corollary 2.21: *Each $\mathcal{V}_\Pi^{(-m)}$ is locally finitely generated and they define a filtration of all smooth vector fields. Let q be the maximal order present in Π. Then we have*

$$\dots \mathcal{V}_\Pi^{(1)} \subset \mathcal{V}_\Pi^{(0)} \subset \mathcal{V}_\Pi^{(-1)} \subset \dots \mathcal{V}_\Pi^{(-q+1)} \subset \mathcal{V}_\Pi^{(-q)} = \mathcal{V}_b(X). \tag{2.55}$$

In fact, such a filtration of the smooth vector fields yields another way of defining a quasihomogeneous structure.

Proposition 2.22: *For each m let $\mathcal{V}_\Pi^{(-m)}$ be a finitely generated C^∞-module of vector fields that satisfy* (2.52) *and* (2.55)*. Then they define a quasihomogeneous structure at $Y \subset X$.*

PROOF. We will not prove this here, since there are a lot of cumbersome details involved. A full proof can be found in [18]. We will however consider a special case later on in Proposition 2.39. $\square$

Proposition 2.23: *The vector fields $V \in \mathcal{V}_\Pi^{(0)}$ lift under β to become smooth on $[X;p]_\Pi$ and elements of the Lie algebra $\mathcal{V}_b([X;p]_\Pi)$. It spans $\mathcal{V}_b([X;p]_\Pi)$ over C^∞ except possibly for a null set of the front face.*

PROOF. By Lemma 2.20 we only have to consider the lifts of $x^\alpha \partial/\partial x_j$ with $\kappa\alpha \geq \kappa_j$. We use projective coordinates to calculate the lift. Recall that for each $i \in \{1, \dots, n\}$ there are associated projective coordinates given by

$$\xi_i = x_i^{1/\kappa_i} \text{ and } \xi_j = x_j x_i^{-\kappa_j/\kappa_i} \text{ for } j \neq i \tag{2.56}$$

with the blow-down map locally taking the form

$$x_i = \xi_i^{\kappa_i} \text{ and } x_j = \xi_j \xi_i^{\kappa_j} \text{ for } j \neq i. \tag{2.57}$$

Next, see that

$$\frac{\partial \xi_k}{\partial x_i} = \begin{cases} \frac{1}{\kappa_i} x_i^{(1/\kappa_i \; -1)} & (k = i) \\ -\frac{\kappa_k}{\kappa_i} x_k x_i^{-\kappa_k/\kappa_i \; -1} & (k \neq i) \end{cases} \text{ and} \tag{2.58}$$

$$\frac{\partial \xi_k}{\partial x_j} = \begin{cases} x_i^{-\kappa_k/\kappa_i} & (k = j) \\ 0 & (k \neq j) \end{cases} \text{ for } j \neq i \tag{2.59}$$

and therefore

$$\beta^*(\frac{\partial}{\partial x_j}) = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_j}\frac{\partial}{\partial \xi_k} = \begin{cases} \frac{1}{\kappa_i}\xi_i^{1-\kappa_j}\frac{\partial}{\partial \xi_j} - \sum_{k\neq i}\frac{\kappa_k}{\kappa_i}\xi_k\xi_i^{-\kappa_i}\frac{\partial}{\partial \xi_k} & (j=i) \\ \xi_i^{-\kappa_j}\frac{\partial}{\partial \xi_j} & (j\neq i) \end{cases}. \tag{2.60}$$

Also recall that

$$\beta^*(x^\alpha) = \xi_i^{\alpha\kappa} \cdot \prod_{j\neq i} \xi_j^{\alpha_j} \tag{2.61}$$

which proves that whenever $\kappa\alpha \geq \kappa_j$ we get

$$\beta^*(x^\alpha\frac{\partial}{\partial x_j}) = \begin{cases} \lambda\cdot\xi_i\frac{\partial}{\partial \xi_i} - \sum_{k\neq i}\lambda_k\frac{\partial}{\partial \xi_k} & (j=i) \\ \lambda\cdot\frac{\partial}{\partial \xi_j} & (j\neq i) \end{cases} \in \mathcal{V}_b([X;p]_\Pi). \tag{2.62}$$

It is also easy to see that wherever all $\xi_j \neq 0$ for all $j\neq i$, $\mathcal{V}_b([X;p]_\Pi)$ is spanned by these vector fields, since $\beta^*(x_j\partial_{x_j}) = \xi_j\partial_{\xi_j}$. □

2.3.4. Exceptional subset. As mentioned in the proposition above, there may occur points on the front face, where $\mathcal{V}_b([X;p]_\Pi)$ is not spanned by $\beta^*(\mathcal{V}^0_\Pi)$. As an example consider the blow-up of $0\in\mathbb{R}^2$ where x_1 is given the weight 2 and x_2 is given the weight 3. Local projective coordinates obtained by scaling by x_1 are given by

$$\xi_1 = x_1^{1/2}\ ,\ \xi_2 = x_1^{-3/2}x_2. \tag{2.63}$$

We then have for $2\alpha_1+3\alpha_2\geq 2$

$$\beta^*(x_1^{\alpha_1}x_2^{\alpha_2}\partial_{x_1}) = \xi_1^{2\alpha_1+3\alpha_2-1}\xi_2^{\alpha_2}\partial_{\xi_1} - \frac{3}{2}\xi_1^{2\alpha_1+3\alpha_2-2}\xi_2^{\alpha_2+1}\partial_{\xi_2} \tag{2.64}$$

where the second summand is zero at $\xi=0$. Furthermore, for $2\alpha_1+3\alpha_2\geq 3$ we have

$$\beta^*(x_1^{\alpha_1}x_2^{\alpha_2}\partial_{x_2}) = \xi_1^{2\alpha_1+3\alpha_2-3}\xi_2^{\alpha_2}\partial_{\xi_2} \tag{2.65}$$

which is also equal to zero at $\xi=0$ since either $\alpha_2\neq 0$ or $\alpha_1\geq 2$. Therefore ∂_{ξ_2} cannot be part of the C^∞-span of $\beta^*(\mathcal{V}^{(0)}_\Pi)$.

Definition 2.24: *Given a quasihomogeneous blow-up $[X;p]_\Pi$ we define the **exceptional subset** of the front face, denoted by $E(ff)$, as the set of points $p\in ff$ at which $\mathcal{V}_b([X;p;\Pi])$ is not spanned over C^∞ by $\beta^*(\mathcal{V}^0_\Pi)$.*

We already have seen that this is always a null set in ff. We explicitly calculate it in local projective coordinates.

Lemma 2.25: *Given a quasihomogeneous blow-up and projective coordinates ξ_i,ξ_j as in (2.56), the intersection of $E(ff)$ with the domain D_i of the projective coordinates is given by the equation*

$$D_i\setminus E(ff) = \left\{\xi \mid \forall l : \kappa_l \in \operatorname{span}_{\mathbb{N}_0}\left(\{\kappa_i\}\cup\{\kappa_j \mid \xi_j\neq 0\}\right)\right\}. \tag{2.66}$$

In particular, whenever $\kappa_i = 1$, then $E(ff)\cap D_i = \emptyset$.

Proof. We have to show that near any ξ lying in the set defined in (2.66) the lift of $\mathcal{V}^{(0)}$ spans $\mathcal{V}_b([X;p;\Pi])$ over C^∞. Consider a fixed $j \neq i$, then there is a multi-index α satisfying $\forall j \neq i: \ \alpha_j \neq 0 \Rightarrow \xi_j \neq 0$ and $\kappa\alpha = \kappa_j$. Therefore $x^\alpha \partial_{x_j} \in \mathcal{V}^{(0)}$ and the lift is given by

$$\beta^*(x^\alpha \partial_{x_j}) = \left(\prod_{\alpha_j \neq 0} \xi_j^{\alpha_j} \right) \partial_{\xi_j}, \tag{2.67}$$

which spans, over C^∞ near ξ, the vector field ∂_{ξ_j}. Furthermore

$$\beta^*(x_i \partial_{x_i}) = \xi_i \partial_{\xi_i} - \sum_{j \neq i} \xi_j \partial_{\xi_j}. \tag{2.68}$$

which proves the statement. $\square$

The exceptional subset can also be expressed directly as a subset of ff as defined in (2.50). Consider a point $v = (v_q, \dots, v_1) \in V_S$, meaning $v_k \in {S_{k+1}^\perp}/{S_k^\perp}$. Then

$$E(\mathrm{ff}) = \left\{ [v] \in \mathrm{ff} \mid \exists l : {S_{l+1}^\perp}/{S_l^\perp} \neq \{0\} \text{ and } l \notin \operatorname{span}_{\mathbb{N}_0}\{k \mid v_k \neq 0\} \right\}. \tag{2.69}$$

In other words, the exceptional subset consist of all those points $[v]$, where the set of all weights k for which $v_k \neq 0$ does not span all the weight that occur in the quasihomogeneous structure.

2.3.5. Main theorem. Lastly, we state the theorem showing that the definition of quasihomogeneous structure is actually meaningful.

Theorem 2.26: *Let $\Pi = ((S_k), [x]_{\sim_S})$ and $\bar{\Pi} = ((\bar{S}_k), [\bar{x}]_{\sim_{\bar{S}}})$ be two quasihomogeneous structures at $p \in X$. Then the identity on X lifts to become a diffeomorphism $[X;p]_\Pi \cong [X;p]_{\bar{\Pi}}$ if and only if $\Pi = \bar{\Pi}$.*

Proof. First, assume that the identity on X lifts to become a diffeomorphism $\Phi : [X;p]_\Pi \xrightarrow{\cong} [X;p]_{\bar{\Pi}}$. Let $\mathrm{ff} \subset [X;p]_\Pi$ and $\bar{\mathrm{ff}} \subset [X;p]_{\bar{\Pi}}$ be the two front faces. Then we clearly have $\Phi(\mathcal{I}^m(\mathrm{ff})) = \mathcal{I}^m(\bar{\mathrm{ff}})$. Now Proposition 2.8 immediately implies that $\mathcal{F}^{(m)} = \bar{\mathcal{F}}^{(m)}$ for each m. Therefore $\Pi = \bar{\Pi}$.

Now, take $\Pi = \bar{\Pi}$, that is $\Pi = ((S_k), [x]_{\sim_s})$ and $\bar{\Pi} = ((S_k), [\bar{x}]_{\sim_s})$ with $(x) \sim_s (\bar{x})$. To start with, we assume that the coordinate transformation is linear. That is, we have $\bar{x} = Ax$ for a real invertible matrix. Since the coordinates both match (S_k), Lemma 2.5 implies that A lifts to a diffeomorphism.
Returning to the general case $x \sim_S \bar{x}$, we now can assume that the coordinate transformation $\bar{x} = \Phi(x)$ has a linear term equal to the identity. We then may connect Φ to the identity by a homotopy

$$\Phi_t := \mathrm{Id} + t(\Phi - \mathrm{Id}) \ , \ t \in [0,1] \tag{2.70}$$

Since $\Phi^* \mathcal{F}^{(m)} = \mathcal{F}^{(m)}$, we also have $\Phi_t^* \mathcal{F}^{(m)} = \mathcal{F}^{(m)}$. Such a parameter dependent family of maps Φ_t is always given by the integration of a (parameter dependent) vector fields W_t

characterized by the equation

$$\frac{d}{dt}\Phi_t^* f = \Phi_t^*(W_t f) \text{ for all } f \in \mathcal{C}^\infty \tag{2.71}$$

If $f \in \mathcal{F}^{(m)}$, we then also have $W_t f \in \mathcal{F}^{(m)}$ because $\mathcal{F}^{(m)}$ is invariant under Φ_t for all t. Thus, $W_t \in \mathcal{V}^{(0)}$ by Definition 2.18. Proposition 2.23 now states that we may lift W_t to a vector field $\tilde{W}_t \in \mathcal{V}_b([NY;Y]_{(S_k)})$. The lift $\tilde{\Phi}_t$ of Φ_t is now again determined by the equation

$$\frac{d}{dt}\tilde{\Phi}_t^* f = \tilde{\Phi}_t^*(\tilde{W}_t f) \text{ for all } f \in \mathcal{C}^\infty \ , \ \tilde{\Phi}_0 = \mathrm{Id}\,. \tag{2.72}$$

Then, the lift of Φ is given by $\tilde{\Phi}_1$.

□

2.4. General p-submanifold

In the case of a single point, we defined a quasihomogeneous structure using both equivalent classes of normal fibrations and in a coordinate-free way using function spaces $\mathcal{F}^{(m)}$ of quasihomogeneous weight m. Both can be generalized to p-submanifolds, we start with the latter one:

Definition 2.27: *Let $\mathcal{F}^{(m)}$ be function spaces for $m \in \mathbb{N}$ that satisfy:*

(1) $S_k = \{df|_{N^*Y} \mid f \in \mathcal{F}^{(k)}\} \subseteq N^*Y$ is a conormal filtration at $Y \subset X$.
(2) $\mathcal{F}^{(m)}$ is a C^∞-Module.
(3) $\mathcal{I}(Y) = \mathcal{F}^{(1)} \supseteq \mathcal{F}^{(2)} \supseteq \dots$.
(4) $\mathcal{F}^{(m)} \cdot \mathcal{F}^{(m')} \subseteq \mathcal{F}^{(m+m')}$
(5) The sequence

$$0 \longrightarrow \mathcal{F}^{(m+1)} + \sum_{\substack{n_1+\dots+n_k=m \\ k\geq 2,\ n_j \geq 1}} \mathcal{F}^{(n_1)} \cdots \mathcal{F}^{(n_k)} \longrightarrow \mathcal{F}^{(m)} \overset{d|_{N^*Y}}{\longrightarrow} C^\infty(Y; {}^{S_m}\!/_{S_{m+1}}) \longrightarrow 0 \tag{2.73}$$

is well-defined and exact for each m.
(6) $\mathcal{F}^{(\infty)} = \mathcal{I}^{(\infty)}$ where $\mathcal{F}^{(\infty)} = \bigcap_m \mathcal{F}^{(m)}$ and $\mathcal{I}^{(\infty)} = \bigcap_m \mathcal{I}^{(m)}$.

Then we call $\mathcal{F}^{(m)}$ a ***quasihomogeneous structure*** *at $Y \subset X$.*

As before, we have

Lemma 2.28: *Given a conormal filtration (S_k) at $Y \subset X$ together with a normal fibration ϕ of Y, the spaces $\mathcal{F}_\phi^{(m)}$ as in Definition 2.10 are a quasihomogeneous structure at Y.*

We move on to the coordinate description of these spaces, although only locally for the beginning:

Definition 2.29: *Local product coordinates (x_i, y_j) of Y near p are said to* ***match*** *the quasihomogeneous structure $\mathcal{F}^{(m)}$ if there are weights κ such that, restricted to the domain of the coordinates, $\mathcal{F}^{(m)} = \mathcal{F}_{(x_i)}^{(m)}$ where the latter function space is only defined locally.*

Lemma 2.30: *Let $\mathcal{F}^{(m)}$ be a quasihomogeneous structure at $Y \subset X$. Then at any $p \in Y$ there are local product coordinates that match $\mathcal{F}^{(m)}$.*

PROOF. Start with any local coordinates y_j in Y. Then the proof of proposition 2.12 then applies. $\square$

We do have a need for a global version of this.

Definition 2.31: *Given a conormal filtration (S_k) at Y and a normal fibration $\phi : U_{\subseteq NY} \to V_{\subseteq X}$, we call the quasihomogeneous structure $\mathcal{F}^{(m)}_\phi$ the **linearized** quasihomogeneous structure of (S_k) under ϕ at Y.*

Definition 2.32: *We say that a normal fibration $\phi : U_{\subseteq NY} \to V_{\subseteq X}$ **matches** a quasihomogeneous structure $\mathcal{F}^{(m)}$ at $Y \subset X$ if $\mathcal{F}^{(m)}_\phi = \mathcal{F}^{(m)}$ for all m, in other words if ϕ linearizes $\mathcal{F}^{(m)}$.*

A matching normal fibration ϕ yields local matching coordinates everywhere by taking a local trivialization of ϕ together with product coordinates where the fiber ones are taken to be linear.
We can now write down the 'coordinate'-version of the definition of a quasihomogeneous structure:

Definition 2.33: *Let $Y \subset X$ be a p-submanifold. A **quasihomogeneous structure** at Y may equivalently be defined as a tuple $((S_k), [\phi])$ where (S_k) is a conormal filtration at Y and $[\phi]$ is a equivalence class of normal fibrations, where two normal fibrations $\phi, \bar{\phi}$ are said to be equivalent if $\mathcal{F}^{(m)}_\phi = \mathcal{F}^{(m)}_{\bar{\phi}}$. We denote a quasihomogeneous structure in any of those two forms by Π.*

The following Proposition yields the equivalence of Definition 2.27 and 2.33.

Proposition 2.34: *Let $\mathcal{F}^{(m)}$ be a quasihomogeneous structure at $Y \subset X$. Then there is a matching normal fibration $\phi : U_{\subseteq NY} \to V_{\subseteq X}$ that linearizes it, i.e. $\mathcal{F}^{(m)} = \mathcal{F}^{(m)}_\phi$.*

PROOF. Start with any tubular neighborhood $\bar{\phi} : U_{\subseteq NY} \to V_{\subset X}$ of Y. The matching condition above is local near any point $p \in Y$. Near such a p take local product coordinates $\bar{x}_i, y_j$ (with $Y = \{\bar{x}_i = 0\}$) that trivialize $\bar{\phi}$. The pullback of the quasihomogeneous structure at NY is then precisely represented by the $\bar{x}_i$. Furthermore, we may choose local coordinates x_i tangent to the fibers that linearizes the original quasihomogeneous structure at Y. Thus, we may compose the coordinate transformation from x_i to $\bar{x}_i$ (extended by the identity away from p) with $\bar{\phi}$ to get a new normal fibration ϕ that linearizes the original $\mathcal{F}^{(m)}$. $\square$

Definition 2.35: *As before we define the vector fields of homogeneous order at least $-m$ as the space*

$$\mathcal{V}_\Pi^{(-m)} = \{V \in \mathcal{V}_b(X)\ Y|\ V\mathcal{F}^{(m')} \subset \mathcal{F}^{(m'-m)} \text{ for all } m'\}. \tag{2.74}$$

Lemma 2.19 still holds. The local coordinate representation in Lemma 2.20 also holds except for additional terms of unrestricted b-vector fields in the y-coordinates. Equation 2.55 still holds. Proposition 2.22 as well.
We can now define the quasihomogeneous blow-up in the general case:

Definition 2.36: *Let $Y \subset X$ be a submanifold and $\mathcal{F}^{(m)}$ be a quasihomogeneous structure at Y. Let the set ff be defined as before in (2.9) as*

$$\begin{aligned} N_S &= {}^{N^+Y}\!\big/_{S_q^{\perp+}} \oplus {}^{S_q^{\perp+}}\!\big/_{S_{q-1}^{\perp+}} \oplus \cdots \oplus {}^{S_3^{\perp+}}\!\big/_{S_2^{\perp+}} \oplus S_2^{\perp+} \\ \mathit{ff} &= {}^{(N_S \setminus \{0\})}\!\big/_{\mathbb{R}_{>0}} \end{aligned} \tag{2.75}$$

with the $\mathbb{R}_{>0}$ action defined as before by the vector field R_S given by the k-th multiple of the standard outgoing vector field on ${}^{S_{k+1}^{\perp+}}\!/_{S_k}$.
We define the blow-up of Y with respect to $\mathcal{F}^{(m)}$ as

$$[X;Y]_\Pi := \mathit{ff} \sqcup (X \setminus Y) \tag{2.76}$$

with C^∞ structure and blow-down map constructed using a matching normal fibration $\phi : U_{\subseteq NY} \to V_{\subseteq X}$.

As before, we have to prove that the C^∞ structure is well-defined. Let $\phi, \psi : U_{\subseteq NY} \to V_{\subseteq X}$ be two normal fibration both matching $\mathcal{F}^{(m)}$. Then there is a diffeomorphism f (given by $\psi \circ \phi^{-1}$) defined on V such that $\psi = f \circ \phi$. Since both ϕ and ψ match the $\mathcal{F}^{(m)}$, f clearly maps $\mathcal{F}^{(m)}$ to itself. Now the proof as in the local case applies. First approximate f to arbitrary high order using vector fields that necessarily are elements of $\mathcal{V}_\Pi^{(0)}$, afterwards connect f to the identity by a homotopy as before. The definition of $\mathcal{V}_\Pi^{(0)}$ is identical to before, as is its local coordinate characterization.

Corollary 2.37: *For any Y, Π the blow-down map $\beta : [X;Y]_\Pi \to X$ is a b-map.*

Proof. Clearly, β is a smooth map between manifolds with corners. If Y is an interior p-submanifold of X then there is nothing else to show. Now let $Y \subseteq \partial X$. Recall from the definition of a conormal filtration 2.1 that for any boundary hypersurface $H \in \mathrm{Hu}(Y)$ that contains Y there is a associated number $\kappa_H \in \mathbb{N}$ such that $N^*H \subseteq S_{\kappa_H}$ but $N^*(H) \nsubseteq S_{\kappa_H+1}$. Now it is easy to see in local projective coordinates that β is in fact a b-map with the $e(H,G)$ from the definition of b-map satisfying $e(H, \mathrm{ff}) = \kappa_H$ for $H \in \mathrm{Hu}(Y)$ and $e(H, \mathrm{ff}) = 0$ for $H \notin \mathrm{Hu}(Y)$. $\square$

Local *projective coordinates* are constructed as before. For any local product coordinates (x, y) (with $Y = \{x = 0\}$) that linearize the quasihomogeneous structure Π, one gets local projective coordinates (ξ, y) as before in Section 2.1.1, by scaling all x coordinates by a fixed x_i.

Product structure: Consider two p-submanifolds $Y, Z \subset X$, such that their intersection $Y \cap Z$ is again a p-submanifold of X. Given quasihomogeneous structures Π_Y at Y and Π_Z at Z, we want to understand under which conditions these define a quasihomogeneous structure at $Y \cap Z$. One simple case where this is true is the following:

Lemma 2.38: *Consider two p-submanifold $Y, Z \subset X$ with quasihomogeneous structure Π_Z, Π_Y given by filtrations $\mathcal{F}_Y^{(m)}, \mathcal{F}_Z^{(m)}$ respectively. If Y and Z intersect transversally,* *then*

$$\mathcal{F}^{(m)}_{Y\cap Z} = \sum_{n+n'=m} \mathcal{F}^{(n)}_Y \mathcal{F}^{(n')}_Z \tag{2.77}$$

is a quasihomogeneous structure at $Y \cap Z$.

Proof. Since Y and Z intersect transversally, one can chose local product coordinates x, x', x'' near each $p \in Y \cap Z$ such that $Y = \{x = 0\}$ and $Z = \{x' = 0\}$. We may also chose x, x' in such a way, that these coordinates linearize both $\mathcal{F}^{(m)}_Y = \sum_{\kappa\alpha\leq m} x^\alpha C^\infty(X)$ and $\mathcal{F}^{(m)}_Z = \sum_{\kappa'\beta\leq m} x'^\beta C^\infty(X)$. In these coordinate we now have

$$\mathcal{F}^{(m)}_{Y\cap Z} = \sum_{\kappa\alpha+\kappa'\beta\leq m} x^\alpha x'^\beta C^\infty(X), \tag{2.78}$$

which is a quasihomogeneous structure at $Y \cap Z$. $\square$

2.5. Defining vector fields

As we have already seen, one may define a quasihomogeneous structure not only by a filtration of the space of smooth functions, $\mathcal{F}^{(m)}$, but also by a filtration of the space of smooth vector fields, $\mathcal{V}^{(-m)}_\Pi$. As we have seen, one simple way to construct such a filtration $\mathcal{F}^{(m)}$ is to choose local coordinates and associate a weight to each coordinate function. We now want to study an analogous way of constructing a filtration $\mathcal{V}^{(-m)}_\Pi$, by choosing a collection of vector fields that span the normal bundle of the submanifold one wants to blow up and associating a negative weight to each of these vector fields. Since we left out the proof of Proposition 2.22, we do want to give a full proof in this special setting.

Setting: Let $Y \subset X$ be a p-submanifold. Let $N_1, \dots N_k$ be smooth vector fields that, projected to NY, span NY everywhere. Each N_i has a negative weight $-\kappa_i$ associated. We assume that the N_i span a Lie algebra and that the Lie bracket is compatible with the associated weights in the sense that

$$[N_i, N_j] \in \operatorname{span}_{C^\infty}\{N_m \mid \kappa_m \leq \kappa_i + \kappa_j\}. \tag{2.79}$$

We introduce some additional notation: Let $I = (i_1, \dots, i_n)$ be a sequence of integers $i_m \in \{1, \dots k\}$. We set $N_I = N_{i_1} \dots N_{i_n}$. We denote the length of I by $|I|$ and furthermore we define the *weight* of I as $\langle I \rangle := \kappa_{i_1} + \dots + \kappa_{i_n}$. We then want to prove the following:

Proposition 2.39: *The function spaces*

$$\mathcal{F}^{(m)} := \{f \in C^\infty \mid \forall I,\ I < m : N_I f \in \mathcal{I}(Y)\} \tag{2.80}$$

define a quasihomogeneous structure at Y where the filtration of the conormal-bundle $N^*Y = S_1 \supseteq S_2 \dots$ is given by

$$S_m = (\operatorname{span}\{N_i|_{NY} \mid \kappa_i < m\})^\circ. \tag{2.81}$$

The weights $-\kappa_i$ associated to N_i then coincide with the weights of N_i with respect to this quasihomogeneous structure.

Preparations for the proof: First we want to replace the indices I by something simpler. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a k-multi-index of arbitrary integers. Set $N^\alpha = N_1^{\alpha_1} \dots N_k^{\alpha_k}$. The weight associated to α is simply $\kappa\alpha$. The condition (2.79) now guarantees the following:

Lemma 2.40: *For all I there are smooth coefficients $\lambda_{I,\alpha}$ such that*

$$N_I = \sum_{\kappa\alpha \le \langle I \rangle} \lambda_{I,\alpha} N^\alpha. \tag{2.82}$$

Proof. In short, we may commute N_I step by step to sort the indexes in I. We may use induction over $|I|$. For $|I| = 1$ the statement is trivial. The induction step reduces to showing that any $N_i N^{\tilde{\alpha}}$ (with $|\tilde{\alpha}| = |\alpha| - 1$) takes the form as above. We may successively commute N_i with all N_j, $j < i$, producing error terms which are the sums of some $\lambda N^{\tilde{I}}$ with $|\tilde{I}| = |I| - 1$, $\langle \tilde{I} \rangle \le \langle I \rangle$. This proves the Lemma by induction. □

Using this result, we can simplify the definition of (2.80) to obtain

$$\mathcal{F}^{(m)} = \{ f \in C^\infty \mid \forall \kappa,\ \kappa\alpha < m : N^\alpha f \in \mathcal{I}(Y) \}. \tag{2.83}$$

Definition 2.41: *We say local product coordinates $y_i, x_1, \dots x_k$, where $Y = \{x = 0\}$, are* ***linearly adapted*** *to $N_1, \dots N_k$ if $N_i|_{\{x=0\}} = \partial_{x_i}|_{\{x=0\}}$. Such coordinates always exist.*

Lemma 2.42: *Let (x_i) be local coordinates that are linearly adapted to the N_i. We then have*

$$N^\alpha(x^\beta)(0) = \begin{cases} \partial_x^\alpha(x^\beta)(0) & (|\alpha| = |\beta|) \\ 0 & (|\alpha| < |\beta|) \end{cases}. \tag{2.84}$$

Proof. Since (x_i) are linearly adapted to the N_i we have for each i

$$N_i = \partial_{x_i} + \sum_{j \ne i} \lambda_{i,j} \partial_{x_j} + \sum_s \mu_s \partial_{y_s}\ ,\ \ \lambda_{i,j} \in \mathcal{I}(Y). \tag{2.85}$$

Since $x^\beta \in \mathcal{I}^{|\beta|}(Y)$ this shows that

$$N_i x^\beta = \partial_{x_i}(x^\beta) + \underbrace{\sum_{j \ne i} \lambda_{i,j} \partial_{x_j}(x^\beta)}_{\in \mathcal{I}^{|\beta|}(Y)}. \tag{2.86}$$

Continuing in this manner proves the Lemma. □

Proof of the proposition 2.39: We need to show that the function spaces $\mathcal{F}^{(m)}$ as in (2.83) satisfy conditions (1) to (5) of Definition 2.27.
Proof of (1),(3): This is clear by definition.
Proof of (5): $\mathcal{I}^\infty(Y) \subset \mathcal{F}^{(\infty)}$ is trivial. The other inclusion follows immediately from Lemma 2.42.
Proof of (2): Let $a \in C^\infty$, $f \in \mathcal{F}^{(m)}$. Then

$$N^\alpha(af) = \sum_{\beta + \beta' \le \alpha} \lambda_{\beta,\beta'} N^\beta a N^{\beta'} f \tag{2.87}$$

for some constants $\lambda_{\beta,\beta'}$. since we always have $\kappa\beta' \le \kappa\alpha < m$, we have $N^{\beta'} f \in \mathcal{I}(Y)$ showing that $N^\alpha(af) \in \mathcal{I}(Y)$.

Proof of (4): This follows for the same reason as (2), by assuming $a \in \mathcal{F}^{(m')}$.
Proof of (5): The fact that the first map is injective is clear, since it is an inclusion (as shown by (4)). Next we show that the map

$$\mathcal{F}^{(m)} \xrightarrow{d|_Y} S_m \tag{2.88}$$

is well-defined and surjective, which then of course implies that the map into ${S_m}/{S_{m+1}}$ is also well-defined and surjective. First take any $f \in \mathcal{F}^{(m)}$. By definition this implies that $N_i f(0, y) = 0$ for all $\kappa_i < m$. This shows that $df|_{NY} \in S_m$ and thus the map is well-defined. For the surjectivity, start with any local coordinates x_i that are linearly adapted to the N_i. Then S_m is spanned by those dx_i with $\kappa_i \geq m$. Of course x_i does not need to be an element of $\mathcal{F}^{(m)}$. Now consider functions of the form

$$\tilde{x}_i = x_i + \sum_{\substack{\kappa\beta < \kappa_i \\ 2 \leq |\beta|}} \lambda_{i,\beta} x^\beta \tag{2.89}$$

for some constants $\lambda_{i,\beta}$. We have $d\tilde{x}_i|_{NY} = dx_i|_{NY}$. We want to prove that there are some constants $\lambda_{i,\beta}$ (which will turn out to be unique) such that $\tilde{x}_i \in \mathcal{F}^{(m)}$. This is true if and only if

$$\forall \alpha,\ \kappa\alpha < m:\ 0 = N^\alpha(\tilde{x}_i)(0) = \left(N^\alpha(x_i) + \sum_{\substack{\kappa\beta < \kappa_i \\ 2 \leq |\beta|}} \lambda_{i,\beta} N^\alpha(x^\beta) \right)(0). \tag{2.90}$$

Using Lemma 2.42, this simplifies to

$$\forall \alpha,\ \kappa\alpha < m : 0 = N^\alpha(x_i)(0) + \alpha! \lambda_{i,\alpha} + \sum_{\substack{\kappa\beta < \kappa_i \\ 2 \leq |\beta| < |\alpha|}} \lambda_{i,\beta} N^\alpha(x^\beta)(0). \tag{2.91}$$

To better focus on what is important, we write $N^\alpha(x_i)(0) = c_\alpha$ and $N^\alpha(x^\beta)(0) = c_{\alpha,\beta}$ since they are simply constants. Thus we obtain that $\tilde{x}_i \in \mathcal{F}^{(m)}$ if and only if

$$\forall \alpha,\ \kappa\alpha < m : 0 = c_\alpha + \alpha! \lambda_{i,\alpha} + \sum_{\substack{\kappa\beta < \kappa_i \\ 2 \leq |\beta| < |\alpha|}} \lambda_{i,\beta} c_{\alpha,\beta}. \tag{2.92}$$

For $|\alpha| = 2$, this is satisfied by setting $\alpha! \lambda_{i,\alpha} = c_\alpha$. We can now inductively over $|\alpha|$ set

$$\alpha! \lambda_{i,\alpha} = -c_\alpha - \sum_{\substack{\kappa\beta < \kappa_i \\ 2 \leq |\beta| < |\alpha|}} \lambda_{i,\beta} c_{\alpha,\beta}. \tag{2.93}$$

This finishes the prove of the surjectivity of $\mathcal{F}^{(m)} \xrightarrow{d|_{NY}} {S_m}/{S_{m+1}}$.
In fact, the functions $(\tilde{x}_i)$ are again local coordinates that now linearize the function space $\mathcal{F}^{(m)}$ in the usual sense (which follows simply from Taylor series). The exactness of the sequence now follows from this coordinate representation.
As mentioned above, we will not go into to much detail regarding the precise relation between the vector fields N_i and the quasihomogeneous structure $\mathcal{F}^{(m)}$. However, there is one easy corollary that we will also need later on:

Corollary 2.43: *The quasihomogeneous structure $\mathcal{F}^{(m)}$ given by vector fields $N_1, \ldots, N_k$ as above is independent of the tangential parts of N_i to Y. To be more precise, adding any vector field that is tangent to Y onto N_i does not change $\mathcal{F}^{(m)}$.*

2.6. Special cases

We want to point out two special cases in which a conormal filtration (S_k) actually uniquely defines a full quasihomogeneous structure. These are rather important, since in a lot of situations where quasihomogeneous blow-ups can be used, the (S_k) appear quite naturally (and intuitively), but the $\mathcal{F}^{(m)}$ do not seem to be given in an equally natural way. Often enough, the reason for this is that the $\mathcal{F}^{(m)}$ already are uniquely defined by the (S_k).

Proposition 2.44: ***The parabolic case:*** *If (S_k) is a conormal filtration at $Y \subset X$ that satisfies $S_3 = \{0\}$, then (S_k) uniquely determines full quasihomogeneous structure.*

PROOF. First notice that $S_3 = \{0\}$ also implies $S_k = \{0\}$ for all $k > 3$. In other words, only the weights 1 and 2 appear. The fact that there is only a single choice for the function spaces $\mathcal{F}^{(m)}$ immediately follows from the local coordinates characterization in Proposition 2.15, since any two coordinate systems (x_i), $(\tilde{x}_i)$ that match such a (S_k) are equivalent. $\square$

Proposition 2.45: ***The boundary case:*** *If (S_k) is a conormal filtration at a boundary p-submanifold $Y \subset \partial X$ that satisfies $S_3 \subseteq N^* \operatorname{Fa}(Y)$, then (S_k) uniquely determines a full quasihomogeneous structure. In other words, if each direction with weight greater then 2 is a normal direction.*

PROOF. With the parabolic case in mind, we may in fact assume $S_2 \subseteq N^* \operatorname{Fa}(Y)$ without loss of generality. Again we use local coordinates. Let $x_i, \bar{x}_i$ be any two sets of local coordinates matching (S_k). Let j denote the index such that S_2 is spanned by $dx_j, \ldots, dx_n$. Then for each $i \geq j$ both x_i and $\bar{x}_i$ are local boundary defining function of a hypersurface $H \in \operatorname{Hu}(Y)$ (potentially after reordering). Thus, each summand in the Taylor series of x_i (with respect to $\bar{x}$) contains $\bar{x}_i$ at least to first order. Now, again using proposition 2.15, we have $(x_i) \sim_S (\bar{x}_i)$, proving the proposition. $\square$

In particular, the quasihomogeneous blow-up of any *boundary face* $Y \in M(X)$ is completely determined by the (S_k). In fact, since the filtration (S_k) is by definition required to decompose over $N^*Y = \bigoplus_{H \in \operatorname{Hu}(Y)} N^*YH$, the filtration (S_k) itself is already fully determined by associating a weight κ_H to each $H \in \operatorname{Hu}(Y)$. This is a special case of a *generalized blow-up* introduced by Kottke and Melrose in [12].

CHAPTER 3

Commutativity of blow-ups

When performing a series of (quasihomogeneous or radial) blow-ups of p-submanifolds in a manifold with corners X, one is frequently concerned about commutativity. We start by shortly recalling the definition of the lift of submanifolds under blow-up and the standard commutativity results in the radial case.

3.1. Lifting submanifolds

Definition 3.1: *Let Z be a closed subset of a manifold with corners X. Furthermore, let Y be a p-submanifold of X. Denote by $\beta : [X;Y] \to X$ the radial blow-up of Y in X. We define the lift of Z to $[X;Y]$, denoted by $\beta^*(Z)$, in two cases:*

(1) If $Z \subset Y$, then we set $\beta^*(Z) = \beta^{-1}(Z)$.

(2) If $Z = \mathrm{cl}(Z \setminus Y)$, then we set $\beta^*(Z) = \mathrm{cl}\,\beta^{-1}(Z \setminus Y)$.

If neither is the case, then the lift is not defined.

In order to perform the blow-up of $\beta^*(Z)$ in $[X;Y]$, it has to be a p-submanifold of $[X;Y]$. In what cases this occurs is well understood but rather lengthy to recall completely. Instead, we only recall a special case that is needed for the commutativity results presented below:

Lemma 3.2: *Let Y, Z be p-submanifolds of X that intersect cleanly, meaning that for each $p \in Y \cap Z$ we have $T_pY \cap T_pZ = T_p(Y \cap Z)$ (or equivalently that near any such p they can be linearized by local coordinates simultaneously). Then Z lifts to become a p-submanifold of $[X;Y]$ (and, by symmetry, vice versa).*

Proof. The definition of cleanly intersecting implies that near each point $p \in Y \cap Z$ one can choose local coordinates x, x', x'', y of X such that locally $Y = \{x = x' = 0\}$, $Z = \{x' = x'' = 0\}$. The statement of the lemma now follows directly calculating the lift of Z in local projective coordinates on $[X;Y]$. □

3.2. Standard commutativity results

We recall three frequently used results concerning commutativity of radial blow-ups.

Theorem 3.3: *Let X be a manifold with corners and $Z \subset Y \subset X$ be two nested p-submanifolds. Then the identity on X lifts to become a diffeomorphism:*

$$[X;Z;Y] \cong [X;Y;Z]. \tag{3.1}$$

Proof. Clearly, this result is local near the preimage of Z. Furthermore, by naturality of blow-up, we may replace X by the normal bundle NZ. We can also choose this normal fibration in such a way, that Y is a subbundle of NZ. To simplify further, by the product structure of blow-up, we may assume that Z is simply a point. Thus it suffices to consider the model

$$X = \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}$$
$$Y = \{0\} \times \mathbb{R}^{n'}_{k'} \tag{3.2}$$
$$Z = \{0\} \times \{0\}$$

with standard coordinates x_i on $\mathbb{R}^n_k$ and x'_i on $\mathbb{R}^{n'}_{k'}$, thus the first k of the x_i and the first k' of the x'_i are ≥ 0. We then have $Y = \{x_i = 0\}$. We may construct projective coordinates on $[X; Y]$ by scaling all x_i by a fixed x_a yielding coordinates

$$\xi_i = \begin{cases} x_a & (i = a) \\ \frac{x_i}{x_a} & (i \neq a) \end{cases} \, , \; x'_i. \tag{3.3}$$

Considering all choices of a, these coordinate domains cover the whole of $[X; Y]$. The lift of Z to $[X; Y]$, denoted by Z^*, is given in these coordinates by

$$Z^* = \{\xi_a = 0, \; x'_i = 0\}. \tag{3.4}$$

Thus, we may construct local projective coordinates on $[X; Y; Z]$ by scaling these coordinates either by ξ_a or a fixed x'_b.
Scaling by ξ_a, we get projective coordinates

$$\xi_i \; , \; \eta_i = \frac{x'_i}{\xi_a} \tag{3.5}$$

on $[X; Y; Z]$. Scaling by x'_b, we get projective coordinates

$$\lambda = \frac{\xi_a}{x'_b} \; , \; \xi_{i \; (i \neq a)} \; , \; \eta_i = \begin{cases} x'_b & (i = b) \\ \frac{x'_i}{x'_b} & (i \neq b) \end{cases} . \tag{3.6}$$

The domain of all these coordinate systems cover $[X; Y; Z]$. We will compute the lift of the identity explicitly for all these coordinate systems: First consider those coordinates where we first scaled by x_a and then by x'_b, yielding coordinates $\lambda, \; \xi_{i \; (i \neq a)}, \; \eta_i$. The blow-down map to $[X; Y]$ is then locally given by

$$\xi_i, \; x'_i = \begin{cases} \eta_b & (i = b) \\ \eta_b \eta_i & (i \neq b) \end{cases} . \tag{3.7}$$

From here, the local blow-down map to X is given by

$$x_i = \begin{cases} \xi_a = \lambda \eta_b & (i = a) \\ \xi_a \xi_i = \lambda \eta_b \xi_i & (i \neq a) \end{cases} \, , \; x'_i = \begin{cases} \eta_b & (i = b) \\ \eta_b \eta_i & (i \neq b). \end{cases} \tag{3.8}$$

We may now get local projective coordinates on $[X; Z]$ by scaling by x'_b, yielding

$$\tilde{\xi}_i = \frac{x_i}{x'_b} \; , \; \tilde{\eta}_i = \begin{cases} x'_b & (i = b) \\ \frac{x'_i}{x'_b} & (i \neq b) \end{cases} . \tag{3.9}$$

The lift of Y to $[X;Z]$, denoted by Y^* is then locally given by $Y^* = \{\tilde{\xi}_i = 0\}$. Scaling by $\tilde{\xi}_a$, we obtain projective coordinates on $[X;Z;Y]$ given by

$$\mu_i = \begin{cases} \tilde{\xi}_a & (i = a) \\ \frac{\tilde{\xi}_i}{\tilde{\xi}_a} & (i \neq a) \end{cases} , \ \tilde{\eta}_i. \tag{3.10}$$

We can now explicitly calculate the lift of the identity in these local coordinate systems:

$$\mu_i = \begin{cases} \tilde{\xi}_a = \frac{x_a}{x'_b} = \frac{\lambda\eta_b}{\eta_b} = \lambda & (i = a) \\ \frac{\tilde{\xi}_i}{\tilde{\xi}_a} = \frac{x_i x'_b}{x_a x'_b} = \frac{\xi_i \lambda \eta_b}{\lambda \eta_b} = \xi_i & (i \neq a) \end{cases} , \ \tilde{\eta}_i = \begin{cases} x'_b = \eta_b & (i = b) \\ \frac{x'_i}{x'_b} = \frac{\eta_i \eta_b}{\eta_b} = \eta_i & (i \neq b) \end{cases} . \tag{3.11}$$

Clearly, this coordinate system is smooth up to the boundary. Lastly, we do the same procedure for the projective coordinates in $[X;Y;Z]$ obtained from first scaling by x_a and then by ξ_a, yielding coordinates ξ_i, η_i. The local blow-down map to $[X;Y]$ is given by

$$\xi_i, \ x'_i = \eta_i \xi_a. \tag{3.12}$$

The local blow-down map to X is then given by

$$x_i = \begin{cases} \xi_a & (i = a) \\ \xi_i \xi_a & (i \neq a) \end{cases} \ x'_i = \eta_i \xi_a. \tag{3.13}$$

We now obtain local coordinates on $[X;Z]$ by scaling with x_a:

$$\tilde{\xi}_i = \begin{cases} x_a = \xi_a & (i = a) \\ \frac{x_i}{x_a} = \frac{\xi_i \xi_a}{\xi_a} = \xi_i & (i \neq a) \end{cases} , \ \tilde{\eta}_i = \frac{x'_i}{x_a} = \frac{\eta_i \xi_a}{\xi_a} = \eta_i. \tag{3.14}$$

The lift of Y to $[X;Z]$ is disjoint from the domain of these coordinates. Thus, they are also coordinates on $[X;Z;Y]$. These coordinates are smooth up to the boundary, finishing the proof of the proposition.

□

Theorem 3.4: *Let X be a manifold with corners and $Y, Z \subset X$ be two p-submanifolds that intersect transversally, meaning for each $p \in Y \cap Z$ we have $T_pY + T_pZ = T_pX$. Then, the identity on X lifts to become a diffeomorphism:*

$$[X;Y;Z] \cong [X;Z;Y]. \tag{3.15}$$

Proof. From the definition of transversal intersection, it follows that locally near any point $p \in Y \cap Z$ there is a local decomposition of X of the form

$$X = X_1 \times X_2 \times X_3 \ , \ Y = \{p_1\} \times X_2 \times X_3 \ , \ Z = X_1 \times \{p_2\} \times X_3. \tag{3.16}$$

Now, the result immediately follows, since both blown-up spaces are simply

$$[X;Y;Z] = [X;Z;Y] = [X_1;\{p_1\}] \times [X_2;\{p_2\}] \times X_3. \tag{3.17}$$

□

The following corollary is also used several times later on.

Theorem 3.5: *Let $Y, Z \subset X$ be two cleanly intersecting p-submanifolds. Then the identity on X lifts to become a diffeomorphism:*

$$[X; Y; (Y \cap Z); Z] \cong [X; Z; (Y \cap Z); Y]. \tag{3.18}$$

Proof. From the two results above it follows that $[X; Y; (Y \cap Z); Z] = [X; (Y \cap Z); Y; Z]$. Since Y, Z lift to become disjoint after the blow-up of their intersection, their blow-up commutes. Putting it all together we get

$$[X; Y; (Y \cap Z); Z] = [X; (Y \cap Z); Z; Y] = [X; Z; (Y \cap Z); Y]. \tag{3.19}$$

□

We have a need for quasihomogeneous versions of theorems 3.3 and 3.4. To address it, we first need to address the issue of lifting not only submanifolds, but also their associated quasihomogeneous structures.

3.3. Lifting quasihomogeneous structures

Let Y, Z be two p-submanifolds of X with associated quasihomogeneous structures Π_Y, Π_Z. Recall that when lifting a submanifold Z to $[X; Y]$ (in the radial case), we asked that they intersect each other cleanly, which can be defined by demanding that near any point in their intersection, they can be linearized simultaneously by a choice of local coordinates. We can strengthen this definition to include the quasihomogeneous structures as well: Recall Definition 2.29 of *matching local coordinates.*

Definition 3.6: *We say that the quasihomogeneous structures Π_Y, Π_Z intersect cleanly, if near each $p \in Y \cap Z$ there are local product coordinates of Y, Z that match both Π_Y and Π_Z as in Definition 2.29. If either Y or Z is contained in the other, we additionally ask that the associated weights κ_i of Π_Y and $\tilde{\kappa}_i$ of Π_Z agree on those coordinates that are zero in both Y and Z.*

We can now define the lift of Π_Z to $[X; Y]_{\Pi_Y}$. As before, we need to differentiate between the two cases $Z \subset Y$ and $Z = \mathrm{cl}(Z \setminus Y)$.

Definition/Theorem 3.7: *Let $Z \subset Y \subset X$ be p-submanifolds with associated cleanly intersecting quasihomogeneous structures Π_Z, Π_Y. Let Π_Z be given by the function spaces $\mathcal{F}_Z^{(m)}$ as in Definition 2.27. Then, Π_Z lifts to a quasihomogeneous structure $\beta^*(\Pi_Z)$ of $\beta^*(Z)$ in $[X; Y]_{\Pi_Y}$, given by the function spaces $\mathcal{F}_Z^{(m)*} = \mathrm{span}_{C^\infty} \beta^*(\mathcal{F}_Z^{(m)})$.*

Note: Recall that the conormal bundle of Y restricted to Z is naturally embedded in the conormal bundle of Z, $N^*Y|_Z \hookrightarrow N^*Z$. In the special case where both Y and Z are blown up at most parabolicly, and thus the quasihomogeneous structures are simply given by subbundles $S_{2,Y} \subset N^*Y$, $S_{2,Z} \subset N^*Z$, the quasihomogeneous structures intersect cleanly if and only if $S_{2,Y}|_Z = S_{2,Z} \cap N^*Y|_Z$.

Proof. Of course, we need to prove that $\mathcal{F}_Z^{(m)*}$ does in fact define a quasihomogeneous structure at $\beta^*(Z)$. Conditions (1) to (4) and (6) in Definition 2.27 immediately follow form the definition of $\mathcal{F}_Z^{(m)*}$. Condition (5) can of course be checked locally, which we will do in projective coordinates. Since the quasihomogeneous structures intersect each other cleanly, we may choose local coordinates x_i, x'_i, y_i near any point $p \in Z$ with associated weights κ_i for x_i and κ'_i for x'_i such that locally $Z = \{x_i = x'_i = 0\}$, $Y = \{x_i = 0\}$ and the quasihomogeneous structures of Y and Z are locally given by $x_i, x'_i,\ \kappa_i, \kappa'_i$. Recall that part of the definition of cleanly intersecting quasihomogeneous structure was the fact that the weights κ_i assigned to x_i with respect to either Π_Y or Π_Z are identical. Thus, we have locally

$$\mathcal{F}_Z^{(m)} = \sum_{\kappa\alpha + \kappa'\alpha' \geq m} x^\alpha x'^{\alpha'} C^\infty(X). \tag{3.20}$$

We can construct local projective coordinates on $[X;Y]_{\Pi_Y}$ by scaling by any fixed x_i, denoted by x_a:

$$\xi_i = \begin{cases} x_a^{1/\kappa_a} & (i = a) \\ x_i x_a^{-\kappa_i/\kappa_a} & (i \neq a) \end{cases},\ x'_i, y_i. \tag{3.21}$$

Therefore, we have $x_a = \xi_a^{\kappa_a}$ and $x_i = \xi_i \xi_a^{\kappa_i}$ for $i \neq a$. This yields $x^\alpha = \xi_a^{\kappa\alpha} \prod_{i \neq a} \xi_i^{\alpha_i}$. Locally, we have $\beta^*(Z) = \{\xi_a = x'_i = 0\}$. On this coordinate domain we get

$$\begin{aligned} \mathcal{F}_Z^{(m)*} &= \operatorname{span}_{C^\infty} \sum_{\kappa\alpha + \kappa'\alpha' \geq m} \xi_a^{\kappa\alpha} \prod_{i \neq a} \xi_i^{\alpha_i} x'^{\alpha'} C^\infty(X) \\ &= \sum_{k + \kappa'\alpha' \geq m} \xi_a^k x'^{\alpha'} C^\infty([X;Y]). \end{aligned} \tag{3.22}$$

These function spaces have again the form of a quasihomogeneous structure, where the x'_i have associated weights κ'_i and ξ_a has associated weight 1. For this, we have already seen in Proposition 2.9 that it satisfies the exact sequence form condition (5) in Definition 2.27 of the quasihomogeneous structure Thus finishing the proof. $\square$

Next, we consider the case where $Z = \operatorname{cl}(Z \backslash Y)$. For this, recall that $\beta^*(Z) = \operatorname{cl} \beta^{-1}(Z \backslash Y)$. Also recall that the blow-down map $\beta : [X;Y]_{\Pi_Y} \to X$ is a diffeomorphism outside of the front face.

Definition/Theorem 3.8: *Let Y, Z be p-submanifolds of X with associated quasihomogeneous structures Π_Y, Π_Z. Furthermore, assume that $Z = \operatorname{cl}(Z \setminus Y)$ and that Π_Y and Π_Z intersect cleanly. Then, Π_Z lifts to a quasihomogeneous structure $\beta^*(\Pi_Z)$ of $\beta^*(Z)$ in $[X;Y]_{\Pi_Y}$ defined by the function spaces $\mathcal{F}_Z^{(m)*}$ consisting of those smooth functions on $[X;Y]_{\Pi_Y}$ that, restricted to any open set disjoint from the front face, are in the lift of $\mathcal{F}_Z^{(m)}$:*

$$\mathcal{F}_Z^{(m)*} = \begin{Bmatrix} f \in C^\infty([X;Y]_{\Pi_Y}) \mid \\ \forall U \subset [X;Y]_{\Pi_Y} \textit{ closed with } U \cap \mathit{ff}_Y = \emptyset : f|_U \in \beta^*(\mathcal{F}_Z^{(m)}) \end{Bmatrix}. \tag{3.23}$$

Notice that $\mathcal{F}_Z^{(m)*}$ is not equal to $\beta^*(\mathcal{F}_Z^{(m)})$. For example, if $Z \supset Y$, functions in $\beta^*(\mathcal{F}_Z^{(m)})$ necessarily vanish on the front face, since they vanish on Y. This is not the case for functions in $\mathcal{F}_Z^{(m)*}$. Also notice that this is in fact the same philosophy as lifting Z: We lift $\mathcal{F}^{(m)}$ 'away form the front face' and then take its closure in C^∞.

Proof. Again, we have to prove that $\mathcal{F}_Z^{(m)*}$ actually defines a quasihomogeneous structure at $\beta^*(Z)$ in $[X;Y]_{\Pi_Y}$. Similarly to the previous case, conditions (1)-(4) and (6) in Definition 2.27 are clear, since they are 'smooth' conditions. For the last condition (5) we again use local coordinates. Since Π_Y and Π_Z intersect cleanly, we may chose local coordinates near any $p \in Y \cap Z$ given as x_i, x_i', x_i'', y_i such that locally $Z = \{x_i = x_i' = 0\}$, $Y = \{x_i' = x_i'' = 0\}$ with the x, x', x'' having associated weights $\kappa, \kappa', \tilde{\kappa}', \kappa''$ (where $\kappa', \tilde{\kappa}'$ are the associated weights of x_i by Π_Y and Π_Z respectively). We therefore have locally

$$\mathcal{F}_Z^{(m)} = \sum_{\kappa\alpha + \tilde{\kappa}'\alpha' \geq m} x^\alpha x'^{\alpha'} C^\infty(X). \tag{3.24}$$

Furthermore, we may construct local projective coordinates by scaling by either x_i' or x_i''. The coordinate domain of those projected coordinates constructed by scaling by x_i' do not intersect the lift of $\beta^*(Z)$. In this case the statement is clear, since we simply (and correctly) get locally $\mathcal{F}_Z^{(m)*} = C^\infty([X;Y])$. Scaling by a fixed x_a'', we get local coordinates

$$x_i \ , \ \xi_i' = x_i' x_a''^{-\kappa_i'/\kappa_a''} \ , \ \xi_i'' = \begin{cases} x_a''^{1/\kappa_a''} & (i = a) \\ x_i'' x_a''^{-\kappa_i''/\kappa_a} & (i \neq a) \end{cases} \ , \ y_i. \tag{3.25}$$

In these coordinates we get

$$\begin{aligned} \beta^*(Z) &= \{x_i = \xi_i' = 0\}, \\ \mathrm{ff}_Y &= \{\xi_a'' = 0\}. \end{aligned} \tag{3.26}$$

Furthermore, we get $x'^{\alpha'} = \xi^{\alpha'} \xi_a''^{\kappa'\alpha'}$ and thus

$$\beta^*(\mathcal{F}_Z^{(m)}) = \sum_{\kappa\alpha + \kappa'\alpha' \geq m} x^\alpha \xi'^{\alpha'} \xi_a''^{\tilde{\kappa}'\alpha'} \beta^* C^\infty(X). \tag{3.27}$$

Those smooth functions that lie in this space for any $\xi_a'' \geq \varepsilon > 0$ are precisely

$$\mathcal{F}_Z^{(m)*} = \sum_{\kappa\alpha + \tilde{\kappa}'\alpha' \geq m} x^\alpha \xi'^{\alpha'} C^\infty([X;Y]_{\Pi_Y}). \tag{3.28}$$

Again, this is locally a quasihomogeneous structure of $\beta^*(F)$ with associated weights κ to x_i and $\tilde{\kappa}'$ to ξ_i', for which we already know condition (5) to be true by Proposition 2.9.

□

3.4. Standard results in the quasihomogeneous case

We are now ready to state the analogous results of Theorems 3.3 and 3.4 in the case of quasihomogeneous blow-ups:

Theorem 3.9: *Let X be a manifold with corners and $Z \subset Y \subset X$ be two nested p-submanifolds with cleanly intersecting quasihomogeneous structures Π_Z, Π_Y. Then, the identity on X lifts to become a diffeomorphism*

$$[[X;Z]_{\Pi_Z};Y]_{\Pi_Y} \cong [[X;Y]_{\Pi_Y};Z]_{\Pi_Z}. \tag{3.29}$$

Proof. Again, this result is local near the preimage of Z. Since the quasihomogeneous structures intersect cleanly, we may reduce to the case of having local coordinates x_i, x_i' with associated weights κ_i, κ_i' such that $Y = \{x_i = 0\}$, $Z = \{x_i = x_i' = 0\}$. We may construct projective coordinates on $[X;Y]$ by scaling all x_i by a fixed x_a yielding coordinates

$$\xi_i = \begin{cases} x_a^{1/\kappa_a} & (i = a) \\ x_i x_a^{-\kappa_i/\kappa_a} & (i \neq a) \end{cases} \ , \ x_i'. \tag{3.30}$$

Considering all choices of a, these coordinate domains cover the whole of $[X;Y]$. In these coordinates, the lift of Z to $[X;Y]$, denoted by Z^*, is given by $Z^* = \{\xi_a = 0,\ x_i' = 0\}$. We already have seen that the quasihomogeneous structure lifts to be locally linearized by these coordinates with the associated weight of ξ_a being 1. Thus, we may construct local projective coordinates on $[X;Y;Z]$ by scaling these coordinates either by ξ_a or a fixed x_b'. Scaling by ξ_a, we get projective coordinates

$$\xi_i \ , \ \eta_i = x_i' \xi_a^{-1/\kappa_i} \tag{3.31}$$

on $[X;Y;Z]$. Scaling by x_b', we get projective coordinates

$$\lambda = \frac{\xi_a}{x_b'^{1/\kappa_b'}} \ , \ \xi_{i\ (i\neq a)} \ , \ \eta_i = \begin{cases} x_b'^{1/\kappa_b'} & (i = b) \\ \frac{x_i'}{x_b'^{\kappa_i'/\kappa_b'}} & (i \neq b) \end{cases} . \tag{3.32}$$

Again, all these coordinate systems together cover $[X;Y;Z]$. We will compute the lift of the identity explicitly for all these coordinate systems: Consider first those coordinates where we first scaled by x_a and then by x_b': Thus we have coordinates λ, $\xi_{i\ (i\neq a)}$, η_i. The blow-down map to $[X;Y]$ is then locally given by

$$\xi_i,\ x_i' = \begin{cases} \eta_b^{\kappa_b'} & (i = b) \\ \eta_b^{\kappa_i'} \eta_i & (i \neq b) \end{cases} . \tag{3.33}$$

From here, the local blow-down map to X is given by

$$x_i = \begin{cases} \xi_a^{\kappa_a} = \lambda^{\kappa_a} \eta_b^{\kappa_a} & (i = a) \\ \xi_a^{\kappa_i} \xi_i = \lambda^{\kappa_i} \eta_b^{\kappa_i} \xi_i & (i \neq a) \end{cases} \ , \ x_i' = \begin{cases} \eta_b^{\kappa_b'} & (i = b) \\ \eta_b^{\kappa_i'} \eta_i & (i \neq b). \end{cases} \tag{3.34}$$

We now may get local projective coordinates on $[X;Z]$ by scaling by x_b', yielding

$$\tilde{\xi}_i = \frac{x_i}{x_b'^{\kappa_i'/\kappa_b'}} \ , \ \tilde{\eta}_i = \begin{cases} x_b'^{1/\kappa_b'} & (i = b) \\ \frac{x_i'}{x_b'^{\kappa_i'/\kappa_b'}} & (i \neq b) \end{cases} . \tag{3.35}$$

The lift of Y to $[X;Z]$, denoted by Y^*, is then locally given by $Y^* = \{\tilde{\xi}_i = 0\}$ with associated weights κ_i to $\tilde{\xi}_i$. Scaling by $\tilde{\xi}_a$, we obtain projective coordinates on $[X;Z;Y]$ given by

$$\mu_i = \begin{cases} \tilde{\xi}_a^{1/\kappa_a} & (i = a) \\ \frac{\tilde{\xi}_i}{\tilde{\xi}_a^{\kappa_i/\kappa_a}} & (i \neq a) \end{cases} \ , \ \tilde{\eta}_i. \tag{3.36}$$

We can now explicitly calculate the lift of the identity in these local coordinate systems:

$$\mu_i = \begin{cases} \tilde{\xi}_a^{1/\kappa_a} = \frac{x_a^{1/\kappa_a}}{x_b'^{1/\kappa_b'}} = \frac{\lambda \eta_b}{\eta_b} = \lambda & (i = a) \\ \frac{\tilde{\xi}_i}{\tilde{\xi}_a^{\kappa_i/\kappa_a}} = \frac{\xi_i(\lambda\eta_b)^{\kappa_i}}{(\lambda\eta_b)^{\kappa_i}} = \xi_i & (i \neq a) \end{cases}, \tag{3.37}$$

$$\tilde{\eta}_i = \begin{cases} x_b'^{1/\kappa_b'} = \eta_b & (i = b) \\ \frac{x_i'}{x_b'^{\kappa_i'/\kappa_b'}} = \frac{\eta_i \eta_b^{\kappa_i'}}{\eta_b^{\kappa_i'}} = \eta_i & (i \neq b) \end{cases}. \tag{3.38}$$

Clearly, this is smooth up to the boundary. Lastly, we have to do the same for the projective coordinates in $[X; Y; Z]$ obtained from first scaling by x_a and then by ξ_a, yielding coordinates $\xi_i,\ \eta_i$. The local blow-down map to $[X; Y]$ is given by

$$\xi_i,\ x_i' = \eta_i \xi_a^{\kappa_i'}. \tag{3.39}$$

The local blow-down map to X is then given by

$$x_i = \begin{cases} \xi_a^{\kappa_a} & (i = a) \\ \xi_i \xi_a^{\kappa_i} & (i \neq a) \end{cases} \quad x_i' = \eta_i \xi_a^{\kappa_i'}. \tag{3.40}$$

Scaling by x_a, we now obtain local coordinates on $[X; Z]$ given by

$$\tilde{\xi}_i = \begin{cases} x_a^{1/\kappa_a} = \xi_a & (i = a) \\ \frac{x_i}{x_a^{\kappa_i/\kappa_a}} = \frac{\xi_i \xi_a^{\kappa_i}}{\xi_a^{\kappa_i}} = \xi_i & (i \neq a) \end{cases}, \quad \tilde{\eta}_i = \frac{x_i'}{x_a^{\kappa_i'/\kappa_a}} = \frac{\eta_i \xi_a^{\kappa_i'}}{\xi_a^{\kappa_i'}} = \eta_i. \tag{3.41}$$

The lift of Y to $[X; Z]$ is disjoint from the domain of these coordinates, thus they are also coordinates on $[X; Z; Y]$. These coordinates are clearly smooth up to the boundary, finishing the proof of the proposition. □

Theorem 3.10: *Let X be a manifold with corners and $Y, Z \subset X$ be two p-submanifolds that intersect transversally, meaning for each $p \in Y \cap Z$ we have $T_pY + T_pZ = T_pX$. Then, any quasihomogeneous structures Π_Y, Π_Z of Y and Z, respectively, intersect each other cleanly and we have*

$$[[X; Z]_{\Pi_Z}; Y]_{\Pi_Y} \cong [[X; Y]_{\Pi_Y}; Z]_{\Pi_Z}. \tag{3.42}$$

Proof. From the definition of transversal intersection, it follows that locally near any point $p \in Y \cap Z$ there are local coordinates x_i, y_i, z_i such that $Y = \{y_i = 0\}$, $Z = \{z_i = 0\}$. Now, one can linearize Π_Y simply by changing y_i into $\tilde{y}_i$ and Π_Z by changing z_i into $\tilde{z}_i$. Thus, they can be linearized simultaneously. The fact that the blow-ups commute now follows for the same reason as in the radial case in Theorem 3.4. □

3.5. Radial extension of a quasihomogeneous structure

Consider again the situation of two nested p submanifolds $Z \subset Y \subset X$. Let Π_Y be a quasihomogeneous structure at Y. Then Π_Y cannot be restricted to a quasihomogeneous structure at Z. One way to see this is that a normal fibration $NY \to X$ of Y does not restrict to become a normal fibration of Z. Put simply, Z has more normal codirections then Y, for which Π_Y does not yield any information.

An extension of Π_Y at Z can simply be viewed as any quasihomogeneous structure of Z that intersects Π_Y cleanly. This is of course far from unique. A canonical way to proceed is to use the radial extension defined below, which requires no additional information. Philosophically, it is constructed by giving all 'new' codirections the weight 1:

Definition/Theorem 3.11: *Let $Z \subset Y \subset X$ be p-submanifolds and Π_Y be a quasihomogeneous structure at Y given by the function spaces $\mathcal{F}_Y^{(m)}$. Then, the function spaces*

$$\mathcal{F}_Z^{(m)} = \sum_{k=0}^{m} \mathcal{I}^k(Z)\mathcal{F}_Y^{(m-k)} \tag{3.43}$$

define a quasihomogeneous structure, denoted by $\Pi_Z^{rad,Y}$, at Z. This quasihomogeneous structure called the radial extension of Π_Y at Z. It intersects Π_Y cleanly.

Proof. Again, we have to check all 6 conditions in Definition 2.27. Condition (1) follows from the fact $\{f \in \mathcal{F}_Z^{(m)} \mid df \neq 0\} = \{f \in \mathcal{F}_Y^{(m)} \mid df \neq 0\}$. Conditions (2)-(4) and (6) follow from the fact that they are true for both $\mathcal{F}_Y^{(m)}$ and $\mathcal{I}^m(Z)$. Condition (5) is again easily checked in local coordinates using proposition 2.9, which also immediately implies that $\Pi_Z^{rad,Y}$ and Π_Y intersect cleanly. □

3.6. Separating submanifolds

There is another concept related to the questions of commutativity of blow-ups, called separating submanifolds. In this section, all blow-ups can be either radial or quasihomogeneous. We start with the special case of blowing up boundary faces. Let X be a manifold with corners and $A, B, C \in M(X)$ be some boundary faces. We denote by $\mathrm{Fa}(B+C)$ the smallest face of X that contains both B and C. We then have the following result:

Lemma 3.12: *If A, B, C satisfy*

$$B \subset A \subsetneq \mathrm{Fa}(B+C), \tag{3.44}$$

then B and C lift do become transversally intersecting on $[X; A]$ and thus

$$[X; A; B; C] \cong [X; A; C; B]. \tag{3.45}$$

In this case, we say that A separates B and C. Note that $A \neq \mathrm{Fa}(B+C)$ is a necessary condition.

Proof. A boundary face is locally always given as the zero-set of a collection of boundary defining functions of some of the boundary hypersurfaces of X. Therefore, we have

$$\mathrm{Fa}(B+C) = \{x_i = 0\} \tag{3.46}$$

for some collection x_i of such boundary defining functions. Since $B \subset A \subset \mathrm{Fa}(B+C)$ and both are again boundary faces, we have

$$\begin{aligned} A &= \{x_i = 0,\ y_i = 0\}, \\ B &= \{x_i = 0,\ y_i = 0,\ z_i = 0\} \end{aligned} \tag{3.47}$$

for some additional boundary defining functions $y_i,\ z_i$. Notice that there has to be at least one y_i since otherwise $A = \mathrm{Fa}(B+C)$. Lastly, $C \subset \mathrm{Fa}(B+C)$ is again a boundary face and, by definition of $\mathrm{Fa}(B+C)$, non of the defining functions y_i, z_i vanish on C, since that would immediately yield a smaller face containing both B and C. Thus, we have

$$C = \{x_i = 0, w_i = 0\} \tag{3.48}$$

for some more defining functions w_i. Next, let us consider the lift of $B,\ C$ under the blow-up of A. The blow-up

$$\beta_A : [X; A] \to X \tag{3.49}$$

introduces a new hypersurface with a new boundary defining function, denoted by ρ. Boundary defining functions of the old boundary faces are given by rescaled versions of the old ones. To be more precise, for any boundary defining function x_i of a boundary hypersurface H_i, the function $\beta_A^*(x_i)/\rho$, defined on the interior of $[X; A]$, extends smoothly to the boundary and becomes a boundary defining function of $\beta^*(H_i)$, which we denote by $\tilde{x}_i$. Since C is not contained in A (since there is at least one y_i), it lifts to

$$\beta_A^*(C) = \{\tilde{x}_i = 0, \tilde{w}_i = 0\}. \tag{3.50}$$

Since $B \subset A$, its lift $\beta_A^*(B)$ is defined as the preimage of B under β_A. Clearly, a point $p \in \beta_A^*(B)$ must satisfy $\rho = 0$, and thus its image under blow-down satisfies $x_i = 0,\ y_i = 0$. Thus, it is easy to see that

$$\beta_A^*(B) = \{\rho = 0, \tilde{z}_i = 0\}. \tag{3.51}$$

Clearly, $\beta_A^*(B)$ and $\beta_A^*(C)$ intersect transversally, which proves the statement. □

Example: Consider

$$X = \mathbb{R}^4_+ \tag{3.52}$$

with its standard boundary defining functions x_1, x_2, x_3, x_4. Set

$$\begin{aligned} B &= \{x_1 = x_2 = x_3 = 0\}, \\ C &= \{x_1 = x_4 = 0\}. \end{aligned} \tag{3.53}$$

Notice that B and C do not intersect transversally. We have $\mathrm{Fa}(B+C) = \{x_1 = 0\}$. Thus, with

$$A = \{x_1 = x_2 = 0\} \tag{3.54}$$

the conditions of the lemma are satisfied and we get

$$[X; A; B; C] = [X; A; C; B]. \tag{3.55}$$

Notice that this does not follow from any of the ‘standard’ commutativity results.

General p-submanifolds: If $A, B, C \subset X$ are cleanly intersecting p-submanifolds, we can write down an analogous result. However, we cannot just copy the condition from above: If B, C are interior p-submanifolds that do not intersect transversally, then $\mathrm{Fa}(B+C) = X$. Thus, we can take for example B, C to be two intersecting lines in $\mathbb{R}^3$ and A

to be the plane they span. These submanifolds satisfy the conditions of the lemma above but the commutativity result clearly does not hold. However, even though the condition of the Lemma above is formulated as a global one, it might as well have been formulated locally, since the transversality of B and C on $[X;A]$ is a local result. Translating the condition into a local one yields the correct condition for p-submanifolds:

Lemma 3.13: *Let $A, B, C \subset X$ be cleanly intersecting p-submanifolds. If $B \subset A$, $B \cap C \neq \emptyset$ and for each $p \in B \cap C$ we have*

$$T_pB \subset T_pA \subsetneq T_pB + T_pC, \tag{3.56}$$

then B and C lift do become transversally intersecting on $[X;A]$ and thus

$$[X;A;B;C] \cong [X;A;C;B]. \tag{3.57}$$

In this case, we say that A separates B and C.

Proof. The proof is now completely analogous. We show that under the blow-up of A, the lift of B and C intersect transversally. This is of course a local property that is trivial away from $B \cap C$ so we only need to consider a neighborhood of a point $p \in B \cap C$. Near such a point we may now chose local coordinates as before. This gives

$$\begin{aligned} A &= \{x_i = 0,\ y_i = 0\} \\ B &= \{x_i = 0,\ y_i = 0,\ z_i = 0\} \\ C &= \{x_i = 0, w_i = 0\}. \end{aligned} \tag{3.58}$$

where the ∂_{x_i} span $T_pB + T_pC$ and then copy the proof from above. □

Stability under additional blow-ups: Given submanifolds $A, B, C \subset X$ such that A separates B and C, it is not clear whether or not the same is true for the lifts of A, B, C under additional blow-ups. In fact, generally this is not the case. We formulate two cases that are sufficient for most settings:

Lemma 3.14: *If A, B, C satisfy $B \subset A \subsetneq \mathrm{Fa}(B + C)$, we have*

$$[X;A;\ldots;B;C] \cong [X;A;\ldots;C;B]. \tag{3.59}$$

Proof. This follows directly form the fact that 'transversal intersection' is stable under lifting to blown-up spaces. □

We also want to analyze the stability under the blow-up of a submanifold F that occurs prior to the blow-up of $A;B;C$. A sufficient condition for stability is the following:

Lemma 3.15: *Let $F, A, B, C \subset X$ with A, B, C satisfying* (3.44) *and all four intersecting cleanly. Furthermore, we assume*

$$B \subseteq F \Rightarrow A \subseteq F, \tag{3.60}$$

meaning F contains either both A and B or neither. Denote the blow-up of F by $\beta_F : [X;F] \to X$. Then, the lifts $\beta_F^*(A)$, $\beta_F^*(B)$, $\beta_F^*(C) \in \mathcal{M}([X;F])$ again satisfy (3.44).

Proof. First assume F contains neither. Recall that the definition of the lift of a submanifold differentiates two cases. This condition assures that the same case applies for A and B. If F contains both A and B, then their lifts are $\beta_F^*(A) = \beta^{-1}(A)$, $\beta_F^*(B) = \beta^{-1}(B)$. If F contains neither, then their lifts are defined as $\beta_F^*(A) = \mathrm{cl}(\beta^{-1}(A \setminus F))$, $\beta_F^*(B) = \mathrm{cl}(\beta^{-1}(B \setminus F))$. The Lemma now follows from simple calculations in local coordinates separately for the cases of C also being contained in F and the case where it is not. $\square$

One can check that the condition (3.60) is itself stable under additional blow-ups. However, writing it down is a bit cumbersome, since one has to differentiate several cases.

Corollary 3.16: *Let $A \subset X$ separate B and C. Let $\mathcal{F} = F_1; \ldots; \mathcal{F}_k$ be a list of p-submanifolds such that for each i F_i lifts to a p-submanifold under blow up of $F_1, \ldots F_{i-1}$ and F_i, A, B, C intersect cleanly and satisfy the condition* (3.60)*. Denote the blow-up of $\mathcal{F}$ by $\beta : [X; \mathcal{F}] \to X$. Then, $\beta^*(A)$ separates $\beta^*(B)$ and $\beta^*(C)$.*

Example Same as before, but replace $\mathbb{R}^4_+$ with $\mathbb{R}^4$.

3.7. Commutativity and boundary structure

Lastly, we want to formulate a result that connects commutativity with the boundary structure of the blown-up spaces. Although it is not used explicitly later on in this thesis, it was very useful during the 'trial and error' part of the constructions done in the following chapters and thus we include it here:

Setting: Let X be a manifold with corners and boundary Hypersurfaces $H_1, \ldots, H_r$. Let $F_1, \ldots, F_k \in \mathcal{M}(X)$ be a sequence of distinct boundary faces of X. Furthermore, let $\sigma = (\sigma_1, \ldots, \sigma_k)$ be a permutation of the indexes. We want to answer the question of whether or not the identity on X lifts to a diffeomorphism

$$[X; \mathcal{F}] := [X; F_1; \ldots; F_k] \cong [X; F_{\sigma_1}; \ldots; F_{\sigma_k}] =: [X; \mathcal{F}_\sigma]. \tag{3.61}$$

Condition: A necessary condition is that the two manifolds $[X; \mathcal{F}]$, $[X; \mathcal{F}_\sigma]$ have the same boundary structure. This can be made precise in the following sense:

For a manifold with corners X, let $\mathcal{M}(X)$ be the set of all boundary faces. It is partially ordered by inclusion. The blow-down map $\beta_{\mathcal{F}} : [X; \mathcal{F}] \to X$ induces a map $\mathcal{M}([X; \mathcal{F}]) \to \mathcal{M}(X)$, since β is a b-map and thus maps boundary faces to boundary faces. This induced map is compatible with the partial order of both sets. We then get the following definition:

Definition 3.17: *We say that $[X; \mathcal{F}]$ and $[X; \mathcal{F}_\sigma]$ have the same boundary structure, written $\mathcal{M}([X; \mathcal{F}]) \cong_\beta \mathcal{M}([X; \mathcal{F}_\sigma])$, if there is a bijection between the two sets that is compatible with the partial order and for which the following diagram commutes:*

$$\begin{tikzcd} \mathcal{M}([X; \mathcal{F}]) \arrow[r, "\beta_{\mathcal{F}}"] \arrow[d, "\cong_\beta"] & \mathcal{M}(X) \\ \mathcal{M}([X; \mathcal{F}_\sigma]) \arrow[ur, "\beta_{\mathcal{F}_\sigma}"] & \end{tikzcd} . \tag{3.62}$$

As mentioned before, this condition is necessary for the two blown-up spaces to be diffeomorphic. We now show that it is fact also sufficient:

Theorem 3.18: *The identity on X lifts to a diffeomorphism*

$$[X;\mathcal{F}] \cong [X;\mathcal{F}_\sigma] \tag{3.63}$$

if and only if $[X;\mathcal{F}]$ and $[X;\mathcal{F}_\sigma]$ have the same boundary structure as defined above:

$$\mathcal{M}([X;\mathcal{F}]) \cong_\beta \mathcal{M}([X;\mathcal{F}_\sigma]). \tag{3.64}$$

Proof. We only need to prove the 'if' part. We use the theory of generalized blow-ups and monoidal complexes introduced by Kottke and Melrose in [12].

> In short, the proof will go as follows: Both blow-ups are generalized blow-ups with respect to refinements given by a series of star subdivisions. Since the same boundary faces are blown up, the same generators are used in these subdivisions, and thus the same minimal (one dimensional) monoids are present in both refinements in the end. All other monoids in the refinements are direct sums of these minimal ones. Which of these sums are present is fully determined by the boundary structure of the blown-up space. Since these are equal, so are the refinements and thus so are the blown-up spaces. Let us fill in the details:

Recall from [12] that a *monoidal complex* is a set of monoids σ_a indexed over a poset $(A,\leq)$ together with monoid isomorphisms $i_{a,b}:\sigma_a\to\tau\leq\sigma_b$ whenever $a\leq b$, called face maps. Any manifold with corners X has an associated 'basic monoidal complex', $\mathcal{P}_X$, consisting of smooth monoids

$$\sigma_F = \bigoplus_{\substack{G\in\mathcal{M}_1(X)\\ F\leq G}} \mathbb{N}e_G,\ F\in\mathcal{M}(X) \tag{3.65}$$

together with morphisms

$$\sigma_{F'}\hookrightarrow\sigma_F,\ F'\leq F\in\mathcal{M}(X). \tag{3.66}$$

In other words, the monoids in $\mathcal{P}_X$ are indexed over $\mathcal{M}(X)$. A morphism ϕ between monoidal complexes consist of a map between the indexing posets $\phi_\#$ together with monoid homomorphisms $\phi_{ab}:\sigma_a\to\sigma_b$ whenever $b=\phi_\#(a)$, that are required to commute with the face maps.
In Corollary 7.3 in [12] it is shown that the iterative blow-up of boundary faces is a generalized blow-up with a refinement given by the subsequent star-subdivision of $\mathcal{P}_X$ by elements v_{F_i} where v_{F_i} is the sum of the generators of the hypersurfaces meeting at F_i. Let us denote these refinements by $\mathcal{R}_\mathcal{F}\to\mathcal{P}_X$, $\mathcal{R}_{\mathcal{F}_\sigma}\to\mathcal{P}_X$. This gives

$$[X;\mathcal{F}]\cong[X;\mathcal{R}_\mathcal{F}]\ ,\ [X;\mathcal{F}_\sigma]\cong[X;\mathcal{R}_{\mathcal{F}_\sigma}]. \tag{3.67}$$

In Theorem 6.2 in [12] it is shown that the blow-down map

$$\beta:[X;\mathcal{R}_\mathcal{F}]\to X \tag{3.68}$$

defines a map $\beta_\sharp : \mathcal{P}_{[X;\mathcal{R}_\mathcal{F}]} \to \mathcal{P}_X$ that factors through an isomorphism

$$\begin{tikzcd} \mathcal{P}_{[X;\mathcal{R}_\mathcal{F}]} \arrow[r] \arrow[d, "\cong"] & \mathcal{P}_X \\ \mathcal{R}_\mathcal{F} \arrow[ur] & \end{tikzcd} \quad . \tag{3.69}$$

We want to show that there is an isomorphism between the monoidal complexes

$$\mathcal{R}_\mathcal{F} \cong \mathcal{P}_{[X;\mathcal{R}_\mathcal{F}]} \overset{\cong}{\to} \mathcal{P}_{[X;\mathcal{R}_{\mathcal{F}_\sigma}]} \cong \mathcal{R}_{\mathcal{F}_\sigma}. \tag{3.70}$$

The identification (3.64) defines a bijection between the posets $\mathcal{M}([X;\mathcal{F}]) \cong \mathcal{M}([X;\mathcal{F}_\sigma])$ over which the two monoidal complexes are indexed. Furthermore, if $F,\ F'$ are identified, then $\sigma_F \in \mathcal{P}_{[X;\mathcal{R}_\mathcal{F}]}$ and $\sigma_{F'} \in \mathcal{P}_{[X;\mathcal{R}_{\mathcal{F}_\sigma}]}$ are naturally identified by (3.65) and the fact that $\{G \in \mathcal{M}_1([[X;\mathcal{R}_\mathcal{F}]]),\ F \leq G\}$ and $\{G' \in \mathcal{M}_1([[X;\mathcal{R}_{\mathcal{F}_\sigma}]]),\ F' \leq G'\}$ are identified by (3.64).
To show that these two monoidal complexes are also isomorphic as refinements of $\mathcal{P}_X$ we need to show that the diagram

$$\begin{tikzcd} \mathcal{R}_\mathcal{F} \arrow[r] \arrow[d, "\cong"] & \mathcal{P}_X \\ \mathcal{R}_{\mathcal{F}_\sigma} \arrow[ur] & \end{tikzcd} \tag{3.71}$$

commutes. The maps between the posets over which these monoidal complexes are indexed commute by assumption, see (3.62). It is left to show that the monoid homomorphisms commute as well. This follows from the definition of star-subdivision: As mentioned earlier, the refinements $\mathcal{R}_\mathcal{F},\ \mathcal{R}_{\mathcal{F}_\sigma}$ were constructed through a series of star subdivisions by elements v_{F_i}. Thus, the minimal (one dimensional) monoid $\mathbb{N}e_G \in \mathcal{R}_\mathcal{F}$ and its identified monoid in $\mathcal{R}_{\mathcal{F}_\sigma}$, where $G = \mathrm{ff}(F_i)$, are both mapped to $\mathbb{N}v_{F_i} \subset \mathcal{R}_X$. All other monoids are direct sums of these minimal ones and, by definition of star-subdivision, are mapped to the direct sum of the images $\mathbb{N}v_{F_i}$, which are identical. This shows that the two refinements $\mathcal{R}_\mathcal{F},\ \mathcal{R}_{\mathcal{F}_\sigma}$ of $\mathcal{P}_X$ are isomorphic and thus Corollary 6.4 in [12] proves that $[X;\mathcal{R}_\mathcal{F}] \cong [X;\mathcal{R}_{\mathcal{F}_\sigma}]$.

□

Note: This result can easily be generalized to the case where the F_i are a cleanly intersecting family of p-submanifolds, since the question of weather or not the identity lifts to a diffeomorphism is a local question, and locally a family of cleanly intersecting p-submanifolds are all boundary faces with respect to a suitable 'slicing' of the manifold into quadrants. Of course this is quite a restrictive condition on the F_i and it would be interesting to gain a deeper understanding in less restrictive situations.

Example: Unfortunately, the set of all boundary faces $\mathcal{M}(X)$ can get quite large. For example, consider a manifold X with two hypersurfaces that meet. Then, the b-resolution of the triple space X_b^3 already has 676 boundary faces. So one should ask if this result can be of any use in 'real life' examples.
Fortunately, the functor that maps X to $\mathcal{M}(X)$ commutes with blow-up. Therefore, it is

very easy to compute $\mathcal{M}([X;F])$ from $\mathcal{M}(X)$ provided F is a boundary face of X (and thus given as an element of $\mathcal{M}(X)$). It is also easy to calculate the lift of any other boundary face $\beta^*(F')$ as an element of $\mathcal{M}([X;F])$. These operations are easily performed by any computer algebra system, and they have been implemented in Python in the special case where the boundary structure is relatively simple: If one assumes that each boundary face is equal to the intersection of some hypersurfaces (and thus does not have several components), one can identify $\mathcal{M}(X)$ as a subset of the power set of $\mathcal{M}_1(X)$ with the partial order being the inclusion.

Using such an implementation, the question of commutativity when one only blows up boundary faces can easily be checked with a computer. Interested readers may contact the author concerning the Python code.

Part 2

Pseudodifferential Calculus on $\overline{\mathrm{SL}}(n, \mathbb{R})$

CHAPTER 4

Semisimple Lie groups

4.1. Introduction

Before we start with the analysis on $\overline{SL}(n,\mathbb{R})$, we will briefly recall some key elements of Lie Group theory that are needed in this part of the thesis. The theory of Lie groups and Lie algebras is very mature and there is an abundance of literature on the topic. To name only one book that combines a good introduction with great depth, see [11]. Here, we only go through what is needed in the analysis later on.
At first glance, a Lie Group and a Lie Algebra are two completely different objects, we recall their definitions and their connection below.

A Lie algebra $\mathfrak{g}$ is a vector space over a field equipped with a bilinear product $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying certain criteria. Before we recall the full definition, we start with some examples:
Example 1: Let M be a real manifold (possibly with corners). Then the space of all smooth vector fields $\mathfrak{X}(M)$, equipped with the standard commutator bracket $[X,Y] := XY - YX$ is a Lie algebra.
Example 2: The space $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{K})$ of all $n\times n$-matrices over a field $\mathbb{K}$ equipped with the standard commutator bracket $[A,B] = AB - BA$ is a Lie algebra. Especially the cases $\mathbb{K} = \mathbb{R},\mathbb{C}$ are of primary interest. $\mathfrak{gl}(n,\mathbb{K})$ has several interesting sub algebras, for example $\mathfrak{sl}(n,\mathbb{K}) = \{a \in \mathfrak{gl} \mid \operatorname{tr} A = 0\}$. These matrix Lie algebras have been studied extensively and whole books have been written on even just one of these.
Example 3: The Lie algebra of a Lie group G. This is defined later on and example 2 is a special case.
Let us continue with the actual definition of a Lie algebra.

Definition 4.1: *Let $\mathfrak{g}$ be a vector space over a field $\mathbb{K}$. $\mathfrak{g}$ is called a* ***Lie algebra*** *if it is equipped with a bilinear product $[\cdot,\cdot]$ that satisfies*

(1) $[X,X] = 0$ *for all* $X \in \mathfrak{g}$.
(2) For any $X,Y,Z \in \mathfrak{g}$ the ***Jacobi identity*** *is satisfied:*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0. \tag{4.1}$$

The first statement implies that $[X,Y] = -[Y,X]$.
We continue with some basic definitions surrounding Lie algebras. We set

$$\operatorname{ad} : \mathfrak{g} \to \operatorname{End}_{\mathbb{K}} \mathfrak{g}\ ,\ (\operatorname{ad} X)(Y) = [X,Y]. \tag{4.2}$$

For an $X \in \mathfrak{g}$, $\operatorname{ad} X$ is called the **adjoint endomorphism** of X. The fact that it is an endomorphism follows from the linearity in the second factor of $[\cdot,\cdot]$. The map ad itself is also linear, which follows from the linearity in the first factor.

A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is called a **homomorphism** if it satisfies

$$\phi([X,Y]) = [\phi(X), \phi(Y)] \text{ for all } X, Y \in \mathfrak{g}. \tag{4.3}$$

For any subsets $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ we set $[\mathfrak{a}, \mathfrak{b}] = \{[X,Y] \mid X \in \mathfrak{a},\ Y \in \mathfrak{b}\}$. A **Lie subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. It is again a Lie algebra. Such an $\mathfrak{h}$ is an **ideal**, if it satisfies $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Next, we recall the definition of a Lie group and its associated Lie algebra: A **Lie group** G is a group equipped with the structure of a smooth manifold, such that multiplication and inversion are smooth maps. The most common examples are the matrix Lie groups $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n \times n) \mid \det A \neq 0\}$ and its subgroups, for example

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det A = 1\}. \tag{4.4}$$

Let $x \in G$. We denote the multiplication with x from the left with $L_x : G \to G$ and from the right with $R_x : G \to G$. A vector field X on G is called right-invariant (and analogously left-invariant) if for any x, y

$$(dR_{x^{-1}y})(X(x)) = X(y), \tag{4.5}$$

thus if X (as an operator on smooth functions) commutes with R_x. By definition, a right-invariant vector field is uniquely determined by its vector at a single point. Thus the map $X \mapsto X(1)$ is a 1-1 map between the right-invariant vector fields and T_1G. The inverse map is simply given by the equation $Xf(x) = X(1)(L_{x^{-1}}f)$. It is easy to see that this space is closed under the Lie bracket. The Lie bracket of these vector fields induces a Lie bracket on T_1G, which therefore becomes a Lie algebra. This $\mathfrak{g} = T_1G$ is simply called the Lie algebra of G. This justifies the notation $\mathfrak{gl}$, $\mathfrak{sl}$ from the example above, since one can easily check (by computing T_1G) that these are the Lie algebras of the general- and special linear group $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$ respectively.
The **centralizer** of a Lie algebra $\mathfrak{g}$ with respect to a subset $\mathfrak{s} \subset \mathfrak{g}$ is defined as

$$Z_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} \mid [X,Y] = 0 \text{ for all } Y \in \mathfrak{s}\}. \tag{4.6}$$

It is a Lie subalgebra of $\mathfrak{g}$. One is commonly interested in the centralizer of a single element $S \in \mathfrak{g}$. Similarly, if $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$, we define the **normalizer** of $\mathfrak{s}$ as

$$N_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} \mid [X,Y] \in \mathfrak{s} \text{ for all } Y \in \mathfrak{s}\}. \tag{4.7}$$

4.1.1. Ideals. Recall that a subset $\mathfrak{h} \subset \mathfrak{g}$ is called an ideal, if it satisfies $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. They play an important role in the analysis of Lie algebras and we briefly recall some concepts surrounding them.

Lemma 4.2: *If $\mathfrak{a}$, $\mathfrak{b}$ are ideals in $\mathfrak{g}$, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$.*

Given an ideal $\mathfrak{a} \subseteq \mathfrak{g}$, The quotient $\mathfrak{g}/\mathfrak{a}$ becomes a Lie algebra under the bilinear form $[X + \mathfrak{a}, Y + \mathfrak{a}] = [X,Y] + \mathfrak{a}$. It is an easy exercise to check that this is well-defined. An important theorem concerning ideals is the *second isomorphism theorem.* It states that given two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$, then $\mathfrak{g}/\mathfrak{a} \cong \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}$. This is easily proven, since

the map is explicitly given by $A + B + \mathfrak{a} \mapsto B + (\mathfrak{a} \cap \mathfrak{b})$. There are some important ideals defined in any Lie algebra. The **center** $Z_{\mathfrak{g}} = Z_{\mathfrak{g}}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$ is an ideal of $\mathfrak{g}$, and so is the **commutator ideal** $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$. Furthermore, for any homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, the kernel $\ker \phi$ is an ideal.
Next we recursively define

$$\begin{aligned} &\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i] \text{ for } i \geq 0, \\ &\mathfrak{g}_0 = \mathfrak{g},\ \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] \text{ for } i \geq 0. \end{aligned} \tag{4.8}$$

The two resulting decreasing sequences

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots, \\ &\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \end{aligned} \tag{4.9}$$

are called the **commutator series** and **lower central series** of $\mathfrak{g}$ respectively. Each $\mathfrak{g}^i$ and $\mathfrak{g}_i$ is an ideal, which follows from the fact that ideals are stable under the Lie bracket. The Lie algebra $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^i = 0$ for some i.[1] We say $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}_i = 0$ for some i. To give an example, the Lie algebra of all upper triangle matrices is solvable, the Lie algebra of all strict upper triangle matrices is nilpotent. Solvable and nilpotent ideals are stable under taking subalgebras and quotients. Furthermore, if an ideal $\mathfrak{a} \subset \mathfrak{g}$ and the quotient $\mathfrak{g}/\mathfrak{a}$ are both solvable, then so is $\mathfrak{g}$. Lastly, the sum of two solvable ideals $\mathfrak{a} + \mathfrak{b}$ is again solvable, which is a simple consequence of the second isomorphism theorem. As a result, any finite dimensional Lie algebra has a unique solvable ideal that contains all solvable ideals. This ideal is called the **radical** of $\mathfrak{g}$, denoted by $\operatorname{rad} \mathfrak{g}$. A Lie algebra is called **simple**, if it is nonabelian and has no proper nonzero ideals. It is called **semisimple** if $\operatorname{rad} \mathfrak{g} = 0$, i.e. if it has no nonzero solvable ideals.
Recall that for each $X \in \mathfrak{g}$, $\operatorname{ad} X$ is an endomorphism on $\mathfrak{g}$. Thus, if $\mathfrak{g}$ is a finite-dimensional vector space, it makes sense to define

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}\ ,\ B(X, Y) = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y). \tag{4.10}$$

This is a symmetric bilinear form, called the **Killing form**. It is connected to the concept of semisimplicity and solvability via Cartan's Criterion:

Theorem 4.3: *(Cartan's Criterion). A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non degenerate, meaning that its radical* $\operatorname{rad} B = \{X \in \mathfrak{g} \mid B(X, Y) = 0$ *for all* $Y \in \mathfrak{g}\}$ *is zero.*
$\mathfrak{g}$ is solvable if and only if its Killing form satisfies $B(X, Y) = 0$ for all $X \in \mathfrak{g}$ and $Y \in [\mathfrak{g}, \mathfrak{g}]$.

A proof can be found for example in [11]. An important consequence of this is the following theorem, that also justifies the name 'semisimple':

Theorem 4.4: *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a direct product $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ of ideals $\mathfrak{g}_i$, each of which is a simple Lie algebra. In this case, the decomposition is unique and all ideals of $\mathfrak{g}$ are sums of these ideals.*

[1] We call a Lie group solvable, if its Lie algebra is solvable. The same is true for all the following properties of Lie algebras.

Again, a proof can be found in [11]. If $\mathfrak{g}$ is semisimple and $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal then the orthogonal complement $\mathfrak{a}^{\perp}$ with respect to the Killing form B is again an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. We say a Lie Algebra is **reductive**, if for any ideal $\mathfrak{a} \subseteq \mathfrak{g}$ there is a corresponding ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Thus any semisimple Lie algebra is reductive. The following statement, which is proven by an iterative scheme and Theorem 4.4, shows a connection the other way around.

Theorem 4.5: *If $\mathfrak{g}$ is reductive, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + Z_{\mathfrak{g}}$ with $[\mathfrak{g}, \mathfrak{g}]$ being semisimple.*

This implies that any reductive Lie Group with center 0 is semisimple. For real or complex matrix Lie algebras, reductiveness can be checked via the following criteria:

Lemma 4.6: *Let $\mathfrak{g}$ be a real Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Then $\mathfrak{g}$ is reductive if it is closed under conjugate transpose.*

Proof. This is a simple consequence of the fact that

$$\langle X, Y \rangle := \operatorname{Re} \operatorname{tr}(X \bar{Y}^{\perp}) \tag{4.11}$$

is a real inner product on any matrix Lie algebra. Therefore the orthogonal complement can be taken with respect to this inner product. □

This equivalent characterization can be used to prove the semisimplicity of a large collection of real and complex matrix Lie algebras. Examples can again be found in [11]. Here, we only prove it for the Lie algebra that is the topic of this chapter:

Lemma 4.7: $\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr} A = 0\}$ *is a semisimple Lie algebra.*

Proof. The fact that $\mathfrak{sl}(n, \mathbb{R})$ is closed under brackets (and thus a Lie algebra) and closed under transpose is clear by definition. Therefore, it is a reductive Lie algebra. The only thing left to show is that it has center 0. This follows directly from that fact that each $0 \neq A \in \mathfrak{sl}(n, \mathbb{R})$ has non zero image and non zero kernel, so for a given A take any B that does not have full rank (yielding $\operatorname{tr} B = 0$), has image contained in the kernel of A and is non zero on the image of A. Then $[A, B] = AB - BA = -BA \neq 0$. □

Lastly, we need a decomposition theorem for semisimple Lie algebras that will play a vital role in the analysis of the following chapters.

4.2. The Iwasawa decomposition

The Iwasawa decomposition is defined for any semisimple Lie Group. It can be viewed as a generalization of the fact that any matrix can be decomposed into an orthogonal matrix and a upper triangle matrix. A comprehensive discussion can be found in [11]. Here we briefly recall its construction.

4.2.1. The Cartan decomposition. We start with recalling the Cartan decomposition, which generalizes the polar decomposition of matrices to any semisimple Lie group.

Definition 4.8: *Let $\mathfrak{g}$ be a semisimple Lie algebra with Killing form $B(\cdot, \cdot)$.*

(1) We call a Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ an involution if $\theta^2 = \mathrm{Id}$.

(2) *We call an involution a Cartan-involution if the bilinear form*

$$B_\theta(X,Y) := -B(X,\theta Y) \tag{4.12}$$

is positive definite.

Lemma 4.9: *Every semisimple Lie Algebra $\mathfrak{g}$ has a Cartan involution. Any two Cartan involutions θ_1, θ_2 are conjugate under some $g \in \mathfrak{g}$, meaning $\theta_1 = g^{-1}\theta_2 g$.*

Proof. A proof can be found in [11]. □

Example: On $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \operatorname{tr} A = 0\}$ the map $\theta : \mathfrak{g} \to \mathfrak{g},\ A \mapsto -A^\top$ is a Cartan involution. On $\mathfrak{sl}(n,\mathbb{R})$, the Killing form is given by $B(X,Y) = 2n\operatorname{tr}(XY)$. Therefore $B_\theta(X,X) = 2n\operatorname{tr}(XX^\top)$ which is positive definite.

Since a Cartan involution of $\mathfrak{g}$ satisfies $\theta^2 = \mathrm{Id}$, it has only the two eigenvalues ± 1. Let $\mathfrak{k}$ denote the 1-eigenspace and $\mathfrak{p}$ denote the -1-eigenspace. Then $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to B_θ. This yields the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{4.13}$$

which is called a **Cartan decomposition** of $\mathfrak{g}$.

Lemma 4.10: *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, then we have*

(1) $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k},\ [\mathfrak{p},\mathfrak{k}] \subseteq \mathfrak{p},\ [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p},\ [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$,
(2) $\mathfrak{k}$ *is a Lie subalgebra,* $\mathfrak{p}$ *is not.*

Proof. Since θ is a Lie algebra automorphism, the Lie bracket of two eigenspaces is contained in the eigenspace corresponding to the product of the two corresponding eigenvalues. □

Example 4.11: *Consider $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ and*

$$\theta : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R}),\ A \mapsto -A^\top. \tag{4.14}$$

We have

$$\begin{aligned} \mathfrak{k} &= \{X \mid X = -X^\top\} = \mathfrak{so}(n), \\ \mathfrak{p} &= \{X \mid X = X^\top\} = \{\textit{pos. def. matrices}\}. \end{aligned} \tag{4.15}$$

The Lie subgroups corresponding to the decomposition $\mathfrak{k}$ and $\mathfrak{p}$ are $\mathrm{SO}(n)$ and P, denoting the positive definite matrices, respectively. Thus, the Cartan decomposition yields the polar decomposition $\mathrm{SL}(n) = \mathrm{SO}(n)P$.
Lastly, we consider the different Cartan involutions one could have chosen for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ instead of the one above. Lemma 4.9 implies that any other Cartan involution θ' is related to θ by

$$\begin{aligned} \theta'(X) &= -g(g^{-1}Xg)^\top g^{-1} = -gg^\top X^\top (g^{-1})^\top g^{-1} \\ &= -hX^\top h^{-1} = h\theta(X)h^{-1} \end{aligned} \tag{4.16}$$

for some fixed $g \in \mathfrak{sl}(n,\mathbb{R})$ and $h = gg^\top$. Additionally, whenever $\theta = \theta'$, we have $X = hXh^{-1}$ and thus $h = \mathrm{Id}$. Therefore a choice of Cartan involution on $\mathfrak{sl}(n)$ is equivalent to the choice of a positive definite matrix h $(= gg^\top)$.

In fact, Lemma 4.9 shows, that any two Cartan decompositions of an arbitrary semisimple Lie group are conjugate.

4.2.2. The Iwasawa decomposition. An Iwasawa decomposition is now derived from a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
Choose a maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Such a subalgebra is automatically abelian (meaning $[x,y] = 0\ \forall x,y \in \mathfrak{a}$), since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and, since $\mathfrak{a}$ is a Lie algebra, $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a} \subset \mathfrak{p}$. We proceed by defining the *root system* relative to $\mathfrak{a}$.

Definition 4.12: *Let $\lambda : \mathfrak{a} \to \mathbb{R}$ be a real linear form on $\mathfrak{a}$ (in other words, $\lambda \in \mathfrak{a}^*$). Then we set*

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X\ \forall H \in \mathfrak{a}\}. \tag{4.17}$$

We then call the set

$$\Sigma = \{\lambda \neq 0 \mid \mathfrak{g}_\lambda \neq \{0\}\} \tag{4.18}$$

the root system relative to $\mathfrak{a}$.

This results in the direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda\in\Sigma} \mathfrak{g}_\lambda. \tag{4.19}$$

Notice that whenever $\lambda \in \Sigma$ we also have $-\lambda \in \Sigma$. A choice of one of these for each such pair, denoted by $\Sigma^+ \subset \Sigma$, is called a *positive root system*, if for any $\lambda_1, \lambda_2 \in \Sigma^+$ that satisfy $\lambda_1 + \lambda_2 \in \Sigma$, we have $\lambda_1 + \lambda_2 \in \Sigma^+$. Let Σ^+ be such a choice. We then set

$$\mathfrak{n} := \sum_{\lambda>0} \mathfrak{g}_\lambda. \tag{4.20}$$

Notice that this is a *subalgebra*, since we again have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Next, we take a closer look at $\mathfrak{g}_0$. The Cartan decomposition of $\mathfrak{g}$ yields

$$\mathfrak{g}_0 = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus (\mathfrak{a} \cap \mathfrak{g}_0) \overset{\mathfrak{a}\ \text{max.}}{=} (\mathfrak{k} \cap \mathfrak{g}_0) \oplus \mathfrak{a} =: \mathfrak{m} \oplus \mathfrak{a}. \tag{4.21}$$

However, we do need to prove that this sum is actually direct. To do so, take $X = K + P \in \mathfrak{g}_0$. Then we have $0 = [H, K+P] = \underbrace{[H,K]}_{\in\mathfrak{p}} + \underbrace{[H,P]}_{\in\mathfrak{k}}$ which yields $[H,K] = 0$ and $[H,P] = 0$. So far, we get the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} + \mathfrak{n} + \sum_{\lambda<0} \mathfrak{g}_\lambda. \tag{4.22}$$

Now take $X \in \sum_{\lambda<0} \mathfrak{g}_\lambda$. Notice that $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ and $\theta(X + \theta X) = \theta X + X$, which shows that $X = X + \theta X - \theta X \in \mathfrak{k} + \mathfrak{n}$. We therefore conclude that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{4.23}$$

which is called the Iwasawa decomposition. We do however still need to show that this is in fact a direct sum. This can be shown by applying θ to (4.23), followed by some straight-forward calculations. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, then let K, A, N denote the Lie subgroups of G generated by $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ respectively. Then $G = KAN$ is called the Iwasawa decomposition of G. Again, this decomposition is unique up to conjugation:

Lemma 4.13: *The Iwasawa decomposition* (4.23) *of a semisimple Lie group is unique up to conjugation by an element* $g \in \mathfrak{g}$.

A proof can be found in [11].

4.2.3. Example: SL(n)**.** Consider $\mathfrak{g} = \mathfrak{sl}(n) = \{A \in \mathbb{R}^{n\times n} \mid \operatorname{tr} A = 0\}$. As shown above, $\theta : X \mapsto -X^\top$ is a Cartan-involution, which results in the following Cartan-decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{4.24}$$

$$\mathfrak{k} = \{x \in \mathfrak{sl} \mid X = -X^\top\} = \text{ orth. matrices in } \mathfrak{sl}, \tag{4.25}$$

$$\mathfrak{p} = \{x \in \mathfrak{sl} \mid X = X^\top\} = \text{ pos. def. matrices in } \mathfrak{sl}. \tag{4.26}$$

We now need to chose a maximal subalgebra of $\mathfrak{p}$: For this we consider the subalgebra of diagonal matrices

$$\mathfrak{a} = \{\operatorname{diag}(d_1, \dots, d_n) \mid \sum d_i = 0\}. \tag{4.27}$$

We now need to find all $\lambda \in \mathfrak{a}^*$ such that $\mathfrak{g}_\lambda \neq \{0\}$. To do so, it is convenient to choose a basis of $\mathfrak{g}$. Let E_{ij} denote the matrix with a 1 at (i, j) and zeros everywhere else. Clearly, $\mathfrak{g}$ is spanned by the E_{ij} with $i \neq j$ together with the elements of $\mathfrak{a}$. Let $D = \operatorname{diag}(d_1, \dots, d_n) \in \mathfrak{a}$, then we have

$$[D, E_{ij}] = DE_{ij} - E_{ij}D = (d_i - d_j)E_{ij} \tag{4.28}$$

and of course $[D, D'] = 0$ for all $D' \in \mathfrak{a}$. Therefore, all roots λ with $\mathfrak{g}_\lambda \neq \{0\}$ are given by $\lambda_{ij} : D \mapsto (d_i - d_j)$ with

$$\mathfrak{g}_{\lambda_{ij}} = \operatorname{span}\{E_{ij}\}, \tag{4.29}$$

$$\mathfrak{g}_0 = \mathfrak{a}. \tag{4.30}$$

We therefore get the root-decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{i\neq j} \operatorname{span}\{E_{ij}\}. \tag{4.31}$$

Next, we need to chose a positive root system. One obvious choice is

$$\Sigma_+ = \{\lambda_{ij} \mid j > i\}. \tag{4.32}$$

This results in

$$\mathfrak{n} = \sum_{j>i} \operatorname{span}\{E_{ij}\} = \{\text{ strictly upper triangle matrices}\}. \tag{4.33}$$

We therefore get the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \tag{4.34}$$

where $\mathfrak{k}$ are the orthogonal matrices, $\mathfrak{a}$ are the diagonal matrices with trace 0 and $\mathfrak{n}$ are the strictly upper diagonal matrices. The corresponding decomposition of $\mathrm{SL}(n)$ into the subgroups generated by these subalgebras is

$$\mathrm{SL}(n) = \mathrm{SO}(n)AN \tag{4.35}$$

where A is the set of positive diagonal matrices and N is the set of upper diagonal matrices with all diagonal entires one.

Any other Iwasawa decomposition of $\mathrm{SL}(n,\mathbb{R})$ is conjugate to this one by an element $g \in \mathrm{SL}(n,\mathbb{R})$. Since $\mathrm{SO}(n)$ is stable under conjugation, they are given by $\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)A_gN_g$ with $A_g = g^{-1}Ag$ and $N_g = g^{-1}Ng$.

CHAPTER 5

Hd-compactification of $\overline{\mathrm{SL}}(n, \mathbb{R})$

In [1] Albin, Dimakis, Melrose and Vogan introduce the hd-compactification of semisimple Lie groups, which can be understood as the real analogue of the wonderful compactification. The right- (or left-) invariant vector fields lift to become a Lie algebra of b-vector fields and thus, allow for the construct a pseudodifferential calculus adapted to the degenerate behavior of these vector fields at the boundary of the compactification. We consider this for the special case of $\mathrm{SL}(n, \mathbb{R})$.
In this case one can write down the construction of the hd-compactification explicitly. Since we only consider $\mathrm{SL}(n, \mathbb{R})$ in the next chapter, we will restrict to it in the discussion of the hd-compactification as well. For details and for the general case, see [1].

Definition 5.1: *An hd-compactification of a real reductive Lie group with compact center is a compact manifold with corners $\bar{G}$ and a diffeomorphism into the interior $G \hookrightarrow \bar{G}$ such that*

(1) (inversion) Inversion extends to a diffeomorphism of $\bar{G}$.
(2) (b-normality) The right action of G extends smoothly to $\bar{G}$ with isotropy algebra at each boundary point containing the b-normal space.
(3) (b-transversality) The combined action of $G \times G$ on left and right is b-transitive, i.e. has Lie algebra spanning ${}^{b}T\bar{G}$.
(4) (minimality) Near the interior of a boundary face of codimension d the span of the Lie algebra for the right action contains vector fields $x_j v_j$ $(j = 1, \ldots, d)$ where the x_j are defining functions for the local boundary hypersurfaces and the v_j are locally independent tangent vector fields not themselves in the span of the Lie algebra.

As one can see, conditions (2)-(3) are all closely linked to the Lie algebra of right-invariant vector fields and therefore can be shown by analysis of the Lie algebra of $\mathrm{SL}(n)$, which is $\mathfrak{g} = T_e \,\mathrm{SL}(n)$.

5.1. Construction and geometric properties

We will now explicitly construct an hd-compactification for $\mathrm{SL}(n, \mathbb{R})$.
Let us start by considering the inclusion

$$\mathrm{SO}(n, \mathbb{R}) \subset \mathrm{SL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{Hom}(n, \mathbb{R}) \tag{5.1}$$

where $\mathrm{Hom}(n, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is the space of linear maps with the standard Hilbert-Schmidt norm of an element $A = (a_{ij})_{ij}$ given by

$$||A|| = \left(\sum a_{ij}^2\right)^{1/2}. \tag{5.2}$$

Moreover, let

$$\mathbb{SH} = \{e \in \mathrm{Hom}(n)\ ;\ ||e|| = 1\} \subset \mathrm{Hom}(n) \tag{5.3}$$

be the unit sphere in $\mathrm{Hom}(n)$[1] and $\mathbb{SI} = \mathrm{GL}(n) \cap \mathbb{SH}(n)$ be the open subset of invertible maps with norm 1. Consider the map

$$\mathrm{SL}(n) \to \mathbb{SI}(n)\ ;\ e \mapsto \frac{e}{||e||} \tag{5.4}$$

given by scaling each element to have norm 1.

Lemma 5.2: *The radial scaling above is a diffeomorphism onto its range*

$$\mathbb{SI}_+(n) = \{e \in \mathbb{SH}(n) \mid \det e > 0\} \subset \mathbb{SI}(n). \tag{5.5}$$

PROOF. The map

$$\mathbb{SI}_+(n) \ni e \mapsto e/(\det(e))^{1/n} \in \mathrm{SL}(n) \tag{5.6}$$

is obviously the inverse, proving the statement. □

One should picture the image on the unit sphere $\mathbb{SI}_+(n)$ to be one open half of $\mathbb{SH}(n)$, where the other open half consists of the matrices with negative determinant. The 'equator' in the middle is the closed set of matrices of determinant zero. This 'equator' has the additional structure of a stratified space over the sets of matrices with a fixed rank

$$S_q = \{e \in \mathbb{SH}(n) \mid e \text{ has corank at most } q\}. \tag{5.7}$$

These are a nested sequence of submanifolds

$$S_{n-1} \subset S_{n-2} \subset \cdots \subset S_1 \subset \mathbb{SH}\,. \tag{5.8}$$

The situation may be visualized as in Figure 5.1 The subsets $S_q \setminus S_{q+1}$, consisting of matrices of corank exactly q, are actually the isotropy types of the action of $\mathrm{SO}(n)$ on $\mathbb{SH}(n)$ by conjugation and therefore, they can be blown up successively to resolve this action.

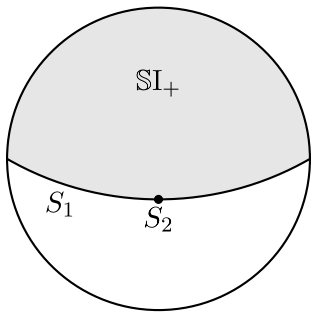

FIGURE 5.1. A schematic visualization of $\mathbb{SH}(3)$, $SI_+(3)$ and S_q. Of course it is actually eight-dimensional.

The hd-compactification will now be constructed by iterative blow-ups of all S_q. By a general result of Albin and Melrose on the resolution of group actions [2], the deepest

[1]The field considered in this thesis is always $\mathbb{R}$, so it is dropped in the notation when convenient.

stratum S_{n-1} is a submanifold and iteratively, after the blow-up of S_q, the lift of the next stratum S_{q-1} becomes a submanifold. Therefore we may define the iterative blown-up space

$$\overline{\mathbb{SH}}(n) := [SH, S_{n-1}, \ldots, S_1]. \tag{5.9}$$

Then, the hd-compactification of $\mathrm{SL}(n)$ is given by

$$\overline{\mathrm{SL}}(n) := \{\text{closure of the lift of } \mathbb{SI}_+ \text{ in } \overline{\mathbb{SH}}(n)\}. \tag{5.10}$$

We denote by H_q the new boundary hypersurface generated by the blow-up of S_q. The situation may be visualized as in Figure 5.2

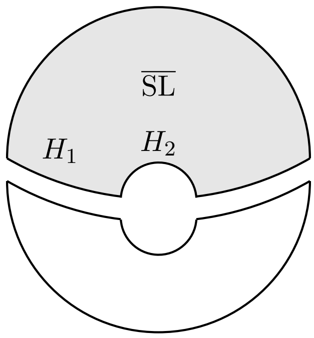

FIGURE 5.2. A schematic visualization of $\overline{\mathbb{SH}}(3)$.

Before we can prove that this is actually an hd-compactification we need to understand the geometry of the newly created boundary faces.
In the inductive argument used below, we generalize the situation slightly for notational convenience. Let W_1, W_2 be two euclidean vector spaces of equal dimension n. Let

$$\mathrm{Hom}(W_1, W_2) \supset \mathbb{SH}(W_1, W_2) \supset \mathbb{SI}(W_1, W_2)\ ,\ S_q(W_1, W_2)\ ,\ \overline{\mathbb{SH}}(W_1, W_2) \tag{5.11}$$

be defined analogously to before.
We start by looking at the structure of each $S_q \setminus S_{q+1}$. An element of $e \in S_q \setminus S_{q+1}$ can be characterized by the following three objects:

(1) The image of e, which is a $n-q$-dimensional subspace $U_2 \subset W_2$.
(2) The kernel of e, or equivalently the orthogonal complement of it, which is a $n-q$-dimensional subspace $U_1 \subset W_1$.
(3) The restricted linear map $e|_{U_1} \in \mathbb{SH}(U_1, U_2)$.

Therefore, $S_q \setminus S_{q+1}$ fibers over the double Grassmannian

$$\begin{tikzcd} \mathbb{SH}(n-q) \arrow[r, dash] & S_q \setminus S_{q+1} \arrow[d] \\ & \mathrm{Gr}(W_1, n-q) \times \mathrm{Gr}(W_2, n-q) \end{tikzcd} \tag{5.12}$$

with the fiber over a point (U_1, U_2) in the base being the space $\mathbb{SH}(U_1, U_2)$. The geometry of H_q is now similar to the one of S_q, but with larger fiber. Namely the spherical normal

bundle is glued in. In order to understand the geometry of the normal bundle, we will specifically construct a normal bundle of $S_q \setminus S_{q+1}$ in $\mathbb{SH}(W_1, W_2)$.
Consider an element $a \in \mathbb{SH}(W_1, W_2)$ sufficiently close to $S_q \setminus S_{q+1}$ and consider its left- and right polar-decompositions

$$a = u_1 p_1 = p_2 u_2. \tag{5.13}$$

Since $u_i \in \mathrm{SO}$ $(i = 1, 2)$, p_i has also norm 1. Since p_i is diagonalizable and is again close to a matrix of rank 1, it must have q eigenvalues near 0 and $n - q$ away from 0 (counted with multiplicity). Let $U_i \subset W_i$ denote the product of the eigenspaces corresponding to the eigenvalues of p_i away from 0 for $i = 1, 2$.

Lemma 5.3: *With a, U_1 and U_2 as above, the map a takes block-diagonal form with respect to U_1, U_2 i.e.*

$$a(U_1) \subset U_2\ ,\ \ a(U_1^\perp) \subset U_2^\perp. \tag{5.14}$$

Proof. Recall, that $p_1 = (a^*a)^{1/2}$, $p_2 = (aa^*)^{1/2}$. Furthermore, recall that for the square-root of a matrix m and $\lambda > 0$ we have

$$\begin{aligned} &v \text{ is eigenvector of } m \text{ with eigenvector } \lambda \\ \Longleftrightarrow\ &v \text{ is eigenvector of } m^{1/2} \text{ with eigenvector } \lambda^{1/2}. \end{aligned} \tag{5.15}$$

Let v be an eigenvector with corresponding eigenvalue λ of a^*a, i.e. $a^*av = \lambda v$. Applying a to this equation, we see that $aa^*(av) = \lambda(av)$. In other words, a maps eigenvectors of p_1 to eigenvectors of p_2 with the same corresponding eigenvalue. This immediately proves the statement. □

Lemma 5.3 yields the following.

Lemma 5.4: *We denote by E the vector bundle with base $S_q \setminus S_{q+1}$ and fiber over (U_1, U_2) (in the base of $S_q \setminus S_{q+1}$), given by $\mathrm{Hom}(U_1^\perp, U_2^\perp)$. Let $U \subset E$ be the open neighborhood of the zero section with point-wise vector norm $< 1/4$. Let*

$$\iota : U \to \mathbb{SH}(W_1, W_2)\ ,\ \ (e, f) \mapsto [e + \rho f \pi] \tag{5.16}$$

where $[\cdot]$ denotes the projection onto $\mathbb{SH}$, $\pi : W_1 \to U_1^\perp$ denotes the projection and $\rho : U_2^\perp \hookrightarrow W_2$ denotes the inclusion. Then ι is a diffeomorphism onto its image, i.e. it defines a tubular neighborhood of $S_q \setminus S_{q+1}$ and therefore identifies the normal bundle of $S_q \setminus S_{q+1}$ with E.

Proof. Clearly, ι is smooth. We show that it is a diffeomorphism by explicitly constructing its inverse. Near the zero section f has norm near 0. For a given a sufficiently close to $S_q \setminus S_{q+1}$ we may use the above lemma to see that a decomposes and we can define the inverse by t $a \mapsto (a|_{U_1}, a|_{U_1^\perp})$. □

Lemma 5.4 shows that the normal bundle of $S_q \setminus S_{q+1}$ can be identified with the bundle E. The normal space at each point $p \in S_q$, with base point (U_1, U_2) of the fibration (5.12), is given by $\mathrm{Hom}(U_1^\top, U_2^\top)$. Thus, the spherical normal bundle is identified as

$$SN_p(S_q \setminus S_{q+1}) \cong {}^{(\mathrm{Hom}(U_1^\top, U_2^\top) \setminus \{0\})}\!\big/_{\mathbb{R}_{>0}} \cong \mathbb{SH}(q). \tag{5.17}$$

In order to formulate the next theorem, we need to introduce some additional notation. Given subspaces $U_1 \subset W_1$ and $U_2 \subset W_2$ of equal dimension, we may interpret an element $f \in \mathbb{SH}(U_1, U_2)$ as an element $\mathbb{SH}(W_1, W_2)$, denoted by $f^{(W_1,W_2)}$, by setting $f^{(W_1,W_2)}$ to be the composition of the orthogonal projection $W_1 \to U_1$ and f and the inclusion $U_2 \to W_2$.

We denote the boundary hypersurface of $\overline{\mathbb{SH}}(W_1, W_2)$ corresponding to the blow-up of S_q by H_q. It is not connected, but rather has several components. Nonetheless, we refer to it as one hypersurface.

Theorem 5.5: *The resolution $\overline{\mathbb{SH}}(W_1, W_2) := [\mathbb{SH}; S_{n-1}; \ldots; S_1]$ is a compact manifold with corners, where the boundary hypersurfaces H_q fibers over the double Grassmannian with fiber modeled (inductively) by resolved spaces*

$$\begin{tikzcd} \overline{\mathbb{SH}}(n-q) \times \overline{\mathbb{SH}}(q) \arrow[r, dash] & H_q \arrow[d] \\ & \mathrm{Gr}(\mathbb{R}^n, n-q) \times \mathrm{Gr}(\mathbb{R}^n, n-q) \end{tikzcd}, \tag{5.18}$$

where the fiber over (U_1, U_2) is $\overline{\mathbb{SH}}(U_1, U_2) \times \overline{\mathbb{SH}}(U_1^\perp, U_2^\perp)$. Each hypersurface H_q has a boundary defining function, denoted by τ_q, that is defined in a neighborhood of H_q. This identifies a neighborhood of H_q in $\overline{\mathbb{SH}}$ with $H_q \times [0, \varepsilon)$. An element p in $H_q^\circ \times [0, \varepsilon)$ is therefore given by a triple (τ_q, e, f), where $\tau_q \in [0, \varepsilon)$, $e \in \mathbb{SH}(U_1, U_2)$ for some subspaces (U_1, U_2), and $f \in \mathbb{SH}(U_1^\perp, U_2^\perp)$. The blow-down map $\beta : \overline{\mathbb{SH}} \to \mathbb{SH}$ is then locally given by

$$(\tau_q, e, f) \mapsto [e^{(W_1,W_2)} + \tau_q f^{(W_1,W_2)}] \tag{5.19}$$

with $[\cdot]$ being the projection onto $\mathbb{SH}$.

Intuitively, both the base and the first factor of the fiber in (5.18) are a result of (5.12), while the second factor of the fiber is a result of (5.16). We also need to analyze the geometry for boundary faces of arbitrary codimension, not only the hypersurfaces, which will be done in Theorem 5.6. Since the statement of Theorem 5.5 is a special case of Theorem 5.6, we omit the prove here and only give a prove of Theorem 5.6.

In order to describe the geometry of the higher-codimension faces, we need some additional notation. For any multi-index $\bar{q} = \{q_1, q_2 \ldots, q_r\} \subset \{1, \ldots, n-1\}$ (where we assume $1 \le q_1 < q_2 < \cdots < q_r \le n-1$) we denote

$$F_{\bar{q}} := H_{q_1} \cap \cdots \cap H_{q_r} \subset \overline{\mathbb{SH}}(W_1, W_2). \tag{5.20}$$

$F_{\bar{q}}$ is a boundary face of codimension r. Again, even though $F_{\bar{q}}$ has several connected components, we refer to it as one boundary face.

For an n-dimensional Euclidean vector space W and a fixed multi-index $\bar{q}$, we denote the partial flag manifold of type $\bar{q}$ of W by $\mathcal{F}_{\bar{q}}(W)$. That is, elements of $\mathcal{F}_{\bar{q}}(W)$ are flags of subspaces

$$V_r \subset V_{r-1} \subset \cdots \subset V_1 \subset W, \tag{5.21}$$

where $\dim V_k = n - q_k$. For later notational convenience we rather denote these flags using the orthogonal complements between the different subspaces, meaning elements of $\mathcal{F}_{\bar{q}}(W)$ are given by flags

$$U_r \subset U_r \oplus U_{r-1} \subset \cdots \subset U_r \oplus \cdots \oplus U_1 \subset W \tag{5.22}$$

with

$$\dim U_k = q_{k+1} - q_k =: p_k, \tag{5.23}$$

where $q_{r+1} := n$. Furthermore, we denote by U_0 the orthogonal complement of $U_r \oplus \cdots \oplus U_1$ in W with dimension $q_1 =: p_0$. Using this notation, we formulate a generalization of Theorem 5.5.

Theorem 5.6: *With the notation from above, the boundary face $F_{\bar{q}} \subset \overline{\mathbb{SH}}(W_1, W_2)$ is the total space of a fibration*

$$\begin{array}{ccc} \prod_{i=0}^{r} \overline{\mathbb{SH}}(p_i) & \longrightarrow & F_{\bar{q}} \\ & & \downarrow \\ & & \mathcal{F}_{\bar{q}}(W_1) \times \mathcal{F}_{\bar{q}}(W_1) \end{array} \tag{5.24}$$

with fiber over the base point

$$\begin{aligned} &(U_{1,r} \subset \cdots \subset U_{1,r} \oplus \cdots \oplus U_{1,0} = W_1) \in \mathcal{F}_{\bar{q}}(W_1) \\ &(U_{2,r} \subset \cdots \subset U_{2,r} \oplus \cdots \oplus U_{2,0} = W_2) \in \mathcal{F}_{\bar{q}}(W_2) \end{aligned} \tag{5.25}$$

given by $\prod_{i=1}^{k+1} \overline{\mathbb{SH}}(U_{1,i}, U_{2,i})$. There are local boundary defining functions τ_{q_i} for each H_{q_i} near $F_{\bar{q}}$, identify a neighborhood of $F_{\bar{q}}$ in $\overline{\mathbb{SH}}$ with $F_{\bar{q}} \times [0, \varepsilon)^r$. This extends the fibration (5.24) into the interior. Elements $p \in F_{\bar{q}}^{\circ} \times [0, \varepsilon)^r$ are given by tuples

$$(\tau_{q_1}, \ldots, \tau_{q_r}, e_0, \ldots, e_r) \tag{5.26}$$

with $\tau_{q_i} \in [0, \varepsilon)$ and $e_i \in \overline{\mathbb{SH}}^{\circ}(U_{1,i}, U_{2,i})$. Since e_i is assumed to lie in the interior of $\overline{\mathbb{SH}}(U_{1,i}, U_{2,i})$, it may be interpreted as an element $e_i \in \mathbb{SI}(U_{1,i}, U_{2,i}) \subset \mathbb{SH}(U_{1,i}, U_{2,i})$. The blow-down map locally takes the form

$$\begin{aligned} &(\tau_{q_1}, \ldots, \tau_{q_r}, e_0, \ldots, e_r) \\ \mapsto &[e_r + \tau_{q_r} \left(e_{r-1} + \tau_{q_{r-1}} \left(\ldots (e_1 + \tau_{q_1} e_0) \ldots\right)\right)] \\ = &\left[\sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}}\right) e_i\right] =: \gamma \end{aligned} \tag{5.27}$$

where $[\cdot]$ is the projection onto $\mathbb{SH}$. To be more precise, each e_i in the blow-down map should be replaced with $e_i^{(W_1,W_2)}$, which is not written out for better readability.

Proof. First of all, notice that the fibration (5.18) and the local representation of the blow-down map (5.19) are special cases of (5.24) and (5.27), respectively. The proof follows an inductive argument over q, starting with the first blow-up of S_{n-1}. Denote by H'_{n-1} the resulting hypersurface in $[\mathbb{SH}(W_1,W_2);S_{n-1}]$. Notice that it is not equal to H_{n-1}, since the subsequent blow-ups of S_q $(q<n-1)$ alter it. Applying Lemma 5.4 and using the naturality of blow-up, we see that H'_{n-1} is the total space of a fibration

$$\begin{tikzcd} \mathbb{SH}(n-1) \arrow[r, dash] & H'_{n-1} \arrow[d] \\ & S_{n-1} \end{tikzcd} . \tag{5.28}$$

Plugging in the fibration (5.12) of S_{n-1}, we get

$$\begin{tikzcd} \mathbb{SH}(1)\times\mathbb{SH}(n-1) \arrow[r, dash] & H'_{n-1} \arrow[d] \\ & \mathrm{Gr}(\mathbb{R}^n,1)\times\mathrm{Gr}(\mathbb{R}^n,1) \end{tikzcd} \tag{5.29}$$

with fiber over (U_1,U_2) being $\mathbb{SH}(U_1,U_2)\times\mathbb{SH}(U_1^\perp,U_2^\perp)$.

Consider a neighborhood of H'_{n-1}. The function $||f||$ from (5.16) lifts to become a boundary defining function

$$\tau_{n-1}:=\beta^*(||f||) \tag{5.30}$$

of H'_{n-1}. Elements of a neighborhood of H'_{n-1} are now parameterized by tuples $(\tau_{n-1},(e,f))$, where $\tau_1\in[0,\epsilon), e\in S_{n-1}, f\in\mathbb{SH}(U_1^\perp,U_2^\perp)$ with (U_1,U_2) being the base point of e under (5.12). Using this parametrization, the blow-down map takes the form

$$(\tau_{n-1},(e,f))[e+\tau_{n-1}f]. \tag{5.31}$$

This matrix has block-diagonal form with respect to $U_1,U_1^\perp,U_2$ and $U_2^\perp$. The matrix $[e+\tau_{n-1}f]$ is an element of S_q (for $q<n-1$) if and only if f has corank q as an element of $\mathbb{SH}(n-1)$. This shows that the blow-ups of the subsequent S_q restrict to H'_{n-1} to be precisely the resolution of the fiber $\mathbb{SH}(n-1)$ to $\overline{\mathbb{SH}}(n-1)$. Therefore, the final boundary hypersurface H_{n-1} in $\overline{\mathbb{SH}}(n)$ is the total space of a fibration as in (5.18).
Notice that this finishes the proof for the case $n=2$, since no further blow-ups occur in that case.
For the remaining steps, we introduce the following notation. For any $q\le n$ we write

$$\overline{\mathbb{SH}}^q(W_1,W_2):=[\mathbb{SH}(W_1,W_2);S_{n-1};\dots;S_q], \tag{5.32}$$

meaning $\overline{\mathbb{SH}}^1(W_1,W_2)=\overline{\mathbb{SH}}(W_1,W_2)$ and $\overline{\mathbb{SH}}^n(W_1,W_2)=\mathbb{SH}(W_1,W_2)$. We now prove the following statements inductively over q.

Fix an index $q\in\{1,\dots n-1\}$. After the blow-up of all $S_{q'}$ with $q'>q$, denote the lift of $S_q\subset\mathbb{SH}(W_1,W_2)$ to the space $\overline{\mathbb{SH}}^{q+1}(W_1,W_2)$ by S_q^*. Then S_q^* is the total space of a *fibration*

$$\begin{tikzcd} \overline{\mathbb{SH}}(n-q) \arrow[r, dash] & S_q^* \arrow[d] \\ & \mathrm{Gr}(W_1, n-q) \times \mathrm{Gr}(W_2, n-q) \end{tikzcd} \tag{5.33}$$

At each point $p \in S_q^*$ with base (U_1, U_2), the normal space is naturally identified with $\mathrm{Hom}(U_1^\perp, U_2^\perp)$*. Denote by H_q' the hypersurface corresponding to S_q^* in $\overline{\mathbb{SH}}^q(W_1, W_2)$. Then it is the total space of a fibration*

$$\begin{tikzcd} \overline{\mathbb{SH}}(n-q) \times \mathbb{SH}(q) \arrow[r, dash] & H_q' \arrow[d] \\ & \mathrm{Gr}(\mathbb{R}^n, n-q) \times \mathrm{Gr}(\mathbb{R}^n, n-q) \end{tikzcd} \tag{5.34}$$

where the norm in the fiber $\mathrm{Hom}(U_1^\perp, U_2^\perp)$ *lifts to a boundary defining function τ_q of H_q', uniquely defined in a neighborhood of H_q'. The remaining blow-ups of $S_{q-1}, \dots, S_1$ restrict to H_q' to resolve the second factor of the fiber and yield the fibration* (5.18)*.*

For $q' > q$, temporarily denote the hypersurface corresponding to $S_{q'}$ in $\overline{\mathbb{SH}}^q(W_1, W_2)$ by $H_{\bar{q}'}$. For any $\bar{q} = \{q_1, \dots, q_r\}$ (with $q_1 \le q_2 \le \dots \le q_r$) that satisfies $q_1 \ge q$, denote by $F_{\bar{q}}' = H_{q_1}' \cap \dots \cap H_{q_k}'$ the codimension r boundary face in $\overline{\mathbb{SH}}^q(W_1, W_2)$. Then it is the total space of a fibration

$$\begin{tikzcd} \prod_{i=1}^r \overline{\mathbb{SH}}(p_i) \times \overline{\mathbb{SH}}^q(p_0) \arrow[r, dash] & F_{\bar{q}}' \arrow[d] \\ & \mathcal{F}_{\bar{q}}(W_1) \times \mathcal{F}_{\bar{q}}(W_2) \end{tikzcd} \tag{5.35}$$

with the remaining blow-ups of $S_{q-1}, \dots, S_1$ restricting to $F_{\bar{q}}'$ to resolve the last factor of the fiber and yield the fibration (5.24)*. The blow-down map takes the form* (5.27)*.*

Notice that we already showed all of these statements for the case of $q = n-1$. This includes the fibrations (5.35), since the only multi-index $\bar{q}$ with $q_1 \ge n-1$ is $\bar{q} = \{n-1\}$. Also notice that for the case of $q = 1$, the statements above prove Theorems 5.5 and 5.6.

Now assume the statements above to be true for all $q' > q$. Consider a neighborhood of an interior point of any boundary face

$$F_{\bar{q}}' \subset \overline{\mathbb{SH}}^{q+1}(W_1, W_2). \tag{5.36}$$

Using (5.35) for the case of $q+1$, we have a local parametrization of such a neighborhood by elements

$$(\tau_{q_1}, \dots \tau_{q_r}, e_0, \dots e_r), \tag{5.37}$$

where $\tau_{q_i} \in [0, \varepsilon)$ and, since we consider an interior point of the boundary face, $e_i \in \mathbb{SI}(U_{1,i}, U_{2,i})$ for $i = r, \dots 1$ and $e_0 \in \mathbb{SH}(U_{1,0}, U_{2,0}) \setminus S^{q+1}(U_{1,0}, U_{2,0})$. The blow-down map

$\overline{\mathbb{SH}}^{q+1}(W_1,W_2) \to \mathbb{SH}(W_1,W_2)$ now takes the local form $(\tau_{q_1},\dots\tau_{q_r},e_0,\dots e_r) \mapsto \gamma$ as in (5.27). For all $\tau_{q_i} > 0$ we then have

$$\gamma \in S_q \subset \mathbb{SH}(W_1,W_2) \iff e_0 \in S^q(U_{1,0},U_{2,0}). \tag{5.38}$$

By continuity of the lift, this shows that, near $F'_{\bar{q}}$, the lift S^*_q is given by

$$\begin{aligned}(\tau_{q_1},\dots\tau_{q_r},e_0,\dots e_r) &\in S^*_q(W_1,W_2) \subset \overline{\mathbb{SH}}^{q+1}(W_1,W_2)\\ &\iff e_0 \in S^*_q(U_{1,0},U_{2,0}) \subset \overline{\mathbb{SH}}^{q}(U_{1,0},U_{2,0})\end{aligned}. \tag{5.39}$$

Using (5.33) inductively over the dimension n, we know that $S^*_q(U_{1,0},U_{2,0})$ fibers over the double Grassmannian $\mathrm{Gr}(U_{1,0},p_0-q)\times\mathrm{Gr}(U_{2,0},p_0-q)$. Denote the base point for an element e_0 by $U_{1,a}\subset U_{1,0},\ U_{2,a}\subset U_{2,0}$. This locally defines a projection

$$S^*_q(W_1,W_2) \to \mathrm{Gr}(W_1,n-q)\times\mathrm{Gr}(W_2,n-q) \tag{5.40}$$

given by

$$\begin{aligned}&(\tau_{q_1},\dots\tau_{q_r},e_0,\dots e_r)\\ \mapsto&(\underbrace{U_{1,r}\oplus\cdots\oplus U_{1,1}\oplus U_{1,a}}_{=:V_1},\ \underbrace{U_{2,r}\oplus\cdots\oplus U_{2,1}\oplus U_{2,a}}_{=:V_2}).\end{aligned} \tag{5.41}$$

Elements $(\tau_{q_1},\dots\tau_{q_r},e_0,\dots e_r)\in S^*_q(W_1,W_2)$ lying in the fiber over (V_1,V_2) are naturally identified with elements in $\overline{\mathbb{SH}}(V_1,V_2)$ that lie near the corresponding face $F_{\bar{q}}(V_1,V_2)\subset\overline{\mathbb{SH}}(V_1,V_2)$, simply by interpreting each $U_{\alpha,i}$ not as a subspace of W_α, but of V_α. Using Theorem 5.6 inductively for lower dimension n, we see that elements near this face $F_{\bar{q}}(V_1,V_2)\subset\overline{\mathbb{SH}}(V_1,V_2)$ are also parameterized precisely by $(\tau_{q_1},\dots\tau_{q_r},e_0,\dots e_r)$.
The fact that this holds for any boundary face $F'_{\bar{q}}$ yields global fibration (5.33) of $S^*_q\subset\overline{\mathbb{SH}}^{q+1}(W_1,W_2)$.
The normal space at each point is given by the normal space of $e_0\in S^*_q(U_{1,0},U_{2,0})\subset\overline{\mathbb{SH}}^{q}(U_{1,0},U_{2,0})$. We denote $U_{1,b}=U_{1,0}\ominus U_{1,a}$ and $U_{2,b}=U_{2,0}\ominus U_{2,a}$. Using induction over the dimension n, we see that the normal space at e_0 is identified with $\mathrm{Hom}(U_{1,b},U_{2,b})$. This proves (5.34) for the blow-up of S^*_q. The intersection of the lifted boundary face $F'_{\bar{q}}$ with the newly created boundary hypersurface H'_q is now a boundary face $F'_{\bar{q}\cup\{q\}}\subset\overline{\mathbb{SH}}^{q}(W_1,W_2)$. The subspace $U_{\alpha,r},\dots,U_{\alpha,1},U_{\alpha,a},U_{\alpha,b}$ yield the fibration (5.35) for the boundary face $F'_{\bar{q}\cup\{q\}}$ with the blow-down map taking the form.

$$\begin{aligned}&(\tau_{q_1},\dots,\tau_{q_r},e_0,\dots,e_r)\\ \mapsto&[e_r+\tau_{q_r}\left(e_{r-1}+\tau_{q_{r-1}}\left(\dots(e_1+\tau_{q_1}[e_a+\tau_q e_b])\dots\right)\right)].\end{aligned} \tag{5.42}$$

Replacing τ_1 by $\tau_1(1+\tau_q)^{-1/2}$ and τ_q by $\tau_q(1+\tau_q)^{-1/2}$ yields (5.27) The fact that the remaining blow-ups, restricted to this face, resolve the last factor of this fibration follows from the analogous argument as before in (5.38). □

Notice that the $U_{\alpha,j}$ (for fixed $i = 1,2$) are pairwise orthogonal and their direct sum is W_i. In terms of these decompositions the element γ in (5.27) has block-diagonal form

$$\tag{5.43} \begin{array}{c} \\ U_{1,r} \\ U_{1,r-1} \\ \vdots \\ U_{1,i} \\ \vdots \\ U_{1,0} \end{array} \begin{array}{c} \begin{array}{cccccc} U_{2,r} & U_{2,r-1} & \cdots & U_{2,i} & \cdots & U_{2,0} \end{array} \\ \begin{pmatrix} e_r & & & & & \\ & \tau_{q_r} e_{r-1} & & & & \\ & & \ddots & & & \\ & & & \tau_{q_r}\cdots\tau_{q_{i+1}} e_i & & \\ & & & & \ddots & \\ & & & & & \tau_{q_r}\cdots\tau_{q_1} e_0 \end{pmatrix} \end{array} \cdot \lambda$$

where λ is a scaling factor close to 1 for small τ. We continue with some further remarks on the fibrations (5.24). By definition of the different fibrations (5.24), they agree in the following sense. For $q \in \bar{q}$, the face $F_{\bar{q}}$ is also a boundary face of the hypersurface H_q. The fibration in (5.18), restricted to $F_{\bar{q}}$, and the fibration in (5.24) form a tower and commute with the projection of the flag manifold $\mathcal{F}^2_{\bar{q}}$ onto the $n-q$ Grassmannian in each factor, i.e.

$$\tag{5.44} \begin{array}{ccc} F_{\bar{q}} & \hookrightarrow & H_q \\ \downarrow & & \downarrow \\ \mathcal{F}^2_{\bar{q}} & \longrightarrow & \mathrm{Gr}(n-q)^2 \end{array} ,$$

where the map $\mathcal{F}^2_{\bar{q}} \to \mathrm{Gr}(n-q)^2$ is given by projection in each factor. In fact, whenever $\bar{q} \geq \bar{q}'$, the boundary face $F_{\bar{q}}$ is also a boundary face of $F_{\bar{q}'}$. The two corresponding fibrations, restricted to the smaller face $F_{\bar{q}}$, form a tower and commute with the natural projection from the double flag variety of type $\bar{q}$ to the one of type $\bar{q}'$, i.e.

$$\tag{5.45} \begin{array}{ccc} F_{\bar{q}} & \hookrightarrow & F_{\bar{q}'} \\ \downarrow & & \downarrow \\ \mathcal{F}^2_{\bar{q}} & \longrightarrow & \mathcal{F}^2_{\bar{q}'} \end{array} .$$

Example: In order to get a better understanding of what the Theorems 5.5 and 5.6 state, we consider the following example. Consider elements of $\overline{\mathbb{SH}}(\mathbb{R}^3, \mathbb{R}^3)$ near the codimension 2 boundary face $F_{\{1,2\}} = H_1 \cap H_2$. Theorem 5.6 states that elements close to $F_{\{1,2\}}$ are parameterized by $(\tau_1, \tau_2, e_1, e_2, e_3)$ with e_i being an element of some two-point space $\overline{\mathbb{SH}}(U_{1,i}, U_{2,i})$. Consider a curve of elements $p_t = (t, t, e_1, e_2, e_3)$. For $t = 0$, this is an element of $F_{\{1,2\}}$. Denote the blow-down of p_t by $\gamma_t \in \overline{\mathbb{SH}}(\mathbb{R}^3, \mathbb{R}^3)$, given as in (5.27). The situation may be visualized as in Figure 5.3. Then Theorem 5.6 implies that (for small t) γ_t decomposes into block-diagonal form with three blocks (in this case each having dimension 1), where the norm of the first block tends to 1 as $t \to 0$, the norm of the second block tends to 0 to first order, and the norm of the third block tends to 0 to second order.

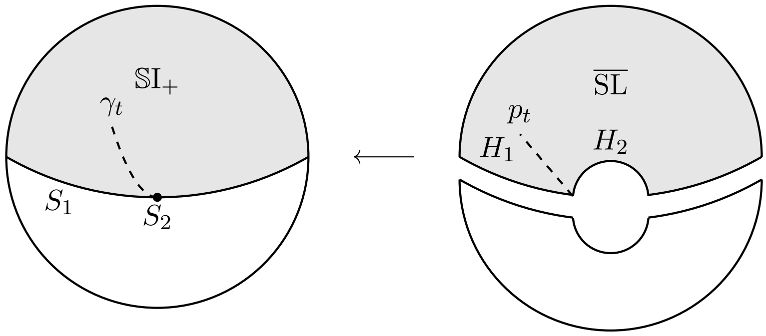

FIGURE 5.3. A visualization of the blow-down map $\overline{\mathbb{S}\mathrm{H}}(3) \to \mathbb{S}\mathrm{H}(3)$.

$\overline{\mathbb{S}\mathbf{H}}(n)$ versus $\overline{\mathbf{SL}}(n)$: Recall that $\overline{\mathbb{S}\mathrm{H}}(W_1, W_2)$ has two connected components. The hd-compactification $\overline{\mathrm{SL}}(W_1, W_2)$ is defined as only that component, which interior consists of the matrices with positive determinant. In other words, each element $p \in \overline{\mathbb{S}\mathrm{H}}(W_1, W_2)$ has an associated sign

$$\operatorname{sgn}(p) := \begin{cases} 1 & (p \in \overline{\mathrm{SL}}(W_1, W_2)) \\ -1 & (\text{otherwise}) \end{cases}. \tag{5.46}$$

In the interior, it is given by the sign of the determinant. The fibration (5.24) in Theorem 5.6 shows that the boundary face $\mathcal{F}_{\bar{q}} \subset \overline{\mathbb{S}\mathrm{H}}(W_1, W_2)$ has $2^{|\bar{q}|+1}$ components. Since the determinant of a block-diagonal matrix is the product of the determinants of each block, precisely half of the components of $\mathcal{F}_{\bar{q}}$ lie in $\overline{\mathrm{SL}}(W_1, W_2)$, given by those $(e_0, \dots e_r) \in \prod_{i=1}^{k+1} \overline{\mathbb{S}\mathrm{H}}(U_{1,i}, U_{2,i})$ for which $\prod_i \operatorname{sgn}(e_i) = 1$.

From now on, $F_{\bar{q}}$ only refers to those connected components that lie in $\overline{\mathrm{SL}}(W_1, W_2)$.

Theorem 5.6 now still holds for $F_{\bar{q}} \subset \overline{\mathrm{SL}}(W_1, W_2)$ simply by replacing the normal fiber in (5.24) with its corresponding positive part

$$\left(\prod_{i=1}^{k+1} \overline{\mathbb{S}\mathrm{H}}(k_i) \right)_+ := \{(e_0, \dots, e_r) \in \prod_{i=1}^{k+1} \overline{\mathbb{S}\mathrm{H}}(k_i) \mid \prod_i \operatorname{sgn}(e_i) = 1\}. \tag{5.47}$$

5.2. Proof of correctness

Theorem 5.6 allows us to prove that the constructed space $\overline{\mathrm{SL}}(n, \mathbb{R})$ is in fact a hd-compactification of $\mathrm{SL}(n, \mathbb{R})$ with respect to Definition 5.1. We start with the inversion.

Theorem 5.7: *The inversion map on* $\mathrm{SL}(n, \mathbb{R})$ *lifts to become a diffeomorphism on* $\overline{\mathrm{SL}}(n, \mathbb{R})$.

PROOF. The only thing to show is that the inversion extends smoothly from the interior of $\overline{\mathrm{SL}}(n)$ to the boundary. Let γ be as in eq. (5.27) with all $\tau_{q_i} > 0$, i.e. the image under the blow-down of an interior point in $\overline{\mathrm{SL}}(n)$ near the interior of a boundary face $F_{\bar{q}}$. This γ is an element of $\mathbb{S}\mathrm{I}(n)$. After rescaling, the corresponding element in $\mathrm{SL}(n)$ is given by

$$g = \det(\gamma)^{1/n}\gamma = \det(\gamma)^{1/n} \sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}} \right) e_i. \tag{5.48}$$

Since g has block-diagonal form, inversion yields

$$\begin{aligned} g^{-1} &= \det(\gamma)^{-\frac{1}{n}} \sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}^{-1}} \right) e_i^{-1} \\ &= \det(\gamma)^{-\frac{1}{n}} \left(\prod_{i=1}^{r} \tau_{q_i}^{-1} \right) \sum_{i=0}^{r} \left(\prod_{i'\leq i} \tau_{q_{i'}} \right) e_i^{-1}. \end{aligned} \tag{5.49}$$

Scaling back to an unit norm element with norm, we get

$$\tilde{\gamma} = a \sum_{i=0}^{r} \left(\prod_{i'\leq i} \tau_{q_{i'}} \right) e_i^{-1}, \tag{5.50}$$

where a is a smooth positive function with values close to 1, since the first summand e_0 has norm close to 1 and all other summands have norm close to 0. Under the fibration (5.24), γ is projected onto the double-flag

$$\{0\} \subset U_{\alpha,r} \subset U_{\alpha,r} \oplus U_{\alpha,r-1} \subset \cdots \subset U_{\alpha,r} \oplus \cdots \oplus U_{\alpha,1} \subset \mathbb{R}^n. \tag{5.51}$$

$\tilde{\gamma}$ is projected onto the double-flag given by the inverse flags of orthogonal complements

$$\{0\} \subset U_{\alpha,1} \subset U_{\alpha,1} \oplus U_{\alpha,2} \subset \cdots \subset U_{\alpha,1} \oplus \cdots \oplus U_{\alpha,r} \subset \mathbb{R}^n. \tag{5.52}$$

Therefore, $\tilde{\gamma}$ lies close to a different boundary face than γ. However, as a map between flag manifolds, mapping the upper flags to the lower ones is smooth. The action on the fibers, i.e. the inversion of the e_i is also smooth. This shows that inversion is smooth up to the boundary. □

Theorem 5.8: *The left- and right action of* $\mathrm{SL}(n)$ *on itself extend smoothly to* $\overline{\mathrm{SL}}(n)$*. Therefore, the combined action*

$$\mathrm{SL}(n) \times \mathrm{SL}(n) \ni (a, b) \mapsto \left[g \mapsto agb^{-1}\right] \tag{5.53}$$

also extends smoothly. Furthermore, the following holds:

(1) The extended combined action (5.53) acts transitively on the interior of each boundary face of $\overline{\mathrm{SL}}(n)$*.*
(2) For a fixed $(a, b) \in \mathrm{SL}(n) \times \mathrm{SL}(n)$ *the lift of the map* $g \mapsto agb^{-1}$ *is a diffeomorphism on* $\overline{\mathrm{SL}}(n)$ *(as a manifold with corners).*

Proof. First, the left- and right action of $\mathrm{SL}(n)$ on all of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ are smooth after projection to $\mathbb{SH}$ and they fix each S_q. Therefore, both actions lift and are smooth on the blown-up space $\overline{\mathbb{SH}}(n)$. Also, for a fixed (a, b), the map $g \mapsto agb^{-1}$ is a diffeomorphism, since it is smooth and its inverse is given by $g \mapsto a^{-1}gb$. The only thing left to show is that the combined action of $\mathrm{SL}(n)$ acts transitively on each boundary face. To see this, consider the combined action on a boundary face $F_{\bar{q}}$. As shown above, $F_{\bar{q}}$ fibers over a double flag-manifold with flags of the form

$$\{0\} \subset U_{\alpha,r} \subset U_{\alpha,r} \oplus U_{\alpha,r-1} \subset \cdots \subset U_{\alpha,r} \oplus \cdots \oplus U_{\alpha,1} \subset \mathbb{R}^n, \tag{5.54}$$

for $\alpha = 1, 2$. Let $p, q \in F^\circ_{\bar{q}}$ where p lies over the double flag corresponding to $\{U_{\alpha,j}\}$ and q over $\{\tilde{U}_{\alpha,j}\}$. Now, elements γ in the interior that lie over the same fiber as p take block-diagonal form as in (5.43). Let e_i be the corresponding block-entries of p and $\tilde{e}_i$ of q, respectively. $\mathrm{SO}(n)$ acts transitively on flag-manifolds (of any type) from the left and the right. Therefore, there exists an element $b \in \mathrm{SL}(n)$ such that the right action of b maps each $\tilde{U}_{1,i}$ to $U_{1,i}$ and, analogously, an element a whose left action maps $U_{2,i}$ to $\tilde{U}_{2,i}$. This reduces the proof to the case where p and q lie in the same fiber, i.e. $U_{\alpha,j} = \tilde{U}_{\alpha,j}$. Since γ has block-diagonal form, we may reduce further to each block, i.e. the right action of $\mathrm{SL}(U_{1,j})$ and the left action of $\mathrm{SL}(U_{2,j})$ each on $\mathbb{SI}(U_{1,j}, U_{2,j})$. The combined action on the left and right is transitive. In fact, just one of these actions is sufficient. The right action of $e_i\tilde{e}^{-1}$ maps e_i to $\tilde{e}_i$, i.e. we may chose $a = (a_i)$ and $b = (b_i)$ such that

$$\begin{aligned} \gamma &= \sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}} \right) e_i \\ \mapsto a^{-1}\gamma b &= \sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}} \right) a_i e_i b_i = \sum_{i=0}^{r} \left(\prod_{i'>i} \tau_{q_{i'}} \right) \tilde{e}_i. \end{aligned} \tag{5.55}$$

Letting each τ_{q_i} tend to zero shows that the extended action maps

$$p \mapsto apb^{-1} = q. \tag{5.56}$$

□

The proof above also implies the following Lemma about the orbits of the left- and right action.

Corollary 5.9: *The orbits of the right action of* $\mathrm{SL}(n)$ *on* $\overline{\mathrm{SL}}(n)$*, restricted to the interior of any boundary face, consist of all points with fixed right flag in the base (i.e. arbitrary left flag and full fibers).*

Proof. The only thing that is left to check is that the orbits are not even larger. In fact, in the interior near a boundary face, the right action does also act on the right flag. However, this vanishes at least to first oder at the boundary. To see this, let $p \in S_q$ be a point with corank exactly equal to q. Associated to p are the two $(n-q)$-dimensional subspaces U_{1q}, being its kernel, and U_{2q} , being the orthogonal complement of its image. These spaces are precisely the subspaces occurring in the flags associated to points in the blown-up space lying over p. The action on the right by any element $a \in \mathrm{SL}$ may change the kernel, but not the image, therefore fixing U_{2q}. □

We can now move on to prove that we actually constructed an hd-compactification:

Theorem 5.10: $\overline{\mathrm{SL}}(n, \mathbb{R})$ *is an hd-compactification of* $\mathrm{SL}(n, \mathbb{R})$.

PROOF. Recall the definition of an hd-compactification, Definition 5.1. The first part of the definition has already been proven by Theorem 5.7, so we move on to the second part.

b-normality: The fact that the right action extends smoothly to $\overline{\mathrm{SL}}(n)$ has also been shown in Theorem 5.8. Now we need to show, that the right-invariant vector fields span the b-normal space at each boundary point. Here, Theorem 5.8 helps again. We only need to show b-normality of an interior point of each boundary face. Using Theorem 5.8, we can then extend the result to all boundary points. To do this, it is sufficient to show the b-normality of a neighborhood of a single point in the boundary face of maximal codimension $F = H_1 \cap \cdots \cap H_{n-1}$, since such a neighborhood intersects the interior of all boundary faces. In fact, not even a whole neighborhood is needed. Showing b-normality for elements near a fixed point in F that lie in the same fiber (of the boundary decomposition near F, as in (5.26)) is sufficient for the same reason. F fibers over the product of two copies of the full flag-manifold $\mathcal{F}(\mathbb{R}^n, n) \times \mathcal{F}(\mathbb{R}^n, n)$ with discrete fiber (being the product of the two-point spaces $\mathbb{S}\mathrm{H}(V_{1,j}, V_{2,j})$ where the $V_{i,j}$ are one-dimensional). We choose the standard flag for which the kth subspace is spanned by the first k elements of the standard basis of $\mathbb{R}^n$ for both flags. This fixes an Iwasawa decomposition

$$\mathrm{SL}(n) = \mathrm{SO}(n)AN \tag{5.57}$$

where A is the set of positive diagonal matrices and N is the set of upper diagonal matrices with all diagonal entires one.

Note: A point in the flag-variety fixes a Iwasawa decomposition in the following way. Consider any flag $\{U_i\}$. We may choose an orthonormal basis (x_i) of $\mathbb{R}^n$ such that U_k is spanned by $x_1, \ldots, x_k$. In fact, such a basis is unique up to the negation of any single x_i, i.e. replacing it with $-x_i$. Therefore, after a consistent choice of signs, these bases are uniquely determined for each point in the flag variety. This orthonormal basis identifies elements of $\mathrm{SL}(n)$ with matrices. In fact, we have seen in Chapter 4 that this actually defines a Iwasawa decomposition $\mathrm{SL}(n) = \mathrm{SO}(n)A_pN_p$ where A_p is the group of positive diagonal matrices and N_p consists of the upper diagonal matrices with all diagonal entries one.

On a Lie algebra level the decomposition (5.57) corresponds to

$$\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{5.58}$$

where

$$\begin{aligned} \mathfrak{so} &= \{A \mid -A^\top = A\}, \\ \mathfrak{a} &= \{\mathrm{diag}(a_1, \ldots, a_n) \mid \operatorname{tr} a = 0\}, \\ \mathfrak{n} &= \{\text{strict upper triangle matrices}\}. \end{aligned} \tag{5.59}$$

The space $\mathfrak{n}$ is spanned by the matrices $E_{i,j}$ with a single non-zero entry 1 at (i, j) for $i > j$. The $E_{i,j}$ are joint eigenvectors of the adjoint action of $\mathfrak{a}$ with $\{\lambda_{i,j} = [a \mapsto (a_i - a_j)]\}$ being the root system. The simple roots are the $\lambda_i := \lambda_{i,i+1}$. These simple roots form a basis of the dual space $\mathfrak{a}'$. The corresponding dual basis of $\mathfrak{a}$, denoted by Λ_i, is characterized by

$$\lambda_i \Lambda_j = \delta_{i,j}. \tag{5.60}$$

Therefore, we have

$$\Lambda_i = \mathrm{diag}(\underbrace{1-\frac{i}{n}, \dots 1-\frac{i}{n}}_{i \text{ times}}, \underbrace{-\frac{i}{n}, \dots -\frac{i}{n}}_{n-i \text{ times}}). \tag{5.61}$$

We therefore have

$$\exp(t\Lambda_i) = \mathrm{diag}(e^{t(1-\frac{i}{n})}, \dots, e^{t(1-\frac{i}{n})}, e^{-t}, \dots, e^{-t}). \tag{5.62}$$

Up to a scalar (t-dependent) multiple, this is equal to

$$D_i(t) := \mathrm{diag}(e^{t(\frac{i}{n}-1)} \exp(t\Lambda_i) = \mathrm{diag}(1, \dots, 1, e^{-t\frac{n}{n-i}}, e^{-t\frac{n}{n-i}}). \tag{5.63}$$

Now consider a point p in the boundary face of maximal codimension $F = H_1 \cap \cdots \cap H_{n-1}$ which has the standard flag of $\mathbb{R}^n$ as its left flag, i.e. $U_{1,i} = \mathrm{span}\{e_i\}$. Let $\tau_1, \dots \tau_{n-1}$ be defined as above. Then, $(\tau_1, \dots, \tau_{n-1}, e_1, \dots e_{n-1})$ as in (5.26) are local fiber coordinates. With respect to these coordinates, elements $\gamma = (\tau_1, \dots, \tau_{n-1}, e_1, \dots e_{n-1})$ that are in the same fiber as p take diagonal form as in (5.43) (where $r = n-1$). The right action of $D_i(t)$ on γ does not change either flag. In fact, this action is given as the multiplication of two diagonal matrices

$$\gamma D_i(t) = \begin{pmatrix} \begin{bmatrix} e_r \end{bmatrix} & & & & & \\ & \begin{bmatrix} \tau_r e_{r-1} \end{bmatrix} & & & & \\ & & \ddots & & & \\ & & & \begin{bmatrix} e^{-t}\tau_r \cdots \tau_{n-i} e_i \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} e^{-t}\tau_r \cdots \tau_1 e_0 \end{bmatrix} \end{pmatrix}. \tag{5.64}$$

In terms of the parametrization (5.26), the projection of γD_i onto the unit sphere is given by

$$\gamma D_i = (\tau_1, \dots, \tau_{n-i-1}, e^{-t}\tau_{n-i}, \tau_{n-i+1}, \dots, \tau_{n-1}, e_1, \dots e_{n-1}). \tag{5.65}$$

resulting from the fact that changing τ_{n-i} also changes λ_{n-1}. Therefore, the corresponding vector field $\partial_t|_{t=0}(\pi(\gamma D_i(t)))$ is equal to $-\tau_{n-i}\partial_{\tau_{n-i}}$ at γ. Since any boundary face has such a γ as an interior point we can use Theorem 5.8 to conclude the b-normality at any boundary point of $\overline{\mathrm{SL}}(n)$. In fact, the analysis above allows for a more explicit result:

> Let $p \in H_i$ be a point in the hypersurface H_i (even in the boundary of H_i) with corresponding subspaces U_1, U_2 of dimension $\dim U_i = n-i$. Let τ_i be a boundary defining function of H_i. Let $D \in \mathfrak{g}$ be the element in the Lie algebra, that is given by $(1-\frac{n-i}{n})\,\mathrm{Id}$ on U_1 and $-\frac{n-i}{n}\,\mathrm{Id}$ on $U_1^\top$ (note that this is in fact an element of $\mathfrak{g}$ since it has trace 0). Then, the corresponding right-invariant vector field is equal to $\tau_i\partial_{\tau_i}$ at p.

b-transversality: Again, consider a point in the boundary face of maximal codimension $p \in F$ with both flags being the standard one in $\mathbb{R}^n$ and each $e_i = 1$. We have already seen that the right action of $\mathrm{SL}(n)$ acts surjectively on the left flag. Moreover, the right action of SL on p is smooth, the identity maps p to itself and an open neighborhood of the identity is mapped surjectively to an open neighborhood of the standard flag in the left flag variety. Therefore, $\mathfrak{g} = T_{\mathrm{Id}}\,\mathrm{SL}$ is mapped surjectively to the tangent space of the left flag manifold at p. Analogously, the Lie algebra of the left action maps surjectively to the tangent space of the right flag manifold. The b-normal space is in the span of the right action as seen above. The fiber at p is discrete. Therefore, b-transversality is given at p. Due to the smoothness of the map from $\overline{\mathrm{SL}} \times \mathfrak{g} \times \mathfrak{g}$ to ${}^bT\overline{\mathrm{SL}}$, the surjectiveness extends to an open neighborhood of p. Since such a neighborhood contains an interior point of every boundary face of $\overline{\mathrm{SL}}$ we may use Theorem 5.8 to extend the result to the whole boundary of $\overline{\mathrm{SL}}(n)$.

Minimality: Again, we only need to show minimality near one point in the maximal codimension boundary face F and then extend the result using the $\mathrm{SL} \times \mathrm{SL}$ action. Consider the same $p \in F$ and a point γ near p lying in the same fiber. As already seen in (5.43), γ takes the form

$$\gamma = \mathrm{diag}(1, \tau_{n-1}, ..., \tau_{n-1} \cdot ... \cdot \tau_1). \tag{5.66}$$

Setting $\sigma_1 := 1,\ \sigma_2 := \tau_{n-1}, ...\ \sigma_i := \tau_{n-1} \cdot ... \cdot \tau_{n-i+1}$ yields $\gamma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. Furthermore, for $j > i$ we set

$$\sigma_{ij} = \frac{\sigma_j}{\sigma_i} = \tau_{n-i} \cdots \tau_{n-j+1}. \tag{5.67}$$

We want to calculate the vector field corresponding to $E_{ij} \in \mathfrak{n}$ at γ. Since $E_{ij}^2 = 0$, we have

$$\gamma \cdot \exp(tE_{ij}) = \gamma \cdot (\mathrm{Id} + tE_{ij}) = \gamma + t\sigma_i E_{ij} =: a. \tag{5.68}$$

Recall that the tangent space at γ consist of the tangent directions in both flag manifolds and the normal direction. Since the fiber is discrete, it contributes no additional tangent directions. To start with, we want to understand the action on the right flag manifold. Therefore, we need to understand the eigenspaces of the right polar decomposition. We have

$$\begin{aligned} aa^* &= (\gamma + t\sigma_i E_{ij})(\gamma + t\sigma_i E_{ji}) \\ &= \gamma^2 + t\sigma_i^2 \sigma_{ij}(E_{ij} + E_{ji}) + t^2 \sigma_i^2 E_{ii}. \end{aligned} \tag{5.69}$$

Notice that this matrix has block-diagonal form corresponding to the one-dimensional spaces $U_k = \mathrm{span}\{e_k\}$ for $k \neq i, j$ and the single 2-dimensional space $U_i \oplus U_j$. The matrix aa^* has only one eigenvalue near 1 with corresponding one-dimensional eigenspace $\tilde{U}_1$ (still being U_1). The remainder lies in the orthogonal complement and, after rescaling, has a single eigenvalue near 1 with corresponding one-dimensional eigenspace $\tilde{U}_2$, and so forth. From (5.69) it follows that all $U_k = \mathrm{span}\{e_k\}$ remain unchanged for $k \neq i, j$, while U_i, U_j change into $\tilde{U}_j, \tilde{U}_j$. However, due to the block-diagonal form of aa^*, we still have

$U_i \oplus U_j = \tilde{U}_j \oplus \tilde{U}_j$.
After rescaling by σ_i^{-2}, the matrix aa^*, restricted to $U_i \oplus U_j$, takes the form

$$\sigma_i^{-2} aa^*|_{U_i \oplus U_j} = \begin{pmatrix} 1+t^2 & \sigma_{ij}t \\ \sigma_{ij}t & \sigma_{ij}^2 \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{ij}t \\ \sigma_{ij}t & \sigma_{ij}^2 \end{pmatrix} + \mathcal{O}(t^2), \tag{5.70}$$

whose has eigenvalue near 1 is given by

$$\lambda = \frac{1}{2}\left(1+\sigma_{ij}^2 + \sqrt{(1-\sigma_{ij}^2)^2 + 4\sigma_{ij}^2 t^2}\right) + +O(t^2). \tag{5.71}$$

The corresponding eigenspace of λ, denoted by $\tilde{U}_i$, is spanned by

$$v_1 = \begin{pmatrix} \frac{1}{2}\left(1-\sigma_{ij}^2 + \sqrt{(1-\sigma_{ij}^2)^2 + 4\sigma_{ij}^2 t^2}\right) \\ \sigma_{ij}t \end{pmatrix} = \begin{pmatrix} 1-\sigma_{ij}^2 \\ \sigma_{ij}t \end{pmatrix} + \mathcal{O}(t^2). \tag{5.72}$$

Consequently, since $\tilde{U}_j$ has to be the orthogonal complement of this in $U_i \oplus U_j$, we have

$$\tilde{U}_j = \operatorname{span}\left\{ \begin{pmatrix} -\sigma_{ij}t \\ 1-\sigma_{ij}^2 \end{pmatrix} + \mathcal{O}(t^2, \sigma_{ij}^2) \right\}. \tag{5.73}$$

Therefore, the action of E_{ij} on the right flag restricted to $U_i \oplus U_j$ is given by

$$\begin{pmatrix} 1-\sigma_{ij}^2 & -\sigma_{ij}t \\ \sigma_{ij}t & 1-\sigma_{ij}^2 \end{pmatrix} + \mathcal{O}(t^2). \tag{5.74}$$

The full flag manifold may be identified with $\mathrm{SL}(n)/NA$ and therefore the tangent space at each point may be identified with $\mathfrak{so}(n)$, again using the Iwasawa decomposition. At least to first order in σ_{ij}, the action above is identical to the action of

$$b = \begin{pmatrix} 0 & -\sigma_{ij} \\ \sigma_{ij} & 0 \end{pmatrix} \in \mathfrak{so}(n) \tag{5.75}$$

since

$$\exp(tb) = \begin{pmatrix} \cos({}_j\tau t) & -\sin(\sigma_{ij}t) \\ \sin(\sigma_{ij}t) & \cos({}_j\tau t) \end{pmatrix} = \begin{pmatrix} 1 & -\sigma_{ij}t \\ \sigma_{ij}t & 1 \end{pmatrix} + \mathcal{O}(t^2). \tag{5.76}$$

To summarize, the vector field corresponding to E_{ij} at γ projected to the tangent space of the right flag manifold is equal (at least to first order in σ_{ij}) to $\sigma_{ij}(E_{ji} - E_{ij}) \in \mathfrak{so}(n)$. In fact, this is true for any γ, that is projected to an arbitrary flag $p \in \mathcal{F}(\mathbb{R}^n)$, when E_{ij} is replaced with the corresponding matrix defined by the Iwasawa decomposition $\mathrm{SL}(n) = \mathrm{SO}(n)A_pN_p$. Of course, E_{ij} may also act on the left flag manifold. However, we have already seen in Corollary 5.9 that all of these individual directions lie in the span of the Lie algebra. For $j = i+1$ this action vanishes at the boundary to first order, so the minimality condition holds.

□

As mentioned in Corollary 5.9, the orbit of the right action of $\mathrm{SL}(n,\mathbb{R})$ that goes through this point consists of all points with identical right flag in the base. All other orbits are conjugate to this choice under the action of $\mathrm{SO}(n)$. The Iwasawa decomposition conjugates under this action as well, yielding $\mathrm{SL}(n,\mathbb{R}) = \mathrm{SO}(n)A_pN_p$ for each right flag p. This decomposition yields elements $(E_{ij})_p$ of the Lie algebra that satisfy the same calculation as above in their respective orbits of the right action. The $(E_{ij})_p$ depend smoothly on p, so we can define the following.

Definition 5.11: *We denote by N_{ij} the smooth vector field, defined semi-globally on a neighborhood of the boundary $\partial\overline{SH}(n)$, given on each orbit of the right action by the $(E_{ij})_p \in \mathfrak{sl}$ defined as above.*

These vector fields are not right-invariant themselves, but restricted to each fiber of the fibrations (5.18), they are equal to a right-invariant vector field. Projected to the right flag $\mathcal{F}(\mathbb{R}^n, \bar{q})$ of a fibration (5.24), the vector field N_{ij} is given by $(E_{ji}) - (E_{ij}) \in \mathfrak{so}(n)$. Thus, it is nonzero on this flag if only if $\exists\ q \in \bar{q}:\ n - j < q \leq n - i$. We also denote this projected vector field on $\mathcal{F}(n)$ by N_{ij}, abusing notation slightly.
As mentioned above, the vector fields $\sigma_I N_I$ are themselves not part of the Lie algebra of right-invariant vector fields. Furthermore, the Lie algebra is not a C^∞-module. After all, it is simply a finite-dimensional real vector space. For these reasons it is convenient to consider a larger Lie algebra for the following analysis, that contains the right-invariant vector fields and consists of those smooth vector fields that have the same asymptotic behavior at the boundaries of $\overline{\mathrm{SL}}(n)$.

Definition 5.12: *We set $\mathcal{V}_e$ to be the C^∞-span of the right-invariant vector fields on $\overline{SL}(n, \mathbb{R})$.*

Lemma 5.13: *Let $p \in F = H_1 \cap \cdots \cap H_{n-1}$ be a point in the boundary face of maximal codimension. Let $\tau_1, \ldots \tau_{n-1}$ be the boundary defining functions as in* (5.30)*. Then, $\mathcal{V}_e$ is locally spanned near p by*

(1) the radial vector fields $\tau_i \partial_{\tau_i}$,
(2) the vector fields spanning the fibers of the projection onto the right flag and
(3) the vector fields $\sigma_{ij} N_{ij}$ defined as above.

This result extends to all boundary faces $F_{\bar{q}}$, via the right action of $\mathrm{SL}(n)$. The lemma above may be formulated analogously, except only the $\sigma_{ij} N_{ij}$, that do not vanish on the partial flag manifold $\mathcal{F}_{\bar{q}}$, are relevant. From the calculation above it follows, that these are precisely those for which σ_{ij} vanishes on $\mathcal{F}_{\bar{q}}$.

Lemma 5.14: *Let p be an interior point of the boundary face $F_{\bar{q}} = H_{\bar{q}_1} \cap \cdots \cap H_{\bar{q}_k}$ corresponding to some multi-index $\bar{q}$. Let $\tau_{\bar{q}_1}, \ldots \tau_{\bar{q}_k}$ be the boundary defining functions as in* (5.30)*. Then, $\mathcal{V}_e$ is locally spanned near p by*

(1) the normal vector fileds $\tau_{q_i} \partial_{\tau_{q_i}}$,
(2) the vector fields spanning the fibers of the projection onto the right flag manifold $F_{\bar{q}} \times [0, \varepsilon)^k \to \mathcal{F}_{\bar{q}}$ and
(3) the vector fields $\sigma_{ij} N_{ij}$ for all $j > i$ such that the condition $\exists\ q_k \in \bar{q} : n - j < q_k \leq n - i$ is satisfied.

Since the vector fields N_{ij} are given explicitly at each point by elements of the Lie algebra, we may calculate their commutators by matrix-calculation. To write down the commutators compactly, it is convenient to switch to a different indexing scheme for the N_{ij}. Since

$1 \leq i < j \leq n$, we may associate the integer interval $I_{ij} = \{n-j+1, \dots n-i\}$ to such a tuple (i,j) and use the set of these intervals as a new index set

$$\mathfrak{I} = \{I_{ij} \mid 1 \leq i < j \leq n\}. \tag{5.77}$$

In other words, $\mathfrak{I}$ is the set of all subintervals of $\{1, \dots, n-1\}$. Furthermore, we set

$$\mathfrak{I}_{\bar{q}} := \{I \in \mathfrak{I} \mid I \cap \bar{q} \neq \emptyset\} \subset \mathfrak{I} \tag{5.78}$$

and set $\sigma_I = \sigma_{I_{ij}} = \sigma_{ij}$. Now, the lemma above becomes

Lemma 5.15: *$\mathcal{V}_e$ is as Lie algebroid. Let p be a point near the boundary face $F_{\bar{q}} = H_{\bar{q}_1} \cap \dots \cap H_{\bar{q}_k}$ corresponding to some multi-index $\bar{q}$. Let $\tau_{\bar{q}_1}, \dots \tau_{\bar{q}_k}$ be the boundary defining functions as in* (5.30)*. Then, $\mathcal{V}_e$ is locally spanned by*

(1) the normal vector fields $\tau_{q_i}\partial_{\tau_{q_i}}$,
(2) the vector fields spanning the fibers of the projection onto the right flag manifold $F_{\bar{q}} \times [0,\varepsilon)^k \to \mathcal{F}_{\bar{q}}$ and
(3) the vector fields $\sigma_I N_I$ for all $I \in \mathfrak{I}_{\bar{q}}$.

Moreover, we obtain the following commutator result:

Lemma 5.16: *Let $I, J \in \mathfrak{I}$. Then, the vector fields N_I satisfy*

$$[N_I, N_J] = \begin{cases} N_{I\setminus J} & \text{if } J \subset I \text{ and } I \setminus J \text{ is an interval,} \\ N_{I\cup J} & \text{if } I \text{ and } J \text{ are adjacent but non overlapping,} \\ 0 & \text{otherwise} \end{cases} \tag{5.79}$$

modulo vector fields tangent to the fibers fibers of $F_{\bar{q}} \times [0,\varepsilon)^k \to \mathcal{F}_{\bar{q}}$. Since, by definition, $N_I\tau_j = 0$ for all I, j, we additionally have

$$[\sigma_I N_I, \sigma_J N_J] = \begin{cases} \sigma_{I\cup J} N_{I\setminus J} & \text{if } J \subset I \text{ and } I \setminus J \text{ is an interval,} \\ \sigma_{I\cup J} N_{I\cup J} & \text{if } I \text{ and } J \text{ are adjacent but non overlapping,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.80}$$

For usage later on, it is convenient to consider the subbundles of the tangent bundle spanned by the N_I:

Each N_I spans a one-dimensional subbundle of $T\mathcal{F}_{\bar{q}}$ for each $\bar{q}$, such that $\bar{q} \cap I \neq \emptyset$, denoted by $E^I_{\bar{q}}$. These subbundles are independent and span $T\mathcal{F}_{\bar{q}}$. Thus, they stratify $T\mathcal{F}_{\bar{q}}$. The commutators of these subbundles are, as in (5.79), given by

$$[E^I_{\bar{q}}, E^J_{\bar{q}}] = \begin{cases} E^{I\setminus J}_{\bar{q}} & \text{if } J \subset I, \\ E^{I\cup J}_{\bar{q}} & \text{if } I \text{ and } J \text{ are adjacent but non overlapping,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.81}$$

The projection $\pi_{\bar{q},\bar{q}'} : \mathcal{F}_{\bar{q}} \to \mathcal{F}_{\bar{q}'}$ $(\bar{q} \geq \bar{q}')$ maps each $E^I_{\bar{q}}$ with $\bar{q}' \cap I \neq \emptyset$ onto $E^I_{\bar{q}'}$ and each $E^I_{\bar{q}}$ with $\bar{q} \cap I \neq \emptyset$, but $\bar{q}' \cap I = \emptyset$, onto 0.
The analysis of the commutators above shows that $\mathcal{V}_e$ is in fact a Lie algebra and, by definition, it is a C^∞-module.

CHAPTER 6

Calculus on $\mathrm{SL}(3,\mathbb{R})$

The goal of this chapter is to develop an elliptic theory for differential operators on $\mathrm{SL}(3,\mathbb{R})$, lifted to the compactification $\overline{SL}(3,\mathbb{R})$, that lie in the universal enveloping algebra of $\mathcal{V}_e$. As seen in the last chapter, these include right-invariant differential operators. Of course the same can be done for left-invariant operators. The main concern of this chapter is the resolution of $\mathcal{V}_e$. We start by summarizing the results of the last chapter for the case of $\mathrm{SL}(3,\mathbb{R})$.

6.1. Geometry of $\overline{\mathrm{SL}}(3,\mathbb{R})$

Let us summarize the results above for the case of $\mathrm{SL}(3)$: Recall that $\overline{\mathrm{SL}}(3)$ is only one of the two connected components of $\overline{\mathbb{SH}}(3)$.
$\overline{\mathrm{SL}}(3)$ is an eight-dimensional manifold with corners that has two boundary hypersurfaces H_1, H_2, meeting in a codimension 2 boundary face $F = H_1 \cap H_2$. We will summarize the necessary results for the calculus about the geometry and the Lie algebra $\mathcal{V}_e$ at each of the boundary faces.

The boundary face F: We know that F fibers over the product of two copies of the full flag manifold $\mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3)$, which we parameterize using two sets of orthogonal one-dimensional spaces $U_{1,j}$ and $U_{2,j}$ for $j = 1, 2, 3$ such that the two flags are given by

$$U_{\alpha,1} \subset U_{\alpha,1} \oplus U_{\alpha,2} \subset U_{\alpha,1} \oplus U_{\alpha,2} \oplus U_{\alpha,3} = \mathbb{R}^3 \text{ for } \alpha = 1, 2. \tag{6.1}$$

The fiber of $F \to \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3)$ over such a double flag consist of those 4 points in the 8-point space $\prod_{j=1}^{3} \mathbb{SI}(U_{1,j}, U_{2,j})$ such that γ in (5.27) has positive determinant. F is six-dimensional, with each of the two factors of the base being three-dimensional and the fiber being discrete. F also fibers over only the right flag manifold, where the left one becomes an additional factor in the fiber. We denote this fibration

$$\begin{array}{ccc} \{{+}{+}{+}, {+}{-}{-}, {-}{+}{-}, {-}{-}{+}\} \times \mathcal{F}(\mathbb{R}^3) & \text{———} & F \\ & & \big\downarrow \phi_F \\ & & \mathcal{F}(\mathbb{R}^3) \end{array} \tag{6.2}$$

by ϕ_F with now both base and fiber being three-dimensional.
Let τ_1, τ_2 be the boundary defining functions of H_1, H_2 respectively, as defined before by the normal fibration (5.16). Thus, near F, $\overline{\mathrm{SL}}(3)$ takes the form $[0,\epsilon)^2 \times F$ (see Figure 6.1).

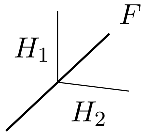

FIGURE 6.1. A visualization of $\overline{\mathrm{SL}}(3)$ near F.

Locally near a point $p \in F$ let z_1, z_2, z_3 be local coordinates tangential to the fibers of ϕ_F. Recall the local characterization of $\mathcal{V}_e$ given in Lemma 5.15. In this chapter the three vector fields $N_{\{1\}},\ N_{\{2\}},\ N_{\{1,2\}}$ are simply denoted by N_1, N_2, N_3, respectively. Then Lemma 5.15 states that $\mathcal{V}_e$ is locally spanned by the vector fields

$$\tau_1\partial_{\tau_1},\ \tau_2\partial_{\tau_2},\ \partial_{z_1},\ \partial_{z_2},\ \partial_{z_3}, \tau_1 N_1,\ \tau_2 N_2,\ \tau_1\tau_2 N_3 \tag{6.3}$$

with commutator satisfying $[\tau_1 N_1, \tau_2 N_2] = \tau_1\tau_2 N_3$.

Some additional structure of F: The geometry of F may be understood even more directly: We may parameterize F by tuples $(U_{1,j}, U_{2,j}, e_j)$ where the $U_{\alpha,j}$ are as above and the e_j are elements of the two-point spaces $e_j \in \mathbb{SI}(U_{1,j}, U_{2,j})$. Consider the map

$$\begin{gathered} \mathrm{SO}(3) \times \mathrm{SO}(3) \longrightarrow F \\ (a, b) \mapsto (U_{1,j} = a(b_j), U_{2,j} = b(b_j), e_j = ab^{-1}|_{U_{1,j}}) \end{gathered} \tag{6.4}$$

where b_j is the subspace of $\mathbb{R}^n$ spanned by the j-th standard vector.
Notice that this map is not $1-1$ but in fact $4-1$ where multiplying both a and b from the right with the same element of $Z = \{d = \mathrm{diag}(\pm 1, \pm 1, \pm 1) \mid \det d = 1\},\ |Z| = 4$, yields the same image. This identifies F with

$$F = \mathrm{SO}(3) \times_Z \mathrm{SO}(3) \tag{6.5}$$

where Z acts diagonally. This has two fibrations, on the left and right, over $\mathrm{SO}(3)/Z$, which bases are the left and right flag manifold.

The boundary face H_1: Recall Theorem 5.5 and Corollary 5.9. H_1 fibers over the right Grassmannian $\phi_1 : H_1 \to \mathrm{Gr}(\mathbb{R}^3, 2)$ which is two-dimensional. Recall that there is a natural projection $\pi_1 : \mathcal{F}(\mathbb{R}^2) \to \mathrm{Gr}(\mathbb{R}^3, 2)$ given by forgetting the one-dimensional subspace. The fibrations ϕ_1, ϕ_F agree on F in the sense that

$$\phi_1|_F = \pi_1 \circ \phi_F. \tag{6.6}$$

$\mathrm{SO}(3)$ acts on $\mathrm{Gr}(\mathbb{R}^3, 2)$. The vector field generated by N_2, as a vector field on $\mathrm{Gr}(\mathbb{R}^3, 2)$. The other two vector fields N_1, N_3, as vector fields on $\mathrm{Gr}(\mathbb{R}^3, 2)$, generate linear independent vector fields on $\mathrm{Gr}(\mathbb{R}^3, 2)$ that therefore span its tangent space everywhere.
For any interior point $p \in H_1$ let $z_1, \ldots z_4$ be local coordinates tangent to the fibers. Then $\mathcal{V}_e$ near p is spanned by

$$\tau_1\partial_{\tau_1}, \partial_{z_i}, \tau_1 N_1, \tau_1 N_3. \tag{6.7}$$

The boundary face H_2: The situation is analogous. H_2 fibers over the right Grassmannian $\phi_2 : H_2 \to \mathrm{Gr}(\mathbb{R}^3, 1)$ which is two-dimensional. Let $\pi_2 : \mathcal{F}(\mathbb{R}^2) \to \mathrm{Gr}(\mathbb{R}^3, 1)$ again denote the projection. We then have

$$\phi_2|_F = \pi_2 \circ \phi_F. \tag{6.8}$$

For any interior point $p \in H_2$ let $z_1, \dots z_4$ be local coordinates tangential the fiber. Then $\mathcal{V}_e$ near p is spanned by

$$\tau_2 \partial_{\tau_2}, \partial_{z_i}, \tau_2 N_2, \tau_2 N_3. \tag{6.9}$$

We may summarize the connection between the three fibrations ϕ_F, ϕ_1, ϕ_2 with the following commutative diagram:

$$\begin{tikzcd}
H_1 \arrow[d, "\phi_1"] & F \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_F"] & H_2 \arrow[d, "\phi_2"] \\
\mathrm{Gr}(\mathbb{R}^3, 2) & \mathcal{F}(\mathbb{R}^3) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathrm{Gr}(\mathbb{R}^3, 1)
\end{tikzcd} \tag{6.10}$$

One may visualize the fibers of the different fibrations as in Figure 6.1, where the fibers on F are represented by points (being the intersection of the different fibers of ϕ_1 and ϕ_2). Recall that the three vector fields N_1, N_2, N_3, as vector fields on the base, span the tangent space of $\mathcal{F}(\mathbb{R}^3)$. The two vector fields N_1, N_3 also span the tangent space of $\mathrm{Gr}(2)$ under projection by π_1 and N_2, N_3 span the tangent space of $\mathrm{Gr}(1)$ under projection by π_2. Thus, near F, we can choose local coordinates y_1, y_2, y_3 that are the pullback of coordinates on $\mathcal{F}(\mathbb{R}^3)$ such that locally $N_3 = \partial_{y_3}$ and

(1) either y_1, y_3 are pullbacks of coordinates on $\mathrm{Gr}(\mathbb{R}^3, 2)$ and thus $N_1 \in \mathrm{span}\{\partial_{y_1}, \partial_{y_3}\}$
(2) or y_2, y_3 are pullbacks of coordinates on $\mathrm{Gr}(\mathbb{R}^3, 1)$ and thus $N_2 \in \mathrm{span}\{\partial_{y_2}, \partial_{y_3}\}$.

We can not do both simultaneously. Furthermore, we can not trivialize N_1, N_2, N_3 simultaneously, since $[N_1, N_2] = N_3$.

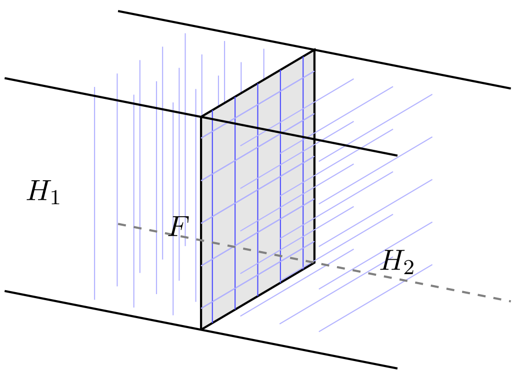

FIGURE 6.2. A visualization of how the fibers on H_1 and H_2 intersect.

6.2. The resolved single space

We want to resolve $\mathcal{V}_e$, i.e. we want to construct a resolution of the double space such that the lift of $\mathcal{V}_e$ from either factor becomes smooth and transversal to the diagonal.

Additionally, the two maps back to the single space (given first by blow-down and then by projection from either the left or right) are required to be b-fibrations. The whole process is governed by (6.3).
However, before passing to the double space, we start by blowing up F in $\overline{\mathrm{SL}}(3)$. Clearly, one should ask why we do so. The main reason for this is that the fibrations $\phi_1,\ \phi_2$ of the two hypersurfaces $H_1,\ H_2$ of $\overline{\mathrm{SL}}(3,\mathbb{R})$ are not *iterated*. This means that, restricted to their intersection F, they do not form a tower. However, as described above, the intersection of the fibers of the two fibrations $\phi_1,\ \phi_2$ restricted to F are the fibers of a common larger fibration ϕ_F. The blow-up of F resolves this and the blown-up space becomes a *manifold with fibered corners*. What this means exactly is described below.
Manifolds that carry such a structure have gained a lot of interest in recent years, for example in [6]. This resolution of the fibrations is used extensively in the resolution of $\mathcal{V}_e$. This does not mean that a resolution is impossible without this extra step. However, any such resolution would certainly come with its own challenges compared to the approach taken here.

We set

$$X = [\overline{\mathrm{SL}}(3); F] \tag{6.11}$$

and denote the new boundary hypersurface by H_F (see Figure 6.3).

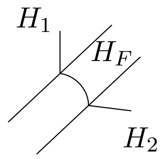

FIGURE 6.3. A visualization of the resolved single space X.

The fibrations $\phi_F : F \to \mathcal{F}(\mathbb{R}^3)$ lifts to become a fibration $\phi_{H_F} : H_F \to \mathcal{F}(\mathbb{R}^3)$ where the fiber gains an additional factor given by the inward pointing part of the spherical normal bundle of F. The three vector fields N_i lift to H_F to become independent vector fields spanning (under projection) the tangent space of the base of ϕ_{H_F}.

Both ϕ_1 and ϕ_2 lift as well by composing with the blow-down map which is a diffeomorphism on H_1 and H_2. By a slight abuse of notation, we denote the lifts of $H_{1,2}$ together with the fibrations $\phi_{1,2}$ again by $H_{1,2},\ \phi_{1,2}$, yielding the following commutative diagram

$$\begin{tikzcd}
& H_1 \cap H_F \arrow[dl, hook] \arrow[dr, hook] & & H_F \cap H_2 \arrow[dl, hook] \arrow[dr, hook] & \\
H_1 \arrow[d, "\phi_1"] & & H_F \arrow[d, "\phi_F"] & & H_2 \arrow[d, "\phi_2"] \\
\mathrm{Gr}(\mathbb{R}^3, 2) & & \mathcal{F}(\mathbb{R}^3) \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & \mathrm{Gr}(\mathbb{R}^3, 1)
\end{tikzcd} . \tag{6.12}$$

Near H_F and away from H_1 we may construct new local coordinates form the old ones used in (6.3) by replacing τ_2 with the projective coordinate $t = \tau_2/\tau_1$. The lifts of the spanning vector fields in (6.3) then become

$$\tau_1\partial_{\tau_1},\ t\partial_t,\ \partial_{z_i},\ \tau_1 N_1,\ t\tau_1 N_2,\ t\tau_1^2 N_3 \tag{6.13}$$

(for $i = 1, \dots, 4$) up to additional terms that lie in the span of the remaining ones and thus can be omitted. Near H_F and away from H_2 the situation is completely analogous by introducing the projective coordinate $t_2 = \tau_1/\tau_2$:

$$t_2\partial_{t_2},\ \tau_2\partial_{\tau_2},\ \partial_{z_i},\ \tau_2\rho N_1,\ \tau_2 N_2,\ t_2\tau_2^2 N_3. \tag{6.14}$$

The lift of of $\mathcal{V}_e$ on X is not a C^∞-module, since new variables have been introduced through blow-up. Thus we move to an even larger Lie algebra.

Definition 6.1: *Let $\mathcal{V}_{\mathrm{SL}}$ denote the C^∞-span of the lift of $\mathcal{V}_e$ on X.*

$\mathcal{V}_{\mathrm{SL}}$ is a Lie algebroid and now locally spanned over C^∞ by the vector fields (6.13). This implies that $\mathcal{V}_{\mathrm{SL}}$ consist of the sections of a vector-bundle ${}^{\mathrm{SL}}TX$ (see for example [22]) that is naturally associated to $\mathcal{V}_{\mathrm{SL}}$, yielding $\mathcal{V}_{\mathrm{SL}} = C^\infty(X, {}^{\mathrm{SL}}TX)$. The bundle ${}^{\mathrm{SL}}TX$ is equipped with a map into the ordinary b-vector bundle $\iota_{\mathrm{SL}} : {}^{\mathrm{SL}}TX \to {}^bTX$ that is induced by the inclusion $\mathcal{V}_{\mathrm{SL}} \to \mathcal{V}_b$. The universal enveloping algebra of $\mathcal{V}_{\mathrm{SL}}$ is the ring of differential operators denoted by $\mathrm{Diff}^*_{\mathrm{SL}}$ that is locally given by operators L of the form

$$L = \sum \lambda_{a,b,c_i,d,e,f}(\tau_1\partial_{\tau_1})^a(t\partial_t)^b(\partial_{z_i})^{c_i}(\tau_1 N_1)^d(t\tau_1 N_2)^e(t\tau_1^2 N_3)^f. \tag{6.15}$$

Notice that the order of different factors does change the coefficients, since they do not commute. As usual, this space is filtered by differential operators of degree m, denoted by $\mathrm{Diff}^m_{\mathrm{SL}}$.

Furthermore, $\mathrm{Diff}^m_{\mathrm{SL}}$ has several natural subalgebras that arise from the construction of the space X. Differential operators on $\overline{\mathrm{SL}}(n)$ arising from $\mathcal{V}_e$ lift to become elements of Diff^m_X. They form a subalgebra of $\mathrm{Diff}^m_{\mathrm{SL}}$, denoted by Diff^m_e. Secondly, there is an even smaller subalgebra of differential operators arising form right-invariant vector fields on $\mathrm{SL}(n)$, denoted by Diff^m_{ri}, yielding

$$\mathrm{Diff}^m_{ri} \subset \mathrm{Diff}^m_e \subset \mathrm{Diff}^m_{\mathrm{SL}}. \tag{6.16}$$

There is a symbol map that associates to each $L \in \mathrm{Diff}^m_{\mathrm{SL}}$, given by a homogeneous polynomial of degree m on the fibers of ${}^{\mathrm{SL}}T^*X$, denoted by ${}^{\mathrm{SL}}\sigma_m(L)$. Locally the vector fields (6.13) yield a basis of ${}^{\mathrm{SL}}T_pX$. Let $\xi, \eta, \zeta_i, \vartheta_j$ denote its dual basis in ${}^{\mathrm{SL}}T^*_pX$. If L has the local form (6.15), then we have

$$\begin{aligned} &{}^{\mathrm{SL}}\sigma_m(L)(p, (\xi, \eta, \zeta_i, \vartheta_j)) = \\ &\sum_{a+b+\sum c_i+d+e+f=m} \lambda_{a,b,c_i,d,e,f}(\xi)^a(\eta)^b(\zeta_i)^{c_i}(\vartheta_1)^d(\vartheta_2)^e(\vartheta_3)^f. \end{aligned} \tag{6.17}$$

This yields the notion of a elliptic operator:

Definition 6.2: *We say $L \in \mathrm{Diff}^m_{\mathrm{SL}}$ is elliptic, if for all $p \in X$ the symbol satisfies ${}^{\mathrm{SL}}\sigma_m(L)(p, \cdot) \neq 0$ on ${}^{\mathrm{SL}}T^*_pX \setminus \{0\}$.*

The next goal is the construction of a resolution of the double space X^2. As a first step in this resolution, we move to the so called *symmetric lexicographic blow-up*. Since we will need it again in the resolution of the triple space and in the discussion of $\mathrm{SL}(n, \mathbb{R})$, we define it in general in the next section.

6.3. Symmetric lexicographic blow-up

Setting: Consider a manifold with corners M that has a partial order on the set of boundary hypersurfaces $(\mathcal{M}_1(M), \leq)$ such that whenever two hypersurfaces $F, G \in \mathcal{M}_1(M)$ have non empty intersection $F \cap G \neq \emptyset$, they are comparable.

We will define the *symmetric lexicographic blow-up* of the double- and triple space M^2, M^3, denoted by M^2_{lex}, M^3_{lex}, respectively, although the same concept translates to any power M^k. We then proceed to show the existence of projections maps $M^3_{lex} \to M^2_{lex}$.

Symmetric lexicographic blow-up of M^2: Let the set of hypersurfaces of M be indexed by some index I, thus giving $\mathcal{M}_1(M) = \{H_a \mid a \in I\}$. Consider those codimension two faces of M^2 that are given by the product of two hypersurfaces of M

$$\mathcal{M}_2(M^2) \supset P := \{H_{ab} = H_a \times H_b \mid a, b \in I\}. \tag{6.18}$$

There is a partial order on P given by $H_{ab} \leq H_{a'b'}$ if and only if $H_a \leq H_{a'}$ and $H_b \leq H_{b'}$. A *linear extension* of this partial order on P is a total order on P that is compatible with this partial order. Such an extension always exist for a partial order on a finite set. It can be constructed by starting with any minimal element og $\mathcal{P}$ and continuing iteratively. Of course, such a linear extension is not unique. Let $\mathcal{P}$ be an exhaustive list of all elements in P that is ordered increasingly according to any linear extension of the partial order on P. This is equivalent to the fact that whenever $H_{ab} \leq H_{a'b'}$, then H_{ab} appears before $H_{a'b'}$ in $\mathcal{P}$. We then define

$$M^2_{lex} = [M^2; \mathcal{P}]. \tag{6.19}$$

Of course we have to check that this is unique, which is shown in the next Lemma.

Lemma 6.3: *The resolution M^2_{lex} is independent of the choice of linear extension. To be precise, whenever $\mathcal{P}$, $\tilde{\mathcal{P}}$ are two comprehensive lists of P that are ordered according to two linear extensions of the partial order in P, then the identity on M lifts to become a diffeomorphism*

$$[M^2; \mathcal{P}] \cong [M^2; \tilde{\mathcal{P}}]. \tag{6.20}$$

Proof. Any two linear extensions $\mathcal{P}$, $\tilde{\mathcal{P}}$ can be transformed into each other by swapping neighboring, non comparable (with respect to the partial order) elements.
Let $H_{ab},\ H_{a'b'} \in P$ be not comparable. By definition of the partial order on P, this can happen in two cases: Either $H_a, H_{a'}$ or $H_b, H_{b'}$ are not comparable. In this case they have to be disjoint and thus $H_{ab},\ H_{a'b'}$ are disjoint. Therefore the order of blow-ups can be switched if they are adjacent in $\mathcal{P}$. If both $H_a,\ H_{a'}$ and $H_b,\ H_{b'}$ are comparable, we

denote by $H_{\tilde a}$, $H_{\tilde b}$ the smaller one of the two, respectively. Then we have $H_{\tilde a \tilde b} \leq H_{ab}$ and thus $H_{\tilde a \tilde b}$ is blown up before H_{ab}, $H_{a'b'}$. Furthermore $H_{ab} \cap H_{a'b'} \subset H_{\tilde a \tilde b}$ with neither of them being included on its own. Thus the lifts of H_{ab}, $H_{a'b'}$ are disjoint and thus their order of blow-up may be switched.

$\square$

Lemma 6.4: *The blow-down map $\beta : M^2_{lex} \to M^2$ composed with either the left or right projection $\pi : M^2 \to M$ is a b-fibration.*

Proof. The map $\pi \circ \beta$ is an interior b-map. We have to check that its b-differential at each point is surjective both as a map between the b-tangent and the b-normal spaces. The first statement is clear, since it is the case for β and π. The second statement is also easy to see since any $\mathrm{ff}(H_{ab})$ is mapped to either H_a or H_b (depending on which projection π is used). $\square$

Symmetric lexicographic blow-up of M^3: Similarly, consider the codimension three faces of M^3 that are the products of three hypersurfaces of M

$$\mathcal{M}_3(M^2) \supset C := \{H_{abc} = H_a \times H_b \times H_c \mid a, b, c \in I\} \tag{6.21}$$

with partial order being defined analogously. Also consider the codimension two faces of M^3 which are the product of two hypersurfaces of M and an additional factor M. For a simpler notation, we write $M = H_0$ (and assume 0 is not an element of the index set I). Set

$$E := \{H_{abc} = H_a \times H_b \times H_c \mid \text{ precisely one of the } a, b, c \text{ is } 0\}. \tag{6.22}$$

Again, there is a partial order on E given by the partial order on $\mathcal{M}_1(M)$ together with $H_0 = M$ added as a new greatest element. In fact, this yields a partial order on $C \cup E$. No element of E can be smaller then an element of C.
Analogously to before, let $\mathcal{E}$ be a exhaustive list of E that is ordered according to a linear extension of the partial order on E. Let $\mathcal{C}$ be an exhaustive list of C that is ordered according to a linear extension of the partial order on C. We define

$$M^3_{lex} = [M^3; \mathcal{C}, \mathcal{E}]. \tag{6.23}$$

The fact that this is well-defined follows analogously to the case of M^2_{lex}.

Projection maps: The set E decomposes into three parts

$$\begin{aligned} E^{LM} &= \{H_{abc} \in E \mid H_c = M\}, \\ E^{LR} &= \{H_{abc} \in E \mid H_b = M\}, \\ E^{MR} &= \{H_{abc} \in E \mid H_a = M\}. \end{aligned} \tag{6.24}$$

Let $\mathcal{E}^{LM}$, $\mathcal{E}^{LR}, \mathcal{E}^{MR}$ be exhaustive lists of these three sets, each of which being ordered with respect to a linear extension of the partial order on E.

Lemma 6.5: *We have*

$$
\begin{aligned}
M^3_{lex} = [M^3;\mathcal{C};\mathcal{E}] &\cong [M^3;\mathcal{C};\mathcal{E}^{LM};\mathcal{E}^{LR};\mathcal{E}^{MR}] \\
&\cong [M^3;\mathcal{E}^{LM};\mathcal{C};\mathcal{E}^{LR};\mathcal{E}^{MR}] \\
&\cong [M^3;\mathcal{E}^{LR};\mathcal{C};\mathcal{E}^{LM};\mathcal{E}^{MR}] \\
&\cong [M^3;\mathcal{E}^{MR};\mathcal{C};\mathcal{E}^{LM};\mathcal{E}^{LR}]
\end{aligned}
\tag{6.25}
$$

PROOF. The first equality follows form the fact that any two faces from different $\mathcal{E}^\alpha$ have intersection contained in one of the codimension three faces in $\mathcal{C}$. Therefore, they lift to become disjoint when blowing up $\mathcal{C}$. By symmetry, we only need to prove the first one of the last three equalities. For this, we successively commute the first element of $\mathcal{E}^{LM}$ with all elements in $\mathcal{C}$, then the second one, and so on. For this, it suffice to show the following. The blow-ups of H_{abc} and H_{ij0} commutes after all $H_{a'b'c'} \leq H_{abc}$ and all $H_{i'j'0} \leq H_{ij0}$ have already been blown up. We show this by distinguishing between three cases.

(1) If $H_a = H_i$ and $H_b = H_j$, then $H_{abc} \subset H_{ij0}$ and since no other F with $H_{abc} \subset F$ is blown up before, the inclusion is also true for the lifts of H_{abc} and H_{ab0}. Thus their blow-ups commutes.
(2) If either $H_a \geq H_i$ (or analogously $H_b \geq H_j$), then $H_{abc} \cap H_{ij0} \subset H_{ibc}$ with neither of them being contained and $H_{ibc} \leq H_{abc}$. Thus H_{abc} and H_{ij0} lift to become disjoint through the blow-up of H_{ibc}.
(3) Lastly, consider the case $H_a \leq H_i$ and $H_b \leq H_j$ with at least one of these inequalities being strict, lets say $H_a \neq H_i$. If additionally $H_b \neq H_j$, then H_{abc} and H_{ij0} are disjoint. Otherwise $H_{ab0} \leq H_{ij0}$ and $H_{abc} \subset H_{ab0} \subsetneq \mathrm{Fa}(H_{abc} + H_{ij0}) = H_{0j0}$ and thus H_{ab0} separates H_{abc} and H_{ij0}, yielding that their lifts intersect transversally. Furthermore, no boundary face containing H_{abc} is blown up before H_{ab0}.

$\square$

Now the definition of the projection maps are clear. We write it down for the first case (corresponding to LM): Denote by $\gamma^{LM} : M^3_{lex} \to [M^3;\mathcal{E}^{LM}]$ the collective blow-down of $\mathcal{C};\mathcal{E}^{LR};\mathcal{E}^{MR}$. We now have $[M^3;\mathcal{E}^{LM}] \cong M^2_{lex} \times M$. Denote by $\phi^{LM} : M^2_{lex} \times M \to M^2_{lex}$ the corresponding projection. Then set

$$
\pi^{LM}_{lex} : M^3_{lex} \to M^2_{lex},\ \pi^{LM}_{lex} = \phi^{LM} \circ \gamma^{LM}.
\tag{6.26}
$$

Corollary 6.6: *The three lifted projection maps $\pi^\alpha_{lex} : M^3_{lex} \to M^2_{lex}$ for $\alpha = LM, LR, MR$ are b-fibrations.*

PROOF. By symmetry, we only need to prove the case $\alpha = LM$. Both γ^{LM} and ϕ^{LM} are surjective b-maps and b-submersion and thus, so is their composition. By a general theorem about b-fibrations (see e.g. [19]), the only thing left to show is that no hypersurface of M^3_{lex} is mapped to a boundary face of codimension 2 or higher by π^{LM}_{lex}. For any $H_{ab0} \in \mathcal{E}^{LM}$ the image of the corresponding hypersurface $\mathrm{ff}(H_{ab0}) \subset M^3_{lex}$ under the lifted projection π^{LM}_{lex} is simply $\mathrm{ff}(H_{ab}) \subset M^2_{lex}$. For any $H_{a,b,c} \in \mathcal{C};\mathcal{E}^{MR};\mathcal{E}^{LR}$ the image of the corresponding hypersurface $\mathrm{ff}(H_{a,b,c})$ under π_{lex}^{LM} is given by the lift of $H_{abc} \subset M^3$ under $[M^3; \mathcal{E}^{LM}] \to M^3$. Since $[M^3; \mathcal{E}^{LM}] \cong M_{lex}^2 \times M$ and by the naturality of blow-up, this lift is given by the lift in each factor. Thus, after projection to the first factor, the image of $\mathrm{ff}(H_{a,b,c})$ is given by the lift of $H_{ab} \subset M^2$ to M_{lex}^2, which is always a hypersurface of M_{lex}^2. □

6.4. The resolved double space

Let us return to SL(3). Recall that $X = [\overline{\mathrm{SL}}(3); F]$. The goal of this section is to construct a resolution of X^2 via a series of blow-ups so that the diagonal $\triangle \subset X^2$ lifts to a p-submanifold and $\mathcal{V}_{\mathrm{SL}}$ lifts to become smooth and transversal to the lifted diagonal. Furthermore, we require the blow-down map composed with the projection from either the left or right back to X to be a b-fibration. The first step in this resolution will be the symmetric lexicographic blow-up.

Consider the lifts (from the left or right factor) of these vector fields to the double space X^2. We fix the notation $\triangle_U = \{(p,p) \mid p \in U\} \subset X^2$ for any $U \subset X$. Furthermore, for any local coordinate function x on X we denote the lifts to the left and right to X^2 by x and x', respectively, although this is a slight abuse of notation. Note that

$$\triangle_X \cap \partial X^2 = \triangle_{H_1} \cup \triangle_{H_F} \cup \triangle_{H_2} \tag{6.27}$$

where

$$\triangle_{H_1} \cap \triangle_{H_F} \neq \emptyset \ , \ \triangle_{H_F} \cap \triangle_{H_2} \neq \emptyset \ , \ \triangle_{H_1} \cap \triangle_{H_2} = \emptyset. \tag{6.28}$$

We move to the symmetric lexicographic blow-up of X^2. The partial order of the boundary faces of X is given by the size (dimension) of the fibers defined on it, meaning $H_1, H_2 \geq H_F$. Thus the symmetric lexicographic blow-up is given by

$$X_{lex}^2 := [X^2; H_{22}; H_{11}; H_{21}; H_{12}; H_{2F}; H_{F2}; H_{1F}; H_{F1}; H_{FF}] \tag{6.29}$$

with blow-down map denoted by

$$\beta_1 : X_{lex}^2 \to X^2 \tag{6.30}$$

where $H_{ab} = H_a \times H_b$ as before. We call the new resulting boundary hypersurfaces $\mathrm{ff}(H_{ab})$. The lift of the vector fields $\mathcal{V}_{\mathrm{SL}}$ from either the left or the right are tangential to the boundary faces H_{ab}, since $\mathcal{V}_{\mathrm{SL}} \subset \mathcal{V}_b$. Therefore, they lift smoothly to X_{lex}^2. From (6.27) we can conclude that the lift of the diagonal (which we again denote by $\triangle_X$[1]) is resolved to a p-submanifold. It intersects the boundary of X_{lex}^2 in $\mathrm{ff}(H_{11})$, $\mathrm{ff}(H_{22})$ and $\mathrm{ff}(H_{FF})$, yielding an interior p-submanifolds in each of these three faces. All other blow-ups in (6.29) occur away from the (lift of the) diagonal. In other words, near the diagonal the symmetric lexicographic resolution and the b-resolution of the double space are identical. Since we are mainly interested in the behavior of $\mathcal{V}_{\mathrm{SL}}$ near the diagonal, one should ask the question whether or not the remaining blow-ups in (6.29) are necessary. While it is

[1]This is a slight abuse of notation. The same will be done in the second step of the resolution. From the context it should always be clear which diagonal is meant.

true that the lifted diagonal does not intersect with these front faces, they do alter the boundary of $\mathrm{ff}(H_{11})$, $\mathrm{ff}(H_{22})$ and $\mathrm{ff}(H_{FF})$, which therefore changes the model operators in a potential large calculus.
We will analyze the lift of $\mathcal{V}_{\mathrm{SL}}$ on each of the three new boundary hypersurfaces that intersect the diagonal individually.

On $\mathrm{ff}(\boldsymbol{H_{FF}})$: Recall from (6.12) that H_F has a fibration $H_F \to \mathcal{F}(\mathbb{R}^3)$. Thus, we have a fibration $H_{FF} \to \mathcal{F}(\mathbb{R}^3)^2$. However, H_{FF} is blown up last in X^2_{lex}, thus we are interested in the structure of the lift of H_{FF} after all previous blow-ups. By the naturality of blow-up, the lift of H_{FF} after blow-up of H_{ab} is equal (diffeomorphic) to $[H_{FF}; (H_{FF} \cap H_{ab})]$. Again recall from (6.12) that for both $i = 1, 2$ the intersection $H_i \cap H_F \hookrightarrow H_F \to \mathcal{F}(\mathbb{R})$ fibers over the same base. From the local product structure of the fibration it follows that the blow-up $[H_{FF}; (H_{FF} \cap H_{ab})]$ occurs in the fiber, thus the lift of H_{FF} again fibers over $\mathcal{F}(\mathbb{R})^2$ with now slightly altered fiber (through blow-up). The new front face $\mathrm{ff}(H_{FF})$ also fibers over this base with the fiber gaining an additional factor, namely the spherical normal bundle of H_{FF}. We denote this fibration by ϕ_{FF}:

$$\begin{tikzcd} \mathrm{ff}(H_{FF}) \arrow[d, "\phi_{FF}"] \\ \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \end{tikzcd} . \tag{6.31}$$

Let $p \in \mathrm{ff}(H_{FF})^\circ$. Then $\beta_1(p)$ lies in H°_{FF}. Local coordinates on X near F may be pulled back from the left and right to give local coordinates on X^2. Starting with local coordinates on X as in (6.13), we can chose local coordinates on X^2 centered at $\beta_1(p)$ given by $t, t', \tau_1, \tau_1', z_i, z_i'$ together with some coordinates y_i, y_i' which are pullbacks of coordinates on left and right copies of the flag manifold in the base. In these coordinates H^2_F is given locally by $\{\tau_1 = \tau_1' = 0\}$. Again, we may construct new coordinates on the blown-up space from the old ones by taking

$$t, t', \tau_1, \psi = \tau_1'/\tau_1, z_i, z_i', y_i, y_i', \tag{6.32}$$

where ψ is tangent to the fibers of the spherical normal bundle of H_{FF}. In these coordinates, the lift from the left of the spanning vector fields (6.13) become

$$\tau_1\partial_{\tau_1} - \psi\partial_\psi,\ t\partial_t,\ \partial_{z_i},\ \tau_1 N_1,\ t\tau_1 N_2,\ t\tau_1^2 N_3. \tag{6.33}$$

In term of these local coordinates, the lift of the diagonal restricted to $\mathrm{ff}(H_F F)$ is given by $\psi = 1,\ t = t', z = z',\ y = y'$. Therefore, on $\triangle_X \cap \mathrm{ff}(H_F F)$ the first vector fields $\tau_1\partial_{\tau_1} - \psi\partial_\psi,\ t\partial_t,\ \partial_{z_i}$ in (6.33) do not vanish and are tangent to the fibers of ϕ_{FF} and the last three vector fields still vanish.

On $\mathrm{ff}(\boldsymbol{H_{22}})$: Analogously to before, recall from (6.12) that H_2 has a fibration $H_2 \to \mathrm{Gr}(\mathbb{R}^3, 1)$ and thus we have a fibration $H_{22} \to (\mathrm{Gr}(\mathbb{R}^3, 1))^2$, which in return yields a fibration of $\mathrm{ff}(H_{22})$. However, unlike in the case of $\mathrm{ff}(H_{FF})$, there are blow-ups occurring in the construction of X^2_{lex} after H_{22} is blown up. These blow-ups restricted to $\mathrm{ff}(H_{22})$

again occur in the fibers, thus yielding a fibration of the final front face

(6.34)
$$\begin{tikzcd} \mathrm{ff}(H_{22}) \arrow[d, "\phi_{22}"] \\ \mathrm{Gr}(\mathbb{R}^3,1)\times \mathrm{Gr}(\mathbb{R}^3,1) \end{tikzcd}.$$

Using local coordinates as in (6.9) and their dashed counterparts, we may chose the projective coordinates

(6.35)
$$\tau_2, \psi_2 = \tau_2'/\tau_2, z_1, \dots, z_4$$

in which the lift of the vector fields (6.9) become

(6.36)
$$\tau_2\partial_{\tau_2} - \psi_2\partial_{\psi_2}, \partial_{z_i}, \tau_2 N_2, \tau_2 N_3.$$

Again, with respect to these local coordinates the lift of the diagonal $\triangle_X$ is given by $\psi_2 = 1$.

Near $\mathrm{ff}(H_{FF}) \cap \mathrm{ff}(H_{22})$: We may use local coordinates as in (6.13) near a point $p \in H_2 \cap H_F$ to get local coordinates $t, t', \tau_1, \tau_t', z_i, z_i', y_i, y_i'$ near a point in $H_{22} \cap H_{FF}$ such that locally $H_{FF} = \{\tau_1 = \tau_1' = 0\}$ and $H_{22} = \{t = t' = 0\}$, so we may chose the projective coordinates

(6.37)
$$t, \rho = t'/t, \tau_1, \psi = \tau_1'/\tau_1, z_i, z_i', y_i, y_i'.$$

In this coordinate system, the spanning vector fields take the form

(6.38)
$$\tau_1\partial_{\tau_1} - \psi\partial_\psi,\ t\partial_t - \rho\partial_\rho,\ \partial_{z_i},\ \tau_1 N_1,\ t\tau_1 N_2,\ t\tau_1^2 N_3.$$

The diagonal is locally given by $\{\psi = \rho = 1\}$.

On $\mathrm{ff}(H_{11})$: On $\mathrm{ff}(H_{11})$, the situation is completely analogous where it has a fibration

(6.39)
$$\begin{tikzcd} \mathrm{ff}(H_{11}) \arrow[d, "\phi_{11}"] \\ \mathrm{Gr}(\mathbb{R}^3,2)\times \mathrm{Gr}(\mathbb{R}^3,2) \end{tikzcd}$$

and local coordinates constructed using (6.14) as a starting point instead of (6.13).

Again, we get a commutative diagram showing the connection between the fibrations:

(6.40)
$$\begin{tikzcd}[column sep=small] & \mathrm{ff}(H_{11})\cap\mathrm{ff}(H_{FF}) \arrow[dl, hook] \arrow[dr, hook] & & \mathrm{ff}(H_{FF})\cap\mathrm{ff}(H_{22}) \arrow[dl, hook] \arrow[dr, hook] & \\ \mathrm{ff}(H_{11}) \arrow[d, "\phi_{11}"] & & \mathrm{ff}(H_{FF}) \arrow[d, "\phi_{FF}"] & & \mathrm{ff}(H_{22}) \arrow[d, "\phi_{22}"] \\ (\mathrm{Gr}(\mathbb{R}^3,2))^2 & & (\mathcal{F}(\mathbb{R}^3))^2 \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & (\mathrm{Gr}(\mathbb{R}^3,1))^2 \end{tikzcd}$$

In the local coordinates (6.37) we have $\mathrm{ff}(H_{FF}) = \{\tau_1 = 0\}$ and $\mathrm{ff}(H_{22}) = \{t = 0\}$. Now let us check how the lift of the diagonal meets the boundary:

$$
\begin{aligned}
\triangle_X \cap \mathrm{ff}(H_{FF}) &= \{\tau_1 = 0, \psi = 1, \rho = 1, y = y', z = z'\} \\
\triangle_X \cap \mathrm{ff}(H_{22}) &= \{t = 0, \psi = 1, \rho = 1, y = y', z = z'\}.
\end{aligned}
\tag{6.41}
$$

The last three vector fields in (6.38) still vanish on $\triangle_X \cap \mathrm{ff}_F$ and the last two still vanish on $\triangle_X \cap \mathrm{ff}_2$. The situation at ff_1 is analogous where the third to last and last vector fields still vanish. Therefore, we need a second step in the resolution where we will blow up a p-submanifold in each of the $\mathrm{ff}(H_{ii})$. Again, we analyze the situation first on ff_F and then on ff_2. For this we use local coordinates as in (6.37).

On ff(H_{FF})**:** Consider the flow-out of $\triangle_X \cap \mathrm{ff}(H_{FF})$ under the lift of $\mathcal{V}_{\mathrm{SL}}$, denoted by S_F. The vector fields in (6.38) that do vanish on $\mathrm{ff}(H_{FF})$ are precisely the last three, all others do not. Thus we get

$$
S_F := (\phi_{FF})^{-1}(\triangle_{\mathcal{F}(\mathbb{R}^3)}) \subset \mathrm{ff}(H_{FF}).
\tag{6.42}
$$

Since $\triangle_{\mathcal{F}(\mathbb{R}^3)} \subset \mathcal{F}(\mathbb{R}^3)^2$ is clearly a p-submanifold, so is $S_F \subset \mathrm{ff}(H_{FF})$. As a submanifold of ff_F, it has codimension 3. In local coordinates as in (6.37) the flow-out is

$$
S_F = \{\tau_1 = 0, y = y'\}.
\tag{6.43}
$$

Notice that the last vector field $t\tau_2^2 N_3$ vanishes to second order on S_F, so we need to blow it up parabolically.

Also note that S_F is in fact the lift under the symmetric lexicographic blow-up of a submanifold (also denoted by) $S_F \subset H_{FF}$, which is again the fiber diagonal under the fibration $H_{FF} \to \mathcal{F}(\mathbb{R}^3)^2$, i.e. the preimage of $\triangle_{\mathcal{F}(\mathbb{R}^3)}$ under $H_{FF} \to \mathcal{F}(\mathbb{R}^3)^2$. We may interpret S_F as a submanifold of X^2_{lex} or X^2 depending on the context.

On ff(H_{22})**:** Analogously, the flow-out of $\triangle_X \cap \mathrm{ff}(H_{22})$ under $\mathcal{V}_{\mathrm{SL}}$ is a p-submanifold of ff_2 given by

$$
S_2 = (\phi_{22})^{-1}(\triangle_{\mathrm{Gr}(\mathbb{R}^3,\ 1)}) \subset \mathrm{ff}(H_{22}).
\tag{6.44}
$$

As a submanifold of ff_2, it has codimension 2. In the local coordinates (6.37) it takes the form

$$
S_2 = \{t = 0, y_2 = y_2', y_3 = y_3'\}.
\tag{6.45}
$$

This blow-up does not need to be parabolic, since the two vanishing vector fields N_2, N_3 both vanish to first order.
Again, notice that S_2 is the lift of a p-submanifold $S_2 \subset H_{22} \subset X^2$ under lexicographic blow-up, given by the fiber diagonal with respect to the fibration of H_2.

On ff(H_{11}): Again, the situation is analogous to the one at ff_2. The flow-out of $\triangle_X \cap \mathrm{ff}(H_{11})$ under $\mathcal{V}_{\mathrm{SL}}$ is given by

$$S_1 = (\phi_{11})^{-1}(\triangle_{\mathrm{Gr}(\mathbb{R}^3,\ 2)}) \subset \mathrm{ff}(H_{11}). \tag{6.46}$$

As a submanifold of ff_1 it has codimension 2. Again, this blow-up will be radial, since the two vanishing vector fields N_1, N_3 both vanish to first order. S_1 is the lift under a submanifold $S_1 \subset H_{11} \subset X^2$ under lexicographic blow-up, given by the fiber diagonal with respect to the fibration of H_1.

Quasihomogeneous structure at S_F: As mentioned earlier, the blow-up of S_F needs to be parabolic, since the last vector field $t\tau_2^2 N_3$ vanishes to second order on S_F. Since $S_F \subset \mathrm{ff}(H_{FF})$ is the fiber diagonal with respect to the fibration $\mathrm{ff}(H_{FF}) \to \mathcal{F}(\mathbb{R}^3)^2$, its normal bundle (as as submanifold of $\mathrm{ff}(H_{FF})$) is identified with the tangent bundle of the base $T\mathcal{F}(\mathbb{R}^3)$. The three vector fields $N_1,\ N_2,\ N_3$ (without the prefactor), projected to this normal bundle, span the normal bundle everywhere. Together with the normal vector field ∂_{τ_1}, they span the normal bundle of S_F as a submanifold of X^2_{lex}. Thus, we may define a quasihomogeneous structure at S_F by associating a negative weight to each of these vector fields, as in Proposition 2.39. We associate the weight -2 to N_3 and -1 the the remaining ones. This is a parabolic structure, since the largest (negative) weight appearing is 2. We denote the resulting quasihomogeneous structure by Π_F.

The second step in the resolution of X^2 will therefore consist of blowing up the three submanifolds $S_i \subset \mathrm{ff}(H_{ii})$ where $i = 1, 2, F$. The S_i are neither disjoint (except S_1 and S_2) nor do they meet transversally, as can be seen from the coordinate representation (6.43) and (6.45), so the order matters.
One may visualize the situation as in Figure 6.4.

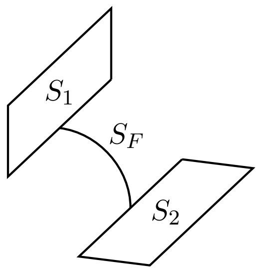

FIGURE 6.4. A schematic visualization of the three submanifolds $S_1,\ S_2,\ S_F$.

As with the lexicographic blow-up, we choose the same order and blow up the ones with bigger fibers ($S_1,\ S_2$) first. We set

$$X^2_e = [[X^2_{lex}; S_1, S_2]; S_F]_{\Pi_F}, \tag{6.47}$$

where the index Π_F on the second blow-up denotes the fact that it is parabolic. We denote the blow-down map by

$$\beta_2 : X^2_e \to X^2_{lex} \tag{6.48}$$

and the three new boundary hypersurfaces by ff_{S_1}, ff_{S_2}, ff_{S_F}, respectively. One may visualize X_e^2 as in Figure 6.5.

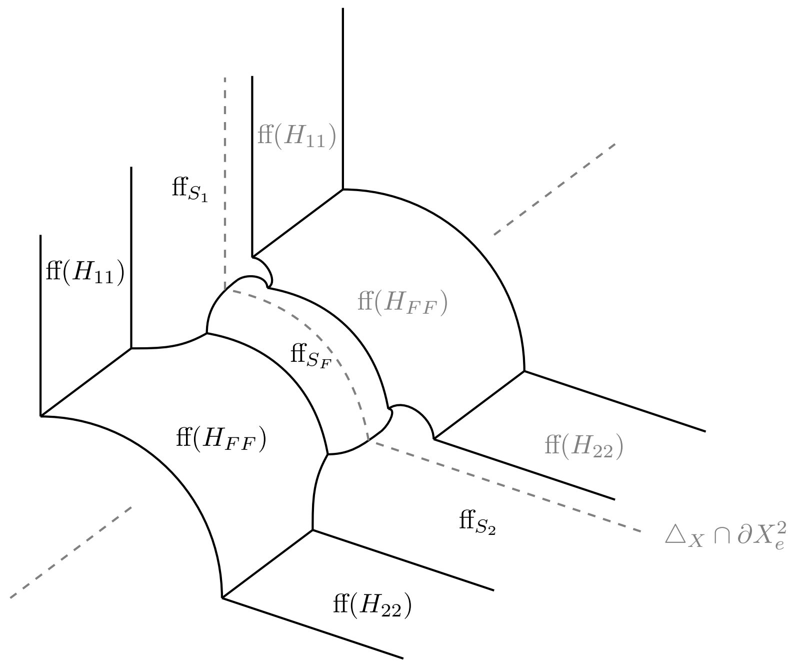

FIGURE 6.5. A schematic visualization of X_e^2.

We analyze this blow-up locally near $S_F \cap S_2$ by constructing new coordinates in which S_F and S_2 take product form. Recall the local coordinates (6.37). Also recall that the coordinates y_i, y_i' are pull-backs from the left and right factor of the base $\phi_{\mathrm{ff}_F} : \mathrm{ff}_F \to \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3)$. Recall from (6.40) that this base fibers again by $\mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \to \mathrm{Gr}(\mathbb{R}^3, 1) \times \mathrm{Gr}(\mathbb{R}^3, 1)$. Thus, we may chose y_i in such a way that y_2, y_3, y_2', y_3' are pullbacks of coordinates on the base under π_2 and, additionally, such that the vector field N_3 is locally given by ∂_{y_3}. Furthermore, we then know that

$$
\begin{aligned}
&N_2 \in \operatorname{span}_{C^\infty}\{\partial_{y_2}, \partial_{y_3}\}, \\
&N_1 \in \operatorname{span}_{C^\infty}\{\partial_{y_1}, \partial_{y_2}, \partial_{y_3}\}.
\end{aligned} \tag{6.49}
$$

Notice that since $[N_1, N_2] = N_3$ we may not trivialize N_1, N_2, N_3 simultaneously. We define new coordinates starting from (6.37) by setting

$$
\begin{aligned}
&t, \tau_1, \bar\rho = \rho - 1, \bar\psi = \psi - 1, \bar z_i = z_i - z_i', \bar z_i' = z_i + z_i', \\
&\bar y_i = y_i - y_i', \bar y_i' = y_i + y_i'.
\end{aligned} \tag{6.50}
$$

. Since locally $N_3 = \partial_{y_3}$, the vector fields on (6.38) take the local form

$$
\tau_1 \partial_{\tau_1} - \partial_{\bar\psi},\ t\partial_t - \partial_{\bar\rho},\ \partial_{\bar z_i} + \partial_{\bar z_i'},\ \tau_1 N_1,\ t\tau_1 N_2,\ t\tau_1^2(\partial_{\bar y_3} + \partial_{\bar y_3'}) \tag{6.51}
$$

and we have $\mathrm{span}_{C^\infty}\{N_1, N_2, N_3\} = \mathrm{span}_{C^\infty}\{\partial_{\bar{y}_1}, \partial_{\bar{y}_2}, \partial_{\bar{y}_3}\}$. The submanifolds take the product form

$$
\begin{aligned}
S_F &= \{\tau_1 = \bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0\}, \\
S_2 &= \{t = \bar{y}_2 = \bar{y}_3 = 0\}, \\
S_F \cap S_2 &= \{t = \tau_1 = \bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0\}, \\
\triangle_X &= \{\bar{\psi} = \bar{\rho} = \bar{y}_i = \bar{z}_i = 0\}.
\end{aligned} \tag{6.52}
$$

On ff_{S_2}: Let us compute the lift of $\mathcal{V}$ after the first blow-up of S_2, again by using projective coordinates. As before, we only need to compute the behavior near the lift of the diagonal, so a single set of projective coordinates suffices. Starting from 6.50), we get local projective coordinates on ff_{S_2} by rescaling in t:

$$\tau_1, \bar{z}_i, \bar{z}_i', \bar{y}_i', \bar{y}_1, \bar{\rho}, \bar{\psi}, t, \tilde{y}_2 = \frac{\bar{y}_2}{t}, \tilde{y}_3 = \frac{\bar{y}_3}{t}. \tag{6.53}$$

The lifts of the vector fields in (6.51) then become

$$\tau_1\partial_{\tau_1} - \partial_{\bar{\psi}},\ t\partial_t - \partial_{\bar{\rho}},\ \partial_{\bar{z}_i} + \partial_{\bar{z}_i'},\ \tau_1\tilde{N}_1,\ \tau_1\tilde{N}_2,\ \tau_1^2\underbrace{(\partial_{\tilde{z}_3} + t\partial_{\bar{z}_3'})}_{=:\tilde{N}_3}, \tag{6.54}$$

where $\mathrm{span}_{C^\infty}\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3\} = \mathrm{span}_{C^\infty}\{\partial_{\bar{y}_1}, \partial_{\tilde{y}_2}, \partial_{\tilde{y}_3}\}$.

On ff_{S_F} (near ff_{S_2}): Analogously, we construct new projective coordinates starting from (6.53) by rescaling by τ_1. Recall that this blow-up is parabolic, where $d\tilde{y}_3$ is given the weight 2:

$$\tau_1, \bar{z}_i, \bar{z}_i', \bar{y}_i', \bar{\rho}, \bar{\psi}, t, \hat{y}_1 = \frac{\bar{y}_1}{\tau_1}, \hat{y}_2 = \frac{\tilde{y}_2}{\tau_1}, \hat{y}_3 = \frac{\tilde{y}_3}{\tau_1^2}. \tag{6.55}$$

The lift of the vector fields in (6.54) then are

$$\tau_1\partial_{\tau_1} - \partial_{\bar{\psi}} - \hat{y}_i\partial_{\hat{y}_i},\ t\partial_t - \partial_{\bar{\rho}},\ \partial_{\bar{z}_i} + \partial_{\bar{z}_i'},\ \hat{N}_1,\ \hat{N}_2,\ \underbrace{\partial_{\hat{z}_3} + t\tau_1^2\partial_{\bar{z}_3'}}_{=:\hat{N}_3}, \tag{6.56}$$

where $\mathrm{span}_{C^\infty}\{\hat{N}_1, \hat{N}_2, \hat{N}_3\} = \mathrm{span}_{C^\infty}\{\partial_{\hat{y}_1}, \partial_{\hat{y}_2}, \partial_{\hat{y}_3}\}$. In these projective coordinates the diagonal is given by

$$\triangle_X = \{\bar{\psi} = \bar{\rho} = \bar{y}_i = \hat{z}_i = 0\}. \tag{6.57}$$

Furthermore, $\mathrm{ff}_{S_2} = \{t = 0\}$ and $\mathrm{ff}_{S_F} = \{\tau_1 = 0\}$. As one can see, the lifted vector fields (6.56) are in fact transversal to the lift of the diagonal.

On ff_{S_1}: Again, the situation is analogous to the one at ff_{S_2}.
Lastly, we need to check that our combined blow-down map is a b-fibration.

Lemma 6.7: *Let $\pi : X^2 \to X$ be either the left or the right projection. Then the map*

$$\pi \circ \beta_1 \circ \beta_2 : X_e^2 \to X \tag{6.58}$$

is a b-fibration.

PROOF. All three maps $\pi,\ \beta_1,\ \beta_2$ are surjective b-maps and b-submersion, and thus so is their composition. By a general theorem about b-fibrations (see e.g. [19]), the only thing left to show is that no hypersurface of X_e^2 is mapped to a boundary face of codimension 2 or higher by $\pi\circ\beta_1\circ\beta_2$. A hypersurface $\mathrm{ff}(H_{ab})$ generated in the first step of the resolution is mapped to H_a or H_b (depending on which projection is used). The hypersurfaces $\mathrm{ff}(S_i)$ are mapped (under both projections) to H_i, since its image in X^2 contains the corresponding diagonal part in $H_i \times H_i$. $\square$

We summarize the needed result in a theorem.

Theorem 6.8: *The Lie algebra $\mathcal{V}_{\mathrm{SL}}$ lifts to X_e^2 from either the left or right to become smooth vector fields that are transversal to the lift of the diagonal $\triangle_X$. The two maps back from X_e^2 to X (first blowing down and then projecting from either the left or right) are b-fibrations.*

This immediately implies the following.

Corollary 6.9: *If $L \in \mathrm{Diff}_X^*$ is elliptic, then its lift to X_e^2 is transversally elliptic to $\triangle_X$.*

6.5. Pseudodifferential operators

We are now ready to define a calculus that consists of a ring of pseudodifferential operators $\Psi_X^*(X)$, which contains Diff_X^* as a subring and also contains basic parametrices for elliptic operators. It will be defined as operators on the interior of X, whose Schwartz kernels are push-forwards of distributions on X_e^2 with very simple structure.

There is however one obstacle to overcome when using Schwartz kernels as a definition. Schwartz kernels act on densities, not functions. There are several approaches to solve this: One is to fix a density μ on X from the beginning and then define the action on a smooth function f to be the action on $f\mu$. This has the disadvantage of having to choose a (arbitrary) density to start with. Another possibility is to simply define the operator to act on densities, which has the disadvantage of the resulting object being a function (and not again a density). There are several other options, the one we choose here is for the operator to act on half-densities (with the resulting object again being a half-density). For this to work, we have to regard the Schwartz kernels as distributional section of the half-density bundle $\Omega^{1/2}(X^2)$ as well.
To motivate the definition below, we consider the lift of Schwartz kernels of elements in Diff_X^* to X_e^2. The simplest such operator is the identity I. Its Schwartz kernel is the delta-function along the diagonal on X^2. However, we want to regard this as a distributional half-density. In terms of coordinates on X^2 that are chosen to be identical coordinates of X on each of the two factors as in (6.13), choose a canonical half-density

$$\mu = \sqrt{d\tau_1\ dt\ dy\ dz\ d\tau_1'\ dt'\ dy'\ dz'}. \tag{6.59}$$

This is well-defined up to a smooth positive function. Denote the Schwartz kernel of the identity operator by K_I. In local coordinates as above it takes the form

$$K_I = \delta(\tau_1 - \tau_1')\delta(t - t')\delta(y - y')\delta(z - z')\mu. \tag{6.60}$$

Any distributional half-density K on X^2 is identified with the distributional half-density $\beta^*(K)$ on X_e^2 via the blow-down map, since they are objects defined on the interior, where β is a diffeomorphism. We may calculate this lift for K_I in local coordinates as in (6.55). The Jacobi-determinant of the map $\beta_e : X_e^2 \to X^2$ is equal to $t^3\tau_1^5$ up to a smooth function that is non-vanishing on ff(S_F) and ff(S_2). Recall that t and τ_1 are locally boundary defining functions of ff(S_2) and ff(S_F), respectively. For the sake of a global definition below, let r_1, r_2, r_F denote global boundary defining functions of ff(S_1), ff(S_2), ff(S_F), respectively. Using both local coordinate systems (6.13),(6.14) it is easy to see that the Jacobi determinant of $\beta_e : X_e^2 \to X^2$ is globally given by

$$r_{\text{det}} := r_1^3 r_2^3 r_F^5 \tag{6.61}$$

up to a smooth, nowhere vanishing function. Thus the lift of μ is given by $r_{\text{det}}^{1/2}\nu$ where ν is a non vanishing standard half-density on X_e^2. The Schwartz kernel of the identity K_I lifts to

$$\begin{aligned} \beta^* K_I &= r_{\text{det}}^{-1}\delta(\bar{\psi})\delta(\bar{\rho})\delta(\bar{y}_i)\delta(\bar{z}_i) r_{\text{det}}^{1/2}\nu \\ &= \delta(\bar{\psi})\delta(\bar{\rho})\delta(\bar{y}_i)\delta(\bar{z}_i) r_{\text{det}}^{-1/2}\nu. \end{aligned} \tag{6.62}$$

Therefore, we may interpret the identity operator as a distributional section of the singular half-density bundle $r_{\text{det}}^{-1/2}\Omega^{1/2}(X_e^2)$, where it becomes a simple delta distribution along the lifted diagonal $\triangle_e$. We may lift any $L \in \text{Diff}_X^*$ to X_e^2 and interpret its kernel as a distributional section K_L of $r_{\text{det}}^{-1/2}\Omega^{1/2}(X_e^2)$. Thus, it is given by $\kappa_L r_{\text{det}}^{-1/2}\nu$, where $\kappa_L = L(\kappa_I)$ is called the *normalized kernel* of K_L with respect to a half-density $r_{\text{det}}^{-1/2}\nu$. Thus, κ_L is given by smooth multiples of derivatives of the delta function along the diagonal. This motivates the following definition of the small calculus.

Definition 6.10: *Consider kernels A that are distributional sections of the half-density bundle $r_{\text{det}}^{-1/2}\Omega^{1/2}(X_e^2)$. Thus A has the form $A = \kappa_A r_{\text{det}}^{-1/2}\nu$, where κ_A is called the normalized kernel of A with respect to the half-density ν. We set $\Psi_X^*(X, \Omega^{1/2})$ to be the space of those A, which normalized kernels κ_A are conormal to the lifted diagonal $\triangle_X \subset X_e^2$, smoothly up the front faces $ff_{S_F}, ff_{S_1}, ff_{S_2}$, and vanishes to infinite order at all other boundary hypersurfaces of X_e^2.*

Note that the class of kernels κ_A described in the definition above is a special case of a large class of function spaces with specific behavior at the boundary, called *polyhomogeneous functions*. These play a vital role when one wants to extend the calculus presented here to include more elaborate parametrices. We will prove in the next section that the space defined above is a ring, meaning it is closed under composition.
The space Ψ_X^* is filtered by the subspaces Ψ_X^m of those elements whose kernels κ_A have singularities at the diagonal of order at most m.

Next, let us define how such an $A \in \Psi_X^*$ acts on half-densities. Let γ be any half-density on X. In order to define the action of A on γ, we need to fix an auxiliary positive nowhere vanishing half-density $\bar{\gamma}$ on X. Denote by $\bar{\pi}_L, \ \bar{\pi}_R : X_e^2 \to X$ the lifts of the left and right projection $\pi_L, \pi_R : X^2 \to X$, i.e. $\bar{\pi}_{L,R} = \beta \circ \pi_{L,R}$. Since γ and $\bar{\gamma}$ are half-densities on

X, $\pi_R^*(\gamma)\pi_L^*(\bar\gamma)$ is a half-density on X^2. Its lift to X_e^2, which is given by $\bar\pi_R^*(\gamma)\bar\pi_L^*(\bar\gamma)$, is therefore a half-density on X_e^2. We now can define the action of A on γ by the equation

$$A(\gamma)\bar\gamma = (\bar\pi_L)_*(A \cdot \bar\pi_R^*(\gamma)\bar\pi_L^*(\bar\gamma)). \tag{6.63}$$

The right-hand side of this equation is a density on X. Therefore, dividing booth sides by $\bar\gamma$ yields a half density $A(\gamma) \in \Omega^{1/2}(X)$. This is defined independent of the choice of auxiliary half density $\bar\gamma$.

The symbol map ${}^{\mathrm{SL}}\sigma_m$ given by (6.17) extends to Ψ_X^m in the following way. Since the associated normalized kernel κ_A of an operator $A \in \Psi_X^m$ is a conormal distribution to the diagonal in X_e^2, the symbol map

$$\kappa_A \mapsto {}^{\mathrm{SL}}\sigma_m(\kappa_A) \in S^{\{m\}}(N^*\triangle_X; \Omega^{1/2}(N^*\triangle_X) \otimes \pi^*(\Omega^{1/2}X)) \tag{6.64}$$

is defined as by Hörmander ([10]). The bracket in the order denotes the fact that the symbols are quotients of symbols of one order lower.

Theorem 6.8 shows that $N^*\triangle_X \cong {}^{\mathrm{SL}}T^*X$, thus we may interpret the symbol as a half-density on ${}^{\mathrm{SL}}T^*X$. This yields the short exact sequence

$$0 \to \Psi_X^{m-1} \hookrightarrow \Psi_X^m \stackrel{{}^{\mathrm{SL}}\sigma_m}{\to} S^{\{m\}}({}^{\mathrm{SL}}T^*X) \to 0. \tag{6.65}$$

We now may extend the definition of an elliptic operator to Ψ_X^*:

Definition 6.11: *We say $A \in \Psi_X^m$ is elliptic, if its symbol $a = {}^{\mathrm{SL}}\sigma_m(A)$ is an invertible element of $S^{\{m\}}({}^{\mathrm{SL}}T^*X)$, i.e. if there is an element $b \in S^{\{-m\}}({}^{\mathrm{SL}}T^*X)$ such that $a \cdot b \equiv b \cdot a \equiv 1$.*

In the next section of this chapter, we will prove that the space Ψ_X^* is closed under composition and that the symbol is multiplicative in the sense that

$${}^{\mathrm{SL}}\sigma_{m+m'}(A \circ B) = {}^{\mathrm{SL}}\sigma_m(A) \cdot {}^{\mathrm{SL}}\sigma_{m'}(B). \tag{6.66}$$

For now, we use this multiplicity to prove the following main theorem of this chapter by the standard iterative inversion scheme.

Theorem 6.12: *Let $A \in \Psi_X^m$ be an elliptic pseudodifferential operator. Then there exist an operator $B \in \Psi_X^{-m}$ such that both $AB - I$ and $BA - I$ are elements of $\Psi_X^{-\infty}$. The parametrix B is unique up to an element of $\Psi_X^{-\infty}$.*

The two remainders $AB-I$ and $BA-I$ are smoothing in the interior, since their Schwartz kernels are smooth functions on X_e^2. However, their push-forward to X^2 is not smooth. Unfortunately, this implies that the remainders are not compact, as was shown for example in [23]. This is the main motivation to carry the calculus further. This is done by adding elements to Ψ_X^m that have weaker conditions imposed at the boundary faces of X_e^2. The geometric foundation for such a larger calculus is already shown in this thesis. However, since a large amount of analytical work is needed for such a calculus, it will not be carried out here. Instead, it opens the door for a great deal of future work.

For now, the construction above still misses a composition theorem and a proof of equation (6.66), which will be the topic of the next section.

6.6. Resolution of the triple space

In the previous section, we left out the proof of the fact that Ψ_X^* is closed under composition and the symbol map is multiplicative in the sense of (6.66). In order to prove such a composition theorem, we use a geometric approach of constructing a resolution of the triple product space X^3, as already used several times in similar calculi, e.g. [17], [15], [9] and many more. The idea behind using the triple space is the following: When composing to operators A, B, we may regard B as acting on the third factor and mapping to the second and A as acting on the second and mapping to the first factor. Thus $A \circ B$ acts on the third and maps to the first factor. We may write this symbolically for the Schwartz kernels $\kappa_A, \kappa_B, \kappa_{A\circ B}$ as

$$\kappa_{A\circ B} = (\pi_M)_*(\pi_R^*\kappa_A \pi_L^*\kappa_B), \tag{6.67}$$

where $\pi_L, \pi_M, \pi_R : X^3 \to X^2$ are the three projections dropping the left, middle and right factor, respectively. Using this equation, one can easily analyze the behavior of $\kappa_{A\circ B}$ using the Pullback- and Pushforward-Theorem as in [19] .

However, the Schwartz kernels of operators in Ψ_X^* are interpreted as distributions on X_e^2, rather than X^2, so the equation above is hard to analyze. To solve this, we *resolve* the triple product space, again by a series of blow-ups, into a space denoted by X_e^3 on which we may interpret the composition as a product of Schwartz kernels lifted from X_e^2. We require that the resolution space X_e^3 has interior diffeomorphic to the interior of X^3. Secondly, we require that it is equipped with three lifted projections maps $\bar{\pi}_\alpha : X_e^3 \to X_e^2$ replacing the projections in (6.67). These maps are required to be *b-fibrations* (as defined in [19]). This is necessary for the Pullback- and Pushforward-Theorem to still apply. Such a space together with the lifted projections is constructed below. We then show that they fulfill our requirements in Proposition (6.16). An immediate consequence of this construction will be the following theorem.

Theorem 6.13: *For any m, m' we have $\Psi_X^m \circ \Psi_X^{m'} \subset \Psi_X^{m+m'}$ with the symbol map being multiplicative, i.e. satisfying* (6.66).

Proof. As said above, the proof uses an analog version of (6.67) on the resolved triple space X_e^3 together with the Pullback- and Pushforward-Theorem by Melrose ([19]). The details of this approach have been carried out several times in the past, for example in [17], [15], [8] . Thus, we omit them here and carry on with the construction of X_e^3. ☐

Construction of $\boldsymbol{X_e^3}$: Recall that $X_e^2 = [X_{lex}^2; S_1; S_2; S_F]$. As a first step in the construction of X_e^3, we move to the symmetric lexicographic blow-up that we already discussed earlier, denoted by X_{lex}^3. As with the double space, there is a second step in the resolution given by the blow-up of double- and triple-fiber-diagonals.

There are three projections $\pi^\alpha : X^3 \to X^2$ for $\alpha = LM, LR, MR$ given by dropping the last, middle and first factor X, respectively. Recall that in the resolution of the double space, the second step consisted of the blow-up of (the lifts of) three submanifolds $S_i \subset H_{ii}$ for $i = 1, 2, F$, where the blow-up of S_F is quasihomogeneous. For each S_i and each of the three π_α we denote the preimage of S_i under π^α by $S_i^\alpha \subset X^3$, giving a total of nine submanifolds. We then have for example $S_F^{LM} \subset H_{FF0}$, $S_2^{MR} \subset H_{022}$ et cetera. In general, $S_i^\alpha \subset H_{abc}$ where two of the three indices a, b, c are equal to i (according to α) and the remaining one is 0.

Quasihomogeneous structure at S_F^α: The quasihomogeneous structure of $S_F \subset X^2$ simply lifts under each projection to become a quasihomogeneous structure of S_F^α. Since it is parabolic, it is simply given by a subbundle of the conormal bundle.

Each S_i^α has a non-empty intersection with precisely those H_{abc} that have two of the indices a, b, c equal to i (again according to α) and the last one being any of the four $1, 2, F, 0$. For example, S_F^{LM} has non-empty intersection with $H_{FF1}, H_{FF2}, H_{FFF}, H_{FF0}$. Thus, the preimage of each of the S_i^α under the symmetric lexicographic blow-up consists of four parts, each being a submanifold of a different front face in X_{lex}^3. These four parts are denoted by $S_i^{\alpha,j} \subset \mathrm{ff}(H_{abc})$ where $j = 0, 1, 2, F$ and two of the indices a, b, c are equal to i (according to α) and the remaining one is equal to j. In fact, $S_i^{\alpha,j}$ is the lift of $S_i^\alpha \cap H_{abc}$ (with H_{abc} as above) under symmetric lexicographic blow-up. For example, we have $S_2^{LR,F} = \beta_{lex}^*(S_2^{LR} \cap H_{2F2}) \subset \mathrm{ff}(H_{2F2})$, et cetera. For the second step in the resolution we will blow up all 36 $S_i^{\alpha,j}$, following lexicographic order with respect to i, j. Before doing so there is one more thing to notice: all $S_i^{\alpha,j}$ with $i \neq j$ lie in different boundary faces $\mathrm{ff}(H_{abc})$. However, when $i = j$ the three $S_i^{\alpha,i}$ (for fixed i) all lie in $\mathrm{ff}(H_{iii})$. It is easy to see that the intersection of any two of the three $S_i^{\alpha,i}$ yields a submanifold denoted by $T_i \subset \mathrm{ff}(H_{iii})$, which is a triple-fiber diagonal.

Quasihomogeneous structure at $S_F^{\alpha,i}$ and T_F: The quasihomogeneous structure of S_F^α can be radially extended according to Definition 3.11 to each $S_F^\alpha \cap H_{abc}$. We need to check that this structure lifts to a quasihomogeneous structure of $S_F^{\alpha,i}$ under the radial blow-up of the corresponding H_{abc}. The cornormal bundles of H_{abc} and S_F^α, restricted to the conormal bundle of $S_F^\alpha \cap H_{abc}$, intersect only in $\{0\}$. Therefore, the radial quasihomogeneous structure of H_{abc} and the radial extension of the quasihomogeneous structure to $S_F^{\alpha,i}$ intersect cleanly according to Theorem 3.7. This theorem then implies that we can lift the quasihomogeneous structure $S_F^{\alpha,i}$ under the blow-up of H_{abc}. Lastly, we construct the quasihomogeneous structure of T_F. Recall that T_F is the intersection of any two of the three $S_F^{\alpha,F}$. Since the quasihomogeneous structures of $S_F^{\alpha,F}$ are parabolic, they are simply given by subbundles of their conormal bundles. We set the quasihomogeneous structure at T_F to be again parabolic and defined by the sum of the three bundles of $S_F^{\alpha,F}$ restricted to T_F. It is easy to see that this quasihomogeneous structure intersects cleanly with any of the three quasihomogeneous structures of $S_F^{\alpha,F}$. The blow-ups of $S_F^{\alpha,i}$ and T_F below are quasihomogeneous according to these structures. All other blow-ups are radial. Theorems 3.9 and 3.10 assure that normal commutativity results apply.

We are now ready to define the second step of our resolution:

Definition 6.14: *With the notation above, we define for fixed* $i \in \{1, 2, F\}$, $j \in \{1, 2, F, 0\}$ *the following lists of submanifolds*

$$\mathcal{S}_i^{*,j} := \begin{cases} S_i^{LM,j}; S_i^{LR,j}; S_i^{MR,j} & (j \neq i), \\ T_i; S_i^{LM,i}; S_i^{LR,i}; S_i^{MR,i} & (j = i). \end{cases} \tag{6.68}$$

For fixed $i = 1, 2, F$ *we define the lists*

$$\mathcal{S}_i^{**} := \mathcal{S}_i^{*,1}; \mathcal{S}_i^{*,2}; \mathcal{S}_i^{*,F}; \mathcal{S}_i^{*,0}. \tag{6.69}$$

We then set the second resolution of the triple space to

$$X_e^3 := [X_{lex}^3; \mathcal{S}_1^{**}; \mathcal{S}_2^{**}; \mathcal{S}_F^{**}]. \tag{6.70}$$

As a first step, we show that this is in fact symmetric in α, meaning that the space X_e^3 is independent of the order of the three submanifolds $S_i^{LM,j}; S_i^{LR,j}; S_i^{MR,j}$ in the definition above. This is a consequence of the following lemma, that is purposely formulated stronger then needed here for later use.

Lemma 6.15: *For* $i \neq j$ *any two* $S_i^{\alpha,j}$, $S_{i'}^{\beta,j'}$ *are separated by the boundary face* H_{abc} *that contains* $S_i^{\alpha,j}$*, as in Lemma 3.13. For example,* $S_1^{LM,F}$, $S_F^{MR,F}$ *are separated by* H_{11F}*. For* $i = j$ *any two* $S_i^{\alpha,i}$, $S_i^{\beta,i}$ *are separated by* T_i*.*

Proof. The second statement is clear by definition of T_i. Notice that in the first statement i and i' (and j and j') need not be identical, even though that would suffice in order to show that X_e^3 is symmetric in α. Recall that we need to show that at any $p \in S_i^{\alpha,j} \cap S_{i'}^{\beta,j'}$ we have $T_p S_i^{\alpha,j} \subset T_p H_{abc} \subset T_p S_i^{\alpha,j} + T_p S_{i'}^{\beta,j'}$. Also recall that $S_i^{\alpha,j}$ is the fiber diagonal corresponding to the fibration ϕ_i of H_i in the two factors of corresponding to α. Restricted to $S_i^{\alpha,j} \cap S_{i'}^{\beta,j'}$, the two fibrations $\phi_i, \phi_{i'}$ either coincide or form a tower. Now the statement of the lemma is a straightforward calculation in local coordinates. In order to not introduce any unnecessary and cumbersome notation, we only do so in an example. The general case works completely analogously. Consider $S_1^{LM,F} \subset H_{11F}$ and $S_F^{MR,F} \subset H_{FFF}$. On $H_1 \cap H_F$ we have $\phi_1 = \phi_F \circ \pi_{F,1}$, thus we may choose local coordinates τ_1, τ_F, x, y, z such that x is tangent to the fibers of ϕ_F and x, y are tangent to the fibers of ϕ_1. Lifting these coordinates (together with their dashed and double dashed counterparts) to the triple space, we locally get $H_{11F} = \{\tau_1 = \tau_1' = \tau_F'' = 0\}$, $S_1^{LM,F} = \{\tau_1 = \tau_1' = \tau_F'' = 0, x = x', y = y'\}$, $S_F^{MR,F} = \{\tau_F = \tau_F' = \tau_F'' = 0, x' = x''\}$, showing that H_{11F} separates $S_1^{LM,F}$ and $S_F^{MR,F}$. □

We are now ready to construct the lifted projections $\pi_r^\alpha : X_e^3 \to X_e^2$ and show that they are b-fibrations in the next proposition.

Proposition 6.16: *For each* $\alpha = LM, LR, MR$ *there is a b-fibration* $\pi_r^\alpha : X_e^3 \to X_e^2$ *fixed by the condition that it commutes with the corresponding projections*

$$\begin{array}{ccccc} X_e^3 & \longrightarrow & X_{lex}^3 & \longrightarrow & X^3 \\ \Big\downarrow{\scriptstyle \pi_r^\alpha} & & \Big\downarrow{\scriptstyle \pi_{lex}^\alpha} & & \Big\downarrow{\scriptstyle \pi^\alpha} \\ X_e^2 & \longrightarrow & X_{lex}^2 & \longrightarrow & X^2. \end{array} \tag{6.71}$$

Proof. By symmetry, we only need to construct π_r^α for $\alpha = LM$. The main idea of the construction is to successively commute the blow-ups in (6.70) such that all blow-ups related to $\alpha = LM$ are made first, which will be shown in equation (6.85) As we will see, this will allow for a straightforward definition of π_r^α as a composition of the collective blow-down of all remaining blow-ups together with a projection.
We introduce the notation

$$\begin{aligned} \mathcal{S}_i^{LM,*} &:= S_i^{LM,1}; S_i^{LM,2}; S_i^{LM,F}; S_i^{LM,0}, \\ \mathcal{S}_i^{\setminus LM} &:= \mathcal{S}_i^{**} \setminus \mathcal{S}_i^{LM}. \end{aligned} \tag{6.72}$$

The list $\mathcal{S}_i^{LM,*}; \mathcal{S}_i^{\setminus LM}$ is therefore a reordering of the list $\mathcal{S}_i^{**}$.

As a first step, we will show that

$$X_e^3 \cong [X_{lex}^3; \mathcal{S}_1^{LM,*}; \mathcal{S}_2^{LM,*}; \mathcal{S}_F^{LM,*}; \mathcal{R}_1] \tag{6.73}$$

with $\mathcal{R}_1 = \mathcal{S}_1^{\setminus LM}; \mathcal{S}_2^{\setminus LM}; \mathcal{S}_F^{\setminus LM}$. For this, we need to 'move' all $S_i^{LM,j}$ to the front, starting with the first one ($S_1^{LM,1}$). Notice that any two $S_i^{\alpha,j}$, $S_{i'}^{\beta,j'}$ with $\alpha \neq \beta$ have been separated as shown in the previous lemma. Furthermore, T_i and $S_i^{LM,i}$ commute since $T_i \subset S_i^{LM,i}$. Any other $S_i^{LM,j}$ (fixed i, $j \neq i$) that is blown up afterwards then commutes with T_i since $S_i^{LM,i}$ separates them. Lastly, T_i and $S_{i'}^{LM,j}$ with $i' \geq i$ commute since they are separated by $S_i^{LM,j}$, as shown in (6.73).

Next, we continue to move the $S_i^{LM,j}$ further to the front by commuting them with as many blow-ups of X_{lex}^3 as possible. Recall the definition of X_{lex}^3 and $\mathcal{C}, \mathcal{E}, \mathcal{E}^\alpha$ and (6.25) where we have shown that

$$X_{lex}^3 = [X^3; \mathcal{E}^{LM}; \mathcal{C}; \mathcal{E}^{LR}; \mathcal{E}^{MR}]. \tag{6.74}$$

Plugging this in, we next show that

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; \mathcal{C}; \mathcal{S}_1^{LM,*}; \mathcal{S}_2^{LM,*}; \mathcal{S}_F^{LM,*}; \mathcal{R}_2] \tag{6.75}$$

with $\mathcal{R}_2 = \mathcal{E}^{LR}; \mathcal{E}^{MR}; \mathcal{R}_1$. This follows from the fact that the intersection of any $S_i^{LM,j}$ with an element of $\mathcal{E}^{LR}; \mathcal{E}^{MR}$ is contained in an element of $\mathcal{C}$ and thus they lift to become disjoint after blowng up all elements of $\mathcal{C}$.

Recall that $\mathcal{C}$ is ordered lexicographically, meaning whenever $H_{abc} \leq H_{a'b'c'}$, then H_{abc} is blown up first. Also recall that $S_i^{LM,j} \subset H_{iij}$. Thus, as already argued in the construction of the symmetric lexicographic resolution, whenever H_{abc} and H_{iij} are not comparable, there is a common larger element of $\mathcal{C}$ that contains their intersection and thus they may be commuted. Furthermore, if $H_{iij} \leq H_{abc}$, then the blow-up of H_{iij} separates H_{abc} and $S_i^{LM,j}$. Thus we may commute $S_i^{LM,j}$ with all H_{abc} except the ones where $H_{abc} \leq H_{iij}$. Consequently, we may commute the blow-ups of $\mathcal{C}; \mathcal{S}_1^{LM,*}; \mathcal{S}_2^{LM,*}; \mathcal{S}_F^{LM,*}$ in the following way: We introduce the notation

$$H_{ab*} := H_{ab1}; H_{ab2}; H_{abF}. \tag{6.76}$$

Then

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; H_{11*}; \mathcal{S}_1^{LM,*}; H_{22*}; \mathcal{S}_2^{LM,*}; H_{FF*}; \mathcal{S}_F^{LM,*}; \mathcal{R}_3] \tag{6.77}$$

with $\mathcal{R}_3$ defined accordingly.

We proceed by some commutations in each of the three blocks $H_{ii*}; S_i^{LM,*}$. Since they all work rather analogously, we only do one in detail. We have

$$H_{FF*}; \mathcal{S}_F^{LM,*} = H_{FF1}; H_{FF2}; H_{FFF}; S_F^{LM,1}; S_F^{LM,2}; S_F^{LM,F}; S_F^{LM,0}. \tag{6.78}$$

Repeating the argument from above, we may commute the elements of this list in (6.77) to

$$H_{FF1}; S_F^{LM,1}; H_{FF2}; S_F^{LM,2}; H_{FFF}; S_F^{LM,F}; S_F^{LM,0}, \tag{6.79}$$

with the result being again diffeomorphic to X_e^3. Next, notice that $S_F^{LM,0}$ is by definition the lift of S_F^{LM} under all previous blow-ups (in particular the blow-up of H_{FF0} in $\mathcal{E}^{LM}$) and $S_F^{LM,F}$ is precisely the lift of $S_F^{LM} \cap H_{FFF}$. Plugging this in, we can write the list (6.79) as

$$H_{FF1}; S_F^{LM,1}; H_{FF2}; S_F^{LM,2}; H_{FFF}; (S_F^{LM} \cap H_{FFF}); S_F^{LM}. \tag{6.80}$$

The lifts of these last three submanifolds under all previous blow-ups still satisfy $(S_F^{LM} \cap H_{FFF})^* = S_F^{LM*} \cap H_{FFF}^*$, since all previous blow-ups either contain both S_F^{LM} and H_{FFF} or neither. We now make use of the standard commutativity result 3.5 which lets us commute the list (6.80) to

$$H_{FF1}; S_F^{LM,1}; H_{FF2}; S_F^{LM,2}; S_F^{LM}; (S_F^{LM} \cap H_{FFF}); H_{FFF}, \tag{6.81}$$

without changing the resulting space X_e^3. Repeating the same argument twice we end up with

$$S_F^{LM}; (S_F^{LM} \cap H_{FF1}); H_{FF1}; (S_F^{LM} \cap H_{FF2}); H_{FF2}; (S_F^{LM} \cap H_{FFF}); H_{FFF}. \tag{6.82}$$

We do the same for the other two blocks $H_{ii*}; S_i^{LM,*}$. In order to write down the result of this commutations, we introduce the notation

$$(H \cap S)_i^{LM} := (S_i^{LM} \cap H_{ii1}); H_{ii1}; (S_i^{LM} \cap H_{ii2}); H_{ii2}; (S_i^{LM} \cap H_{iiF}); H_{iiF}. \tag{6.83}$$

Then the commutations above yield

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; S_1^{LM}; (H \cap S)_1^{LM}; S_2^{LM}; (H \cap S)_2^{LM}; S_F^{LM}; (H \cap S)_F^{LM}; \mathcal{R}_3]. \tag{6.84}$$

Next, notice that S_2^{LM} is disjoint from all elements in $(H \cap S)_1^{LM}$. Furthermore, S_F^{LM} and H_{iij} $(i = 1, 2)$ have been separated by the blow-up of $H_{ii0} \in \mathcal{E}^{LM}$. S_F^{LM} and $(S_i^{LM} \cap H_{iij})$ have been separated by the blow-up of S_i^{LM}. Thus we may commute to obtain

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; S_1^{LM}; S_2^{LM}; S_F^{LM}; \mathcal{R}_4] \tag{6.85}$$

with $\mathcal{R}_4 = (H \cap S)_1^{LM}; (H \cap S)_2^{LM}; (H \cap S)_F^{LM}; \mathcal{R}_3$.

Denote by $\beta^{LM}_{\mathcal{R}_4} : X^3_e \to [X^3; \mathcal{E}^{LM}; S^{LM}_1; S^{LM}_2; S^{LM}_F]$ the collective blow-down of $\mathcal{R}_4$. Notice that

$$[X^3; \mathcal{E}^{LM}; S^{LM}_1; S^{LM}_2; S^{LM}_F] \cong X^2_e \times X. \tag{6.86}$$

Denote by $\tilde{\gamma} : X^2_e \times X \to X^2_e$ the projection onto the first factor. We then set

$$\pi^{LM}_e = \tilde{\gamma} \circ \beta^{LM}_{\mathcal{R}_4} : X^3_e \to X^2_e. \tag{6.87}$$

The fact that this commutes with the other blow-downs and projection maps is clear by construction. It remains to show that π^{LM}_e is a b-fibration, which follows again by noticing that the image of each hypersurface of X^3_e is a hypersurface in X^2_e. $\square$

CHAPTER 7

Calculus on $\mathrm{SL}(n,\mathbb{R})$

In this Chapter, we generalize the results of the previous Chapter to the case of $\overline{\mathrm{SL}}(n,\mathbb{R})$ for arbitrary n. All major obstacles already occurred in some form in the case of $\mathrm{SL}(3,\mathbb{R})$ and the structure of this Chapter is largely analogous. We start with recalling the geometric properties of the hd-compactification $\overline{\mathrm{SL}}(n,\mathbb{R})$ and the Lie algebra of vector fields $\mathcal{V}_e$.

7.1. Geometry of $\overline{\mathrm{SL}}(n,\mathbb{R})$

Recall from Chapter 5 that $\overline{\mathrm{SL}}(n,\mathbb{R})$ has $n-1$ boundary hypersurfaces $H_1,\dots H_{n-1}$. Each hypersurface H_q has a fibration over two copies of the Grassmannian $H_q \to \mathrm{Gr}(n-q)\times \mathrm{Gr}(n-q)$, as shown in (5.18). Composing this with the projection onto the right factor yields another fibration, denoted by

$$\begin{tikzcd} \overline{\mathbb{SH}}(n-q)\times\overline{\mathbb{SH}}(q)\times\mathrm{Gr}(n-q) \arrow[r, dash] & H_q \arrow[d, "\phi_q"] \\ & \mathrm{Gr}(n-q). \end{tikzcd} \tag{7.1}$$

Let $\bar{q}=\{q_1,\dots,q_r\}\subseteq\{1,\dots n-1\}$ be any nonempty subset. The intersection of the corresponding hypersurfaces

$$F_{\bar{q}} := H_{q_1}\cap\dots\cap H_{q_r} \tag{7.2}$$

is non-empty and a boundary face of $\overline{\mathrm{SL}}(n,\mathbb{R})$. Recall from (5.24) that $F_{\bar{q}}$ fibers over two copies of the flag manifold $\mathcal{F}_{\bar{q}}$ of type $\bar{q}$. Again, we get another fibration of $F_{\bar{q}}$ by composing this with the projection onto the right factor denoted by

$$\begin{tikzcd} \left(\prod_{i=0}^{r}\overline{\mathbb{SH}}(p_i)\right)_+\times\mathcal{F}_{\bar{q}} \arrow[r, dash] & F_{\bar{q}} \arrow[d, "\phi_{\bar{q}}"] \\ & \mathcal{F}_{\bar{q}}. \end{tikzcd} \tag{7.3}$$

Elements of the flag manifold $\mathcal{F}_{\bar{q}}=\mathcal{F}(\mathbb{R}^n,\bar{q})$ of type $\bar{q}$ are flags of the form

$$V_{q_r}\subset V_{q_{r-1}}\subset\dots\subset V_{q_1}\subset\mathbb{R}^n\ ,\ \dim(V_{q_i}=n-q_i). \tag{7.4}$$

For notational convenience we denote such flags by

$$U_{q_r}\subset U_{q_r}\oplus U_{q_{r-1}}\subset\dots\subset\bigoplus_{r\dots1}U_{q_i}\ ,\ \dim(U_{q_i})=q_{i+1}-q_i \tag{7.5}$$

where $q_r+1=n$.

Whenever $\bar{q}\supseteq\bar{q}'$ there is a natural projection $\pi_{\bar{q},\bar{q}'}:\mathcal{F}_{\bar{q}}\to\mathcal{F}_{\bar{q}'}$. Furthermore, whenever $F_{\bar{q}}\hookrightarrow F_{\bar{q}'}$ is a boundary face, we have $\bar{q}\supseteq\bar{q}'$. In this case, the two fibrations are compatible in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
F_{\bar q} & \hookrightarrow & F_{\bar q'} \\
\downarrow{\scriptstyle \phi_{\bar q}} & & \downarrow{\scriptstyle \phi_{\bar q'}} \\
\mathcal{F}_{\bar q} & \xrightarrow{\pi_{\bar q, \bar q'}} & \mathcal{F}_{\bar q'}
\end{array} \tag{7.6}
$$

Notice that the map $\pi_{\bar q,\bar q'}$ defines a fibration of the flag manifold $\mathcal{F}_{\bar q}$. The fibers are again partial flag manifolds in appropriate orthogonal complements between the U_{q_i}.
Later on we will need the following Lemma and its Corollary:

Lemma 7.1: *Fix $\bar q$ and take any $\bar q'$ with $\bar q \supseteq \bar q'$. Then the fibers of all fibrations $\mathcal{F}_{\bar q} \overset{\pi_{\bar q,\bar q'}}{\to} \mathcal{F}_{\bar q'}$ intersect cleanly.*

Proof. This again follows directly by the global existence of the vector fields N_I but it can also be shown directly from the definition of the $\pi_{\bar q,\bar q'}$. Take a point $p \in \mathcal{F}_{\bar q}$ with fiber S of $\pi_{\bar q,\bar q'}$ running through it. Then S is equal to all flags with identical subspaces U_q to p for all $q \in \bar q'$. These intersect cleanly. □

Corollary 7.2: *Let $\bar q \supseteq \bar q'$. Denote by $TS_{\bar q,\bar q'} \subset T\mathcal{F}_{\bar q}$ the subbundle given by the tangent space of the fiber of $\mathcal{F}_{\bar q} \overset{\pi_{\bar q,\bar q'}}{\to} \mathcal{F}_{\bar q'}$ at each point. Then for any collection of $\bar q_i' \subseteq \bar q$ the sum*

$$\sum_i TS_{\bar q,\bar q_i'} \subset T\mathcal{F}_{\bar q} \tag{7.7}$$

is a vector subbundle (of fixed dimension).

Recall form Lemma 5.15 that we are interested in resolving the Lie algebra of vector fields $\mathcal{V}_e$. Let p be a point near the boundary face $F_{\bar q} = H_{\bar q_1} \cap \cdots \cap H_{\bar q_k}$. Let $\tau_{\bar q_1}, \ldots \tau_{\bar q_k}$ be the boundary defining functions as in (5.30). Then $\mathcal{V}_e$ is locally spanned by

(1) The normal vector fields $\tau_{q_i}\partial_{\tau_{q_i}}$,
(2) The vector fields spanning the fibers of the projection onto the right flag manifold $F_{\bar q} \times [0,\varepsilon)^k \to \mathcal{F}_{\bar q}$,
(3) and the vector fields $\sigma_I N_I$ for all $I \in \mathfrak{I}_{\bar q}$.

The vector fields N_I where defined for each interval $I \in \mathfrak{I}$ of integers between 1 and $n-1$. They are (semi) globally defined vector fields near the boundary of $\overline{\mathrm{SL}}(n,\mathbb{R})$. When restricted $F_{\bar q}$, these vector fields are non-zero if and only if $I \cap \bar q \neq \emptyset$. The set of all such intervals is denoted by $\mathfrak{I}_{\bar q}$. The projection of these N_I, $i \in \mathfrak{I}_{\bar q}$, to the base $\mathcal{F}_{\bar q}$ of $\phi_{\bar q}$, span the tangent space $T\mathcal{F}_{\bar q}$. They satisfy

$$[N_I, N_J] = \begin{cases} N_{I\setminus J} & \text{if } J \subset I, \\ N_{I\cup J} & \text{if } I \text{ and } J \text{ are adjacent but non overlapping,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

7.2. The resolved single space

As in the case of $\overline{\mathrm{SL}}(3)$ the different fibrations ϕ_q of the hypersurfaces H_q do not form a *iterated fibration structure.* Therefore we resolve the single space. In this case, we need to perform the total boundary blow-up of $\overline{\mathrm{SL}}(n,\mathbb{R})$, denote by

$$X = \overline{\mathrm{SL}}(n,\mathbb{R})_{\mathrm{tb}}. \tag{7.9}$$

The total boundary blow-up of a manifold with corners M consist of blowing up all of its boundary faces $F \in \mathcal{M}(M)$ of codimension at least 2[1] in the following order: Start with all boundary faces of maximal codimension k. By definition, these are manifolds without boundary and disjoint from each other. Therefore one can blow them up in any order, resulting in the same space. The boundary faces of codimension $k-1$ of M lift under theses blow-ups and become disjoint from each other. Thus they can be blown up in any order. Continuing in this matter yields the total boundary blow-up. The details of this construction can be found in [22].
In our case the resulting space X has boundary hypersurfaces $H_{\bar{q}}$ for each multi-index $\bar{q}$ corresponding to the blow-up of the boundary face $F_{\bar{q}}$ of $\overline{\mathrm{SL}}(n,\mathbb{R})$.

Lemma 7.3: *Denote by $S^+F_{\bar{q}}$ the normal fiber of the inwards pointing part of the spherical normal bundle of $F_{\bar{q}}$. Then the fibration $\phi_{\bar{q}}$ lifts under the total boundary blow-up to become a fibration of the hypersurface $H_{\bar{q}}$*

$$\begin{tikzcd} \left(\prod_{i=0}^r \overline{\mathbb{S}\mathrm{H}}(p_i)\right)_{+,tb} \times \mathcal{F}_{\bar{q}} \times (S^+F_{\bar{q}})_{tb} \arrow[r, dash] & H_{\bar{q}} \arrow[d, "\psi_{\bar{q}}"] \\ & \mathcal{F}_{\bar{q}} \end{tikzcd} \tag{7.10}$$

where $\left(\prod_{i=0}^r \overline{\mathbb{S}\mathrm{H}}(p_i)\right)_{+,tb}$ is the total boundary blow-up of $\left(\prod_{i=0}^r \overline{\mathbb{S}\mathrm{H}}(p_i)\right)_{+}$ and $(S^+F_{\bar{q}})_{tb}$ is the total boundary blow-up of the normal fiber $S^+F_{\bar{q}}$.

Proof. Consider a boundary face $F_{\bar{q}}$ of codimension $|\bar{q}|$. The base of the fibration (7.3) is a smooth manifold with corners, so is the factor $\mathcal{F}_{\bar{q}}$ of the normal fiber. Therefore, the collection of all boundary faces $F_{\bar{q}'} \subset \overline{\mathrm{SL}}(n,\mathbb{R})$ with fixed codimension $|\bar{q}'| > |\bar{q}|$ restricts to $F_{\bar{q}}$ to become the collection of all codimension $|\bar{q}'| - |\bar{q}|$ boundary faces of the first factor of the fiber $\left(\prod_{i=0}^r \overline{\mathbb{S}\mathrm{H}}(p_i)\right)_{+}$. Denote by $F^*_{\bar{q}}$ the lift of $F_{\bar{q}}$ under the blow-up of all $F_{\bar{q}'}$ with $|\bar{q}'| > |\bar{q}|$. Then $F^*_{\bar{q}}$ is the total space of a fibration

$$\begin{tikzcd} \left(\prod_{i=0}^r \overline{\mathbb{S}\mathrm{H}}(p_i)\right)_{+,tb} \times \mathcal{F}_{\bar{q}} \arrow[r, dash] & F^*_{\bar{q}} \arrow[d, "\phi_{\bar{q}}"] \\ & \mathcal{F}_{\bar{q}}. \end{tikzcd} \tag{7.11}$$

The front face generated by the blow-up of $F^*_{\bar{q}}$ fibers over $F^*_{\bar{q}}$ with normal fiber $S^+F_{\bar{q}}$. The remaining blow-ups of the total boundary fibration restrict to this hypersurface to resolve $S^+F_{\bar{q}}$ into its total boundary blow-up, proving the statement. $\square$

Furthermore, we get a whole family of fibrations of $H_{\bar{q}}$ by composing $\psi_{\bar{q}}$ with any of the projections $\pi_{\bar{q},\bar{q}'} : \mathcal{F}_{\bar{q}} \to \mathcal{F}_{\bar{q}'}$. These fibrations do not form a tower nor a tree, but they are

[1] Blowing up a boundary face of codimension 1 (i.e. a hypersurface) does not change the manifold, i.e. the blown-up space is diffeomorphic to the original one.

partially ordered. To give an example, take $n = 5$, $\bar{q} = \{1, 2, 4\}$, we then have:

(7.12)
$$
\begin{tikzcd}
 & H_{\bar{q}} \arrow[d] & \\
 & \mathcal{F}_{\bar{q}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{F}_{\{1,2\}} \arrow[d] \arrow[dr] & \mathcal{F}_{\{1,4\}} \arrow[dl] \arrow[dr] & \mathcal{F}_{\{2,4\}} \arrow[dl] \arrow[d] \\
\mathcal{F}_{\{1\}} & \mathcal{F}_{\{2\}} & \mathcal{F}_{\{4\}}.
\end{tikzcd}
$$

The boundary hypersurfaces $H_{\bar{q}}$ of X are partially ordered by the codimension of the original boundary face $F_{\bar{q}}$ of $\overline{\mathrm{SL}}(n, \mathbb{R})$. This partial order coincides with the partial order of the indices $\bar{q}$. A general result concerning the total boundary blow-up (see [22]) states that whenever a boundary face

(7.13)
$$F = H_{\bar{q}_1} \cap \cdots \cap H_{\bar{q}_k}$$

is non-empty, the boundary hypersurfaces $H_{\bar{q}_1}, \ldots, H_{\bar{q}_k}$ are totally ordered. Thus we may assume $\bar{q}_1 \geq \cdots \geq \bar{q}_k$. Therefore the k fibrations of F, given by the restriction of the fibration $\psi_{\bar{q}_i}$ of $H_{\bar{q}_i}$ to F, form a tower of fibrations

(7.14)
$$
\begin{tikzcd}
F \arrow[r, "\psi_{\bar{q}_1}"] \arrow[rr, bend right, "\psi_{\bar{q}_2}"'] \arrow[rrr, bend right, "\psi_{\bar{q}_3}"'] \arrow[rrrrr, bend right, "\psi_{\bar{q}_k}"'] & \mathcal{F}_{\bar{q}_1} \arrow[r, "\pi_{\bar{q}_1, \bar{q}_2}"] & \mathcal{F}_{\bar{q}_2} \arrow[r, "\pi_{\bar{q}_2, \bar{q}_3}"] & \mathcal{F}_{\bar{q}_3} \arrow[r] & \ldots \arrow[r] & \mathcal{F}_{\bar{q}_k}
\end{tikzcd}
$$

or briefly, for any $\bar{q}_i \geq \bar{q}_j$ appearing above we have

(7.15)
$$\psi_{\bar{q}_i}|_F = \pi_{\bar{q}_i, \bar{q}_j} \circ \psi_{\bar{q}_j}|_F.$$

In other words, X is a *manifold with fibered corners* as in [6].

Vector fields: Let us consider the lift of the Lie Algebra $\mathcal{V}_e$ from $\overline{\mathrm{SL}}(n, \mathbb{R})$ to X. Since $\mathcal{V}_e \subset \mathcal{V}_b$, vector fields in $\mathcal{V}_e$ lift to become smooth on X. Consider a point in a boundary face $H_{\bar{q}}$. Let $\tau_{\bar{q}}$ denote the boundary defining function of $H_{\bar{q}}$ as before. When blowing up a boundary face of a manifold with corners, the normal vector fields $\tau_{q_i} \partial_{\tau_{q_i}}$ lift to become smooth and span, over C^∞, both the normal vector field $\tau_{\bar{q}} \partial_{\tau_{\bar{q}}}$ and those vector fields that are tangent to the fibers of the spherical normal bundle. Thus the normal vector fields together with the vector fields tangent to the fibers of $\phi_{\bar{q}} : F_{\bar{q}} \to \mathcal{F}_{\bar{q}}$ lift to span, over C^∞, the normal vector fields and those tangent to the fibers of $\psi_{\bar{q}} : H_{\bar{q}} \to \mathcal{F}_{\bar{q}}$. Next consider the lifts of the vector fields N_I. Recall that I is an interval of the form $I = I_{ij} = \{n - j + 1, \ \ldots, \ n - i\}$. The vector fields N_I are not themselves element of $\mathcal{V}_e$, but $\sigma_I N_I = \tau_{n-i} \cdots \tau_{n-j+1} N_I$ are. However, the N_I are tangent to the boundary, hence they lift to smooth vector fields on X and, when restricted to $H_{\bar{q}}$ and projected to the base $\mathcal{F}_{\bar{q}}$, span the tangent space $T\mathcal{F}_{\bar{q}}$ everywhere. The only thing left to to is calculate the lift of the prefactor σ_I:

We denote by $\tau_{\bar{q}}$ a boundary defining function of the new hypersurface $H_{\bar{q}}$. This is of course not unique. One possibility is to take the lift of the sum $\tau_{\bar{q}} = \tau_{\bar{q}_1} + \cdots + \tau_{\bar{q}_k}$. However the only thing we have to assume is that $\tau_{\bar{q}}$ is a function of the τ_i on $\overline{\mathrm{SL}}(n,\mathbb{R})$, since then we still have that $N_I \tau_{\bar{q}} = 0$ for any $I, \bar{q}$. We do not need to calculate the exact lift, but only its order of vanishing at each boundary face $H_{\bar{q}}$. First, consider the lift of a single τ_k from $\overline{\mathrm{SL}}(n,\mathbb{R})$ to X. Clearly, the lift vanishes to first order at precisely those $H_{\bar{q}}$ for which $k \in \bar{q}$. Thus we get (up to higher order terms)

$$\beta^*(\sigma_I) = \prod_{k=i}^{j-1} \prod_{n-k\in\bar{q}} \tau_{\bar{q}} = \prod_{\bar{q}} \tau_{\bar{q}}^{|\bar{q}\cap I|} =: \bar{\tau}_I. \tag{7.16}$$

In other words, the order of vanishing of $\bar{\tau}_I$ at a boundary face $H_{\bar{q}}$ is given by $\#\{q_k \in \bar{q} \mid n-j < q_k \leq n-i\} = |I \cap \bar{q}|$.

Therefore all vector fields $V \in \mathcal{V}_e$ lift to X to become smooth vector fields in $\mathcal{V}_b$.

Definition 7.4: *We denote by $\mathcal{V}_{\mathrm{SL}}$ the C^∞-span of the lift of $\mathcal{V}_e$ to X.*

The calculations above show that this is a Lie algebroid and locally given in the following way:

Lemma 7.5: *Let p be an interior point of a boundary hypersurface $H_{\bar{q}}$. Then $\mathcal{V}_{\mathrm{SL}}$ is a Lie algebroid contained in $\mathcal{V}_b(X)$ and is locally spanned in by*

(1) The normal vector field $\tau_{\bar{q}}\partial_{\tau_{\bar{q}}}$,
(2) the vector fields spanning the fibers of $\psi_{\bar{q}}$,
(3) and the vector fields $\bar{\tau}_I N_I$ for all $I \in \mathfrak{I}_{\bar{q}}$.

Since the new boundary defining functions are chosen to only depend on the old ones, we still have $N_I \bar{\tau}_J = 0$ and thus the commutativity-result (5.80) lifts to become

$$[\bar{\tau}_I N_I, \bar{\tau}_J N_J] = \begin{cases} \bar{\tau}_{I\cup J} N_{I\setminus J} & \text{if } J \subset I \\ \bar{\tau}_{I\cup J} N_{I\cup J} & \text{if } I \text{ and } J \text{ are adjacent but non overlapping} \\ 0 & \text{otherwise} \end{cases}. \tag{7.17}$$

We also need to describe the behavior of $\mathcal{V}_{\mathrm{SL}}$ at boundary faces of higher codimension of X. Fortunately, there is hardly any work to do: The first two points of the Lemma above still hold at higher dimensional boundary by the general phenomena of the total boundary blow-up that the normal vector fields lift to span the normal vector fields together with those tangent to the fibers of the spherical normal bundles. Recall that at a boundary face $F = H_{\bar{q}_1} \cap \cdots \cap H_{\bar{q}_k}$ the multi-indices are totally ordered, i.e. $\bar{q}_1 \supset \cdots \supset \bar{q}_k$. The corresponding fibrations form a tower. By continuity we get the following result.

Lemma 7.6: *Let p be an interior point of a boundary face $F = H_{\bar{q}_1} \cap \cdots \cap H_{\bar{q}_k}$ where we assume $\bar{q}_1 \geq \cdots \geq \bar{q}_k$. Then $\mathcal{V}_{\mathrm{SL}}$ is locally spanned near p by*

(1) The normal vector field $\tau_{\bar{q}_1}\partial_{\tau_{\bar{q}_1}}, \ldots, \tau_{\bar{q}_k}\partial_{\tau_{\bar{q}_k}}$
(2) The vector fields spanning the fibers of $\psi_{\bar{q}_1}$,
(3) and the vector fields $\bar{\tau}_I N_I$ for $I \in \mathfrak{I}_{\bar{q}_1}$.

Let $1 < l \leq k$. Then the vector fields spanning the fibers of $\psi_{\bar{q}_1}$ together with those N_I with $I \in \mathfrak{I}_{\bar{q}_1} \setminus \mathfrak{I}_{\bar{q}_l}$ span the vector fields tangent to the fibers of $\psi_{\bar{q}_l}$.

As in the case of SL(3), $\mathcal{V}_{\mathrm{SL}}$ is a Lie algebroid and thus has a naturally associated vector bundle ${}^{\mathrm{SL}}TX$ such that $\mathcal{V}_{\mathrm{SL}} = C^\infty(X, {}^{\mathrm{SL}}TX)$. Again, the bundle ${}^{\mathrm{SL}}TX$ is equipped with a map into the b-vector bundle $\iota_X : {}^{\mathrm{SL}}TX \to {}^{b}TX$ induced by the inclusion $\mathcal{V}_{\mathrm{SL}} \hookrightarrow \mathcal{V}_b$.

Local coordinates: We have a need for adapted local coordinates near $p \in F$. Recall equation (7.14). We may chose local coordinates $(y_1, \ldots, y_k, z)$ on F centered at p that locally decompose all fibrations in (7.14)

$$\begin{tikzcd}
F \arrow[r, "\psi_{\bar{q}_1}"] \arrow[d, dash] & \mathcal{F}_{\bar{q}_1} \arrow[r, "\pi_{\bar{q}_1,\bar{q}_2}"] \arrow[d, dash] & \mathcal{F}_{\bar{q}_2} \arrow[r] \arrow[d, dash] & \ldots \arrow[r] & \mathcal{F}_{\bar{q}_{k-1}} \arrow[r, "\pi_{\bar{q}_{k-1},\bar{q}_k}"] \arrow[d, dash] & \underset{y_k}{\mathcal{F}_{\bar{q}_k}}. \\
\underset{z}{\mathfrak{F}_{\bar{q}_1}} & \underset{y_1}{\mathfrak{F}_{\bar{q}_1,\bar{q}_2}} & \underset{y_2}{\mathfrak{F}_{\bar{q}_2,\bar{q}_3}} & & \underset{y_{k-1}}{\mathfrak{F}_{\bar{q}_{k-1},\bar{q}_k}} &
\end{tikzcd} \tag{7.18}$$

Combined with the $\tau_{\bar{q}}$, these form local coordinates on X near p. We may chose them in such a way that

$$\operatorname{span}\{\partial_{y_l}\} = \operatorname{span}\{N_I \mid I \in \mathfrak{I}_{\bar{q}_l} \setminus \mathfrak{I}_{\bar{q}_{l+1}}\} \text{ for all } 1 \leq l \leq k. \tag{7.19}$$

As in the case of SL(3), we set $\mathrm{Diff}^*_{\mathrm{SL}}$ to be the universal enveloping algebra of $\mathcal{V}_{\mathrm{SL}}$. It is locally given by

$$L \in \mathrm{Diff}^*_{\mathrm{SL}} \Leftrightarrow L = \sum_{a_i,\beta,c_I} (\tau_{\bar{q}_i}\partial_{\tau_{\bar{q}_i}})^{a_i} (\partial_z)^\beta (\bar{\tau}_I N_I)^{c_I} \tag{7.20}$$

$\mathrm{Diff}^*_{\mathrm{SL}}$ is filtered by differential operators of degree at most m, denoted by $\mathrm{Diff}^m_{\mathrm{SL}}$. As before in the case of SL(3), $\mathrm{Diff}^m_{\mathrm{SL}}$ has naturally occurring sub-algebras given by the enveloping algebras of the subbundles $\mathcal{V}_e$ and $\mathcal{V}_{ri}$ lifted to X, denoted by

$$\mathrm{Diff}^m_{ri} \subset \mathrm{Diff}^m_e \subset \mathrm{Diff}^m_{\mathrm{SL}}. \tag{7.21}$$

Each $L \in \mathrm{Diff}^m_{\mathrm{SL}}$ has an associated symbol ${}^{\mathrm{SL}}\sigma_m(L) \in C^\infty(X, {}^{\mathrm{SL}}T^*X)$ defined analogously in comparison to (6.17).

Definition 7.7: *We say $L \in \mathrm{Diff}^m_{\mathrm{SL}}$ is elliptic, if for all $p \in X$ the symbol satisfies ${}^{\mathrm{SL}}\sigma_m(L)(p, \cdot) \neq 0$ on ${}^{\mathrm{SL}}T^*_pX \setminus \{0\}$.*

Following the scheme presented in the previous chapter, we continue to construct a resolution of the double space X^2, denoted by X^2_e, on which the Schwartz kernels of operators in $\mathrm{Diff}^m_{\mathrm{SL}}$ take a relatively simple form and thus admits the construction of a pseudodifferential calculus.

7.3. The resolved double space

We follow the construction from the case of $\overline{\mathrm{SL}}(3)$ closely. From now on, we will only consider the lift (from either the left or the right) of $\mathcal{V}_{\mathrm{SL}}$. As before, for any local coordinate function x on X we denote the pullback from the left and right to X^2 by x and x'.
Set $\triangle_U = \{(p,p) \mid p \in U\} \subset X^2$ for any $U \subset X$. We then have

$$\triangle_X \cap \partial X^2 = \bigcup_{\bar{q}} \triangle_{H_{\bar{q}}} \tag{7.22}$$

where $\triangle_{H_{\bar{q}}} \subset H_{\bar{q}}^2$. As before, we move to the symmetric lexicographic blow-up defined as in Section 6.3 as a first step in the resolution:

$$\beta_{lex} : X^2_{lex} \to X^2. \tag{7.23}$$

The codimension 2 faces that are blown up in this step are now indexed by two multi-indices $\bar{q}, \bar{w} \subset \{1, \dots n-1\}$. We denote the newly generated front face, generated by the blow-up of $H_{\bar{q},\bar{w}}$, by $\mathrm{ff}(H_{\bar{q},\bar{w}})$. We continue to calculate the lift of $\mathcal{V}_{\mathrm{SL}}$ near the lift of the diagonal. The lift of the diagonal only intersects the boundary in the hypersurfaces $\mathrm{ff}(H_{\bar{q},\bar{q}})$.
Let $F = H_{\bar{q}_1} \cap \dots \cap H_{\bar{q}_k}$ be a boundary face of X as in Lemma 7.6. Let p be an interior point of $F \times F \subset X^2$. We then may chose local coordinates $\tau_{\bar{q}_i}, \tau'_{\bar{q}_i}, y_1, y'_1, \dots y_k, y'_k, z, z'$ as in (7.18) centered at $p \in X^2$. The Lie algebra $\mathcal{V}_{\mathrm{SL}}$ is then locally spanned by

$$\tau_{\bar{q}_i}\partial_{\tau_{\bar{q}_i}}, \partial_{z_j}, \bar{\tau}_I N_I \ (I \in \mathfrak{I}_{\bar{q}_1}). \tag{7.24}$$

In local the coordinates above, $H_{\bar{q}_1,\bar{q}_1}$ is given by $\{\tau_{\bar{q}_1} = \tau'_{\bar{q}_1} = 0\}$, thus we may get projective coordinates by replacing $\tau'_{\bar{q}_1}$ with $\xi_{\bar{q}_1} = \frac{\tau'_{\bar{q}_1}}{\tau_{\bar{q}_1}}$. Doing this successively for the blow-ups of all $H_{\bar{q}_i,\bar{q}_i}$, we get coordinates on X^2_{lex} whose domain cover an open neighborhood of $\beta^{-1}_{lex}(p)$ given by

$$\tau_{\bar{q}_i}, \xi_{\bar{q}_i} = \frac{\tau'_{\bar{q}_i}}{\tau_{\bar{q}_i}}, y_1, y'_1, \dots y_k, y'_k, z, z' \tag{7.25}$$

The lift of the vector fields (7.24) is locally given by

$$\tau_{\bar{q}_i}\partial_{\tau_{\bar{q}_i}} - \xi_{\bar{q}_i}\partial_{\xi_{\bar{q}_i}}, \partial_{z_j}, \bar{\tau}_I N_I. \tag{7.26}$$

The lift of the diagonal is locally given by

$$\triangle_X = \{\xi_{\bar{q}_i} = 1, y_i = y'_i, z = z'\}. \tag{7.27}$$

Thus, on $\mathrm{ff}(H_{\bar{q}_l,\bar{q}_l})$ near p the flow-out of $\triangle_X$ under $\mathcal{V}_e$ is given by

$$S_{\bar{q}_l} = \{\tau_{\bar{q}_l} = 0,\ y_{l+1} = y'_{l+1}, \dots, y_k = y'_k, z = z'\} \subset \mathrm{ff}(H_{\bar{q}_l,\bar{q}_l}). \tag{7.28}$$

This equals the fiber diagonal under the lifted fibration $\mathrm{ff}(H_{\bar{q}_l,\bar{q}_l}) \to \mathcal{F}_{\bar{q}_l} \times \mathcal{F}_{\bar{q}_l}$. Thus on each $\mathrm{ff}(H_{\bar{q},\bar{q}})$ the flow out of $\triangle_X \cap \mathrm{ff}(H_{\bar{q},\bar{q}})$ under $\mathcal{V}_{\mathrm{SL}}$ is precisely the fiber diagonal

$$S_{\bar{q}} := \psi^{-1}_{\bar{q},\bar{q}}(\triangle_{\mathcal{F}_{\bar{q}}}) \subset \mathrm{ff}(H_{\bar{q}_l,\bar{q}_l}) \tag{7.29}$$

where $\psi_{\bar{q},\bar{q}} : \mathrm{ff}(H_{\bar{q}_l,\bar{q}_l}) \to \mathcal{F}_{\bar{q}} \times \mathcal{F}_{\bar{q}}$ is the lift of the fibration $H_{\bar{q}_l,\bar{q}_l} \to \mathcal{F}_{\bar{q}} \times \mathcal{F}_{\bar{q}}$ given on each factor by (7.10). It is the lift of the fiber diagonal in $H_{\bar{q}_l,\bar{q}_l}$ under the lexicographic blow-up. As a second step of the resolution, we will blow up all $S_{\bar{q}}$, again in a symmetric

lexicographic order according to the indices $\bar{q}$.
These blow-ups need to be quasihomogeneous, since we aim to resolve the $\bar{\tau}_I N_I$, which vanish to different orders. Thus we need to construct quasihomogeneous structures at each $S_{\bar{q}}$ and show that they lift when blowing up all $S_{\bar{q}}$ in lexicographic order.

Quasihomogeneous structure at $S_{\bar{q}}$: Recall that the vector fields N_I as in Lemma 7.5 span, at each point, the normal space of $S_{\bar{q}}$. Thus we may define a quasihomogeneous structure at $S_{\bar{q}}$ by associating negative weights to each of these N_I as described in Proposition 2.39. The weight we associate to N_I at $S_{\bar{q}}$ is given by $-|\bar{q} \cap I|$. We denote this quasihomogeneous structure by $\mathcal{F}^{(m)}_{\bar{q}}$.

Denote by $(S_{\bar{q}})_{\bar{q}}$ any increasing ordering of the $S_{\bar{q}}$. We then define the second step in our resolution by

$$X^2_e = [X^2_{lex}; (S_{\bar{q}})_{\bar{q}}] \tag{7.30}$$

where the blow-up of each $S_{\bar{q}}$ is quasihomogeneous with respect to its quasihomogeneous structure defined above. Whenever some of the $S_{\bar{q}}$ have nonempty intersection, they are totally ordered. This X^2_e is well-defined. Lastly, we need to calculate that this does in fact resolve $\mathcal{V}_X$. We start with the local coordinates (7.25) near $\mathrm{ff}(H_{\bar{q}_1,\bar{q}_1}) \cap \cdots \cap \mathrm{ff}(H_{\bar{q}_k,\bar{q}_k})$. Recall that we assume $\bar{q}_1 \geq \bar{q}_2 \geq \cdots \geq \bar{q}_k$, thus $S_{\bar{q}_k}$ is blown up first, followed by the remaining $S_{\bar{q}_i}$ in decreasing order. We define new local coordinates from (7.25) by setting

$$\tau_{\bar{q}_i},\ \xi_{\bar{q}_i}\ ,\ \begin{matrix} \bar{y}_1 := y_1 - y_1' \\ \bar{y}_1' := y_1 + y_1' \end{matrix}, \ldots \begin{matrix} \bar{y}_k := y_k - y_k' \\ \bar{y}_k' := y_k + y_k' \end{matrix},\ \begin{matrix} \bar{z} := z - z' \\ \bar{z}' := z + z' \end{matrix} \tag{7.31}$$

In these, we locally have

$$\begin{aligned} \triangle_X &= \{\xi_{\bar{q}_i} = 1,\ \bar{y}_i = 0, \bar{z} = 0\}, \\ S_{\bar{q}_l} &= \{\tau_{\bar{q}_i} = 0,\ \bar{y}_{l+1} = \ldots \bar{y}_k = \bar{z} = 0\}. \end{aligned} \tag{7.32}$$

Equation (7.19) becomes

$$\begin{aligned} &\mathrm{span}\{\partial_{\bar{z}_i} + \partial_{\bar{z}_i'}\} = \mathrm{span}\{N_I \mid I \in \mathfrak{I}_{\bar{q}_k}\} \\ &\mathrm{span}\{\partial_{\bar{y}_l} + \partial_{\bar{y}_l'}\} = \mathrm{span}\{N_I \mid I \in \mathfrak{I}_{\bar{q}_{l-1}} \setminus \mathfrak{I}_{\bar{q}_l}\} \text{ for all } 1 \leq l \leq k \end{aligned} \tag{7.33}$$

We first blow up $S_{\bar{q}_k} = \{\tau_{\bar{q}_k} = 0, \bar{z} = 0\}$ with respect to the quasihomogeneous structure $\mathcal{F}^{(m)}_{\bar{q}_k}$. We define new projective coordinates on $\mathrm{ff}_{S_{\bar{q}_k}}$ starting with (7.31) and scaling by $\tau_{\bar{q}_k}$:

$$\tau_{\bar{q}_i},\ \xi_{\bar{q}_i}, \bar{y}_1,\ \bar{y}_1', \ldots \bar{y}_k^* = \frac{\bar{y}_k}{\tau_{\bar{q}_k}^{\kappa_*}}, \bar{y}_k', \bar{z}, \bar{z}' \tag{7.34}$$

where κ_* is a suitable power. Recall that the negative weight associated to N_I on $S_{\bar{q}_k}$ was $-|I \cap \bar{q}_k|$. Thus the vector field $\tau_{\bar{q}_k}^{|I \cap \bar{q}_k|} N_I$ has weight zero and thus lifts to a smooth vector field, which we denote by N_I^*. Also recall that $\tau_I = \prod_{\bar{q}} \tau_{\bar{q}}^{|\bar{q} \cap I|}$. Thus the vector fields (7.26) lift to become

$$\tau_{\bar{q}_i} \partial_{\tau_{\bar{q}_i}} - \xi_{\bar{q}_i} \partial_{\xi_{\bar{q}_i}}, \partial_{z_i}, \prod_{\bar{q} \neq \bar{q}_k} \tau_{\bar{q}}^{|\bar{q} \cap I|} N_I^*. \tag{7.35}$$

The lift of $S_{\bar{q}_{k-1}}$ is given by $S_{\bar{q}_{k-1}} = \{\tau_{\bar{q}_{k-1}} = 0,\ \bar{y}_k^* = 0,\ \bar{z} = 0\}$. The lifted vector fields N_I^* still satisfy the commutativity result (5.79). Thus they still define quasihomogeneous structures at the other $S_{\bar{q}'}$. We now may get projective coordinates on $\mathrm{ff}(S_{\bar{q}_{k-1}})$ by scaling by $\tau_{\bar{q}_{k-1}}$, giving new coordinates $\bar{y}_{k-1}^* = \frac{\bar{y}_{k-1}}{\tau_{\bar{q}_{k-1}}^{\kappa*}}$ and (by abuse of notation) $\bar{y}_k^* = \frac{\bar{y}_k^*}{\tau_{\bar{q}_{k-1}}^{\kappa*}}$. Following the same pattern, we get local projective coordinates

$$\tau_{\bar{q}_i},\ \xi_{\bar{q}_i}, \bar{y}_1^*,\ \bar{y}_1', \ldots \bar{y}_k^*, \bar{y}_k', \bar{z}^*,\ \bar{z}'. \tag{7.36}$$

on the final space X_e^2 where in each step, when blowing up $S_{\bar{q}_l}$, the coordinates $\bar{z}^*,\ \bar{y}_l^*,\ \ldots,\ \bar{y}_r^*$ are scaled by a suitable power of $\tau_{\bar{q}_l}$ with respect to the quasihomogeneous structure of $S_{\bar{q}_l}$. The lift of the vector fields (7.35) take the form

$$\tau_{\bar{q}_i}\partial_{\tau_{\bar{q}_i}} - \xi_{\bar{q}_i}\partial_{\xi_{\bar{q}_i}}, \partial_{z_i}, \prod_{\bar{q} \notin \{\bar{q}_1, \ldots \bar{q}_k\}} \tau_{\bar{q}}^{|\bar{q} \cap I|} N_I^* \tag{7.37}$$

where (again by slight abuse of notation) N_I^* denotes the lift of of $\tau_{\bar{q}_l}^{|I \cap \bar{q}_l|} N_I^*$ when blowing up $S_{\bar{q}_l}$ after each step. After all blow-ups the lift of the diagonal takes the form

$$\triangle_X = \{\xi_{\bar{q}_i} = 1,\ \bar{y}_i^* = 0, \bar{z}^* = 0\} \tag{7.38}$$

and we have

$$\mathrm{span}\{\partial_{\bar{y}_l^*} + \partial_{\bar{y}_l'}\} = \mathrm{span}\{N_I^* \mid I \in \mathfrak{I}_{\bar{q}_l} \setminus \mathfrak{I}_{\bar{q}_{l+1}}\} \text{ for all } 1 \leq l \leq k. \tag{7.39}$$

This certainly implies the following result:

Proposition 7.8: *The vector fields in the Lie algebra $\mathcal{V}_X$ lift (from either the left or the right) to become smooth b-vector fields on X_e^2 and are transversal to the lift of the diagonal $\triangle_X \subset X_e^2$. The two maps back from X_e^2 to X (first blow down to X^2 followed by projecting from either left or right) are b-fibrations.*

Proof. The only thing not shown in the calculation above is the fact that the two lifted projections $\pi_{L,R} \circ \beta : X_e^2 \to X$ are b-fibrations. This follows again form the fact that it is a b-submersion and the fact that each hypersurface of X_e^2 is mapped to a hypersurface of X. $\square$

Corollary 7.9: *If $L \in \mathrm{Diff}_{\mathrm{SL}}^*$ is elliptic then its lift to X_e^2 is transversally elliptic to $\triangle_X$.*

7.4. Pseudodifferential operators

As for SL(3), we want to define pseudodifferential operators in terms of their Schwartz kernels, which we will view as distributional sections of a singular half-density bundle on X_e^2. We need to understand what singular factor we need to pull out in order for differential operators to have (normalized) Schwartz kernels that are simple derivatives of delta distributions on the lifted diagonal $\triangle_X \subset X_e^2$. For this, we analyze the lift of the Identity kernel, again interpreted as a half-density. In terms of local coordinates on X^2 that are given by identical ones on each factor as in (7.18) consider a canonical half-density on X^2 given by $\mu = \sqrt{d\tau_{\bar{q}_i} dy dz d\tau_{\bar{q}_i}' dy' dz'}$. Then the Schwartz kernel K_I of the identity operator is given locally as

$$K_I = \delta(\tau_{\bar{q}_I} - \tau_{\bar{q}_I})\delta(y - y')\delta(z - z')\mu. \tag{7.40}$$

Next, we compute the order of vanishing of the Jacobi Determinant of the blow-down map $\beta : X_e^2 \to X^2$ at the different boundary hypersurfaces. Recall from the local coordinate computations (7.31), (7.34) and (7.36) that $\tau_{\bar{q}_i}$ was locally a boundary defining function of the front face $\mathrm{ff}(S_{\bar{q}_i})$. Let $r_{\bar{q}}$ denote a global boundary defining function of $\mathrm{ff}(S_{\bar{q}})$. Then from the local coordinate calculations (7.31) - (7.36) one can see that the Jacobi Determinant is given by

$$r_{\det} := \prod_{\bar{q}} r_{\bar{q}}^{\left(1+\sum_{I\in\mathfrak{I}}|I\cap\bar{q}|\right)} \tag{7.41}$$

up to a smooth function that is non vanishing on any $\mathrm{ff}(S_{\bar{q}})$. Note that the 1 in the exponent is a result of the blow-up of $H_{\bar{q},\bar{q}}$ and each $|I \cap \bar{q}|$ coming from the projective coordinate change in the coordinate function corresponding to N_I. [2]
Thus the lift of μ is given by $r_{\det}^{1/2}\nu$ where ν is a non vanishing standard half-density on X_e^2. The Schwartz kernel of the identity operator then lifts to

$$\begin{aligned} \beta^* K_I &= r_{\det}^{-1}\delta(\xi_{\bar{q}_i} - 1)\delta(\bar{y}_i^*)\delta(\bar{z}^*)r_{\det}^{1/2}\nu \\ &= \delta(\xi_{\bar{q}_i} - 1)\delta(\bar{y}_i^*)\delta(\bar{z}^*)r_{\det}^{-1/2}\nu. \end{aligned} \tag{7.42}$$

Therefore we may interpret the identity as a distributional section of the singular half-density bundle $r_{\det}^{-1/2}\Omega^{1/2}(X_e^2)$ where it becomes a simple delta distribution along the lifted diagonal. As before, we now may interpret any $L \in \mathrm{Diff}_e^*$ as such a section with the normalized kernel being a sum of derivatives of delta distributions along the diagonal. We may now define the small calculus:

Definition 7.10: *Consider kernels A that are distributional sections of the half-density bundle $r_{\det}^{-1/2}\Omega^{1/2}(X_e^2)$. Thus A has the form $A = \kappa_A r_{\det}^{-1/2}\nu$, where κ_A is called the normalized kernel of A with respect to the half-density ν. We set $\Psi_X^*(X, \Omega^{1/2})$ to be the space of those A, which normalized kernels κ_A are conormal to the lifted diagonal $\triangle_X \subset X_e^2$, smoothly up the front faces $\mathrm{ff}(S_{\bar{q}})$ and vanish to infinite order at all other boundary hypersurfaces of X_e^2.*

It is again filtered by subspaces Ψ_X^m of operators of oder m. The action of an $A \in \Psi_X^m$ on a half-density is defined analogously to (6.63), as is the symbol map $A \mapsto {}^X\sigma_m(A) \in S^{\{m\}}({}^XT^*X) \subset C^\infty(X, {}^XT^*X)$. Again, this results in the short exact sequence

$$0 \to \Psi_X^{m-1} \hookrightarrow \Psi_X^m \xrightarrow{{}^X\sigma_m} S^{\{m\}}({}^XT^*X) \to 0. \tag{7.43}$$

As before, we say an element $A \in \Psi_X^m$ is elliptic, if its symbol is an invertible element of $S^{\{m\}}({}^XT^*X)$. In the next section we will prove the following

[2] Note that for a fixed $\bar{q} \subseteq \{1, \ldots, n-1\}$, we have $\sum_{I\in\mathfrak{I}} |I \cap \bar{q}| = \sum_{k\in\bar{q}} k(n-k)$.

Theorem 7.11: *For any m, m' we have $\Psi_X^m \circ \Psi_X^{m'} \subset \Psi_X^{m+m'}$ with the symbol map being multiplicative, i.e. it satisfies*

$$^{\mathrm{SL}}\sigma_{m+m'}(A \circ B) = {}^{\mathrm{SL}}\sigma_m(A) \cdot {}^{\mathrm{SL}}\sigma_{m'}(B). \tag{7.44}$$

Problem 1: *This theorem is proven in Section 7.5.*

By the standard iterative scheme we get the following main theorem of this chapter:

Theorem 7.12: *Let $A \in \Psi_X^m$ be an elliptic pseudodifferential operator. Then there exist an element $B \in \Psi_X^{-m}$ such that both $AB - I$ and $BA - I$ are elements of $\Psi_X^{-\infty}$. The parametrix B is a unique up to an element of $\Psi_X^{-\infty}$.*

As for SL(3), the remainders $AB - I$ and $BA - I$ are not compact, so a larger calculus should be investigated in future work. For now, we finish with the proof of the composition theorem in the next section.

7.5. Resolution of the triple space

In this section, we prove the composition Theorem 7.11. The proof is analogous to the one of Theorem 6.13, meaning it is a direct consequence of the existence of a triple space resolution denoted by X_e^3 together with three lifted projection maps $\bar{\pi}_\alpha : X_e^3 \to X_e^2$ that are be b-fibrations. The construction of this space together with these maps is the content of the section and is the last piece in the construction of the small calculus.

Construction of X_e^3: As a first step we move to the symmetric lexicographic blow-up as defined in Section 6.3, denoted by X_{lex}^3. As with the double space, there is a second step in the resolution given by the blow-up of double- and triple-fiber-diagonals: There are three projections $\pi^\alpha : X^3 \to X^2$ for $\alpha = LM, LR, MR$ given by dropping the last, middle and first factor X, respectively. Recall that in the resolution of the double space, the second step consisted of the blow-up of (the lifts of) submanifolds $S_{\bar{q}} \subset H_{\bar{q}\bar{q}}$ where these blow-ups were quasihomogeneous. For each $S_{\bar{q}}$ and each of the three π_α we denote the preimage of $S_{\bar{q}}$ under π^α by $S_{\bar{q}}^\alpha \subset X^3$.

Quasihomogeneous structure at $S_{\bar{q}}^\alpha$: The quasihomogeneous structure of $S_{\bar{q}} \subset X^2$ lifts under each projection to become a quasihomogeneous structure of $S_{\bar{q}}^\alpha$.

Each $S_{\bar{q}}^\alpha$ has a non-empty intersection with precisely those $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ that have two of the indices $\bar{q}_1, \bar{q}_2, \bar{q}_3$ equal to $\bar{q}$ (according to α) and the last one being arbitrary. For example, $S_{\bar{q}}^{LM}$ has non-empty intersection with all $H_{\bar{q},\bar{q},*}$ where $*$ is either 0 (meaning $X = H_0$) or any $\bar{q}'$. Thus the preimage of each of the $S_{\bar{q}}^\alpha$ under the symmetric lexicographic blow-up consists of submanifolds, one for each $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$. These parts are denoted by $S_{\bar{q}}^{\alpha,\bar{p}} \subset \mathrm{ff}(H_{\bar{q}_1,\bar{q}_2,\bar{q}_3})$ where two of the indices $\bar{q}_1, \bar{q}_2, \bar{q}_3$ are equal to $\bar{q}$ (according to α) and the remaining one is equal to $\bar{p}$. In fact $S_{\bar{q}}^{\alpha,\bar{p}}$ is the lift of $S_{\bar{q}}^\alpha \cap H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ under the symmetric lexicographic blow-up, with $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ as above.
For the second step in the resolution we will blow up all $S_{\bar{q}}^{\alpha,\bar{p}}$ again following lexicographic

order with respect to $\bar{q}, \bar{p}$. Before doing so there is one more thing to notice: All $S_{\bar{q}}^{\alpha,\bar{p}}$ with $\bar{q} \neq \bar{p}$ lie in different boundary faces $\mathrm{ff}(H_{\bar{q}_1,\bar{q}_2,\bar{q}_3})$. However, when $\bar{q} = \bar{p}$ the three corresponding $S_{\bar{q}}^{\alpha,\bar{q}}$ (for different α) all lie in $\mathrm{ff}(H_{\bar{q},\bar{q},\bar{q}})$. The intersection of any two of these three $S_{\bar{q}}^{\alpha,\bar{q}}$ yields a submanifold denoted by $T_{\bar{q}} \subset \mathrm{ff}(H_{\bar{q},\bar{q},\bar{q}})$ which is the lift of the triple-fiber diagonal in $H_{\bar{q},\bar{q},\bar{q}}$ with respect to the fibration $H_{\bar{q}} \to \mathcal{F}_{\bar{q}}$.

Quasihomogeneous structure at $S_{\bar{q}}^{\alpha,\bar{p}}$ and $T_{\bar{q}}$: The quasihomogeneous structure of $S_{\bar{q}}^{\alpha}$ can be extended radially according to Definition 3.11 to each $S_{\bar{q}}^{\alpha} \cap H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$. We need to check that this structure lifts to a quasihomogeneous structure of $S_{\bar{q}}^{\alpha,\bar{p}}$ under the radial blow-up of the corresponding $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$. The conormal bundles of $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ and $S_{\bar{q}}^{\alpha}$, restricted to the conormal bundle of $S_{\bar{q}}^{\alpha} \cap H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$, intersect only in $\{0\}$. Therefore, the radial quasihomogeneous structure of $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ and the radial extension of the quasihomogeneous structure to $S_{\bar{q}}^{\alpha,\bar{p}}$ intersects cleanly according to Theorem 3.7, which implies that the quasihomogeneous structure of $S_{\bar{q}}^{\alpha} \cap H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ lifts under the blow-up of the corresponding $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$.
The submanifold $T_{\bar{q}}$ is the intersection of any two $S_{\bar{q}}^{\alpha,\bar{q}}$. As submanifolds of $H_{\bar{q}\bar{q}\bar{q}}$ these pairwise intersect transversally, which is easily seen in local product coordinates with respect to the fibration $H_{\bar{q}} \to \mathcal{F}_{\bar{q}}$. Therefore, the quasihomogeneous structures of any such two $S_{\bar{q}}^{\alpha,\bar{q}}$ define a quasihomogeneous structure at $T_{\bar{q}}$ by Lemma 2.38. By symmetry, this is independent of the choice of $S_{\bar{q}}^{\alpha,\bar{q}}$ among these three. The blow-ups of $S_{\bar{q}}^{\alpha,\bar{p}}$ and $T_{\bar{q}}$ below are always meant to be quasihomogeneous according to these structures. Theorems 3.9 and 3.10 assure that normal commutativity results apply.

We are now ready to define the second step of our resolution:

Definition 7.13: *For a fixed multi index $\bar{q}$ and a fixed $\bar{p}$ being either a multi index or $\emptyset$, define the following lists*

$$\mathcal{S}_{\bar{q}}^{*,\bar{p}} := \begin{cases} S_{\bar{q}}^{LM,\bar{p}}; S_{\bar{q}}^{LR,\bar{p}}; S_{\bar{q}}^{MR,\bar{p}} & (\bar{q} \neq \bar{p}), \\ T_{\bar{q}}; S_{\bar{q}}^{LM,\bar{q}}; S_{\bar{q}}^{LR,\bar{q}}; S_{\bar{q}}^{MR,\bar{q}} & (\bar{q} = \bar{p}). \end{cases} \tag{7.45}$$

For fixed $\bar{q}$ we define

$$\mathcal{S}_{\bar{q}}^{**} := (\mathcal{S}_{\bar{q}}^{*,\bar{p}})_{\bar{p},lex} \tag{7.46}$$

the list of all $\mathcal{S}_{\bar{q}}^{*,\bar{p}}$ (for fixed $\bar{q}$) in any lexicographic order of $\bar{p}$. This includes the case of $\bar{p} = \emptyset$ corresponding to the full manifold X rather then any boundary hypersurface. We then set the second resolution of the triple space to

$$X_e^3 := [X_{lex}^3; (\mathcal{S}_{\bar{q}}^{**})_{\bar{q},lex}]. \tag{7.47}$$

Note that since the order of this list is lexicographic, $\mathcal{S}_{\bar{q}}^{*,\emptyset}$ is the last element of the list $\mathcal{S}_{\bar{q}}^{**}$. As a first step, we show that this is in fact symmetric in α, meaning that the space X_e^3 is independent of the order of the three submanifolds $S_{\bar{q}}^{LM,\bar{p}}; S_{\bar{q}}^{LR,\bar{p}}; S_{\bar{q}}^{MR,\bar{p}}$ in the definition above. This is a consequence of the following lemma, which is purposely formulated stronger than required here.

Lemma 7.14: *For $\bar q \neq \bar p$ any two $S_{\bar q}^{\alpha,\bar p}$, $S_{\bar q'}^{\beta,\bar p'}$ are separated by the boundary face $H_{\bar q_1,\bar q_2,\bar q_3}$ containing $S_{\bar q}^{\alpha,\bar p}$.*
For $\bar q = \bar p$ any two $S_{\bar q}^{\alpha,\bar q}$, $S_{\bar q}^{\beta,\bar q}$ are separated by $T_{\bar q}$.

Proof. The proof is analogous to the one of Lemma 6.15. □

We are now ready to construct the three lifted projections $\pi_e^\alpha : X_e^3 \to X_e^2$ and show that they are b-fibrations.

Proposition 7.15: *For each $\alpha = LM, LR, MR$ there is a b-fibration $\pi_e^\alpha : X_e^3 \to X_e^2$ fixed by the condition that it commutes with the corresponding projections*

$$\begin{tikzcd} X_e^3 \arrow[r] \arrow[d, "\pi_e^\alpha"] & X_{lex}^3 \arrow[r] \arrow[d, "\pi_{lex}^\alpha"] & X^3 \arrow[d, "\pi^\alpha"] \\ X_e^2 \arrow[r] & X_{lex}^2 \arrow[r] & X^2 \end{tikzcd} . \tag{7.48}$$

Proof. By symmetry, we only need to construct π_r^α for $\alpha = LM$. The main idea of the construction is to successively commute the blow-ups in (6.70) such that all blow-ups related to $\alpha = LM$ are made first. As we will see, this will allow for a straightforward definition of π_e^α as a composition the collective blow-down of all remaining blow-ups together with a projection.
We introduce the notation

$$\begin{aligned} \mathcal{S}_{\bar q}^{LM,*} &:= (S_i^{LM,\bar p})_{\bar p,\text{lex}}, \\ \mathcal{S}_{\bar q}^{\setminus LM} &:= \mathcal{S}_{\bar q}^{**} \setminus \mathcal{S}_{\bar q}^{LM}. \end{aligned} \tag{7.49}$$

Therefore, the list $\mathcal{S}_{\bar q}^{LM,*}; \mathcal{S}_{\bar q}^{\setminus LM}$ is a commutation of the list $\mathcal{S}_{\bar q}^{**}$.

As a first step, we show that

$$X_e^3 \cong [X_{\text{lex}}^3; (\mathcal{S}_{\bar q}^{LM,*})_{\bar q,\text{lex}}; \mathcal{R}_1] \tag{7.50}$$

with $\mathcal{R}_1 = (\mathcal{S}_{\bar q}^{\setminus LM})_{\bar q,\text{lex}}$. For this, we need to 'move' all $S_{\bar q}^{LM,\bar p}$ to the front, starting with the first one. Notice that any two $S_{\bar q}^{\alpha,\bar p}$, $S_{\bar q'}^{\beta,\bar p'}$ with $\alpha \neq \beta$ have been separated as shown in the previous lemma. Furthermore, $T_{\bar q}$ and $S_{\bar q}^{LM,\bar q}$ commute since $T_{\bar q} \subset S_{\bar q}^{LM,\bar q}$. Any other $S_{\bar q}^{LM,\bar p}$ (fixed $\bar q$, $\bar p \neq \bar q$) that is blown up afterwards then commutes with $T_{\bar q}$ since $S_{\bar q}^{LM,\bar q}$ separates them. Lastly, $T_{\bar q}$ and $S_{\bar q'}^{LM,\bar p}$ with $\bar q' \geq \bar q$ commute since they are separated by $S_{\bar q}^{LM,\bar p}$, as shown in (7.50). We want to continue to move the $S_{\bar q}^{LM,\bar p}$ further to the front by commuting them with as many blow-ups of X_{lex}^3 as possible. Recall the definition of X_{lex}^3 and $\mathcal{C}, \mathcal{E}, \mathcal{E}^\alpha$ and (6.25) where we saw that

$$X_{lex}^3 = [X^3; \mathcal{E}^{LM}; \mathcal{C}; \mathcal{E}^{LR}; \mathcal{E}^{MR}]. \tag{7.51}$$

Plugging this in, we show that

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; \mathcal{C}; (\mathcal{S}_{\bar q}^{LM,*})_{\bar q,\text{lex}}; \mathcal{R}_2] \tag{7.52}$$

with $\mathcal{R}_2 = \mathcal{E}^{LR}; \mathcal{E}^{MR}; \mathcal{R}_1$. This follows from the fact that the intersection of any $S_{\bar q}^{LM,\bar p}$ with an element of $\mathcal{E}^{LR}; \mathcal{E}^{MR}$ is contained in an element of $\mathcal{C}$ and thus they lift to become disjoint after the blow-up of all elements in $\mathcal{C}$.

Next, recall that $\mathcal{C}$ is ordered lexicographically, that is whenever $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3} \leq H_{\bar{q}_1',\bar{q}_2',\bar{q}_3'}$, then $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ is blown up first. Also recall that $S_{\bar{q}}^{LM,\bar{p}} \subset H_{\bar{q}\bar{q}\bar{p}}$. Thus, as already argued in the construction of the symmetric lexicographic resolution, whenever $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ and $H_{\bar{q}\bar{q}\bar{p}}$ are not comparable, there is a common larger element of $\mathcal{C}$ that contains their intersection and thus they may be commuted. Furthermore, if $H_{\bar{q}\bar{q}\bar{p}} \leq H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$, then the blow-up of $H_{\bar{q}\bar{q}\bar{p}}$ separates $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ and $S_{\bar{q}}^{LM,\bar{p}}$. Thus we may commute $S_{\bar{q}}^{LM,\bar{p}}$ with all $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3}$ except the ones where $H_{\bar{q}_1,\bar{q}_2,\bar{q}_3} \leq H_{\bar{q}\bar{q}\bar{p}}$. Consequently, we may commute the blow-ups of $\mathcal{C}; (\mathcal{S}_{\bar{q}}^{LM,*})_{\bar{q},\mathrm{lex}}$ in the following way: We introduce the notation

$$H_{\bar{q}_1,\bar{q}_2,*} = (H_{\bar{q}_1,\bar{q}_2,\bar{q}})_{\substack{\bar{q},\mathrm{lex}\\ \bar{q}\neq\emptyset}}. \tag{7.53}$$

Then

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; (H_{\bar{q}\bar{q}*}; \mathcal{S}_{\bar{q}}^{LM,*}; \mathcal{S}_{\bar{q}}^{LM,\emptyset})_{\substack{\bar{q},\mathrm{lex}\\ \bar{q}\neq\emptyset}}; \mathcal{R}_3]. \tag{7.54}$$

We proceed by some commutations in each of the blocks $H_{\bar{q}\bar{q}*}; \mathcal{S}_{\bar{q}}^{LM,*}; \mathcal{S}_{\bar{q}}^{LM,\emptyset}$. Repeating the argument from above, we may commute this block in (7.54) further to

$$(H_{\bar{q}\bar{q}\bar{p}}; \mathcal{S}_{\bar{q}}^{LM,\bar{p}})_{\bar{p},\mathrm{lex}}; \mathcal{S}_{\bar{q}}^{LM,\emptyset} \tag{7.55}$$

with the result still being diffeomorphic to X_e^3. Notice that $S_{\bar{q}}^{LM,\emptyset}$ is by definition the lift of $S_{\bar{q}}^{LM}$ under all previous blow-ups (in particular the blow-up of $H_{\bar{q}\bar{q}0}$ in $\mathcal{E}^{LM}$) and $S_{\bar{q}}^{LM,\bar{p}}$ is precisely the lift of $S_{\bar{q}}^{LM} \cap H_{\bar{q}\bar{q}\bar{p}}$. Plugging this in we get

$$(H_{\bar{q}\bar{q}\bar{p}}; \mathcal{S}_{\bar{q}}^{LM} \cap H_{\bar{q}\bar{q}\bar{p}})_{\bar{p},\mathrm{lex}}; \mathcal{S}_{\bar{q}}^{LM}. \tag{7.56}$$

For each $\bar{p}$ in the list, we use the standard commutativity result and 'swap' $H_{\bar{q}\bar{q}\bar{p}}$ and $\mathcal{S}_{\bar{q}}^{LM}$. Repeating this for all $\bar{p}$, starting with the last one in the list, we get

$$\mathcal{S}_{\bar{q}}^{LM}; (\mathcal{S}_{\bar{q}}^{LM} \cap H_{\bar{q}\bar{q}\bar{p}}; H_{\bar{q}\bar{q}\bar{p}})_{\bar{p},\mathrm{lex}}. \tag{7.57}$$

Plugging this in yields

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; (\mathcal{S}_{\bar{q}}^{LM}; (\mathcal{S}_{\bar{q}}^{LM} \cap H_{\bar{q}\bar{q}\bar{p}}; H_{\bar{q}\bar{q}\bar{p}})_{\bar{p},\mathrm{lex}})_{\bar{q},\mathrm{lex}}; \mathcal{R}_3]. \tag{7.58}$$

We want to commute $\mathcal{S}_{\bar{q}}^{LM}$ further to the front. Whenever $\bar{q}$ and $\bar{q}'$ are not comparable $S_{\bar{q}}^{LM}$ is disjoint from all elements in $(\mathcal{S}_{\bar{q}'}^{LM} \cap H_{\bar{q}'\bar{q}'\bar{p}}; H_{\bar{q}'\bar{q}'\bar{p}})_{\bar{p},\mathrm{lex}}$. When $\bar{q}' \subseteq \bar{q}$, $S_{\bar{q}}^{LM}$ and $H_{\bar{q}'\bar{q}'\bar{p}}$ have been separated by the blow-up of $H_{\bar{q}'\bar{q}'0} \in \mathcal{E}^{LM}$ and $S_{\bar{q}}^{LM}$ and $(S_{\bar{q}'}^{LM} \cap H_{\bar{q}'\bar{q}'\bar{p}})$ have been separated by the blow-up of $S_{\bar{q}'}^{LM}$. Thus we may commute the blow-ups further to obtain

$$X_e^3 \cong [X^3; \mathcal{E}^{LM}; (\mathcal{S}_{\bar{q}}^{LM})_{\bar{q},\mathrm{lex}}; \mathcal{R}_4]. \tag{7.59}$$

with $\mathcal{R}_4 = (\mathcal{S}_{\bar{q}}^{LM} \cap H_{\bar{q}\bar{q}\bar{p}}; H_{\bar{q}\bar{q}\bar{p}})_{\bar{p},\mathrm{lex}})_{\bar{q},\mathrm{lex}}; \mathcal{R}_3$.

Denote by

$$\beta^{LM}_{\mathcal{R}_4} : X^3_e \to [X^3; \mathcal{E}^{LM}; (\mathcal{S}^{LM}_{\bar{q}})_{\bar{q},\mathrm{lex}}] \tag{7.60}$$

the collective blow-down of $\mathcal{R}_4$. Notice that

$$[X^3; \mathcal{E}^{LM}; (\mathcal{S}^{LM}_{\bar{q}})_{\bar{q},\mathrm{lex}}] \cong X^2_e \times X. \tag{7.61}$$

Denote by $\tilde{\gamma} : X^2_e \times X \to X^2_e$ the projection onto the first factor. We then set

$$\pi^{LM}_e = \tilde{\gamma} \circ \beta^{LM}_{\mathcal{R}_4} : X^3_e \to X^2_e. \tag{7.62}$$

The fact that this commutes with the other blow-downs and projection maps is clear by construction. It remains to show that π^{LM}_e is a b-fibration, which follows again from the fact that the image of each hypersurface of X^3_e is a hypersurface in X^2_e. □

Theorem 7.11 is now a direct consequence of Theorem 7.15. Therefore, this also finishes the proof of Theorem 7.12.

CHAPTER 8

Outlook - where to continue

We finish this thesis with a short outlook on some of the questions remaining unanswered and potential future work.

Regarding Part 1: Quasihomogeneous blow-ups.

The treatment of quasihomogeneous blow-ups in Part 1 of this thesis aimed to be comprehensive and ready to use in a variety of future work. However, there is always more to learn about the fundamentals.

Recall that in the case of radial blow-up, the Lie algebra of vector fields tangent to $Y \subset X$ lifts to span, over C^∞, the space $\mathcal{V}_b([X;Y])$. The analogous result for quasihomogeneous blow-ups, Theorem 2.23, only states this spanning result almost everywhere on the front face. To be more precise, the front face has an exceptional subset, given in Definition 2.24, on which the lift of the 0-homogeneous vector fields $\mathcal{V}_\Pi^{(0)}$ does not span, over C^∞, the space $\mathcal{V}_b([X;Y]_\Pi)$. This is related to the fact, that in general the lift of m-quasihomogeneous functions $\mathcal{F}^{(m)}$ does not span, over C^∞, the space $\mathcal{I}^m(\mathrm{ff})$. As stated in Proposition 2.8, $\mathcal{F}^{(m)}$ consist of all smooth functions on the manifold X, which lift to $[X;Y]_\Pi$ lie in $\mathcal{I}^m(\mathrm{ff})$. Intuitively, the reason that the lift of $\mathcal{F}^{(m)}$ does not span $\mathcal{I}^m(\mathrm{ff})$ is, that there are 'not enough' *smooth* function of weighted homogeneity *precisely* m. A deeper understanding of this phenomena would be desirable.

In Chapter 3, we stated a collection of commutativity results for both radial and quasihomogeneous blow-ups. In Lemma 3.13, we defined the concept of a *separating submanifold* A, given two intersecting submanifolds B and C and showed, that the blow-up of B and C commutes after the blow-up of A, i.e. $[X;A;B;C] \cong [X;A;C;B]$. In Lemma 3.14 and Lemma 3.15 we gave sufficient conditions, under which the relationship 'A separates B and C' is stable under additional blow-ups. We would like to have a comprehensive understanding of this stability.

Regarding part 2: Pseudodifferential calculus on $\mathrm{SL}(n,\mathbb{R})$.

In Chapter 7, we constructed an algebra of pseudodifferential operators Ψ_X^*, that contains right-invariant operators on $\mathrm{SL}(n,\mathbb{R})$ together with basic parametrices for these operators. In Theorem 6.12 we have shown, that these parametrices exist for elliptic operators and that the error term is a smoothing operator on the resolved space X. The elliptic theory

for these operators developed in this thesis is far from finished. While the parametrix constructed in Theorem 6.12 does remove the singularity at the diagonal, it is not compact, since it is only smooth on the resolved space.
This yields the smoothness of solutions on the resolved spaces X. However, it is not enough to formulate a more general result on the polyhomogeneity of solutions.
Thus, a larger calculus, that aims to refine the parametrix constructed in Theorem 6.12, is desirable. To do this, one need to analyze (and invert) model operators on the boundary faces $\mathrm{ff}(S_{\bar{q}}) \subset X_e^2$, that intersect the lifted diagonal. Theorem 5.6 and Lemma 7.3 suggest, that inverting these model operators can be done (at least partially), by an iterative scheme of using the small calculus developed here on the lower dimensional factors $\overline{\mathbb{SH}}(p_i)$ in the fibers (see Equation (7.10)). As a first step in the direction of such an iterative scheme, a detailed description of the lift of the fibration (7.10) to a fibration of $\mathrm{ff}(S_{\bar{q}})$ would be needed. A comprehensive description and analysis of these model operators and a potential iterative scheme to invert them would be desirable.

Furthermore, a generalization to a general semisimple Lie groups G is a possibility for future work. The hd-compactification, constructed in [1], is applicable for these more general groups. In [1], the boundary faces of the constructed compactification $\bar{G}$ are shown to be in 1-1 correspondence with the conjugacy classes of parabolic subgroups, which are indexed over subsets S of the Dynkin diagram D. For each such subset $S \subset D$, $\bar{G}$ has a boundary face F_S that fibers over two copies of a corresponding flag variety $\mathcal{F}_S$. However, a more detailed description of the geometry and the behavior of right-invariant vector fields near any boundary face of the compactification, as in Theorem 5.6 and Lemma 5.15, would be needed.

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Declaration

Hereby, I declare, that the presented dissertation with the title

Quasihomogeneous Blow-ups and Pseudodifferential Calculus on $\overline{\mathbf{SL}}(\boldsymbol{n}, \mathbb{R})$

is my own work and, that I have not used other sources than the sources stated in the text or in the bibliography. The dissertation has, neither as a whole, nor in part, been submitted for assessment in a doctoral procedure at another university.

I confirm that I am aware of the guidelines of good scientific practice of the Carl von Ossietzky University Oldenburg and that I observed them. Furthermore, I declare that I have not availed myself of any commercial placement or consulting services in connection with my doctoral procedure.

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