Spectral theory on manifolds with fibred boundary metrics

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"A mathematician who is not somewhat of a poet, will never be a true mathematician." Karl Weierstrass.

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Contents

1. Introduction and statement of main results 3
2. Fundamentals of geometric microlocal analysis 7
3. Review of the pseudo-differential b-calculus 11
4. Review of the pseudo-differential ϕ-calculus 15
5. Outline of the heat kernel construction for a ϕ-metric 19
6. Step 1: Construction of the heat blowup space 20
7. Step 2: Construction of an initial heat kernel parametrix 23
8. Step 3: Triple space construction and composition of operators 26
9. Resolvent of ϕ Hodge Laplace 37
10. Hodge Laplacian for ϕ-metrics, split parametrix construction 41
11. Review of the resolvent construction on scattering manifolds 57
12. Low energy resolvent for ϕ-metrics, proof of main theorem 63
13. Triple space construction and composition theorems 76
14. Analytic torsion on fibred boundary ϕ metric manifolds 83
15. Heat kernel in short time regime and long time regime 85
16. Analytic torsion 95
References 99

1. Introduction and statement of main results

Consider a compact smooth Riemannian manifold $\overline{M}$ with boundary ∂M, which is the total space of a fibration $\phi : \partial M \to B$ over a closed manifold B with fibres given by copies of a closed manifold F. In the open interior M of such a manifold there are various possible complete Riemannian metrics. We shall recall here the main three classes of these complete Riemannian metrics.

1.1. **Fibred boundary and scattering metrics.** In this work we are interested in the fibred boundary metrics, also called ϕ-metrics. Ignoring cross-terms for the purpose of a clear exposition, these metrics are asymptotically given near the boundary ∂M by

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F,$$

where g_B is a Riemannian metric on the base B, and g_F is a symmetric bilinear form on ∂M, restricting to Riemannian metrics on fibres F. In case of trivial fibres, such a metric is called scattering. A trivial example of a scattering metric is the Euclidean space, with the metric written in polar coordinates as $dr^2 + r^2 d\theta^2$. After a change of variables $x = r^{-1}$ we obtain

$$g_{Sc} = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2}.$$

Such metrics arise naturally in various geometric examples. Complete Ricci flat metrics are often ϕ-metrics. Scattering metrics include metrics of locally Euclidean (ALE) manifolds. Products of these spaces with any compact manifold provide natural examples of ϕ-metrics. Furthermore, common classes of gravitational instantons, such as the Taub-NUT metrics and reduced 2-monopole moduli space metric, are ϕ-metrics under appropriate coordinate change, cf. [HHM04, p.2].

While there are various approaches to Euclidean scattering theory, a microlocal approach has been taken by Melrose [Mel94], where elliptic theory of scattering metrics has been developed. Elliptic theory of ϕ-metrics has been studied by Mazzeo and Melrose in [MaMe98]. This work later was generalized to the case of towers of fibrations with so called a-metrics by Grieser and Hunsicker [GrHu09]. Elliptic theory of [MaMe98] has also been extended by Grieser and Hunsicker [GrHu14] to include not fully elliptic operators. Hodge theory of ϕ-metrics has been developed by Hausel, Hunsicker and Mazzeo [HHM04]. Index theory (bypassing usual heat operator approach and using adiabatic limit methods instead) in this setting has been addressed by Leichtnam, Mazzeo and Piazza [LMP06].

1.2. **Fibred boundary cusp and b-metrics.** Fibred (boundary) cusp metrics, also referred to as d-metrics are conformally equivalent to ϕ-metrics by a conformal factor x^2. Ignoring as before the cross-terms for the purpose of a clear exposition, these metrics are asymptotically given near the boundary ∂M by

$$g_d = \frac{dx^2}{x^2} + x^2 g_F + \phi^* g_B.$$

In case of trivial fibres, such a metric is called a b-metric. Under the coordinate change $x = e^{-t}$, a b-metric becomes a cylindrical metric $dt^2 + g_B$. Same change of coordinates turns a genuine fibred cusp metric into $dt^2 + \phi^* g_B + e^{-2t} g_F$, which is a Q-rank one cusp when ϕ is a fibration of tori over a torus. Other examples include products of compact manifolds with locally symmetric spaces with finite volume hyperbolic cusps.

Elliptic theory of b-metrics was pioneered by Melrose [Mel93]. Since ϕ-metrics and fibred cusp metrics differ by a conformal change, elliptic theory of ϕ-metrics is suited for fibred cusp metrics as well. Vaillant [Vai01] has utilized elliptic theory of ϕ-metrics as well as a microlocal heat kernel construction in order to establish an index theorem for fibred cusp metrics. We emphasize that his heat kernel construction refers to the Hodge Dirac and Hodge Laplacian of a fibred cusp metric, not a ϕ-metric.

1.3. **Complete edge and conformally compact metrics.** The third class of complete Riemannian metrics on manifolds with fibred boundary, that has been of focal relevance in recent geometric analysis developments are complete edge metrics that by definition are given asymptotically near the boundary ∂M by

$$g_e = \frac{dx^2 + \phi^* g_B}{x^2} + g_F.$$

In case of trivial fibres, such metrics are also called conformally compact with the classical example being the hyperbolic space $\mathbb{H}^n$. The edge metrics also generalize the b-metrics that arise as special case of edge metrics with trivial base. The significance of edge metrics also lies in their conformal equivalence to the incomplete singular wedge metrics. These metrics appeared prominently in the resolution of the Calabi-Yau conjecture on Fano manifolds, cf. Donaldson [Don11], Tian [Tia15] as well as Jeffres, Mazzeo, Rubinstein [JMR11].

Elliptic theory of edge metrics has been developed by Mazzeo [Maz91]. Prior to that, the zero-calculus containing geometric operators associated to conformally compact metrics, as well as Hodge theory have been studied by Mazzeo [Maz86]. Meromorphic extension of the resolvent of conformally compact spaces is due to Mazzeo and Melrose [MaMe87]. Heat kernel and the (renormalized) Gauss Bonnet index theorem on general edge metrics is due to Albin [Alb07]. Let us also mention the work by Mazzeo and Vertman [MaVe12] on analytic torsion and by Vertman [Ver16] on incomplete wedge spaces, both of which are based on a microlocal heat kernel construction on wedge manifolds.

1.4. **Main results of the Thesis.** In this Thesis we deal with spectral geometry of manifolds with ϕ metric structures. After reviewing of geometric microlocal analysis in section 2 and also b- and ϕ-pseudodifferential theory 3, 4, we outline the construction of polyhomogeneous kernel of heat kernel of ϕ manifolds in section 5 and construct the heat kernel in finite time regime on the heat space HM_ϕ. In order to obtain polyhomogeneous description of heat kernel in long time regime, we continue in 9 and study the resolvent of ϕ Hodge

Laplace, i.e we construct the resolvent at low energy $(\Delta + k^2)^{-1}$ as $k \longrightarrow 0^+$. Our second main result is that in fact the resolvent at low energy lifts to the polyhomogeneous conormal distribution on the blown up space $M^2_{k,\phi}$. Wee formulate the characterisation of this result in terms of split (k, ϕ) calculus. Having polyhomogeneity of resolvent at low energy we use residue theorem and express heat kernel in terms of resolvent and obtain the polyhomogeneity of heat kernel in long time regime as third result in section 9. As a corollary we obtain the polyhomogeneity of heat kernel along diagonal both in finte time and long time regimes. As a last result, we define renormalized zeta function and show that this has meromorphic expansion on the whole complye plane $\mathbb{C}$ and consequently we will be able to define analytic torsion in the set up of manifold with fibred boundary ϕ metric structures.

We formulate now the main results of this Thesis in the following 4 statements. We remark that these theorems are obtained under certain assumptions that we especify later.

Theorem 1.1. *(to be published in [TaVe20])*[1] *The heat kernel of Δ_ϕ lifts to a polyhomogeneous function on the heat space HM_ϕ, vanishing to infinite order at ff, tf, rf and lf, smooth at fd, and of order $(-m)$ at td. More precisely, $e^{-t\Delta_\phi} \in \Psi^{3,0}_\phi(M)$.*

Theorem 1.2. *(to be published in [GTV20])* [2] *Consider a rescaled operator*

$$\Box_\phi := x^{-\frac{b+1}{2}} \Delta_\phi x^{\frac{b+1}{2}}.$$

Then the resolvent $(\Box_\phi + k^2)^{-1}$ is an element of the split calculus $\Psi^{-2,\mathcal{E}}_{k,\phi,\mathcal{H}}(M)$, defined in Definition 12.4, where the individual index sets satisfy

$$\mathcal{E}_{sc} \geq 0, \quad \mathcal{E}_{\phi f_0} \geq 0, \quad \mathcal{E}_{bf_0} \geq -2, \quad \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \quad \mathcal{E}_{zf} \geq -2.$$

The leading terms at sc, ϕf_0, bf_0 and zf are of orders $0, 0, -2, -2$, respectively, and are given by the constructions in Section 12.3.

Theorem 1.3. *(to be published in [Tal20])* [3] *The heat kernel which is given by*

$$H^M(t, x, x') = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\Delta_\phi + \lambda)^{-1} d\lambda,$$

is polyhomogeneous conormal at $t = \omega^{-\frac{1}{2}}$ at $\omega \longrightarrow 0$ on $M^2_{\omega,\phi}$ with index sets given in terms of index sets of resolvent $(\Delta_\phi + \lambda)^{-1}$ at low energy level. More explicitly the asymptotics of heat kernel in long time regime are of leading order 0 at sc face and of order 0 at zf and bf_0 faces. More over the leading order at the face ϕf_0 is 2. In long

[1] See Theorem 8.2 for the precise statement
[2] See Theorem 12.11 for the precise statement
[3] See Theorem 15.7 for the precise statement

time regime the heat kernel vanishes to infinite order at lb, rb, and bf faces of $M^2_{\omega,\phi}$. *The explicit index sets are as follows*

$$\mathcal{E}_{sc} \geq 0,\ \mathcal{E}_{\phi f_0} \geq 2,\ \mathcal{E}_{bf_0} \geq 0,\ \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \mathcal{E}_{zf} \geq 0.$$

Theorem 1.4. *(to be published in* [Tal20]*). On manifold endowed with* ϕ *structure* (M, g_ϕ) *the renormalized zeta function defined in terms of renormalized heat trace's integral along diagonal, admits meromorphic expansion to whole of complex plane* $\mathbb{C}$ *and consequently one may define the analytic torsion by*

$$\log^R T_{M,g_\phi} := \frac{1}{2}\sum_q (-1)^q q \frac{d}{ds}\, {}^R\zeta^q_{M,\phi}(s)|_{s=0}. \tag{1.1}$$

2. Fundamentals of geometric microlocal analysis

We briefly recall here the main concepts and tools of geometric microlocal analysis that will later be used for the construction of the resolvent kernel for Δ_ϕ. The main reference is [Mel93]; see [Gri01] for an introduction.

2.1. Manifolds with corners.

A compact manifold with corners X, of dimension N, is by definition modelled near each point $p \in X$ diffeomorphically by $(\mathbb{R}^+)^k \times \mathbb{R}^{N-k}$ for some $k \in \mathbb{N}_0$, where $\mathbb{R}^+ = [0, \infty)$. If p corresponds to 0 then k is called the codimension of p. A face of X, of codimension k, is the closure of a connected component of the set of points of codimension k. A boundary hypersurface is a face of codimension one, a corner is a face of codimension at least two. We assume that each boundary hypersurface H is embedded, i.e. it has a defining function ρ_H, that is, a smooth function $X \to \mathbb{R}^+$ with $H = \{\rho_H = 0\}$ and $d\rho_H$ nowhere vanishing on H. The set of boundary hypersurfaces of X is denoted $\mathcal{M}_1(X)$. In this section, we always work in the category of manifolds with corners.

2.2. Blowup of p-submanifolds.

Assume $P \subset X$ is a p-submanifold of a manifold with corners X, that is, near any $p \in P$ there is a local model for X in which P is locally a coordinate subspace. The blowup space $[X; P]$ is constructed by gluing $X \backslash P$ with the inward spherical normal bundle of $P \subset X$. The latter is called the front face of the blowup. The resulting space is equipped with a natural topology and the unique minimal differential structure with respect to which smooth functions with compact support in the interior of $X \backslash P$ and polar coordinates around P in X are smooth, cf. [Mel93, §4.1].

The canonical blowdown map

$$\beta : [X; P] \longrightarrow X,$$

is defined as the identity on $X\backslash P$ and as the bundle projection on the inward spherical normal bundle of $P \subset X$. Finally, given a p-submanifold $Z \subset X$, we define its lift under β to a submanifold of $[X; P]$ as follows.

(1) if $Z \subseteq P$ then $\beta^*(Z) := \beta^{-1}(Z)$,
(2) if $Z \nsubseteq P$ then $\beta^*(Z) :=$ closure of $\beta^{-1}(Z\backslash P)$.

2.3. **b-vector fields, polyhomogeneous functions and conormal distributions.** Let X be a manifold with corners.

Definition 2.1. (b-vector fields) A b-vector field on X is a smooth vector field which is tangential to all boundary hypersurfaces of X. The space of b-vector fields on X is denoted $\mathcal{V}_b(X)$.

Definition 2.2. (Polyhomogeneous functions)

(1) A subset $E = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}_0$ is called an index set if
  (a) the real parts $\mathrm{Re}(\gamma)$ accumulate only at $+\infty$.
  (b) For each γ there exists P_γ such that $(\gamma, p) \in E$ implies $p \leq P_\gamma$.
  (c) If $(\gamma, p) \in E$ then $(\gamma + j, p') \in E$ for all $j \in \mathbb{N}_0$ and $0 \leq p' \leq p$.

If $a \in \mathbb{R}$ then a also denotes the index set $(a+\mathbb{N}_0)\times\{0\}$. Addition of index sets is addition in $\mathbb{C} \times \mathbb{N}_0$. For example, $a + E = \{(\gamma + a, p) \mid (\gamma, p) \in E\}$. The *extended union* of two index sets E and F is defined as

$$E \,\overline{\cup}\, F = E \cup F \cup \{((\gamma, p + q + 1) : \exists\, (\gamma, p) \in E, \text{ and } (\gamma, q) \in F\}. \tag{2.1}$$

If E is an index set and $a \in \mathbb{R}$ then we write

$$\begin{aligned} E > a &:\iff (\gamma, k) \in E \text{ implies } \mathrm{Re}\,\gamma > a \\ E \geq a &:\iff (\gamma, k) \in E \text{ implies } \mathrm{Re}\,\gamma \geq a, \text{ and } k = 0 \text{ if } \mathrm{Re}\,\gamma = a. \end{aligned}$$

(2) An index family $\mathcal{E} = (\mathcal{E}_H)_{H\in\mathcal{M}_1(X)}$ for X is an assignment of an index set $\mathcal{E}_H$ to each boundary hypersurface H.

(3) A smooth function ω on the interior of X is called polyhomogeneous on X with index family $\mathcal{E}$, we write $\omega \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X)$, if ω admits an asymptotic expansion near each $H \in \mathcal{M}_1(X)$ of the form

$$\omega \sim \sum_{(\gamma,p)\in\mathcal{E}_H} a_{\gamma,p}\rho_H^{\gamma}(\log \rho_H)^p, \quad \rho_H \longrightarrow 0,$$

for some tubular neighborhood of H and defining function ρ_H, where the coefficients $a_{\gamma,p}$ are polyhomogeneous functions on H with index family $\mathcal{E}^H$. Here $\mathcal{E}^H$ is the index family for H which to any $H \cap H' \in \mathcal{M}_1(H)$, where $H' \in \mathcal{M}_1(X)$ has non-trivial intersection with H, assigns the index set $\mathcal{E}_{H'}$. Asymptotic expansions are always assumed to be preserved under iterated application of b-vector fields.

Definition 2.3 (Conormal distributions)**.** Let $P \subset X$ be a p-submanifold which is interior, i.e. not contained in ∂X. A distribution u on X is **conormal** of order $m \in \mathbb{R}$ with respect to P if it is smooth on $X \setminus P$ and near any point of P, with X locally modelled by $(\mathbb{R}^+)^k_x \times \mathbb{R}^{N-k}_{y',y''}$ and $P = \{y'' = 0\}$ locally,

$$u(x, y', y'') = \int e^{iy''\eta''} a(x, y'; \eta'') \, d\eta'' \tag{2.2}$$

for a symbol a of order $\mu = m + \frac{1}{4} \dim X - \frac{1}{2} \operatorname{codim} P$.

We only need the case $\dim X = 2 \dim P$, then $\mu = m$. If $X = M \times M$ for a closed manifold M and $P \subset X$ is the diagonal then conormal distributions are precisely the Schwartz kernels of pseudodifferential operators on M, with m equal to the order of the operator.

Polyhomogeneous sections of and conormal distributions valued in vector bundles over X are defined analogously.

2.4. **b-maps and b-fibrations.** The contents of this subsection are due to Melrose [Mel92], [Mel93], see also [Maz91, §2.A].

A smooth map between manifolds with corners is one which locally is the restriction of a smooth map on a domain of $\mathbb{R}^N$. We single out two classes of smooth maps, such that polyhomogeneous functions behave nicely under the pullback and the push-forward by these maps. We begin with the definition of a b-map.

Definition 2.4. Consider two manifolds with corners X and X'. Let ρ_H, $H \in \mathcal{M}_1(X)$ and $\rho_{H'}$, $H' \in \mathcal{M}_1(X')$ be defining functions. A smooth map $f : X' \to X$ is called b-map if for every $H \in \mathcal{M}_1(X)$, $H' \in \mathcal{M}_1(X')$ there exists $e(H, H') \in \mathbb{N}_0$ and a smooth non vanishing function h_H such that

$$f^*(\rho_H) = h_H \prod_{H' \in \mathcal{M}_1(X')} r_{H'}^{e(H,H')}. \tag{2.3}$$

The crucial property of a b-map f is that the pullback of polyhomogeneous functions under f is again polyhomogeneous, with an explicit control on the transformation of the index sets.

Proposition 2.5. *Let* $f : X' \longrightarrow X$ *be a b-map and* $u \in \mathcal{A}^{\mathcal{F}}_{phg}(X)$. *Then* $f^*(u) \in \mathcal{A}^{\mathcal{E}}_{phg}(X')$ *with index set* $\mathcal{E} = f^b(\mathcal{F})$, *where* $f^b(\mathcal{F})$ *defined as in* [Maz91, A12].

In order to obtain a polyhomogeneous function under pushforward by f, one needs additional conditions on f. On any manifold with corners X, we associate to the space of b-vector fields $\mathcal{V}_b(X)$ the b-tangent bundle ${}^bTX$, such that $\mathcal{V}_b(X)$ forms the space of its smooth sections. There is a natural bundle map ${}^bTX \to TX$ (see Subsection 3.1 for details in case X is a manifold with boundary). The differential $d_x f : T_x X' \longrightarrow T_{f(x)} X$ of a b-map f lifts under this map to the b-differential $d^b_x f : {}^bT_x X' \longrightarrow {}^bT_{f(x)} X$ for each $x \in X'$. We can now proceed with the following definition.

Definition 2.6.

- A b-map $f : X' \to X$ is a b-submersion if $d^b_x f$ is surjective for all $x \in X'$.
- f is called b-fibration if f is a b-submersion and, in addition, does not map boundary hypersurfaces of X' to corners of X, i.e for each H there exists at most one H' such that $e(H, H') \neq 0$ in (2.3).

We now formulate the Pushforward theorem due to Melrose [Mel93]. The pushforward map acts on densities instead of functions, and hence we consider the density bundle $\Omega(X)$ of X, and the corresponding b-density bundle

$$\Omega_b(X) := \left(\prod_{H \in \mathcal{M}_1(X)} \rho_H^{-1} \right) \Omega(X). \tag{2.4}$$

Then we write $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X, \Omega_b(X))$ for polyhomogeneous sections of the b-density bundle $\Omega_b(X)$ over X, with index set $\mathcal{E}$. The precise result is now as follows.

Proposition 2.7. *Let $f : X' \longrightarrow X$ be a b-fibration. Then for any index family $\mathcal{E}'$ for X', such that for each H' with $e(H, H') = 0$ for all H we have*[4] *$\mathcal{E}'_{H'} > 0$, the pushforward map is well-defined and acts as*

$$f_* : \mathcal{A}^{\mathcal{E}'}_{phg}(X', \Omega_b(X')) \longrightarrow \mathcal{A}^{f_b(\mathcal{E}')}_{phg}(X, \Omega_b(X)).$$

Here, $f_b(\mathcal{E}')$ is defined as in [Maz91, A.15].

2.5. **Operators acting on half-densities.** We will always identify an operator with its Schwartz kernel via integration, so it is natural to consider densities. The most symmetric way to do this is using half-densities: if the Schwartz kernel is a half-density then the operator it defines maps half-densities to half-densities naturally. However, differential operators are not typically given as acting on half-densities.

The connection is made by fixing a real density ν (in this paper, typically the volume form associated to g_ϕ or a related density) on X. This defines an isometry

$$L^2(X, \nu) \to L^2(X, \Omega^{\frac{1}{2}}) \quad u \mapsto u\, \nu^{\frac{1}{2}}$$

where $L^2(X, \nu) := \{u : X \to \mathbb{C} \mid \int_X |u|^2 \nu < \infty\}$ and $L^2(X, \Omega^{\frac{1}{2}})$ is the space of square-integrable half-densities on X.[5]

[4]This condition means that $\mathcal{E}'_{H'} > 0$ for any H' which maps into interior of X.

[5]Note that $L^2(X, \Omega^{\frac{1}{2}})$ is naturally identified with $L^2(X, \Omega_b^{\frac{1}{2}})$ (if X has corners) – one could also write any other rescaled density bundle here – since square integrability is intrinsic for half-densities.

Then if A is an operator acting in $L^2(X,\nu)$ (i.e. on functions), the operator on half-densities induced by this identification is given by

$$\widetilde{A}(u\nu^{\frac{1}{2}}) := (Au)\nu^{\frac{1}{2}}.$$

Note that

$$A \text{ symmetric in } L^2(X,\nu) \iff \widetilde{A} \text{ symmetric in } L^2(X,\Omega^{\frac{1}{2}})$$

since by definition of $\widetilde{A}$ we have $\int_X Au_1 \cdot \overline{u_2}\,\nu = \int_X \widetilde{A}(u_1\nu^{\frac{1}{2}}) \cdot \overline{u_2\nu^{\frac{1}{2}}}$.

Also, if A is given by an integral kernel K with respect to ν, i.e. $(Au)(p) = \int K(p,p')u(p')\,\nu(p')$ then $\widetilde{A}$ is given by the integral kernel $\widetilde{K}$ where $\widetilde{K}$ is the half-density

$$\widetilde{K}(p,p') = K(p,p')\,\nu(p)^{\frac{1}{2}}\,\nu(p')^{\frac{1}{2}}. \tag{2.5}$$

In practice we often write A instead of $\widetilde{A}$.

3. Review of the pseudo-differential b-calculus

In this section we review elements of the b-calculus [Mel93]. In this section $\overline{M}$ is a compact manifold with boundary ∂M, of dimension n. In contrast to the rest of the paper, ∂M need not be fibred.

3.1. b-vector fields and b-differential operators.

We provide a brief exposition of central elements of the b-calculus and refer the reader to [Mel93] for further details. Recall that the space of b-vector fields $\mathcal{V}_b = \mathcal{V}_b(\overline{M})$ is defined as the space of smooth vector fields on $\overline{M}$ which are tangential to ∂M. Fix local coordinates (x,θ) near a boundary point, where x defines the boundary, so that $\theta = \{\theta_i\}_i$ define local coordinates on ∂M. Then, $\mathcal{V}_b$ is spanned, locally freely over $C^\infty(\overline{M})$, by

$$(x\partial_x, \partial_{\theta_i}).$$

The b-tangent bundle ${}^bTM$ over $\overline{M}$ is defined by requiring its space of smooth sections to be $\mathcal{V}_b$. Interpreting an element of $\mathcal{V}_b$ as a section of TM rather than of ${}^bTM$ defines a vector bundle map ${}^bTM \to TM$ which is an isomorphism over the interior of M but has kernel $\operatorname{span}\{x\partial_x\}$ over ∂M. The dual bundle of ${}^bTM$, the b-cotangent bundle, is denoted by ${}^bT^*M$. It has local basis $(\frac{dx}{x}, d\theta_i)$. Let us also consider some Hermitian vector bundle E over $\overline{M}$.

The space of b-differential operators $\mathrm{Diff}_b^m(M;E)$ of order $m \in \mathbb{N}_0$ with values in E, consists of differential operators of m-th order on M, given locally near the boundary ∂M by the following differential expression (we use the convention $D_x = \frac{1}{i}\partial_x$ etc.)

$$P = \sum_{q+|\alpha|\le m} P_{\alpha,q}(x,\theta)(xD_x)^q(D_\theta)^\alpha, \tag{3.1}$$

where the coefficients $P_{\alpha,q} \in C^\infty(\overline{\mathcal{U}}, \mathrm{End}(E))$ are smooth sections of $\mathrm{End}(E)$. Its b-symbol is given at a base point $(x, \theta) \in \overline{\mathcal{U}}$ by the homogeneous polynomial in $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}^{n-1}$

$$\sigma_b(P)(x, \theta; \xi, \zeta) = \sum_{j+|\alpha|=m} P_{\alpha,q}(x, \theta)\xi^q \zeta^\alpha$$

Invariantly this is a function on ${}^bT^*M$ (valued in $\mathrm{End}(E)$) if we identify (ξ, ζ) with $\xi\frac{dx}{x} + \zeta \cdot d\theta \in {}^bT^*_{(x,\theta)}M$. An operator $P \in \mathrm{Diff}^m_b(M; E)$ is said to be b-elliptic if $\sigma_b(P)$ is invertible on ${}^bT^*M\backslash\{0\}$. Writing $P^m({}^bT^*M; E)$ for the space of homogeneous polynomials of degree m on the fibres of ${}^bT^*M$ with values in $\mathrm{End}(E)$, the b-symbol map defines a short exact sequence

$$0 \longrightarrow \mathrm{Diff}^{m-1}_b(M; E) \hookrightarrow \mathrm{Diff}^m_b(M; E) \xrightarrow{\sigma_b} P^m({}^bT^*M; E) \longrightarrow 0. \tag{3.2}$$

3.2. b**-Pseudodifferential operators.** Parametrices to b-elliptic b-differential operators are polyhomogeneous conormal distributions on the b-double space that we now define. Consider the double space $\overline{M} \times \overline{M}$ and blow up the codimension two corner $\partial M \times \partial M$. This defines the b-double space

$$M^2_b = [\overline{M} \times \overline{M}; \partial M \times \partial M].$$

We may illustrate this blowup as in Figure 1, where θ, θ' are omitted. As usual, this blowup can be described in projective local coordinates. If $(x, \theta), (x', \theta')$ are local coordinates on the two copies of $\overline{M}$ near the boundary then local coordinates near the upper corner of the resulting front face bf are given by

$$s = \frac{x}{x'},\ x',\ \theta,\ \theta', \tag{3.3}$$

where s defines lb and x' defines bf locally. Interchanging the roles of x and x', we get projective local coordinates near the lower corner of bf. Pullback by the blowdown map β_b is simply a change of coordinates from standard to projective coordinates. We will always fix a boundary defining function x for ∂M and choose $x' = x$ as functions on $\overline{M}$. Then s is defined on a full neighborhood of bf $\setminus$ rb, and if in addition $\theta' = \theta$ as (local) functions on $\overline{M}$ then the b-diagonal,

$$\mathrm{Diag}_b := \beta_b^* \,\mathrm{Diag}_M, \quad \mathrm{Diag}_{\overline{M}} = \{(p, p) \mid p \in \overline{M}\} \subset \overline{M} \times \overline{M},$$

is locally $s = 1, \theta = \theta'$. It is a p-submanifold of M^2_b.

We defined b-densities in (2.4). The b-density bundle $\Omega_b(\overline{M}^2)$ on the double space $\overline{M}^2$ has local basis

$$\frac{dx}{x}\frac{dx'}{x'}d\theta d\theta'.$$

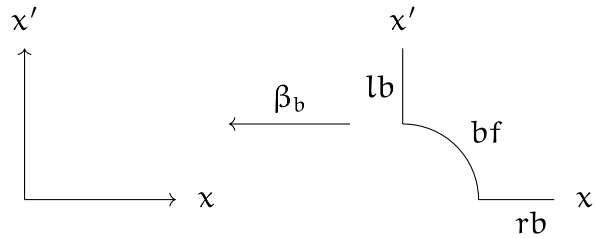

FIGURE 1. b-double space M_b^2 and $\beta_b : M_b^2 \to \overline{M}^2$.

The b-density bundle on M_b^2 is, in coordinates (3.3), spanned by

$$\frac{ds}{s}\frac{dx'}{x'}d\theta d\theta'.$$

Note that $\Omega_b(M_b^2) = \beta_b^*\Omega_b(\overline{M}^2)$. The corresponding half b-density bundle is denoted by $\Omega_b^{1/2}(M_b^2)$.

We can now define the *small and full calculus* of pseudo-differential b-operators, following [Mel93]. Note that we identify the operators with the lifts of their Schwartz kernels to M_b^2. Recall from Subsection 2.5 that operators act on half-densities, so their Schwartz kernels are half-densities.

Definition 3.1.

1. The small calculus $\Psi_b^m(M;E)$ of b-pseudodifferential operators is the space of $\Omega_b^{1/2}(M_b^2)\otimes\mathrm{End}(E)$-valued distributions on M_b^2 which are conormal with respect to the b-diagonal and vanish to infinite order at lb, rb.
2. The full calculus $\Psi_b^{m,\mathcal{E}}(M;E)$ of b-pseudodifferential operators is defined as

$$\Psi_b^{m,\mathcal{E}}(M;E) := \Psi_b^m(M;E) + \mathcal{A}_b^{\mathcal{E}}$$
$$\text{where } \mathcal{A}_b^{\mathcal{E}} := \mathcal{A}_{phg}^{\mathcal{E}}(M_b^2, \Omega_b^{1/2}(M_b^2)\otimes \mathrm{End}(E)),$$

if $\mathcal{E}$ is an index family for M_b^2 with $\mathcal{E}_{bf} \geq 0$. We also write $\mathcal{A}_b^{\mathcal{E}}(M)$ or $\mathcal{A}_b^{\mathcal{E}}(M,E)$ instead of simply $\mathcal{A}_b^{\mathcal{E}}$, if we want to specify the underlying space or bundle involved. [6]

[6]This is a coarse version of the Definition 5.51 given in [Mel93]: there, the index sets are given for lb and rb only, and then the $\mathcal{A}_{phg}$ term is replaced by $\mathcal{A}_{phg}^{\mathcal{E}_{lb},0,\mathcal{E}_{rb}}(M_b^2) + \mathcal{A}_{phg}^{\mathcal{E}_{lb},\mathcal{E}_{rb}}(\overline{M}^2)$. If $\mathcal{E}_{lb} + \mathcal{E}_{rb} > 0$, which is true for all index sets of operator appearing in our paper, this is contained in the given definition by the pull-back theorem, with $\mathcal{E}_{bf} = 0 \cup (\mathcal{E}_{lb} + \mathcal{E}_{rb})$. Also, this notation, and similar notation used below for other calculi, is not the same as that, e.g., in [GrHu09, Definition 12]. Here we assume the conormal singularity to have smooth coefficients up to bf, while there they have index set $\mathcal{E}_{bf}$. This is especially relevant when the index set at the front face is allowed to contain negative exponents, as in the definition of the (k, sc) calculus.

Here $\mathrm{End}(E)$ is the vector bundle over M^2_b which is the pullback of the bundle over $\overline{M}\times\overline{M}$ that has fibre $\mathrm{Hom}(E_{p'},E_p)$ over $(p,p')\in\overline{M}\times\overline{M}$. Note that $\Psi_b^{-\infty,\mathcal{E}}(M;E)=\mathcal{A}_b^{\mathcal{E}}$ if $0\subset\mathcal{E}_{bf}$.

3.3. **Fredholm properties of b-operators.** For any $P\in\mathrm{Diff}^m_b(M;E)$ of the form (3.1) locally near the boundary, we define the corresponding indicial operator $I(P)$ and indicial family $I_\lambda(P)$ by

$$\begin{aligned} I(P) &= \sum_{q+|\alpha|\leq m} P_{\alpha,q}(0,\theta)(xD_x)^q(D_\theta)^\alpha, \\ I_\lambda(P) &= \sum_{q+|\alpha|\leq m} P_{\alpha,q}(0,\theta)(\tfrac{1}{i}\lambda)^q(D_\theta)^\alpha, \end{aligned}$$

where the latter is a family of differential operators on ∂M, acting in $L^2(\partial M,E\restriction\partial M)$. The set of indicial roots $\mathrm{spec}_b(P)$ is defined as

$$\mathrm{spec}_b(P):=\{\lambda\in\mathbb{C}\mid I_\lambda(P)\text{ is not invertible}\}.$$

Before we can proceed with stating the Fredholm theory results for b-operators, let us define weighted b-Sobolev spaces for $m\in\mathbb{R}$ and $\ell\in\mathbb{R}$

$$x^\ell H^m_b(M;E):=\{u=x^\ell\cdot v\,|\,\forall P\in\Psi^m_b(M;E):Pv\in L^2(M;E)\}.$$

Note that we define $L^2(M;E)\equiv L^2(M;E;\mathrm{dvol}_b)$ with respect to the b-density dvol_b, which is a non-vanishing section of the b-density bundle $\Omega_b(\overline{M})$.

Theorem 3.2 (Parametrix in the b-calculus)**.** *Let $P\in\mathrm{Diff}^m_b(M;E)$ be b-elliptic. Then for each $\alpha\notin\mathrm{Re}(\mathrm{spec}_b(P))$ there is an index family $\mathcal{E}(\alpha)$ for M^2_b satisfying*

$$\mathcal{E}(\alpha)_{lb}>\alpha,\quad \mathcal{E}(\alpha)_{rb}>-\alpha,\quad \mathcal{E}(\alpha)_{bf}\geq 0,$$

and a parametrix $Q_\alpha\in\Psi_b^{-m,\mathcal{E}}(M;E)$, inverting P up to remainders

$$P\circ Q_\alpha=\mathrm{Id}-R_{r,\alpha},\quad Q_\alpha\circ P=\mathrm{Id}-R_{l,\alpha},$$

where the remainders satisfy

$$R_{r,\alpha}\in x^\infty\Psi_b^{-\infty,\mathcal{E}(\alpha)}(M,E),\quad R_{l,\alpha}\in\Psi_b^{-\infty,\mathcal{E}(\alpha)}(M,E)\,x^\infty.$$

The restriction of the Schwartz kernel of Q_α to bf *is given by the inverse of the indicial operator $I(P)$ in $x^\alpha L^2(\mathbb{R}^+\times\partial M,E)$, with weight α, i.e. having asymptotics as dictated by $\mathcal{E}(\alpha)$ at* lb *and* rb*.*

The index family $\mathcal{E}(\alpha)$ is determined by $\mathrm{spec}_b(P)$ and satisfies

$$\begin{aligned} \pi\mathcal{E}(\alpha)_{lb} &= \{z+r\mid z\in\mathrm{spec}_b(P),\mathrm{Re}\,z>\alpha,r\in\mathbb{N}_0\}, \\ \pi\mathcal{E}(\alpha)_{rb} &= \{-z+r\mid z\in\mathrm{spec}_b(P),\mathrm{Re}\,z<\alpha,r\in\mathbb{N}_0\}, \end{aligned}\tag{3.4}$$

where $\pi:\mathbb{C}\times\mathbb{N}_0\to\mathbb{C}$ is the projection onto the first factor, i.e. we neglect logarithms.

Note that $x^\infty \Psi_b^{-\infty,\mathcal{E}(\alpha)}(M,E) = \mathcal{A}_b^{\mathcal{E}(\alpha)_{|rb}}$ where $\mathcal{E}(\alpha)_{|rb}$ is the index family with index sets equal to $\mathcal{E}(\alpha)$ at rb and empty otherwise. Similarly $\Psi_b^{-\infty,\mathcal{E}(\alpha)}(M,E)\, x^\infty = \mathcal{A}_b^{\mathcal{E}(\alpha)_{|lb}}$.

By standard boundedness results this implies (cf. [Mel93, Theorem 5.60 and Prop. 5.61]) the following Fredholm and regularity result.

Theorem 3.3 (Fredholmness and regularity of elliptic b-operators)**.**
Let $P \in \mathrm{Diff}_b^m(M;E)$ be b-elliptic. Then P is Fredholm as a map

$$P : x^\alpha H_b^{s+m}(M;E) \longrightarrow x^\alpha H_b^s(M;E),$$

for any $\alpha \notin \mathrm{Re}(\mathrm{spec}_b(P))$ and any $s \in \mathbb{R}$. The Fredholm inverse of P is in the full b-calculus $\Psi_b^{-m,\mathcal{E}(\alpha)}(M;E)$ with $\mathcal{E}(\alpha)$ as in Theorem 3.2. Moreover, if $u \in x^\alpha H_b^s(M;E)$ for some $\alpha, s \in \mathbb{R}$ and $Pu \in \mathcal{A}_{phg}^I(M;E)$ for some index set I then $u \in \mathcal{A}_{phg}^J(M;E)$, where $J = I\overline{\cup}K$ for some index set $K > \alpha$, determined by $\mathrm{spec}_b(P)$.

In particular, if u has only Sobolev regularity, but is mapped by a differential b-operator to a section with an asymptotic expansion at ∂M, for instance if u is in the kernel of P, then u must also have a full asymptotic expansion at ∂M.

Recall that the Fredholm inverse is defined as follows: if $K = \ker P$ and $R = \mathrm{Ran}\, P$, then the Fredholm inverse of P is zero on $R^\perp$ and equals $(P_{|K^\perp \to R})^{-1}$ on R.

4. Review of the pseudo-differential ϕ-calculus

In this section we review elements of the ϕ-calculus, following [MaMe98]. We are now in the setting of a compact manifold $\overline{M}$ with a fibration $\phi : \partial M \to B$ of the boundary. We also fix a boundary defining function $x \geq 0$ and collar neighborhood $\overline{\mathcal{U}} \cong [0,\varepsilon]_x \times \partial M$ of a neighborhood $\overline{\mathcal{U}} \subset \overline{M}$ of ∂M.

4.1. ϕ-vector fields and ϕ-differential operators.

Definition 4.1. A b-vector field V on $\overline{M}$ is called ϕ-vector field, $V \in \mathcal{V}_\phi \equiv \mathcal{V}_\phi(\overline{M})$, if at the boundary it is tangent to the fibres of the fibration $\phi : \partial M \to B$ and if it satisfies $Vx \in x^2 C^\infty(\overline{M})$ for the chosen boundary defining function x. Near a boundary point we use coordinates $\{x, y_i, z_j\}$ with $y = \{y_i\}_i$ being local coordinates on the base B, lifted to ∂M and extended to $[0,\varepsilon) \times \partial M$, and $z = \{z_j\}_j$ restricting to local coordinates on the fibres F. Then $\mathcal{V}_\phi$ is spanned, locally freely over $C^\infty(\overline{M})$, by the vector fields

$$x^2\frac{\partial}{\partial x}, x\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j}.$$

We introduce the so called ϕ-tangent space by requiring $\mathcal{V}_\phi(M)$ to be its smooth sections

$$C^\infty(\overline{M}, {}^\phi TM) = \mathcal{V}_\phi = C^\infty(\overline{M})\text{-span}\left\langle x^2\frac{\partial}{\partial x}, x\frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j}\right\rangle,$$

where the second equality obviously holds only locally near ∂M. Note that the metric g_ϕ extends to a smooth positive definite quadratic form on ${}^\phi TM$ over all of $\overline{M}$. The dual bundle ${}^\phi T^*M$, the so-called ϕ-cotangent space, satisfies

$$C^\infty(\overline{M}, {}^\phi T^*M) = C^\infty(\overline{M})\text{-span}\left\langle \frac{dx}{x^2}, \frac{dy_i}{x}, dz_j\right\rangle.$$

The space of ϕ-vector fields $\mathcal{V}_\phi$ is closed under brackets, hence is a Lie algebra, and is a $C^\infty(\overline{M})$-module. Hence it leads to the definition of ϕ-differential operators $\mathrm{Diff}^*_\phi(M;E)$, where E is some fixed Hermitian vector bundle. Explicitly, $P \in \mathrm{Diff}^m_\phi(M;E)$ if it is an m-th order differential operator in the open interior M, and has the following structure locally near the boundary ∂M

$$P = \sum_{|\alpha|+|\beta|+q\leq m} P_{\alpha,\beta,q}(x,y,z)(x^2D_x)^q(xD_y)^\beta D_z^\alpha, \tag{4.1}$$

with coefficients $P_{\alpha,\beta,q} \in C^\infty(\overline{\mathcal{U}}, \mathrm{End}(E))$ smooth up to the boundary. The ϕ-symbol $\sigma_\phi(P)$ is then locally given over the base point $(x,y,z) \in \overline{\mathcal{U}}$ by the homogeneous polynomial in $(\xi,\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^{\dim B} \times \mathbb{R}^{\dim F}$

$$\sigma_\phi(P)(x,y,z;\xi,\eta,\zeta) = \sum_{|\alpha|+|\beta|+q=m} P_{\alpha,\beta,q}(x,y,z)\xi^q\eta^\beta\zeta^\alpha. \tag{4.2}$$

Invariantly this is a function (valued in $\mathrm{End}(E)$) if we identify (ξ,η,ζ) with $\xi\frac{dx}{x^2} + \eta \cdot \frac{dy}{x} + \zeta \cdot dz \in {}^\phi T^*_{(x,y,z)}M$. We say that P is ϕ-elliptic if $\sigma_\phi(P)$ is invertible off the zero-section of $T^*_\phi M$. Writing $P^m(T^*_\phi M;E)$ for the space of homogeneous polynomials of degree k on the fibres of $T^*_\phi M$ valued in $\mathrm{End}(E)$, the ϕ-symbol map defines a short exact sequence

$$0 \longrightarrow \mathrm{Diff}^{m-1}_\phi(M;E) \hookrightarrow \mathrm{Diff}^m_\phi(M;E) \xrightarrow{\sigma_\phi} P^m(T^*_\phi M;E) \longrightarrow 0. \tag{4.3}$$

4.2. **ϕ-Pseudodifferential operators.** We now recall the notion of ϕ-pseudo differential operators $\Psi^*_\phi(M;E)$ from Mazzeo and Melrose [MaMe98]. These will be operators whose Schwartz kernels lift to polyhomogeneous distributions with conormal singularity along the lifted diagonal on the ϕ-double space M^2_ϕ obtained from the b-double space M^2_b by an additional blowup: consider the interior fibre diagonal and its boundary

$$\begin{aligned}\mathrm{diag}_{\phi,\mathrm{int}} &= \{(p,p') \in \overline{\mathcal{U}} \times \overline{\mathcal{U}} : \phi(p) = \phi(p')\},\\ \mathrm{diag}_\phi &:= \partial(\beta_b^*(\mathrm{diag}_{\phi,\mathrm{int}})) = \{s = 1, y = y', x' = 0\}.\end{aligned}$$

The ϕ-double space is now defined by

$$M^2_\phi := [M^2_b; \mathrm{diag}_\phi], \quad \beta_{\phi-b} : M^2_\phi \to M^2_b. \tag{4.4}$$

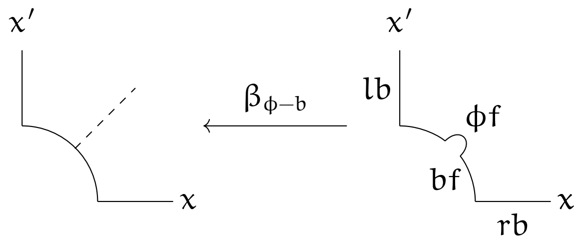

FIGURE 2. ϕ-double space M^2_ϕ

This blowup is illustrated in Figure 2, with the y, z coordinates omitted. Projective coordinates near the interior of ϕf can be given using (3.3) by

$$T = \frac{s-1}{x'}, \ Y = \frac{y-y'}{x'}, z, \ x', \ y', \ z', \tag{4.5}$$

where x' defines ϕf locally and bf lies in the limit $|(T, Y)| \to \infty$. Here, the roles of x and x' can be interchanged freely. Pullback by the blowdown map is again simply a change of coordinates from standard to e.g. the projective coordinates above. The total blowdown map is given by

$$\beta_\phi = \beta_b \circ \beta_{\phi-b} : M^2_\phi \longrightarrow \overline{M}^2.$$

We now define the *small calculus* and the *full calculus* of pseudo-differential ϕ-operators, following [MaMe98] and [GrHu09]. As always we identify operators with the lifts of their Schwartz kernels to M^2_ϕ, and let operators act on half-densities. It is convenient[7] to normalize to the $b\phi$-density bundle

$$\Omega_{b\phi}(M^2_\phi) := \rho_{\phi f}^{-(b+1)} \Omega_b(M^2_\phi) = \rho_{\phi f}^{-2(b+1)} \beta^*_\phi \Omega_b(\overline{M}^2). \tag{4.6}$$

The corresponding half $b\phi$-density bundle is denoted by $\Omega^{1/2}_{b\phi}(M^2_\phi)$.

Definition 4.2. We define small and full calculi of ϕ-operators.

1. The small calculus $\Psi^m_\phi(M; E)$ of ϕ-pseudodifferential operators is the space of $\Omega^{1/2}_{b\phi}(M^2_\phi) \otimes \mathrm{End}(E)$-valued distributions on M^2_ϕ which are conormal with respect to the lifted diagonal and vanish to infinite order at lb, rb and bf.

[7] The reason for this is that ϕ-differential operators, as well as operators in the small ϕ-calculus, have kernels which are conormal to the diagonal uniformly up to ϕf, when considered as sections of $\Omega^{\frac{1}{2}}_{b\phi}(M^2_\phi)$.

2. The full calculus $\Psi_\phi^{m,\mathcal{E}}(M;E)$ of ϕ-pseudodifferential operators is defined as

$$\Psi_\phi^{m,\mathcal{E}}(M;E) := \Psi_\phi^m(M;E) + \mathcal{A}_\phi^{\mathcal{E}},$$
$$\text{where } \mathcal{A}_\phi^{\mathcal{E}} := \mathcal{A}_{phg}^{\mathcal{E}}(M_\phi^2, \Omega_{b\phi}^{1/2}(M_\phi^2) \otimes \mathrm{End}(E)),$$

if $\mathcal{E}$ is an index family for M_ϕ^2 with $\mathcal{E}_{\phi f} \geq 0$. We also write $\mathcal{A}_\phi^{\mathcal{E}}(M)$ instead of simply $\mathcal{A}_\phi^{\mathcal{E}}$, if we want to specify the underlying space.

4.3. **Fredholm theory of ϕ-operators.** The normal operator of a ϕ-differential operator $P \in \mathrm{Diff}_\phi^m(M)$ is defined as follows. Write P in coordinates near the boundary as

$$P = \sum_{|\alpha|+|\beta|+q\leq m} P_{\alpha,\beta,q}(x,y,z)(x^2D_x)^q(xD_y)^\beta D_z^\alpha.$$

Then we define

$$N_\phi(P)_{y'} := \sum_{|\alpha|+|\beta|+q\leq m} P_{\alpha,\beta,q}(0,y',z)D_T^qD_Y^\beta D_z^\alpha. \tag{4.7}$$

The normal operator $N_\phi(P)_{y'}$ is a family of differential operators acting on $\mathbb{R} \times \mathbb{R}^b \times F$, parametrized by $y' \in B$. The Schwartz kernel of $N_\phi(P)$ can be identified with the restriction of the Schwartz kernel of P to ϕf, using coordinates (4.5).

Note that $N_\phi(P)_{y'}$ is translation invariant (constant coefficient) in (T,Y). Performing Fourier transform in $(T,Y) \in \mathbb{R} \times \mathbb{R}^b$ we define the normal family

$$\widehat{N}_\phi(P)_{y'} := \sum_{|\alpha|+|\beta|+q\leq m} P_{\alpha,\beta,q}(0,y',z)\tau^q\xi^\beta D_z^\alpha. \tag{4.8}$$

This is a family of differential operators on F, parametrized by $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^b$ and $y' \in B$.

Definition 4.3 (Full ellipticity)**.** An elliptic differential ϕ-operator $P \in \mathrm{Diff}_\phi^m(M)$ is said to be fully elliptic if additionally the operator family $\widehat{N}_\phi(P)_{y'}(\tau,\xi)$ is invertible for all $(\tau,\xi;y')$.

We can now state Fredholm results for fully elliptic ϕ-differential operators, due to [MaMe98].

Theorem 4.4 (Invertibility up to smooth kernel)**.** [MaMe98] *(Proposition 8). If $P \in \mathrm{Diff}_\phi^m(M;E)$ is fully elliptic in the sense of Definition 4.3, then there exists a small calculus parametrix $Q \in \Psi_\phi^{-m}(M;E)$ satisfying*

$$PQ - \mathrm{Id},\ QP - \mathrm{Id} \in x^\infty \Psi_\phi^{-\infty}(M;E) = \mathcal{A}_\phi^{\varnothing}.$$

In order to state continuity and Fredholm results we introduce weighted ϕ-Sobolev spaces. We write for any $m, \ell \in \mathbb{R}$

$$x^\ell H^m_\phi(M;E) := \{u = x^\ell \cdot v \mid \forall\, P \in \Psi^m_\phi(M;E) :\ Pv \in L^2(M;E)\}.$$

As before, we define $L^2(M;E) \equiv L^2(M;E;\mathrm{dvol}_b)$ with respect to the b-density dvol_b. However, as derivatives in the definition of $H^m_\phi(M,E)$, e.g. for $m \in \mathbb{N}$, we use ϕ-derivatives.

Theorem 4.5. [MaMe98, Proposition 9, 10]. *Let* $P \in \mathrm{Diff}^m_\phi(M;E)$ *be fully elliptic. Then for any* $\alpha, s \in \mathbb{R}$, P *is Fredholm as a map*

$$P : x^\alpha H^{s+m}_\phi(M;E) \to x^\alpha H^s_\phi(M;E).$$

The Fredholm inverse lies in $\Psi^{-m}_\phi(M;E)$.

5. Outline of the heat kernel construction for a ϕ-metric

Let Δ_ϕ be the unique self-adjoint extension of the Hodge Laplacian on the $\phi-$manifold M with fibred boundary ∂M and a ϕ-metric g_ϕ. The heat operator of Δ_ϕ is denoted by $e^{-t\Delta_\phi}$ and solves for any given function ω_0 in the domain of Δ_ϕ by definition the homogeneous heat problem

$$\begin{aligned}(\partial_t + \Delta_\phi)\omega(t,p) = 0, &\quad (t,p) \in [0,\infty) \times M,\\ \omega(0,p) = \omega_0(p), &\quad p \in M,\end{aligned} \tag{5.1}$$

with $\omega = e^{-t\Delta_\phi}\omega_0$. The heat operator is an integral operator

$$e^{-t\Delta_\phi}\omega_0(p) = \int_M H(t,p,\widetilde{p})\,\omega_0(\widetilde{p})\mathrm{dvol}_{g_\phi}(\widetilde{p}), \tag{5.2}$$

with the heat kernel H being a smooth function on the open interior of the heat space $M^2_h := [0,\infty) \times \overline{M}^2$, acting pointwise on the exterior algebra $\Lambda^{*\phi}T^*M$ and taking values in $\Lambda^{*\phi}T^*M$. Viewing the heat kernel as a section in $\Lambda^{*\phi}T^*M \boxtimes \Lambda^{*\phi}T^*M$, we may also equivalently rewrite (5.2) as

$$e^{-t\Delta_\phi}\omega_0(p) = \int_M \Big(H(t,p,\widetilde{p}), \omega_0(\widetilde{p}) \Big)_{g_\phi} \mathrm{dvol}_{g_\phi}(\widetilde{p}). \tag{5.3}$$

Consider local coordinates $(t,(x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z}))$ near the highest codimension corner in the heat space M^2_h, where (x,y,z) and $(\widetilde{x},\widetilde{y},\widetilde{z})$ are two copies of the local coordinates on M near the boundary ∂M, as before. Then the heat kernel H has a non-uniform behaviour at the diagonal $D = \{(0,p,p) \mid p \in \overline{M}\}$ and at the submanifold

$$A = \{(t,(0,y,z),(0,\widetilde{y},\widetilde{z})) \in M^2_h : y = \widetilde{y}\}.$$

The asymptotic behaviour of H near the submanifolds D and A of M^2_h is conveniently studied using the blowup procedure of §2. We proceed in this section with the following 2 steps:

Step 1: We construct the so called heat blowup space HM_ϕ by an additional blowup in $[0,\infty) \times M^2_\phi$, where M^2_ϕ is the ϕ-double space introduced in §4. More specifically, we lift the diagonal $D \in M^2_h$ to $[0,\infty) \times M^2_\phi$ and blow it up parabolically, which simply means that we treat $\sqrt{t}$ as a smooth function. We then may define the heat calculus of smoothing operators with Schwartz kernels that lift to polyhomogeneous conormal functions on the heat blowup space HM_ϕ.

Step 2: We obtain an initial parametrix for H inside the heat calculus, solving the heat equation up to first order. This requires us to lift the heat equation to HM_ϕ and to solve the resulting equations (normal problems) at the various boundary faces of the heat blowup space.

Step 3: The exact heat kernel is then obtained by a Neumann series argument, which requires the triple space construction and the composition formula of the final section §8.

6. Step 1: Construction of the heat blowup space

Consider the ϕ-double space M^2_ϕ with the blowdown map $\beta_\phi : M^2_\phi \to \overline{M}^2$. We obtain an intermediate heat blowup space by taking its product with the time axis $[0,\infty)$. We treat the square root of the time variable $\tau := \sqrt{t}$ as a smooth variable. The resulting intermediate heat blowup space is illustrated in Figure 3.

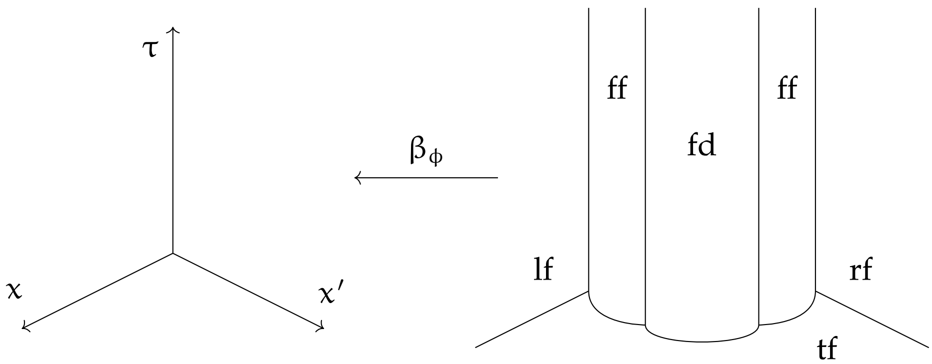

Figure 3. Intermediate heat blowup space $[0,\infty) \times M^2_\phi$.

Let us explain the abbreviations for the boundary hypersurfaces in the intermediate heat blowup space: ff stands for front face, fd – the fibre diagonal, lf and rf — the left and right faces, respectively, and finally tf stands for the temporal face. The projective coordinates on $[0,\infty) \times M^2_\phi$ near the various boundary hypersurfaces are as follows.

Projective coordinates near the intersection of rf with ff, away from fd. We have the following projective coordinates, which are valid uniformly up to tf, away from an open neighborhood of fd.

$$s = \frac{x}{x'}, \, y, \, z, \, x', \, y', \, z', \tau = \sqrt{t}. \tag{6.1}$$

In these coordinates, s is a defining function of rf, τ is a defining function of tf, x' is a defining function of ff. Interchanging the roles of x and x' yields projective coordinates near the intersection of lf with ff, where $s' = x'/x$ is a defining function of lf.

Projective coordinates near the fd up to tf. We have the following projective coordinates, which are valid in open neighborhood of fd uniformly up to tf, away from lf and rf. Here the roles of x and x' can be interchanged, leading to an equivalent system of coordinates.

$$S = \frac{s-1}{x'}, \, U = \frac{y-y'}{x'}, \, z, \, x', \, y', \, z', \, \tau = \sqrt{t}. \tag{6.2}$$

In these coordinates, τ is a defining function of tf, x' is a defining function of fd, and ff lies in the limit $|(S,U)| \to \infty$.

Heat blowup space as a blowup of temporal diagonal. The final heat blowup space HM_ϕ is obtained by blowing up the lift of the diagonal $D = \{(0,p,p) \mid p \in \overline{M}\}$ to $[0,\infty) \times M^2_\phi$. In the local coordinates in an open neighborhood $\mathcal{U}_{fd}$ of fd, its lift is given by

$$\beta_\phi^{-1}(D) \restriction \mathcal{U}_{fd} = \{S = 0, U = 0, z = z', \tau = 0\}. \tag{6.3}$$

The heat blowup space HM_ϕ is then defined as a parabolic blowup

$$HM_\phi := \left[[0,\infty) \times M^2_\phi, \beta_\phi^{-1}(D) \right]. \tag{6.4}$$

This blowup space is illustrated in 19.

The full blow down map is defined by

$$\beta := \beta_\phi \circ \beta'_\phi : HM_\phi \to [0,\infty) \times \overline{M}^2.$$

Let us now describe the resulting heat blowup space in projective coordinates. The previous coordinate systems (6.1) and (6.2) remain valid away from an open neighborhood of the new boundary face td. Near td we have up to the

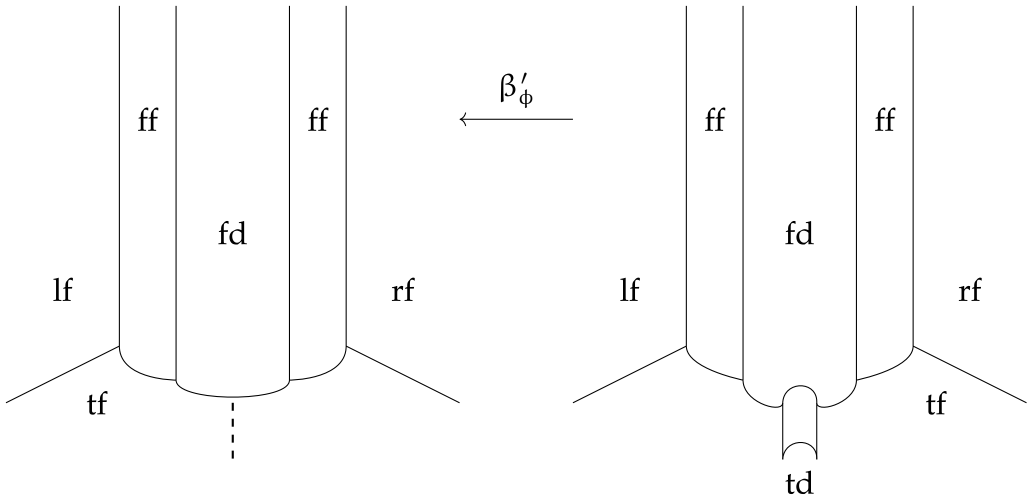

FIGURE 4. The heat blowup space HM_ϕ.

intersection with fd the following projective coordinates (with respect to the notation of (6.2))

$$\mathcal{S} = \frac{S}{\tau} = \frac{x - x'}{(x')^2\sqrt{t}}, \mathcal{U} = \frac{U}{\tau} = \frac{y - y'}{x'\sqrt{t}}, \mathcal{Z} = \frac{z - z'}{\sqrt{t}}, x', y', z', \tau = \sqrt{t}. \tag{6.5}$$

In these coordinates, x' is a defining function of fd, τ is a defining function of td, and tf lies in the limit $|(\mathcal{S}, \mathcal{U}, \mathcal{Z})| \to \infty$. In these coordinates, the roles of x and x' may be interchanged freely. The pullback by the blowdown map β is locally simply a change between standard and projective coordinates (6.1), (6.2) and (6.2).

We conclude the heat blow up space construction by singling out a class of polyhomogeneous conormal functions on it, that define the "heat calculus" in our setting.

Definition 6.1 (Heat calculus)**.** We write $m = \dim M$ and $b = \dim B$. For any $a, \ell \in \mathbb{R}$, the space $\Psi^{a,\ell}_\phi(M)$ is defined as the space of operators A acting on compactly supported smooth sections of $\Lambda^* T^* M$, given by Schwartz kernels K_A, taking values in $\Lambda^{*\phi}T^*M \boxtimes \Lambda^{*\phi}T^*M$, which lift to polyhomogeneus conormal functions $\beta^* K_A$ on HM_ϕ such that for any defining functions ρ_{fd} and ρ_{td} of fd and td, respectively, we have

$$\beta^* K_A = \rho_{fd}^{-3+a} \cdot \rho_{td}^{-m+\ell} G_A,$$

for some polyhomogeneous conormal function G_A, smooth at fd and td, and vanishing to infinite order at lf, rf, ff and tf. We further define

$$\Psi^\infty_\phi(M) := \bigcap_{a,\ell,\in\mathbb{R}} \Psi^{a,\ell}_\phi(M).$$

Remark 6.2. We point out that our Schwartz kernels are not multiplied with half-densities here, which is common in many other references. This simplifies our presentation here, but leads to some shifts in the asymptotics later on, when we study compositions of the Schwartz kernels in §8.

7. Step 2: Construction of an initial heat kernel parametrix

We construct an initial heat parametrix by solving the heat equation, lifted to HM_ϕ, to leading order at fd and td. The solutions of the heat equation at fd and td can be extended off these boundary faces with any power of the respective defining functions. The correct powers are determined by studying the lift of the delta distribution at fd and td as well.

Solving the heat equation near fd. Let us consider the relevant geometric quantities written in projective coordinates near fd. Recall the projective coordinates (6.2), that are valid near fd, away from td. We can interchange the roles of x and x', as well as the roles of y and y', and still get projective coordinates[8] that are valid near fd away from td (recall $s' = x'/x$)

$$S' = \frac{s'-1}{x},\ U' = \frac{y'-y}{x},\ z,\ x,\ y,\ z',\ \tau = \sqrt{t}. \tag{7.1}$$

In these coordinates, τ is a defining function of tf, x is a defining function of fd, and ff lies in the limit $|(S', U')| \to \infty$. We compute in these coordinates

$$\begin{aligned} \beta^*\partial_x &= -x^{-2}\left((1+2xS')\partial_{S'} - x^2\partial_x + xU'\partial_{U'} \right), \\ \beta^*\partial_y &= -x^{-1}\left(\partial_{U'} - \partial_y \right), \quad \beta^*\partial_z = \partial_z. \end{aligned} \tag{7.2}$$

Let us point out that fd is the total space of fibration over B with fibres $\mathbb{R}\times\mathbb{R}^b\times F^2$. Here, $y \in B$ denotes the base point of the fibration, $(S', U', z, z') \in \mathbb{R}\times\mathbb{R}^b\times F^2$ coordinates on the fibres. In view of (7.2) and submersion's assumption we compute

$$\beta^*\Delta_\phi = (1+2xS')^2\partial_{S'}^2 + \Delta_{U',y} + \Delta_{F,y} + \mathcal{O},$$

where $\Delta_{U',y}$ is the Laplacian on $\mathbb{R}^b$ with Euclidean coordinates U', defined with respect to the metric $g_B(y)$ at $T_yB \cong \mathbb{R}^b$; $\Delta_{F,y}$ is the Laplacian on the fibre $(F, g_F(y))$ at the base point y. The non-explicit term $\mathcal{O}$ is a differential operator in (S', U', z) with coefficients that vanish at $x = 0$. We conclude

$$\beta^*\left(t\left(\partial_t + \Delta_\phi\right)\right) \restriction \mathrm{fd} = \frac{1}{2}\tau\partial_\tau + \tau^2\left(-\partial_{S'}^2 + \Delta_{U',y} + \Delta_{F,y}\right) =: \mathcal{L}_{\mathrm{fd}}.$$

[8]That change makes the lifts of the partial derivatives in (7.2) more complicated, but it allows us to write the lift of the volume form in (7.5) in a more convenient form.

Note that the parameter y simply indicates the base point of the fibration fd, and for each fixed base point, the equation $\mathcal{L}_{fd} u = 0$ is an partial differential equation on the fibres of fd. A solution to $\mathcal{L}_{fd} u = 0$ is given by

$$N_{fd}(H)(\tau, S', U', z, z'; y) := \frac{e^{-\frac{|S'|^2}{4\tau^2}}}{(4\pi\tau^2)^{\frac{m}{2}}} H_{\Delta_{U',y}}(\tau, U', 0) H_{\Delta_{F,y}}(\tau, z, z'). \tag{7.3}$$

We need to extend it off the front face fd as

$$\beta^* H_0 := x^\alpha N_{fd}(H)(\tau, S', U', z, z'; y)\, \psi(x),$$

with a cutoff function $\psi \in C_0^\infty[0, \infty)$ with compact support in $[0, 2\varepsilon)$ such that $\psi \equiv 1$ on $[0, \varepsilon]$. At the moment $\alpha \in \mathbb{R}$ can be chosen freely, since $x > 0$ does not appear in the equation $\mathcal{L}_{fd} u = 0$. We need to fix a particular value of α such that for the volume form $dvol_{g_\phi}$ we have as $\tau \to 0$ the following equality of distributions

$$\lim_{\tau \to 0} \beta^* \left(H_0 dvol_{g_\phi} \right) |_{fd} = \delta(S') \cdot \delta(U') \cdot \delta(z, z'). \tag{7.4}$$

We compute for the lift of volume form near fd as $x \to 0$, writing $\mathcal{O}$ for smooth functions on HM_ϕ that vanish at fd

$$\begin{aligned} \beta^* dvol_{g_\phi} &= \beta^* \left((x')^{-b-2} \sqrt{det g_B} \sqrt{det g_F}\, dx' dy' dz' \right) \cdot (1 + \mathcal{O}) \\ &= \sqrt{det g_B(y)} \sqrt{det g_F(y)} dS' dU' dz' \cdot (1 + \mathcal{O}), \end{aligned} \tag{7.5}$$

where $g_B(y)$ is the restriction of g_B to $T_y B$, and $g_F(y)$ the restriction of g_F to the vertical subspace $T_p^V \partial M$, see the notation in submersion's assumption. Setting $\alpha = 0$, we compute from (7.3) and (7.5)

$$\begin{aligned} \lim_{\tau \to 0} \beta^* \left(H_0 dvol_{g_\phi} \right) |_{fd} = \lim_{\tau \to 0} &\left(\frac{e^{-\frac{|S'|^2}{4\tau^2}}}{(4\pi\tau^2)^{\frac{m}{2}}} dS' \right) \\ &\times \lim_{\tau \to 0} \left(H_{\Delta_{U',y}}(\tau, U', 0) \sqrt{det g_B(y)} dU' \right) \\ &\times \lim_{\tau \to 0} \left(H_{\Delta_{F,y}}(\tau, z, z') \sqrt{det g_F(y)} dz' \right) \\ &= \delta(S') \cdot \delta(U') \cdot \delta(z, z'). \end{aligned}$$

Hence the following heat parametrix solves the heat equation to first order at fd, and satisfies the initial condition (7.4). Writing $H_{\Delta_{U',y}}$ for the heat kernel of $\Delta_{U',y}$, and $H_{\Delta_{F,y}}$ for the heat kernel of $\Delta_{F,y}$, we set

$$\begin{aligned} \beta^* H_0' :&= N_{fd}(H)(\tau, S', U', z, z'; y)\, \psi(x) \\ &\equiv \frac{e^{-\frac{|S'|^2}{4\tau^2}}}{(4\pi\tau^2)^{\frac{m}{2}}} H_{\Delta_{U',y}}(\tau, U', 0) H_{\Delta_{F,y}}(\tau, z, z')\, \psi(x). \end{aligned} \tag{7.6}$$

Solving the heat equation near $\mathrm{td} \cap \mathrm{fd}$. Our heat parametrix H_0' in (7.6) does not solve the heat equation to any order at td. Here we explain the standard procedure how H_0' is corrected to provide a heat parametrix, solving the heat equation to higher order at td as well. Recall, near td we have up to the intersection with fd the coordinates (6.5)

$$\mathcal{S} = \frac{S}{\tau} = \frac{x - x'}{(x')^2\sqrt{t}}, \ \mathcal{U} = \frac{U}{\tau} = \frac{y - y'}{x'\sqrt{t}}, \ \mathcal{Z} = \frac{z - z'}{\sqrt{t}}, \ x', \ y', \ z', \ \tau = \sqrt{t}. \tag{7.7}$$

In these coordinates, x' is a defining function of fd, τ is a defining function of td, and tf lies in the limit $|(\mathcal{S}, \mathcal{U}, \mathcal{Z})| \to \infty$. In these coordinates, the roles of x and x' may be interchanged freely. We fix coordinates near $\mathrm{td} \cap \mathrm{fd}$. In these coordinates the individual partial derivatives are written as follows

$$\beta^*\partial_x = \frac{1}{\tau x'^2}\partial_{\mathcal{S}}, \quad \beta^*\partial_y = \frac{1}{\tau x'}\partial_{\mathcal{U}}, \quad \beta^*\partial_z = \frac{1}{\tau}\partial_{\mathcal{Z}}. \tag{7.8}$$

Let us point out that td is the total space of fibration over M with fibres $\mathbb{R} \times \mathbb{R}^b \times \mathbb{R}^f$. Here, $(x', y', z') \in M$ denotes the base point of the fibration, $(\mathcal{S}, \mathcal{U}, \mathcal{Z}) \in \mathbb{R} \times \mathbb{R}^b \times \mathbb{R}^f$ coordinates on the fibres. In view of (7.8) we compute

$$\beta^* t\Delta_\phi = -(1 + \tau x'\mathcal{S})^4 \partial_{\mathcal{S}}^2 + (1 + \tau x'\mathcal{S})^2 \Delta_{\mathcal{U}, y'} + \Delta_{\mathcal{Z}, y', z'} + \mathcal{O},$$

where $\Delta_{\mathcal{U}, y'}$ is the Laplacian on $\mathbb{R}^b$ with Euclidean coordinates $\mathcal{U}$, defined with respect to the metric $g_B(y')$ at $T_{y'}B \cong \mathbb{R}^b$; $\Delta_{\mathcal{Z}, y', z'}$ is the Laplacian on defined with respect to the metric $g_F(y', z')$ at $T^V_{(y', z')}\partial M \cong \mathbb{R}^f$. The non-explicit term $\mathcal{O}$ is a differential operator in $(\mathcal{S}, \mathcal{U}, \mathcal{Z})$ with coefficients that vanish at $\tau = 0$. We conclude

$$\beta^* \left(t\left(\partial_t + \Delta_\phi\right)\right) \restriction \mathrm{td} = \frac{1}{2}\tau\partial_\tau + \Delta_{(\mathcal{S}, \mathcal{U}, \mathcal{Z})} =: \mathcal{L}_{\mathrm{td}},$$

where $\Delta_{(\mathcal{S}, \mathcal{U}, \mathcal{Z})} = -\partial_{\mathcal{S}}^2 + \Delta_{\mathcal{U}, y'} + \Delta_{\mathcal{Z}, y', z'}$ is in fact the Laplacian on $\mathbb{R}^m$ defined with respect to the metric $\overline{g} = dx^2 + \phi^* g_B + g_F + h$ at $T_{(x', y', z')}M$. Note that the parameters $(x', y', z') \in M$ simply indicate the base point of the fibration td, and for each fixed base point, the equation $\mathcal{L}_{\mathrm{td}} u = 0$ is an partial differential equation on the fibres of td. A solution to $\mathcal{L}_{\mathrm{td}} u = 0$ is given by

$$N_{\mathrm{td}}(H)(\tau, \mathcal{S}, \mathcal{U}, \mathcal{Z}, x', y', z') := \frac{1}{(4\pi\tau^2)^{\frac{m}{2}}} \exp\left(-\|(\mathcal{S}, \mathcal{U}, \mathcal{Z})\|^2_{\overline{g} \restriction T_{(x', y', z')}M}\right). \tag{7.9}$$

Constructing an initial heat parametrix. In view of the previous discussion, we need to extend H_0' to a neighborhood of td, such that (7.9) is a leading order term in its td asymptotics. This would define a heat parametrix H_0'' which solves the heat equation to first order at fd and at td, satisfying the usual initial condition. Now the solutions $N_{fd}(H)$ and $N_{td}(H)$ are related as follows

$$\begin{aligned} N_{fd}(H) &\sim \tau^{-m} N_{td}(H)|_{fd}, &&\text{as } \tau \to 0, \\ N_{td}(H) &\sim N_{fd}(H)|_{td}, &&\text{as } x' \to 0. \end{aligned} \tag{7.10}$$

Thus there exists a well-defined operator $H_0'' \in \Psi_\phi^{3,0}(M)$ with the Schwartz kernel still denoted by H_0'' that lifts to a polyhomogeneous function $\beta^* H_0''$ on HM_ϕ, compactly supported in an open neighborhood of fd and td, such that

$$\begin{aligned} \beta^* H_0'' &\sim \tau^{-m} N_{td}(H), &&\text{as } \tau \to 0, \\ \beta^* H_0'' &\sim N_{fd}(H), &&\text{as } x' \to 0. \end{aligned} \tag{7.11}$$

By construction, $t(\partial_t + \Delta_\phi)H_0'' \in \Psi_\phi^{4,1}(M)$ and we can solve the error at td away by the usual argument, which is outlined in various cases, cf. Melrose [?], and Grieser [Gri04] as a basic reference, as well as Albin [Alb07], Mazzeo and Vertman [MaVe12] for the same argument in different geometric settings. This defines an initial heat parametrix $H_0 \in \Psi_\phi^{3,0}(M)$ such that $(\partial_t + \Delta_\phi)H_0 \in \Psi_\phi^{4,\infty}(M)$ and hence proves the following result.

Theorem 7.1. *There exists an initial heat parametrix $H_0 \in \Psi_\phi^{3,0}(M)$, solving the heat equation to first order at fd, and to infinite order at td, i.e. $(\partial_t + \Delta_\phi)H_0 \in \Psi_\phi^{4,\infty}(M)$. The restriction of H_0 to fd is given by $N_{fd}(H)$. The leading term in the asymptotic expansion of H_0 at td is given by $N_{td}(H)$, and hence in particular, H_0 converges to the delta distribution as $t \to 0$.*

8. Step 3: Triple space construction and composition of operators

In this section we use the initial heat parametrix H_0 in Theorem 7.1 to construct the exact heat kernel as a polyhomogeneous function on the heat space HM_ϕ. The construction is based on the following composition result, which is the main technical result of this section.

Theorem 8.1. ***(Composition Theorem)*** *Assume that, $A \in \Psi_\phi^{a,\ell}(M)$, $B \in \Psi_\phi^{a',\infty}(M)$. We denote the corresponding Schwartz kernels of A and B by K_A and K_B, respectively. Then the composition $A \circ B$ with the Schwartz kernel given by*

$$K_{A\circ B}(t, p, p') := \int_0^t K_A(t - t', p, p'')K_B(t', p'', p')dt' \, d\mathrm{vol}_{g_\phi}(p''),$$

is well defined and $A \circ B \in \Psi_\phi^{a+a',\infty}(M)$.

We will prove this theorem below, and assuming it for the moment we explain the heat kernel construction, leading up to the actual heat kernel. Consider the initial heat parametrix H_0 of Theorem 7.1, with the error term

$$P := (\partial_t + \Delta_\phi)H_0 \in \Psi_\phi^{4,\infty}(M).$$

Below we view H_0 as an operator acting on any $\omega \in C^\infty(M \times [0,t])$, for simplicity assumed to be compactly supported in the space variable, by additional convolution in time

$$H_0 * \omega(t,p) := \int_0^t H_0(t-t',p,p')\omega(t',p')dt'\,dvol_{g_\phi}(p').$$

Then we compute, using convergence of H_0 to the delta distribution as $t \to 0$

$$(\partial_t + \Delta_\phi)H_0 * \omega = (\mathrm{Id} + P)\omega. \tag{8.1}$$

Hence, viewing the operators H_0 and P as integral operators acting by additional convolution in time, we obtain the heat kernel formally by inverting the error term $(\mathrm{Id} + P)$ on the right hand side of (8.1)

$$e^{-t\Delta_\phi} = H_0 \circ (\mathrm{Id} + P)^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell H_0 \circ P^\ell, \tag{8.2}$$

where the Neumann series is a priori just a formal sum at the moment. By the composition result in Theorem 8.1 we have for any $\ell \in \mathbb{N}$

$$H_0 \circ P^\ell \in \Psi_\phi^{3+4\ell,\infty}(M).$$

From here we conclude $e^{-t\Delta_\phi} \in \Psi_\phi^{3,0}(M)$ once convergence of the sum of Schwartz kernels is established: $\beta^* K_P$ vanishes to infinite order at td, and hence K_P lifts to be continuous on $M^2_\phi \times [0,\infty)$ without blowing up the temporal diagonal. Thus, by the usual Volterra series argument, K_{P^ℓ} at $t \in [0,\infty)$ can be estimated (up to a uniform constant) against t^ℓ for any $\ell \in \mathbb{N}$. In fact this is a general feature of Volterra series, cf. [BGV03], [Mel93], [Gri04]. Thus, (8.2) converges. Thus we may conclude our main result.

Theorem 8.2. *The heat kernel of Δ_ϕ lifts to a polyhomogeneous function on the heat space HM_ϕ, vanishing to infinite order at ff, tf, rf and lf, smooth at fd, and of order $(-m)$ at td. More precisely, $e^{-t\Delta_\phi} \in \Psi_\phi^{3,0}(M)$.*

8.1. **Proof of the composition theorem.** In this subsection we prove Theorem 8.1, where $\beta^* K_B$ is assumed to be vanishing to infinite order at td, but $\beta^* K_A$ is not necessarily. We begin with a proposition, that will allow us to assume without loss of generality that in this case both $\beta^* K_A$ and $\beta^* K_B$ are vanishing to infinite order at td.

Proposition 8.3. *Consider $A \in \Psi_\phi^{a,\ell}(M), B \in \Psi_\phi^{a',\infty}(M)$. Consider the composition $A \circ B$ with the Schwartz kernel $K_{A\circ B}$. Then $\beta^* K_{A\circ B}$ vanishes to infinite order at td.*

Proof. Consider the composition kernel

$$K_{A\circ B}(t,p,p') = \int_0^t \int_M K_A(t-\widetilde{t},p,\widetilde{p})K_B(\widetilde{t},\widetilde{p},p')d\widetilde{t}dvol_{g_\phi}(\widetilde{p}). \tag{8.3}$$

Note that as $\widetilde{p} \to \partial M$, the kernel $K_A(t-\widetilde{t}, p, \widetilde{p})$ approaches to the left face and $K_B(\widetilde{t}, \widetilde{p}, p')$ approaches to right face. In both cases, these kernels vanish to infinite order uniformly in the time variable. Hence the composition is well-defined.

We fix $p, p' \in M$. Since K_B vanishes to infinite order at td, td and rf, there exists for each N there is a constant $C > 0$, depending only on p' and N, such that

$$|\beta^* K_B(\widetilde{t}, \widetilde{p}, p')| \leq C \cdot \widetilde{t}^N.$$

Using projective coordinates near td with $\tau = \sqrt{t - \widetilde{t}}$, we find $\beta^* \mathrm{dvol}_{g_\phi} \sim \tau^m$ and $\beta^* K_A \sim \tau^{-m+a}$ as $\tau \to 0$. Hence $\beta^*(K_A \mathrm{dvol}_{g_\phi})$ is uniformly bounded as $\tau \to 0$ and thus its M-integral can be estimates against a constant, independent of τ. Thus we find for a constant $\widetilde{C} > 0$, depending only on p, p' and N

$$|K_{A \circ B}(t, p, p')| \leq \widetilde{C} \cdot \int_0^t \widetilde{t}^N d\widetilde{t} = \frac{\widetilde{C}}{N+1} \cdot t^{N+1}.$$

Since N can be taken arbitrarily large, we conclude that $\beta^* K_{A \circ B}$ vanishes to infinite order[9] at td. □

As a consequence, we can assume without loss of generality that both $\beta^* K_A$ and $\beta^* K_B$ are vanishing to infinite order at td and prove the composition Theorem 8.1 in that case. We proceed with the proof. We write the composition integral (8.3) using pullback and push forward as follows. We write $\mathbb{R}_+ \equiv \mathbb{R}^+ := [0, \infty)$ and define the maps

$$\begin{aligned}
\pi_C : \overline{M}^3 \times \mathbb{R}^+_{t'} \times \mathbb{R}^+_{t''} &\longrightarrow \overline{M}^2 \times \mathbb{R}^+_{t'+t''}, \quad (p, p', p'', t', t'') \to (p, p'', t' + t''), \\
\pi_L : \overline{M}^3 \times \mathbb{R}^+_{t'} \times \mathbb{R}^+_{t''} &\longrightarrow \overline{M}^2 \times \mathbb{R}^+_{t''}, \quad (p, p', p'', t', t'') \to (p, p', t''), \\
\pi_R : \overline{M}^3 \times \mathbb{R}^+_{t''} \times \mathbb{R}^+_{t''} &\longrightarrow \overline{M}^2 \times \mathbb{R}^+_{t'}, \quad (p, p', p'', t', t'') \to (p', p'', t').
\end{aligned}$$

Then we can write (8.3) by pulling back K_A and K_B to $\overline{M}^3 \times \mathbb{R}^2_+$ via π_L, π_R and pushing forward (integrate) along $t = t' + t''$ and in p' with respect to dvol_{g_ϕ}

$$K_C = (\pi_C)_* \left((\pi_L^* K_A) \cdot (\pi_R^* K_B) \right). \tag{8.4}$$

We prove the composition theorem by constructing the so called heat triple space, denoted by HM^3_ϕ and obtained by a resolution process from $\overline{M}^3 \times \mathbb{R}^2_+$, with blow down map

$$\beta_3 : HM^3_\phi \longrightarrow \overline{M}^3 \times \mathbb{R}^2_+.$$

[9]The proof of this proposition does not make any statement, whether the vanishing at td is uniform up to fd. That is the content of the composition Theorem 8.1.

We show that the maps π_C, π_L, π_R lift to b-fibrations Π_C, Π_L, Π_R in the sense of Melrose [**?**], i.e. in the commutative diagram diagram (8.5)

$$\begin{array}{ccc} HM^3_\phi & \longrightarrow & M^2_\phi \times \mathbb{R}^+ \\ {\scriptstyle \beta_{Tr}}\big\downarrow & & \big\downarrow {\scriptstyle \beta_\phi} \\ \overline{M}^3 \times \mathbb{R}^2_+ & \longrightarrow & \overline{M}^2 \times \mathbb{R}_+ \end{array} \tag{8.5}$$

the maps π_C, π_L, π_R in bottom arrow lift to b-fibrations Π_C, Π_L, Π_R in the upper arrow. Here we use the intermediate heat space $M^2_\phi \times \mathbb{R}^+$, introduced in Figure 3, together with the corresponding blowdown map β_ϕ, since the additional blowup of the temporal diagonal will not be necessary due to Proposition 8.3. Defining $\kappa_{A,B,C} := \beta^*_\phi(K_{A,B,C})$, we obtain using the commutativity of diagram (8.5) a new version of the relation (8.4)

$$\kappa_C \equiv \beta^*_\phi(K_C) = (\Pi_C)_*(\Pi^*_L \kappa_A \otimes \Pi^*_R \kappa_B). \tag{8.6}$$

The idea is now to see that $\Pi^*_L \kappa_A, \Pi^*_R \kappa_B$ are indeed polyhomogeneous and the pushforward under Π_C preserves the polyhomogeneity. In the rest of this section we follow this strategy more concretely, first construct the triple space HM^3_ϕ, compute the lift of boundary defining functions under projections to compute explicitly the asymptotics of $\Pi^*_L \kappa_A, \Pi^*_R \kappa_B$ and also the pushfoward, $(\Pi_C)_*(\Pi^*_L \kappa_A \otimes \Pi^*_R \kappa_B)$.

Construction of the triple space. In order to apply Melrose's pushforward theorem [Mel92], to conclude polyhomogeneity κ_C, the maps Π_C, Π_L, Π_R need to be b-fibrations. This dictates the construction of the triple space HM^3_ϕ as a blowup of $\overline{M}^3 \times \mathbb{R}^2_+$. We describe the blowups using local coordinates $p = (x, y, z)$ and their copies $p' = (x', y', z')$ and $p'' = (x'', y'', z'')$ on M. The time coordinates on each $\mathbb{R}^2_+$ are written as t and t'. The first submanifold to blow up is then

$$F := \{t' = t'' = 0, x = x' = x'' = 0\} \subset \overline{M}^3 \times \mathbb{R}^2_+.$$

We refer the reader to §**??** for the basic elements of the blowup procedure. As before we blow up parabolically in the time direction, i.e. we treat $\sqrt{t}$ and $\sqrt{t'}$ as smooth. The resulting blowup space $M^3_b = [\overline{M}^3 \times \mathbb{R}^2_+; F]$ is illustrated in Figure 5 and comes with the blow down map

$$\beta_1 : M^3_b \longrightarrow \overline{M}^3 \times \mathbb{R}^2_+.$$

We denote the resulting new boundary face, which is the inward spherical normal bundle of $F \subset \overline{M}^3 \times \mathbb{R}^2_+$ by (111), where the first 1 indicates that the boundary face corresponds to $x = 0$, the second 1 corresponds to $x' = 0$, and

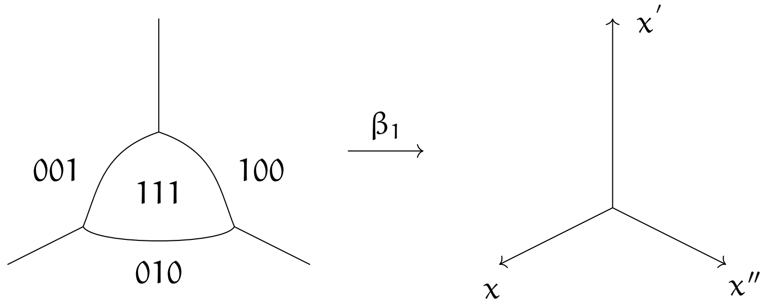

FIGURE 5. Illustration of M^3_b in spatial direction.

the third 1 to $x'' = 0$. This principle is also used in the namesgiving for other boundary faces, e.g. (100) is the lift of $\{x = 0\}$.

We then blow up M^3_b at

$$F_O := \beta_1^*\{t' = t'' = 0\}.$$

The next submanifolds to blow up are submanifolds of the codimension 2 in spatial direction corresponding to each projection π_C, π_L, π_R. Accordingly we denote these submanifolds as F_C, F_L, F_R, which are defined as

$$\begin{aligned} F_C &:= \beta_1^*\{t' = t'' = 0, x = x'' = 0\}, \\ F_L &:= \beta_1^*\{t'' = 0, x' = x'' = 0\}, \\ F_R &:= \beta_1^*\{t' = 0, x = x' = 0\}. \end{aligned}$$

We point out that the order of blowing of submanifolds F_C, F_L, F_R after blowing up of F and F_O is immaterial as they become disjoint. As before we blow up parabolically in the time direction. The resulting blowup space

$$M^3_{b,t} = [[M^3_b; F_O]; F_C, F_L, F_R],$$

is illustrated in Figure 6 and comes with the blow down map

$$\beta_2 : M^3_{b,t} \longrightarrow M^3_b.$$

The triple elliptic space of the ϕ-calculus, see Grieser and Hunsicker [?], includes the fibre-diagonal blow up in each direction. Here, we need to perform the same blowups combined with blowing up the time direction. More precisely, using local coordinates, we blow up the following submanifolds

$$\begin{aligned} F_{C,Sc} &:= (F \cup F_C) \cap (\beta_2 \circ \beta_1)^*\{x = x'', y = y''\}, \\ F_{L,Sc} &:= (F \cup F_L) \cap (\beta_2 \circ \beta_1)^*\{x' = x'', y' = y''\}, \\ F_{R,Sc} &:= (F \cup F_R) \cap (\beta_2 \circ \beta_1)^*\{x = x', y = y'\}. \end{aligned}$$

as well as their intersection

$$O := F_{C,Sc} \cap F_{L,Sc} \cap F_{R,Sc}.$$

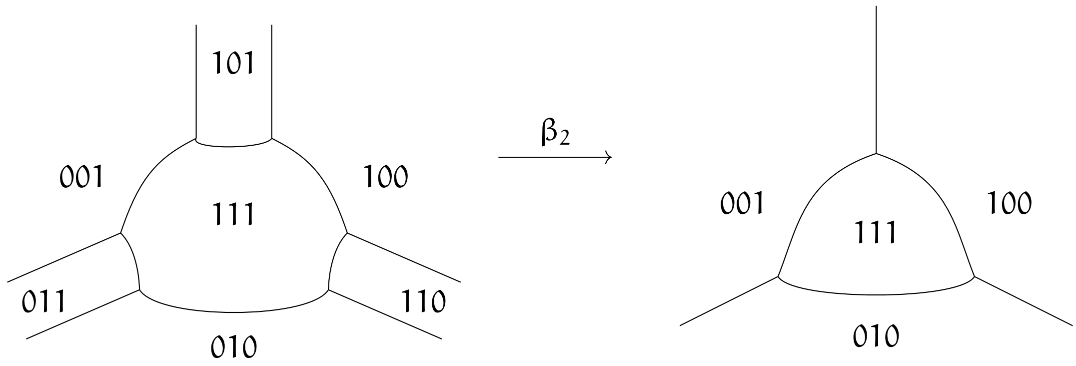

FIGURE 6. Illustration of $M^3_{b,t}$ in spatial direction.

This defines the triple space in the heat calculus, illustrated in Figure 7 as

$$HM^3_\phi := [\overline{M}^3_{b,t}, O; F_{C,Sc}, F_{L,Sc}, F_{R,Sc}].$$

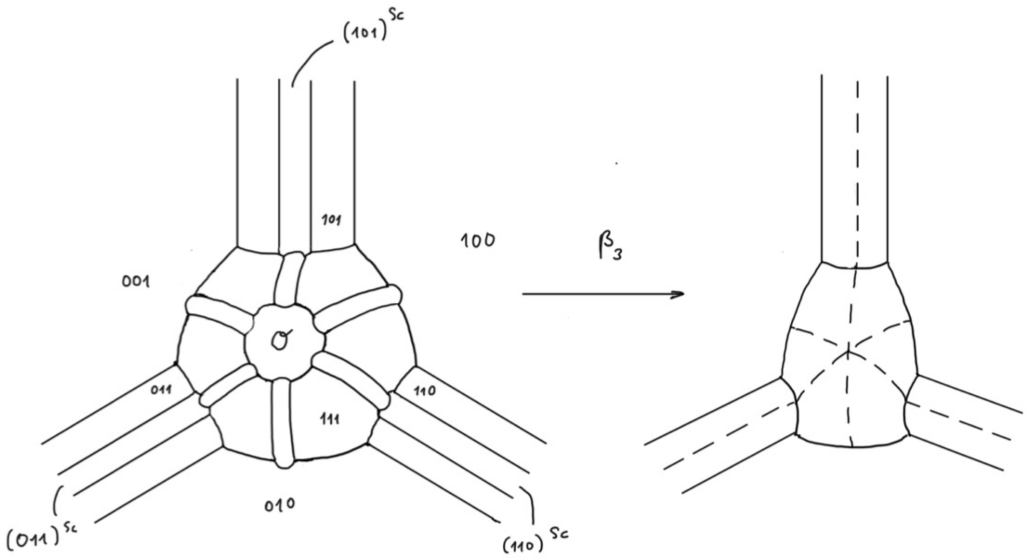

FIGURE 7. Illustration of HM^3_ϕ in spatial direction.

The triple space comes with the intermediate blow down map $\beta_3 : HM^3_\phi \to M^3_{b,t}$ and we define the total blowdown map as

$$\beta_{Tr} := \beta_3 \circ \beta_2 \circ \beta_1 : HM^3_\phi \longrightarrow \overline{M}^3 \times \mathbb{R}^2_+.$$

From the construction, we compute immediately

$$
\begin{aligned}
\beta_{Tr}^*(x) &= \rho_{111}\rho_{O}\rho_{101}\rho_{101}^{sc}\rho_{110}\rho_{110}^{sc}\rho_{100},\\
\beta_{Tr}^*(x') &= \rho_{111}\rho_{O}\rho_{011}\rho_{011}^{sc}\rho_{110}\rho_{110}^{sc}\rho_{010},\\
\beta_{Tr}^*(x'') &= \rho_{111}\rho_{O}\rho_{101}\rho_{101}^{sc}\rho_{011}\rho_{011}^{sc}\rho_{001}.
\end{aligned}
\tag{8.7}
$$

Lifts of boundary defining functions under various projections. We shall now study the lifts of boundary defining functions on the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$ to the triple heat space under the maps $\Pi_C, \Pi_L, \Pi_R : HM_\phi^3 \to M_\phi^2 \times \mathbb{R}^+$. We use the following notation: we denote a boundary defining function of any boundary face (ijk), in HM_ϕ^3 by $\rho_{ijk}; i, j, k \in \{0, 1\}$. Boundary defining functions of $(110)^{Sc}, (101)^{Sc}$ and $(011)^{Sc}$ are denoted by $\rho_{110}^{Sc}, \rho_{101}^{Sc}$ and ρ_{011}^{Sc}, respectively. The boundary face $\mathcal{O}$, arising from the blowup of O, comes with a defining function ρ_O. The boundary face, arising from the blowup of F_O, comes with a defining function τ_O. Let τ and τ' be defining functions for the two boundary faces in HM_ϕ^3, corresponding to $\{t' = 0\}$ and $\{t'' = 0\}$, respectively. The notation is according to the labels in Figures 8 and 7.

We rename the boundary faces lf, rf and ff in the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$ as $(ij), i, j \in \{0, 1\}$, according to the labels in Figure 8 and write for the corresponding boundary defining functions ρ_{ij}. The boundary face fd is renamed $(11)^{Sc}$ and its defining function is written as ρ_{11}^{Sc}. Defining function of tf in $M_\phi^2 \times \mathbb{R}^+$ is denoted by τ. The notation is according to the labels in Figure 8, and corresponds closely to the notation of boundary faces on the triple space.

We compute the pullback of $\rho_{11}, \rho_{01}, \rho_{10}, \rho_{11}^{sc}$ under Π_C, Π_L, Π_R. Here, Figure 8 provides a helpful orientation.

$$
\begin{array}{lll}
\Pi_C^*(\rho_{11}) = \rho_{111} \cdot \rho_{101} & \Pi_L^*(\rho_{11}) = \rho_{111} \cdot \rho_{110} & \Pi_R^*(\rho_{11}) = \rho_{111} \cdot \rho_{011} \\
\Pi_C^*(\rho_{01}) = \rho_{011} \cdot \rho_{011}^{Sc} \cdot \rho_{001} & \Pi_L^*(\rho_{01}) = \rho_{011} \cdot \rho_{011}^{Sc} \cdot \rho_{010} & \Pi_R^*(\rho_{01}) = \rho_{101} \cdot \rho_{101}^{Sc} \cdot \rho_{001} \\
\Pi_C^*(\rho_{10}) = \rho_{110} \cdot \rho_{110}^{Sc} \cdot \rho_{100} & \Pi_L^*(\rho_{10}) = \rho_{101} \cdot \rho_{101}^{Sc} \cdot \rho_{100} & \Pi_R^*(\rho_{10}) = \rho_{110} \cdot \rho_{110}^{Sc} \cdot \rho_{010} \\
\Pi_C^*(\rho_{11}^{Sc}) = \rho_{101}^{Sc} \cdot \rho_O & \Pi_L^*(\rho_{11}^{Sc}) = \rho_{110}^{Sc} \cdot \rho_O & \Pi_R^*(\rho_{11}^{Sc}) = \rho_{011}^{Sc} \cdot \rho_O
\end{array}
\tag{8.8}
$$

Now the lifts of the time variable τ are somewhat more intricate to argue. Let us first compute the lifts of time direction boundary defining functions under the blow down map β_{Tr}. We find

$$
\begin{aligned}
\beta_{Tr}^*(t') &= \tau' \cdot \tau_O \cdot \rho_{110}^2 \cdot (\rho_{110}^{Sc})^2 \cdot \rho_{101}^2 \cdot (\rho_{101}^{Sc})^2 \cdot \rho_{111}^2 \cdot \rho_O^2,\\
\beta_{Tr}^*(t'') &= \tau'' \cdot \tau_O \cdot \rho_{101}^2 \cdot (\rho_{101}^{Sc})^2 \cdot \rho_{011}^2 \cdot (\rho_{011}^{Sc})^2 \cdot \rho_{111}^2 \cdot \rho_O^2,\\
\beta_{Tr}^*(t' + t'') &= \tau_O \cdot \rho_{111}^2 \cdot \rho_{101}^2 \cdot \rho_{F_0}^2 \cdot (\rho_{101}^{Sc})^2
\end{aligned}
\tag{8.9}
$$

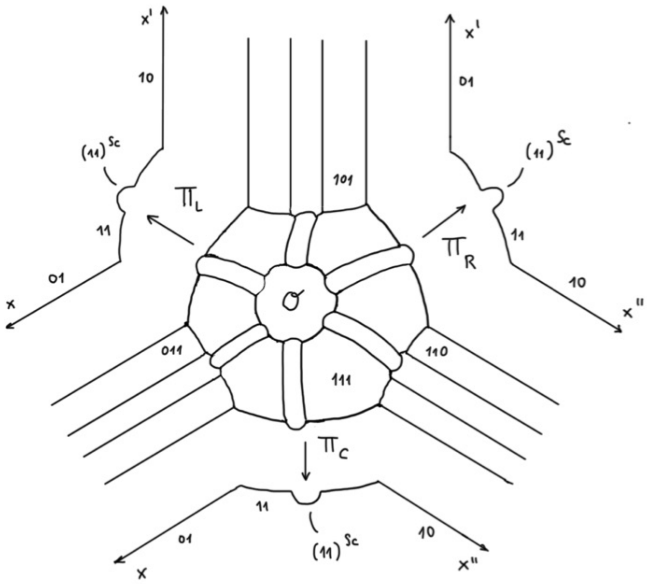

FIGURE 8. Illustration of projections in spatial direction.

From the commutative diagram (8.5), we have

$$\pi_C \circ \beta_{Tr} = \beta_\phi \circ \Pi_C. \tag{8.10}$$

Now the lifts

$$\beta_\phi^*(t') \subset \Pi_L(HM^3_\phi), \quad \beta_\phi^*(t'') \subset \Pi_R(HM^3_\phi), \quad \beta_\phi^*(t'+t'') \subset \Pi_C(HM^3_\phi),$$

are equal to $\tau \cdot (\rho_{11}^{Sc})^2 \cdot \rho_{11}^2$. Therefore we compute, in view of (8.9) and (8.10)

$$\begin{aligned}\Pi_C^*(\tau(\rho_{11}^{Sc})^2\rho_{11}^2) &= \Pi_C^*(\tau)\rho_{111}^2(\rho_{101}^{Sc})^2 \cdot \rho_O^2, \\ &= \tau_O \cdot \rho_{111}^2 \cdot (\rho_{101}^{Sc})^2 \cdot \rho_O^2.\end{aligned}$$

From here we conclude $\Pi_C^*(\tau) = \tau_O$. Similarly one can compute the other lifts and we arrive at the following identities

$$\begin{aligned}\Pi_L^*(\tau) &= \tau' \cdot \tau_O \cdot (\rho_{101}^{Sc})^2 \cdot \rho_{101}^2, \\ \Pi_R^*(\tau) &= \tau' \cdot \tau_O \cdot (\rho_{101}^{Sc})^2 \cdot \rho_{101}^2, \\ \Pi_C^*(\tau) &= \tau_O.\end{aligned} \tag{8.11}$$

Projections Π_C, Π_L, Π_R *are b-fibrations.* Crucial condition for the application of Melrose's pushforward theorem [MEL92] is that the maps Π_C, Π_L, Π_R are b-fibrations. Let us recall the notion of b-fibrations here briefly.

Definition 8.4. Assume M and M' be two compact manifolds with corners and let $f : M \longrightarrow M'$ be smooth map. Let $\{H_i\}_{i\in I}$ and $\{H'_i\}_{i\in I'}$ be enumerations of the codimension one boundary faces of M and M' respectively and let ρ_i and ρ'_j be global defining functions for H_i and H'_j, respectively.

(i) f is called b-map if for any $i \in I'$

$$f^*\rho'_i = A_{ij} \cdot \prod_{j\in I} \rho_j^{e(i,j)} \quad \text{for some } A_{ij} > 0, e(i,j) \in \mathbb{N} \cup \{0\}.$$

(ii) f is called b-submersion if the total differential f_* induces a surjective map between b-tangent bundles of M and M' where the the b-tangent space $T^b_p M$ at a corner p of codimension k, with local coordinates $(x_1, ..., x_k) \in \mathbb{R}^k_+$ and $(y_1, ..., y_{n-k}) \in \mathbb{R}^{n-k}$, is spanned by

$$\{x_1\partial_{x_1}, \cdots x_k\partial_{x_k}, \partial_{y_j}\}.$$

(iii) f is called b-fibration if for each j there is at most one i such that $e(i,j) \neq 0$. which means that H_i in M gets mapped into at most one H'_j in M'.

By discussion above, (8.8) and (8.11), Π_C, Π_L, Π_R are indeed b-fibrations.

Lifts of kernels and densities to the triple space. Consider $A \in \Psi^{\ell,q}_\phi(M)$, $B \in \Psi^{\ell',\infty}_\phi(M)$. By Proposition 8.3, we may assume $q = \infty$. We write K_A and K_B for the Schwartz kernels of A and B, respectively. We consider the composition $C = A \circ B$ with Schwartz kernel K_C. We have by construction

$$\begin{aligned} \pi_L^* K_A &= K_A(t', x, y, z, x', y', z'), \\ \pi_R^* K_B &= K_B(t'', x', y', z', x'', y'', z''). \end{aligned} \tag{8.12}$$

We also write $K_C \equiv K_C(t, x, y, z, x'', y'', z'')$ and set

$$\begin{aligned} \nu_3 &:= dt'dt''\mathrm{dvol}_{g_\phi}(x, y, z)\, \mathrm{dvol}_{g_\phi}(x', y', z')\, \mathrm{dvol}_{g_\phi}(x'', y'', z''), \\ \nu_2 &:= dt\, \mathrm{dvol}_{g_\phi}(x, y, z)\, \mathrm{dvol}_{g_\phi}(x'', y'', z''). \end{aligned} \tag{8.13}$$

Then we obtain by construction (making the relation (8.4) precise)

$$K_C \cdot \nu_2 = (\pi_C)_* \left(\pi_L^* K_A \cdot \pi_R^* K_A \cdot \beta_{Tr}^* \nu_3\right).$$

Writing this relation in terms of the lifts $\kappa_{A,B,C} = \beta_\phi^* K_{A,B,C}$, we obtain

$$\kappa_C \cdot \beta_\phi^* \nu_2 = (\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A \cdot \beta_{Tr}^* \nu_3\right). \tag{8.14}$$

This formula makes clear how to proceed

Step 1): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A$.
Step 2): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3$.
Step 3): Apply Pushfoward Theorem to study pushforward by $(\Pi_C)_*$
Step 4): Compute the asymptotics of $\beta_\phi^* \nu_2$
Step 5): Compare the asymptotics of both sides in (8.14) to study κ_C.

Step 1): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A$. Since $A \in \Psi_\phi^{\ell,p}(M)$ and $B \in \Psi_\phi^{\ell',\infty}(M)$, where we assume by Proposition 8.3 that $q = \infty$, we find by definition

$$\kappa_A \sim \rho_{fd}^{\ell-3} \left(\rho_{ff}\rho_{rf}\rho_{lf}\rho_{tf}\right)^\infty \equiv \left(\rho_{11}^{Sc}\right)^{\ell-3} \left(\tau\rho_{11}\rho_{10}\rho_{01}\right)^\infty,$$
$$\kappa_B \sim \rho_{fd}^{\ell'-3} \left(\rho_{ff}\rho_{rf}\rho_{lf}\rho_{tf}\right)^\infty \equiv \left(\rho_{11}^{Sc}\right)^{\ell'-3} \left(\tau\rho_{11}\rho_{10}\rho_{01}\right)^\infty.$$

From now on we adopt a notation, where for the kernels on the triple space HM_ϕ^3, we write out only the asymptotics at those boundary faces, where the kernel is not vanishing to infinite order, suppressing the other boundary faces from the formula. We find in view of the (8.8) and (8.11) after an explicit counting of exponents

$$\Pi_L^* \kappa_A \sim \left(\rho_{110}^{Sc}\rho_O\right)^{\ell-3},$$
$$\Pi_R^* \kappa_B \sim \left(\rho_{011}^{Sc}\rho_O\right)^{\ell'-3}.$$

Consequently, we arrive under the notation which suppresses defining functions of infinite order

$$\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \sim \rho_O^{\ell+\ell'-6}. \tag{8.15}$$

Step 2): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3$.

The density ν_3 is given in local coordinates by

$$\begin{aligned}\nu_3 &= dt'dt'' \left(x^{-b-2}dxdydz\right)\left(x'^{-b-2}dx'dy'dz'\right)\left(x''^{-b-2}dx''dy''dz''\right) \\ &= t't''(xx'x'')^{-1}\frac{dt'}{t'}\frac{dt''}{t''}\left(\frac{dx}{x}\frac{dy}{x^b}dz\right)\left(\frac{dx'}{x'}\frac{dy'}{x'^b}dz'\right)\left(\frac{dx''}{x''}\frac{dy''}{x''^b}dz''\right) \\ &=: t't''(xx'x'')^{-1}\nu_3',\end{aligned}$$

up to a smooth bounded function. The lift $\beta_{Tr}^*\nu_3'$ equals $h \cdot \nu_b^{(3)}$, where $\nu_b^{(3)}$ is a b-density on HM_ϕ^3, that is a smooth density on HM_ϕ^3, divided by a product of all its boundary defining functions. The factor h is a polyhomogeneous function on HM_ϕ^3, smooth at the boundary face $\mathcal{O}$; its asymptotics at other boundary faces is irrelevant, since $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B$ vanishes to infinite order there. Thus we compute in view of (8.7), (8.9) and (8.15)

$$\begin{aligned}&\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3 \\ &= \Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \left(t't''(xx'x'')^{-1}\right) \beta_{Tr}^* \nu_3' \\ &\sim \rho_O^{\ell+\ell'-5}\beta_{Tr}^*\nu_3' \sim \rho_O^{\ell+\ell'-5}\nu_b^{(3)},\end{aligned} \tag{8.16}$$

where as before we suppressed the other boundary defining functions from the notation, where the kernels vanish to infinite order.

Step 3): Apply Pushfoward Theorem to study pushforward by $(\Pi_C)_*$.

The next step is applying the pushforward theorem of Melrose [Mel92]. We use the notation of Definition 8.4. Then the pushforward theorem of Melrose says the following.

Theorem 8.5. *Let M, M' be two compact manifolds with corners and ν_b, ν_b' are b-densities on M, M', respectively. Let u be a polyhomogeneous function on M with index sets E_j at the faces H_j of M. Suppose that each $(z,p) \in E_j$ has $Re(z) > 0$ if the index j satisfies $e(i,j) = 0$ for all j. Then the pushforward $f_*(u\nu_b)$ is well-defined and equals $h\nu_b'$ where h is polyhomogeneous on M' and has an index family $f_b(\mathcal{E})$ given by an explicit formula in terms of the index family $\mathcal{E}$ for M.*

We refer the reader to [Mel92, Mel93] for the explicit definition of the index family $f_b(\mathcal{E})$, and just say that in our specific case we have for any $\alpha > 0$ (all other suppressed boundary functions enter with infinite order)

$$(\Pi_C)_* \left(\rho_O^\alpha \nu_b^{(3)} \right) = (\rho_{11}^{Sc})^\alpha \nu_b^{(2)},$$

where $\nu_b^{(2)}$ is a b-density on the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$, that is a smooth density on $M_\phi^2 \times \mathbb{R}^+$, divided by a product of all its boundary defining functions. Hence we arrive in view of (8.16) at

$$\begin{aligned}(\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3\right) &= (\rho_{11}^{Sc})^{\ell+\ell'-5} \nu_b^{(2)} \\ &\equiv (\rho_{11}^{Sc})^{\ell+\ell'-5} \left(\tau \rho_{11}\rho_{10}\rho_{01}\right)^\infty \nu_b^{(2)}.\end{aligned} \tag{8.17}$$

Step 4): Compute the asymptotics of $\beta_\phi^* \nu_2$.

The density ν_2 is given in local coordinates by

$$\begin{aligned}\nu_2 &= dt \left(x^{-b-2} dx dy dz\right) \left(x''^{-b-2} dx'' dy'' dz''\right) \\ &= t' t'' (xx'')^{-1} \frac{dt}{t} \left(\frac{dx}{x}\frac{dy}{x^b} dz\right)\left(\frac{dx''}{x''}\frac{dy''}{x''^b} dz''\right) \\ &=: t(xx'')^{-1} \nu_2',\end{aligned}$$

up to a smooth bounded function. The lift $\beta_\phi^* \nu_2'$ equals $h' \cdot \nu_b^{(2)}$, where h' is a polyhomogeneous function on $M_\phi^2 \times \mathbb{R}^+$, smooth at the boundary face fd; its asymptotics at other boundary faces is irrelevant, since it will be multiplied with a kernel that vanishes to infinite order there. Hence we find

$$\begin{aligned}\beta_\phi^* \nu_2 &= \beta_\phi^* \left(t(xx'')^{-1}\right) \nu_b^{(2)} \\ &= \tau \left(\rho_{11}^{Sc}\rho_{11}\right)^{-2} \left(\rho_{10}\rho_{01}\right)^{-1} h' \nu_b^{(2)}\end{aligned} \tag{8.18}$$

Step 5): Compare the asymptotics of both sides to study κ_C. Combining (8.17) and (8.18), we find

$$
\begin{aligned}
(\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3\right) &= (\rho_{11}^{Sc})^{\ell+\ell'-5} \left(\tau\rho_{11}\rho_{10}\rho_{01}\right)^\infty \nu_b^{(2)} \\
&= (\rho_{11}^{Sc})^{\ell+\ell'-3} \left(\tau\rho_{11}\rho_{10}\rho_{01}\right)^\infty \beta_\phi^* \nu_2 \\
&= \kappa_C \cdot \beta_\phi^* \nu_2.
\end{aligned}
\tag{8.19}
$$

This proves the composition Theorem 8.1.

9. Resolvent of ϕ Hodge Laplace

9.1. **Setting of manifolds with fibred boundary metrics.** Consider a compact smooth manifold $\overline{M}$ with interior M and boundary $\partial\overline{M}$ (also denoted here by ∂M), which is the total space of a fibration $\phi : \partial M \to B$ over a closed manifold B with fibres given by copies of a closed manifold F. We fix a boundary defining function $x : \overline{M} \to \mathbb{R}^+ = [0, \infty)$, i.e. $x^{-1}(0) = \partial M$ and $dx \neq 0$ at ∂M, and an identification of an open neighborhood of ∂M with a cylinder $[0, \varepsilon)_x \times \partial M$, for some $\varepsilon > 0$. We consider a ϕ-metric (also called fibred boundary metric) on M, i.e. a Riemannian metric g_ϕ which on $\mathcal{U} = (0, \varepsilon)_x \times \partial M$ has the form

$$
g_\phi \restriction \mathcal{U} = g_0 + h, \quad g_0 = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F \tag{9.1}
$$

where g_B is a Riemannian metric on the base B, g_F is a symmetric bilinear form on the total space ∂M, restricting to Riemannian metrics on the fibres F, and the perturbation h is a two tensor on $\mathcal{U}$ satisfying the bound given in Assumption 9.1 below. If $h \equiv 0$ then g_ϕ is called an exact ϕ-metric.

Recall that the Hodge Laplace operator associated to g_ϕ is $\Delta_\phi = (d + d^*)^2$ where d denotes the exterior derivative and d^* its adjoint. This is a self-adjoint and non-negative operator in $L^2(M, \Lambda T^*M; d\mathrm{vol}_\phi)$, where $d\mathrm{vol}_\phi$ is the volume form for g_ϕ, so the resolvent

$$
(\Delta_\phi + k^2)^{-1}
$$

is defined for all $k \neq 0$. We always take $k > 0$. The objective of this part is the precise analysis of the Schwartz kernel of the resolvent, in particular of its asymptotic behavior as $k \to 0$. More generally, our results apply to the Hodge Laplacian acting on sections of a vector bundle over $\overline{M}$ equipped with a flat connection.

If the fibration ϕ is trivial, $\partial M = B \times F$ has the product metric, and $h \equiv 0$ then Δ_ϕ acting on functions is, over $\mathcal{U}$, given explicitly by

$$
\Delta_\phi \restriction \mathcal{U} = -x^4 \partial_x^2 + x^2 \Delta_B + \Delta_F - (2 - b) x^3 \partial_x, \tag{9.2}
$$

where $b = \dim B$, Δ_B is the Laplace Beltrami operator of (B, g_B) and Δ_F is the Laplace Beltrami operator on (F, g_F). The full Hodge Laplacian admits a similar structure. In the non-product case we recover under additional assumptions a similar structure, where Δ_F is now replaced by a family of Hodge Laplacians Δ_{F_y}, associated to the family of Riemannian metrics $g_F(y)$ on F, with parameter $y \in B$. We will be explicit below.

The operator $\Delta_\phi + k^2$ is, for any parameter $k \geq 0$, an elliptic element in a general class of differential operators having a structure similar to (9.2), called ϕ-differential operators. Mazzeo and Melrose in [MaMe98] developed a pseudo-differential calculus, denoted by Ψ^*_ϕ, which contains parametrices for such operators P. Therein they gave a stronger condition, called full ellipticity, that is equivalent to P being Fredholm between naturally associated Sobolev spaces, and is necessary for P to have an inverse in the calculus. The operator $\Delta_\phi + k^2$ is fully elliptic for $k > 0$, but not for $k = 0$ unless Δ_{F_y} is invertible for all $y \in B$, which may happen in the case of the Hodge Laplacian twisted by a flat vector bundle. In that case, the behavior of the resolvent can be deduced directly from [MaMe98]. The contribution of our paper is that we allow the fibrewise Laplacians to have non-trivial kernel.

If the fibre F is a point then fibred boundary metrics are called scattering or asymptotically conic metrics. In this case the low energy resolvent has been studied before, with application to the boundedness of Riesz transforms: for the scalar Laplacian this was first done in the asymptotically Euclidean case, i.e. for $B = \mathbb{S}^b$, by Carron, Coulhon and Hassell [CCH06], then for general base and using a different construction of the resolved double space (see below) by Guillarmou and Hassell [GuHa08, GuHa09], who also allow a potential. Guillarmou and Sher [GuSh15] then used the calculus of [GuHa08] to analyze the low energy resolvent of the Hodge Laplacian and used this to study the behavior of analytic torsion under conic degeneration.

9.2. **Assumptions and the main result.** We study the Schwartz kernel of the resolvent $(\Delta_\phi + k^2)^{-1}$ as $k \to 0$ under additional assumptions that we shall list here. We shall also explain why we make these assumptions.

Assumption 9.1. *The higher order term* h *satisfies* $|h|_{g_0} = O(x^3)$ *as* $x \to 0$.

This assumption guarantees that the terms resulting from h do not alter the leading order behaviour of the Hodge Laplacian, more precisely of the terms P_{ij} in (10.27).

Assumption 9.2. *We assume that* $\phi : (\partial M, g_F + \phi^* g_B) \to (B, g_B)$ *is a Riemannian submersion.*

This assumption is used in order to obtain the expression (10.4) for the Hodge de Rham operator.

The next assumption requires some preparation: The spaces $\mathcal{H}_y = \ker \Delta_{F_y}$ of harmonic forms on the fibres have finite dimension independent of the base point $y \in B$, since they are isomorphic to the cohomology of F. It is a standard fact that these spaces then form a vector bundle $\mathcal{H}$ over B. The metric g induces a flat connection on the bundle $\mathcal{H}$, see [HHM04, Proposition 15]. We denote the twisted Gauss-Bonnet operator on B with values in this flat vector bundle by $D_B = \eth + \eth^*$. Its definition will be recalled explicitly in (10.5). Write N_B for the number operator on differential forms $\Omega^*(B,\mathcal{H})$, multiplying any $\omega \in \Omega^\ell(B,\mathcal{H})$ by ℓ. We define the family of operators

$$I_\lambda(P_{00}) := -\lambda^2 + \begin{pmatrix} D_B^2 + \left(\frac{b-1}{2} - N_B\right)^2 & 2\eth \\ 2\eth^* & D_B^2 + \left(\frac{b+1}{2} - N_B\right)^2 \end{pmatrix} \tag{9.3}$$

acting on $\Omega^*(B,\mathcal{H}) \oplus \Omega^{*-1}(B,\mathcal{H})$. This is the indicial family of a b-operator P_{00} that will be introduced in (10.27). Its set of indicial roots is defined as

$$\mathrm{spec}_b(P_{00}) := \{\lambda \in \mathbb{C} \mid I_\lambda(P_{00}) \text{ is not invertible}\}. \tag{9.4}$$

Noting that $I_\lambda(P_{00})$ is symmetric, we find that $\mathrm{spec}_b(P_{00}) \subset \mathbb{R}$ is real. We can now formulate our next assumption.

Assumption 9.3. *Assume that* $\mathrm{spec}_b(P_{00}) \cap [-1,1] = \varnothing$. *Due to symmetry of* $I_\lambda(P_{00})$ *under the reflection* $\lambda \mapsto -\lambda$, *this is equivalent to* $\mathrm{spec}_b(P_{00}) \cap [-1,0] = \varnothing$.

Assumption 9.3 can be reformulated in terms of spectral conditions on D_B precisely as in [GuSh15, (21)] with d_N and Δ_N replaced by $\eth$ and D_B^2. It is satisfied if D_B has a sufficiently large spectral gap around zero. As we will see in Corollary 10.8, Assumption 9.3 implies the non-resonance assumption

$$\ker_{x^{-1}L^2(M,g_\phi)} \Delta_\phi = \ker_{L^2(M,g_\phi)} \Delta_\phi. \tag{9.5}$$

imposed also by Guillarmou and Sher [GuSh15], in addition to asking for $0 \notin \mathrm{spec}_b(P_{00})$. These conditions are needed in order to construct the Fredholm inverse for the Hodge Laplacian. Specifically, assumption $0 \notin \mathrm{spec}_b(P_{00})$ is used to obtain a Fredholm inverse in (12.12). The no zero-resonances assumption is used in Lemma 12.6, that jointly with the functional analytic observation of Lemma 12.7, yields the zf parametrix in (12.18) and (12.19). Now, the full Assumption 9.3 is used to ensure that such constructed parametrices actually act boundedly on $L^2(M,g_\phi)$.

Assumption 9.4. *The twisted Gauss Bonnet operator* D_B, *defined in* (10.5), *commutes with the orthogonal projection* Π *in* (10.15) *onto fibre-harmonic forms.*

This assumption implies that the off-diagonal terms in the decomposition of Δ_ϕ with respect to the bundles $\mathcal{H}$ and its orthogonal complement in $C^\infty(F)$, near ∂M, vanish quadratically at the boundary as ϕ-operators. This assumptions is needed to analyze the structure of the Fredholm inverse of Δ_ϕ, as used by Grieser and Hunsicker in [GrHu14, Theorem 2], cf. Remark 10.6.

Assumption 9.5. *The base* B *of the fibration* $\phi : \partial M \to B$ *is of dimension* $\dim B \geq 2$*.*

This assumption has also been imposed by Guillarmou and Hassel [GuHa08] and [GuHa09], where in case of trivial fibres F it means $\dim M \geq 3$. If $\dim B < 2$ then the resolvent has a different behavior as $k \to 0$. We use the assumption explicitly in (10.29). Our main result is now as follows.

Theorem 9.6. *Under the Assumptions 9.1, 9.2, 9.3, 9.4 and 9.5, the Schwartz kernel of the resolvent* $(\Delta_\phi + k^2)^{-1}$, $k > 0$, *lifts to a polyhomogeneous conormal distribution on an appropriate manifold with corners, with a conormal singularity along the diagonal.*

We also determine the exponents and the leading terms in the asymptotics of the resolvent kernel, and make the different asympotic behavior with respect to the decomposition into fibrewise harmonic forms and their orthogonal complement explicit. The precise statement is given in Theorem 12.11.

9.3. **Key points of the proof.** As in the papers by Guillarmou, Hassell and Sher, our strategy to prove polyhomogeneity of the resolvent kernel is as follows: first, we construct an appropriate manifold with corners, which we call $M^2_{k,\phi}$, on which we expect the resolvent to be polyhomogeneous. The Hodge Laplacian behaves like a scattering Hodge Laplacian on the space of fibrewise harmonic forms and like a fully elliptic ϕ-operator on the orthogonal complement of this space. The low energy resolvents for these two types of operators are described by two different blowup spaces, denoted $M^2_{k,sc,\phi}$ and $M^2_\phi \times \mathbb{R}_+$. Therefore, $M^2_{k,\phi}$ is chosen as a common blowup of these two spaces.

The boundary hypersurfaces of $M^2_{k,\phi}$ then correspond to limiting regimes, so that the leading asymptotic term(s) of the resolvent kernel at each of them can be determined by solving a simpler model problem. The model problem at the $k = 0$ face called zf involves the Fredholm inverse of Δ_ϕ, so we need to analyze this first. Combining these solutions and an interior parametrix, obtained by inverting the principal symbol (which corresponds to the 'freezing coefficients' model problem at each interior point), we obtain an initial parametrix for the resolvent. We then improve this parametrix using a Neumann series argument and finally show by a standard argument that the exact resolvent has actually the same structure as the improved parametrix. We analyze the low energy resolvent of Δ_ϕ in the non-fully elliptic case, i.e. allowing the fibrewise Laplacians to have non-trivial kernel. A parametrix construction for a closely related operator, the Hodge Laplacian for the cusp metric $x^2 g_\phi$, was given by Grieser and Hunsicker in [GrHu14], and a similar construction for the spin Dirac operator (assuming constant dimension for the kernel of the operators induced on the fibres) was given by Vaillant in [Vai01]. Vaillant also analyzed the low energy behavior of the resolvent for this operator.

For the construction of the Fredholm inverse of Δ_ϕ we introduce and analyze a pseudodifferential calculus, the 'split' calculus (see Definition 10.3), which contains this Fredholm inverse, and reflects the fact that this operator exhibits different asymptotic behavior on the subspaces of fibre-harmonic forms and on its orthogonal complement. This construction is close to the construction by Grieser and Hunsicker in [GrHu14], where the notion of split operator was first introduced. For the proof of our main theorem we then construct a split resolvent calculus that additionally encodes the asymptotic behavior of the resolvent as $k \to 0$. This split resolvent calculus combines the ϕ-calculus of Mazzeo and Melrose, the resolvent calculus of Guillarmou and Hassell and the split calculus.

In Section 10 we analyze the structure of the Hodge Laplacian and display its split structure with respect to the bundle of fibrewise harmonic forms and its orthogonal complement. We introduce the split pseudodifferential calculus, which is a variant of the ϕ-calculus that reflects this splitting. We give an improved version of the parametrix construction for the Hodge Laplacian in that calculus, building on the construction by Grieser and Hunsicker [GrHu14].

In Section 11 we review the low energy resolvent construction of Guillarmou, Hassell and Sher [GuHa08], [GuSh15] on scattering manifolds. In particular, we present the blowup space $M^2_{k,sc}$ due to Melrose and Sá Barreto [MeSa], which is used in those works, and its slightly more general cousin $M^2_{k,sc,\phi}$ that we need.

The proof of our main theorem is given in Section 12. We first construct the blowup space $M^2_{k,\phi}$ and then construct the initial parametrix. The final step towards the resolvent kernel requires a composition theorem, that we prove in Section 13.

10. Hodge Laplacian for ϕ-metrics, split parametrix construction

In this section we analyze the structure of the Hodge Laplacian for a ϕ-metric and exhibit its 'split' structure with respect to fibre harmonic and perpendicular forms. We define a pseudodifferential calculus which reflects this structure, carry out the parametrix construction (building on and extending [GrHu14]) and show that the Fredholm inverse of the Hodge Laplacian lies in this calculus. We continue in the setting of a compact manifold $\overline{M}$ with fibred boundary ∂M, with a choice of boundary defining function $x \geq 0$ and collar neighborhood $\overline{\mathcal{U}} = [0, \varepsilon]_x \times \partial M$. In this section we always consider $E := \Lambda^{\phi} T^* M$ and omit the vector bundle from the notation.

10.1. **Structure of the Hodge Laplacian Δ_ϕ.** Recall the definition of a Riemannian submersion $\phi : (\partial M, g_{\partial M}) \to (B, g_B)$: we split the tangent bundle $T\partial M$ into subbundles $\mathcal{V} \oplus T^H \partial M$, where at any $p \in \partial M$ the vertical subspace $\mathcal{V}_p$ is the tangent space to the fibre of ϕ through p, and the horizontal subspace $T^H_p \partial M$ is its orthogonal complement with respect to $g_{\partial M}$. Then ϕ is a Riemannian submersion if the restriction of the differential $d\phi : T^H \partial M \to TB$ is an isometry. In this case, one can write $g_{\partial M} = g_F + \phi^* g_B$ where g_F equals $g_{\partial M}$ on $\mathcal{V}$ and vanishes on $T^H_p \partial M$. We also write $\phi^* TB$ for $T^H_p \partial M$.

With respect to the fibred boundary metric ϕ we obtain on the collar neighborhood $\mathcal{U}$

$$ {}^{\phi}T\mathcal{U} := {}^{\phi}TM \restriction_{\overline{\mathcal{U}}} = \operatorname{span}\left\{x^2 \partial_x\right\} \oplus x\phi^* TB \oplus \mathcal{V}. \tag{10.1} $$

This splitting induces an orthogonal splitting of the ϕ−cotangent bundle ${}^{\phi}T^*M$:

$$ {}^{\phi}T^*\mathcal{U} := {}^{\phi}T^*M \restriction_{\overline{\mathcal{U}}} = \operatorname{span}\left\{\frac{dx}{x^2}\right\} \oplus x^{-1}\phi^* T^* B \oplus \mathcal{V}^*, \tag{10.2} $$

where $\mathcal{V}^*$ is the dual of $\mathcal{V}$, see (10.11). Recall that the ϕ-tangent bundle ${}^{\phi}TM$ is spanned locally over $\overline{\mathcal{U}}$ by $x^2\partial_x$, $x\partial_{y_i}$, ∂_{z_j}. In terms of this basis the metric g_ϕ takes the form

$$ g_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_{00}(y) & xA_{01}(y,z) \\ 0 & xA_{10}(y,z) & A_{11}(y,z) \end{pmatrix} + O(x^3) \tag{10.3} $$

with A_{ij} smooth and A_{00} not depending on z because of the Riemannian submersion condition on g_0. However, note that in local coordinates $\mathcal{V}^*$ is usually not spanned by the dz_j except at $x = 0$, unless the off-diagonal terms in (10.3) vanish.

With respect to the corresponding decomposition of $\Lambda^{\phi} T^* M$ over $\mathcal{U}$ and assuming that the higher order term $h \equiv 0$ is trivially zero for the moment, we compute as in Hausel, Hunsicker and Mazzeo [HHM04, §5.3.2] for the Hodge Dirac operator D_ϕ over the collar neighborhood $\mathcal{U}$

$$ D_\phi = x^2 D_x + x\mathbb{A} + D_F + xD_B - x^2\mathcal{R}, \tag{10.4} $$

where we set $\mathbb{A} = A + A^*$, $D_F = d_F + d_F^*$, $\mathcal{R} = R + R^*$ and

$$ D_B = (d_B - \mathbb{I}) + (d_B - \mathbb{I})^*. \tag{10.5} $$

Here, $\mathbb{I}$ and R are the second fundamental form and the curvature of the Riemannian submersion ϕ, respectively; d_B is the sum of the lift of the exterior derivative on B to ∂M plus the action of the derivative in the B-direction on the $\mathcal{V}^*-$components of the form. The term $x^2 D_x$ acts for any $\omega \in \Lambda^\ell \left(x^{-1}\phi^* T^* B\right) \oplus \Lambda\mathcal{V}^*$ as follows

$$\left(x^2 D_x\right)\omega = \frac{dx}{x^2} \wedge \left(x^2 \partial_x\right)\omega, \quad x^2 D_x \left(\frac{dx}{x^2} \wedge \omega\right) = -\left(x^2 \partial_x\right)\omega.$$

To be precise, $x^2\partial_x$ is the lift of the corresponding differential operator on $(0, \varepsilon)$ under the projection $\pi : (0, \varepsilon) \times \partial M \longrightarrow (0, \varepsilon)$. Finally, $\mathcal{A}$ is a $0-$th order differential operator, acting for any $\omega \in \Lambda^\ell \left(x^{-1}\phi^* T^* B\right) \oplus \Lambda\mathcal{V}^*$ by

$$\mathcal{A}\omega = -\ell \cdot \frac{dx}{x^2} \wedge \omega, \quad \mathcal{A}\left(\frac{dx}{x^2} \wedge \omega\right) = (b - \ell) \cdot \omega.$$

We can now take the square of D_ϕ to compute the Hodge Laplacian

$$\Delta_\phi = D_\phi^2 = (x^2 D_x)^2 + D_F^2 + x^2 D_B^2 + Q, \; Q \in x \cdot \mathrm{Diff}_\phi^2(M). \tag{10.6}$$

Since D_B^2 equals Δ_B plus terms in $x \cdot \mathrm{Diff}_\phi^2(M)$, we conclude for the normal operator and normal family, defined in (4.7), (4.8)

$$N_\phi(\Delta_\phi)_y = \Delta_{T,Y} + \Delta_{F_y} \tag{10.7}$$

$$\widehat{N}_\phi(\Delta_\phi)_y = \tau^2 + |\xi|^2 + \Delta_{F_y} \in \mathrm{Diff}^2(F_y). \tag{10.8}$$

Here $\Delta_{T,Y}$ is the Euclidean Laplacian on $\mathbb{R}^{b+1} \cong \mathbb{R} \times T_y B$, where the scalar product on $T_y B$ as well as the norm $|\xi|$ for $\xi \in T_y^* B$ are defined by g_B. Also, $\Delta_{F_y} \equiv D_{F_y}^2$. Thus the normal family of the Hodge Laplacian is invertible for each $(\tau, \xi) \neq 0$, and hence Δ_ϕ is fully elliptic, only if $\ker(\Delta_{F_y}) = \{0\}$. Note that we do not impose this restriction in this paper and hence require additional methods for the low energy resolvent construction.

Remark 10.1. In case the higher order term h in (9.1) is non-trivial, additional terms of the form $x \cdot \mathrm{Diff}_\phi^2(M)$ appear, which do not contribute to the normal operator. They do, however, contribute non-trivially to the split structure of Δ_ϕ in the next section, unless Assumption 9.1 is imposed.

10.2. **Splitting into fibrewise harmonic and perpendicular forms.** The explicit form of $\widehat{N}_\phi(\Delta_\phi)$ in (10.8) shows that the Hodge Laplacian is not fully elliptic, unless Δ_{F_y} is invertible. Thus, near ∂M we split the differential forms into forms which are fibrewise harmonic and the perpendicular bundle. In the former, Δ_ϕ acts as a scattering operator, in the latter Δ_ϕ is fully elliptic, in a sense made precise below.

To be precise, let us write $V := [0,\varepsilon) \times B$. Then the $\phi-$cotangent bundle (with trivial fibre) reduces over V to the *scattering* cotangent bundle, cf. (10.2)

$$ {}^{sc}T^*V = \operatorname{span}\left\{\frac{dx}{x^2}\right\} \oplus x^{-1}T^*B. \tag{10.9}$$

Consequently we find from (10.2)

$$ {}^{\phi}T^*\mathcal{U} = \phi^*\left({}^{sc}T^*V\right) \oplus \mathcal{V}^*. \tag{10.10}$$

If $p \in \overline{\mathcal{U}}$ and F is the fibre through p, then pull-back under the inclusion of F into $\overline{\mathcal{U}}$ is a map $({}^{\phi}T^*\mathcal{U})_p \to T^*_pF$, which restricts to an isomorphism $\mathcal{V}^*_p \to T^*_pF$, and this is an isometry for g_0^* and g_F^* since ϕ is a Riemannian submersion. Hence we have the isometry

$$ \mathcal{V}^* \cong T^*F. \tag{10.11}$$

The decomposition (10.10) induces a decomposition for the exterior algebras

$$ \Lambda\,{}^{\phi}T^*\mathcal{U} = \Lambda\,\phi^*\left({}^{sc}T^*V\right) \otimes \Lambda\mathcal{V}^*. \tag{10.12}$$

Together with (10.9), this allows us to write

$$ C^\infty(\overline{\mathcal{U}}, \Lambda^{\phi}T^*\mathcal{U}) = C^\infty(V, \Lambda^{sc}T^*V) \otimes C^\infty(F, \Lambda T^*F), $$

where $C^\infty(F, \Lambda T^*F)$ is considered as a bundle over B, the fibre over $y \in B$ being $C^\infty(F_y, \Lambda T^*F_y)$. We decompose this bundle as

$$ C^\infty(F, \Lambda T^*F) = \widetilde{\mathcal{H}} \oplus \widetilde{\mathcal{C}}, \tag{10.13}$$

where for each $y \in B$ the space $\widetilde{\mathcal{H}}_y$ is the kernel of the Hodge Laplacian Δ_{F_y} on the fibre $(F_y, g_F(y))$, and $\widetilde{\mathcal{C}}_y$ is its orthogonal complement with respect to the L^2-scalar product. Note that $\dim \widetilde{\mathcal{H}}_y = \operatorname{rank} H^*(F_y)$ is finite and independent of $y \in B$ by the Hodge theorem. It is a classical result [BGV03, Corollary 9.11] that $\widetilde{\mathcal{H}}$ is a smooth vector bundle over B. For each $(x,y) \in V$ let

$$ \mathcal{H}_{(x,y)} = \Lambda({}^{sc}T^*V)_{(x,y)} \otimes \widetilde{\mathcal{H}}_y, \quad \mathcal{C}_{(x,y)} = \Lambda({}^{sc}T^*V)_{(x,y)} \otimes \widetilde{\mathcal{C}}_y. $$

We define a smooth section of $\mathcal{H}$ to be a smooth differential form on $\overline{\mathcal{U}}$ whose restriction to the fibre over $(x,y) \in V$ lies in $\mathcal{H}_{(x,y)}$ for each (x,y), and similarly for $\mathcal{C}$. Summarizing, we get

$$ C^\infty(\overline{\mathcal{U}}, \Lambda^{\phi}T^*\mathcal{U}) = C^\infty(V, \mathcal{H}) \oplus C^\infty(V, \mathcal{C}). \tag{10.14}$$

Corresponding to this decomposition we have projections

$$ \begin{aligned} &\Pi : C^\infty(\overline{\mathcal{U}}, \Lambda^{\phi}T^*\mathcal{U}) \to C^\infty(V, \mathcal{H}), \\ &\Pi^\perp := \mathrm{Id} - \Pi : C^\infty(\overline{\mathcal{U}}, \Lambda^{\phi}T^*\mathcal{U}) \to C^\infty(V, \mathcal{C}), \end{aligned} \tag{10.15}$$

These maps are defined fibrewise from the projections of $C^\infty(F_y, \Lambda T^*F_y)$ to $\widetilde{\mathcal{H}}_y$ and to $\widetilde{\mathcal{C}}_y$, respectively, for each $y \in B$.

10.3. **Split structure of the Hodge Laplacian.** We now turn to the structure of the Hodge Laplacian with respect to the decomposition above. Recall from (10.4) that the Hodge Dirac operator $D_\phi = d + d^*$ over $\mathcal{U}$ (still with respect to g_0, setting $h \equiv 0$ for the moment) takes with respect to the decomposition (10.12) the following form

$$D_\phi = x^2 D_x + x\mathbb{A} + D_F + xD_B - x^2\mathcal{R}. \tag{10.16}$$

We consider now as in [HHM04] the decomposition of D_ϕ with respect to the decomposition (10.14), which leads to the following matrix representation

$$\begin{aligned} D_\phi &= \begin{pmatrix} (D_\phi)_{00} & (D_\phi)_{01} \\ (D_\phi)_{10} & (D_\phi)_{11} \end{pmatrix} \\ &:= \begin{pmatrix} \Pi D_\phi \Pi & \Pi D_\phi \Pi^\perp \\ \Pi^\perp D_\phi \Pi & \Pi^\perp D_\phi \Pi^\perp \end{pmatrix} : \begin{matrix} \Gamma(V, \mathcal{H}) \\ \oplus \\ \Gamma(V, \mathcal{C}) \end{matrix} \longrightarrow \begin{matrix} \Gamma(V, \mathcal{H}) \\ \oplus \\ \Gamma(V, \mathcal{C}). \end{matrix} \end{aligned} \tag{10.17}$$

The individual terms in the matrix (10.17) are given as follows. We define the operator $\eth = \Pi(d_B - \mathbb{I})\Pi$, which in [HHM04, Proposition 15] is shown to act as a differential, i.e. $\eth^2 = 0$ and $(\eth^*)^2 = 0$. Then we find from (10.16)

$$(D_\phi)_{00} = x^2 D_x + x(\eth + \eth^*) + x\mathbb{A} - x^2 \Pi\mathcal{R}\Pi, \tag{10.18}$$

$$(D_\phi)_{11} = x^2 D_x + x\Pi^\perp D_B \Pi^\perp + x\mathbb{A} - x^2 \Pi^\perp \mathcal{R}\Pi^\perp + \Pi^\perp D_F \Pi^\perp, \tag{10.19}$$

$$(D_\phi)_{01} = \Pi \left(xD_B - x^2\mathcal{R} \right) \Pi^\perp, \tag{10.20}$$

$$(D_\phi)_{10} = \Pi^\perp \left(xD_B - x^2\mathcal{R} \right) \Pi. \tag{10.21}$$

Note that in (10.20) and (10.21) we used that $x^2 D_x$ and $\mathbb{A}$ commute with Π. We can express the matrix in (10.17) as follows

$$D_\phi = \begin{pmatrix} (D_\phi)_{00} & (D_\phi)_{01} \\ (D_\phi)_{10} & (D_\phi)_{11} \end{pmatrix} = \begin{pmatrix} xA_{00} & xA_{01} \\ xA_{10} & A_{11} \end{pmatrix}, \tag{10.22}$$

where the individual entries are given by

$$\begin{aligned} A_{00} &= xD_x + (\eth + \eth^*) + \mathbb{A} - x\Pi\mathcal{R}\Pi, & A_{01} &= \Pi \left(D_B - x\mathcal{R} \right) \Pi^\perp, \\ A_{10} &= \Pi^\perp \left(D_B - x\mathcal{R} \right) \Pi, & A_{11} &= (D_\phi)_{11}. \end{aligned} \tag{10.23}$$

We point out that A_{00} acts as an elliptic differential b-operator on sections of $\mathcal{H}$. Similarly, A_{11} acts as a ϕ-operator and A_{01}, A_{10} as b-operators. However, below it is useful to think of the latter as ϕ-operators.

For the Hodge Laplacian $\Delta_\phi = (D_\phi)^2$ (with respect to g_0 in (9.1) with $h \equiv 0$) one computes from (10.22) that

$$\Delta_\phi = \begin{pmatrix} (\Delta_\phi)_{00} & (\Delta_\phi)_{01} \\ (\Delta_\phi)_{10} & (\Delta_\phi)_{11} \end{pmatrix}, \tag{10.24}$$

where the individual entries are given in terms of (10.23) by

$$\begin{aligned}(\Delta_\phi)_{00} &= (xA_{00})^2 + (xA_{01})(xA_{10}),\\ (\Delta_\phi)_{01} &= (xA_{00})(xA_{01}) + xA_{01}A_{11},\\ (\Delta_\phi)_{10} &= (xA_{10})(xA_{00}) + A_{11}(xA_{10}),\\ (\Delta_\phi)_{11} &= (xA_{10})(xA_{01}) + A_{11}^2.\end{aligned}$$

10.3.1. *Unitary transformation of Δ_ϕ to an operator in $L^2(M; dvol_b)$.* The Hodge Laplacian Δ_ϕ is identified with its unique self-adjoint extension in $L^2(M; dvol_\phi)$, where $dvol_\phi$ is the volume form induced by g_ϕ. It is convenient to transform Δ_ϕ to a self-adjoint operator in $L^2(M; dvol_b)$, where

$$dvol_b = x^{b+1} dvol_\phi. \tag{10.25}$$

We can pass between $L^2(M; dvol_b)$ and $L^2(M; dvol_\phi)$ by an isometry

$$W : L^2(M; dvol_\phi) \to L^2(M; dvol_b), \quad \omega \mapsto x^{-\frac{b+1}{2}}\omega. \tag{10.26}$$

We use that isometry to define the operator

$$\Box_\phi := W \circ \Delta_\phi \circ W^{-1} \equiv x^{-\frac{b+1}{2}} \Delta_\phi x^{\frac{b+1}{2}},$$

which is self-adjoint in $L^2(M; dvol_b)$ instead of $L^2(M; dvol_\phi)$. Below, we will deal only with $\Box_\phi$, which is unitarily equivalent to the Hodge Laplacian.

Recall that we impose Assumption 9.4 so that $[D_B, \Pi] = 0$. This condition implies that $\Pi^\perp (D_B) \Pi = 0$ and $\Pi (D_B) \Pi^\perp = 0$, hence $xA_{01} = -x^2 \Pi \mathcal{R} \Pi^\perp$ and $xA_{10} = -x^2 \Pi^\perp \mathcal{R} \Pi$ are x^2 times zero-order operators. This implies that $\Box_\phi = x^{-\frac{b+1}{2}} \Delta_\phi x^{\frac{b+1}{2}}$ can be written

$$\Box_\phi = \begin{pmatrix} xP_{00}x & xP_{01}x \\ xP_{10}x & P_{11} \end{pmatrix}. \tag{10.27}$$

where $P_{00} \in \mathrm{Diff}^2_b(M)$, $P_{11} \in \mathrm{Diff}^2_\phi(M)$ and $P_{01}, P_{10} \in \mathrm{Diff}^2_\phi(M)$ (in fact, $P_{01}, P_{10} \in \mathrm{Diff}^1_\phi(M)$ in the special case where the metric perturbation $h \equiv 0$), with each P_{ij} sandwiched by appropriate Π and $\Pi^\perp$ factors. We call (10.27) the *split structure* of $\Box_\phi$.

All of the computations above are done for $h \equiv 0$ a priori. Now, for a non-trivial higher order term h, the Assumption 9.1 with a stronger decay of $|h|_{g_0} = O(x^3)$, guarantees that (10.27) still holds with higher order contributions to all P_{ij}.

10.3.2. *Normal operator of $\Box_\phi$ under the splitting.* Here we in particular explain in what sense $P_{11} \in \mathrm{Diff}^2_\phi(M)$ is fully elliptic. The normal operator of $\Box_\phi$ has the following form under the splitting above (cf. (10.7))

$$N(\Box_\phi)_y = \begin{pmatrix} N(xP_{00}x)_y & 0 \\ 0 & N(P_{11})_y \end{pmatrix} = \begin{pmatrix} \Delta_{\mathbb{R}^{b+1}} \otimes \mathrm{Id}_{\widetilde{\mathcal{H}}_y} & 0 \\ 0 & \Delta_{\mathbb{R}^{b+1}} \oplus (\Delta_{F_y})_{|\widetilde{\mathcal{C}}_y} \end{pmatrix} \tag{10.28}$$

Note that the lower right corner is invertible as a map on sections of $\mathcal{C}$, with inverse a convolution operator in the (T, Y)-variables on $\mathbb{R}^{b+1}$, rapidly decaying as $|(T, Y)| \to \infty$ (the class of operators with this property is called the *suspended calculus*). This can be shown by using a spectral decomposition of Δ_F, or directly as in [GrHu14, Proposition 2.3]. In this sense P_{11} is fully elliptic on sections of $\mathcal{C}$.

The upper left corner of (10.28) behaves differently: If $b \geq 2$ then the standard fundamental solution defines a bounded operator (acting by convolution)

$$\Delta^{-1}_{\mathbb{R}^{b+1}} = c|(T, Y)|^{1-b} * : x^{\gamma+2}L^2(\mathbb{R}^{b+1}, \mathrm{dvol}_b) \to x^\gamma H^2_b(\mathbb{R}^{b+1}, \mathrm{dvol}_b), \tag{10.29}$$

if $\gamma \in (0, b-1)$, where $x > 0$ is smooth and equals $\frac{1}{r}$ for large values of the radial function r on $\mathbb{R}^{b+1}$. For $b < 2$ there are no values γ for which $\Delta_{\mathbb{R}^{b+1}}$ is invertible between these spaces. See for example [ARS14, Proposition A.2]. In any case, the convolution only decays polynomially at infinity.

Structure of P_{00}. Computations verbatim to [GuSh15, (15)] yield

$$P_{00} = -(x\partial_x)^2 + L + x^2\,\Pi\,\mathcal{R}\,\Pi^\perp\,\mathcal{R}\,\Pi. \tag{10.30}$$

where the action of L on $\Lambda^\ell(x^{-1}\phi^*T^*B) \otimes \mathcal{H} \oplus \frac{dx}{x^2} \wedge \Lambda^{\ell-1}(x^{-1}\phi^*T^*B) \otimes \mathcal{H}$ is given by (do not confuse this matrix representation with the matrix formula under the splitting into fibrewise harmonic and perpendicular forms)

$$L = \begin{pmatrix} \Pi D_B^2 \Pi + \left(\frac{b-1}{2} - N_B\right)^2 & 2(d_B - \mathbb{I}) \\ 2(d_B - \mathbb{I})^* & \Pi D_B^2 \Pi + \left(\frac{b+1}{2} - N_B\right)^2 \end{pmatrix}, \tag{10.31}$$

where N_B denotes the number operator on $\Lambda^*(x^{-1}\phi^*T^*B)$, multiplying elements in $\Lambda^\ell(x^{-1}\phi^*T^*B)$ by ℓ. The operator P_{00} is a differential b-operator. Its indicial family $I_\lambda(P_{00})$, given in (9.3), and the set of indicial roots $\mathrm{spec}_b(P_{00})$ are defined in §3.3.

Note that $xP_{00}x$ is, up to $O(x^4)$ and modulo the unitary transformation (10.26), the Hodge Laplace operator on $(V, \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2})$ twisted by the vector bundle $\mathcal{H}$, equipped with the flat connection $\tilde{\partial} = \Pi(d_B - \mathbb{I})\Pi$.

10.4. **Parametrix construction for the split Hodge Laplacian.** We now review the parametrix construction for the operator $\Box_\phi$, which is the main result of [GrHu14]. We begin with the definition of the split ϕ-calculus.

A section u of $\Lambda^\phi T^*M$ can be decomposed over the collar neighborhood $\overline{\mathcal{U}}$ of the boundary ∂M into fibrewise harmonic and perpendicular forms as in (10.14). With respect to that decomposition, we can write u over $\overline{\mathcal{U}}$ as $\Pi u + \Pi^\perp u$ or as a two-vector

$$u \restriction \overline{\mathcal{U}} = \begin{pmatrix} \Pi u \\ \Pi^\perp u \end{pmatrix}. \tag{10.32}$$

The different parts of the parametrix of $\Box_\phi$ with respect to this decomposition have different index sets in their asymptotics on M^2_ϕ. We introduce notation to describe this behavior. First we consider sections over M.

Definition 10.2. For an index set E, we define $\mathcal{A}^E_{\mathcal{H}}(M)$ to be the space of $u \in \mathcal{A}^E_{\mathrm{phg}}(M, \Lambda^\phi T^*M)$, whose decomposition (10.32) has index sets $\begin{pmatrix} E \\ E+2 \end{pmatrix}$. In other words, the leading terms to two orders in the asymptotics of u at ∂M have values in $\mathcal{H}$.

We extend this concept to sections on the ϕ-double space M^2_ϕ. Note that for a section on $\overline{M}^2$ we can distinguish its $\mathcal{H}$- and $\mathcal{C}$-parts in both factors near the corner $\partial M \times \partial M$ and thus represent it by a 2×2 matrix. However, near $\mathrm{lb} = \partial M \times \overline{M}$ and $\mathrm{rb} = \overline{M} \times \partial M$ we can do this only in the first (resp. second) factor, so we get a 2×1 resp. 1×2 vector. The boundary hypersurfaces of M^2_ϕ lying over $\partial M \times \partial M$ are bf and ϕf. Thus we define the split ϕ-calculus, identifying operators with lifts of their Schwartz kernels, as follows.

Definition 10.3 (Split ϕ-calculus)**.** Let $\mathcal{E}$ be an index family for M^2_ϕ and consider $K \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M^2_\phi, \Omega^{1/2}_{b\phi}(M^2_\phi) \otimes \mathrm{End}(\Lambda^\phi T^*M))$. We write ΠK meaning that Π acts on the first component in M^2, and $K\Pi$ meaning that Π acts on the second component – the notation suggested by interpreting K as an operator. Then

$$K \in \mathcal{A}^{\mathcal{E}}_{\mathcal{H}}(M^2_\phi)$$

if the following holds:

(1) at bf, when K is written with respect to the $\mathcal{H}$-$\mathcal{C}$ decomposition as a 2×2 matrix $\begin{pmatrix} \Pi K \Pi & \Pi K \Pi^\perp \\ \Pi^\perp K \Pi & \Pi^\perp K \Pi^\perp \end{pmatrix}$, it has index sets $\begin{pmatrix} \mathcal{E}_{\mathrm{bf}} & \mathcal{E}_{\mathrm{bf}}+2 \\ \mathcal{E}_{\mathrm{bf}}+2 & \mathcal{E}_{\mathrm{bf}}+4 \end{pmatrix}$.

(2) at lb, when K is written as a vector $\begin{pmatrix} \Pi K \\ \Pi^\perp K \end{pmatrix}$, it has index sets $\begin{pmatrix} \mathcal{E}_{\mathrm{lb}} \\ \mathcal{E}_{\mathrm{lb}}+2 \end{pmatrix}$.

(3) at rb, when K is written as a vector $(K\Pi, K\Pi^\perp)$, it has index sets $\begin{pmatrix} \mathcal{E}_{\mathrm{rb}} & \mathcal{E}_{\mathrm{rb}}+2 \end{pmatrix}$.

(4) at ϕf, when K is written as a 2×2 matrix, it has index sets

$$\begin{pmatrix} \mathcal{E}_{\phi f} & \mathcal{E}_{\phi f} + 2 \\ \mathcal{E}_{\phi f} + 2 & \mathcal{E}_{\phi f} \end{pmatrix}$$

The split ϕ-calculus is then defined in view of Definition 4.2 by

$$\Psi^{m,\mathcal{E}}_{\phi,\mathcal{H}}(M) = \Psi^m_\phi(M, \Lambda^\phi T^*M) + \mathcal{A}^{\mathcal{E}}_{\mathcal{H}}(M^2_\phi).$$

We call the matrices and vectors of index sets in (1)-(4) the *split index family* associated with $\mathcal{E}$.

In short, the definition says that terms with a $\Pi^\perp$ factor on the left have an extra x^2 factor from the left (affecting all faces except rb), and terms with a $\Pi^\perp$ factor on the right have an extra x^2 on the right (affecting all faces except lb) – except for the $\Pi^\perp K \Pi^\perp$ term at ϕf, which is no better than the $\Pi K \Pi$ term. Essentially, this latter fact says that K is diagonal to two leading orders at ϕf.

Note that the proof of our Composition Theorem 13.4 also yields a composition theorem for $\Psi^{m,\mathcal{E}}_{\phi,\mathcal{H}}(M)$, by restriction to zf. However, we do not need it, so we do not state it explicitly here.

The following result is similar to [GrHu14, Theorem 12], which is the main result therein. Our statement here is slightly stronger and we write out the proof in detail, since it will be adapted to the resolvent construction below.

Theorem 10.4. *For each $\alpha \notin \mathrm{spec}_b(P_{00})$, $\Box_\phi$ has a parametrix Q_α such that*

$$\Box_\phi Q_\alpha = \mathrm{Id} - R_\alpha, \ \ Q_\alpha \Box_\phi = \mathrm{Id} - R'_\alpha,$$

where $Q_\alpha \in \Psi^{-2,\mathcal{E}}_{\phi,\mathcal{H}}(M)$ and remainders $R_\alpha \in x^\infty \Psi^{-\infty,\mathcal{E}}_{\phi,\mathcal{H}}(M)$ and $R'_\alpha \in \Psi^{-\infty,\mathcal{E}}_{\phi,\mathcal{H}}(M)\, x^\infty$. The index family $\mathcal{E}$ is given in terms of $\mathcal{E}(\alpha)$ (that is determined by $\mathrm{spec}_b(P_{00})$ and satisfies (3.4)) by

$$\mathcal{E}_{\mathrm{lb}} = \mathcal{E}(\alpha)_{\mathrm{lb}} - 1, \ \mathcal{E}_{\mathrm{rb}} = \mathcal{E}(\alpha)_{\mathrm{rb}} - 1, \ \mathcal{E}_{\mathrm{bf}} \geq -2, \ \mathcal{E}_{\phi f} \geq 0.$$

Index sets for Q_α with respect to the $\mathcal{H}$-$\mathcal{C}$ decomposition are illustrated schematically in Figure 9 (ℓ shall run through elements of $\mathcal{E}(\alpha)_{\mathrm{lb}}$ and r through elements of $\mathcal{E}(\alpha)_{\mathrm{rb}}$)

As preparation for the proof we need two considerations for dealing with the splitting of $\Box_\phi$ in (10.27). Both arise from the need to compose a parametrix of P_{00} with ϕ-operators coming from the other entries of the matrix. Recall that P_{00} is a b-operator in the base $V = [0, \varepsilon) \times B$, so its Schwartz kernel, and the Schwartz kernel of its parametrix Q_{00}, lift to distributions on the b-double space V^2_b, valued in $\mathrm{End}(\mathcal{H})$ (and half-densities). On the other hand, the kernels of ϕ-operators are distributions on the ϕ-double space $\mathcal{U}^2_\phi$. Thus, in order to

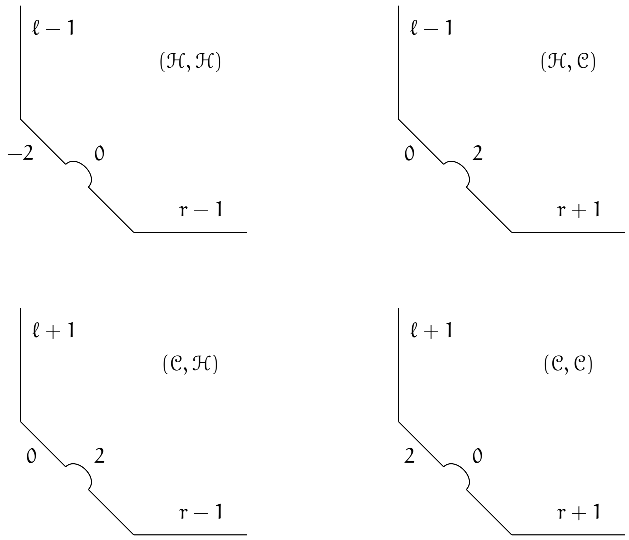

FIGURE 9. Schematic structure of index sets of Q_α near $\mathrm{bf}\cap\phi\mathrm{f}$

analyze the compositions, we first lift Q_{00} to V^2_ϕ and[10] then to $\mathcal{U}^2_\phi$. To do this we need the following facts.

Fact 1: Lifting from b*-double to* ϕ*-double space.* For the lift of a kernel from V^2_b to V^2_ϕ under the blow-down map $\beta_{\phi-b}$ in Figure 2, an elementary calculation (compare [GrHu14, Proposition 1]) implies that for any index family $\mathcal{F}$ on V^2_b and any $m\in\mathbb{R}$ (we use the notation introduced in Definitions 3.1 and 4.2 and omit the vector bundle from the notation)

$$\beta^*_{\phi-b}:\Psi_b^{-m}(V)\to\rho^m_{\phi f}\Psi_\phi^{-m}(V)+\mathcal{A}_\phi^{\mathcal{F}_m}(V),\quad \beta^*_{\phi-b}:\mathcal{A}_b^{\mathcal{F}}(V)\to\mathcal{A}_\phi^{\mathcal{F}'}(V),\qquad(10.33)$$

where $\mathcal{F}_{m,\mathrm{bf}}=0$, $\mathcal{F}_{m,\phi f}=m\,\overline{\cup}\,(b+1)$, $\mathcal{F}_{m,\mathrm{lb}}=\mathcal{F}_{m,\mathrm{rb}}=\varnothing$ and $\mathcal{F}'$ has the same index sets as $\mathcal{F}$ at lb, rb and bf, and in addition $\mathcal{F}'_{\phi f}=\mathcal{F}_{\mathrm{bf}}+(b+1)$ (the shift arises from the density factor, see (4.6)). We use this to lift the b-parametrix Q_{00}, where $m=2$. Note that, since $b\geq 2$ by Assumption 9.5, the leading term in $\mathcal{F}'_{\phi f}$ is 2, without logarithm. If $b=1$, there would be an additional logarithm. If $b=0$, the leading term would be 1.

Fact 2: Extended calculus. The lift $\beta^*_{\phi-b}Q_{00}$ is a distribution on V^2_ϕ, valued in $\mathrm{End}(\mathcal{H})$ (and half-densities). Since $\mathcal{H}\subset C^\infty(F,\Lambda T^*F)$ and $\overline{\mathcal{U}}$ is an F-bundle over V, this distribution can also be interpreted as a distribution on $\mathcal{U}^2_\phi$. It is conormal with respect to the interior submanifold $\{x=x',y=y'\}$, which is the

[10]The notation V^2_ϕ indicates the double space constructed in Subsection 4.2, with M replaced by V and with the trivial fibration (whose fibres are points).

diagonal in V^2_ϕ, but is the fibre diagonal (thus larger than the diagonal) in $\mathcal{U}^2_\phi$. Thus $\beta^*_{\phi-b}Q_{00}$ does not define a pseudo-differential operator in $\Psi^{-2,*}_\phi(\mathcal{U})$.[11]

Since in our argument sums of such operators and operators in $\Psi^{-2,*}_\phi(\mathcal{U})$ will be considered, we define for any pseudo-differential calculus on M or $\mathcal{U}$, the corresponding *extended calculus* as the space of operators whose kernels are sums of two terms, one conormal with respect to the diagonal and one conormal with respect to the fibre diagonal, the latter term being supported near the corner $(\partial M)^2$ of $\overline{M}^2$. We denote the extended calculus by $\overline{\Psi}^{-2,*}_\phi(\mathcal{U})$.

Note that the fibre diagonal hits the boundary of M^2_ϕ only in the interior of the ϕ-face ϕf, just like the diagonal, because $\mathcal{U}^2$ is a fibre bundle with fibre F^2. In particular, the extension does not affect asymptotic behavior at other faces. In [GrHu14, Proposition 2] it is shown that the extended calculus enjoys the same composition properties as the standard calculus.

The extended calculus is only used in the intermediate steps, the final parametrix lies in the standard (non-extended) calculus.

Proof of Theorem 10.4. We follow the parametrix construction given in [GrHu14]. We explain the main steps since our notation is somewhat different and since we will generalize the construction to the k-dependent case. We also simplify the construction slightly and give more details of the second step. We construct the right parametrix, it then turns out to be a left parametrix also, by standard arguments. The construction proceeds in four steps:

Step 1: We first construct a parametrix Q_1, with $\Box_\phi Q_1 = \mathrm{Id} - R_1$ where the remainder R_1 vanishes at the boundary faces bf and ϕf suitably.
Step 2: We improve Q_1 to a parametrix Q_2 whose remainder also vanishes at lb.
Step 3: Using a parametrix in the small ϕ-calculus, obtained by inverting the principal symbol, we remove the interior singularity of the remainder.
Step 4: Iteration gives a remainder as in the theorem.

Step 1: We first work near the boundary, i.e. in $\mathcal{U}$. We use the fact that the diagonal terms of $\Box_\phi$ in (10.27) are elliptic resp. fully elliptic in the b-calculus and (extended) ϕ-calculus sense, and that the product of the off-diagonal terms is higher order compared to the product of the diagonal terms in terms of x. Abstractly, an approximate right inverse for a block matrix $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with

[11] Note that it is the fibre diagonal in the ϕ-double space, unlike the fibre diagonal in the b-double space which was used in the definition of M^2_ϕ.

this property is given by[12]

$$Q_1 = \begin{pmatrix} \widehat{A} & -\widehat{A}B' \\ -\widehat{D}C' & \widehat{D} \end{pmatrix}, \tag{10.34}$$

where $\widehat{A}$, $\widehat{D}$ are right parametrices for A, D, say

$$A\widehat{A} = \mathrm{Id} - R,\ D\widehat{D} = \mathrm{Id} - S \quad \text{and} \quad B' = B\widehat{D},\ C' = C\widehat{A}.$$

A short calculation gives $PQ_1 = \mathrm{Id} - R_1$ where

$$R_1 = R_1' + R_1'', \quad R_1' = \begin{pmatrix} R & -RB' \\ -SC' & S \end{pmatrix},\ R_1'' = \begin{pmatrix} B'C' & 0 \\ 0 & C'B' \end{pmatrix}.$$

We apply this with $A = xP_{00}x$, $B = xP_{01}x$, $C = xP_{10}x$, $D = P_{11}$. Since $\alpha \notin \operatorname{spec}_b(P_{00})$ there is a b-calculus parametrix Q_{00}, obtained from a small calculus parametrix (near the diagonal) and inversion of the indicial operator $I(P_{00})$

$$P_{00}Q_{00} = \mathrm{Id} - R_{00}, \quad Q_{00} \in \Psi_b^{-2,\mathcal{E}(\alpha)}(V, \mathcal{H}),\ R_{00} \in \rho_{bf}\mathcal{A}_b^{\mathcal{E}(\alpha)}(V, \mathcal{H}). \tag{10.35}$$

Note that Theorem 3.2 actually yields a better parametrix with remainder in $\rho_{bf}^\infty \rho_{lb}^\infty \mathcal{A}_b^{\mathcal{E}(\alpha)}(V, \mathcal{H})$. However, an extension of that result to the resolvent is not straightforward, and in fact the rather crude parametrix Q_{00} with remainder R_{00} is fully sufficient for our purposes.

Also, there is a ϕ-calculus parametrix for P_{11}, i.e.

$$P_{11}Q_{11} = \mathrm{Id} - R_{11}, \quad Q_{11} \in \overline{\Psi}_\phi^{-2}(\mathcal{U}),\ R_{11} \in \mathcal{A}_\phi^{\varnothing}(\mathcal{U}). \tag{10.36}$$

(We leave out the bundle $\Lambda^\phi T^*M$ from notation in this proof.) We refer the reader to [GrHu14, Proposition 2] for details why this works in the extended calculus, and emphasize that we removed higher order terms at ϕf right away.

Then $\widehat{A} = x^{-1}Q_{00}x^{-1}$ lifts to an element of $\Psi_\phi^{-2,\mathcal{E}}(V, \mathcal{H}) \subset \overline{\Psi}_\phi^{-2,\mathcal{E}}(\mathcal{U})$ by (10.33), with $\mathcal{E}$ as in the statement. With $\widehat{D} = Q_{11}$ we get

$$\begin{aligned} B' &= (xP_{01}x)Q_{11} \in x^2\,\overline{\Psi}_\phi^0(\mathcal{U}) = \overline{\Psi}_\phi^0(\mathcal{U})x^2, \\ C' &= (xP_{10}x)\widehat{A} \in x^2\,\overline{\Psi}_\phi^{0,\mathcal{E}}(\mathcal{U}). \end{aligned}$$

The extra x^2 factors in B', C' give $Q_1 \in \overline{\Psi}_{\phi,\mathcal{H}}^{-2,\mathcal{E}}(\mathcal{U})$. We now analyze the remainder terms R_1' and R_1''. We shall use the notation for index families

$$\begin{aligned} &\mathcal{F} = (a, b, l + \lambda, r + \rho) \text{ means:} \\ &\mathcal{F}_{bf} \geq a,\ \mathcal{F}_{\phi f} \geq b,\ \mathcal{F}_{lb} = \mathcal{E}(\alpha)_{lb} + \lambda,\ \mathcal{F}_{rb} = \mathcal{E}(\alpha)_{rb} + \rho. \end{aligned} \tag{10.37}$$

[12]This formula arises from taking leading terms in the Schur complement formula for the inverse of a block matrix.

First, $R = xR_{00}x^{-1}$ lifts by (10.33) to be in $\mathcal{A}_\phi^{(1,4,l+1,r-1)}(\mathcal{U})$. Then by the composition result [GrHu14, Theorem 9] we find $RB' \in \mathcal{A}_\phi^{(3,6,l+1,r+1)}(\mathcal{U})$. Similarly, we conclude with $S = R_{11}$ that $SC' \in \mathcal{A}_\phi^{(\varnothing,\varnothing,\varnothing,r-1)}(\mathcal{U})$. The diagonal terms in R_1'' are

$$\begin{aligned} B'C' &\in x^4\overline{\Psi}_\phi^{0,\mathcal{E}}(\mathcal{U}) = \rho_{\phi f}^4\overline{\Psi}_\phi^{0,(2,0,l+3,r-1)}(\mathcal{U}), \\ C'B' &\in x^2\overline{\Psi}_\phi^{0,\mathcal{E}}(\mathcal{U})x^2 = \rho_{\phi f}^4\overline{\Psi}_\phi^{0,(2,0,l+1,r+1)}(\mathcal{U}). \end{aligned}$$

In summary we get $R_1 \in \rho_{\phi f}^4\overline{\Psi}_\phi^{0,\mathcal{R}_1}(\mathcal{U})$, where

$$\mathcal{R}_1 = \begin{pmatrix} (1,0,l+1,r-1) & (3,2,l+1,r+1) \\ (\varnothing,\varnothing,\varnothing,r-1) & (2,0,l+1,r+1) \end{pmatrix}. \tag{10.38}$$

Note that all terms in R_1 vanish at bf and ϕf. If we had simply inverted the diagonal terms then we would have got a remainder whose $\Pi^\perp R\Pi$ component does not vanish at bf, which would not be good enough for the iteration argument in the resolvent construction below.

At this point the Schwartz kernels of Q_1 and R_1 are defined over $\mathcal{U} \times \mathcal{U}$ only. Using a cutoff function we modify Q_1 to an element of $\overline{\Psi}_{\phi,\mathcal{H}}^{-2,\mathcal{E}}(M)$, without changing it near $\mathrm{bf} \cup \phi\mathrm{f}$, so that $R_1 = \mathrm{Id} - PQ_1$ is in $\rho_{\phi f}^4\overline{\Psi}_\phi^{0,\mathcal{R}_1}(M)$ with $\mathcal{R}_1$ as above.

Step 2: We want to refine our parametrix Q_1 from Step 1 so that the remainder vanishes to infinite order at lb. We accomplish this by determining Q_1' supported near lb so that $\Box_\phi Q_1'$ agrees with R_1 at lb to infinite order. Then

$$\Box_\phi Q_1' = R_1 - R_2, \tag{10.39}$$

where R_2 has the same index sets as R_1 except for $\varnothing$ at lb, and also has order zero. The construction is essentially the same as that in [Mel93, Lemma 5.44], but we need to be careful with the correct exponents in the $\mathcal{H} - \mathcal{C}$ splitting. Finding Q_1' amounts to solving

$$\begin{pmatrix} xP_{00}x & xP_{01}x \\ xP_{10}x & P_{11} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \equiv \begin{pmatrix} r_0 \\ r_1 \end{pmatrix} \tag{10.40}$$

as an equation in the (x, y, z) variables, with (x', y', z') as parameters, to infinite order as $x \to 0$, with control of the $x' \to 0$ behavior which corresponds to the asymptotics at the intersection of lb with bf. Here

- $\begin{pmatrix} r_0 \\ r_1 \end{pmatrix}$ runs through the two columns of R_1,
- $\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$ runs through the corresponding two columns of Q_1'.

By (10.38) the exponents at lb occuring in $\begin{pmatrix} r_0 \\ r_1 \end{pmatrix}$ are $\begin{pmatrix} l+1 \\ l+1 \end{pmatrix}$ for $l \in \pi\mathcal{E}(\alpha)_{\mathrm{lb}}$, for either column of R_1 (in fact, slightly better in the first column). Then (10.40) can be solved to leading order for $\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$ having leading exponents $\begin{pmatrix} l-1 \\ l+1 \end{pmatrix}$ at lb by first choosing[13] q_0 satisfying (always to leading order) $(xP_{00}x)q_0 = r_0$ (using b-ellipticity of P_0). Note that the term $(xP_{01}x)q_1$ will be of higher order. Then we choose q_1 satisfying $P_{11}q_1 = r_1 - (xP_{10}x)q_0$ (using full ellipticity of P_{11}). Doing this iteratively removes all orders step by step.

Uniformity at $\mathrm{lb} \cap \mathrm{bf}$ (as $x' \to 0$) is shown as in [Mel93, Lemma 5.44], we only need to check the exponents at bf. For the left column of R_1, they are $(1, \varnothing)$ by (10.38), which implies they are $(-1, 1)$ for the left column of Q_1' by the explanation above. For the right column of R_1 they are $(3, 2)$, which implies similarly $(1, 2)$ for the right column of Q_1'. In summary, Q_1' has index sets (only listing bf, lb)

$$\begin{pmatrix} (-1, l-1) & (1, l-1) \\ (1, l+1) & (2, l+1) \end{pmatrix},$$

so we get that $Q_1' \in \Psi_{\phi,\mathcal{H}}^{-\infty,\mathcal{E}}(M)$ after possibly adjusting $\mathcal{E}$ by allowing more log terms at lb. More precisely, Q_1' is higher order at bf than Q_1.

Step 3: We now have $\Box_\phi Q_2 = \mathrm{Id} - R_2$ with $Q_2 = Q_1 + Q_1'$ and R_2 from (10.39). Next we remove the interior conormal singularity of the remainder. First, note that $(R_2\Pi, R_2\Pi^\perp)$ has index sets (notation as in (10.37))

$$\Big((1, 4, \varnothing, r-1), (2, 4, \varnothing, r+1)\Big). \tag{10.41}$$

Since $\Box_\phi$ is ϕ-elliptic, we may find a 'small' parametrix Q_σ arising from inverting the ϕ-symbol, so $\Box_\phi Q_\sigma = \mathrm{Id} - R_\sigma$, with $Q_\sigma \in \Psi_\phi^{-m}(M)$ and $R_\sigma \in \Psi_\phi^{-\infty}(M)$. Then we obtain

$$\Box_\phi Q_3 = \mathrm{Id} - R_3, \text{ with } Q_3 := Q_2 + Q_\sigma R_2,\ R_3 := R_\sigma R_2$$

Since $Q_\sigma R_2$ has the same index sets as R_2, see (10.41), it is also $\Psi_{\phi,\mathcal{H}}^{-2,\mathcal{E}}(M)$ like Q_2 and does not contribute to the leading terms of Q_3 at bf and ϕf. Also, R_3 has the same index sets as R_2 and in addition is smoothing.

A priori, our construction only yields that Q_3 lies in the extended ϕ-calculus. The following standard regularity argument shows that Q_3 actually lies in the ϕ-calculus: since Q_σ is also a small left parametrix, $Q_\sigma \Box_\phi = \mathrm{Id} - R_\sigma'$ with $R_\sigma' \in \Psi_\phi^{-\infty}$, we have

$$Q_\sigma - Q_\sigma R_3 = Q_\sigma \Box_\phi Q_3 = Q_3 - R_\sigma' Q_3$$

[13]If l is an indicial root of P_{00} then q_0 gets extra logarithmic terms.

and since both remainders are smoothing, Q_3 has the same singularity as Q_σ.

Step 4: The remainder term R_3 is smoothing and vanishes at $\mathrm{lb}, \mathrm{bf}, \phi\mathrm{f}$, so that the Neumann series $\mathrm{Id} + R_3 + R_3^2 + \dots$ can be summed asymptotically by the composition theorem for the full ϕ-calculus. Denote the sum by $\mathrm{Id} + S$. Multiplying this from the right gives $Q_\alpha = Q_3(\mathrm{Id} + S)$ and $R_\alpha = R_3(\mathrm{Id} + S)$ as required.

Finally, we note that Q_α has the same leading terms as Q_1, to two orders at bf and ϕf. This concludes the proof of the theorem. □

Remark 10.5. Note that, as in [GrHu14], we first constructed the boundary parametrix and then combined it with the interior parametrix arising from inverting the ϕ-symbol. This is the opposite order from what is done, e.g., by Vaillant [Vai01] in the same context. Our approach has the advantage of giving control of the $\mathcal{H} - \mathcal{C}$ decomposition of the parametrix both on the domain and range side (the opposite order would give only control on the range side of the right parametrix). This gives more precise information on the parametrix and is needed, for example, for the proof of the Fredholm property stated below.

Remark 10.6. The fact that the off-diagonal terms of $\Box_\phi$ in (10.27) are higher order in terms of $x \to 0$ is the reason that we impose the assumption $[\Pi, D_B] = 0$. Without this assumption the off-diagonal terms would only be in $\mathrm{Diff}^2_\phi(M) \cap x\,\mathrm{Diff}^2_b(M)$ (rather than $x^2\,\mathrm{Diff}^2_\phi(M)$), and it is not clear what the result would be in this case.

10.5. **Mapping properties and absence of resonances.** By standard arguments the existence of a parametrix allows to deduce mapping, in particular Fredholm, results for $\Box_\phi$, as well as asymptotic information on elements in its kernel. In our context we obtain different regularity for the $\mathcal{H}$ and the $\mathcal{C}$ components.

The *split Sobolev space* is the Sobolev space analogue of Definition 10.2

$$H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) = x^{-2}\,H^2_{b,0}(V, \mathcal{H}, \mathrm{dvol}_b) + H^2_\phi(M; \mathrm{dvol}_b) \tag{10.42}$$

where $H^2_{b,0}(V, \mathcal{H}, \mathrm{dvol}_b)$ is the space of H^2_b sections of $\mathcal{H}$ compactly supported in $V = [0, \varepsilon) \times B$.[14] This space has a natural Hilbert space topology, see [GrHu14, §6.1, Definition 7]. Note that

$$\begin{aligned} &H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) \not\subseteq L^2(M; \mathrm{dvol}_b), \\ &x^2 H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) \subseteq L^2(M; \mathrm{dvol}_b). \end{aligned} \tag{10.43}$$

[14] Note that $H^2_b \subset H^2_\phi \subset x^{-2} H^2_b$.

Split Sobolev spaces could also be defined for orders other than two, but this is not natural since they do not form a scale of spaces and the following result does not hold in general for other orders.

Corollary 10.7. *The operator $\Box_\phi$ is bounded*

$$\Box_\phi : x^{\alpha+1}H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) \to x^{\alpha+1}L^2(M; \mathrm{dvol}_b) \tag{10.44}$$

for all $\alpha \in \mathbb{R}$ and Fredholm if $\alpha \notin \mathrm{spec}_b(P_{00})$. Its Fredholm inverse $G_{\phi,\alpha}$ lies in $\Psi^{-2,\mathcal{E}}_{\phi,\mathcal{H}}(M)$ with the same index family $\mathcal{E}$ as in Theorem 10.4. The leading asymptotic terms of the (lift of the) Schwartz kernel of $G_{\phi,\alpha}$ are as follows:

(1) *at* bf*: it is $(xx')^{-1}$ times the inverse of $I(P_{00})$, acting on $x^\alpha L^2$, pulled back from the* b*-face of M^2_b to the* bf*-face of M^2_ϕ. In particular, only the $\mathcal{HH}$ component of the Fredholm inverse $G_{\phi,\alpha}$ is non-zero at* bf*.*

(2) *at ϕf: it is the inverse of the normal operator of $\Box_\phi$, where the $\mathcal{H}$ part of the inverse is given by (10.29).*

The Fredholm inverse was defined after Theorem 3.3. We now continue with the proof of Corollary 10.7.

Proof. Boundedness follows easily from (10.27) and the Fredholm property follows from boundedness and compactness of the parametrix and remainder on the appropriate spaces, see [GrHu14, Theorem 13]. The shift in weight arises since Q_{00} in (10.35) is bounded $x^\alpha L^2 \to x^\alpha H^2_b$, so $x^{-1}Q_{00}x^{-1} : x^{\alpha+1}L^2 \to x^{\alpha-1}H^2_b$. The statement on the Fredholm inverse then follows by standard arguments as in [Mel93, Propositions 5.42 and 5.64].

The leading terms at bf and ϕf only arise from the first parametrix Q_1 constructed in Step 1 of the proof of Theorem 10.4. The leading (i.e. order -2) contribution at bf occurs only in the $\mathcal{HH}$-component and is $x^{-1}Q_{00}x^{-1}$, which implies the claim for bf. The leading (i.e. order 0) term at ϕf is the direct sum of the pull-back of $x^{-1}Q_{00}x^{-1}$ to ϕf and of the inverse of the normal operator of P_{11}. Since $b \geq 2$ by Assumption 9.5, the singularity of Q_{00} at the diagonal is of the type $\left(|s-1|^2 + |y-y'|^2\right)^{-\frac{b-1}{2}}$ where $s = x/x'$ (recall projective coordinates (3.3)). Thus $x^{-1}Q_{00}x^{-1}$ is given by the half-density

$$(xx')^{-1}\left(|s-1|^2 + |y-y'|^2\right)^{-\frac{b-1}{2}}\sqrt{\frac{dx}{x}ds\,dy\,dy'}.$$

Pulling back this half-density to M^2_ϕ gives in projective coordinates (4.5)

$$\left(|T|^2 + |Y|^2\right)^{-\frac{b-1}{2}}\sqrt{dT\,dY\frac{dx}{x^2}\frac{dy'}{x^b}},$$

which shows that the standard fundamental solution on $\mathbb{R}^{b+1}$ indeed appears at ϕf as in (10.29). ∎

We now obtain information on the kernel of $\Box_\phi$.

Corollary 10.8. *Under the Assumption 9.3 the following holds: for any solution $u \in x^{-1}L^2(M; \mathrm{dvol}_b)$ to $\Box_\phi u = 0$, we have that $u \in \mathcal{A}^{\mathcal{E}(1)-1}_{\mathcal{H}}(M)$. In particular, $u \in L^2(M; \mathrm{dvol}_b)$ and thus there are no resonances, i.e.*

$$\ker_{x^{-1}L^2(M;\mathrm{dvol}_b)} \Box_\phi = \ker_{L^2(M;\mathrm{dvol}_b)} \Box_\phi.$$

Equivalently, we have for the unitarily equivalent Hodge Laplacian Δ_ϕ

$$\ker_{x^{-1}L^2(M;\mathrm{dvol}_\phi)} \Delta_\phi = \ker_{L^2(M;\mathrm{dvol}_\phi)} \Delta_\phi.$$

Proof. Taking the dual spaces of (10.44) with respect to L^2 we get $\Box_\phi : x^{\alpha-1}L^2 \to x^{\alpha-1}H^{-2}_{\mathcal{H}}$ (compare [GrHu14, Theorem 13]) if $\alpha \notin \mathrm{spec}_b(P_{00})$. We take $\alpha = 0$ and use the left parametrix Q_0 from Theorem 10.4. Applying Q_0 to $\Box_\phi u = 0$ we get $u = R_0' u$. Since R_0' has index sets

$$\begin{pmatrix} \mathcal{E}(0)_{\mathrm{lb}} - 1 \\ \mathcal{E}(0)_{\mathrm{lb}} + 1 \end{pmatrix},$$

at lb and $\varnothing$ at all other faces, and since $\mathcal{E}(0)_{\mathrm{lb}} = \mathcal{E}_{\mathrm{lb}}(1)$ by Assumption 9.3, we conclude $u \in \mathcal{A}^{\mathcal{E}(1)-1}_{\mathcal{H}}(M)$. The claim on no resonances follows. □

11. Review of the resolvent construction on scattering manifolds

The operator $(\Delta_\phi + k^2)$ and its unitary transformation $(\Box_\phi + k^2)$ are fully elliptic ϕ-differential operators and invertible for $k > 0$, so the Schwartz kernels of their inverses are polyhomogeneous distributions on $M^2_\phi \times (0,\infty)_k$, where the ϕ-double space M^2_ϕ is defined in (4.4), with a conormal singularity at $\mathrm{Diag}_\phi \times (0,\infty)$ where Diag_ϕ is the lifted diagonal in M^2_ϕ. However, that description is not uniform up to $k = 0$. In case of $\dim F = 0$ (scattering manifolds), the behavior of the resolvent $(\Box_\phi + k^2)^{-1}$ as $k \to 0$ was analyzed by Guillarmou-Hassell [GuHa08, GuHa09], as well as Guillarmou-Sher [GuSh15], who define a blowup $M^2_{k,sc}$ of $\overline{M}^2 \times \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$, on which the resolvent is polyhomogeneous and conormal.

In this section we review this construction and generalize it slightly to obtain a space $M^2_{k,sc,\phi}$, which in case $\dim F = 0$ reduces to $M^2_{k,sc}$ and in the general case serves as intermediate step in our construction of the resolvent space $M^2_{k,\phi}$ for $\Box_\phi$, which is carried out in Section 12. In Subsection 11.4 we review the results of Guillarmou, Hassell and Sher in the case $\dim F = 0$.

In this section M is a manifold with fibred boundary, and local coordinates near the boundary are as in Definition 4.1.

11.1. **Blowup of the codimension 3 corner.** We consider $\overline{M}^2 \times \mathbb{R}^+$ with copies of local coordinates (x, y, z) and (x', y', z') on the two factors $\overline{M}$ near ∂M. The highest codimension corner in $\overline{M}^2 \times \mathbb{R}^+$ is given by

$$C_{111} = \partial M \times \partial M \times \{0\},$$

and writes in local coordinates as $C_{111} = \{x = x' = k = 0\}$. Here the lower index (111) indicates that C_{111} is defined by setting each of x, x', k equal to zero. A similar convention will be applied below for the codimension 2 corners. The blowup of this corner and blow-down map are denoted by

$$\beta_1 : [\overline{M}^2 \times \mathbb{R}^+, C_{111}] \to \overline{M}^2 \times \mathbb{R}^+$$

This leads to a new boundary hypersurface that we call bf_0, as illustrated in Figure 10, where all other variables are omitted. We may introduce local projective coordinates near each corner of bf_0.

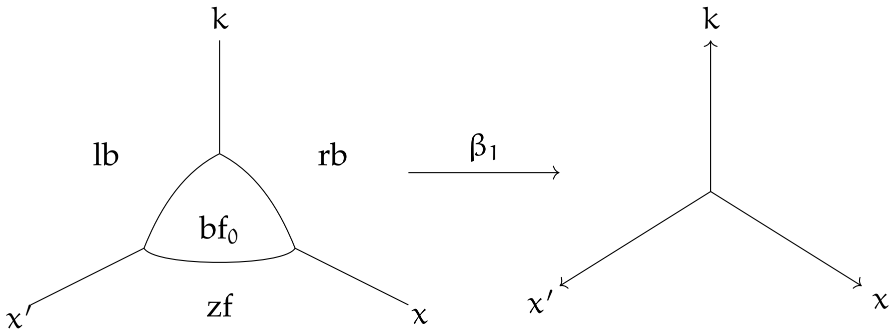

FIGURE 10. Blowup of C_{111}

Near top corner. Away from zf we may introduce,

$$\xi = \frac{x}{k}, \ \xi' = \frac{x'}{k}, \ y, \ z, \ y', \ z', \ k. \tag{11.1}$$

Here, ξ' is a local boundary defining function of the right boundary face rb (we write $\rho_{\mathrm{rb}} = \xi'$), ξ of the left boundary face lb (we write $\rho_{\mathrm{lb}} = \xi$), and k of the new boundary face bf_0 (we write $\rho_{\mathrm{bf}_0} = k$).

Near right corner. In a similar way, away from lb we may introduce,

$$s' = \frac{x'}{x}, \ \kappa = \frac{k}{x}, \ x, \ y, \ z, \ y', \ z'. \tag{11.2}$$

Here, local boundary defining functions are given by $\rho_{\mathrm{bf}_0} = x$, $\rho_{\mathrm{rb}} = s'$, $\rho_{\mathrm{zf}} = \kappa$. Projective coordinates near the left corner are obtained by interchanging the roles of x and x', and replacing rb by lb.

11.2. **Blowup of the codimension 2 corners.** The next step is to blow up the codimension 2 corners that are given by

$$C_{011} = \overline{M} \times \partial M \times \{0\}, \quad C_{101} = \partial M \times \overline{M} \times \{0\}, \quad C_{110} = \partial M \times \partial M \times \mathbb{R}^+.$$

More precisely we blow up their lifts to $[\overline{M}^2 \times \mathbb{R}^+, C_{111}]$, which we still denote by the same $C_{011}, C_{101}, C_{110}$ notational convention. This defines

$$M^2_{k,b} := \left[\left[\overline{M}^2 \times \mathbb{R}^+, C_{111}\right], C_{011}, C_{101}, C_{110}\right]$$

with blow-down map β_2 and new front faces lb_0, rb_0 and bf. This blowup is illustrated in Figure 11. We keep the notation bf_0 for the lift of the face bf_0.

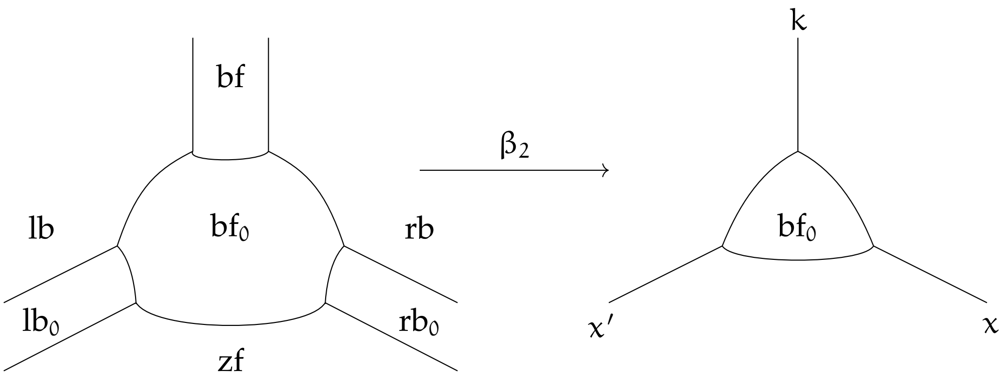

FIGURE 11. Blow up of $C_{110}, C_{011}, C_{101}$.

The associated (full) blowdown map to $\overline{M}^2 \times \mathbb{R}^+$ is given by

$$\beta_b = \beta_2 \circ \beta_1 : M^2_{k,b} \longrightarrow \overline{M}^2 \times \mathbb{R}^+.$$

We now describe the blowup in terms of projective coordinates that are valid near some of the intersections of the various boundary hypersurfaces in $M^2_{k,b}$.

Projective coordinates near $\mathrm{bf}_0 \cap \mathrm{bf}$*.* We use the projective coordinates (11.1). The new projective coordinates near the left corner on the top are now given by

$$\zeta = \frac{\xi}{\xi'} = \frac{x}{x'}, \ \xi' = \frac{x'}{k}, \ y, \ z, \ y', \ z', \ k. \tag{11.3}$$

where $\rho_{\mathrm{lb}} = \zeta$, $\rho_{\mathrm{bf}} = \xi'$, $\rho_{\mathrm{bf}_0} = k$. Interchanging the roles of x and x' gives a set of coordinates near the left top corner, i.e. near $\mathrm{bf}_0 \cap \mathrm{bf}$, valid away from rf.

Projective coordinates near $\mathrm{bf}_0 \cap \mathrm{rb}_0$. We use the projective coordinates (11.2). The new projective coordinates in the lower right corner, where $\mathrm{bf}_0, \mathrm{rb}_0$ and zf meet, are now given by

$$s' = \frac{x'}{x},\ \tau = \frac{\kappa}{s'} = \frac{k}{x'},\ x,\ y,\ z,\ y',\ z'. \tag{11.4}$$

where $\rho_{\mathrm{zf}} = \tau, \rho_{\mathrm{rb}_0} = s', \rho_{\mathrm{bf}_0} = x$. Projective coordinates near the left lower corner, where $\mathrm{bf}_0, \mathrm{lb}_0$ and zf meet, are obtained by interchanging the roles of x and x'. Projective coordinates near the other corners are obtained similarly.

11.3. **Blowup of the fibre diagonal.** The final step is to blow up the lifted fibre diagonal in the face bf. In projective coordinates (11.3) it is expressed by

$$\mathrm{diag}_{k,sc,\phi} := \{\zeta = 1,\ y = y',\ \xi' = 0\} \subseteq \mathrm{bf}.$$

The new front face is denoted sc and the resulting space $M^2_{k,sc,\phi} := [M^2_{k,b}, \mathrm{diag}_{k,sc,\phi}]$ with blow-down map β_3 is illustrated in Figure 12.

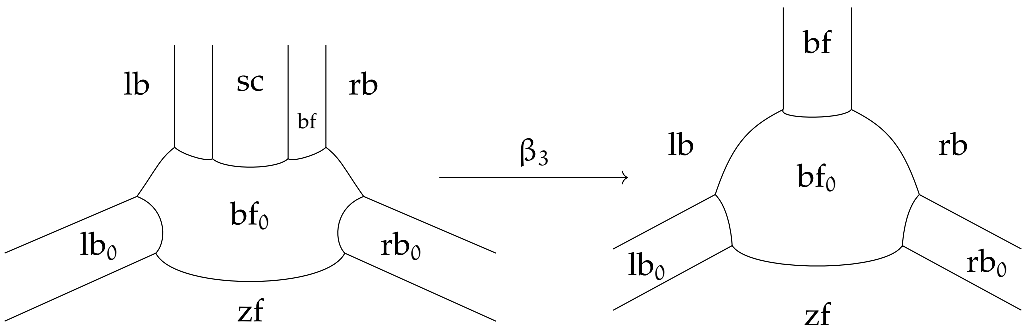

FIGURE 12. Final bowup $M^2_{k,sc,\phi} := [M^2_{k,b}, \mathrm{diag}_{k,sc,\phi}]$.

The associated (full) blowdown map to $\overline{M}^2 \times \mathbb{R}^+$ is given by

$$\beta_{k,sc,\phi} = \beta_3 \circ \beta_2 \circ \beta_1 : M^2_{k,sc,\phi} \longrightarrow \overline{M}^2 \times \mathbb{R}^+.$$

In the same pattern as before, we may introduce projective coordinates near the intersection $\mathrm{bf}_0 \cap \mathrm{sc}$. Using the coordinate system (11.3), we define new projective coordinates as follows

$$X := \frac{\zeta - 1}{\xi'} = kT,\ U := \frac{y - y'}{\xi'} = kY,\ \xi' = \frac{x'}{k},\ y',\ z,\ z',\ k, \tag{11.5}$$

with (T, Y) as in (4.5). Here, the boundary defining functions are as follows: $\rho_{\mathrm{bf}_0} = k, \rho_{\mathrm{sc}} = \xi'$ and the boundary face bf lies in the limit $|(X, U)| \to \infty$. We illustrate these coordinates in Figure 13.

The lifted diagonal in $M^2_{k,sc,\phi}$ is, by definition, the set

$$\mathrm{Diag}_{k,sc,\phi} := \beta^*_{k,sc,\phi}(\mathrm{Diag}_{\overline{M}} \times \mathbb{R}^+)\,.$$

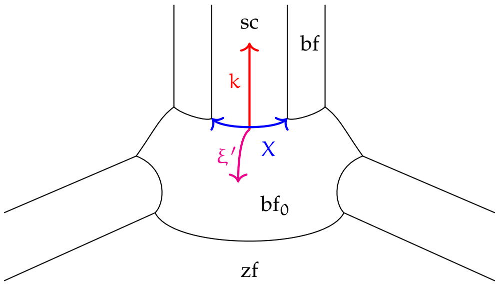

FIGURE 13. Illustration of projective coordinates in $M^2_{k,sc}$.

Remark 11.1. For each fixed $k_0 > 0$ the level set $\{k = k_0\}$ in the blowup manifold $M^2_{k,sc,\phi}$ is simply the ϕ-space M^2_ϕ, introduced in (4.4). The face zf is diffeomorphic to the b-space M^2_b, and in fact in case of trivial fibres F, the Hodge Laplacian Δ_ϕ can be reduced to a b-operator near zf. That observation has been crucial in the resolvent construction by Guillarmou, Hassell and Sher.

If the fibres of ϕ are points then we write Δ_{sc}, ${}^{sc}TM$, $M^2_{k,sc}$ instead of Δ_ϕ, ${}^\phi TM$, $M^2_{k,sc,\phi}$ respectively. Note that, if M is a general ϕ-manifold with trivialization $\overline{\mathcal{U}} \cong \partial M \times [0, \varepsilon)$ near the boundary then ϕ defines a fibration $\overline{\mathcal{U}} \to V = B \times [0, \varepsilon)$, and this induces a fibration

$$\mathcal{U}^2_{k,sc,\phi} \to V^2_{k,sc} \tag{11.6}$$

with fibres F^2. Therefore, a distribution on $V^2_{k,sc}$ valued in $\mathrm{End}(\mathcal{H})$ for a subbundle $\mathcal{H} \subset C^\infty(F, E)$ and some bundle $E \to \overline{\mathcal{U}}$ can instead be considered as a distribution on $\mathcal{U}^2_{k,sc,\phi}$ valued in E.

11.4. **Asymptotics of the resolvent on scattering manifolds.** We close the section with a short review of the main result by Guillarmou-Hassell [GuHa08], as well as Guillarmou-Sher [GuSh15] on the resolvent kernel of the Hodge Laplacian in the special case where fibres are points, usually referred to as *scattering* manifolds.

Note that Δ_{sc} basically corresponds to the action of Δ_ϕ on the fibre harmonic forms $\mathcal{H}$, that is, only the top left corner in (10.27) is present. Similarly, transforming Δ_{sc} under (10.26) defines $\Box_{sc}$, which corresponds to the action of $\Box_\phi$ on $\mathcal{H}$. In view of (10.27) we define (identify x with the corresponding multiplication operator)

$$P := x^{-1} \circ \Box_{sc} \circ x^{-1}. \tag{11.7}$$

This is an elliptic b-differential operator on $\overline{M}$, corresponding to P_{00} in (10.27). By [GuSh15, (15)], parallel to the formulae in (10.30) and (10.31), we have up to higher order terms coming from the higher order terms in the metric

$$P = -(x\partial_x)^2 + L_{sc}, \tag{11.8}$$

where the action of L_{sc} on $\Lambda^*(x^{-1}T^*B) \oplus x^{-2}dx \wedge \Lambda^{*-1}(x^{-1}T^*B)$ is given by

$$L_{sc} = \begin{pmatrix} \Delta_B + \left(\frac{b-1}{2} - N_B\right)^2 & 2d_B \\ 2d_B^* & \Delta_B + \left(\frac{b+1}{2} - N_B\right)^2 \end{pmatrix}, \tag{11.9}$$

where Δ_B is the Hodge Laplacian of B, and d_B denotes here the exterior differential on B. N_B denotes as before the number operator on $\Lambda^*(x^{-1}T^*B)$, multiplying elements in $\Lambda^\ell(x^{-1}T^*B)$ by ℓ. The indicial family $I_\lambda(P)$ and the set of indicial roots $\operatorname{spec}_b(P)$ are defined as in §3.3.

Before we can state the main theorem of Guillarmou and Sher [GuSh15], let us introduce the (k, sc)-calculus of pseudo-differential operators with Schwartz kernels lifting to $M^2_{k,sc}$. As usual we identify the operators with the lifts of their Schwartz kernels.

Definition 11.2. Let $\overline{M}$ be a compact manifold with boundary. We define small and full (k, sc)-calculi as follows. Consider the following density bundle (compare to (4.6))

$$\Omega_{b\phi}(M^2_{k,sc}) := \rho_{sc}^{-(b+1)}\Omega_b(M^2_{k,sc}) = \rho_{sc}^{-2(b+1)}\beta^*_{k,sc}\Omega_b(\overline{M}^2 \times \mathbb{R}^+). \tag{11.10}$$

The corresponding half-density bundle is denoted by[15] $\Omega^{1/2}_{b\phi}(M^2_{k,sc})$.

1. The small (k, sc)-calculus, denoted $\Psi^m_{k,sc}(M)$ for $m \in \mathbb{R}$, is the space of distributions on $M^2_{k,sc}$, valued in $\Omega^{1/2}_{b\phi}(M^2_{k,sc}) \otimes \operatorname{End}(\Lambda^{sc}T^*M)$, which are conormal with respect to the lifted diagonal and which vanish to infinite order at all boundary hypersurfaces except bf_0, zf and sc.

2. Consider $(a_{bf_0}, a_{zf}, a_{sc}) \in \mathbb{R}^3$ and an index family $\mathcal{E}$ for $M^2_{k,sc}$ such that $\mathcal{E}_{bf}, \mathcal{E}_{lb}, \mathcal{E}_{rb} = \varnothing$. The full (k, sc)-calculus is then defined by

$$\Psi^{m,(a_{bf_0},a_{zf},a_{sc}),\mathcal{E}}_{k,sc}(M) := \rho_{bf_0}^{a_{bf_0}}\rho_{zf}^{a_{zf}}\rho_{sc}^{a_{sc}}\Psi^m_{k,sc}(M) + \mathcal{A}^{\mathcal{E}}_{k,sc}(M), \tag{11.11}$$

where we simplified notation by setting

$$\mathcal{A}^{\mathcal{E}}_{k,sc}(M) := \mathcal{A}^{\mathcal{E}}_{phg}\Big(M^2_{k,sc}, \Omega^{1/2}_{b\phi}(M^2_{k,sc}) \otimes \operatorname{End}(\Lambda^{sc}T^*M)\Big). \tag{11.12}$$

Theorem 11.3. [GuSh15, Theorem 18] *Let (M, g) be a scattering (asymptotically conic) manifold, satisfying Assumption 9.1, 9.3 and 9.5, where*

(1) Assumption 9.3 can be replaced with (9.5) and $0 \notin \operatorname{spec}_b(P)$.
(2) Assumption 9.5 is equivalent to $\dim M \geq 3$.

[15] This is precisely the half-density bundle $\widetilde{\Omega}^{\frac{1}{2}}(M^2_{k,sc})$ introduced in [GuHa08, §2.2.2].

Then $(\Box_{sc} + k^2)^{-1}$ *lies in the full* (k, sc)*-calculus* $\Psi^{-2,(-2,0,0),\mathcal{E}}_{k,sc}(M)$*, where*

$$\mathcal{E}_{zf} \geq -2, \quad \mathcal{E}_{bf_0} \geq -2, \quad \mathcal{E}_{sc} \geq 0,$$
$$\mathcal{E}_{lb_0} = \mathcal{E}_{rb_0} > 0.$$

Moreover, the leading terms of the resolvent at all boundary hypersurfaces of $M^2_{k,sc}$ *are given by solutions of explicit model problems.*

12. Low energy resolvent for ϕ-metrics, proof of main theorem

Recall that our aim is the construction of the inverse for $\Delta_\phi + k^2$, which is a self-adjoint operator in $L^2(M; dvol_\phi)$. We equivalently describe the parametrix construction for

$$x^{-\frac{b+1}{2}} \circ (\Delta_\phi + k^2) \circ x^{\frac{b+1}{2}} = \Box_\phi + k^2,$$

which is a self-adjoint operator in $L^2(M; dvol_b)$. In the collar neighborhood $\mathcal{U}$ of the boundary, $\Box_\phi$ acts with respect to the splitting into fibre harmonic forms $\mathcal{H}$ and the perpendicular bundle $\mathcal{C}$ by a 2×2 matrix (see (10.27))

$$\Box_\phi = \begin{pmatrix} xP_{00}x & xP_{01}x \\ xP_{10}x & P_{11} \end{pmatrix}.$$

As in the parametrix construction for $\Box_\phi$ in Section Recall from Subsection 10.3 that $xP_{00}x$ is, up to higher order terms, a Hodge Laplacian on $V = B \times [0, \varepsilon)$ for a scattering metric, twisted by the bundle $\mathcal{H}$. Therefore it has a resolvent parametrix which is well-behaved (i.e. polyhomogeneous and conormal) on the space $V^2_{k,sc}$ and valued in $\mathrm{End}(\mathcal{H})$. By (11.6) and the subsequent explanation it is therefore a well-behaved distribution on $\mathcal{U}_{k,sc,\phi} \subset M^2_{k,sc,\phi}$. On the other hand, for operators like P_{11}, i.e. fully elliptic ϕ-Laplacians, the resolvent is well-behaved on $\mathcal{U}^2_\phi \times \mathbb{R}^+ \subset M^2_\phi \times \mathbb{R}^+$. These two spaces are illustrated along each other for comparison in Figure 14. In order to describe the resolvent of $\Box_\phi$ we therefore need to construct a blowup of $\overline{M}^2 \times \mathbb{R}^+$ that blows down to both $M^2_{k,sc,\phi}$ and $M^2_\phi \times \mathbb{R}^+$. We now construct such a space, which we call $M^2_{k,\phi}$.

12.1. Construction of the blowup space $M^2_{k,\phi}$. The space $M^2_{k,\phi}$ is constructed from $M^2_{k,sc,\phi}$ by the blowup of the fibre diagonal $\mathrm{diag}_{k,\phi}$ in bf_0, which is defined as the closure of the subset given in projective coordinates (11.5) by $\{X = 0, U = 0, k = 0\}$. The blowup space $M^2_{k,\phi}$ is then defined by

$$M^2_{k,\phi} := [M^2_{k,sc,\phi}, \mathrm{diag}_{k,\phi}], \tag{12.1}$$

with the blowdown map $\beta_{\phi-sc} : M^2_{k,\phi} \longrightarrow M^2_{k,sc,\phi}$, and total blowdown map

$$\beta_{k,\phi} = \beta_{k,sc,\phi} \circ \beta_{\phi-sc} : M^2_{k,\phi} \longrightarrow \overline{M}^2 \times \mathbb{R}^+.$$

The resulting blowup space is illustrated in Figure 15.

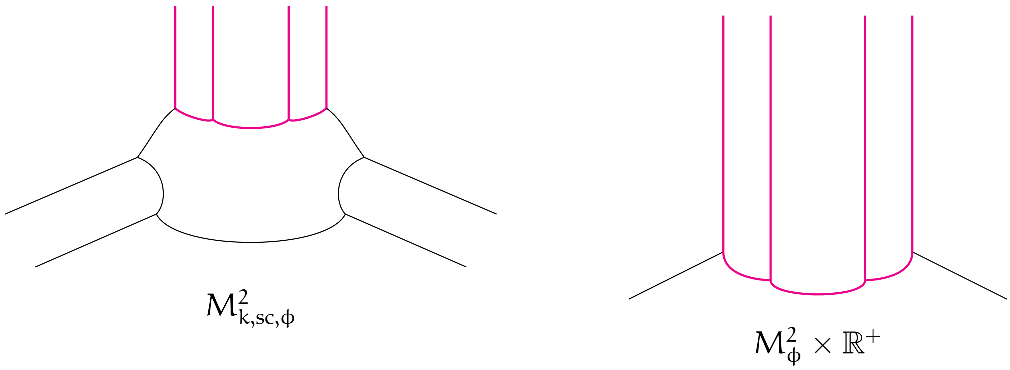

FIGURE 14. Comparison of the blowup spaces $M^2_{k,sc,\phi}$ and $M^2_\phi \times \mathbb{R}^+$.

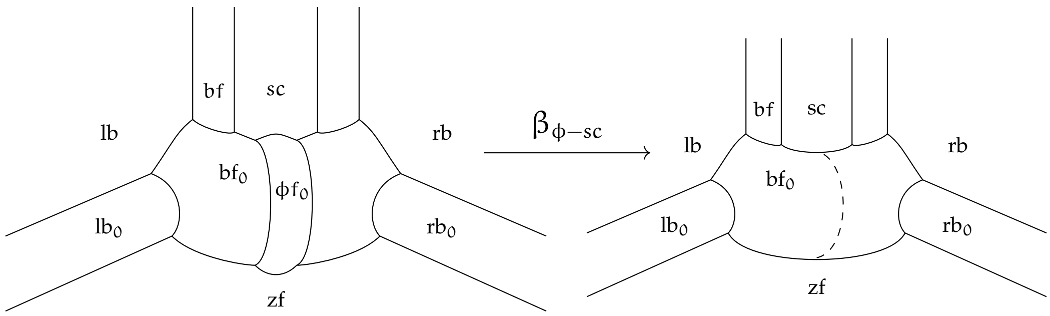

FIGURE 15. Blowup space $M^2_{k,\phi}$

By construction, the blowup space $M^2_{k,\phi}$ blows down to $M^2_{k,sc,\phi}$. However, existence of a blowdown map to $M^2_\phi \times \mathbb{R}^+$ is non-trivial and is the subject of the following lemma.

Lemma 12.1. *The identity in the interior extends to a b-map*

$$\beta'_{k,\phi} : M^2_{k,\phi} \longrightarrow M^2_\phi \times \mathbb{R}^+$$

arising from blowing up the products $f \times \{0\}$ *where* f *runs through the faces* bf, ϕf, lb, rb *of* M^2_ϕ *in this order. The face* ϕf_0 *of* $M^2_{k,\phi}$ *is the lift of the front face of the blow-up of* $\phi f \times \{0\}$.

For the proof we use the following result on interchanging the order of blowups.

Lemma 12.2. *Let* Z *be a manifold with corners and* $A, B \subset Z$ *be two p-submanifolds which intersect cleanly*[16]*. We shall blow up both submanifolds in different order and write e.g.* $[Z; A, B] := [[Z, A], \widetilde{B}]$*, where* $\widetilde{B}$ *is the lift of* B *under the blow-down map* $[Z, A] \to Z$*. Then there are natural diffeomorphisms as follows.*

[16] This means that near any $p \in A \cap B$ one can find coordinates adapted to the corners of Z so that both A and B are coordinate subspaces locally.

(a) *If* A, B *are transversal or disjoint, or one is contained in the other, then interchanging the order of blowups of* A *and* B *yields diffeomorphic manifolds with corners*

$$[Z; A, B] \cong [Z; B, A].$$

(b) *In general, interchanging the order of blowups of* A *and* B *yields diffeomorphic results if additionally the intersection* $A \cap B$ *is blown up:*

$$[Z; A, B, A \cap B] \cong [Z; B, A, A \cap B].$$

In both cases diffeomorphy holds in the sense that the identity on $Z \setminus (A \cup B)$ *extends smoothly to a b-map between the two spaces with smooth inverse.*

Proof. For the proof of statement (a) we refer to [HMM95, Lemma 2.1]. For the proof of statement (b) we shall illustrate the idea on a specific example that models the blowup of ϕf_0 in $M^2_{k,sc}$. Consider $Z := \mathbb{R}_y \times \mathbb{R}^+_x \times \mathbb{R}^+_z$ with coordinates (x, y, z) as indicated in the lower indices. We set

$$A := \{x = z = 0\} \text{ and } B := \{x = y = 0\}.$$

Their intersection is $A \cap B = \{x = 0, y = 0, z = 0\}$. We shall consider projective coordinates on the blowups $[Z; A, B, A \cap B]$ and $[Z; B, A, A \cap B]$.

Blowup $[Z; A, B, A \cap B]$: The blowup is obtained in three steps, blowing up A first, then the lift of B second, and finally the lift of the intersection $A \cap B$ as the third step. This is illustrated in Figure 16.

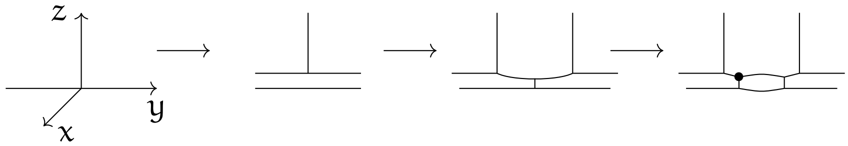

FIGURE 16. $[Z; A, B, A \cap B]$.

Blowup $[Z; B, A, A \cap B]$: The blowup is obtained in three steps, blowing up B first, then the lift of A second, and finally the lift of the intersection $A \cap B$ as the third step. This is illustrated in Figure 17.

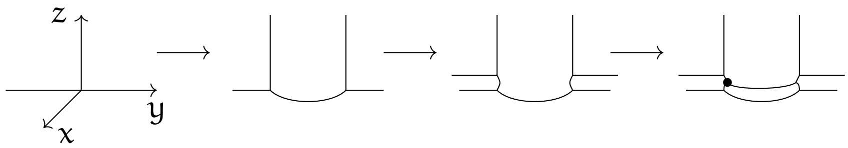

FIGURE 17. $[Z; B, A, A \cap B]$.

There is an obvious bijective correspondence between the face lattices of the two spaces. We shall write out explicitly the projective coordinates near the corners indicated by a bullet in Figures 16 and 17. Straightforward computations show that projective coordinates in both blowups are the same and given by

$$s_1 = \frac{zy}{x}, \quad s_2 = \frac{x}{y}, \quad s_3 = \frac{x}{z}. \tag{12.2}$$

These projective coordinates are illustrated in Figure 18.

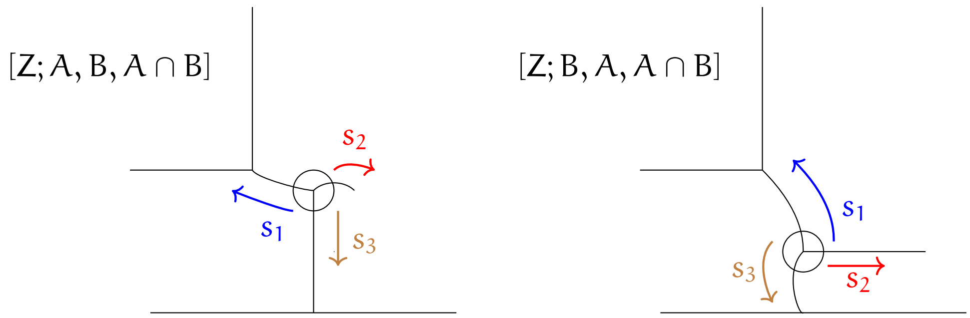

FIGURE 18. Coordinates in corners of $[Z; A, B, A \cap B]$ and $[Z; B, A, A \cap B]$.

Similarly we may check that projective coordinates near any pair of corresponding corners of $[Z; A, B, A \cap B]$ and $[Z; B, A, A \cap B]$ coincide, proving the statement in this model case. The general case is studied along the same lines. $\square$

We can now prove Lemma 12.1.

Proof of Lemma 12.1. We proceed in the notation of §11 and denote for example by C_{111} the highest codimension corner in $\overline{M}^2 \times \mathbb{R}^+$. Here the lower index (111) indicates that C_{111} is defined by setting each of x, x', k equal to zero. Similarly, the codimension 2 corners are given by

$$C_{011} = M \times \partial M \times \{0\}, \quad C_{101} = \partial M \times M \times \{0\}, \quad C_{110} = \partial M \times \partial M \times \mathbb{R}^+.$$

By a slight abuse of notation, we shall use the same notation for the lifts of these corners as well. We set in accordance with notation of Lemma 12.2

$$Z = [\overline{M}^2 \times \mathbb{R}^+; C_{110}], \quad A = C_{111}, \quad B = \mathrm{diag}_\phi \times \mathbb{R}^+$$

where diag_ϕ is the fibre diagonal in the b-face of M^2_b, see (4.4). In local projective coordinates $(s = x/x', x', y, y', z, z', k)$ near the bottom of the resulting front face in Z, we have $A = \{x = x' = k = 0\}$ and $B = \{s = 1, y = y'\}$, which shows that A, B intersect cleanly but not transversally. We obtain

$$M^2_{k,\phi} = \Big[[Z; A, B, A \cap B], C_{101}, C_{011}\Big] \cong \Big[[Z; B, A, A \cap B], C_{101}, C_{011}\Big]. \qquad (12.3)$$

Now $[Z, B] = M^2_\phi \times \mathbb{R}^+$, so the latter space is simply

$$\Big[[M^2_\phi \times \mathbb{R}^+, A, A \cap B], C_{101}, C_{011}\Big].$$

Also, the lifts of A, $A \cap B$, C_{101}, C_{011} are $f \times \{0\}$ with $f = \mathrm{bf}, \phi\mathrm{f}, \mathrm{lb}, \mathrm{rb}$. This proves the lemma. $\square$

12.2. **Definition of the** (k, ϕ)**-calculus.** The lifted diagonal $\mathrm{Diag}_{k,\phi}$, i.e. the closure of the preimage of $\{(p, p, k) \mid p \in M, k > 0\} \subset \overline{M}^2 \times \mathbb{R}^+$ under the map $\beta_{k,\phi} : M^2_{k,\phi} \to \overline{M}^2 \times \mathbb{R}^+$, is a p-submanifold of $M^2_{k,\phi}$ and hits its boundary only in the faces sc, ϕf_0 and zf. The Schwartz kernel of the operator $\Box_\phi + k^2$ lifts to $M^2_{k,\phi}$ to be conormal to $\mathrm{diag}_{k,\phi}$, uniformly to the boundary with a non-vanishing delta type singularity, when written as a section of the half-density bundle

$$\Omega^{\frac{1}{2}}_{b\phi}(M^2_{k,\phi}) := \rho_{\mathrm{sc}}^{\frac{-(b+1)}{2}} \rho_{\phi f_0}^{\frac{-(b+1)}{2}} \Omega^{\frac{1}{2}}_b(M^2_{k,\phi}). \qquad (12.4)$$

This is because the same is true for $\Box_\phi$ on M^2_ϕ, hence on $M^2_\phi \times \mathbb{R}^+$, with respect to its diagonal $\mathrm{diag}_\phi \times \mathbb{R}^+$, and this diagonal hits $\mathrm{bf} \times \{0\}$ transversally, so blowing up that face (which results in ϕf_0 away from bf_0) does not affect conormality. Equivalently, in local coordinates we blow up $\{x' = k = 0\}$, and this does not affect conormality with respect to $\{(T, Y) = 0, z = z'\}$. We also use here that the half-density bundles are compatible in the sense that

$$\Omega^{\frac{1}{2}}_{b\phi}(M^2_{k,\phi}) = (\beta'_\phi)^* \Big(\Omega^{\frac{1}{2}}_{b\phi}(M^2_\phi) \otimes \Omega^{\frac{1}{2}}_b(\mathbb{R}^+)\Big).$$

This motivates the following definition. As usual we define operators (in our case, families of operators on M depending on the parameter $k > 0$) by their Schwartz kernels and identify kernels on $\overline{M}^2 \times (0, \infty)$ with those lifted to the interior of $M^2_{k,\phi}$.

Definition 12.3. Let $\overline{M}$ be a compact manifold with fibred boundary. We define small and full (k, ϕ)-calculi as follows.

1. The small (k, ϕ)-calculus, denoted by $\Psi^m_{k,\phi}(M)$ for $m \in \mathbb{R}$, is the set of distributions on $M^2_{k,\phi}$, valued in $\Omega^{\frac{1}{2}}_{b\phi}(M^2_{k,\phi}) \otimes \mathrm{End}(\Lambda^\phi T^* M)$, which are conormal of order m with respect to the lifted diagonal and which vanish at all boundary hypersurfaces except ϕf_0, zf and sc.

2. Consider $(a_{\phi f_0}, a_{zf}, a_{sc}) \in \mathbb{R}^3$, and an index family $\mathcal{E}$ for $M^2_{k,\phi}$, such that
$$\mathcal{E}_{bf} = \mathcal{E}_{lb} = \mathcal{E}_{rb} = \varnothing, 0 \subset \mathcal{E}_{sc} \cap \mathcal{E}_{\phi f_0} \cap \mathcal{E}_{zf}\,. \tag{12.5}$$
Full (k, ϕ)-calculus is then defined by
$$\Psi^{m,(a_{\phi f_0}, a_{zf}, a_{sc}),\mathcal{E}}_{k,\phi}(M) = \rho^{a_{\phi f_0}}_{\phi f_0} \rho^{a_{zf}}_{zf} \rho^{a_{sc}}_{sc} \Psi^m_{k,\phi}(M) + \mathcal{A}^{\mathcal{E}}_{k,\phi}, \tag{12.6}$$
where we simplified notation by setting
$$\mathcal{A}^{\mathcal{E}}_{k,\phi} := \mathcal{A}^{\mathcal{E}}_{phg}\Big(M^2_{k,\phi}, \Omega^{1/2}_{b\phi}(M^2_{k,\phi}) \otimes \mathrm{End}(\Lambda^{\phi}T^*M)\Big).$$
If the triple $(a_{\phi f_0}, a_{zf}, a_{sc}) = (0, 0, 0)$, we simply write $\Psi^{m,\mathcal{E}}_{k,\phi}(M)$.

Because of the different behavior of $\mathcal{H}$- and $\mathcal{C}$-valued sections we also need a split version of the (k, ϕ)-calculus.

Definition 12.4 (Split (k, ϕ)-calculus)**.** Let $\mathcal{E}$ be an index family for $M^2_{k,\phi}$ satisfying (12.5). Let $\mathcal{A}^{\mathcal{E}}_{\mathcal{H}}(M^2_{k,\phi})$ be the space of sections $K \in \mathcal{A}^{\mathcal{E}}_{k,\phi}$ which satisfy the conditions in Definition 10.3, with bf, lb, rb and ϕf replaced by bf_0, lb_0, rb_0 and ϕf_0, respectively. The split (k, ϕ)-calculus is defined as
$$\Psi^{m,\mathcal{E}}_{k,\phi,\mathcal{H}}(M) = \Psi^m_{k,\phi}(M) + \mathcal{A}^{\mathcal{E}}_{\mathcal{H}}(M^2_{k,\phi}).$$
The space $\Psi^{m,(a_{\phi f_0}, a_{zf}, a_{sc}),\mathcal{E}}_{k,\phi,\mathcal{H}}(M)$ is defined in an analogous way.

12.3. Initial parametrix construction on $M^2_{k,\phi}$.

Theorem 12.5. *There exists resolvent parametrix* $G(k) \in \Psi^{-2,(0,0,0),\mathcal{E}}_{k,\phi,\mathcal{H}}(M)$, *such that*
$$(\Box_\phi + k^2) G(k) = \mathrm{Id} - R(k),$$
with remainder $R(k) \in \Psi^{-\infty,(1,1,1),\mathcal{R}}_{k,\phi,\mathcal{H}}(M)$ *where the index set* $\mathcal{E}$ *satisfies*
$$\begin{aligned} &\mathcal{E}_{lb_0} = \mathcal{E}(-1)_{lb} - 1, \ \mathcal{E}_{rb_0} = \mathcal{E}(-1)_{rb} - 1, \\ &\mathcal{E}_{bf_0} \geq -2, \ \mathcal{E}_{\phi f_0} \geq 0, \ \mathcal{E}_{sc} = 0, \ \mathcal{E}_{zf} = -2, \end{aligned} \tag{12.7}$$
with $\mathcal{E}(-1)$ *determined by* $I(P_{00})$ *and satisfying* (3.4) *with* $P = P_{00}$, $\alpha = -1$, *and an index set* $\mathcal{R}$ *positive at all faces. Moreover,* $\mathcal{E}_f$ *and* $\mathcal{R}_f$ *are empty at* $f = \mathrm{bf}, \mathrm{lb}, \mathrm{rb}$. *Note that by Assumption 9.3,* $\mathcal{E}(-1)_{lb} = \mathcal{E}(1)_{lb} > 1$ *and similarly at* rb. *Hence*
$$\mathcal{E}_{lb_0} > 0, \quad \mathcal{E}_{rb_0} > 0. \tag{12.8}$$

The proof of this result occupies this subsection. Constructing $G(k)$ requires solving model problems at the various boundary hypersurfaces of $M^2_{k,\phi}$, leading to the construction of the leading terms at these faces. We first find the leading terms at zf and then at the faces sc, bf_0, ϕf_0 lying over $\partial M \times \partial M$. Along the way we check that the constructions match near all intersections, and also with a small parametrix at the diagonal.

The construction at zf is global on $M \times M$ and uses Corollary 10.7. The construction at the faces sc, bf_0, ϕf_0 closely follows the route taken in Step 1 in the proof of Theorem 10.4, with $\Box_\phi$ replaced by $\Box_\phi + k^2$ and M^2_ϕ by $M^2_{k,\phi}$. In the intermediate steps of the construction we need the extended calculus as explained after Theorem 10.4. The definition given there carries over to the (k,ϕ) case since near the preimage of $(\partial M)^2$ the space $M^2_{k,\phi}$ is an F^2-bundle over $V^2_{k,\phi}$. The lifting property (10.33) also carries over and now reads

$$\beta^*_{\phi-sc} : \Psi^{-m}_{k,sc}(V) \to \rho^m_{\phi f_0}\Psi^{-m}_{k,\phi}(V) + \mathcal{A}^{\mathcal{F}_m}_{k,\phi}(V), \quad \mathcal{A}^{\mathcal{F}}_{k,sc}(V) \to \mathcal{A}^{\mathcal{F}'}_{k,\phi}(V) \tag{12.9}$$

for any index set $\mathcal{F}$ for $M^2_{k,sc}$, where $\mathcal{F}_{m,\mathrm{bf}_0} = 0$, $\mathcal{F}_{m,\phi f_0} = m\,\overline{\cup}\,(b+1)$, and $\mathcal{F}_{m,f} = \varnothing$ at all other faces f, and where $\mathcal{F}'$ has the same index sets as $\mathcal{F}$ at all faces that already exist on $V^2_{k,sc}$, and in addition $\mathcal{F}'_{\phi f_0} = \mathcal{F}_{\mathrm{bf}_0} + (b+1)$. Note that, since the boundary fibration for V has point fibres, we have $V^2_{k,sc} = V^2_{k,sc,\phi}$.

We also need to pull back via $\beta'_{k,\phi} : M^2_{k,\phi} \to M^2_\phi \times \mathbb{R}^+$:

$$(\beta'_{k,\phi})^* : C^\infty(\mathbb{R}^+_k, \Psi^{-m}_\phi(M)) \to \Psi^{-m}_{k,\phi}(M)\,. \tag{12.10}$$

This follows from the fact that ϕf_0 arises as front face of the blow-up of the corner $\phi f \times \{0\} \subset M^2_\phi \times \mathbb{R}^+_k$ (see Figure 17) and the fact that this face is transversal to the diagonal in $M^2_\phi \times \mathbb{R}^+$.

In the following construction we denote the leading term of order m at a face f by $G_m(f)$. That is, the resolvent behaves like $\rho^m_f G_m(f) + o(\rho^m_f)$ near the interior of f. Here ρ_f is a defining function for the interior of f. We use k as interior defining function for all faces 'at $k = 0$' i.e. zf, lb_0, bf_0, ϕf_0. At zf we need to construct two leading terms, $G_{-2}(\mathrm{zf})$ and $G_0(\mathrm{zf})$.

12.3.1. *Leading terms at* zf. We shall define (a fibred cusp operator)

$$\Box_{c\phi} := x^{-1}\Box_\phi x^{-1}.$$

From Corollary 10.7 it follows that the following operators are Fredholm:

$$\begin{aligned} &\Box_\phi : \ H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) \to L^2(M; \mathrm{dvol}_b), \quad \text{if } -1 \notin \mathrm{spec}_b(P_{00}) \\ &\Box_{c\phi} : \ x^2 H^2_{\mathcal{H}}(M; \mathrm{dvol}_b) \to L^2(M; \mathrm{dvol}_b), \quad \text{if } 0 \notin \mathrm{spec}_b(P_{00}). \end{aligned} \tag{12.11}$$

We abbreviate $L^2 = L^2(M; \mathrm{dvol}_b)$ and $H^2_{\mathcal{H}} = H^2_{\mathcal{H}}(M; \mathrm{dvol}_b)$. Assume $0 \notin \mathrm{spec}_b(P_{00})$[17]. Recall that by (10.43) only the second map is an operator in L^2. Let $G_{c\phi}$ be the Fredholm inverse of $\Box_{c\phi}$ in L^2. Because $\Box_{c\phi}$ is self-adjoint in L^2 we have

$$\Box_{c\phi} G_{c\phi} = \mathrm{Id} - \Pi_{c\phi}, \tag{12.12}$$

[17] This is of course a consequence of Assumption 9.3.

where $\Pi_{c\phi}$ is the orthogonal projection in L^2 onto the L^2-kernel of $\Box_{c\phi}$, which has finite dimension. Conjugating by x and inserting $x \cdot x^{-1}$ between $\Box_{c\phi}$ and $G_{c\phi}$ we obtain

$$\Box_\phi x^{-1} G_{c\phi} x^{-1} = \mathrm{Id} - x\Pi_{c\phi} x^{-1}. \tag{12.13}$$

This is an identity in xL^2, since the operators on the left hand side in (12.13) map

$$xL^2 \xrightarrow{x^{-1}G_{c\phi}x^{-1}} xH^2_{\mathcal{H}} \xrightarrow{\Box_\phi} xL^2.$$

We now show that (12.13) is also an identity in L^2 (without weight x!). First, we consider the right hand side of (12.13).

Lemma 12.6. *If Assumption 9.3 is satisfied, then $x\Pi_{c\phi}x^{-1}$ extends to a projection* [18] *in L^2, and the orthogonal projection Π_ϕ of L^2 onto the L^2-kernel of $\Box_\phi$ satisfies*

$$\ker \Pi_\phi = \ker \left(x\Pi_{c\phi} x^{-1} \right).$$

Proof. We have by Assumption 9.3

$$\ker_{L^2} \Box_{c\phi} = \ker_{L^2} \Box_\phi x^{-1} = x \ker_{x^{-1}L^2} \Box_\phi = x \ker_{L^2} \Box_\phi \tag{12.14}$$

where the last equality is by Corollary 10.8 that follows from Assumption 9.3. In particular, $\ker_{L^2} \Box_{c\phi} \subset xL^2$. Writing the Schwartz kernel of $\Pi_{c\phi}$ as $\sum_{j=1}^N \phi_j \otimes \overline{\phi_j}$ for an orthonormal basis (ϕ_j) of $\ker_{L^2} \Box_{c\phi}$, we see that the Schwartz kernel of $x\Pi_{c\phi}x^{-1}$ equals $\sum_{j=1}^N (x\phi_j) \otimes (x^{-1}\overline{\phi_j})$, and from $x^{-1}\phi_j \in L^2$ we conclude that this extends to an operator on L^2, and that (for $u \in L^2$)

$$u \in \ker \left(x\Pi_{c\phi} x^{-1} \right) \iff u \perp x^{-1} \ker_{L^2} \Box_{c\phi}.$$

Now (12.14) gives $x^{-1} \ker_{L^2} \Box_{c\phi} = \ker_{L^2} \Box_\phi$, and the claim follows from $\ker_{L^2} \Box_\phi = \operatorname{Ran} \Pi_\phi = (\ker \Pi_\phi)^\perp$ by the following sequence of identities

$$\ker \Pi_\phi = (\ker_{L^2} \Box_\phi)^\perp = \left(x^{-1} \ker_{L^2} \Box_{c\phi} \right)^\perp = \ker \left(x\Pi_{c\phi} x^{-1} \right).$$

$\square$

Next, we consider the left hand side of (12.13). Since $x^{-1}G_{c\phi}x^{-1}$ is a parametrix of $\Box_\phi$ in xL^2 with finite rank remainder, the argument in Corollary 10.7, with $\alpha = 0$, shows that its index set at rb is $\mathcal{E}(0)_{\mathrm{rb}} - 1$. Now $x^{-1}G_{c\phi}x^{-1}$ extends to L^2 iff its index set at rb is positive, and by (3.4) this is equivalent to

$$[-1, 0] \cap \operatorname{spec}_b(P_{00}) = \varnothing. \tag{12.15}$$

Thus by Assumption 9.3, (12.13) is indeed an identity on L^2. In view of (12.13) as an identity on L^2 and Lemma 12.6 we can obtain a formula for a Fredholm inverse of $\Box_\phi$, using the following simple functional analytic result, relating projections and orthogonal projections.

[18] Note that the projection $x\Pi_{c\phi}x^{-1}$ is not orthogonal.

Lemma 12.7. *Let* H_1 *and* H_2 *be Hilbert spaces,* $P : H_1 \to H_2$ *and* $G : H_2 \to H_1$ *be operators such that* $P \circ G = \mathrm{Id} - \Pi$ *for a continuous projection* Π *in* H_2*. Then the operator* $\mathrm{Id} - \Pi + \Pi^*$ *in* H_2 *is invertible, and the orthogonal projection* Π_o *in* H_2 *with* $\ker \Pi = \ker \Pi_o$ *is given by*

$$\Pi_o = \Pi^* \circ (\mathrm{Id} - \Pi + \Pi^*)^{-1}. \tag{12.16}$$

Moreover, setting $G_o := G \circ (\mathrm{Id} - \Pi + \Pi^*)^{-1}$*, we have*

$$P \circ G_o = \mathrm{Id} - \Pi_o. \tag{12.17}$$

Proof. Since Π is a projection, $\ker \Pi = \mathrm{Ran}\,(\mathrm{Id} - \Pi)$. The property $\ker \Pi = \ker \Pi_o$ implies that Π_o also vanishes on $\mathrm{Ran}\,(\mathrm{Id} - \Pi)$, so

$$\Pi_o \circ (\mathrm{Id} - \Pi) = 0.$$

Next, Π_o is the identity on the orthogonal complement $(\ker \Pi_o)^\perp = (\ker \Pi)^\perp = \mathrm{Ran}\,\Pi^*$. Thus $\Pi_o \circ \Pi^* = \Pi^*$. Adding this to the property $\Pi_o \circ (\mathrm{Id} - \Pi) = 0$ above, we conclude

$$\Pi_o \circ (\mathrm{Id} - \Pi + \Pi^*) = \Pi^*.$$

The operator $\Pi - \Pi^*$ is skew-adjoint, thus has purely imaginary spectrum. Consequently, $\mathrm{Id} - \Pi + \Pi^*$ is invertible and (12.16) follows. Finally, (12.17) follows by a straightforward computation with $S = \mathrm{Id} - \Pi + \Pi^*$:

$$P \circ G_o = P \circ G \circ S^{-1} = (\mathrm{Id} - \Pi) \circ S^{-1} = (S - \Pi^*) \circ S^{-1} = \mathrm{Id} - \Pi^* \circ S^{-1}.$$

□

We apply Lemma 12.7 by setting (in the notation therein)

$$P = \Box_\phi,\ \ G = x^{-1} G_{c\phi} x^{-1},\ \ \Pi = x \Pi_{c\phi} x^{-1}, H_1 = H^2_{\mathcal{H}}, H_2 = L^2.$$

We then obtain a Fredholm inverse (G_o) to $\Box_\phi$ by setting (note $\Pi_{c\phi}^* = \Pi_{c\phi}$)

$$G_\phi := x^{-1} G_{c\phi} x^{-1} \circ (\mathrm{Id} - x\Pi_{c\phi} x^{-1} + x^{-1}\Pi_{c\phi} x)^{-1}. \tag{12.18}$$

Remark 12.8. Formulas like (12.16) have appeared in the literature before, e.g. in [BLZ09, Lemma 3.5]. Our functional analytic approach is different from the approach by [GuSh15] and [GuHa08], where the Fredholm parametrix is obtained by algebraic computation with bases of $\ker_{L^2} \Box_\phi$.

We can now define the two leading terms of the resolvent parametrix at zf as

$$G_{-2}(\mathrm{zf}) = \Pi_\phi\,, \quad G_0(\mathrm{zf}) = G_\phi\,. \tag{12.19}$$

Note that $\Box_\phi G_\phi = \mathrm{Id} - \Pi_\phi$ and $\Box_\phi \Pi_\phi = 0$ imply that this defines indeed a parametrix:

$$(\Box_\phi + k^2)(k^{-2}\Pi_\phi + G_\phi) = \mathrm{Id} + O(k^2)\,. \tag{12.20}$$

12.3.2. *Leading terms at* sc, bf_0, ϕf_0. In this step we will construct a parametrix

$$Q_1 \in \overline{\Psi}^{-2,\mathcal{E}'}_{k,\phi,\mathcal{H}}(M), \tag{12.21}$$

defined near the boundary of $M^2_{k,\phi}$ and extended smoothly to the interior, with remainder term vanishing at the boundary, see (12.24) below for the precise statement. Here $\mathcal{E}'$ coincides with the index set $\mathcal{E}$ in (12.7) except at zf, where $\mathcal{E}'_{zf} = 0$. We follow Step 1 in the proof of Theorem 10.4, replacing $\Box_\phi$ by $\Box_\phi + k^2$ and taking the weight $\alpha = -1$. That is, with $\Box_\phi$ written as in (10.27) and in the notation of the proof of that theorem we now set

$$A = xP_{00}x + k^2,\ B = xP_{01}x,\ C = xP_{10}x,\ D = P_{11} + k^2.$$

Diagonal terms in (10.34):

Comparing (11.8), (11.9) with (10.30), (10.31) shows in view of (11.7) that we can apply the Guillarmou-Hassel resolvent construction to find a parametrix $Q_{00,k}$ for $A = xP_{00}x + k^2$, but only near the boundary, i.e. on $V^2_{k,sc}$. That is, we use the solutions of the model problems at the diagonal, at sc and bf_0 (but not at zf) to obtain

$$\begin{aligned} &(xP_{00}x + k^2)Q_{00,k} = \mathrm{Id} - R_{00,k}, \\ &Q_{00,k} \in \Psi^{-2,(-2,0,0),\mathcal{E}_0}_{k,sc}(V),\ R_{00,k} \in \rho_{\mathrm{bf}_0}\rho_{sc}\mathcal{A}^{\mathcal{E}_0}_{k,sc}(V), \end{aligned} \tag{12.22}$$

where $\mathcal{E}_0$ is the index set $\mathcal{E}$ but without the ϕf_0 part. By (12.9) $Q_{00,k}$ lifts to an element of $\Psi^{-2,\mathcal{E}}_{k,\phi}(V,\mathcal{H}) \subset \overline{\Psi}^{-2,\mathcal{E}}_{k,\phi}(\mathcal{U})$. In the notation of (10.34) we set $\widetilde{A} = Q_{00,k}$ and $R = R_{00,k}$.

Next we construct a parametrix for $D = P_{11} + k^2$. As discussed after (10.28), the normal operator of $P_{11} = \Pi^\perp \Box_\phi \Pi^\perp$ is invertible, and by [GrHu14, Proposition 2.3] its inverse lies in the extended suspended calculus of M^2_ϕ. The argument given there shows that the same holds for $P_{11} + k^2$, with smooth dependence on $k \geq 0$. So the arguments of loc. cit. apply to give $Q_{11,k} \in C^\infty(\mathbb{R}^+_k, \overline{\Psi}^{-2}_\phi(\mathcal{U}))$, $R_{11,k} \in C^\infty(\mathbb{R}_{+,k}, \mathcal{A}^{\varnothing}_\phi(\mathcal{U}))$, satisfying $(P_{11}+k^2)Q_{11,k} = \mathrm{Id} - R_{11,k}$ as an identity in $C^\infty(V,\mathcal{C})$. Pulling them back under the map $\beta'_\phi : \mathcal{U}^2_{k,\phi} \longrightarrow \mathcal{U}^2_\phi \times \mathbb{R}^+$ we obtain by the extended analogue of (12.10)

$$Q_{11,k} \in \overline{\Psi}^{-2}_{k,\phi}(\mathcal{U}),\ R_{11,k} \in \mathcal{A}^{\varnothing}_{k,\phi}(\mathcal{U}).$$

Off-diagonal terms in (10.34):

With $\widehat{A} = Q_{00,k}$ and $\widehat{D} = Q_{11,k}$ we get $B' = xP_{01}xQ_{11,k} \in x^2\overline{\Psi}^0_{k,\phi}(\mathcal{U}) = \overline{\Psi}^0_{k,\phi}(\mathcal{U})x^2$ and $C' = xP_{10}xQ_{00,k} \in x^2\overline{\Psi}^{0,\mathcal{E}}_{k,\phi}(\mathcal{U})$. The extra x^2 factors in B', C' give $Q_1 \in \overline{\Psi}^{-2,\mathcal{E}}_{k,\phi,\mathcal{H}}(\mathcal{U})$.

Remainder:

The analysis of the remainders is analogous to the proof of Theorem 10.4. For index families we use a notation analogous to (10.37):

$$\begin{aligned}&\mathcal{F} = (a, b, c, l+\lambda, r+\rho) :\Longleftrightarrow \\ &\mathcal{F}_{\mathrm{bf}_0} \geq a,\ \mathcal{F}_{\phi f_0} \geq b,\ \mathcal{F}_{\mathrm{sc}} \geq c, \mathcal{F}_{\mathrm{lb}_0} = \mathcal{E}(0)_{\mathrm{lb}} + \lambda,\ \mathcal{F}_{\mathrm{rb}_0} = \mathcal{E}(0)_{\mathrm{rb}} + \rho\end{aligned} \tag{12.23}$$

(and all other index sets empty). By (12.9) $R_{00,k}$ lifts to $\mathcal{A}^{(1,4,1,l+1,r-1)}_{k,\phi}(\mathcal{U})$. Then $R_{00,k}B' \in \mathcal{A}^{(3,6,3,l+1,r+1)}_{k,\phi}(\mathcal{U})$ and $R_{11,k}C' \in \mathcal{A}^{(\varnothing,\varnothing,\varnothing,\varnothing,r-1)}_{k,\phi}(\mathcal{U})$. Also, $B'C' \in x^4\overline{\Psi}^{0,\mathcal{E}}_{k,\phi}(\mathcal{U}) = \rho^4_{\phi f_0}\rho^4_{\mathrm{sc}}\overline{\Psi}^{0,(2,0,0,l+3,r-1)}_{k,\phi}(\mathcal{U})$ and $C'B' \in x^2\overline{\Psi}^{0,\mathcal{E}}_{k,\phi}(\mathcal{U})x^2 = \rho^4_{\phi f_0}\rho^4_{\mathrm{sc}}\overline{\Psi}^{0,(2,0,0,l+1,r+1)}_{k,\phi}(\mathcal{U})$. In summary we get $(\Box_\phi + k^2)Q_1 = \mathrm{Id} - R_1$ where $R_1 \in \rho^4_{\phi f_0}\rho^4_{\mathrm{sc}}\overline{\Psi}^{0,\mathcal{R}_1}_{k,\phi}(\mathcal{U})$ with

$$\mathcal{R}_1 = \begin{pmatrix} (1,0,0,l+1,r-1) & (3,2,-1,l+1,r+1) \\ (\varnothing,\varnothing,\varnothing,\varnothing,r-1) & (2,0,0,l+1,r+1) \end{pmatrix} \tag{12.24}$$

This has positive index sets at $\mathrm{bf}_0, \mathrm{sc}, \phi f_0$.

Matching at the intersections of boundary faces:

The terms at $\mathrm{sc}, \mathrm{bf}_0, \phi f_0$ match between each other and the faces $\mathrm{bf}, \mathrm{lb}, \mathrm{rb}$ by construction. We now show that the terms at bf_0 and ϕf_0 also match with the leading terms at zf.

First, the coefficient of the k^{-2} term is the orthogonal projection Π_ϕ to $K = \ker_{L^2}\Box_\phi$, whose integral kernel is $\sum_{i=1}^N \psi_i \otimes \overline{\psi_i}$ (times b-half densities) for an orthonormal basis (ψ_i) of K. By Corollary 10.8, $\psi_i \in \mathcal{A}^F_{\mathcal{H}}(M)$ for an index set $F > 0$, and this implies $\Pi_\phi \in \mathcal{A}^{\mathcal{F}}_{\mathcal{H}}(M^2_\phi)$ with $\mathcal{F}$ positive at all faces (and even with an index set $> b+1$ at ϕf by (4.6)). Since the leading orders at bf_0 and ϕf_0 in $\mathcal{E}$ are $-2, 0$ respectively, it follows that $k^{-2}\Pi_\phi = (\kappa')^{-2}(x')^{-2}\Pi_\phi$ (recall from (11.2) that $\kappa' = k/x'$) is lower order than the leading terms of $G_1(k)$ at these faces, in each of the components of the $\mathcal{H}-\mathcal{C}$ decomposition.

Next, the k^0 coefficient at zf is G_ϕ in (12.18). The fact that $G_{c\phi}$ is a Fredholm inverse of $\Box_{c\phi}$ implies as in the proof of Corollary 10.7 that $G_{c\phi}$ is pseudodifferential, and more precisely that $x^{-1}G_{c\phi}x^{-1} \in \Psi^{-2,\mathcal{E}_0}_{\phi,\mathcal{H}}(M)$ with $\mathcal{E}_0$ the index set for zf induced by $\mathcal{E}$ in (12.7), and has the same leading terms at the intersection with bf_0 and ϕf_0 as $Q_1(k)$ restricted to $k = 0$, in each component of the $\mathcal{H}-\mathcal{C}$ decomposition. Also, the terms $S = x\Pi_{c\phi}x^{-1} + x^{-1}\Pi_{c\phi}x$ vanish at the boundary as in the argument above for Π_ϕ, so $(\mathrm{Id}+S)^{-1} = \mathrm{Id} + S'$ where S' has the same vanishing orders by standard arguments, so the factor $(\mathrm{Id}+S)^{-1}$ does not change the leading term.

12.3.3. *Singularity at the diagonal.* The fact that the distribution kernel of $\Box_\phi + k^2$ has a delta type singularity on $\mathrm{diag}_{k,\phi}$, uniformly and non-vanishing at the boundary, means that its (k,ϕ)-principal symbol, which is an endomorphism of $N^*\mathrm{diag}_{k,\phi}$, is uniformly invertible. By inverting this symbol and applying the standard parametrix construction one obtains a 'small' parametrix $Q_\triangle$ satisfying

$$(\Box_\phi + k^2)Q_\triangle = \mathrm{Id} - R_\triangle, \quad Q_\triangle \in \Psi^{-2}_{k,\phi}(M),\ R_\triangle \in \Psi^{-\infty}_{k,\phi}(M).$$

The same argument as in the proof of Theorem 10.4 shows two things: that the parametrix constructed so far, which was only in the extended ϕ-calculus near the boundary, is actually in the ϕ-calculus itself, and that Q_Δ can be adjusted to match with the parametrices at the faces hit by the diagonal.

12.3.4. *Remainder term of initial parametrix.* We choose our initial parametrix $G(k)$ to be an element of $\Psi^{-2,\mathcal{E}}_{k,\phi,\mathcal{H}}(M)$ which matches the models at zf, sc, bf_0, $\phi\mathrm{f}_0$ as explained above. We now analyze the term $R(k)$ in $(\Box_\phi + k^2)G(k) = \mathrm{Id} - R(k)$ and in particular track the different terms in the $\mathcal{H} - \mathcal{C}$ decomposition.

First, consider lb_0 near its intersection with zf, away from bf_0. Here

$$G(k) = k^{-2}G_{-2}(\mathrm{zf}) + G_0(\mathrm{zf}) + \tilde{G},$$

where $\tilde{G}$ has index sets $\mathcal{L}, 1$ at lb_0, zf, respectively, with $\mathcal{L} = \begin{pmatrix} \mathcal{E}_{\mathrm{lb}_0} \\ \mathcal{E}_{\mathrm{lb}_0} + 2 \end{pmatrix}$. Since κ (recall $\kappa = k/x$ as defined in (11.2)) defines zf near $\mathrm{zf} \cap \mathrm{lb}_0$, this means $\tilde{G} = \kappa G'$ with G' having index sets $\mathcal{L}, 1$ at lb_0, zf, respectively. Then by (12.20)

$$(\Box_\phi + k^2)G(k) = \mathrm{Id} + k^2 G_0(\mathrm{zf}) + (\Box_\phi + k^2)(\kappa G').$$

The main term here is $\Box_\phi \kappa G' = k\,\Box_\phi k^{-1}\kappa G' = \kappa x\,\Box_\phi x^{-1} G'$. Now $x\,\Box_\phi x^{-1}$ has the same structure (10.27) as $\Box_\phi$, and x defines lb_0, so applying it to G' yields index set

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{E}_{\mathrm{lb}_0} \\ \mathcal{E}_{\mathrm{lb}_0} + 2 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{\mathrm{lb}_0} + 2 \\ \mathcal{E}_{\mathrm{lb}_0} + 2 \end{pmatrix} \tag{12.25}$$

where $\otimes$ is the *tropical* matrix product, which is the usual matrix product with $+$ replaced by $\cup$ (respectively replacing $+$ by 'min' for lower bounds on index sets) and $\cdot$ by $+$.

The index set of $\kappa G'$, and hence of $R(k)$, at rb_0 is the same as that of G' since P does not act on the local defining function x' of rb_0, so it is $\begin{pmatrix} \mathcal{E}_{\mathrm{rb}_0} & \mathcal{E}_{\mathrm{rb}_0} + 2 \end{pmatrix}$. Now (12.8) implies that $R(k) \in \mathcal{A}^{\mathcal{R}}_{\mathcal{H}}(M^2_{k,\phi})$ near $(\mathrm{lb}_0 \cup \mathrm{rb}_0) \cap \mathrm{zf}$ with $\mathcal{R}_{\mathrm{lb}}, \mathcal{R}_{\mathrm{rb}}$ positive. Similar arguments and (12.24) show that $R(k) \in \mathcal{A}^{(1,1,1),\mathcal{R}}_{\mathcal{H}}(M^2_{k,\phi})$ with $\mathcal{R}$ positive at all faces. This finished the proof of Theorem 12.5.

Remark 12.9. Our Assumption 9.3 implies that the remainder term, after constructing model solutions at zf, sc, bf_0, $\phi\mathrm{f}_0$, is positive at all faces, including lb_0 and rb_0. For this reason, and since we do not require finer information for the Riesz transform, we do not need a construction of leading terms at these side faces as in the works by Guillarmou, Hassell and Sher.

12.4. **Statement and proof of the main result.** Now we have all tools in place to finish our microlocal construction of the resolvent for the Hodge Laplacian Δ_ϕ on ϕ-manifolds at low energy. Recall that we work under the rescaling (10.26) and thus Δ_ϕ is replaced with the unitarily equivalent operator $\square_\phi$ acting in $L^2(M; \mathrm{dvol}_b)$. We recall the initial parametrix $Q(k)$ for $(\square_\phi + k^2)$, constructed in Theorem 12.5

$$(\square_\phi + k^2)Q(k) = I - R(k).$$

In order to invert the right hand side we begin with a lemma that is parallel to [GuHa08, Corollary 2.11].

Lemma 12.10. *For* $N > \dim M$, $R(k)^N$ *is Hilbert-Schmidt for each* $k > 0$, *with Hilbert-Schmidt norm*

$$\|R(k)^N\|_{HS} \longrightarrow 0, \text{ as } k \longrightarrow 0.$$

Proof. Since the Schwartz kernel for the error term $R(k)$ is a polyhomogeneous conormal distribution when lifted to $M^2_{k,\phi}$, vanishing to positive order at all boundary faces, there exists a positive lower bound $\varepsilon > 0$ for all its index sets. Then the Composition Theorem 13.4 implies that $R(k)^N$ has index sets that are bounded below by $N\varepsilon > 0$. Since the order of $R(k)$ as a pseudo-differential operator is (-1), the order of the conormal singularity of $R(k)^N$ is $(-N) < -\dim M$, so its Schwartz kernel is continuous across the lifted diagonal. Thus $R(k)^N$ is a Hilbert Schmidt operator in $L^2(M; \mathrm{dvol}_b)$. Finally, its Hilbert-Schmidt norm tends to zero as $k \to 0$ since its Schwartz kernel vanishes at the $k = 0$ faces zf, lb_0 and rb_0. $\square$

Thus $I - R(k)$ is invertible as an operator in $L^2(M; \mathrm{dvol}_b)$ for $k > 0$ sufficiently small. We can now state and prove our main theorem.

Theorem 12.11 (Main Theorem)**.** *The resolvent* $(\square_\phi + k^2)^{-1}$ *is an element of the split calculus* $\Psi^{-2,\mathcal{E}}_{k,\phi,\mathcal{H}}(M)$, *defined in Definition 12.4, where the individual index sets satisfy*

$$\mathcal{E}_{\mathrm{sc}} \geq 0, \quad \mathcal{E}_{\phi\mathrm{f}_0} \geq 0, \quad \mathcal{E}_{\mathrm{bf}_0} \geq -2, \quad \mathcal{E}_{\mathrm{lb}_0}, \mathcal{E}_{\mathrm{rb}_0} > 0, \quad \mathcal{E}_{\mathrm{zf}} \geq -2. \tag{12.26}$$

The leading terms at sc, $\phi\mathrm{f}_0$, bf_0 *and* zf *are of orders* $0, 0, -2, -2$, *respectively, and are given by the constructions in Section 12.3.*

Proof. Fix $N > \dim M$. By Lemma 12.10 there is $k_0 > 0$ so that $R(k)^N$ has operator norm less than one for $k \in (0, k_0]$. Therefore, $\mathrm{Id} - R(k)^N$ and hence $\mathrm{Id} - R(k)$

is invertible for these k, with inverse given by the Neumann series $\sum_{j=0}^{\infty} R(k)^j$. As in the proof of Lemma 12.10, all index sets of $R(k)^j$ are bounded below by $j\varepsilon$, and since this tends to ∞ as $j \to \infty$, it follows by standard arguments that $(\mathrm{Id} - R(k))^{-1} = \mathrm{Id} + S(k)$ where $S(k)$ lies in the calculus with the same lower bounds for the index sets as $R(k)$.

By Proposition 12.5 the initial parametrix $Q(k)$ satisfies the claims of the theorem. Hence the same holds for $(\Box_\phi + k^2)^{-1} = Q(k)(\mathrm{Id} + S(k))$. Since $S(k)$ has positive index sets everywhere, the leading terms of $(\Box_\phi + k^2)^{-1}$ are the same as those of $Q(k)$. This proves the statement when k is restricted to $k \leq k_0$. Since $(\Box_\phi + k^2)$ is fully elliptic for all $k > 0$, with smooth dependence on k, the statement holds for all k. $\Box$

13. Triple space construction and composition theorems

The following results hold for any vector bundle E. We applied these composition results above for the particular case where $E = \Lambda^\phi T^*M$.

Theorem 13.1. *Consider operators A and B with integral kernels lifting to*

$$\begin{aligned} \beta_\phi^* K_A &\in \mathcal{A}^{\mathcal{E}}_{phg}(M^2_{k,\phi}, \Omega^{1/2}_{b\phi} \otimes \mathrm{End}(E)) = \mathcal{A}^{\mathcal{E}}_{k,\phi}, \\ \beta_\phi^* K_B &\in \mathcal{A}^{\mathcal{F}}_{phg}(M^2_{k,\phi}, \Omega^{1/2}_{b\phi} \otimes \mathrm{End}(E)) = \mathcal{A}^{\mathcal{F}}_{k,\phi}. \end{aligned} \tag{13.1}$$

Assume that both lifts vanish to infinite order at bf, lb and rb. Then, provided $\mathcal{E}_{rb_0} + \mathcal{F}_{lb_0} > 0$, the composition of operators $A \circ B$ is well-defined and has integral kernel lifting to

$$\beta_\phi^* K_{A\circ B} \in \mathcal{A}^{\mathcal{C}}_{phg}(M^2_{k,\phi}, \Omega^{1/2}_{b\phi} \otimes \mathrm{End}(E)) = \mathcal{A}^{\mathcal{C}}_{k,\phi}. \tag{13.2}$$

Furthermore, the analogous statement holds for the split spaces:

$$\beta_\phi^* K_A \in \mathcal{A}^{\mathcal{E}}_{\mathcal{H}},\ \beta_\phi^* K_B \in \mathcal{A}^{\mathcal{F}}_{\mathcal{H}} \implies \beta_\phi^* K_{A\circ B} \in \mathcal{A}^{\mathcal{C}}_{\mathcal{H}}. \tag{13.3}$$

The index family $\mathcal{C}$ is given by

$$\begin{aligned} \mathcal{C}_{\phi f_0} &= (\mathcal{E}_{bf_0} + \mathcal{F}_{bf_0} + (b+1))\,\overline{\cup}\,(\mathcal{E}_{lb_0} + \mathcal{F}_{rb_0} + (b+1))\,\overline{\cup}\,(\mathcal{E}_{\phi f_0} + \mathcal{F}_{\phi f_0}), \\ \mathcal{C}_{bf_0} &= (\mathcal{E}_{bf_0} + \mathcal{F}_{bf_0})\overline{\cup}(\mathcal{E}_{lb_0} + \mathcal{F}_{rb_0})\overline{\cup}(\mathcal{E}_{\phi f_0} + \mathcal{F}_{bf_0})\overline{\cup}(\mathcal{E}_{bf_0} + \mathcal{F}_{\phi f_0}), \\ \mathcal{C}_{rb_0} &= (\mathcal{E}_{zf} + \mathcal{F}_{rb_0})\overline{\cup}(\mathcal{E}_{rb_0} + \mathcal{F}_{bf_0})\overline{\cup}(\mathcal{E}_{rb_0} + \mathcal{F}_{\phi f_0}), \\ \mathcal{C}_{lb_0} &= (\mathcal{E}_{lb_0} + \mathcal{F}_{zf})\overline{\cup}(\mathcal{E}_{bf_0} + \mathcal{F}_{lb_0})\overline{\cup}(\mathcal{E}_{\phi f_0} + \mathcal{F}_{lb_0}), \\ \mathcal{C}_{zf} &= (\mathcal{E}_{zf} + \mathcal{F}_{zf})\overline{\cup}(\mathcal{E}_{rb_0} + \mathcal{F}_{lb_0}), \\ \mathcal{C}_{bf} &= \mathcal{C}_{lb} = \mathcal{C}_{rb} = \varnothing, \\ \mathcal{C}_{sc} &= (\mathcal{E}_{sc} + \mathcal{F}_{sc}). \end{aligned} \tag{13.4}$$

Proof. The Schwartz kernel $K_{A\circ B}$ may be expressed using projections

$$\begin{aligned}
\pi_R &: \overline{M}^3 \times \mathbb{R}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}^+, \quad (p,p',p'',k) \mapsto (p,p',k),\\
\pi_L &: \overline{M}^3 \times \mathbb{R}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}^+, \quad (p,p',p'',k) \mapsto (p',p'',k),\\
\pi_C &: \overline{M}^3 \times \mathbb{R}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}^+, \quad (p,p',p'',k) \mapsto (p,p'',k).
\end{aligned}$$

With this notation we can write (provided the pushforward is well-defined)

$$K_{A\circ B} = (\pi_C)_* \Big(\pi_R^* K_A \otimes \pi_L^* K_B \Big).$$

To prove the theorem, we need to define a triple space $M^3_{k,\phi}$ given by a blowup of $\overline{M}^3 \times \mathbb{R}^+$, such that the projections π_L, π_R, π_C lift to b-fibrations Π_L, Π_R, Π_C on the triple space $M^3_{k,\phi}$. More precisely, writing $\beta^2_\phi \equiv \beta_\phi : M^2_{k,\phi} \to \overline{M}^2 \times \mathbb{R}^+$ for the blowdown map on $M^2_{k,\phi}$, we are looking for a space $M^3_{k,\phi}$ and a smooth map $\beta^3_\phi : M^3_{k,\phi} \to \overline{M}^3 \times \mathbb{R}^+$, such that the following diagram commutes for $* \in \{L, R, C\}$.

$$\begin{tikzcd}
M^3_{k,\phi} \arrow[r, "\Pi_*"] \arrow[d, "\beta^3_\phi"] & M^2_{k,\phi} \arrow[d, "\beta^2_\phi"] \\
\overline{M}^3 \times \mathbb{R}^+ \arrow[r, "\pi_*"] & \overline{M}^2 \times \mathbb{R}^+
\end{tikzcd}$$

Once we know that the projections lift to b-fibrations, we deduce that

$$\beta^*_\phi K_{A\circ B} = (\Pi_C)_* \Big(\Pi_R^* \left(\beta^*_\phi K_A \right) \cdot \Pi_L^* \left(\beta^*_\phi K_B \right) \Big),$$

is polyhomogeneous on $M^3_{k,\phi}$ by the pullback and pushforward theorems of Melrose. We proceed in four steps.

- Step 1: Construct the triple space $M^3_{k,\phi}$.
- Step 2: Show that projections π_L, π_R, π_C lift to b-fibrations Π_L, Π_R, Π_C.
- Step 3: Prove (13.2) via pushforward theorem of Melrose.
- Step 4: Prove (13.3).

Step 1: Construction of the triple space $M^3_{k,\phi}$. We set up a notation for the various corners in $\overline{M}^3 \times \mathbb{R}^+$. We write (x, x', x'') for the defining functions on the three copies of M. As before, $k \geq 0$ is the coordinate on the $\mathbb{R}^+$-component. We set

$$(u_1, u_2, u_3, u_4) := (x, x', x'', k).$$

Now for any quadruple of binary indices $i_1, i_2, i_3, i_4 \in \{0, 1\}$ we define

$$C_{i_1 i_2 i_3 i_4} := \{u_j = 0 : \text{ for all } j \text{ with } \ i_j = 1\}.$$

For example the highest codimension corner in $\overline{M}^3 \times \mathbb{R}^+$ is given bs

$$C_{1111} = \{x = x' = x'' = k = 0\},$$

The four codimension 3 corners are given by

$$C_{1110} := \{x = x' = x'' = 0\}, \quad C_{0111} := \{x' = x'' = k = 0\},$$
$$C_{1011} := \{x = x'' = k = 0\}, \quad C_{1101} := \{x = x' = k = 0\}.$$

We will slightly abuse notation, by denoting the lifts of $C_{i_1 i_2 i_3 i_4}$ as $C_{i_1 i_2 i_3 i_4}$ again. We will also denote any boundary face arising from blowing up $C_{i_1 i_2 i_3 i_4}$ by $\rho_{i_1 i_2 i_3 i_4}$.

Construction of the b-triple space. We can now begin with blowing up individual corners in $\overline{M}^3 \times \mathbb{R}^+$. We first blow up highest codimension corner C_{1111} and then the codimension 3 corners. Note that the order of blow up for the codimension 3 corners is immaterial after the blow up of C_{1111}, since the lifted submanifolds become disjoint. The last step is blow up of the six codimension 2 corners, i.e. C_{1010}, C_{1100}, C_{0011}, C_{0110}, C_{0101}, C_{1001}. This defines the b-triple space

$$\begin{aligned} M^3_{k,b} = [\overline{M}^3 \times [0, k_0]; C_{1111}, C_{1110}, C_{0111}, C_{1011}, C_{1101}; \\ C_{1100}, C_{0011}, C_{1010}, C_{1001}, C_{0110}, C_{0101}], \end{aligned}$$

with the blowdown map $\beta^{(3)}_b : M^3_{k,b} \longrightarrow \overline{M}^3 \times \mathbb{R}^+$. The projections π_L, π_C, π_R lift to maps $M^3_{k,b} \to M^2_{k,b}$ (the latter space is constructed in Figure 11), which are denoted by $\pi_{b,L}, \pi_{b,C}, \pi_{b,R}$, respectively.

Blow up of fibre diagonals $\mathrm{diag}_{k,sc,\phi}$ *in* bf *faces.* Now consider the fibre diagonal $\mathrm{diag}_{k,sc,\phi}$ in $M^2_{k,b}$. The preimage of this submanifold under each of the projections $\pi_{b,*}$ is given by the union of two fibre diagonals. Namely one has the following

$$\begin{aligned} \pi^{-1}_{b,R}(\mathrm{diag}_{k,sc,\phi}) &= D_{y,y'}(C_{1100}) \cup D_{y,y'}(C_{1110}), \\ \pi^{-1}_{b,L}(\mathrm{diag}_{k,sc,\phi}) &= D_{y',y''}(C_{0110}) \cup D_{y',y''}(C_{1110}), \\ \pi^{-1}_{b,C}(\mathrm{diag}_{k,sc,\phi}) &= D_{y,y''}(C_{1010}) \cup D_{y,y''}(C_{1110}), \end{aligned}$$

Here, e.g. $D_{y,y'}(C_{1100})$ denotes the fibre diagonal $\{y = y'\}$ lifted to $(\beta^{(3)}_b)^*(C_{1100})$. These three preimages intersect at

$$\mathcal{O} = D_{y,y'}(C_{1110}) \cap D_{y',y''}(C_{1110}) \cap D_{y,y''}(C_{1110}).$$

We define the triple scattering space which we denote as $M^3_{k,sc}$ by

$$\begin{aligned} M^3_{k,sc} := [M^3_{k,b}; \mathcal{O}, D_{y,y'}(C_{1110}), D_{y',y''}(C_{1110}), D_{y,y''}(C_{1110}), \\ D_{y,y'}(C_{1100}), D_{y',y''}(C_{0110}), D_{y,y''}(C_{1010})], \end{aligned}$$

with blow down map $\beta^3_{k,sc} : M^3_{k,sc} \longrightarrow \overline{M}^3 \times \mathbb{R}^+$ and lifts of the projections π_L, π_C, π_R denoted by $\pi_{sc,L}, \pi_{sc,C}, \pi_{sc,R}$, respectively. We denote defining functions for the boundary face resulting from blowing up $\mathcal{O}$ by $\rho_{\mathcal{O}}$. Defining functions for the other boundary faces are denoted as follows: e.g. blow up of $D_{y,y'}(C_{1110})$ yields a new boundary face with defining functions denoted by $\rho^{D_{y,y'}}_{1110}$.

Blow up of fibre diagonals ϕf_0 *in* bf_0 *faces.* Now consider the fibre diagonal $\mathrm{diag}_{k,\phi} \subset M^2_{k,sc,\phi}$. Again, the preimage of ϕf_0 under each of the projections $\pi_{sc,*}$ is given by the union of two fibre diagonals. Namely one has (identifying the submanifolds notationally with their lifts to $M^3_{k,sc}$) the following

$$\begin{aligned}
\pi^{-1}_{sc,R}(\mathrm{diag}_{k,\phi}) &= D_{y,y'}(C_{1101}) \cup D_{y,y'}(C_{1111}), \\
\pi^{-1}_{sc,L}(\mathrm{diag}_{k,\phi}) &= D_{y',y''}(C_{0111}) \cup D_{y',y''}(C_{1111}), \\
\pi^{-1}_{sc,C}(\mathrm{diag}_{k,\phi}) &= D_{y,y''}(C_{1011}) \cup D_{y,y''}(C_{1111}),
\end{aligned}$$

These three lifts intersect at

$$\mathcal{O}' = D_{y,y'}(C_{1111}) \cap D_{y',y''}(C_{1111}) \cap D_{y,y''}(C_{1111}).$$

We can now define the final (phi) triple space as follows

$$\begin{aligned}
M^3_{k,\phi} := [M^3_{k,sc}; \mathcal{O}', D_{y,y'}(C_{1111}), D_{y',y''}(C_{1111}), D_{y,y''}(C_{1111}),& \\
D_{y,y'}(C_{1101}), D_{y,y''}(C_{1011}), D_{y',y''}(C_{0111})],&
\end{aligned}$$

with blow down map, $\beta^3_{k,\phi} : M^3_{k,\phi} \longrightarrow \overline{M}^3 \times \mathbb{R}^+$. The lifts of the projections π_L, π_C, π_R are denoted by Π_L, Π_C, Π_R, respectively. Similar to the previous step, we denote defining functions for the boundary face resulting from blowing up $\mathcal{O}'$ by $\rho_{\mathcal{O}'}$. Defining functions for the other boundary faces are denoted as follows: e.g. blow up of $D_{y,y'}(C_{1111})$ yields a new boundary face with defining functions denoted by $\rho^{D_{y,y'}}_{1111}$.

Step 2: lifts Π_L, Π_C, Π_R are b-fibrations.

Theorem 13.2. *The projections π_L, π_R, π_C lift to b-fibrations*

$$\Pi_L, \Pi_R, \Pi_C : M^3_{k,\phi} \longrightarrow M^2_{k,\phi}. \tag{13.5}$$

Proof. We provide a brief proof idea. First we observe that π_* lift to b-fibrations $\pi_{*,b}$ on b-triple space $M^3_{k,b}$. As the argument is symmetric in x, x', x'', we argue just for π_R. Second, we apply the commuting lemma of blowup of submanifolds [GuHa08] to obtain a presentation of $M^3_{k,\phi}$ in the following way

$$\begin{aligned}
M^3_{k,\phi} = [M^2_{k,\phi} \times M; C_{1111}, D_{y',y''}, D^{1111}_{y',y''}, C_{1110}, C_{1011}, C_{0111}, C_{0011}, C_{1010}, C_{0110},& \\
\mathcal{O}, D_{y,y''}, D_{1010}, D_{y,y'}, D_{1100}, \mathcal{O}', D^{1111}_{y,y''}, D_{1011}, D^{1111}_{y,y'}, D_{1101}].&
\end{aligned}$$

Now we use the extension lemma of b-fibrations [GuHa08] to conclude that there exists a b-fibration $\Pi_R : M^3_{k,\phi} \longrightarrow M^2_{k,\phi}$. $\square$

Step 3: Proof via Melrose's pushforward theorem. We compute explicitly the lift of boundary defining functions of $M^2_{k,\phi}$ under Π_L, Π_C, Π_R to $M^3_{k,\phi}$. Let us start with $\Pi_R^*(\rho_{\mathrm{bf}_0})$. Here, bf_0 is the front face in $M^2_{k,\phi}$ coming from the lift of $\{x = x' = k = 0\}$. This corresponds to the corners C_{1101} and C_{1111}. Thus $\Pi_R^*(\rho_{\mathrm{bf}_0})$ is given by a product of defining functions for front faces that arise from blowing up C_{1101} and C_{1111} and their fibre diagonals away from $\{y = y'\}$. This leads to

$$\Pi_R^*(\rho_{\mathrm{bf}_0}) = \rho_{1101}\rho_{1111}\rho_{1111}^{D_{y,y''}}\rho_{1111}^{D_{y',y''}}.$$

Arguing similarly for the other boundary defining functions, we compute

$$\begin{aligned}
&\Pi_R^*(\rho_{\phi f_0}) = \rho_{1111}^{D_{y,y'}}\rho_{1101}^{D_{y,y'}}\rho_{\mathcal{O}'}, && \Pi_R^*(\rho_{\mathrm{sc}}) = \rho_{1110}^{D_{y,y'}}\rho_{1100}^{D_{y,y'}}\rho_{\mathcal{O}},\\
&\Pi_R^*(\rho_{\mathrm{rb}_0}) = \rho_{0101}\rho_{0111}\rho_{0111}^{D_{y',y''}}, && \Pi_R^*(\rho_{\mathrm{lb}_0}) = \rho_{1001}\rho_{1011}\rho_{1011}^{D_{y,y''}},\\
&\Pi_R^*(\rho_{\mathrm{zf}}) = \rho_{0011}\rho_{0001}, && \Pi_R^*(\rho_{\mathrm{lb}}) = \rho_{1010}\rho_{1000}\rho_{1010}^{D_{y,y''}},\\
&\Pi_R^*(\rho_{\mathrm{rb}}) = \rho_{0110}\rho_{0110}^{D_{y',y''}}\rho_{0100}, && \Pi_R^*(\rho_{\mathrm{bf}}) = \rho_{1100}\rho_{1110}\rho_{1110}^{D_{y',y''}}\rho_{1110}^{D_{y,y''}},
\end{aligned}$$

By exactly the same arguments with compute for the other projections

$$\begin{aligned}
&\Pi_L^*(\rho_{\mathrm{bf}_0}) = \rho_{0111}\rho_{1111}\rho_{1111}^{D_{y,y'}}\rho_{1111}^{D_{y,y''}} && \Pi_C^*(\rho_{\mathrm{bf}_0}) = \rho_{1011}\rho_{1111}\rho_{1111}^{D_{y,y'}}\rho_{1111}^{D_{y',y''}},\\
&\Pi_L^*(\rho_{\mathrm{bf}}) = \rho_{1110}\rho_{0110}\rho_{1110}^{D_{y,y'}}\rho_{1110}^{D_{y,y''}}, && \Pi_C^*(\rho_{\mathrm{bf}}) = \rho_{1010}\rho_{1110}\rho_{1110}^{D_{y,y'}}\rho_{1110}^{D_{y',y''}},\\
&\Pi_L^*(\rho_{\phi f_0}) = \rho_{1111}^{D_{y',y''}}\rho_{0111}^{D_{y',y''}}\rho_{\mathcal{O}'} && \Pi_C^*(\rho_{\phi f_0}) = \rho_{1111}^{D_{y,y''}}\rho_{1011}^{D_{y,y''}}\rho_{\mathcal{O}'},\\
&\Pi_L^*(\rho_{\mathrm{sc}}) = \rho_{0110}^{D_{y',y''}}\rho_{1110}^{D_{y',y''}}\rho_{\mathcal{O}}, && \Pi_C^*(\rho_{\mathrm{sc}}) = \rho_{1010}^{D_{y,y''}}\rho_{1110}^{D_{y,y''}}\rho_{\mathcal{O}},\\
&\Pi_L^*(\rho_{\mathrm{rb}_0}) = \rho_{0011}\rho_{1011}\rho_{1011}^{D_{y,y''}}, && \Pi_C^*(\rho_{\mathrm{rb}_0}) = \rho_{0011}\rho_{0111}\rho_{0111}^{D_{y',y''}},\\
&\Pi_L^*(\rho_{\mathrm{lb}_0}) = \rho_{0101}\rho_{1101}\rho_{1101}^{D_{y,y'}}, && \Pi_C^*(\rho_{\mathrm{lb}_0}) = \rho_{1001}\rho_{1101}\rho_{1101}^{D_{y,y'}},\\
&\Pi_L^*(\rho_{\mathrm{lb}}) = \rho_{1100}\rho_{0100}\rho_{1100}^{D_{y,y'}}, && \Pi_C^*(\rho_{\mathrm{lb}}) = \rho_{1100}\rho_{1000}\rho_{1100}^{D_{y,y'}},\\
&\Pi_L^*(\rho_{\mathrm{rb}}) = \rho_{1010}\rho_{1010}^{D_{y,y''}}\rho_{0010}, && \Pi_C^*(\rho_{\mathrm{rb}}) = \rho_{0010}\rho_{0110}^{D_{y,y'}}\rho_{0110},\\
&\Pi_L^*(\rho_{\mathrm{zf}}) = \rho_{1001}\rho_{0001}, && \Pi_C^*(\rho_{\mathrm{zf}}) = \rho_{0101}\rho_{0001}.
\end{aligned}$$

From here we can read off the index family $\mathcal{C}$. We write out the argument for $\mathcal{C}_{\phi f_0}$: The relation $\Pi_C^*(\rho_{\phi f_0}) = \rho_{1111}^{D_{y,y''}}\rho_{1011}^{D_{y,y''}}\rho_{\mathcal{O}'}$ shows asymptotics of which boundary faces contributes to $\mathcal{C}_{\phi f_0}$. Now we compute that $\Pi_R^*\left(\beta_\phi^* K_A\right) \otimes \Pi_L^*\left(\beta_\phi^* K_B\right)$ has the following asymptotics (note that both $\beta_\phi^* K_A$ and $\beta_\phi^* K_B$ vanish to infinite order at bf, rb and lb)

(1) index set $\mathcal{E}_{\mathrm{bf}_0} + \mathcal{F}_{\mathrm{bf}_0}$ as $\rho_{1111}^{D_{y,y''}} \to 0$
(2) index set $\mathcal{E}_{\mathrm{lb}_0} + \mathcal{F}_{\mathrm{rb}_0}$ as $\rho_{1011}^{D_{y,y''}} \to 0$
(3) index set $\mathcal{E}_{\phi f_0} + \mathcal{F}_{\phi f_0}$ as $\rho_{\mathcal{O}'} \to 0$

From there we conclude (taking into account the lifting properties of $\Omega^{1/2}_{b\phi}$ as in [GrHu14, Theorem 9])

$$\mathcal{C}_{\phi f_0} = (\mathcal{E}_{bf_0} + \mathcal{F}_{bf_0} + (b+1)) \,\overline{\cup}\, (\mathcal{E}_{lb_0} + \mathcal{F}_{rb_0} + (b+1)) \,\overline{\cup}\, (\mathcal{E}_{\phi f_0} + \mathcal{F}_{\phi f_0})$$

Proceeding similarly at other boundary faces we conclude the rest of (13.4).

Step 4: Composition of split operators. We now prove (13.3). It is useful to introduce the 'tropical' operations on index sets, $E \oplus F := E \cup F$ and $E \odot F := E + F$. Then we have for any $u \in \mathcal{A}^E_{\mathrm{phg}}(M)$ and $v \in \mathcal{A}^F_{\mathrm{phg}}(M)$

$$u + v \in \mathcal{A}^{E\oplus F}_{\mathrm{phg}}(M), \; u \cdot v \in \mathcal{A}^{E\odot F}_{\mathrm{phg}}(M)\,.$$

This implies a similar rule for 2×2 matrices of functions $\hat{u} = (u_{ij})$ on M and of index sets $\widehat{E} = (E_{ij})$ where $i,j \in \{0,1\}$: if we define $\hat{u} \in \mathcal{A}^{\widehat{E}}_{\mathrm{phg}}(M)$ to mean $u_{ij} \in \mathcal{A}^{E_{ij}}_{\mathrm{phg}}(M)$ for all i,j then for any $\hat{u} \in \mathcal{A}^{\widehat{E}}_{\mathrm{phg}}(M)$ and $\hat{v} \in \mathcal{A}^{\widehat{F}}_{\mathrm{phg}}(M)$

$$\hat{u} + \hat{v} \in \mathcal{A}^{\widehat{E}\oplus\widehat{F}}_{\mathrm{phg}}(M), \; \hat{u}\cdot\hat{v} \in \mathcal{A}^{\widehat{E}\otimes\widehat{F}}_{\mathrm{phg}}(M),$$

where $\otimes$ denotes the tropical matrix product, which is the usual matrix product with $+, \cdot$ replaced by $\oplus, \odot$, respectively.

We now turn to index families $\mathcal{E}$, $\mathcal{F}$ for $M^2_{k,\phi}$ and integral kernels $K_A \in \mathcal{A}^{\mathcal{E}}_{k,\phi}$ and $K_B \in \mathcal{A}^{\mathcal{F}}_{k,\phi}$, omitting β^*_ϕ for simplicity. Define the tropical sum $\mathcal{E} \oplus \mathcal{F}$ face by face and the ϕ-tropical product as

$$\mathcal{E} \odot_\phi \mathcal{F} := \mathcal{C} \tag{13.6}$$

if $\mathcal{C}$ is defined from $\mathcal{E}$, $\mathcal{F}$ as in (13.4). Then linearity and (13.2) imply

$$K_A \in \mathcal{A}^{\mathcal{E}}_{k,\phi},\; K_B \in \mathcal{A}^{\mathcal{F}}_{k,\phi} \Longrightarrow K_{A+B} \in \mathcal{A}^{\mathcal{E}\oplus\mathcal{F}}_{k,\phi},\; K_{A\circ B} \in \mathcal{A}^{\mathcal{E}\odot_\phi\mathcal{F}}_{k,\phi}\,.$$

Again, this implies for 2×2 matrices of kernels and index families, with notation analogous to above,

$$\widehat{K}_A \in \mathcal{A}^{\widehat{\mathcal{E}}}_{k,\phi},\; \widehat{K}_B \in \mathcal{A}^{\widehat{\mathcal{F}}}_{k,\phi} \Longrightarrow \widehat{K}_{A+B} \in \mathcal{A}^{\widehat{\mathcal{E}}\oplus\widehat{\mathcal{F}}}_{k,\phi},\; \widehat{K}_{A\circ B} \in \mathcal{A}^{\widehat{\mathcal{E}}\otimes_\phi\widehat{\mathcal{F}}}_{k,\phi}\,. \tag{13.7}$$

Recall the definition of the split (k,ϕ)-calculus, Definitions 10.3 and 12.4. Given an index family $\mathcal{E}$ for M^2_ϕ, let $\widehat{\mathcal{E}}$ be the associated split index family, considering it as 2×2 matrix at each face. Consider kernels K_A, K_B supported over $\mathcal{U}\times\mathcal{U}$ (see below for the general case), and let $\widehat{K}_A, \widehat{K}_B$ be the associated 2×2 matrices of kernels as in Definition 10.3. Now if $K_A \in \mathcal{A}^{\mathcal{E}}_{\mathcal{H}}(M^2_{k,\phi})$, $K_B \in \mathcal{A}^{\mathcal{F}}_{\mathcal{H}}(M^2_{k,\phi})$ then

$$\widehat{K}_A \in \mathcal{A}^{\widehat{\mathcal{E}}}_{k,\phi}, \quad \widehat{K}_B \in \mathcal{A}^{\widehat{\mathcal{F}}}_{k,\phi}.$$

Thus (13.7) implies $\widehat{K}_{A\circ B} \in \mathcal{A}^{\widehat{\mathcal{E}}\otimes_\phi\widehat{\mathcal{F}}}_{k,\phi}$. So (13.3) will follow if we prove the

$$\text{Claim:} \quad \widehat{\mathcal{E}} \otimes_\phi \widehat{\mathcal{F}} \subset (\mathcal{E} \odot_\phi \mathcal{F})\widehat{\ }. \tag{13.8}$$

This is to be understood as inclusion of index sets at each face and in each matrix component. We prove (13.8) at the face bf_0, the proof at the other faces is analogous. We abbreviate bf_0, $\phi\mathrm{f}_0$, lb_0, rb_0 by b, ϕ, l, r respectively. For example, $\widehat{\mathcal{E}}_{01,b}$ denotes the 01 component of $\widehat{\mathcal{E}}$ at bf_0 (which is $\mathcal{E}_{\mathrm{bf}_0}+2$).

For any $i, j \in \{0,1\}$ we have by definition of $\mathcal{C}$ in (13.4), with sums over $k \in \{0,1\}$,

$$
\begin{aligned}
(\widehat{\mathcal{E}} \otimes_\phi \widehat{\mathcal{F}})_{ij,b} &= \bigoplus_k (\widehat{\mathcal{E}}_{ik} \odot_\phi \widehat{\mathcal{F}}_{kj})_b \\
&= \bigoplus_k \left[(\widehat{\mathcal{E}}_{ik,b} \odot \widehat{\mathcal{F}}_{kj,b}) \,\overline{\cup}\, (\widehat{\mathcal{E}}_{ik,l} \odot \widehat{\mathcal{F}}_{kj,r}) \,\overline{\cup}\, (\widehat{\mathcal{E}}_{ik,\phi} \odot \widehat{\mathcal{F}}_{kj,b}) \,\overline{\cup}\, (\widehat{\mathcal{E}}_{ik,b} \odot \widehat{\mathcal{F}}_{kj,\phi}) \right] \\
&\subset \left[\bigoplus_k (\widehat{\mathcal{E}}_{ik,b} \odot \widehat{\mathcal{F}}_{kj,b}) \right] \,\overline{\cup}\, \left[\bigoplus_k (\widehat{\mathcal{E}}_{ik,l} \odot \widehat{\mathcal{F}}_{kj,r}) \right] \,\overline{\cup}\, \dots
\end{aligned}
\tag{13.9}
$$

where in the last line we used part (a) of the following lemma (recall that $\oplus = \cup$). For an index set E denote

$$
E^b = \begin{pmatrix} E & E+2 \\ E+2 & E+4 \end{pmatrix}.
$$

Lemma 13.3.

(a) If $E_1, \dots, E_N$ *and* $F_1, \dots, F_N$ *are index sets then*

$$
(E_1 \,\overline{\cup}\, \dots \,\overline{\cup}\, E_N) \cup (F_1 \,\overline{\cup}\, \dots \,\overline{\cup}\, F_N) \subset (E_1 \cup F_1) \,\overline{\cup}\, \dots \,\overline{\cup}\, (E_N \cup F_N)
$$

(b) If $\mathcal{E}$, $\mathcal{F}$ *are index families for* $M^2_{k,\phi}$ *then, with notation as introduced above,*

$$
\widehat{\mathcal{E}}_b \otimes \widehat{\mathcal{F}}_b = (\mathcal{E}_b + \mathcal{F}_b)^b, \qquad \widehat{\mathcal{E}}_l \otimes \widehat{\mathcal{F}}_r = (\mathcal{E}_l + \mathcal{F}_r)^b,
$$
$$
\widehat{\mathcal{E}}_\phi \otimes \widehat{\mathcal{F}}_b = (\mathcal{E}_\phi + \mathcal{F}_b)^b, \qquad \widehat{\mathcal{E}}_b \otimes \widehat{\mathcal{F}}_\phi = (\mathcal{E}_b + \mathcal{F}_\phi)^b.
$$

Proof. (a) follows immediately from the definition of the extended union. For (b) we calculate

$$
\begin{aligned}
\widehat{\mathcal{E}}_b \otimes \widehat{\mathcal{F}}_b &= \begin{pmatrix} \mathcal{E}_b & \mathcal{E}_b+2 \\ \mathcal{E}_b+2 & \mathcal{E}_b+4 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{F}_b & \mathcal{F}_b+2 \\ \mathcal{F}_b+2 & \mathcal{F}_b+4 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{E}_b+\mathcal{F}_b & \mathcal{E}_b+\mathcal{F}_b+2 \\ \mathcal{E}_b+\mathcal{F}_b+2 & \mathcal{E}_b+\mathcal{F}_b+4 \end{pmatrix} = (\mathcal{E}_b+\mathcal{F}_b)^b,
\end{aligned}
$$

where, for example, the upper left entry arises as $(\mathcal{E}_b \odot \mathcal{F}_b) \oplus ((\mathcal{E}_b+2) \odot (\mathcal{F}_b+2)) = (\mathcal{E}_b+\mathcal{F}_b) \cup ((\mathcal{E}_b+2)+(\mathcal{F}_b+2)) = \mathcal{E}_b+\mathcal{F}_b$. The other calculations are similar. □

We can now finish the proof of the claim (13.8), continuing from (13.9). By the definition of $\otimes$ and by part (2) of the lemma we have $\bigoplus_k (\widehat{\mathcal{E}}_{ik,b} \odot \widehat{\mathcal{F}}_{kj,b}) = (\widehat{\mathcal{E}}_b \otimes \widehat{\mathcal{F}}_b)_{ij} = (\mathcal{E}_b+\mathcal{F}_b)^b_{ij}$. Rewriting the other terms of (13.9) similarly using the other identities in part (2) of the lemma we get

$$
(\widehat{\mathcal{E}} \otimes_\phi \widehat{\mathcal{F}})_b \subset (\mathcal{E}_b+\mathcal{F}_b)^b \,\overline{\cup}\, (\mathcal{E}_l+\mathcal{F}_r)^b \,\overline{\cup}\, (\mathcal{E}_\phi+\mathcal{F}_b)^b \,\overline{\cup}\, (\mathcal{E}_b+\mathcal{F}_\phi)^b.
$$

Since this last expression equals $(\mathcal{E} \odot_\phi \mathcal{F})\widehat{\ }_b$, we obtain the bf_0 part of the claim (13.8). The proof at the other boundary faces is analogous.

Finally, the restriction that K_A was supported over $\mathcal{U} \times \mathcal{U}$ was only made to simplify the notation. One way to remove it is to write $M = \mathcal{U} \cup \mathcal{U}'$ where $\mathcal{U}' = M \setminus [(0, \varepsilon/2) \times \partial M]$, then any K_A is a sum of four terms, supported over $\mathcal{U} \times \mathcal{U}$, $\mathcal{U} \times \mathcal{U}'$, $\mathcal{U}' \times \mathcal{U}$ and $\mathcal{U}' \times \mathcal{U}'$ respectively. We have dealt with the first term; the others are treated similarly. $\square$

We can now prove the general composition formula, where the operators have a conormal singularity along the diagonal. Note that the lifted diagonal in $M^2_{k,\phi}$ intersects only the boundary faces ϕf_0, zf and sc.

Theorem 13.4. *Consider ϕ-operators*

$$A \in \Psi^{m,(a_{\phi f_0}, a_{zf}, a_{sc}),\mathcal{E}}_{k,\phi}(M; E),$$
$$B \in \Psi^{m',(a'_{\phi f_0}, a'_{zf}, a'_{sc}),\mathcal{F}}_{k,\phi}(M; E).$$

Assume that $\mathcal{E}_{\phi f_0}, \mathcal{E}_{zf}, \mathcal{E}_{sc}$ contains the index sets $a_{\phi f_0}$, a_{zf}, a_{sc}, respectively. Similarly, assume that $\mathcal{F}_{\phi f_0}, \mathcal{F}_{zf}, \mathcal{F}_{sc}$ contains the index sets $a'_{\phi f_0}$, a'_{zf}, a'_{sc}, respectively. Then, provided $\mathcal{E}_{rb_0} + \mathcal{F}_{lb_0} > 0$, the composition of operators $A \circ B$ is well-defined with

$$A \circ B \in \Psi^{m+m',(a_{\phi f_0}+a'_{\phi f_0}, a_{zf}+a'_{zf}, a_{sc}+a'_{sc}),\mathcal{C}}_{k,\phi}(M; E).$$

Furthermore, the analogous statement holds for the split calculi:

$$A \in \Psi^{m,(a),\mathcal{E}}_{k,\phi,\mathcal{H}}(M; E),\ B \in \Psi^{m',(a'),\mathcal{F}}_{k,\phi,\mathcal{H}}(M; E) \implies A \circ B \in \Psi^{m+m',(a+a'),\mathcal{C}}_{k,\phi,\mathcal{H}}(M; E)\,.$$

The index family $\mathcal{C}$ is given by (13.4).

Proof. The statement follows from Theorem 13.1 exactly as in [GuHa08, §6]. $\square$

14. Analytic torsion on fibred boundary ϕ metric manifolds

Analytic torsion was introduced by Ray and Singer [RaSi71] as analytic counterpart of Reidemeister torsion in topology. Ray and Singer conjectured that these two torsions are equivalent on closed manifolds. Cheeger and Müller proved this conjecture independently. Assume that (M, g) is a closed Riemannian manifold and $e^{-t\Delta_g}(x, y) := H(t, x, y)$ is the heat kernel with respect to Hodge Laplacian $\Delta^q_g : \Omega^q(M) \longrightarrow \Omega^q(M)$, acting on the space of q forms. i.e fundamental solution to heat equation,

$$\begin{aligned} &\partial_t H(t, x, y) + \Delta^q_{g,x} H(t, x, y) = 0, \\ &H(t = 0, x, y) = \delta(x - y). \end{aligned} \tag{14.1}$$

One define the heat trace to be, $\mathrm{Tr}(e^{-t\Delta_g^q}) = \int_X e^{-t\Delta_g}(x,x)d\mathrm{vol}_g$. Assume Δ_g^q is Hodge Laplacian acting on the space of q-forms. The corresponding zeta function is defined to be,

$$\zeta_q^M(s) := \frac{1}{\Gamma(s)} \int_{\mathbb{R}^+} \mathrm{Tr}(e^{-t\Delta_g^q}) t^{s-1} dt. \tag{14.2}$$

So defined (14.2) is defined on $\mathrm{Re}(s) > \frac{n}{2}$ where $n = \dim(M)$, but has holomorphic continuation to whole complex plane $\mathbb{C}$ and in particular regular at $s = 0$ and the determinant of Laplacian is defined to be,

$$\det(\Delta_g^q) := e^{-\zeta_q'(0)}.$$

One defines the analytic torsion as,

$$\log T(M) := \frac{1}{2} \sum_q (-1)^q q \zeta_q'(0),$$

typically one would like to generalize the definition of zeta function and analytic torsion on non compact set up or non smooth set up and obtain Cheeger Müller type statements on the new set up. In order to count some previous works in this direction briefly, one can refer to the Doctoral Thesis of Vertman in which he studied analytic torsion on conic manifolds [Ver09]. Mazzeo and Vertman continued the study of analytic torsion on edge manifolds [MaVe12]. Hartmann, Spreafico worked on analytic torsion and relation to the Reidemeister torsion [HaSp17] and Albin, Rochon, Sher obtained Cheeger Müller type statement on the set up manifolds with wedge singularities [ARS18]. At the end we refer to the PhD thesis of Sher [She13] in which he constructed the zeta function in the set up of asymptotic conic manifolds.

In this work we consider manifold $\overline{M}$ with boundary ∂M with fibration boundary structure, i.e ∂M is fibred over closed base B with typical closed fibre F. On such a topological structure one may consider metric g_ϕ defined as,

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F + h(x),$$

where x is a boundary defining function of ∂M, and ϕ is fibration and g_B is a Riemannian metric over base B and g_F is a symmetric bilinear form which restricts to Riemannian metric on fibre F and additional assumptions as in [GTV20]. In this set up, we are going to define the concept of analytic torsion. The main difficulty arises when we consider the heat trace,

$$\mathrm{Tr}(e^{-t\Delta_{g_\phi}^q}) = \int_{\mathbb{R}^+} e^{-t\Delta_{g_\phi}}(x,x) d\mathrm{vol}_\phi,$$

i.e in ϕ set up, the boundary is located at infinity and therefore the integration over diagonal diverges. To address this problem, the renormalized heat trace is described in the Hadamard manner, which essentially takes into account the integration of the heat kernel along the diagonal on the finite component. The heat kernel structure theorem may be employed to take the finite part of this integral at zero to be heat trace renormalized . In order to describe analytical torsion in the set up of ϕ manifolds, we can explicitly describe renormalized zeta function and Laplacian determinant by means of renormalized heat trace.

In section 15 we describe structure theorems of heat kernel both in finite time regimes and long time regimes on appropriate space. Namely on manifolds with corners. In finite time regime this description is explicit [TaVe20] and in long time regime we use functional calculus and asymptotic of ϕ resolvent of ϕ Hodge Laplacian at low energy [GTV20] to relate heat kernel and resolvent via,

$$H^M(t, z, z') = \int_\Gamma e^{-t\lambda}(\Delta_\phi - \lambda)^{-1} d\lambda,$$

where Γ is an curve around spectrum the spectrum of Hodge Laplacian Δ_ϕ. oriented counter-clockwise.

In section 16 we use these structure theorems in order to determine the asymptotics of renormalized heat trace and show that the renormalized zeta function admits meromorphic continuation on whole of complex plane $\mathbb{C}$. One defines then the renormalized determinant of Laplacian and analytic torsion in the set up of ϕ manifolds. Our work is a generalization of [She13].

15. Heat kernel in short time regime and long time regime

We start this section by defining the heat kernel on a compact Riemannian manifold and continue to determine the heat kernel asymptotic on the manifold with fibred boundary, ϕ metric in finite time and long time. Using these asymptotics analytic torsion with respect to ϕ Hodge Laplacian will be constructed in section 16.

Definition 15.1. Assume (M, g) is a closed Riemannian manifold and $\Delta_g^q : \Omega^q(M) \longrightarrow \Omega^q(M)$ is the Hodge Laplace acting on the space of q forms. A heat kernel is a function, $H_{g,q} : M \times M \times [0, \infty) \longrightarrow M$ that satisfies,

- $H_{g,q}(x, y, t)$ is C^1 in t and C^2 in (x, y).
- $\frac{\partial H}{\partial t} + \Delta_{g,x}^q(H) = 0$, where $\Delta_{g,x}^q$ is Hodge Laplacian on manifold (M, g).
- $\lim_{t \longrightarrow 0^+} H_{g,q}(x, y, t) = \delta(x - y)$ where by $\delta(x - y)$ we mean delta distribution.

Then the solution of heat equation with initial condition $u(x, 0) = f(x)$ is given by, $u(x, t) = \int_M H(x, y, t)f(y)dy$.
Recall that for (M, g) smooth closed manifold and $\{\lambda_j\}$ spectrum of Δ_g on M

and ψ_j the associated eigenfunctions one may write,

$$H(x,y,t)=\sum_i e^{-\lambda_j t}\psi_i(x)\psi_i(y),$$

and the heat trace is defined as,$\mathrm{Tr}(H(t))=\int_M H(x,x,t)dx=\sum_i e^{-\lambda_j t}$. The heat trace has a asymptotic expansion as,

$$\mathrm{Tr}H(t)\sim_{t\longrightarrow 0^+}(4\pi t)^{-\frac{\dim(M)}{2}}\sum_{j=1}^{\infty}a_j t^j,$$

where a_j are integrals over M of universal homogeneous polynomials in the curvature and its covariant derivatives.

We observe that the heat kernel $H(t,x,x')$ as integral kernel is supported on $M^2\times\mathbb{R}^+$. We switch now to manifold with fibred boundary endowed with ϕ metric $(\overline{M},g_\phi)$. One determines asymptotics of heat kernel in short and long time regimes by blow up process.
For finite-time regime this integral kernel lifts on so called heat space to be polyhomogeneous conormal kernel and for long-time regime the statement is to determine polyhomogeneous kernel of resolvent of ϕ Hodge Laplacian at low energy level and then residue theorem applies in order to calculate long time heat kernel asymptotics.

Finite time regime. In order to determine the behaviour of heat kernel in short time regime, we observe that the heat kernel initially is supported on $\overline{M}^2\times\mathbb{R}^+$. One may use resolution process to obtain a manifold with corners which we denote it as HM_ϕ. One constructs in [TaVe20] the integral kernel of heat equation which lifts to polyhomogeneous conormal distribution on the heat space HM_ϕ.
The heat space is constructed from $\overline{M}^2\times\mathbb{R}^+$, by resolution process. Namely, one takes the elliptic ϕ-space in time and blow up additionally the diagonal of $\overline{M}^2_\phi\times\mathbb{R}$ at $t=0$. The space HM_ϕ can be visualized as in figure 19, where we denote $\beta_{\phi-h}$ to be the blow down map. The main result of [TaVe20] reads as,

Theorem 15.2. [TaVe20]*(Theorem 7.2) The fundamental solution of the heat equation, for finite time* $t<\infty$,

$$\partial_t H(t,x,x')+\Delta^q_{g_\phi,x}H(t,x,x')=0,$$
$$H(t=0,x,x')=\delta(x-x'),$$

under the assumption (9.2), lifts to polyhomogeneous conormal distribution on HM_ϕ *with leading asymptotics* 0 *at* fd *and* $-n$ *at td and vanishing to infinite order on other hypersurfaces of* HM_ϕ. *Here* $n=\dim\overline{M}$.

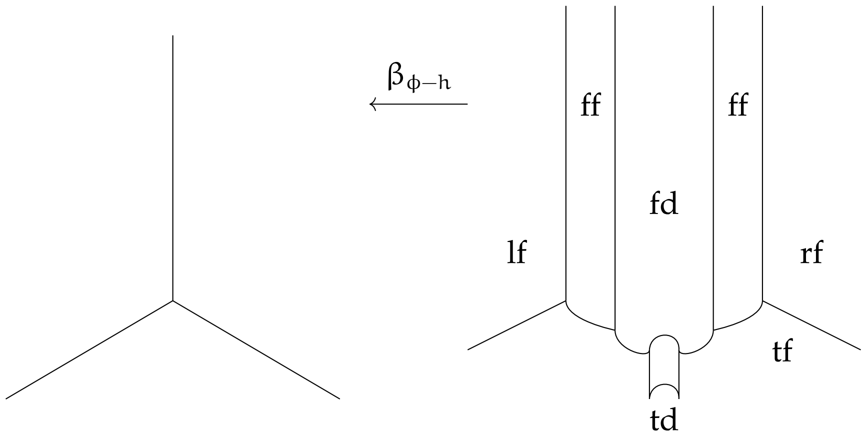

FIGURE 19. The heat blowup space HM_ϕ.

Consequently, Theorem 15.2 completely determines behaviour of heat kernel in finite time regime.

Long time regime. The relation between heat kernel and resolvent can be read from Residue Theorem . Namely, we may express heat kernel as,

$$H^M_{\phi,q}(t,z,z') = \frac{1}{2\pi i}\int_{\Gamma'} e^{-t\lambda}(\Delta^q_\phi - \lambda)^{-1} d\lambda, \tag{15.1}$$

Where Γ is the counter- clockwise path around the spectrum and with $(\Delta_\phi - \lambda)^{-1}$ we understand the integral kernel of resolvent $(\Delta_\phi - \lambda)^{-1}$. Note that Δ_ϕ admits positive continuous real spectrum and consequently one visualize Γ' as, figure 20. Or with the change of variable $\lambda = -\lambda$ in (15.1) one obtains,

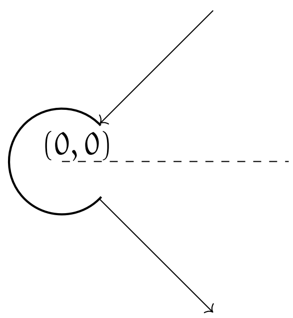

FIGURE 20. Γ'

$$H^M_{\phi,q}(t,z,z') = \frac{1}{2\pi i}\int_{\Gamma} e^{t\lambda}(\Delta^q_\phi + \lambda)^{-1} d\lambda, \tag{15.2}$$

But now Γ is visualized as figure 21.

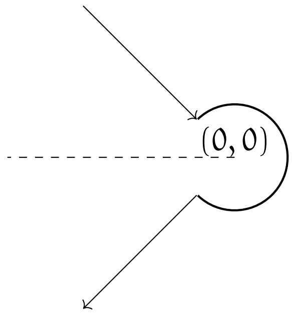

FIGURE 21. Γ

In order to explain the behaviour of heat kernel as $t \longrightarrow \infty$ one applies (15.2).
In [GTV20] authors constructed the polyhomogeneous integral kernel of resolvent of Δ_ϕ on the space $M^2_{\lambda,\phi}$ at low energy level i.e as $\lambda \longrightarrow 0^+$. The space $M^2_{\lambda,\phi}$ is manifold with corners and may be illustrated as in figure 22.

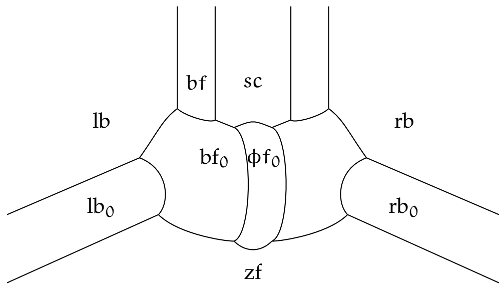

FIGURE 22. Blowup space $M^2_{\lambda,\phi}$

The main result of [GTV20] , expresses low energy of resolvent of ϕ Hodge Laplacian in terms of polyhomogeneous conormal distribution on $M^2_{\lambda,\phi}$. The result reads as follows,

Theorem 15.3. [GTV20] *(Theorem 7.11) Under assumptions (9.1, 9.2, 9.3, 9.4, 9.5), the resolvent* $(\Delta_\phi + k^2)^{-1}$ *as* $k \longrightarrow 0^+$ *is an element of the split calculus (defined as in* [GTV20]*) where,*

$$\mathcal{E}_{sc} \geq 0,\ \mathcal{E}_{\phi f_0} \geq 0,\ \mathcal{E}_{bf_0} \geq -2,\ \mathcal{E}_{lb_0}, \mathcal{E}_{rb_0} > 0, \mathcal{E}_{zf} \geq -2.$$

The leading terms at sc, ϕf_0, bf_0 *and* zf *are of order* $0, 0, -2, -2$.

Remark 15.4. Theorem 15.3 is obtained in parallel work [KoRo20].

We apply (15.2) and determine the asymptotics of heat kernel in large time regime by integration over path Γ. One parametrize Γ as, For $\frac{\pi}{2} < \varphi < \pi$ and $a \in \mathbb{R}^+$ fixed, Γ splits in two paths $\Gamma_{1,a}$ and $\Gamma_{2,a}$ where $\Gamma_{1,a}$ and $\Gamma_{2,a}$ are parametrised as,

$$\begin{aligned} \Gamma_{1,a} &= ae^{i\theta}, \qquad -\varphi \leq \theta \leq \varphi, \\ \Gamma_{2,a} &= re^{i\theta}, \qquad a \leq r < \infty. \end{aligned} \tag{15.3}$$

Assume that $\mathcal{R}(\lambda, z, z')$ is the Schwartz kernel of resolvent $(\Delta_\phi + e^{i\theta}\lambda)^{-1}$, $\lambda > 0$. Recall that in short time regime we used $\tau := t^{\frac{1}{2}}$. Consequently fix $\omega = \tau^{-1}$ and $a = \omega^2$. $\omega \longrightarrow 0$ corresponds to $t \longrightarrow \infty$ and the resolvent at low energy i.e as $a \longrightarrow 0^+$. We may split (15.2) into two parts and write, (by suppressing x, y, x', y')),

$$\begin{aligned} H(\omega^{-2}, z, z') &= \frac{1}{2\pi i}\int_\Gamma e^{\frac{\lambda}{\omega^2}}\mathcal{R}(\lambda, z, z')d\lambda \\ &= \frac{1}{2\pi i}\left(\int_{\Gamma_{1,\omega^2}} e^{\frac{\lambda}{\omega^2}}\mathcal{R}(\lambda, z, z')d\lambda + \int_{\Gamma_{2,\omega^2}} e^{\frac{\lambda}{\omega^2}}\mathcal{R}(\lambda, z, z')d\lambda\right). \end{aligned}$$

Integration on Γ_{1,ω^2}. One may use, $\lambda = \omega^2 e^{i\theta}$, $-\varphi \leq \theta \leq \varphi$, and therefore,

$$\frac{1}{2\pi i}\int_{\Gamma_{1,\omega^2}} e^{\frac{\lambda}{\omega^2}}\mathcal{R}(\lambda, z, z')d\lambda = \frac{\omega^2}{2\pi}\int_{-\varphi}^{\varphi} e^{e^{i\theta}}\mathcal{R}(\theta, \omega, z, z')d\theta. \tag{15.4}$$

Firstly we observed that the integral converges by the boundedness of integrand with respect to θ. By Theorem 15.3 for each fixed θ, the integrand in (15.4) is polyhomogeneous on $M^2_{\omega,\phi}$ with conormal singularity at diagonal $\triangle_\phi$ and all coefficients depend smoothly on $\theta \in [-\varphi, \varphi]$. Consequently the integral (15.4) is polyhomogeneous on $M^2_{\omega,\phi}$.
The index set of $H(\omega^{-2}, z, z')$ is computed from index sets $\mathcal{R}(\theta, \omega, z, z')$ and from ω^2, i.e as ω is boundary defining function for faces zf, bf_0, ϕf_0, lb_0, rb_0, on the faces zf, bf_0, ϕf_0, lb_0, rb_0 we add $+2$ to the index sets of the resolvent in Theorem 15.3 and the index sets on other faces remain the same as those index sets of $\mathcal{R}(\theta, \omega, z, z')$.

Integration on Γ_{2,ω^2}. The path Γ_{2,ω^2} consists of two rays at angels φ and $-\varphi$. The integration over Γ_{2,ω^2} refers therefore to the integration over these two rays. We claim for convergence and polyhomogeneity of integration along φ ray. The argument for integration at $-\varphi$ ray is the same. Parametrising $\lambda = re^{i\varphi}$, we express the integral over Γ_{2,ω^2} to be,

$$\int_{\Gamma_{2,\omega^2}} e^{\frac{\lambda}{\omega^2}}\mathcal{R}(\lambda, z, z')d\lambda = \int_{\omega^2}^{\infty} e^{\frac{r}{\omega^2}e^{i\varphi}}\mathcal{R}(r, z, z')dr, \tag{15.5}$$

And use change of variable $s = \sqrt{r}$, $dr = 2sds$ in (15.5) to write,

$$\int_{\Gamma_{2,\omega^2}} e^{\frac{\lambda}{\omega^2}} \mathcal{R}(\lambda, z, z') d\lambda = \int_{\omega}^{\infty} 2s e^{(\cos\varphi)\frac{s^2}{\omega^2}} e^{i(\sin\varphi)\frac{s^2}{\omega^2}} \mathcal{R}(s^2, z, z') ds. \tag{15.6}$$

As $\frac{\pi}{2} < \varphi < \pi$ is fixed, $\cos(\varphi) < 0$ and therefore for $s \longrightarrow +\infty$ the $e^{(\cos\varphi)\frac{s^2}{\omega^2}}$ and consequently entire integrand decays to infinite order at $s = \infty$, which mean that the integral (15.6) converges. In order to analyse the polyhomogeneity of (15.6), we split $\mathcal{R}(s^2, z, z')$ into two pieces. Namely we use partition of unity and write, $\mathcal{R} = \mathcal{R}_D + \mathcal{R}_C$, where $\mathcal{R}_D$ is supported in a neighborhood of $\triangle_\phi$ and $\mathcal{R}_C$ is supported away from $\triangle_\phi$. Recall that the diagonal $\triangle_\phi$ intersects the sc, ϕf_0 and zf faces of $M^2_{s,\phi}$. Accordingly we may split $\mathcal{R}_D$ into three pieces and write, $\mathcal{R}_D = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$. $\mathcal{R}_1$ is supported away from sc and near zf and $\mathcal{R}_2$ is supported away from zf and near sc and $\mathcal{R}_3$ is supported in interior of bf_0 near diagonal of ϕf_0. We argue for the polyhomogeneity of integral (15.6) for each of $\mathcal{R}_i, i = 1, 2, 3$. Compare to figure 23.

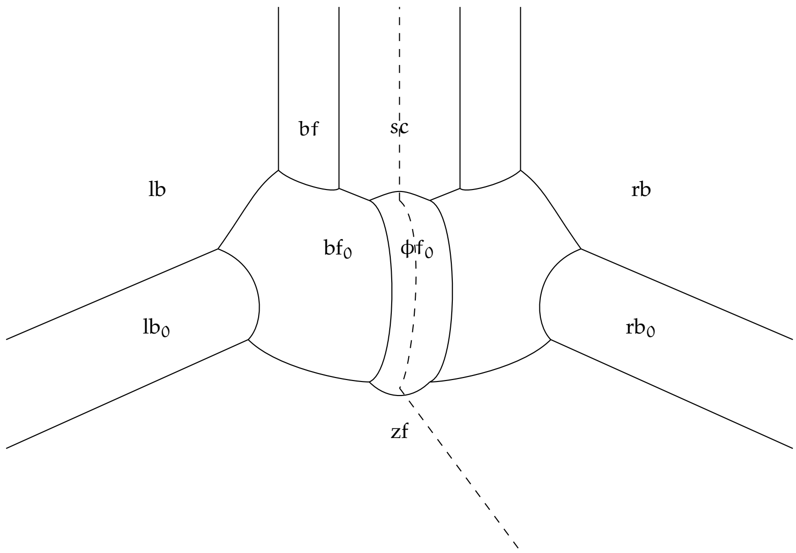

FIGURE 23. Diagonal of $M^2_{s,\phi}$

Polyhomogeneity of $\mathcal{R}_1$. On the support of $\mathcal{R}_1$ near zf, away from $x \neq 0$, we may use projective coordinates, $\hat{X} = \frac{x-x'}{x}, \hat{Y} = \frac{y-y'}{x}, \hat{Z} = \frac{z-z'}{x}, \mu = \frac{s}{x}$. By definition of conormal singularity along diagonal $\triangle_\phi$, we may express,

$$\mathcal{R}_1 \sim \int_{\mathbb{R}^n} e^{i(\hat{X},\hat{Y},\hat{Z})(\zeta_1,\zeta_2,\zeta_3)} \sum_{j=0}^{\infty} a_j(\frac{s}{x}, x, y, z, \frac{\zeta}{|\zeta|})|\zeta|^{2-j} d\zeta, \tag{15.7}$$

where by $\sim$ we mean that $\mathcal{R}_1$ may be expressed locally by (15.7) on $M^2_{s,\phi}$ up to smooth function. Thus we may absorb the smooth reminder into $\mathcal{R}_C$ and plug

(15.7) into (15.6) to obtain,

$$\int_{\mathbb{R}^n} e^{i(\hat{X},\hat{Y},\hat{Z})(\zeta_1,\zeta_2,\zeta_3)} \sum_{j=0}^{\infty} (\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} a_j(\frac{s}{x}, x, y, z, \frac{\zeta}{|\zeta|})|\zeta|^{2-j} d\zeta ds.$$

From definition of conormal singularity, one has that the coefficients a_j are polyhomogeneous in x and $\frac{s}{x}$ with index set independent from j and a_j is smooth in $(y, z, \frac{\zeta}{|\zeta|})$. Observe that the pullback of a_j via projection to, $M_b^3(s, x, \omega) \times N_y \times N_f \times \mathbb{S}^{n-1}_{\frac{\zeta}{|\zeta|}}$, is polyhomogeneous conormal with index set independent of j. Therefore, $2se^{(\frac{s}{\omega})^2} e^{i\varphi} a_j(\frac{s}{x}, x, y, z, \frac{\zeta}{|\zeta|})$ is polyhomogeneous conormal on $M_b^3(s, \omega, x) \times N_y \times N_f \times \mathbb{S}^{n-1}_{\frac{\zeta}{|\zeta|}}$ and it admits cut off singularity at $\frac{s}{\omega} = 1$. The integrand has infinite decay in s due to the fact that $\cos\varphi < 0$ and therefore integration is well defined. Integration in s corresponds to a projection map,

$$M_b^3(s, \omega, x) \longrightarrow M_b^2(\omega, x),$$

which is b-fibration [Mel96] by the following lemma from [MeSi08].

Lemma 15.5. *If $m < n$ each of the projections off $n - m$ factors of X, $\pi : X^n \longrightarrow X^m$, fixes a unique b-stretched projection π_b giving a commutative diagram,*

$$\begin{array}{ccc} X_b^n & \longrightarrow & X_b^m \\ \downarrow & & \downarrow \\ X^n & \longrightarrow & X^m \end{array} \tag{15.8}$$

and furthermore π_b is b-fibration.

Extension to $N_y \times N_z \times \mathbb{S}^{n-1}_{\frac{\zeta}{|\zeta|}}$ by $(y, z, \frac{\zeta}{|\zeta|})$ does not change the b-fibration property of this map and consequently by pushforward Theorem of Melrose $\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} a_j(\frac{s}{x}, x, y, z, \frac{\zeta}{|\zeta|})ds$, is polyhomogeneous conormal on,
$M_b^2(\omega, x) \times N_y \times N_z \times \mathbb{S}^{n-1}_{\frac{\zeta}{|\zeta|}}$, with index set independent of j and consequently on $M^2_{\omega,\phi}$.

Polyhomogeneity of $\mathcal{R}_2$. We need to prove the polyhomogeneity of the integration of kernel $\mathcal{R}_2$ along diagonal $\triangle_\phi \cap \mathrm{sc}$ of $M^2_{s,\phi}$.
$\mathcal{R}_2$ is supported near sc face of $M^2_{s,\phi}$. The coordinates we employ are in the form, $X = s(\frac{1}{x} - \frac{1}{x'}), Y = \frac{s}{x}(y - y'), \mu = \frac{x}{s}, s, Z = \frac{s}{x}(z - z')$. The definition of conormality implies that,

$$\mathcal{R}_2 \sim \int_{\mathbb{R}^n} e^{i(X,Y,Z)(\zeta_1,\zeta_2,\zeta_3)} \sum_{j=0}^{\infty} b_j(\frac{x}{s}, s, y, z, \frac{\zeta}{|\zeta|})|\zeta|^{2-j} d\zeta. \tag{15.9}$$

Where b_j are polyhomogeneous conormal in $\frac{x}{s}$ and s with index sets independent of j and b_j smoothly depend on $(y, z, \frac{\zeta}{|\zeta|})$. We plug (15.9) into (15.6) and analyse the integral by arguing in two different regimes which correspond to $\omega > \frac{x}{2}$ and $\omega < \frac{x}{2}$.
First we assume that $\omega > \frac{x}{2}$ and we introduce projective coordinates, on sc face of $M^2_{\omega,\phi}$, $\bar{X} = \omega(\frac{1}{x} - \frac{1}{x'}), \bar{Y} = \frac{\omega}{x}(y - y'), \bar{Z} = \frac{\omega}{x}(z - z'), \bar{\lambda} = \frac{x}{\omega}$. We need expression of conormal singularity at, $\bar{X} = \bar{Y} = \bar{Z} = 0$, in this region. We note that, $(X, Y, Z) = (\frac{s}{\omega}\bar{X}, \frac{s}{\omega}\bar{Y}, \frac{s}{\omega}\bar{Z})$ and take new variable η which is defined as, $\eta = (\eta_1, \eta_2, \eta_3) = (\frac{s}{\omega}\zeta_1, \frac{s}{\omega}\zeta_2, \frac{s}{\omega}\zeta_3)$. We plug (15.9) in (15.6) and employ new variables $\bar{X}, \bar{Y}, \bar{Z}$. Interchanging the integral and the sum to obtain,

$$\int_{\mathbb{R}^n} \int_{\omega}^{\infty} e^{i(\bar{X},\bar{Y},\bar{Z})(\eta_1,\eta_2,\eta_3)} \sum_{j=0}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} b_j(\frac{x}{s}, s, y, z, \frac{\eta}{|\eta|})(\frac{s}{\omega})^{j-2-(n)} |\eta|^{2-j} ds d\eta.$$

We argue that, $\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} b_j(\frac{x}{s}, s, y, z, \frac{\eta}{|\eta|})(\frac{s}{\omega})^{j-2-(n)} ds$, is polyhomogeneous conormal in $\frac{x}{\omega}, \omega$ independent of j. The argument is similar to the last step as the integrand decays to infinite order at $s = \infty$ and is polyhomogeneous conormal in $\frac{x}{s}, s$ independent of j. Integration with respect to s yields the polyhomogeneity on $X^2_b(x, \omega) \times N_y \times N_f \times \mathbb{S}^{n-1}_{\frac{\eta}{|\eta|}}$ by Melrose pushforward Theorem.
The second region corresponds to $\omega < \frac{x}{2}$. Here one use coordinates, $\hat{X} = \frac{x-x'}{x}, \hat{Y} = y - y', \hat{Z} = z - z', \frac{s}{x}, x$, and one expects that conormal singularities arise on, $\hat{X} = \hat{Y} = \hat{Z} = 0$. One notes that $(X, Y, Z) = (\frac{s}{x})(\hat{X}, \hat{Y}, \hat{Z})$. One introduce variables $(\zeta_1', \zeta'_2, \zeta_3') = \frac{s}{x}(\zeta_1, \zeta_2, \zeta_3)$ and plug these variables in 15.9 to obtain,

$$\int_{\mathbb{R}^n} e^{i(\hat{X},\hat{Y},\hat{Z})(\zeta_1',\zeta_2',\zeta_3')} \sum_{j=0}^{\infty} (\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2 e^{i\varphi}} b_j(\frac{x}{s}, s, y, z, \frac{\zeta'}{|\zeta'|})(\frac{s}{x})^{j-2-n} ds |\zeta'|^{2-j}) d\zeta',$$

The j-coefficient may be written as,

$$(\frac{\omega}{x})^{j-2-n} \int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} b_j(\frac{x}{s}, s, y, z, \frac{\zeta'}{|\zeta'|})(\frac{s}{\omega})^{j-2-n} ds, \tag{15.10}$$

which is $(\frac{\omega}{x})^{j-2-n}$ times (15.7). The $(\frac{\omega}{x})^{j-2-n}$ is polyhomogeneous conormal on $M^2_b(\omega, x)$ and consequently (15.10) is polyhomogeneous conormal on $M^2_b(\omega, x)$ for each j. As $\omega < \frac{x}{2}$ by increasing j the order of polyhomogeneity increase and hence all coefficients are polyhomogeneous conormal on $M^2_b(\omega, x)$ with respect to the index set of the $j = 0$ coefficient.
We conclude that the integration on $\mathcal{R}_2$ yields to the polyhomogeneity of (15.9) on $M^2_{\omega,\phi}$.

Polyhomogeneity of $\mathcal{R}_3$. The polyhomogeneity integration (15.5) with respect to $\mathcal{R}_3$ follows from the fact that $\mathcal{R}_3$ is supported in compact region namely on diagonal of ϕf_0, and by conormality we may express by adequate local coordinates on ϕf_0,

$$\mathcal{R}_3 \sim \int_{\mathbb{R}^n} e^{i(z-z')(\eta)} \sum_{j=0}^{\infty} c_j(s,z,\frac{\eta}{|\eta|})|\eta|^{2-j} d\eta, \tag{15.11}$$

where c_j are polyhomogeneous conormal at $s = 0$ and $s = \infty$, with index sets independent of j at $s = 0$ and $s = \infty$. We plug (15.11) into (15.6) and we get,

$$\int_{\mathbb{R}^n} e^{i(z-z')\eta} \sum_{j=0}^{\infty} (\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} c_j(s,z,\frac{\eta}{|\eta|}) ds |\eta|^{2-j} d\eta. \tag{15.12}$$

Note that the j-th coefficient, $\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} c_j(s,z,\frac{\eta}{|\eta|}) ds$, is polyhomogeneous conormal on $M_b^2(s,\omega)$ with index sets independent of $j, z, \frac{\eta}{|\eta|}$ with infinite decay at $s = \infty$. Consequently (15.12) is polyhomogeneous on $M^2_{\omega,\phi}$.

Polyhomogeneity of $\mathcal{R}_{\mathcal{C}}$. Now we argue the polyhomogeneity of $\mathcal{R}_{\mathcal{C}}$ term. Explicitly one may write, by suppressing (x,y,z) in z and (x',y',z') in z', $\int_{\omega}^{\infty} 2se^{(\frac{s}{\omega})^2} e^{i\varphi} \mathcal{R}_s(s^2,z,z') ds$. By assumption $\mathcal{R}_s(s^2,x,y,z,x',y',z')$ is polyhomogeneous conormal on $M^2_{s,\phi}$ and smooth across the diagonal $\triangle_{s,\phi}$. We apply the following lemma parallel to [She13] (Lemma 10) to the integral kernel,

$$\mathcal{R}_s(s^2,x,y,z,x',y',z'). \tag{15.13}$$

to conclude that (15.13) is polyhomogeneous on $M^2_{\omega,\phi}$.

Lemma 15.6. *Assume $T(k,z,z')$ be a function which is polyhomogeneous conormal on $M^2_{k,\phi}$ and smooth in the interior and decaying to infinite order at lb, rb, and bf. Then,*

$$\int_{\omega}^{\infty} 2ke^{(\frac{k}{\omega})^2} e^{i\varphi} T(k,z,z') dk, \tag{15.14}$$

is polyhomogeneous on $M^2_{\omega,\phi}$ for ω bounded from below.

Proof. In the following proof we use explicit local coordinates in each region of $M^2_{k,\phi}$ in order to plug the polyhomogeneity expression of $T(k,z,z')$ and evaluate directly the integral (15.14) to show that the resulting function is polyhomogeneous on $M^2_{\omega,\phi}$.

Near $\mathrm{bf} \cap \mathrm{bf}_0 \cap \mathrm{lb}$. We may use coordinates, $(\zeta = \frac{x}{k}, s = \frac{x'}{x}, k)$ in this region and the polyhomogeneity of $T(k,x,x')$ with respect to these coordinates becomes

$T \sim \sum a_{ijl}\zeta^i s^j k^l$. We plug this expression into (15.14) and use change of variable $\frac{k}{\omega} = t$ and obtain,

$$\int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}T(k,x,x')dk = \sum \int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}a_{ijk}k^{l-i}x^is^jdk =$$
$$\sum a'_{ijl}2x^is^j\int_1^\infty \omega te^{-t^2}(\omega t)^{l-i}\omega dt = \sum 2a'_{ijl}x^is^j\omega^{l-i+2}\int_1^\infty e^{-t^2}t^{l-i+1}dt$$
$$\sim \sum a'_{ijl}(\frac{x}{\omega})^is^j\omega^{l+2},$$

i.e the integral is polyhomogeneous with respect to coordinates $(\frac{x}{\omega}, s, \omega)$ that corresponds to region $\mathrm{bf} \cap \mathrm{bf}_0 \cap \mathrm{lb}$ of $M^2_{\phi,\omega}$. As the argument is symmetric near $\mathrm{bf} \cap \mathrm{bf}_0 \cap \mathrm{rb}$ we leave the proof.

Near $\mathrm{lb} \cap \mathrm{lb}_0 \cap \mathrm{bf}_0$. In this region, we may use coordinates $(x, s' = \frac{x'}{x}, \kappa = \frac{k}{x})$. Plugging the definition of polyhomogeneity of $T \sim \sum a_{ijl}x^is'^j\kappa^l$ into integral (15.14) and using change of variable $\frac{k}{\omega} = t$ yields to

$$\int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}T(k,x,x')dk = \sum \int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}a_{ijl}x^is'^j\kappa^ldk =$$
$$\sum a'_{ijl}2x^is'^j\int_1^\infty \omega te^{-t^2}e^{i\varphi}(\omega t)^l\omega dt = \sum 2a'_{ijl}x^{i-l}s^j\omega^l\int_1^\infty e^{-t^2}t^ldt$$
$$\sim \sum a'_{ijl}x^is'^j(\frac{\omega}{x})^l,$$

which means the integral is polyhomogeneous with respect to coordinates $(x, \frac{x'}{x}, s, \frac{\omega}{x})$. That means the polyhomogeneity on the region $\mathrm{lb} \cap \mathrm{lb}_0 \cap \mathrm{bf}_0$. The similar argument shows also the polyhomogeneity of integral on the region $\mathrm{rb} \cap \mathrm{rb}_0 \cap \mathrm{bf}_0$.

Near $\mathrm{sc} \cap \mathrm{bf}_0$. We use the coordinates $(S = \frac{k(x-x')}{x'^2}, S' = \frac{x'}{k}, k)$ and the polyhomogeneity of $T(k,x,x')$ with respect to these coordinates $T \sim \sum a_{ijl}S^iS'^jk^l$ in the integral (15.14), and use change of variable $\frac{k}{\omega} = t$,

$$\int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}T(k,x,x')dk = \sum \int_\omega^\infty 2ke^{(\frac{k}{\omega})^2}e^{i\varphi}a_{ijl}S^iS'^jk^ldk =$$
$$\sum a'_{ijl}2(\frac{x-x'}{x'^2})^ix'^j\int_1^\infty \omega^{i-j+1}t^{i-j+1}e^{-t^2}e^{i\varphi}(\omega t)^{i-j+1+l}\omega dt =$$
$$\sum 2a'_{ijl}S^iS'^j\omega^l\int_1^\infty e^{-t^2}t^{l+i-j+1}dt \sim \sum a'_{ijl}S^iS'^j\omega^l,$$

which yields to polyhomogeneity of integral in the region $\mathrm{sc} \cap \mathrm{bf}_0$ of $M^2_{\omega,\phi}$. Similar argument shows the polyhomogeneity near the face $\phi\mathrm{f}_0$ of $M^2_{\omega,\phi}$ as well. $\square$

In the following theorem, we summarize the polyhomogeneity of the heat kernel in long time regime.

Theorem 15.7. *The heat kernel which is given by (15.2), i.e*

$$H^M(t,x,x') = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\Delta_\phi+\lambda)^{-1}d\lambda,$$

is polyhomogeneous conormal at $t=\omega^{-\frac{1}{2}}$ at $\omega \longrightarrow 0$ on $M^2_{\omega,\phi}$ with index sets given in terms of index sets of resolvent $(\Delta_\phi+\lambda)^{-1}$ at low energy level. More explicitly the asymptotics of heat kernel in long time regime are of leading order 0 at sc face and of order 0 at zf and bf_0 faces. More over the leading order at the face ϕf_0 is 2. In long time regime the heat kernel vanishes to infinite order at lb, rb, and bf faces of $M^2_{\omega,\phi}$. The explicit index sets are as follows,

$$\mathcal{E}_{sc}\geq 0,\ \mathcal{E}_{\phi f_0}\geq 2,\ \mathcal{E}_{bf_0}\geq 0,\ \mathcal{E}_{lb_0},\mathcal{E}_{rb_0}>0, \mathcal{E}_{zf}\geq 0.$$

16. Analytic torsion

For (M,g) compact oriented C^∞ Riemannian manifold of dimension $\dim(M)=n$ and Δ_g^q Hodge Laplacian acting on $\Omega^q(M)$, let

$$0\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\longrightarrow+\infty,$$

be the sequence of eigenvalues of Δ_g^q. The zeta function Δ^q is then defined by,

$$\zeta_q(s)=\sum_{\lambda_i>0}\lambda_i^{-s},$$

and it turns out that the zeta function converges in the half plane $\mathrm{Re}(s)>\frac{n}{2}$. One can explicitly express the zeta function by integration of heat kernel along diagonal. Namely for $s>\frac{n}{2}$,

$$\zeta_{M,q}(s)=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(H^M(t))t^{s-1}dt.$$

$\zeta_{M,q}(s)$ admits meromorphic continuation to $\mathbb{C}$ with regular value at 0 and one may define the determinant of Hodge Laplacian as $\exp(-\frac{d}{ds}\zeta_{M,q}(s)|_{s=0})$ and analytic torsion is defined to be,

$$\mathrm{Log}T_M=\frac{1}{2}\sum_q(-1)^q q\frac{d}{ds}\zeta_{M,q}(s)|_{s=0}.$$

Consider now exact ϕ metric, $g_\phi=\frac{dx^2}{x^4}+\frac{\phi^*g_B}{x^2}+g_F$. One observe that the integration along diagonal of heat kernel diverges as $x\longrightarrow 0$. The way to overcome to this problem is to define renormalization. In this section, We describe two ways of renormalization. Renormalization together with structure Theorems of heat kernel 15.2, 15.7 give rise to the definition of renormalized heat trace and renormalized zeta function. We show that the renormalized zeta function admits meromorphic continuation to $\mathbb{C}$. We define determinant of Laplacian and renormalized zeta function in the setup of fibred boundary manifold (M, g_ϕ).

16.1. **Renormalized heat trace.** We follow [AAR13] in order to define two ways of renormalization.

Riesz renormalization. The first method of renormalization is called the renormalization in the sense of Riesz. Assume M is a manifold with boundary and x is a boundary defining function. Assume further that f admits an asymptotic expansion in terms of x and $\log x$ i.e,

$$f \sim \sum a_{s,p} x^s (\log(x))^p \tag{16.1}$$

Assume that μ is smooth non-vanishing density on M. The expansion of f (16.1) implies the meromorphic continuation of the function, $z \mapsto \int_M x^z f d\mu$, Now the Riesz renormalized integral of f is defined to be, ${}^R\int_M f d\mu = \underset{\epsilon=0}{\mathrm{FP}} \int_M x^z f d\mu$.

Hadamard renormalization. The second method is due to Hadamard and is called in [Mel93] as b-integral. Assume M is manifold with boundary and x is boundary defining function. One may define,

$$\epsilon \mapsto \int_{x \geq \epsilon} f d\mu, \tag{16.2}$$

and show that (16.2) has asymptotic expansion as $\epsilon \longrightarrow 0$ when f admits expansion (16.1). We define then ${}^h\int_M f d\mu = \underset{\epsilon=0}{\mathrm{FP}} \int_{x\geq\epsilon} f d\mu$, where by finite part we mean taking out the divergent part.

Heat kernel renormalization in the sense of Hadamard.

Definition 16.1. Assume $\overline{M}$ is a fibred boundary manifold and assume further that $H^M_\phi(x, y, z, x', y', z')$ is the heat kernel. One defines the renormalized heat trace to be,

$$^R\mathrm{Tr}(H^M_\phi(t)) = {}^h\int_M H^M_\phi(t, x, y, z, x, y, z) d\mathrm{vol}_\phi(x, y, z).$$

Equivalently one may take a sharp cutoff function $\chi(r)$ which is supported on $[0, 1]$ and is equal to 1 for $0 \leq r \leq 1$. For fix $\epsilon > 0$ consider $\chi(\frac{\epsilon}{x})$. The renormalized heat trace is defined as,

$$^R\mathrm{Tr}(H^M_\phi(t)) = \underset{\epsilon=0}{\mathrm{FP}} \int_M \chi(\frac{\epsilon}{x}) H^M_\phi(t, x, y, z, x, y, z) d\mathrm{vol}_\phi(x, y, z). \tag{16.3}$$

In order to show the polyhomogeneity of renormalized heat trace (16.3), we need the polyhomogeneity of the heat kernel along diagonal.

Lemma 16.2. *Assume* $(\overline{M}, g_\phi)$ *is a fibred boundary manifold and* Δ_ϕ^q *is the Hodge Laplacian acting on the space of* q *forms. The corresponding heat kernel is polyhomogeneous conormal along diagonal* $(x, y, z; x, y, z)$ *in both short time regime and long time regime. Precisely,*

- *For* t *bounded above,* $H_\phi^M(t, x, y, z, x, y, z)$, *is polyhomogeneous conormal in* $(\tau = \sqrt{t}, x)$ *with smooth dependence on* y *and* z.
- *For* t *bounded below and* $\omega = t^{-\frac{1}{2}}$, $H_\phi^M(t, x, y, z, x, y, z)$ *is polyhomogeneous conormal as a function of* (ω, x) *on* b*-space* $(\mathbb{R}_+(\omega) \times M)_b$ *with smooth dependence on* y, z.

Proof. The lemma follows from the structure theorems of heat kernel in last section i.e from Theorems 15.2 and 15.7.

□

We employ 16.2 and formulate the polyhomogeneity of renormalized heat trace in the following theorem,

Theorem 16.3. *The renormalized heat trace defined by (16.3) has polyhomogeneous expansion in* ϵ *for each fixed* t. *Moreover the finite part at* $\epsilon = 0$ *has leading asymptotics at* $t = 0$ *and* $t = \infty$. *More precisely for coefficients* a_j *and* b_{jl},

$$ {}^R\mathrm{Tr}(H_\phi^M(t)) \sim_{t \longrightarrow 0} \sum_{j \geq 0} a_j t^{\frac{-m+j}{2}}, \tag{16.4} $$

$$ {}^R\mathrm{Tr}(H_\phi^M(t)) \sim_{t \longrightarrow \infty} \sum_{j \geq 0} \sum_{l=0}^{p_j} b_{jl} t^{-\frac{j}{2}} \log^l(t). \tag{16.5} $$

Proof. By definition of renormalized heat trace (16.3), we may write,

$$ \int_M \chi(\frac{\epsilon}{x}) H_\phi^M(t, x, y, z, x, y, z) d\mathrm{vol}_\phi(x, y, z) = \int_{x \geq \epsilon} H_\phi^M(t, x, y, z, x, y, z) d\mathrm{vol}_\phi(x, y, z). \tag{16.6} $$

Polyhomogeneity of (16.6) in (t, ϵ). Consider the integral,

$$ \int_{x \geq \epsilon} H_\phi^M(t, x, y, z, x, y, z) dg_\phi(x, y, z). $$

The integral converges as the integral is away from $x = 0$ and $H_\phi^M(t, x, y, z, x, y, z)$ is polyhomogeneous conormal in both finite and long time regimes as stated in Lemma 16.2. As cutoff function is equal to 1 on this region there is no need to argue for the polyhomogeneity of cutoff function χ. The projection π_x lift to b-fibration Π_x on HM_ϕ in short time regime [TaVe20] and b-fibration $\Pi_{\phi,x}$ on

$M^2_{\omega,\phi}$ in long time regime [GTV20]. Consequently integration yields to polyhomogeneous function in $(\sqrt{t}, \epsilon)$ both in short time and long time by Melrose push forward Theorem.

Asymptotic in (16.4). By Lemma 16.2 one can express the diagonal of heat kernel in short time regime, for $N \in \mathbb{N}$ as,

$$\mathrm{tr}_x H^M_\phi(t, x) = \sum_{j=0}^{N} x^j a_j(t) + H_N(x, t), \tag{16.7}$$

where $a_j(t) \sim_{t \longrightarrow 0} \sum_{k \geq 0} \tau^{-m+k}$ and $H_N(x, t) = O(\tau^{-m} x^{N+1})$. As $\epsilon \longrightarrow 0$ we may Plug (16.7) into (16.6) and evaluate the integral to obtain (16.4).

Asymptotic in (16.5). The diagonal of heat kernel in long time regime is of leading order 0 at sc and 2 at ϕf_0 and 0 at zf face of $M^2_{\omega,\phi}$. As $\epsilon \longrightarrow 0$ one may express explicit diagonal of heat kernel i.e Lemma 16.2 at $\omega^{-\frac{1}{2}} = t = \infty$. On ϕf_0 face one obtain,

$$\mathrm{tr}_x H^M_\phi(x, \omega) \sim \sum h_j(x, \omega),$$

for $h_i(x, \omega)$ homogeneous of order 2. Similarly on zf face we have,

$$\mathrm{tr}_x H^M_\phi(x, \omega) \sim \sum a_j x^j.$$

Plugging these expressions into (16.6) with respect to ϕ-volume form $d\mathrm{vol}_\phi = x^{-b-2} dx dy dz$ we obtain (16.5).

∎

Now we define the renormalized zeta function of Hodge Laplacian. Let (M, g_ϕ) be fibred boundary manifold and fix k the degree of the corresponding Hodge Laplacian, Δ^k_ϕ. Formally, the renormalized zeta function is defined as,

$$\frac{1}{\Gamma(s)} \int_0^\infty {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt. \tag{16.8}$$

Apriori (16.8) is not defined for any $s \in \mathbb{C}$ but one may break (16.8) at say 1 and obtain two integrals. The first integral is denoted as ${}^R_0\zeta^k_{M,\phi}(s)$ and is well defined for $\mathrm{Re}(s) > \frac{n}{2}$,

$${}^R_0\zeta^k_{M,\phi}(s) := \frac{1}{\Gamma(s)} \int_0^1 {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt, \tag{16.9}$$

and the second integral is denoted by ${}^R_\infty\zeta^k_{M,\phi}(s)$ and is well defined for $\mathrm{Re}(s) < 0$,

$${}^R_\infty\zeta^k_{M,\phi}(s) := \frac{1}{\Gamma(s)} \int_1^\infty {}^R\mathrm{Tr}(H^k_{M,\phi})(t) t^{s-1} dt. \tag{16.10}$$

By asymptotics (16.4) and (16.5) of renormalized heat trace both in short time regime $t = 0$ and long time regime $t = \infty$ as stated in Theorem 16.3, one implies that the renormalized zeta functions (16.9), (16.10) extends meromorphically to entire complex plane $\mathbb{C}$.

Definition 16.4. Assume $(\overline{M}, g_\phi)$ is fibred boundary manifold and Δ^k_ϕ is the Hodge Laplace acting on the space of k forms. One defines,

- The renormalized zeta function on $(\overline{M}, g_\phi)$ at degree k, denoted as ${}^R\zeta^k_{M,\phi}(s)$ is defined to be,

$$ {}^R\zeta^k_{M,\phi}(s) :=^R_0 \zeta^k_{M,\phi}(s) +^R_\infty \zeta^k_{M,\phi}(s) \tag{16.11}$$

- The renormalized determinant of the Laplacian on $\overline{M}$ is denoted as $-\frac{d}{ds}{}^R\zeta_{M,\phi}(s)|_{s=0}$, where $\frac{d}{ds}{}^R\zeta_{M,\phi}(s)|_{s=0}$ is the coefficient of s in the Laurent series for ${}^R\zeta_{M,\phi}(s)$ at $s = 0$.

One may define renormalized analytic torsion on fibred boundary manifold as,

Definition 16.5. For $(\overline{M}, g_\phi)$ fibred boundary manifold, denote Δ^k_ϕ to be Hodge Laplacian acting on the space of k forms. One may define the renormalized analytic torsion by,

$$ \mathrm{Log}\, {}^R T_{M,g_\phi} := \frac{1}{2}\sum_q (-1)^q q \frac{d}{ds}\, {}^R\zeta^q_{M,\phi}(s)|_{s=0}. \tag{16.12}$$

We conclude the discussion of this section pointing out that the renormalized analytic torsion as defined in (16.5) may be studied further and one may ask for the statement similar to Cheeger Müller Theorem in the fibred boundary manifold setup.

Open problem 16.6. *(Cheeger Müller in the set up of ϕ manifolds)*

1. Define the toplogical torsion in the set up of manifold with fibred boundary. Is this trivial extension from closed manifolds?
2. Prove Cheeger Müller type statement.

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