Examination of the Closedness of Spaces of Stochastic Integrals

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Abstract

We study stochastic hedging concepts for a general semimartingale model in incomplete markets. To ensure the existence of solutions, one wants the spaces of terminal values of stochastic integrals $\int \vartheta dX$ (representing the cumulative gains from trade) to lie in a closed subset of L^2. In addition, we consider constrained hedging strategies with the notion of correspondences (i.e. the trading strategies are restricted to lie in a predefined set) and ensure the according closedness as well. This is a main contribution of this work. Using the closedness and a change of the according probability measure (using a so-called *variance-minimizing martingale measure* $\tilde{P}$) to transfer the problem to a (local) martingale setting then allows us, to determine explicit hedging strategies for the respective spaces.

Zusammenfassung

In der vorliegenden Dissertation beschäftigen wir uns mit stochastischen Hedgingkonzepten in einem allgemeinen Semimartingal Modell. Um die Existenz einer Lösung zu gewährleisten, ist es naheliegend, zunächst die L^2-Abgeschlossenheit des Raumes der Endwerte stochastischer Integrale $\int \vartheta dX$ (welche die kummulierten Gewinne einer Handelsstrategie repräsentieren) sicherzustellen. Zusätzlich betrachten wir Handelsbeschränkungen mit Hilfe von Korrespondenzen (d.h., dass die Handelsstrategien in soweit eingeschränkt sind, dass sie in eine vorher festgelegte Menge abbilden müssen) und untersuchen auch in diesem Kontext die entsprechende L^2-Abgeschlossenheit. Dies ist ein wesentlicher Beitrag dieser Arbeit.

Unter Verwendung der L^2-Abgeschlossenheit und eines entsprechenden Maßwechsels hin zu dem sogenannten *varianzminimierenden lokalen Martingalmaß* $\tilde{P}$, mit dem wir unser Hedgingproblem in ein (lokales) Martingal-Setting überführen, bestimmen wir abschließend explizite Hedgingstrategien für die entsprechenden Räume.

Contents

1 Introduction

1.1 Motivation

Due to recent crises at global financial markets and thereupon strengthened regulatory requirements for investments, the question of how reassuring oneself against potential losses is of utmost importance. Having a profound look on current hedging strategies, one recognizes that although much work on this field has already been done, developing hedging strategies for a huge variety of investments is continuously one of the main issues of research in Financial Mathematics. Impressively, the answers to very fundamental questions on how to protect oneself from financial losses have not been given completely yet. For example, the following and very intuitive question stays to be of concern:

Having sold a European contingent claim, how does one secure oneself against the risk that may arise from the sale in the future?

One of the reasons why this question is quite difficult to answer is probably that one wants to work in incomplete markets, this means that we have to pay attention to claims that are not replicable. This way, it is not always possible to create a portfolio that replicates the payoff of the sold claim exactly. In this context our question turns into an optimization problem, where we have to solve the problem of minimizing the incurring hedging error. In addition, the answer highly depends on the chosen financial model, i.e. different conditions on price processes and trading strategies may as well very likely lead to different results for according hedging strategies.

Facing the above question in a mathematical way, the claim can be modelled as a square integrable random variable H describing a certain payoff that is generated at time T. In addition, each terminal portfolio value achieved by a self-financing trading strategy θ can be represented by a stochastic integral $\int \vartheta dX$, whereas X describes the price process of the underlying stock. For a non-replicable claim, hedging can now be interpreted as finding the terminal portfolio value, that is closest (i.e. minimizing the L^2-norm) to the random variable H, which models the claim. In the Hilbert space L^2, this means to orthogonally project the claim onto the subset of L^2 of terminal portfolio values. Clearly, the respective subspaces of stochastic integrals should be closed in L^2. In this thesis, for subspaces modelling terminal portfolio values we consider the three different spaces $G_T(\Theta_S)$, $G_T(\Theta_v)$ and $G_T(\Theta_p^{eq})$.

In conclusion, this thesis aims to establish the L^2-closedness of the respective subspaces, as well for unrestricted as for restricted trading strategies, and to find risk-minimizing strategies for all of these subspaces. Furthermore, to weaken the dependency of the chosen financial market model, we provide conditions under which as well the respective hedging strategies as the respective subspaces coincide.

1.2 Theoretical Background and Objective

Subsequently, we want to give a brief overview about what has already been established in the literature and what this work contributes to the current research. We will specify this in the respective sections.

The space $G_T(\Theta_S)$ has been established by Delbaen/Monat/Stricker/Schweizer/Schachermayer (1991) who prove the L^2-closedness in the case that the price process X is a continuous semimartingale, where it turns out that their results do not hold in the discontinuous case. Delbaen et al. (1997) then found necessary and sufficient conditions for the closedness. Further results have been found by Grandits/Krawczyk (1998) and by Choulli/Krawczyk/Stricker (1999). The problem of finding explicit hedging strategies has been worked on by Schweizer (1996), Rheinlaender/Schweizer (1997), Hou/Karatzas (2005) and Arai (2005). In addition, $G_T(\Theta_S)$ is the only space where the closedness under trading constraints already has been established. This can be found in Czichowsky/Schweizer (2012).

For $G_T(\Theta_S)$, the contribution of this work will be providing a counterexample for the closedness in the discontinuous case and subsequently providing conditions to ensure the closedness in the discontinuous case, to deepen the results on the closedness of $G_T(\Theta_S(C))$, i.e the closedness of $G_T(\Theta_S)$ under trading constraints and to specify the shape of the hedging strategy.

$G_T(\Theta_v)$ has been established in Xia/Yan (2006). They provide conditions for the closedness and develop a duality between the closedness of $G_T(\Theta_v)$ and signed martingale measures. Concerning the explicit hedging strategy, they consider the problem of finding mean-variance efficient portfolios. We expand those results by solving the more general mean-variance hedging problem.

$G_T(\Theta_p^{eq})$ is implicitly developed in Delbaen/Schachermayer (1996) whereas the corresponding hedging strategy for solving the mean-variance hedging problem is worked on in Gouriéroux/Laurent/Pham (1998). We will expand the results on that hedging strategy and in addition solve another type of hedging problem similar to the problem of finding mean-variance efficient portfolios. We also provide conditions under which $G_T(\Theta_p^{eq})$ and $G_T(\Theta_v)$ coincide and solve the closedness question under constraints for these spaces.

In conclusion, under special conditions we will establish the closedness with and without constraints, solve the mean-variance hedging problem and the problem of finding mean-variance efficient portfolios for all three spaces and draw a variety of relationships between them.

1.3 Outline

The thesis is divided into five main sections. In section 2 and 3, we establish the necessary mathematical framework, where we provide fundamental results from stochastic analysis and define the financial market model which we will work on. Section 4 is dedicated to establish the L^2-closedness of all of our three subspaces which will enable us in section 6 to find hedging strategies by projecting the non-replicable claim onto our closed subsets.

In addition, in section 5 we analyze the L^2-closedness under trading constraints imposed on our strategies $\vartheta \in \Theta$ and the respective closedness of the occurring spaces $G_T(\Theta)$ of stochastic integrals. In section 6, we use those results from section 4 and 5 to find the solutions to different hedging problems by using Hilbert space methods. In sections 4-6, we will not only state results for each space separately, but also develop connections between them and transfer conclusions from one space to another.

2 Preliminaries

Before we can start working on our actual topic, namely the examination of the closedness of spaces consisting of stochastic integrals representing terminal portfolio values, we will first provide some basic definitions used in the context of stochastic analysis.
The aim of this section is to clarify the setting in terms of the probability space that we work on and to give the reader exact definitions of the different stochastic processes that we will use during this work. Here, the latter case is of utmost importance since those definitions are not uniformly given in the literature and all our results may only be valid given this exact set of definitions and framework. Particularly, the different types of martingales and how they are connected shall be pointed out in this section, as they will be used to model price processes of stocks and therefore will play a very important role throughout all of this work.
The section will only give a brief overview to establish all of the necessary framework for our thesis. For further information the reader may have a look at Protter (2005) which we clearly orientate us on. Especially for the construction and properties of the stochastic integral, we refer to it.

2.1 Stochastic Processes

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$. By a filtration we mean a family of σ-algebras $(\mathcal{F}_t)_{0\leq t\leq\infty}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. For convenience, we will usually write $\mathbb{F}$ for the filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$.

Definition 1. *A filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to satisfy the usual hypothesis if*

$\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$;

$\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, all t, $0 \leq t \leq \infty$; that is, the filtration $\mathbb{F}$ is right continuous.

We always assume that the usual hypothesis holds.

Definition 2. *A random variable $T : \Omega \to [0, \infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$, for every t, $0 \leq t \leq \infty$.*

One important consequence of the right continuity of the filtration is the following theorem.

Theorem 1. *The event $\{T < t\} \in \mathcal{F}_t$, $0 \leq t \leq \infty$, if and only if T is a stopping time.*

A stochastic process X on $(\Omega, \mathcal{F}, P)$ is a collection of $\mathbb{R}$-valued or $\mathbb{R}^d$-valued random variables $(X_t)_{0 \leq t \leq \infty}$. The process X is said to be adapted if $X_t \in \mathcal{F}_t$ (that is, is $\mathcal{F}_t$ measurable) for each t. We must take care to be precise about the concept of equality of two stochastic processes.

Definition 3. *Two stochastic processes X and Y are modifications if $X_t = Y_t$ a.s., each t. Two processes X and Y are indistinguishable if a.s., for all t, $X_t = Y_t$.*

Remark 1. *If X and Y are modifications there exists a null set, N_t, such that if $\omega \notin N_t$, then $X_t(\omega) = Y_t(\omega)$. The null set N_t depends on t.*

Definition 4. *A stochastic process X is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits. Similarly, a stochastic process X is said to be càglàd if it a.s. has sample paths which are left continuous, with right limits.*
(The nonsensical words càdlàg and càglàd are acronyms from the French for continu à droite, limité à gauche and continu à gauche, limité à droite, respectively.)

Theorem 2. *Let X and Y be two stochastic processes, with X a modification of Y. If X and Y have right continuous paths a.s., then X and Y are indistinguishable.*

Corollary 1. *Let X and Y be two stochastic processes which are càdlàg. If X is a modification of Y, then X and Y are indistinguishable.*

Càdlàg processes provide natural examples of stopping times.

Definition 5. *Let X be a stochastic process and let Λ be Borel set in $\mathbb{R}$.*
Define

$$T(\omega) = \inf\{t > 0 : X_t \in \Lambda\}.$$

Then T is called the hitting time of Λ for X.

Theorem 3. *Let X be an adapted càdlàg stochastic process, and let Λ be an open set. Then the hitting time of Λ is a stopping time.*

Theorem 4. *Let X be an adapted càdlàg stochastic process, and let Λ be a closed set. Then the random variable*

$$T(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda \text{ or } X_{t-}(\omega) \in \Lambda\}$$

is a stopping time.

Theorem 5. *Let S, T be stopping times. Then the following are stopping times:*

(i) $S \wedge T = \min(S,T)$*;*

(ii) $S \vee T = \max(S,T)$*;*

(iii) $S + T$*;*

(iv) αS*, where* $\alpha > 1$.

The σ-algebra $\mathcal{F}_t$ can be thought of as representing all (theoretically) observable events up to and including time t. We would like to have an analogous notion of events that are observable before a random time.

Definition 6. *Let T be a stopping time. The stopping time σ-algebra $\mathcal{F}_T$ is defined to be*

$$\{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t, \ \text{all } t \geq 0\}.$$

The previous definition is not especially intuitive. However it does well represent 'knowledge' up to time T, as the next theorem illustrates.

Theorem 6. *Let T be a finite stopping time. Then $\mathcal{F}_T$ is the smallest σ-algebra containing all càdlàg adapted processes sampled at T. That is,*

$$\mathcal{F}_T = \sigma\{X_T; X \text{ all adapted càdlàg processes}\}.$$

Theorem 7. *Let X be adapted and càdlàg. If $\Delta X_T 1_{\{T<\infty\}} = 0$ a.s. for each stopping time T, then ΔX is indistinguishable from the zero process.*

2.2 Martingales

In this section we give, mostly without proofs, only the essential results from the theory of continuous time martingales.
Also, recall that we will always assume as given a filtered, complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$ is assumed to be right continuous.

Definition 7. *A real-valued, adapted process $X = (X_t)_{0 \leq t < \infty}$ is called a martingale (resp. supermartingale, submartingale) with respect to the filtration $\mathbb{F}$ if*

(i) $X_t \in L^1(dP)$*; that is,* $E\{|X_t|\} < \infty$*;*

(ii) if $s \le t$, then $E\{X_t|\mathcal{F}_s\} = X_s$, a.s. (resp. $E\{X_t|\mathcal{F}_s\} \le X_s$, resp. $\ge X_s$).

Note that martingales are only defined on $[0, \infty)$; that is, for finite t and not $t = \infty$. It is often possible to extend the definition to $t = \infty$.

Definition 8. *A martingale X is said to be closed by a random variable Y if $E\{|Y|\} < \infty$ and $X_t = E\{Y|\mathcal{F}_t\}$, $0 \le t < \infty$.*

Theorem 8. *Let X be a supermartingale. The function $t \mapsto E\{X_t\}$ is right continuous if and only if there exists a modification Y of X which is càdlàg. Such a modification is unique.*

Corollary 2. *If $X = (X_t)_{0 \le t < \infty}$ is a martingale then there exists a unique modification Y of X which is càdlàg.*

Definition 9. *A family of random variables $(U_\alpha)_{\alpha \in A}$ is uniformly integrable if*

$$\lim_{n \to \infty} \sup_{\alpha} \int_{\{|U_\alpha| \ge n\}} |U_\alpha| dP = 0.$$

Theorem 9. *Let $(U_\alpha)_{\alpha \in A}$ be a subset of L^1. The following are equivalent:*

(i) $(U_\alpha)_{\alpha \in A}$ is uniformly integrable.

(ii) $\sup_{\alpha \in A} E\{|U_\alpha|\} < \infty$, and for every $\epsilon > 0$ there exists $\delta > 0$ such that $\Lambda \in \mathcal{F}$, $P(\Lambda) \le \delta$, imply $E\{|U_\alpha 1_\Lambda|\} < \epsilon$.

(iii) There exists a positive, increasing, convex function $G(x)$ defined on $[0, \infty)$ such that $\lim_{x \to \infty} \frac{G(x)}{x} = +\infty$.

Theorem 10. *Let X be a right continuous martingale which is uniformly integrable. Then $Y = \lim_{t \to \infty} X_t$ a.s. exists, $E\{|Y|\} < \infty$, and Y closes X as a martingale.*

Theorem 11. *Let X be a (right continuous) martingale. Then $(X_t)_{t \ge 0}$ is closed if and only if $(X_t)_{t \ge 0}$ is uniformly integrable, and if and only if $Y = \lim_{t \to \infty} X_t$ exists a.s., $E\{|Y|\} < \infty$, and $(X_t)_{0 \le t \le \infty}$ is a martingale, where $X_\infty = Y$.*

Remark 2. *If X is a uniformly integrable martingale, then X_t converges to $X_\infty = Y$ in L^1 as well as almost surely.*

Definition 10. *Let X be a stochastic process and let T be a random time. X^T is said to be the process stopped at T if $X_t^T = X_{t \wedge T}$.*

Note that if X is adapted and càdlàg and if T is a stopping time, then

$$X_t^T = X_{t\wedge T} = X_t 1_{\{t<T\}} + X_T 1_{\{t\geq T\}}$$

is also adapted. A martingale stopped at a stopping time is still a martingale, as the next theorem shows; see its corollary.

Theorem 12. *Let X be a uniformly integrable right continuous martingale, and let T be a stopping time. Then $X^T = (X_{t\wedge T})_{0\leq t\leq\infty}$ is also a uniformly integrable right continuous martingale.*

Corollary 3. *Let M be a martingale, and T a finite valued stopping time. Then M^T, the martingale stopped at T, is still a martingale.*

Corollary 4. *Let Y be an integrable random variable and let S, T be stopping times. Then*

$$\begin{aligned} E\{E\{Y|\mathcal{F}_S\}|\mathcal{F}_T\} &= E\{E\{Y|\mathcal{F}_T\}|\mathcal{F}_S\} \\ &= E\{Y|\mathcal{F}_{S\wedge T}\}. \end{aligned}$$

The next inequality is elementary, but indispensable.

Theorem 13. *(Jensen's Inequality)*
Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex, and let X and $\phi(X)$ be integrable random variables. For any σ-algebra $\mathcal{G}$,

$$\phi \ \circ E\{X|\mathcal{G}\} \leq E\{\phi(X)|\mathcal{G}\}.$$

Corollary 5. *Let X be a martingale, and let ϕ be convex such that $\phi(X_t)$ is integrable, $0 \leq t < \infty$. Then $\phi(X)$ is a submartingale. In particular, if M is a martingale, then $|M|$ is a submartingale.*

Corollary 6. *Let X be a submartingale and let ϕ be convex, non-decreasing, and such that $\phi(X_t)_{0\leq t<\infty}$ is integrable. Then $\phi(X)$ is also a submartingale.*

Theorem 14. *Let X be a positive submartingale. For all $p > 1$, with q conjugate to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), we have*

$$||\sup_t X_t||_{L^p} \leq q \ \sup_t ||X_t||_{L^p}.$$

For a real valued process, we let X^* denote $\sup_s |X_s|$. Note that if M is a martingale with $M_\infty \in L^2$, then $|M|$ is a positive submartingale, and taking $p = 2$ we have

$$E\{(M^*)^2\} \leq 4E\{M_\infty^2\}.$$

This last inequality is called Doob's maximal quadratic inequality.

Definition 11. *An adapted counting process N is a Poisson process if*

(i) for any s, t, $0 \leq s < t < \infty$, $N_t - N_s$ is independent of $\mathcal{F}_s$;

(ii) for any s, t, u, v, $0 \leq s < t < \infty$, $0 \leq u < v < \infty$, $t - s = v - u$, then the distribution of $N_t - N_s$ is the same as that of $N_v - N_u$.

Properties (i) and (ii) are known respectively as increments independent of the past, and stationary increments.

Definition 12. *An adapted process $B = (B_t)_{0 \leq t < \infty}$ taking values in $\mathbb{R}^n$ is called an n-dimensional $\mathbb{F}$ Brownian motion if*

(i) for $0 \leq s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$ (increments are independent of the past);

(ii) for $0 < s < t$, $B_t - B_s$ is a Gaussian random variable with mean zero and variance matrix $(t - s)C$, for a given, non-random matrix C.

The Brownian motion starts at x if $P(B_0 = x) = 1$.

Definition 13. *An adapted process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is a Lévy process if*

(i) X has increments independent of the past; that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$; and

(ii) X has stationary increments; that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$; and

(iii) X_t is continuous in probability; that is, $\lim_{t \to s} X_t = X_s$, where the limit is taken in probability.

Definition 14. *A process H is said to be simple predictable if H has a representation*

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t)$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$. The collection of simple predictable processes is denoted by $\mathbb{S}$.

Note that we can take $T_1 = T_0 = 0$ in the above definition, so there is no 'gap' between T_0 and T_1. We can topologize $\mathbb{S}$ by uniform convergence in (t, ω), and we denote $\mathbb{S}$ endowed with this topology by $\mathbb{S}_u$. We also write $\mathbb{L}^0$ for the space of finite-valued random variables topologized by convergence in probability.
Let X be a stochastic process. An operator I_X, induced by X should have two fundamental properties to earn the name 'integral'. The operator I_X should be linear, and it should satisfy some version of the Bounded Convergence Theorem. A particularly weak form of the Bounded Convergence Theorem is that the uniform convergence of processes H^n to H implies only the convergence in probability of $I_X(H^n)$ to $I_X(H)$.
Inspired by the above considerations, for a given process X we define a linear mapping $I_X : S \to \mathbb{L}^0$ by letting

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i}),$$

where $H \in S$ has the representation

$$H_t = H_0 1_{\{0\}} + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}.$$

Since this definition is a path-by-path definition for the step functions $H(\omega)$, it does not depend on the choice of the representation of H in $\mathbb{S}$.

Definition 15. *A process X is a total semimartingale if X is càdlàg, adapted, and $I_X : \mathbb{S}_u \to \mathbb{L}^0$ is continuous.*

Recall that for a process X and a stopping time T, the notation X^T denotes the process $(X_{t \wedge T})_{t \geq 0}$.

Definition 16. *A process X is called a semimartingale if, for each $t \in [0, \infty)$, X^t is a total semimartingale.*

Definition 17. *An adapted, càdlàg process A is a finite variation process (FV) if almost surely the paths of A are of finite variation on each compact interval of $[0, \infty)$. We write $\int_0^\infty |dA_s|$ or $|A|_\infty$ for the random variable which is the total variation of the paths of A.*

Theorem 15. *Let X be a semimartingale. If X has a decomposition*

$$X_t = X_0 + M_t + A_t$$

with M a local martingale and A a predictably measurable FV process, $M_0 = A_0 = 0$, then such a decomposition is unique.

Theorem 16. *The set of (total) semimartingales is a vector space.*

Theorem 17. *If Q is a probability and absolutely continuous with respect to P, then every (total) P semimartingale X is a (total) Q semimartingale.*

Theorem 18. *Let $(P_k)_{k\geq 1}$ be sequence of probabilities such that X is a P_k semimartingale for each k. Let $R = \sum_{k=1}^{\infty} \lambda_k P_k$, where $\lambda_k \geq 0$, each k, and $\sum_{k=1}^{\infty} \lambda_k = 1$. Then X is a semimartingale under R as well.*

Theorem 19. *(Stricker's Theorem)*
Let X be a semimartingale for the filtration $\mathbb{F}$. Let $\mathbb{G}$ be a subfiltration of $\mathbb{F}$, such that X is adapted to the $\mathbb{G}$ filtration. Then X is a $\mathbb{G}$ semimartingale.

Theorem 20. *Each square integrable martingale with càdlàg paths is a semimartingale.*

Corollary 7. *Each càdlàg, locally square integrable local martingale is a semimartingale.*

Corollary 8. *A local martingale with continuous paths is a semimartingale.*

Corollary 9. *The Wiener process is a semimartingale.*

Definition 18. *An adapted, càdlàg process X is decomposable if there exist processes N, A such that*

$$X_t = X_0 + N_t + A_t$$

with $N_0 = A_0 = 0$, N a locally square integrable local martingale, and A an FV process.

Theorem 21. *A decomposable process is a semimartingale.*

Corollary 10. *A Lévy process is a semimartingale.*

Definition 19. *Let $A = [a, b]$ be an interval. The variation of paths of a process B is defined to be*

$$V_A(\omega) = \sup_{\pi \in \mathcal{P}} \sum_{t_i \in \pi} |B_{t_{i+1}} - B_{t_i}|$$

where $\mathcal{P}$ are all finite partitions of $[a, b]$.

Definition 20. *An adapted, càdlàg process Y is a classical semimartingale if there exist processes N, B with $N_0 = B_0 = 0$ such that*

$$Y_t = Y_0 + N_t + B_t$$

where N is a local martingale and B is an FV process.

Theorem 22. *Let X be an adapted, càdlàg process. The following are equivalent:*
(i) X is a semimartingale;
(ii) X is decomposable;
(iii) given $\beta > 0$, there exist M, A with $M_0 = A_0 = 0$, M a local martingale with jumps bounded by β, A an FV process, such that $X_t = X_0 + M_t + A_t$;
(iv) X is a classical semimartingale

Definition 21. *Let X be a semimartingale. If X has a decomposition*

$$X_t = X_0 + M_t + A_t$$

with $M_0 = A_0 = 0$, M a local martingale, A an FV process, and with A predictable, then X is said to be a special semimartingale.

Definition 22. *If X is a special semimartingale, then the unique decomposition*

$$X = M + A$$

with $M_0 = X_0$ and $A_0 = 0$ and A predictable is called the canonical decomposition.

Theorem 23. *If X is a special semimartingale, then its decomposition*

$$X = M + A$$

with A predictable is unique.

Definition 23. *Let X be a stochastic process. A property π is said to hold locally if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ increasing to ∞ a.s. such that $X^{T_n} 1_{\{T_n > 0\}}$ has property π each $n \geq 1$.*

Theorem 24. *Let X be a semimartingale. X is special if and only if the process*

$$X_t^* = \sup_{s \leq t} |X_s|$$

is locally integrable.

Definition 24. *An adapted, càdlàg process X is a quasimartingale on $[0, \infty]$ if $E[|X_t|] < \infty$, for each t, and if $Var[X] < \infty$.*

Theorem 25. *(Rao's Theorem)*
A quasimartingale X has a unique decomposition $X = M + A$, where M is a local martingale and A is a predictable process with paths of locally integrable variation and $A_0 = 0$.

2.3 The Quadratic Variation of a Semimartingale

The quadratic variation process of a semimartingale, also known as the bracket process, is a simple object that nevertheless plays a fundamental role.

Definition 25. *Let X, Y be semimartingales. The quadratic variation process of X, denoted $[X, X] = ([X, X]_t)_{t \geq 0}$, is defined by*

$$[X, X] = X^2 - 2 \int X_- dX$$

(recall that $X_{0-} = 0$). The quadratic covariation of X, Y, also called the bracket process of X, Y, is defined by

$$[X, Y] = XY - \int X_- dY - \int Y_- dX.$$

It is clear that the operation $(X, Y) \to [X, Y]$ is bilinear and symmetric. We therefore have a polarization identity

$$[X, Y] = \frac{1}{2}([X + Y, X + Y] - [X, X] - [Y, Y]).$$

The next theorem gives some elementary properties of $[X, X]$. (X is assumed to be a given semimartingale throughout this section).

Theorem 26. *The quadratic variation process of X is a càdlàg, increasing, adapted process. Moreover, it satisfies the following.*
(i) $[X, X]_0 = X_0^2$ and $\Delta[X, X] = (\Delta X)^2$.
(ii) If T is a stopping time, then

$$[X^T, X] = [X, X^T] = [X^T, X^T] = [X, X]^T.$$

Corollary 11. *The bracket process $[X,Y]$ of two semimartingales has paths of finite variation on compacts, and it is also a semimartingale.*

Corollary 12. *(Integration by parts)*
Let X, Y be two semimartingales. Then XY is a semimartingale and

$$XY = \int X_- dY + \int Y_- dX + [X,Y].$$

In the integration by parts formula above, we have $(X_-)_0 = (Y_-)_0 = 0$. Hence evaluating at 0 yields

$$X_0Y_0 = (X_-)_0Y_0 + (Y_-)_0X_0 + [X,Y]_0.$$

Since $[X,Y]_0 = \Delta X_0 \Delta Y_0 = X_0Y_0$, the formula is valid. Without the convention that $(X_-)_0 = 0$, we could have written the formula

$$X_tY_t = \int_{0^+}^t X_{s-}dY_s + \int_{0^+}^t Y_{s-}dX_s + [X,Y]_t.$$

2.4 Compensators

Let A be a process of locally integrable variation, hence a FV process. A is then locally a quasimartingale, and hence by Rao's Theorem, there exists a unique decomposition

$$A = M + \tilde{A}$$

where $\tilde{A}$ is a predictable FV process. In other words, there exists a unique, predictable FV process $\tilde{A}$ such that $A - \tilde{A}$ is a local martingale.

Definition 26. *Let A be an FV process with $A_0 = 0$, with locally integrable total variation. The unique FV predictable process $\tilde{A}$ such that $A - \tilde{A}$ is a local martingale is called the compensator of A.*

Theorem 27. *Let A be an increasing process of integrable variation, and let M be a bounded martingale. The compensator of $\int_0^t M_s dA_s$ is $\int_0^t M_{s-}dA_s$.*

Definition 27. *Let X be a semimartingale such that its quadratic variation process $[X,X]$ is locally integrable, that is, there exist stopping times T_n increasing to ∞ a.s. such that $E\{[X,X]_{T_n}\} < \infty$, each n. Then the conditional quadratic variation of X,* *denoted* $< X, X >= (< X, X >_t)_{t\geq 0}$, *exists and it is defined to be the compensator of* $[X, X]$. *That is* $< X, X >= \widetilde{[X, X]}$.

If X is a continuous semimartingale then $[X, X]$ is also continuous and hence already predictable; thus

$$[X, X] =< X, X >$$

when X is continuous. In particular for a standard Brownian motion B,

$$[B, B]_t =< B, B >_t = t,$$

all $t \geq 0$. The conditional quadratic variation is also known in the literature by its notation. It is sometimes called the sharp bracket, the angle bracket, or the oblique bracket. It has properties analogous to that of the quadratic variation processes. For example, if X and Y are two semimartingales such that $< X, X >$, $< Y, Y >$, and $< X + Y, X + Y >$ all exist, then $< X, Y >$ exists and can be defined by polarization

$$< X, Y >= \frac{1}{2}(< X + Y, X + Y > - < X, X > - < Y, Y >).$$

However, $< X, Y >$ can be defined independently as the compensator of $[X, Y]$ which provides of course that $[X, Y]$ is locally of integrable variation. In other words, there exist stopping times $(T^n)_{n\geq 1}$ increasing to ∞ a.s. such that $E\{\int_0^{T^n} |d[X, Y]_s|\} < \infty$ for each n. Also, $< X, X >$ is a non-decreasing process by the preceding discussion, since $[X, X]$ is non-decreasing. The conditional quadratic variation is inconvenient since unlike the quadratic variation it doesn't always exist. Moreover, while $[X, X]$, $[X, Y]$ and $[Y, Y]$ all remain invariant with a change to an equivalent probability measure, the sharp brackets in general change with a change to an equivalent probability measure and may even no longer exist. Although the angle bracket is ubiquitous in the literature it is sometimes unnecessary as one can often use the quadratic variation instead, and indeed whenever possible we use the quadratic variation rather than the conditional quadratic variation. $< X, X >$ occurs naturally in extensions of Girsanov's theorem for example, and it has become indispensable in many areas of advanced analysis in the theory of stochastic processes.

2.5 The Bichteler-Dellacherie Theorem

The next theorem shows us that a process X is a classical semimartingale if and only if it is a semimartingale.

Theorem 28. *(Bichteler-Dellacherie Theorem)*
An adapted, càdlàg process X is a semimartingale if and only if it is a classical semimartingale. That is, X is a semimartingale if and only if it can be written $X = M + A$, where M is a local martingale and A is an FV process.

Theorem 29. *Let X be an adapted, càdlàg process. The following are equivalent:*
(i) X is a semimartingale;
(ii) X is decomposable;
(iii) given $\beta > 0$, there exist M, A with $M_0 = A_0 = 0$, M a local martingale with jumps bounded by β, A an FV process, such that $X_t = X_0 + M_t + A_t$;
(iv) X is a classical semimartingale.

2.6 Local Martingales

Recall that for a process X and a stopping time T we further recall that X^T denotes the stopped process

$$X_t^T = X_{t\wedge T} = X_t 1_{\{t<T\}} + X_T 1_{\{t\geq T\}}.$$

Definition 28. *An adapted, càdlàg process X is a local martingale if there exists a sequence of increasing stopping times, T_n, with $\lim_{n\to\infty} T_n = \infty$ a.s. such that $X_{t\wedge T_n} 1_{\{T_n>0\}}$ is a uniformly integrable martingale for each n. Such a sequence (T_n) of stopping times is called a fundamental sequence.*

Definition 29. *A stopping time T reduces a process M if M^T is a uniformly integrable martingale.*

Theorem 30. *Let M, N be local martingales and let S and T be stopping times.*
(a) If T reduces M and $S \leq T$ a.s., then S reduces M.
(b) The sum $M + N$ is also a local martingale.
(c) If S, T both reduce M, then $S \vee T$ also reduces M.
(d) The processes M^T, $M^T 1_{\{T>0\}}$ are local martingales.
(e) Let X be a càdlàg process and let T_n be a sequence of stopping times increasing to ∞ a.s. such that $X^{T_n} 1_{\{T_n>0\}}$ is a local martingale for each n. Then X is a local martingale.

Theorem 31. *Let X be a process which is locally a square integrable martingale. Then X is a local martingale.*

Theorem 32. *Let M be adapted, càdlàg and let $(T_n)_{n\geq 1}$ be a sequence of stopping times increasing to ∞ a.s.. If $M^{T_n} 1_{\{T_n>0\}}$ is a martingale for each n, then M is a local martingale.*

It is often of interest to determine when a local martingale is actually a martingale. A simple condition involves the maximal function. Recall that $X_t^* = \sup_{s \le t} |X_s|$ and $X^* = \sup_s |X_s|$.

Theorem 33. *Let X be a local martingale such that $E\{X_t^*\} < \infty$ for every $t \ge 0$. Then X is a martingale. If $E\{X^*\} < \infty$, then X is a uniformly integrable martingale.*

Remark 3. *Note that in particular a bounded local martingale is a uniformly integrable martingale.*

Definition 30. *Two probability laws P, $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if $P << \mathcal{Q}$ and $\mathcal{Q} << P$. (Recall that $P << \mathcal{Q}$ denotes that P is absolutely continuous with respect to $\mathcal{Q}$.) We write $\mathcal{Q} \approx P$ to denote equivalence.*

If $\mathcal{Q} << P$, then there exists a random variable Z in $L^1(dP)$ such that $\frac{d\mathcal{Q}}{dP} = Z$ and $E_P\{Z\} = 1$, where E_P denotes expectation with respect to the law P. We let

$$Z_t = E_P\{\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t\}$$

be the right continuous version. Then Z is a uniformly integrable martingale and hence a semimartingale. Note that if $\mathcal{Q}$ is equivalent to P then $\frac{dP}{d\mathcal{Q}} \in L^1(d\mathcal{Q})$ and $\frac{dP}{d\mathcal{Q}} = (\frac{d\mathcal{Q}}{dP})^{-1}$.

Lemma 1. *Let $\mathcal{Q} \approx P$, and $Z_t = E_P\{\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t\}$. An adapted, càdlàg process M is a $\mathcal{Q}$ local martingale if and only if MZ is a P local martingale.*

Theorem 34. *(Girsanov-Meyer Theorem)*
Let P and $\mathcal{Q}$ be equivalent. Let X be a semimartingale under P with decomposition $X = M + A$. Then X is also a classical semimartingale under $\mathcal{Q}$ and has a decomposition $X = L + C$, where

$$L_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a $\mathcal{Q}$ local martingale, and $C = X - L$ is a $\mathcal{Q}$ FV process.

Theorem 35. *(Girsanov-Meyer Theorem: Predictable Version)*
Let X be a P local martingale with $X_0 = 0$. Let $\mathcal{Q}$ be another probability equivalent to P and let $Z_t = E\{\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t\}$. If $< X, Z >$ exists for the probability P, then the canonical $\mathcal{Q}$ decomposition of X is

$$X_t = (X_t - \int_0^t \frac{1}{Z_{s-}} d < X, Z >_s) + \int_0^t \frac{1}{Z_{s-}} d < X, Z >_s .$$

Theorem 36. *Let* X *be a* P *local martingale with* $X_0 = 0$*. Let* $\mathcal{Q}$ *be another probability absolutely continuous with respect to* P*, and let* $Z_t = E\{\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t\}$*. Assume that* $< X, Z >$ *exists for* P*. Then* $A_t = \int_0^t \frac{1}{Z_{s-}} d < X, Z >_s$ *exists a.s. for the probability* $\mathcal{Q}$*, and* $X_t - \int_0^t \frac{1}{Z_{s-}} d < X, Z >_s$ *is a* $\mathcal{Q}$ *local martingale.*

Theorem 37. *(Lenglart-Girsanov Theorem)*
Let X *be a* P *local martingale with* $X_0 = 0$*. Let* $\mathcal{Q}$ *be a probability absolutely continuous with respect to* P*, and let* $Z_t = E_P\{\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t\}$*,* $R = \inf\{t > 0 : Z_t = 0, Z_{t-} > 0\}$*, and* $U_t = \Delta X_R 1_{\{t \geq R\}}$*. Then*

$$X_t - \int_0^t \frac{1}{Z_s} d[X, Z]_s + \tilde{U}_t$$

is a $\mathcal{Q}$ *local martingale.*

3 Financial Market Model

To describe a financial market operating in continuous time, we begin with a probability space $(\Omega, \mathcal{F}, P)$, a time horizon $T \in (0, \infty)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. Additionally, we assume that $\mathcal{F}_0$ is trivial and we set $\mathcal{F}_T = \mathcal{F}$. We have $d+1$ basic *primary* assets available for trade with price processes $S^i = (S^i_t)_{0 \leq t \leq T}$ for $i = 0, 1, \ldots, d$. To simplify the notation, we assume that S^0 has a strictly positive price. Then we use S^0 as a numéraire and describe all quantities in this paper discounted with S^0. This way asset 0 has (discounted) price 1 at all times and the other assets (discounted) prices are $X^i = \frac{S^i}{S^0}$ for $i = 1, \ldots, d$. Furthermore, we assume that time t prices X_t are $\mathcal{F}_t$-measurable, i.e. observable at time t and that X is a càdlàg semimartingale.

In this setting a contingent claim is an $\mathcal{F}_T$-measurable random variable H describing a random payoff to be made at time T of some financial instrument. This way our claims are of European type because the maturity is fixed.

As far as hedging is now concerned, the following question arises: Having sold H, how can one insure against the random loss at time T?

To approach this question we consider self-financing portfolio strategies $(V_0, \vartheta) = (V_0, \vartheta_t)_{0 \leq t \leq T}$, where V_0 is an $\mathcal{F}_0$-measurable random variable that can be interpreted as the initial outlay to start the strategy and ϑ is a d-dimensional predictable process. In such a strategy, ϑ^i_t describes the number of units of asset i held at time t and predictability of ϑ is a mathematical formulation of the informational constraint that ϑ is not allowed to anticipate the movement of X.

Throughout this work we refer to the theory of stochastic integration established in Protter (2005).

At any time t, the cumulative gains from trade up to time t are given by

$$G_t(\vartheta) = \int_0^t \vartheta_s dX_s$$

and the value process V is given by

$$V_t = V_0 + \int_0^t \vartheta_s dX_s = V_0 + G_t(\vartheta). \tag{1}$$

The strategy being self-financing means that the cost process which is given by

$$C_t = V_t - \int_0^t \vartheta_s dX_s = V_t - G_t(\vartheta)$$

is constant so that it is not allowed to put external money into the portfolio.
Since X is a semimartingale it makes sense to speak of stochastic integrals with respect to X and we denote by $L(X)$ the linear space of all $\mathbb{R}^d$-valued predictable X-integrable processes ϑ.
A claim H is called attainable if there exists a self-financing strategy with $V_T = H$ P-a.s. By equation 1 this means that H can be written as

$$H = H_0 + \int_0^T \vartheta_s^H dX_s \; P - \text{f.s.},$$

i.e. as the sum of a constant H_0 and a stochastic integral with respect to X. We speak of a complete market if every claim is attainable; otherwise the market is called incomplete. We define

$$\mathcal{M}_q^e(P) = \left\{ \mathcal{Q} \approx P : \frac{d\mathcal{Q}}{dP} \in L^q(P), \; X \text{ is a local } \mathcal{Q} - \text{martingale} \right\}$$

and

$$\mathcal{M}_q(P) = \left\{ \mathcal{Q} << P : \frac{d\mathcal{Q}}{dP} \in L^q(P), \; X \text{ is a local } \mathcal{Q} - \text{martingale} \right\}$$

with $1 \leq q \leq \infty$ and we assume that $\mathcal{M}_q(P) \neq \emptyset$. Note that since we do not assume that $\mathcal{M}_q^e(P) \neq \emptyset$ we do not necessarily have an arbitrage-free market.

Since we want to examine different spaces of stochastic integrals and see if they are closed or not, we will vary the subspaces $\Theta \subseteq L(X)$ of integrands as well as the conditions on our semimartingale X that describes the basic stock prices.
First we need to establish some notation that will be used throughout this paper. Suppose we have chosen a linear subspace Θ of $L(X)$. Then a Θ-strategy is a pair $(V_0, \vartheta) \in \mathbb{R} \times \Theta$ with value process $V_0 + \mathcal{G}(\vartheta)$ and the linear space

$$\mathcal{G} := G_T(\Theta) = \left\{ \int_0^T \vartheta_u dX_u | \vartheta \in \Theta \right\}$$

describes all outcomes of self-financing Θ-strategies with initial wealth $V_0 = 0$ and

$$\mathcal{A} := \mathbb{R} + \mathcal{G} = \left\{ V_0 + \int_0^T \vartheta_u dX_u | (V_0, \vartheta) \in \mathbb{R} \times \Theta \right\}$$

is the space of contingent claims replicable by self-financing Θ-strategies.
Throughout the paper, $p \in [1, \infty]$ will be arbitrary (but fixed) and q will denote the conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. On $L^p(P)$ we shall consider the norm topology, if $1 \leq p < \infty$ and the weak-star topology $\sigma(L^\infty, L^1)$ if $p = \infty$.

4 Examination of the Closedness of Spaces of Stochastic Integrals in L^2

To be able to find suitable hedging strategies for the sale of an option in an incomplete financial market, we will try to find the orthogonal projection of the claim onto different spaces of terminal portfolio values. Therefore we will analyze the closedness of those subspaces of the Hilbert space L^2 in this section.

For trading strategies we will consider the sets Θ_p^{eq}, Θ_v and Θ_S, i.e. we will examine the L^2-closedness of the spaces $G_T(\Theta_2^{eq})$, $G_T(\Theta_v)$ and $G_T(\Theta_S)$. We will give explicit information on when and by whom the respective spaces have been introduced and what this work contributes to mathematical research in each subsection, but we want to give a brief overview in advance. The notion of $G_T(\Theta_v)$ has first been mentioned by Xia/Yan (2006) and $G_T(\Theta_S)$ was first described and worked on by Delbaen/Monat/Schachermayer/Schweizer/Stricker (1991). $G_T(\Theta_2^{eq})$ has only been described implicitly by Theorem 1.2 in Delbaen/Schachermayer (1996). This work firstly provides a summary of the existing results on the different spaces in a uniform setting and gives an explicit definition of the space $G_T(\Theta_2^{eq})$ and provides a proof for the closedness in L^2. In addition, we develop sufficient conditions under which $G_T(\Theta_2^{eq})$ and $G_T(\Theta_v)$ coincide, so that we can transfer all the results from one space to another and vice versa. Lastly, we give a counterexample for the closedness of $G_T(\Theta_S)$ in the discontinuous case.

4.1 The Space $G_T(\Theta_p^{eq})$

The first space that we will take a look at is $G_T(\Theta_p^{eq})$, which is a very intuitive construction when we take a look at the space D_p in Delbaen/Schachermayer (1996). It is based on the idea of taking linear combinations of stochastic integrals with so-called simple integrands and then taking a look at the corresponding L^2-closure. We will give an explicit definition of the elements of the space (Definition 32) and with the help of Theorem 1.2 in Delbaen/Schachermayer (1996) we will show that it is closed in L^2 (Corollary 13). Furthermore, we state an identity between $G_T(\Theta_p^{eq})$ and another space in the case that X is continuous (Theorem 39).

We consider X to be an $\mathbb{R}^d$-valued càdlàg semimartingale based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ which is locally in the following sense: There exists a sequence $(U_s)_{s=1}^{\infty}$ of localizing stopping times increasing to T such that, for each $s \in \mathbb{N}$, the family

$$\{X_U : U \text{ stopping time, } U \leq U_s\}$$

is bounded in $L^p(P)$ (and uniformly integrable in the case $p = 1$).

A predictable $\mathbb{R}^d$-valued process will be called a simple p-admissible integrand for X if it has a representation

$$H_t = H_0 \chi_{\{0\}}(t) + \sum_{i=1}^{n} H_i \chi_{]]T_i, T_{i+1}]]}(t)$$

where $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$ is a finite sequence of stopping times dominated by some corresponding U_s and where H_i is in $L^\infty(\Omega, \mathcal{F}_{T_i}, P)$. The space of all those processes is denoted by Θ_p^s.

Using the concepts of Protter (2005) for those processes we then may form the stochastic integral

$$\int_0^t H dX = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{t \wedge T_{i+1}} - X_{t \wedge T_i})$$

and the random variable

$$\int_0^T H dX = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i}).$$

For every summand the interpretation here is that H_i defines the trading strategy of buying at time T_i the amount of $H_i(\omega) = (H_i^1(\omega), \ldots, H_i^d(\omega))$ units of the (discounted) stocks $X = (X^1, \ldots, X^d)$ and selling it at time T_{i+1}. At time $t \in [0, \infty]$ the random variable $(\int_0^t H dX)(\omega)$ describes the cumulated gain (or loss) up to time t and $(\int_0^T H dX)(\omega)$ the final result, if an agent follows the trading strategy H. We call the linear space consisting of the above random variables $\int_0^T H dX$ the space of claims attainable by simple integrands or by simple trading strategies for obvious reasons.

Since $G_T(\Theta_p^s)$ will not necessarily be closed, we are led to the following concept:

Definition 31. *In the above setting let*

$$K_p = \overline{G_T(\Theta_p^s)} \text{ and}$$
$$D_p = \overline{G_T(\Theta_p^s) - L_+^p} \cap \overline{G_T(\Theta_p^s) + L_+^p},$$

where the bar denotes the closure with respect to the norm topology of $L^p(P)$, for $1 \leq p < \infty$, and with respect to the σ^*-topology, for $p = \infty$.

The interpretation of the elements in K_p is obvious: a random variable $f \in L^p$ is in K_p if it can be approximated by random variables $G_T(\Theta_p^s)$. The space D_p (which clearly contains K_p) has a more intriguing interpretation: a random variable $f \in L^p$ is in D_p if it may be approximately "sandwiched" between elements of $G_T(\Theta_p^s)$, i.e. if there are simple p-admissible strategies H^+ and H^- such that $(\int_0^T H^+ dX) - f)_-$ as well as $(\int_0^T H^- dX) - f)_+$ both are small with respect to the topology on $L^p(P)$. An economic agent will wish to approximate f by either $\int_0^T H^+ dX$ or $\int_0^T H^- dX$ depending on whether she wants to buy or sell the contingent claim modelled by the random variable f.

Before we state the next theorem we need a lemma that will be needed for the proof of the theorem.

Lemma 2. *If a stochastic process X_n converges to X in L^p, p an integer, then $E[X_n^p]$ converges to $E[X^p]$.*

Proof 1. *Note that by Hölder's inequality, for each $1 \leq j \leq p$, we have*

$$\lim_{n \to +\infty} E[|X_n - X|^j |X|^{p-j}] = 0.$$

Now we can conclude the result using the expansion

$$X_n^p = (X_n - X + X)^p = \sum_{j=0}^{p} \binom{p}{j} (X_n - X)^j X^{p-j} = X^p + \sum_{j=1}^{p} \binom{p}{j} (X_n - X)^j X^{p-j}.$$

The following theorem is a partial statement of Theorem 1.2 in Delbaen/Schachermayer (1996). Since we focus on measures $\mathcal{Q} \in \mathcal{M}_q(P)$, we do not state all their equivalences and therefore give a slightly different proof.

Theorem 38. *Let $1 \leq p \leq \infty$, q its conjugate exponent, X a semimartingale locally in $L^p(P)$ such that $\mathcal{M}_q^e(P) \neq \emptyset$, and $f \in L^p(P)$. The following assertions are equivalent:*
(i) $f \in D_p$.

(ii) There is an X-integrable predictable process H such that, for each $\mathcal{Q} \in \mathcal{M}_q(P)$, the process $\int_0^t HdX$ is a uniformly integrable $\mathcal{Q}$-martingale converging to f in the norm of $L^1(\mathcal{Q})$.
(iii) $E_{\mathcal{Q}}[f] = 0$ for each $\mathcal{Q} \in \mathcal{M}_q(P)$.

Proof 2. *(ii) $\Rightarrow$ (iii) is obvious*
(iii) $\Rightarrow$ (i)
If $f \notin \overline{G_T(\Theta_p^s) - L_+^p(P)}$, then the Hahn-Banach theorem provides us with an element g of $L_+^q(P)$ vanishing on $G_T(\Theta_p^s)$ - so that, after normalization, it is the density of a non-negative probability measure $\mathcal{R}$ in $\mathcal{M}_q(P)$ - and such that $E_P[fg] = E_{\mathcal{R}}[f] > 0$.
The case $f \notin \overline{G_T(\Theta_p^s) + L_+^p(P)}$ is similar.
Since we need the equivalence from (i) and (iii) for the proof of (i) $\Rightarrow$ (ii) we also show (i) $\Rightarrow$ (iii)
For $\mathcal{Q} \in \mathcal{M}_q(P)$, we have that $\mathcal{Q}$ takes values ≤ 0 on $\overline{G_T(\Theta_p^s) - L_+^p(P)}$:
For $u \in G_T(\Theta_p^s)$ and $v \in L_+^p(P)$, we have

$$E[u - v] = E[u] - E[v]$$

and since $E[u]$ is 0 and $E[v]$ is ≥ 0, we can conclude with lemma 2 that $\mathcal{Q}$ takes values ≤ 0 on $\overline{G_T(\Theta_p^s) - L_+^p(P)}$. That $\mathcal{Q}$ takes values ≥ 0 on $\overline{G_T(\Theta_p^s) + L_+^p(P)}$ can be shown in a similar way. Hence $\mathcal{Q}$ vanishes on D_p.
(i) $\Rightarrow$ (ii)
Fix $f \in D_p$ and $\mathcal{Q} \in \mathcal{M}_q^e(P)$; first note that the identity mapping considered as an operator from $L^p(P)$ to $L^1(\mathcal{Q})$ is well defined and continuous. Hence we have that,

$$f \in \overline{G_T(\Theta_p^s) + L_+^1(\mathcal{Q})}^{L^1(\mathcal{Q})}$$

which means that there is a sequence $(f_n)_{n=1}^{\infty} = (\int_0^T H_n dX)_{n=1}^{\infty}$ in $G_T(\Theta_p^s)$ such that

$$\lim_{n\to\infty} E_{\mathcal{Q}}[f - f_n]_+ = 0.$$

But from the martingale property and (iii) - which we have already proven to being equivalent to (i) - we get for each $n \in \mathbb{N}$

$$E_{\mathcal{Q}}[f_n] = E_{\mathcal{Q}}[f] = 0,$$

which implies that

$$\lim_{n\to\infty} E_{\mathcal{Q}}[|f - f_n|] = 0,$$

i.e. f *is in the* $L^1(\mathcal{Q})$*-closure of* $G_T(\Theta_p^s)$.

The rest of the proof follows an argument of C. Stricker (1990): We may identify f *- as well as each* f_n *- with a uniformly integrable martingale* $(f_t)_{t\in\mathbb{R}_+}$ *by letting*

$$f_t = E_{\mathcal{Q}}[f|\mathcal{F}_t].$$

We are now in a position to apply the theorem of M. Yor (1978, cor. 2.5.2) to exhibit a predictable integrand H *with the desired properties w.r.t.* $\mathcal{Q}$.

We still have to show that H *also has the desired properties w.r.t. each* $\mathcal{R} \in \mathcal{M}_q(P)$*. We have to show that* $\int_0^t HdX = E_{\mathcal{R}}[f|\mathcal{F}_t]$ *for each* $t \in \mathbb{R}_+$*, which will readily show that the* $\mathcal{R}$*-almost surely defined stochastic integral* $\int HdX$ *is indeed a* $\mathcal{R}$*-uniformly integrable martingale.*

As $\mathcal{R} \in \mathcal{M}_q(P)$ *we have that each* $\int H_n dX$ *is an* $\mathcal{R}$*-uniformly integrable martingale so that* $\int_0^t H_n dX = E[\int_0^T H_n dX|\mathcal{F}_t]$*. By the same argument as above,* $(\int_0^T H_n dX)_{n=1}^\infty$ *converges to* f *in* $L^1(\mathcal{R})$ *and therefore* $(\int_0^t H_n dX)_{n=1}^\infty$ *is a Cauchy sequence in* $L^1(\mathcal{R})$ *that converges to* $\int_0^t HdX$ *in* $L^1(\mathcal{Q})$ *and therefore also to* $\int_0^t HdX$ *in* $L^1(\mathcal{R})$*. This shows that* $\int HdX$ *is indeed a* $\mathcal{R}$*-uniformly integrable martingale converging to* f *in* $L^1(\mathcal{R})$*, thus finishing the proof of the implication (i)* $\Rightarrow$ *(ii).*

Definition 32. Θ_p^{eq} *consists of all* $\vartheta \in L(X)$ *such that* $G_T(\vartheta)$ *is in* $L^p(P)$ *and the process* $G(\vartheta) = \int \vartheta dX$ *is a uniformly* $\mathcal{Q}$*-integrable* $\mathcal{Q}$*-martingale for every* $\mathcal{Q} \in \mathcal{M}_q^e(P)$.

Corollary 13. *Let* X *be a* $\mathbb{R}^d$*-valued semimartingale which is locally in* $L^2(P)$ *and* $\mathcal{M}_2^e(P) \neq \emptyset$*. Then the space* $G_T(\Theta_2^{eq})$ *is closed in* $L^2(P)$.

Proof 3. *This is due to Theorem 38 and follows from the equivalence of (i) and (ii). Note that the equivalence also holds true if we assume* $\mathcal{Q} \in \mathcal{M}_q^e(P)$ *(see Delbaen/Schachermayer (1996)).*

Remark 4. *As one might have wondered that we do not explicitly require a no-arbitrage condition in our market model, we will see in section 5.2 that the condition* $\mathcal{M}_2^e(P) \neq \emptyset$ *can be interpreted as one.*

Remark 5. *Due to Theorem 38 we also have the following, more general assertion:*
Let $1 \leq p \leq \infty$*,* q *its conjugate exponent,* X *a* $\mathbb{R}^d$*-valued semimartingale which is locally in* $L^p(P)$ *and* $\mathcal{M}_q^e(P) \neq \emptyset$*. Then the space* $G_T(\Theta_p^{eq})$ *is closed in* $L^p(P)$.

But since we want to find hedging strategies in Hilbert spaces, we focus on the case of $p = 2$.

Of course, the question arises, whether the concept of 'sandwichable' contingent claims is vacuous in the sense that we always have $\overline{G_T(\Theta_p^s)} = G_T(\Theta_p^{eq})$. The next example which has been established in Delbaen/Schachermayer (1996) (for the case $p = 2$), shows that this is not always the case, i.e. there are situations where $G_T(\Theta_p^{eq})$ strictly contains $\overline{G_T(\Theta_p^s)}$. But after that, we supply a sufficient condition for X so that we always have $\overline{G_T(\Theta_p^s)} = G_T(\Theta_p^{eq})$.

Example 1. *We construct a uniformly bounded discrete adapted stochastic process* $X = (X_t)_{t=0}^{\infty}$ *defined on* $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^{\infty}, P)$ *with the following properties.*
(i) There exists an equivalent martingale measure $\mathcal{Q}$ *for* X *with density function* $\frac{d\mathcal{Q}}{d\mathcal{P}} \in L^{\infty}(\Omega, \mathcal{F}, P)$.
(ii) $\overline{G_T(\Theta_2^s)}$ *is strictly contained in* D_2.
We work on $\Omega = \mathbb{N}$. *Denote, for* $t \in \mathbb{N}_0$,

$$\begin{aligned}
A_1^t &= \{3t+1\}\,, \\
A_2^t &= \{3t+2\}\,, \\
A_3^t &= \{3t+3\}\,, \\
B^t &= \{3t+1, 3t+2, \ldots\} = \bigcup_{s \geq t} (A_1^s \cup A_2^s \cup A_3^s), \\
C^t &= \{3t+2, 3t+3, \ldots\} = B^t \setminus A_1^t.
\end{aligned}$$

In order to keep track of the right order of magnitude of the sequences constructed below we shall use the following notation: For sequences $(a_t)_{t=0}^{\infty}$ *and* $(b_t)_{t=0}^{\infty}$ *of positive numbers we write* $a_t \approx b_t$ *if there are constants* $c, C > 0$ *such that* $ca_t < b_t < Ca_t$ *for all* t *sufficiently big.*
$\mathcal{F}$ *will denote the sigma-algebra of all subsets of* Ω *and we shall define measures* P *and* $\mathcal{Q}$ *on* $\mathcal{F}$*: let (formally)* $P(A_1^{-1}) = -1$ *and define recursively, for* $t \geq 0$,

$$\begin{aligned}
P(A_1^t) &= 2^{-1}(1 - 2^{-(2t-1)})P(A_1^{t-1}) = 2^{-(t+1)} \prod_{s=1}^{t} (1 - 2^{-(2s-1)}) \approx 2^{-t}, \\
P(A_2^t) = P(A_3^t) &= 2^{-(2t+2)} P(A_1^t) = 2^{-(3t+3)} \prod_{s=1}^{t} (1 - 2^{-(2s-1)}) \approx 2^{-3t},
\end{aligned}$$

and (formally) $\mathcal{Q}(A_1^{-1}) = 4$ *and, for* $t \geq 0$,

$$\begin{aligned}
\mathcal{Q}(A_1^t) &= 2^{-3}\mathcal{Q}(A_1^{t-1}) = 2^{-(3t+1)} \approx 2^{-3t}, \\
\mathcal{Q}(A_2^t) &= 2^{-(t+2)}\mathcal{Q}(A_1^t) = 2^{-(4t+3)} \approx 2^{-4t}, \\
\mathcal{Q}(A_3^t) &= (1 - 2^{-2} - 2^{-(t+2)})\mathcal{Q}(A_1^t) = (1 - 2^{-2} - 2^{-(t+2)})2^{-(3t+1)} \approx 2^{-3t}.
\end{aligned}$$

Let us try to explain the idea behind this definition: we start with letting $P(A_1^0) = \mathcal{Q}(A_1^0) = 2^{-1}$ *so that* $P(C^0) = \mathcal{Q}(C^0) = 2^{-1}$. *For each* $t \in \mathbb{N}_0$ *the set* C^t *is broken into*

$$C^t = A_2^t \cup A_3^t \cup A_1^{t+1} \cup C^{t+1}.$$

The mass of C^t *is divided among these 4 sets such that*

$$P(A_1^{t+1}) = P(C^{t+1}) \text{ and } \mathcal{Q}(A_1^{t+1}) = \mathcal{Q}(C^{t+1}). \tag{2}$$

In the case of P *the mass of* C^t *is distributed among the 4 sets above with the weights* $\left\{2^{-(2t+2)}, 2^{-(2t+2)}, 2^{-1}(1-2^{-(2t+1)}), 2^{-1}(1-2^{-(2t+1)})\right\}$ *and in the case of* $\mathcal{Q}$ *with the weights* $\left\{2^{-(t+2)}, (1-2^{-2}-2^{-(t+2)}), 2^{-3}, 2^{-3}\right\}$.
Clearly the measures P *and* $\mathcal{Q}$ *are equivalent and* $\frac{d\mathcal{Q}}{dP}$ *is uniformly bounded.*
Now we define a sequence $(f_t)_{t=0}^{\infty}$ *of functions on* Ω *by*

$$f_t = \begin{cases} 1, & \text{on } A_1^t, \\ -1, & \text{on } C^t, \\ 0, & \text{elsewhere.} \end{cases}$$

In view of 2 we have

$$E_P[f_t] = E_{\mathcal{Q}}[f_t] = 0. \tag{3}$$

Let, for $t \in \mathbb{N}_0$,

$$a_t = 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) \approx 2^t,$$

and define the function f *on* Ω *by*

$$f = \sum_{t=0}^{\infty} a_t f_t.$$

As for each $\omega \in \Omega$ *the values* $f_t(\omega)$ *are eventually zero, the above sum converges every-*

where on Ω. It is elementary to calculate explicitly the values of f:

$$f = \begin{cases} c_t & \textit{on } A_1^t, \\ -b_t & \textit{on } A_2^t \cup A_3^t, \end{cases}$$

where

$$b_t = \sum_{s=0}^{t} a_s = \sum_{s=0}^{t} (2^s \prod_{r=s+1}^{\infty} (1 - 2^{-(2r-1)})) \approx 2^t$$

and, more precisely, $\lim_{t\to\infty} \frac{b^t}{2^t} = 2$,

$$c_t = a_t - b_{t-1} = 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) - \sum_{s=0}^{t-1} 2^s \prod_{r=s+1}^{\infty} (1 - 2^{-(2r-1)}) \approx 1.$$

Note that $f \in L^2(P)$ and that, for all $t \in \mathbb{N}_0$,

$$(f_t, f)_P = 0, \tag{4}$$

where $(\cdot,\cdot)_P$ denotes the inner product in $L^2(P)$. Indeed, letting F_n denote the n-th partial sum of f

$$F_n = \sum_{t=0}^{n} a_t f_t$$

and noting the biorthogonality of $(f_t)_{t=0}^{\infty}$, we have, for $n \geq t$,

$$\begin{aligned} (f_t, F_n)_P &= a_t (f_t, f_t)_P \\ &= a_t 2P(A_1^t) \\ &= 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) 2 (2^{-(t+1)} \prod_{s=1}^{t} (1 - 2^{-(2s-1)})) \\ &= \prod_{s=1}^{\infty} (1 - 2^{-(2s-1)}). \end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{n\to\infty}(f_t, f-F_n)_P &= \lim_{n\to\infty} E_P[(F_n - f)\chi_{C^n}] \\
&= \lim_{n\to\infty}\Big(\sum_{m=n+1}^{\infty} E_P[F_n - f]\chi_{A_1^m} + \sum_{m=n}^{\infty} E_P[F_n - f]\chi_{A_2^m\cup A_3^m}\Big) \\
&= \lim_{n\to\infty}\Big(\sum_{m=n+1}^{\infty}(-b_n - c_m)P(A_1^m) + \sum_{m=n}^{\infty}(-b_n + b_m)P(A_2^m\cup A_3^m)\Big) \\
&= -\lim_{n\to\infty} b_n \sum_{m=n+1}^{\infty} P(A_1^m) \\
&= -\lim_{n\to\infty} 2^{n+1}\sum_{m=n+1}^{\infty}\Big(2^{-(m+1)}\prod_{s=1}^{m}(1-2^{-(2s-1)})\Big) \\
&= -\prod_{s=1}^{\infty}(1-2^{-(2s-1)}).
\end{aligned}$$

Combining these two equalities we obtain 4.
For later use we observe that

$$E_P(f\chi_{B^t}) = \sum_{s=t}^{\infty} c_t P[A_1^t] - \sum_{s=t}^{\infty} b_t P[A_2^t\cup A_3^t] \approx 2^{-t}. \tag{5}$$

Also note that

$$\begin{aligned}
\lim_{n\to\infty} ||(F_n - f)_+||^2_{L^2(P)} &= \lim_{n\to\infty}\sum_{m=n+1}^{\infty}(b_m - b_n)^2 P(A_2^m\cup A_3^m) \\
&= \lim_{n\to\infty}\sum_{m=n+1}^{\infty} 2^{2m}\cdot 2^{-3m} = 0.
\end{aligned} \tag{6}$$

Now define, for $t\in\mathbb{N}_0$,

$$g_t = \begin{cases} M_t & \text{on } A_2^t, \\ -m_t & \text{on } A_3^t, \\ 1 & \text{on } B^{t+1}, \end{cases}$$

where the real numbers M_t and m_t will be chosen such that the relations

$$(f, g_t)_P = 0 \text{ and } E_Q[g_t] = 0 \tag{7}$$

hold true. Clearly these equations are satisfied iff M_t and m_t solve the two linear equa-

tions

$$M_t \cdot P(A_2^t)(-b_t) + m_t \cdot P(A_3^t)b_t = -E_P[f\chi_{B^{t+1}}]$$
$$M_t \cdot \mathcal{Q}(A_2^t) - m_t \cdot \mathcal{Q}(A_3^t) = -\mathcal{Q}(B^{t+1}).$$

We can rearrange these equations to get

$$M_t - m_t = \frac{E_P(f\chi_{B^{t+1}})}{P(A_2^t)b_t} \approx 2^t$$
$$M_t \cdot \left(-\frac{\mathcal{Q}(A_2^t)}{\mathcal{Q}(A_3^t)}\right) + m_t = \frac{\mathcal{Q}(B^{t+1})}{\mathcal{Q}(A_3^t)} \approx 1$$

which yields, in view of $\frac{\mathcal{Q}(A_2^t)}{\mathcal{Q}(A_3^t)} \approx 2^{-t}$,

$$M_t \approx 2^t \text{ and } m_t \approx 1.$$

Now we are ready to define the process X*:*
Let $X_0 = 0$ *and, for* $t \geq 0$,

$$X_{2t+1} - X_{2t} = 2^{-t} f_t,$$
$$X_{2t+2} - X_{2t+1} = 2^{-2t} g_t.$$

Clearly, $(X_t)_{t=0}^{\infty}$ *is a uniformly bounded process and it follows from 3 and 7 that* X *is a* $\mathcal{Q}$*-martingale with respect to its natural filtration* $(\mathcal{F}_t)_{t=0}^{\infty}$*. Note that each* F_n *is a simple integral on the process* X*, hence* $F_n \in G_T(\Theta_2^s)$ *and we obtain from 6 that*

$$f \in \overline{G_T(\Theta_2^s) - L_+^2(P)}. \tag{8}$$

On the other hand, we claim that for

$$G_n = F_n + b_n g_n$$

we have that

$$\lim_{n\to\infty} ||(G_n - f)_-||_{L^2(P)} = 0 \tag{9}$$

which will readily imply that

$$f \in \overline{G_T(\Theta_2^s) + L_+^2(P)}, \tag{10}$$

and therefore, combining 8 and 10,

$$f \in D_2.$$

To prove 9, note that

$$\begin{aligned}||(G_n - f)_-||^2_{L^2(P)} = ||(G_n - f) - \chi_{A_3^n}||^2_{L^2(P)} &\approx (2b_n)^2 P(A_3^t) \\ &\approx 2^{2n} 2^{-3n}.\end{aligned}$$

Finally, we shall show that f is orthogonal to $G_T(\Theta_2^s)$, whence in particular

$$f \in D_2 \setminus K_2. \tag{11}$$

Indeed, as for each $t \geq 0$ the support of $X_{t+1} - X_t$ is contained in an atom of $\mathcal{F}_t$, the space $G_T(\Theta_2^s)$ of simple integrals consists of the linear span of $(f_t)_{t=0}^{\infty}$ and $(g_t)_{t=0}^{\infty}$ and therefore we obtain the assertion from 4 and 7. The construction of the example is now completed. □

So we have seen that $\overline{G_T(\Theta_p^s)} = G_T(\Theta_p^{eq})$ does not hold in general. But the next theorem shows that we have $\overline{G_T(\Theta_p^s)} = G_T(\Theta_p^{eq})$ for the case of continuous processes. The content of the theorem has been established in Theorem 2.2 of Delbaen/Schachermayer (1996), but here we will adjust it to our setting and apply it to our newly constructed space $G_T(\Theta_p^{eq})$.

Theorem 39. *Suppose that $X = (X_t)_{t \in \mathbb{R}_+}$ is a continuous $\mathbb{R}^d$-valued semimartingale which is locally in L^p, let $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that $\mathcal{M}_q^e(P) \neq \emptyset$. Then we have $\overline{G_T(\Theta_p^s)} = G_T(\Theta_p^{eq})$.*

Proof 4. *Suppose first that $1 \leq p < \infty$. Let $f \in G_T(\Theta_p^{eq})$; by theorem 38 there is an X-integrable predictable process H such that $\int_0^t HdX$ is a uniformly integrable martingale with respect to each $\mathcal{Q} \in \mathcal{M}_q^e(P)$ and such that almost surely*

$$f = \lim_{t \to \infty} \int_0^t HdX.$$

The definition of $G_T(\Theta_p^{eq})$ respectively D_p implies the existence of sequences $(H^{+,n})_{n=1}^{\infty}$ and $(H^{-,n})_{n=1}^{\infty}$ of simple p-admissible integrands such that, for

$$f^{+,n} = \int_0^T H^{+,n} dX \text{ and } f^{-,n} = \int_0^T H^{-,n} dX$$

we have that $((f - f^{+,n})_+)_{n=1}^\infty$ *and* $((f - f^{-,n})_-)_{n=1}^\infty$ *tend to zero in* $L^p(P)$. *By the argument used in the proof of theorem 38 we deduce that, in fact,* $(f - f^{+,n})_{n=1}^\infty$ *and* $(f-f^{+,n})_{n=1}^\infty$ *tend to zero in* $L^1(\mathcal{Q})$*, for each* $\mathcal{Q} \in \mathcal{M}_q^e(P)$. *It follows from the martingale property that, more generally, for each* $\mathcal{Q} \in \mathcal{M}_q^e(P)$ *and each stopping time* S*, we have*

$$\lim_{n\to\infty} || \int_0^S (H - H^{+,n}) dX ||_{L^1(\mathcal{Q})} = 0 \text{ and } \lim_{n\to\infty} || \int_0^S (H - H^{-,n}) dX ||_{L^1(\mathcal{Q})} = 0. \qquad (12)$$

We have to show that, for $\epsilon > 0$*, there is a simple p-admissible integrand* H^ϵ *such that*

$$||f - \int_0^T H^\epsilon dX||_{L^p(P)} < \epsilon.$$

For $C \in \mathbb{R}_+$ *let*

$$S_C = \inf \left\{ t : | \int_0^t H dX | \geq C \right\},$$

so that

$$\lim_{C\to\infty} P(\{S_C < \infty\}) = 0.$$

Let $\delta > 0$ *to be specified below and find* $C = C(\delta) > 1$ *such that*

$$P(\{S_C < \infty\}) < \delta.$$

Let

$$h = \int_0^{S_C} H dX$$

and define, for $n \in \mathbb{N}$

$$h^{+,n} = \int_0^{S_C} H^{+,n} dX \text{ and } h^{-,n} = \int_0^{S_C} H^{-,n} dX.$$

We deduce from 12 that $(h^{+,n})_{n=1}^\infty$ *and* $(h^{-,n})_{n=1}^\infty$*, converge to* h *in* $L^1(\mathcal{Q})$ *and therefore in measure. For* $\eta > 0$*, again to be specified below, we therefore have that, for* $n = n(\eta)$ *sufficiently big,*

$$P(\{|h - h^{+,n}| > 1\}) < \eta \text{ and } P(\{|h - h^{-,n}| > 1\}) < \eta.$$

Fix n *such that the above inequalities hold true and define the* $\mathcal{F}_{S_C}$*-measurable sets* A^+

and A^- *by*

$$A^+ = \{S_C < \infty\} \cap \{h = -C\} \cap \left\{|h - h^{+,n}| \leq 1\right\} \text{ and}$$
$$A^- = \{S_C < \infty\} \cap \{h = +C\} \cap \left\{|h - h^{-,n}| \leq 1\right\},$$

so that A^+ *and* A^- *are disjoint subsets of* $\{S_C < \infty\}$ *covering this set up to a set* $B = \{S_C < \infty\} \setminus (A^+ \cup A^-)$ *of* P*-measure at most* $P(B) < 2\eta$.
Define the predictable integrand $\tilde{H}$

$$\tilde{H} = H\chi_{]0,S_C]} + H^{+,n}\chi_{]S_C,\infty[}\chi_{A^+} + H^{-,n}\chi_{]S_C,\infty[}\chi_{A^-},$$

and define the stopping time S *as the first moment after* S_C *when* $\int_0^t \tilde{H}dX = 0$. *We want to show that*

$$||f - \tilde{f}||_{L^p(P)} < \epsilon \tag{13}$$

if $\delta = \delta(\epsilon) > 0$ *and* $\eta = \eta(C(\delta), \epsilon) > 0$ *are sufficiently small, where*

$$\tilde{f} = \int_0^S \tilde{H}dX.$$

As

$$||f - \tilde{f}||_{L^p(P)} \leq ||(f - \tilde{f})_{\chi_{A^+}}||_{L^p(P)} + ||(f - \tilde{f})_{\chi_{A^-}}||_{L^p(P)} + ||(f - \tilde{f})_{\chi_B}||_{L^p(P)}, \tag{14}$$

it will suffice to show that each of the three terms on the right hand side is less than $\frac{\epsilon}{3}$.
As regards the last one note that

$$||f_{\chi_B}||_{L^p(P)} < \frac{\epsilon}{6} \text{ and } ||\tilde{f}_{\chi_B}||_{L^p(P)} \leq C(2\eta)^{\frac{1}{p}} < \frac{\epsilon}{6}$$

if $\eta = \eta(C(\delta), \epsilon)$ *is small enough.*
As regards the first two terms in 14 we only estimate the first one (the second being analogous): we split the set A^+ *into* $A^+ \cap \{S < \infty\}$ *and* $A^+ \cap \{S = \infty\}$. *For the former set we may estimate*

$$||(f - \tilde{f})\chi_{A^+\cap\{S<\infty\}}||_{L^p(P)} = ||f\chi_{A^+\cap\{S<\infty\}}||_{L^p(P)} \leq ||f\chi_{A^+}||_{L^p(P)},$$

which is smaller than $\frac{\epsilon}{6}$ *if* $\delta = \delta(\epsilon) > 0$ *is sufficiently small as* $f \in L^p(P)$.

For the second set we may estimate

$$\begin{aligned}
&||(f-\tilde{f})\chi_{A^+\cap\{S=\infty\}}||_{L^p(P)}\\
&\leq ||(1+|f-f^{+,n}|)\chi_{A^+\cap\{S=\infty\}}||_{L^p(P)}\\
&\leq ||(1+(f-f^{+,n})_+)\chi_{A^+\cap\{S=\infty\}}||_{L_p(P)} + ||(1+(f-f^{+,n})_-)\chi_{A^+\cap\{S=\infty\}}||_{L^p(P)}\\
&\leq ||(1+(f-f^{+,n})_+)\chi_{A^+\cap\{S=\infty\}}||_{L_p(P)} + ||(2+f_-)\chi_{A^+\cap\{S=\infty\}}||_{L^p(P)}.
\end{aligned}$$

In the last line we have used the fact $f^{+,n}$ is less than or equal to 1 on $\{T=\infty\}$. If we choose $\delta=\delta(\epsilon)>0$ small enough and $n=n(\epsilon,\eta)$ big enough the above expression is smaller than $\frac{\epsilon}{6}$.

Summing up, we have shown 13: given $\epsilon>0$ choose $\delta=\delta(\epsilon)>0$, then $C=C(\delta)>0$, $\eta=\eta(C,\epsilon)>0$ and finally $n=n(\epsilon,\eta)\in\mathbb{N}_0$. However, we are not yet finished, as $\tilde{H}$ is a simple p-admissible integrand only after the stopping time S_C. But it is standard to approximate $\int_0^{S_C} HdX$ by the stochastic integral of a simple p-admissible integrand $\hat{H}$ supported by $[0,S_C]$ such that $|\int_0^t \hat{H}dX|$ is bounded by C and

$$||\int_0^{S_C}\hat{H}dX-\int_0^{S_C}HdX||_{L^p(P)}<\epsilon.$$

Modifying $\tilde{H}$ on $[0,S_C]$ in the indicated way we obtain the desired simple integrand H^ϵ for which

$$||f-\int_0^T H^\epsilon dX||_{L^p(P)}<2\epsilon,$$

thus finishing the proof.

The case $p=\infty$ is easy: simply note that, for $\mathcal{Q}\in\mathcal{M}_1^e(P)$, $(L^\infty(P),\sigma(L^\infty(P),L^1(P)))$, may be identified with $(L^\infty(\mathcal{Q}),\sigma(L^\infty(\mathcal{Q}),L^1(\mathcal{Q})))$. □

4.2 The Space $G_T(\Theta_v)$

The next set of trading strategies which we will take a profound look at is Θ_v which was first mentioned in Xia/Yan (2006). They used signed martingale measures to establish the L^2-closedness of the space. We will expand their ideas in the sense that we will develop the identity $G_T(\Theta_2^{eq}) = G_T(\Theta_v)$ under certain requirements in our Corollary 14. Therefore we get an explicit description of the elements in $G_T(\Theta_v)$ as well. The reader may notice that this is crucial for finding hedging strategies later on, since all (existing) results on the strategies for the respective space can be used for the other space and vice versa. We want to clearly indicate that Lemma 3, Lemma 4 and Theorem 41 were established in Xia/Yan (2006). Our contribution to this section is using their results to provide an assertion about relating $G_T(\Theta_v)$ to our previously constructed space $G_T(\Theta_p^{eq})$ in Corollary 14 and Corollary 15 which will be crucial for finding explicit hedging strategies later on.

For a process X we establish a duality relation between K_p, the $L^p(P)$-closure of the space of claims in $L^p(P)$, which are attainable by 'simple' strategies, and $\mathcal{M}^{q,s}$, all signed martingale measures $\mathcal{Q}$ with $\frac{d\mathcal{Q}}{dP} \in L^q(P)$, where $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

First of all we want to make the following hypothesis (under which X is not necessarily a semimartingale):

Hypothesis 1. *X is locally in $L^p(P)$.*

As in the previous chapter we consider $G_T(\Theta_p^s)$, i.e. the subspace of $L^p(P)$ spanned by the simple stochastic integrals of the form

$$f = h(X_{T_2} - X_{T_1})$$

where $T_1 \leq T_2$ are stopping times. We define

$$K_p = \overline{G_T(\Theta_p^s)},$$

where the bar denotes the closure in $L^p(P)$.

It is clear that a probability measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ makes X a local martingale under $\mathcal{Q}$ if $\frac{d\mathcal{Q}}{dP} \in L^q(P)$ and

$$E_{\mathcal{Q}}[f] = E[\frac{d\mathcal{Q}}{dP} f] = 0$$

for all $f \in G_T(\Theta_p^s)$. We set

$$\mathcal{M}^{q,s} := \left\{g \in L^q(P) : E[gf] = 0 \text{ for all } f \in G_T(\Theta_p^s) \text{ and } E[g] = 1\right\},$$
$$\mathcal{M}^{q,e} := \{g \in \mathcal{M}^{q,s} : g > 0 \text{ a.s.}\}.$$

Any element in $\mathcal{M}^{q,s}$ (resp. $\mathcal{M}^{q,e}$) is called a signed (resp. equivalent) martingale measure for X.

Before we can prove our first lemma we need the following corollary from the Hahn-Banach theorem:

Theorem 40. *If X is a normed space, U a subspace and if $x \in X$ is not contained in the closure of U, then there exists a linear functional f with norm 1 that vanishes on U and for which we have $f(x) = ||x||$.*

Lemma 3. *Under Hypothesis 1, we have*
(a) $\mathcal{M}^{q,s} \neq \emptyset \Leftrightarrow 1 \notin K_p$;
(b) For $g \in L^q(P)$, $g \in \mathcal{M}^{q,s} \Leftrightarrow E[g] = 1$ and $E[gf] = 0$ for all $f \in K_p$.

Proof 5. *(a) First of all we define the space*

$$\hat{K} = \overline{span(G(\Theta_p^s), 1)}.$$

Now according to Theorem 40 the linear functional φ on $\hat{K}$ that satisfies $\varphi|_{K_p} = 0$ and $\varphi(1) = 1$ is well defined and continuous on the closed space $\hat{K}$ iff $1 \notin K_p$.
(b) This is an immediate consequence of the very definition of $\mathcal{M}^{q,s}$.

□

Lemma 3 (b) and the following lemma give a duality relation between K_p and $\mathcal{M}^{q,s}$.

Lemma 4. *Under Hypothesis 1 and that $\mathcal{M}^{q,s} \neq \emptyset$, we have*

$$f \in K_p \Leftrightarrow f \in L^p(P) \text{ and } E[fg] = 0 \text{ for all } g \in \mathcal{M}^{q,s}. \tag{15}$$

To prove this lemma we first need the following proposition:

Proposition 1. *Let φ be a continuous linear functional on $L^p(P)$. Then there exists a $g \in L^q(P)$ such that*

$$\varphi(k) = E[gk]$$

for all $k \in L^p(P)$.

Now we have all the tools to prove the lemma.

Proof 6. *(proof of Lemma 4)*
"⇒" is clear. It is left to show "⇐". Assume $f \in L^p(P)$ and $E[fg] = 0$ for all $g \in \mathcal{M}^{q,s}$. We want to show $f \in K_p$.
Therefore firstly we will show

$$1 \notin \overline{span(G_T(\Theta_p^s), f)}.$$

Actually, we suppose that $1 \in \overline{span(G_T(\Theta_p^s), f)}$. Then there exist a sequence $(f^n, n \geq 1) \subset G_T(\Theta_p^s)$ and a sequence $(\delta^n, n \geq 1) \subset \mathbb{R}$ such that $f^n + \delta^n f \to 1$ in $L^p(P)$. Since we have $\mathcal{M}^{q,s} \neq \emptyset$, for any $g \in \mathcal{M}^{q,s}$, we have

$$0 = E[(f^n + \delta^n f)g] \to E[g] = 1,$$

a contradiction. Because of that, $1 \in \overline{span(G_T(\Theta_p^s), f)}$ is impossible and we have

$$1 \notin \overline{span(G_T(\Theta_p^s), f)}.$$

By Theorem 40, there exists a continuous linear functional φ_1 on $L^p(P)$ such that $\varphi_1(1) \neq 0$ and $\varphi_1(k) = 0$ for all $k \in \overline{span(G_T(\Theta_p^s), f)}$.
Now we want to show that f must be in

$$\overline{span(G_T(\Theta_p^s), 1)}.$$

Therefore we assume again that $f \notin \overline{span(G_T(\Theta_p^s), 1)}$.
Then we also have by the Hahn-Banach Theorem, that there exists a continuous linear functional φ_2 on $L^p(P)$ such that $\varphi_2(f) \neq 0$ and $\varphi_2(k) = 0$ for all $k \in \overline{span(G_T(\Theta_p^s), 1)}$.
Let

$$\varphi := \varphi_1 + \varphi_2,$$

then φ is a continuous linear functional on $L^p(P)$ and hence according to the previous proposition there exists a $g \in L^q(P)$ such that

$$\varphi(k) = E[gk]$$

for all $k \in L^p(P)$. *It is clear that*

$$E[gk] = \varphi(k) = 0$$

for all $k \in K_p$, *and*

$$E[g] = \varphi_1(1) \neq 0,$$

and

$$E[gf] = \varphi_2(f) \neq 0.$$

So after a normalization, there exists a $g \in \mathcal{M}^{q,s}$ *such that*

$$E[gf] \neq 0.$$

This is a contradiction to the assumption on the right hand side of 15. So $f \notin \overline{span(G_T(\Theta_p^s), 1)}$ *is impossible and we have* $f \in \overline{span(G_T(\Theta_p^s), 1)}$.
Now we are able to prove "$\Leftarrow$*":*
Since $f \in \overline{span(G_T(\Theta_p^s), 1)}$, *there exists a sequence* $(k^n) \subset K_p^s$ *and a sequence* $(\delta^n) \subset \mathbb{R}$ *such that*

$$k^n + \delta^n \to f$$

in $L^p(P)$. *For any* $g \in \mathcal{M}^{q,s}$, *we have*

$$\delta^n = E[(k^n + \delta^n)g] \to E[fg] = 0.$$

Here the last equality obviously follows from the assumption on the right-hand side of 15. Since we know that δ^n *converges to* 0 *we can conclude that*

$$k^n \to f$$

in $L^p(P)$, *which implies*

$$f \in K_p.$$

□

Throughout the rest of this section, we always assume X to be a semimartingale satisfying Hypothesis 1. Denoted by Θ_v all $\mathbb{R}^m$-valued predictable X-integrable processes ϑ such that

$$G_T(\vartheta) := \int_0^T \vartheta dX \in L^p(P)$$

and

$$E[G_T(\vartheta)g] = 0$$

for all $g \in \mathcal{M}^{q,s}$. As usual we have

$$G_T(\Theta_v) := \{G_T(\vartheta) : \vartheta \in \Theta_v\}.$$

If we assume $\mathcal{M}^{q,s} \neq \emptyset$, then by Lemma 4, we have

$$G_T(\Theta_v) \subset K_p. \tag{16}$$

Furthermore, we make another assumption that

Hypothesis 2.

$$\mathcal{M}^{q,e} \neq \emptyset,$$

and get the following theorem:

Theorem 41. *Under Hypotheses 1 and 2 we have*

$$K_p = G_T(\Theta_v).$$

In order to proof this theorem we will use the following proposition which is a result of Corollary 2.2 in Yor (1978).

Proposition 2. *Let M be a local martingale under P. Then*

$$\left\{\int_0^T HdM : H \text{ is predictable and } M\text{-integrable and } \int HdM \text{is uniformly integrable}\right\}$$

is closed in $L^1(\Omega, \mathcal{F}, P; \mathbb{R})$.

Now we are able to prove the theorem.

Proof 7. *(proof of Theorem 41)*
By 16, we only need to show that

$$K_p \subset G_T(\Theta_v).$$

For any $w \in G_T(\Theta_p^s)$, we have

$$w = \sum_{i=1}^{n} h_i(X_{T_{2i}} - X_{T_{1i}})$$

where $T_{1i} \leq T_{2i}$ *are stopping times such that, for* $i = 1, \ldots, n$, *the family*

$$\{X_U : U \text{ stopping time }, \ U \leq T_{2i}\} \subset L^p(P)$$

and $h_i \in L^\infty(P)$.
Let

$$w_t = \sum_{i=1}^n h_i(X_{T_{2i}\wedge t} - X_{T_{1i}\wedge t}) = \int_0^t \vartheta dX,$$

where

$$\vartheta = \sum_{i=1}^n h_i 1_{]]T_{1i},T_{2i}]]}.$$

Clearly

$$w_t = G_t(\vartheta)$$

is a uniformly integrable $\mathcal{Q}$*-martingale for each* $\mathcal{Q} \in \mathcal{M}^{q,e}$.
Now we assume $f \in K_p$. *Then there exists a sequence* $(w^j)_{j\geq 1} \subset G_T(\Theta_p^s)$ *converging to* f *in* $L^p(P)$, *whence by the Hölder inequality, in* $L^1(\mathcal{Q})$ *for any* $\mathcal{Q} \in \mathcal{M}^{q,e}$. *The desired* $\vartheta \in \Theta_v$ *can be obtained with the argument in the previous proposition.*

□

With Theorem 39 and 41 we get the following corollary that gives an intriguing connection between $G_T(\Theta_v)$ and $G_T(\Theta_p^{eq})$.

Corollary 14. *Suppose that* $X = (X_t)_{t\in\mathbb{R}_+}$ *is a continuous* $\mathbb{R}^d$*-valued semimartingale which is locally in* L^p, *let* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and suppose that* $\mathcal{M}_q^e(P) \neq \emptyset$ *and* $\mathcal{M}^{q,e} \neq \emptyset$.
Then we have $G_T(\Theta_v) = G_T(\Theta_p^{eq})$.

From Corollary 14 and Theorem 39 we can easily derive the next assertion:

Corollary 15. *Suppose that* $X = (X_t)_{t\in\mathbb{R}_+}$ *is a continuous* $\mathbb{R}^d$*-valued semimartingale which is locally in* L^p, *let* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and suppose that* $\mathcal{M}_q^e(P) \neq \emptyset$ *and* $\mathcal{M}^{q,e} \neq \emptyset$. *Then we have the following:*
(i) Let $G_T(\vartheta) \in G_T(\Theta_p^{eq})$. *Then* ϑ *is an* $\mathbb{R}^m$*-valued predictable* X*-integrable process such that*

$$G_T(\vartheta) = \int_0^T \vartheta dX \in L^p(P)$$

and

$$E[G_T(\vartheta)g] = 0$$

for all $g \in \mathcal{M}^{q,s}$.

(ii) Let $G_T(\vartheta) \in G_T(\Theta_v)$. *Then we have* $\vartheta \in L(X)$ *such that* $G_T(\vartheta)$ *is in* $L^p(P)$ *and the process* $G(\vartheta) = \int \vartheta dX$ *is a uniformly* $\mathcal{Q}$*-integrable* $\mathcal{Q}$*-martingale for every* $\mathcal{Q} \in \mathcal{M}_q^e(P)$.

4.3 The Space $G_T(\Theta)$

The space Θ has been introduced in Delbaen/Monat/Schachermayer/Schweizer/Stricker (1991) where the reader also gets a first intuition why the space $G_T(\Theta)$ is not necessarily closed in the case where X is not continuous by their Example 3.9. Further, the space has been considered with the concepts of $\mathcal{E}$-martingales by Czichowsky/Schweizer (2012) who found requirements for X under which $G_T(\Theta)$ is closed even if X is not continuous.

The main contribution of our work in this section is establishing a connection between the closedness of $G_T(\Theta)$ and the so-called variance-optimal martingale measure in Corollary 16 based on Theorems 2.17, 2.18 and 3.7 in Delbaen/Monat/Schachermayer/Schweizer/Stricker (1991), to state a counterexample for the discontinuous case based on Example 3.9 also in Delbaen/Monat/Schachermayer/Schweizer/Stricker (1991) at the very end of this section and to expand their results to the discontinuous case in Theorem 47 with the help of Theorem 5.2 in Choulli/Krawczyk/Stricker (1998). The content of all other results in this section is just a reproduction of those in Delbaen/Monat/Schachermayer/Schweizer/Stricker (1991).

We denote by $\mathcal{M}^2_{0,loc}$ the space of all locally square integrable local martingales null at 0. Let $X = (X_t)_{0\leq t\leq T}$ be an $\mathbb{R}^d$-valued semimartingale in $\mathcal{S}^2_{loc}$. This means that if

$$X = X_0 + M + A$$

is the canonical decomposition of X, then $M \in \mathcal{M}^2_{0,loc}$ and the variation $|A^i|$ of the predictable finite variation process of X^i is locally square integrable for each $i = 1, \ldots, d$.

Definition 33. *X satisfies the structure condition (SC) if there exists a predictable $\mathbb{R}^d$-valued process $\lambda = (\lambda_t)_{0\leq t\leq T}$ such that*

$$dA_t = d\langle M\rangle_t \lambda_t \ P\text{-}a.s. \quad for \quad all \quad t \in [0,T],$$

and

$$K_t := \int_0^t \lambda_s' d\langle M\rangle_s \lambda_s < +\infty \ P\text{-}a.s. \quad for \quad all \quad t \in [0,T],$$

where $'$ denotes the transposition.

Definition 34. *A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0\leq t\leq T}$ belongs to $L^2(M)$ if*

$$E[\int_0^T \theta_t' d\langle M\rangle_t \theta_t] < +\infty.$$

We define on the space $L^2(M)$ the norm $||\cdot||_{L^2(M)}$ by

$$||\theta||^2_{L^2(M)} := ||(\theta \cdot M)_T||^2_{\mathcal{L}^2(P)} = E[\int_0^T \theta'_t d\langle M\rangle_t \theta_t].$$

Definition 35. *A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0\le t\le T}$ belongs to $L^2(A)$ if the process*

$$(\int_0^t |\theta'_s dA_s|)_{0\le t\le T}$$

is square integrable. We define on the space $L^2(A)$ the norm $||\cdot||_{L^2(A)}$ by

$$||\theta||_{L^2(A)} := ||\int_0^T |\theta'_s dA_s|||_{\mathcal{L}^2(P)}.$$

Finally, Θ is the space defined by $\Theta := L^2(M) \cap L^2(A)$; $\theta \in \Theta$ is called an $\mathcal{L}^2$-strategy. If the structure condition holds, then clearly

$$||\theta||^2_{L^2(A)} = E[(\int_0^T |\theta'_s d\langle M\rangle_s \lambda_s|)^2].$$

Definition 36. *We say that X admits an equivalent local martingale measure if there exists a probability Q equivalent to P such that X is a local martingale under Q.*

Definition 37. *The space $\mathcal{R}^2(P)$ is the space of all càdlàg adapted processes H such that*

$$||H||_{\mathcal{R}^2(P)} := ||\sup_{0\le t\le T} |H_t|||_{\mathcal{L}^2(P)} =: ||H^*_T||_{\mathcal{L}^2(P)}$$

is finite.

Definition 38. *We say that M has the predictable representation property under P, denoted by PRP(P), if each martingale N relative to $(\mathcal{F}_t)_{0\le t\le T}$ and P can be written*

$$N = N_0 + \theta \cdot M$$

where N_0 is $\mathcal{F}_0$-measurable and θ is M-integrable.

Definition 39. *Let $Y = (Y_t)_{0\le t\le T}$ be a uniformly integrable martingale. Then Y belongs to BMO if there is a constant $C > 0$ such that*

$$E[|Y_T - Y_{S-}|^2|\mathcal{F}_S] \le C \ P\text{-a.s.}$$

for every stopping time S.

Definition 40. *Let $Y = (Y_t)_{0\leq t\leq T}$ be a locally square integrable, local martingale. Then Y belongs to bmo_2 if there is a constant $C > 0$ such that*

$$E[\langle Y\rangle_T - \langle Y\rangle_S|\mathcal{F}_S] \leq C \ P\text{-a.s.}$$

for every stopping time S.

Definition 41. *We say that X satisfies the inequality $D_2(P)$ if there is a constant $C > 0$ such that*

$$||\theta||_{L^2(A)} \leq C||\theta||_{L^2(M)} \ , \ \forall\theta \in \Theta.$$

Definition 42. *If L is a uniformly integrable martingale such that $L_0 = 1$ and $L_T > 0$ P-a.s., then we say that L satisfies the reverse Hölder inequality under P, denoted by $R_p(P)$, where $1 < p \leq +\infty$, if and only if there is a constant C such that for every t, we have*

$$E[(\frac{L_T}{L_t})^p|\mathcal{F}_t] \leq C.$$

For $p = +\infty$, we require that $\frac{L_T}{L_t}$ is bounded by C.

A condition dual to $R_p(P)$ is the inequality $A_q(P)$.

Definition 43. *If L is a uniformly integrable martingale such that $L_0 = 1$ and $L_T > 0$ P-a.s., we say that L satisfies the Muckenhoupt inequality denoted by $A_q(P)$ for some $1 \leq q < +\infty$, if and only if there is a constant C such that for every t*

$$E[(\frac{L_t}{L_T})^{(\frac{1}{q-1})}|\mathcal{F}_t] \leq C.$$

If $q = 1$, we require that $\frac{L_t}{L_T}$ is bounded by C.

Definition 44. *Let Z be a positive process. Z satisfies condition (J) if there exists a constant $C > 0$ such that*

$$\frac{1}{C}Z_- \leq Z \leq CZ_-.$$

If Y is a semimartingale, $Y_0 = 0$, then its stochastic exponential, denoted by $\mathcal{E}(Y)$, is the semimartingale

$$\mathcal{E}(Y)_t := \exp(Y_t - \frac{1}{2}\langle Y^c\rangle_t)\Pi_{0<s\leq t}(1 + \Delta Y_s)e^{-\Delta Y_s}.$$

If Z is a semimartingale such that

$$\inf_{0 \leq s \leq T} Z_s > 0$$

(for instance if Z is a strictly positive local martingale), then its stochastic logarithm, denoted by $\mathcal{L}(Z)$, is the semimartingale

$$\mathcal{L}(Z) := \frac{1}{Z_-} \cdot Z.$$

Now let $\mathcal{Q}$ be an equivalent probability measure and define

$$Z_t := E_P[\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t] \text{ and } \hat{Z}_t = E_{\mathcal{Q}}[\frac{dP}{d\mathcal{Q}}|\mathcal{F}_t] = \frac{1}{Z_t}.$$

From Bayes‘ rule

$$E_{\mathcal{Q}}[f|\mathcal{F}_t]Z_t = E_P[fZ_T|\mathcal{F}_t]$$

it easily follows that Z satisfies $R_p(P)$ if and only if $\hat{Z}$ satisfies $A_q(\mathcal{Q})$ where of course $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq +\infty$.

The following theorem relates BMO and $R_p(P)$.

Theorem 42. *The following assertions are equivalent for a strictly positive martingale Z, $Z_0 = 1$:*
(1) $\mathcal{L}(Z)$ is in $BMO(P)$ and there exists a constant $h > 0$ such that $1 + \Delta\mathcal{L}(Z) \geq h$.
(2) $\mathcal{L}(\hat{Z})$ is in $BMO(\mathcal{Q})$ and there exists a constant $h > 0$ such that $1 + \Delta\mathcal{L}(\hat{Z}) \geq h$.
(3) Z satisfies condition (J) and $R_p(P)$ for some $p > 1$.
(4) $\hat{Z}$ satisfies condition (J) and $A_q(\mathcal{Q})$ for some $q < +\infty$.
In addition, (3) is satisfied for $1 < p < \infty$ iff (4) is satisfied for $q = \frac{p}{p-1}$.

The next theorem states that the set of exponents p such that Z satisfies $R_p(P)$ is necessarily open.

Theorem 43. *Assume Z is a strictly positive martingale with $Z_0 = 1$. If Z satisfies condition (J) and $R_p(P)$ ($p > 1$), then there is $p' > p$ such that Z satisfies $R_{p'}(P)$.*

A basic property that will be needed later on, is that if Z satisfies $R_p(P)$ then the conditional expectation with respect to $\mathcal{Q}$ is a continuous operator on $L^q(P)$. More precisely, we have the subsequent result:

Theorem 44. *Assume Z is a strictly positive martingale with $Z_0 = 1$. For $1 < p < +\infty$, assertions (1) and (2) below are equivalent*

(1) Z satisfies $R_p(P)$.
(2) There is a constant C *such that for each* $\mathcal{Q}$*-martingale* N, *and for* $q = \frac{p}{p-1}$ *and* $\lambda > 0$

$$\lambda^q P[N_T^* > \lambda] \leq C E_P[|N_T|^q].$$

Moreover, under the additional assumption that Z satisfies condition (J), the weak inequality in (2) implies the following strong inequality
(3) There is a constant K *such that for each* $\mathcal{Q}$*-martingale* N, *and for* $q = \frac{p}{p-1}$

$$E_P[(N_T^*)^q] \leq K E_P[|N_T|^q].$$

Definition 45. *The symbol* $\mathcal{V}$ *denotes a vector space of bounded continuous adapted processes. If* $Y \in \mathcal{V}$, *we suppose that* $Y_0 = 0$. *We require* $\mathcal{V}$ *to be stable for stopping, i.e. if* S *is a stopping time and if* Y *is in* $\mathcal{V}$, *then* $Y^s \in \mathcal{V}$. *For each stopping time* S, *we denote by* $\mathcal{V}_S$ *the vector space* $\{Y_S | Y \in \mathcal{V}\}$. *The space* $S^{\mathcal{V}}$ *is the space* $\{Y_T - Y_S | Y \in \mathcal{V}\}$.

We remark that $\mathcal{V}$ denotes a vector space of adapted processes while $\mathcal{V}_S$ and $S^{\mathcal{V}}$ denote spaces of ($\mathcal{F}_S$ -resp. $\mathcal{F}_T$-measurable) random variables. Since $\mathcal{V}$ is stable for stopping, we have for every stopping time S and every set $A \in \mathcal{F}_S$ that $1_A S^{\mathcal{V}} \subset S^{\mathcal{V}} \subset \mathcal{V}_T$. Clearly $\mathcal{V}_0 = \{0\}$. The set $\mathbb{M}(\mathcal{V})$ denotes the set of all probability measures $\mathcal{Q}$ that are absolutely continuous with respect to P and for which the elements $Y \in \mathcal{V}$ become $\mathcal{Q}$-martingales. The symbol $\mathbb{M}^e(\mathcal{V})$ is reserved for the elements of $\mathbb{M}(\mathcal{V})$ that are equivalent to P.
We shall simply write $\mathbb{M}^e$ and $\mathbb{M}$ instead of $\mathbb{M}^e(\mathcal{V})$ and $\mathbb{M}(\mathcal{V})$ if there is no risk of confusion.
It is easily seen that if $\mathcal{Q}$ is absolutely continuous with respect to P and if L denotes the càdlàg martingale

$$L_t = E_P[\frac{d\mathcal{Q}}{dP} | \mathcal{F}_t],$$

then $\mathcal{Q} \in \mathbb{M}(\mathcal{V})$ if and only if for every $Y \in \mathcal{V}$, the process YL is a martingale or, what is the same because $\mathcal{V}$ is stable for stopping, $E[L_T Y_T] = 0$. More generally, we define $\mathbb{M}^s$ as the affine space of measures μ absolutely continuous with respect to P such that $\mu(\Omega) = 1$ and

$$E_P[Y_T \frac{d\mu}{dP}] = 0$$

for all $Y \in \mathcal{V}$. If we denote by L the càdlàg martingale

$$L_t = E_P[\frac{d\mu}{dP} | \mathcal{F}_t],$$

then this is equivalent to the property that $E[L_t] = 1$ and LY is a martingale for each $Y \in \mathcal{V}$.

Without further notice, we will identify an absolutely continuous measure μ with its Radon-Nikodym derivative $\frac{d\mu}{dP}$. In this setting, $\mathbb{M}$ and $\mathbb{M}^s$ are closed sets of $\mathcal{L}^1(P)$ and if $\mathbb{M}^e$ is non empty, then it is $\mathcal{L}^1(P)$-dense in $\mathbb{M}$.

An important role will be played by the element of $\mathbb{M}^s \bigcap \mathcal{L}^2$ that has minimal $\mathcal{L}^2(P)$-norm, which we call the variance optimal measure and which we denote by $\mathcal{Q}^{opt}$.

This measure was previously studied by Schweizer (1996) as well as by Delbaen/Schachermayer (1996). It is shown that $\mathbb{M}^s \cap \mathcal{L}^2(P)$ is non empty if and only if the constant function 1 is not in the $\mathcal{L}^2$-closure of $\mathcal{V}_T$. If we adopt the convention that a bar denotes the closure in $\mathcal{L}^2(P)$, then $\mathbb{M}^s \cap \mathcal{L}^2(P)$ is non empty if and only if $1 \notin \overline{\mathcal{V}}_T$. In this case, there is an element μ in $\mathbb{M}^s \cap \mathcal{L}^2(P)$ with minimal norm and it is given by

$$\frac{d\mu}{dP} = \frac{1-f}{1-E[f]},$$

where f is the orthogonal projection of 1 onto the closed subspace $\overline{\mathcal{V}}_T$ of $\mathcal{L}^2(P)$. The $\mathcal{L}^2$-norm of $\frac{d\mu}{dP}$ is given by

$$||\frac{d\mu}{dP}||_{\mathcal{L}^2(P)} = \frac{1}{dist(1, \mathcal{V}_T)} = \frac{1}{(1-E[f])^{\frac{1}{2}}} = \frac{1}{\sin\phi},$$

where ϕ is the positive angle between 1 and $\overline{\mathcal{V}}_T$.

Lemma 5. *If the variance optimal measure $\mathcal{Q}^{opt} \in \mathbb{M}^e(\mathcal{V})$ exists and the càdlàg martingale L defined as*

$$L_t = E[\frac{d\mathcal{Q}^{opt}}{dP}|\mathcal{F}_t]$$

satisfies $R_2(P)$, then L satisfies condition (J).

Theorem 45. *If $\mathcal{V}$ is a space of bounded continuous adapted processes such that for each $Y \in \mathcal{V}$ we have $Y_0 = 0$, if $\mathcal{V}$ is stable for stopping (as described above), if $\mathcal{F}_0$ is trivial, the following are equivalent:*

(1) The variance optimal measure $\mathcal{Q}^{opt} \in \mathbb{M}^e(\mathcal{V})$ exists and the càdlàg martingale L defined as

$$L_t = E[\frac{d\mathcal{Q}^{opt}}{dP}|\mathcal{F}_t]$$

satisfies $R_2(P)$.

(2) There is $\mathcal{Q} \in \mathbb{M}^e(\mathcal{V}) \cap \mathcal{L}^2(P)$ such that the càdlàg martingale Z defined as

$$Z_t = E[\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t]$$

satisfies the inequality $R_2(P)$.
(3) There is a constant C such that for every $Y \in \mathcal{V}$

$$||Y_T^*||_{\mathcal{L}^2(P)} \leq C||Y_T||_{\mathcal{L}^2(P)}.$$

(4) There is a constant C such that for every $Y \in \mathcal{V}$ and every $\lambda \geq 0$

$$\lambda P[Y_T^* > \lambda]^{\frac{1}{2}} \leq C||Y_T||_{\mathcal{L}^2(P)}.$$

(5) There is a constant $C > 0$ such that for every stopping time S, every $A \in \mathcal{F}_S$ and every $U_T \in S^{\mathcal{V}}$

$$||1_A - U_T||_{\mathcal{L}^2(P)} \geq CP[A]^{\frac{1}{2}}.$$

In addition, if one of the above equivalent conditions is fulfilled, then $\mathcal{Q}^{opt}$ satisfies $R_p(P)$ for some $p > 2$.

Proof 8. *It is clear that (1) implies (2). By Theorem 43 and Lemma 5, (1) implies (3) and (2) implies (4), the constant (2) being valid for every $\mathcal{Q}$-uniformly integrable martingale. The strong inequality in (3) certainly implies the weak inequality in (4). We now prove the equivalence of (4) and (5), after which we show that (5), together with (4), implies (1).*
(4) $\Rightarrow$ (5)
This is done by using a reflection argument. Fix a stopping time S, $A \in \mathcal{F}_S$ and a process U of the form

$$U = X - X^{S_A} = 1_A(X - X^S)$$

where $X \in \mathcal{V}$. Define

$$v := \inf\left\{t|U_t > \frac{1}{2}\right\} \wedge T$$

and let

$$Y_t = \begin{cases} U_t, \ \textit{for } t \leq v \\ 2U_v - U_t, \ \textit{for } t > v, \end{cases}$$

i.e. Y is U reflected at time v. Then $Y \in \mathcal{V}$ and

$$|Y_T| = |U_T| 1_{\{v=T\}} + |1 - U_T| 1_{\{v<T\}} \leq |1 - U_T|$$

since $U_T \leq \frac{1}{2}$ on $\{v = T\}$. On A^c, we have $U = 0$, hence $v = T$ and $Y_T = 0$; thus we obtain

$$|Y_T| \leq |1_A - U_T|,$$

and the weak inequality in (4) implies

$$\begin{aligned} ||1_A - U_T||_{\mathcal{L}^2(P)} &\geq ||Y_T||_{\mathcal{L}^2(P)} \\ &\geq C^{-1} \frac{1}{2} P[Y_T^* \geq \frac{1}{2}]^{\frac{1}{2}} \\ &\geq \frac{C^{-1}}{2} P[v < T]^{\frac{1}{2}} \\ &= \frac{C^{-1}}{2} P[U_T^* > \frac{1}{2}]^{\frac{1}{2}}. \end{aligned}$$

On the other hand,

$$||U_T - 1_A||_{\mathcal{L}^2(P)} \geq \frac{1}{2} P[A \bigcap \left\{ U_T^* \leq \frac{1}{2} \right\}]^{\frac{1}{2}}$$

and hence

$$||U_T - 1_A||_{\mathcal{L}^2(P)} \geq \delta P[A]^{\frac{1}{2}}$$

where $\delta = \frac{1}{\sqrt{2}} \min(\frac{C^{-1}}{2}, \frac{1}{2})$.
(5)⇒(4)
For fixed $Y \in \mathcal{V}$ and $\lambda > 0$, let us define

$$S = \inf \{t | |Y_t| > \lambda\}.$$

The element $U_T = -sign(Y_S)(Y_T - Y_S)$ is clearly in $S^{\mathcal{V}}$ and hence for $A = \{S < T\} = \{Y_T^* > \lambda\}$ we have

$$||1_A - \frac{U_T}{\lambda}||_{\mathcal{L}^2(P)} \geq C P[A]^{\frac{1}{2}}$$

or, what is the same

$$C \lambda P[Y_T^* > \lambda]^{\frac{1}{2}} \leq ||\lambda 1_A - U_T||_{\mathcal{L}^2(P)}.$$

But

$$\lambda 1_A - U_T = \lambda 1_A + sign(Y_S)(Y_T - Y_S) = Y_T 1_A sign(Y_S)$$

and hence

$$C\lambda P[Y_T^* > \lambda]^{\frac{1}{2}} \leq ||Y 1_A - U_T||_{\mathcal{L}^2(P)} \leq ||Y_T||_{\mathcal{L}^2(P)}.$$

(5)⇒(1) Since this is a very technical proof we refer to Delbaen/Monat/Schachermayer/ Schweizer (1997). □

From now on we will suppose that X is a continuous semimartingale. The symbol $\mathcal{V}$ stands for the space of stochastic integrals $\int \theta dX$ such that θ is a simple integrand and $\int \theta dX$ remains bounded. The following theorem solves the problem of the closedness of $G_T(\Theta)$ for continuous semimartingales completely.

Theorem 46. *Let X denote a continuous semimartingale. Then the following are equivalent.*

(1) There is an equivalent local martingale measure with square integrable density and $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$.

(2) There is a square integrable local martingale measure $\mathcal{Q}$ that satisfies the inequality $R_2(P)$.

(3) The variance optimal measure $\mathcal{Q}^{opt}$ is in $\mathbb{M}^e \cap \mathcal{L}^2(P)$ and satisfies $R_2(P)$.

(4) $\exists C$ such that for all $Y \in \mathcal{V}$ we have

$$||Y_T^*||_{\mathcal{L}^2(P)} \leq C||Y_T||_{\mathcal{L}^2(P)}.$$

(4') $\exists C$ such that for all $\theta \in \Theta$ we have

$$||G_T(\theta)^*||_{\mathcal{L}^2(P)} = ||\theta||_{G(\Theta)} \leq C||G_T(\theta)||_{\mathcal{L}^2(P)}.$$

(5) $\exists C$ such that for all $Y \in \mathcal{V}$ and all $\lambda \geq 0$ we have

$$\lambda P[Y_T^* > \lambda]^{\frac{1}{2}} \leq C||Y_T||_{\mathcal{L}^2(P)}.$$

(5') $\exists C$ such that for all $\theta \in \Theta$ and all $\lambda \geq 0$ we have

$$\lambda P[(G_T(\theta)^* > \lambda]^{\frac{1}{2}} \leq C||G_T(\theta)||_{\mathcal{L}^2(P)}.$$

(6) $\exists C > 0$ such that for every stopping time S and every $A \in \mathcal{F}_S$ we have

$$||1_A - U||_{\mathcal{L}^2(P)} \geq CP[A]^{\frac{1}{2}}$$

for every $U \in S^{\mathcal{V}}$.

(6') $\exists C > 0$ such that for every stopping time S, every $A \in \mathcal{F}_S$ and every $\theta \in \Theta$ with $\theta = \theta 1_{]]S,T]]}$ we have

$$||1_A - G_T(\theta)||_{\mathcal{L}^2(P)} \geq CP[A]^{\frac{1}{2}}.$$

For the proof of this theorem we need the following definition and two lemmas.

Definition 46. *For $\theta \in \Theta$, we define*

$$|||\theta||| = ||\theta||_{L^2(M)} + ||\theta||_{L^2(A)}$$

and

$$||\theta||_{G(\Theta)} = ||\theta \cdot X||_{\mathcal{R}^2(P)}.$$

Lemma 6. *Assume that X is a (not necessarily continuous) semimartingale in $\mathcal{S}^2_{loc}$.*
(i) The normed space $(\Theta, ||\cdot||_{G(\Theta)})$ is complete if and only if there is a constant $C > 0$ such that

$$\forall \theta \in \Theta,\ |||\theta||| \leq C||\theta||_{G(\Theta)}.$$

(ii) Assume in addition that there is an equivalent local martingale measure $\mathcal{Q}$ for X with square integrable density. Then the normed space $(G_T(\Theta), ||\cdot||_{\mathcal{L}^2(P)})$ is complete, that is, $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$, if and only if there is a constant $C > 0$ such that

$$\forall \theta \in \Theta,\ |||\theta||| \leq C||G_T(\Theta)||_{\mathcal{L}^2(P)}.$$

Lemma 7. *Suppose that $N := \lambda \cdot M$ is in bmo_2. If A is continuous, then there is a constant C such that*

$$E[\langle \theta \cdot X + Z \rangle_T] \leq C||G(\theta) + Z||^2_{\mathcal{R}^2(P)}$$

for all $\theta \in \Theta$ and $Z \in \mathcal{M}^2_0$ strongly orthogonal to M.

Proof 9. *Proof of Theorem 46.*
The theorem is almost a reformulation of the results of Theorem 45. A local martingale measure for X is the same as a martingale measure for $\mathcal{V}$. Since the appropriate spaces of simple stochastic integrals are dense in the spaces of stochastic integrals, we simply deduce from Theorem 45 that the properties (2), (3), (4), (4'), (5), (5'), (6), (6') are all equivalent. Let us now show that (1) implies all the other properties. If there is an equivalent martingale measure with square integrable density, then Lemma 6 applies and the $\mathcal{R}^2(P)$-norm and the $\mathcal{L}^2(P)$-norm are equivalent (both to the $L^2(M)$-norm in fact).

As a result one obtains (4‘) and hence all the other equivalent conditions. Conversely, if (2) up to (6‘) hold, we have to deduce that the space $G_T(\Theta)$ is closed. By assumption, there is a local martingale measure with square integrable density that satisfies the inequality $R_2(P)$. So let $\mathcal{Q}$ be this martingale measure and put

$$E[\frac{d\mathcal{Q}}{dP}|\mathcal{F}_t] = L_t.$$

Then L_t is necessarily of the form

$$L = \epsilon(-\lambda \cdot M + U)$$

where U is a local martingale strongly orthogonal to M, i.e. $\langle M, U\rangle = 0$. It follows with Lemma 4.2 in Delbaen/Monat/Schachermayer/Schweizer (1997) that $-\lambda \cdot M + U$ is in bmo_2. Since M and U are strongly orthogonal, we have

$$\langle -\lambda \cdot M + U\rangle = \langle \lambda \cdot M\rangle + \langle U\rangle$$

and hence the local martingale $-\lambda \cdot M$ is also in bmo_2, which by the way is the same as BMO since M is continuous. Therefore, X satisfies $D_2(P)$ and the norm on Θ is equivalent to the $L^2(M)$-norm. From Lemma 7 we deduce that the $L^2(M)$-norm on Θ is dominated by the $R^2(P)$-norm on $G(\Theta)$. This norm is by hypothesis equivalent to the $\mathcal{L}^2(P)$-norm on $G_T(\Theta)$ and hence by proposition 6, the space $G_T(\Theta)$ is closed.
This completes the proof.

Before we can state a corollary of the previous theorem, we first need to recall the following definition.

Definition 47. *The set of all probability measures $\mathcal{Q}$ that are equivalent local martingale measures for X with $\frac{d\mathcal{Q}}{dP} \in L^2(P)$ is denoted by $\mathcal{M}_2^e(X)$.*

Corollary 16. *Let X be a continuous $\mathbb{R}^d$-valued semimartingale. Then the following statements are equivalent:*
(1) $\mathcal{M}_2^e(X) \neq \emptyset$ and $G_T(\Theta)$ is closed in $L^2(P)$.
(2) There exists some $\mathcal{Q} \in \mathcal{M}_2^e(X)$ satisfying $R_2(P)$.
(3) The variance-optimal martingale measure $\mathcal{Q}^{opt}$ is in $\mathcal{M}_2^e(X)$ and satisfies $R_2(P)$.

Proof 10. *Follows from Theorem 46.*

From Corollary 16 the reader may ask if it is really necessary that we assume the continuity of the semimartingale X. The following example shows that this is the case and that for a process with jumps the corollary does no longer hold. To state the example we first need to clarify the setting and some definitions (more detailed information on $\mathcal{E}$-martingales will be given in the next section):

For a semimartingale Y, we denote its stochastic exponential by $\mathcal{E}(Y)$. Let N be a fixed local P-martingale starting at zero. For any stopping time τ, we denote the process Y stopped at τ by Y^τ and the process Y started at τ by ${}^\tau Y = Y - Y^\tau$, but we set ${}^\tau\mathcal{E} = {}^\tau \mathcal{E}(N) = \mathcal{E}(N - N^\tau)$. We define an increasing sequence of stopping times by $\hat{T}_0 = 0$ and $\hat{T}_{n+1} = \inf\{t > \hat{T}_n | {}^{\hat{T}_n}\mathcal{E}(N)_t = 0\} \wedge T = \inf\{t > \hat{T}_n | \Delta N_t = -1\} \wedge T$, and $(\hat{T}_n)$ tends to T stationarily. An adapted RCLL process Y is an $\mathcal{E}$ local martingale if the product of ${}^{\hat{T}_n}Y$ and ${}^{\hat{T}_n}\mathcal{E}$ is a local P-martingale for any $n \in \mathbb{N}$. It is an $\mathcal{E}$-martingale if for any $n \in \mathbb{N}$, we have $E[|Y_{\hat{T}_n} \; {}^{\hat{T}_n}\mathcal{E}_{\hat{T}_{n+1}}|] < \infty$ and the above product is a (true) P-martingale. Furthermore, we say that $\mathcal{E}$ is regular if ${}^{\hat{T}_n}\mathcal{E}$ is a P-martingale for any n.

In Example 3.9 in Delbaen/Monat/Schachermeyer/Stricker/Schweizer (1997), it is shown that there exists a bounded process $X = (X_0,\ X_1,\ X_2)$ admitting a bounded equivalent martingale measure such that we have the following:

1. $X = M + \lambda \cdot < M >$ where $M \in M_0^2$, $\lambda \in L^2(M)$ and $\lambda \cdot M \notin bmo_2$;
2. $G_T(\Theta)$ is closed in L^2.

Now assume that there is $N \in M_0^2$ such that X is an $\mathcal{E}(N)$-local martingale and $\mathcal{E}(N)$ satisfies $R_2(P)$. From Proposition 1.64 of Jacod/Shiryaev (1987) we deduce that $\mathcal{E}(N)$ is regular. Then by Theorem 3.10 of Choulli/Krawczyk/Stricker (2008), $N \in bmo_2$. Since X is a bounded $\mathcal{E}(N)$-local martingale, X is a special semimartingale and $< N, M >= \lambda \cdot < M >$. It follows that $N = \lambda \cdot M + L$ where $L \in M_0^2$ and $< L, M >= 0$. Thus $\lambda \cdot M \in bmo_2$. This is a contradiction and $\mathcal{E}(N)$ does not satisfy $R_2(P)$.

By the next theorem we will extend the implication (2) $\Rightarrow$ (1) of Corollary 16 to the discontinuous case; as we will see we therefore have to assume that $\mathcal{E}$ is regular and satisfies $R_2(P)$. We clearly indicate that this assertion is very similar to Theorem 5.1 in Choulli/Krawczyk/Stricker (1998) who worked on a space similar to $G_T(\Theta)$.

Theorem 47. *Assume $\mathcal{E}$ is regular, satisfies $R_2(P)$ and that X is an $\mathcal{E}$-local martingale. Then $G_T(\Theta)$ is closed in L^2.*

Proof 11. *Follows directly from Lemma 6 (ii) and Theorem 4.9 in Choulli/Krawczyk/Stricker (1998).*

Example 2. *Let X be a standard Poisson process with intensity λ, stopped at time T, and set $N_t := -X_t + t \wedge T$ for $t \in [0,T]$. Then by Proposition 3.7 in Choulli/Krawczyk/Stricker (1998), $\mathcal{E}$ is regular and satisfies $R_2(P)$, by Proposition 3.17 in Choulli/Krawczyk/Stricker (1998), X is an $\mathcal{E}$-martingale, and for any bounded predictable process θ, $G_T(\theta)$ belongs to $G_T(\Theta)$. Moreover, by the previous theorem, $G_T(\Theta)$ is closed in L^2.*

5 Examination of the Closedness of Spaces of Stochastic Integrals in L^2 under Convex Trading Constraints

Since we have now determined the requirements under which our spaces of terminal portfolio values are closed in the L^2-topology, we will now impose constraints on our trading strategies. We will use the concepts of correspondences which are mappings from Ω into the power set of $\mathbb{R}^d$ to model those constraints. It may not appear intuitive to consider this rather complicated idea of imposing constraints on our trading strategies. Nevertheless, at the end of this section we will see that if we want our trading strategies to map into any convex subset of $\mathbb{R}^d$, the resulting space of terminal portfolio values $G_T(\Theta)$ will be closed in L^2 if and only if there exists a duality between the chosen convex subset and a correspondence. So there will be a close connection between requiring the trading strategies to map into an arbitrary convex set or a correspondence.

The key ideas on correspondences and their properties used in this section can all be found in Karatzas/Kardaras (2007) whereas their first usage in the question of whether a space of terminal portfolio values provided with trading constraints is closed or not is examined in Czichowsky/Schweizer (2012). They examine a space very similar to our $G_T(\Theta)$. Since we use their results to prove the closedness under constraints for $G_T(\Theta_v)$ and $G_T(\Theta_2^{eq})$, we will restate their argumentation while adopting their proofs and argumentation to our space $G_T(\Theta)$. The reader may notice that Proposition 11 provides a slightly different proof for the closedness of $G_T(\Theta)$ to ours in the previous section in Theorem 47. Then, in the following sections, we will use those techniques of Czichowsky/Schweizer (2012) to prove the closedness of $G_T(\Theta_v)$ and $G_T(\Theta_2^{eq})$ under similar constraints and widely extend their results even for $G_T(\Theta)$. To the best of our knowledge, those results are new.

5.1 The Space $G_T(\Theta_S(C))$

We want to clearly outline, that the results in this section are taken from Czichowsky/Schweizer (2012) whereas we only impose slight modifications to their notation and proofs to stay uniformly in the previously used setting for our space $G_T(\Theta)$; the reader may notice that the space Θ_S in this section is defined very similarly to the space Θ in the previous section. It is easily verified that both definitions lead to the same space of stochastic integrals, i.e. we have $G_T(\Theta_S) = G_T(\Theta)$.

For imposing trading constraints, we start with the space $G_T(\Theta_S(C))$, i.e. the space $G_T(\Theta)$ where we endow our trading strategies with convex constraints and previously recalling some important definitions and establishing an adequate notation and setting.

Definition 48. *Let $\mathcal{H}^2(P)$ denote the Banach space of all square integrable semimartingales, i.e. special semimartingales Y with canonical decomposition*

$$Y = Y_0 + M^Y + A^Y,$$

where M^Y is a square integrable martingale null at zero, $M^Y \in \mathcal{M}_0^2(P)$, and A^Y is a predictable finite variation RCLL process null at zero, such that

$$||Y||_{\mathcal{H}^2(P)} := ||Y_0||_{L^2(P)} + ||([M^Y]_T)^{\frac{1}{2}}||_{L^2(P)} + ||\int_0^T |dA_s^Y|||_{L^2(P)} < \infty.$$

We suppose that X is a locally square integrable semimartingale, for short $X \in \mathcal{H}^2_{loc}(P)$ (note that $\mathcal{H}^2_{loc}(P)$ coincides with the semimartingale space $S^2_{loc}(P)$ from the previous section), with canonical decomposition $X = X_0 + M + A$. Then there exists a predictable increasing $RCLL$ process B, e.g.

$$B = \sum_{i=1}^{d}(\langle M^i \rangle + \int |dA^i|),$$

with $\langle M^i, M^j \rangle \ll B$ and $A^i \ll B$ for $i, j = 1, \dots, d$. We define an $\mathbb{R}^{d\times d}$-valued predictable process c^M and an $\mathbb{R}^d$-valued predictable process a by

$$(c^M)^{ij} = \frac{d\langle M^i, M^j \rangle}{dB}$$

and

$$a^i = \frac{dA^i}{dB}.$$

We set $\overline{\Omega} := \Omega \times [0, T]$, $P_B := P \bigotimes B$ and view $\mathbb{R}^d$-valued predictable processes as $\mathcal{P}$-measurable random variables, i.e. elements of $\mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d)$. For trading strategies, we take

$$\Theta := \Theta_S := \left\{\vartheta \in L(X) | \int \vartheta dX \in \mathcal{H}^2(P)\right\}.$$

Since X is a special semimartingale, the canonical decomposition is unique and we have

$$\Theta_S = \mathcal{L}^2(M) \cap \mathcal{L}^2(A)$$

with

$$\mathcal{L}^2(M) := \{\vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) | ||\vartheta||_{\mathcal{L}^2(M)} := (E[\int_0^T \vartheta_s^\top c_s^M \vartheta_s dB_s])^{\frac{1}{2}} < \infty\},$$
$$\mathcal{L}^2(A) := \{\vartheta \in \mathcal{L}^0(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d) | ||\vartheta||_{\mathcal{L}^2(A)} := (E[\int_0^T |\vartheta_s^\top a_s | dB_s)^2])^{\frac{1}{2}} < \infty\}.$$

The wealth process generated up to time $t \in [0, T]$ by a self-financing trading strategy ϑ with initial capital $x \in \mathbb{R}$ is

$$V_t(x, \vartheta) := x + \int_0^t \vartheta_s dX =: x + G_t(\vartheta),$$

where the process $G(\vartheta)$ denotes the cumulative gains from trading.

Proposition 3. *Let Y be a semimartingale in $\mathcal{H}^2(P)$ and $Y_t^* := \sup_{0 \le s \le t} |Y_s|$. Then*

$$E[(Y_T^*)^2] \le 8||Y||^2_{\mathcal{H}^2(P)}.$$

Proof 12. *Let $X = \overline{N} + \overline{A}$ be the canonical decomposition of X. Then*

$$X^* \le \overline{N}^* + \int_0^T |d\overline{A}_s|.$$

Doob‘s maximal quadratic inequality yields

$$E[(\overline{N}^*)^2] \le 4E[\overline{N}_T^2] = 4E[[\overline{N}, \overline{N}]_T],$$

and using $(a+b)^2 \le 2a^2 + 2b^2$ we have

$$\begin{aligned} E[(X^*)^2] &\le 2E[(\overline{N}^*)^2] + 2E[(\int_0^T |d\overline{A}_s|^2] \\ &\le 8E[[\overline{N}, \overline{N}]_T] + 2||\int_0^T |d\overline{A}_s|||^2_{\mathcal{L}^2(P)} \\ &\le 8||X||^2_{\mathcal{H}^2(P)}. \end{aligned}$$

□

Corollary 17. *The sets $G_T(\Theta_s)$ and $\mathcal{A}(\Theta_S) := \mathbb{R} + G_T(\Theta_S)$ are linear subspaces of $L^2(P)$.*

Proof 13. *Follows directly from Proposition 3.*

Definition 49. *A mapping $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is called a correspondence. We say that a correspondence C is predictable if $C^{-1}(F) = \{(\omega,t)|C(\omega,t) \cap F \neq \emptyset\}$ is a predictable set for all closed $F \subseteq \mathbb{R}^d$. The domain $dom(C)$ of a correspondence is given by $dom(C) = \{(\omega,t)|C(\omega,t) \neq 0\}$. A (predictable) selector of a (predictable) correspondence C is a (predictable) process ψ with $\psi(\omega,t) \in C(\omega,t)$ for all $(\omega,t) \in dom(C)$.*

For convenience, we recall some results on predictable correspondences which ensure the existence of predictable selectors in all situations relevant for us.

Proposition 4. *(Castaing)*
For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with closed values, the following are equivalent:
1) C is predictable.
2) $dom(C)$ is predictable and there exists a Castaing representation of C, i.e. a sequence (ψ^n) of predictable selectors of C such that

$$C(\omega,t) = \overline{\{\psi^1(\omega,t), \psi^2(\omega,t), \dots\}} \text{ for each } (\omega,t) \in dom(C).$$

In particular, every closed-valued predictable C admits a predictable selector ψ.

Proposition 5. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^m$ Carathéodory functions, which means that $f(\omega,t,y)$ and $g(\omega,t,x)$ are predictable with respect to (ω,t) and continuous in y and x. Then the mappings C' and C'' given by $C'(\omega,t) = \{y \in \mathbb{R}^m | f(\omega,t,y) \in C(\omega,t)\}$ and $C''(\omega,t) = \overline{\{g(\omega,t,x)|x \in C(\omega,t)\}}$ are predictable correspondences (from $\overline{\Omega}$ to $2^{\mathbb{R}^m}$) with closed values.*

Proposition 6. *Let $C^n : \overline{\Omega} \to 2^{\mathbb{R}^d}$ for each $n \in \mathbb{N}$ be a predictable correspondence with closed values and define the correspondences C' and C'' by $C'(\omega,t) = \bigcap_{n\in\mathbb{N}} C^n(\omega,t)$ and $C''(\omega,t) = \bigcup_{n\in\mathbb{N}} C^n(\omega,t)$. Then C' and C'' are predictable and C' is closed-valued.*

For a predictable correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$, we denote by

$$\mathcal{C} := \mathcal{C}^X := \{\psi \in L(X) | \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t) \in \overline{\Omega}\}$$

the set of $C-$valued or C-constrained integrands for X and by

$$\Theta_S(C) = \Theta_S \cap \mathcal{C} = \{\vartheta \in \Theta_S | \vartheta(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t) \in \overline{\Omega}\}$$

the set of all C-constrained trading strategies.

For a semimartingale Y, we denote its stochastic exponential by $\mathcal{E}(Y)$. Let N be a fixed local P-martingale starting at zero. For any stopping time τ, we denote the process Y stopped at τ by Y^τ and the process Y started at τ by ${}^\tau Y = Y - Y^\tau$, but we set ${}^\tau\mathcal{E} = {}^\tau\mathcal{E}(N) = \mathcal{E}(N - N^\tau)$. So for the stochastic exponential, ${}^\tau\mathcal{E}(N)$ denotes a multiplicative rather than an additive restarting. Since N is P-a.s. RCLL, it has at most a finite number of jumps with $\Delta N = -1$, and moreover each ${}^\tau\mathcal{E}(N)$ can jump to zero at most once. Therefore we can define an increasing sequence of stopping times by $\hat{T}_0 = 0$ and $\hat{T}_{n+1} = \inf\{t > \hat{T}_n | {}^{\hat{T}_n}\mathcal{E}(N)_t = 0\} \wedge T = \inf\{t > \hat{T}_n | \Delta N_t = -1\} \wedge T$, and $(\hat{T}_n)$ tends to T stationarily.

Definition 50. *An adapted RCLL process Y is an $\mathcal{E}$-local martingale if the product of ${}^{\hat{T}_n}Y$ and ${}^{\hat{T}_n}\mathcal{E}$ is a local P-martingale for any $n \in \mathbb{N}$. It is an $\mathcal{E}$-martingale if for any $n \in \mathbb{N}$, we have $E[|Y_{\hat{T}_n} \, {}^{\hat{T}_n}\mathcal{E}_{\hat{T}_{n+1}}|] < \infty$ and the above product is a (true) P-martingale.*

The next two propositions provide some information about the structure of $\mathcal{E}$-martingales.

Proposition 7. *Let Y be a special semimartingale and $Y = Y_0 + M^Y + A^Y$ its canonical decomposition. Then Y is an $\mathcal{E}$-local martingale if and only if $[M^Y, N]$ is locally P-integrable and $A^Y = -\langle M^Y, N\rangle$.*

Proposition 8. *A semimartingale $Y = Y_0 + M^Y - \langle M^Y, N\rangle$ with $E[Y^*_T ({}^{\hat{T}_n}\mathcal{E})^*_T] < \infty$ for any $n \in \mathbb{N}$ is an $\mathcal{E}$-martingale.*

We also need the following definitions.

Definition 51. *We say that $\mathcal{E}$ is regular if ${}^{\hat{T}_n}\mathcal{E}$ is a P-martingale for any n.*

Definition 52. *We say that $\mathcal{E}$ satisfies the reverse Hoelder inequality $R_2(P)$ if there exists a constant $c \geq 1$ such that $E[|{}^t\mathcal{E}_T|^2 | \mathcal{F}_t] \leq c$ for any t.*

Proposition 9. *Assume that $\mathcal{E}$ satisfies $R_2(P)$. Then $\mathcal{E}$ is regular if and only if ${}^\tau\mathcal{E}$ is a P-martingale for any stopping time τ, and in that case, ${}^\tau\mathcal{E}$ is a P-square integrable P-martingale.*

Proposition 10. *Assume that $\mathcal{E}$ is regular and satisfies $R_2(P)$. Then there exists a constant c such that*

$$\frac{1}{c}||Y||_{\mathcal{H}^2(P)} \leq ||Y_T||_{L^2(P)} \leq c||Y||_{\mathcal{H}^2(P)}$$

for every $\mathcal{E}$-martingale Y. We write this for short as $||Y||_{\mathcal{H}^2(P)} \sim ||Y_T||_{L^2(P)}$.

Note that when ${}^0\mathcal{E}(N)$ is a strictly positive P-martingale, the definition of an $\mathcal{E}$-local martingale coincides with the notion of a local martingale under the measure $\mathcal{Q}$ defined by $d\mathcal{Q} = {}^0\mathcal{E}(N)_T dP$. This will be called the *classical case*. We suppose that there exists $N \in \mathcal{M}_{0,loc}(P)$ such that X is an $\mathcal{E}$-local martingale. By Proposition 7, this implies that $\langle M, N\rangle$ exists and $A = -\langle M, N\rangle$. Moreover, we assume that $\mathcal{E}(N)$ satisfies $R_2(P)$, which gives that N is locally P-square integrable and in bmo_2, i.e. there exists a constant $c > 0$ such that $E[\langle N\rangle_T - \langle N\rangle_t|\mathcal{F}_t] \leq c$ for all $t \in [0, T]$. An application of the Kunita-Watanabe decomposition yields $N = -\int \lambda dM + L$ with $\lambda \in \mathcal{L}^2(M)$ and $L \in \mathcal{M}_0^2(P)$ strongly P-orthogonal to M, and hence X satisfies the *structure condition* (SC), i.e.

$$X = X_0 + M + \int d\langle M\rangle\lambda.$$

Since N is in bmo_2, $\int \lambda dM$ is also in bmo_2, which implies by Theorem 3.3 in Delbaen/Monat/Schachermayer/Schweizer/Stricker (1997) the inequality $D_2(P)$, i.e. there exists a constant $c > 0$ such that $||\vartheta||_{\mathcal{L}^2(A)} \leq c||\vartheta||_{\mathcal{L}^2(M)}$. As a consequence, we have $\Theta_S = \mathcal{L}^2(M)$.

Proposition 11. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Then the following hold:*
1) For each σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$ and each $Y_0 \in L^2(\mathcal{B}_0)$, the process $Y_0 + \int \vartheta dX \in \mathcal{H}^2(P)$ is an $\mathcal{E}$-martingale.
2) The spaces $G_T(\Theta_S)$, $\mathcal{A}(\Theta_S)$ and $L^2(\mathcal{B}_0) + G_T(\Theta_S)$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.

Proof 14. *1) The stochastic integral $\int \vartheta dX$ is for each $\vartheta \in \Theta_S$ a special semimartingale in $\mathcal{H}^2(P)$ with canonical decomposition $\int \vartheta dX = \int \vartheta dM + \int \vartheta dA$. Since X is an $\mathcal{E}$-local martingale, we have $A = -\langle M, N\rangle$ by Proposition 7; so $\int \vartheta dA = -\langle \int \vartheta dM, N\rangle$ and therefore $\int \vartheta dX$ is an $\mathcal{E}$-local martingale again by Proposition 7. Since $\mathcal{E}$ is regular and satisfies $R_2(P)$, Proposition 9 states that ${}^\tau\mathcal{E}$ is a square integrable martingale for each stopping time τ. By Doob's inequality and Proposition 3, $\{{}^\tau\mathcal{E}\}^*_T$ and $\{G(\vartheta)\}^*_T$ are in $L^2(P)$ so that $\{G(\vartheta)\}^*_T\{{}^\tau\mathcal{E}\}^*_T$ is in $L^1(P)$ for every stopping time τ. Proposition 8 now implies that $G(\vartheta)$ is an $\mathcal{E}$-martingale. Replacing $G(\vartheta)$ by Y_0 shows in the same way*

that the constant process Y_0 is an $\mathcal{E}$-martingale for any $Y_0 \in L^2(\mathcal{B}_0)$, and hence so is $Y_0 + G(\vartheta)$.
2) Let $(Y_0^n + G_T(\vartheta^n))$ be a sequence in $L^2(\mathcal{B}_0) + G_T(\Theta_S)$ converging to H in $L^2(P)$. By part 1), each $Y_0^n + G(\vartheta^n)$ is an $\mathcal{E}$-martingale and therefore the sequence $(Y_0^n + G(\vartheta^n))$ is a Cauchy sequence in the Banach space $\mathcal{H}^2(P)$ by Proposition 10:

$$||(Y_0^n + G(\vartheta^n)) - (Y_0^m + G(\vartheta^m))||_{\mathcal{H}^2(P)} = ||Y_0^n - Y_0^m \int \vartheta^n - \vartheta^m dX||_{\mathcal{H}^2(P)}$$
$$\leq c \cdot ||Y_0^n - Y_0^m \int_0^T \vartheta^n - \vartheta^m dX||_{L^2(P)} \to 0$$

for $m, n \in \mathbb{N}$ big enough and a constant c.
Hence $(Y_0^n + G(\vartheta^n))$ is convergent to some $Y \in \mathcal{H}^2(P)$ which satisfies $Y_T = H$. Since the space (of processes) $Y_0 + G(\Theta_S)$ is closed in $\mathcal{H}^2(P)$ by construction of the stochastic integral, there exists some $\vartheta \in \Theta_S$ with $Y = Y_0 + G(\vartheta)$, and therefore $L^2(\mathcal{B}_0) + G_T(\Theta_S)$ is closed in $L^2(P)$. Choosing above $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ then implies the closedness of $\mathcal{A}(\Theta_S)$ and $G_T(\Theta_S)$ in $L^2(P)$, which completes the proof. □

Proposition 12. *For each $\mathbb{R}^d$-valued semimartingale Y, there exists an $\mathbb{R}^{d\times d}$-valued predictable process Π^Y, called the projection on the predictable range of Y, which takes values in the orthogonal projections in $\mathbb{R}^d$ and has the following property:*
If $\vartheta \in L(Y)$ and φ is predictable, then φ is in $L(Y)$ with $\int \varphi dY = \int \vartheta dY$ (up to indistinguishability) if and only if $\Pi^Y \vartheta = \Pi^Y \varphi$ P_B-a.e.. We choose and fix one version of Π^Y.

Proof 15. *Lemma 5.3 in Czichowsky/Schweizer (2011).*

The next theorem will play an important role for the closedness of $G_T(\Theta_S(C))$ in $L^2(P)$ since it says something about the closedness of $G(\Theta_S(C))$ in $\mathcal{H}^2(P)$ which is of course connected to our original question. It was first stated as Theorem 4.5 in Czichowsky/Schweizer (2011). To have all the necessary definitions for that, we first add the following one:

Definition 53. *The Émery distance of two semimartingales X and Y is*

$$d(X,Y) = \sup_{|\vartheta|\leq 1} (\sum_{n\in\mathbb{N}} 2^{-n} E[1 \wedge |(\vartheta \cdot (X-Y))_n|]),$$

where $(\vartheta \cdot X)_t := \int_0^t \vartheta_s dX_s$ stands for the vector stochastic integral, which is by construction a real-valued semimartingale, and the supremum is taken over all $\mathbb{R}^d$-valued predictable processes ϑ bounded by 1. With this metric, the space of all $\mathbb{R}^d$-valued semimartingales, denoted by $\mathcal{S}(P)$, is a complete topological vector space, and the corresponding topology is called the semimartingale topology.

Theorem 48. *Let $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed values. Then $C^X \cdot X$ is closed in the semimartingale topology if and only if the projection of C on the predictable range of X is closed, i.e. $\Pi^X(\omega, t)C(\omega, t)$ is closed P_B-a.e.. Equivalently: There exists a C-valued integrand ψ with $H = \psi \cdot X$ for any sequence (ψ^n) of C-valued integrands with $\psi^n \cdot X \mapsto H$ in the semimartingale topology if and only if the projection of C on the predictable range of X is closed.*

Proof 16. *See Theorem 4.5 in (Czichowsky/Schweizer, 2012).*

Now we are able to proof the closedness of $G_T(\Theta_S(C))$ in $L^2(P)$.

Theorem 49. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable coorespondence with closed values and such that $\Theta_S(C)$ is non-empty. Then the spaces $G_T(\Theta_S(C))$, $\mathcal{A}(\Theta_S(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_S(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$ if and only if the projection of C on the predictable range of X is closed, i.e. $\Pi^X(\omega, t)C(\omega, t)$ is closed P_B-a.e..*

Proof 17. *$\Leftarrow$:*
Let $(Y_0^n + G_T(\vartheta^n))$ be a sequence in $L^2(\mathcal{B}_0) + G_T(\Theta_S(C))$ converging to H in $L^2(P)$. Then there exist some $Y_0 \in L^2(\mathcal{B}_0)$ and some $\vartheta \in \Theta_S$ such that $Y_0 + G_T(\vartheta) = H$ P-a.s. and $(Y_0^n + G(\vartheta^n))$ converges to $Y_0 + G(\vartheta)$ in $\mathcal{H}^2(P)$, by the same argument as in the proof of Proposition 11. The convergence in $\mathcal{H}^2(P)$ implies the convergence in the semimartingale topology by Theorem V.14 and the lemma preceding Theorem IV.12 in (Protter, 2005). The space of stochastic integrals of C-valued integrands is closed in the semimartingale topology if the projection of C on the predictable range of X is closed by Theorem 48. Thus there exists $\tilde{\vartheta} \in \Theta_S(C)$ such that $G(\tilde{\vartheta}) = G(\vartheta)$, and therefore $L^2(\mathcal{B}_0) + G_T(\Theta_S(C))$ is closed in $L^2(P)$. As in the proof of Proposition 11, choosing $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ gives the closedness of $\mathcal{A}(\Theta_S)$ and $G_T(\Theta_S)$.
$\Rightarrow$:
First note that for any stopping time τ, the projection Π^{X^τ} on the predictable range of X^τ is simply $\Pi^X 1_{[[0,\tau]]}$. Recall that $X \in \mathcal{H}^2_{loc}(P)$. Arguing by contradiction, we choose

a stopping time τ such that $X^\tau \in \mathcal{H}^2(P)$ and Π^{X^τ} is not closed. Applying Lemma 4.4 in (Czichowsky/Schweizer, 2011) with X^τ and using that $\int \varphi dX^\tau = \int \varphi 1_{[[0,\tau]]} dX$ for any $\varphi \in L(X)$ implies that there exists $\vartheta \in L(X)$ and a sequence (ψ^n) of C-valued integrands such that $(\int \psi^n 1_{[[0,\tau]]} dX)$ converges to $\int \vartheta 1_{[[0,\tau]]} dX$ in the semimartingale topology, but there is no C-valued integrand ψ such that $\int \psi 1_{[[0,\tau]]} dX = \int \vartheta 1_{[[0,\tau]]} dX$. An inspection of the proof of Lemma 4.4 in (Czichowsky/Schweizer, 2011) shows that we can choose ϑ and (ψ^n) such that $(\Pi^X \vartheta) 1_{[[0,\tau]]}$ and $(\Pi^X \psi^n) 1_{[[0,\tau]]}$ are uniformly bounded and $(\Pi^X \psi^n) 1_{[[0,\tau]]} \to (\Pi^X \vartheta) 1_{[[0,\tau]]}$ uniformly in (ω, t). Since $\int \psi^n 1_{[[0,\tau]]} dX = \int (\Pi^X \psi^n) 1_{[[0,\tau]]} dX$ and $\int \vartheta 1_{[[0,\tau]]} dX = \int (\Pi^X \vartheta) 1_{[[0,\tau]]} dX$, we have by dominated convergence that $\int (\psi^n 1_{[[0,\tau]]} + \varphi 1_{]]\tau,T]]}) dX \to \int \tilde{\vartheta} dX$ in $\mathcal{H}^2(P)$ for any $\varphi \in \Theta_S(C)$, with $\tilde{\vartheta} = \vartheta 1_{[[0,\tau]]} + \varphi 1_{]]\tau,T]]}$, and hence also that $G_T(\psi^n 1_{[[0,\tau]]} + \varphi 1_{]]\tau,T]]}) \to G_T(\tilde{\vartheta})$ in $L^2(P)$ by Proposition 3. But because there exists by construction of $\tilde{\vartheta}$ no C-valued integrand ψ with $G(\psi) = G(\tilde{\vartheta})$ and since $G(\tilde{\vartheta})$ is uniquely determined in $\mathcal{H}^2(P)$ by its terminal value $G_T(\tilde{\vartheta})$ by Proposition 10, there cannot be any $\psi \in \Theta_S(C)$ with $G_T(\psi) = G_T(\tilde{\vartheta})$. This contradicts the closedness of $G_T(\Theta_S(C))$ in $L^2(P)$ and therefore completes the proof. □

Having established the closedness of $G_T(\Theta_S(C))$ and $\mathcal{A}(\Theta_S(C))$, we are able to prove the existence of a solution to the mean-variance hedging problem under trading constraints by using the Best approximation theorem in Hilbert spaces which we want to state here for completeness.

Theorem 50. *Let V be a prehilbert space, M a complete convex subset of V, and $x \in V$. We denote the scalar product in the Hilbert space V by $((\ ,\ ))$. Then the following properties are equivalent:*
(a) $tx \in M$ satisfies $||x - tx|| = \min_{y \in M} ||x - y||$.
(b) $tx \in M$ satisfies $((tx - x, tx - y)) \leq 0$ for all $y \in M$.
Moreover, we can associate a unique element $tx \in M$ with every $x \in V$ satisfying either one of these properties.

Remark 6. *If V is a Hilbert space, it is sufficient to suppose that M is a closed convex subset.*

Definition 54. *The mapping t that associates with each element $x \in V$ its Best approximation $tx \in M$ defined in Theorem 50 is called the Best approximation projector from V onto M.*

In the next theorem, we now prove the existence of a solution.

Theorem 51. *Assume $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d}$ be a predictable correspondence with closed convex values such that $\Theta_S(C)$ is non-empty. Then the following hold for every $H \in L^2(P)$:*

(a) There exists a solution $\hat{\vartheta}(x) \in \Theta_S(C)$ to the problem

$$E[|x + G_t(\vartheta) - H|^2] = \min_{\vartheta \in \Theta_S(C)}!$$

(b) There exists a solution $(\hat{x}, \hat{\vartheta}(\hat{x})) \in \mathbb{R} \times \Theta_S(C)$ to the problem

$$E[|x + G_t(\vartheta) - H|^2] = \min_{(x,\vartheta) \in \mathbb{R} \times \Theta_S(C)}!$$

Proof 18. *By Theorem 49, $G_T(\Theta_S(C))$ and $\mathcal{A}(\Theta_S(C))$ are closed and convex subsets of $L^2(P)$. Therefore, Theorem 50 implies the existence of a unique Best approximation of $H - x$ by an element in $G_T(\Theta_S(C))$. This can be identified uniquely with an element $G(\hat{\vartheta}(x))$ in $G(\Theta_S(C))$ which gives some $\hat{\vartheta}(x) \in \Theta_S(C)$ and proves (a). In the same way, we get a unique element $\hat{v}$ in $\mathcal{A}(\Theta_S(C))$ which is the Best approximation to H in $L^2(P)$, and $\hat{v}$ can again be identified with an element $(\hat{x}, \hat{\vartheta}(\hat{x}))$ in $\mathbb{R} \times \Theta_S(C)$.*

5.2 The Space $G_T(\Theta_2^{eq}(C))$

Since we have now established the closedness of the space $G_T(\Theta_S(C))$, we will use very similar concepts to ensure the closedness of $G_T(\Theta_2^{eq}(C))$, i.e. of the space $G_T(\Theta_2^{eq})$ endowed with trading constraints. For modelling the trading constraints we use again the concept of correspondences. The reader may notice that the results for this space are completely new and that we do not just get analogue results to the previous chapter, but are able to even expand those results.

We will start by restating some basic definitions and establishing an adequate setting. Firstly, recall that we defined

$$\mathcal{M}_2^e(P) = \left\{ \mathcal{Q} \approx P : \frac{d\mathcal{Q}}{dP} \in L^2(P),\ X \text{ is a local } \mathcal{Q}\text{-martingale} \right\}.$$

Definition 55. *Θ_2^{eq} consists of all $\vartheta \in L(X)$ such that $G_T(\vartheta)$ is in $L^2(P)$ and the process $G(\vartheta) = \int \vartheta dX$ is a uniformly $\mathcal{Q}$-integrable $\mathcal{Q}$-martingale for every $\mathcal{Q} \in \mathcal{M}_2^e(P)$.*

As in the previous section it will be crucial that the space $G(\Theta_2^{eq}(C))$ is closed in $\mathcal{H}^2(P)$. Therefore we need to make some assumptions.

Definition 56. *We define*

$$\mathcal{M}_2^{eq}(P) = \left\{ \mathcal{Q} \approx P : \frac{d\mathcal{Q}}{d\mathcal{P}} \in L^2(P),\ X \textit{ is a square integrable } \mathcal{Q}\textit{-martingale} \right\}.$$

Hypothesis 3. *We assume $P \in \mathcal{M}_2^{eq}(P)$ and $\mathcal{M}_2^{eq}(P) = \mathcal{M}_2^e(P)$.*

Remark 7. *As the previous hypothesis may look rather restrictive to our setting, the reader should notice that the requirement is satisfied if we assume X to be bounded. In those cases, it is clear that all local martingales identify as martingales. In practical use, we could assume that X is bounded by the amount of money in the market.*

For the main result in this section, i.e. the proof of the closedness of $G_T(\Theta_2^{eq}(C))$, we need the following Lemma.

Lemma 8. *Let $X \in \mathcal{H}^2$ be a (square integrable) martingale, and H a predictable process. Then $\int HdX$ is a square integrable martingale.*

Theorem 52. *For $p > 1$, let $\mathcal{X}$ be a nonempty family of random variables bounded in $\mathcal{L}^p$, i.e. such that*

$$\sup_{X \in \mathcal{X}} ||X||_{\mathcal{L}^p} < \infty.$$

Then $\mathcal{X}$ is uniformly integrable.

With the results from the previous section we can now conclude in an analogous way that the space $G_T(\Theta_2^{eq}(C))$ is closed in $L^2(P)$.

Theorem 53. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and such that $\Theta_2^{eq}(C)$ is non-empty. If the projection of C on the predictable range of X is closed, i.e. $\Pi^X(\omega,t)C(\omega,t)$ is closed P_B-a.e., then the spaces $G_T(\Theta_2^{eq}(C))$, $\mathcal{A}(\Theta_2^{eq}(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_2^{eq}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.*

Proof 19. *Let $(Y_0^n + G_T(\vartheta^n))$ be a sequence in $L^2(\mathcal{B}_0)+G_T(\Theta_2^{eq}(C))$ converging to H in $L^2(P)$. Then there exists some $Y_0 \in L^2(\mathcal{B}_0)$ and some $\vartheta \in \Theta_2^{eq}$ such that $Y_0+G_T(\vartheta) = H$ P-a.s. by the L^2-closedness of $L^2(\mathcal{B}_0) + G_T(\Theta_2^{eq}(C))$.*
We first want to show that $(Y_0^n + G(\vartheta^n))$ converges to $Y_0 + G(\vartheta)$ in $\mathcal{H}^2(P)$:
By Proposition 11, each $Y_0^n + G(\vartheta^n)$ is an $\mathcal{E}$-martingale (note that $G(\Theta_2^{eq}) \subseteq G(\Theta_S)$ because of Lemma 8 and Theorem 52) and therefore the sequence $(Y_0^n + G(\vartheta^n))$ is a Cauchy sequence in the Banach space $\mathcal{H}^2(P)$ by Proposition 10, hence convergent to some $Y \in \mathcal{H}^2(P)$ which satisfies $Y_T = H$. Since the space (of processes) $Y_0 + G(\Theta_2^{eq})$ is closed in $\mathcal{H}^2(P)$ (again because of $G(\Theta_2^{eq}) \subseteq G(\Theta_S)$ and Lemma 8), there exists some $\vartheta \in \Theta_2^{eq}$ with $Y = Y_0 + G(\vartheta)$.
So we have that $(Y_0^n + G(\vartheta^n))$ converges to $Y_0 + G(\vartheta)$ in $\mathcal{H}^2(P)$. The convergence in $\mathcal{H}^2(P)$ implies the convergence in the semimartingale topology by Theorem V.14 and the lemma preceding Theorem IV.12 in Protter (2005). The space of stochastic integrals of C-valued integrands is closed in the semimartingale topology if the projection of C on the predictable range of X is closed by Theorem 48.
Thus, there exists $\tilde{\vartheta} \in \Theta_2^{eq}(C)$ such that $G(\tilde{\vartheta}) = G(\vartheta)$, and therefore $L^2(\mathcal{B}_0)+G_T(\Theta_2^{eq}(C))$ is closed in $L^2(P)$. As in the proof of Proposition 11, choosing $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ gives the closedness of $\mathcal{A}(\Theta_2^{eq}(C))$ and $G_T(\Theta_2^{eq}(C))$.

Since we have now established the closedness of $G_T(\Theta_p^{eq}(C))$ and $\mathcal{A}(\Theta_p^{eq}(C))$, we are able to prove the existence of a solution to the mean-variance hedging problem under trading constraints by using the Best approximation theorem in Hilbert spaces; see Theorem 50.

Theorem 54. *Assume $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d}$ be a predictable correspondence with closed*

convex values such that $\Theta_2^{eq}(C)$ *is non-empty. Then the following hold for every* $H \in L^2(P)$*:*
(a) There exists a solution $\hat{\vartheta}(x) \in \Theta_2^{eq}(C)$ *to the problem*

$$E[|x + G_t(\vartheta) - H|^2] = \min_{\vartheta \in \Theta_2^{eq}(C)}!$$

(b) There exists a solution $(\hat{x}, \hat{\vartheta}(\hat{x})) \in \mathbb{R} \times \Theta_2^{eq}(C)$ *to the problem*

$$E[|x + G_t(\vartheta) - H|^2] = \min_{(x,\vartheta) \in \mathbb{R} \times \Theta_2^{eq}(C)}!$$

Proof 20. *By Theorem 49,* $G_T(\Theta_2^{eq}(C))$ *and* $\mathcal{A}(\Theta_2^{eq}(C))$ *are closed and convex subsets of* $L^2(P)$*. Therefore, Theorem 50 implies the existence of a unique Best approximation of* $H - x$ *by an element in* $G_T(\Theta_2^{eq}(C))$*. This can be identified uniquely with an element* $G(\hat{\vartheta}(x))$ *in* $G(\Theta_2^{eq}(C))$ *which gives some* $\hat{\vartheta}(x) \in \Theta_2^{eq}(C)$ *and proves (a). In the same way, we get a unique element* $\hat{v}$ *in* $\mathcal{A}(\Theta_2^{eq}(C))$ *which is the Best approximation to* H *in* $L^2(P)$*, and* $\hat{v}$ *can again be identified with an element* $(\hat{x}, \hat{\vartheta}(\hat{x}))$ *in* $\mathbb{R} \times \Theta_2^{eq}(C)$*.*

In Theorem 53, we saw that $G_T(\Theta_2^{eq}(C))$ is closed if the projection of C on the predictable range of X is closed. With the following three theorems we will encounter some other conditions under which $G_T(\Theta_2^{eq}(C))$ is closed. Note that these conditions are only placed on C and work for arbitrary semimartingales Y. We clearly want to indicate that the theorems were first stated in Czichowsky/Schweizer (2011).

Theorem 55. *Let* $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *be a predictable correspondence with closed values. Then* $\mathcal{C}^Y \cdot Y$ *is closed in the semimartingale topology for all semimartingales* Y *if with probability* 1*, for all* $t \geq 0$*, all projections* $\Pi C(\omega, t)$ *of* $C(\omega, t)$ *are closed in* $\mathbb{R}^d$*.*
In particular, if with probability 1*, every* $C(\omega, t)$*,* $t \geq 0$*, is compact, or polyhedral, or a continuous and convex set, then* $C^Y \cdot Y$ *is closed in the semimartingale topology for all semimartingales* Y*.*

Proof 21. *If a set is compact or polyhedral, all its projections have the same property (see Corollary 2.15 in Klee (1959)) and are thus closed. For a continuous convex set, every projection is closed by Theorem 1.3 in Gale/Klee (1959). Now if with probability* 1*, for all* $t \geq 0$*, all projections* $\Pi C(\omega, t)$ *of* $C(\omega, t)$ *are closed, the projection* $\Pi^Y C$ *of* C *on the predictable range of every semimartingale* Y *is closed* $P \otimes B^Y$*-a.e.. So* $C^Y \cdot Y$ *is closed in the semimartingale topology by Theorem 48.*

Theorem 56. *Fix $K \subseteq \mathbb{R}^d$ and define $\mathcal{K}^Y = \{\psi \in \mathcal{L}(Y) | \psi(\omega,t) \in K \text{ for all } (\omega,t)\}$. Then $\mathcal{K}^Y \cdot Y$ is closed in the semimartingale topology for all $\mathbb{R}^d$-valued semimartingales Y if and only if all projections ΠK of K in $\mathbb{R}^d$ are closed.*

Proof 22. *The "if"-part follows immediately from Theorem 48. For the converse, assume by way of contradiction that there is a projection Π in $\mathbb{R}^d$ such that ΠK is not closed. Let W be a d-dimensional Brownian motion and set $Y = \Pi^\top \cdot W$. Then Π is the projection on the predictable range of Y, and therefore $\mathcal{K}^Y \cdot Y$ is not closed by Theorem 48.*

Theorem 57. *Let $K \subseteq \mathbb{R}^d$ be a closed convex cone. Then $\mathcal{K}^Y \cdot Y$ is closed in the semimartingale topology for all $\mathbb{R}^d$-valued semimartingales Y if and only if K is polyhedral.*

Proof 23. *By Theorem 56, $\mathcal{K}^Y \cdot Y$ is closed in the semimartingale topology if and only if all projections ΠK are closed in $\mathbb{R}^d$. But Theorem 4.11 in Klee (1959) says that all projections of a convex cone are closed in $\mathbb{R}^d$ if and only if that cone is polyhedral.*

With these three theorems we can derive some corollaries from Theorem 49.

Corollary 18. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable coorespondence with closed values and such that $\Theta^{eq}_2(C)$ is non-empty. If with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega,t)$ of $C(\omega,t)$ are closed in $\mathbb{R}^d$, then the spaces $G_T(\Theta^{eq}_2(C))$, $\mathcal{A}(\Theta^{eq}_2(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta^{eq}_2(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$. In particular, if with probability 1, every $C(\omega,t)$, $t \geq 0$, is compact, or polyhedral or a continuous and convex set, then the spaces $G_T(\Theta^{eq}_2(C))$, $\mathcal{A}(\Theta^{eq}_2(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta^{eq}_2(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.*

Proof 24. *Follows immediately from Theorem 53 and Theorem 55.*

Corollary 19. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Fix $K \subseteq \mathbb{R}^d$ and define*

$$\mathcal{K}^X = \{\psi \in L(X) | \psi(\omega,t) \in K \text{ for all } (\omega,t)\}$$

and

$$\Theta^{eq}_2(K) = \Theta^{eq}_2 \cap \mathcal{K}^X.$$

We assume $\mathcal{K}^X \subseteq \Theta^{eq}_2$. If all projection ΠK of K in $\mathbb{R}^d$ are closed, then the spaces $G_T(\Theta^{eq}_2(K))$, $\mathcal{A}(\Theta^{eq}_2(K))$ and $L^2(\mathcal{B}_0) + G_T(\Theta^{eq}_2(K))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.

Proof 25. *Follows immediately from Theorem 53 and Theorem 56.*

Corollary 20. *Assume that* $\mathcal{E} = \mathcal{E}(N)$ *is regular and satisfies* $R_2(P)$*, and that* $X \in \mathcal{H}^2_{loc}(P)$ *is an* $\mathcal{E}$*-local martingale. Let* $K \subseteq \mathbb{R}^d$ *be a closed convex cone. We assume* $\mathcal{K}^X \subseteq \Theta_2^{eq}$*. If* K *is polyhedral, then the spaces* $G_T(\Theta_2^{eq}(K))$*,* $\mathcal{A}(\Theta_2^{eq}(K))$ *and* $L^2(\mathcal{B}_0) + G_T(\Theta_2^{eq}(K))$*, for any* σ*-field* $\mathcal{B}_0 \subseteq \mathcal{F}_0$*, are closed in* $L^2(P)$*.*

Proof 26. *Follows immediately from Theorem 53 and Theorem 57.*

Of course, in this context the question arises whether we could use more general convex sets than only closed convex cones. In addition, one could ask in which way the closedness of $G_T(\Theta_2^{eq}(K))$ for a general convex set $K \subseteq \mathbb{R}^d$ is related to the subject of predictable correspondences. The following two assertions give answers to both questions. Please note that the theorem has been stated in Czichowsky/Schweizer (2012), whereas the corollary is new.

Theorem 58. *Let* $\mathcal{D} \subseteq L(X)$ *be non-empty. Then* $\mathcal{D}\cdot X$ *is predictably convex and closed in the semimartingale topology if and only if there exists a predictable correspondence* $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *with closed convex values such that the projection of* C *on the predictable range of* X *is closed, i.e.* $\Pi^X(\omega,t)C(\omega,t)$ *is closed* P_B*-a.e., and such that we have* $\mathcal{D}\cdot X = \mathcal{C}^X \cdot X$*, i.e.*

$$\begin{aligned} \mathcal{D}\cdot X &= \{\psi \cdot X | \psi \in \mathcal{D}\} \\ &= \{\psi \cdot X | \psi \in L(X) \text{ and } \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t)\}\,. \end{aligned}$$

Proof 27. $\Leftarrow$:
The pointwise convexity of C *immediately implies that* $\mathcal{C}^X \cdot X$ *is predictably convex, and closedness follows from Theorem 48.*
$\Rightarrow$: *See Theorem 4.11 in Czichowsky/Schweizer (2011).*

Now we are able to state the next assertion that gives information concerning the closedness of $G_T(\Theta_2^{eq}(K))$ for a general convex set $K \subseteq \mathbb{R}^d$.

Corollary 21. *Let* $K \subseteq \mathbb{R}^d$ *be a convex set and let* $\Theta_2^{eq}(K)$ *be non-empty and* $\mathcal{K}^X \subseteq \Theta_2^{eq}$*. Then* $G_T(\Theta_2^{eq}(K))$ *is closed if there exists a predictable correspondence* $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *with closed convex values such that the projection of* C *on the predictable range of* X *is closed, i.e.* $\Pi^X(\omega,t)C(\omega,t)$ *is closed* P_B*-a.e., and such that we have* $\mathcal{K}^X \cdot X = \mathcal{C}^X \cdot X$*,* *i.e.*

$$\begin{aligned}\mathcal{K}^X \cdot X &= \left\{\psi \cdot X | \psi \in \mathcal{K}^X\right\} \\ &= \left\{\psi \cdot X | \psi \in L(X) \textit{ and } \psi(\omega, t) \in C(\omega, t) \textit{ for all } (\omega, t)\right\}.\end{aligned}$$

Proof 28. *Follows immediately from Theorems 49 and 58.*

So the previous corollary tells us, that when we have a convex set K with $K \subseteq \mathbb{R}^d$ and we want our trading strategies only to take values in this set, the set of stochastic integrals $G_T(\Theta_2^{eq}(K))$ is only closed if there exists a related correspondence that somehow "replicates" our convex set, i.e. we have

$$\mathcal{K}^X \cdot X = \mathcal{C}^X \cdot X$$

with

$$\mathcal{K}^X = \left\{\psi \in L(X) | \psi(\omega, t) \in K \text{ for all } (\omega, t)\right\}.$$

5.3 The Space $G_T(\Theta_v(C))$

Since we have now proven the L^2-closedness of the spaces $G_T(\Theta_S(C))$ and $G_T(\Theta_2^{eq}(C))$, we want to consider our third space from the previous chapter, namely $G_T(\Theta_v)$, also endowed with constraints on the corresponding trading strategies. We use again very similar techniques to those already used in the two previous subsections.

This time, contrary to the two previous spaces, we have to make some adjustments to the space taken under consideration to ensure the closedness under trading constraints, i.e. we will not exactly establish the closedness for $G_T(\Theta_v)$ under constraints but for a very similar space. The reader may notice once again that the results in this section are completely new.

First, we want to recall the setting and some basic definitions. Let $G_T(\Theta_p^s)$ again denote the subspace of $L^p(P)$ spanned by the simple stochastic integrals of the form

$$f = h^\tau (X_{T_2} - X_{T_1}),$$

where $T_1 \leq T_1$ are stopping times. Obviously, a probability measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ makes X a local martingale under $\mathcal{Q}$ if $\frac{d\mathcal{Q}}{dP} \in L^p(P)$ and

$$E_{\mathcal{Q}}[f] = E[\frac{d\mathcal{Q}}{dP} f] = 0$$

for all $f \in G_T(\Theta_p^s)$. Recall

$$\mathcal{M}^{q,s} := \{g \in L^q(P) : E[gf] = 0 \text{ for all } f \in G_T(\Theta_p^s) \text{ and } E[g] = 1\}$$

and

$$\mathcal{M}^{q,e} := \{g \in \mathcal{M}^{q,s} : g > 0 \ a.s.\}.$$

Any element in $\mathcal{M}^{q,s}$ (resp. $\mathcal{M}^{q,e}$) is called a signed (resp. equivalent) martingale measure for X.

The space of integrands has been defined as follows (for $p = q = 2$):

Definition 57. Θ_v *consists of all* $\mathbb{R}^m$*-valued predictable* X*-integrable processes* ϑ *such that* $G_T(\vartheta) := \int_0^T \vartheta dX \in L^2(P)$ *and* $E[G_T(\vartheta)g] = 0$ *for all* $g \in \mathcal{M}^{2,s}$.

Of course every signed martingale measure $\mathcal{Q}$ makes X a local martingale but we can not guarantee that $G_t(\vartheta)$ is a local martingale, for $0 \leq t \leq T$, for all $\vartheta \in \Theta_v$. This way we won't be able to prove the $\mathcal{H}^2$-closedness of $G(\Theta_v)$ which is important for our goal of finding a mean-variance hedging strategy under constraints.

But we know that under certain conditions we have that $\int \vartheta dX$ is a local martingale if X is a local martingale by the following theorem:

Theorem 59. *Let X be a local martingale, and let H be a locally bounded process. Then the stochastic integral $\int HdX$ is a local martingale.*

So it makes sense to modify our space of integrands:

Definition 58. *$\Theta_{v'}$ consists of all $\mathbb{R}^m$-valued predictable X-integrable processes ϑ such that $G_T(\vartheta) := \int_0^T \vartheta dX \in L^2(P)$ and $G(\vartheta)$ is a local martingale for every signed martingale measure for X.*

Hypothesis 4. *We assume that P is a signed martingale measure for X, that $\int \vartheta dX$ is P square integrable and that all processes ϑ in $\Theta_{v'}$ are locally bounded.*

Remark 8. *To elaborate on the need of those conditions and on how restrictive they might be to our setting, the reader may notice the following: For the closedness of $G_T(\Theta_{v'})$ it will again be essential that we ensure the closedness of $G(\Theta_{v'})$ in $\mathcal{H}^2$. Since we know that a space of stochastic Integrals is closed in $\mathcal{H}^2$ if X is a local martingal and the integrands are locally bounded (by Theorem 59), it is quite intuitive to use those assumptions. Furthermore, they are not very restrictive since we will see that the existence of an equivalent local martingale measure can be interpreted as a no-arbitrage condition (which is reasonable to assume) and the condition of the trading strategies being locally bounded seems natural in a financial market.*

Theorem 60. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Then $G(\Theta_{v'})$ is closed in $\mathcal{H}^2(P)$.*

Proof 29. *So if $\int \vartheta^n dX$ is a sequence in $G(\Theta_{v'})$, it is also a sequence in $G(\Theta_S)$ since we assume that $\int \vartheta^n dX$ is P-square integrable. In the proof of Proposition 11, we have seen that $G(\Theta_S)$ is closed in $\mathcal{H}^2$. So if $\int \vartheta^n dX$ is a convergent sequence in $G(\Theta_{v'})$, we want the limit $G(\vartheta)$ to be in $G(\Theta_{v'})$ and this is true since $G(\vartheta)$ is P-square integrable and a local martingale for every signed martingale measure for X because of Theorem 59 since X is a local martingale under every signed martingale measure by definition.*

Corollary 22. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Then $G_T(\Theta_{v'})$ is closed in $L^2(P)$.*

Proof 30. *Follows directly from Theorem 60.*

Theorem 61. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and such that $\Theta_{v'}(C)$ is non-empty. Then the spaces $G_T(\Theta_{v'}(C))$, $\mathcal{A}(\Theta_{v'}(C))$ and $L^2(\mathcal{B}_0)+G_T(\Theta_{v'}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$ if and only if the projection of C on the predictable range of X is closed, i.e. $\Pi^X(\omega,t)C(\omega,t)$ is closed P_B-a.e..*

Proof 31. *Let $(Y_0^n + G_T(\vartheta^n))$ be a sequence in $L^2(\mathcal{B}_0)+G_T(\Theta_{v'}(C))$ converging to H in $L^2(P)$. Then there exists some $Y_0 \in L^2(\mathcal{B}_0)$ and some $\vartheta \in \Theta_{v'}$ such that $Y_0+G_T(\vartheta) = H$ P-a.s..*
We first want to show that $(Y_0^n+G(\vartheta^n))$ converges to $Y_0+G(\vartheta)$ in $\mathcal{H}^2(P)$ but this follows directly from Theorem 60.
So we have that $(Y_0^n + G(\vartheta^n))$ converges to $Y_0 + G(\vartheta)$ in $\mathcal{H}^2(P)$. The convergence in $\mathcal{H}^2(P)$ implies the convergence in the semimartingale topology by Theorem V.14 and the lemma preceding Theorem IV.12 in Protter (2005). The space of stochastic integrals of C-valued integrands is closed in the semimartingale topology if the projection of C on the predictable range of X is closed by Theorem 4.5 in Czichowsky/Schweizer (2011). Thus, there exists $\tilde{\vartheta} \in \Theta_{v'}(C)$ such that $G(\tilde{\vartheta}) = G(\vartheta)$, and therefore $L^2(\mathcal{B}_0)+G_T(\Theta_{v'}(C))$ is closed in $L^2(P)$. As in the proof of Proposition 11, choosing $\mathcal{B}_0 = \{\emptyset, \Omega\}$ and $Y_0^n = 0$ for all $n \in \mathbb{N}$ gives the closedness of $\mathcal{A}(\Theta_{v'})$ and $G_T(\Theta_{v\prime})$.

Having established the closedness of $G_T(\Theta_{v'}(C))$ and $\mathcal{A}(\Theta_{v'}(C))$, we are able to prove the existence of a solution to the mean-variance hedging problem under trading constraints by using again the Best approximation theorem in Hilbert spaces.

Theorem 62. *Assume $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d}$ be a predictable correspondence with closed convex values such that $\Theta_{v'}(C)$ is non-empty. Then the following hold for every $H \in L^2(P)$:*
(a) There exists a solution $\hat{\vartheta}(x) \in \Theta_{v'}(C)$ to the problem

$$E[|x + G_t(\vartheta) - H|^2] = \min_{\vartheta \in \Theta_{v'}(C)} !$$

(b) There exists a solution $(\hat{x}, \hat{\vartheta}(\hat{x})) \in \mathbb{R} \times \Theta_{v'}(C)$ to the problem

$$E[|x + G_t(\vartheta) - H|^2] = \min_{(x,\vartheta) \in \mathbb{R} \times \Theta_{v'}(C)} !$$

Proof 32. *By Theorem 49, $G_T(\Theta_{v'}(C))$ and $\mathcal{A}(\Theta_{v'}(C))$ are closed and convex subsets of $L^2(P)$. Therefore, Theorem 50 implies the existence of a unique Best approximation of $H - x$ by an element in $G_T(\Theta_{v'}(C))$. This can be identified uniquely with an element $G(\hat{\vartheta}(x))$ in $G(\Theta_{v'}(C))$ which gives some $\hat{\vartheta}(x) \in \Theta_{v'}(C)$ and proves (a). In the same way, we get a unique element $\hat{v}$ in $\mathcal{A}(\Theta_{v'}(C))$ which is the Best approximation to H in $L^2(P)$, and $\hat{v}$ can again be identified with an element $(\hat{x}, \hat{\vartheta}(\hat{x}))$ in $\mathbb{R} \times \Theta_{v'}(C)$.*

Of course we are able to formulate the corollaries from the previous chapter also for the space $G_T(\Theta_{v'}(C))$ in a completely analogous way. Furthermore, we are also able to state an analogous assertion to Corollary 21 what we will do here for the purpose of completeness.

Corollary 23. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and such that $\Theta_{v'}(C)$ is non-empty. If with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega, t)$ of $C(\omega, t)$ are closed in $\mathbb{R}^d$, then the spaces $G_T(\Theta_{v'}(C))$, $\mathcal{A}(\Theta_{v'}(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_{v'}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$. In particular, if with probability 1, every $C(\omega, t)$, $t \geq 0$, is compact, or polyhedral or a continuous and convex set, then the spaces $G_T(\Theta_{v'}(C))$, $\mathcal{A}(\Theta_{v'}(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_{v'}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.*

Proof 33. *Follows immediately from Theorem 61, Theorem 48 and Theorem 55.*

Corollary 24. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Fix $K \subseteq \mathbb{R}^d$ and define*

$$\mathcal{K}^X = \{\psi \in L(X) | \psi(\omega, t) \in K \text{ for all } (\omega, t)\}$$

and

$$\Theta_{v'}(K) = \Theta_{v'} \cap \mathcal{K}^X.$$

We assume $\mathcal{K}^X \subseteq \Theta_{v'}$. If all projection ΠK of K in $\mathbb{R}^d$ are closed, then the spaces $G_T(\Theta_{v'}(C))$, $\mathcal{A}(\Theta_{v'}(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_{v'}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.

Proof 34. *Follows immediately from Theorem 61, Theorem 48 and Theorem 56.*

Corollary 25. *Assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$, and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Let $K \subseteq \mathbb{R}^d$ be a closed convex cone. We assume*

$\mathcal{K}^X \subseteq \Theta_{v'}$. *If K is polyhedral, then the spaces $G_T(\Theta_{v'}(C))$, $\mathcal{A}(\Theta_{v'}(C))$ and $L^2(\mathcal{B}_0) + G_T(\Theta_{v'}(C))$, for any σ-field $\mathcal{B}_0 \subseteq \mathcal{F}_0$, are closed in $L^2(P)$.*

Proof 35. *Follows immediately from Theorem 61, Theorem 48 and Theorem 57.*

Theorem 63. *Let $K \subseteq \mathbb{R}^d$ be a convex set, $\Theta_{v'}(K)$ be non-empty and $\mathcal{K}^X \subseteq \Theta_{v'}$. Then $G_T(\Theta_{v'}(K))$ is closed if there exists a predictable correspondence $C : \overline{\Omega} \mapsto 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with closed convex values such that the projection of C on the predictable range of X is closed, i.e. $\Pi^X(\omega, t)C(\omega, t)$ is closed P_B-a.e., and such that we have $\mathcal{K}^X \cdot X = \mathcal{C}^X \cdot X$, i.e.*

$$\begin{aligned}\mathcal{K}^X \cdot X &= \left\{\psi \cdot X | \psi \in \mathcal{K}^X\right\} \\ &= \{\psi \cdot X | \psi \in L(X) \text{ and } \psi(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t)\}.\end{aligned}$$

6 Calculation of Explicit Hedging Strategies

Since we have analyzed the closedness of spaces of stochastic integrals in the previous chapters, we now want to find explicit hedging strategies for a claim H. Of course, this means to project our claim which is a square integrable random variable in L^2 onto our closed sets of stochastic integrals and then to somehow find the explicit integrand, i.e. the trading strategy that corresponds to the projection.

It will turn out, the idea of signed martingale measures will play an intriguing role in every single one of our three different spaces, when we try to find hedging strategies. Working with those measures, we will see that exactly the element that minimizes the variance over all signed martingale measures, the so-called variance-optimal signed martingale measure, will be crucial for finding risk-minimizing trading strategies.

We want to clearly indicate that techniques for establishing the risk-minimizing trading strategy for the space $G_T(\Theta_p^{eq})$ can be found in Schweizer (1999), for $G_T(\Theta_S)$ in Hou/Karatzas (2004) and in Xia/Yan (2006) for $G_T(\Theta_v)$. But all authors solve different hedging problems for the corresponding spaces whereby this work solves all of those three hedging problems for every space, what has not been done before in the existing literature. To do so, we use those results of this work, that provide conditions under which the three different spaces coincide.

First of all, we want to give a more detailed description of what our understanding of a hedging strategy is. We will differ between four different types of strategies and so there are four different questions that we will have to ask ourselves. So if we denote a space of integrands resp. trading strategies by $\Theta \subseteq L(X)$ and $X = \{X(t) : 0 \leq t \leq T\}$ is a square integrable d-dimensional process defined on the finite time horizon $[0, T]$, which is a semimartingale on the filtered probability space $(\Omega, \mathcal{F}, P)$, $F = \{\mathcal{F}(t)\}_{0 \leq t \leq T}$, those questions are:

(1) How do we compute

$$V(c) := \inf_{\vartheta \in \Theta} E[H - c - \int_0^T \vartheta' dX]^2 \tag{17}$$

if $c \in \mathbb{R}$ is given and we have the freedom to choose ϑ over the class Θ?

(2) How do we find

$$V := \inf_{(c,\vartheta)\in\mathbb{R}\times\Theta} E[H - c - \int_0^T \vartheta' dX]^2 = \inf_{c\in\mathbb{R}} V(c) \tag{18}$$

when we have the freedom to select both c and ϑ?
(3) How do we

$$\text{minimize the variance } Var(H - \int_0^T \vartheta' dX) \tag{19}$$

over all $\vartheta \in \Theta$?
(4) How do we

$$\text{minimize the variance } Var(H - \int_0^T \vartheta' dX) \tag{20}$$
$$\text{over } \vartheta \in \Theta \text{ with } E[\int_0^T \vartheta' dX] = \mu$$

for some given $\mu \in \mathbb{R}$?

The first three problems will be solved for all spaces whereas we solve problem 20 only for $G_T(\Theta_S)$. Furthermore, we solve the problem of Mean-Variance Portfolio Selection (which will be defined later) for all spaces.

6.1 The space $G_T(\Theta_S)$

We will see that all those questions are connected very tightly and that the solution for question 1 leads the way to the solutions to all the other questions.
So at first, we want to approach the solution to our four problems for the space $G_T(\Theta_S)$ and we do this without using the closedness of $G_T(\Theta_S)$ at the beginning. We recognize that problem 2 has a rather obvious solution, if it is known that the random variable H is of the form

$$H = h + G_T(\zeta^H) \tag{21}$$

for some $h \in \mathbb{R}$ and $\zeta^H \in \Theta_S$; because then we can take $\hat{c} = h$, $\hat{\vartheta} = \zeta^H$, and deduce that $V = 0$ in 18.
For example, it is well known that every $H \in L^2(P)$ can be written in the form 21, in fact with $h = E[H]$, if X is a Brownian motion and if F is the filtration F^X generated by X itself. One can then also describe the integrand ζ^H in terms of the famous Clark (1970) formula, under suitable conditions on the random variable $H \in L^2(P) = L^2(\Omega, \mathcal{F}^X(T), P)$. Thus, in this special case, we can take $\hat{c} = E[H]$, $\hat{\vartheta} = \zeta^H$, and have $V = 0$ in 18.
A little more generally, if we suppose that X is a square integrable martingale, then again it is well known that every $H \in L^2(P)$ admits the so-called Kunita-Watanabe (1967) decomposition

$$H = h + G_T(\zeta^H) + L^H(T) \tag{22}$$

for $h = E[H]$, some $\zeta^H \in \Theta_S$, and some square integrable martingale L^H which is strongly orthogonal to $G(\vartheta)$ for every $\vartheta \in \Theta_S$; in particular,

$$E[L^H(T) \cdot G_T(\vartheta)] = 0 \;\forall \vartheta \in \Theta_S. \tag{23}$$

Then

$$E[H - c - G_T(\vartheta)]^2 = (h - c)^2 + E[G_T(\zeta^H - \vartheta)]^2 + E[L^H(T)]^2,$$

and it is clear that problem 2 admits again the solution $\hat{c} = E[H]$, $\vartheta \hat{=} \zeta^H$, but now with $V = E[L^H(T)]^2$ in 18.
Now we want to approach the question what happens for a general, square integrable semimartingale X. In view of the above discussion it is tempting to try to reduce this general problem to the case where X is a martingale. This can be accomplished by an absolutely continuous change of the probability measure P as we have already done before.

Definition 59. *A signed measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ is called a signed $G_T(\Theta_S)$-martingale measure, if $\mathcal{Q}(\Omega) = 1$, $\mathcal{Q} << P$ with $(\frac{d\mathcal{Q}}{dP}) \in L^2(P)$, and*

$$E[\frac{d\mathcal{Q}}{dP} \cdot G_T(\vartheta)] = 0, \ \forall \vartheta \in \Theta_S. \tag{24}$$

We shall denote by $M_2^s(G_T(\Theta_S))$ the set of all such signed $G_T(\Theta_S)$-martingale measures, and introduce the closed, convex set

$$\begin{aligned} \mathcal{D} &= \{D \in L^2(P) : \mathcal{D} = \frac{d\mathcal{Q}}{d\mathcal{P}}, \mathcal{Q} \in M_2^s(G_T(\Theta_S))\} \\ &= \{D \in L^2(P) : E[D] = 1 \text{ and } E[DG_T(\vartheta)] = 0, \ \forall \vartheta \in \Theta_S\}. \end{aligned} \tag{25}$$

We shall assume from now onwards, that

$$M_2^s(G_T(\Theta_S)) \neq \emptyset \ (\textit{equivalently}, \ \mathcal{D} \neq \emptyset). \tag{26}$$

Remark 9. *The assumption in 26 is equivalent to the requirement that*

$$\textit{the closure in } L^2(P) \textit{ of } G_T(\Theta_S) \textit{ does not contain the constant } 1. \tag{27}$$

The duality approach to Problem 17 is simple, and it is based on the elementary observation

$$\begin{aligned} \min_{x \in \mathbb{R}} [(H - x)^2 + yx] &= (H - (H - \frac{y}{2}))^2 + y(H - \frac{y}{2}) \\ &= yH - \frac{y^2}{4}, \forall y \in \mathbb{R}. \end{aligned} \tag{28}$$

The idea is now, to read 28 with

$$\begin{aligned} x &= c + G_T(\vartheta), \\ y &= 2kD \end{aligned}$$

for a given $c \in \mathbb{R}$ and arbitrary $\vartheta \in \Theta_S$, $D \in \mathcal{D}$ as in 25, and with arbitrary $k \in \mathbb{R}$, to obtain

$$(H - c - G_T(\vartheta))^2 + 2kD(c + G_T(\vartheta)) \geq 2kDH - k^2D^2. \tag{29}$$

Note also that 29 holds as equality for some $\vartheta = \vartheta^{(c)} \in \Theta_S$, $D = D^{(c)} \in \mathcal{D}$ and

$k = k^{(c)} \in \mathbb{R}$, if and only if

$$c + G_T(\vartheta^{(c)}) = H - k^{(c)} D^{(c)}. \tag{30}$$

Now we take expectations in 29 to obtain with the properties of 25:

$$E[H - c - G_T(\vartheta)]^2 \geq -k^2 E[D^2] + 2k[E[DH] - c], \tag{31}$$

for every $k \in \mathbb{R}$, $D \in \mathcal{D}$ and $\vartheta \in \Theta_S$. Clearly,

$$E[D^2] - 1 = Var(D) \geq 0, \ \forall D \in \mathcal{D} \tag{32}$$

and the mapping $k \mapsto -k^2 E[D^2] + 2k[E[DH] - c]$ attains over $\mathbb{R}$ its maximal value

$$\frac{(E[DH] - c)^2}{E[D^2]}$$

at the point

$$k_{D,c} = \frac{E[DH] - c}{E[D^2]}. \tag{33}$$

Thus, we obtain from 31 the inequality

$$\begin{aligned} V(c) &= \inf_{\vartheta \in \Theta_S} E[H - c - G_T(\vartheta)]^2 \\ &\geq \sup_{D \in \mathcal{D}} \sup_{k \in \mathbb{R}} [-k^2 E[D^2] + 2k(E[DH] - c)] \\ &= \sup_{D \in \mathcal{D}} \frac{(E[DH] - c)^2}{E[D^2]} =: \tilde{V}(c), \end{aligned} \tag{34}$$

which is the basis of our duality approach. Here $V(c)$ is the value of our original optimization problem 17, whereas $\tilde{V}(c)$ is the value of an auxiliary optimization problem. This kind of duality is useful, only if the dual problem is easier to solve than the problem 17, and if there is no duality gap, i.e. equality holds in 34, so that by computing the value of the dual problem we also compute the value of problem 17. Both these features hold for our setting, as we are about to show. Furthermore, the duality is strong, in the sense that we can identify an optimal $\tilde{D}_c \in \mathcal{D}$ for the dual problem, namely

$$\tilde{V}(c) = \frac{(E[\tilde{D}_c H] - c)^2}{E[\tilde{D}_c^2]} \tag{35}$$

for all but a critical value of the parameter c, and then obtain from this an optimal process $\vartheta^{(c)}$ for the problem 17 via 30.
In order to start, let us introduce the projection operator

$$\pi : L^2(P) \to (G_T(\Theta_S))^\perp$$

with the property

$$E[(H - \pi(H)) \cdot D] = 0,\ \forall H \in L^2(P),\ \forall D \in (G_T(\Theta))^\perp. \tag{36}$$

In particular,

$$E[(H_1 - \pi(H_1)) \cdot \pi(H_2)] = 0,\ \forall H_1, H_2 \in L^2(P), \tag{37}$$

and from 37 and 27 we have

$$E[\pi(1)] = E[\pi^2(1)] > 0. \tag{38}$$

Proposition 13. *The value of the dual problem in 34, namely*

$$\tilde{V}(c) := \sup_{D \in \mathcal{D}} \frac{(E[DH] - c)^2}{E[D^2]} \tag{39}$$

can be computed as

$$\tilde{V}(c) = E[\pi^2(H - c)],\ \forall c \in \mathbb{R}. \tag{40}$$

The supremum in 39 is attained by

$$\tilde{D}_c := \frac{\pi(H - c)}{E[\pi(H - c)]},\ \forall c \neq \hat{c} := \frac{E[\pi(H)]}{E[\pi(1)]}; \tag{41}$$

it is not attained for $c = \hat{c}$.

For every $c \neq \hat{c}$, we shall call the random variable $\tilde{D}_c \in \mathcal{D}$ of 41 the density of the dual-optimal signed martingale measure in $M_2^s(G_T(\Theta_S))$, namely

$$\tilde{Q}_c(A) := \int_A \tilde{D}_c dP,\ A \in \mathcal{F}. \tag{42}$$

Proof 36. *For every $D \in \mathcal{D}$, we have*

$$E[DH] - c = E[D(H - c)] = E[D \cdot \pi(H - c)]$$

thanks to 36. Thus, from the Cauchy-Schwarz inequality,

$$(E[DH] - c)^2 = (E[D \cdot \pi(H - c)])^2 \leq E[D^2] \cdot E[\pi^2(H - c)],$$

which implies

$$\tilde{V}(c) \leq E[\pi^2(H - c)].$$

Now these last two inequalities are valid as equalities, if and only if we can find $\tilde{D}_c \in \mathcal{D}$ of the form

$$\tilde{D}_c = const \cdot \pi(H - c), \tag{43}$$

where the constant has to be chosen so that $E[\tilde{D}_c] = 1$. This is impossible to do if

$$E[\pi(H - c)] = E[\pi(H) - c \cdot \pi(1)] = 0,$$

i.e. if $c = \hat{c}$ as in 41; in other words, the supremum of 39 cannot be attained in this case. But for $c \neq \hat{c}$, the normalizing constant in 43 can be taken as

$$\frac{1}{E[\pi(H - c)]},$$

leading to the expression of 41 and to 40 as well.
It remains to show that 40 holds even for $c = \hat{c}$. For this, let $c_n := c - \frac{1}{n}$, $n \in \mathbb{N}$ and

$$\varphi_n := \pi(H - c_n),\ \varphi := \pi(H - c),\ \tilde{D}_{c_n} = \frac{\varphi_n}{E[\varphi_n]} \in \mathcal{D}$$

so that

$$\begin{aligned}
\frac{(E[\tilde{D}_{c_n}H] - c)^2}{E[\tilde{D}^2_{c_n}]} &= \frac{(E[\tilde{D}_{c_n}H] - c_n - \frac{1}{n})^2}{E[\tilde{D}^2_{c_n}]} \\
&= \frac{(E[\tilde{D}_{c_n}H] - c_n)^2}{E[\tilde{D}^2_{c_n}]} + \frac{1/n^2}{E[\tilde{D}^2_{c_n}]} - \frac{2}{n} \cdot \frac{E[\tilde{D}_{c_n}(H - c_n)]}{E[\tilde{D}^2_{c_n}]} \\
&= E[\pi^2(H - c_n)] + \frac{1/n^2}{E[\tilde{D}^2_{c_n}]} - \frac{2}{n}E[\pi(H - c_n)] \\
&= E[\varphi_n^2] + \frac{1/n^2}{E[\tilde{D}^2_{c_n}]} - \frac{2}{n}E[\varphi_n] \\
&\to E[\varphi^2] = E[\pi^2(H - c)]
\end{aligned}$$

as $n \to \infty$.

Remark 10. *Suppose that for some $h \in \mathbb{R}$ we have*

$$E[DH] = h, \ \forall D \in \mathcal{D}. \tag{44}$$

Then the dual value function of 39 becomes

$$\tilde{V}(c) = \frac{(h-c)^2}{\inf_{D \in \mathcal{D}} E[D^2]} \tag{45}$$

and, for $c \neq h$, the dual-optimal $\tilde{D}_c$ of 41 coincides with

$$\tilde{D} := argmin_{D \in \mathcal{D}} E[D^2] = \frac{\pi(1)}{E[\pi(1)]} = E[\tilde{D}^2] + \mathcal{R} \in \mathcal{D} \tag{46}$$

for some $\mathcal{R} \in (G_T(\Theta_S))^{\perp\perp}$, and $E[\tilde{D}^2] = \frac{1}{E[\pi(1)]} \geq 1$, as we shall establish below. We call $\tilde{D}$ the density of the variance-optimal signed martingale measure

$$\tilde{Q}(A) := \int_A \tilde{D} dP, \ A \in \mathcal{F} \tag{47}$$

in $M_2^s(G_T(\Theta_S))$.

Proof 37. *(of equation 46)*
For any $D \in \mathcal{D}$, we have

$$1 = (E[D \cdot 1])^2 = (E[D \cdot \pi(1)])^2 \leq E[D^2] \cdot E[\pi^2(1)],$$

from 36 and the Cauchy-Schwarz inequality. Equality holds if and only if

$$D = \tilde{D} := const \cdot \pi(1),$$

and the normalizing constant has to be chosen so that $E[\tilde{D}] = 1$, namely, equal to

$$\frac{1}{E[\pi(1)]}.$$

We conclude that $\tilde{D} = \pi(1)/E[\pi(1)]$ satisfies

$$E[D^2] \geq \frac{1}{E[\pi^2(1)]} = \frac{1}{E[\pi(1)]} = E[\tilde{D}^2], \ \forall D \in \mathcal{D}. \tag{48}$$

On the other hand, since

$$1 = \pi(1) + \eta \tag{49}$$

for some $\eta \in (G_T(\Theta_S))^{\perp\perp}$, *we have* $\tilde{D} = (1-\eta)/E[\pi(1)] = E[\tilde{D}^2] + \mathcal{R}$, *with* $\mathcal{R} = -\eta/E[\pi(1)]$.

Now we can apply the developed duality to find solutions to the problems 17 and 18.

Lemma 9. *Suppose that the infimum in 17 is attained by some* $\vartheta^{(c)} \in \Theta_S$. *Then this process satisfies*

$$E[H - c - G_T(\vartheta^{(c)})] = \frac{E[\tilde{D}H] - c}{E[\tilde{D}^2]} \text{ and} \tag{50}$$

$$E[H - c - G_T(\vartheta^{(c)})]^2 = \frac{c^2 - 2c \cdot E[\tilde{D}H]}{E[\tilde{D}^2]} + E[\pi^2(H)], \tag{51}$$

in the notation of 36 and 46.

Proof 38. *(of equation 50)*
The assumption implies that, for any given $\xi \in \Theta_S$, *the function*

$$\begin{aligned} f(\xi) &:= E[H - c - G_T(\vartheta^{(c)} + \epsilon\xi)]^2 \\ &= \epsilon^2 \cdot E[G_T^2(\xi)] - 2\epsilon \cdot E[(H - c - G_T(\vartheta^{(c)})) \cdot G_T(\xi)] + V(c) \end{aligned}$$

attains its minimum over $\mathbb{R}$ *at* $\epsilon = 0$. *This gives* $f'(0) = 0$, *or equivalently,*

$$E[(H - c - G_T(\vartheta^{(c)})) \cdot G_T(\xi)] = 0, \ \forall \xi \in \Theta_S. \tag{52}$$

Let us also notice that the mapping

$$D \mapsto E[\tilde{D}D] \text{ is constant on } \mathcal{D} \tag{53}$$

since we have

$$E[\tilde{D}^2] - E[\tilde{D}D] = E[\tilde{D}(\tilde{D} - D)] = E[\pi(1)(\tilde{D} - D)]/E[\pi(1)] = E[\tilde{D} - D]/E[\pi(1)] = 0$$

thanks to 36.
Now denote $\gamma := E[H - c - G_T(\vartheta^{(c)})]$. *If* $\gamma = 0$, *then the random variable*

$$D_1 := \tilde{D} + (H - c - G_T(\vartheta^{(c)}))$$

belongs to $\mathcal{D}$ because of 52, and 53 implies

$$0 = E[\tilde{D}(D_1 - \tilde{D})] = E[\tilde{D}(H - c - G_T(\vartheta^{(c)}))] = E[\tilde{D}H] - c,$$

so that 50 holds. If $\gamma \neq 0$, then $D_2 := [H - c - G_T(\vartheta^{(c)})]/\gamma$ is in $\mathcal{D}$, and by 53 once again:

$$E[\tilde{D}^2] = E[\tilde{D}D_2] = \frac{1}{\gamma}(E[\tilde{D}H] - c),$$

and so 50 holds in this case, too.

Proof 39. *(of equation 51)*
From 52, the random variable $H-c-G_T(\vartheta^{(c)})$ belongs to the closed subspace $(G_T(\Theta_S))^{\perp}$ with

$$(G_T(\Theta_S))^{\perp} := \{D \in L^2(P) : E[DG_T(\vartheta)] = 0, \forall \vartheta \in \Theta_S\}$$

so we have

$$\begin{aligned}
&E[H - c - G_T(\vartheta^{(c)})]^2 \\
&\quad = E[(H - c - G_T(\vartheta^{(c)})) \cdot (H - \pi(H) + \pi(H) - c - G_T(\vartheta^{(c)}))] \\
&\quad = E[(H - c - G_T(\vartheta^{(c)})) \cdot (\pi(H) - c)] \\
&\quad = E[\pi^2(H)] - cE[\pi(H)] - c\frac{E[\tilde{D}H] - c}{E[\tilde{D}^2]}.
\end{aligned}$$

51 follows from

$$E[\tilde{D}H] = E[\tilde{D}^2] \cdot E[\pi(H)]. \tag{54}$$

Theorem 64. *(i) Suppose that there exists some $\vartheta^{(c)} \in \Theta_S$ which attains the infimum in problem 17. Then this $\vartheta^{(c)}$ satisfies*

$$H - c - G_T(\vartheta^{(c)}) = \pi(H - c), \tag{55}$$

and there is no duality gap in 34, namely

$$\begin{aligned}
V(c) = \tilde{V}(c) &= E[\pi^2(H - c)] \\
&= \frac{(E[\tilde{D}H] - c)^2}{E[\tilde{D}^2]} + E[\pi^2(H)] - \frac{(E[\pi(H)])^2}{E[\pi(1)]}.
\end{aligned} \tag{56}$$

(ii) Conversely, suppose there exists some $\vartheta^{(c)} \in \Theta_S$ that satisfies 55; then this $\vartheta^{(c)}$ attains the infimum in problem 17, and the equalities of 56 *hold.*

Proof 40. *Proof.* (of equation 56)
Under the assumption of (i), we claim that

$$V(c) = E[H - c - G_T(\vartheta^{(c)})]^2$$
$$= \frac{(E[\tilde{D}H] - c)^2}{E[\tilde{D}^2]} + E[\pi^2(H)] - (E[\pi(H)])^2 \cdot E[\tilde{D}^2] \tag{57}$$
$$= E[\pi^2(H - c)] = \tilde{V}(c) \tag{58}$$

which proves 56.

□

Proof. (of equation 55; $c \neq \hat{c}$)
Let us write 29 with $\vartheta = \vartheta^{(c)}$, $D = \tilde{D}_c$ as in 41, and

$$k = k_c := k_{\tilde{D}_c, c} = \frac{E[\tilde{D}_c H] - c}{E[\tilde{D}_c^2]}$$

as in 33: namely,

$$(H - c - G_T(\vartheta^{(c)}))^2 + 2k_c\tilde{D}_c(c + G_T(\vartheta^{(c)})) \geq 2k_c\tilde{D}_c H - (k_c\tilde{D}_c)^2, \ a.s. \tag{59}$$

Taking expectations in 59, and recalling the optimality of $\vartheta^{(c)}$ as well as Proposition 13, we obtain

$$V(c) = E[H - c - G_T(\vartheta^{(c)})]^2$$
$$\geq -k_c^2 E[\tilde{D}_c^2] + 2k_c(E[\tilde{D}_c H] - c) \tag{60}$$
$$= \frac{(E[\tilde{D}_c H)] - c)^2}{E[\tilde{D}_c^2]} = \tilde{V}(c).$$

But from 56 we know that 60 actually holds as equality, which means that the left-hand side and the right-hand side of 59 must hold as equality, which we know happens only if 30 holds, namely only if

$$H - c - G_T(\vartheta^{(c)}) = k_c\tilde{D}_c = \frac{E[\tilde{D}_c H] - c}{E[\tilde{D}_c^2]} \frac{\pi(H - c)}{E[\pi(H - c)]} = \pi(H - c)$$

holds a.s.. □

Proof. (of equation 55; $c = \hat{C}$)

In this case we shall need a new kind of duality, namely with

$$\mathcal{L} := \{L \in (G_T(\Theta_S))^\perp : E[L] = 0\} \tag{61}$$

replacing the space $\mathcal{D}$ of 25; the elements of $\mathcal{L}$ will be the dual variables in this new duality. We begin by writing 28 with $x = c + G_T(\vartheta)$, $y = 2L$ for arbitrary $\vartheta \in \Theta_S$, $L \in \mathcal{L}$:

$$(H - c - G_T(\vartheta))^2 + 2L(c + G_T(\vartheta)) \geq 2LH - L^2, \tag{62}$$

with equality if and only if

$$H - c - G_T(\vartheta) = L \tag{63}$$

holds a.s.. Taking expectations in 62, we obtain

$$\begin{aligned} E[H - c - G_T(\vartheta)]^2 &\geq E[2L(H - c) - L^2] \\ &= E[2L \cdot \pi(H - c) - L^2] \\ &= E[\pi^2(H - c)] - E[L - \pi(H - c)]^2. \end{aligned} \tag{64}$$

This suggests that we would read 62 - 64 with $\vartheta = \vartheta^{(c)}$, the element of Θ_S that attains the infimum in problem 17, and $L = \tilde{L} := \pi(H - c)$, noting that $E[\tilde{L}] = 0$ since $c = \hat{c} = E[\pi(H)]/E[\pi(1)]$. With these choices, the left-most member of 64 becomes

$$E[H - c - G_T(\vartheta^{(c)})]^2 = V(c),$$

whilst its right-most member is $E[\pi^2(H - c)]\tilde{V}(c)$. From 56 we know that these two quantities are equal, so the two sides of 62 have the same expectation. This means that 62 must hold as equality, which happens only if 63 is valid, namely

$$H - c - G_T(\vartheta^{(c)}) = \pi(H - c), \ a.s..$$

□

Proof. (of part (ii))
Suppose there exists some $\vartheta^{(c)} \in \Theta_S$ that satisfies 55; then

$$E[H - c - G_T(\vartheta^{(c)})]^2 = E[\pi^2(H - c)] = \tilde{V}(c)$$

from Proposition 13. But we also have $\tilde{V}(c) \leq V(c) \leq E[H - c - G_T(\vartheta^{(c)})]^2$. In other words, these last two inequalities are valid as equalities, $\vartheta^{(c)}$ attains the infimum in problem 17, and 56 holds. □

Remark 11. *The case of $G_T(\Theta_S)$ closed:*
If $G_T(\Theta_S)$ is closed in $L^2(P)$, then the infimum in problem 17 is attained, as was assumed in Lemma 9 and in Theorem 64 (i). In this case we have of course $G_T(\Theta_S) = (G_T(\Theta_S))^{\perp\perp}$, and every $H \in L^2(P)$ admits a decomposition of the form

$$H = \pi(H) + G_T(\xi^H) \text{ for some } \xi^H \in \Theta_S. \tag{65}$$

In particular, there exists $\xi^1 \in \Theta_S$ so that 49 holds, and thus

$$H - c - \pi(H - c) = [H - \pi(H)] - c[1 - \pi(1)] = G_T(\xi^H - c\xi^1).$$

Comparing this expression with 55, we conclude that 55 is satisfied with the choice

$$\vartheta^{(c)} = \xi^H - c\xi^1. \tag{66}$$

According to Theorem 64 (ii), the process $\vartheta^{(c)} \in \Theta_S$ of 66 is then optimal for problem 17, and 56 holds.

Theorem 65. *Suppose that $G_T(\Theta_S)$ is closed in $L^2(P)$. Then the value of problem 18 is given as*

$$V = V(\hat{c}) = E[\pi^2(H)] - \frac{(E[\pi(H)])^2}{E[\pi(1)]} \tag{67}$$

with the notation

$$\hat{c} = \frac{E[\pi(H)]}{E[\pi(1)]} = E[\tilde{D}H] = E[\frac{d\mathcal{Q}}{dP}H] \tag{68}$$

of 41. Furthermore, the infimum for problem 18 is attained by the pair $(\hat{c}, \hat{\vartheta})$ with $\hat{c}$ as in 68 and with

$$\hat{\vartheta} := \vartheta^{\hat{c}} = \xi^H - \hat{c}\xi^1. \tag{69}$$

Proof 41. *Immediate from Theorem 64 and Remark 11, when it is observed that the number $\hat{c}$ of 68 minimizes the expression of 56 over $c \in \mathbb{R}$.*

Remark 12. *If $G_T(\Theta_S)$ is closed in $L^2(P)$, then the process $\hat{\vartheta}$ of 69 also*

$$\text{minimizes } Var(H - G_T(\vartheta)), \text{ over all } \vartheta \in \Theta_S. \tag{70}$$

This is because for any $\vartheta \in \Theta_S$*, and with* $c_\vartheta := E[H - G_T(\vartheta)]$*, we have:*

$$\begin{aligned} Var[H - G_T(\vartheta)] &= E[H - c_\vartheta - G_T(\vartheta)]^2 \\ &\geq E[H - \hat{c} - G_T(\hat{\vartheta})]^2 = Var[H - G_T(\hat{\vartheta})], \end{aligned}$$

from Theorem 65. More generally, for any given $c \in \mathbb{R}$*, the process* $\vartheta^{(c)} \in \Theta_S$ *of 73 has the mean-variance efficiency property*

$$Var[H - G_T(\vartheta^{(c)})] \leq Var[H - G_T(\vartheta)], \text{ for any } \vartheta \in \Theta_S \text{ that satisfies } E[H - G_T(\vartheta)] = E[H - G_T(\vartheta^{(c)})]. \tag{71}$$

Indeed, let $\mu_c := E[H - G_T(\vartheta^{(c)})]$ *and observe that, for any* $\vartheta \in \Theta_S$ *with* $E[H - G_T(\vartheta)] = \mu_c$*, we have*

$$\begin{aligned} Var[H - G_T(\vartheta)] = Var[H - c - G_T(\vartheta)] &= E[H - c - G_T(\vartheta)]^2 - (\mu_c - c)^2 \\ &\geq E[H - c - G_T(\vartheta^{(c)})]^2 - (E[H - c - G_T(\vartheta^{(c)})])^2 \\ &= Var[H - c - G_T(\vartheta^{(c)})] = Var[H - G_T(\vartheta^{(c)})]. \end{aligned}$$

With the notation of Proposition 11 we get the following theorem:

Theorem 66. *Assume that* $\mathcal{E} = \mathcal{E}(N)$ *is regular and satisfies* $R_2(P)$ *and that* $X \in \mathcal{H}^2_{loc}(P)$ *is an* $\mathcal{E}$*-local martingale. Then the following hold:*
(i) The infimum in problem 17 is attained , as was assumed in Lemma 9 and in Theorem 64 (i). In this case we have of course $G_T(\Theta_S) = (G_T(\Theta_S))^{\perp\perp}$*, and every* $H \in L^2(P)$ *admits a decomposition of the form*

$$H = \pi(H) + G_T(\xi^H) \text{ for some } \xi^H \in \Theta_S. \tag{72}$$

In particular, there exists $\xi^1 \in \Theta_S$ *so that 49 holds, and thus*

$$H - c - \pi(H - c) = [H - \pi(H)] - c[1 - \pi(1)] = G_T(\xi^H - c\xi^1).$$

Comparing this expression with 55, we conclude that 55 is satisfied with the choice

$$\vartheta^{(c)} = \xi^H - c\xi^1. \tag{73}$$

According to Theorem 64 (ii), the process $\vartheta^{(c)} \in \Theta_S$ *of 73 is then optimal for problem*

17, and 56 holds.
(ii) The value of problem 2 is given as

$$V = V(\hat{c}) = E[\pi^2(H)] - \frac{(E[\pi(H)])^2}{E[\pi(1)]} \tag{74}$$

with the notation

$$\hat{c} = \frac{E[\pi(H)]}{E[\pi(1)]} = E[\tilde{D}H] = E[\frac{d\mathcal{Q}}{dP}H] \tag{75}$$

of 41. Furthermore, the infimum for problem 18 is attained by the pair $(\hat{c}, \hat{\vartheta})$ with $\hat{c}$ as in 75 and with

$$\hat{\vartheta} := \vartheta^{\hat{c}} = \xi^H - \hat{c}\xi^1. \tag{76}$$

(iii) The process $\hat{\vartheta}$ of 69 also

$$\text{minimizes } Var(H - G_T(\vartheta)), \text{ over all } \vartheta \in \Theta_S. \tag{77}$$

Proof 42. *Follows directly from Proposition 11, Remark 11, Remark 12 and Theorem 65.*

6.2 The Space $G_T(\Theta_2^{eq})$

In this section we want to find explicit hedging strategies for the space $G_T(\Theta_2^{eq})$. First we are able to state some obvious results like we already did in the previous section for the space $G_T(\Theta_S)$. We again have a very obvious solution for our problem 18, if it is known that the random variable H is of the form

$$H = h + G_T(\zeta^H) \tag{78}$$

for some $h \in \mathbb{R}$ and $\zeta^H \in \Theta_2^{eq}$; because then we can take $\hat{c} = h$, $\hat{\vartheta} = \zeta^H$, and deduce that $V = 0$.
If X is a Brownian motion and $\mathcal{F}$ the filtration generated by X, then H can be written in the form of 78 with $h = E[H]$.

To face our problems 17-19 for a general semimartingale X, we first need a few definitions. We state them under the assumption, that X is a continuous $\mathbb{R}^d$-valued semimartingale which is locally in $L^2(P)$ and assume that $\mathcal{M}_2^e(X) \neq \emptyset$. Note that $G_T(\Theta_2^{eq})$ is closed then.

Definition 60. *We say that $G_T(\Theta_2^{eq})$ admits no approximate profits in L^2 if it does not contain the constant* 1.

This notion is very intuitive:
It says that one cannot approximate (in the L^2-sense) the riskless payoff 1 by a self-financing strategy with initial wealth 0. This is a no-arbitrage condition on the financial market underlying $G_T(\Theta_2^{eq})$.
In our next definition we establish an analogous notation for the space of signed $G_T(\Theta_2^{eq})$-martingale measures as we did in the previous section for the signed $G_T(\Theta_S)$-martingale measures.

Definition 61. *A signed $G_T(\Theta_2^{eq})$-martingale measure is a signed measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$ with $\mathcal{Q}(\Omega) = 1$, $\mathcal{Q} << P$ with $\frac{d\mathcal{Q}}{dP} \in L^2$ and*

$$E_{\mathcal{Q}}[g] = E[\frac{d\mathcal{Q}}{dP} g] = 0$$

for all $g \in G_T(\Theta_2^{eq})$.
$M_2^s(G_T(\Theta_2^{eq}))$ denotes the convex set of all signed $G_T(\Theta_2^{eq})$-martingale measures and an

element $\tilde{P}_{(G_T(\Theta_2^{eq}))}$ *of* $\mathcal{M}_2^s(G_T(\Theta_2^{eq}))$ *is called variance-optimal if it minimizes*

$$||\frac{d\mathcal{Q}}{dP}||_{L^2} = \sqrt{1 + Var[\frac{d\mathcal{Q}}{dP}]}$$

over all $\mathcal{Q} \in \mathcal{M}_2^s(G_T(\Theta_2^{eq}))$.

Lemma 10. *The following assertions hold:*
(i) $G_T(\Theta_2^{eq})$ *admits no approximate profits in* L^2 *if and only if* $\mathcal{M}_2^s(G_T(\Theta_2^{eq})) \neq \emptyset$.
(ii) If $\mathcal{M}_2^s(G_T(\Theta_2^{eq})) \neq \emptyset$ *then* $\bar{\mathcal{A}} = \mathbb{R} + \overline{G_T(\Theta_2^{eq})}$.
(iii) If $G_T(\Theta_2^{eq})$ *admits no approximate profits in* L^2, *then the variance-optimal signed* $G_T(\Theta_2^{eq})$*-martingale measure* $\tilde{P}_{G_T(\Theta_2^{eq})}$ *exists, is unique and satisfies*

$$\frac{d\tilde{P}_{G_T(\Theta_2^{eq})}}{dP} \in \mathcal{A}. \tag{79}$$

Proof 43. *(i) An element* $\mathcal{Q}$ *of* $\mathcal{M}_2^s(G_T(\Theta_2^{eq}))$ *can be identified with a continuous linear functional* ψ *on* $\mathcal{L}^2$ *satisfying* $\psi = 0$ *on* $G_T(\Theta_2^{eq})$ *and* $\psi(1) = 1$ *by setting*

$$\psi(U) = E[\frac{d\mathcal{Q}}{dP}U] = (\frac{d\mathcal{Q}}{dP}, U).$$

Hence (i) is clear from the Hahn-Banach theorem.
(ii) Any $g \in G_T(\Theta_2^{eq})$ *is the limit in* L^2 *of a sequence* (g_n) *in* $G_T(\Theta_2^{eq})$; *hence* $c+g_n = a_n$ *is a Cauchy sequence in* $\mathcal{A}$ *and thus converges in* L^2 *to a limit* $a \in \mathcal{A}$ *so that* $c+g = a \in \bar{\mathcal{A}}$. *This gives the inclusion* $\supseteq$ *in general. For the converse, we use the assumption that* $\mathcal{M}_2^s(G_T(\Theta_2^{eq})) \neq \emptyset$ *to obtain a signed* $G_T(\Theta_2^{eq})$*-martingale measure* $\mathcal{Q}$. *The random variable* $Z := \frac{d\mathcal{Q}}{dP}$ *is then in* $G_T(\Theta_2^{eq})^{\perp}$ *and satisfies* $(Z, 1) = \mathcal{Q}(\Omega) = 1$. *For any* $a \in \bar{\mathcal{A}}$, *there is a sequence* $a_n = c_n + g_n$ *in* $\mathcal{A}$ *converging to* a *in* L^2. *Since* $c_n + g_n \in \mathbb{R} + G_T(\Theta_2^{eq})$ *for all* n, *we conclude that* $c_n = (c_n + g_n, Z) = (a_n, Z)$ *converges in* $\mathbb{R}$ *to* $(a, Z) =: c$. *Therefore* $g_n = a_n - c_n$ *converges in* L^2 *to* $g := a - c$ *and since this limit is in* $G_T(\Theta_2^{eq})$ *(because of the closedness of* $G_T(\Theta_2^{eq})$*), we have* $a = c + g \in \mathbb{R} + G_T(\Theta_2^{eq})$ *which proves the inclusion* $\subseteq$.
(iii) Existence and uniqueness of $\tilde{P}_{G_T(\Theta_2^{eq})}$ *are clear once we observe that we have to minimize* $||Z||$ *over the closed convex set*

$$\mathcal{Z} := \{Z = \frac{d\mathcal{Q}}{dP} | \mathcal{Q} \in \mathcal{M}_2^s(G_T(\Theta_2^{eq}))\}$$

which is non-empty thanks to (i). For any fixed $Z_0 \in \mathcal{Z}$, *the projection* $\mathcal{Z}$ *of* Z_0 *in* L^2

on $\mathcal{A}$ is again in $\mathcal{Z}$; in fact, one easily verifies that $\tilde{\psi}(U) := (\tilde{Z}, U)$ is 0 on $G_T(\Theta_2^{eq})$ and has $\tilde{\psi}(1) = 1$. Since we have (because of the closedness of $\mathcal{A}$)

$$\tilde{Z} = \tilde{c} + \tilde{g}$$

with $\tilde{g} \in G_T(\Theta_2^{eq})$, we obtain $(Z, \tilde{Z}) = \tilde{c} = (\tilde{Z}, \tilde{Z})$ for all $Z \in \mathcal{Z}$ and therefore

$$||Z||^2 = ||\tilde{Z}||^2 + ||Z - \tilde{Z}||^2 \geq ||\tilde{Z}||^2$$

for all $Z \in \mathcal{Z}$.
Hence we conclude that $\frac{d\tilde{P}_{G_T(\Theta_2^{eq})}}{dP} = \tilde{Z}$ is in $\mathcal{A}$.

For any $g \in G_T(\Theta_2^{eq})$ and any $\mathcal{Q} \in \mathcal{M}_2^s(G_T(\Theta_2^{eq}))$, we have

$$1 = E_{\mathcal{Q}}[1 - g] = E[\frac{d\mathcal{Q}}{dP}(1 - g)] \leq ||\frac{d\mathcal{Q}}{dP}||_{L^2}||1 - g||_{L^2}$$

by the Cauchy-Schwarz inequality and therefore

$$\begin{aligned} \frac{1}{\inf_{\mathcal{Q}\in\mathcal{M}_2^s(G_T(\Theta_2^{eq}))} ||\frac{d\mathcal{Q}}{dP}||_{L^2}} &= \sup_{\mathcal{Q}\in\mathcal{M}_2^s(G_T(\Theta_2^{eq}))} \frac{1}{||\frac{d\mathcal{Q}}{dP}||_{L^2}} \\ &\leq \inf_{g\in G_T(\Theta_2^{eq})} ||1 - g||_{L^2}. \end{aligned}$$

This indicates that finding the variance-optimal signed $G_T(\Theta_2^{eq})$-martingale measure is the dual problem to approximating in L^2 the constant 1 by elements of $G_T(\Theta_2^{eq})$. This duality is reflected in the next result which gives the $G_T(\Theta_2^{eq})$-approximation price as an expectation under $\tilde{P}_{G_T(\Theta_2^{eq})}$.

Proposition 14. *Suppose $\mathcal{M}_2^s(G_T(\Theta_2^{eq})) \neq \emptyset$. If a contingent claim $H \in L^2$ admits a $G_T(\Theta_2^{eq})$-mean-variance optimal pair $(\tilde{V_0}, \tilde{g})$ (i.e. $(\tilde{V_0}, \tilde{g})$ minimizes $||H - V_0 - G_T(\vartheta)||_{L^2}$ over all $(V_0, g) \in \mathbb{R} \times G_T(\Theta_2^{eq})$), the $G_T(\Theta_2^{eq})$-approximation price of H is given by*

$$\tilde{V_0} = \tilde{E}_{G_T(\Theta_2^{eq})}[H]$$

where $\tilde{E}_{G_T(\Theta_2^{eq})}$ denotes the expectation under the variance-optimal signed $G_T(\Theta_2^{eq})$-martingale measure $\tilde{P}_{G_T(\Theta_2^{eq})}$.

Proof 44. *If H admits a $G_T(\Theta_2^{eq})$-mean variance optimal pair $(\tilde{V_0}, \tilde{g})$, then $\tilde{V_0} + \tilde{g}$ is the projection in L^2 of H on $\bar{\mathcal{A}}$ by Lemma 10. Since $H - \tilde{V_0} - \tilde{g}$ is then in the orthogonal* *complement of $\bar{\mathcal{A}}$, 79 implies that*

$$E[(H-\tilde{V}_0-\tilde{g})\frac{d\tilde{P}_{G_T(\Theta_2^{eq})}}{dP}]=0$$

and so we obtain

$$\tilde{V}_0=E[(H-\tilde{g})\frac{d\tilde{P}_{G_T(\Theta_2^{eq})}}{P}]=\tilde{E}_{G_T(\Theta_2^{eq})}[H]$$

because $\tilde{P}_{G_T(\Theta_2^{eq})}$ is in $\mathcal{M}_2^s(G_T(\Theta_2^{eq}))$.

Definition 62. *The variance-optimal signed martingale measure $\tilde{P}$ for X is defined as the variance-optimal signed $G_T(\Theta_2^s)$-martingale measure.*

In general, $\tilde{P}$ is unfortunately a signed measure. But for a continuous process X, the situation is better. Note that we use the notation

$$\mathcal{M}_2^e(X):=\mathcal{M}_2^e(P)$$

to be in line with the existing literature.

Theorem 67. *If X is a continuous $\mathbb{R}^d$-valued semimartingale and $\mathcal{M}_2^e(X)\neq\emptyset$, then $\tilde{P}$ is in $\mathcal{M}_2^e(X)$. In other words, the variance-optimal signed martingale measure for X is then automatically equivalent to P and in particular a probability measure.*

Now we are able to find a solution to our problem 18. For that, we assume that X is continuous and $\mathcal{M}_2^e(X)\neq\emptyset$. By Theorem 67, the variance-optimal martingale measure $\tilde{P}$ for X then exists and is equivalent to P. Moreover, one can show that the process

$$\tilde{Z}_t:=\tilde{E}[\frac{d\tilde{P}}{dP}|\mathcal{F}_t],$$

$0\leq t\leq T$, can be written as

$$\tilde{Z}_t=\tilde{Z}_0+\int_0^t\tilde{\zeta}_u dX_u,$$

$0\leq t\leq T$, for some $\tilde{\zeta}\in\Theta_2^{eq}$. We also note that by Lemma 10 $\tilde{Z}_0$ is a non-random constant. As the next result shows, $\tilde{P}$, $\tilde{Z}$ and $\tilde{\zeta}$ all turn up in the solution of the mean-variance hedging problem.

Theorem 68. *Suppose that X is a continuous process such that $\mathcal{M}_2^e(X)\neq\emptyset$. Let $H\in L^2(P)$ be a contingent claim and write the Galtchouk-Kunita-Watanabe decomposition* *of H under $\tilde{P}$ with respect to X as*

$$H = \tilde{E}[H|\mathcal{F}_0] + \int_0^T \xi_u^{H,\tilde{P}} dX_u + L_T^{H,\tilde{P}} = V_T^{H,\tilde{P}} \tag{80}$$

with

$$V_t^{H,\tilde{P}} := \tilde{E}[H|\mathcal{F}_t] = \tilde{E}[H, \mathcal{F}_0] + \int_0^t \xi_u^{H,\tilde{P}} dX_u + L_t^{H,\tilde{P}},$$

for $0 \leq t \leq T$.

Then the solution to problem 18 is given by

$$\tilde{V}_0 = \tilde{E}[H] \tag{81}$$

and

$$\tilde{\vartheta}_t = \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}(V_{t-}^{H,\tilde{P}} - \tilde{E}[H] - \int_0^t \tilde{\vartheta}_u dX_u) \tag{82}$$

$$= \xi_t^{H,\tilde{P}} - \tilde{\zeta}_t(\frac{V_0^{H,\tilde{P}} - \tilde{E}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u} dL_u^{H,\tilde{P}}), \tag{83}$$

for $0 \leq t \leq T$.

Proof 45. *Thanks to the closedness of $G_T(\Theta_2^{eq})$, 81 follows immediately from Proposition 14. According to Corollary 16 of Schweizer (1996), $\tilde{\vartheta}$ is obtained by projecting the random variable $H - \tilde{E}[H]$ on $G_T(\Theta_2^{eq})$ and this is in particular dealt with in Rheinländer/Schweizer (1997). The representation 82 is very similar to their Theorem 6, but we cannot directly use their results since they work with a different space of integrands. Thus we appeal to some results from Gouriéroux/Laurent/Pham (1998) and this involves a second change of measure. Because $\tilde{Z}$ is a strictly positive P-martingale and $\tilde{Z}_0$ is deterministic, we can define a new probability measure $\tilde{R} \approx \tilde{P} \approx P$ by setting*

$$\frac{d\tilde{R}}{d\tilde{P}} := \frac{\tilde{Z}_T}{\tilde{Z}_0}.$$

Clearly, the $\mathbb{R}^{d+1}$-valued process

$$Y = \frac{1/\tilde{Z}}{X/\tilde{Z}}$$

is then a continuous local $\tilde{R}$-martingale since $\tilde{P} \in \mathcal{M}_2^e(X)$. The density of $\tilde{R}$ with respect to P is $\tilde{Z}_T^2/\tilde{Z}_0$ and because $\tilde{Z}_0$ is deterministic, H is in $L^2(P)$ if and only if $H/\tilde{Z}_T$ is *in $L^2(\tilde{R})$. The basic idea of Gouriéroux/laurent/Pham (1998) is now to use $\tilde{Z}/\tilde{Z}_0$ as a new numéraire, rewrite the original problem in terms of the corresponding new quantities and apply the Galtchouk-Kunita-Watanabe decomposition theorem to $H/\tilde{Z}_T$ under $\tilde{R}$ with respect to Y. This yields*

$$\frac{H}{\tilde{Z}_T} = E_{\tilde{R}}[\frac{H}{\tilde{Z}_T}|\mathcal{F}_0] + \int_0^T \psi_u dY_u + L_T \tag{84}$$

for some $\mathbb{R}^{d+1}$-valued $\psi \in L(Y)$ such that $\int \psi dY$ is a square integrable martingale with null at 0 under $\tilde{R}$ and L being a square integrable martingale with null at 0 under $\tilde{R}$ strongly $\tilde{R}$-orthogonal to Y. According to Theorem 5.1 and the subsequent remark in Gouriéroux/Laurent/Pham (1998), $\tilde{\vartheta}$ is then given by

$$\tilde{\vartheta}_t^i = \psi_t^i + \tilde{\zeta}_t^i(\frac{\tilde{E}[H]}{\tilde{Z}_0} + \int_0^t \psi_u dY_u - \psi_t^{tr} Y_t), \tag{85}$$

for $0 \leq t \leq T$ and $i = 1, \dots, d$, if we note that the relation between their terminology and ours is given by $V(\tilde{a}) = \tilde{Z}/\tilde{Z}_0$, $X^i(\tilde{a}) = \tilde{Z}_0 Y^i$ and $\tilde{a} = -\tilde{\zeta}/\tilde{Z}$. By using Proposition 8 of Rheinländer/Schweizer (1997), 85 can be rewritten as

$$\tilde{\vartheta} = \frac{\tilde{E}[H]}{\tilde{Z}_0}\tilde{\zeta} + \vartheta \tag{86}$$

with ϑ corresponding to ψ from 84 via equation 85 in Rheinländer/Schweizer (1997). Hence it only remains to obtain ϑ or ψ in terms of the decomposition 84 and this is basically already contained in Rheinländer/Schweizer (1997).
More precisely, we start from 84 and argue as in Proposition 10 of Rheinländer/Schweizer (1997) to express the quantities in the decomposition 80 in terms of ψ and L. Note that as long as we make no integrability assertions, the argument only uses Proposition 8 of Rheinländer/Schweizer (1997) which holds as soon as $\mathcal{M}_2^e(X) \neq \emptyset$. The uniqueness of the Galtchouk-Kunita-Watanabe decomposition then implies that

$$L_t^{H,\tilde{P}} = \int_0^t \tilde{Z}_u dL_u,$$

for $0 \leq t \leq T$, and

$$\xi_t^{H,\tilde{P}} = \frac{V_0^{H,\tilde{P}}}{\tilde{Z}_0}\tilde{\zeta}_t + \vartheta_t + L_{t-}\tilde{\zeta}_t,$$

for $0 \leq t \leq T$.

Solving this for ϑ and plugging the result into 86 yields the second expression in 82. The first then follows similarly as in the proof of Theorem 6 of Rheinländer/Schweizer (1997).

The next example is taken from Mavuso (2014) but we adjust it to our setting.

Example 3. *We consider two correlated assets with discounted price processes X and $\bar{U}$. While trading of X is possible at all times, U can only be traded at time 0. Furthermore, we consider a derivative $H \in L^2(P)$ on $\bar{U}$, which we want to hedge by trading strategies in $\bar{U}$ and dynamic hedging strategies in X and in the risk-free asset (whose discounted value is 1 at all times). By ϑ, η and θ, we describe the shares of X, the risk-free asset and the shares of $\bar{U}$. So our goal is to find $\vartheta \in \Theta_2^{eq}$, such that*

$$E[(\hat{V}_0 + \int_0^T \hat{\vartheta}_t dX_t + \hat{\theta}\bar{U}_T - H)^2] = \min_{(\theta,\vartheta,V_0)\in\mathbb{R}\times\Theta_2^{eq}\times\mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t + \theta\bar{U}_T - H)^2] \quad (87)$$

holds. Note that equation 87 can also be written as

$$\min_{\theta\in\mathbb{R}}(\min_{(\vartheta,V_0)\in\Theta_2^{eq}\times\mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t + \theta\bar{U}_T - H)^2)] = \min_{\theta\in\mathbb{R}} g(\theta)$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ for a fixed $\theta \in \mathbb{R}$ is defined as

$$\begin{aligned} g(\theta) &:= \min_{(\vartheta,V_0)\in\Theta_2^{eq}\times\mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t + \theta\bar{U}_T - H)^2] \\ &= \min_{(\vartheta,V_0)\in\Theta_2^{eq}\times\mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t - (H - \theta\bar{U}_T))^2]. \end{aligned}$$

This way, we can first determine the mean-variance hedging strategy for the modified claim $H - \theta\bar{U}_T$ for a fixed θ and then minimize g over all $\theta \in \mathbb{R}$. The function g is well defined, since the minimum

$$\min_{(\vartheta,V_0)\in\Theta_2^{eq}\times\mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t - (H - \theta\bar{U}_T))^2] \quad (88)$$

exists, if $H - \theta\bar{U}_T \in L^2(P)$. We proceed according to the following two steps:
Step 1: *We minimize $g(\theta)$ over all $\theta \in \mathbb{R}$.*
Step 2: *We minimize g over all $\theta \in \mathbb{R}$ to find the mean-variance optimal strategy.*

We will carry out those two steps now:

Step 1: *To minimize $g(\theta)$ for a fixed θ, we start by describing the Galtchouk-Kunita-*

Watanabe decomposition of $H - \theta \bar{U}_T$ with respect to the local martingale X under the variance-optimal martingale measure $\tilde{P}$:

$$H - \theta \bar{U}_T = \tilde{E}[H - \theta \bar{U}_T] + \int_0^T \vartheta_t^{H,\theta} dX_t + L_Z^{H,\theta}.$$

It follows from Theorem 68, that the minimum (for a fixed $\theta \in \mathbb{R}$) is attained in the point $(\vartheta^\theta, V_0^\theta)$, where

$$V_0^\theta = \tilde{E}[H - \theta \bar{U}_T] = \tilde{E}[H] - \theta \tilde{E}[\bar{U}_T]$$

and

$$\vartheta_t^\theta = \vartheta_t^{H,\theta} - \frac{\zeta_t}{\tilde{Z}_t}(V_{t-}^{H,\theta} - V_0^\theta - \int_0^t \vartheta_s^\theta dX_s), \; 0 \leq t \leq T.$$

Furthermore, we have $V_t^{H,\theta} = \tilde{E}[H - \theta \bar{U}_T | \mathcal{F}_t]$. Then we get

$$\begin{aligned}
g(\theta) &= \min_{(\vartheta, V_0) \in \Theta_2^{eq} \times \mathbb{R}} E[(V_0 + \int_0^T \vartheta_t dX_t - (H - \theta \bar{U}_T))^2] \\
&= E[(V_0^\theta + \int_0^T \vartheta_t^\theta dX_t - (H - \theta \bar{U}_T))^2] \\
&= E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{H,\theta}\rangle_t],
\end{aligned}$$

where $Z_t^{\tilde{P}} := E[\frac{d\tilde{P}}{dP}|\mathcal{F}_t]$ is the density process of $\tilde{P}$ according to P. This way, we have found $g(\theta)$ for a fixed $\theta \in \mathbb{R}$.

Step 2: *In the next step, we will determine θ, which minimizes g. Therefore, we will first provide the Galtchouk-Kunita-Watanabe decompositions of H and $\bar{U}_T$ with respect to the continuous local $\tilde{P}$-martingal X:*

$$H = \tilde{E}[H] + \int_0^T \vartheta_t^{H,\tilde{P}} dX_t + L_T^{H,\tilde{P}} \text{ and } \bar{U}_T = \tilde{E}[\bar{U}_T] + \int_0^T \vartheta_t^{\bar{U},\tilde{P}} dX_t + L_T^{\bar{U},\tilde{P}}.$$

Due to the linearity of the projection we get

$$L^{H,\theta} = L^{H,\tilde{P}} - \theta L^{\bar{U},\tilde{P}} \text{ and } \vartheta^{H,\theta} = \vartheta^{H,\tilde{P}} - \theta \vartheta^{\bar{U},\tilde{P}}.$$

With the properties of the quadratic variation we conclude

$$\langle L^{H,\theta}\rangle = \langle L^{H,\tilde{P}}\rangle - 2\theta \langle L^{H,\tilde{P}}, L^{\bar{U},\tilde{P}}\rangle + \theta^2 \langle L^{\bar{U},\tilde{P}}\rangle.$$

Thus we get

$$
\begin{aligned}
g(\theta) &= E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{H,\theta}\rangle_t] \\
&= E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d(\langle L^{H,\tilde{P}}\rangle - 2\theta\langle L^{H,\tilde{P}}, L^{\bar{U},\tilde{P}}\rangle + \theta^2\langle L^{\bar{U},\tilde{P}}\rangle)_t] \\
&= E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{H,\tilde{P}}\rangle_t] - 2\theta E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{H,\tilde{P}}, L^{\bar{U},\tilde{P}}\rangle_t] + \theta^2 E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{\bar{U},\tilde{P}}\rangle_t].
\end{aligned}
$$

So g is a quadratic function in θ and the minimum is attained at

$$
\hat{\theta} = \frac{E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{H,\tilde{P}}, L^{\bar{U},\tilde{P}}\rangle_t]}{E[\int_0^T \frac{Z_t^{\tilde{P}}}{\tilde{Z}_t} d\langle L^{\bar{U},\tilde{P}}\rangle_t]}. \tag{89}
$$

So the mean-variance hedging strategy is implicitly given by

$$
\begin{aligned}
\hat{\vartheta}_t = \vartheta^{\hat{\theta}} &= \vartheta_t^{H,\hat{\theta}} - \frac{\zeta_t}{\tilde{Z}_t}(V_{t-}^{H,\hat{\theta}} - \tilde{E}[H - \hat{\theta}\bar{U}_T] - \int_0^t \hat{\vartheta}_s dX_s),\ 0 \le t \le T \\
&= (\vartheta_t^{H,\tilde{P}} - \hat{\theta}\vartheta_t^{\bar{U},\tilde{P}}) - \frac{\zeta_t}{\tilde{Z}_t}(V_{t-}^{H,\hat{\theta}} - \tilde{E}[H - \hat{\theta}\bar{U}_T] - \int_0^t \hat{\vartheta}_s dX_s),\ 0 \le t \le T
\end{aligned} \tag{90}
$$

and

$$
\hat{V}_0 = V_0^{\hat{\theta}} = \tilde{E}[H] - \hat{\theta}\tilde{E}[\bar{U}_T]. \tag{91}
$$

Since we have seen in Chapter 5, that we have $G_T(\Theta_S) = G_T(\Theta_2^{eq})$ under certain conditions, we are able to solve the problems 17 and 19 for $G_T(\Theta_2^{eq})$ with the help of Theorem 66 as follows:

Theorem 69. *Let X be a continuous $\mathbb{R}^d$-valued semimartingale which is locally in $L^2(P)$ and $P \in \mathcal{M}_2^{eq} = \mathcal{M}_2^{e} \neq \emptyset$. Furthermore, assume that $\mathcal{E} = \mathcal{E}(N)$ is regular and satisfies $R_2(P)$ and that $X \in \mathcal{H}^2_{loc}(P)$ is an $\mathcal{E}$-local martingale. Then the following assertions hold:*

(i) The infimum in problem 17 is attained. In this case we have of course $G_T(\Theta_2^{eq}) = (G_T(\Theta_2^{eq}))^{\perp\perp}$, and every $H \in L^2(P)$ admits a decomposition of the form

$$
H = \pi(H) + G_T(\xi^H) \text{ for some } \xi^H \in \Theta_2^{eq}. \tag{92}
$$

In particular, there exists $\xi^1 \in \Theta_2^{eq}$ so that

$$
H - c - \pi(H-c) = [H - \pi(H)] - c[1 - \pi(1)] = G_T(\xi^H - c\xi^1).
$$

Comparing this expression with 55, we conclude that 55 is satisfied with the choice

$$\vartheta^{(c)} = \xi^H - c\xi^1. \tag{93}$$

The process $\vartheta^{(c)} \in \Theta_2^{eq}$ *of 73 is then optimal for problem 17.*
(ii) The value of problem 2 is given as

$$V = V(\hat{c}) = E[\pi^2(H)] - \frac{(E[\pi(H)])^2}{E[\pi(1)]} \tag{94}$$

with the notation

$$\hat{c} = \frac{E[\pi(H)]}{E[\pi(1)]} = E[\tilde{D}H] = E[\frac{d\mathcal{Q}}{dP}H] \tag{95}$$

of 41. Furthermore, the infimum for problem 18 is attained by the pair $(\hat{c}, \hat{\vartheta})$ *with* $\hat{c}$ *as in 75 and with*

$$\hat{\vartheta} := \vartheta^{\hat{c}} = \xi^H - \hat{c}\xi^1. \tag{96}$$

(iii) The process $\hat{\vartheta}$ *of 69 also minimizes*

$$Var(H - G_T(\vartheta)), \text{ over all } \vartheta \in \Theta_2^{eq}. \tag{97}$$

Proof 46. *Follows directly from Theorem 66.*

Furthermore, with Theorems 66 and 68 we get the following identity:

Theorem 70. *Let* X *be a continuous* $\mathbb{R}^d$*-valued semimartingale which is locally in* $L^2(P)$ *and* $P \in \mathcal{M}_2^{eq} = \mathcal{M}_2^e \neq \emptyset$*. Furthermore, assume that* $\mathcal{E} = \mathcal{E}(N)$ *is regular and satisfies* $R_2(P)$ *and that* $X \in \mathcal{H}^2_{loc}(P)$ *is an* $\mathcal{E}$*-local martingale. With the same notation as in the previous theorem and in Theorem 68, we get the following identities for the pair* $(\hat{c}, \hat{\vartheta})$ *that attains the infimum for problem 18:*

$$\hat{c} = \frac{E[\pi(H)]}{E[\pi(1)]} = E[\tilde{D}H] = E[\frac{d\mathcal{Q}}{dP}H] = \tilde{E}[H] \tag{98}$$

and

$$\tilde{\vartheta}_t = \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}(V_{t-}^{H,\tilde{P}} - \tilde{E}[H] - \int_0^t \tilde{\vartheta}_u dX_u) \tag{99}$$

$$= \xi_t^{H,\tilde{P}} - \tilde{\zeta}_t(\frac{V_0^{H,\tilde{P}} - \tilde{E}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u} dL_u^{H,\tilde{P}}) \tag{100}$$

$$= \xi^H - \hat{c}\xi^1, \tag{101}$$

Proof 47. *Follows directly from Theorems 68 and 69 and Corollary 13.*

6.3 The Space $G_T(\Theta_v)$

If one tries to find trading strategies that solve the problems 17-20 for the space $G_T(\Theta_v)$, one quickly recognizes that there have not been found results in literature yet. In this paper, we will use those results established in the previous section to provide suitable hedging strategies for $G_T(\Theta_v)$ and solve the problems 17-19. Nevertheless, since a different problem from 17-19 has been taken a look at by Xia/Jan (2006) for the space $G_T(\Theta_v)$, we will first restate their respective results before addressing problems 17-19 for $G_T(\Theta_v)$. On the practical side, we will even be able to use the results for the problem solved by Xia/Jan (2006) to solve an equivalent problem again for the space $G_T(\Theta_2^{eq})$.

So we want to apply the closedness results for the space $G_T(\Theta_v)$ corresponding to $p = q = 2$ to investigate the new problem of Mean-Variance Portfolio Selection which will be specified below. We again consider a financial market consisting of $m+1$ assets whose price processes X^j, $j = 0, 1, \ldots, m$, are assumed to be strictly positive semimartingales. We can take asset 0 as the numéraire and we will assume $X^0 = 1$ without loss of generality. Then X^j is as usual the discounted price process of asset j.
Recall

$$\mathcal{M}^{q,s} := \{g \in L^q(P) : E[gf] = 0 \text{ for all } f \in G_T(\Theta_p^s) \text{ and } E[g] = 1\}$$

and

$$\mathcal{M}^{q,e} := \{g \in \mathcal{M}^{q,s} : g > 0 \ a.s.\}\,.$$

Furthermore, recall that we will suppose $p = q = 2$ in the definition of $G_T(\Theta_v)$ and use the following two assumptions for the complete section:

Hypothesis 5. *X is locally in $\mathcal{L}^2(P)$.*

Hypothesis 6. *$\mathcal{M}^{2,e} \neq \emptyset$.*

It is easy to see that, under the previous assumptions and that $M^{2,s} \neq \emptyset$, $\mathcal{M}^{2,s}$ is a nonempty, closed, convex subset of $L^2(P)$. In the same way as in the two previous sections, we can conclude that there exists a unique element of minimal $L^2(P)$-norm in $\mathcal{M}^{2,s}$, which is denoted by g^{opt}. g^{opt} is called the variance-optimal signed martingale measure (VSMM) for S, since it minimizes $Var[g] = E[g^2] - 1$ over $g \in \mathcal{M}^{2,s}$. By the optimality of g^{opt}, for any $g \in \mathcal{M}^{2,s}$, $g - g^{opt}$ is orthogonal to g^{opt} in $L^2(P)$. Thus, we have

$$\hat{a} := E[(g^{opt})^2] = E[g^{opt} g], \tag{102}$$

for all $g \in \mathcal{M}^{2,s}$.
If $K = span(G_T(\Theta_2^s), 1)$ and $\bar{K}$ the $L^2(P)$-closure of K, then we have the following lemma.

Lemma 11. *If $\mathcal{M}^{2,s} \neq \emptyset$, then g^{opt} is the unique element of $\bar{K}$ (as a linear functional on $L^2(P)$) vanishing on K_2 and equaling 1 on the constant function 1.*

Let us now shortly recall the definition of a self-financing trading strategy.

A self-financing trading strategy with initial capital x_0 is defined as a pair (x_0, ϑ), where $\vartheta = (\vartheta^1, \ldots, \vartheta^m)$ is a predictable X-integrable process. For $1 \leq j \leq m$ and $0 \leq t \leq T$, ϑ_t^j represents the number of shares of the asset j held at time t. The (discounted) wealth process $V^{x_0,\vartheta}$ of a self-financing strategy (x_0, ϑ) is as usual given by

$$V_t^{x_0,\vartheta} = x_0 + \int_0^t \vartheta dX, \ 0 \leq t \leq T.$$

Furthermore, the strategy is said admissible if $\vartheta \in \Theta_v$. For a better understanding we denote the space of admissible wealth processes in this chapter by

$$\mathcal{X}(x_0) = \{S : S = V^{x_0,\vartheta}, \vartheta \in \Theta_v\}.$$

Then $\mathcal{X}(x_0)$ is convex and

$$\mathcal{X}(x_0) = x_0 + \mathcal{X}(0).$$

The next lemma is of great importance for our investigation of the problem of mean-variance portfolio selection.

Lemma 12. *Assume Hypotheses 5 and 6. Then we have $g^{opt} \in \mathcal{X}(\hat{a})$, where $\hat{a}$ is defined in equation 102.*

The problem of mean-variance portfolio selection is now to maximize the mean terminal wealth $E[S_T]$ and at the same time to minimize the variance of the terminal wealth $Var[S_T]$ over $S \in \mathcal{X}(x_0)$. This is a multiobjective optimization problem with two conflicting criteria.

Definition 63. *The wealth process $\hat{S} \in \mathcal{X}(x_0)$ is said to be efficient if there exists no wealth process $S \in \mathcal{X}(x_0)$ such that*

$$E[S_T] \geq E[\hat{S}_T]$$

and

$$Var[S_T] \leq Var[\hat{S}_T],$$

with at least one inequality holding strictly.
In this case, $(\sqrt{Var[\hat{S}_T]}, E[\hat{S}_T]) \in \mathbb{R}^2$ *is called an efficient point. The set of all efficient points is called the efficient frontier.*

Remark 13. *Since* $S = x_0 \in \mathcal{X}(x_0)$*, it is clear that* $(\sigma, z) \in \mathbb{R}^+ \times \mathbb{R}$ *is efficient only if* $z \geq x_0$*.*

In order to find out the efficient wealth processes, we consider the following optimization problem parameterized by z:

$$\text{Minimize } Var[S_T] = E[S_T^2] - z^2 \text{ subject to } E[S_T] = z \text{ and } S \in \mathcal{X}(x_0). \tag{103}$$

By Theorem 41, $\{X_T : S \in \mathcal{X}(x_0)\}$ is convex and closed in $L^2(P)$, thus problem 103 admits a unique solution.

Remark 14. *It is easy to see that if* $\hat{S} \in \mathcal{X}(x_0)$ *is efficient, then* $\hat{S}$ *must be the unique solution to problem 103 corresponding to* $z = E[\hat{S}_T] \geq x_0$*.*

The unique solution of 103 for $z = x_0$ is obviously $S = x_0$. In order to solve problem 103 for $z \neq x_0$, we consider, as usual, the following expected quadratic utility problem parameterized by $\lambda \in \mathbb{R}$:

$$\text{Maximize } E[U_\lambda(S_T)] = E[-(S_T - \lambda)^2] \text{ subject to } S \in \mathcal{X}(x_0), \tag{104}$$

where $U_\lambda(x) = -(x - \lambda)^2$. By the convexity and closedness of $\{S_T : S \in \mathcal{X}(x_0)\}$ in $L^2(P)$, problem 104 admits a unique solution. The following lemma reveals the relation between problems 103 and 104, which can be proved in the same way as in Xia (2005, Lemma 3.1).

Lemma 13. *(i) The unique solution* $\hat{S} \in \mathcal{X}(x_0)$ *of problem 103 for some* $z > x_0$ *(resp.* $z < x_0$*) also solves problem 104 for some* $\lambda > x_0$ *(resp.* $\lambda < x_0$*).*
(ii) If $\hat{S} \in \mathcal{X}(x_0)$ *solves problem 104 for some* $\lambda > x_0$ *(resp.* $\lambda < x_0$*), then* $E[\hat{S}_T] > x_0$ *(resp.* $E[\hat{S}_T] < x_0$*) and it also solves problem 103 with* $z = E[\hat{S}_T]$*.*

Now we are going to give a characterization for the solution of problem 104 for $\lambda > x_0$ in terms of VSMM via martingale/convex duality method. For the general theory of utility maximization via this method, we refer the reader to Kramkov and Schachermayer (1999, 2003), among others.

Lemma 14. *Let $\lambda \neq x_0$. If there exist $\tilde{S} \in \mathcal{X}(x_0)$, $\tilde{g} \in \mathcal{M}^{2,s}$ and $\tilde{y} \in \mathbb{R}$ such that the following duality relationship:*

$$\tilde{S}_T = \lambda - \tilde{y}\tilde{g} \tag{105}$$

holds, then $\tilde{S}$ solves problem 104, $\tilde{g} = g^{opt}$, and $\tilde{y} = \frac{\lambda - x_0}{E[(g^{opt})^2]}$.

Proof 48. *By the convexity of function $f(x) = x^2$ we have for all $y \in \mathbb{R}$, $g \in \mathcal{M}^{2,s}$, and $S \in \mathcal{X}(x_0)$ that*

$$\begin{aligned} E[(-yg)^2 - (S_T - \lambda)^2] &\leq E[-2yg(-yg + \lambda - S_T)] \\ &= -2y(E[g(-yg + \lambda)] - x_0). \end{aligned} \tag{106}$$

Assume equation 105 holds, then taking $y = \tilde{y}$ and $S = \tilde{S}$ in equation 106, we get for all $g \in \mathcal{M}^{2,s}$ that

$$-\tilde{y}^2 E[\tilde{g}^2] \leq \tilde{y}^2 E[g^2] - 2\tilde{y}(\lambda - x_0). \tag{107}$$

Multiplying both sides of 105 by $\tilde{g}$ and taking expectation, we have $\tilde{y}E[\tilde{g}^2] = \lambda - x_0$, which implies $\tilde{Y} \neq 0$, since $\lambda \neq x_0$. Substituting the previous equality into 107, we get $E[\tilde{g}^2] \leq E[g^2]$, implying $\tilde{g} = g^{opt}$. Finally, taking $y = \tilde{y}$ and $g = \tilde{g}$ in 106, by 105, we have

$$E[(\tilde{S}_T - \lambda)^2] - E[(S_T - \lambda)^2] \leq -2\tilde{y}(E[\tilde{g}\tilde{S}_T] - x_0) = 0,$$

which implies $\tilde{S}$ solves problem 104.

Theorem 71. *Assume that Hypotheses 5 and 6 hold and let $\lambda > x_0$. The optimization problem 104 admits a unique solution $\hat{S} \in \mathcal{X}(x_0)$. In addition, $\hat{S}$ and the VSMM g^{opt} satisfy the following duality relationship:*

$$\hat{S}_T = \lambda - \hat{y}g^{opt}, \tag{108}$$

where $\hat{y} := \frac{\lambda - x_0}{\hat{a}} > 0$ and $\hat{a}$ is defined in 102.

Proof 49. *$\hat{y} > 0$ is clear. By Lemma 12, there exists a predictable X-integrable, $\mathbb{R}^m$-valued process $\hat{\beta}$ such that*

$$g^{opt} = \hat{a} + (\hat{\beta}X)_T$$

which implies that

$$\lambda - \hat{y}g^{opt} = x_0 - \hat{y}(\hat{\beta}X)_T.$$

Taking $\hat{\vartheta} = -\hat{y}\hat{\beta}$ *and* $\hat{S} = V^{x_0,\hat{\vartheta}}$*, then 108 follows. By 102,* $\hat{S} \in \mathcal{X}(x_0)$ *and therefore the conclusion follows from Lemma 16.*

Now we can derive the efficient frontier for the mean-variance problem.

Theorem 72. *Assume Hypotheses 5 and 6. The set of all efficient terminal wealths is*

$$\{\lambda - \frac{\lambda - x_0}{\hat{a}} g^{opt} : \lambda \geq x_0\}$$

and the set of all efficient portfolio is

$$\{-\frac{\lambda - x_0}{\hat{a}}\hat{\beta} : \lambda \geq x_0\},$$

where $\hat{a}$ *is defined in 102 and* $\hat{\beta}$ *as before.*

Proof 50. *First of all, by convexity of* $\mathcal{X}(x_0)$*, it is easy to see that the set* $D := \{E[S_T] : S \in \mathcal{X}(x_0)\}$ *is convex. Thus by the same method as in Bielecki et al. (2005, Lemma 6.1), we can show that the optimal value* $J^*(z)$ *of problem 103 is strictly increasing for* $z \in D \cap [x_0, \infty)$*, and strictly decreasing for* $z \in D \cap (-\infty, x_0]$*. Consequently, from Remark 14 we see that the set of all efficient wealth processes consists of solutions of problem 103 corresponding to* $z \in D \cap [x_0, \infty)$*. Observing that problem 103 (resp. 104) admits a unique solution* $S = x_0$ *for* $z = x_0$ *(resp.* $\lambda = x_0$*), the above results further imply that the set of all efficient wealth processes consists of solutions of quadratic optimization problem 104 corresponding to* $\lambda \geq x_0$*. Thus by the proof of Theorem 71, we can conclude the proof of the theorem.*

By 108, we have

$$\begin{aligned}
E[\hat{S}_T] &= \lambda - (\lambda - x_0)\hat{a}^{-1} = \lambda(1 - \hat{a}^{-1}) + x_0\hat{a}^{-1} \\
Var[\hat{S}_T] &= E[\hat{S}_T^2] - (E[\hat{S}_T])^2 \\
&= \lambda^2 - 2\lambda(\lambda - x_0)\hat{a}^{-1} + (\lambda - x_0)^2\hat{a}^{-2}E[(g^{opt})^2] - (E[\hat{S}_T])^2 \\
&= \lambda^2(1 - \hat{a}^{-1}) + x_0^2\hat{a}^{-1} - (E[\hat{S}_T])^2.
\end{aligned}$$

Eliminating λ, we get the efficient frontier parametrized by z as follows (note

$Var[g^{opt}] = \hat{a} - 1$):

$$E[\hat{S}_T] = z,\ Var[\hat{S}_T] = \frac{1}{Var[g^{opt}]}(z - x_0)^2, \tag{109}$$

with z varying in $[x_0, \infty)$.
If we denote by $\sigma(g^{opt})$ and $\sigma(\hat{S}_T)$ the standard deviations of g^{opt} and $\hat{S}_T$, respectively, then 109 gives

$$E[\hat{S}_T] = x_0 + \sigma(g^{opt}) \cdot \sigma(\hat{S}_T). \tag{110}$$

Therefore, the efficient frontier in the mean-standard-deviation diagram is a straight line, which is also called the capital market line. The slope of the capital market line is called the market price of risk. By 110, we have an interesting result, which provides a financial meaning of the VSMM, as follows.

Theorem 73. *Assume Hypotheses 5 and 6. The slope of the capital market line, i.e. the market price of risk, is just the standard deviation of the VSMM.*

As an illustration of our previous results, we state an example for solving the problem of Mean-Variance Portfolio Choice for a market driven by a Lévy process.

Example 4. *We assume a finite time horizon T. Let (L_t) be a d-dimensional Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $(\mathcal{F}_t)$ is the natural filtration of L and $\mathcal{F} = \mathcal{F}_T$. Let μ denote the jump measure of L. Its dual predictable projection v has the form $v(dt, dx) = dt \times y(dx)$ with $y(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)y(dx) < \infty$. The (L_t) is square integrable and has the following Lévy decomposition (see Jacod/Shiryaev (1987))*

$$L_t = \alpha t + cB_t + \int_{[0,t]\times\mathbb{R}^d} x(\mu(ds, dx) - v(ds, dx)), \tag{111}$$

where α and c are a constant d-dimensional vector and a $(d \times d)$-matrix and (B_t) is a standard d-dimensional Brownian motion.
Denote by $\mathcal{P}$ the predictable σ-algebra on $\Omega \times [0, T]$, which is generated by all left-continuous and $(\mathcal{F}_t)$-adapted processes, and $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$. We introduce the following notations:

- $L_{\mathcal{F}}([0, T]; \mathbb{R}^d)$*: the set of all $(\mathcal{F}_t)$-predictable, $\mathbb{R}^d$-valued processes θ_1 such that $\int_0^T |\theta_1(s)|^2 ds < \infty$ a.s.*

- $L_{\mathcal{F}}([0,T]\times\mathbb{R}^d)$: *the set of all* $\tilde{\mathcal{P}}$*-measurable,* $\mathbb{R}$*-valued functions* θ_2 *defined on* $\Omega\times[0,T]\times\mathbb{R}^d$ *such that* $\sqrt{\sum_{0<s\le t}\theta_2^2(s,\Delta L_s)I_{[\Delta L_s\neq 0]}}$ *is a locally integrable increasing process and for all* $t\in[0,T]$, $\int_{\mathbb{R}^d}|\theta_2(t,x)|y(dx)<\infty$ *a.s.*

- $\Xi := \{\theta=(\theta_1,\theta_2):\theta_1\in L_{\mathcal{F}}([0,T];\mathbb{R}^d)$ *and* $\theta_2\in L_{\mathcal{F}}([0,T]\times\mathbb{R}^d)\}$.

For any $\theta\in\Xi$, *we can define the Doléans-Dade exponential* $Z(\theta)$ *by*

$$dZ(\theta) = -Z_-(\theta)[\theta_1^\tau dB + d(\theta_2 * (\mu - v))],\ \text{with } Z_0(\theta)=1, \tag{112}$$

where $\theta_2 * (\mu - v) := \int \theta_2 d(\mu - v)$.

We consider a financial market consisting of $(m+1)$ *assets:* m *stocks and a bond, where* $1\le m\le d$. *We assume the bond price process is the constant* 1. *For each* $1\le i\le m$, *the price process* S_t^i *of the i-th stock satisfies the following stochastic differential equation:*

$$dS_t^i = S_{t-}^i(\phi_i^\tau(t)dL_t + b_i(t)dt), \tag{113}$$

where L *is defined by 111,* $\phi_i(t):=(\phi_{i1}(t),\ldots,\phi_{id}(t))^\tau$ *and* $b_i(t)$ *are* $(\mathcal{F}_t)$*-predictable,* $\mathbb{R}^d$*-valued, and* $\mathbb{R}$*-valued processes, respectively. In order to ensure the strict positivity of* S^i, *we assume that* $\phi_i^\tau\Delta L > -1$ *for all* $1\le i\le m$. *Instead of replacing* $b(t)$ *by* $b(t)+\phi(t)\alpha$, *we assume that* $\alpha = 0$ *in 111. Denote*

$$\phi(t) := (\phi_1(t),\ldots,\phi_m(t))^\tau,$$
$$b(t) := (b_1(t),\ldots,b_m(t))^\tau.$$

For a self-financing trading strategy (x_0,ϑ), *we let* $\pi_t^i := \vartheta_t^i S_{t-}^i$, $\pi := (\pi^1,\ldots,\pi^n)^\tau$. *We still call* (x_0,π) *a self-financing trading strategy. Then by 113, the wealth process* (X_t) *of such a self-financing strategy* (x_0,π) *is given by*

$$X_t = x_0 + \int_0^t \pi_t^\tau(\phi(t)dL_t + b(t)dt),\ 0\le t\le T. \tag{114}$$

Proposition 15. *Let* $\theta=(\theta_1,\theta_2)\in\Xi$ *be such that*

$$\int_0^T\int_{\mathbb{R}^d}|x\theta_2(t,x)|y(dx)dt<\infty,\ a.s. \tag{115}$$

If

$$\phi(t)[c\theta_1(t) + \int_{\mathbb{R}^d} x\theta_2(t,x)y(dx)] = b(t),\ a.s. \tag{116}$$

then for each $1 \le i \le m$, $S^i Z(\theta)$ is a local martingale.

Proof 51. *By Itô's formula, we have for each $1 \le i \le m$ that*

$$d(S^i_t Z_t(\theta)) = S^i_{t-} Z_{t-}(\theta)[(b_i(t) - \phi^\tau_i(t)c\theta_1(t))dt - d(\sum_{0<s\le t} \theta_2(s,\Delta L_s)\phi^\tau_i(s)\Delta L_s)].$$

On the other hand, by 115 and Jacod/Shiryaev (1987), we have

$$\sum_{0<s\le t} \theta_2(s,\Delta L_s)\phi^\tau_i(s)\Delta L_s = \int_0^t \int_{\mathbb{R}^d} \theta_2(s,x)\phi^\tau_i(s)xy(dx)ds.$$

Thus if 116 holds, then $S^i Z(\theta)$ is a local martingale.

For determining the efficient portfolios, we put

$$v := cc^\tau + \int_{\mathbb{R}^d} xx^\tau y(dx),$$
$$V(t) := \phi(t)v\phi^\tau(t).$$

Obviously, V and $V(t)$ are symmetric matrices. In this subsection, we make the following assumptions:

- *$\phi(t)$ and $b(t)$ are deterministic;*

- *$det(V(t)) > 0$ a.s. and $y^\tau V^{-1}(t)y \le \epsilon|y|^2$ a.s. for all $t \in [0,T]$ and $y \in \mathbb{R}^m$, where $\epsilon > 0$ is a constant.*

We put

$$\hat\theta_1(t) := c^\tau\phi^\tau(t)V^{-1}(t)b(t),\ \hat\theta_2(t,x) := x^\tau\phi^\tau(t)V^{-1}(t)b(t), \tag{117}$$

and

$$A_t := \exp(-\int_0^t b^\tau(s)V^{-1}(s)b(s)ds). \tag{118}$$

Under the previous assumptions we get the following theorem:

Theorem 74. *Put*

$$\hat\pi_t := (\lambda - x_0)V^{-1}(t)b(t)A_t Z_{t-}(\hat\theta), \tag{119}$$

$$\hat{X}_t := x_0 + \int_0^t \hat{\pi}_s^\tau(\phi(s)dL_s + b(s)ds), \tag{120}$$

then $\hat{X}$ solves the problem 104 with portfolio $\hat{\pi}$, and the VSMM $g^{opt} = Z_T(\hat{\theta})$.

Proof 52. *Let $\hat{Z}_t = Z_t(\hat{\theta})$. By 111 and 112, we have*

$$d\hat{Z}_t = -\hat{Z}_{t-}b^\tau(t)V^{-1}(t)\phi(t)dL_t.$$

Then $\hat{Z}$ is by Novikov's criterion for exponential martingales a square integrable martingale and $\hat{\theta}$ satisfies 115 and 116. By 15, $\hat{Z}_T \in \mathcal{M}^{2,s}$. Since $d < L, L >_t = vdt$, we get

$$E[\hat{Z}_T^2] = E[< \hat{Z}, \hat{Z} >_T] = \int_0^T E[\hat{Z}_s^2]b^\tau(s)V^{-1}(s)b(s)ds,$$

which implies $E[\hat{Z}_T^2] = A_T^{-1}$. Take $\hat{y} = (\lambda - x_0)A_T$, then applying Itô formula gives

$$\lambda - \hat{y}\hat{Z}_T = \lambda - (\lambda - x_0)\hat{Z}_T A_T = x_0 + \int_0^T \hat{\pi}_t^\tau(\phi(t)dL_t + b(t)dt) = \hat{X}_T. \tag{121}$$

It is easy to see that the predictable quadratic variation of $\hat{X}$ is

$$\begin{aligned} < \hat{X} >_T &= \int_0^T \hat{\pi}_t^\tau \phi(t) v \phi^\tau(t)\hat{\pi}_t dt = \int_0^T \hat{\pi}_t^\tau V(t)\hat{\pi}_t dt \\ &= (\lambda - x_0)^2 \int_0^T A_t^2 \hat{Z}_t^2 b^\tau(t)V^{-1}(t)b(t)dt \le C\int_0^T \hat{Z}_t^2 dt, \end{aligned}$$

where $C > 0$ is a constant. Thus we have

$$E[< \hat{X} >_T] \le C\int_0^T E[\hat{Z}_t^2] \le C\int_0^T E[\hat{Z}_T^2] < \infty,$$

which implies that $E[\hat{X}_T g] = x_0$ for all $g \in \mathcal{M}^{2,s}$ and therefore $\hat{X} \in \mathcal{X}(x_0)$. By 121 and Lemma 16, the theorem is proven.

Finally we have the following theorem:

Theorem 75. *Under the assumptions in this example, the set of all efficient portfolios consist of all $\hat{\pi}$, defined by 119, with λ varying in $[x_0, \infty)$.*

Finally, this finishes our example.

Since we stated a result about the conditions under which the two spaces $G_T(\Theta_p^{eq})$ and $G_T(\Theta_v)$ coincide in Corollary 14, we are now able to transfer our results and solve the problem of finding the efficient frontier for the space $G_T(\Theta_2^{eq})$ as well.

We assume for the rest of this section that X is a continuous $\mathbb{R}^d$-valued semi-martingale and we suppose Hypotheses 5 and 6. Furthermore, assume $M_2^e(P) \neq \emptyset$. If we define

$$\mathcal{X}^*(x_0) = \{S : S = V^{x_0,\vartheta}, \vartheta \in \Theta_2^{eq}\}$$

and consider the optimization problems

$$\text{Minimize } Var[S_T] = E[S_T^2] - z^2 \text{ subject to } E[S_T] = z \text{ and } S \in \mathcal{X}^*(x_0), \tag{122}$$

parametrized by z and

$$\text{Maximize } E[U_\lambda(S_T)] = E[-(S_T - \lambda)^2] \text{ subject to } S \in \mathcal{X}^*(x_0), \tag{123}$$

parametrized by $\lambda \in \mathbb{R}$ where $U_\lambda(x) = -(x - \lambda)^2$, we get the following results:

Lemma 15. *(i) The unique solution $\hat{S} \in \mathcal{X}^*(x_0)$ of problem 122 with $S \in \mathcal{X}^*(x_0)$ for some $z > x_0$ (resp. $z < x_0$) also solves problem 123 for some $\lambda > x_0$ (resp. $\lambda < x_0$).*
(ii) If $\hat{S} \in \mathcal{X}^*(x_0)$ solves problem 123 with $S \in \mathcal{X}^*(x_0)$ for some $\lambda > x_0$ (resp. $\lambda < x_0$), then $E[\hat{S}_T] > x_0$ (resp. $E[\hat{S}_T] < x_0$) and it also solves problem 122 with $z = E[\hat{S}_T]$.

Lemma 16. *Let $\lambda \neq x_0$. If there exist $\tilde{S} \in \mathcal{X}^*(x_0)$, $\tilde{g} \in \mathcal{M}^{2,s}$ and $\tilde{y} \in \mathbb{R}$ such that the following duality relationship:*

$$\tilde{S}_T = \lambda - \tilde{y}\tilde{g} \tag{124}$$

holds, then $\tilde{S}$ solves problem 123, $\tilde{g} = g^{opt}$, and $\tilde{y} = \frac{\lambda - x_0}{E[(g^{opt})^2]}$.

Theorem 76. *Let $\lambda > x_0$. The optimization problem 123 admits a unique solution $\hat{S} \in \mathcal{X}^*(x_0)$. In addition, $\hat{S}$ and the VSMM g^{opt} satisfy the following duality relationship:*

$$\hat{S}_T = \lambda - \hat{y} g^{opt}, \tag{125}$$

where $\hat{y} := \frac{\lambda - x_0}{\hat{a}} > 0$ and $\hat{a}$ is defined in 102.

Furthermore, if we look for the efficient frontier, i.e. all efficient wealth processes$\hat{X} \in \mathcal{X}^*(x_0)$ for Θ_2^{eq}-strategies, we get the following result:

Theorem 77. *The set of all efficient terminal wealths is*

$$\{\lambda - \frac{\lambda - x_0}{\hat{a}} g^{opt} : \lambda \geq x_0\}$$

and the set of all efficient portfolio is

$$\{-\frac{\lambda - x_0}{\hat{a}} \hat{\beta} : \lambda \geq x_0\},$$

where $\hat{a}$ is defined in 102 and β is defined in Lemma 13(ii).

Since we have now transferred the results from the space $G_T(\Theta_v)$ for finding the efficient frontier for $G_T(\Theta_2^{eq})$, we will now transfer our results for hedging strategies in the $G_T(\Theta_2^{eq})$ context to the space $G_T(\Theta_v)$.

The reader may remember that we continue assuming that X is a continuous $\mathbb{R}^d$-valued semimartingale and we suppose Hypotheses 5 and 6. Furthermore, we assume$M_2^e(P) \neq \emptyset$.

Then by Corollary 14 and Theorem 68, we can solve the problems 17 and 18 for the space $G_T(\Theta_v)$ ($\tilde{P}$ defined as in Theorem 68):

Theorem 78. *Let $H \in L^2(P)$ be a contingent claim and write the Galtchouk-Kunita-Watanabe decomposition of H under $\tilde{P}$ with respect to X as*

$$H = \tilde{E}[H|\mathcal{F}_0] + \int_0^T \xi_u^{H,\tilde{P}} dX_u + L_T^{H,\tilde{P}} = V_T^{H,\tilde{P}} \tag{126}$$

with

$$V_t^{H,\tilde{P}} := \tilde{E}[H|\mathcal{F}_t] = \tilde{E}[H, \mathcal{F}_0] + \int_0^t \xi_u^{H,\tilde{P}} dX_u + L_t^{H,\tilde{P}},$$

for $0 \leq t \leq T$.
Then the solution to problem 18 for the space $G_T(\theta_v)$ is given by

$$\tilde{V}_0 = \tilde{E}[H] \tag{127}$$

and

$$\tilde{\vartheta}_t = \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}(V_{t-}^{H,\tilde{P}} - \tilde{E}[H] - \int_0^t \tilde{\vartheta}_u dX_u) \tag{128}$$

$$= \xi_t^{H,\tilde{P}} - \tilde{\zeta}_t(\frac{V_0^{H,\tilde{P}} - \tilde{E}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u} dL_u^{H,\tilde{P}}), \tag{129}$$

for $0 \leq t \leq T$.

7 Conclusion

In this section, we want to summarize our results and to recap the concepts used. The thesis consists of three main parts:

- ensuring the L^p-closedness of spaces of stochastic integrals (section 4)
- ensuring the L^p-closedness of those spaces under trading constraints (section 5)
- determining risk-minimizing trading strategies (section 6)

For all of the three parts we considered the spaces $G_T(\Theta_S)$, $G_T(\Theta_v)$ and $G_T(\Theta_2^{eq})$. Concerning the first part, for the spaces $G_T(\Theta_S)$ and $G_T(\Theta_2^{eq})$ we established a duality relation between the space of (local) martingale measures for X and the respective L^p-closedness. More precisely, we found out that $G_T(\Theta_2^{eq})$ and $G_T(\Theta_S)$ are closed in L^p if the spaces of those measures are not empty. For $G_T(\Theta_v)$ we established a very similar relation: The space is closed if there exists a signed martingale measure $\mathcal{Q}$ with $\frac{dP}{d\mathcal{Q}} \in L^q(P)$, where $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, if the process X is continuous and the respective measures exist, some identities have been established:

$$D_p = G_T(\Theta_2^{eq}) = G_T(\Theta_v) = \overline{G_T(\Theta_p^s)} \tag{130}$$

where the bar $^-$ denotes the L^p-closure and D_p is sourced from Theorem 1.2 in Delbaen/Schacermeyer (1996). Those identities helped us to characterize the elements in the respective spaces: Under the upper assumptions, $G_T(\Theta_p^{eq})$ and $G_T(\Theta_v)$ consist of exactly all L^p-limits of the terminal wealth of 'simple' strategies.

In section 5, the L^2-closedness of spaces of stochastic integrals has again been examined but this time whilst imposing constraints on the trading strategies. For modelling those trading constraints we used the concepts of correspondences, meaning that the strategies were required to map into predefined sets represented by those correspondences. To finally establish the closedness, two main results were necessary: the L^2-closedness of the spaces $G_T(\Theta_2^{eq})$, $G_T(\Theta_v)$ and $G_T(\Theta_S)$ which had previously been established in section 4 and the result of Schweizer/Czichowsky (2012) that the space $G_T(\Theta_S(C))$ is closed in the semimartingale topology iff the projection of C on the predictable range of X is closed.
Then the procedure to prove the L^2-closedness under trading constraints was the same

for all three spaces: We proved that the according spaces $G(\Theta(C))$ were closed in the $\mathcal{H}^2$-topology and concluded by the properties of the stochastic integral that they were also closed in the semimartingale topology. With the statement of Schweizer/Czichowsky (2011), we were then easily able to prove the closedness of the spaces $G_T(\Theta_2^{eq}(C))$, $G_T(\Theta_v(C))$ and $G_T(\Theta_S(C))$ under the condition that the projection of C on the predictable range of X is closed. In addition, with the help of Schweizer/Czichowsky (2011), we then found conditions under which the projection of C on the predictable range of X is closed and again linked those conditions to the L^2-closedness of the spaces $G_T(\Theta_2^{eq}(C))$, $G_T(\Theta_v(C))$ and $G_T(\Theta_S(C))$ in a canonical way. In conclusion, we even saw that using the concepts of correspondences is a very general approach since we proved that if the spaces $G_T(\Theta_2^{eq})$, $G_T(\Theta_v)$ and $G_T(\Theta_S)$ are closed under trading constraints, the constraining set can always be modelled as a correspondence.

In section 6, we determined explicit hedging strategies and solved different types of mean-variance hedging problems for the respective (L^2-closed) spaces. For the space $G_T(\Theta_S)$ we addressed the problem with a duality approach: the main tool in this approach was the set $M_2^s(G_T(\Theta_S))$; the set of signed measures, absolutely continuous with respect to P, under which X behaves like a martingale. We showed that this duality is useful, in the sense that the value of an appropriately formulated dual problem can be computed easily; that it 'has no gap' (i.e. the values of the primal and dual problems coincide); that the signed measure which is optimal for the dual problem can be easily identified whenever it exists; and that the identity is also 'strong' in the sense that one can identify the optimal stochastic integral for the primal problem. So, by solving the easier dual problem, we also solved our primal problem.

For $G_T(\Theta_v)$ we used an absolutely similar approach by first stating an analogue dual problem and transferring the results to the primal one. For both spaces the so-called variance-optimal signed martingale measure turned out to be of great importance: it appeared as well in the formulation of the dual problem as in the main solution for both of the spaces. Furthermore, we also used it for solving mean-variance hedging problems for $G_T(\Theta_2^{eq})$: since X is a martingale under that measure, we were able to determine the according Kunita-Watanabe decomposition of our claim H which directly led us to the risk-minimizing hedging strategy.
In the aftermath, we then used the identities in 130 to transfer each solution of the different mean-variance hedging problems from one space to another. So in conclusion, we were able to solve all given problems in this thesis for each space.

8 Outlook

Even though this thesis gave answers to questions like L^2-closedness (under constraints), finding explicit hedging strategies and the connection of the different spaces, it simultaneously evolved many new ones. For example, how does one find explicit hedging strategies in the constrained case? Naturally, this is trivial if the hedging strategy in the unconstrained case already suffices the requirement of being an element of a predefined correspondence, but what if it does not? Furthermore, one could ask how we can find risk-minimizing strategies, if the respective spaces are not closed. In addition, one could of course ask how our results behave in a practical context: are those requirements, that we impose on the price process to ensure L^2-closedness reasonable? Over and above, it is of course necessary to verify how good our strategies really are in terms of minimizing the hedging error.

Consecutively, we want to summarize and outline some key questions and working ideas for further consideration:

- How does one find risk-minimizing hedging strategies in the constraint case, if the minimizing strategy in the general case does not satisfy the constraining condition?
- How effective are the determined strategies, i.e. how good is the approximation of the claim?
- How does one implement the strategies into the common software?
- Comparison of the determined strategies for the three spaces in terms of how good they approximate a given claim.
- Detecting weaknesses of the strategies in simulations and then retrospectively improving the strategies at the detected errors.
- Are the imposed conditions on the price process X and on the trading strategies necessary or just sufficient for the closedness of the respective spaces?

A sophisticated approach for further work on our subject would probably start with implementing and simulating different scenarios with suitable software. This would enable to draw conclusions on the quality of the approximation and then, in a next step, going back to the mathematical theory and to fix possible deficiencies detected as a result from the simulations.

So there is still much work to be done and this thesis can only supply the theoretical background on which those ideas should be based on.

9 List of Notation

Abbreviations

ELMM: equivalent local martingale measure

VSMM: variance-optimal signed martingale measure

Preliminaries

$(\Omega, \mathcal{F}, P)$: probability space

$\{\mathcal{F}_t\}_{t\geq 0}$: filtration, which is an increasing sequence of σ-algebras with

τ: stopping time

$\wedge$: operator which outputs the minimum of two real scalar arguments

$\mathcal{E}(X)$: stochastic exponential of a semimartingale X

$[X_1, X_2\ ]$: quadratic covariation of two semimartingales X_1 and X_2

$<X_1, X_2>$: predictable quadratic covariation of two semimartingales X_1 and X_2

Symbols

$L(X)$: linear space of all $\mathbb{R}^d$-valued predictable X-integrable processes ϑ

$\mathcal{M}_q^e(P)$: $\left\{\mathcal{Q} \approx P : \frac{d\mathcal{Q}}{dP} \in L^q(P),\ X \text{ is a local } \mathcal{Q} - \text{martingale}\right\}$

$\mathcal{M}_q(P)$: $\left\{\mathcal{Q} << P : \frac{d\mathcal{Q}}{dP} \in L^q(P),\ X \text{ is a local } \mathcal{Q} - \text{martingale}\right\}$

$\mathcal{M}^{q,s}$: $\left\{g \in L^q(P) : E[gf] = 0 \text{ for all } f \in G_T(\Theta_p^s) \text{ and } E[g] = 1\right\}$

$\mathcal{M}^{q,e}$: $\{g \in \mathcal{M}^{q,s} : g > 0 \text{ a.s.}\}$

$\mathcal{M}_2^{eq}(P)$: $\left\{\mathcal{Q} \approx P : \frac{d\mathcal{Q}}{d\mathcal{P}} \in L^2(P),\ X \text{ is a square integrable } \mathcal{Q}\text{-martingale}\right\}$

Θ_S: $\left\{\vartheta \in L(X) : \int \vartheta dX \in \mathcal{H}^2(P)\right\}$

Θ_p^{eq}: $\left\{\vartheta \in L(X) : G(\vartheta) \text{ is a uniformly } \mathcal{Q}-\text{integrable } \mathcal{Q}-\text{martingale for every } \mathcal{Q} \in \mathcal{M}_q^e(P)\right\}$

Θ_v: $\left\{\vartheta \in L(X) : G_T(\vartheta) = \int_0^T \vartheta dX \in L^p(P) \text{ and } E[G_T(\vartheta)g] = 0 \text{ for all } g \in \mathcal{M}^{q,s}\right\}$

$\tilde{P}$: variance-optimal signed martingale measure

Π^X: projection on the predictable range of X

$\mathcal{H}^2$: $\{X : X$ square integrable martingale $\}$

$\mathcal{S}^2$: $\{X : X$ square integrable semimartingale $\}$

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