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Introduction

The complex interpolation method, which assigns to an interpolation couple $[E_0, E_1]$ of complex Banach spaces and $0 < \theta < 1$ a so-called "intermediate space" $[E_0, E_1]_\theta$, has its roots in the famous Riesz–Thorin Interpolation Theorem for operators acting between L_p-spaces, and was introduced around 1960 by Calderón and Lions; a related approach is due to Krein. Throughout the last thirty years the theory of complex interpolation of Banach spaces has proved to be a useful tool in several branches of functional analysis, in particular in the theory of operator ideals. As a first starting point in this direction a famous result of Kwapień [Kwa68] in 1968 may be seen: Using classical interpolation techniques (e. g. the Three-Lines-Theorem which motivated large parts of the theory of complex interpolation of Banach spaces), he showed that for $1 \le p \le \infty$ and $1 \le r \le 2$ defined by $1/r = 1 - |1/2 - 1/p|$ every continuous operator T on ℓ_1 with values in ℓ_p is $(r, 1)$-summing, i. e. there exists $C > 0$ such that for all $x_1, \dots, x_n \in \ell_1$ the inequality

$$\left(\sum_{k=1}^{n} \|Tx_k\|_{\ell_p}^r \right)^{1/r} \le C \cdot \sup_{\|x'\|_{\ell_1'} \le 1} \sum_{k=1}^{n} |\langle x', x_k \rangle|$$

holds. This improved upon a well-known result of Littlewood [Lit30] in 1930 which says that there exists a constant $C > 0$ such that for every bilinear and continuous operator $\varphi : c_0 \times c_0 \to \mathbb{R}$ (where c_0 denotes the space of all zero sequences) the inequality

$$\left(\sum_{k,\ell=1}^{\infty} |\varphi(e_k, e_\ell)|^{4/3} \right)^{3/4} \le C \cdot \|\varphi\|$$

holds; here e_k denotes the k-th standard unit vector in c_0. Littlewood's inequality in modern terminology means exactly that the embedding $\mathrm{id} : \ell_1 \hookrightarrow \ell_{4/3}$ is $(4/3, 1)$-summing, and obviously $3/4 = 1 - |1/2 - 3/4|$ as stated in Kwapień's result. Moreover, the number $4/3$ occurs by a naive interpolation between 1 and 2: If we put $\theta = 1/2$, then $3/4 = (1-\theta)/1 + \theta/2$. This suggests to derive Kwapień's formula via interpolation from the following two well-known "border cases": Every operator from ℓ_1 to ℓ_1 or ℓ_∞ is $(2, 1)$-summing, and the famous Grothendieck Inequality [Gro56] implies that every operator from ℓ_1 to ℓ_2 is $(1, 1)$-summing; this is exactly the idea of Kwapień's proof.

The above leads us directly to the aim of this thesis: We show that many important results occurring in the theory of absolutely summing operators and related fields are of interpolative nature (concerning the complex interpolation functor) which has not been covered by literature yet, and we use the same interpolation techniques to give generalizations of these results or to derive completely new ones.

An essential question which arises within this context is whether spaces of operators behave "well" under complex interpolation; more precisely, if $0 < \theta < 1$ and interpolation couples $[E_0, E_1]$ and $[F_0, F_1]$ of complex Banach spaces are given, then do formulas such as

$$[\mathcal{L}(E_0, F_0), \mathcal{L}(E_1, F_1)]_\theta = \mathcal{L}([E_0, E_1]_\theta, [F_0, F_1]_\theta)$$

hold algebraically and topologically (where $\mathcal{L}(E,F)$ denotes the space of all continuous linear operators between Banach spaces E and F, endowed with the usual operator norm)? Kouba proved in [Kou91] that under certain geometric assumptions on the Banach spaces E_i and F_i related formulas for the injective tensor product of these Banach spaces such as

$$[E_0\tilde{\otimes}_\varepsilon F_0, E_1\tilde{\otimes}_\varepsilon F_1]_\theta = [E_0,E_1]_\theta\tilde{\otimes}_\varepsilon[F_0,F_1]_\theta$$

hold (where $E\tilde{\otimes}_\varepsilon F$ denotes the completion of the injective tensor product of the Banach spaces E and F). Kouba also treated the special case of Banach function spaces. In the first section of this work we extend and unify Kouba's results to tensor products of vector-valued Banach function spaces as follows: We find conditions on the Banach function spaces $X_i(\mu), Y_i(\mu)$ and the Banach spaces E_i, F_i such that

$$[X_0(E_0)\tilde{\otimes}_\varepsilon Y_0(F_0), X_1(E_1)\tilde{\otimes}_\varepsilon Y_1(F_1)]_\theta = [X_0(E_0), X_1(E_1)]_\theta\tilde{\otimes}_\varepsilon[Y_0(F_0), Y_1(F_1)]_\theta$$

holds. Moreover, based on variants of the Maurey–Rosenthal Factorization Theorem, our approach offers an alternate proof of Kouba's interpolation formula for tensor products of Banach function spaces. Main ingredients are upper estimates of

$$\|\mathcal{L}(\ell_2,[N_0,N_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2,N_0),\mathcal{L}(\ell_2,N_1)]_\theta\| \tag{0.1}$$

for an interpolation couple $[N_0,N_1]$ of finite-dimensional complex Banach spaces.

In the two sections hereafter we apply "uniform estimates" of (0.1) to the theory of summing operators. More precisely, we show how results for the complex interpolation of spaces of operators can be used in order to obtain asymptotic upper estimates of summing norms of single operators acting between finite-dimensional Banach spaces by complex interpolation. It turns out that in many concrete cases these asymptotic upper estimates are precise—they coincide with the associated asymptotic lower estimates which are derived by various other methods.

Section 2 is devoted to the study of so-called "Bennett–Carl inequalities", which were independently proved by Bennett [Ben73] and Carl [Car74] in 1973/74: For $1 \le u \le 2$ and $1 \le u \le v \le \infty$ the identity operator $\mathrm{id} : \ell_u \hookrightarrow \ell_v$ is absolutely $(r,2)$-summing, i. e. there is a constant $C > 0$ such that for each set of finitely many $x_1, \dots, x_n \in \ell_u$

$$\left(\sum_{k=1}^n \|x_k\|_{\ell_v}^r\right)^{1/r} \le C \cdot \sup_{\|x'\|_{\ell_u'}\le 1}\left(\sum_{k=1}^n |\langle x', x_k\rangle|^2\right)^{1/2},$$

if and only if $1/r \le 1/u - \max(1/v,1/2)$. This result improved upon older ones of Littlewood (see above) and Orlicz, and is nowadays of extraordinary importance in the theory of eigenvalue distribution of power compact operators; a consequence of the above is e. g. that every continuous operator $T : \ell_2 \to \ell_2$ with values in ℓ_u $(1 \le u < 2)$ lies in the Schatten-r-class, $1/r = 1/u - 1/2$, which by Weyl's inequality implies that T has an absolutely r-summing sequence $(\lambda_n(T))$ of eigenvalues. Later in 1992 the Bennett–Carl inequalities were extended within the setting of so-called mixing operators (originally invented by Maurey [Mau74]) by Carl and Defant [CD92].

The crucial step in the proofs of Bennett and Carl is to establish the case $1 \le u \le v = 2$ which in terms of finite-dimensional spaces reads as follows: For $1 \leq u \leq v = 2$ and $2 \leq r \leq \infty$ such that $1/r = 1/u - 1/2$

$$\sup_n \pi_{r,2}(\ell_u^n \hookrightarrow \ell_2^n) < \infty,$$

where $\pi_{r,2}(\ell_u^n \hookrightarrow \ell_2^n)$ denotes the $(r,2)$-summing norm of the embedding $\ell_u^n \hookrightarrow \ell_2^n$. Note that the formula $1/r = 1/u - 1/2$ occurs by "naive interpolation" of the parameter r between the two well-known border cases

$$\sup_n \pi_{\infty,2}(\ell_2^n \hookrightarrow \ell_2^n) = \sup_n \|\ell_2^n \hookrightarrow \ell_2^n\| < \infty$$

$$\sup_n \pi_{2,2}(\ell_1^n \hookrightarrow \ell_2^n) = \sup_n \pi_2(\ell_1^n \hookrightarrow \ell_2^n) < \infty$$

(π_2 the 2-summing norm): For $1 \leq u \leq 2$ choose $0 \leq \theta \leq 1$ with $1/u = (1-\theta)/2 + \theta/1$, then $1/r = (1-\theta)/\infty + \theta/2 = 1/u - 1/2$.

While Bennett and Carl used "Hardy–Littlewood techniques", our proof given in this thesis is heavily based on complex interpolation theory; in contrast to Kwapień's result, here the main point is complex interpolation in the range spaces, for which uniform upper estimates of (0.1) turn out to be crucial. Although our proof—more precisely, the complex interpolation theory behind it—is admittedly far from being simpler than the original ones, the used techniques turn out to be quite fruitful in order to obtain various new results within the framework of summing operators as can be seen in the sections afterwards. As a first example our approach yields a "non-commutative" analogue for identities between finite-dimensional Schatten classes $\mathcal{S}_u^n$: For u and r as above we obtain

$$\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \asymp n^{1/r}.$$

Moreover, our techniques lead us to a more general study of Bennett–Carl inequalities within the setting of symmetric Banach sequence spaces and unitary ideals, with applications to Lorentz and Orlicz sequence spaces.

Section 3 focuses on so-called "(B,q,p)-summing operators", a generalization of the class Π_γ of all "Gaussian-summing operators", which was introduced by Linde and Pietsch in 1974. For $2 < p < \infty$ an infinite orthonormal system B in some $L_2(\mu) \cap L_p(\mu)$ (where μ is a probability measure) is called a $\Lambda(p)$-system if the L_2-norm and the L_p-norm are equivalent on the span of B, and then we denote $K_p(B) := \|(\text{span}B, \|\cdot\|_2) \hookrightarrow (\text{span}B, \|\cdot\|_p)\|$; for simplicity we set $K_2(B) := 1$. This notion goes back as "p-lacunary" to Kadec and Pełczyński [KP62]; for sets of characters on a compact abelian group it coincides with that of $\Lambda(p)$-sets, which were investigated e. g. by Rudin [Rud60] and Bourgain [Bou89], who solved the long-standing "$\Lambda(p)$-set problem". Now for an infinite orthonormal system B and $2 \leq q < \infty$ such that $K_q(B) < \infty$, an operator T between Banach spaces X and Y is said to belong to the class of $(B,q,2)$-summing operators, $\Pi_{B,q,2}$, if there exists a constant $C > 0$ (the least of all these constants is denoted by $\pi_{B,q,2}(T)$) such that for all choices of $b_1, \ldots, b_n$ in B and $x_1, \ldots, x_n$ in X

$$\left(\int_\Omega \| \sum_{i=1}^n b_i \cdot Tx_i \|^q \, d\mu \right)^{1/q} \leq C \cdot K_q(B) \cdot \sup_{x' \in B_{X'}} \left(\sum_{i=1}^n |\langle x', x_i \rangle|^2 \right)^{1/2}. \qquad (0.2)$$

For $q = 2$ and a sequence $(g_i)_{i\in\mathbb{N}}$ of independent Gaussian variables one obtains the ideal of all Gaussian-summing operators. Using similar interpolation techniques as before in Section 2 we show that if B is a $\Lambda(p)$-system for all $2 < p < \infty$, then the limit order of the ideal $\Pi_{B,q,2}$ coincides with the limit order of the ideal of Gaussian-summing operators for all $2 \le q < \infty$; here, for $1 \le u, v \le \infty$ the limit order $\lambda(\Pi_{B,q,2}, u, v)$ is defined as usual:

$$\lambda(\Pi_{B,q,2}, u, v) := \sup\{\lambda > 0 \,|\, \exists\, \rho > 0\, \forall\, n : \pi_{B,q,2}(\ell_u^n \hookrightarrow \ell_v^n) \le \rho \cdot n^\lambda\}.$$

König [Kön74] showed a close connection of the limit order of a Banach operator ideal to the behavior of embedding maps of Sobolev spaces and weakly singular integral operator concerning this operator ideal. In the special case where B consists of characters on a compact abelian group even a certain equivalence holds: We conclude—with the help of results due to Baur—that B is a $\Lambda(p)$-system for all $2 < p < \infty$ if and only if $\lambda(\Pi_{B,2,2}, u, v) = \lambda(\Pi_\gamma, u, v)$ for all $1 \le u, v \le \infty$. Furthermore, we obtain precise asymptotic estimates for the Gaussian-summing norm of identities between finite-dimensional Schatten classes $\mathcal{S}_u^n$ as well as extensions of the Bennett–Carl inequalities within the setting of $(B, 2, p)$-summing operators (just substitute in (0.2) the weak-2-norm by the weak-p-norm, $1 \le p \le 2$).

In the last section we return to Kwapień's result which was discussed above, and consider complex interpolation of spaces of operators on ℓ_1. Pisier in [P79] gave an extension to Banach lattices which satisfy certain convexity and concavity assumptions, and Carl and Defant in [CD92] extended Kwapień's result within the framework of mixing operators. Kwapień and Pisier already used complex interpolation, whereas Carl and Defant's result is based on a certain tensor product trick. We show that the latter result can also be proved by the use of complex interpolation techniques. Furthermore, we offer another generalization of Kwapień's result within the framework of (γ, p)-summing operators (this stands for $(\mathcal{G}, 2, p)$-summing operators, where $\mathcal{G}$ is a sequence of independent Gaussian variables): For $1 \le p \le 2$ every continuous operator on ℓ_1 with values in a p-convex Banach function space X with non-trivial cotype is (γ, p)-summing, i. e.

$$\mathcal{L}(\ell_1, X) = \Pi_{\gamma,p}(\ell_1, X);$$

for $p = 2$ this is well-known. We conclude with a remark on a close relationship of the above result to the type number $p(X) := \sup\{1 \le p \le 2 \,|\, X \text{ is of type } p\}$ of a Banach function space X.

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Preliminaries

Banach spaces, operators and tensor products of Banach spaces

With $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we denote all natural, real and complex numbers, respectively, and $\mathbb{K}$ stands either for $\mathbb{R}$ or for $\mathbb{C}$. If (a_n) and (b_n) are scalar sequences we write $a_n \prec b_n$ whenever there is some $c \geq 0$ such that $a_n \leq c \cdot b_n$ for all n, and $a_n \asymp b_n$ whenever $a_n \prec b_n$ and $b_n \prec a_n$. For $1 \leq p \leq \infty$ the number p' is defined by $1/p + 1/p' = 1$.

The n-th Rademacher function r_n on $[0,1]$ is defined as usual: $r_n(t) := (-1)^k$ if $t \in [k/2^n, (k+1)/2^n)$; we often use the fact that if we define $D_n := \{-1,+1\}^n$, $\mu_n(\{\omega\}) := 1/2^n$ for $\omega \in D_n$ and ε_i to be the i-th projection, then $(r_1, \dots, r_n)$ and $(\varepsilon_1, \dots, \varepsilon_n)$ have the same distribution. In particular, in the left hand side of the forthcoming inequality (0.3) the r_i's, $\int_0^1$ and $d\lambda$ can be replaced by the ε_i's, $\int_{D_n}$ and $d\mu_n$, respectively.

We use standard notation and notions from Banach space theory, as presented e. g. in [DJT95], [LT77], [LT79] and [TJ89]. If E is a Banach space, then B_E is its (closed) unit ball and E' its dual. As usual $\mathcal{L}(E,F)$ denotes the Banach space of all (bounded and linear) operators from E into F endowed with the operator norm $\|\cdot\|$; furthermore, if $E_1, \dots, E_n$ and F are Banach spaces, then $\mathcal{L}(E_1, \dots, E_n; F)$ stands for the collection of all n-linear and continuous operators $T : E_1 \times \cdots \times E_n \to F$ together with the norm

$$\|T\| := \sup\{\|T(x_1, \dots, x_n)\|_F \,|\, x_i \in B_{E_i}, 1 \leq i \leq n\}.$$

For $1 \leq p \leq 2 \leq q < \infty$ a Banach space E is said to be of type p and cotype q if there exist constants $C_p, C_q > 0$ such that for all finite sequences $x_1, \dots, x_n$ in E

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i x_i \right\|_E^2 d\lambda \right)^{1/2} \leq C_p \cdot \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p} \tag{0.3}$$

and

$$\left(\sum_{i=1}^n \|x_i\|_E^q \right)^{1/q} \leq C_q \cdot \left(\int_0^1 \left\| \sum_{i=1}^n r_i x_i \right\|_E^2 d\lambda \right)^{1/2}, \tag{0.4}$$

respectively; with $\mathbf{T_p}(E)$ and $\mathbf{C_q}(E)$ we denote the smallest constants C_p and C_q which satisfy (0.3) and (0.4), respectively.

We call a Banach space $E \subset c_0$ (the space of all zero sequences) a symmetric Banach sequence space if the i-th standard unit vectors e_i form a symmetric basis, i. e. the e_i's form a Schauder basis such that $\|x\|_E = \|\sum_{i=1}^\infty \varepsilon_i x_{\pi(i)} e_i\|_E$ for each $x \in E$, each permutation π of $\mathbb{N}$ and each choice of scalars ε_i with $|\varepsilon_i| = 1$. Moreover, denote for each n the subspace $\operatorname{span}\{e_i \,|\, 1 \leq i \leq n\}$ of E by E_n. Together with its natural order a symmetric Banach sequence space E forms a Banach lattice, and clearly its basis is 1-unconditional. The associated unitary ideal $\mathcal{S}_E$ is the Banach space of all compact operators $T \in \mathcal{L}(\ell_2, \ell_2)$ with singular numbers $(s_i(T))_i$ in E endowed with the norm $\|T\|_{\mathcal{S}_E} := \|(s_i(T))_i\|_E$; with $\mathcal{S}_E^n$ we denote $\mathcal{L}(\ell_2^n, \ell_2^n)$ together with the norm $\|T\|_{\mathcal{S}_E^n} := \|(s_i(T))_{i=1}^n\|_{E_n}$. For $E = \ell_u$ $(1 \leq u < \infty)$ one gets the well-known Schatten-u-class $\mathcal{S}_u$; for simplicity put $\mathcal{S}_\infty := \mathcal{L}(\ell_2, \ell_2)$.

For all information on Banach operator ideals see e. g. [DF93], [DJT95] and [Pie80], and for the theory of tensor norms on tensor products of Banach spaces we refer to [DF93]. Here we would just like to introduce the injective tensor norm ε: For Banach spaces E, F and $z \in E \otimes F$ we define

$$\|z\|_{E \otimes_\varepsilon F} := \sup\{|\langle x' \otimes y', z\rangle| \,|\, x' \in B_{E'}, y' \in B_{F'}\},$$

and with $E \tilde{\otimes}_\varepsilon F$ we denote the completion of $E \otimes F$ with respect to the norm $\| \cdot \|_{E \otimes_\varepsilon F}$. We will extensively use the fact that $E' \otimes_\varepsilon F = \mathcal{L}(E, F)$ isometrically whenever one of the involved spaces is finite-dimensional.

For a Banach operator ideal $(\mathcal{A}, A)$ and $1 \leq u, v \leq \infty$ the limit order $\lambda(\mathcal{A}, u, v)$ is defined as follows:

$$\lambda(\mathcal{A}, u, v) := \inf\{\lambda > 0 \,|\, \exists\, \rho > 0 \,\forall\, n \in \mathbb{N} : A(\ell_u^n \hookrightarrow \ell_v^n) \leq \rho \cdot n^\lambda\}.$$

This notion was introduced by Pietsch; König showed in [Kön74] a non-trivial connection to embedding maps of Sobolev spaces and weakly singular integral operators (see also [Pie80, 22.7]).

Banach function spaces

Let (Ω, Σ, μ) be a σ-finite and complete measure space, and denote all μ-a.e. equivalence classes of real-valued measurable functions on Ω by $L_0(\mu)$. A Banach space $X = X(\mu)$ of (equivalence classes of) functions in $L_0(\mu)$ is said to be a Banach function space if it satisfies the following conditions:

(I) If $|f| \leq |g|$, with $f \in L_0(\mu)$ and $g \in X(\mu)$, then $f \in X(\mu)$ and $\|f\|_X \leq \|g\|_X$.

(II) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to $X(\mu)$.

For $1 \leq p \leq \infty$ we write as usual $L_p(\mu)$ for the space of measurable functions whose p-th power is integrable if $p < \infty$, and essentially bounded if $p = \infty$. These are Banach function spaces with respect to the norms

$$\|f\|_p := \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}, \quad (1 \leq p < \infty)$$

and

$$\|f\|_\infty := \text{ess-sup}|f(\omega)| < \infty.$$

For $1 \leq p \leq q < \infty$ a Banach function space $X(\mu)$ is said to be p-convex and q-concave if there exist constants $C_p, C_q > 0$ such that for all $f_1, \ldots, f_n \in X(\mu)$

$$\left\| \left(\sum_{i=1}^{n} |f_i|^p \right)^{1/p} \right\|_X \leq C_p \cdot \left(\sum_{i=1}^{n} \|f_i\|_X^p \right)^{1/p} \tag{0.5}$$

and

$$\left(\sum_{i=1}^{n} \|f_i\|_X^q \right)^{1/q} \leq C_q \cdot \left\| \left(\sum_{i=1}^{n} |f_i|^q \right)^{1/q} \right\|_X ; \tag{0.6}$$

we denote by $\mathbf{M}^{(\mathbf{p})}(X)$ and $\mathbf{M}_{(\mathbf{q})}(X)$ the smallest constants C_p and C_q which satisfy (0.5) and (0.6), respectively. Each Banach function space X is 1-convex with $\mathbf{M}^{(\mathbf{1})}(X) = 1$, and for $1 \leq p < \infty$ the space $L_p(\mu)$ is p-convex and p-concave with $\mathbf{M}^{(\mathbf{p})}(L_p(\mu)) = \mathbf{M}_{(\mathbf{p})}(L_p(\mu)) = 1$; $L_\infty(\mu)$ is p-convex for all $1 \leq p < \infty$, but in general not p-concave for any $1 \leq p < \infty$.

Let $X_0(\mu), X_1(\mu)$ be Banach function spaces and $0 < \theta < 1$. Define the space $X_0^{1-\theta} X_1^\theta$ as the set of functions $f \in L_0(\mu)$ for which there exist $g \in X_0$ and $h \in X_1$ such that $|f| = |g|^{1-\theta} \cdot |h|^\theta$. Together with the norm

$$\|f\|_{X_0^{1-\theta} X_1^\theta} := \inf\{\|g\|_{X_0}^{1-\theta} \cdot \|h\|_{X_1}^\theta \mid |f| = |g|^{1-\theta} \cdot |h|^\theta, g \in X_0, h \in X_1\},$$

$X_0^{1-\theta} X_1^\theta$ becomes a Banach function space (with respect to (Ω, Σ, μ)). This space, originally invented by Calderón in [Cal64], became an important tool within the framework of complex interpolation of vector-valued Banach function spaces. It can be easily seen (see e. g. [TJ89, p. 218/219]) that if for $1 \leq r < \infty$ the lattices X_0, X_1 are both r-convex or both r-concave, then $X_0^{1-\theta} X_1^\theta$ also has this property, with

$$\mathbf{M}^{(\mathbf{r})}(X_0^{1-\theta} X_1^\theta) \leq \mathbf{M}^{(\mathbf{r})}(X_0)^{1-\theta} \cdot \mathbf{M}^{(\mathbf{r})}(X_1)^\theta, \tag{0.7}$$

$$\mathbf{M}_{(\mathbf{r})}(X_0^{1-\theta} X_1^\theta) \leq \mathbf{M}_{(\mathbf{r})}(X_0)^{1-\theta} \cdot \mathbf{M}_{(\mathbf{r})}(X_1)^\theta, \tag{0.8}$$

respectively.

A finite-dimensional real Banach space $X = (\mathbb{R}^n, \|\cdot\|_X)$ is called an n-dimensional lattice if $\|\cdot\|_X$ is a lattice norm in the above sense. For $0 < r < \infty$ and an n-dimensional lattice X with $\mathbf{M}^{(\max(\mathbf{1},\mathbf{r}))}(X) = 1$ we define the lattice norm

$$\|x\|_r := \||x|^{1/r}\|_X^r, \quad x \in \mathbb{R}^n;$$

the n-dimensional lattice $(\mathbb{R}^n, \|\cdot\|_r)$ is denoted by X^r. Such "powers" of finite-dimensional lattices will play an important role in Sections 1 and 4. Finally note that if X is an n-dimensional lattice, then its dual X' is also an n-dimensional lattice, and one has

$$\|x\|_X = \sup_{\|y\|_{X'} \leq 1} \left| \sum_{i=1}^{n} x_i y_i \right| = \sup_{\|y\|_{X'} \leq 1} \|xy\|_{\ell_1^n}.$$

Let $X(\mu)$ be a Banach function space and E a Banach space. A function x defined on Ω with values in E is said to be strongly measurable if there exists a sequence of strictly simple functions on Ω converging to x almost everywhere; here a function y on Ω with values in E is called strictly simple if it assumes only finitely many non-zero values, each on a measurable set with finite measure. Then by $X(E)$ we denote the collection of strongly measurable functions x with values in E for which $\|x(\cdot)\|_E \in X$. Together with the norm $\|x\|_{X(E)} := \|\|x(\cdot)\|_E\|_X$ this becomes a Banach space ($\mathbb{K}$-linear whenever E is $\mathbb{K}$-linear). A complex Banach space Y of the form $Y = X(\mathbb{C})$ with some real Banach function space X is called a complex Banach function space; for $1 \leq p < \infty$ it is defined to be p-convex or p-concave whenever X has this property, respectively.

Complex interpolation of Banach spaces

In the following we give a short introduction to the theory of interpolation of Banach spaces. We extensively use the complex interpolation method, and therefore we keep the general case short; for an introduction to interpolation theory we refer to [BL78] and [KPS82].

A pair $[X_0, X_1]$ of Banach spaces is called an interpolation couple if there is a Hausdorff topological vector space Y such that X_0 and X_1 are both continuously embedded in Y. With X_Δ and X_Σ we denote $X_0 \cap X_1$ and $X_0 + X_1$, respectively, equipped with their natural norms. A Banach space X which is continuously embedded in Y is called an intermediate space with respect to $[X_0, X_1]$ whenever $X_\Delta \subset X \subset X_\Sigma$ continuously. If we speak of a finite-dimensional interpolation couple $[X_0, X_1]$, we always assume that X_0 and X_1 have the same finite dimension.

From now on all Banach spaces are meant to be complex. Given an interpolation couple $[X_0, X_1]$, we consider the space $\mathcal{F}(X_0, X_1)$ of all functions f with values in X_Σ, which are bounded and continuous on the strip $S := \{z \,|\, 0 \leq \text{Re } z \leq 1\}$ and analytic on the open strip $\{z \,|\, 0 < \text{Re } z < 1\}$, and moreover, the functions $t \mapsto f(j + it)$ $(j = 0, 1)$ are continuous functions from the real line into X_j, which tend to zero as $|t| \to \infty$ (for $X_0 = X_1 = \mathbb{C}$ we denote the set of all these functions by $A(S)$). Equipped with the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max\left(\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right),$$

$\mathcal{F}(X_0, X_1)$ becomes a Banach space. Then for $0 < \theta < 1$ the space $[X_0, X_1]_\theta$ which consists of all $x \in X_\Sigma$ such that $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$, provided with the quotient norm

$$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} \,|\, f(\theta) = x, f \in \mathcal{F}(X_0, X_1)\},$$

is an intermediate space with respect to $[X_0, X_1]$; it is called the complex interpolation space with respect to $[X_0, X_1]$ and θ. Note that for every complex Banach space X the isometric equality $[X, X]_\theta = X$ holds (see [BL78, 4.2.1]).

The duality theorem [BL78, 4.5.2] stated next is only needed for finite-dimensional interpolation couples, but it also holds in the infinite-dimensional case provided that one of the involved spaces is reflexive.

Proposition 0.1. *Let* $[X_0, X_1]$ *be a finite-dimensional interpolation couple and* $0 < \theta < 1$. *Then* $[X_0, X_1]'_\theta = [X'_0, X'_1]_\theta$ *holds isometrically.*

The following mapping property is often referred to as the "usual interpolation theorem" ([BL78, 4.1.2]).

Proposition 0.2. *Let* $[X_0, X_1]$, $[Y_0, Y_1]$ *be interpolation couples and* $T \in \mathcal{L}(X_\Sigma, Y_\Sigma)$ *such that* $T|_{X_j} \in \mathcal{L}(X_j, Y_j)$, $j = 0, 1$. *Then* $T|_{[X_0,X_1]_\theta} \in \mathcal{L}([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)$, *and*

$$\|T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta\| \leq \|T : X_0 \to Y_0\|^{1-\theta} \cdot \|T : X_1 \to Y_1\|^\theta.$$

Another main tool is the following extension of the preceding proposition to multilinear mappings ([BL78, 4.4.1]). Since in all our applications the involved couples consist of two spaces which coincide algebraically with equivalent norms, our formulation is only for this setting.

Proposition 0.3. *Let* $[X_0^{(1)}, X_1^{(1)}], \ldots, [X_0^{(n)}, X_1^{(n)}]$ *and* $[Y_0, Y_1]$ *be interpolation couples for which each consists of algebraically equal and norm-equivalent spaces, and let* $T \in \mathcal{L}(X_\Delta^{(1)}, \ldots, X_\Delta^{(n)}; Y_\Delta)$. *Then*

$$\begin{aligned}\|T : [X_0^{(1)}, X_1^{(1)}]_\theta \times \cdots \times [X_0^{(n)}, X_1^{(n)}]_\theta \to [Y_0, Y_1]_\theta\| \\ \leq \|T : X_0^{(1)} \times \cdots \times X_0^{(n)} \to Y_0\|^{1-\theta} \cdot \|T : X_1^{(1)} \times \cdots \times X_1^{(n)} \to Y_1\|^\theta.\end{aligned}$$

The most common examples of complex interpolation spaces are (vector-valued) L_p-spaces and Schatten classes: For $1 \leq p_0, p_1 \leq \infty$, a σ-complete measure space (Ω, Σ, μ), an interpolation couple $[E_0, E_1]$ and $0 < \theta < 1$

$$[L_{p_0}(\mu, E_0), L_{p_1}(\mu, E_1)]_\theta = L_p(\mu, [E_0, E_1]_\theta) \tag{0.9}$$

and

$$[\mathcal{S}_{p_0}, \mathcal{S}_{p_1}]_\theta = \mathcal{S}_p \tag{0.10}$$

hold isometrically, where $1/p = (1-\theta)/p_0 + \theta/p_1$; but in the case $p_0 = p_1 = \infty$ we have to assume that $L_\infty(\mu) = \ell_\infty^n$ for some n. For (0.9) see [BL78, 5.1.2], whereas (0.10) can be deduced from e. g. [PT68, Satz 8] and the complex reiteration theorem [BL78, 4.6.1]. For $0 < \theta < 1$ a θ-Hilbert space is a complex interpolation space $[E_0, E_1]_\theta$ where E_1 is a Hilbert space (this notion goes back to Pisier); in particular, $L_p(\mu)$ and $\mathcal{S}_p$ for $1 < p < \infty$ are θ-Hilbert spaces for $\theta = 1 - |1 - 2/p|$.

The following interpolation formula for vector-valued Banach function spaces is due to Calderón; note that in [Cal64, 13.6] the space $X_0^{1-\theta} X_1^\theta$ is assumed to be σ-order continuous, but in [KPS82, p. 245] it is shown that this requirement is satisfied if at least one of the spaces X_0 or X_1 is σ-order continuous.

Proposition 0.4. *Let $X_0(\mu), X_1(\mu)$ be Banach function spaces such that at least one is σ-order continuous. Then for each interpolation couple $[E_0, E_1]$ and $0 < \theta < 1$*

$$[X_0(E_0), X_1(E_1)]_\theta = (X_0^{1-\theta} X_1^\theta)([E_0, E_1]_\theta) \tag{0.11}$$

holds isometrically.

Finally we would like to point out that, since we extensively use complex interpolation, the underlying field is always $\mathbb{C}$—important exceptions are mentioned explicitly. However, many of our main results can be easily transferred to the real case; we leave this work to the interested reader.

1 A complex interpolation formula for tensor products of vector-valued Banach function spaces

The following theorem for the complex interpolation of injective tensor products of vector-valued Banach function spaces is proved:

Theorem 1.1. *Let $X_0(\mu), X_1(\mu), Y_0(\nu), Y_1(\nu)$ be real-valued Banach function spaces and $[E_0, E_1]$ and $[F_0, F_1]$ interpolation couples of complex Banach spaces with dense intersections. Then for $0 < \theta < 1$ the equality*

$$[X_0(E_0)\tilde{\otimes}_\varepsilon Y_0(F_0), X_1(E_1)\tilde{\otimes}_\varepsilon Y_1(F_1)]_\theta = [X_0(E_0), X_1(E_1)]_\theta \tilde{\otimes}_\varepsilon [Y_0(F_0), Y_1(F_1)]_\theta, \qquad (1.1)$$

holds algebraically and topologically whenever the Banach lattices X_0, X_1, Y_0, Y_1 are 2-concave and the Banach spaces E_i and F_i satisfy one of the following conditions:

(1) *E_0', E_1', F_0' and F_1' are type 2 spaces.*
(2) *E_0', E_1' are type 2 spaces and $F_0 = F_1$ is a cotype 2 space.*
(3) *$E_0 = E_1$ and $F_0 = F_1$ are cotype 2 spaces.*

This is an extension of deep results due to Kouba [Kou91] who proved the preceding interpolation formula if one of the couples $[X_0, X_1]$ and $[E_0, E_1]$, and one of the couples $[Y_0, Y_1]$ and $[F_0, F_1]$ is trivial (i. e. either $X_0 = X_1 = \mathbb{R}$ or $E_0 = E_1 = \mathbb{C}$, and either $Y_0 = Y_1 = \mathbb{R}$ or $F_0 = F_1 = \mathbb{C}$). Moreover, following an idea of Pisier [P90] and based on variants of the Maurey–Rosenthal Factorization Theorem (see [Def99]), our approach offers an alternate proof of Kouba's interpolation formula for complex-valued Banach function spaces: For 2-concave complex-valued Banach function spaces $X_0(\mu), X_1(\mu), Y_0(\nu), Y_1(\nu)$ and $0 < \theta < 1$

$$[X_0\tilde{\otimes}_\varepsilon Y_0, X_1\tilde{\otimes}_\varepsilon Y_1]_\theta = [X_0, X_1]_\theta \tilde{\otimes}_\varepsilon [Y_0, Y_1]_\theta. \qquad (1.2)$$

The main ingredients of the proof will be "uniform estimates" of

$$d_\theta[M_0, M_1] := \|\mathcal{L}(\ell_2, [M_0, M_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2, M_0), \mathcal{L}(\ell_2, M_1)]_\theta\|, \qquad (1.3)$$

where $[M_0, M_1]$ is a finite-dimensional interpolation couple. Such estimates proved to be of independent interest: The facts $\sup_n d_\theta[\ell_1^n, \ell_2^n] < \infty$ (see [P90] and [Kou91]; here it is a consequence of Proposition 1.5) and $\sup_n d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] < \infty$ (due to Junge in [Jun96, 4.2.6] and based on an extension of Kouba's formulas for the Haagerup tensor product of operator spaces due to [P96]) are used in Section 2 in order to study so-called "Bennett–Carl inequalities" for identity operators between finite-dimensional symmetric Banach sequence spaces as well as their "non-commutative analogues" for identity operators between finite-dimensional unitary ideals. Part of this section is contained in [DM99].

1.1 The approximation lemma

First we show—similar to [Kou91, Section 4]—that equalities as stated in the above theorem are of finite-dimensional nature. In order to make the following more readable, let us introduce the following notation: If $[E_0, E_1]$ is an interpolation couple, $E \subset E_\Delta$ a subspace which is dense in E_0, E_1 and $\mathcal{A} \subset FIN(E)$ is cofinal (i. e. for every $G \in FIN(E)$ there exists $M \in \mathcal{A}$ with $G \subset M$), then the triple $([E_0, E_1], E, \mathcal{A})$ is called a *cofinal interpolation triple.* For $M \in FIN(E)$ we denote by M_0 and M_1 the subspace M of E_0 and E_1 endowed with the induced norm, respectively.

The content of the following lemma is a well-known fact within the theory of complex interpolation of Banach spaces (see e. g. [KPS82, Remark on p. 220] together with [KPS82, Theorem 1.3 on p. 223]).

Lemma 1.2. *Let $[E_0, E_1]$ be an interpolation couple and E be a subspace of E_Δ which is dense in both E_0 and E_1. Then for all $x \in E$*

$$\|x\|_{[E_0,E_1]_\theta} = \inf_{f(\theta)=x} \|f\|_{\mathcal{F}(E_0,E_1)},$$

where the infimum is taken over all $f \in \mathcal{F}(E_0, E_1)$ of the form

$$f(z) = \sum_{n=1}^{N} \psi_n(z) \cdot x_n, \qquad x_n \in E, \psi_n \in A(S).$$

In particular, E is also dense in $[E_0, E_1]_\theta$.

The following two crucial lemmas are only slight modifications of [Kou91, 4.1 and 4.2], but we state their proofs for the convenience of the reader.

Lemma 1.3. *Let $([E_0, E_1], E, \mathcal{A})$ be a cofinal interpolation triple. Then for $0 < \theta < 1$, $\varepsilon > 0$ and each $G \in FIN(E)$ there exists $M \in \mathcal{A}$ such that $G \subset M$ and for all $x \in G$*

$$(1-\varepsilon) \cdot \|x\|_{[M_0,M_1]_\theta} \le \|x\|_{[E_0,E_1]_\theta} \le \|x\|_{[M_0,M_1]_\theta}. \tag{1.4}$$

Proof. Without loss of generality we may assume $\varepsilon < 1$. Then let $0 < \delta < \varepsilon/2$ and $\mathcal{R} = \{x_1, \ldots, x_n\}$ be a δ-net in the unit sphere of the finite-dimensional space $(G, \|\cdot\|_{[E_0,E_1]_\theta})$. By Lemma 1.2 there exists for every $1 \le k \le n$ a function $F_k : S \to E$ of the form

$$F_k(z) = \sum_{r=1}^{n_k} \psi_{r,k}(z) \cdot x_{r,k}$$

where $x_{r,k} \in E$, $\psi_{r,k} \in A(S)$, $F_k(\theta) = x_k$ and $\|F_k\|_{\mathcal{F}(E_0,E_1)} \le 1 + \delta$. Define the finite-dimensional space

$$\tilde{G} := \operatorname{span}\{x_{r,k} \,|\, 1 \le k \le n, 1 \le r \le n_k\},$$

and choose $M \in \mathcal{A}$ such that $\tilde{G} \subset M$; clearly we have $G \subset M$. Now take $x \in G$ with $\|x\|_{[E_0,E_1]_\theta} = 1$. Then one may write $x = y_0 + \sum_{k=1}^{\infty} \lambda_k \cdot y_k$ with $0 \le \lambda_k < \delta^k$ and $y_k \in \mathcal{R}$. If $y_k = x_j$ for some $1 \le j \le n$ we put $H_k = F_j$ and define $F := H_0 + \sum_{k=1}^{\infty} \lambda_k \cdot H_k$. Then $F \in \mathcal{F}(M_0, M_1)$, $F(\theta) = x$ and

$$\|F\|_{\mathcal{F}(M_0,M_1)} \le \left(\sum_{k=0}^{\infty} \delta^k \right) (1+\delta) = \frac{1+\delta}{1-\delta},$$

hence

$$\|x\|_{[M_0,M_1]_\theta} \le \frac{1}{1-\varepsilon},$$

which gives the first inequality. The second one is clear by the usual interpolation theorem. $\square$

If $[M_0, M_1]$ and $[N_0, N_1]$ are finite-dimensional interpolation couples, then we define for $0 < \theta < 1$

$$d_\theta[M_0, M_1; N_0, N_1] := \|[M_0, M_1]_\theta \otimes_\varepsilon [N_0, N_1]_\theta \hookrightarrow [M_0 \otimes_\varepsilon N_0, M_1 \otimes_\varepsilon N_1]_\theta\|.$$

The following lemma—which for obvious reasons is called "approximation lemma"—reduces the proof of Kouba type formulas (1.1) or (1.2) to uniform estimates of $d_\theta[M_0, M_1; N_0, N_1]$ for cofinally many suitable finite-dimensional subspaces of the underlying infinite-dimensional spaces.

Approximation Lemma 1.4. *Let* $([E_0, E_1], E, \mathcal{A})$ *and* $([F_0, F_1], F, \mathcal{B})$ *be cofinal interpolation triples and* $0 < \theta < 1$. *If*

$$d_\theta[E_0, E_1; F_0, F_1] := \sup_{M\in\mathcal{A}} \sup_{N\in\mathcal{B}} d_\theta[M_0, M_1; N_0, N_1] < \infty,$$

then

$$[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta = [E_0, E_1]_\theta \tilde{\otimes}_\varepsilon [F_0, F_1]_\theta.$$

Proof. From the density assumptions we conclude that $E \otimes F$ is dense in $[E_0, E_1]_\theta \tilde{\otimes}_\varepsilon [F_0, F_1]_\theta$ and in $[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta$, hence it is sufficient to show that for a given $z \in E \otimes F$

$$\|z\|_{[E_0,E_1]_\theta \tilde{\otimes}_\varepsilon [F_0,F_1]_\theta} \le \|z\|_{[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta} \tag{1.5}$$

$$\le d_\theta[E_0, E_1; F_0, F_1] \cdot \|z\|_{[E_0,E_1]_\theta \tilde{\otimes}_\varepsilon [F_0,F_1]_\theta}. \tag{1.6}$$

We start with a simple observation to show (1.5). If $[M_0, M_1]$ and $[N_0, N_1]$ are finite-dimensional interpolation couples, then for each $0 < \theta < 1$

$$\|[\mathcal{L}(M_0, N_0), \mathcal{L}(M_1, N_1)]_\theta \hookrightarrow \mathcal{L}([M_0, M_1]_\theta, [N_0, N_1]_\theta)\| \le 1; \tag{1.7}$$

indeed, consider for $i = 0, 1$ the bilinear mapping

$$\phi_i : \mathcal{L}(M_i, N_i) \times M_i \to N_i, \quad (T, x) \mapsto Tx,$$

which clearly has norm 1, hence (1.7) follows from the fact that by bilinear interpolation (see [BL78, 4.4.1]) the interpolated mapping

$$\phi_\theta : [\mathcal{L}(M_0, N_0), \mathcal{L}(M_1, N_1)]_\theta \times [M_0, M_1]_\theta \to [N_0, N_1]_\theta$$

also has norm ≤ 1. Now (1.5) is a straightforward consequence: Obviously $\mathcal{C} := \{M \otimes N \mid M \in \mathcal{A}, N \in \mathcal{B}\} \subset FIN(E \otimes F)$ is cofinal, hence, by Lemma 1.3 and the fact that the injective norm respects subspaces, there exist $M \in \mathcal{A}$ and $N \in \mathcal{B}$ such that $z \in M \otimes N$ and

$$\|z\|_{[M_0 \otimes_\varepsilon N_0, M_1 \otimes_\varepsilon N_1]_\theta} \le (1 + \varepsilon) \cdot \|z\|_{[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta}.$$

Finally, by the mapping property of the injective norm and (1.7),

$$\begin{aligned}\|z\|_{[E_0,E_1]_\theta \tilde{\otimes}_\varepsilon [F_0,F_1]_\theta} &\leq \|z\|_{[M_0,M_1]_\theta \otimes_\varepsilon [N_0,N_1]_\theta} \\ &\leq \|z\|_{[M_0 \otimes_\varepsilon N_0, M_1 \otimes_\varepsilon N_1]_\theta} \\ &\leq (1+\varepsilon) \cdot \|z\|_{[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta}.\end{aligned}$$

In order to show (1.6) let $z \in G \otimes H$ for some $G \in FIN(E), H \in FIN(F)$, and choose by Lemma 1.3 subspaces $M \in \mathcal{A}$ and $N \in \mathcal{B}$ such that $G \subset M, H \subset N$ and

$$\|(G, \|\cdot\|_{[E_0,E_1]_\theta}) \hookrightarrow [M_0, M_1]_\theta\| \leq \sqrt{1+\varepsilon},$$

$$\|(H, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow [N_0, N_1]_\theta\| \leq \sqrt{1+\varepsilon}.$$

Then, by the mapping property,

$$\|(G, \|\cdot\|_{[E_0,E_1]_\theta}) \otimes_\varepsilon (H, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow [M_0, M_1]_\theta \otimes_\varepsilon [N_0, N_1]_\theta\| \leq 1+\varepsilon,$$

hence, since the injective norm respects subspaces,

$$\|z\|_{[M_0,M_1]_\theta \otimes_\varepsilon [N_0,N_1]_\theta} \leq (1+\varepsilon) \cdot \|z\|_{[E_0,E_1]_\theta \otimes_\varepsilon [F_0,F_1]_\theta}.$$

By the usual interpolation theorem we obtain

$$\begin{aligned}\|z\|_{[E_0 \tilde{\otimes}_\varepsilon F_0, E_1 \tilde{\otimes}_\varepsilon F_1]_\theta} &\leq \|z\|_{[M_0 \otimes_\varepsilon N_0, M_1 \otimes_\varepsilon N_1]_\theta} \\ &\leq d_\theta[M_0, M_1; N_0, N_1] \cdot \|z\|_{[M_0,M_1]_\theta \otimes_\varepsilon [N_0,N_1]_\theta} \\ &\leq (1+\varepsilon) \cdot d_\theta[E_0, E_1; F_0, F_1] \cdot \|z\|_{[E_0,E_1]_\theta \otimes_\varepsilon [F_0,F_1]_\theta}.\end{aligned}$$

□

1.2 The Hilbert space case

Recall for a finite-dimensional interpolation couple $[E_0, E_1]$ the definition of $d_\theta[E_0, E_1]$ from (1.3), and note that by the approximation lemma

$$d_\theta[E_0, E_1] = \sup_n d_\theta[\ell_2^n, \ell_2^n; E_0, E_1].$$

The main step in the proof of (1.1) is the following estimate:

Proposition 1.5. *Let X_0, X_1 be n-dimensional lattices and $[E_0, E_1]$ a finite-dimensional interpolation couple. Then for $0 < \theta < 1$*

$$\begin{aligned}&d_\theta[X_0(E_0), X_1(E_1)] \\ &\qquad \leq \sqrt{2} \cdot \mathbf{C_2}([E_0, E_1]_\theta) \cdot \mathbf{M_{(2)}}(X_0)^{1-\theta} \cdot \mathbf{M_{(2)}}(X_1)^\theta \cdot d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)].\end{aligned} \tag{1.8}$$

Before giving the proof we collect some facts about powers of finite-dimensional lattices.

Lemma 1.6. *Let X, X_0, X_1 be n-dimensional lattices, E a Banach space, $\lambda \in \mathbb{R}^n$ and $0 < \theta < 1$.*

(a) If $\mathbf{M}_{(\mathbf{2})}(X) = 1$, then $\|D_\lambda \otimes \mathrm{id} : \ell_2^n(E) \to X(E)\| \le \|D_\lambda\| = \|\lambda\|_{(((X')^2)')^{1/2}}$, where $D_\lambda : \ell_2^n \to X$ denotes the diagonal operator associated with λ.

(b) $(X_0^{1-\theta} X_1^\theta)' = (X_0')^{1-\theta}(X_1')^\theta$ holds isometrically.

(c) For $0 < r < \infty$ let $\mathbf{M}^{(\max(\mathbf{1},\mathbf{r}))}(X_0) = \mathbf{M}^{(\max(\mathbf{1},\mathbf{r}))}(X_1) = 1$. Then $\left(X_0^{1-\theta} X_1^\theta\right)^r = (X_0^r)^{1-\theta}(X_1^r)^\theta$ holds isometrically.

Proof. (a) For $x \in \ell_2^n(E)$

$$\|(D_\lambda \otimes \mathrm{id})x\|_{X(E)} = \|(\lambda_k \cdot \|x_k\|)_k\|_X \le \|D_\lambda : \ell_2^n \to X\| \cdot \left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2},$$

and (note that $\mathbf{M}^{(\mathbf{2})}(X') = \mathbf{M}_{(\mathbf{2})}(X) = 1$)

$$\begin{aligned}
\|\lambda\|_{(((X')^2)')^{1/2}} &= \|\lambda^2\|^{1/2}_{((X')^2)'} = \sup_{\|\mu\|_{(X')^2} \le 1} \|\lambda^2 \mu\|^{1/2}_{\ell_1^n} = \sup_{\||\mu|^{1/2}\|_{X'} \le 1} \|\lambda^2 \mu\|^{1/2}_{\ell_1^n} \\
&= \sup_{\|\nu\|_{X'} \le 1} \|\lambda \nu\|_{\ell_2^n} = \sup_{\|\nu\|_{X'} \le 1} \sup_{\|\mu\|_{\ell_2^n} \le 1} \left| \sum_{i=1}^n \lambda_i \nu_i \mu_i \right| \\
&= \sup_{\|\mu\|_{\ell_2^n} \le 1} \sup_{\|\nu\|_{X'} \le 1} \left| \sum_{i=1}^n \lambda_i \mu_i \nu_i \right| = \sup_{\|\mu\|_{\ell_2^n} \le 1} \|\lambda \mu\|_X = \|D_\lambda : \ell_2^n \to X\|.
\end{aligned}$$

(b) By the Calderón formula (0.11), the duality theorem (see Proposition 0.1) and the fact that $Y(\mathbb{C})' = Y'(\mathbb{C})$ holds isometrically for every finite-dimensional lattice Y, one arrives at the isometric identity

$$(X_0^{1-\theta} X_1^\theta)'(\mathbb{C}) = ((X_0')^{1-\theta}(X_1')^\theta)(\mathbb{C}),$$

which clearly implies the above statement.

(c) First note that by $\mathbf{M}^{(\mathbf{r})}(X_0^{1-\theta} X_1^\theta) = 1$ (see (0.7)) the power $(X_0^{1-\theta} X_1^\theta)^r$ is normed. Let $V := (X_0^{1-\theta} X_1^\theta)^r$ and $W := (X_0^r)^{1-\theta}(X_1^r)^\theta$. Then, if $|f|^{1/r} = |g|^{1-\theta} \cdot |h|^\theta$,

$$\|f\|_W \le \||g|^r\|^{1-\theta}_{X_0^r} \cdot \||h|^r\|^\theta_{X_1^r} = \left(\|g\|^{1-\theta}_{X_0} \cdot \|h\|^\theta_{X_1}\right)^r,$$

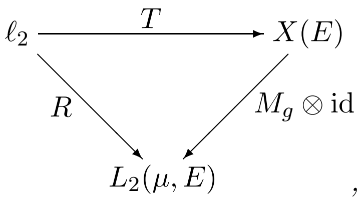
which clearly implies $\|f\|_W \le \|f\|_V$. Conversely, let $|f| = |g|^{1-\theta} \cdot |h|^\theta$. Then

$$\|f\|_V = \||f|^{1/r}\|^r_{X_0^{1-\theta} X_1^\theta} \le \||g|^{1/r}\|^{r(1-\theta)}_{X_0} \cdot \||h|^{1/r}\|^{r\theta}_{X_1} = \|g\|^{1-\theta}_{X_0^r} \cdot \|h\|^\theta_{X_1^r},$$

hence $\|f\|_V \le \|f\|_W$. □

Another important tool is a variant of the Maurey–Rosenthal Factorization Theorem ([Mau74]) for vector-valued Banach function spaces given in [Def99].

Lemma 1.7. *Let $X(\mu)$ be a 2-concave Banach function space and E a Banach space of cotype 2. Then each $T \in \mathcal{L}(\ell_2, X(E))$ factorizes as follows:*



,

where $R : \ell_2 \to L_2(\mu, E)$ is an operator and $M_g : L_2(\mu) \to X$ a multiplication operator with respect to $g \in L_0(\mu)$ such that $\|R\| \cdot \|M_g\| \le \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \mathbf{M_{(2)}}(X) \cdot \|T\|$.

Proof. For $x_1, \ldots, x_n \in \ell_2$

$$\begin{aligned}
\left\| \left(\sum_{i=1}^n \|Tx_i(\cdot)\|_E^2 \right)^{1/2} \right\|_X &\le \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \left\| \int_{D_n} \| \sum_{i=1}^n \varepsilon_i(\omega) \cdot Tx_i(\cdot) \|_E \, \mu_n(d\omega) \right\|_X \\
&\le \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \int_{D_n} \left\| \| (\sum_{i=1}^n \varepsilon_i(\omega) \cdot Tx_i)(\cdot) \|_E \right\|_X \mu_n(d\omega) \\
&= \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \int_{D_n} \left\| T(\sum_{i=1}^n \varepsilon_i(\omega) \cdot x_i) \right\|_{X(E)} \mu_n(d\omega) \\
&\le \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \|T\| \cdot \left(\sum_{i=1}^n \|x_i\|_{\ell_2}^2 \right)^{1/2}
\end{aligned}$$

(the constant $\sqrt{2}$ comes from the Khinchine–Kahane inequality for the case "L_2 versus L_1"), hence by [Def99, 4.4] there exists $0 \le \omega \in L_0(\mu)$ with

$$\sup_{y \in B_{L_2(\mu)}} \|\omega^{1/2} \cdot y\|_X \le \sqrt{2} \cdot \mathbf{M_{(2)}}(X) \cdot \mathbf{C_2}(E) \cdot \|T\| \tag{1.9}$$

such that for all $x \in \ell_2$

$$\left(\int_\Omega \|Tx(\cdot)\|_E^2 / \omega \, d\mu \right)^{1/2} \le \|x\|_{\ell_2}. \tag{1.10}$$

Define the operator $R \in \mathcal{L}(\ell_2, L_2(\mu, E))$ by $Rx := Tx/\omega^{1/2}$ for $x \in \ell_2$ (well-defined by (1.10)) and the multiplication operator $M_g : L_2(\mu) \to X$ with $g := \omega^{1/2}$ (well-defined by (1.9)). Clearly, this produces the desired factorization. □

Now we are prepared for the *Proof* of Proposition 1.5. Its main idea—the use of factorizations of Maurey–Rosenthal type—is taken from [P90].

Without loss of generality we may assume that $\mathbf{M_{(2)}}(X_0) = \mathbf{M_{(2)}}(X_1) = 1$; indeed, let Y_0 and Y_1 be the associated renormed spaces such that $\mathbf{M_{(2)}}(Y_i) = 1$ and

$\|X_i \hookrightarrow Y_i\| \cdot \|Y_i \hookrightarrow X_i\| \leq \mathbf{M_{(2)}}(X_i)$ for $i = 0, 1$ (see [LT79, 1.d.8]). Now consider the factorization

$$\begin{array}{ccc} \ell_2 \otimes_\varepsilon [X_0(E_0), X_1(E_1)]_\theta & \xrightarrow{\ \mathrm{id} \otimes \mathrm{id}\ } & [\ell_2 \otimes_\varepsilon X_0(E_0), \ell_2 \otimes_\varepsilon X_1(E_1)]_\theta \\ {\scriptstyle u := \mathrm{id} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle v := \mathrm{id} \otimes \mathrm{id}} \\ \ell_2 \otimes_\varepsilon [Y_0(E_0), Y_1(E_1)]_\theta & \xrightarrow[\ \mathrm{id} \otimes \mathrm{id}\ ]{} & [\ell_2 \otimes_\varepsilon Y_0(E_0), \ell_2 \otimes_\varepsilon Y_1(E_1)]_\theta \end{array}$$

and observe that $\|u\| \cdot \|v\| \leq \mathbf{M_{(2)}}(X_0)^{1-\theta} \cdot \mathbf{M_{(2)}}(X_1)^\theta$.

Put $X_\theta := X_0^{1-\theta} X_1^\theta$. Since $[X_0(E_0), X_1(E_1)]_\theta = X_\theta([E_0, E_1]_\theta)$ holds isometrically (see (0.11)) and $\mathbf{M_{(2)}}(X_\theta) = 1$ (see (0.8)), by Lemma 1.7 every operator $T \in \mathcal{L}(\ell_2, X_\theta([E_0, E_1]_\theta))$ factors

$$\begin{array}{ccc} \ell_2 & \xrightarrow{\ T\ } & X_\theta([E_0, E_1]_\theta) \\ & {\scriptstyle R} \searrow \quad \swarrow {\scriptstyle D_\lambda \otimes \mathrm{id}} & \\ & \ell_2^n([E_0, E_1]_\theta) & \end{array} ,$$

with $\|R\| \cdot \|D_\lambda\| \leq \sqrt{2} \cdot \mathbf{C_2}([E_0, E_1]_\theta) \cdot \|T : \ell_2 \to [X_0(E_0), X_1(E_1)]_\theta\|$. Define $Y_\eta := (((X'_\eta)^2)')^{1/2}$ for $\eta = 0, 1, \theta$; by Lemma 1.6 (b),(c) and the Calderón formula (0.11) we have $[Y_0(\mathbb{C}), Y_1(\mathbb{C})]_\theta = Y_\theta(\mathbb{C})$. By Lemma 1.6 (a) the mapping

$$\Phi_\eta : Y_\eta(\mathbb{C}) \to \mathcal{L}(\ell_2^n(E_\eta), X_\eta(E_\eta)), \quad \mu \mapsto D_\mu \otimes \mathrm{id}$$

has norm ≤ 1, and consequently the interpolated mapping

$$[\Phi_0, \Phi_1]_\theta : [Y_0(\mathbb{C}), Y_1(\mathbb{C})]_\theta \to V := [\mathcal{L}(\ell_2^n(E_0), X_0(E_0)), \mathcal{L}(\ell_2^n(E_1), X_1(E_1))]_\theta$$

has norm ≤ 1. Moreover, by bilinear interpolation (see Proposition 0.3) the mapping

$$U \times V \to W, \quad (u, v) \mapsto v \circ u,$$

where

$$U := [\mathcal{L}(\ell_2, \ell_2^n(E_0), \mathcal{L}(\ell_2, \ell_2^n(E_1))]_\theta \text{ and } W := [\mathcal{L}(\ell_2, X_0(E_0)), \mathcal{L}(\ell_2, X_1(E_1))]_\theta,$$

also has norm ≤ 1. Since by definition $\|R\|_U \leq d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)] \cdot \|R\|$, we obtain altogether

$$\begin{aligned} \|T\|_W = \|(D_\lambda \otimes \mathrm{id}) \circ R\|_W \leq \|R\|_U \cdot \|D_\lambda \otimes \mathrm{id}\|_V &= \|R\|_U \cdot \|[\Phi_0, \Phi_1]_\theta(\lambda)\|_V \\ &\leq d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)] \cdot \|R\| \cdot \|\lambda\|_{Y_\theta} \\ &\leq d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)] \cdot \sqrt{2} \cdot \mathbf{C_2}([E_0, E_1]_\theta) \cdot \|T\|, \end{aligned}$$

the desired inequality. □

A quick look at (1.8) reveals that in the case $E = E_0 = E_1$ one has

Corollary 1.8. *Let X_0, X_1 be n-dimensional lattices and E a finite-dimensional normed space. Then for* $0 < \theta < 1$

$$d_\theta[X_0(E), X_1(E)] \leq \sqrt{2} \cdot \mathbf{C_2}(E) \cdot \mathbf{M_{(2)}}(X_0)^{1-\theta} \cdot \mathbf{M_{(2)}}(X_1)^{\theta}. \tag{1.11}$$

For the case that E_0 and E_1 have different norms, one can use the following upper estimate for $d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)]$ in terms of type 2 constants, which is taken from [Kou91, 3.5]: Let $[F_0, F_1]$ be a finite-dimensional interpolation couple. Then

$$d_\theta[F_0, F_1] \leq \mathbf{T_2}(F_0')^{1-\theta} \cdot \mathbf{T_2}(F_1')^{\theta}. \tag{1.12}$$

Note that the estimate given in (1.12) is slightly different from that in Kouba's work. It follows from a short analysis of his proof in the finite-dimensional case: Since in our setting a Hilbert space is involved, Kouba's formula (3.8) on p. 47 can be changed into $\gamma_z(T) \leq |||T|||_z$. Moreover, calculating the term $W^E(z)$ defined in Kouba's Lemma 3.2 (use two spaces instead of a family of Banach spaces, see also [CCRSW82, p. 218]), one obtains

$$W^E(z) = \tilde{\mathbf{T}}_\mathbf{2}(E_0)^{1-\theta(z)} \cdot \tilde{\mathbf{T}}_\mathbf{2}(E_1)^{\theta(z)}$$

(with $\theta(z)$ as in [CCRSW82, Corollary 5.1]), respectively. Together with the fact that the Gaussian type 2 constant is smaller than the Rademacher type 2 constant, this leads to the above estimates (note that E_0 and E_1 in Kouba's formula, in our context have to be replaced by F_0' and F_1').

Using the simple fact that $\mathbf{T_2}(\ell_2^n(E_i')) = \mathbf{T_2}(E_i')$ for $i = 0, 1$ (see e. g. [DJT95, 11.12]), (1.12) gives $d_\theta[\ell_2^n(E_0), \ell_2^n(E_1)] \leq \mathbf{T_2}(E_0')^{1-\theta} \cdot \mathbf{T_2}(E_1')^{\theta}$. Furthermore, by the duality of type and cotype (see e. g. [DJT95, 11.10]) and the interpolative nature of the type 2 constants (see e. g. [TJ89, (3.8)]) $\mathbf{C_2}([E_0, E_1]_\theta) \leq \mathbf{T_2}([E_0', E_1']_\theta) \leq \mathbf{T_2}(E_0')^{1-\theta} \cdot \mathbf{T_2}(E_1')^{\theta}$. Altogether we arrive at

Corollary 1.9. *Let X_0, X_1 be n-dimensional lattices and $[E_0, E_1]$ a finite-dimensional interpolation couple. Then for* $0 < \theta < 1$

$$d_\theta[X_0(E_0), X_1(E_1)] \leq \sqrt{2} \cdot \mathbf{M_{(2)}}(X_0)^{1-\theta} \cdot \mathbf{M_{(2)}}(X_1)^{\theta} \cdot (\mathbf{T_2}(E_0')^{1-\theta} \cdot \mathbf{T_2}(E_1')^{\theta})^2. \tag{1.13}$$

1.3 The finite-dimensional case in general

Our estimates for $d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)]$ are as follows:

Proposition 1.10. *Let X_0, X_1 and Y_0, Y_1 be n-dimensional and m-dimensional lattices, respectively, and $[E_0, E_1]$, $[F_0, F_1]$ two arbitrary finite-dimensional interpolation couples. Then for* $0 < \theta < 1$

$$\begin{aligned} &d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)] \\ &\quad \leq 16 \cdot [(\mathbf{M_{(2)}}(X_0) \cdot \mathbf{M_{(2)}}(Y_0))^{1-\theta} (\mathbf{M_{(2)}}(X_1) \cdot \mathbf{M_{(2)}}(Y_1))^{\theta}]^{5/2} \cdot t_\theta[E_0, E_1] \cdot t_\theta[F_0, F_1], \end{aligned} \tag{1.14}$$

where, if G represents either E or F,

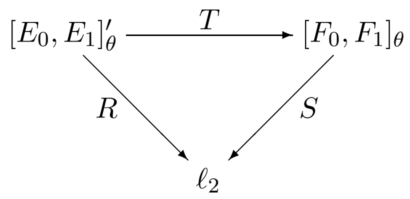
$$t_\theta[G_0, G_1] := \begin{cases} \mathbf{C_2}(G)^{5/2} & \text{if } G{=}G_0 = G_1, \\ (\mathbf{T_2}(G_0')^{1-\theta} \cdot \mathbf{T_2}(G_1')^{\theta})^{7/2} & \text{else.} \end{cases} \tag{1.15}$$

The proof is based on the following "factorization lemma", which will enable us to use the estimates from the Hilbert space case derived in (1.11) and (1.13) in order to obtain estimates for the general case. As usual we denote by Γ_2 the Banach operator ideal of all operators T which allow a factorization $T = RS$ through a Hilbert space, together with the norm $\gamma_2(T) := \inf \|R\| \cdot \|S\|$.

Lemma 1.11. *Let $[E_0, E_1]$ and $[F_0, F_1]$ be finite-dimensional interpolation couples. Then for $0 < \theta < 1$*

$$\|\Gamma_2([E_0, E_1]_\theta', [F_0, F_1]_\theta) \hookrightarrow [\Gamma_2(E_0', F_0), \Gamma_2(E_1', F_1)]_\theta\| \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1].$$

Proof. Let $T : [E_0, E_1]_\theta' \to [F_0, F_1]_\theta$ factorize as follows:



,

and consider by bilinear interpolation the norm 1 mapping

$$U \times V \to W, \quad (u, v) \mapsto v \circ u',$$

where

$$U := [\mathcal{L}(\ell_2, E_0), \mathcal{L}(\ell_2, E_1)]_\theta, \quad V := [\mathcal{L}(\ell_2, F_0), \mathcal{L}(\ell_2, F_1)]_\theta$$

and

$$W := [\Gamma_2(E_0', F_0), \Gamma_2(E_1', F_1)]_\theta.$$

Then

$$\|T\|_W = \|SR\|_W \leq \|R'\|_U \cdot \|S\|_V \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1] \cdot \|R'\| \cdot \|S\|,$$

which clearly gives $\|T\|_W \leq d_\theta[E_0, E_1] \cdot d_\theta[F_0, F_1] \cdot \gamma_2(T)$. ☐

Another ingredient needed for the proof of Proposition 1.10 is a simple fact about the cotype 2 constant of vector-valued Banach function spaces.

Lemma 1.12. *Let X be a 2-concave Banach function space and E a Banach space of cotype 2. Then $X(E)$ has cotype 2, and $\mathbf{C_2}(X(E)) \leq \sqrt{2} \cdot \mathbf{M_{(2)}}(X) \cdot \mathbf{C_2}(E)$.*

Proof. Let $x_1, \dots, x_n \in X(E)$. Then

$$\begin{aligned}
\left(\sum_{i=1}^n \|x_i\|_{X(E)}^2\right)^{1/2} &= \left(\sum_{i=1}^n \|\|x_i(\cdot)\|_E\|_X^2\right)^{1/2} \leq \mathbf{M_{(2)}}(X) \cdot \left\|\left(\sum_{i=1}^n \|x_i(\cdot)\|_E^2\right)^{1/2}\right\|_X \\
&\leq \sqrt{2} \cdot \mathbf{M_{(2)}}(X) \cdot \mathbf{C_2}(E) \cdot \left\| \int_{D_n} \|\sum_{i=1}^n \varepsilon_i(\omega) \cdot x_i(\cdot)\|_E \, \mu_n(d\omega) \right\|_X \\
&\leq \sqrt{2} \cdot \mathbf{M_{(2)}}(X) \cdot \mathbf{C_2}(E) \cdot \int_{D_n} \left\| \|\sum_{i=1}^n \varepsilon_i(\omega) \cdot x_i(\cdot)\|_E \right\|_X \mu_n(d\omega) \\
&= \sqrt{2} \cdot \mathbf{M_{(2)}}(X) \cdot \mathbf{C_2}(E) \cdot \int_{D_n} \|\sum_{i=1}^n \varepsilon_i(\omega) \cdot x_i\|_{X(E)} \, \mu_n(d\omega),
\end{aligned}$$

which clearly gives the claim. □

With this the proof of Proposition 1.10 is straightforward:

Proof of Proposition 1.10. For the moment denote by D_γ the norm of the embedding

$$\Gamma_2([X_0(E_0), X_1(E_1)]'_\theta, [Y_0(F_0), Y_1(F_1)]_\theta) \hookrightarrow [\Gamma_2(X_0(E_0)', Y_0(F_0)), \Gamma_2(X_1(E_1)', Y_1(F_1))]_\theta$$

and $d_\theta := d_\theta[X_0(E_0), X_1(E_1); Y_0(F_0), Y_1(F_1)]$. Using Pisier's Factorization Theorem ([P86a, 4.1] or [DF93, 31.4]), the Calderón formula (0.11), Lemma 1.12 and the interpolative nature of the 2-concavity constants (see (0.8)) one has

$$\begin{aligned}
d_\theta &\leq (2 \cdot \mathbf{C_2}([X_0(E_0), X_1(E_1)]_\theta) \cdot \mathbf{C_2}([Y_0(F_0), Y_1(F_1)]_\theta))^{3/2} \cdot D_\gamma \\
&= (2 \cdot \mathbf{C_2}((X_0^{1-\theta} X_1^\theta)([E_0, E_1]_\theta)) \cdot \mathbf{C_2}((Y_0^{1-\theta} Y_1^\theta)([F_0, F_1]_\theta)))^{3/2} \cdot D_\gamma \\
&\leq 8 \cdot (\mathbf{M_{(2)}}(X_0^{1-\theta} X_1^\theta) \cdot \mathbf{M_{(2)}}(Y_0^{1-\theta} Y_1^\theta) \cdot \mathbf{C_2}([E_0, E_1]_\theta) \cdot \mathbf{C_2}([F_0, F_1]_\theta))^{3/2} \cdot D_\gamma \\
&\leq 8 \cdot ((\mathbf{M_{(2)}}(X_0) \cdot \mathbf{M_{(2)}}(Y_0))^{1-\theta} \cdot (\mathbf{M_{(2)}}(X_1) \cdot \mathbf{M_{(2)}}(Y_1))^\theta \\
&\qquad \cdot \mathbf{C_2}([E_0, E_1]_\theta) \cdot \mathbf{C_2}([F_0, F_1]_\theta))^{3/2} \cdot D_\gamma.
\end{aligned}$$

Now the estimates stated in the proposition follow from Lemma 1.11 together with (1.11) and (1.13). □

1.4 The proof of Theorem 1.1

To prove Theorem 1.1 we need some additional notation. For a σ-finite measure space (Ω, Σ, μ) let $FIN_\chi(\mu)$ be the set of all subspaces of $S(\mu)$—the linear space of all strictly simple functions—which are generated by a finite sequence of characteristic functions of measurable, pairwise disjoint sets with finite non-zero measures, and with $S(\mu, E)$ we denote the linear space of all strictly simple functions with values in a normed space E.

Now let us start the proof of Theorem 1.1. First observe that if we define

$$\mathcal{A} := \{U(M) \,|\, U \in FIN_\chi(\mu), M \in FIN(E_\Delta)\}$$

and

$$\mathcal{B} := \{V(N) \,|\, V \in FIN_\chi(\nu), N \in FIN(F_\Delta)\},$$

then $([X_0(E_0), X_1(E_1)], S(\mu, E_\Delta), \mathcal{A})$ and $([Y_0(F_0), Y_1(F_1)], S(\nu, F_\Delta), \mathcal{B})$ are cofinal interpolation triples whenever X_0, X_1 and Y_0, Y_1 have non-trivial concavity. Indeed (we only treat one of the two cases), these assumptions together with [LT79, 1.a.5] and [LT79, 1.a.7] imply that X_0 and X_1 are σ-order continuous, and by [KPS82, p. 211] it follows that $S(\mu, E_\Delta)$ is dense in $X_0(E_0)$ and $X_1(E_1)$; obviously each $G \in FIN(S(\mu, E_\Delta))$ is contained in some $U(M)$ with $U \in FIN_\chi(\mu)$ and $M \in FIN(E_\Delta)$. Moreover, if U is generated by measurable, pairwise disjoint sets $A_1, \dots, A_n$ with finite non-zero measures, then $\chi_{A_1}, \dots, \chi_{A_n}$ is a 1-unconditional basis for U, hence U is order isometric to $\mathbb{R}^n$ endowed with a lattice norm under the canonical order.

This now puts us in the position to apply the Approximation Lemma 1.4 together with our estimates obtained in the finite-dimensional case: For $U \in FIN_\chi(\mu)$, $V \in FIN_\chi(\nu)$, $M \in FIN(E_\Delta)$ and $N \in FIN(F_\Delta)$

$$\begin{aligned} d_\theta[U_0(M_0), U_1(M_1); &V_0(N_0), V_1(N_1)] \\ &\leq 16 \cdot [(\mathbf{M_{(2)}}(U_0) \cdot \mathbf{M_{(2)}}(V_0))^{1-\theta} (\mathbf{M_{(2)}}(U_1) \cdot \mathbf{M_{(2)}}(V_1))^\theta]^{5/2} \\ &\quad \cdot t_\theta[M_0, M_1] \cdot t_\theta[N_0, N_1] \\ &\leq 16 \cdot [(\mathbf{M_{(2)}}(X_0) \cdot \mathbf{M_{(2)}}(Y_0))^{1-\theta} (\mathbf{M_{(2)}}(X_1) \cdot \mathbf{M_{(2)}}(Y_1))^\theta]^{5/2} \\ &\quad \cdot t_\theta[E_0, E_1] \cdot t_\theta[F_0, F_1], \end{aligned}$$

where the latter inequality follows from the fact that $\mathbf{M_{(2)}}$ respects sublattices, $\mathbf{C_2}$ subspaces and $\mathbf{T_2}$ quotients; recall from (1.15) the definition of $t_\theta[\cdot, \cdot]$. □

2 Bennett–Carl inequalities

Recall from the introduction that in [Ben73] and [Car74] Bennett and Carl independently proved the following inequalities: For $1 \le u \le 2$ and $1 \le u \le v \le \infty$ the identity operator $\mathrm{id} : \ell_u \hookrightarrow \ell_v$ is absolutely $(r,2)$-summing, i. e. there is a constant $c > 0$ such that for each set of finitely many $x_1, \dots, x_n \in \ell_u$

$$\Big(\sum_{k=1}^{n} \|x_k\|_{\ell_v}^r\Big)^{1/r} \le c \cdot \sup_{\|x'\|_{\ell_{u'}} \le 1} \Big(\sum_{k=1}^{n} |\langle x', x_k\rangle|^2\Big)^{1/2},$$

if (and only if) $1/r \le 1/u - \max(1/v, 1/2)$.

Later in [CD92] the "Bennett–Carl inequalities" were extended within the setting of so-called mixing operators (originally invented by Maurey [Mau74]): For $1 \le u \le 2$ and $1 \le u \le v \le \infty$ every s-summing operator T defined on ℓ_v has a 2-summing restriction to ℓ_u if (and only if) $1/s \ge 1/2 - 1/u + \max(1/v, 1/2)$.

Nevertheless literature so far has not offered an approach to the Bennett–Carl inequalities within the framework of interpolation theory. We prove an abstract interpolation formula for the mixing norm of a fixed operator, and obtain as an application not only the original Bennett–Carl inequalities but also their "non-commutative" analogues for finite-dimensional Schatten classes. Moreover, we consider Bennett–Carl inequalities in a more general setting of symmetric Banach sequence spaces and unitary ideals, and apply these results to Orlicz and Lorentz sequence spaces. Part of this section is contained in [DM98]. Further extensions of the Bennett–Carl inequalities within the framework of Orlicz sequence spaces can be found in a recent paper of Maligranda and Mastyło [MM99].

For all information on summing and mixing operators see e. g. [DF93], [DJT95] and [Pie80]. An operator $T \in \mathcal{L}(E,F)$ is called absolutely (r,p)-summing $(1 \le p \le r \le \infty)$ if there is a constant $\rho \ge 0$ such that

$$\Big(\sum_{i=1}^{n} \|Tx_i\|^r\Big)^{1/r} \le \rho \cdot \sup\Big\{ \Big(\sum_{i=1}^{n} |\langle x', x_i\rangle|^p\Big)^{1/p} \,|\, x' \in B_{E'}\Big\}$$

for all finite sets of elements $x_1, \dots, x_n \in E$ (with the obvious modifications for p or $r = \infty$). In this case, the infimum over all possible $\rho \ge 0$ is denoted by $\pi_{r,p}(T)$, and the Banach operator ideal of all absolutely (r,p)-summing operators by $(\Pi_{r,p}, \pi_{r,p})$; the special case $r = p$ gives the ideal (Π_p, π_p) of all absolutely p-summing operators.

An operator $T \in \mathcal{L}(E,F)$ is called (s,p)-mixing $(1 \le p \le s \le \infty)$ whenever its composition with an arbitrary operator $S \in \Pi_s(F,Y)$ is absolutely p-summing; with the norm

$$\mu_{s,p}(T) := \sup\{\pi_p(ST) \,|\, \pi_s(S) \le 1\}$$

the class $\mathcal{M}_{s,p}$ of all (s,p)-mixing operators forms again a Banach operator ideal. Obviously, $(\mathcal{M}_{p,p}, \mu_{p,p}) = (\mathcal{L}, \|\cdot\|)$ and $(\mathcal{M}_{\infty,p}, \mu_{\infty,p}) = (\Pi_p, \pi_p)$. Recall that due to [Mau74] (see also [DF93, 32.10–11]) summing and mixing operators are closely related:

$$(\mathcal{M}_{s,p}, \mu_{s,p}) \subset (\Pi_{r,p}, \pi_{r,p}) \qquad \text{for } 1/s + 1/r = 1/p,$$

and "conversely"

$$(\Pi_{r,p}, \pi_{r,p}) \subset (\mathcal{M}_{s_0,p}, \mu_{s_0,p}) \qquad \text{for } 1 \le p \le s_0 < s \le \infty \text{ and } 1/s + 1/r = 1/p.$$

Moreover, it is known that the identity map of a cotype 2 space is $(2,1)$-mixing and therefore every $(s,2)$-mixing operator on a cotype 2 space is even $(s,1)$-mixing (see again [Mau74] and [DF93, 32.2]). Finally, a quick investigation of [TJ70] shows that if $1 \le q \le 2 \le r \le \infty$ with $1/r = 1/q - 1/2$ are given, then $\Pi_{r,2}(X,\cdot) = \Pi_{q,1}(X,\cdot)$ for every Banach space X such that id_X is $(2,1)$-mixing; by the above, this holds in particular for cotype 2 spaces, hence, most of our main results in this section can also be formulated in terms of $(q,1)$-summing norms. However, we prefer using the $(r,2)$-summing norm since on the one hand some of our techniques require it, and on the other hand the $(r,2)$-summing norm is deeply connected to the eigenvalue distribution of power compact operators (as can be seen e. g. in (2.1) right below).

For an operator $T \in \mathcal{L}(E,F)$ the n-th Weyl number $x_n(T)$ of T is defined by

$$x_n(T) := \sup\{a_n(TS) \,|\, S \in \mathcal{L}(\ell_2, E) \text{ with } \|S\| = 1\},$$

where $a_n(TS)$ denotes the n-th approximation number of TS: For $T \in \mathcal{L}(E,F)$

$$a_n(T) := \inf\{\|T - T_n\| \,|\, T_n \in \mathcal{L}(E,F), \operatorname{rank} T_n < n\}.$$

We will use the following important inequality of König to obtain lower estimates:

$$n^{1/r} \cdot x_n(T) \le \pi_{r,2}(T), \qquad T \in \Pi_{r,2} \tag{2.1}$$

(for all details on s-numbers and this inequality see [Kön86, 2.a.3] or [Pie87]).

2.1 Complex interpolation of mixing operators

The aim of this section is to prove the following complex interpolation formula for the mixing norm of a fixed operator acting between two complex interpolation spaces:

Theorem 2.1. *Let $2 \le s_0, s_1 \le \infty$, $0 \le \theta \le 1$ and s_θ given by $1/s_\theta = (1-\theta)/s_0 + \theta/s_1$. Then for two finite-dimensional interpolation couples $[E_0, E_1]$, $[F_0, F_1]$ and each $T \in \mathcal{L}([E_0,E_1]_\theta, [F_0,F_1]_\theta)$*

$$\mu_{s_\theta,2}(T : [E_0,E_1]_\theta \to [F_0,F_1]_\theta) \le d_\theta[E_0,E_1] \cdot \mu_{s_0,2}(T : E_0 \to F_0)^{1-\theta} \cdot \mu_{s_1,2}(T : E_1 \to F_1)^\theta.$$

Proof. For the moment let $E_\theta := [E_0,E_1]_\theta$ and $F_\theta := [F_0,F_1]_\theta$, and consider for $\eta = 0, \theta, 1$ the bilinear mapping

$$\begin{array}{cccccc} \Phi_\eta^{n,m}: & \ell_{s_\eta}^n(F'_\eta) & \times & \mathcal{L}(\ell_2^m, E_\eta) & \longrightarrow & \ell_2^m(\ell_{s_\eta}^n) \\ & (y'_1,\dots,y'_n) & \times & S & \longmapsto & ((\langle y'_k, TSe_j\rangle)_k)_j \end{array},$$

where (e_j) denotes the canonical basis in $\mathbb{C}^m$. By the discrete characterization of the mixing norm (see [Mau74] or [DF93, 32.4]) $\mu_{s_\eta}(T : E_\eta \to F_\eta)$ is the infimum over all $c \geq 0$ such that for all n, m, all $y'_1, \ldots, y'_n \in F'_\eta$ and all $x_1, \ldots, x_m \in E_\eta$

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle y'_k, T x_j \rangle|^{s_\eta} \right)^{2/s_\eta} \right)^{1/2} \leq c \cdot \left(\sum_{k=1}^n \|y'_k\|_{F'_\eta}^{s_\eta} \right)^{1/s_\eta} \cdot \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j \rangle|^2 \right)^{1/2}.$$

Since for each $S = \sum_{j=1}^m e_j \otimes x_j \in \mathcal{L}(\ell_2^m, E_\eta)$

$$\|S\| = \sup_{x' \in B_{E'_\eta}} \left(\sum_{j=1}^m |\langle x', x_j \rangle|^2 \right)^{1/2},$$

it clearly follows that

$$\mu_{s_\eta,2}(T : E_\eta \to F_\eta) = \sup_{n,m} \|\Phi_\eta^{n,m}\|.$$

Now the proof follows by bilinear complex interpolation: For the interpolated bilinear mapping

$$[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta : [\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta \times [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta \longrightarrow [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta$$

by Proposition 0.3

$$\|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\| \leq \|\Phi_0^{n,m}\|^{1-\theta} \cdot \|\Phi_1^{n,m}\|^\theta.$$

Since by the interpolation theorem for $\ell_p(E)$'s (see (0.9)) together with the duality theorem (see Proposition 0.1)

$$[\ell_{s_0}^n(F'_0), \ell_{s_1}^n(F'_1)]_\theta = \ell_{s_\theta}^n([F_0, F_1]'_\theta) \qquad \text{and} \qquad [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta = \ell_2^m(\ell_{s_\theta}^n)$$

(isometrically), we obtain

$$\|\Phi_\theta^{n,m}\| \leq \|\mathcal{L}(\ell_2^m, [E_0, E_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\| \cdot \|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\|.$$

Consequently

$$\begin{aligned} \mu_{s_\theta,2}(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) &= \sup_{n,m} \|\Phi_\theta^{n,m}\| \\ &\leq \sup_{n,m} \{ \|\mathcal{L}(\ell_2^m, [E_0, E_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta\| \cdot \|\Phi_0^{n,m}\|^{1-\theta} \cdot \|\Phi_1^{n,m}\|^\theta \} \\ &\leq d_\theta[E_0, E_1] \cdot \mu_{s_0,2}(T : E_0 \to F_0)^{1-\theta} \cdot \mu_{s_1,2}(T : E_1 \to F_1)^\theta, \end{aligned}$$

the desired result. □

In the same way an analogous result for the $(r, 2)$-summing norm can be obtained.

Applications of Theorem 2.1 come from "uniform estimates" for $d_\theta[E_0, E_1]$. From Proposition 1.5 we obtain

$$d_\theta[\ell_1, \ell_2] := \sup_n d_\theta[\ell_1^n, \ell_2^n] \leq \sqrt{2}; \tag{2.2}$$

just take $X_0 = \ell_1^n$, $X_1 = \ell_2^n$, $E_0 = E_1 = \mathbb{C}$ and note that (trivially) $\mathbf{M}_{(\mathbf{2})}(\ell_1^n) = \mathbf{M}_{(\mathbf{2})}(\ell_2^n) = 1$. Using an extension of Kouba's formulas for the Haagerup tensor product of operator spaces due to [P96], Junge in [Jun96, 4.2.6] proved an analogue of (2.2) for Schatten classes:

$$d_\theta[\mathcal{S}_1, \mathcal{S}_2] := \sup_n d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] < \infty. \tag{2.3}$$

Finally we state a corollary on θ-Hilbert spaces which together with (2.2) and (2.3) is crucial for our purposes.

Corollary 2.2. *Let $0 \le \theta \le 1$, $E = [E_0, \ell_2^n]_\theta$ be an n-dimensional θ-Hilbert space and $2 \le s_\theta \le \infty$ given by $s_\theta = 2/\theta$. Then*

$$\mu_{s_\theta,2}(E \hookrightarrow \ell_2^n) \le d_\theta[E_0, \ell_2^n] \cdot \pi_2(E_0 \hookrightarrow \ell_2^n)^{1-\theta}.$$

2.2 Bennett–Carl inequalities for symmetric Banach sequence spaces

As indicated above the preceding interpolation theorem implies the Bennett–Carl result and its extension of Carl–Defant as an almost immediate consequence:

Corollary 2.3. *Let $1 \le u \le 2$ and $1 \le u \le v \le \infty$. Then for $2 \le s \le \infty$ such that $1/s = 1/2 - 1/u + \max(1/v, 1/2)$*

$$\sup_n \mu_{s,2}(\ell_u^n \hookrightarrow \ell_v^n) < \infty.$$

In particular, for $2 \le r \le \infty$ such that $1/r = 1/u - \max(1/v, 1/2)$

$$\sup_n \pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) < \infty.$$

Proof. Only the case $1 \le u < v \le 2$ has to be considered; the case $2 \le v \le \infty$ then easily follows by factorization through ℓ_2^n, and the case $u = v$ is trivial anyway. In what follows we use the complex interpolation formula for ℓ_p^n's without further reference.
i) Take first $v = 2$. It is well-known (see e. g. [Pie80, 22.4.8] or (2.7)) that

$$\pi_2(\ell_1^n \hookrightarrow \ell_2^n) = 1.$$

For $1 \le u \le 2$ choose $0 \le \theta \le 1$ such that $1/u = (1-\theta)/1 + \theta/2$. Then $s_\theta := 2/\theta = u'$, and by Corollary 2.2 together with (2.2)

$$\mu_{u',2}(\ell_u^n \hookrightarrow \ell_2^n) \le d_\theta[\ell_1, \ell_2] < \infty.$$

ii) Let $1 \le u < v < 2$. Combining case i),

$$\mu_{u',2}(\ell_u^n \hookrightarrow \ell_2^n) \le d_\theta[\ell_1, \ell_2],$$

and

$$\mu_{2,2}(\ell_u^n \hookrightarrow \ell_u^n) = \|\ell_u^n \hookrightarrow \ell_u^n\| = 1,$$

we arrive at

$$\mu_{s_{\tilde\theta},2}(\ell_u^n \hookrightarrow \ell_v^n) \le \sup_n d_{\tilde\theta}[\ell_u^n,\ell_u^n]\cdot d_\theta[\ell_1,\ell_2]^{1-\tilde\theta} < \infty,$$

with $\tilde\theta := (1/v-1/2)/(1/u-1/2)$ and $1/s_{\tilde\theta} := (1-\tilde\theta)/u' + \tilde\theta/2 = 1/2 - 1/u + 1/v = 1/s$. □

As in the original proofs of Bennett and Carl, the crucial step in the preceding proof is to show that for the symmetric Banach sequence space $E = \ell_u$

$$\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty, \tag{2.4}$$

where $1 \le u \le 2$ and $1/r = 1/u - 1/2$. We will now prove a result within the framework of symmetric Banach sequence spaces which shows that (2.4) is sharp in a very strong sense. Take an arbitrary 2-concave and u-convex symmetric Banach sequence space E—these geometric assumptions in particular imply that the continuous inclusions $\ell_u \subset E \subset \ell_2$ hold—which satisfies (2.4). The following result shows that there is only one such space:

Theorem 2.4. *Let $1 \le u \le 2$ and $1/r = 1/u - 1/2$. For each 2-concave and u-convex symmetric Banach sequence space E the following are equivalent:*

(1) $\sup_n \mu_{u',2}(E_n \hookrightarrow \ell_2^n) < \infty$.

(2) $\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty$.

(3) $E = \ell_u$.

Clearly we only have to deal with the implication (2)⇒(3); its proof is based on two lemmas. For the first one we invent the notion of "enough symmetries in the orthogonal group". Let $E = (\mathbb{C}^n, \|\cdot\|)$ be an n-dimensional Banach space. We say that E has *enough symmetries in $\mathcal{O}(n)$* if there is a compact subgroup G in $\mathcal{O}(n)$ such that

$$\forall\, u \in \mathcal{L}(E)\, \forall\, g, g' \in G : \|u\| = \|gug'\| \tag{2.5}$$

and

$$\forall\, u \in \mathcal{L}(E) \text{ with } ug = gu \text{ for all } g \in G\, \exists\, c \in \mathbb{K} : u = c\cdot \mathrm{id}_E. \tag{2.6}$$

Basic examples of spaces with enough symmetries in the orthogonal group are the finite-dimensional spaces E_n and $\mathcal{S}_E^n$ associated with a symmetric Banach sequence space E. The following lemma extends the corresponding results in [CD97, p. 233, 236].

Lemma 2.5. *Let E_n and F_n have enough symmetries in $\mathcal{O}(n)$. Then*

$$\pi_2(E_n \hookrightarrow F_n) = n^{1/2}\cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|}, \tag{2.7}$$

and for $1 \le k \le n$

$$\left(\frac{n-k+1}{n}\right)^{1/2}\cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|} \le x_k(E_n \hookrightarrow F_n) \le \left(\frac{n}{k}\right)^{1/2}\cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|}. \tag{2.8}$$

Proof. (2.7): Trace duality allows to deduce the lower estimate from the upper one:

$$n \le \pi_2(\ell_2^n \hookrightarrow F_n) \cdot \pi_2(F_n \hookrightarrow \ell_2^n) \le \|\ell_2^n \hookrightarrow E_n\| \cdot \pi_2(E_n \hookrightarrow F_n) \cdot n^{1/2} \cdot \|\ell_2^n \hookrightarrow F_n\|^{-1}.$$

For the proof of the upper estimate it may be assumed without loss of generality that $F_n = \ell_2^n$ (factorize through ℓ_2^n). In this case it suffices to show that

$$\|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot B_{\ell_2^n}$$

is John's ellipsoid $D_{\max}$ of maximal volume in B_{E_n} (see e. g. [P89, 3.8] or [DJT95, 6.30]). By definition there is a linear bijection $u : \ell_2^n \to E_n$ such that $u(B_{\ell_2^n}) = D_{\max}$. In particular, $\|u\| = 1$ and $N(u^{-1}) = n$ (N denotes the nuclear norm, see e. g. [P89, 3.7] or [DJT95, 6.30]). On the other hand by a standard averaging argument there is a linear bijection $v : \ell_2^n \to E_n$ with $\|v\| = 1$, $N(v^{-1}) = n$ and $vg = gv$ for all $g \in G$, where G is a compact group in $\mathcal{O}(n)$ satisfying (2.5) and (2.6) (see [P89, 3.5] which also holds in the complex case). By property (2.6) of G and the fact that $\|v\| = 1$ we have $v = \|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot \mathrm{id}$. Then by Lewis' uniqueness theorem $v^{-1}u \in \mathcal{O}(n)$ ([P89, 3.7] or [DJT95, 6.25]). Altogether we finally obtain

$$\|\ell_2^n \hookrightarrow E_n\|^{-1} \cdot B_{\ell_2^n} = v(B_{\ell_2^n}) = v[v^{-1}u(B_{\ell_2^n})] = u(B_{\ell_2^n}) = D_{\max}.$$

(2.8): Recall from (2.1) that $k^{1/2} \cdot x_k(T) \le \pi_2(T)$ for every 2-summing operator T acting between two Banach spaces. Together with (2.7) this gives the second inequality. The first then follows from the basic properties of the Weyl numbers (see e. g. [Kön86]):

$$\begin{aligned} 1 &= x_n(\mathrm{id}_{\ell_2^n}) \\ &\le x_k(\ell_2^n \hookrightarrow F_n) \cdot x_{n-k+1}(F_n \hookrightarrow \ell_2^n) \\ &\le \|\ell_2^n \hookrightarrow E_n\| \cdot x_k(E_n \hookrightarrow F_n) \cdot \left(\frac{n}{n-k+1}\right)^{1/2} \cdot \|\ell_2^n \hookrightarrow F_n\|^{-1}. \end{aligned}$$

□

The following obvious examples will be useful later.

Corollary 2.6. *For* $1 \le u, v \le \infty$

$$\pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) = n \cdot \frac{\max(1, n^{1/v-1/2})}{\max(1, n^{1/u-1/2})} \tag{2.9}$$

and

$$x_{[n^2/2]}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp \frac{\max(1, n^{1/v-1/2})}{\max(1, n^{1/u-1/2})}. \tag{2.10}$$

The preceding lemma turns out to be of special interest in combination with a result due to Szarek and Tomczak-Jaegermann [STJ80, Proposition 2.2] which states that for each 2-concave symmetric Banach sequence space E

$$\|\ell_2^n \hookrightarrow E_n\| \asymp n^{-1/2} \cdot \|\sum_1^n e_i\|_{E_n}. \tag{2.11}$$

The second lemma, which we need for the proof of Theorem 2.4, is based on (2.11) and an important result about the interpolation of Banach lattices due to Pisier [P79].

Lemma 2.7. *For $1 \le u \le 2$ let E be a u-convex and u'-concave symmetric Banach sequence space. Then*

$$\|E_n \hookrightarrow \ell_u^n\| \asymp \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}}. \tag{2.12}$$

In particular, if E is even 2-concave, then

$$\|E_n \hookrightarrow \ell_u^n\| \asymp \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}} \asymp \frac{n^{1/u-1/2}}{\|\ell_2^n \hookrightarrow E_n\|}. \tag{2.13}$$

Proof. (2.13) follows directly from (2.12) and (2.11), and clearly $n^{1/u} \le \|E_n \hookrightarrow \ell_u^n\| \cdot \|\sum_1^n e_i\|_{E_n}$. For the upper estimate in (2.12) we only have to consider $1 < u < 2$: The case $u = 1$ is stated below in (2.14), and a 2-convex and 2-concave symmetric Banach sequence space necessarily equals ℓ_2 with equivalent norms. Without loss of generality we may assume $\mathbf{M}^{(\mathbf{u})}(E) = \mathbf{M}_{(\mathbf{u}')}(E) = 1$ (see [LT79, 1.d.8]). Then by [P79, Theorem 2.2] there exists a symmetric Banach sequence space E_0 such that $E = [E_0, \ell_2]_\theta$ with $\theta = 2/u'$; moreover, we have $E_n = [E_0^n, \ell_2^n]_\theta$ with equal norms. The conclusion now follows by interpolation: It can be shown easily that

$$\|E_0^n \hookrightarrow \ell_1^n\| \le \frac{n}{\|\sum_1^n e_i\|_{E_0^n}} \tag{2.14}$$

(see e. g. [STJ80, Proposition 2.5]), hence

$$\|E_n \hookrightarrow \ell_u^n\| \le \|E_0^n \hookrightarrow \ell_1^n\|^{1-\theta} \cdot \|\ell_2^n \hookrightarrow \ell_2^n\|^\theta \le \frac{n^{1-\theta}}{\|\sum_1^n e_i\|_{E_0^n}^{1-\theta}}.$$

Since $E_n = [E_0^n, \ell_2^n]_\theta$ is of J-type θ (i. e. $\|x\|_{E_n} \le \|x\|_{E_0^n}^{1-\theta} \cdot \|x\|_{\ell_2^n}^\theta$ for all $x \in E_n$), we have

$$\|\textstyle\sum_1^n e_i\|_{E_n} \le \|\sum_1^n e_i\|_{E_0^n}^{1-\theta} \cdot n^{\theta/2},$$

and consequently

$$\|E_n \hookrightarrow \ell_u^n\| \le \frac{n^{1-\theta/2}}{\|\sum_1^n e_i\|_{E_n}} = \frac{n^{1/u}}{\|\sum_1^n e_i\|_{E_n}}.$$

□

Proof of the implication (2)⇒(3) in Theorem 2.4: Assume that $\sup_n \pi_{r,2}(E_n \hookrightarrow \ell_2^n) < \infty$. By (2.1), (2.8) and (2.13)

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \ge [n/2]^{1/r} \cdot x_{[n/2]}(E_n \hookrightarrow \ell_2^n) \succ \frac{n^{1/r}}{\|\ell_2^n \hookrightarrow E_n\|} \asymp \|E_n \hookrightarrow \ell_u^n\|, \tag{2.15}$$

which by assumption shows that $\sup_n \|E_n \hookrightarrow \ell_u^n\| < \infty$. This clearly gives the claim. □

Note that (2.15) (except of the latter asymptotic) does not depend on the special choice of r.

If E is a 2-concave and u-convex ($1 \le u \le 2$) symmetric Banach sequence space different from ℓ_u (i. e. the inclusion $\ell_u \subset E$ is strict), then by Theorem 2.4 for $1/r = 1/u - 1/2$

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \nearrow \infty.$$

The following result gives the precise asymptotic order of the sequence $(\pi_{r,2}(E_n \hookrightarrow \ell_2^n))_n$:

Corollary 2.8. *For $1 \le u \le 2$ let E be a 2-concave and u-convex symmetric Banach sequence space. Then for $2 \le r, s \le \infty$ such that $1/r = 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(E_n \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(E_n \hookrightarrow \ell_2^n) \asymp \frac{n^{1/r+1/2}}{\| \sum_1^n e_i \|_{E_n}}.$$

Proof. The lower estimate has already been shown in (2.15), and the upper estimate simply follows by factorization through ℓ_u^n, the Bennett–Carl inequalities and (2.13). □

Remark 2.9. (a) Since a u-convex Banach lattice is p-convex for all $1 \le p \le u$ (see [LT79, 1.d.5]), the formula in the preceding theorem even holds for all $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/2$.

(b) For $1 \le u \le 2$ let E be a 2-concave and u-convex symmetric Banach sequence space, F an arbitrary symmetric Banach sequence space, and let $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/2$. Then—by factorization through ℓ_2^n for the upper estimate and (2.15) for the lower one—the following formula holds:

$$\pi_{r,2}(E_n \hookrightarrow F_n) \asymp n^{1/r} \cdot \frac{\|\ell_2^n \hookrightarrow F_n\|}{\|\ell_2^n \hookrightarrow E_n\|};$$

in particular, if F is 2-concave, then

$$\pi_{r,2}(E_n \hookrightarrow F_n) \asymp n^{1/r} \cdot \frac{\| \sum_1^n e_i \|_{F_n}}{\| \sum_1^n e_i \|_{E_n}}.$$

Note that these results can be considered as extensions of (2.7).

(c) For the special case $F = \ell_v$ ($1 \le u \le v \le 2$) the formulas in (b) even hold for all $2 \le r \le \infty$ such that $1/r \ge 1/u - 1/v$; simply repeat the proof of Corollary 2.8 for $1/r = 1/u - 1/v$ and use the argument from remark (a).

(d) Using the right-hand side of [STJ80, (2.2)] and [P89, 10.4], one can see that for (2.11) it is sufficient to assume that E is of weak cotype 2 (for this notion we refer to [P89, 10.1]; note that 2-concavity (=cotype 2) implies weak cotype 2), but we decided to keep on using the more common notion of 2-concavity.

2.3 Bennett–Carl inequalities for unitary ideals

We now use Junge's counterpart (2.3) of (2.2) and Theorem 2.1 to show a "non-commutative" analogue. Note first that for all $1 \le u, v \le \infty$ and $2 \le r \le \infty$

$$n^{1/r} \le \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n), \tag{2.16}$$

and hence also for $2 \le s \le \infty$

$$n^{1/2-1/s} \le \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n);$$

this is a consequence of the trivial estimate $\pi_{r,2}(\ell_2^n \hookrightarrow \ell_2^n) \geq n^{1/r}$ (insert e_k's) and the fact that ℓ_2^n is 1-complemented in each $\mathcal{S}_u^n$ (assign to each $x \in \ell_2^n$ the matrix $x \otimes e_1 \in \mathcal{S}_u^n$). For u, v considered in Corollary 2.3 this lower bound is optimal:

Corollary 2.10. *Let $1 \leq u \leq 2$ and $1 \leq u \leq v \leq \infty$. Then for $2 \leq s \leq \infty$ such that $1/s = 1/2 - 1/u + \max(1/v, 1/2)$*

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/2-1/s}.$$

In particular, for $2 \leq r \leq \infty$ and $1/r = 1/u - \max(1/v, 1/2)$

$$\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/r}.$$

Proof. The proof of the upper bound is analogous to that of Corollary 2.3: Of course the complex interpolation formula for $\mathcal{S}_p^n$'s is needed instead of that for ℓ_p^n's, and in i) use $\pi_2(\mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = n^{1/2}$ (see (2.9)) and Junge's result (2.3) in order to obtain

$$\mu_{u',2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \leq d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] \cdot n^{(1-\theta)/2} \leq d_\theta[\mathcal{S}_1, \mathcal{S}_2] \cdot n^{1/u-1/2},$$

where $\theta = 2/u'$. Then in ii) one arrives at

$$\mu_{s_{\tilde{\theta}},2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \prec n^{(1-\tilde{\theta})(1/u-1/2)} = n^{1/u-1/v},$$

with $\tilde{\theta} := (1/v - 1/2)/(1/u - 1/2)$ and $1/s_{\tilde{\theta}} = (1 - \tilde{\theta})/u' + \theta/2 = 1/2 - 1/u + 1/v$=1/s. □

Exploiting the ideas of the preceding section one easily obtains the asymptotic order of the $(r, 2)$-summing and the $(s, 2)$-mixing norm of identities between finite-dimensional unitary ideals $\mathcal{S}_E^n$ and $\mathcal{S}_2^n$:

Corollary 2.11. *For $1 \leq u \leq 2$ let E be a 2-concave and u-convex symmetric Banach sequence space. Then for all $2 \leq r, s \leq \infty$ such that $1/r \geq 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}}.$$

Proof. Recall the simple fact that for all symmetric Banach sequence spaces E and F

$$\|\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n\| = \|E_n \hookrightarrow F_n\|, \tag{2.17}$$

and by the same reasoning as in Remark 2.9 (a) it is enough to deal with the case $1/r = 1/u - 1/2$. Then factorization through $\mathcal{S}_u^n$ and (2.13) give

$$\mu_{u',2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \prec \|\mathcal{S}_E^n \hookrightarrow \mathcal{S}_u^n\| \cdot n^{1/u-1/2} \asymp \frac{n^{2/u-1/2}}{\|\sum_1^n e_i\|_{E_n}} = \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}},$$

and in order to obtain the lower estimate apply again (2.8) together with (2.1) and the second asymptotic in (2.13):

$$\pi_{r,2}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \geq [n^2/2]^{1/r} \cdot x_{[n^2/2]}(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_2^n) \succ \frac{n^{2/r}}{\|\ell_2^n \hookrightarrow E_n\|} \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{E_n}}.$$

□

2.4 Applications

Weyl numbers

The results of the preceding sections can be used to improve on the estimates for Weyl numbers of identities on symmetric Banach sequence spaces and unitary ideals in (2.8): The exponent $1/2$ in each of the two inequalities there can be replaced by $1/u - 1/2$ whenever u-convexity and 2-concavity assumptions are made.

Corollary 2.12. *For $1 \leq u, v \leq 2$ let E and F be 2-concave symmetric Banach sequence spaces where E is u-convex and F is v-convex. Then there exist constants $C_u, C_v > 0$ such that for all $1 \leq k \leq n$*

$$C_v^{-1} \cdot \left(\frac{n-k+1}{n}\right)^{1/v-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}} \leq x_k(E_n \hookrightarrow F_n) \leq C_u \cdot \left(\frac{n}{k}\right)^{1/u-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}},$$

and all $1 \leq k \leq n^2$

$$C_v^{-1} \cdot \left(\frac{n^2-k+1}{n^2}\right)^{1/v-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}} \leq x_k(\mathcal{S}_E^n \hookrightarrow \mathcal{S}_F^n) \leq C_u \cdot \left(\frac{n^2}{k}\right)^{1/u-1/2} \cdot \frac{\|\sum_1^n e_i\|_{F_n}}{\|\sum_1^n e_i\|_{E_n}}.$$

Proof. The upper estimates follow by using the inequality (2.1) and the results from the preceding two sections, and the lower estimates then are immediate consequences of the upper ones—simply repeat the proof of (2.8) with a different exponent. □

Recall that for the embedding $\ell_u^n \hookrightarrow \ell_2^n$, $1 \leq u \leq 2$ by [CD92, 2.3.3] even the following equality is known: $x_k(\ell_u^n \hookrightarrow \ell_2^n) = k^{1/2-1/u}$, $1 \leq k \leq n$. The second estimate in Corollary 2.12 implies that for $1 < u < 2$

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) = x_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) \geq C_u^{-1} \cdot \left(\frac{n^2-k+1}{n}\right)^{1/u-1/2};$$

this disproves the conjecture

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) \asymp \max\left(1, \left(\frac{n^2-k+1}{n^2}\right)^{1/2} \cdot n^{1/u-1/2}\right)$$

from [CD97, p. 249] (put $k := [n^2 - n^\alpha + 1]$, $1 < \alpha < 2$). We conjecture that

$$a_k(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_u^n) \asymp \max\left(1, \left(\frac{n^2-k+1}{n}\right)^{1/u-1/2}\right);$$

note that by the interpolation property of the Gelfand numbers [Pie80, 11.5.8] it would suffice to prove the case $u = 1$.

Identities on Orlicz and Lorentz sequence spaces

In the following we apply our results, in particular the corollaries 2.8 and 2.11, to two natural examples of symmetric Banach sequence spaces: Orlicz and Lorentz sequence spaces. We only treat the case where the image space of the embedding is the finite-dimensional Hilbert space and leave the formulation for other spaces and the corollaries for Weyl numbers to the reader.

We start with Orlicz sequence spaces ℓ_M. An Orlicz function M is a continuous non-decreasing and convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. It is said to satisfy the Δ_2-condition at zero if $\lim_{t\to 0} \sup M(2t)/M(t) < \infty$. To any Orlicz function M we associate the space ℓ_M of all sequences of scalars $x = (x_1, x_2, \dots)$ such that $\sum_{i=1}^n M(|x_n|/\rho) < \infty$ for some $\rho > 0$. The space ℓ_M equipped with the norm

$$\|x\| := \inf \left\{ \rho > 0 \,\Big|\, \sum_{n=1}^{\infty} M(|x_n|/\rho) \leq 1 \right\}$$

is a Banach space usually called an Orlicz sequence space; it is a symmetric Banach sequence space if M satisfies the Δ_2-condition at zero.

Corollary 2.13. *Let $1 < u < 2$ and M be a strictly increasing Orlicz function which satisfies the Δ_2-condition at zero. Assume that there exists $K > 0$ such that for all $s, t \in (0, 1]$*

$$K^{-1} \cdot s^2 \leq M(st)/M(t) \leq K \cdot s^u. \tag{2.18}$$

Then for $2 < r, s < \infty$ such that $1/r > 1/u - 1/2$ and $1/s = 1/2 - 1/r$

$$\pi_{r,2}(\ell_M^n \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(\ell_M^n \hookrightarrow \ell_2^n) \asymp \frac{n^{1/r+1/2}}{\|\sum_1^n e_i\|_{\ell_M^n}} \asymp n^{1/r+1/2} \cdot M^{-1}(1/n)$$

and

$$\pi_{r,2}(\mathcal{S}_{\ell_M}^n \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}_{\ell_M}^n \hookrightarrow \mathcal{S}_2^n) \asymp \frac{n^{2/r+1/2}}{\|\sum_1^n e_i\|_{\ell_M^n}} \asymp n^{2/r+1/2} \cdot M^{-1}(1/n).$$

Note that (2.18) together with the Δ_2-condition assures that ℓ_M is 2-concave and p-convex for all $1 \leq p < u$ (see [LT79, 2.b.5]).

Now we state an analogue for Lorentz sequence spaces $d(w, u)$. Let $1 \leq u < \infty$ and $w = (w_n)_n$ be a non-increasing sequence of positive numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. The Banach space of all sequences of scalars $x = (x_1, x_2, \dots)$ for which

$$\|x\| := \sup_{\pi} \left(\sum_{n=1}^{\infty} |x_{\pi(n)}|^u w_n \right)^{1/u} < \infty,$$

where π ranges over all the permutations of the integers, is denoted by $d(w, u)$ and it is called a Lorentz sequence space.

Corollary 2.14. *Let $1 < u < 2$ and w be such that $n \cdot w_n^q \asymp \sum_{i=1}^n w_i^q$, where $q = 2/(2-u)$. Then for $2 < r, s < \infty$ such that $1/r \geq 1/u - 1/2$ and $1/s = 1/2 - 1/r$*

$$\pi_{r,2}(d_n(w,u) \hookrightarrow \ell_2^n) \asymp \mu_{s,2}(d_n(w,u) \hookrightarrow \ell_2^n) \asymp n^{1/r+1/2-1/u} \cdot w_n^{-1/u}.$$

and

$$\pi_{r,2}(\mathcal{S}^n_{d(w,u)} \hookrightarrow \mathcal{S}_2^n) \asymp \mu_{s,2}(\mathcal{S}^n_{d(w,u)} \hookrightarrow \mathcal{S}_2^n) \asymp n^{2/r+1/2-1/u} \cdot w_n^{-1/u}.$$

Recall that the space $d(w,u)$ is u-convex, and if $1 \leq u < 2$, it is 2-concave if and only if w satisfies the condition in the assumption of the corollary (see [Rei81, p. 245–247]).

Schatten Limit Orders

Finally, we consider the asymptotic order of the sequences $(\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n))_n$ for arbitrary $2 \leq r \leq \infty$, $1 \leq u, v \leq \infty$. Define the limit orders

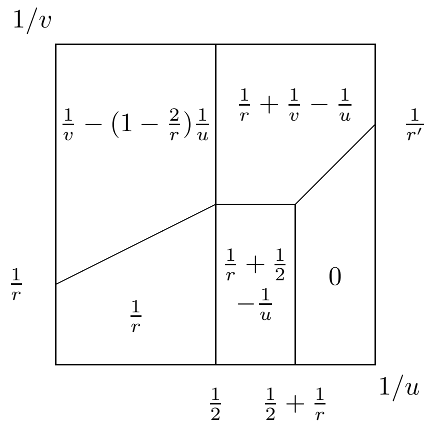
$$\lambda(\Pi_{r,2}, u, v) := \inf\{\lambda > 0 \,|\, \exists\, \rho > 0\, \forall\, n : \pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) \leq \rho \cdot n^\lambda\}$$

and

$$\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) := \inf\{\lambda > 0 \,|\, \exists\, \rho > 0\, \forall\, n : \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \leq \rho \cdot n^\lambda\}.$$

Here we only handle the limit order of summing operators since—using the fact that $\Pi_{r,2}$ and $\mathcal{M}_{s,2}$ for $1/s+1/r = 1/2$ are almost equal—one can easily see that $\lambda(\Pi_{r,2}, u, v) = \lambda(\mathcal{M}_{r,2}, u, v)$ and $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = \lambda_{\mathcal{S}}(\mathcal{M}_{r,2}, u, v)$ (with the obvious definition for the right sides of these equalities; see [Pie80, 22.3.7]).

The calculation of the limit order $\lambda(\Pi_{r,2}, u, v)$ was completed in [CMP78]:



Moreover, the proof in [CMP78] shows that the limit order is attained: $\pi_{r,2}(\ell_u^n \hookrightarrow \ell_v^n) \asymp n^{\lambda(\Pi_{r,2},u,v)}$. In view of Corollary 2.10 the following conjecture seems to be natural:

Conjecture: $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = 1/r + \lambda(\Pi_{r,2}, u, v)$.

For the border cases $r = 2$ (the 2-summing norm) and $r = \infty$ (the operator norm) this conjecture by (2.7) and (2.9) is true. In the following corollary we confirm the upper estimates of this conjecture for all u, v and the lower ones for all u, v except those in the upper left corner of the picture.

Corollary 2.15. *Let $1 \le u, v \le$ and $2 < r < \infty$.*

(a) $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) = 1/r + \lambda(\Pi_{r,2}, u, v)$ *for $1 \le u \le 2$.*

(b) $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v) \le 1/r + \lambda(\Pi_{r,2}, u, v)$ *for $2 \le u \le \infty$, with equality whenever $1/v \le 1/r + (1 - 2/r)(1/u)$.*

Proof. Let $1/s := 1/2 - 1/r$. The upper estimates for the case $1 \le u \le 2$ follow from Corollary 2.10: Consider for $u_0 := (1/2 - 1/r)^{-1}$ the following alternative: (i) $1/u \le 1/u_0$ or (ii) $1/u > 1/u_0$. Then the conclusion in case (i) is a consequence of Corollary 2.10 and the following factorization:

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{u_0}^n\| \cdot \mu_{s,2}(\mathcal{S}_{u_0}^n \hookrightarrow \mathcal{S}_2^n) \cdot \|\mathcal{S}_2^n \hookrightarrow \mathcal{S}_v^n\| \prec n^{2/r+1/2-1/u+\max(0,1/v-1/2)},$$

and for (ii) look with $v_0 := (1/u - 1/r)^{-1} \le 2$ at

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_0}^n) \cdot \|\mathcal{S}_{v_0}^n \hookrightarrow \mathcal{S}_v^n\| \prec n^{1/r+\max(0,1/r+1/v-1/u)}.$$

Now let $2 \le u \le \infty$. Although this part is very close to the calculations made in [CMP78, Lemma 6], we give an outline of the proof for the convenience of the reader. By (2.9) and Theorem 2.1 (with no interpolation in the range or the image),

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n) \le \pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{2/r} \cdot \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n\|^{1-2/r} = n^{1/r+1/2-(1-2/r)(1/u)},$$

hence, by factorization, for $1 \le v \le 2$

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le n^{1/r+1/v-(1-2/r)(1/u)}.$$

Furthermore, for $1/v_1 := 1/r + (1 - 2/r)(1/u)$

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_1}^n) \le \pi_2(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{2/r} \cdot \|\mathcal{S}_u^n \hookrightarrow \mathcal{S}_u^n\|^{1-2/r} = n^{2/r},$$

hence

$$\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le n^{2/r}$$

for all $v_1 \le v \le \infty$. Finally, for all $2 < v < v_1$ and $0 < \theta < 1$ such that $1/v = (1-\theta)/v_1 + \theta/2$

$$\begin{aligned}\mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) &\le \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_{v_1}^n)^{1-\theta} \cdot \mu_{s,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_2^n)^{\theta} \\ &\le n^{1/r+(1-\theta)/r+\theta(1/2-(1-2/r)(1/u))} = n^{1/r+1/v-(1-2/r)(1/u)}.\end{aligned}$$

Looking at the picture for $\lambda(\Pi_{r,2}, u, v)$ one can see that these are the desired results. For the lower estimates recall (2.1):

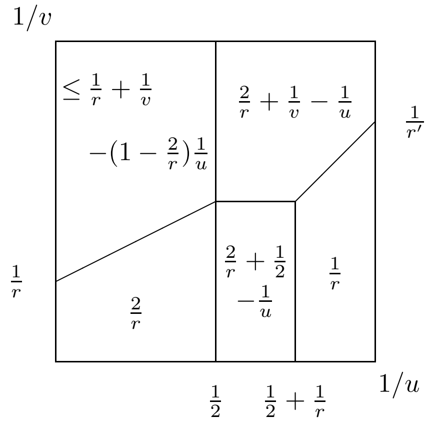
$$[n^2/2]^{1/r} \cdot x_{[n^2/2]}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le \pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n),$$

hence (2.10) implies

$$\pi_{r,2}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \succ \begin{cases} n^{2/r+1/v-1/u} & \text{if } 1 \le u, v \le 2, \\ n^{2/r+1/2-1/u} & \text{if } 1 \le u \le 2 \le v \le \infty, \\ n^{2/r} & \text{if } 2 \le u, v \le \infty. \end{cases}$$

Using (2.16), these estimates can be improved upon, for those u, v for which $\lambda(\Pi_{r,2}, u, v) = 0$. □

Our results for $\lambda_{\mathcal{S}}(\Pi_{r,2}, u, v)$ can be summarized in the following diagram:



3 Complex interpolation of operators generated by orthonormal systems

In this section we continue our work on interpolating ideal norms within the framework of (B,q,p)-summing operators ($1 \le p \le 2 \le q < \infty$), where B is an orthonormal system in some $L_2(\mu)$, and either $q = 2$ or $p = 2$ and B is a so-called "$\Lambda(q)$-system"; for the case $p = q = 2$ these Banach operator ideals became of interest recently in the theses of Baur [Bau97] and Seigner [Sei95], and for the more special case of an orthonormal system generated by Gaussian variables this notion goes back to Linde and Pietsch [LP74]. Our main result will be a characterization of sets of characters on a compact abelian group which are $\Lambda(p)$-sets for all $2 < p < \infty$ by means of the limit orders of the associated ideals of B-summing operators. Moreover, we compute the asymptotic order of the Gaussian-summing norm of Schatten class identities and shed some new light on the Bennett–Carl inequalities.

Henceforth, we only consider probability spaces (Ω, Σ, μ) such that $L_2(\mu)$ is infinite-dimensional. Moreover, $B \subset L_2(\mu)$ always denotes an infinite orthonormal system. For $2 < p < \infty$ an orthonormal system $B \subset L_2(\mu)$ is said to be a $\Lambda(p)$-*system* if $B \subset L_p(\mu)$, and there exists a constant $C > 0$ such that for each finite sequence $a_1, \ldots, a_n$ of scalars and $b_1, \ldots, b_n \in B$

$$\left\| \sum_{i=1}^{n} a_i b_i \right\|_{L_p(\mu)} \le C \cdot \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2}. \tag{3.1}$$

We write $K_p(B)$ for the smallest constant C satisfying (3.1). For simplicity we set $K_p(B) := \infty$ whenever B is not a $\Lambda(p)$-system, and $K_2(B) := 1$ for any orthonormal system B.

The so-called "Khinchine inequalities" (see e. g. [DJT95, 1.10] or [DF93, 8.5]) imply that the system of the Rademacher functions is a $\Lambda(p)$-system for all $2 < p < \infty$: Let $1 \le p < \infty$. Then there exist constants a_p and $b_p \ge 1$ such that for each finite sequence $a_1, \ldots, a_n$ of scalars

$$a_p^{-1} \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2} \le \left\| \sum_{i=1}^{n} a_i r_i \right\|_{L_p[0,1]} \le b_p \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2}. \tag{3.2}$$

For Gaussian variables instead of Rademacher functions, the above inequalities even turn into an equality (see e. g. [DF93, 8.7]).

For $1 \le p \le 2 \le q < \infty$ and an orthonormal system $B \subset L_2(\mu)$ such that either $q = 2$, or $p = 2$ and $K_q(B) < \infty$, an operator $T : X \to Y$ between Banach spaces X and Y is said to belong to the class of (B,q,p)-*summing operators*, $\Pi_{B,q,p}$, if there exists a constant $c > 0$ such that for all finite sequences $b_1, \ldots, b_n$ in B and $x_1, \ldots, x_n$ in X

$$\left(\int_\Omega \| \sum_{i=1}^{n} b_i \cdot Tx_i \|^q d\mu \right)^{1/q} \le c \cdot K_q(B) \cdot \sup_{x' \in B_{X'}} \left(\sum_{i=1}^{n} |\langle x', x_i \rangle|^p \right)^{1/p}. \tag{3.3}$$

We write $\pi_{B,q,p}(T)$ for the smallest constant c satisfying (3.3). In this way we obtain the injective and maximal Banach operator ideal $(\Pi_{B,q,p}, \pi_{B,q,p})$; note that $\Pi_{B,2,p}$ is defined for every (infinite) orthonormal system B. Trivially, $\Pi_{B,2,p}(\mathrm{id}_{\mathbb{K}}) = \Pi_{2,p}(\mathrm{id}_{\mathbb{K}}) = 1$, and for an easy argument which shows $\Pi_{B,q,2}(\mathrm{id}_{\mathbb{K}}) = 1$ we refer to [Sei95, 4.4]. The parameter q in the above definition together with the $\Lambda(q)$-system assumption was inspired by the "q-Parseval-Normen" in [Sei95], whereas the parameter p (instead of 2) seems to be new in this context.

The ideal $(\Pi_{B,2,2}, \pi_{B,2,2})$ will also be denoted by (Π_B, π_B), the ideal of all B-summing operators; this definition goes back to [Bau97] and [Sei95], and if $\mathcal{G}$ is a sequence of independent standard Gaussian random variables (on $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ whenever needed explicitly), then $(\Pi_{\mathcal{G}}, \pi_{\mathcal{G}})$ is the ideal (Π_γ, π_γ) of Gaussian-summing operators introduced in [LP74]. It is well-known (see e. g. [DJT95, 12.12]) that (Π_γ, π_γ) coincides with the ideal (Π_{as}, π_{as}) of almost summing operators which is generated by the system of Rademacher functions. Note that for $q = 2$ and the orthonormal system $B := \{2^{n/2} \cdot \chi_n \,|\, n \in \mathbb{N}\} \subset L_2((0,1])$, where χ_n denotes the characteristic function of the interval $(2^{-n}, 2^{1-n}]$, one has $\Pi_{B,2,p} = \Pi_{2,p}$. We will also write $(\Pi_{\gamma,p}, \pi_{\gamma,p})$ and $(\Pi_{as,p}, \pi_{as,p})$ instead of $(\Pi_{\gamma,2,p}, \pi_{\gamma,2,p})$ and $(\Pi_{as,2,p}, \pi_{as,2,p})$.

Note that—compare to the case of mixing operators (see the proof of Theorem 2.1)—the (B, q, p)-summing norm of an operator $T : E \to F$ can be computed in the following way: Consider for $\mathcal{F} = \{b_1, \dots, b_m\} \subset B$ the mapping

$$\begin{array}{rccc} \Phi^{m,\mathcal{F}}: & \mathcal{L}(\ell_{p'}^m, E) & \to & L_q(\mu, F) \\ & S & \mapsto & K_q(B)^{-1} \cdot \sum_{i=1}^m b_i \cdot TSe_i; \end{array}$$

then $T \in \Pi_{B,q,p}$ if and only if $c := \sup\{\|\Phi^{m,\mathcal{F}}\| \,|\, m \in \mathbb{N}, \mathcal{F} \subset B \text{ with } |\mathcal{F}| = m\} < \infty$, and in this case $\pi_{B,q,p}(T) = c$.

Some simple consequences of the above definition are as follows:

Proposition 3.1. *Let $1 \le p \le 2 \le q_0 \le q_1 \le q < \infty$, B an orthonormal system with $K_q(B) < \infty$ and X, Y Banach spaces.*

(a) $\Pi_{B,q_1,2} \subset \Pi_{B,q_0,2}$, and $\pi_{B,q_0,2}(T) \le (K_{q_1}(B)/K_{q_0}(B)) \cdot \pi_{B,q_1,2}(T)$ for all $T \in \Pi_{B,q_1,2}$.

(b) $\Pi_{as,1} = \mathcal{L}$, and $\Pi_{as,1}(T) \le \sqrt{2} \cdot \|T\|$ for all $T \in \mathcal{L}$.

(c) $\Pi_{\gamma,p} \subset \Pi_{as,p}$, and $\Pi_{\gamma,p}(X,Y) = \Pi_{as,p}(X,Y)$ whenever Y has a finite cotype.

(d) $\Pi_2 \subset \Pi_q \subset \Pi_{B,q,2} \subset \Pi_B \subset \Pi_\gamma$, and all inclusions have norm 1, except the third one (see (a)*).*

(e) If Y has cotype 2, then $\Pi_2(X,Y) = \Pi_{B,q,2} = \Pi_\gamma(X,Y)$ and $\Pi_{\gamma,p}(X,Y) \subset \Pi_{B,2,p}(X,Y)$.

Proof. (a) This is obvious since $L_{q_1}(\mu) \subset L_{q_0}(\mu)$.

(b) Let $T \in \mathcal{L}(X,Y)$ and $x_1,\dots,x_n \in X$. Then

$$\begin{aligned}\left(\int_0^1 \|\sum_{i=1}^n r_i \cdot Tx_i\|^2 d\lambda\right)^{1/2} &\le \sqrt{2}\cdot\int_0^1 \|\sum_{i=1}^n r_i\cdot Tx_i\|\,d\lambda \\ &\le \sqrt{2}\cdot\|T\|\cdot\int_0^1 \sup_{x'\in B_{X'}} |\sum_{i=1}^n r_i\cdot\langle x',x_i\rangle|\,d\lambda \\ &\le \sqrt{2}\cdot\|T\|\cdot\sup_{x'\in B_{X'}}\sum_{i=1}^n |\langle x',x_i\rangle|.\end{aligned}$$

(c) This follows from [DJT95, 12.11] and [DJT95, 12.27].

(d) The first and the third inclusion are trivial, the second follows from the Pietsch Domination Theorem (see e.g. [DJT95, 2.12 and 12.5] and [Bau97, 7.10] or [Sei95, 4.9]), and the last one can be found in [PW98, 4.15.3].

(e) The first part is a simple consequence of the definition together with (d), and the second part follows from [LT91, 9.24 and 9.25]. □

3.1 Complex Interpolation of (B,q,p)-summing operators

Our main tool will be the following interpolation theorem which is a natural counterpart to Theorem 2.1.

Theorem 3.2. *Let $0<\theta<1$ and $1\le p_0,p_1\le 2\le q<\infty$ such that either $p_0=p_1=2$ or $q=2$. Then for two finite-dimensional interpolation couples $[E_0,E_1]$ and $[F_0,F_1]$, each $T\in\mathcal{L}([E_0,E_1]_\theta,[F_0,F_1]_\theta)$ and each orthonormal system $B\subset L_2(\mu)$ with $K_q(B)<\infty$*

$$\begin{aligned}&\pi_{B,q,p_\theta}(T:[E_0,E_1]_\theta\to[F_0,F_1]_\theta)\\ &\qquad\le d_\theta[\ell_{p_0},\ell_{p_1};E_0,E_1]\cdot\pi_{B,q,p_0}(T:E_0\to F_0)^{1-\theta}\cdot\pi_{B,q,p_1}(T:E_1\to F_1)^\theta,\end{aligned}$$

where $1/p_\theta := (1-\theta)/p_0+\theta/p_1$.

Proof. For the moment set $E_\theta := [E_0,E_1]_\theta$, $F_\theta := [F_0,F_1]_\theta$, and consider for $\eta=0,\theta,1$ and $\mathcal{F}=\{b_1,\dots,b_m\}\subset B$ the mapping

$$\begin{array}{cccc}\Phi_\eta^{m,\mathcal{F}}: & \mathcal{L}(\ell_{p_\eta'}^m,E_\eta) & \to & L_q(\mu,F_\eta)\\ & S & \mapsto & K_q(B)^{-1}\cdot\sum_{i=1}^m b_i\cdot TSe_i;\end{array}$$

then, as explained above,

$$\pi_{B,q,p_\eta}(T:E_\eta\to F_\eta)=\sup\{\|\Phi_\eta^{m,\mathcal{F}}\|\,|\,m\in\mathbb{N},\mathcal{F}\subset B\text{ with }|\mathcal{F}|=m\}.$$

For the interpolated mapping

$$[\Phi_0^{m,\mathcal{F}},\Phi_1^{m,\mathcal{F}}]_\theta : [\mathcal{L}(\ell_{p_0'}^m,E_0),\mathcal{L}(\ell_{p_1'}^m,E_1)]_\theta\to[L_q(\mu,F_0),L_q(\mu,F_1)]_\theta$$

by the usual interpolation theorem

$$\|[\Phi_0^{m,\mathcal{F}}, \Phi_1^{m,\mathcal{F}}]_\theta\| \leq \|\Phi_0^{m,\mathcal{F}}\|^{1-\theta} \cdot \|\Phi_0^{m,\mathcal{F}}\|^{\theta}.$$

Since $[L_q(\mu, F_0), L_q(\mu, F_1)]_\theta = L_q(\mu, [F_0, F_1]_\theta)$ (isometrically) we obtain

$$\|\Phi_\theta^{m,\mathcal{F}}\| \leq \|\mathcal{L}(\ell_{p'_\theta}^m, [E_0, E_1]_\theta) \hookrightarrow [\mathcal{L}(\ell_{p'_0}^m, E_0), \mathcal{L}(\ell_{p'_1}^m, E_1)]_\theta\| \cdot \|[\Phi_0^{m,\mathcal{F}}, \Phi_1^{m,\mathcal{F}}]_\theta\|.$$

Consequently

$$\begin{aligned}
&\pi_{B,q,p_\theta}(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) \\
&\qquad = \sup\{\|\Phi_\theta^{m,\mathcal{F}}\| \mid m \in \mathbb{N}, \mathcal{F} \subset B \text{ with } |\mathcal{F}| = m\} \\
&\qquad \leq \sup\{d_\theta[\ell_{p_0}^m, \ell_{p_1}^m; E_0, E_1] \cdot \|\Phi_0^{m,\mathcal{F}}\|^{1-\theta} \cdot \|\Phi_0^{m,\mathcal{F}}\|^{\theta} \mid m \in \mathbb{N}, \mathcal{F} \subset B \text{ with } |\mathcal{F}| = m\} \\
&\qquad \leq d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1] \cdot \pi_{B,q,p_0}(T : E_0 \to F_0)^{1-\theta} \cdot \pi_{B,q,p_1}(T : E_1 \to F_1)^{\theta},
\end{aligned}$$

the desired result. □

In order to simplify future work, we will formulate two immediate corollaries of this theorem.

Corollary 3.3. *Let $0 < \theta < 1$ and $[E_0, E_1]$, $[F_0, F_1]$ be two finite-dimensional interpolation couples. Then for each $T \in \mathcal{L}([E_0, E_1]_\theta, [F_0, F_1]_\theta)$*

$$\pi_\gamma(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) \leq d_\theta[E_0, E_1] \cdot \pi_\gamma(T : E_0 \to F_0)^{1-\theta} \cdot \pi_\gamma(T : E_1 \to F_1)^{\theta}.$$

Corollary 3.4. *Let $1 < p < 2$ and $[E_0, E_1]$, $[F_0, F_1]$ be two finite-dimensional interpolation couples. Then for $\theta := 2/p'$ and each $T \in \mathcal{L}([E_0, E_1]_\theta, [F_0, F_1]_\theta)$*

$$\pi_{as,p}(T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta) \leq d_\theta[\ell_1, \ell_2; E_0, E_1] \cdot \|T : E_0 \to F_0\|^{1-\theta} \cdot \pi_{as}(T : E_1 \to F_1)^{\theta}.$$

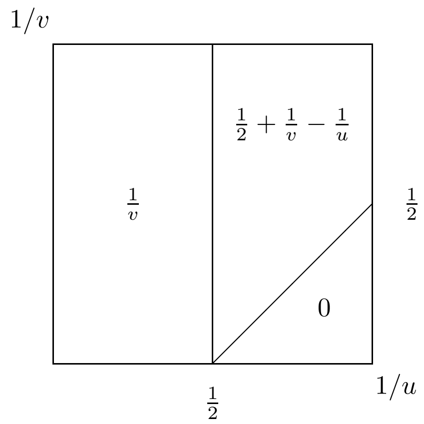
3.2 $\Lambda(p)$-systems and the limit order of $(B, q, 2)$-summing operators

In this part we show that for each $2 \leq q < \infty$ and each orthonormal system B which is a $\Lambda(p)$-system for all $2 < p < \infty$ the limit order of the ideal $\Pi_{B,q,2}$ coincides with that of Π_γ:

Theorem 3.5. *Let $B \subset L_2(\mu)$ be a $\Lambda(p)$-system for all $2 < p < \infty$. Then for all $2 < q < \infty$ and $1 \leq u, v \leq \infty$*

$$\lambda(\Pi_{B,q,2}, u, v) = \lambda(\Pi_B, u, v) = \lambda(\Pi_\gamma, u, v) = \begin{cases} 1/v & \text{if } 2 \leq u \leq \infty, \\ \max(0, 1/2 + 1/v - 1/u) & \text{if } 1 \leq u \leq 2. \end{cases}$$

As a diagram:



The proof of the theorem requires the following lemma which gives for $2 \le u \le \infty$ and $2 < q < \infty$ lower and upper estimates of $\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_q^m)$ whenever $B \subset L_2(\mu)$ is a $\Lambda(q)$-system:

Lemma 3.6. *For $2 \le u \le \infty$ and $2 < q < \infty$ let $B \subset L_2(\mu)$ be a $\Lambda(q)$-system. Then*

$$K_q(B)^{-1} \cdot m^{1/q} \le \pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_q^m) \le m^{1/q}.$$

Proof. Let $x^{(1)} = (x_i^{(1)})_1^m, \dots, x^{(n)} = (x_i^{(n)})_1^m \in \ell_u^m$. Then for all $1 \le i \le m$

$$\left(\sum_{k=1}^n |x_i^{(k)}|^2\right)^{1/2} \le \left(\sum_{k=1}^n |\langle e_i, x^{(k)}\rangle|^2\right)^{1/2} \le \sup_{x' \in B_{\ell_{u'}^m}} \left(\sum_{k=1}^n |\langle x', x^{(k)}\rangle|^2\right)^{1/2}. \tag{3.4}$$

Hence, for $b_1, \dots, b_n \in B$,

$$\begin{aligned}
\left(\int_\Omega \left\|\sum_{k=1}^n b_k \cdot x^{(k)}\right\|_{\ell_q^m}^q d\mu\right)^{1/q} &= \left(\int_\Omega \sum_{i=1}^m \left|\sum_{k=1}^n b_k \cdot x_i^{(k)}\right|^q d\mu\right)^{1/q} \\
&= \left(\sum_{i=1}^m \int_\Omega \left|\sum_{k=1}^n b_k \cdot x_i^{(k)}\right|^q d\mu\right)^{1/q} \\
&\le K_q(B) \cdot \left(\sum_{i=1}^m \left(\sum_{k=1}^n |x_i^{(k)}|^2\right)^{q/2}\right)^{1/q} \\
&\le K_q(B) \cdot m^{1/q} \cdot \sup_{x' \in B_{\ell_{u'}^m}} \left(\sum_{k=1}^n |\langle x', x^{(k)}\rangle|^2\right)^{1/2}.
\end{aligned}$$

Consequently $\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_q^m) \le m^{1/q}$. For the lower estimate use Proposition 3.1 (d):

$$\begin{aligned}\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_q^m) &\ge K_q(B)^{-1} \cdot \pi_B(\ell_u^m \hookrightarrow \ell_q^m) \ge K_q(B)^{-1} \cdot \|\ell_q^m \hookrightarrow \ell_2^m\|^{-1} \cdot \pi_2(\ell_u^m \hookrightarrow \ell_2^m)\\ &= K_q(B)^{-1} \cdot m^{1/q}.\end{aligned}$$

(see also [PW98, 3.11.11] for the main part of this proof). □

Proof of Theorem 3.5: For $1 \le v \le 2$ the statement follows from the fact that ℓ_v has cotype 2 and consequently, by Proposition 3.1 (e), $\Pi_{B,q,2}(\cdot, \ell_v) = \Pi_2(\cdot, \ell_v)$. Now let $2 < v < \infty$ and $2 \le u \le \infty$. Then if $v \ge q$, we conclude from Lemma 3.6

$$\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_v^m) \le (K_v(B)/K_q(B)) \cdot \pi_{B,v,2}(\ell_u^m \hookrightarrow \ell_v^m) \le (K_v(B)/K_q(B)) \cdot m^{1/v},$$

and if $v < q$, one has

$$\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_v^m) \le \|\ell_q^m \hookrightarrow \ell_v^m\| \cdot \pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_q^m) \le m^{1/v},$$

hence $\lambda(\Pi_{B,q,2}, u, v) \le 1/v$; from the continuity of the limit order it follows that $\lambda(\Pi_{B,q,2}, u, \infty) = 0$.

Now consider the case $1 \le u \le 2 \le v \le \infty$. Theorem 3.2 with $p_0 = p_1 = 2$ and the fact that $d_\theta[\ell_1^m, \ell_2^m] \le \sqrt{2}$ give, for u, v_0 such that $1/v_0 = 1/u - 1/2$ and $\theta := 2/u'$,

$$\lambda(\Pi_{B,q,2}, u, v_0) \le (1-\theta) \cdot \lambda(\Pi_{B,q,2}, 1, 2) + \theta \cdot \lambda(\Pi_{B,q,2}, 2, \infty) = 0.$$

For arbitrary $1 \le u \le 2 \le v \le \infty$ factorize through $\ell_{v_0}^m$:

$$\pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_v^m) \le m^{\max(0,1/v+1/2-1/u)} \cdot \pi_{B,q,2}(\ell_u^m \hookrightarrow \ell_{v_0}^m),$$

hence $\lambda(\Pi_{B,q,2}, u, v) \le \max(0, 1/v + 1/2 - 1/u)$. The upper estimates for $\lambda(\Pi_B, u, v)$ now follow by Proposition 3.1 (d); using the same argument it is sufficient to prove the lower estimates for Π_B only. These can be obtained by factorization through ℓ_2^m:

$$\pi_B(\ell_u^m \hookrightarrow \ell_v^m) \ge \|\ell_v^m \hookrightarrow \ell_2^m\|^{-1} \cdot \pi_B(\ell_u^m \hookrightarrow \ell_2^m) = m^{\max(1/v+\min(1/2-1/u,0),0)}$$

(recall that we only have to treat the case $2 \le v \le \infty$). □

Note that we have calculated the limit order of Π_γ in particular, which is already known by the results of [LP74].

For the case $q = 2$ Theorem 3.5 is sharp in the following sense: Let G be a compact abelian group, m_G its normalized Haar measure and Γ its dual group. Then every subset $\Lambda \subset \Gamma$ is an orthonormal system in $L_2(G, m_G)$, and we have the following characterization:

Corollary 3.7. *For every infinite subset $\Lambda \subset \Gamma$ the following are equivalent:*

(a) Λ is a $\Lambda(p)$-set for all $2 < p < \infty$.

(b) $\lambda(\Pi_\Lambda, u, v) = \lambda(\Pi_\gamma, u, v)$ for all $1 \le u, v \le \infty$.

Proof. The implication (a)$\Rightarrow$(b) has already been proved in Theorem 3.5, and the reverse one follows from [Bau97]: (b) trivially implies $\lambda(\Pi_B, \infty, \infty) = 0$, and the comments after [Bau97, 7.12] then tell us that $\Pi_p \subset \Pi_\Lambda$ for all $2 < p < \infty$. This in turn gives by [Bau97, 9.6] (see also [Bau99, 5.1]) that Λ is a $\Lambda(p)$-set for all $2 < p < \infty$. $\square$

Note that this corollary is a natural counterpart to a recent result of Baur [Bau97, 9.5] (see also [Bau99, 4.2]): Recall that a subset $\Lambda \subset \Gamma$ is said to be a *Sidon set* if there exists $\theta > 0$ such that for all $Q = \sum_{\gamma\in\Lambda} \alpha_\gamma \cdot \gamma \in \operatorname{span}(\Lambda)$ we have $\sum_{\gamma\in\Lambda} |\alpha_\gamma| \leq \theta \cdot \|Q\|_\infty$.

Proposition 3.8. *For every infinite subset $\Lambda \subset \Gamma$ the following are equivalent:*

(a') Λ is a Sidon set.

(b') $\Pi_\Lambda = \Pi_\gamma$.

In order to see the strong relationship of Corollary 3.7 to Baur's result the following characterization of a Sidon set due to Pisier [P78] may be helpful: (a') is equivalent to

(a") Λ is a $\Lambda(p)$-set with constant $K_p(\Lambda) \leq \kappa\sqrt{p}$ for all $2 < p < \infty$ and some $\kappa > 0$.

Moreover, Proposition 3.8 shows that condition (b) in Corollary 3.7 cannot be replaced by condition (b'): Γ contains subsets which are $\Lambda(p)$-sets for all $2 < p < \infty$ but fail to be Sidon sets (see e. g. [LR75, 5.14]). Hence, Corollary 3.7 may be viewed as a weak version of Baur's result. However, while the implication (a)$\Rightarrow$(b) in Corollary 3.7 also holds in the general case, it seems to be unknown whether (a") implies (b') for arbitrary orthonormal systems B. Baur has recently informed us that her results (and therefore Corollary 3.7) also hold in the non-abelian case.

Note that by using [Bau99, 5.1] the case $q = 2$ in Theorem 3.5 (and therefore Corollary 3.7) can be proved without complex interpolation: If B is a $\Lambda(p)$-system for all $2 < p < \infty$, then $\Pi_B(\cdot, X) = \Pi_\gamma(\cdot, X)$ holds for any X quotient of a subspace of some $L_p(\mu)$ and some $1 \leq p < \infty$, which—together with the continuity of the limit order—clearly implies that $\lambda(\Pi_B, u, v) = \lambda(\Pi_\gamma, u, v)$ for all $1 \leq u, v \leq \infty$. Nevertheless, our concept of interpolating (B, q, p)-summing norms will lead us in the following to some more new results for which we do not have any alternative proofs without interpolation yet.

3.3 Schatten limit orders and Bennett–Carl inequalities revisited

Next we pick up the ideas from the proof of Theorem 3.5 in order to compute the Schatten limit order of the ideal Π_γ; more precisely, we obtain asymptotic estimates for the sequence $(\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n))_n$. It turns out that theses quantities are closely connected to the limit order of Π_γ:

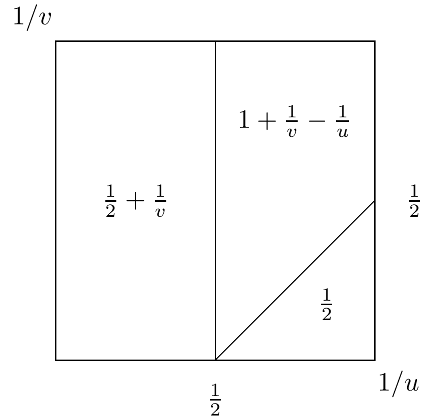
Proposition 3.9. *For* $1 \le u, v \le \infty$

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp \begin{cases} n^{1/2+1/v} & \text{if } 2 \le u \le \infty, \\ n^{1/2+\max(0,1/2+1/v-1/u)} & \text{if } 1 \le u \le 2. \end{cases}$$

In particular,

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/2+\lambda(\Pi_\gamma,u,v)}.$$

As a diagram for the Schatten limit order $\lambda_{\mathcal{S}}(\Pi_\gamma, u, v)$ we now get the following:



Proof. Since by [TJ74] $\mathcal{S}_v$ is of cotype 2 for $1 \le v \le 2$ only the case $2 \le v \le \infty$ has to be considered. First let $u = v = \infty$. Then by [PW98, 4.15.18] (a result of Pisier, see also [P89, 4.4] and [P86b]) and [FLM77, 3.3] for each $\varepsilon > 0$

$$\pi_\gamma(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_\infty^n) \asymp \sqrt{D(\mathcal{S}_\infty^n, \varepsilon)} \asymp n^{1/2}, \tag{3.5}$$

where $D(X, \varepsilon)$ denotes the Dvoretzky dimension of a Banach space X, i. e. the largest m such that there exists an m-dimensional subspace X_m of X with Banach–Mazur distance $d(X_m, \ell_2^m) \le 1 + \varepsilon$ (see [PW98, 4.15.15]). Now the general case $2 \le u, v \le \infty$ follows by factorization:

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le n^{1/v} \cdot \pi_\gamma(\mathcal{S}_\infty^n \hookrightarrow \mathcal{S}_\infty^n) \asymp n^{1/v+1/2},$$

and conversely

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \ge n^{1/v-1/2} \cdot \pi_\gamma(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_2^n) = n^{1/v+1/2}.$$

The case $1 \le u \le 2 \le v \le \infty$ is done by interpolation: We have (recall that $\pi_2(\mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) = n^{1/2}$)

$$\pi_\gamma(\mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n) \asymp \pi_\gamma(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n) \asymp n^{1/2},$$

hence for $1 < u < 2 < v_0 < \infty$ and $0 < \theta < 1$ such that $1/v_0 = 1/u - 1/2$ and $\theta = 2/u'$

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \le d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] \cdot \pi_\gamma(\mathcal{S}_1^n \hookrightarrow \mathcal{S}_2^n)^{1-\theta} \cdot \pi_\gamma(\mathcal{S}_2^n \hookrightarrow \mathcal{S}_\infty^n)^\theta \asymp n^{1/2};$$

recall that due to [Jun96] $\sup_n d_\theta[\mathcal{S}_1^n, \mathcal{S}_2^n] < \infty$. The remaining estimates now follow easily from

$$\pi_\gamma(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \ge \pi_\gamma(\ell_2^n \hookrightarrow \ell_2^n) = n^{1/2}$$

and analogous factorizations as in the corresponding part of the proof of Theorem 3.5. $\square$

Next we present an extension of the Bennett–Carl inequalities within the framework of (γ, p)-summing operators.

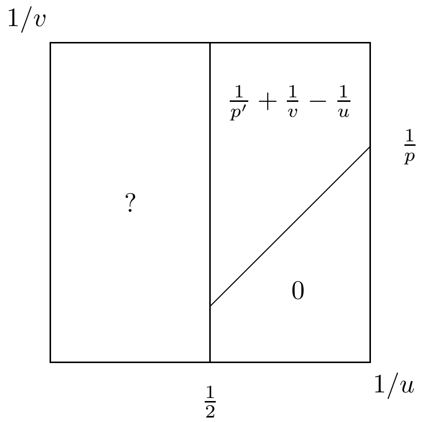
Theorem 3.10. *Let $1 \le u < 2$, $1 \le v \le \infty$ and $1 \le p \le 2$. Then*

$$\pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_v^n) \asymp n^{\max(0,1/p'+1/v-1/u)}.$$

In particular, for $1 \le u < 2$, $1 \le u \le v \le \infty$ and $1 \le p \le 2$

$$(\mathrm{id} : \ell_u \hookrightarrow \ell_v) \in \Pi_{\gamma,p} \quad \text{if and only if} \quad 1/p \ge 1/u' + 1/v.$$

As an (incomplete) diagram for the limit order of $\Pi_{\gamma,p}$ this gives the following:



Proof. The case $p = 1$ is trivial, and the case $p = 2$ is due to [LP74]. Now fix $1 < p < 2$ and $1 \le u < 2$, and let $1/v_2 := 1/u - 1/2$, $1/v_p := 1/u - 1/p'$ and $\theta := 2/p'$. Then

$$(1-\theta)/u + \theta/v_2 = 1/u - \theta \cdot (1/u - 1/v_2) = 1/u - 1/p' = 1/v_p$$

and

$$(1-\theta)/1 + \theta/2 = 1 - \theta/2 = 1/p.$$

Recall from (1.14) that $d_\theta[\ell_1, \ell_2; \ell_u^n, \ell_u^n] \le 16$, hence by Corollary 3.4

$$\pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_{v_p}^n) \asymp \pi_{as,p}(\ell_u^n \hookrightarrow \ell_{v_p}^n) \le d_\theta[\ell_1, \ell_2; \ell_u^n, \ell_u^n] \cdot \|\ell_u^n \hookrightarrow \ell_u^n\|^{1-\theta} \cdot \pi_{as}(\ell_u^n \hookrightarrow \ell_{v_2}^n)^\theta \prec 1.$$

Now let $1 \le u < 2$ and $1 \le u \le v \le \infty$. Then

$$\pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_v^n) \le \|\ell_{v_p}^n \hookrightarrow \ell_v^n\| \cdot \pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_{v_p}^n) \asymp n^{\max(0,1/p'+1/v-1/u)}.$$

For the lower estimates consider first the case $1 \le v \le 2$:

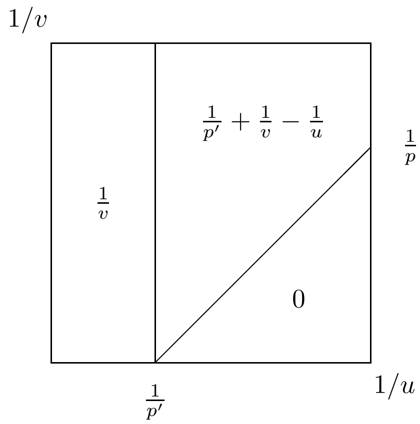
$$\pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_v^n) \succ \pi_{2,p}(\ell_u^n \hookrightarrow \ell_v^n) \asymp n^{\max(0,1/p'+1/v-1/u)},$$

and for $2 \leq v \leq \infty$ such that $1/v \geq 1/u - 1/p'$

$$\pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_v^n) \geq \|\ell_v^n \hookrightarrow \ell_2^n\|^{-1} \cdot \pi_{\gamma,p}(\ell_u^n \hookrightarrow \ell_2^n) \asymp n^{\max(0,1/p'+1/v-1/u)}.$$

□

We conjecture the following diagram for the limit order of $\Pi_{\gamma,p}$:



The "if-part" of Theorem 3.10 now gives the Bennett–Carl inequalities within the more general setting of $(B, 2, p)$-summing operators. Recall that there exists an orthonormal system B such that $\Pi_{2,p} = \Pi_{B,2,p}$ for all $1 \leq p \leq 2$.

Corollary 3.11. *Let $1 \leq u < 2$, $1 \leq u \leq v < \infty$ and $1 \leq p \leq 2$ such that $1/p \geq 1/u' + 1/v$. Then for all orthonormal systems B with $K_{\max(2,v)}(B) < \infty$*

$$(\mathrm{id} : \ell_u \hookrightarrow \ell_v) \in \Pi_{B,2,p}.$$

Proof. For the case $v \leq 2$ the statement immediately follows from Theorem 3.10 together with Proposition 3.1 (e). Now let $2 < v < \infty$ and B be a $\Lambda(v)$-system. Then by [Bau99, 5.1] and the definition of $\Pi_{B,2,p}$ and $\Pi_{\gamma,p}$ it follows that $\Pi_{\gamma,p}(\cdot, \ell_v) \subset \Pi_{B,2,p}(\cdot, \ell_v)$, and together with the preceding theorem this gives the claim. □

We conjecture that the above corollary even holds for all $1 \leq u < \infty$.

Remark 3.12. Let $1 \leq u \leq 2$. By (1.14) and the fact that $\mathcal{S}_u$ is of cotype 2 we know that $\sup_n d_\theta[\ell_1, \ell_2; \mathcal{S}_u^n, \mathcal{S}_u^n] < \infty$, hence, using Proposition 3.9 and similar arguments as in the proof of Theorem 3.10, for $1 \leq u \leq 2$, $1 \leq p \leq 2$ and $1 \leq u \leq v \leq \infty$ such that $1/v \leq 1/p - 1/u'$ we obtain

$$\pi_{\gamma,p}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \asymp n^{1/p'} \tag{3.6}$$

(note that for all $1 \leq u, v \leq \infty$ one has $\pi_{\gamma,p}(\mathcal{S}_u^n \hookrightarrow \mathcal{S}_v^n) \geq \pi_{\gamma,p}(\ell_2^n \hookrightarrow \ell_2^n) = n^{1/p'}$).

4 Complex interpolation of spaces of operators on ℓ_1

In [Kwa68] it was shown that for $1 \le p \le \infty$ and $1 \le r \le 2$ defined by $1/r = 1 - |1/2 - 1/p|$ every continuous operator on ℓ_1 with values in ℓ_p is $(r,1)$-summing, i. e. $\mathcal{L}(\ell_1, \ell_p) = \Pi_{r,1}(\ell_1, \ell_p)$. Carl and Defant in [CD92] gave a generalization of Kwapień's result within the framework of mixing operators: For $2 \le s \le \infty$ such that $1/s = |1/2 - 1/p|$ we have $\mathcal{L}(\ell_1, \ell_p) = \mathcal{M}_{s,1}(\ell_1, \ell_p)$. While Kwapień used interpolation techniques (e. g. the Three-Lines-Theorem), Carl and Defant used a certain tensor product trick. Our aim in this section is to show that their result can be proved by complex interpolation. Moreover, we show that the same ideas can be carried over to the notion of (γ, p)-summing operators, and from this we obtain another generalization of Kwapień's result (at least for $1 \le p \le 2$) which also gives a link to the so-called type indices of Banach function spaces.

The following lemma is a reformulating of Kwapień's interpolation trick for arbitrary maximal Banach operator ideals, with an additional approximation part.

Lemma 4.1. *Let $([F_0, F_1], F, \mathcal{B})$ be a cofinal interpolation triple, $0 < \theta < 1$ and $(\mathcal{A}, A)$ a maximal Banach operator ideal with*

$$c_\theta := \sup_n \sup_{M \in \mathcal{B}} \|\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) \hookrightarrow \mathcal{A}(\ell_1^n, [M_0, M_1]_\theta)\| < \infty.$$

Then

$$\mathcal{L}(\ell_1, [F_0, F_1]_\theta) = \mathcal{A}(\ell_1, [F_0, F_1]_\theta).$$

Proof. Denote by ε the injective tensor norm, by α the finitely generated tensor norm associated to $(\mathcal{A}, A)$ and by $\overleftarrow{\alpha}$ its cofinite hull in the sense of [DF93, 17.3 and 12.4]. We prove that

$$\|\ell_\infty^n \otimes_\varepsilon (F, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow \ell_\infty^n \otimes_{\overleftarrow{\alpha}} (F, \|\cdot\|_{[F_0,F_1]_\theta})\| \le c_\theta. \tag{4.1}$$

Then by density (see [DF93, 13.4])

$$\|\ell_\infty^n \otimes_\varepsilon [F_0, F_1]_\theta \hookrightarrow \ell_\infty^n \otimes_{\overleftarrow{\alpha}} [F_0, F_1]_\theta\| \le c_\theta,$$

hence the claim follows by the Embedding Theorem [DF93, 17.6] and local techniques [DF93, 23.1].

For $z \in \ell_\infty^n \otimes F$ choose by Lemma 1.3 a subspace $M \in \mathcal{B}$ such that $z \in \ell_\infty^n \otimes M$ and $\|(M, \|\cdot\|_{[F_0,F_1]_\theta}) \hookrightarrow [M_0, M_1]_\theta\| \le 1 + \varepsilon$. Then by the mapping properties of $\overleftarrow{\alpha}$ and ε and the fact that the injective tensor norm respects subspaces we have

$$\|z\|_{\ell_\infty^n \otimes_{\overleftarrow{\alpha}} (F, \|\cdot\|_{[F_0,F_1]_\theta})} \le \|z\|_{\ell_\infty^n \otimes_{\overleftarrow{\alpha}} [M_0,M_1]_\theta} \quad \text{and} \quad \|z\|_{\ell_\infty^n \otimes_\varepsilon [M_0,M_1]_\theta} \le (1+\varepsilon) \cdot \|z\|_{\ell_\infty^n \otimes_\varepsilon (F, \|\cdot\|_{[F_0,F_1]_\theta})},$$

hence (4.1) follows from the assumption and the Embedding Theorem. □

In our applications the approximation lemma will be combined with what we call the "interpolation trick", due to Kwapień and based on complex interpolation of vector-valued ℓ_p's [BL78, 5.1.2]:

$$\mathcal{L}(\ell_1^n, [M_0, M_1]_\theta) = \ell_\infty^n([M_0, M_1]_\theta) = [\ell_\infty^n(M_0), \ell_\infty^n(M_1)]_\theta = [\mathcal{L}(\ell_1^n, M_0), \mathcal{L}(\ell_1^n, M_1)]_\theta$$

(isometrically).

4.1 Mixing operators on ℓ_1

Proposition 4.2. *Let $[E_0, E_1]$ and $[F_0, F_1]$ be finite-dimensional interpolation couples and $2 \le s_0, s_1 \le \infty$. Then for $0 \le \theta \le 1$ and $2 \le s_\theta \le \infty$ defined by $1/s_\theta = (1-\theta)/s_0 + \theta/s_1$*

$$\|[\mathcal{M}_{s_0,2}(E_0, F_0), \mathcal{M}_{s_1,2}(E_1, F_1)]_\theta \hookrightarrow \mathcal{M}_{s_\theta,2}([E_0, E_1]_\theta, [F_0, F_1]_\theta)\| \le d_\theta[E_0, E_1].$$

Proof. Consider for $\eta = 0, 1$ the trilinear norm 1 mappings

$$\begin{array}{rccccccc} \Phi_\eta^{n,m}: & \mathcal{M}_{s_\eta,2}(E_\eta, F_\eta) & \times & \ell_{s_\eta}^n(F_\eta') & \times & \mathcal{L}(\ell_2^m, E_\eta) & \longrightarrow & \ell_2^m(\ell_{s_\eta}^n) \\ & T & \times & (y_1', \dots, y_n') & \times & S & \longmapsto & ((\langle y_k', TSe_\ell\rangle)_k)_\ell \end{array}.$$

Then for the interpolated mapping

$$[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta : [\mathcal{M}_{s_0,2}(E_0, F_0), \mathcal{M}_{s_1,2}(E_1, F_1)]_\theta \times [\ell_{s_0}^n(F_0'), \ell_{s_1}^n(F_1')]_\theta$$
$$\times [\mathcal{L}(\ell_2^m, E_0), \mathcal{L}(\ell_2^m, E_1)]_\theta \to [\ell_2^m(\ell_{s_0}^n), \ell_2^m(\ell_{s_1}^n)]_\theta,$$

by multilinear interpolation (Proposition 0.3) we also have $\|[\Phi_0^{n,m}, \Phi_1^{n,m}]_\theta\| \le 1$. It follows that for each $T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta$, each $S \in \mathcal{L}(\ell_2^m, [E_0, E_1]_\theta)$ and $y_1', \dots, y_n' \in [F_0, F_1]_\theta'$

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle y_k', TSe_j\rangle|^{s_\theta} \right)^{2/s_\theta} \right)^{1/2}$$
$$\le \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta} \cdot \|S\|_{[\mathcal{L}(\ell_2^m,E_0), \mathcal{L}(\ell_2^m,E_1)]_\theta} \cdot \|(y_k)_k\|_{[\ell_{s_0}^n(F_0'), \ell_{s_1}^n(F_1')]_\theta}$$
$$\le d_\theta[E_0, E_1] \cdot \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta} \cdot \|S\|_{\mathcal{L}(\ell_2^m, [E_0,E_1]_\theta)} \cdot \|(y_k)_k\|_{\ell_{s_\theta}^n([F_0,F_1]_\theta')},$$

hence

$$\|T\|_{\mathcal{M}_{s_\theta,2}([E_0,E_1]_\theta, [F_0,F_1]_\theta)} \le d_\theta[E_0, E_1] \cdot \|T\|_{[\mathcal{M}_{s_0,2}(E_0,F_0), \mathcal{M}_{s_1,2}(E_1,F_1)]_\theta}.$$

□

A very profitable situation is given if $E_0 = E_1 = \ell_1$ or ℓ_∞:

Corollary 4.3. *Let F be a θ-Hilbert space, $0 \le \theta < 1$.*

(a) $\mathcal{L}(\ell_1, F) = \mathcal{M}_{\frac{2}{1-\theta},2}(\ell_1, F) = \Pi_{\frac{2}{\theta},2}(\ell_1, F)$.

(b) $\mathcal{L}(\ell_\infty, F) = \mathcal{M}_{\frac{2}{1-\theta},2}(\ell_\infty, F) = \Pi_{\frac{2}{\theta},2}(\ell_\infty, F)$.

(c) Every $\frac{2}{1-\theta}$-summing operator on F factorizes through a Hilbert space.

By local techniques (see e. g. [DF93, 23.1]) the spaces ℓ_1 and ℓ_∞ can be replaced by $\mathcal{L}_{1,\lambda}$-spaces and $\mathcal{L}_{\infty,\lambda}$-spaces, respectively. Since every 2-summing operator on ℓ_1 is 1-summing (a) implies *(a') $\mathcal{L}(\ell_1, F) = \mathcal{M}_{\frac{2}{1-\theta},1}(\ell_1, F) = \Pi_{\frac{2}{1+\theta},1}(\ell_1, F)$.*

Proof. (a) Let $F = [F_0, F_1]_\theta$ where F_1 is a Hilbert space and F_Δ is dense in F_0 and F_1, and let $M \in FIN(F_\Delta)$. Then by the Little Grothendieck Theorem (see e. g. [DF93, 11.11])

$$\|\mathcal{L}(\ell_1^n, M_1) \hookrightarrow \mathcal{M}_{\infty,2}(\ell_1^n, M_1) = \Pi_2(\ell_1^n, M_1)\| \leq K_{LG},$$

and trivially $\|\mathcal{L}(\ell_1^n, M_0) \hookrightarrow \mathcal{M}_{2,2}(\ell_1^n, M_0) = \mathcal{L}(\ell_1^n, M_0)\| \leq 1$, hence, by the usual interpolation theorem together with Proposition 4.2, we have

$$\|[\mathcal{L}(\ell_1^n, M_0), \mathcal{L}(\ell_1^n, M_1)]_\theta \hookrightarrow \mathcal{M}_{2/(1-\theta),2}(\ell_1^n, [M_0, M_1]_\theta)\| \leq K_{LG}^\theta.$$

The claim now follows by Lemma 4.1.

(c) follows from (a') by trace duality: By local techniques (see again [DF93, 23.1]) statement (a') in terms of quotient ideals (see e. g. [DF93, 25.6]) reads as follows:

$$\Pi_{\frac{2}{1-\theta}}(F, \cdot) \subset (\Pi_1 \circ \Gamma_1^{-1})(F, \cdot),$$

where Γ_p for $1 \leq p \leq \infty$ stands for the Banach operator ideal of all $T : F \to Y$ such that $F \xrightarrow{T} Y \hookrightarrow Y''$ factorizes through some $L_p(\mu)$. Hence the abstract quotient formula from [DF93, 25.7] and the fact that the adjoint Γ_2^* of Γ_2 is contained in $\Gamma_1 \circ \Gamma_\infty$ (a result of Kwapień, see e. g. [DJT95, 7.12]) imply the conclusion:

$$\Pi_{\frac{2}{1-\theta}}(F, \cdot) \subset (\Gamma_1 \circ \Gamma_\infty)^* \subset \Gamma_2.$$

Finally, (b) is an immediate consequence of (c): Take $T \in \mathcal{L}(\ell_\infty, F)$ and some $S \in \Pi_{\frac{2}{1-\theta}}(F, F) \subset \Gamma_2(F, F)$ (by (c)). Then ST by the little Grothendieck theorem is 2-summing. □

For $\theta = 1$ and $F = \ell_2$ the statements (a) and (b) are the "Little Grothendieck Theorems". The special cases $\mathcal{L}(\ell_1, \ell_p) = \Pi_{r,1}(\ell_1, \ell_p)$, $1/r = 1 - |1/2 - 1/p|$, and $\mathcal{L}(\ell_\infty, \ell_p) = \Pi_{p,2}(\ell_\infty, \ell_p)$, $2 \leq p \leq \infty$, are due to Kwapień [Kwa68] and Lindenstrauss–Pełczyński [LP68], respectively; recall that ℓ_p is $1 - |1 - 2/p|$-Hilbertian and that for $1 \leq p \leq 2$ every $T : \ell_\infty \to \ell_p$ is even 2-summing. For mixing norms and θ-Hilbert spaces (a) was proved by Lermer [Ler94] (see also [Hau97])—for $F = \ell_p$ this result can be found in [CD92]. The observations (b) and (c) in the present form seem to be new (in the case $F = \ell_p$ see [DF93, Ex. 34.12] and [DJT95, p. 168]).

We will finish with another result on Schatten classes.

Corollary 4.4. *Let $2 \leq r, s \leq \infty$ and $1/r = 1/2 - 1/s$.*

(a) $\mathcal{S}_r = \Pi_{r,2}(\ell_2) = \mathcal{M}_{s,2}(\ell_2)$.

(b) $\Pi_{r,2}(E, \ell_2) = \mathcal{M}_{s,2}(E, \ell_2)$ for every Banach space E.

The first equality in (a) is due to Mitiagin and was first published in [Kwa68] (see e. g. [Kön86, 1.d.12] or [DJT95, 10.3] for an elementary proof). The second equality in (a) was proved in [CD92]—here is an alternative proof by interpolation: By Proposition 4.2 and the first equality for θ defined by $1/r = (1-\theta)/2$ the embeddings in

$$\mathcal{S}_r^n = [\mathcal{S}_2^n, \mathcal{S}_\infty^n]_\theta = [\mathcal{M}_{\infty,2}(\ell_2^n), \mathcal{M}_{2,2}(\ell_2^n)]_\theta \hookrightarrow \mathcal{M}_{s,2}(\ell_2^n) \hookrightarrow \Pi_{r,2}(\ell_2^n) = \mathcal{S}_r^n$$

all have norm ≤ 1, and by localization this gives the claim. Now it is easy to prove (b), which is a kind of extension of (a): Since $\mathcal{M}_{s,2} = \mathcal{I}_{s'} \circ \Pi_2^{-1}$ (see e. g. [DF93, 32.1]; $\mathcal{I}_{s'}$ denotes the ideal of s'-integral operators), it suffices to show that $TS \in \mathcal{I}_{s'}$ whenever $T \in \Pi_{r,2}(E, \ell_2)$ and $S \in \Pi_2(X, E)$. But by the Grothendieck–Pietsch factorization theorem (see e. g. [DF93, 11.3]) we know that $S = UV$ where $V \in \Pi_2(X, H)$, $U \in \mathcal{L}(H, E)$ and H a Hilbert space. Then by (a) and local techniques $TU \in \Pi_{r,2}(H, \ell_2) = \mathcal{M}_{s,2}(H, \ell_2)$, which gives $TS \in \mathcal{I}_{s'}$.

4.2 (γ, p)-summing operators on ℓ_1

In this part we show that every continuous operator on ℓ_1 with values in a p-convex Banach function space ($1 \leq p \leq 2$) with non-trivial cotype is (γ, p)-summing.

Proposition 4.5. *Let $[E_0, E_1]$ and $[F_0, F_1]$ be finite-dimensional interpolation couples. Then for each orthonormal system B, all $1 \leq p_0, p_1 \leq 2$ and $0 < \theta < 1$*

$$\|[\Pi_{B,2,p_0}(E_0, F_0), \Pi_{B,2,p_1}(E_1, F_1)]_\theta \hookrightarrow \Pi_{B,2,p_\theta}([E_0, E_1]_\theta, [F_0, F_1]_\theta)\| \leq d_\theta[\ell_{p_0}, \ell_{p_1}; E_0, E_1],$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Proof. Consider for $\eta = 0, 1$ and $\mathcal{F} = \{b_1, \ldots, b_m\} \subset B$ the bilinear mapping

$$\begin{array}{cccccc} \Phi_\eta^{m,\mathcal{F}} : & \Pi_{B,2,p_\eta}(E_\eta, F_\eta) & \times & \mathcal{L}(\ell_{p'_\eta}^m, E_\eta) & \to & L_2(\mu, F_\eta) \\ & T & \times & S & \mapsto & \sum_{i=1}^m b_i \cdot TSe_i. \end{array}$$

By definition $\|\Phi_\eta^{m,\mathcal{F}}\| \leq 1$, hence for the interpolated mapping

$$[\Phi_0^{m,\mathcal{F}}, \Phi_1^{m,\mathcal{F}}]_\theta : [\Pi_{B,2,p_0}(E_0, F_0), \Pi_{B,2,p_1}(E_1, F_1)]_\theta \times [\mathcal{L}(\ell_{p'_0}^m, E_0), \mathcal{L}(\ell_{p'_1}^m, E_1)]_\theta$$
$$\to [L_2(\mu, F_0), L_2(\mu, F_1)]_\theta (= L_2(\mu, [F_0, F_1]_\theta))$$

we obtain $\|[\Phi_0^{m,\mathcal{F}}, \Phi_1^{m,\mathcal{F}}]_\theta\| \leq 1$. It follows that for each $T : [E_0, E_1]_\theta \to [F_0, F_1]_\theta$ and each $S \in \mathcal{L}(\ell_{p'_\theta}^m, [E_0, E_1]_\theta)$

$$\begin{aligned} &\left\| \sum_{i=1}^m b_i \cdot TSe_i \right\|_{L_2(\mu, [F_0, F_1]_\theta)} \\ &\qquad \leq \|T\|_{[\Pi_{B,2,p_0}(E_0,F_0), \Pi_{B,2,p_1}(E_1,F_1)]_\theta} \cdot \|S\|_{[\mathcal{L}(\ell_{p'_0}^m, E_0), \mathcal{L}(\ell_{p'_1}^m, E_1)]_\theta} \\ &\qquad \leq d_\theta[\ell_{p_0}^m, \ell_{p_1}^m; E_0, E_1] \cdot \|T\|_{[\Pi_{B,2,p_0}(E_0,F_0), \Pi_{B,2,p_1}(E_1,F_1)]_\theta} \cdot \|S\|_{\mathcal{L}(\ell_{p'_\theta}^m, [E_0,E_1]_\theta)}, \end{aligned}$$

hence

$$\|T\|_{\Pi_{B,2,p_\theta}([E_0,E_1]_\theta,[F_0,F_1]_\theta)} \leq d_\theta[\ell_{p_0},\ell_{p_1};E_0,E_1]\cdot \|T\|_{[\Pi_{B,2,p_0}(E_0,F_0),\Pi_{B,2,p_1}(E_1,F_1)]_\theta}.$$

□

We first state the following result for "θ-type 2 spaces":

Theorem 4.6. *Let $[F_0,F_1]$ be a dense interpolation couple such that F_1 is a type 2 space. Then for $1 \leq p \leq 2$ and $\theta := 2/p'$*

$$\mathcal{L}(\ell_1,[F_0,F_1]_\theta) = \Pi_{as,p}(\ell_1,[F_0,F_1]_\theta).$$

Proof. The cases $p=1$ (trivial) and $p=2$ (see e. g. [DJT95, 12.10]) are clear, so let $1<p<2$ and $M \in FIN(F_\Delta)$. By [DJT95, p. 245] we know that

$$\|\mathcal{L}(\ell_1^n,M_1)\hookrightarrow \Pi_{as,2}(\ell_1^n,M_1)\| \leq K_G\cdot \mathbf{T_2}(F_1),$$

where K_G denotes the Grothendieck constant, and by Lemma 3.1 (b)

$$\|\mathcal{L}(\ell_1^n,M_0)\hookrightarrow \Pi_{as,1}(\ell_1^n,M_0)\| \leq \sqrt{2},$$

hence, by the usual interpolation theorem together with Proposition 4.5, we obtain (recall from (1.14) that $d_\theta[\ell_1,\ell_2;\ell_1^n,\ell_1^n]\leq 16$)

$$\|[\mathcal{L}(\ell_1^n,M_0),\mathcal{L}(\ell_1^n,M_1)]_\theta \hookrightarrow \Pi_{as,p}(\ell_1^n,[M_0,M_1]_\theta)\| \leq 2^{(7-\theta)/2}\cdot (K_G\cdot \mathbf{T_2}(F_1))^\theta.$$

The claim now follows by Lemma 4.1. □

As an immediate consequence of Theorem 4.6 we get an extension of Kwapień's result for the case $1 \leq p \leq 2$ within the setting of $(B,2,p)$-summing operators—recall that there exists an orthonormal system B such that $\Pi_{2,p}=\Pi_{B,2,p}$, and $\Pi_{2,p}(\ell_1,\ell_p)=\Pi_{r,1}(\ell_1,\ell_p)$ for $1/r = 1/2+1/p'$ (see [TJ70]).

Corollary 4.7. *Let $1\leq p\leq 2$. Then for each orthonormal system B*

$$\mathcal{L}(\ell_1,\ell_p)=\Pi_{B,2,p}(\ell_1,\ell_p) \quad \text{and} \quad \mathcal{L}(\ell_1,\mathcal{S}_p)=\Pi_{B,2,p}(\ell_1,\mathcal{S}_p).$$

Proof. By Theorem 4.6 the statement is clear for the system of Rademacher functions, hence—since ℓ_p and $\mathcal{S}_p$ have cotype 2—also for $\Pi_{\gamma,p}$, and consequently for all orthonormal systems B by Proposition 3.1 (e). □

The following "extrapolation lemma" will be useful to prove the result for Banach function spaces indicated above.

Lemma 4.8. *Let X_0 and X_1 be finite-dimensional lattices, and for $0<r<1<p<\infty$ assume that $\mathbf{M^{(p)}}(X_0)=1$. Then for $0<\theta<1$ such that $p(1-\theta)+r\theta=1$*

$$(X_0^p)^{1-\theta}(X_1^r)^\theta = X_0^{p(1-\theta)}X_1^{r\theta}$$

holds isometrically. In particular, if X is a finite-dimensional lattice with $\mathbf{M^{(p)}}(X)=1$ for $1<p<2$, then for $\theta:=2/p'$

$$X=(X^p)^{1-\theta}(X^{p/2})^\theta$$

holds isometrically.

Proof. Let $V := (X_0^p)^{1-\theta}(X_1^r)^\theta$ and $W := X_0^{p(1-\theta)}X_1^{r\theta}$. Then, if $|f| = |g|^{1-\theta}\cdot|h|^\theta$,

$$\|f\|_W = \|(|g|^{1/p})^{p(1-\theta)}\cdot(|h|^{1/r})^{r\theta}\|_W \leq \||g|^{1/p}\|_{X_0}^{p(1-\theta)}\cdot\||h|^{1/r}\|_{X_1}^{r\theta} = \|g\|_{X_0^p}^{1-\theta}\cdot\|h\|_{X_1^r}^{\theta},$$

hence $\|f\|_W \leq \|f\|_V$. Conversely, let $|f| = |g|^{p(1-\theta)}\cdot|h|^{r\theta}$. Then

$$\|f\|_V = \|(|g|^p)^{1-\theta}\cdot(|h|^r)^\theta\|_V \leq \||g|^p\|_{X_0^p}^{1-\theta}\cdot\||h|^r\|_{X_1^r}^{\theta} = \|g\|_{X_0}^{p(1-\theta)}\cdot\|h\|_{X_1}^{r\theta},$$

hence $\|f\|_V \leq \|f\|_W$. For the rest observe that $X = X^{1-\eta}X^\eta$ holds isometrically with equal norms for each $0<\eta<1$; this follows from the abstract Hölder inequality (see e.g. [LT79, 1.d.2]): Let $|f| = |g|^{1-\eta}\cdot|h|^\eta$. Then

$$\|f\|_X = \||g|^{1-\eta}\cdot|h|^\eta\|_X \leq \|g\|_X^{1-\eta}\cdot\|h\|_X^\eta.$$

Conversely, we have $|f| = |f|^{1-\eta}\cdot|f|^\eta$, hence

$$\|f\|_{X^{1-\eta}X^\eta} \leq \|f\|_X^{1-\eta}\cdot\|f\|_X^\eta = \|f\|_X.$$

Clearly $\theta := 2/p'$ satisfies $p(1-\theta)+p\theta/2 = 1$. Altogether we obtain that

$$(X^p)^{1-\theta}(X^{p/2})^\theta = X^{p(1-\theta)}X^{p\theta/2} = X$$

holds isometrically. □

Theorem 4.9. *For $1\leq p\leq 2$ let $X(\mu)$ be a p-convex Banach function space with non-trivial cotype. Then*

$$\mathcal{L}(\ell_1, X) = \Pi_{\gamma,p}(\ell_1, X).$$

Proof. Since X has finite cotype, it is enough to deal with $\Pi_{as,p}(\ell_1,X)$ instead of $\Pi_{\gamma,p}(\ell_1,X)$. The case $p=1$ is trivial anyway, and for $p=2$ note that a 2-convex Banach function space with finite cotype has type 2 (see e.g. [LT79, 1.f.13]). Fix $1<p<2$, and without loss of generality we may assume that $\mathbf{M^{(p)}}(X)=1$. Since X has finite cotype it is σ-order continuous by [LT79, 1.a.5] and [LT79, 1.a.7], hence $S(\mu)$ is dense in X. Now, by Lemma 4.1 (with $F_0 = F_0 = X(\mathbb{C})$ and $\mathcal{B} = FIN_\chi(\mu)$), we can reduce the problem to the finite-dimensional case, i.e. it is sufficient to show that there exists $C>0$ such that for all n and $M \in FIN_\chi(\mu)$

$$\|\mathcal{L}(\ell_1^n, M(\mathbb{C})) \hookrightarrow \Pi_{as,p}(\ell_1^n, M(\mathbb{C}))\| \leq C.$$

From Lemma 4.8 together with the Calderón formula (0.11) we know that $M(\mathbb{C}) = [M^p(\mathbb{C}), M^{p/2}(\mathbb{C})]_\theta$ holds isometrically for $\theta = 2/p'$, and consequently—using the same arguments as in the proof of Theorem 4.6—we arrive at

$$\|\mathcal{L}(\ell_1^n, M(\mathbb{C})) \hookrightarrow \Pi_{as,p}(\ell_1^n, M(\mathbb{C}))\| \leq 2^{(7-\theta)/2}\cdot(K_G\cdot\mathbf{T_2}(M^{p/2}(\mathbb{C})))^\theta.$$

What now remains to be given is an appropriate estimate for $\mathbf{T_2}(M^{p/2}(\mathbb{C}))$. Using [LT79, 1.d.6] one can easily see that for each $q>2$

$$\mathbf{T_2}(M^{p/2}(\mathbb{C})) \leq 2\cdot\mathbf{T_2}(M^{p/2}) \leq 2\cdot b_{2q/p}\cdot\mathbf{M_{(2q/p)}}(M^{p/2})\cdot\mathbf{M^{(2)}}(M^{p/2})$$

(with $b_{2q/p}$ from the Khinchine inequality (3.2)). By the definition of $M^{p/2}$ and the assumption $\mathbf{M}^{(\mathbf{p})}(M) = 1$ a straightforward calculation shows that $\mathbf{M}^{(\mathbf{2})}(M^{p/2}) = 1$, and if X is q-concave for some $2 < q < \infty$ (by assumption there exists such q), then

$$\mathbf{M}_{(\mathbf{2q/p})}(M^{p/2}) \leq \mathbf{M}_{(\mathbf{q})}(M)^{p/2} \leq \mathbf{M}_{(\mathbf{q})}(X)^{p/2}$$

(see also [LT79, p. 54]). Altogether we obtain $\mathbf{T_2}(M^{p/2}(\mathbb{C})) \leq 2 \cdot b_{2q/p} \cdot \mathbf{M}_{(\mathbf{q})}(X)^{p/2}$. This proves the theorem in the complex case; the real case now follows by complexification. □

A Banach space X has type 2 if and only if $\mathcal{L}(\ell_1, X) = \Pi_\gamma(\ell_1, X)$ (see e. g. [DJT95, 12.10]); Theorem 4.9 now reveals that the ideal $\Pi_{\gamma,p}$ might play the same role for the notion of type p $(1 < p < 2)$, at least for Banach function spaces. If we define as usual

$$p(X) := \sup\{1 \leq p \leq 2 \,|\, X \text{ has type } p\},$$

and in addition

$$p_\gamma(X) := \sup\{1 \leq p \leq 2 \,|\, \mathcal{L}(\ell_1, X) = \Pi_{\gamma,p}(\ell_1, X)\}$$

(with $\sup \emptyset := 1$ if necessary), then $p(X)$ and $p_\gamma(X)$ coincide for a Banach function space X:

Corollary 4.10. *Let X be a Banach function space. Then $p(X) = p_\gamma(X)$.*

Proof. Let $p(X) > 1$. Then by [LT79, 1.f.9] and [LT79, 1.f.13] X is p-convex for all $1 \leq p < p(X)$ and q-concave for some $q < \infty$, hence by Theorem 4.9 the equality $\mathcal{L}(\ell_1, X) = \Pi_{\gamma,p}(\ell_1, X)$ holds for all $1 \leq p < p(X)$, and consequently $p_\gamma(X) \geq p(X)$. Conversely, if $p_\gamma(X) > 1$, then by similar arguments as in [DJT95, p. 237] the Banach space X is of type p for all $1 \leq p < p_\gamma(X)$, hence $p(X) \geq p_\gamma(X)$. □

The preceding result leads to the following natural questions: Does $p(X) = p_\gamma(X)$ hold for arbitrary Banach spaces X? Furthermore: Is for $1 < p < 2$ a Banach space X of type p if and only if $\mathcal{L}(\ell_1, X) = \Pi_{\gamma,p}(\ell_1, X)$?

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