Homological Classification of Monoids by Projectivities of Right Acts

Vom Fachbereich Mathematik der
Carl-von-Ossietzky Universität Oldenburg
zur Erlangung des Grades einer
Doktorin der Naturwissenschaften
angenommene Dissertation, vorgelegt von

Helga Oltmanns

aus

Westrhauderfehn

Erstreferent: Prof. Dr. Dr. h.c. Ulrich Knauer
Korreferent: Prof. Dr. Mati Kilp

Tag der mündlichen Prüfung: 7. Juni 2000

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Contents

1 Introduction

In order to examine properties of mathematical objects, it is often helpfull to represent them by other objects, which are better known. For instance, in the investigations of rings properties of modules over rings, i.e., of representations of rings by endomorphisms of abelian groups, can be utilized.

The representation of monoids by transformations of sets, which leads to acts over monoids, is the foundation of the dissertation in hand.
Like in many other categories, projective and injective objects play an important role in the category of acts. Several different properties of acts over monoids can be found in the literature. They can roughly be divided in two parts: properties around injectivity and properties around projectivity.

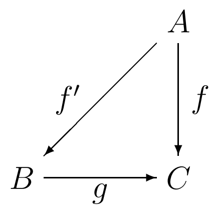
The first part — properties weaker than injectivity — includes, for example, weak injectivity, fg-injectivity and principally weak injectivity. These have been considered, for example, by Gould (cf. [9]). Injective acts themselves have been treated by Berthiaume ([2]), Burgess ([7]), Skornjakov ([20]) et al.

Properties like weak projectivities, which had been introduced in [17], as well as projectivity itself (cf. Knauer [15]) belong to the second part (properties weaker than projectivity). Further concepts in this part are properties like strong flatness (cf. Bulman-Fleming, [3]), equalizer flatness (cf. Normak, [19]), flatness (cf. Kilp, [11]). A survey on flatness properties of acts has been given by Bulman-Fleming in [4].

A detailed exposition of these concepts can be found in "Monoids, acts and categories" ([14]). Here, the authors Kilp, Knauer and Mikhalev consider both parts. After presenting elementary properties of monoids, acts and constructions like products, pullbacks, etc., various classes of acts are described in detail, and it is shown, that the classes are different. The 4. Chapter of this book is called "Homological classification". It includes a large number of results about acts over monoids, which concern questions of homological classification, where properties of the representations, i.e., properties of acts, lead to internal properties of the underlying monoid.

In the following, the definition of acts, some properties of acts, special elements and subsets of monoids, results concerning flat acts, etc., are needed. This preliminaries will be given in **Chapter 2**.

In this dissertation, properties weaker than projectivity will be considered. Recall, that an act A_S is called **projective**, if for arbitrary acts B_S and C_S every homomorphism $f : A \longrightarrow C$ can be lifted with respect to every epimorphism $g : B \longrightarrow C$, i.e., there exists a homomorphism $f' : A \longrightarrow B$, such that the following diagram is commutative:



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A question arises in a quite natural way: What happens, if we restrict the class of epimorphisms in the definition of projectivity? I.e., what happens, if we require f to be liftable with respect to epimorphisms $g : X \longrightarrow Y$, where X and Y are elements of particular subclasses of all acts?

This question leads to $(\mathcal{X}, \mathcal{Y})$**-projectivity**. A right act A_S is called $(\mathcal{X}, \mathcal{Y})$-projective, if every homomorphism from A_S to an act $Y_S \in \mathcal{Y}$ can be lifted with respect to every epimorphism from $X_S \in \mathcal{X}$ onto Y_S, where various classes $\mathcal{X}$ and $\mathcal{Y}$ will be considered, for example the class of all free acts or the class of all generators. This definition within the respective classes is presented at the beginning of **Chapter 3**.

By definition we initially get 226 formally different concepts of $(\mathcal{X}, \mathcal{Y})$-projectivity. In a certain sense, the third Chapter contains a filtration of these properties:

Some of the properties turn out to be trivial, i.e., all acts have the respective property (Lemma 3.2). Others are equivalent or coincide with projectivity (see Lemma 3.26). At the end, thirty six concepts remain for further investigations.

The second topic of the third Chapter is to present various implications between the remaining properties. The results are illustrated in an implication scheme at the end of Chapter 3.

It turns out, that projectivity itself is included in the concept of $(\mathcal{X}, \mathcal{Y})$-projectivities (see for example Theorem 4.3), and it is the strongest projectivity at all.

Since zeros in monoids play an important role with respect to $(\mathcal{X}, \mathcal{Y})$-projectivities of acts, the investigation of the concepts introduced in Chapter 3 will be the subject matter of **Chapter 4**, concerning acts over monoids with left zero. The main statements of this Chapter contain equivalences in the case of monoids with left zero as well as a summary of the results of Chapter 3

concerning equivalences in the case of arbitrary monoids. The results of this Chapter are tools in the derivation of results in the Chapters 5 and 6.
Note, that these results also describe the respective situation in the category of acts with zeros.

As mentioned before, a large number of results about acts over monoids refer to questions of homological classification, where properties of acts lead to internal properties of the underlying monoid. In **Chapter 5** internal properties of monoids will be described by properties of their representations under the viewpoint of homological classification with respect to $(\mathcal{X}, \mathcal{Y})$-projectivities of acts.
The first Section contains characterizations of monoids over which all acts are $(\mathcal{X}, \mathcal{Y})$-projective. For example, all acts are $(S/sS, S/I)$-projective if and only if S is the disjoint union of a special submonoid R of S and a simple semigroup or S is a group or, in particular, S is the disjoint union of a group and a zero (Theorem 5.52, Theorem 5.53, Corollary 5.56).
Furthermore, in Section 5.2, implications between the concepts are considered. For example, all $(S/sS, S/tS)$-projective acts are $(S/\varrho(x,y), S/tS)$-projective if and only if S fulfills condition (Moz) (Theorem 5.73), which is introduced at the beginning of Chapter 5.
The results will be summarized in tables given at the end of Chapter 5.

The assertions of Chapter 5 will be used in **Chapter 6** to point out the differences between the concepts of $(\mathcal{X}, \mathcal{Y})$-projectivity. Concrete (counter-) examples will be given.
In the same way, distinctions between $(\mathcal{X}, \mathcal{Y})$-projectivities and other properties weaker than projectivity will be demonstrated. The considered properties in Section 6.2 are flatness properties. The needed respective results concerning homological classification are included in Chapter 2 — Preliminaries.

Acknowledgements:

Many people deserve my thanks. I am very grateful to Professor Ulrich Knauer for supervising this work, for his support and valuable advices. I thank Professor Mati Kilp for being a reviewer of my dissertation, for his valuable hints and for the opportunity to spend some time at the University of Tartu. A big thank for the hospitality and the inspiring atmosphere to all members of the Algebraic Institute of the University of Tartu.
Finally thanks to my family and my friends who accompanied me through the ups and downs.

2 Preliminaries

Most of the definitions and results given in this chapter are taken from [14]. Otherwise the respective paper/book will be especially cited.

Throughout this dissertation let S be a monoid with identity 1.

Definition 2.1 *A nonempty set A is called a* **right S-act (or right act over S)***, if there exists a mapping*

$$\mu : A \times S \longrightarrow A$$

$$(a, s) \longrightarrow a\,s := \mu(a, s),$$

such that

(a) $a\,1 = a$ *and*

(b) $a\,(s\,t) = (a\,s)\,t$ *for* $a \in A,\ s,\ t \in S$.

A right S-acts A is denoted by A_S or by A, if the context allows to omit the index S, and is simply called an act.
Analogously, left S-acts ${}_SA$ are defined.

For right S-acts the following names are also used: S-sets, S-polygons, S-systems, S-automata ([20], [12], [21] and others).
Notice, that the monoid S with the internal multiplication of elements is a right S-act.
The one element right S-act $\{\Theta\}_S$ is shortly denoted by Θ_S.

Definition 2.2 *If A_S and B_S are right S-acts, then a mapping $f : A_S \longrightarrow B_S$ is called a* **homomorphism of right S-acts, or S-homomorphism***, if*

$$f(as) = f(a)s \quad \text{for all } a \in A \text{ and for all } s \in S.$$

An S-homomorphism will simply be called homomorphism, if the context allows to omit S. In this case we sometimes shortly write $f : A \longrightarrow B$.
Note that the composition of S-homomorphisms is an S- homomorphism.
The category of all right S-acts is denoted by **Act** $-S$.

In **Act** $-S$ *the* **epimorphisms** *are exactly the surjective homomorphisms and the* **monomorphisms** *are the injective homomorphisms. Thus* **isomorphisms** *are bijective homomorphisms. If $f : A_S \longrightarrow B_S$ is an isomorphism, we write $A_S \cong B_S$.*
An epimorphism $f : A_S \longrightarrow B_S$ is called a **retraction***, if there exists $g : B_S \longrightarrow A_S$, such that $f\,g = id_B$. In this case, B_S is called a* **retract** *of A_S.*

Definition 2.3 *Let A_S be a right S-act. An equivalence relation ϱ on A is called an* S**-act congruence** *or* **congruence on** A_S*, if $a\,\varrho\,a'$ implies $(a\,s)\,\varrho\,(a'\,s)$ for $a,\ a' \in A,\ s \in S$.*
For $(a, a') \in A \times A$ the smallest congruence on A_S, containing (a, a'), is denoted by $\varrho(a, a')$ and is called a **principal** *or* **monocyclic** *congruence.*

Let ϱ be a congruence on A_S. Define a right multiplication on the factor set $A/\varrho = \{[a]_\varrho \mid a \in A\}$ by elements of S by:

$$[a]_\varrho\, s = [as]_\varrho \text{ for every } s \in S.$$

Then A/ϱ becomes a right S-act, which is called a **factor act of** A **by** ϱ*.*
The canonical surjection

$$\pi : A_S \longrightarrow (A/\varrho)_S$$
$$a \longrightarrow [a]_\varrho$$

is an S-homomorphism and is called a **canonical epimorphism***.*

For the monoid S, every right (semigroup) congruence ϱ on S is an S-act congruence on S_S.
A factor act of S by a right principal (monocyclic) congruence on S is called a **monocyclic act***.*

Note, that the kernel equivalence $ker\ f$ of an S-homomorphism f is an S-act congruence, which is called a **kernel congruence of** f.

Definition 2.4 *Let A_S be a right S-act. An element $\theta \in A$ is called a* **zero of** A_S **(a fixed element, a sink)** *if $\theta\, s = \theta$ for all $s \in S$.*

An act could have more then one zero. If the monoid S has a left zero z, then every element $a\,z \in A$ is a zero of A_S.

Definition 2.5 *Let A_S be a right S-act. A subset $B \subseteq A$ of A is called a* **subact** *of A_S, if $b\,s \in B$ for all $b \in B,\ s \in S$.*
A subact B_S of A_S is called a **proper subact***, if $B \neq A$.*
Any subact $B_S \subseteq A_S$ defines the **Rees congruence** *ϱ_B on A_S by setting*

$$a\,\varrho_B\,a' \ \text{ iff } \ a,\ a' \in B \text{ or } a = a'.$$

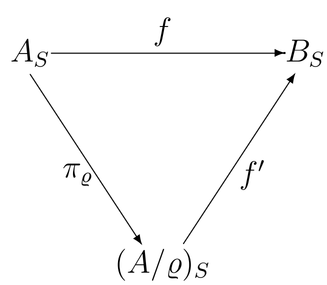
The resulting factor act is denoted by $(A/B)_S$ and is called the **Rees factor act of** A_S **by the subact** B_S*.*

In particular, if $I \subseteq S$ is a right ideal of S, then $(S/I)_S$ is called the Rees factor act of S_S by the right ideal I_S.

Notice, that if $(A/B)_S$ is the Rees factor act of A_S by the subact B_S, then the class $[b]_{\varrho_B}$ of an element $b \in B$ is a zero in $(A/B)_S$ and the class $[a]_{\varrho_B}$ of $a \in A \setminus B$ is the one-element classes $\{a\}$. Thus the Rees factor act $(A/B)_S$ could be considered as $(A/B)_S = ((A \setminus B) \cup \{\theta\})_S$.

Definition 2.6 *A right S-act A_S is called* **simple***, if A_S has no proper subacts.*

Theorem 2.7 *(***Homomorphism Theorem for right S-acts***)*
*Let $A_S,\ B_S \in$ **Act** $-S$, let $f : A_S \longrightarrow B_S$ be an S-homomorphism and let ϱ be a right congruence on A_S, such that $a\,\varrho\,a'$ implies $f(a) = f(a')$. Then $f' : (A/\varrho)_S \longrightarrow B_S$ with $f'([a]_\varrho) := f(a),\ a \in A$, is the unique S-homomorphism, such that the following diagram is commutative:*



If $\varrho = ker\, f$, then f' is injective, and if f is surjective, then f' is surjective.

Note that this implies, that every epimorphic image of a right S-act A_S is isomorphic to a factor act of A_S, especially epimorphic images of S_S are isomorphic to factor acts $(S/\varrho)_S$.

Definition 2.8 *The right S-act A_S is called* **cyclic***, if there exists an element $a \in A$, such that $A_S = aS = \{as \mid s \in S\}$.*

Proposition 2.9 *An right S-act A_S is cyclic if and only if there exists a right congruence ϱ on S, such that $A_S \cong (S/\varrho)_S$.*

If $f : A_S \longrightarrow B_S,\ A_S$ a cyclic right S-act, is an S-homomorphism, then $f(A)$ is a cyclic subact of B_S.
If f is an epimorphism and $A_S = aS$, then $B_S = f(a)S$.

Remark 2.10 *Let $I \neq \emptyset$ be a set and let $(X_i)_{i\in I}$ be a family of right S-acts. The* **coproduct** $\coprod\limits_{i\in I} X_i$ *in* $\mathbf{Act}-S$ *is the disjoint union* $\dot{\bigcup\limits_{i\in I}} X_i$ *with the injections* $u_i : X_i \longrightarrow \dot{\bigcup\limits_{i\in I}} X_i$ *defined by* $u_i = id_{\dot{\bigcup\limits_{i\in I}} X_i} |_{X_i}, (i \in I)$*, i.e., for every* $B_S \in \mathbf{Act}-S$ *and for every family* $(f_i)_{i\in I}$ *of homomorphisms* $f_i : (X_i)_S \longrightarrow B_S$ *the mapping* $f : \dot{\bigcup\limits_{i\in I}} (X_i)_S \longrightarrow B_S$ *with* $f(x) = f_i(x)$ *for* $x \in X_i$ *is the unique* S*-homomorphism, such that* $f\, u_i = f_i$ *for all* $i \in I$*.*
f *is called* **the coproduct induced morphism by** $(f_i)_{i\in I}$ **with respect to** $\dot{\bigcup\limits_{i\in I}} (X_i)_S$ *and is denoted by* $[(f_i)_{i\in I}]$*.*

Definition 2.11 *A right S-act is called* **decomposable***, if there exist two subacts B_S and C_S, such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. In this case, $A_S = B_S \cup C_S$ is called a* **decomposition** *of A_S. Otherwise A_S is called* **indecomposable**.

If A_S is an indecomposable right S-act and $f : A_S \longrightarrow B_S$ is an epimorphism, then B_S is indecomposable.

Theorem 2.12 *Every right S-act A_S has a unique decomposition into indecomposable subacts.*

Definition 2.13 *An element $z \in S$ is called* **left (right) zero of** S*, if $zt = z$ $(tz = z)$ for all $t \in S$ and* **zero** *of S, if z is a left and right zero.*
S is called **(left) (right) zero monoid***, if all elements of $S \setminus \{1\}$ are (left) (right) zeros of S.*

Definition 2.14 *An element s of S is called* **right (left) invertible***, if there exists $s' \in S$, such that $ss' = 1$ $(s's = 1)$. In this case s' is called a* **right (left) inverse** *of s.*

Definition 2.15 *An element c of a monoid S is called* **right (left) cancellable***, if $rc = tc$ $(cr = ct)$ for $r,\ t \in S$ implies $r = t$.*
The monoid S is called **right (left) cancellative***, if all its elements are right (left) cancellable.*

Definition 2.16 *An element e of a monoid S is called* **idempotent***, if $e^2 = e$. The set of all idempotents of S is denoted by $E(S)$. If $E(S) = S$, then S is called an idempotent monoid or a* **band**.

Definition 2.17 *Let* $S =< a >^1 := \{1, a, a^2, ...\}$ *be a* **monogenic** *semigroup with generating element a and identity adjoined. If there exist natural numbers* k, l *such that* $k \neq l$ *but* $a^k = a^l$, *then let* m *be the smallest natural number with* $a^m = a^{m+q}$ *for some* $q \in \mathbb{N}$. *Then* m *is called the* **index or height** *of* a.

Definition 2.18 *A monoid is called* **periodic**, *if all its monogenic submonoids are finite.*

Definition 2.19 *An element* s *of a monoid* S *is called* **(left) nilpotent**, *if there exists* $n \in \mathbb{N}$, *such that* $s^n = z \in S$, *where* z *is a (left) zero of* S.
A monoid is called **(left) nil**, *if all elements of* S *are (left) nilpotent.*

Definition 2.20 *A monoid* S *is called* **left (right) reversible**, *if any two right (left) ideals of* S *have a non-empty intersection. This is equivalent to the following condition: for all* $s,\ t \in S$ *there exist* $u,\ v \in S$ *such that* $su = tv$ $(us = vt)$.

Note, that this condition is sometimes called **left (right) Ore condition**.

Definition 2.21 *A monoid* S *is called* **left (right) collapsible**, *if for every* $s,\ t \in S$ *there exists* $z \in S$ *such that* $zs = zt$ $(sz = tz)$.

Definition 2.22 *Let* S *be a monoid and let* $s \in S$. *The smallest right (left) ideal of* S *containing* s *is* sS (Ss). *It is called the* **principal right (left) ideal generated by** s.
S *is called a* **principal right (left) ideal monoid**, *if all its right (left) ideals are principal right (left) ideals.*

Recall, that for monoids the Green's relations $\mathcal{R}$ and $\mathcal{J}$ are defined by

$$s\mathcal{R}\text{t if and only if sS = tS and}$$

$$s\mathcal{J}\text{t if and only if SsS = StS.}$$

Remark 2.23 *If* S *contains a left zero* z, *then for all* $s \in S$ *the Rees congruence* ϱ_{sS} *on* S, *given by the principal right ideal* sS, *is the principal congruence* $\varrho(sz, s)$.
Furthermore, even if in general Δ *is not a principal right congruence on* S, *if* S *contains a zero* z, *then* $\varrho(sz, z) = \varrho(z, z) = \Delta$ *is considered as a principal right congruence.*

Definition 2.24 *A monoid S is called* **(right) simple***, if S has no proper (right) ideals (i.e., no (right) ideals I with $I \neq S$.)*
A right ideal I of S is called **simple***, if I contains no proper right subideals.*

Definition 2.25 *A monoid S is called*

- *a* **right (left) Rees monoid***, if all right (left) congruences on S are Rees congruences,*
- *a* **right (left) principal Rees monoid***, if all right (left) principal congruences on S are Rees congruences.*

Definition 2.26 *([18], [10], [4]) A right ideal I of S is called* **left stabilizing***, if for every $i \in I$ there exists $k \in I$ with $ki = i$.*

Note that the condition of Definition 2.26 is called **Condition (LU)** in [14].

The categorical definition of free acts is not given here. It can be found for instance in [14]. The categorical definition of projective acts is included in the definition of $(\mathcal{X}, \mathcal{Y})$-projective acts given at the beginning of Chapter 3.

Result 2.27 A right S-act A_S is

- **free** , if it is isomorphic to a coproduct $\coprod_{i \in I} S$ of copies of S for some non-empty set I.
- ([15]) **projective**, if it is isomorphic to a coproduct $\coprod_{j \in J} e_j S$ of cyclic right S-acts, each of them generated by an idempotent $e_j \in E(S),\ j \in J \neq \emptyset$.

Proposition 2.28 *For every right S-act A_S there exists a free right S-act F_S, such that A_S is an epimorphic image of F_S.*

Proposition 2.29 *Every retract of a projective act is projective.*

Definition 2.30 *([13] and others) A right S-act G_S is called a* **generator** *in* **Act** $-S$ *if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$ (which in this case is a retraction with $\gamma : S_S \longrightarrow G_S$ such that $\pi\gamma = id_S$, i.e. $G_S \underset{\gamma}{\overset{\pi}{\rightleftarrows}} S_S$).*

Result 2.31 *([13],[17]) For every $A_S \in$* **Act** $-S$ *the coproduct $S_S \coprod A_S$ is a generator in* **Act** $-S$ *(with retraction $\pi : S_S \coprod A_S \longrightarrow S_S$ such that $\pi|_S = id_S$, $\pi(A_S) =: z$, and the natural injection $S_S \longrightarrow S_S \coprod A_S$ as coretraction) if and only if z is a left zero of S.*

For later use, we recall the definitions of flatness properties and torsion freeness. For the definition of pullbacks and of the tensor product in **Act** $-S$ see [14].

Definition 2.32 *A right S-act A_S is called*

- **pullback flat***, if the tensor functor $A_S \otimes_S -$ preserves pullbacks.*
- **flat***, if the tensor functor $A_S \otimes_S -$ preserves monomorphisms.*
- **weakly flat***, if the tensor functor $A_S \otimes_S -$ preserves all monomorphisms from left ideals of S into S.*
- **principally weakly flat***, if the tensor functor $A_S \otimes_S -$ preserves all monomorphisms from principal left ideals of S into S.*
- **torsionfree***, if $ac = a'c$ implies $a = a'$ for all $a,\ a' \in A,\ c \in S,\ c$ right cancellable.*

Furthermore, A_S **satisfies condition** (P)*, if for $a,\ a' \in A,\ s,\ s' \in S$ the equation $as = a's'$ implies the existence of $a'' \in A,\ u,\ v \in S$, such that $a = a''u,\ a' = a''v$ and $us = vs'$.*

Now we recall some results concerning properties of Rees factor acts and homological classification, which will be used in Chapter 6.

Proposition 2.33 *Let I be a right ideal of S. The Rees factor act $(S/I)_S$ is*

(a) free iff $|I| = 1$ ([14]).

(b) projective iff $|I| = 1$ or $I = S$ and S has a left zero ([14]).

(c) pullback flat iff $|I| = 1$ or $I = S$ and S is left collapsible ([14]).

(d) satisfies condition (P) iff $|I| = 1$ or $I = S$ and S is right reversible ([18],[10]).

(e) (weakly) flat iff S is right reversible and I is left stabilizing ([18],[10]).

(f) principally weakly flat iff I is left stabilizing ([18],[10]).

(g) torsionfree iff $sc \in I$ implies $s \in I$ for every $s,\ c \in S,\ c$ right cancellable ([14]).

Corollary 2.34 *[14] The one element right S-act Θ_S is*

(a) free iff $S = \{1\}$.

(b) projective iff S has a left zero.

(c) pullback flat iff S is left collapsible.

(d) satisfies condition (P) iff S is right reversible.

(e) (weakly) flat iff S is right reversible.

(f) principally weakly flat.

(g) torsionfree.

In [14] the assertion (b) of Proposition 2.33 is not explicitly given in this form, but is given by 2.33 (b) for proper right ideals of S and 2.34 (b).
The same is valid for condition 2.33 (d).
Conditions (c) of Proposition 2.33 and Corollary 2.34 are proved in [14] by using that for cyclic right S-acts pullback flatness is equivalent to a so-called condition (E).

Result 2.35 *All right S-acts*

(a) are free iff $S = \{1\}$ ([20]).

(b) are projective iff $S = \{1\}$ ([12]).

(c) are pullback flat iff $S = \{1\}$ ([12]).

(d) satisfy condition (P) iff S is a group ([19]).

(e) are (weakly) flat iff S is regular and satisfies condition (R), i.e., for all $s,\ t \in S\ \exists w \in Ss \cap St$ such that $w\,\varrho(s,t)\,s$ ([14]).

(f) are principally weakly flat iff S is regular.

(g) are torsionfree iff every right cancellable element of S is right invertible ([16]).

Note that these results can also be found in [14].

3 $(\mathcal{X},\mathcal{Y})$-projectivities of acts

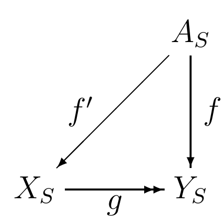
In this Chapter we come back to the question of the introduction: What happens, if we restrict in the definition of projectivity the class of epimorphisms? The answer is introduced at first, namely the concepts of $(\mathcal{X},\mathcal{Y})$-projectivity (Section 3.1).
During this Chapter equivalences between these properties and to projectivity itself are proved (see for instance Proposition 3.31) as well as implications between the remaining concepts.

The results with respect to equivalences are summarized in Section 3.6, Table 1, which leads to the restricted table (Table 2). Table 2 shows the $(\mathcal{X},\mathcal{Y})$-projectivities remaining for further investigations.
The implications between these properties are illustrated in the scheme at the end of this chapter (Section 3.8).

Moreover, subsection 3.3.3 and 3.3.4 contain results relative to $(\mathcal{X},\mathcal{Y})$-projectivity of coproducts of acts, which will be used in Chapter 5.

3.1 Notation and basic informations

Let S be a monoid, and let $\mathcal{X}$, $\mathcal{Y}$ denote classes of right S-acts.
A right S-act A_S is called **$(\mathcal{X},\mathcal{Y})$-projective**, if for every $X_S \in \mathcal{X}$, $\mathrm{Y_S} \in \mathcal{Y}$ every S-homomorphism $f : A_S \longrightarrow Y_S$ can be lifted in **Act** $-S$ with respect to every epimorphism $g : X_S \longrightarrow Y_S$, i.e., there exists an S-homomorphism f' such that the following diagram is commutative:



A right S-act A_S is called **trivially-$(\mathcal{X},\mathcal{Y})$-projective**, if there does not exist an epimorphism from X_S onto Y_S or if there does not exist a homomorphism from A_S to Y_S, i.e., if the requirement above is empty.

A right S-act A_S is called **non-trivially-$(\mathcal{X},\mathcal{Y})$-projective**, if A_S is $(\mathcal{X},\mathcal{Y})$-projective and not trivially-$(\mathcal{X},\mathcal{Y})$-projective.

In the following we will study $(\mathcal{X}, \mathcal{Y})$-projectivity for different classes $\mathcal{X}$ and $\mathcal{Y}$.

The classes $\mathcal{X}$, $\mathcal{Y}$ considered here are the classes of

- all right S-acts,
- free right S-acts $F_S \cong \coprod_{i \in I} S_S$,
- projective right S-acts $P_S \cong \coprod_{j \in J} (e_j S)_S$ for $e_j^2 = e_j \in S$,
- generators G_S, i.e., right S-acts G_S such that there exists an epimorphism $\pi : G_S \longrightarrow S_S$ (which in this case is a retraction with $\gamma : S_S \longrightarrow G_S$ such that $\pi\gamma = id_S$, i.e., $G_S \overset{\pi}{\underset{\gamma}{\rightleftarrows}} S_S$),
- factor acts $(S/\varrho)_S$ of S by
  - (arbitrary) right congruences ϱ,
  - right principal congruences, i.e., $\varrho = \varrho(x, y)$ for some $x,\ y \in S$,
  - right Rees congruences ϱ_I for right ideals $I \subseteq S$,
  - right principal Rees congruences ϱ_{sS} for $s \in S$,
- coproducts $\coprod_{m \in M} (S/\varrho_m)_S$ of factor acts of S by right congruences on S, where different sets of congruences will be considered,
- and finally A_S itself,

where the index sets I, J and M are supposed to be non-empty sets.

In the following, I, J and M denote non-empty sets. This will not be mentioned explicitly in every situation.

Instead of $(\mathcal{X}, \mathcal{Y})$-projectivity of special classes of acts, the abbreviation **(X, Y)-projectivity** with $X \in \mathcal{X}$, $\mathrm{Y} \in \mathcal{Y}$, will be used.
In a concrete situation it will be clear by the structure of X_S and Y_S used in this abbreviation, which classes $\mathcal{X}$ and $\mathcal{Y}$ are considered, respectively.
The acts G_S and H_S will represent the class of generators in $\mathbf{Act}-S$, B_S and C_S the class of all acts.
In this context we sometimes call X_S **first** and Y_S **second component** of (X, Y)-projectivity.

Whenever an assertion is made for an arbitrary class $\mathcal{X}$ of acts, we simply write $X_S \in \mathbf{Act}-S$, i.e., the class $\mathcal{X}$ is not specified. (Analogously: $Y_S \in \mathbf{Act}-S$.)

Some of the properties have already been considered but named differently. These are:

Remark 3.1 *Let S be a monoid, let $A_S,\ B_S,\ C_S \in$ **Act** $-S$, let $s,\ x,\ y \in S$, let I denote a right ideal of S and let ϱ be a right congruence on S. Then*

- *(B,C)-projectivity is known as projectivity (cf. [14] et al.),*
- *$(S,S/\varrho)$-projectivity is called is weak projectivity (wp) in [17],*
- *$(S,S/\varrho(x,y))$-projectivity is called principal weak projectivity ((pwp) in [17],*
- *$(S,S/I)$-projectivity is called Rees weak projectivity (Rwp) in [17],*
- *$(S,S/sS)$-projectivity is called principal Rees weak projectivity (pRwp) in [17] and*
- *(A,C)-projectivity is called quasi projectivity (qp) in [1].*

We will use these terms simultaneously to the respective (X,Y)-projectivity. The notation x-**congruence** with $x \in \{\phi,p,R,pR\}$ will be used in the following for right, right principal, right Rees or right principal Rees congruences, respectively, where ϕ denotes the empty symbol.

If we consider coproducts $\coprod\limits_{i\in I} X_S$ of copies of the same act X_S we denote the i-th copy of X_S in this coproduct by X_S^i and the i-th natural injection by u_i.

Replacing $\mathcal{X}$ and $\mathcal{Y}$ in $(\mathcal{X},\mathcal{Y})$-projectivity by the special acts mentioned before leads to 226 formally different combinations: There are 15 different first/second components X_S and Y_S, which leads to 225 pairs (X,Y). Furthermore, for arbitrary acts B_S we obtain (B,B)-projectivity.

Our first aim is to exclude those $(\mathcal{X},\mathcal{Y})$-projectivities, which are trivially-$(\mathcal{X},\mathcal{Y})$-projectivities, especially this means that we suppose the existence of an epimorphism $g : X_S \longrightarrow Y_S$ for some $X_S \in \mathcal{X},\ \mathrm{Y_S} \in \mathcal{Y}$.
Then the amount of combinations is reduced to 196 by the following observations:

- If the first component is an indecomposable act then the second component is indecomposable, too, for it is the epimorphic image of the first act.

- If the first component is S itself, the second component has to be isomorphic to a factor act of S, since it is an epimorphic image of S. Moreover, epimorphic images of factor acts of S are isomorphic to factor acts of S itself.

As a consequence of this observations we get for instance, that (S, C)-projectivity has not to be considered anymore, since it is equivalent to $(S, S/\varrho)$-projectivity. Equivalences of this kind cause the reduction to the amount of 196 pairs (X, Y). During the next sections further equivalences will be proved.

3.2 (X,Y)-projectivities with projective second component

In this section it will be shown, that some of the projectivities defined before, are properties of **all** acts $A_S \in \mathbf{Act}-S$, and are in this sense trivial.

Proposition 3.2 *Let $B_S \in \mathbf{Act}-S$ and let $e_j \in E(S)$, $j \in J$. Then every right S-act A_S is $(B, \coprod_{j\in J} e_jS)$-projective.*

Proof. Since $\coprod_{j\in J} e_jS$ is projective, every epimorphism π from an act B_S onto $\coprod_{j\in J}(e_jS)_S$ is a retraction, i.e., there exists a homomorphism γ such that $B_S \overset{\pi}{\underset{\gamma}{\rightleftarrows}} \coprod_{j\in J}(e_jS)_S$, where $\pi\gamma = id_{\coprod_{j\in J} e_jS}$.
Let now A_S be an act.
If $f : A_S \longrightarrow \coprod_{j\in J}(e_jS)_S$ is an homomorphism and if $\pi : B_S \longrightarrow \coprod_{j\in J}(e_jS)_S$ is an epimorphism, then for $\gamma f : A_S \longrightarrow B_S$ the equality $\pi(\gamma f) = f$ holds. Thus A_S is $(B, \coprod_{j\in J} e_jS)$-projective.

□

As a consequence, all $(\mathcal{X}, \mathcal{Y})$-projectivities with projective second component will be left out of consideration in what follows.

3.3 Equivalences of conditions

During this section equivalences between (formally) different (X, Y)-projectivities will be presented. In particular, equivalences to the concepts of weak projectivities (see Remark 3.1) will be given.

As preparation of some proofs at first the following general results are given. Note that general results concerning (X,Y)-projectivity of coproducts and products will be given right before using them.

Lemma 3.3 *Let* $\mathcal{X}$, $\mathcal{Y}$ denote classes of right S-acts. If for all $Y_S \in \mathcal{Y}$ there exists an $X'_S \in \mathcal{X}$ and an epimorphism $\beta : X'_S \longrightarrow Y_S$, then $(\mathcal{X},\mathcal{Z})$-projectivity implies $(\mathcal{Y},\mathcal{Z})$-projectivity for all classes $\mathcal{Z}$ of right S-acts.

Proof. Let $X \in \mathcal{X}$, Y $\in \mathcal{Y}$, Z $\in \mathcal{Z}$, let A_S be an (X,Z)-projective act, let $\alpha : Y \longrightarrow Z$ be an epimorphism and let $f : A \longrightarrow Z$ be an homomorphism. By assumption, there exists $X'_S \in \mathcal{X}$ and an epimorphism $\beta : X' \longrightarrow Y$. Since A_S is $(\mathcal{X},\mathcal{Z})$-projective, there exists $f' : A \longrightarrow X'$ with $(\alpha\beta)f' = f$ and thus $\beta f' : A \longrightarrow Y$ is an homomorphism with $\alpha(\beta f') = f$, i.e., A_S is $(\mathcal{Y},\mathcal{Z})$-projective.

□

Furthermore, since every act is the epimorphic image of a free act (Proposition 2.28), we obtain by Lemma 3.3:

Corollary 3.4 *Let $X_S,\ Y_S \in$ **Act** $-S$. Then $(\coprod_{i\in I} S, Y)$-projectivity implies (X,Y)-projectivity.*

□

Since coproducts of cyclic acts appear as first component, it is useful to get some more information about epimorphisms from these coproducts onto S-acts Y_S. The results will be used for example to prove Proposition 3.12.

Lemma 3.5 *Let $A_S = \coprod_{j\in J} a_jS$ be a coproduct of cyclic acts. If $g : \coprod_{i\in I} X^i_S \longrightarrow A_S$ is an epimorphism, then for every $j \in J$ there exists an $i \in I$ such that $g\mid_{X^i} = gu_i$ is an epimorphism onto a_jS.*

Proof. Let $j \in J$ and $x \in g^{-1}(a_j),\ a_j \in A_j$. Then $x \in X^i$ for some $i \in I$, and every $\tilde{a} \in a_jS$ is the image of an element xs of X^i because there exists $s \in S$, such that the equality $\tilde{a} = a_js = g(x)s = g(xs)$ holds. Therefore $g|_{X^i}$ is an epimorphism from X^i onto a_jS.

□

Corollary 3.6 *Let $X_S,\ A_S = aS \in$ **Act** $-S$. If $g : \coprod_{i\in I} X^i_S \longrightarrow A_S$ is an epimorphism, then there exists an $i \in I$ such that $g\mid_{X^i} = gu_i$ is an epimorphism onto A_S.*

In Proposition 2.29 we saw that retracts of projective acts are projective. The analogue is true for (X, Y)-projectivity:

Lemma 3.7 *Let $\mathcal{X}$, $\mathcal{Y}$ denote classes of right S-acts. Then retracts of $(\mathcal{X}, \mathcal{Y})$-projective right S-acts are $(\mathcal{X}, \mathcal{Y})$-projective.*

Proof. Let $X \in \mathcal{X}$, Y $\in \mathcal{Y}$, let A_S, $\bar{A}_S \in \mathbf{Act}-S$ let $\pi : A \longrightarrow \bar{A}$ be a retraction with coretraction γ and let A_S be $(\mathcal{X}, \mathcal{Y})$-projective. Let $g : X \longrightarrow Y$ be an epimorphism and let $f : \bar{A} \longrightarrow Y$ be a homomorphism. Then $f\pi : A \longrightarrow X$ is a homomorphism, which by assumption can be lifted with respect to g, i.e., there exists $(f\pi)' : A \longrightarrow X$ with $g(f\pi)' = f\pi$. Thus for $(f\pi)'\gamma : \bar{A} \longrightarrow X$ the equation $g(f\pi)'\gamma = f\pi\gamma = f$ holds, i.e., $\bar{A}_S$ is (X, Y)-projective.

□

After this general results with respect to $(\mathcal{X}, \mathcal{Y})$-projectivity of acts, in the following we consider the situation for special first/second components.

3.3.1 Cyclic acts and coproducts of cyclic acts as second component

The following implication will be used to prove further equivalences:

Lemma 3.8 *Let $x \in \{\phi, p, R, pR\}$, let ϱ be a right x-congruence on S and let $X_S \in \mathbf{Act}-S$. If A_S is $(S, S/\varrho)$-projective, then A_S is $(X, S/\varrho)$-projective.*

Proof. Let $g : X \longrightarrow S/\varrho$ be an epimorphism and let $\pi_\varrho : S \longrightarrow S/\varrho$ be the canonical epimorphism. Since S_S is projective, there exists an homomorphism π'_ϱ with $g\pi'_\varrho = \pi_\varrho$. Since A_S is $(S, S/\varrho)$-projective, every homomorphism $f : A_S \longrightarrow S/\varrho$ can be lifted with respect to π_ϱ, i.e., there exists f' with $\pi_\varrho f' = f$. This yields $g(\pi'_\varrho f') = \pi_\varrho f' = f$, and thus f can be lifted relative to g.

□

Corollary 3.9 *Let $x \in \{\phi, p, R, pR\}$ and let ϱ be a right x-congruence on S. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent*

(i) A_S *is* $(\coprod_{i \in I} S, S/\varrho)$*-projective,*

(ii) A_S *is* $(S, S/\varrho)$*-projective.*

Proof. $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (i)$ is Lemma 3.8.

□

The analogous equivalence is true even for coproducts of arbitrary acts as first component, but surely this can not be directly derived by Lemma 3.8:

Proposition 3.10 *Let $x \in \{\phi, p, R, pR\}$, let ϱ be a right x-congruence on S, let $\mathcal{X}$ be a class of right S-acts and let $X_S,\ X_S^i \in \mathcal{X},\ \mathrm{i} \in \mathrm{I}$.*
Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent:

(i) A_S is $(\coprod_{i\in I} X^i, S/\varrho)$-projective,

(ii) A_S is $(X, S/\varrho)$-projective.

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$. By Corollary 3.6 for every epimorphism $g : \coprod_{i\in I} X^i \longrightarrow S/\varrho$ there exists an $i \in I$ such that $g\mid_{X^i} = gu_i$ is an epimorphism. By (ii) for $f : A \longrightarrow S/\varrho$ there exists an homomorphism $f' : A_S \longrightarrow X^i$ with $gu_if' = f$ and thus f can be lifted with respect to g, i.e., (i) is valid.

□

Remark 3.11 *In the following, $(\coprod_{i\in I} X^i, S/\varrho)$-projectivity will be considered only by using the equivalent property of $(X, S/\varrho)$-projectivity.*

For cyclic acts we obtain another chain of equivalent conditions:

Proposition 3.12 *Let $x \in \{\phi, p, R, pR\}$, let $\varrho,\ \varrho_m,\ m \in M,\ \sigma$ be right x-congruences on S, let $\mathcal{X}$ be a class of right S-acts and let $X_S,\ X_S^i \in \mathcal{X}, \mathrm{i} \in \mathrm{I}$.*
Then for a cyclic act S/σ the following assertions are equivalent:

(i) S/σ is $(\coprod_{i\in I} X^i, \coprod_{m\in M} S/\varrho_m)$-projective,

(ii) S/σ is $(\coprod_{i\in I} X^i, S/\varrho)$-projective,

(iii) S/σ is $(X, S/\varrho)$-projective.

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Leftrightarrow (iii)$ is Proposition 3.10.
$(iii) \Longrightarrow (i)$. Let $f : S/\sigma \longrightarrow \coprod_{m\in M} S/\varrho_m$ be a homomorphism and $g : \coprod_{i\in I} X^i \longrightarrow$

$\coprod_{m\in M} S/\varrho_m$ an epimorphism. Since S/σ is cyclic, there exists a $k \in M$ with $f(S/\sigma) \subseteq S/\varrho_k$. By Lemma 3.5, for S/ϱ_k there exists a $j \in I$, such that $g\,v_j$ is an epimorphism from X^j onto S/ϱ_k, where $v_j : X^j \longrightarrow \coprod_{i\in I} X^i$ denotes the j-th canonical injection. By (iii) there exists $f' : S/\sigma \longrightarrow X^j$, such that $g\,v_j\,f' = f$. Thus f can be lifted with respect to g by using $v_j\,f'$.

□

3.3.2 Cyclic acts and coproducts of cyclic acts as first component

Proposition 3.13 *Let ϱ_m, $m \in M$, be right congruences on S, let $e_j \in E(S)$, $j \in J$, and let $Y_S \in$ **Act** $-S$. Then for every $A_S \in$ **Act** $-S$ the following assertions are equivalent:*

(i) A_S *is* $(\coprod_{i\in I} S, Y)$*-projective,*

(ii) A_S *is* $(\coprod_{m\in M} S/\varrho_m, Y)$*-projective,*

(iii) A_S *is* $(\coprod_{j\in J} e_jS, Y)$*-projective.*

Proof. $(i) \Longrightarrow (ii)$ is valid by Corollary 3.4.
$(ii) \Longrightarrow (iii)$. Every cyclic act e_jS is isomorphic to a factor act S/ϱ_j of S by a right congruence (Proposition 2.9). Thus (ii) yields (iii).
$(iii) \Longrightarrow (i)$ since $1 \in E(S)$.

□

Corollary 3.14 *Let ϱ be a right congruence on S, let $e \in E(S)$ and let $Y_S \in$* **Act** $-S$*. Then for $A_S \in$* **Act** $-S$ *the following assertions are equivalent:*

(i) A_S *is* (S, Y)*-projective,*

(ii) A_S *is* $(S/\varrho, Y)$*-projective,*

(iii) A_S *is* (eS, Y)*-projective.*

□

Remark 3.15 *The equivalences presented in Proposition 3.13 use* $1S = S = S/\Delta$.
If S *does not contain a zero, then* $\varrho(x,y) \neq \Delta$ *for all* $x,\ y \in S$. *Thus for right principal congruences* $(\coprod_{n\in N} S/\varrho(x_n,y_n), Y)$*-projectivity in general does not imply* $(\coprod_{i\in I} S, Y)$*-projectivity, i.e., for coproducts of monocyclic acts as first component, one can not expect equivalences like in Proposition 3.13.*
If S *is a monoid without left zero, then* $S/I \ncong S$ *for every right ideal* I *of* S *and thus one also can not expect equivalences like those of Proposition 3.13 for coproducts of Rees factor acts as first component.*

Since $(S, S/\sigma)$-projectivities are known as weak projectivities, we summarize the results given in Proposition 3.13 and Corollary 3.9 for $Y = S/\sigma$ in the next Corollary.

Corollary 3.16 *Let* $x \in \{\phi, p, R, pR\}$, *let* $\varrho,\ \varrho_m,\ m \in M$, *be right congruences on* S, *let* σ *be a right* x*-congruence on* S *and let let* $e,\ e_j \in E(S),\ j \in J$. *Then the following assertions are equivalent*

(i) A_S *is* $(\coprod_{m\in M} S/\varrho_m, S/\sigma)$*-projective,*

(ii) A_S *is* $(S/\varrho, S/\sigma)$*-projective,*

(iii) A_S *is* $(\coprod_{i\in I} S, S/\sigma)$*-projective,*

(iii) A_S *is* $(S, S/\sigma)$*-projective,*

(iv) A_S *is* $(\coprod_{j\in J} e_j S, S/\sigma)$*-projective,*

(v) A_S *is* $(eS, S/\sigma)$*-projective.*

□

3.3.3 Coproducts of $(\mathcal{X},\mathcal{Y})$-projective acts

Since coproducts of acts appear as components in some (X,Y)-projectivities it is useful to study properties of coproducts of acts. This will be done in this subsection. The importance of this investigation will become clear in Lemma 5.18. For the respective properties of products see the next subsection.

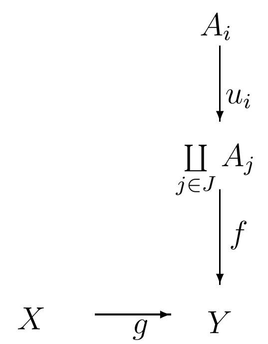
Proposition 3.17 *Let $X,\ Y \in \mathbf{Act}-S$ and let $A_j,\ j \in J$, be indecomposable right S-acts. If*

$$A_j \text{ is } (X,Y) - \text{projective}$$

for all $j \in J$, then

$$\coprod_{j\in J} A_j \text{ is } (X,Y) - \text{projective.}$$

Proof. Let $A_i \in \mathbf{Act}-S,\ i \in J$, let $u_i : A_i \longrightarrow \coprod_{j\in J} A_j$ denote the natural injections, and let A_i be (X,Y)-projective for all $i \in J$. Consider for every $i \in J$ the following diagram:



Then $f u_i$ is a homomorphism from A_i into Y, which can be lifted by assumption, i.e, there exists $(f\,u_i)'$ with $g\,(f\,u_i)' = f\,u_i$.
By the universal property of the coproduct f can be lifted with respect to g by the coproduct induced homomorphism $[((f\,u_i)')_{i\in J}]$, i.e. $\coprod_{j\in J} A_j$ is (X,Y)-projective.

□

The next example shows, that in general, the converse is not true. But in the case of monoids with left zero we finally get Proposition 3.19.

Example 3.18 [17] Let S be the three element right zero semigroup $\{a,b,c\}$ with identity adjoined. Then the coproduct $S/aS \coprod S/\varrho(a,b)$ is $(S, S/\varrho(x,y))$-projective but $(S/\varrho(a,b))_S$ is not.

Proposition 3.19 *Let S be a monoid with left zero, let $X,\ Y \in \mathbf{Act}-S$ and let $A_j,\ j \in J$, be indecomposable right S-acts. Then*

$$A_j \text{ is } (X,Y) - \text{projective}$$

for all $j \in J$, iff

$$\coprod_{j\in J} A_j \text{ is } (X,Y) - \text{projective.}$$

Proof. Necessity has been proved in Proposition 3.17.
Sufficiency. Since S contains a left zero, every act A_j has at least one zero z_j. Then for every $k \in J$ the mapping $p_k : \coprod_{j\in J} A_j \longrightarrow A_k$ with $p_k(a) = a,\ a \in A_k$, and $p_k(a) = z_k,\ a \in A_j,\ j \neq k$, is an epimorphism with $p_k u_k = id_{A_k}$, i.e. , every A_j is a retract of $\coprod_{j\in J} A_j$. Since $\coprod_{j\in J} A_j$ is $(\mathcal{X},\mathcal{Y})$-projective, by Lemma 3.7 every A_j is $(\mathcal{X},\mathcal{Y})$-projective.

□

If $\mathcal{Y}$ is the class of coproducts of Rees factor acts, we obtain the analogous equivalence in the case of arbitrary monoids because of the existence of zeros in Rees factor acts.

Lemma 3.20 *Let $I_k,\ k \in K$, be right ideals of S, let $X_S \in$ **Act** $-S$ and let $A_j,\ j \in J$, be indecomposable right S-acts. Then*

$$\coprod_{j\in J} A_j \textit{ is } (X, \coprod_{k\in K} S/I_k) - \textit{projective}$$

iff

$$A_j \textit{ is } (X, \coprod_{k\in K} S/I_k) - \textit{projective}$$

for all $j \in J$.

Proof. Sufficiency is included in Proposition 3.17.
Necessity. Let $\coprod_{j\in J} A_j$ be $(X, \coprod_{k\in K} S/I_k)$-projective and let $f : A_j \longrightarrow \coprod_{k\in K} S/I_k$ be an homomorphism. Since A_j is an indecomposable act there exists $k \in K$ such that $f(A_j) \subseteq S/I_k$.
Let $\widehat{0_k}$ denote the zero in S/I_k. Define a homomorphism $h : \coprod_{j\in J} A_j \longrightarrow \coprod_{k\in K} S/I_k$ by $h(a) = f(a)$ if $a \in A_j$ and $h(a) = \widehat{0_k}$ otherwise. Then $h\, u_j = f$.
By assumption for every epimorphism $g : X \longrightarrow \coprod_{k\in K} S/I_k$ there exists h' with $gh' = h$. Therefore there exists a homomorphism $h'u_j : A_j \longrightarrow X$ with $gh'u_j = hu_j = f$, i.e., A_j is $(X, \coprod_{k\in K} S/I_k)$-projective.

□

3.3.4 Products of $(\mathcal{X}, \mathcal{Y})$-projective acts

In this subsection we shortly consider products of (X, Y)-projective acts. Recall that the product $\prod_{j\in J} A_j$ of acts is the cartesian product of the acts A_j with the natural projections and componentwise multiplication.

Products are the categorically dual construction to coproducts and are therefore in some sense related to the categorically dual concepts to projectivities, namely to injectivities. Indeed, for injective acts the analogue assertion to Proposition 3.17 is true, i.e., products of injective acts are injective [14], which can be proved by using the universell property of the product.
Since this property yields morphisms **into** the product we can not expect the product of $(\mathcal{X}, \mathcal{Y})$-projective acts to be $(\mathcal{X}, \mathcal{Y})$-projective. But in the case of monoids with left zero we obtain:

Proposition 3.21 *Let S be a monoid with left zero, let $X,\ Y \in \textbf{Act}-S$ and let $A_j,\ j \in J$, be indecomposable right S-acts. If*

$$\prod_{j\in J} A_j \text{ is } (X, Y) - \text{projective},$$

then

$$A_j \text{ is } (X, Y) - \text{projective}$$

for all $j \in J$.

Proof. Let $A_j \in \textbf{Act}-S,\ j \in J$, and let $p_j : \prod_{j\in J} A_j \longrightarrow A_j$ denote the natural projections.
Since S contains a left zero, every act A_j has at least one zero. Let z_j denote a zero in A_j. Then for $k \in J$ the mapping $i_k : A_k \longrightarrow \prod_{j\in J} A_j$ with $i_k(a) = (x_j)_{j\in J}$ with $x_k = a$ and $x_j = z_j,\ j \neq k$, is a homomorphism with $p_k i_k = id_{A_k}$, i.e., every A_k is a retract of the product $\prod_{j\in J} A_j$.
Thus by Lemma 3.7 we obtain, that (X, Y)-projectivity of the product $\prod_{j\in J} A_j$ implies (X, Y)-projectivity of $A_j,\ j \in J$.

□

Even if S contains a left zero, the converse is not true:

Example 3.22 Consider the monoid $(\mathbb{Z}_2, \cdot)$. Then $\mathbb{Z}_{2_{\mathbb{Z}_2}}$ is projective in $\textbf{Act}-\mathbb{Z}_2$. The product $\mathbb{Z}_2 \prod \mathbb{Z}_2$ is a four-element, indecomposable right S-act and is therefore not isomorphic to a coproduct of cyclic acts. Thus $\mathbb{Z}_2 \prod \mathbb{Z}_2$ is not projective.

Remark 3.23 *From the proof of Proposition 3.21 it became clear, that for coproducts of right S-acts $(A_j)_S$, each of them containing a zero, the implication holds even if S itself has no (left) zero (for instance, if every A_j is a Rees factor act of S).*
Since products of right S-acts do not occur as first or second component of the projectivities considered her, we do not study the analogue situations to those in Lemma 3.20.

3.3.5 Rees factor acts and coproducts of Rees factor acts as second component

Recall that by Proposition 3.10 we do not have to distinguish between $(\coprod_{i\in I} X^i, S/\varrho)$-projectivity and $(X, S/\varrho)$-projectivity. In the case of coproducts of Rees factor acts as second component and arbitrary first ones we obtain an in some sense analoguous assertion :

Proposition 3.24 *Let $I,\ I_k,\ k \in K$, be right ideals of S and let $X \in \mathbf{Act}-S$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent:*

- *(i) A_S is $(X, \coprod_{k\in K} S/I_k)$-projective,*
- *(ii) A_S is $(X, S/I)$-projective.*

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$ Consider in $\mathbf{Act}-S$ the following diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ X & \xrightarrow[g]{} & \coprod_{k\in K} S/I_k \end{array}$$

Then $X = \coprod_{k\in K} g^{-1}(S/I_k)$ and $A = \coprod_{k\in K} f^{-1}(S/I_k)$.
Now consider for every $k \in K$ the diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow \tilde{f}_k \\ X & \xrightarrow[\tilde{g}_k]{} & S/I_k \end{array}$$

with $\tilde{f}_k(a) = f(a)$ if $a \in f^{-1}(S/I_k)$, and $\tilde{f}_k(a) = \widehat{0_k}$ otherwise, where $\widehat{0_k}$ denotes the zero in S/I_k, and $\tilde{g}_k(x) = g(x)$, if $x \in g^{-1}(S/I_k)$, $\tilde{g}_k(x) = \widehat{0_k}$ otherwise.
Since by (ii) A_S is $(X, S/I)$-projective, for every $k \in K$ there exist a homomorphism $\tilde{f}'_k : A \longrightarrow X$, such that $\tilde{f}_k = \tilde{g}_k \tilde{f}'_k$. This implies that $\tilde{f}'_k(f^{-1}(S/I_k)) \subseteq g^{-1}(S/I_k)$. Therefore $f' : A \longrightarrow X$ defined by $f'(a) = \tilde{f}'_k(a),\ a \in f^{-1}(S/I_k)$, is a well defined homomorphism with $gf' = f$. Hence, (i) is valid.

□

Corollary 3.25 *Let $sS,\ s_lS \subseteq S,\ l \in L$, be principal right ideals of S, and let $X_S \in \mathbf{Act}-S$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* $(X, \coprod_{l \in L} S/s_lS)$*-projective,*

(ii) A_S *is* $(X, S/sS)$*-projective.*

□

Noticing, that the zero in Rees factor acts is an essential part of the proof of Proposition 3.24, we get the same equivalence for arbitrary factor acts of S if we demand S to be a monoid with left zero:

Lemma 3.26 *Let S be a monoid with left zero, let $\varrho,\ \varrho_m,\ m \in M$, be right (principal) congruences on S and let $X_S \in \mathbf{Act}-S$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* $(X, \coprod_{m \in M} S/\varrho_m)$*-projective,*

(ii) A_S *is* $(X, S/\varrho)$*-projective.*

Proof. Analogously to the proof of Proposition 3.24. □

Note, that in Chapter 5 monoids fulfilling condition (Moz) will be treated and this condition will be proved to be weaker than the existence of a left zero. By taking a look to the proof of Lemma 3.24, it becomes clear, that this condition is sufficient for the equivalence presented in Lemma 3.26.

3.3.6 Rees factor acts and coproducts of Rees factor acts as first component

In the previous subsection we saw, that we do not have to distinguish between Rees factor acts and coproduct of Rees factor acts as second component. If these acts are considered as first component in (X, Y)-projectivity, we do not get an analogue assertion, but in this case we obtain:

Proposition 3.27 *Let $I,\ I_k,\ k\in K$, be right ideals of S, let $s,\ s_l\in S,\ l\in L$, and let $Y_S\in\mathbf{Act}-S$. Then for $A_S\in\mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* $(\coprod_{k\in K} S/I_k, Y)$*-projective,*

(ii) A_S *is* $(\coprod_{l\in L} S/s_lS, Y)$*-projective.*

Proof. $(i)\Longrightarrow(ii)$ is obvious.
$(ii)\Longrightarrow(i)$ For every I_k let $s_k\in I_k$. Then $s_kS\subseteq I_k$. Thus there exists an epimorphism $\pi:\coprod_{k\in K} S/s_kS\longrightarrow\coprod_{k\in K} S/I_k$ and Lemma 3.3 completes the proof.
□

Corollary 3.28 *Let I be a right ideal of S, let $s\in S$, and let $Y_S\in\mathbf{Act}-S$. Then for $A_S\in\mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* $(S/I,Y)$*-projective,*

(ii) A_S *is* $(S/sS,Y)$*-projective.*

□

Thus by Proposition 3.27 and Corollary 3.28 $(S/I,Y)$- and $(\coprod_{k\in K} S/I_k,Y)$-projectivities can be left out of consideration in what follows.

3.3.7 Various first/second components

In this subsection we consider $(\mathcal{X},\mathcal{Y})$-projectivities for various classes $\mathcal{X}$ and $\mathcal{Y}$, and equivalences between them. Note that, for later use, we include even those equivalences in the following results, which have already been given before.

Lemma 3.29 *Let $\varrho_m,\ m\in M$, be right congruences on S, let G_S be a generator in $\mathbf{Act}-S$, let $e_j\in E(S),\ j\in J$, and let $B_S,\ Y_S\in\mathbf{Act}-S$. Then for $A_S\in\mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* (B,Y)*-projective,*

(ii) A_S *is* (G,Y)*-projective,*

(iii) A_S *is* $(\coprod_{i\in I} S,Y)$*-projective,*

(iv) A_S *is* $(\coprod_{m\in M} S/\varrho_m, Y)$*-projective*

(v) A_S *is* $(\coprod_{j\in J} e_jS, Y)$*-projective.*

Proof. The equivalences $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ are parts of Proposition 3.13.
The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are obvious.
The implication $(iii) \Longrightarrow (i)$, valid by Corollary 3.4, completes the proof.

□

Especially for weak projectivity like defined in [17] Corollary 3.9 together with Lemma 3.29 yields:

Corollary 3.30 *Let* $x \in \{\phi, p, R, pR\}$*, let* ϱ *be a right* x*-congruence on* S*, and let* G_S *be a generator in* $\mathbf{Act}-S$*. Then for* $A_S \in \mathbf{Act}-S$ *the following assertions are equivalent:*

(i) A_S *is* $(G, S/\varrho)$*-projective,*

(ii) A_S *is* $(S, S/\varrho)$*-projective.*

□

In the next result the most surprising equivalence is $(xi) \Leftrightarrow (vi)$:

Proposition 3.31 *Let* $\varrho_m,\ m \in M$*, be right congruences on* S*, let* $B_S,\ C_S \in \mathbf{Act}-S$*, let* G_S *denote a generator in* $\mathbf{Act}-S$ *and let* $e_j \in E(S),\ j \in J$*. Then for* $A_S \in \mathbf{Act}-S$ *the following assertions are equivalent:*

(i) A_S *is* (B, A)*-projective,*

(ii) A_S *is* (G, A)*-projective,*

(iii) A_S *is* $(\coprod_{i\in I} S, A)$*-projective,*

(iv) A_S *is* $(\coprod_{m\in M} S/\varrho_m, A)$*-projective,*

(v) A_S *is* $(\coprod_{j\in J} e_jS, A)$*-projective,*

(vi) A_S *is projective, i.e.,* A_S *is* (B, C)*-projective,*

(vii) A_S *is* (G, C)*-projective,*

(viii) A_S *is* $(\coprod_{i\in I} S, C)$*-projective,*

(ix) A_S *is* $(\coprod_{m\in M} S/\varrho_m, C)$*-projective,*

(x) A_S *is* $(\coprod_{j\in J} e_jS, C)$*-projective,*

(xi) A_S *is* (B,B)*-projective.*

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ and $(vi) \Leftrightarrow (vii) \Leftrightarrow (viii) \Leftrightarrow (ix) \Leftrightarrow (x)$ are a part of Lemma 3.29.
The implications $(vi) \Longrightarrow (v)$ and $(vi) \Longrightarrow (xi)$ are obvious.
$(v) \Longrightarrow (vi)$ By Proposition 2.28 there exists a free act $F(A)_S \cong \coprod_{i\in I} S$, such that there exists an epimorphism $\alpha : \coprod_{i\in I} S \longrightarrow A$. For this consider in **Act** $-S$ the following diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle id} \\ \coprod_{i\in I} S & \xrightarrow{\alpha} & A \end{array}$$

By (v) there exist an homomorphism $\gamma : A \longrightarrow \coprod_{i\in I} S$ such that the diagram commutes, i.e., A_S is a retract of a free and thus projective act and is therefore projective by Proposition 2.29.

$(xi) \Longrightarrow (vi)$ By Proposition 2.28, A_S is the factor act of a free act $F(A)_S$, i.e., there exists an epimorphism $\alpha : F(A) \longrightarrow A$. Consider in **Act** $-S$ the following diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle id} \\ (\coprod_{n\in\mathbb{N}} F^n(A) \sqcup \coprod_{n\in\mathbb{N}} A^n) & \xrightarrow{g} & (\coprod_{n\in\mathbb{N}} F^n(A) \sqcup \coprod_{n\in\mathbb{N}} A^n) \end{array}$$

with $id : A \longrightarrow A^1,\ id(a) = a^1$ denotes the identity on A_S and where the epimorphism

$$g : \coprod_{n\in\mathbb{N}} F^n(A) \sqcup \coprod_{n\in\mathbb{N}} A^n \longrightarrow \coprod_{n\in\mathbb{N}} F^n(A) \sqcup \coprod_{n\in\mathbb{N}} A^n$$

given by

$$g : \begin{cases} F^1 \longrightarrow A^1 & g(f^1) = \alpha(f^1) \\ A^n \longrightarrow A^{n+1}, & g(a^n) = a^{n+1}, \quad n > 1 \\ F^n \longrightarrow F^{n-1}, & g(f^n) = f^{n-1}, \quad n > 1 \end{cases}$$

where f^i, $i \in \mathbb{N}$, denotes the element $f \in F(A)_S$ in the i-th copy of $F(A)_S$ in the coproduct $\coprod_{n\in\mathbb{N}} F^n(A)$ and a^i denotes the element $a \in A_S$ in the i-th copy of A_S in the coproduct $\coprod_{n\in\mathbb{N}} A$.
By (xi) the diagram is commutative, i.e., there exists $id' : A \longrightarrow \coprod_{n\in\mathbb{N}} F^n(A) \sqcup \coprod_{n\in\mathbb{N}} A^n$ with $g\, id' = id$. This implies $id'(A) \subseteq F^1(A)$. Thus $g \mid_{F^1} id' = \alpha\, id' = id$, i.e., A_S is a retract of a free, and therefore projective act, and is therefore projective.

□

3.4 Projectivities which are trivial or equivalent to other projectivities

In this section it turns out, that some of the $(\mathcal{X}, \mathcal{Y})$-projectivities are equivalent to others, whenever they are non-trivially-$(\mathcal{X}, \mathcal{Y})$-projectivities.

Lemma 3.32 *Let $A_S \in$ **Act** $-S$.*
A_S is (S, A)-projective iff A_S is cyclic projective or trivially-(S, A)-projective.

Proof. Let $g : S \longrightarrow A$ be an epimorphism. Then $A \cong S/ker\, g$, i.e., A_S is a cyclic act. Since A_S is (S, A)-projective, for id_A there exists a homomorphism id'_A with $g\, id'_A = id_A$, i.e., A_S is a retract of S and is therefore projective.
If there is no epimorphism from S onto A_S, A_S is trivially-(S, A)-projective.

□

Lemma 3.33 *If $A_S \in$ **Act** $-S$ is a non-trivially-(A, G)-projective right S-act, then A_S is a generator in* **Act** $-S$.

Proof. Let G_S be a generator in **Act** $-S$, i.e., $G_S \overset{\pi}{\underset{\gamma}{\leftrightarrows}} S_S$. If there exist an epimorphims $g : A \longrightarrow G$, then $\pi g : A \longrightarrow S$ is an epimorphism, i.e., A_S is a generator in **Act** $-S$.

□

For (X, G)-projectivity with $X \neq A$, we need some preparing results to finally get Lemma 3.37. These are given right now.

Lemma 3.34 *If there exists a generator $G_S \ncong S_S$ in* **Act** $-S$*, which is an epimorphic image of S, then S is infinite.*

Proof. Suppose S is finite. Let G_S be a generator in $\mathbf{Act}-S$, i.e., $G_S \overset{\pi}{\underset{\gamma}{\rightleftarrows}} S_S$, then $|S| \leq |G|$. Let $g : S \longrightarrow G$ be an epimorphism. Then $|G| \leq |S|$. Thus g is an S-isomorphism, i.e. $G_S \cong S_S$.

□

Lemma 3.35 *If S is finite, then all right-S-acts A_S are (S,G)-projective.*

Proof. Since S is finite, S_S is the unique generator in $\mathbf{Act}-S$, which is an epimorphic image of S by Lemma 3.34. Thus by the projectivity of S every morphism from an arbitrary act A_S to G_S can be lifted with respect to every epimorphism from S onto a generator.

□

We now change the first component to be a Rees factor act by a principal right ideal of S. In this case it turns out that a non-trivially-$(S/sS,G)$-projective act is already (S,G)-projective.

Proposition 3.36 *There exists an epimorphism π from a Rees factor act $(S/sS)_S$ of S by a principal right ideal onto a generator G_S in $\mathbf{Act}-S$ iff S contains a left zero.*

Proof. Let $s \in S$, let $\alpha : S/sS \longrightarrow G$ be an epimorphism and let $\bar{0}$ denote the zero in S/sS. Then $\alpha(\bar{0})$ is a zero element in G_S. Since G_S is a generator, there exists an epimorphism $\pi : G \longrightarrow S$. Thus $\pi(\alpha(\bar{0})) =: z$ is a left zero in S.
If z is a left zero in S, then $S/zS = S$ and S itself is a generator.

□

Lemma 3.37 *Let S be a monoid with left zero and let $s \in S$.*
For $A_S \in \mathbf{Act}-S$ the following assertions are equivalent

(i) *A_S is $(S/sS,G)$-projective,*

(ii) *A_S is (S,G)-projective.*

Proof. $(i) \Longrightarrow (ii)$ Let A_S be $(S/sS,G)$-projective and let z denote the left zero in S.
Then $S/zS = S$ leads to A_S is (S,G)-projective.
$(ii) \Longrightarrow (i)$ by Lemma 3.3.

□

Corollary 3.38 *Let S be a monoid with left zero and let $s_l \in S,\ l \in L$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent* *(i)* A_S *is* $(\coprod_{l \in L} S/s_l S, G)$*-projective,*

(ii) A_S *is* $(\coprod_{i \in I} S, G)$*-projective.*

□

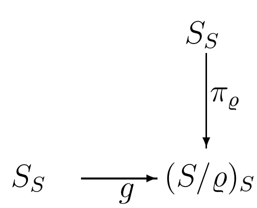
Summarizing, we get, that non-trivially-$(S/sS, G)$-projective acts are already (S, G)-projective, and non-trivially-$(\coprod_{l \in L} S/s_l S, G)$-projective acts are $(\coprod_{i \in I} S, G)$-projective.
Note, that therefore these projectivities will not be included in the implication scheme at the end of this chapter.

3.5 When $(S, S/\sigma)$- and $(S/\sigma, S/\varrho)$-projective acts are $(S, S/\varrho)$-projective

We start with a useful proposition concerning $(S, S/\varrho)$-projectivities. This result has also been formulated in another way in [17].

Proposition 3.39 *([17]) Let* ϱ *be a right congruence on* S *and let* $f : A_S \longrightarrow (S/\varrho)_S$ *be a homomorphism. Then* f *can be lifted with respect to every epimorphism* $g : S_S \longrightarrow (S/\varrho)_S$ *iff* f *can be lifted with respect to the canonical epimorphism* $\pi_\varrho : S_S \longrightarrow (S/\varrho)_S$*.*

Proof. For an epimorphism $g : S \longrightarrow S/\varrho$ consider in **Act** $-S$ the following diagram:



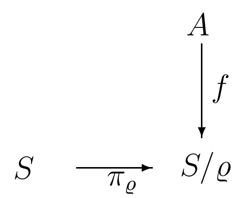
Since S is projective, there exists an endomorphism π'_ϱ of S such that $g\,\pi'_\varrho = \pi_\varrho$.
Now let $A_S \in$ **Act** $-S$ and let $f : A \longrightarrow S/\varrho$ be a homomorphism, which can be lifted with respect to π_ϱ, i.e. there exists $f' : A \longrightarrow S$ with $\pi_\varrho\, f' = f$. Then for $\pi'_\varrho\, f' : A \longrightarrow S$ the equation $g\,\pi'_\varrho\, f' = \pi_\varrho\, f' = f$ holds, i.e. f can be lifted with respect to g.
The converse is obvious.

□

Remark 3.40 *Let ϱ_m, $m \in M$, be right congruences on S and let $f : A_S \longrightarrow \coprod_{m\in M} (S/\varrho_m)_S$ be a homomorphism. Then f can be lifted with respect to every epimorphism $g : \coprod_{i\in I} S_S \longrightarrow \coprod_{m\in M} (S/\varrho_m)_S$ iff f can be lifted with respect to the coproduct of the canonical epimorphism $(\coprod_{m\in M} \pi_{\varrho_m}) : \coprod_{m\in M} S_S \longrightarrow \coprod_{m\in M} (S/\varrho_m)_S$. This assertion can be proved by using the projectivity of coproduct of copies of S in the same way like before.*

Lemma 3.41 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ denote different sets of right congruences on S, such that for every $\varrho \in \mathcal{K}_1$ there exists an $\sigma \in \mathcal{K}_2$ with $\sigma \subseteq \varrho$. If $A_S \in$ **Act** $-S$ is $(S, S/\sigma)$- and $(S/\sigma, S/\varrho)$-projective, then A_S is $(S, S/\varrho)$-projective.*

Proof. By Proposition 3.39 it is sufficient to consider the canonical epimorphisms, respectively.
Let $\varrho \in \mathcal{K}_1$ and let $\pi_\varrho : S \longrightarrow S/\varrho$ denote the canonical epimorphism.
Consider in **Act** $-S$ the following diagram:



By assumption there exists $\sigma \in \mathcal{K}_2$ with $\sigma \subseteq \varrho$. Let π_σ denote the canonical epimorphism from S onto S/σ. By the Homomorphism Theorem for acts (Theorem 2.7) there exists a unique homomorphism π'_ϱ with $\pi'_\varrho \pi_\sigma = \pi_\varrho$.
Since A_S is $(S/\sigma, S/\varrho)$-projective, there exists $f' : A \longrightarrow S/\sigma$ with $\pi'_\varrho f' = f$.
Since A_S is $(S, S/\sigma)$-projective, there exists $f'' : A \longrightarrow S$ with $\pi_\sigma f'' = f'$.

Thus there exists a homomorphism $f'' : A \longrightarrow S$ with

$$\pi_\varrho f'' = \pi'_\varrho \pi_\sigma f'' = \pi'_\varrho f' = f,$$

i.e. f can be lifted with respect to π_ϱ and therefore with respect to every epimorphism $g : S \longrightarrow S/\varrho$, i.e. A_S is $(S, S/\varrho)$-projective.

□

An interesting aspect of this Lemma turns out, if the consequences for weak projectivities are explicitly given:

Corollary 3.42 *Let $s,\ x,\ y \in S$, let ϱ be a right congruence on S and let $\hat{I}$ be a right ideal of S. Then for every $A_S \in \mathbf{Act}-S$ the following assertions hold:*

(a) If A_S is $(S, S/\varrho(x,y))$- and $(S/\varrho(x,y), S/\varrho)$-projective, then A_S is weakly projective.

(b) If A_S is $(S, S/\varrho(x,y))$- and $(S/\varrho(x,y), S/\hat{I})$-projective, then A_S is Rees weakly projective.

(c) If A_S is $(S, S/\varrho(x,y))$- and $(S/\varrho(x,y), S/sS)$-projective, then A_S is principally Rees weakly projective.

(d) If A_S is $(S, S/sS)$- and $(S/sS, S/\hat{I})$-projective, then A_S is Rees weakly projective.

Proof. By Lemma 3.41 in each case the existence of suitable congruences σ remains to be shown:
(a) If ϱ is an arbitrary right congruence on S with $x\,\varrho\,y$, then $\varrho(x,y) \subseteq \varrho$, for it is the smallest right congruence on S such that x and y are in the same congruence class.
(b), (c) Let $\hat{J}$ be a (principal) right ideal of S, let $\varrho_{\hat{J}}$ denote the Rees congruence on S by $\hat{J}$, let $j \in \hat{J}$ and $s \in S \setminus \{1\}$. Then $\varrho(j, js) \subseteq \varrho_{\hat{J}}$.
(d) Furthermore for the Rees congruence on S by the principal right ideal jS with $j \in \hat{J}$ we have $\varrho_{jS} \subseteq \varrho_{\hat{J}}$.

□

In general we can not expect assertions analogue to those of Corollary 3.42. For instance, if ϱ is a right congruence on S, such that $(S/\varrho)_S$ does not have a zero, there does not exist a Rees congruence ϱ_I on S with $\varrho_I \subseteq \varrho$:
If $\varrho_I \subseteq \varrho$, then there exists an epimorphism $g : S/I \longrightarrow S/\varrho$, which yields a zero in $(S/\varrho)_S$ contradicting $(S/\varrho)_S$ being an act without zero.

3.6 Table of results

We tabularize the results obtained so far.
The acts in the first column of the table denote the first component of the pair (X, Y), the acts in the first row the second one. Thus every array in the table belongs to a pair (X, Y) and therefore to (X, Y)-projectivity. We start with 180 pairs, omitting the pairs with projective second component (Proposition 3.2) and the pair (B, B), since (B, B)-projectivity is equivalent to projectivity (Proposition 3.31).

Arrays belonging to (X, Y)-projectivities, which are already excluded by the observations in section 3.1, are marked by 'o'.
Arrays belonging to (X, Y)-projectivities, which are equivalent to others, are marked by the number of the respective Proposition/Lemma/Corollary. If the related property is equivalent to one of the properties projectivity (proj), quasi projectivity (qp), wp, pwp, Rwp, pRwp, this is marked by the respective abbreviations.
By $(*)$ arrays are marked, which belong to projectivities, which are equivalent to others, whenever they are non-trivially-projectivities (see section 3.4).

Notations:

- $I,\ J,\ K,\ L,\ M,\ N$ are non-empty sets,
- $\varrho,\ \varrho_m,\ m \in M$, are right congruences on S,
- $\sigma,\ \sigma_n,\ n \in N$, are right principal congruences on S,
- $e,\ e_j \in E(S),\ j \in J$,
- $\hat{I},\ I_k,\ k \in K$, are right ideals of S,
- $s,\ s_l \in S,\ l \in L$.

	C_S	A_S	G_S	$\coprod_{m\in M} S/\varrho_m$	S/ϱ	$\coprod_{n\in N} S/\sigma_n$	S/σ	$\coprod_{k\in K} S/I_k$	$S/\hat{I}$	$\coprod_{l\in L} S/s_lS$	S/sS
B_S	proj	proj	3.29	3.29	wp	3.29	pwp	3.24	Rwp	3.25	pRwp
A_S	qp		gen/ triv					3.24		3.25	
G_S	proj	proj	3.29	3.29	wp	3.29	pwp	Rwp	Rwp	3.25	pRwp
$\coprod_{i\in I} S$	proj	proj			wp		pwp	Rwp	Rwp	pRwp	pRwp
S_S	o	(*)		o	wp	o	pwp	o	Rwp	o	pRwp
$\coprod_{m\in M} S/\varrho_m$	proj	proj	3.29	3.29	wp	3.29	pwp	Rwp	Rwp	pRwp	pRwp
S/ϱ	o	3.14	3.14	o	wp	o	pwp	o	Rwp	o	pRwp
$\coprod_{j\in J} e_jS$	proj	proj	3.29	3.29	wp	3.29	pwp	Rwp	Rwp	pRwp	pRwp
eS	o	3.14	3.14	o	wp	o	pwp	o	Rwp	o	pRwp
$\coprod_{n\in N} S/\sigma_n$					3.10		3.10	3.24	3.10	3.25	3.10
S/σ	o			o		o		o		o	
$\coprod_{k\in K} S/I_k$	3.27	3.27	3.27	3.27	3.10	3.27	3.10	3.24	3.10	3.25	3.10
$S/\hat{I}$	o	3.28	3.28	o	3.28	o	3.28	o	3.28	o	3.28
$\coprod_{l\in L} S/s_lS$			(*)		3.10		3.10	3.24	3.10	3.25	3.10
S/sS	o		(*)	o		o		o		o	

Table 1: Table of Results

For further investigations, the concepts of projectivity, weak projectivity, principal weak projectivity, Rees weak projectivity, principal Rees weak projectivity and (X, Y)-projectivities, where the pair (X, Y) belongs to an empty place in this table, remain.

For the aim of lucidity, we restrict our following investigations to those projectivities with first component $\neq A$, except of Chapter 4, where further investigations on projectivities of acts over monoids with left zero will be made. A_S as first component is left out, because (A, X)-projective acts should be treated in the context of quasi projectivity (see Remark 3.1) and, furthermore, since all other remaining first components are coproducts of different kinds of cyclic acts and these are in some sense related to each other.

Thus, for further investigations, we can reduce Table 1 in the following way:

- we delete all rows and columns, which are totally filled in Table1, taking care, that at least one of the equivalent conditions, respectively, remains,
- we delete the second row, where A is the first component.

The resulting table is Table 2. In this table, the abbreviation xwp, $x \in \{\Phi, p, R, pR\}$, is used for x weakly projective and it marks the places in the table, which belong to the pair (X, Y), for which (X, Y)-projectivity will be used as defining property for x weak projectivity in the following. Note that these are exactly the equivalences given by the original definition in [17]. Then we obtain the following:

	C_S	A_S	G_S	$\coprod_{m\in M} S/\varrho_m$	S/ϱ	$\coprod_{n\in N} S/\sigma_n$	S/σ	$S/\hat{I}$	S/sS
$\coprod_{i\in I} S$	proj.	proj			3.9		3.9	3.9	3.9
S_S	o	(∗)		o	wp	o	pwp	Rwp	pRwp
$\coprod_{n\in N} S/\sigma_n$					3.10		3.10	3.10	3.10
S/σ	o			o		o			
$\coprod_{l\in L} S/s_lS$			(∗)		3.10		3.10	3.10	3.10
S/sS	o		(∗)	o		o			

Table 2: Remaining pairs (X, Y)

3.7 Further implications

Some implications are quiet clear by definition or by the results obtained so far. Following, implications which are not obvious will be proved, for giving an implication scheme at the end of this chapter.

Lemma 3.43 *Let ϱ be a right congruence on S and let G_S denote a generator in* $\textbf{Act}-S$*. Then $(S, S/\varrho)$-projectivity implies (S, G)-projectivity.*

Proof. If there exists an epimorphism g from S onto G_S, then G_S is a factor act of S. Thus, since A_S is $(S, S/\varrho)$-projective, every homomorphism from A_S to G_S can be lifted with respect to g. Thus A_S is (S, G)-projective.
Otherwise, A_S is trivially-(S, G)-projective and is therefore (S, G)-projective.

□

Lemma 3.44 *Let $X_S \in \textbf{Act}-S$ and let $s,\ x,\ y \in S$. Then $(S/\varrho(x, y), X)$-projectivity implies $(S/sS, X)$-projectivity.*

Proof. If there exist an $s \in S$ with $|sS| = 1$, then s is a left zero in S. Then every right principal Rees congruence $\varrho_{s'S}$ is the principal congruence $\varrho(s's, s')$ and the implication holds.
Let $s \in S$ with $|sS| \geq 2$. Then for $u \in S$ we get: $\varrho(s, su) \subseteq \varrho_{sS}$ is valid. Thus there exists an epimorphism $\pi : S/\varrho(s, su) \longrightarrow S/sS$ and Lemma 3.3 completes the proof.

□

Corollary 3.45 *Let $X_S \in \textbf{Act}-S$, let $s_l \in S,\ l \in L,\ x_n,\ y_n \in S,\ n \in N$. Then $(\coprod_{n\in N} S/\varrho(x_n, y_n), X)$-projectivity implies $(\coprod_{l\in L} S/s_lS, X)$-projectivity.*

□

Remark 3.46 *In the following, $(\coprod_{n\in N} S/\varrho(x_n, y_n), G)$-projectivity and $(S/\varrho(x, y), G)$-projectivity will left out of consideration (see Lemma 3.37 and Corollary 3.38 together with Lemma 3.44 and Corollary 3.45).*

Lemma 3.47 *Let $s \in S$, let $A_S \in \textbf{Act}-S$ and let ϱ be a right congruence on S. If A_S is $(S/sS, S/\varrho)$-projective, then A_S is $(S/sS, A)$-projective.*

Proof. If there exists an epimorphism $g : S/sS \longrightarrow A$, then $A_S \cong S/\varrho$ for a right congruence ϱ on S and thus the implications holds. Otherwise A_S is trivially-$(S/sS, A)$-projective and there is nothing to show.

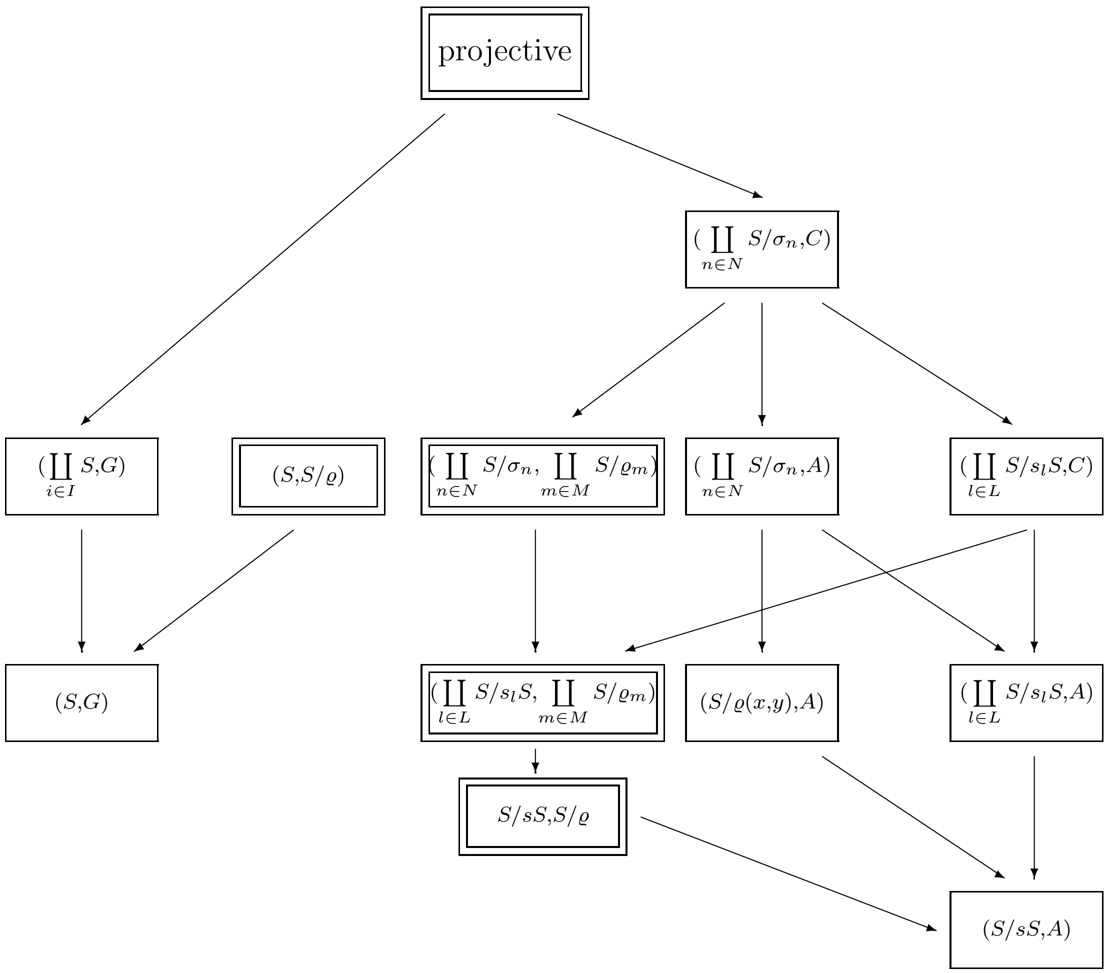
□

3.8 Implication scheme

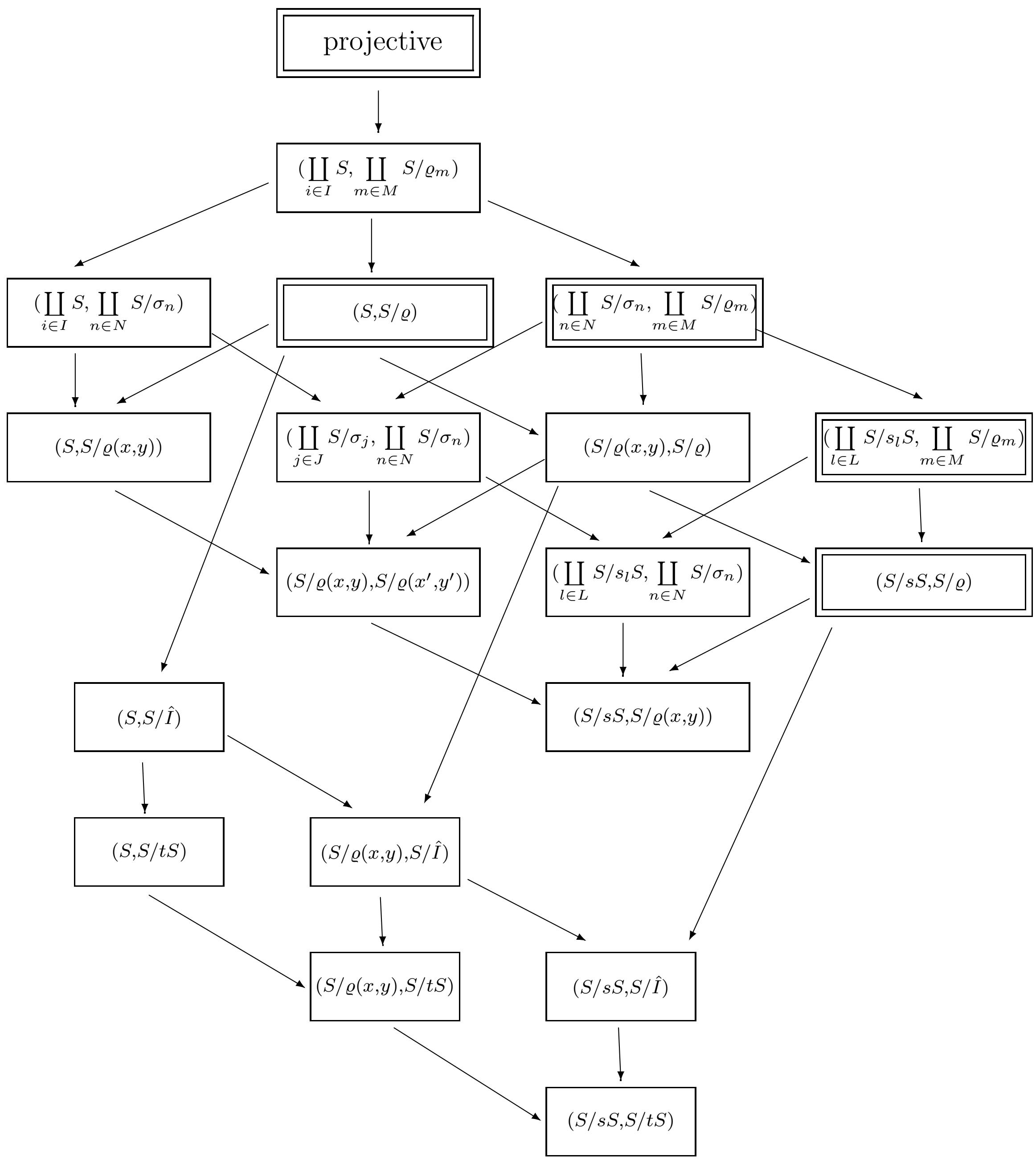
The implications between the projectivities introduced before are presented in the following scheme, where (X,Y)-projectivity is abbreviated by the pair (X,Y). The notations are equal to those in Table 1 (2), except in the case of monocyclic acts $(S/\sigma)_S$, which will be denoted by $(S/\varrho(x,y))_S,\ x,\ y \in S$. Additionally, let $x',\ y',\ t \in S$ and let $\sigma_j,\ j \in J$ be principal right congruences on S.

By reason of lucidity, the implication scheme is divided in part 1 and part 2. Projectivities included in both parts are marked by double framed boxes. These are the places, where part 1 an part 2 can be combined to one scheme.

Remind, that concepts like $(S/sS,G)$-projectivity, which are trivially or equivalent to other concepts as well as $(\coprod_{n\in N} S/\varrho(x_n,y_n),G)$- and $(S/\varrho(x,y),G)$-projectivity, are left out. Furthermore, projectivities with first component A, which in some sense belong to quasi projectivity, will be left out, for they won't be considered any further, except of Chapter 4.
Then we obtain:



Implication scheme — part 1



Implication scheme — part 2

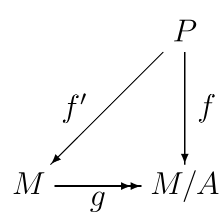
3.9 Comments

The representation of monoids by endomorphisms of sets has lead to acts and vice versa. It is well known, that the representations of rings by endomorphisms of abelian groups leads to modules over rings and vice versa. Thus, in some sense they are related to acts over monoids.

In this Chapter various projectivity properties have been introduced for acts over monoids. What is the situation like in the category of (right) modules?
Also in the theory of modules over rings some projectivities have been studied. *Projective* modules are defined analogously to projective acts by the categorical definition, i.e, P is called projective, if the functor $Mor_{Mod}(P, -)$ preserves epimorphisms.
In 1966, Y. Miyashita ([24]) introduced a generalization of projective modules, so called *quasi-projective* modules and many authors followed him. Some of the weaker concepts studied by now are given by the following definitions (cf. [26], [23]):

Let $M,\ N,\ A$ be modules. A module P is called

(1) *M-projective*, if for every submodule A of M, for every epimorphism $g : M \longrightarrow M/A$ and for every homomorphism $f : P \longrightarrow M/A$ there exists f', such that the following diagram commutes:



(2) *self-projective (quasi projective),* if P is P-projective.

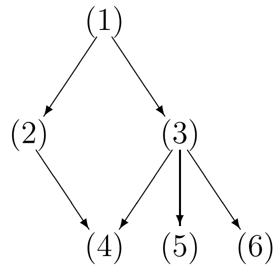
(3) *epi-projective relative to M,* if P is M-projective with respect to all epimorphisms $f : P \longrightarrow M/A$.

(4) *self-epi-projective,* if P is epi-projective relative to P.

(5) *semi-projective,* if P is self-projective with respect to epimorphisms $g : P \longrightarrow N,\ N$ a submodule of P.

(6) *direct-projective,* if every direct summand of a decomposition $P = A_1 \oplus A_2$ is a retract of of P.

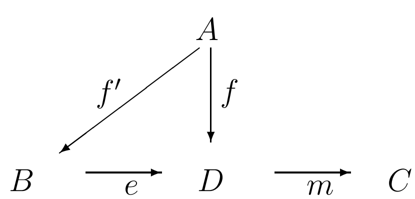
For these properties we get the following implication scheme:



Most of these concepts could be considered as $(\mathcal{X},\mathcal{Y})$-projectivities with $\mathcal{X}$ consisting of one element. A more general investigation with respect to modules, comparable with $(\mathcal{X},\mathcal{Y})$-projectivities with different classes $\mathcal{X}$ and $\mathcal{Y}$, is not known to me.

In [23] the notation 'weak projectivity' occurs, which is called 'kostetig' in [26]. There is no connection to weak projectivity of acts, as defined in [17].
The same is true for weak projective covers of a module, introduced by Park and Kim in [25] in 1991. Here the adjective 'weak' is related to the whole expression 'projective cover of' (the respective module P is supposed to be projective), weak projective modules do not occur in this article.

Another concept of projectivity occurs in [8] for arbitrary categories:
In a category $\mathcal{C}$ a class $\mathcal{P} \subseteq \mathrm{Ob}\,(\mathcal{C})$ is called (E,M)-*pseudoprojective,* if for arbitrary objects $A,\ B \in \mathcal{P}$ and for every $D \in \mathcal{C}$, for which there exists $C \in \mathcal{P}$ and a monomorphism $m : D \longrightarrow C$, every homomorphism $f : A \longrightarrow D$ can be lifted with respect to every epimorphism $e : B \longrightarrow D$, i.e., there exists f', such that the following diagram is commutative:



In the category $\mathbf{Act}-S$ we see, that (E,M)-pseudoprojectivity is not directly comparable with $(\mathcal{X},\mathcal{Y})$-projectivity of acts, since it is a property of a class of acts, and even if we suppose the class to consist of one element only, we do not obtain a concept like the one introduced in Chapter 3.

Nevertheless, the following equivalence holds:

A class $\mathcal{P}$ of S-acts is (E, M)-pseudoprojective if and only if every $P \in \mathcal{P}$ is $(\mathcal{P}, \mathcal{P}')$-projective, where the class $\mathcal{P}'$ is given by $\mathcal{P}' = \{\mathrm{D} \in \mathbf{Act}-\mathrm{S} \mid \exists \mathrm{Q} \in \mathcal{P}$ such that there exists a monomorphism $\mathrm{m} : \mathrm{D} \longrightarrow \mathrm{Q}\}$.

As indicated by the respective properties of modules, it is a quite natural way to ask for weaker concepts of projectivity by restricting the definition to special homomorphisms f, i.e., to look at properties like epi-projectivity.
In the category of acts, epi-$(\mathcal{X}, \mathcal{Y})$-projectivity, i.e., $(\mathcal{X}, \mathcal{Y})$-projectivity with respect to epimorphisms f, leads to weaker concepts than considered in this dissertation.
Another interesting property is $(B, B/A)$-projectivity for arbitrary acts B_S and Rees factor acts $(B/A)_S$ by subacts A_S of B_S (see the definition of M-projectivity of modules).

Moreover, (E, M)-pseudoprojective classes of acts could be studied, where the results concerning homological classification of monoids by $(\mathcal{X}, \mathcal{Y})$-projectivities of acts (Chapter 5) could be used because of the equivalence presented before.

A further interesting idea is, to consider $(\mathcal{X}, \mathcal{Y})$-projectivity of acts, where $\mathcal{Y}$ is a class of amalgamated coproducts of acts.
These are not considered here.

4 Summary and Supplements

As mentioned at the end of Chapter 3, page 38, in this Chapter we consider (X, Y)-projectivities, where A_S itself as first component is permitted. Thus, in this situation the restriction of Table 1 is given by the following table (Table 3). (For the used abbreviations and for informations see the explanations with respect to Table 1 in Section 3.6. (pages 36, 37)

	C_S	A_S	G_S	$\coprod_{m\in M} S/\varrho_m$	S/ϱ	$\coprod_{n\in N} S/\sigma_n$	S/σ	$S/\hat{I}$	S/sS
A_S	qp		gen or triv						
$\coprod_{i\in I} S$	proj	proj			3.9		3.9	3.9	3.9
S_S	o	(∗)		o	wp	o	pwp	Rwp	pRwp
$\coprod_{n\in N} S/\sigma_n$					3.10		3.10	3.10	3.10
S/σ	o			o		o			
$\coprod_{l\in L} S/s_lS$			(∗)		3.10		3.10	3.10	3.10
S/sS	o		(∗)	o		o			

Table 3: Remaining pairs (X, Y) for monoids with left zero

In the following we will summarize the results of Chapter 3 with special attention to projectivities with first component S. Recall, that these are called weak, principally weak, Rees weak and principally Rees weak projectivity in [17].

Nevertheless, the **main** aspect is, to investigate $(\mathcal{X}, \mathcal{Y})$-projectivities of acts over a monoid with left zero. For $(\mathcal{X}, \mathcal{Y})$-projectivities of acts the presence of

a left zero plays an important role, among other things, since in this case every act contains a zero. We obtain further equivalences, which are tabularized in Table 4 of Section 4.2.
Questions concerning homological classification with respect to acts over monoids with left zero are discussed in the general context in Chapter 5.

It will turn out, that in the case of a monoid with left zero, the remaining properties are mainly the weak projectivities and those projectivities with first component A, i.e., (X, Y)-projectivities which are related to the concept of quasi-projectivity.

After proving further implications, these are collected in an implication scheme in Section 4.3.

4.1 Further equivalences in the case of monoids with left zero

Recall, that if S has a left zero z, then every right principal Rees congruence by the right ideal sS is the right principal congruence $\varrho(sz, s)$ by Remark 2.23.

First of all, equivalences of projectivities with arbitrary second component Y will be given. They will be used in the proofs of the following (five) theorems.

Lemma 4.1 *Let S be a monoid with left zero z, and let $s_l,\ x_n,\ y_n \in S$ $(l \in L,\ n \in N)$. Then for all $Y_S \in$* **Act** *$-S$ the following assertions are equivalent:*

(i) A_S is $(\coprod_{i\in I} S, Y)$-projective,

(ii) A_S is $(\coprod_{n\in N} S/\varrho(x_n, y_n), Y)$-projective,

(iii) A_S is $(\coprod_{l\in L} S/s_lS, Y)$-projective.

Proof. (i) $\Longrightarrow$ (ii) is a part of Corollary 3.4.
(ii) $\Longrightarrow$ (iii) is valid, since every Rees factor act by a right principal ideal is a factor act by a right principal congruence by the existence of a left zero in S (Remark 2.23).
(iii) $\Longrightarrow$ (i). For the left zero z the Rees factor act S/zS is S itself.

□

Corollary 4.2 *Let S be a monoid with left zero z, and let $s,\ x,\ y \in S$. Then for all $Y_S \in \mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* (S, Y)*-projective,*

(ii) A_S *is* $(S/\varrho(x, y), Y)$*-projective,*

(iii) A_S *is* $(S/sS, Y)$*-projective.*

In the following, we consider equivalences to projectivity and to (S, X)-projectivity, starting with the strongest property in 4.1.1, i.e. with projectivity.

4.1.1 Equivalences to projectivity

Theorem 4.3 *Let $\varrho_m,\ m \in M$, be right congruences on S, let G_S denote a generator in $\mathbf{Act}-S$, let $B_S,\ C_S \in \mathbf{Act}-S$ and let $e_j \in E(S),\ j \in J$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent:*

(i) A_S *is* (B, C)*-projective (i.e.,* A_S *is projective),*

(ii) A_S *is* (B, A)*-projective,*

(iii) A_S *is* (G, A)*-projective,*

(iv) A_S *is* $(\coprod_{i \in I} S, A)$*-projective,*

(v) A_S *is* $(\coprod_{m \in M} S/\varrho_m, A)$*-projective,*

(vi) A_S *is* $(\coprod_{j \in J} e_j S, A)$*-projective,*

(vii) A_S *is* (G, C)*-projective,*

(viii) A_S *is* $(\coprod_{i \in I} S, C)$*-projective,*

(ix) A_S *is* $(\coprod_{m \in M} S/\varrho_m, C)$*-projective,*

(x) A_S *is* $(\coprod_{j \in J} e_j S, C)$*-projective,*

(xi) A_S *is* (B, B)*-projective.*

Furthermore, let $I_k,\ k \in K$, be right ideals of S and let $s_l,\ x_n,\ y_n \in S$ $(l \in L,\ n \in N)$. If S contains a left zero, the assertions (i) to (xi) are equivalent to:

(xii) A_S *is* $(\coprod_{k\in K} S/I_k, A)$*-projective,*

(xiii) A_S *is* $(\coprod_{l\in L} S/s_lS, A)$*-projective,*

(xiv) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), A)$*-projective,*

(xv) A_S *is* $(\coprod_{k\in K} S/I_k, C)$*-projective,*

(xvi) A_S *is* $(\coprod_{l\in L} S/s_lS, C)$*-projective,*

(xvii) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), C)$*-projective,*

(xviii) A_S *is* (B, G)*-projective,*

(xix) A_S *is* $(\coprod_{i\in I} S, G)$*-projective,*

(xx) A_S *is* $(\coprod_{k\in K} S/I_k, G)$*-projective,*

(xxi) A_S *is* $(\coprod_{l\in L} S/s_lS, G)$*-projective,*

(xxii) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), G)$*-projective.*

Proof. The equivalences (i) to (xi) are given in Proposition 3.31.
$(i) \Longrightarrow (xix)$ is obvious.
$(xix) \Longrightarrow (i)$ Let $g : \coprod_{i\in I} S \longrightarrow A$ be an epimorphism. Consider in **Act** $-S$ the following diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle id_A} \\ \coprod_{i\in I} S & \xrightarrow[g]{} & A \end{array}$$

Since S contains a left zero, by Result 2.31 the coproduct $A \coprod S$ is a generator in **Act** $-S$. Consider now in **Act** $-S$ the following diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle id_A} \\ (\coprod_{i\in I} S) \coprod S & \xrightarrow[g \sqcup id_S]{} & A \coprod S \end{array}$$

where $g \sqcup id_S : (\coprod_{i\in I} S) \coprod S \longrightarrow A \coprod S$ with $g \sqcup id_S(s) = g(s)$ if $s \in \coprod_{i\in I} S$ and $g \sqcup id_S(s) = s$ if $s \in S$ is the coproduct of the morphisms id_S and g. By (xix) there exists $id'_A : A \longrightarrow (\coprod_{i\in I} S) \coprod S$, such that the diagram is commutative, i.e., $(g \sqcup id_S)\, id'_A = id_A$. By the definition of $g \sqcup id_S$, this implies $id'_A(A) \subseteq \coprod_{i\in I} S$. Thus id'_A is a homomorphism from A to $\coprod_{i\in I} S$ with $g\, id'_A = id_A$, i.e., A_S is a retract of $\coprod_{i\in I} S$ and is therefore projective by Proposition 2.29.

The equivalences $(xii) \Leftrightarrow (xiii)$, $(xv) \Leftrightarrow (xvi)$ and $(xx) \Leftrightarrow (xxi)$ are parts of Proposition 3.27.

The equivalences $(xiii) \Leftrightarrow (xiv)$, $(xvi) \Leftrightarrow (xvii)$ and $(xxi) \Leftrightarrow (xxii)$ are parts of Lemma 4.1.

The implications $(xix) \Longrightarrow (xv)$ and $(xix) \Longrightarrow (xviii)$ follow from $(xix) \Longrightarrow (i)$.

The implications $(xv) \Longrightarrow (xii)$, $(xvii) \Longrightarrow (xxii)$ and $(xviii) \Longrightarrow (xix)$ are obvious.

$(xiii) \Longrightarrow (xix)$ and $(xxi) \Longrightarrow (xix)$: Since S contains a left zero z, by Lemma 4.1 (xxi) implies (xix) and $(xiii) \Longrightarrow (iv) \Longrightarrow$ projective, which has been proved to be equivalent to (xix).

□

4.1.2 Equivalences to $(S, S/\varrho)$-projectivity

Theorem 4.4 *Let ϱ, ϱ_m, $m \in M$, σ be right congruences on S, let G_S denote a generator in* **Act** $-S$*, let $B_S \in$* **Act** $-S$ *and let e, $e_j \in E(S)$, $j \in J$. Then for $A_S \in$* **Act** $-S$ *the following assertions are equivalent*

(i) *A_S is $(S, S/\varrho)$-projective (i.e., A_S is weakly projective),*

(ii) *A_S is $(B, S/\varrho)$-projective,*

(iii) *A_S is $(G, S/\varrho)$-projective,*

(iv) *A_S is $(\coprod_{i\in I} S, S/\varrho)$-projective,*

(v) *A_S is $(\coprod_{m\in M} S/\varrho_m, S/\varrho)$-projective,*

(vi) *A_S is $(S/\sigma, S/\varrho)$-projective,*

(vii) *A_S is $(\coprod_{j\in J} e_j S, S/\varrho)$-projective,*

(viii) A_S *is* $(eS, S/\varrho)$*-projective.*

Furthermore, let $R \neq \emptyset$ *be a set, let* $\varrho_r,\ r \in R$*, be right congruences on* S*, let* $\hat{I},\ I_k,\ k \in K$*, be right ideals of* S*, and let* $s,\ s_l,\ x,\ y,\ x_n,\ y_n \in S$ $(l \in L,\ n \in N)$*. If* S *contains a left zero, the assertions* (*i*) *to* (*viii*) *are equivalent to:*

(ix) A_S *is* $(B, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(x) A_S *is* $(G, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xi) A_S *is* $(\coprod_{i\in I} S, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xii) A_S *is* $(\coprod_{m\in M} S/\varrho_m, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xiii) A_S *is* $(\coprod_{j\in J} e_jS, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xiv) A_S *is* $(\coprod_{k\in K} S/I_k, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xv) A_S *is* $(\coprod_{k\in K} S/I_k, S/\varrho)$*-projective,*

(xvi) A_S *is* $(S/\hat{I}, S/\varrho)$*-projective,*

(xvii) A_S *is* $(\coprod_{l\in L} S/s_lS, \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xviii) A_S *is* $(\coprod_{l\in L} S/s_lS, S/\varrho)$*-projective,*

(xix) A_S *is* $(S/sS, S/\varrho)$*-projective,*

(xx) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), \coprod_{r\in R} S/\varrho_r)$*-projective,*

(xxi) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), S/\varrho)$*-projective,*

(xxii) A_S *is* $(S/\varrho(x, y), S/\varrho)$*-projective,*

Proof. The equivalences $(i) \Leftrightarrow (vi) \Leftrightarrow (viii)$ are parts of Corollary 3.14.

The equivalences $(i) \Leftrightarrow (iv)$, $(v) \Leftrightarrow (vi)$, $(vii) \Leftrightarrow (viii)$, $(xv) \Leftrightarrow (xvi)$, $(xviii) \Leftrightarrow (xix)$ and $(xxi) \Leftrightarrow (xxii)$ are parts of Proposition 3.10.

The equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are given by Lemma 3.29.

Thus we get that for an arbitrary monoid S the equivalence of the assertions (i) to $(viii)$ holds.

The equivalences $(ix) \Leftrightarrow (x) \Leftrightarrow (xi) \Leftrightarrow (xii) \Leftrightarrow (xiii)$ are a part of Lemma 3.29.

The equivalences $(xiv) \Leftrightarrow (xvii)$ and $(xv) \Leftrightarrow (xviii)$ are given by Proposition 3.27.

$(xviii) \Leftrightarrow (xxi)$ and $(xvii) \Leftrightarrow (xx) \Leftrightarrow (xi)$ are given by Lemma 4.1, since S has a left zero.

$(xx) \Leftrightarrow (xxi)$ is valid by Lemma 3.26, since S contains a left zero.

Thus for a monoid S containing a left zero, the assertions (ix) to $(xxii)$ are equivalent.

The equivalence $(iv) \Leftrightarrow (xviii)$, given by Lemma 4.1, completes the proof.

□

4.1.3 Equivalences to $(S, S/\varrho(x,y))$-projectivity

Theorem 4.5 *Let ϱ, ϱ_m, $m \in M$, be right congruences on S, let G_S denote a generator in* **Act** $-S$, *let $B_S \in$* **Act** $-S$, *let x, $y \in S$, and let e, $e_j \in E(S)$, $j \in J$. Then for $A_S \in$* **Act** $-S$ *the following assertions are equivalent*

(i) A_S *is* $(S, S/\varrho(x,y))$*-projective (i.e.,* A_S *is principally weakly projective),*

(ii) A_S *is* $(B, S/\varrho(x,y))$*-projective,*

(iii) A_S *is* $(G, S/\varrho(x,y))$*-projective,*

(iv) A_S *is* $(\coprod_{i\in I} S, S/\varrho(x,y))$*-projective,*

(v) A_S *is* $(\coprod_{m\in M} S/\varrho_m, S/\varrho(x,y))$*-projective,*

(vi) A_S *is* $(S/\varrho, S/\varrho(x,y))$*-projective,*

(vii) A_S *is* $(\coprod_{j\in J} e_jS, S/\varrho(x,y))$*-projective,*

(viii) A_S *is* $(eS, S/\varrho(x,y))$*-projective.*

Furthermore, let $R \neq \emptyset$ be a set, let x_r, y_r, x_n, y_n, s, $s_l \in S$ ($r \in R$, $n \in N$, $l \in L$) and let $\hat{I}$, I_k, $k \in K$, be right ideals of S. If S contains a left zero, the assertions (i) *to* (viii) *are equivalent to:*

(ix) A_S *is* $(B, \coprod_{n\in N} S/\varrho(x_n,y_n))$*-projective,*

(x) A_S *is* $(G, \coprod_{n\in N} S/\varrho(x_n,y_n))$*-projective,*

(xi) A_S *is* $(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n,y_n))$*-projective,*

(xii) A_S *is* $(\coprod_{m\in M} S/\varrho_m, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(xiii) A_S *is* $(\coprod_{j\in J} e_jS, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(xiv) A_S *is* $(\coprod_{k\in K} S/I_k, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(xv) A_S *is* $(\coprod_{k\in K} S/I_k, S/\varrho(x, y))$*-projective,*

(xvi) A_S *is* $(S/\hat{I}, S/\varrho(x, y))$*-projective,*

(xvii) A_S *is* $(\coprod_{l\in L} S/s_lS, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(xviii) A_S *is* $(\coprod_{l\in L} S/s_lS, S/\varrho(x, y))$*-projective,*

(xix) A_S *is* $(S/sS, S/\varrho(x, y))$*-projective,*

(xx) A_S *is* $(\coprod_{r\in R} S/\varrho(x_r, y_r), \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(xxi) A_S *is* $(\coprod_{r\in R} S/\varrho(x_r, y_r), S/\varrho(x, y))$*-projective,*

(xxii) A_S *is* $(S/\varrho(x_r, y_r), S/\varrho(x, y))$*-projective,*

Proof. In analogy to the proof of Theorem 4.4, using Corollary 3.14, Proposition 3.27, Lemma 3.29 and Lemma 4.1 for $Y = S/\varrho(x, y)$ and Proposition 3.10 and Lemma 3.26 for right principal congruences ϱ and $\varrho_m,\ m \in M$.

□

4.1.4 Equivalences to $(S, S/I)$-projectivity

Theorem 4.6 *Let* $\varrho,\ \varrho_m,\ m \in M$*, be right congruences on* S*, let* G_S *denote a generator in* $\mathbf{Act}-S$*, let* $B_S \in \mathbf{Act}-S$*, let* $\hat{I},\ I_k,\ k \in K$*, be right ideals of* S*, and let* $e,\ e_j \in E(S),\ j \in J$*. Then for* $A_S \in \mathbf{Act}-S$ *the following assertions are equivalent*

(i) A_S *is* $(S, S/\hat{I})$*-projective (i.e.,* A_S *is Rees weakly projective),*

(ii) A_S *is* $(B, \coprod_{k\in K} S/I_k)$*-projective,*

(iii) A_S *is* $(B, S/\hat{I})$*-projective,*

(iv) A_S *is* $(G, \coprod_{k\in K} S/I_k)$*-projective,*

(v) A_S *is* $(G, S/\hat{I})$*-projective,*

(vi) A_S *is* $(\coprod_{i\in I} S, \coprod_{k\in K} S/I_k)$*-projective,*

(vii) A_S *is* $(\coprod_{i\in I} S, S/\hat{I})$*-projective,*

(viii) A_S *is* $(\coprod_{m\in M} S/\varrho_m, \coprod_{k\in K} S/I_k)$*-projective,*

(ix) A_S *is* $(\coprod_{m\in M} S/\varrho_m, S/\hat{I})$*-projective,*

(x) A_S *is* $(S/\varrho, S/\hat{I})$*-projective,*

(xi) A_S *is* $(\coprod_{j\in J} e_jS, \coprod_{k\in K} S/I_k)$*-projective,*

(xii) A_S *is* $(\coprod_{j\in J} e_jS, S/\hat{I})$*-projective,*

(xiii) A_S *is* $(eS, S/\hat{I})$*-projective.*

Furthermore, let $R \neq \emptyset$ *be a set, let* $s,\ s_l,\ x,\ y,\ x_n,\ y_n \in S$ $(l \in L,\ n \in N)$, *and let* $O,\ I_r,\ r \in R$, *be right ideals of* S. *If* S *contains a left zero, the assertions* (i) *to* (xiii) *are equivalent to:*

(xiv) A_S *is* $(\coprod_{r\in R} S/I_r, \coprod_{k\in K} S/I_k)$*-projective,*

(xv) A_S *is* $(\coprod_{r\in R} S/I_r, S/\hat{I})$*-projective,*

(xvi) A_S *is* $(S/O, S/\hat{I})$*-projective,*

(xvii) A_S *is* $(\coprod_{l\in L} S/s_lS, \coprod_{k\in K} S/I_k)$*-projective,*

(xviii) A_S *is* $(\coprod_{l\in L} S/s_lS, S/\hat{I})$*-projective,*

(xix) A_S *is* $(S/sS, S/\hat{I})$*-projective,*

(xx) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), \coprod_{k\in K} S/I_k)$*-projective,*

(xxi) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), S/\hat{I})$*-projective,*

(xxii) A_S *is* $(S/\varrho(x, y), S/\hat{I})$*-projective.*

Proof. The equivalences $(i) \Leftrightarrow (x) \Leftrightarrow (xiii)$ are parts of Corollary 3.14.

The equivalences $(ii) \Leftrightarrow (iii)$, $(iv) \Leftrightarrow (v)$, $(vi) \Leftrightarrow (vii)$, $(viii) \Leftrightarrow (ix)$, $(xi) \Leftrightarrow (xii)$, $(xiv) \Leftrightarrow (xv)$, $(xvii) \Leftrightarrow (xviii)$ and $(xx) \Leftrightarrow (xxi)$ are included in Proposition 3.24.

The equivalences $(i) \Leftrightarrow (vii)$, $(ix) \Leftrightarrow (x)$, $(xv) \Leftrightarrow (xvi)$, $(xviii) \Leftrightarrow (xix)$ and $(xxi) \Leftrightarrow (xxii)$ are given by Proposition 3.10.

The equivalences $(iii) \Leftrightarrow (v) \Leftrightarrow (vii)$ and $(viii) \Leftrightarrow (xi)$ are parts of Lemma 3.29.

So far for an arbitrary monoid S the assertions (i) to $(xiii)$ are equivalent.

The equivalence $(xv) \Leftrightarrow (xviii)$ is given by Proposition 3.27.

$(xviii) \Leftrightarrow (xxi)$ is given by Lemma 4.1.

The last equivalence $(xx) \Leftrightarrow (vi)$, given by Lemma 4.1, completes the proof.

□

4.1.5 Equivalences to $(S, S/sS)$-projectivity

Theorem 4.7 *Let ϱ, ϱ_m, $m \in M$, be right congruences on S, let G_S denote a generator in $\mathbf{Act}-S$, let e, $e_j \in E(S)$, $j \in J$, let $B_S \in \mathbf{Act}-S$, and let s, $s_l \in S$, $l \in L$. Then for $A_S \in \mathbf{Act}-S$ the following assertions are equivalent*

(i) A_S *is* $(S, S/sS)$*-projective (i.e.,* A_S *is principally Rees weakly projective),*

(ii) A_S *is* $(B, \coprod_{l\in L} S/s_lS)$*-projective,*

(iii) A_S *is* $(B, S/sS)$*-projective,*

(iv) A_S *is* $(G, \coprod_{l\in L} S/s_lS)$*-projective,*

(v) A_S *is* $(G, S/sS)$*-projective,*

(vi) A_S *is* $(\coprod_{i\in I} S, \coprod_{l\in L} S/s_lS)$*-projective,*

(vii) A_S *is* $(\coprod_{i\in I} S, S/sS)$*-projective,*

(viii) A_S *is* $(\coprod_{m\in M} S/\varrho_m, \coprod_{l\in L} S/s_lS)$*-projective,*

(ix) A_S *is* $(\coprod_{m\in M} S/\varrho_m, S/sS)$*-projective,*

(x) A_S *is* $(S/\varrho, S/sS)$*-projective,*

(xi) A_S *is* $(\coprod_{j\in J} e_jS, \coprod_{l\in L} S/s_lS)$*-projective,*

(xii) A_S *is* $(\coprod_{j\in J} e_jS, S/sS)$*-projective,*

(xiii) A_S *is* $(eS, S/sS)$*-projective.*

Furthermore, let $R \neq \emptyset$ *be a set, let* $x,\ y,\ x_n,\ y_n,\ t,\ s_r \in S$ $(n \in N,\ r \in R)$*, and let* $\hat{I},\ I_k,\ k \in K$*, be right ideals of* S*. If* S *contains a left zero, the assertions* (*i*) *to* (*xiii*) *are equivalent to:*

(xiv) A_S *is* $(\coprod_{r\in R} S/s_rS, \coprod_{l\in L} S/s_lS)$*-projective,*

(xv) A_S *is* $(\coprod_{r\in R} S/s_rS, S/sS)$*-projective,*

(xvi) A_S *is* $(S/tS, S/sS)$*-projective,*

(xvii) A_S *is* $(\coprod_{k\in K} S/I_k, \coprod_{l\in L} S/s_lS)$*-projective,*

(xviii) A_S *is* $(\coprod_{k\in K} S/I_k, S/sS)$*-projective,*

(xix) A_S *is* $(S/\hat{I}, S/sS)$*-projective,*

(xx) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), \coprod_{l\in L} S/s_lS)$*-projective,*

(xxi) A_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), S/sS)$*-projective,*

(xxii) A_S *is* $(S/\varrho(x, y), S/sS)$*-projective.*

Proof. Using Corollary 3.14, Proposition 3.27, Lemma 3.29 and Lemma 4.1 with $Y = S/sS$ or $Y = \coprod_{l\in L} S/s_lS$, respectively, for right Rees congruences ϱ and $\varrho_m,\ m \in M$, Proposition 3.10 with $X \cong S/\hat{I},\ X^i \cong S/I_k$, and Corollary 3.25, one gets the equivalences in analogy to the proof of Theorem 4.6.

□

4.2 Summarizing table

By the results of section 4.1, in the case of S being a monoid with left zero, Table 3 of page 46 turns into the following table (Table 4).
Arrays belonging to a pair (X, Y), for which (X, Y)-projectivity is equivalent to quasi-projectivity (qp) or x weak projectivity with $x \in \{\Phi, p, R, pR\}$, are marked by the respective abbreviation (see informations to Table 1).

If (X, Y)-projectivity has been proved to be equivalent to another projectivity, this is marked by the number of the respective Proposition/Lemma/Corollary. Like before, an array is marked by $(*)$, if it belongs to a pair (X, Y), for which (X, Y)-projectivity is equivalent to another, whenever it is a non-trivially-(X, Y)-projectivity.

	C_S	A_S	G_S	$\coprod_{m\in M} S/\varrho_m$	S/ϱ	$\coprod_{n\in N} S/\sigma_n$	S/σ	$S/\hat{I}$	S/sS
A_S	qp		gen or triv						
$\coprod_{i\in I} S$	proj	proj	proj	wp	wp	pwp	pwp	Rwp	pRwp
S_S	o	$(*)$		o	wp	o	pwp	Rwp	pRwp
$\coprod_{n\in N} S/\sigma_n$	proj	proj	proj	wp	wp	pwp	pwp	Rwp	pRwp
S/σ	o	$(*)$	4.2	o	wp	o	pwp	Rwp	pRwp
$\coprod_{l\in L} S/s_lS$	proj	proj	proj	wp	wp	pwp	pwp	Rwp	pRwp
S/sS	o	$(*)$	4.2	o	wp	o	pwp	Rwp	pRwp

Table 4: Table of results for monoids with left zero

As indicated in the introduction of Chapter 4, in the case of S being a monoid with left zero, the concepts of (A, Y)-projectivity with $Y_S \in \mathbf{Act}-S$, the x-weak projectivities and (S, G)-projectivity remain for further investigations.

4.3 Further implications, implication scheme

To summarize the remaining properties in the case of monoids with left zero in an implication scheme, we prove further implications:

Lemma 4.8 *Let S be a monoid with left zero, let $X_S \in \mathbf{Act}-S$ and let $s,\ x,\ y \in S$. Then A_S is $(X, S/\varrho(x,y))$-projective implies A_S is $(X, S/sS)$-projective.*

Proof. Since S has a left zero z, every right principal Rees congruence ϱ_{sS} is the right principal congruence $\varrho(s, sz)$ by Remark 2.23.

□

Lemma 4.9 *Let S be a monoid with left zero and let $\varrho,\ \varrho_m,\ m \in M$, be right congruences on S. If A_S is $(S, S/\varrho)$-projective then A_S is $(A, \coprod_{m\in M} S/\varrho_m)$-projective.*

Proof. Let $g : A \longrightarrow \coprod_{m\in M} S/\varrho_m$ be an epimorphism, let $f : A \longrightarrow \coprod_{m\in M} S/\varrho_m$ be a homomorphism and let $\pi := \coprod_{m\in M} \pi_{\varrho_m} : \coprod_{m\in M} S \longrightarrow \coprod_{m\in M} S/\varrho_m$ be the coproduct of the canonical epimorphisms. By the projectivity of $\coprod_{m\in M} S$, there exists $\pi' : \coprod_{m\in M} S \longrightarrow A$, such that $g\pi' = \pi$. Since A_S is $(S, S/\varrho)$-projective, which by Theorem 4.4 is equivalent to A_S being $(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$-projective, there exists $f' : A \longrightarrow \coprod_{m\in M} S$ with $\pi f' = f$. Thus for $\pi' f' : A \longrightarrow A$ we get $g(\pi' f') = (g\pi')f' = \pi f' = f$ and thus A_S is $(A, \coprod_{m\in M} S/\varrho_m)$-projective.

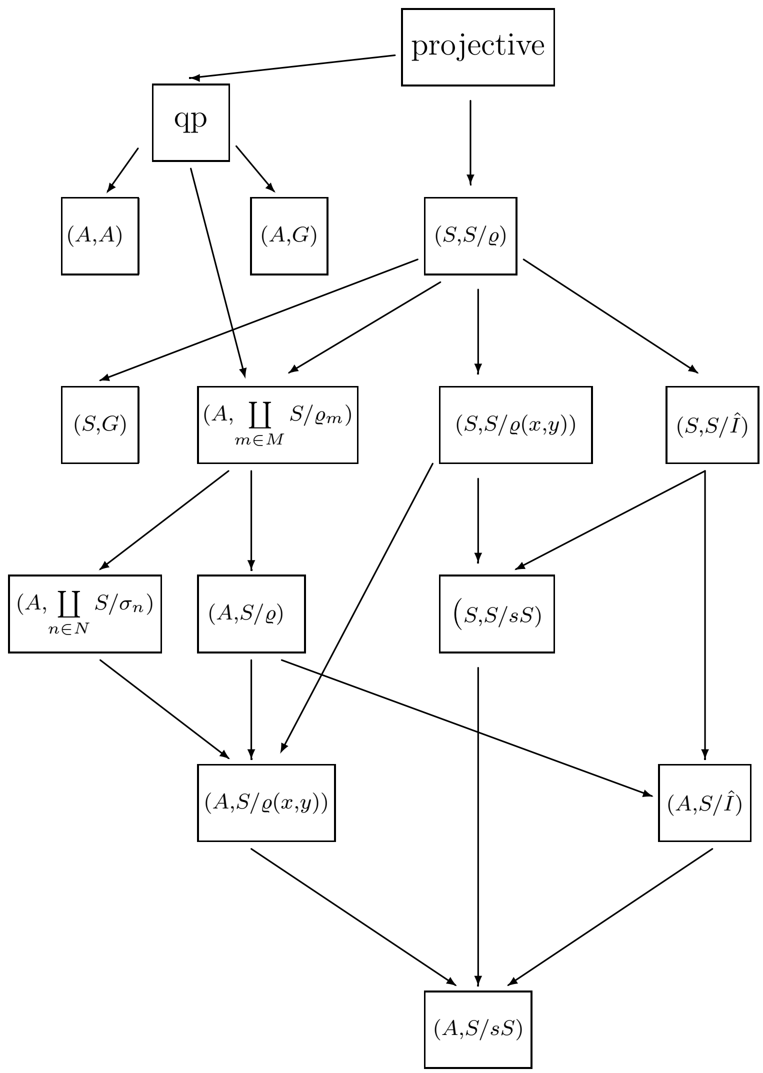
□

Lemma 4.10 *Let S be a monoid with left zero, let $x \in \{\phi, p, R, pR\}$ and let ϱ be a right x-congruence on S. If A_S is $(S, S/\varrho)$-projective then A_S is $(A, S/\varrho)$-projective.*

Proof. This implication is obtained by using the projectivity of S in the same way like in the proof of the previous Lemma.

□

Thus, if S is a monoid with left zero z, the implication scheme part 1 and part 2, giving at the end of Chapter 3, turns into:



Implication scheme for S with left zero

5 Homological classification

In this Chapter questions concerning homological classification will be discussed, i.e., questions like “When do all right S-acts have property A?” or “When does property A imply property B?” will be answered by characterizing internal properties of the monoid S.
The properties A and B considered here are all $(\mathcal{X}, \mathcal{Y})$-projectivities as defined in Chapter 3.

Whenever studying homological classification of monoids, it is useful to have a look at the one element act Θ_S at first. For instance, in Proposition 5.8 it will be proved, that Θ_S is $(S, S/sS)$-projective if and only if S contains a left zero. Since "all right S-acts are ..." implies "Θ_S is ... ", by the observations concerning Θ_S we get necessary conditions on monoids over which all acts have a special property. Therefore the second paragraph deals with $(\mathcal{X}, \mathcal{Y})$-projectivities of Θ_S.
In this context a new kind of monoids occurs: Monoids **fulfilling condition** *(Moz)*. They are introduced in the first part and some basic results concerning these monoids are given (see for instance Proposition 5.4).

In the first section we begin with studying the question of all acts being $(\mathcal{X}, \mathcal{Y})$-projective, starting with the highest properties of the implication scheme part 1 (page 41), going to the highest one of part 2 (page 42) and finally ending at the bottom, i.e., at all acts being $(S/sS, S/tS)$-projective.
For instance, in turns out, that all right S-acts are $(S, S/sS)$-projective if and only if S is the disjoint union of a group and a left zero semigroup or $S = \{1\}$ (Corollary 5.68).

The next step is to study the conditions, under which further implications between the concepts are valid. We will see, for example, that $(S/sS, S/tS)$-projectivity implies $(S/\varrho(x, y), S/tS)$-projectivity if and only if S fulfills condition (Moz) (Theorem 5.73). The respective results are the main content of Section 5.2.

The results will be summarized in tables at the end of this chapter. They will be used in Chapter 6 to make sure, that there are differences between the concepts of $(\mathcal{X}, \mathcal{Y})$-projectivities and to prove, that there are no implications between $(\mathcal{X}, \mathcal{Y})$-projectivities and flatness properties as introduced in Definition 2.32. Flatness properties are considered in this context, since like (X, Y)-projectivities, they are properties of acts, which are weaker then projectivity.

A more detailed clue will be given right at the beginning of each subsection.

Condition (Moz)

In this paragraph monoids fulfilling condition (Moz) are introduced and some results concerning these monoids are presented. In the case of monoids with at least two idempotents a characterization of these monoids is given in Proposition 5.4. For monoids S with $E(S) = \{1\}$, there is no general characterization so far. In the case of the groups $(\mathbb{Z}_n, +)$, $n \in \mathbb{N}$, we get a necessary condition for fulfilling (Moz) in Lemma 5.5.

Definition 5.1 *A monoid S is said to* **fulfill condition** (Moz), *if every monocyclic act has a zero.*

Remark 5.2 *If $\varrho \neq \Delta$ is a right congruence on S, then for $(x, y) \in \varrho$ we get $\varrho(x, y) \subseteq \varrho$. Thus there exists an epimorphism from $(S/\varrho(x, y))_S$ onto $(S/\varrho)_S$. Therefore, if S is a monoid, which fulfills (Moz), then every factor act of S by a right congruence $\varrho \neq \Delta$ has a zero.*

Note, that if S is a monoid with left zero, then all acts have zeros and thus S fulfills condition (Moz).

Examples 5.3 (a) Every left zero semigroup with identity adjoined fulfills condition (Moz).

(b) The group $(\mathbb{Z}_3, +)$ fulfills condition (Moz), since every monocyclic act $(\mathbb{Z}_3/\varrho(x, y))_{\mathbb{Z}_3}$ is isomorphic to $\Theta_{\mathbb{Z}_3}$.

(c) The group $(\mathbb{Z}_4, +)$ does not fulfill condition (Moz), since the monocyclic act $(\mathbb{Z}_4/\varrho(1, 3))_{\mathbb{Z}_4}$ has no zero.

Proposition 5.4 *For a monoid S the following assertions are equivalent:*

(i) $|E(S)| > 1$ *and S fulfills condition* (Moz),

(ii) S contains a left zero.

Proof. $(i) \Longrightarrow (ii)$ For $e \in E(S)$, $e \neq 1$ consider the right principal congruence $\varrho(e, 1)$. By Lemma 2.2 of [5], for x, $y \in S$ and any $e \in S$ the relation $x\,\varrho(e, 1)\,y$ is equivalent to $e^m x = e^n y$ for some m, $n \geq 0$. Since $e \in E(S)$ this yields:

$$x\,\varrho(e, 1)\,y \Leftrightarrow ex = ey.$$

Let now $[z]_{\varrho(e,1)}$ denote the zero in $S/\varrho(e,1)$, i.e., $[z]_{\varrho(e,1)} = [z]_{\varrho(e,1)}s = [zs]_{\varrho(e,1)}$ for every $s \in S$. Thus for every $s \in S$ we get $z\,\varrho(e,1)\,zs$, which is equivalent to $ez = ezs$ for every $s \in S$. Therefore ez is a left zero in S.

$(ii) \implies (i)$ If z is a left zero in S, then $z^2 = z$, i.e., $z \in E(S)$ and thus $|E(S)| > 1$. Furthermore, if S contains a left zero, then every right S-act has a zero, which implies condition (Moz) for the monoid S.

□

Lemma 5.5 *Let $n \in \mathbb{N}$, $n \geq 5$. If $(\mathbb{Z}_n, +)$ fulfills condition (Moz) then $n = 2k+1$ for $k \in \mathbb{N}$.*

Proof. Let $\{0, 1, 2, \ldots, n-1\}$ denote a representing system of the residue classes of $\mathbb{Z}$ by n and let $x,\ y \in \mathbb{Z}_n$. Without loss of generality let $y \leq x$.

By Lemma 2.2 of [5] we get the following chain of equivalences:

$$\begin{aligned} x\,\varrho(0,2)\,y \quad &\Leftrightarrow\ \exists i,\ j \in \mathbb{N}_0 :\ i2 + x = j2 + y \\ &\Leftrightarrow\ \exists i,\ j \in \mathbb{N}_0 :\ x - y = 2(j-i) \\ &\Leftrightarrow\ \exists k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{N}_0 : \\ &\qquad (x = 2k_1\ \wedge\ y = 2k_2) \\ &\quad\ \vee (x = 2k_3 + 1\ \wedge\ y = 2k_4 + 1) \end{aligned}$$

Let now $[z]$ be a zero in $\mathbb{Z}_n/\varrho(0,2)$. Then $[z] = [z + (n-1)]$ and thus:

$$\begin{aligned} z\,\varrho(0,2)\,(z + (n-1)) \quad &\Leftrightarrow\ \exists k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{N}_0 : \\ &\qquad (z = 2k_1\ \wedge\ z + (n-1) = 2k_2) \\ &\quad\ \vee (z = 2k_3 + 1\ \wedge\ z + (n-1) = 2k_4 + 1) \\ &\Leftrightarrow \exists k \in \mathbb{N} :\ n - 1 = 2k \\ &\Leftrightarrow\ \exists k \in \mathbb{N} :\ n = 2k + 1 \end{aligned}$$

□

(As I presume, even the converse is true, but it has not been proved so far.) A more general result in view to monoids without idempotents $e,\ e \neq 1$, fulfilling condition (Moz) remains to be proved.

A useful property of monocyclic acts over monoids fulfilling condition (Moz) is proved in the following Lemma, which will be used in the second paragraph of subsection 5.1.3.

Lemma 5.6 *Let S be a monoid, which fulfills condition (Moz) and let $u,\ x,\ y \in S$. Then every monocyclic act $(S/\varrho(x,y))_S$ is the epimorphic image of a Rees factor act $(S/uS)_S$ of S_S by a principal right ideal uS of S.*

Proof. Let $[u]_{\varrho(x,y)}$ be the zero in $S/\varrho(x,y)$. Then we get $us \in [u]_{\varrho(x,y)}$ for every $s \in S$, i.e., $I = uS \subseteq [u]_{\varrho(x,y)}$. Therefore $\varrho_{uS} \subseteq \varrho(x,y)$ and thus by the Homomorphism Theorem for acts (2.7) there exists an epimorphism from S/uS onto $S/\varrho(x,y)$.

□

Θ_S is $(\mathcal{X}, \mathcal{Y})$-projective

Like indicated before, in some of the proofs concerning results with respect to homological classification of monoids, the one element act Θ_S, having the respective property, will be used. Therefore, in this paragraph we study Θ_S with respect to different $(\mathcal{X}, \mathcal{Y})$-projectivities.

Lemma 5.7 *Let $\mathcal{X}, \mathcal{Y}$ be classes of right S-acts. If $\Theta_S \in \mathcal{Y}$, then Θ_S is $(\mathcal{X}, \mathcal{Y})$-projective if for all $X_S \in \mathcal{X}$ the zero morphism $z : X_S \longrightarrow \Theta_S$ is a retraction.*
Moreover, this implies that every $X_S \in \mathcal{X}$ has a zero.

Proof. If $\Theta_S \in \mathcal{Y}$, then $(\mathcal{X}, \mathcal{Y})$-projectivity of Θ_S implies, that for every $X \in \mathcal{X}$ the identity id_Θ can be lifted with respect to the zero morphism $z : X \longrightarrow \Theta$, i.e., there exists $id'_\Theta : \Theta \longrightarrow X$, such that $z\, id'_\Theta = id_\Theta$, i.e, z is a retraction and $id'(\Theta)$ is a zero in X.

□

Proposition 5.8 *Let $\varrho,\ \varrho_m,\ m \in M$, be right congruences on S, let $\hat{I},\ I_k,\ k \in K$, be right ideals of S and let $s,\ s_l \in S,\ l \in L$. Then the following assertions are equivalent:*

(i) Θ_S *is* $(\coprod_{i \in I} S, \coprod_{m \in M} S/\varrho_m)$*-projective,*

(ii) Θ_S *is* $(S, S/\varrho)$*-projective,*

(iii) Θ_S *is* $(\coprod_{i \in I} S, \coprod_{k \in K} S/I_k)$*-projective,*

(iv) Θ_S *is* $(S, S/\hat{I})$*-projective,*

(v) Θ_S *is* $(\coprod_{i \in I} S, \coprod_{l \in L} S/s_l S)$*-projective,*

(vi) Θ_S *is* $(S, S/sS)$*-projective,*

(vii) S *contains a left zero,*

(viii) Θ_S *is* (S, Θ)*-projective,*

(ix) Θ_S *is projective*

Proof. Since $\Theta_S \cong S/\varrho = S/1S,\ \varrho = S \times S$, is a cyclic act, by Proposition 3.12 the equivalences $(i) \Leftrightarrow (ii),\ (iii) \Leftrightarrow (iv)$ and $(v) \Leftrightarrow (vi)$ are valid.

The implications $(ii) \Longrightarrow (iv) \Longrightarrow (vi)$ are obvious.

$(vi) \Longrightarrow (vii)$ and $(viii) \Longrightarrow (vii)$. Since $\Theta_S \cong S/1S$, by (vi) (or $(viii)$, respectively) S contains a left zero by Lemma 5.7.

$(vii) \Leftrightarrow (ix)$ is a part of Corollary 2.34.

$(ix) \Longrightarrow (i)$ and $(ix) \Longrightarrow (viii)$ are obvious.

□

Proposition 5.9 *Let $x,\ y,\ x_n,\ y_n \in S,\ n \in N$. Then the following assertions are equivalent:*

(i) Θ_S *is* $(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projective,*

(ii) Θ_S *is* $(S, S/\varrho(x, y))$*-projective,*

(iii) S *is a monoid with left zero or no monocyclic act has a zero.*

Proof. Since $\Theta_S \cong S/\varrho = S/1S,\ \varrho = S \times S$, is a cyclic act, by Proposition 3.10 the equivalence $(i) \Leftrightarrow (ii)$ is valid.

$(ii) \Longrightarrow (iii)$. If there exists a monocyclic act $S/\varrho(x, y)$ with zero $\tilde{z}$, then $f : \Theta \longrightarrow S/\varrho(x, y)$ with $f(\Theta) = \tilde{z}$ is a homomorphism. By (ii) there exists a homomorphism $f' : \Theta \longrightarrow S$, which implies, that $f'(\Theta)$ is a left zero in S.

$(iii) \Longrightarrow (i)$. If S contains a left zero z, then $\Theta_S \cong zS$, which is projective and thus (i) is valid.
Otherwise, there does not exist a homomorphism from Θ_S into a coproduct of monocyclic acts, i.e., Θ_S is trivially-$(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n, y_n))$-projective and therefore $(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n, y_n))$-projective.

□

Note, that (iii) of Proposition 5.9 implies that S contains a left zero or S does not fulfill condition (Moz).

Proposition 5.10 *Let $s,\ s_l \in S,\ l \in L$, let $x,\ y,\ x_n,\ y_n \in S,\ n \in N$, and let $I,\ I_k,\ k \in K$, be right ideals of S. Then the following assertions are equivalent:*

(i) Θ_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), \Theta)$ *-projective,*

(ii) Θ_S *is* $(S/\varrho(x, y), \Theta)$*-projective,*

(iii) Θ_S *is* $(\coprod_{n\in N} S/\varrho(x_n, y_n), \coprod_{k\in K} S/I_k)$*-projective,*

(iv) Θ_S *is* $(S/\varrho(x,y), S/I)$*-projective,*

(v) Θ_S *is* $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{l\in L} S/s_lS)$*-projective,*

(vi) Θ_S *is* $(S/\varrho(x,y), S/sS)$*-projective,*

(vii) S *fulfills condition* (Moz)*.*

Proof. Since $\Theta_S \cong S/(S\times S) = S/1S$ is a cyclic act, by Proposition 3.12 the equivalences $(i) \Leftrightarrow (ii)$, $(iii) \Leftrightarrow (iv)$ and $(v) \Leftrightarrow (vi)$ are valid.

The implication $(iv) \Longrightarrow (vi)$ is obvious.

$(ii) \Longrightarrow (vii)$ and $(vi) \Longrightarrow (vii)$. Since $\Theta_S \cong S/1S$, for $x,\ y \in S$ by (vi) or (ii), respectively, Lemma 5.7 gives the existence of a zero in $S/\varrho(x,y)$, i.e., (vii).

$(vii) \Longrightarrow (ii)$ and $(vii) \Longrightarrow (iv)$. If S contains a left zero z, then $\Theta_S \cong zS$, which is projective, since $z \in E(S)$. Thus $(vii) \Longrightarrow (ii)$ and $(vii) \Longrightarrow (iv)$.
If S is a monoid without left zero fulfilling condition (Moz), then for $x,\ y \in S$ the zero morphism $z : S/\varrho(x,y) \longrightarrow \Theta$ is a retraction, i.e., $(vii) \Longrightarrow (ii)$ by Lemma 5.7.
Furthermore, since in this case every Rees factor act S/I has a unique zero $\bar{0}$, then $f : \Theta \longrightarrow S/I$ being a homomorphism implies $f(\Theta) = \bar{0}$. Let $g : S/\varrho(x,y) \longrightarrow S/I$ be an epimorphism, let $\tilde{z}$ be a zero in $S/\varrho(x,y)$. Then $g(\tilde{z}) = \bar{0}$ and the mapping $f' : \Theta \longrightarrow S/\varrho(x,y)$ defined by $f'(\Theta) = \tilde{z}$ is a homomorphism with $g\,f' = f$. Thus $(vii) \Longrightarrow (iv)$.

□

Note that the assertions made in the previous Proposition are not equivalent to Θ_S being $(S/\varrho(x,y), S/\varrho)$- and $(S/sS, S/\varrho)$-projective. The case of $(S/sS, S/\varrho)$-projectivity will be discussed later on (Lemma 5.12 and Proposition 5.14).

Proposition 5.11 *Let* $s,\ t,\ s_l \in S,\ l \in L$*, and let* I *be a right ideal of* S*. The one element right* S*-act* Θ_S *is*

(a) $(\coprod_{l\in L} S/s_lS, \Theta)$*-projective,*

(b) $(S/sS, \Theta)$*-projective,*

(c) $(S/sS, S/I)$*-projective,*

(d) $(S/sS, S/tS)$*-projective.*

Proof. Since every coproduct of Rees factor acts has a zero, the zero morphism $z : \coprod_{l \in L} S/s_l S \longrightarrow \Theta$ is a retraction. By Lemma 5.7 this yields (a), which implies (b).

If S contains a left zero z, then $\Theta_S \cong zS$ and is therefore projective, which implies (c) and (d).
If S is a monoid without left zero, then every Rees factor act contains a unique zero. Let 0_{sS} denote the zero in S/sS and 0_I the zero in S/I. Then for an epimorphism $g : S/sS \longrightarrow S/I$ we get $g(0_{sS}) = 0_I$. Furthermore, for a homomorphism $f : \Theta \longrightarrow S/I$ the equation $f(\Theta) = 0_I$ holds. Then f can be lifted with respect to g by $f' : \Theta \longrightarrow S/sS$ with $f'(\Theta) = 0_{sS}$, i.e., (c), which implies (d).

□

If the first component is the coproduct of Rees factor acts by principal right ideals, one could not use the same argumentation like in the proof of Proposition 5.11, if the second component is a factor act of S by a (principal) right congruence ϱ (or a coproduct of these), because there could exist two zeros in $(S/\varrho)_S$, one of them the image of Θ_S by the zero morphism, the other the image of the zero element of $(S/sS)_S$ under an epimorphism. Then this morphism could not be lifted, i.e., Θ_S is not $(S/sS, S/\varrho)$-projective.

In the following we will see, that for left reversible monoids this case can not arise.

Lemma 5.12 *Let S be a monoid without left zero. Then every epimorphic image of a Rees factor act by a proper right ideal contains a unique zero iff S is left reversible.*

Proof. Sufficiency. Let I be a proper right ideal of S, let $\bar{0}$ denote the zero in S/I and let $g : S/I \longrightarrow S/\varrho$ be an epimorphism. Then $g(\bar{0}) =: \tilde{z}$ is a zero of S/ϱ. Suppose $\tilde{z'}$ is a zero in S/ϱ. Since g is an epimorphism, there exists $\bar{s} \in S/I$ such that $g(\bar{s}) = \tilde{z'}$. Then for all $t \in S$ we get:

$$g(\bar{st}) = g(\bar{s}\,t) = g(\bar{s})\,t = \tilde{z'}\,t = \tilde{z'}.$$

Thus $g(\bar{s}S) = \tilde{z'}$.

If S is left reversible, there exists $j \in sS \cap I$. Thus $\tilde{z'} = g(\bar{j}) = \tilde{z}$, i.e., the zero in S/I is unique.

Necessity. Suppose S not to be left reversible, i.e., there exist right ideals I, J with $I \cap J = \emptyset$. Then $I \cup J \neq S$, since this would imply $1 \in J$, contradicting $I \cap J = \emptyset$. Then $\bar{J} = \{\bar{j} \in S/I \mid j \in J\}$ is a subact of $(S/I)_S$ and the Rees factor act $((S/I)/\bar{J})_S$ is an epimorphic image of $(S/I)_S$, which contains two

zeros.
Thus the uniqueness of the zero implies left reversibility of the monoid.

□

Together with Lemma 3.5 this yields:

Corollary 5.13 *Let S be a monoid without left zero, let I_m, $m \in M$, be right ideals of S, let ϱ_m, $m \in M$, be right congruences on S and let $g : \coprod_{m\in M}(S/I_m)_S \longrightarrow \coprod_{m\in M}(S/\varrho_m)_S$ be an epimorphism. Then every factor act $(S/\varrho_m)_S$ has a unique zero iff S is left reversible.*

□

Proposition 5.14 *Let S be a monoid, which fulfills condition (Moz), let $s,\ s_l \in S$, $l \in L$ and let ϱ, ϱ_m, $m \in M$, be right congruences on S. Then the following assertions are equivalent:*

(i) Θ_S *is* $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$*-projective,*

(ii) Θ_S *is* $(S/sS, S/\varrho)$*-projective,*

(iii) *S is left reversible or S contains a left zero.*

Proof. $(i) \Leftrightarrow (ii)$ is valid by Proposition 3.12, since Θ_S is a cyclic act.
$(ii) \Longrightarrow (iii)$. Let S be a monoid without left zero. For every proper right ideal I of S there exists a right congruence ϱ on S, such that $\varrho_I \subseteq \varrho$. Thus there exists an epimorphism $g : S/I \longrightarrow S/\varrho$. For $i \in I$ there exists an epimorphism $h : S/iS \longrightarrow S/I$. Thus $gh : S/iS \longrightarrow S/\varrho$ is an epimorphism, too.
Let $\bar{0}_{iS}$ denote the zero in S/iS, which is unique, since S is a monoid without left zero. Then $gh(\bar{0}_{iS}) =: z$ is a zero in S/ϱ.
For every zero z' in S/ϱ the mapping $f : \Theta \longrightarrow S/\varrho$ with $f(\Theta) = z'$ is a homomorphism. By (ii), there exists a homomorphism $f' : \Theta \longrightarrow S/iS$, such that $(gh)f' = f$. By the uniqueness of the zero in S/iS we get $f'(\Theta) = \bar{0}_{iS}$. This implies:

$$z' = f(\Theta) = (gh)f'(\Theta) = (gh)(\bar{0}_{iS}) = z,$$

i.e., the zero in S/ϱ is unique. Thus by Proposition 5.12, S is left reversible.
$(iii) \Longrightarrow (ii)$. If S is a monoid with left zero, then by Proposition 5.8 Θ_S is projective and thus $(S/sS, S/\varrho)$-projective.
Let S be a left reversible monoid without left zero, let $g : S/sS \longrightarrow S/\varrho$ be an epimorphism and let $\bar{0}_{sS}$ denote the zero in S/sS. Then, by Lemma 5.12, S/ϱ contains a unique zero z. If $f : \Theta \longrightarrow S/\varrho$ is a homomorphism, then $f(\Theta) = z = g(\bar{0}_{sS})$. Thus for $f' : \Theta \longrightarrow S/sS$ with $f'(\Theta) = \bar{0}_{sS}$ the equality $gf' = f$ holds, i.e., (ii).

□

Remark 5.15 *Note, that by Corollary 2.34 the one-element right S-act fulfills condition (P) if and only if Θ_S is flat, which is the case if and only if S is right reversible.*

An important part of the proof of Proposition 5.14 was the fact, that epimorphic images of Rees factor acts are factor acts of S by some right congruences on S. These congruences need not to be right principal congruences. Thus for factor acts by right principal congruences as second component, one could not use Lemma 5.12.

5.1 All acts are $(\mathcal{X}, \mathcal{Y})$-projective

In this section properties of monoids over which all acts are $(\mathcal{X}, \mathcal{Y})$-projective are studied. The results concerning $(\mathcal{X}, \mathcal{Y})$-projectivity of the one-element act Θ_S will be used in various proofs.
At the beginning, we recall some results concerning homological classification of monoids by projectivity, which are also important for the following proofs For details, see ([14], [5] et al). For the same reason, the equivalences of Lemma 5.18 are given.

Result 5.16 *(cf. [14]) All right S-acts are projective if and only if $S = \{1\}$.*

Result 5.17 *([5]) All monocyclic right S-acts, i.e., all factor acts of the form $(S/\varrho(x,y))_S$ for $x,\ y \in S$, are projective if and only if $S = \{1\}$ or $S = \{0,1\}$.*

We start with a general result, which will be used in the next proofs:

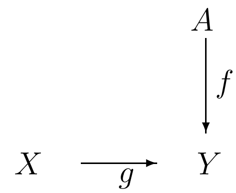
Lemma 5.18 *Let $\mathcal{X}$, $\mathcal{Y}$ denote classes of right S-acts and let $X \in \mathcal{X}$, $\mathrm{Y} \in \mathcal{Y}$. Then the following assertions are equivalent:*

(i) All right S-acts are (X, Y)-projective,

(ii) all $Z \in \mathcal{Y}$ are (X, Y)-projective,

(iii) for all $Z \in \mathcal{Y}$ every epimorphism $g : X_S \longrightarrow Z_S$ is a retraction.

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (iii)$ Let $Z \in \mathcal{Y}$ and $g : X \longrightarrow Z$ be an epimorphism. Then by (ii) the identity id_Z on Z can be lifted with respect to g, i.e., there exists a homomorphism γ with $g\,\gamma = id_Z$. This yields (iii).
$(iii) \Longrightarrow (i)$ Let $A_S \in$ **Act** $-S$. Consider in **Act** $-S$ the following diagram:



By (iii) there exists a homomorphism γ with $g\,\gamma = id_Y$, which implies $g\,(\gamma\, f) = f$. Thus A_S is (X, Y)-projective.

□

5.1.1 All are (X, G)-projective

This subsection deals with $(\mathcal{X}, \mathcal{Y})$-projectivities, where the second component is the class of all generators in $\mathbf{Act}-S$. Starting with first components being coproducts of copies of S, we shortly consider copies of Rees factor acts of S by principal right ideals as first components in Corollary 5.20 again, even if they were left out in the implication scheme (part 1). At the end, (S, G)-projectivity of all acts is studied.

Theorem 5.19 *Let G_S be a generator in $\mathbf{Act}-S$. Then the following assertions are equivalent:*

(i) All right S-acts are $(\coprod_{i\in I} S, G)$-projective,

(ii) all generators in $\mathbf{Act}-S$ are $(\coprod_{i\in I} S, G)$-projective,

(iii) all generators in $\mathbf{Act}-S$ are projective,

(iv) S is a group,

(v) all right S-acts fulfill condition (P).

Proof. $(i) \Leftrightarrow (ii)$ is a part of Lemma 5.18.
$(ii) \Longrightarrow (iii)$ Let H_S be a generator in $\mathbf{Act}-S$. By Proposition 2.28, there exists a free act $\coprod_{i\in I} S$ and an epimorphism $\pi : \coprod_{i\in I} S \longrightarrow H_S$.
By Lemma 5.18, π is a retraction. Thus H_S is a retract of a free and therefore projective act and is therefore projective by Proposition 2.29.
$(iii) \Longrightarrow (ii)$ is obvious.
$(iii) \Leftrightarrow (iv)$ has been proved in [13], Theorem 3.8.
$(iv) \Leftrightarrow (v)$ by Result 2.35.

□

Remember that by Proposition 3.30 there exists a non-trivially-$(\coprod_{l\in L} S/s_lS, G)$-projective act if and only if S has a left zero. This leads to:

Corollary 5.20 *Let S be a monoid with left zero and let $s_l \in S,\ l \in L$. Then the following assertions are equivalent:*

(i) All right S-acts are $(\coprod_{l\in L} S/s_lS, G)$-projective,

(ii) $S = \{1\}$.

Proof. $(i) \Longrightarrow (ii)$. Since S is a monoid with left zero, by Corollary 3.38 we obtain that $(\coprod_{l\in L} S/s_lS, G)$-projectivity is equivalent to $(\coprod_{i\in I} S, G)$-projectivity. Thus (i) implies that all acts are $(\coprod_{i\in I} S, G)$-projective. By Theorem 5.19, this is equivalent to S being a group. Since S has a left zero, this yields $S = \{1\}$.
$(ii) \Longrightarrow (i)$. If $S = \{1\}$ then all acts are projective by Result 5.16 and thus $(\coprod_{l\in L} S/s_lS, G)$-projective.

□

Although it will turn out, that (S, G)-projectivity of all acts does not give much informations about the internal properties of S, for the aim of completeness the respective result is given in Theorem 5.21.
In Lemma 3.35 it has been proved, that for a finite monoid S all acts are (S, G)-projective. Thus studying all acts being (S, G)-projective is mainly interesting in the case of infinite monoids.

By Lemma 5.18 and Proposition 3.18.6 of [14] we get:

Theorem 5.21 *Let G_S be a generator in* **Act** $-S$*. Then the following assertions are equivalent:*

(i) All right S-acts are (S, G)-projective,

(ii) all generators in **Act** $-S$ *are (S, G)-projective,*

(iii) every epimorphism $\pi : S_S \longrightarrow H_S$ onto a generator H_S in **Act** $-S$ *is a retraction,*

(iv) all cyclic generators in **Act** $-S$ *are projective,*

(v) all cyclic generators in **Act** $-S$ *are isomorphic to cyclic acts generated by an idempotent e with $e\mathcal{J}1$.*

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are parts of Lemma 5.18.
The equivalence $(iv) \Leftrightarrow (v)$ follows from Proposition 3.18.6 of [14].
$(iii) \Longrightarrow (iv)$. Let $H_S = (S/\varrho)_S$ be a cyclic generator. Then by (iii) the canonical epimorphism is a retraction. Thus H_S is a retract of a projective act and thus H_S is projective by Proposition 2.29.
$(iv) \Longrightarrow (iii)$. Let $\pi : S \longrightarrow H$ be an epimorphism onto a generator in $\mathbf{Act}-S$. Then H_S is cyclic and therefore by (iv) projective. Thus the identity id_H can be lifted with respect to π, i.e., π is a retraction.

□

Recall that by Lemma 3.34 for finite monoids S every cyclic generator is isomorphic to S_S, and that by Theorem 5.21 in this case all acts are obviously (S, G)-projective.
By the following example it becomes clear, that in general condition (v) of Theorem 5.21 is not only satisfied in a trivial way if S is infinite, i.e., that $e\mathcal{J}1$ does not imply $(eS)_S \cong S_S$.

Example 5.22 Cyclic projective generators need not to be isomorphic to S itself:
Let S be the monoid generated by the elements $e,\ k,\ k'$ and the relations $e^2 = e,\ ek = k,\ k'k = 1$. Then eS is a cyclic projective generator in $\mathbf{Act}-S$ but eS and S are not isomorphic as right S-acts ([15]).
(Another example, due to B. M. Schein, can also be found in [15]).

Notes:

- In Proposition 3.18.9 of [14] it has been proved, that if S is a group, then every projective generator in $\mathbf{Act}-S$ is isomorphic to S. Thus, condition (iv) of Theorem 5.19 implies condition (v) of Theorem 5.21, which is related to the implication $(\coprod_{i \in I} S, G)$-projectivity $\Longrightarrow$ (S, G)-projectivity.
  The converse is not true (see Lemma 3.35).

- Example 5.22 has already been used in Chapter 3 (Example 18.11) of [14] to make sure, that there exist cyclic projective generators, which are not isomorphic to S.
  Furthermore, in Chapter 5 (Example 3.23) of [14] the same example gives rise to the monoids S and eSe, which are Morita equivalent (i.e., the categories $\mathbf{Act}-S$ and $\mathbf{Act}-eSe$ are equivalent) but neither isomorphic nor anti-isomorphic.

In general, condition (v) of Theorem 5.21 is hard to handle with respect to the decision, if (i) is true. If S is periodic, then by Proposition 1.3.26 of [14] for $e \in E(S)$ the relation $e\mathcal{J}1$ implies $e = 1$. Thus in this case we obtain:

Corollary 5.23 *Let S be a periodic semigroup. Then all right S-acts are (S, G)-projective iff all cyclic generators are isomorphic to S_S.*

Note, that Lemma 3.35 could also be proved by using the fact that every finite monoid is periodic.

5.1.2 All are (X, A)-projective

This part deals with projectivities with second component A. The concepts of $(S, S/\varrho)$- and $(S, S/\varrho(x, y))$-projectivity are also included, since all acts having one of these properties is equivalent to all acts being (S, A)-projective.

Even if (S, A)-projectivity has been left out in the implication scheme because of Lemma 3.32, it yields an interesting result:

Theorem 5.24 *Let $x,\ y \in S$ and let ϱ be a right congruence on S. Then the following assertions are equivalent:*

(i) All right S-acts A_S are (S, A)-projective,

(ii) all cyclic acts are $(S, S/\varrho)$-projective,

(iii) all right S-acts are $(S, S/\varrho)$-projective,

(iv) all cyclic acts are projective,

(v) $S = \{1\}$ or $S = \{0, 1\}$,

(vi) all monocyclic acts are $(S, S/\varrho(x, y))$-projective,

(vii) all right S-acts are $(S, S/\varrho(x, y))$-projective.

Proof. $(i) \Longrightarrow (iv)$. Let $(S/\tau)_S$ be a cyclic act. $(S/\tau)_S$ is $(S, S/\tau)$-projective by (i). Thus by Lemma 5.18 the canonical epimorphism is a retraction, i.e., $(S/\tau)_S$ is a retract of a projective act and is therefore projective, i.e., (iv) is valid.

$(iv) \Longrightarrow (ii)$ is obvious.

$(ii) \Leftrightarrow (iii)$ is included in Lemma 5.18.

$(ii) \Longrightarrow (iv)$. If $(S/\tau)_S$ is $(S, S/\varrho)$-projective, then by Lemma 5.18 the canonical epimorphism is a retraction. Since S is projective, this implies the projectivity of $(S/\tau)_S$.

$(iv) \Longrightarrow (v)$. If all cyclic acts are projective, then all monocyclic acts are projective and Result 5.17 yields (v).

$(v) \Longrightarrow (i)$. If $S = \{1\}$, then by Result 5.16 all acts are projective and therefore (S, A)-projective.

Let $S = \{0, 1\}$. If $g : S \longrightarrow A$ is an epimorphism, then A is isomorphic to a factor act of S. Thus $A \cong S$ or $A \cong \Theta$. S_S is projective and therefore (S, A)-projective. Since S contains a left zero, Θ_S is (S, A)-projective by Proposition 5.8.

$(v) \Leftrightarrow (vi)$ is Result 5.17.

$(vi) \Leftrightarrow (vii)$ is a part of Lemma 5.18.

□

Remark 5.25 *The equivalences $(iii) \Leftrightarrow (v) \Leftrightarrow (vii)$ have already been proved in [17] in terms of principally weak and weak projectivity.*

Changing the first component to be a Rees factor act by a principal right ideal of S yields:

Proposition 5.26 *Let $i,\ s \in S$ and let $A_S \in$ **Act** $-S$.*
If all right S-acts are $(S/sS, A)$-projective, then all Rees factor acts $(S/I)_S$ by right ideals I of S are isomorphic to Rees factor acts by principal right ideals iS, i.e., S is a principal right ideal monoid.

Proof. Let I be a right ideal of S.
If $|I| = 1$, then I is generated by a left zero, i.e., I is a principal right ideal.
If $I = S$, then I is generated by the identity.

Let $|I| > 1,\ I \neq S$.
All acts are $(S/sS, A)$-projective implies that all Rees factor acts S/I are $(S/sS, S/I)$-projective. For $i \in I$, $iS \subseteq I$ holds. Thus the mapping $\pi : S/iS \longrightarrow S/I$ with $\pi(1_{S/iS}) = 1_{S/I}$ is an epimorphism, which is a retraction by Lemma 5.18, i.e., there exists a homomorphism $\gamma : S/I \longrightarrow S/iS$ with $\pi\gamma = id_{S/I}$. Since $1_{S/I} = \{1\} = 1_{S/iS}$, this implies $\gamma(1_{S/I}) = 1_{S/iS}$. Thus for $\bar{u} \in S/iS$ we get:

$$\gamma\pi(\bar{u}) = \gamma\pi(1_{S/iS}u) = \gamma(\pi(1_{S/iS})u) = \gamma(1_{S/I}u) = \gamma(1_{S/I})u = 1_{S/iS}u = \bar{u},$$

i.e., $\gamma\pi = id_{S/iS}$. Thus π is an isomorphism of acts.
Since $iS \subseteq I$, this implies $iS = I$, i.e., every right ideal of S is a principal right ideal.

□

Corollary 5.27 *All right S-acts are $(S/sS, A)$-projective implies the equivalence of $(S, S/tS)$- and $(S, S/I)$-projectivity.*

□

Example 5.28 All right S-acts are $(S/sS, A)$-projective does not imply, that **all** factor acts $(S/\varrho)_S$ by (principal) right congruences of S are isomorphic to Rees factor acts by principal right ideals:

Let S be the two element right zero semigroup $\{a, b\}$ with identity 1 adjoined. Then all Rees factor acts $(S/xS)_S$ of S by right congruences are $(S/1S)_S = \Theta_S$ and $(S/aS)_S = (S/bS)_S$. If there exists an epimorphism $g : (S/xS)_S \longrightarrow A_S$, then $A_S = \Theta_S$ or $A_S = (S/aS)_S$ and thus every right S-act A_S is $(S/sS, A)$-projective.
Now consider the right congruence $\varrho(1, a)$. Since $(S/\varrho(1, a))_S$ has no zero, it can not be isomorphic to a Rees factor act of S.

Especially, this means, that S need not to be a right (principal) Rees monoid in this case.

5.1.3 All are $(X, \coprod_{m\in M} S/\varrho_m)$- or $(X, S/\varrho)$-projective

Now we start studying properties, which are included in the implication scheme — part 2, i.e., $(\mathcal{X}, \mathcal{Y})$-projectivities, where $\mathcal{Y}$ is the class of cyclic S-acts or of coproducts of cyclic S-acts, with regard to the question of all acts having this property.

The first lemma shows, that the question of $(\coprod_{k\in K} S/\tau_k, \coprod_{m\in M} S/\varrho_m)$-projectivity of all acts, can be reduced to the question of all acts being $(S/\tau, S/\varrho)$-projective. Nevertheless, the main statements with respect to $(\mathcal{X}, \mathcal{Y})$-projectivity for various classes $\mathcal{X}$ will be giving in detail, i.e., in each case they will include both kinds of projectivity. The respective titles of the paragraphs will use the coproduct variant.

Lemma 5.29 *Let $x,\ y \in \{\phi, p, R, pR\}$ and let $\tau,\ \tau_k,\ k \in K$, be right x-congruences and $\sigma,\ \sigma_j,\ j \in J,\ \varrho,\ \varrho_m,\ m \in M$, be right y-congruences on S. Then the following assertions are equivalent:*

(i) All right S-acts are $(\coprod_{k\in K} S/\tau_k, \coprod_{m\in M} S/\varrho_m)$-projective,

(ii) *all coproducts* $\coprod_{j\in J}(S/\sigma_j)_S$ *of factor acts of* S *by right y-congruences are* $(\coprod_{k\in K} S/\tau_k, \coprod_{m\in M} S/\varrho_m)$*-projective,*

(iii) *every epimorphism* $g: \coprod_{k\in K}(S/\tau_k)_S \longrightarrow \coprod_{j\in J}(S/\sigma_j)_S$ *is a retraction,*

(iv) *all cyclic acts* $(S/\sigma)_S$ *are* $(\coprod_{k\in K} S/\tau_k, \coprod_{m\in M} S/\varrho_m)$*-projective,*

(v) *all cyclic acts* $(S/\sigma)_S$ *are* $(S/\tau, S/\varrho)$*-projective,*

(vi) *all right* S*-acts are* $(S/\tau, S/\varrho)$*-projective,*

(vii) *every epimorphism* $g: (S/\tau)_S \longrightarrow (S/\sigma)_S$ *is a retraction.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(v) \Leftrightarrow (vi) \Leftrightarrow (vii)$ are parts of Lemma 5.18.
$(ii) \Longrightarrow (iv)$ and $(iv) \Longrightarrow (v)$ are obvious.
$(v) \Longrightarrow (iv)$ is Proposition 3.12.
$(iv) \Longrightarrow (ii)$ is Proposition 3.17.

□

The previous Lemma will be used in various proofs of the next subsections, where special sets of congruences τ and ϱ, respectively, are considered.

all are $(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$-projective

In this part the second component is supposed to be a factor act of S by a right (principal) congruence or a coproduct of those and the first a free act. It turns out, that in this case all acts have the respective property if and only if S is trivial or $S = \{0, 1\}$ (Theorem 5.30).
Rees congruences will be considered later on, when $(S/sS, S/I)$- and $(S/sS, S/tS)$-projectivity will be studied.

Theorem 5.30 *Let* $\sigma,\ \sigma_j,\ j \in J,\ \varrho,\ \varrho_m,\ m \in M$, *be right (principal) congruences on* S.
Then the following assertions are equivalent:

(i) *All right* S*-acts are* $(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$*-projective,*

(ii) *all coproducts* $\coprod_{j\in J}(S/\sigma_j)_S$ *of factor acts of* S *by right (principal) congruences are* $(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$*-projective,*

(iii) *all coproducts* $\coprod_{j\in J}(S/\sigma_j)_S$ *of factor acts of* S *by right (principal) congruences are projective,*

(iv) *all cyclic acts* $(S/\sigma)_S$ *are projective,*

(v) $S=\{1\}$ *or* $S=\{0,1\}$*,*

(vi) *all right* S*-acts are* $(S,S/\varrho)$*-projective.*

Proof. $(i)\Leftrightarrow(ii)$ is Lemma 5.18.
$(ii)\Longrightarrow(iii)$. Let $p_j : S \longrightarrow S/\tau_j$ denote the j-th canonical epimorphism. Then $\coprod_{j\in J} p_j : \coprod_{j\in J} S \longrightarrow \coprod_{j\in J} S/\tau_j$ is an epimorphism, which is a retraction by (ii) and Lemma 5.18. Thus $\coprod_{j\in J} S/\tau_j$ is a retract of a projective act and is therefore projective.
$(i)\Leftrightarrow(vi)$ is a part of Lemma 5.29.
$(iii)\Longrightarrow(ii)$ is obvious.
$(iv)\Longrightarrow(iii)$ is a part of Lemma 5.18.
$(iv)\Leftrightarrow(v)\Leftrightarrow(vi)$ have been proved in Theorem 5.24.

□

Like mentioned before, the equivalence $(v)\Leftrightarrow(vi)$ has already been proved in [17] in terms of (principally) weak projectivity.

all are $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$-projective
all are $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$-projective

In this paragraph, at first we consider $(X,S/\varrho)$-projectivities with monocyclic acts as first component. We already know, that these are equivalent to the respective coproduct version (Lemma 5.29).
It turns out that all acts being $(S/\varrho(x,y),S/\varrho)$-projective implies that S fulfills condition (Moz), like defined in Definition 5.1. Moreover, we will see, that all acts being $(S/\varrho(x,y),S/\varrho)$-projective is equivalent to S fulfilling condition (Moz) together with all acts being $(S/sS,S/\varrho)$-projective, i.e., the question is reduced to $(S/sS,S/\varrho)$-projectivity of all acts.

For monoids with left zero a complete answer will be given in Corollary 5.36 for factor acts of S by right (principal) congruences on S as second component. For Rees factor acts as second component see the following subsections.

As indicated before, we have to study monoids over which all acts are $(S/\varrho(x,y), S/\varrho)$-projective. As preparation of the proof of the main theorem in this context, we need:

Lemma 5.31 *Let $u \in \{\phi, R, pR\}$ and let ϱ be a right u-congruence on S. If all right S-acts are $(S/\varrho(x,y), S/\varrho)$-projective, then S fulfills condition (Moz).*

Proof. Consider the right congruence $\varrho = S \times S$. Then $(S/\varrho)_S \cong \Theta_S$ and for all $x,\ y \in S$ the zero morphism $g : S/\varrho(x,y) \longrightarrow \Theta$ is an epimorphism, which is a retraction by Lemma 5.18. Thus there exists a monomorphism $\gamma : \Theta \longrightarrow S/\varrho(x,y)$, which implies, that $\gamma(\Theta)$ is a zero in $(S/\varrho(x,y))_S$.
For right (principal) Rees congruences take $\varrho = \varrho_I$ with $I = 1S = S$.

□

The converse is not true:

Example 5.32 Let S be the monoid $\{1, a\}$ with $a1 = 1a = a$ and $a^2 = a$ and zero 0 externally adjoined.
Since S contains a left zero, every (cyclic) right S-act has a zero, i.e., S fulfills condition (Moz).
Furthermore, since S contains a left zero, $(S/\varrho(x,y), S/\varrho)$- projectivity is equivalent to $(S, S/\varrho)$-projectivity (Lemma 4.4, 4.5, 4.6, 4.7). Since $(S/aS)_S$ is not a retract of S, it is not $(S, S/\varrho)$-projective, i.e., not $(S/\varrho(x,y), S/\varrho)$-projective.

Note that monocyclic acts as second component are not included in Lemma 5.31. The reason turns out if we take a look at the proof: The main aspect of this proof is, that Θ_S is isomorphic to a factor act of S by a right u-congruence. Since in general this is not true for right principal congruences, one can not use Θ_S in the same way for monocyclic acts.

Proposition 5.33 *Let $s,\ x,\ y \in S$, let S be a monoid, which fulfills condition (Moz), let $u \in \{\phi, p, R, pR\}$ and let ϱ be a right u-congruence on S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/\varrho(x,y), S/\varrho)$-projective,

(ii) all right S-acts are $(S/sS, S/\varrho)$-projective.

Proof. $(i) \Longrightarrow (ii)$ is Lemma 3.44.

$(ii) \Longrightarrow (i)$. Let $(S/\varrho(x,y))_S$ be a monocyclic act. By assumption, there exists a zero $[z]_{\varrho(x,y)}$ in $(S/\varrho(x,y))_S$. Since $[z]_{\varrho(x,y)}$ is a zero, we obtain $zS \subseteq$

$[z]_{\varrho(x,y)}$ and thus $\varrho_{zS} \subseteq \varrho(x,y)$. Therefore every factor act $(S/\varrho(x,y))_S$ is the epimorphic image of a Rees factor act $(S/zS)_S$ and Lemma 3.3 completes the proof.

□

Now by Lemma 5.29 and Proposition 5.33 we obtain:

Corollary 5.34 *Let S be a monoid, which fulfills condition (Moz), let $u \in \{\phi, p, R, pR\}$, let $\sigma,\ \sigma_j,\ j \in J,\ \varrho,\ \varrho_m,\ m \in M$, be right u-congruences on S and let $s,\ s_l,\ x,\ y,\ x_n,\ y_n \in S,\ l \in L,\ n \in N$. Then the following assertions are equivalent:*

(i) All right S-acts are $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$-projective,

(ii) all right S-acts are $(S/\varrho(x,y), S/\varrho)$-projective,

(iii) all right S-acts are $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$ -projective,

(iv) all right S-acts are $(S/sS, S/\varrho)$-projective.

□

Thus by Lemma 5.31 and Proposition 5.33, we have:

Theorem 5.35 *Let $s,\ x,\ y \in S$, let $u \in \{\phi, R, pR\}$ and let ϱ be a right u-congruence on S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/\varrho(x,y), S/\varrho)$-projective,

(ii) S is a monoid, which fulfills condition (Moz), and all right S-acts are $(S/sS, S/\varrho)$-projective.

□

At first, we saw that the question of all acts being $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$-projective can be reduced to the one of all acts being $(S/\varrho(x,y), S/\varrho)$-projective. Now we obtained, that for characterizing monoids S over which all acts have the respective property, we have to investigate monoids over which all acts are $(S/sS, S/\varrho)$-projective.

For monoids with left zero, we can give a complete answer by Theorem 5.30 and Lemma 4.4 to 4.5, for ϱ not being a Rees congruence, right now; for Rees congruences I refer to the next subsection.

Corollary 5.36 *Let S be a monoid with left zero, let ϱ be a right congruence on S, and let $s,\ x,\ y,\ \tilde{x},\ \tilde{y} \in S$. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/\varrho))$-projective,

(ii) all right S-acts are $(S/sS, S/\varrho(x,y))$-projective,

(iii) all right S-acts are $(S/\varrho(x,y), S/\varrho)$-projective,

(iv) all right S-acts are $(S/\varrho(\tilde{x},\tilde{y}), S/\varrho(x,y))$-projective,

(v) all right S-acts are $(S, S/\varrho)$-projective,

(vi) all right S-acts are $(S, S/\varrho(x,y))$-projective,

(vii) $S=\{1\}$ or $S=\{0,1\}$,

(viii) all cyclic acts are projective.

□

From now on, we change the second component to be a Rees factor act of S by a (principal) right ideal.

5.1.4 All are $(S/\varrho(x,y), S/I)$-projective

In this subsection the question, whether all acts are $(S/\varrho(x,y), S/I)$- projective, will be discussed. Starting with a useful Proposition, we get a necessary condition on S in Proposition 5.40, which leads to Proposition 5.41, which finally gives Theorem 5.57.

We will use the notation "S is the **disjoint union of the semigroup T and the monoid R**". This means, that $T \neq \emptyset$ is a subsemigroup of S, $R \neq \emptyset$ is a submonoid of S and the set S is the disjoint union of the sets T and R, the products $tt',\ t,\ t' \in T$ and $rr',\ r,\ r' \in R$ are the respective products in R and T, whereas the products $t\,r$ and $r\,t$ with $t \in T$ and $r \in R$ may be in R or in T. (Although in some situations it would be more comfortable to permit T and R to be empty, we follow the tradition to suppose semigroups to be not empty (cf. [22].)

Since the set of all right invertible elements will be used in the following, we introduce it explicitly by:

Definition 5.37 *For every monoid S let* $\mathrm{R} = \{s \in S \mid \exists s' \in S : \; s\,s' = 1\}$ *denote the* **submonoid of S, consisting of all right invertible elements of S.**

Remark 5.38 *The monoid* R*, introduced in Definition 5.37, is right cancellative (since* R *consists of all right invertible (and thus right cancellable) elements of S).*
The elements of the submonoid R *need not to be right invertible in* R*, since the right inverse s' of an element $s \in \mathrm{R}$ is in general not right invertible, i.e., $s' \notin \mathrm{R}$ is possible. Thus* R *need not to be a group.*

Lemma 5.39 *Let $x,\ y \in S$ and let J be a right ideal of S. If all right S-acts are $(S/\varrho(x,y), S/J)$-projective, then all elements $u,\ v \in S$ satisfy one of the following conditions:*

(a) $uS \cup vS = S$, i.e., u or v is right invertible,

(b) the principal congruence $\varrho(u,v)$ generated by u and v is the Rees congruence ϱ_I by the right ideal $I = uS \cup vS$.

Proof. Let $u,\ v \in S$ and let $I = uS \cup vS \neq S$. Then $\varrho(u,v) \subseteq \varrho_I$. Thus $g : S/\varrho(u,v) \longrightarrow S/I$ with $g(1_{S/\varrho(u,v)}) = 1_{S/I}$ is an epimorphism. By Lemma 5.18 , g is a retraction, i.e., there exists $id' : S/I \longrightarrow S/\varrho(u,v)$ with $g\,id' = id_{S/I}$.
Since $1_{S/\varrho(u,v)} = \{1\} = 1_{S/I}$, we get $id'(1_{S/I}) = 1_{S/\varrho(u,v)}$. Thus, for every $i \in I$ we obtain the following chain of equations:

$$id'([i]_{\varrho_I}) = id'(1_{S/I}\, i) = id'(1_{S/I})\, i = 1_{S/\varrho(u,v)}\, i = [i]_{\varrho(u,v)}.$$

Since $[i]_{\varrho_I} = [u]_{\varrho_I}$ for every $i \in I$, this implies $[i]_{\varrho(u,v)} = [u]_{\varrho(u,v)}$, i.e., $i \in [u]_{\varrho(u,v)}$ for all $i \in I$ and thus $\varrho_I \subseteq \varrho(u,v)$. This yields $\varrho_I = \varrho(u,v)$, i.e., (b).
If $uS \cup vS = S$, then $1 \in uS \cup vS$, i.e., there exists $u' \in S$ with $uu' = 1$ or $v' \in S$ with $vv' = 1$, i.e., u or v is right invertible.

□

This Proposition leads to the following necessary condition on S with all right S-acts being $(S/\varrho(x,y), S/I)$-projective.

Lemma 5.40 *Let $x,\ y \in S$ and let I be a right ideal of S. If all right S-acts are $(S/\varrho(x,y), S/I)$-projective, then S is the disjoint union of a right principal Rees semigroup and the submonoid* R *of S, consisting of all right invertible elements of S, and fulfills condition (Moz).*

Proof. The subsemigroup of S, consisting of all non right invertible elements of S, is a right principal Rees semigroup by Lemma 5.39.
S fulfills condition (Moz) by Proposition 5.31.

□

Together with Theorem 5.36 this yields:

Proposition 5.41 *For a monoid S the following assertions are equivalent:*

(i) All right S-acts are $(S/\varrho(x,y), S/I)$-projective,

(ii) S is the disjoint union of a right principal Rees semigroup and the submonoid R of S, consisting of all right invertible elements of S, S fulfills condition (Moz), and all right S-acts are $(S/sS, S/I)$-projective.

□

Thus we have to consider $(S/sS, S/I)$-projectivity next. We will come back to $(S/\varrho(x,y), S/I)$-projectivity in Theorem 5.57.

5.1.5 All are $(S/sS, S/I)$-projective

We already know, that $(S/sS, X)$-projectivity is equivalent to (S, X)-projectivity, if S has a left zero (see Lemma 4.4, 4.5, 4.6, 4.7). Thus in the following subsections by considering $(S/sS, X)$-projectivity of all acts, we also get characterizations of monoids S with all acts being (S, X)-projective. The respective results will be given separately.

By Lemma 5.18 we know, that all acts are $(S/sS, S/I)$-projective if and only if all Rees factor acts $(S/J)_S$ are. Therefore, we start with some useful propositions concerning Rees factor acts of S, which prepare the main statements of this subsection.
By doing this, we obtain for instance Lemma 5.44, which indicates, that it is useful to divide the following investigations in those for monoids with and those for monoids without left zero. We will obtain necessary conditions on S, over which all acts are $(S/sS, S/I)$-projective, and we will see, that in this situations S has to be left reversible (Proposition 5.47, Proposition 5.48). Thus the next step is to consider left reversible monoids in view to $(S/sS, S/I)$-projectivity of all acts and it turns out, that in this case the necessary conditions on S developed before (Lemma 5.44), are already sufficient for all acts

being $(S/sS, S/I)$-projective (Theorem 5.51).

Finally, this leads to the characterizations of monoids, over which all acts are $(S/sS, S/I)$-projective in Theorem 5.52 for S without left zero and in Theorem 5.53 for monoids with left zero. They will be summarized in a general characterization in Corollary 5.56. Note, that the characterization with respect to all acts being $(S, S/I)$-projective, which has been proved in [17] in terms of Rees weak projectivity, is obtained as a corollary of Theorem 5.53.

Like promised before, at the end of this subsection we get an answer to the question of all acts being $(S/\varrho(x,y), S/I)$-projective in Theorem 5.57.

Now we are starting by considering Rees factor acts with respect to $(S/sS, S/I)$-projectivity.

Lemma 5.42 *Let I, J be right ideals of S. If S is left reversible or a monoid without left zero then the mapping $\alpha : (S/I)_S \longrightarrow (S/J)_S$ with $\alpha(1_{S/I}) = 1_{S/J}$ is an S-epimorphism iff $I \subseteq J$.*

Proof. Sufficiency is clear, since $I \subseteq J$ implies $\varrho_I \subseteq \varrho_J$.
Necessity. Let S be a monoid without left zero. Then every Rees factor act contains a unique zero. Let $\tilde{0}$ denote the zero in S/I, i.e., $\tilde{0} = I$, and let $\bar{0}$ denote the zero in S/J, i.e., $\bar{0} = J$. By α being an S-homomorphism we get $\alpha(\tilde{0}) = \bar{0}$.
Thus for every $i \in I$ we have:

$$\bar{0} = \alpha(\tilde{0}) = \alpha([i]_{\varrho_I}) = \alpha(1_{S/I}\, i) = 1_{S/J}\, i = [i]_{\varrho_J},$$

i.e., $[i]_{\varrho_J} = \bar{0}$ and thus $i \in J$.
If S is left reversible, then there exists $i \in I \cap J$. Then $\alpha(\tilde{0}) = \alpha([i]_{\varrho_I}) = [i]_{\varrho_J} = \bar{0}$, since $i \in J$. Thus for every $i' \in I$ we obtain: $\alpha([i']_{\varrho_I}) = \alpha([i]_{\varrho_I}) = \bar{0}$ and therefore $I \subseteq J$.

□

Note, that if J is a one-element right ideal of S, then $(S/J)_S \cong S_S$ is projective and thus $(S/sS, S/I)$-projective.

Lemma 5.43 *Let J be a right ideal of S with $|J| \neq 1$. If the Rees factor act $(S/J)_S$ is $(S/sS, S/I)$-projective, then J satisfies one of the following conditions:*

(a) $J = S$,

(b) J has no proper subideals, i.e., $jS = J$ for all $j \in J$.

Proof. Let J be a right ideal of S with $|J| \neq 1$. Since for all $j \in J$ the relation $jS \subseteq J$ is valid, $\alpha : S/jS \longrightarrow S/I$ with $\alpha(1_{S/jS}) = 1_{S/J}$ is an epimorphism. Now consider the following diagram:

$$\begin{array}{ccc} & & S/J \\ & & \Big\downarrow id \\ S/jS & \xrightarrow[\alpha]{} & S/J \end{array}$$

By assumption there exists id' with $\alpha\, id' = id$.

If $J \neq S$, then every congruence class of an element $s \notin J$ is a one-element class, especially $1_{S/jS} = \{1\} = 1_{S/J}$. This implies $id'(1_{S/J}) = 1_{S/jS}$. Thus for $i \in J$ we get the following chain of equations:

$$id'([i]_{\varrho_J}) = id'(1_{S/J}\, i) = id'(1_{S/J})\, i = 1_{S/jS}\, i = [i]_{\varrho_{jS}}.$$

Since $[i]_{\varrho_J} = [j]_{\varrho_J}$ for every $i \in J$, we get $[i]_{\varrho_{jS}} = [j]_{\varrho_{jS}}$.
Thus $i\, \varrho_{jS}\, j$ which is equivalent to $i \in jS$. This implies $J \subseteq jS$ and thus $J = jS$, i.e., J has no proper subideals.

□

Lemma 5.44 *Let $s \in S$ and let $I \subseteq S$ denote a right ideal of S. Then all right S-acts are $(S/sS, S/I)$-projective, iff all Rees factor acts $(S/J)_S$ are $(S/sS, S/I)$-projective.*

Furthermore, this implies that every proper right ideal J of S is simple (i.e., J has no proper subideals, i.e., $J = jS$ for every $j \in J$).
In this case, S is a principal right ideal monoid.

Proof. The equivalence is given by Lemma 5.18.
If all Rees factor acts $(S/J)_S$ are $(S/sS, S/I)$-projective, then by Lemma 5.43 every right ideal of S fulfills (a), (b) or (c).
By definition, this implies that S is a principal right ideal monoid.

□

Remark 5.45 *If there exists a right ideal $I \neq S$ with $|I| \neq 1$, which is simple, then S is a monoid without left zero z, since otherwise for all $i \in I$ the implication $izS = \{iz\} \subseteq iS = I$ is valid, i.e., izS would be a proper subideal of I.*

Thus, if S is a monoid with left zero, then for $I \neq S$ the right ideal I is simple if and only if $|I| = 1$.

Lemma 5.46 *Let S be a monoid with left zero. If all Rees factor acts are $(S/sS, S/I)$-projective, then the left zero in S is unique and is therefore a zero.*

Proof. If z, z' are left zeros in S, then $J = \{z, z'\}$ is a right ideal of S, which does not fulfill one of the conditions of Lemma 5.44, contradicting S/J is $(S/sS, S/I)$-projective. Thus S contains a unique left zero z.
Let $s \in S$. Then for all $t \in S$ the equation $(sz)t = s(zt) = sz$ implies, that sz is a left zero in S. By the uniqueness of the left zero, this yields $sz = z$, i.e., z is a right zero of S and is therefore a zero.

□

Conditions (a) to (c) of Lemma 5.44 in general do not give the $(S/sS, S/I)$-projectivity for all acts. In the following, we differentiate between monoids with left zero and those without a left zero (as indicated by Remark 5.45). First we will see, that in both cases the $(S/sS, S/I)$-projectivity of all acts leads to a special class of monoids, namely to left reversible monoids.

Proposition 5.47 *Let S be a monoid without left zero, let $s \in S$ and let I be a right ideal of S. If all right S-acts are $(S/sS, S/I)$-projective, then S is the disjoint union of a right simple semigroup and the submonoid* R *of S, consisting of all right invertible elements of S or S is a group.*
Furthermore, this implies that S is left reversible.

Proof. Let S be a monoid without left zero. If all acts are $(S/sS, S/I)$-projective, then by Lemma 5.44 every proper right ideal of S is simple. Thus every element of S is right invertible or generates a simple right ideal of S.

Let $I = \{s \mid sS \text{ is a proper simple right ideal of } S\}$.

If $I = \emptyset$, then $S = \mathrm{R}$, which implies, that S is a group and is therefore left reversible.

Let $I \neq \emptyset$. Then obviously $\mathrm{R} \cap I = \emptyset$.
I is a right ideal of S, since $s \in I,\ u \in S$ implies $suS = sS$ and is therefore simple, i.e., I is a right simple subsemigroup of S. Thus S is the disjoint union of R and a right simple semigroup.

Furthermore, for $s,\ t \in \mathrm{R}$ we get: $sS \cap tS = S \neq \emptyset$. For $s \in R,\ t \in I,\ sS \cap tS = tS \neq \emptyset$ and for $s,\ t \in I$ we finally get $sS = tS = I \neq \emptyset$. Thus S is left reversible.

□

Proposition 5.48 *Let S be a monoid with left zero z, let $s \in S$ and let I be a right ideal of S. If all right S-acts are $(S/sS, S/I)$-projective, then S is the disjoint union of a group and a zero or $S = \{1\}$.*
Furthermore, this implies, that S is left reversible.

Proof. By Lemma 5.46 the left zero z in S is a zero.
Without loss of generality, let $|S| \geq 2$. Since all acts are $(S/sS, S/I)$-projective, we get by Lemma 5.44 for every $s \in S \setminus \{z\}$, that $sS = S$, i.e., s is right invertible, i.e., $\mathrm{R} = S \setminus \{z\}$. Let $r \in \mathrm{R}$ and let $r' \in S$ with $rr' = 1$. Then $r' \in \mathrm{R}$ and thus there exists $r'' \in \mathrm{R}$ such that $r'r'' = 1$. This implies $r = r(r'r'') = (rr')r'' = r''$,, i.e., $r'r = 1$. Thus every $r \in \mathrm{R}$ is invertible and therefore R is a group.

Since z is a zero in S, we get $z \in I$ for every right ideal I of S. Thus every two right ideals of S have a non empty intersection, i.e., S is left reversible.

□

Note, that the left reversibility of S is also sufficient for the uniqueness of a left zero in S.
Propositions 5.47 and 5.48 imply, that to get more informations about internal properties of the sublying monoid S, we have to study $(S/sS, S/I)$-projectivities of acts over a left reversible monoid S.
The respective results in Lemma 5.49 and Lemma 5.50 will be needed to prove Theorem 5.51.

Lemma 5.49 *Let S be a left reversible monoid. If every proper right ideal of S is simple, then S has at least one proper right ideal $I = S \setminus \mathrm{R}$.*

Proof. Let $I,\ J \subseteq S$ be proper right ideals of S. Since S is left reversible, there exists $i \in I \cap J$. This implies $iS \subseteq I$ and $iS \subseteq J$. Since I and J are simple, this yields $I = iS = J$, i.e., S has at least one proper right ideal $I = iS$.
Since $I = iS \neq S$, for all $s \in S$ we get $is \neq 1$, i.e., $I = iS \subseteq S \setminus \mathrm{R}$.
Let $s \in S \setminus \mathrm{R}$. By the uniqueness of the proper right ideal, we get $sS = iS$. Thus there exists $s' \in S$, such that $s = i\,s'$, i.e., $S \setminus \mathrm{R} \subseteq iS = I$ and thus $S \setminus \mathrm{R} = I$.

□

Notice, that if S is a left reversible monoid with left zero z, then by Lemma 5.49 the right ideal $S \setminus \mathrm{R} = \{z\}$ is the unique proper subideal of S.

Lemma 5.50 *Let S be a monoid and let $I = S \setminus \mathrm{R}$ be the proper right ideal of S, consisting of all non right invertible elements of S. Then every epimorphism $g : (S/I)_S \longrightarrow (S/I)_S$ is an isomorphism.*

Proof. For $s \in S \setminus I,\ s \neq 1$, the class of s in S/I is a one element class and will be denoted by s itself, the class of the identity will be denoted by $1_{S/I}$, the class of elements of I will be denoted by 0.

Since $1 \notin I$, we get $1_{S/I} = \{1\}$ and $|S/I| \geq 2$.
Suppose, there exists $v \in S \setminus I = \mathrm{R}$ such that $g(v) = 0$. Then $g(1_{S/I}) =$

$g(v\,v^{-1}) = g(v)\,v^{-1} = 0\,v^{-1} = 0$. This implies $g(S/I) = 0$, i.e., g is not surjective contradicting g being an epimorphism.
Thus the image of an element of $S \setminus I$ is an element of $S \setminus I$.
Moreover, this implies $g(0) = 0$, since g is surjective.

Since g is an epimorphism, there exists $s \in S \setminus I$, such that $g(s) = 1_{S/I}$. Let $v,\ w \in S/I$. Then

- $g(v) = g(w) = 0 \ \Leftrightarrow v = 0 = w$,
- $g(v) = g(w) \neq 0 \ \Leftrightarrow v,\ w \in \mathrm{R}$.
  Then $g(v) = g(s\,s^{-1}\,v) = g(s)\,s^{-1}\,v = 1_{S/I}\,s^{-1}\,v = s^{-1}\,v$ and analogously $g(w) = s^{-1}\,w$. Thus $g(v) = g(w) \Leftrightarrow s^{-1}\,v = s^{-1}\,w \Leftrightarrow s^{-1} = s^{-1}\,w\,v^{-1}$, which implies $1_{S/I} = s\,s^{-1} = s\,s^{-1}\,w\,v^{-1} = 1_{S/I}\,w\,v^{-1}$ which is equivalent to $v = w$.

Therefore g is injective, i.e., g is an S-isomorphism.

□

For left reversible monoids S we are now able to give a characterization with respect to all acts being $(S/sS, S/I)$-projective.

Theorem 5.51 *Let S be a left reversible monoid, let $s \in S$ and let I be a right ideal of S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/I)$-projective,

(ii) all Rees factor acts $(S/J)_S$ are $(S/sS, S/I)$-projective,

(iii) every proper right ideal J of S is simple (i.e., J has no proper subideals, i.e., $J = jS$ for every $j \in J$).

Proof. $(i) \Leftrightarrow (ii)$ is a part of Lemma 5.18.
$(ii) \Longrightarrow (iii)$ follows from Lemma 5.44.
$(iii) \Longrightarrow (ii)$ If S contains a left zero z, then by (iii) and Remark 5.45, for all right ideals I we get $|I| = 1$ or $I = S$. Thus all Rees factor acts of S are isomorphic to S_S or to Θ_S. The epimorphisms, which have to be considered in this case, are: the zeromorphism $g_1 : S \longrightarrow \Theta$, the identity id_Θ and epimorphisms $g_2 : S \longrightarrow S$.
The identity is obviously a retraction, g_1 is a retraction, since S contains a left zero and g_2 is a retraction, since S is projective.
Thus every epimorphism from a Rees factor act by a principal right ideal onto a Rees factor act by a right ideal is a retraction and Lemma 5.18 gives (ii).

If S is a monoid without left zero, then for every right ideal I of S we have $I = S$ or I is simple. By Lemma 5.49 we obtain $I = S$ or $I = S \setminus \mathrm{R}$. Thus

$S/I \cong (\mathrm{R} \cup \{\bar{0}\})$ or $S/I = S/S = \Theta$.
The zero morphism $g_1 : (\mathrm{R} \cup \bar{0}\}) \longrightarrow \Theta$ is a retraction, since $(\mathrm{R} \cup \bar{0}\})$ contains a zero. The identity id_Θ is obviously a retraction. By Lemma 5.50, every epimorphism $g_2 : (\mathrm{R} \cup \bar{0}\}) \longrightarrow (\mathrm{R} \cup \bar{0}\})$ is an isomorphism, which is a retraction. Thus we get, that even in this case the respective epimorphisms are retractions and Lemma 5.18 completes the proof.

□

Now we can characterize monoids, over which all acts are $(S/sS, S/I)$-projective, where we start with monoids without left zero. By Lemma 5.44, Remark 5.45, Proposition 5.47, Lemma 5.49 and Theorem 5.51 we get:

Theorem 5.52 *Let S be a monoid without left zero, let $s \in S$ and let I be a right ideal of S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/I)$-projective,

(ii) all Rees factor acts $(S/J)_S$ are $(S/sS, S/I)$-projective,

(iii) S is left reversible and every proper right ideal J of S is simple (i.e., J has no proper subideals, i.e., $J = jS$ for every $j \in J$),

(iv) if I is a proper right ideal of S, then $I = S \setminus \mathrm{R}$,

(v) S is the disjoint union of the submonoid R of S, consisting of all right invertible elements of S, and a right simple semigroup, or S is a group.

Proof. $(i) \Leftrightarrow (ii)$ is given by Lemma 5.18.
$(ii) \Longrightarrow (iii)$. By Proposition 5.47, the monoid S is left reversible, and thus Theorem 5.51 yields (iii).
$(iii) \Longrightarrow (ii)$ is Theorem 5.51.
$(iii) \Longrightarrow (iv)$ is Lemma 5.49.
$(iv) \Longrightarrow (iii)$. Without loss of generality, suppose that $S \setminus \mathrm{R} = I$ is a proper right ideal of S and is unique by (iv). Then obviously S is left reversible. Furthermore, the uniqueness of I yields that I is simple and thus (iii) holds.
$(ii) \Longrightarrow (v)$ is Proposition 5.47.
$(v) \Longrightarrow (iii)$. If S is a group, then S does not contain a proper right ideal. Thus (iii) is obviously true.
Let S be the disjoint union of R and a simple semigroup H and let I be a right ideal of S. If $I \cap \mathrm{R} \neq \emptyset$, then $I = S$, i.e., $(iii)(a)$.
Let $I \cap \mathrm{R} = \emptyset$, i.e., $I \subseteq H$. Since H is simple, this implies $I = H$, i.e., I has no subideals, i.e., $(iii)(b)$.
The left reversibility is obvious now.

□

Note, that if S is the disjoint union of a monoid T and a zero, we also talk about S as a *monoid with zero adjoined*. For monoids with left zero, we obtain the following theorem, where the analogous assertions to (ii), (iii) and (iv) of Theorem 5.52 are left out, because they will not be used anymore.

Theorem 5.53 *Let S be a monoid with left zero z, let $s \in S$ and let I be a right ideal of S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/I)$-projective,

(ii) S is a group with zero adjoined or $S = \{1\}$.

Proof. $(i) \Longrightarrow (ii)$ is Proposition 5.48.

$(ii) \Longrightarrow (i)$ If S is a group with zero adjoined, then all Rees factor acts of S are isomorphic to S or to Θ. S is projective and thus $(S, S/I)$-projective and so is $\Theta_S \cong 0S$.

If $S = \{1\}$, then by Result 2.35 all acts are projective and thus $(S/sS, S/I)$-projective.

□

Recall, that if all acts are $(S, S/I)$-projective, then the respective projectivity of Θ_S yields a left zero in S by Proposition 5.8. Moreover, if S contains a left zero, then by Theorem 4.6 $(S, S/I)$-projectivity is equivalent to $(S/sS, S/I)$-projectivity. Together with Theorem 5.53 we now obtain the respective results in view to Rees weak projectivity as:

Corollary 5.54 *Let $s \in S$ and let I denote a right ideal of S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S, S/I)$-projective (Rees weakly projective),

(ii) S is a group with zero adjoined or $S = \{1\}$.

□

The previous result has been proved directly in [17] in terms of Rees weak projectivity. Remark, that this characterization is given by a corollary in this context.

Remark 5.55 *Each of the conditions of Theorem 5.53 implies that all cyclic acts fulfill condition (P) (cf [14]).*

Summarizing, for arbitrary monoids S we get:

Corollary 5.56 *Let $s \in S$ and let I be a right ideal of S. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/I)$-projective,

(ii) S satisfies one of the following conditions:

- *(a) S is the disjoint union of the submonoid R of S, consisting of all right invertible elements of S, and a right simple semigroup.*
- *(b) S is a group.*
- *(c) S is a group with zero adjoined.*

□

For arbitrary monoids —with or without left zero — we obtained characterizations, whether all acts are $(S/sS, S/I)$-projective. The next section deals with $(S/sS, S/tS)$-projectivity, where some properties can be directly developed from the conditions in this section.

Before going any further, we come back to monoids, over which all acts are $(S/\varrho(x,y), S/\varrho)$-projective. By Proposition 5.41 and Corollary 5.56 we get :

Theorem 5.57 *For a monoid S the following assertions are equivalent:*

(i) All right S-acts are $(S/\varrho(x,y), S/I)$-projective,

(ii) S satisfies one of the following conditions:

- *(a) S fulfills condition (Moz) and S is the disjoint union of the submonoid R of S, consisting of all right invertible elements of S, and a right simple right principal Rees semigroup.*
- *(b) S fulfills condition (Moz) and S is a group.*
- *(c) S is a group with zero adjoined.*

□

Example 5.58 In Example 5.3 it has been shown, that the group $(\mathbb{Z}_3, +)$ fulfills condition (Moz). Thus every right $\mathbb{Z}_3$-act is $(S/\varrho(x,y), S/I)$-projective.

Like said before, in the following subsection the investigations are reduced to the case of second components being Rees factor acts by principal right ideals of S.

5.1.6 All are $(S/sS, S/tS)$-projective

In this part we study monoids, over which all acts are $(S/sS, S/tS)$-projective, i.e., we arrive at the bottom of the implication scheme part 2. Some of the results can be proved in analogy to the respective results concerning $(S/sS, S/I)$-projectivity.

It will turn out, that we can not use left reversibility of S in the same way like before, since left reversibility is not necessary for all acts being $(S/sS, S/tS)$-projective (see Remark 5.61).

Since $(S/sS, S/tS)$-projectivity of all acts is equivalent to $(S/sS, S/tS)$-projectivity of all Rees factor acts by principal right ideals (Lemma 5.18), we start with studying $(S/rS)_S$ with respect to this property.

Note that if r is a left zero in S, then $(S/rS)_S \cong S_S$ is projective and thus $(S/sS, S/tS)$-projective.

Lemma 5.59 *Let $s,\ t \in S$ and let $r \in S$ be not a left zero.*
If the Rees factor act $(S/rS)_S$ is $(S/sS, S/tS)$-projective, then r satisfies one of the following conditions:

(a) r is right invertible,

(b) rS has no proper subideals.

Proof. Since r is not a left zero, $|rS| \neq 1$. Then we obtain in analogy to the proof of Proposition 5.43:
$rS = S$, i.e., there exists $r' \in S$ such that $rr' = 1$, i.e., r is right invertible, or rS contains no proper subideal.

□

Lemma 5.60 *Let $r,\ s,\ t \in S$. Then all right S-acts are $(S/sS, S/tS)$-projective, iff all Rees factor acts $(S/rS)_S$ are $(S/sS, S/tS)$-projective.*

Furthermore, this implies that every $r \in S$ satisfies one of the following conditions:

(a) r is right invertible,

(b) rS has no proper subideals, i.e., $ru\,\mathcal{R}\,\mathrm{r}$ for every $u \in S$.

Proof. The equivalence is a part of Lemma 5.18.
The implication is a consequence of Lemma 5.59.

□

Remark 5.61 *Remark 5.45 is valid even in this situation, i.e., if there exists a principal right ideal rS of S with $|rS| \neq 1$, which fulfills condition (b) of Lemma 5.60, then S is a monoid without left zero.*
Note, that unlike in the case of all right S-acts being $(S/sS, S/I)$-projective, $(S/sS, S/tS)$-projectivity of all right S-acts does not yield the uniqueness of a left zero in S.
The reason is, that for left zeros $z,\ z'$ in S the right ideal $\{z, z'\}$ is in general not principal. Thus in this situation we do not have an analogon to Lemma 5.46.

Proposition 5.62 *Let $s,\ t \in S$. If all right S-acts are $(S/sS, S/tS)$-projective, then S is the disjoint union of a semigroup I, the submonoid R of S, consisting of all right invertible elements of S, and a left zero semigroup L, where L and I are possibly empty.*
Moreover, in this case I is the disjoint union of right simple subsemigroups $i'S$ of S, where I' is a representing system for the classes of $\mathcal{R}$ on S of elements of I.

Proof. Let L denote the set of left zeros of S and let R be the submonoid of right invertible elements of S.
Let $I = \{s \in S \mid sS$ is a proper principal simple right ideal of $S\} \setminus L$.
Then $R \cap L = \emptyset,\ R \cap I = \emptyset$ and by definition $L \cap I = \emptyset$. Since all acts are $(S/sS, S/tS)$-projective, by Lemma 5.60, an element of S is a left zero or is right invertible or generates a simple principal ideal of S, i.e., $S = I \cup L \cup R$. $I,\ L$ and R are closed under multiplication, i.e., are subsemigroups of S.
If $i,\ j \in I$, then either $iS \cap jS = \emptyset$ or $iS = jS$, since both ideals are simple. Let I' contain a representant $i \in [i]_{\mathcal{R}} \in I/\mathcal{R}$ for every $\mathcal{R}$-class. Then $I = \bigcup\limits_{i' \in I'} i'S$ is the disjoint union of right simple subsemigroups of S.

□

By Remark 5.45, we get again, that we have to distinguish between monoids with left zero(s) and those without left zero, since in either case one of the conditions of Lemma 5.60 is unnecessary.
By Proposition 5.62 we obtain:

Proposition 5.63 *Let $s,\ t \in S$ and let S be a monoid, such that all right S-acts are $(S/sS, S/tS)$-projective.*

If S contains a left zero, then S is the disjoint union a left zero semigroup and the group R or $S = \{1\}$.

If S is a semigroup without left zero, then S is the disjoint union of right simple semigroups of the form $i'S$, $i' \in S$, and the submonoid R *of S, consisting of all right invertible elements of S, or S is a group.*

Proof. If S contains a left zero, then every simple principal right ideal of S is a one-element right ideal, i.e., it is generated by a left zero. Thus by Proposition 5.62, S is the disjoint union of R and the left zero semigroup L.
Without loss of generality suppose $|\mathrm{R}| \geq 2$. Let $r \in \mathrm{R}$, $r \neq 1$. Since r is right invertible, there exists $r' \in S$ with $rr' = 1$. If r' is a left zero in S, then the equality $r = (rr')r = r(r'r) = rr' = 1$ yields a contradiction. Thus $r' \in \mathrm{R}$, i.e., there exists $r'' \in \mathrm{R}$ such that $r'r'' = 1$. This implies: $r = r(r'r'') = (rr')r'' = r''$, i.e., $r'r = 1$. Thus R is a monoid, such that every element has a unique inverse, i.e., R is a group.
If $|R| = 1$ and if $1z = 1$ for a left zero $z \in S$, then $z = 1z = 1$ yields $S = \{1\}$.

The second statement follows directly from Proposition 5.62 with $L = \emptyset$ or $L = \emptyset = I$.

□

Remark 5.64 *By the previous Proposition it becomes clear, that all right S-acts being $(S/sS, S/tS)$-projective does not imply S to be left reversible. For example, if S is the two element left zero semigroup $\{a, b\}$ with identity adjoined, then S is the disjoint union of a group and a left zero semigroup, but since $aS \cap bS = \emptyset$, it is not left reversible. (Indeed, we will see later on, that for S all right S-acts are $(S/sS, S/tS)$-projective.)*

Nevertheless, we want to study left reversible monoids in this context in the following.
In analogy to the proof of Proposition 5.49, we obtain:

Corollary 5.65 *Let S be a left reversible monoid without left zero and let $i \in S$. If all principal right ideals of S are simple, then S has at least one proper principal right ideal $iS = S \setminus$* R.

□

Theorem 5.66 *Let s, $t \in S$ and let S be a left reversible monoid. Then the following assertions are equivalent:*

(i) All right S-acts are $(S/sS, S/tS)$-projective,

(ii) all Rees factor acts by principal right ideals are $(S/sS, S/tS)$-projective,

(iii) every $r \in S$ *satisfies one of the following conditions:*

(a) r *is right invertible,*

(b) rS *has no proper subideals,i.e.,* $ru\,\mathcal{R}\,\mathrm{r}$ *for every* $u \in S$*,*

(iv) S *is the disjoint union of a right simple semigroup and the submonoid* R *of* S*, consisting of all right invertible elements of* S*, or* S *is a group or* S *is the disjoint union of a group and a zero,*

(v) all right S*-acts are* $(S/sS, S/I)$*-projective.*

Proof. $(i) \Leftrightarrow (ii)$ is a part of Lemma 5.18.
$(ii) \Longrightarrow (iii)$ follows from Lemma 5.60.
$(iii) \Longrightarrow (ii)$ in analogy to the proof of Theorem 5.51.
$(ii) \Longrightarrow (iv)$. Let S be a monoid with left zero. By Proposition 5.63 we get that S is the disjoint union of a group and a left zero semigroup. Since S is left reversible, for left zeros $z,\ z'$ in S we obtain: $zS \cap z'S \neq \emptyset$, which implies $z = z'$. Thus the left zero in S is unique and is therefore a zero, i.e., S is the disjoint union of a group and a zero.

Let S be a monoid without left zero. Again by Proposition 5.63 we obtain, that S is a group or S is the disjoint union of right simple semigroups $i'S,\ i' \in I'$, and R. For $i',\ i'' \in I'$ the left reversibility of S yields the existence of $j \in i'S \cap i''S$. Then $jS \subseteq i'S$ and $jS \subseteq i''S$. Since $i'S$ and $i''S$ are right simple, this implies $i'S = jS = i''S$. Therefore the disjoint union of the right simple semigroups $i'S,\ i' \in I'$, is a right simple semigroup iS.

Thus S is a group or the disjoint union of R and a right simple semigroup.

$(iv) \Leftrightarrow (v)$ is Corollary 5.56.
$(v) \Longrightarrow (ii)$ is obvious.

□

Even if we can not get a more general characterization by considering left reversible monoids, in the case of S containing a left zero, we obtain:

Theorem 5.67 *Let* S *be a monoid with left zero and let* $s,\ t \in S$*. Then the following assertions are equivalent:*

(i) All right S*-acts are* $(S/sS, S/tS)$*-projective,*

(ii) all elements r *of* S*, which are not left zeros, are right invertible, i.e.,* S *is the disjoint union of a group and a left zero semigroup or* $S = \{1\}$*.*

Proof. $(i) \Longrightarrow (ii)$ follows from Proposition 5.63.
$(ii) \Longrightarrow (i)$ If $S = \{1\}$, then by Result 2.35 all acts are projective and therefore $(S/sS, S/tS)$-projective.

If all non left zero elements are right invertible, then all Rees factor acts of S by principal right ideals are isomorphic to S_S or Θ_S. Since S has a left zero, Θ_S is projective and so is S. Thus all Rees factor acts of S by principal right ideals are $(S, S/tS)$-projective and Lemma 5.18 completes the proof.

□

By Theorem 5.67 and by the fact, that Θ_S is $(S, S/tS)$-projective if and only if S contains a left zero, we obtain the following characterization of monoids S, over which all acts are $(S, S/tS)$-projective. Note, that in this Corollary also the question of all acts being $(S/\varrho(x, y), S/tS)$-projective is anwered.

Corollary 5.68 *Let $s,\ t,\ x,\ y \in S$.*

(i) All right S-acts are $(S, S/tS)$-projective,

(ii) S contains a left zero and all right S-acts are $(S/\varrho(x, y), S/tS)$-projective,

(iii) S contains a left zero and all right S-acts are $(S/sS, S/tS)$-projective,

(iv) S contains a left zero and all elements r of S, which are not left zeros, are right invertible, i.e., S is the disjoint union of a group and a left zero semigroup or $S = \{1\}$.

Proof. $(i) \Longrightarrow (ii)$. By (i), the one element act Θ_S is $(S, S/I)$-projective, which gives a left zero in S by Proposition 5.8. The second part is obvious.
$(ii) \Longrightarrow (iii)$ is obvious.
$(iii) \Leftrightarrow (iv)$ is Theorem 5.67.
$(iii) \Longrightarrow (i)$ is a part of Theorem 4.6.

□

Remark 5.69 *The equivalence $(i) \Leftrightarrow (iv)$ has already been proved in [17] in terms of principally Rees weak projectivity.*

So far, during this chapter we studied monoids S, over which <u>all</u> acts have a special property, whereas we went from the strongest to the weakest projectivity. The results will be summarized in tables at the end of this chapter beneath those results, which will be developed in the next section.

5.2 Characterization of monoids by implications between the concepts

Some implications between different properties of acts yield further properties of the respective monoid. In this section we are searching for conditions, under which a concept of projectivity implies a stronger one.
We will see, that most of those implications leads to S being a monoid with left zero or to S fulfilling condition (Moz).

First we study implications $(\mathcal{X}, \mathcal{Y})$-projectivity $\Longrightarrow$ $(\mathcal{X}', \mathcal{Y}')$-projectivity with $\mathcal{X}$ being the class of all Rees factor acts by principal right ideals of S and $\mathcal{X}' = \{\mathrm{S}\}$, and coproducts of these, respectively.

Theorem 5.70 *Let $s,\ t,\ s_l \in S,\ l \in L$, let $\hat{I}$ be a right ideal of S and let $A_S \in \mathbf{Act}-S$. Then the following assertions are equivalent:*

(i) $(S/sS, A)$*-projectivity implies* (S, A)*-projectivity,*

(ii) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* $(\coprod_{i\in I} S, A)$*-projectivity (which is equivalent to projectivity),*

(iii) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* (S, A)*-projectivity,*

(iv) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* $(S, S/\hat{I})$*-projectivity,*

(v) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* $(S, S/tS)$*-projectivity,*

(vi) $(S/sS, S/tS)$*-projectivity implies* $(S, S/tS)$*-projectivity,*

(vii) $(S/sS, S/\hat{I})$*-projectivity implies* $(S, S/\hat{I})$*-projectivity,*

(viii) $(S/sS, S/\hat{I})$*-projectivity implies* $(S, S/tS)$*-projectivity,*

(ix) S *contains a left zero.*

Proof. For $j \in \{(i), (ii), (iii), (vi), (vii), (viii)\}$ the implications $(ix) \Longrightarrow j$ are obvious, recalling that for a left zero z in S the Rees factor act $(S/zS)_S$ is isomorphic to S_S.
$(ix) \Longrightarrow (iv), (v)$ is valid, since $(ix) \Longrightarrow (ii)$ and projectivity $\Longrightarrow (S, S/\hat{I})$-projectivity $\Longrightarrow (S, S/tS)$-projectivity.

$(i) \Longrightarrow (ix)$: Θ_S is $(S/sS, \Theta)$-projective (Proposition 5.11) and thus by (i) Θ_S is (S, Θ)-projective. By Proposition 5.8 the monoid S contains a left zero.

The same argument yields the implications $j \Longrightarrow (ix)$ for every $j \in \{(ii), (iii), (iv), (v), (vi), (vii), (viii)\}$.

□

In general, Θ_S can not be used in the same way, if the second component is an arbitrary cyclic act. In this case we get:

Theorem 5.71 *Let S be a left reversible monoid, let $s,\ s_l \in S,\ l \in L$, and let $\varrho,\ \varrho_m,\ m \in M$, denote right congruences on S. Then the following assertions are equivalent:*

(i) $(S/sS, S/\varrho)$*-projectivity implies* $(S, S/\varrho)$*-projectivity,*

(ii) $(\coprod_{l\in L} S/s_lS,\ \coprod_{m\in M} S/\varrho_m)$*-projectivity implies* $(\coprod_{i\in I} S,\ \coprod_{m\in M} S/\varrho_m)$*-projectivity,*

(iii) $(\coprod_{l\in L} S/s_lS,\ \coprod_{m\in M} S/\varrho_m)$*-projectivity implies* $(S, S/\varrho)$*-projectivity,*

(iv) S *contains a zero.*

Proof. $(i) \Longrightarrow (iv)$ Since S is left reversible, Θ is $(S/sS, S/\varrho)$-projective by Proposition 5.14 and thus by (i) $(S, S/\varrho)$-projective. By Proposition 5.8 there exists a left zero z in S. Suppose z' is a left zero in S. Then $zS = \{z\} = \{z'\} = z'S$ by the left reversibility of S, i.e., z is unique and is therefore a zero in S.
$(ii) \Longrightarrow (iv)$ and $(iii) \Longrightarrow (iv)$ can be proved in the same way.
$(iv) \Longrightarrow (ii)$ (which implies (iii)) and $(iv) \Longrightarrow (i)$ have been proved in Theorem 4.4.

□

To use the one element act Θ_S in the same way for monocyclic acts as second components, we have to make sure, that there are homomorphism, which have to be lifted, i.e., that Θ_S is not trivially-$(S/sS, S/\varrho(x,y))$-projective. For this situation we obtain the following assertion, which can be proved in the same way like Theorem 5.71 (applying Proposition 5.10 and Theorem 4.5).

Corollary 5.72 *Let S be a left reversible monoid, which fulfills condition (Moz), let $s,\ s_l,\ x,\ y,\ x_n,\ y_n \in S,\ (l \in L,\ n \in N)$. Then the following assertions are equivalent:*

(i) $(S/sS, S/\varrho(x,y))$*-projectivity implies* $(S, S/\varrho(x,y))$*-projectivity,*

(ii) $(\coprod_{l\in L} S/s_lS,\ \coprod_{n\in N} S/\varrho(x_n,y_n))$*-projectivity implies* $(\coprod_{i\in I} S,\ \coprod_{n\in N} S/\varrho(x_n,y_n))$*-projectivity,*

(iii) $(\coprod_{l\in L} S/s_lS, \coprod_{n\in N} S/\varrho(x_n, y_n))$*-projectivity implies* $(S, S/\varrho(x,y))$*-projectivity,*

(iv) S *contains a zero.*

□

Now we change the first component in $(\mathcal{X}', \mathcal{Y}')$-projectivity to the class $\mathcal{X}'$ consisting of all monocyclic acts (or coproducts of these, respectively).

Theorem 5.73 *Let* $s,\ s_l,\ t,\ x,\ y,\ x_n,\ y_n \in S$ $(l \in L,\ n \in N)$ *let* I *denote a right ideal of* S *and let* $A_S \in \mathbf{Act}-S$*. Then the following assertions are equivalent:*

(i) $(S/sS, S/tS)$*-projectivity implies* $(S/\varrho(x,y), S/tS)$*-projectivity,*

(ii) $(S/sS, S/I)$*-projectivity implies* $(S/\varrho(x,y), S/I)$*-projectivity,*

(iii) $(S/sS, S/I)$*-projectivity implies* $(S/\varrho(x,y), S/tS)$*-projectivity*

(iv) $(S/sS, A)$*-projectivity implies* $(S/\varrho(x,y), A)$*-projectivity,*

(v) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* $(\coprod_{n\in N} S/\varrho(x_n, y_n), A)$*-projectivity,*

(vi) $(\coprod_{l\in L} S/s_lS, A)$*-projectivity implies* $(S/\varrho(x,y), A)$*-projectivity,*

(vii) S *fulfills condition* (Moz).

Proof. $(i) \Longrightarrow (vii)$. Θ_S is $(S/sS, S/tS)$-projective (Proposition 5.11) and thus by (i) Θ_S is $(S/\varrho(x,y), S/tS)$-projective. By Proposition 5.10 the monoid S fulfills condition (Moz).
$(vii) \Longrightarrow (i)$. Let $\bar{u}$ denote the zero in $S/\varrho(x,y)$. Then for the right principal Rees congruence ϱ_{uS}, generated by the principal right ideal uS, we get $\varrho_{uS} \subseteq \varrho(x,y)$. Thus there exists an epimorphism $\alpha : S/uS \longrightarrow S/\varrho(x,y)$ and Lemma 3.3 yields (i).

For $j \in \{(ii), (iii), (iv), (v), (vi)\}$ the equivalences $j \Leftrightarrow (vii)$ can be proved in analogy to the proof of $(i) \Leftrightarrow (vii)$.

□

Like Theorem 5.71, for arbitrary cyclic acts or coproducts of those as second component, we obtain the following characterization in the case of left reversible monoids:

Theorem 5.74 *Let S be a left reversible monoid, let $s,\ x,\ y,\ s_l,\ x_n,\ y_n \in S,\ l \in L,\ n \in N$, and let $\varrho,\ \varrho_m,\ m \in M$, denote right congruences on S. Then the following assertions are equivalent:*

(i) $(S/sS, S/\varrho)$*-projectivity implies* $(S/\varrho(x,y), S/\varrho)$*-projectivity,*

(ii) $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$*-projectivity implies* $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$*-projectivity,*

(iii) $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$*-projectivity implies* $(S/\varrho(x,y), S/\varrho)$*-projectivity,*

(iv) S *fulfills condition* (Moz).

Proof. $(i) \Longrightarrow (iv)$. Since S is left reversible, Θ_S is $(S/sS, S/\varrho)$-projective and thus by (i) it is $(S/\varrho(x,y), S/\varrho)$-projective. Thus for $x,\ y \in S$ the identity on Θ_S has to be lifted with respect to the zero morphism $g : S/\varrho(x,y) \longrightarrow \Theta \cong S/S^2$. Then $id_\Theta(\Theta)$ is a zero in $S/\varrho(x,y)$, i.e., S fulfills condition (Moz).

$(iv) \Longrightarrow (i)$ can be proved analogously to $(vii) \Longrightarrow (i)$ in the proof of Theorem 5.73.

The remaining equivalence can be proved in the same way.

□

In the next theorem, we demand $\mathcal{X}$ to be the class of all monocyclic acts (coproduct of monocyclic acts) and $\mathcal{X}'$ to consist of S_S or of coproducts of copies of S. In this case, we suppose, that S fulfills condition (Moz).

Theorem 5.75 *Let S be a monoid, which fulfills condition (Moz), let $s,\ x,\ y,\ x_n,\ y_n \in S,\ n \in N$, let $\hat{I}$ be a right ideal of S and let $A_S \in$ **Act** $-S$. Then the following assertions are equivalent:*

(i) $(S/\varrho(x,y), S/sS)$*-projectivity implies* $(S, S/sS)$*-projectivity,*

(ii) $(S/\varrho(x,y), S/\hat{I})$*-projectivity implies* $(S, S/\hat{I})$*-projectivity,*

(iii) $(S/\varrho(x,y), S/\hat{I})$*-projectivity implies* $(S, S/sS)$*-projectivity,*

(iv) $(S/\varrho(x,y), A)$*-projectivity implies* (S, A)*-projectivity,*

(v) $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$*-projectivity implies* $(\coprod_{i\in I} S, A)$*-projectivity,*

(vi) $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$*-projectivity implies* (S, A)*-projectivity,*

(vii) $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$*-projectivity implies* $(S, S/\hat{I})$*-projectivity,*

(viii) $(\coprod_{n\in N} S/\varrho(x_n, y_n), A)$*-projectivity implies* $(S, S/tS)$*-projectivity,*

(ix) S *contains a left zero.*

Proof. $(i) \Longrightarrow (ix)$ Since S fulfills condition (Moz), by Proposition 5.10 the one element act Θ_S is $(S/\varrho(x, y), S/sS)$-projective and thus by (i) $(S, S/sS)$-projective. Then Proposition 5.8 yields the existence of a left zero in S.

$(ix) \Longrightarrow (i)$ $(S/\varrho(x, y), X)$-projectivity implies $(S/sS, X)$-projectivity. By (ix) and Theorem 5.70 we get (i).

The remaining equivalences can be proved analogously.

□

We saw, that some implications between the concepts introduced in Chapter 3 lead to characterization of the monoid S. The respective results will be summarized in the tables given in section 5.3.

5.3 Summarizing Tables

The following tables summarize the results of the previous sections. The same abbreviations as in the implication scheme — part1 and part 2 — are used here. In addition, let R denote the submonoid of S, consisting of all right invertible elements of S. The abbreviation "l.z." denotes the situation, that S contains a left zero, "zero" and "(Moz)" analogously. The respective properties appear in the same order as they came up in Chapter 5.

The first tables include the results concerning characterizations of monoids by properties of all acts, whereas in Table 5 the general results are summarized, in Table 6 the results for left reversible monoids are collected and in Table 7 those for monoids with left zero, not including those characterizations, which in general lead to S containing a left zero.

In Table 8 necessary conditions on a monoid S, over which all acts have a special property, are collected.

The Tables 9 to 12 due to Section 5.2, where implications between different projectivities yielded further characterizations. Table 9 includes the general case, Table 10 the situation for left reversible monoids, Table 11 the one for monoids fulfilling condition (Moz) and finally in Table 12 the results concerning left reversible monoids, which also fulfill condition (Moz), are summarized.
Note, that in tables, which belong to Section 5.2, the empty places could be filled by the necessary (but not sufficient) property A, where in every case A is the same property, which can be found in the same column like the array under consideration.

Recall, that R is the submonoid of S, consisting of all right invertible elements of S (Definition 5.37).

All right S-acts are	if and only if
$(\coprod_{i\in I} S, G)$-projective	S is a group
(S, G)-projective	all cyclic generators are isomorphic to eS with $e^2 = e$ and $e\mathcal{J}1$
(S, A)-projective	$S = \{1\}$ or $S = \{0, 1\}$
$(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$-projective	$S = \{1\}$ or $S = \{0, 1\}$
$(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n, y_n))$- projective	$S = \{1\}$ or $S = \{0, 1\}$
$(S, S/\varrho)$-projective	$S = \{1\}$ or $S = \{0, 1\}$
$(S, S/\varrho(x, y))$-projective	$S = \{1\}$ or $S = \{0, 1\}$
$(S/\varrho(x, y), S/\varrho)$-projective	S fullfills condition (Moz) and all right S-acts are $(S/sS, S/\varrho)$-projective
$(S/sS, S/I)$-projective	S is the disjoint union of R and a right simple semigroup or S is a group or S is a group with zero adjoined
$(S, S/I)$-projective	S is a group with zero adjoined or $S = \{1\}$
$(S/\varrho(x, y), S/I)$-projective	S fullfills condition (Moz) and S is the disjoint union of R and a right simple right principal Rees semigroup or S is a group, which fulfills condition (Moz) or S is a group with zero adjoined
$(S/\varrho(x, y), S/tS)$-projective	S fullfills condition (Moz) and all right S-acts are $(S/sS, S/tS)$-projective
$(S, S/tS)$-projective	S is the disjoint union of a group and a left zero semigroup or $S = \{1\}$

Table 5: Characterization of monoids by properties of all right S-acts

All right S-acts are	if and only if
$(S/sS, S/tS)$-projective	if S is the disjoint union of a right simple semigroup and R or S is a group or S is a group with zero adjoined
$(S/\varrho(x,y), S/tS)$-projective	S fulfills condition (Moz) and (S is a group or S is the disjoint union of a right simple semigroup and R) or S is group with zero adjoined

Table 6: Characterization of left reversible monoids by properties of all right S-acts

All right S-acts are	if and only if
$(\coprod_{l\in L} S/s_lS, G)$-projective	$S=\{1\}$
$(S/sS, G)$-projective	All cyclic generators are isomorphic to $eS,\ e\in E(S),\ e\mathcal{J}1$
$(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$-proj.	$S=\{1\}$ or $S=\{0,1\}$
$(S/sS, S/\varrho)$-projective	$S=\{1\}$ or $S=\{0,1\}$
$(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$-proj.	$S=\{1\}$ or $S=\{0,1\}$
$(S/\varrho(x,y), S/\varrho))$-projective	$S=\{1\}$ or $S=\{0,1\}$
$(S/sS, S/\varrho(x,y))$-projective	$S=\{1\}$ or $S=\{0,1\}$
$(S/\varrho(x,y), S/\varrho(\tilde{x},\tilde{y}))$-projective	$S=\{1\}$ or $S=\{0,1\}$
$(S/sS, S/I)$-projective	S is a group with zero adjoined or $S=\{1\}$
$(S/\varrho(x,y), S/I)$-projective	S is group with zero adjoined or $S=\{1\}$
$(S/sS, S/tS)$-projective	S is the disjoint union of a group and a left zero semigroup or $S=\{1\}$
$(S/\varrho(x,y), S/tS)$-projective	S is the disjoint union of a group and a left zero semigroup or $S=\{1\}$

Table 7: Characterization of monoids with left zero

All right S-acts are	implies
$(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$ -projective (*)	S fulfills condition (Moz)
$(S/\varrho(x,y), S/\varrho)$-projective (*)	S fulfills condition (Moz)
$(S/sS, A)$-projective	S is a principal right ideal monoid
$(S/sS, S/tS)$-projective	S is the disjoint union of R and right simple semigroups $i'S$ or S is a group or S is the disjoint union of a group and a left zero semigroup

Table 8: Necessary conditions on a monoid S

((*) where ϱ is a right x-congruence, $x\in\{\phi,R,pR\}$)

$\Longrightarrow$	(S,A)	$(\coprod_{i\in I} S,A)$	$(S,S/I)$	$(S,S/tS)$	$(S/\sigma,A)$	$(\coprod_{n\in N} S/\sigma_n,A)$	$(S/\sigma,S/I)$	$(S/\sigma,S/tS)$
$(S/sS,A)$	l.z.				(Moz)			
$(\coprod_{l\in L} S/s_lS,A)$	l.z.	l.z.	l.z.	l.z.	(Moz)	(Moz)		
$(S/sS,S/I)$			l.z.	l.z.			(Moz)	(Moz)
$(S/sS,S/tS)$				l.z.				(Moz)

Table 9: Characterization of monoids by implications

$\Longrightarrow$	$(S,S/\varrho)$	$(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$	$(S/\sigma,S/\varrho)$	$(\coprod_{n\in N} S/\sigma_n, \coprod_{m\in M} S/\varrho_m)$
$(S/sS,S/\varrho)$	zero		(Moz)	
$(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$	zero	zero	(Moz)	(Moz)

Table 10: Characterization of left reversible monoids by implications

$\Longrightarrow$	(S,A)	$(\coprod_{i\in I} S,A)$	$(S,S/I)$	$(S,S/tS)$
$(S/\sigma,A)$	l.z.			
$(\coprod_{n\in N} S/\sigma_n,A)$	l.z.	l.z.	l.z.	l.z.
$(S/\sigma,S/I)$			l.z.	l.z.
$(S/\sigma,S/tS)$				l.z.

Table 11: Characterization of monoids fulfilling condition (Moz) by implications

$\Longrightarrow$	$(S,S/\sigma)$	$(\coprod_{i\in I} S, \coprod_{n\in N} S/\sigma_n)$
$(S/sS,S/\sigma)$	zero	
$(\coprod_{l\in L} S/s_lS, \coprod_{n\in N} S/\sigma_n)$	zero	zero

Table 12: Characterization of left reversible monoids, which fulfill condition (Moz), by implications

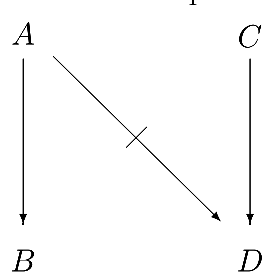
6 Differences between the concepts

In the first Section of this Chapter the $(\mathcal{X}, \mathcal{Y})$-projectivities are proved to be different from each other (we did not show this by now). Some distinctions are already clear by the results concerning homological classification. In this cases, these Results will be used for putting out the respective differences, sometimes concrete examples will be added. Other differences will be proved only by giving concrete examples.

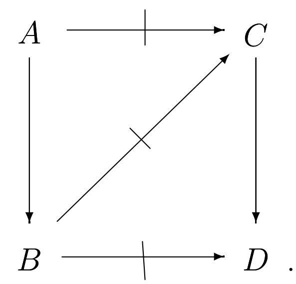
Subsection 1 treats differences between the various branches of the implication scheme (part 1 and part 2), which has been presented at the end of Chapter 3. Subsection 2 is divided in three subsubsections: the first one deals with projectivities, where the second component is the class of all generators in **Act** $-S$, the second one with (X, A)- and (X, C)-projectivities and the third with those projectivities, where the second component is a cyclic act or a coproduct of cyclic acts.

The second Section of this Chapter treats differences between $(\mathcal{X}, \mathcal{Y})$-projectivities and other properties underneath projectivity. The considered properties — flatness properties and torsionfreeness — have shortly been introduced in Chapter 2 (Preliminaries). Some results concerning homological classification relative to these properties have also been recalled (Results 2.35), and beneath Corollary 2.34 they will be used to prove distinctions.

Note that an implication scheme of the form



implies



Consequently we only prove the first situation in the following.

At the beginning of each example, the abbreviation $A \not\Longrightarrow B$ indicates the situation, for which an example is given, i.e., the example shows that property A does not imply property B.

During this Chapter let G_S be a generator in **Act** $-S$, let $A_S,\ B_S,\ C_S$ be right S-acts, let $s,\ t,\ s_l,\ x_n,\ y_n \in S$, let $\varrho,\ \varrho_m,\ m \in M$, denote right congruences on S and let $\hat{I},\ I_k,\ k \in K$, be right ideals of S.

Note, that we will use the properties of the one element act Θ_S, which have been studied in Propositions 5.8, 5.9, 5.10, 5.11 and in Corollary 5.14. Before considering the respective situations, we treat four special monoids and their factor acts.

Examples 6.1 1. Let $S = \{a, 1\}$ be the two element group.

Then $\varrho(1,a) = S \times S = \varrho_{aS}$ is the unique principal (Rees) congruence on S.
Thus $(S/\varrho)_S = (S/\Delta)_S \cong S_S$ and $(S/\varrho)_S = (S/S^2)_S \cong \Theta_S$ are all factor acts of S by right congruences.

2. Let $S = \{1, a, 0\}$ be the two element group with zero 0 adjoined.

Then all monocyclic acts are given by $(S/\varrho(1,0))_S \cong (S/\varrho(a,0))_S \cong \Theta_S$ and $(S/\varrho(1,a))_S$ with $\varrho(1,a) = \{(1,a),(a,1)\} \cup \Delta$.
$(S/1S)_S = (S/aS)_S \cong \Theta_S$ and $(S/0S)_S \cong S_S$ are all Rees factor acts of S.
There are no other factor acts of S by right congruences.

3. Let $S = \{1, a, b\}$ be the two element right zero semigroup $\{a, b\}$ with identity adjoined.

$(S/\varrho(1,a))_S \cong (S/\varrho(1,b))_S$ and $(S/\varrho(a,b))_S$ are all monocyclic acts.
$(S/1S)_S \cong \Theta_S$ and $S/aS \cong (S/bS)_S \cong (S/\{a,b\})_S$ are all Rees factor acts.
Moreover, $(S/\Delta)_S \cong S_S$ is a factor act of S.

4. Let $S = \{1, a, b\}$ be the two element left zero semigroup $\{a, b\}$ with identity adjoined.

$(S/\varrho(1,a))_S \cong (S/\varrho(1,b))_S \cong \Theta_S,\ (S/\Delta)_S \cong S_S$ and $(S/\varrho(a,b))_S$ are all monocyclic acts.
$(S/1S)_S \cong \Theta_S$ and $(S/aS)_S \cong (S/bS)_S \cong S_S$ and $(S/\{a,b\})_S$ are all Rees factor acts.

There are no further factor acts of S by right congruences on S.

6.1 Differences between $(\mathcal{X}, \mathcal{Y})$-projectivities

6.1.1 Differences between various parts of the diagram

As mentioned before, in this subsection differences between various branches of the implication scheme (part 1 and part 2), given in Chapter 3, will be proved.

Example 6.2 $(\coprod_{i \in I} S, \coprod_{m \in M} S/\varrho_m)$ **-projective** $\not\Longrightarrow$ $(\coprod_{i \in I} S, G)$**-projective**

Let $S = \{0, 1\}$ be the monoid consisting of a zero and an identity. By Theorem 5.30, all right S-acts are $(\coprod_{i \in I} S, \coprod_{m \in M} S/\varrho_m)$-projective. Since S is not a group, by Theorem 5.19, there exists a right S-act, which is not $(\coprod_{i \in I} S, G)$-projective.

Example 6.3 $(\coprod_{i \in I} S, \coprod_{m \in M} S/\varrho_m)$ **-projective** $\not\Longrightarrow$ $(\coprod_{l \in L} S/s_l S, A)$**-projective**

Let $S = \{0, 1\}$ and let A_S be the amalgamated coproduct of two copies of S by the zero. By Theorem 5.30 all right S-acts are $(\coprod_{i \in I} S, \coprod_{m \in M} S/\varrho_m)$ -projective and so is A_S. Since S has a left zero, $(\coprod_{l \in L} S/s_l S, A)$-projectivity is equivalent to $(\coprod_{i \in I} S, A)$-projectivity. Let $g : S \coprod S \longrightarrow A$ where $g|S$ is the identity. Then the identity id_A on A_S can not be lifted with respect to g, i.e., A_S is not $(\coprod_{l \in L} S/s_l S, A)$-projective.

Example 6.4 $(\coprod_{i \in I} S, G)$ **-projective** $\not\Longrightarrow$ $(S, S/tS)$**-projective**
$(\coprod_{i \in I} S, G)$ **-projective** $\not\Longrightarrow$ (S, A)**-projective**

Let S be a group with at least two elements. By Theorem 5.19 all right S-acts are $(\coprod_{i \in I} S, G)$-projective. Since S has no left zero, the one-element act Θ_S is not $(S, S/tS)$-projective and not (S, A)-projective by Proposition 5.8.

As a concrete example take $(S, \cdot) = (\mathbb{Z}_3, +)$ (or $(\mathbb{Z}_n, +)$ for an arbitrary natural number $n \geq 2$).

Example 6.5 $(\coprod_{n\in N} S/\varrho(x_n,y_n),C)$ **-projective** $\not\Longrightarrow$ $(S,S/\varrho(x,y))$**-projective**
$(\coprod_{n\in N} S/\varrho(x_n,y_n),C)$ **-projective** $\not\Longrightarrow$ $(S,S/sS)$**-projective**

Consider Example 6.1.1. Since S has no left zero and fulfills condition (Moz), by Propositions 5.8 and 5.9 the one element act Θ_S is not $(S,S/\varrho(x,y))$-projective and not $(S,S/sS)$-projective.

Since every monocyclic act is isomorphic to Θ_S, every coproduct of monocyclic acts is isomorphic to $\coprod_{n\in N}\Theta_S$ for some set $N\neq\emptyset$. Thus, if $g:\coprod_{n\in N}(S/\varrho(x_n,y_n))_S\longrightarrow C_S$ is an epimorphism, then $C_S\cong\coprod_{k\in K}\Theta_S$ for some set $K\neq\emptyset$.
If now $f:\Theta_S\longrightarrow C_S$ is a homomorphism, then there exists $k\in K$, such that $f(\Theta)=\Theta^k$, where Θ^k denotes the element of the k-th copy of Θ_S of the coproduct. Since g is an epimorphism, there exists an $n\in N$, such that $g(\Theta^n)=\Theta^k$ (Θ^n analogously to Θ^k). Then $f':\Theta_S\longrightarrow\coprod_{n\in N}\Theta_S$ with $f'(\Theta)=\Theta^n$ is a homomorphism with $gf'=f$, i.e., Θ_S is $(\coprod_{n\in N} S/\varrho(x_n,y_n),C)$-projective.

Example 6.6 $(\coprod_{l\in L} S/s_lS,C)$ **-projective** $\not\Longrightarrow$ $(S/\varrho(x,y),S/tS)$**-projective**

Consider Example 6.1.3.
The Rees factor act $(S/\{a,b\})_S=(S/aS)_S$ is $(\coprod_{l\in L} S/s_lS,C)$- projective:
Every Rees factor act of S is isomorphic to $(S/aS)_S$ or to Θ_S. Thus $\coprod_{l\in L}(S/s_lS)_S$ is the disjoint union of copies of $(S/aS)_S$ and Θ_S. Therefore an epimorphic image C_S of coproducts of Rees factor acts of S by right principal ideals is the disjoint union of copies of $(S/aS)_S$, copies of Θ_S and amalgamated coproducts of copies of $(S/aS)_S$ by the zero in each copy. Thus if $f:(S/aS)_S\longrightarrow C_S$ is a homomorphism, then $f(S/aS)\cong(S/aS)_S$ or $f(S/aS)\cong\Theta_S$. Thus there exists f' with $g\,f'=f$.

$(S/aS)_S$ is not $(S/\varrho(x,y),S/tS)$-projective:
Since a and b are right zeros of S, for every element $\bar{x}\in S/\varrho(1,b)$ we have $\bar{x}a=\bar{a}$ and $\bar{x}b=\bar{b}$, which implies, that $(S/\varrho(1,b))_S$ has not a zero. Thus the zero morphism $f:(S/aS)_S\longrightarrow(S/1S)_S$ can not be lifted with respect to the epimorphism $g:(S/\varrho(1,b))_S\longrightarrow(S/1S)_S$, since otherwise the zero $\tilde{0}$ of the Rees factor act $(S/aS)_S$ would give rise to a zero $f'(\tilde{0})$ in $(S/\varrho(1,b))_S$.

Example 6.7 $(\coprod_{l\in L} S/s_lS,C)$ **-projective** $\not\Longrightarrow$ $(S/\varrho(x,y),S/\varrho(\tilde{x},\tilde{y}))$**-projective**

Consider the group $(S,\cdot)=(\mathbb{Z}_4,+)$. Then every coproduct of Rees factor acts $(S/s_lS)_S$ of S is isomorphic to a coproduct of copies of Θ_S and so is C_S, if

there is an epimorphism from $\coprod_{l\in L}(S/s_lS)_S$ onto C_S. Thus Θ_S is $(\coprod_{l\in L} S/s_lS, C)$-projective.
The monocyclic act $(S/\varrho(1,2))_S$ is isomorphic to Θ_S. Thus there exists an epimorphism $g:(S/\varrho(1,3))_S \longrightarrow (S/\varrho(1,2))_S$. Since $(S/\varrho(1,3))_S$ does not contain a zero (see Example 5.3.(c)), the identity id_Θ can not be lifted with respect to g, i.e., Θ_S is not $(S/\varrho(x,y), S/\varrho(\tilde{x},\tilde{y}))$-projective.

Example 6.8 $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$ **-projective** $\not\Longrightarrow$ $(S/sS, S/\varrho(x,y))$**-projective**

Consider for the monoid in Example 6.1 2 the monocyclic act $A_S := (S/\varrho(1,a))_S$.

Every coproduct of monocyclic acts is isomorphic to a coproduct of copies of $(S/\varrho(1,a))_S$ and Θ_S.
If $g:\coprod_{n\in N}(S/\varrho(x_n,y_n))_S \longrightarrow A_S$ is an epimorphism, then by Corollary 3.6 there exists $k\in N$, such that $g\mid_{S/\varrho(x_k,y_k)}$ is an epimorphism onto A_S, which implies $(S/\varrho(x_k,y_k))_S \cong A_S$.
Then the identitiy id_A could be lifted by id'_A with $id'_A(A) = S/\varrho(x_k,y_k)$. The homomorphism $f: A_S \longrightarrow A_S$ with $f(A)=\bar{a}$, where $\bar{a}$ denotes the class of a in A_S, can also be lifted by id'_A. Since there are no further homomorphisms from A_S into itself, this implies, that A_S is $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$-projective.

Since S has a zero, $(S/sS, S/\varrho(x,y)$-projectivity is equivalent to $(S, S/\varrho(x,y))$-projectivity by Theorem 4.4. Since A_S is a monocyclic act, A_S being $(S, S/\varrho(x,y))$-projective would imply, that A_S is a retract of S_S and is therefore projective. Since $A_S \ncong S_S$ and $A_S \ncong (0S)_S$, by Result 2.27 this leads to a contradiction. Thus A_S is not $(S, S/\varrho(x,y))$-projective, i.e., not $(S/sS, S/\varrho(x,y)$-projective.

6.1.2 Generators as second component

Example 6.9 (S,G)**-projective** $\not\Longrightarrow$ $(\coprod_{i\in I} S, G)$**-projective**

Let S be a monoid, such that all cyclic generators are isomorphic to eS with $e\in E(S)$ and $e\mathcal{J}1$ and S be not a group. Then by Theorem 5.21, all right S-acts are (S,G)-projective, but by Theorem 5.19, there exists at least one right S-act, which is not $(\coprod_{i\in I} S, G)$-projective.

As a concrete example, take $S=\{0,1\}$. Then the amalgamated coproduct $G_S = (S\coprod^0 S)_S$ of two copies of S by the zero is a generator in **Act**$-S$. Since G_S is an indecomposable right S-act, the identity id_G can not be lifted with respect to the canonical epimorphism $g: S\coprod S \longrightarrow G$, i.e., G_S is not $(\coprod_{i\in I} S, G)$-projective.

Since S is finite, by Lemma 3.35 all right S acts are (S, G)-projective and so is G_S.

6.1.3 Second component A or C

Example 6.10 $(\coprod_{l\in L} S/s_lS, C)$**-projective** $\not\Longrightarrow$ $(S/\varrho(x,y), A)$**-projective**

As an example take Example 6.7.

Example 6.11 $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$**-projective** $\not\Longrightarrow$ $(\coprod_{l\in L} S/s_lS, C)$**-projective**
$(S/\varrho(x,y), A)$**-projective** $\not\Longrightarrow$ $(\coprod_{l\in L} S/s_lS, A)$**-projective**

Consider in Example 6.1.4 the monocyclic act $(S/\varrho(a,b))_S$. Then the identity $id_{S/\varrho(a,b)}$ could not be lifted with respect to $g : (S/bS)_S \longrightarrow (S/\varrho(a,b))_S$, since $(S/\varrho(a,b))_S$ is not a retract of $(S/bS)_S$. Thus $(S/\varrho(a,b))_S$ is not $(\coprod_{l\in L} S/s_lS, A)$-projective and therefore not $(\coprod_{l\in L} S/s_lS, C)$-projective.
Since for all $x,\ y \in S$ the monocyclic act $(S/\varrho(x,y))_S$ is isomorphic to $(S/\varrho(a,b))_S$ or to Θ_S, we get that $(S/\varrho(a,b))_S$ is a retract of every coproduct of monocyclic acts. Thus, $(S/\varrho(a,b))_S$ is $(\coprod_{n\in N} S/\varrho(x_n,y_n), A)$-projective and therefore $(S/\varrho(x,y), A)$-projective .

6.1.4 Cyclic acts and coproducts of cyclic acts as second component

Example 6.12 $(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n,y_n))$**- projective** $\not\Longrightarrow$ $(S/sS, S/tS)$**-projective**

Let $(S, \cdot) = (\mathbb{N}, max)$ and let $A_{\mathbb{N}} = (\mathbb{N}/2\mathbb{N})_{\mathbb{N}}$.
Since no monocyclic act contains a zero, there does not exist a homomorphism f from a Rees factor act of $\mathbb{N}$ into a monocyclic act, i.e., $A_{\mathbb{N}}$ is trivially-$(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n,y_n))$-projective and thus $(\coprod_{i\in I} S, \coprod_{n\in N} S/\varrho(x_n,y_n))$-projective.

$A_{\mathbb{N}}$ is not $(S/sS, S/tS)$-projective, since the identity on $A_{\mathbb{N}}$ could not be lifted with respect to the epimorphism $g : (\mathbb{N}/3\mathbb{N})_{\mathbb{N}} \longrightarrow A_{\mathbb{N}},\ g(1_{\mathbb{N}/3\mathbb{N}}) = 1_{\mathbb{N}/2\mathbb{N}}$:
Let $id'_A : A_{\mathbb{N}} \longrightarrow (\mathbb{N}/3\mathbb{N})_{\mathbb{N}}$ be a mapping with $g\, id'_A = id_A$, which implies $id'_A(1_{\mathbb{N}/2\mathbb{N}}) = 1_{\mathbb{N}/3\mathbb{N}}$. Let $\tilde{0}$ denote the zero in $A_{\mathbb{N}}$, $\bar{0}$ the zero in $(\mathbb{N}/3\mathbb{N})_{\mathbb{N}}$ and $\bar{2}$ the class of 2 in $(\mathbb{N}/3\mathbb{N})_{\mathbb{N}}$. If id'_A is a homomorphism, the equation $\bar{0} = id'_A(\tilde{0}) = id'_A(1_{\mathbb{N}/2\mathbb{N}}\, 2) = id'_A(1_{\mathbb{N}/2\mathbb{N}})\, 2 = 1_{\mathbb{N}/3\mathbb{N}}\, 2 = \bar{2}$ yields the contradiction $\bar{0} = \bar{2}$. Thus the identity id_A could not be lifted with respect to g.

Example 6.13 $(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$**-projective** $\not\Longrightarrow$ $(S, S/\varrho(x,y))$**-projective**
$(\coprod_{n\in N} S/\varrho(x_n,y_n), \coprod_{m\in M} S/\varrho_m)$**-projective** $\not\Longrightarrow$ $(S, S/tS)$**-projective**

Consider $(\mathbb{Z}_3,+)$. Then every monocyclic act is isomorphic to Θ_S. Thus, if $g : \coprod_{n\in N}(S/\varrho(x_n,y_n))_S \longrightarrow \coprod_{m\in M}(S/\varrho_m)_S$ is an epimorphism, $\coprod_{m\in M}(S/\varrho_m)_S$ is isomorphic to a coproduct of copies of Θ_S. Thus the one element act Θ_S is $(\coprod_{n\in N}(S/\varrho(x_n,y_n))_S, \coprod_{m\in M} S/\varrho_m)$-projective.
Since S fulfills condition (Moz) and S does not contain a left zero, Θ_S is not $(S, S/\varrho(x,y))$-projective by Proposition 5.9. Moreover, since S has not a left zero, by Proposition 5.8 we get that Θ_S is not $(S, S/tS)$-projective.

Example 6.14 $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$**-projective** $\not\Longrightarrow$ $(S/\varrho(x,y), S/tS)$**-projective**

Consider Example 6.1.3. Then Θ_S is $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$-projective by Corollary 5.14, since S is left reversible. Since $(S/\varrho(1,a))_S$ has no zero, S does not fulfill condition (Moz). Thus by Lemma 5.10, Θ_S is not $(S/\varrho(x,y), S/tS)$-projective.

As a further example one could consider Θ_S in $\mathbf{Act}-\mathbb{Z}_4$ for the group $(\mathbb{Z}_4,+)$.

Example 6.15 $(\coprod_{l\in L} S/s_lS, \coprod_{n\in N} S/\varrho(x_n,y_n))$**- projective**
$\not\Longrightarrow$ $(S/\varrho(x,y), S/\varrho(\tilde{x},\tilde{y}))$**-projective**
$(S/sS, S/\varrho)$**-projective** $\not\Longrightarrow$ $(S/\varrho(x,y), S/\varrho(\tilde{x},\tilde{y}))$**-projective**

Is included in Example 6.7.

Example 6.16 $(\coprod_{l\in L} S/s_lS, \coprod_{m\in M} S/\varrho_m)$**-projective**
$\not\Longrightarrow$ $(\coprod_{j\in J} S/\varrho(x_j,y_j), \coprod_{n\in N} S/\varrho(x_n,y_n))$**-projective**

Is included in Example 6.7.

Example 6.17 $(S, S/I)$**-projective** $\not\Longrightarrow$ $(S/sS, S/\varrho(x,y))$**-projective**

Consider the monoid in Example 6.1.2. Since S is a group with zero adjoined, by Corollary 5.54 all right S-acts are $(S, S/I)$-projective. Since S contains a left zero, all right S-acts are $(S/sS, S/\varrho(x,y))$-projective iff all are $(S, S/\varrho(x,y))$-projective, and thus by Theorem 5.24 iff $S = \{1\}$ or $S = \{0,1\}$. Therefore there exists a non $(S/sS, S/\varrho(x,y))$-projective right S-act.

Example 6.18 $(S/\varrho(x,y), S/I)$**-projective** $\not\Longrightarrow$ $(S, S/tS)$**-projective**

Consider the group $(\mathbb{Z}_3, +)$. Since S fulfills condition (Moz), by Proposition 5.10 the one element act Θ_S is $(S/\varrho(x,y), S/I)$-projective. Since S does not contain a left zero, by Proposition 5.8 we get, that Θ_S is not $(S, S/tS)$-projective.

Example 6.19 $(S, S/tS)$**-projective** $\not\Longrightarrow$ $(S/sS, S/I)$**-projective**

Consider the monoid in Example 6.1.4.
Since S has a left zero, $(S/sS, S/I)$-projectivity is equivalent to $(S, S/I)$-projectivity. Consider $A_S = (S/\{a,b\})_S$. Since A_S is not a retract of S, it is not $(S, S/I)$-projective.

But A_S is $(S, S/tS)$-projective:
Every Rees factor act of S by a principal right ideal is isomorphic to Θ_S or to S_S. $f : A_S \longrightarrow \Theta_S$ can be lifted with respect to the zero morphism $g : S_S \longrightarrow \Theta_S$ by $f' : A_S \longrightarrow S_S$ with $f'(A) = a$.
$f : A_S \longrightarrow S_S$ with $f(A) = a$ (or b, respectively) can be lifted with respect to $g = id_S$ by itself.

Example 6.20 $(S/\varrho(x,y), S/I)$**-projective** $\not\Longrightarrow$ $(S, S/tS)$**-projective**

is included in Example 6.5.

Example 6.21 $(S/sS, S/I)$**-projective** $\not\Longrightarrow$ $(S/\varrho(x,y), S/tS)$**-projective**

For every monoid S, the one element act Θ_S is $(S/sS, S/I)$-projective by Lemma 5.11.
Let S be a monoid, which does not fulfill condition (Moz). Then by Lemma

5.10, Θ_S is not $(S/\varrho(x,y), S/tS)$-projective.

As a concrete example take the monoid in Example 6.1.3. The monocyclic act $(S/\varrho(1,a))_S$ has no zero, i.e., S does not fulfill condition (Moz) and thus by Lemma 5.10, Θ_S is not $(S/\varrho(x,y), S/tS)$-projective.

6.2 Differences from other concepts

Example 6.22 pullback flat $\not\Longrightarrow$ $(S, S/tS)$-projective

Let S be a left collapsible monoid without left zero. Then Θ_S is pullback flat by Corollary 2.34, but by Lemma 5.8 not $(S, S/tS)$-projective.

Example 6.23 (see also [17]) **$(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$-projectivity $\not\Longrightarrow$ condition (P)**

Take $\mathbb{Z}_3 = \{0,1,2\}$ with the usual multiplication of residue classes. Since $\mathbb{Z}_3$ has a left zero, $(\coprod_{i\in I} S, \coprod_{m\in M} S/\varrho_m)$-projectivity is equivalent to $(S, S/\varrho)$-projectivity by Lemma 4.4.
Consider in $\mathbf{Act}-\mathbb{Z}_3$ the coproduct $\mathbb{Z}_3 \coprod^0 \mathbb{Z}_3$ with amalgamated zero whose elements we denote as $\{0, 1_1, 2_1, 1_2, 2_2\}$ on which $\mathbb{Z}_3$ acts as follows: $1_i x = (1x)_i$, $2_i x = (2x)_i$ for $x \in \mathbb{Z}_3\backslash\{0\}$, $1_i 0 = 2_i 0 = 0$, $i \in \{1,2\}$, and 0 is fixed.
All factor acts of $\mathbb{Z}_3$ are trivial or isomorphic to $\mathbb{Z}_3/\varrho(1,2)$.

It is sufficient to prove the lifting property in the following three cases:
For all right congruences ϱ the zero homomorphism $f : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3/\varrho$ with $f(\mathbb{Z}_3 \coprod^0 \mathbb{Z}_3) = \{\bar{0}\}$ can be lifted by $f' : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ with $f'(\mathbb{Z}_3 \coprod^0 \mathbb{Z}_3) = \{0\}$.
The homomorphism $f : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3/\varrho(1,2)$ with $f(1_i) = \bar{1} = \bar{2} = f(2_i)$, $i \in \{1,2\}$, $f(0) = \bar{0}$ can be lifted by taking $f' : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ defined by $f'(1_i) = 1$, $f'(2_i) = 2$ $(i \in \{1,2\})$ and $f'(0) = 0$.
The homomorphism $f : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3/0\mathbb{Z}_3$ with $f(1_i) = \bar{1}$, $f(2_i) = \bar{2}$, $i \in \{1,2\}$, $f(0) = \bar{0}$ can be lifted by taking $f' : \mathbb{Z}_3 \coprod^0 \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ defined by $f'(1_i) = 1$, $f'(2_i) = 2$ $(i \in \{1,2\})$ and $f'(0) = 0$.
Thus $\mathbb{Z}_3 \coprod^0 \mathbb{Z}_3$ is weakly projective.
When considering $1_1 0 = 1_2 0$ one can easily verify that $\mathbb{Z}_3 \coprod^0 \mathbb{Z}_3$ does not fulfill condition (P).

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