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Abstract

The concept of Schur rings was introduced in 1933 by I. Schur. For several decades applications of Schur rings were restricted to the investigation of permutation groups. Starting in the fifties, similar concepts like association schemes, cellular algebras and coherent configurations were introduced independently by different authors. They were used for various questions in algebraic combinatorics and statistics. In this thesis three different tasks which are related to these concepts are considered: (1) characterization of commuting graphs, (2) consideration of strongly regular graphs and partial difference sets and (3) investigation of cyclotomic schemes. The first part deals with graphs with commuting adjacency matrices. Here, we give results for commuting regular graphs and discuss the case of non-regular graphs. The second part deals with the construction of partial difference sets by using strongly regular Cayley graphs. Theoretical and computational approaches are discussed and all regular partial difference sets in groups up to order 49 are determined. Moreover, regular partial difference sets for strongly regular graphs up to 255 vertices which have primitive automorphism group, are constructed. In the third part an algorithm for the determination of cellular subrings of cellular rings is adopted for cyclotomic schemes. This algorithm uses the information given by cyclotomic numbers for the complete theoretical determination of all subschemes. The determination of subschemes for cyclotomic schemes with three, four and six classes are described in detail.

Zusammenfassung

Die Theorie der Schur Ringe, eingeführt 1933 von I. Schur, spielte über mehrere Jahrzehnte nur auf dem Gebiet der Permutationsgruppen eine Rolle. Seit den fünfziger Jahren wurden unabhängig voneinander ähnliche Konzepte wie Assoziationsschemata, zellulare Ringe oder kohärente Konfigurationen für verschiedene Probleme in der Algebraischen Kombinatorik und Statistik entwickelt. In dieser Dissertation werden Fragestellungen aus den Gebieten (1) vertauschbare Graphen, (2) streng reguläre Graphen und partielle Differenzenmengen und (3) zyklotomische Schemata vor dem Hintergrund dieser Konzepte betrachten. Der erste Teil behandelt Graphen mit kommutierenden Adjazenzmatrizen. Ergebnisse für vertauschbare reguläre Graphen werden präsentiert und die Fragestellung für nicht reguläre Graphen diskutiert. Der zweite Teil beschäftigt sich mit der Konstruktion von partiellen Differenzenmengen durch streng reguläre Cayley Graphen. Dazu werden theoretische und computer-basierte Verfahren diskutiert und eine vollständige Liste regulärer partieller Differenzenmengen in allen Gruppen mit bis zu 49 Elementen konstruiert. Außerdem werden reguläre partielle Differenzenmengen für streng reguläre Graphen bis zu 255 Ecken mit primitiver Automorphismengruppe bestimmt. In einem dritten Teil der Dissertation wird eine Methode zur Bestimmung alle Unterschemata in einem zyklotomischen Schema beschrieben. Dazu wird ein Algorithmus, der alle zellularen Unterringe eines zellularen Ringes bestimmt, verwendet. Dieser Algorithmus benutzt Informationen, die durch die zyklotomischen Zahlen des Schemas gegeben werden. Abschließend werden Unterschemata der zyklotomischen Schemata mit drei, vier und sechs Klassen beschrieben.

Chapter 1

Introduction

In 1933 I. Schur introduced a new concept for the investigation of permutation groups [Sch33]: To each permutation group which contains a regular permutation group H as a subgroup he associated a subring of the group ring $\mathbb{Z}(H)$. This was the starting point for a notion which we nowadays call Schur ring and which Schur's student H. Wielandt later pointed out explicitly ([Wie35], [Wie49], [Wie64]). For several decades Schur rings were regarded as a tool for purely group theoretical problems. However, in the seventies M. H. Klin und R. Pöschel elaborated first applications for problems in graph theory (e.g., [KliP78]). Presently, Schur rings are also considered as an important tool in algebraic combinatorics.

In the last fifty years several concepts similar to Schur rings were introduced. On the one hand, there is the notion of association scheme and its associated Bose-Mesner algebra which was introduced by R. C. Bose and T. Shimamoto resp. R. C. Bose and D. M. Mesner ([BosS52], [BosM59]). On the other hand, in 1970 D. G. Higman published his concept of coherent configurations [Hig70]. The coherent configurations coincide up to language with a special class of cellular algebras which were already introduced in 1968 by B. Y. Weisfeiler and A. A. Leman in [WeiL68]. The fact that similar concepts were introduced independently by different mathematicians for different problems indicates that the underlying idea is as well natural as important. We will discuss these notions in Section 2.2.

If we consider a very special class of Schur rings, the primitive, symmetric Schur rings of rank 3, then we have objects which, on the one hand, correspond to nontrivial strongly regular Cayley graphs (see Proposition 3.1.26) and, on the other hand, to nontrivial, regular partial difference sets (see Proposition 3.1.27).

Strongly regular graphs were first studied by R. C. Bose in connection with partial geometries and symmetric association schemes with two classes [Bos63]. A short time later, D. G. Higman initiated the study of rank 3 permutation groups using strongly regular graphs [Hig64]. Both, the combinatorial and the group theoretical aspects have been developed over the years (see for example [HesH71], [Hub75], [CamGS78], [Bro96]). Strongly regular graphs have become a popular area in algebraic combinatorics during the last decades. A strong increase of the interest in these graphs at the end of the sixties was based on the discovery of new simple finite groups in connection to strongly regular graphs. Until today the search for unknown strongly regular graphs is one of the most challenging tasks for

mathematicians working on this field.
Partial difference sets were named by I. M. Chakravarti in 1969 [Cha69]. However, they were introduced some years earlier by R. C. Bose and J. M. Cameron in their studies of calibration designs and the bridge tournament problem [BosC65]. A systematical study of partial difference sets as a generalization of difference sets was started by S. L. Ma ([Ma84], [Ma89]). Many results were obtained already in the seventies in terms of strongly regular graphs since strongly regular Cayley graphs are directly connected to partial difference sets. The development of the theory of partial difference sets is on a similar stage as in case of strongly regular graphs: The determination of unknown partial difference sets resp. the discovering of new connections between the known results is the challenge. An extended survey of partial difference sets is given in [Ma94]. It includes a table of all known parameter sets for partial difference sets in abelian groups.
The present thesis is dedicated to three subjects: the main part deals with techniques for the determination of partial difference sets and, after that, the complete determination of partial difference sets for certain groups, in particular, by strongly regular graphs. A second part presents a method for the determination of subschemes in cyclotomic schemes where the focus is on subschemes with two classes. Each 2-class subscheme corresponds to a partial difference set. The third part is concerned with the investigation of commuting graphs.

In the past, research in the area of partial difference sets refers mainly to a qualitative level, i.e., proving of existence and non-existence results for partial difference sets with certain parameters, connections between known partial difference sets and so on. By the complete determination of partial difference sets in certain groups resp. for certain strongly regular graphs this thesis follows a more quantitative conception. In Section 4 the determination of all nontrivial, regular partial difference sets in all groups of order up to 49 is presented. The partial difference sets were obtained by theoretical and computational approaches. The techniques of the determination of partial difference sets are described in Chapter 3. Basis for this work was the correspondence between strongly regular Cayley graphs and partial difference sets. The main source was a complete catalogue of strongly regular Cayley graphs which was obtained from the list of strongly regular graphs on the website of E. Spence [Spe01] in combination with some theoretical results. For each of these Cayley graphs all partial difference sets were determined.
In our work we were restricted by the bound 49 since a complete catalogue for strongly regular Cayley graphs with more than 49 vertices does not exists. This fact is not surprising, since the phenomenon of "combinatorial explosion", the enormous growth of the number of objects by increasing order, also affects the class of strongly regular graphs. Only for some special parameter sets all strongly regular graphs are known. To extend our work to graphs with more than 49 vertices without loosing the idea of completeness we restricted ourselves to strongly regular graphs with a certain property. Since we were concerned with graphs with transitive automorphism group, from both algebraic and combinatorial points of view it was natural to restrict the investigation to graphs with primitive automorphism group. A nearly complete catalogue of strongly regular graphs up to 255 vertices with primitive

automorphism group was created by C. Pech one year ago with the aid of computers (C. Pech, private communication). We were able to determine the missing graphs (with 121 and 169 vertices) theoretically. Using the same techniques as for the small cases we succeeded to determine almost all partial difference sets for strongly regular graphs up to 255 vertices with primitive automorphism groups (see Section 4.2). However, we were not successful for seven out of 95 graphs, since the computer facilities were not sufficient.

As already mentioned before, in this thesis a second subject is treated. In Chapter 5 a special class of association schemes, the class of cyclotomic schemes, is considered. We present a method for the determination of all subschemes of a cyclotomic scheme. Here we are mainly interested in 2-class schemes which correspond to partial difference sets and strongly regular graphs. The method is based on an algorithm developed by I. A. Faradžev and M. H. Klin (cf. [FarKM94]) which computes all cellular subrings of a given cellular ring. A program implementation, the computer package COCO, is briefly described in Section 3.3.1.
We adopted this algorithm for cyclotomic schemes. This allows us to determine all subschemes of a cyclotomic scheme purely theoretically provided we have formulas for the so-called cyclotomic numbers of the scheme. In Chapter 5 we determine all subschemes for cyclotomic schemes with three and four classes and give a brief idea for the case of cyclotomic schemes with six classes.

Besides partial difference sets and cyclotomic schemes this thesis contains a section which is dedicated to the starting point of the authors scientific interest (see Section 2.3). The very first task which finally led to the present thesis was the investigation of pairs of commuting graphs, i.e., pairs of graphs with commuting adjacency matrices. The aim was to find a characterization for such graphs which is independent of the vertex labelings. One approach is to consider properties of the structure of commuting graphs. A second approach which is more natural is to investigate matrix algebras which contain adjacency matrices of commuting graphs. Following the second approach the author became acquainted with cellular rings, Bose-Mesner algebras, etc. which, finally, changed his research interests from a purely graph theoretical task to a task in algebraic combinatorics.

There are several places in this thesis where a wider presentation of the theoretical background would be nice. Especially in Chapter 3 and 4 we use objects and results from coding theory, design theory, and other areas without presenting a comprehensive explanation. However, presenting all these topics in detail would go far beyond the scope of this thesis.

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study distance regular graphs and Bose-Mesner algebras. Financial support of the DAAD and the Ben-Gurion-University of the Negev made it possible for me to visit M. H. Klin in Beer-Sheva in February/March 2000. At that time I started my studies on partial difference sets in connection to strongly regular graphs. A grant of the Carl von Ossietzky University Oldenburg allowed a visit of M. H. Klin in October 2000 in Oldenburg. I am indebted to U. Knauer and M. H. Klin for their support all over the time.
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Chapter 2

Groups and Graphs

2.1 Preliminaries

In this section we will give necessary preliminaries and notations which we use throughout this thesis. We restrict ourselves to the presentation of definitions and results, proofs are omitted. Basic definitions and results in classical group theory are not presented here, we refer to [Hup67] or [Hal76]. For information about special finite groups (names, symbols, notations), which are not explained in this thesis, we refer to the Atlas of finite groups [CCNPW85] and the Atlas of finite group representations [BLNPRSTWW01]. A more general introduction into the theory of permutation groups is the book of Wielandt [Wie64]. An overview of combinatorial objects can be found in [vLiW92], for further results in graph theory see [Har95]. For more information about cellular rings (resp. coherent algebras) we refer to [FarKM94], a detailed introduction is also given in [KliRRT99].

2.1.1 Permutation groups

Let V be a set of order v. A one-to-one mapping of V onto itself is called a **permutation on** V. The image of an element $x \in V$ with respect to a permutation α is denoted by x^α. For two permutations α, β on V we define the product $\alpha\beta$ by $x^{\alpha\beta} := (x^\alpha)^\beta$ for all $x \in V$. Clearly, this product is again a permutation on V. The set of all permutations on V forms a group which is called the **symmetric group**. The symmetric group on V is denoted by S_V; if $V = \{1, \ldots, v\}$, then we will also write S_v instead of S_V. Each subgroup H of order n of the symmetric group S_V is called a **permutation group on** V **of order** n **and degree** v, it will be denoted by (H, V) or simply by H if it is clear that H acts on V.
Notice that by this definition permutation groups are always finite and act on finite sets. It is possible to give an extended definition, e.g., permutation groups acting on infinite sets. However, in this thesis we only consider the finite case.

Each permutation group (H, V) of degree v can be represented as a group of $v \times v$-matrices with entries 0 and 1: For each permutation $\sigma \in H$ we define a matrix $P(\sigma) = (p_{ij})_{i,j \in V}$ by $p_{ij} := 1$ if $i^\sigma = j$ and $p_{ij} := 0$ otherwise. The matrix $P(\sigma)$ is called the **permutation**

matrix of σ. It is easy to see that each permutation matrix has exactly one entry 1 in each row and in each column.

For the permutation groups of order n with degree v we define an equivalence relation:

Definition 2.1.1 *Let V, W be two sets of order v and let $(H, V), (K, W)$ be two permutation groups. The groups $(H, V), (K, W)$ are called* **similar permutation groups***, if there exists a bijection $\sigma : V \to W$, such that $K = \{\sigma^{-1} h \sigma | h \in H\}$.*

In the following we will give some results for permutation groups which are important for this work. For further results on permutation groups we refer to [Wie64].
Let (H, V) be a permutation group. Then for $x \in V$ the set $\{x^\alpha | \alpha \in H\}$ is called the **orbit of** x **under** H. The permutation group (H, V) induces a permutation group $\tilde{H}$ on $V \times \ldots \times V$ (k times, $k \geq 1$) by $(x_1, \ldots, x_k)^{\tilde{\alpha}} := (x_1^\alpha, \ldots, x_k^\alpha)$ for all $(x_1, \ldots, x_k) \in V \times \ldots \times V$ and $\alpha \in H$. The orbits of $\tilde{H}$ are called k**-orbits** of H. The group (H, V) also induces a permutation group $\widehat{H}$ on the set of k-element subsets of V by $\{x_1, \ldots, x_k\}^{\widehat{\alpha}} := \{x_1^\alpha, \ldots, x_k^\alpha\}$ for distinct $x_i \in V, 1 \leq i \leq k$ and $\alpha \in H$. The orbits of the group $\widehat{H}$ are called $\{k\}$**-orbits**.

Definition 2.1.2 *A permutation group (H, V) is called* **transitive***, if it has exactly one orbit in V, i.e., $\{x^\alpha | \alpha \in H\} = V$ for all $x \in V$.*

Definition 2.1.3 *Let (H, V) be a permutation group. Then for an element $x \in V$ the set $H_x := \{\alpha \in H | x^\alpha = x\}$ is called the* **stabilizer of** x **in** H.

It is easy to check that for all $x \in V$ the stabilizer H_x is a subgroup of (H, V).

Definition 2.1.4 *A permutation group (H, V) is called* **semiregular***, if for each $x \in V$ we have $H_x = \{e\}$. If (H, V) is semiregular and transitive, then it is called* **regular**.

There exists a helpful characterization of regular permutation groups:

Proposition 2.1.5 *([Wie64], Proposition 4.2)*
A transitive permutation group (H, V) acting on a set V with $|V| = v$ is regular if and only if H is of order v.

By a well-known Theorem of Cayley each abstract group H has a representation as a regular permutation group (H, H), where the action is defined by multiplication from the right (cf. [Hal76], Theorem 1.4.2). In this case the permutation group (H, H) is called the **right regular representation of** H.

In the following we give some results from classical group theory which contain a complete characterization for classes of groups of a certain order (cf. [Hal76], Section 4.4).

Lemma 2.1.6 *Each group H of order p, p a prime, is isomorphic to $\mathbb{Z}_p$, i.e. H is cyclic.*

Lemma 2.1.7 *Each group of order p^2, p a prime, is isomorphic to $\mathbb{Z}_{p^2}$ or to $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Lemma 2.1.8 *Each group H of order pq, where p, q are primes, $p > q$ and $p \not\equiv 1 \bmod q$, is isomorphic to $\mathbb{Z}_{pq}$, i.e. H is cyclic.*

Lemma 2.1.9 *Let H be a group of order p^3, p a prime. Then for H one of the following cases holds:*

1) *The group H is abelian and isomorphic to one of the groups $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$;*

2) *The group H is non-abelian, has order 8 and is isomorphic to the dihedral group D_4 or the quaternion group Q;*

3) *The group H is non-abelian, has order $p^3 \neq 8$ and is isomorphic to the group $\langle a, b\rangle$ with $a^{p^2} = b^p = 1$, $ab = ba^{p+1}$ or to the group $\langle a, b, c\rangle$ with $a^p = b^p = c^p = 1$, $ab = bac, ac = ca, bc = cb$.*

The next proposition gives another important characterization of a special class of groups:

Proposition 2.1.10 *(cf. [Hal76], Theorem 2.5.2)*
A group H is isomorphic to the direct product of subgroups K_i, $i = 1, \ldots, k$, if

1) *the subgroups K_i are normal, $i = 1, \ldots, k$,*

2) *$H = K_1 \cdot \ldots \cdot K_k := \{x_1 \cdot \ldots \cdot x_k | x_i \in K_i,\ i = 1, \ldots, k\}$ and*

3) *$K_i \cap (K_1 \cdot \ldots \cdot K_{i-1} \cdot K_{i+1} \cdot \ldots \cdot K_k) = \{e\}$ for $i = 1, \ldots, k$.*

The definition of transitive and regular permutation groups given above can be generalized:

Definition 2.1.11 *A permutation group (H, V) is called* **k-transitive**, *$k \geq 1$, if the induced permutation group $(\tilde{H}, V \times \ldots \times V)$ acts transitively on the set of k-tuples with all coordinates different. If this action is regular, then H is called* **sharply k-transitive**. *The group (H, V) is called* **k-homogeneous**, *$k \geq 1$, if the induced permutation group $\widehat{H}$ on the set of k-element subsets of V is transitive. If this action is regular, then (H, V) is called* **sharply k-homogeneous**.

Notice that the 1-transitive resp. 1-homogeneous permutation groups are the transitive permutation groups. It is easy to see that a k-transitive permutation group is k-homogeneous and that a k-transitive permutation group is $(k-1)$-transitive ($k \geq 2$). The fact that each k-homogeneous permutation group of degree n is also $(k-1)$-homogeneous ($\frac{n}{2} \geq k \geq 2$) is more difficult to prove (cf. [LivW65]).

Definition 2.1.12 *Let (H, V) be a transitive permutation group. Then the number of 2-orbits of (H, V) is called the* **rank of** *(H, V).*

It is not difficult to check the following remark:

Remark 2.1.13 *For an element $x \in V$ the rank of a transitive permutation group (H, V) is equal to the number of orbits of H_x on V.*

Now we turn to special transitive permutation groups.

Definition 2.1.14 *Let (H, V) be a transitive permutation group. The group (H, V) is called* **primitive** *if each equivalence relation $\mathcal{R} \subseteq V \times V$, which satisfies $(x, y) \in \mathcal{R} \Rightarrow (x^\alpha, y^\alpha) \in \mathcal{R}$ for all $\alpha \in H$, is trivial, i.e., $\mathcal{R} = \Delta$ or $\mathcal{R} = V \times V$ where Δ is the diagonal $\{(x, x) | x \in V\}$. A transitive permutation which is not primitive is called* **imprimitive**.

For primitive permutation groups exists an important characterization:

Theorem 2.1.15 *([Wie64], Theorem 8.2)*
Let V be a set with $|V| > 1$ and let $x \in V$. A transitive permutation group (H, V) is primitive if and only if H_x is a maximal subgroup of H.

The following statements give sufficient conditions for the primitivity of permutation groups.

Proposition 2.1.16 *([Wie64], Theorem 8.3)*
A transitive group of prime degree is primitive.

Proposition 2.1.17 *([Wie64], Theorem 9.6)*
Every 2-transitive permutation group is primitive.

Finally, we give a result for normal subgroups of primitive permutation groups.

Proposition 2.1.18 *([Wie64], Theorem 8.8)*
Let N be a normal subgroup of a primitive permutation group (H, V) with $|N| > 1$. Then N acts transitively on V.

Another special class of transitive permutation groups are the so-called generously transitive permutation groups:

Definition 2.1.19 *A transitive permutation group (H, V) is called a* **generously transitive permutation group**, *if for all $i, j \in V$ there exists a permutation $\sigma \in H$, such that $i^\sigma = j$ and $j^\sigma = i$.*

The terminology of generously transitive permutation groups was introduced by P. M. Neumann in [Neu75].

Next we introduce the semidirect product $(H \rtimes K, H)$ for groups H and $K \leq Aut(H)$.

Definition 2.1.20 *Let (H, H) be a permutation group, i.e., the group H acts on itself by right multiplication (right regular representation). Let $K \leq Aut(H)$ be a subgroup of the automorphism group of H. Then the permutation group $(H \rtimes K, H)$ with the action $h^{(\alpha,\beta)} := h^\beta \alpha$ for $h \in H$, $(\alpha, \beta) \in H \rtimes K$ is called the* **semidirect product of** *H* **and** *K.*

As an example for the semidirect product of two permutation groups we consider the normalizer of a permutation group in the symmetric group:

Definition 2.1.21 *Let H be a group and (H, H) its right regular representation. Then the normalizer $N_{S_v}((H, H))$ of (H, H) in the symmetric group S_v is called the* **holomorph of** *H.*

Remark 2.1.22 *(cf. [Cam99], Chapter 1.7)*
Let H be a group. Then $H \rtimes Aut(H)$ is the holomorph of H.

There are different binary operations for two permutation groups $(H, V), (K, W)$ like the product $(H \times K, V \times W)$, the wreath product $(H, V) \wr (K, W) = (K^V \rtimes H, V \times W)$ where K^V is the set of all mappings $V \to K$, the exponentiation $(K, W) \uparrow (H, V) = (K^V \rtimes H, W^V)$, etc. For details about their definitions we refer to [KliPR88], Chapter 1.7. Here we only want to mention the exponentiation of two permutation groups:

Definition 2.1.23 *Let (H, V) and (K, W) be two permutation groups. The permutation group $(K, W) \uparrow (H, V) = (K^V \rtimes H, W^V)$ is called* **exponentiation of** *(K, W)* **with** *(H, V) and is defined as follows: For $h, h' \in H$ and $\alpha, \alpha' \in K^V$ we have $(\alpha, h) \cdot (\alpha', h') := (\alpha'', hh')$, where $\alpha'' : V \to K$ with $\alpha''(x) := \alpha(x)\alpha'(x^h), x \in V$. For the action of $K^V \rtimes H$ on W^V we consider $f \in W^V$ and $(\alpha, h) \in K^V \rtimes H$. We have*

$$f^{(\alpha,h)} : V \to W \text{ with } f^{(\alpha,h)}(x) := (f(x^{h^{-1}}))^{\alpha(x^{h^{-1}})}, x \in V.$$

We give an example in order to explain the action of the exponentiation of two permutation groups.

Example 2.1.24 *Let $V := \{0, 1\}$ and $W := \{0, 1, 2\}$ and consider the groups (S_2, V) and (S_3, W). The exponentiation of (S_3, W) with (S_2, V) acts on the nine elements of W^V which can be described as pairs $(f(0), f(1)) := (w_1, w_2)$, $w_1, w_2 \in W$, where $f \in W^V$. An element $(\alpha, h) \in (S_3^V \rtimes, S_2)$ acts as follows: The element h permutes $\{0, 1\}$, i.e., the coordinates of $(f(0), f(1)) := (w_1, w_2)$ and for $i \in V$ the permutation $\alpha(i) \in S_3$ acts on the i^{th} coordinate.*

In Example 3.2.48 the exponentiation $(S_n \uparrow S_2)$, $n \geq 2$, occurs as the automorphism group of the lattice graph $L_2(n)$.
For the investigation of permutation groups we have some further theory. A strong tool to describe permutation groups is the notion of its centralizer ring.

Definition 2.1.25 *(cf. [FarKM94], p. 7)*
Let (H, V) be a permutation group acting on the set V of order v and for all $\sigma \in H$ let $P(\sigma)$ be the permutation matrix corresponding to the permutation σ. The ring of $v \times v$ integer matrices, which commute with $P(\sigma)$ for every $\sigma \in H$, is called the **centralizer ring of** *(H, V) and is denoted by $\mathcal{V}(H, V)$.*

In the next section we give a generalization of the centralizer ring which is not defined by a permutation group but by axioms.

2.1.2 Graphs

Definition 2.1.26 *Let V be a finite set. A pair of sets (V, R), where $R \subseteq V \times V$, is called a* **directed, simple graph**. *The set V is called the* **vertex set** *and the set R is called the* **arc set**.
For $v \in V$ the arc (v, v) is called a **loop**.
A graph Γ without loops is called **undirected**, *if for each arc (v, w) of Γ there exists also the arc (w, v) of Γ. In this case we will identify the opposite arcs (v, w) and (w, v) with the* **edge** *$\{v, w\}$. The set of all edges E of an undirected graph $\Gamma = (V, E)$ is a set of two-element subsets of V and is called the* **edge set**.

In this work we only consider simple, undirected graphs without loops, i.e., the edge set consists of 2-subsets of the vertex set. If we need in some cases directed graphs, like in Section 2.2, then we will mention this. Moreover, in general for a (directed) graph $\Gamma = (V, E)$ with n vertices we assume $V = \{1, \ldots, n\}$, i.e., the vertices of the graph are labeled by the numbers $1, \ldots, n$.
For $\sigma \in S_n$ we say that the graph $\Gamma^\sigma = (V, E^\sigma)$ with $E^\sigma := \{\{v^\sigma, w^\sigma\} | \{v, w\} \in E\}$ is obtained from Γ by a **relabeling** σ. The class of graphs Γ^σ, $\sigma \in S_n$ is called an **unlabeled graph** (cf. [Cam99], p. 32). The unlabeled graphs with n vertices are orbits of S_n on the class of graphs with n vertices (using the notion of graph isomorphisms (see below) it is easy to see that the unlabeled graphs are the isomorphism classes of graphs). Relabelings and unlabeled graphs can be defined for directed graphs analogously.

Definition 2.1.27 *Let $\Gamma = (V, E)$ be a graph. Two vertices $v, w \in V$ are called* **adjacent**, *if there exists an edge $\{v, w\} \in E$. Then v is called a* **neighbour** *of w and vice versa.*
The number of neighbours of a vertex $v \in V$ is called the **valency of the vertex** *v. If all vertices of Γ are of the same valency k, then Γ is called* **regular of valency** *k.*
A sequence of $(k+1)$ vertices $v_0, v_1, \ldots, v_k \in V$ with $\{v_i, v_{i+1}\} \in E, i = 0, ..., k-1$ is called a **path of length** *k. The length of the shortest path from a vertex v to a vertex w is called the* **distance** *$d_\Gamma(v, w)$* **of** *v* **and** *w. If for $v, w \in V$ such a path does not exists, then we define $d_\Gamma(v, w) := \infty$. The maximal distance occurring in a graph Γ is called the* **diameter of** *Γ.*
A graph $\Gamma = (V, E)$ is called **connected**, *if for each $v, w \in \Gamma$ there exist a path from v to w.*

For a disconnected graph consisting of i components, each of these isomorphic to a graph Γ, we will write $i \circ \Gamma$.

For a graph $\Gamma = (V, E)$ we define the **complement** $\overline{\Gamma}$ as follows: It has also vertex set V, but two vertices are adjacent if and only they are non-adjacent in Γ.
The **line graph** $L(\Gamma)$ of a graph $\Gamma = (V, E)$ has as vertex set the edge set E of Γ. Two vertices are adjacent in $L(\Gamma)$ if and only if they have one common vertex in Γ.
The graph K_n with n vertices, which has all possible edges between its vertices, is called a **complete graph**. The graph denoted by $K_{n,m}$ is called a **complete bipartite graph** and can be described as follows: For its vertex set V there exists a partition $V = V_1 \cup V_2$ with

$|V_1| = n$ and $|V_2| = m$, such that all vertices in V_i are non-adjacent, $i = 1, 2$, and all possible edges between the vertices from V_1 and the vertices from V_2 exist.

Definition 2.1.28 *Let $\Gamma = (V, E)$ be a graph. For $V' \subseteq V$ the graph $\Gamma' = (V', E(V'))$, where $E(V')$ consists of all edges $\{v, w\} \in E$ with $v, w \in V'$, is called the* **subgraph of Γ induced by V'**.

There are different ways to primitive graphs. Here we follow the definition given in [BroCN89]:

Definition 2.1.29 *([BroCN89], p. 437)*
Let $\Gamma = (V, E)$ be a connected graph of diameter d. For $i \in \{1, \ldots, d\}$ let Γ_i be the graph with vertex set V and edge set $E_i := \{\{v, w\} | d_\Gamma(v, w) = i\}$. If all graphs Γ_i, $i = 1, \ldots, d$, are connected, then the graph Γ is called **primitive**. *Otherwise, it is called* **imprimitive**.

Since we can consider the edge set of a graph as a binary relation on the vertex set, it is possible to describe a graph by a $(0, 1)$-matrix.

Definition 2.1.30 *Let $\Gamma = (V, E)$ be a (directed) graph. Then the matrix $A = (a_{ij})_{i,j \in V}$ with*

$$a_{ij} := \begin{cases} 1 & if \quad (i, j) \in E \\ 0 & \quad\quad otherwise \end{cases}$$

is called the **adjacency matrix of Γ**.

Notice that the adjacency matrix for an undirected graph is symmetric.
In many cases the results concerning the adjacency matrix of a graph do not depend on the concrete labeling of this graph, because a relabeling yields a graph with adjacency matrix similar to that of the original graph.
In this thesis we have to take care of the labeling of graphs only in Section 2.3.

Important properties of (directed) graphs are their internal symmetries. The symmetries of combinatorial objects can be described by permutation groups under which these objects are invariant.

Definition 2.1.31
Let Γ and Γ' be two (directed) graphs. A bijective mapping $\varphi : V(\Gamma) \to V(\Gamma')$ is called an **graph isomorphism**, *if for all vertices $v, w \in V(\Gamma)$ holds*

$$(v, w) \in E(\Gamma) \Longleftrightarrow (v^\varphi, w^\varphi) \in E(\Gamma').$$

An isomorphism from Γ onto itself is called an **automorphism of Γ**.

Remark 2.1.32 *Each automorphism of a (directed) graph $\Gamma = (V, E)$ is a permutation on V. The automorphisms of Γ form a permutation group. It is called the* **automorphism group of Γ** *and will be denoted by $Aut(\Gamma)$.*

If the automorphism group of a (directed) graph acts transitively, then the graph is called **vertex transitive**.
Notice that we introduced the concepts of automorphisms and adjacency matrices also for directed graphs. We need this more general definition, including directed graphs, in Section 2.2.
There is a connection between the primitivity of a graph and primitivity properties of its automorphism group.

Proposition 2.1.33 *([BroCN89], Proposition A.5.1)*
Let Γ be a connected graph, and let G be a subgroup of the automorphism group of Γ. Then Γ is primitive, if G is primitive.

2.1.3 Designs, projective geometries and two-graphs

In this subsection we will give a brief introduction to the notions of designs, Steiner systems, projective geometries and two-graphs. These objects are not in the center of this thesis, however, they are connected to the construction of strongly regular graphs, difference sets and partial difference sets (see Chapter 3 and 4).

Definition 2.1.34 *Let V be a finite set, $\mathcal{B}$ be a set of nonempty subsets of V and $I \subseteq V \times \mathcal{B}$ be a relation. Then the triple $(V, \mathcal{B}, I)$ is called an* **incidence structure** *with* **point set** *V,* **block set** *$\mathcal{B}$ and* **incidence relation** *I.*
Let $S = (V, \mathcal{B}, I)$ be an incidence structure. For integers v, k, t, λ with $v \geq k \geq t \geq 0$ and $\lambda \geq 1$ the incidence structure S is called a **t-(v, k, λ)-design***, if $|V| = v$, $|B| = k$ for all $B \in \mathcal{B}$ and if for each subset $T \subseteq V$ with $|T| = t$ there exist exactly λ blocks in $\mathcal{B}$ each of them containing the elements of T.*
A t-(v, k, λ)-design is called **symmetric***, if $v = |\mathcal{B}|$, i.e., the number of points is equal to the number of blocks.*

There are several classes of designs which are of special interest in design theory. In the following we give some examples which are used in this thesis. For an introduction to design theory we refer to [HugP88].

Definition 2.1.35 *A t-(v, k, λ)-design is called a* **Steiner system***, if $\lambda = 1$. It will be denoted by $S(t, k, v)$.*
A Steiner system $S(2, 3, v)$, i.e., a 2-$(v, 3, 1)$-design, is called **Steiner triple system** *and will be denoted by $STS(v)$.*

Another class of designs we get by the so-called projective geometries:

Definition 2.1.36 *Let W be a vector space of dimension n over a field $\mathbb{F}_q$, where q is a prime power. Let V be the set of all one-dimensional subspaces of W and let $\mathcal{B}$ be the set of all two-dimensional subspaces of W. We can define an incidence relation I as follows: an element $x \in V$ is incident to an element $B \in \mathcal{B}$, if x is contained in B. The incidence* *structure* $(V, \mathcal{B}, I)$ *is called a* **projective geometry** *and is denoted by* $PG(n-1, q)$ *or* $PG(W)$*. The set* V *is called the* **point set of** $PG(W)$ *and each element of the set* $\mathcal{B}$ *is called a* **line***.*

A projective geometry $PG(2, q)$ is a 2-$(q^2+q+1, q+1, 1)$-design. Is is also called a **projective plane**.
For more information about projective geometries and associated permutations groups we refer to [BigW79], Chapter 2.

Besides the Steiner systems and projective geometries a third special class of incidence structures plays a role in the present work. These are the so-called two-graphs.

Definition 2.1.37 *Let* $(V, \mathcal{B}, I)$ *be an incidence structure where each block has size 3 and an even number of 3-subsets of each 4-subset of* V *are blocks in* $\mathcal{B}$*. Then* $(V, \mathcal{B}, I)$ *is called a* **two-graph** *and is denoted by* $(V, \mathcal{B})$*.*
A two-graph $(V, \mathcal{B})$ *is called* **regular***, if there exists an integer* k *such that each 2-subset of* V *is a subset in exactly* k *blocks.*

From each graph $\Gamma = (V, E)$, we can derive a two-graph as follows: a 3-subset of V is a block, if an odd number of its 2-subsets are edges in E. The constructed two-graph will be denoted by $(V, \mathcal{B}(\Gamma))$. The **switching class** of a two-graph $(V, \mathcal{B})$ consists of all graphs $\Gamma = (V, E)$ with $\mathcal{B} = \mathcal{B}(\Gamma)$.
Let $(V, \mathcal{B})$ be a two-graph. For each $\omega \in V$ there is in the switching class of $(V, \mathcal{B})$ a graph Γ which has ω as an isolated vertex. The graph with $|V| - 1$ vertices we get by deleting the vertex ω in Γ is called a **descendant of** $(V, \mathcal{B})$.
For more details about two-graphs we refer to [Sei76].

2.2 Cellular rings, association schemes and Schur rings

In this section we give an introduction to the notions of cellular rings, association schemes and Schur rings; three concepts which are very close to each other.

2.2.1 Cellular rings

The concept of cellular rings is the most general of these three notions. It was introduced by B. Y. Weisfeiler and A. A. Leman in [WeiL68]. For the definition of cellular rings and their basic properties presented below we refer to [FarKM94], Chapter 1.

Definition 2.2.1 *(cf. [FarKM94], p.4)*
A **cellular ring** W **of degree n and rank** r *is a ring of* $n \times n$ *integer matrices which satisfies the following conditions:*

1. W *as a* $\mathbb{Z}$*-module has a basis* $\{A_1, \dots, A_r\}$ *of* $(0,1)$*-matrices;*

2. $\forall i \in \{1, \dots, r\} \, \exists j \in \{1, \dots, r\} : A_i^t = A_j$*;*

3. $\sum_{i=1}^{r} A_i = J_n$*, where* J_n *is the matrix with all entries equal to 1;*

4. $I_n \in W$*, where* I_n *is the unit matrix.*

In general, the last condition is not necessary for the definition of a cellular ring; cellular rings can also be defined without unit matrix. However, in this work we only consider cellular rings with unit matrix.
The basis in Condition 1 is called the **standard basis of** W. If W is a cellular ring with standard basis $\{A_1, \dots, A_r\}$, we will also write $W := \langle A_1, \dots, A_r \rangle$. We will call the elements of the standard basis of a cellular ring W the **basis matrices of** W.
Since the basis matrices are $(0,1)$-matrices, they can be interpreted as adjacency matrices of directed graphs $\Gamma(A_i)$, $i = 1, \dots, r$. These graphs are called the **basis graphs of** W. If it is convenient, we will use the notation $W := \langle \Gamma(A_1), \dots, \Gamma(A_r) \rangle$ instead of $W := \langle A_1, \dots, A_r \rangle$.
If all basis graphs of a cellular ring W are regular, i.e., all vertices of a graph have the same number of incoming arcs and the same number of outgoing arcs, then W is called a **cell**.
By Condition 3 all basis graphs together form a complete graph. It can be considered as a complete colored graph, if the arcs of a basis graph have the same color and we have different colors for different basis graphs.
The multiplication in a cellular ring $W := \langle A_1, \dots, A_r \rangle$ is completely determined by nonnegative integers p_{ij}^k, $1 \le i, j, k \le r$, which are called the **structure constants of the cellular ring** W: For basis matrices A_i, A_j we have

$$A_i A_j = \sum_{k=1}^{r} p_{ij}^k A_k.$$

For given numbers $i, j, k \in \{1, \dots, r\}$, the number p_{ij}^k can be interpreted as the number of paths of length 2 in the complete colored graph of W connecting the ends of a fixed arc (u, w) of color k along an arc (u, v) of color i and an arc (v, w) of color j:

$$p_{ij}^k = |\{v \,|(u,v) \text{ has color } i, (v,w) \text{ has color } j\}|, \text{ where } (u,w) \text{ has color } k.$$

If the cellular ring is commutative, then we have $p_{ij}^k = p_{ji}^k$ for all $1 \le i, j, k \le r$. For commutative cellular rings we have a sufficient condition:

Proposition 2.2.2 *Let $W := \langle A_1, \ldots, A_r \rangle$ be a cellular ring. If all basis matrices $A_1, \ldots, A_r$ are symmetric, i.e., all basis graphs are undirected, then W is commutative.*

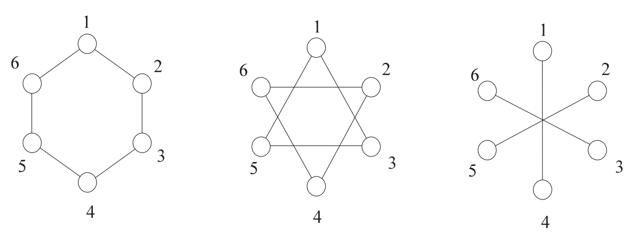
Proof: Since the basis matrices are symmetric, all matrices in W are symmetric. Thus, for $A, B \in W$ we have $AB \in W$ and $AB = (AB)^t = B^t A^t = BA$. □

Remark 2.2.3 *If in Definition 2.2.1 we take matrices with entries from the complex field $\mathbb{C}$, then we get a matrix algebra which is called a* **coherent algebra** *(cf. [Hig70]). Coherent algebras are of importance for investigations where computations in a field are necessary. For our purposes it is sufficient to consider cellular rings with matrices over $\mathbb{Z}$.*

For historical remarks about cellular rings and coherent algebras see Section 2.2.4.

A simple example of a cellular ring is $W = \langle I_n, J_n - I_n \rangle$ which is a cellular ring of rank 2. Cellular rings of rank 2 will be called **trivial**. Examples of nontrivial cellular rings are the centralizer rings of permutation groups which are not 2-transitive (see Definition 2.1.25). The rank of a cellular ring which is a centralizer ring of a permutation group (H, V) equals the rank of the permutation group (H, V). The arcs of the basis graphs of this cellular ring are the 2-orbits of (H, V). If the group H acts transitively on V, then the centralizer ring $\mathcal{V}(H, V)$ is a cell.

Example 2.2.4 *The non-reflexive basis graphs of the centralizer ring $\mathcal{V}(D_6)$ of the dihedral group D_6 acting on $\{1, \ldots, 6\}$:*



Since all basis graphs are undirected, the cellular ring $\mathcal{V}(D_6)$ is commutative.

It turns out that, in general, a cellular ring is not a centralizer ring of a suitable permutation group. An example of H. Wielandt is given in [Wie64] (Theorem 26.4). This fact leads to the following definition:

Definition 2.2.5 *A cellular ring is called* **Schurian***, if it is the centralizer ring of a suitable permutation group.*

Now we introduce cellular subrings:

Definition 2.2.6 *A subring W' of a cellular ring W is called a* **cellular subring***, if it satisfies the conditions in Definition 2.2.1.*

It is easy to see that the basis elements of a cellular subring $W' \leq W$ can be obtained by summing basis elements of W. However, in general, we cannot sum up basis matrices arbitrarily for the construction of basis matrices of cellular subrings. This is only in exceptional cases possible:

Definition 2.2.7 *(cf. [GolIK94], p. 169)*
A cellular ring $W = \langle I_n, A_1, \ldots, A_r \rangle$ is called an **amorphic cellular ring***, if for each partition $M_1 \cup \ldots \cup M_s = \{1, \ldots, r\}$ and matrices $B_i := \sum_{j \in M_i} A_j$, $i = 1, \ldots, s$, the set $\{I_n, B_1, \ldots, B_s\}$ is a basis of a cellular subring of W.*

The amorphic cellular rings have a lot of nice properties: Each cellular subring of an amorphic cellular ring is again amorphic. It is clear that cellular rings of rank 3 are amorphic; first nontrivial cases appear for cellular rings of rank 4. Each basis matrix of an amorphic cellular ring of rank 4 is the adjacency matrix of a (possibly trivial) strongly regular graph (strongly regular graphs are defined in Definition 3.1.1). For further results for amorphic cellular rings we refer to [GolIK94].

Since the matrices of the standard basis of a cellular ring can be interpreted as (basis) graphs, we define the automorphism group of a cellular algebra $W := \langle A_1, \ldots, A_r \rangle$ by

$$Aut(W) := \bigcap_{i=1}^{r} Aut(\Gamma(A_i)).$$

With the concept of cellular rings a Galois correspondence between permutation groups and cellular rings can be described. We do not want to go into details here and only mention two simple facts:

Remark 2.2.8 *([FarKM94], p. 16)*
Let $(H, V), (H', V)$ be two permutation groups with $H \leq H'$. Then for the corresponding centralizer rings we have $\mathcal{V}(H', V) \leq \mathcal{V}(H, V)$.
Let W, W' be two cellular rings with $W \leq W'$. Then we have $Aut(W') \leq Aut(W)$.
Moreover, for a permutation group (H, V) and a cellular ring W we have the following inclusions:

$$Aut(\mathcal{V}(H, V)) \geq (H, V) \text{ and } \mathcal{V}(Aut(W)) \geq W.$$

For cells, i.e., cellular rings where all basis graphs are regular, we consider the special case when all non-reflexive basis graphs are connected. Notice, that we call a directed graph $\Gamma = (V, R)$ **connected**, if the undirected graph $\Gamma' = (V, E)$ with $E := \{\{u, v\} | (u, v) \in R\}$ is connected.

Definition 2.2.9 *(FarKM94], Definition 2.3.1)*
The cell $W = \langle I_n, A_1, \ldots, A_r \rangle$ is called **primitive***, if all non-reflexive basis graphs $\Gamma(A_1), \ldots, \Gamma(A_r)$ are connected. Otherwise, W is called* **imprimitive***.*

Proposition 2.2.10 *(FarKM94], Proposition 2.3.1)*
The permutation group (H, V) is primitive if and only if the centralizer ring $\mathcal{V}(H, V)$ is primitive.

Corollary 2.2.11 *(FarKM94], Corollary 2.3.2)*
Let W be a cellular ring. If $Aut(W)$ is a primitive permutation group, then W is also primitive.

2.2.2 Association schemes

In this section we will introduce association schemes which can be considered as special cellular rings. However, the theory of association schemes was developed independently from the theory of cellular rings and was based on a different motivation (see Section 2.2.4).

Definition 2.2.12 *([FarKM94], pp. 9-10)*
Let V be a finite set and let $R_i \subseteq V \times V$, $i = 0, \ldots, d$, satisfy the following conditions:

1. $R_0 = \Delta := \{(x, x) | x \in V\}$;
2. $R_0 \cup R_1 \cup \ldots \cup R_d = V \times V$, $R_i \cap R_j = \emptyset$ *for* $i \neq j$;
3. $R_i^t := \{(y, x) | (x, y) \in R_i\} = R_j$ *for some* $j \in \{0, \ldots, d\}$;
4. *For all* $i, j, k \in \{0, \ldots, d\}$ *and for* $(u, w) \in R_k$ *the number*
$$p_{ij}^k = |\{v \in V | (u, v) \in R_i, (v, w) \in R_j\}|$$
does not depend on the choice of the pair $(u, w) \in R_k$.

Then $\mathcal{M} = (V, \{R_i\}_{i=0}^d)$ *is called an* **association scheme on V with d classes**. *The numbers* p_{ij}^k *are called the* **intersection numbers of $\mathcal{M}$**.

If we have $R_i^t = R_i$ for each $i \in \{0, \ldots, d\}$ in Condition 3, then the association scheme is called **symmetric**.
For an association scheme $\mathcal{M} = (V, \{R_i\}_{i=0}^d)$ we denote by $A_i = A(R_i)$ the adjacency matrix of the graph $\Gamma_i := (V, R_i)$, $i \in \{0, \ldots, d\}$. The matrices $A_0, \ldots, A_d$ generate a coherent algebra $\mathrm{A}(\mathcal{M})$ (over $\mathbb{C}$) which is called the **adjacency algebra of $\mathcal{M}$** or the **Bose-Mesner algebra of $\mathcal{M}$**.
One can check that the concept of association schemes coincides with the concept of cells, i.e., with the class of cellular rings, where all basis graphs are regular ($\mathrm{A}(\mathcal{M})$ over $\mathbb{Z}$).
On the basis of this fact we can switch between these two concepts in case of applications for different tasks. Traditionally, the language of association schemes is often used in the symmetric case. In this work we will use the language of association schemes in Chapter 5. Otherwise we will use the notion of cellular rings.

2.2.3 Schur rings

Schur rings first were studied by I. Schur [Sch33] and H. Wielandt [Wie35], [Wie49] in their work about permutation groups. The idea of Schur was to investigate a permutation group (G, H) containing a regular subgroup (H, H), by its transitivity module $\mathcal{B}(G, H)$ in the group ring $\mathbb{Z}(H)$.

Definition 2.2.13 *(cf. [Wie64], Chapter 21)*
Let H be a group. Then the set of all formal sums $\sum_{h\in H} a_h h$ with integer coefficients a_h and the below given operations is called the **group ring** $\mathbb{Z}(H)$*. For $T \subseteq H$ the element $\underline{T} := \sum_{h\in T} h$ is called a* **simple quantity***.*

In a group ring $\mathbb{Z}(H)$ we have the following operations for elements $\sum_{h\in H} a_h h$, $\sum_{h\in H} b_h h$:

$$\sum_{h\in H} a_h h + \sum_{h\in H} b_h h := \sum_{h\in H} (a_h + b_h)h, \qquad c\sum_{h\in H} a_h h := \sum_{h\in H} (ca_h)h,\ c \in \mathbb{Z},$$

$$\left(\sum_{h\in H} a_h h\right)\left(\sum_{h\in H} b_h h\right) := \sum_{h\in H}\left(\sum_{g\cdot g'=h} a_g b_{g'}\right) h.$$

Definition 2.2.14 *(cf. [FarKM94], p.8)*
Let H be a group. A subring $\mathcal{S}$ of the group ring $\mathbb{Z}(H)$ is called a **Schur ring over** H **of rank** r*, if the following conditions are satisfied:*

1. *$\mathcal{S}$ as a $\mathbb{Z}$-module has a basis $\{\underline{T_1}, \ldots, \underline{T_r}\}$ formed of simple quantities;*
2. *$T_1 = \{e\}$ and $\sum_{i=1}^{r} \underline{T_i} = \underline{H}$, i.e., $\{T_1, \ldots, T_r\}$ is a partition of H;*
3. *$\forall i \in \{1, \ldots, r\}\, \exists j \in \{1, \ldots, r\} : T_i^{-1} := \{h^{-1} | h \in T_i\} = T_j$.*

For a Schur ring $\mathcal{S}$ with basis $\{\underline{T_1}, \ldots, \underline{T_r}\}$ we write $\mathcal{S} = \langle \underline{T_1}, \ldots, \underline{T_r}\rangle$; in this case the simple quantities $\underline{T}_i$, $1 \leq i \leq r$ are called **basis quantities**. If we have $T_i^{-1} = T_i$ for all $i \in \{1, \ldots, r\}$, then the Schur ring is called **symmetric**. The Schur rings $\mathcal{S}_1 = \langle \underline{\{e\}}, \underline{H \setminus \{e\}}\rangle$ and $\mathcal{S}_2 = \langle \underline{\{x\}} | x \in H\rangle$ over a group H are called **trivial**.

Example 2.2.15 *We consider the Schur rings over the group $(\mathbb{Z}_5, +)$; notice that the group is given in additive notation. There exist three Schur rings over $(\mathbb{Z}_5, +)$, two trivial Schur rings and $\mathcal{S} := \langle \underline{\{0\}}, \underline{\{1, 4\}}, \underline{\{2, 3\}}\rangle$. As an example for the manipulation with basis quantities of $\mathcal{S}$ consider:*

$$\underline{\{1, 4\}} + \underline{\{2, 3\}} = \underline{\{1, 2, 3, 4\}},$$

$$\underline{\{1, 4\}}\ \underline{\{1, 4\}} = \underline{\{1 + 1, 4 + 1, 1 + 4, 4 + 4\}} = 2\underline{\{0\}} + \underline{\{2, 3\}}.$$

Definition 2.2.16 *Let H be a group and let (G, H) be a permutation group which has a regular subgroup (H, H). Let $\{T_1, \ldots, T_r\}$ be the orbits of the stabilizer G_e of the identity element where $T_1 := \{e\}$. Then $\mathcal{B}(G, H) := \langle \underline{T}_1, \ldots, \underline{T}_r \rangle$ is called the* **transitivity module of** *(G, H).*

The following proposition is an important result of Schur:

Proposition 2.2.17 *(cf. [Sch33], §2, Satz E)*
Let H be a group and let (G, H) be a permutation group which has a regular subgroup (H, H). The transitivity module $\mathcal{B}(G, H)$ is a Schur ring over H.

We have a relation between the transitivity module $\mathcal{B}(G, H)$ and the centralizer ring $\mathcal{V}(G, H)$:

Theorem 2.2.18 *([Wie64], Theorem 28.8)*
Let H be a group and let (G, H) be a permutation group which has a regular subgroup (H, H). Then the transitivity module $\mathcal{B}(G, H) = \langle \underline{T}_1, \ldots, \underline{T}_r \rangle$ is isomorphic to the centralizer ring $\mathcal{V}(G, H) := \langle A_1, \ldots, A_r \rangle$. The basis matrices $A_1, \ldots, A_r$ correspond to the basis quantities $\underline{T}_1, \ldots, \underline{T}_r$.

In particular, $\mathcal{B}(H, H)$ and $\mathcal{V}(H, H)$ are isomorphic and one can check that there is a one-to-one-correspondence between the cellular subrings of the centralizer ring $\mathcal{V}(H, H)$ and the Schur rings over H, i.e., the Schur subrings of $\mathcal{B}(H, H)$.
As mentioned in Section 2.2.1 it is not true that every cellular ring is the centralizer ring of a suitable permutation group. The same result holds for Schur rings, in general, a Schur ring over a group H is not the transitivity module of a suitable permutation group (G, H). As in Definition 2.2.5 we will call a Schur ring which is the transitivity module of a suitable permutation group **Schurian**.

Definition 2.2.19 *([Wie64], Definition 23.2)*
A Schur ring $\mathcal{S}$ over a group H is called **primitive***, if $K = \{e\}$ and $K = H$ are the only subgroups of H for which $\underline{K} \in \mathcal{S}$ holds. Otherwise, $\mathcal{S}$ is called* **imprimitive***.*

As in the case of cellular rings it turns out that primitive Schur rings which are Schurian correspond to primitive permutation groups:

Theorem 2.2.20 *([Wie64], Theorem 24.12)*
Let H be a group and let (G, H) be a permutation group which has a regular subgroup (H, H). The group (G, H) is primitive if and only if the transitivity module $\mathcal{B}(G, H)$ is a primitive Schur ring.

An important theorem about the basis quantities of Schur rings over abelian groups is due to Schur and Wielandt.

Theorem 2.2.21 (Schur-Wielandt) *(cf. [Wie64], Theorem 23.9)*
Let $\mathcal{S}$ be a Schur ring over an abelian group H of order n. For a basis quantity $\underline{T}$ the quantity $\underline{T}^{(m)} := \sum_{h \in T} h^m$ is a basis quantity, if $gcd(m, n) = 1$.

For $gcd(m, n) = 1$ the quantity $\underline{T}^{(m)}$ is called a **quantity conjugated to** $\underline{T}$. The sum of all quantities conjugated to $\underline{T}$ is called the **trace of** $\underline{T}$.

Definition 2.2.22 *Let $\mathcal{S}$ be a Schur ring over a group H of order n and let (K, H) be a permutation group acting on the elements of H. We say that (K, H)* **preserves basis quantities** *of the Schur ring $\mathcal{S} = \langle \underline{T}_1, \ldots, \underline{T}_r \rangle$, if for $T_i \in \{T_1, \ldots, T_r\}$ and for all $\alpha \in K$ holds $\underline{T}_i^\alpha := \underline{\{h^\alpha | h \in T_i\}} \in \mathcal{S}$.*

Proposition 2.2.23 *([FarKM94], Proposition 2.4.1)*
Let $(H, +)$ be an abelian group of order n and let $\mathbb{Z}_n^*$ be the multiplicative group of residues mutually prime with n. For $h \in H$ and $\mu \in \mathbb{Z}_n^*$ let $h\mu := h + \ldots + h$ (μ times). Then the permutation group $(\mathbb{Z}_n^*, H)$ preserves all basis quantities for any Schur ring over H.

The statement of Proposition 2.2.23 follows with the Theorem of Schur-Wielandt (notice that we changed to additive notation $(H, +)$). Moreover, we get that for each basis quantity $\underline{T}$ of a Schur ring $\mathcal{S}$ over a group $(H, +)$ the image $\mu\underline{T}$, $\mu \in \mathbb{Z}_n^*$ is again a basis quantity in $\mathcal{S}$. Thus, we can say that the permutation group $(\mathbb{Z}_n^*, H)$ preserves all Schur rings over $(H, +)$.

Proposition 2.2.24 *([FarKM94], Proposition 2.4.6)*
Let $(H, +)$ be an abelian group of order n and $(\mathbb{Z}_n^*, H)$ be a permutation group preserving all Schur rings over H. Let $O_0 = \{0\}, O_1, \ldots, O_k$ be all distinct orbits of $(\mathbb{Z}_n^*, H)$. Then $\langle \underline{O}_0, \ldots, \underline{O}_k \rangle$ is a Schur ring over H.

Remark 2.2.25 *In the previous proposition we get the same Schur ring $\langle \underline{O}_0, \ldots, \underline{O}_k \rangle$ if we take the orbits of the permutation group $(\mathbb{Z}_m^*, H)$ instead of $(\mathbb{Z}_n^*, H)$, where m is the exponent of the group $(H, +)$ (i.e., $m := min\{\mu \in \mathbb{N} | \mu h = 0$ for all $h \in H\}$).*

Definition 2.2.26 *([FarKM94], Definition 2.4.3)*
The Schur ring $\langle \underline{O}_0, \ldots, \underline{O}_k \rangle$ over the abelian group H in Proposition 2.2.24 is called the **complete Schur ring of traces over** *H.*

The complete Schur ring of traces allows us to determine some classes of primitive Schur rings over certain abelian groups (see Section 3.2.3).

2.2.4 Historical remarks

In this Section 2.2 we gave the definitions of three concepts, namely, cellular rings, association schemes and Schur rings, which are very close to each other: The concept of association schemes coincides with the class of cells and the class of cells corresponds by Theorem 2.2.18 to the class of Schur rings. The reason for the existence and for the application of three similar objects we find in the development of these three concepts by different authors working on different problems.
The mathematician I. Schur and his student H. Wielandt developed the theory of Schur rings for the investigation of permutation groups (cf. [Sch33], [Wie35], [Wie49], [Wie64]). With aid

of the theory of Schur rings they wanted to find sufficient conditions that each permutation group (G, H), which contains a given regular group (H, H) as subgroup, is either imprimitive or doubly transitive, i.e., that (H, H) is a B-group (Burnside group). Later in the 50's H. Wielandt intensified the consideration of the centralizer ring of a permutation group. His main results are published in [Wie64].

Association schemes were introduced by R. C. Bose and T. Shimamoto in [BosS52]. The motivation came from the investigation of special kinds of partitions of the cartesian square for the construction of partially balanced block designs. In [BosM59] R. C. Bose and D. M. Mesner gave a definition of an associated matrix algebra for each association schemes, which nowadays is called Bose-Mesner algebra. The investigation of association schemes was intensified after a connection to algebraic codes, strongly regular graphs and distance regular graphs was discovered (e.g., [Del73]). Association schemes became one of the most important objects in algebraic combinatorics.

The concept of cellular rings, or more general, the concept of cellular algebras was introduced by B. Y. Weisfeiler and A. A. Leman from Moscow in [WeiL68]. Their motivation was based on various questions like identification of graphs, computation of automorphism groups of graphs, enumeration of special classes of graphs, etc. (see [Wei76]). Some time after Weisfeiler and Leman, in 1970, D. G. Higman independently introduced his concept of coherent configurations (see [Hig70]). Higman's main aim was to extend certain results of group representation theory to the theory of coherent configurations. Later, when the results of Weisfeiler and Leman were published in the western world, it turned out that the concept of coherent configurations coincides (up to language) with the concept of cellular algebras with identity. Due to the fact that these two concepts were developed independently, in this thesis we will use the name coherent algebra in the complex case and the name cellular ring in the integer case.

For a more detailed historical survey we refer to [FarKM94], Section 1.4 or [KliRRT99], Section 10.

2.3 Graphs, commuting with a given graph

In this section the author wants to give a brief overview about the initial point of his scientific interest: the investigation of commuting graphs. At first view, the distance from this initial point, a pure graph-theoretical problem, to the present thesis, the investigation of special classes of Schur rings, seems to be far. However, the way the author has gone resp. his interests developed is not unnatural. In fact, most of the result that were obtained by arguments based on structural properties of graphs, can be also explained in terms of cellular rings.

In Proposition 2.2.2 we have shown that a cellular ring with symmetric basis matrices is commutative. If we consider these basis matrices as (basis-)graphs, we get a property for sets resp. pairs of graphs, which one can investigate as a graph-theoretical problem independently of the theory of cellular rings. The investigation of pairs of graphs with commuting adjacency matrices may give new information about eigenvectors of graphs (see below). This was the

initial motivation for the author.
A part of the results in this Section are published in [Hei99].

Definition 2.3.1 *Let Γ_A, Γ_B be graphs with n vertices. We say that* **the graph Γ_B commutes with** Γ_A*, if there exists a relabeling $\sigma \in S_n$ such that the adjacency matrices of Γ_A and Γ_B^σ commute, i.e., one has $A(\Gamma_A)A(\Gamma_B^\sigma) = A(\Gamma_B^\sigma)A(\Gamma_A)$. In this case the graphs Γ_A, Γ_B are also called* **commuting graphs**.

Recall that we assume that the vertices of a graph are labeled by $\{1, \dots, n\}$ and that a relabeling is a permutation $\sigma \in S_n$ (see Page 18).

The investigation of commuting graphs is connected with the study of eigenvalues and eigenvectors of graphs. One question that is being investigated for a long time is the question of relations between structural properties of a graph and its eigenvalues and eigenvectors (cf. [CveDS80], [CveRS97]). One strategy to get results in the case of eigenvalues is to consider **p**airs of **i**sospectral **n**onisomorphic **g**raphs (PINGs), i.e., pairs of graphs which have the same spectrum. It follows immediately that on the one hand, different properties of these graphs cannot depend on the spectrum. On the other hand, common properties of these isospectral graphs may depend on the spectrum. In the case of eigenvectors of graphs we can consider pairs of graphs with "same" eigenvectors. Here we have to explain the meaning of "same" eigenvectors. Since we find for each graph on n vertices an orthonormal basis of $\mathbb{R}^n$ consisting of the eigenvectors of the graph, it is natural to consider pairs of graphs Γ_A and Γ_B with n vertices for which an orthonormal basis of $\mathbb{R}^n$ exists such that all vectors of this basis are eigenvectors of Γ_A and eigenvectors of Γ_B. These graphs are exactly the commuting graphs, because we have the following statement from linear algebra:

Proposition 2.3.2 *(cf. [Gan86], Kapitel 8.5, Folgerung 3)*
Let $A_1, ..., A_m \in M(n \times n, \mathbb{R})$ be symmetric matrices. Then the following statements are equivalent:

(i) $A_iA_j = A_jA_i$, *for all* $i, j \in \{1, ..., m\}$.

(ii) There exists an orthonormal basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ such that $v_1, ..., v_n$ are eigenvectors of A_i, $i = 1, ..., m$.

We can immediately give some trivial examples of pairs of commuting graphs:

Remark 2.3.3
1. Every graph commutes with itself.
2. Every graph with n vertices commutes with $\overline{K}_n$.

If we interpret the product of two adjacency matrices of graphs in a graph-theoretical way, we get another characterization of commuting graphs.

Let $\Gamma_A = (V_A, E_A)$ be a graph. Then for a vertex $i \in V_A$ and a permutation $\sigma \in S_n$ we denote by $N_A^\sigma(i) := \{j \in V_A | \{i, j\} \in E_A^\sigma\}$ the **set of neighbours of i in the graph Γ_A^σ**. If σ is the identity in S_n, then we also write $N_A(i)$ instead of $N_A^\sigma(i)$.

Remark 2.3.4 *([Hei99], Remark 5)*
Let $\Gamma_A = (V_A, E_A), \Gamma_B = (V_B, E_B)$ be graphs with adjacency matrices $A = (a_{ij})_{ij}, B = (b_{ij})_{ij}$. Then for all $i, j \in V_A = V_B = \{1, \ldots, n\}$ we have

$$\sum_{k=1}^{n} a_{ik} b_{kj} = |N_A(i) \cap N_B(j)|.$$

Proof: Since

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{ij},$$

we obtain for all $i, j \in V_A = V_B$:

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k \in N_A(i) \cap N_B(j)}^{n} 1 \quad = |N_A(i) \cap N_B(j)|.$$

□

Proposition 2.3.5 *([Hei99], Lemma 6)*
Two graphs $\Gamma_A = (V_A, E_A), \Gamma_B = (V_B, E_B)$ commute if and only if a relabeling σ of Γ_B exist, such that for all $i, j \in V_A = V_B$ the following holds

$$|N_A(i) \cap N_B^\sigma(j)| = |N_A(j) \cap N_B^\sigma(i)|.$$

Proof: Two symmetric matrices commute if and only if the matrix product is symmetric:

$$AB = ((AB)^t)^t = (B^t A^t)^t = (BA)^t.$$

The assertion follows with Remark 2.3.4. □

The characterization of commuting graphs in Proposition 2.3.5 depends on the relabeling of a graph. Thus, this proposition is not very helpful for checking directly if two given graphs commute, because it takes too much time to check the condition $|N_A(i) \cap N_B^\sigma(j)| = |N_A(j) \cap N_B^\sigma(i)|$ for all $i, j \in V_A = V_B$ and for each relabeling σ of Γ_B. However, this proposition is a useful theoretical result for proving further statements.

At this point we have two main questions:

1. Is there a way of finding out, if two given graphs commute, without considering relabelings of these graphs?

2. Is there a possibility to construct pairs of commuting graphs?

In case of regular graphs there are some results for the first question.

Proposition 2.3.6 *([Hei99], Proposition 7)*

1. *A graph Γ with n vertices commutes with the complete graph K_n if and only if Γ is regular.*

2. *Let Γ_1 and Γ_2 be commuting graphs and let Γ_2 be regular. Then Γ_2 commutes with $\overline{\Gamma}_1$, the complement of Γ_1.*

3. *A graph $\Gamma = (V, E)$ with $|V| = 2n$ commutes with the complete bipartite graph $K_{n,n}$, if Γ is a regular subgraph of $K_{n,n}$.*

Proof:

1. For all vertices i, j of the graphs Γ and K_n we have:

$$|N_{K_n}(i) \cap N_\Gamma(j)| = |N_{K_n}(j) \cap N_\Gamma(i)| \Leftrightarrow |N_\Gamma(j)| = |N_\Gamma(i)|.$$

   Since Γ is regular, the equality $|N_\Gamma(j)| = |N_\Gamma(i)|$ is satisfied for all vertices i, j and by Proposition 2.3.5 we get the assertion.

2. By hypothesis we assume that the graph Γ_2 is relabeled such that the adjacency matrices $A(\Gamma_1), A(\Gamma_2)$ commute. By the definition of the complementary graph we know that $A(K_n) - A(\Gamma_1)$ is the adjacency matrix of $\overline{\Gamma}_1$. By 1 and the hypothesis it follows:

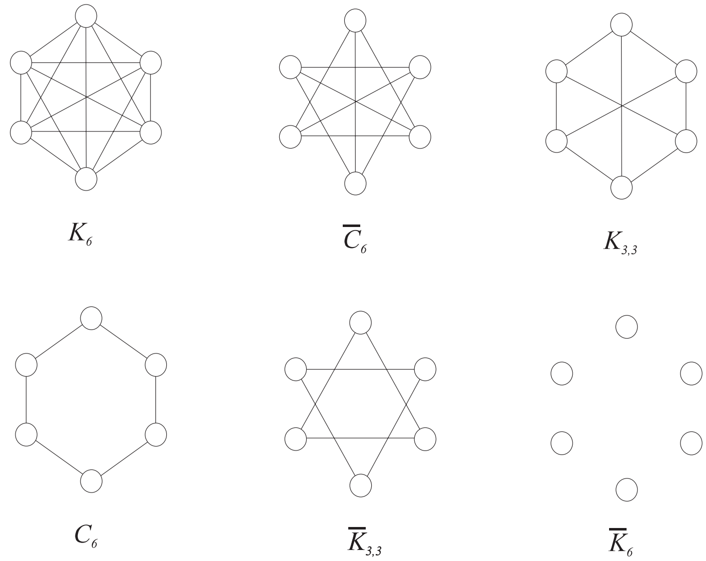
$$\begin{aligned} A(\overline{\Gamma}_1)A(\Gamma_2) &= (A(K_n) - A(\Gamma_1))A(\Gamma_2) &&= A(K_n)A(\Gamma_2) - A(\Gamma_1)A(\Gamma_2) \\ &= A(\Gamma_2)A(K_n) - A(\Gamma_2)A(\Gamma_1) &&= A(\Gamma_2)(A(K_n) - A(\Gamma_1)) \\ &= A(\Gamma_2)A(\overline{\Gamma}_1). \end{aligned}$$

3. Since Γ is a subgraph of $K_{n,n}$ there exists a common relabeling for Γ and $K_{n,n}$ such that $A(\Gamma) = \begin{pmatrix} 0 & C_1 \\ C_1^t & 0 \end{pmatrix}$ and $A(K_{n,n}) = \begin{pmatrix} 0 & C_2 \\ C_2^t & 0 \end{pmatrix}$, where C_1, C_2 are $(0, 1)$-matrices with n rows and columns. All row sums and column sums of the matrix C_1 are equal because Γ is regular. All the entries of the matrix C_2 are 1 because $K_{n,n}$ is completely bipartite. Thus, the equations $C_1C_2^t = C_2C_1^t$ and $C_2C_1^t = C_1C_2^t$ hold and Assertion 3 follows.

□

With Proposition 2.3.6 and Remark 2.3.3 we can construct examples of families of commuting graphs:

Example: Below, we have a family of six pairwise commuting graphs. Clearly, by Remark 2.3.3 each graph with six vertices commutes with $\overline{K_6}$. Since all graphs are regular, by Proposition 2.3.6.1 they all commute with K_6. The graph C_6 is a regular subgraph of $K_{3,3}$, hence, they commute by Proposition 2.3.6.3. The remaining cases follow by Proposition 2.3.6.2.



The fact that the graphs in the previous example commute pairwise can be explained from another point of view: Notice that all the graphs of this family can be described as elements in the centralizer ring of the dihedral group D_6. Since the graphs of the non-reflexive basis matrices of this centralizer ring are the undirected graphs $C_6, 2 \circ K_3, 3 \circ K_2$ (see Example 2.2.4), it is clear that the above given graphs commute (Proposition 2.2.2).

The main result of the following proposition is that a connected, regular graph does not commute with a non-regular graph.

Proposition 2.3.7 *([Hei99], Proposition 8)*
Let Γ be a graph. The graph Γ is regular if and only if there exists a connected, regular graph that commutes with Γ.

Proof:
"$\Rightarrow$" By Proposition 2.3.6 every regular graph with n vertices commutes with K_n, which is connected and regular.
"$\Leftarrow$" Let Γ_1, Γ_2 be commuting graphs and let Γ_2 be connected and regular. Then we know that Γ_2 has a one-dimensional eigenspace $\{\lambda(1, ..., 1) | \lambda \in \mathbb{R}\}$ ([CveDS80], Theorem 3.23, [ColS57], Satz 2). Since Γ_1 and Γ_2 commute, by Proposition 2.3.2 there exists an orthonormal basis of $\mathbb{R}^n$ consisting of common eigenvectors of Γ_1 and Γ_2. One vector of this basis is $\frac{1}{\sqrt{n}}(1, ..., 1)$. Thus, $(1, ..., 1)$ is eigenvector of Γ_1 and the graph Γ_1 is regular.

□

With aid of Lemma 2.3.5 we can proof a sufficient condition, when two given graphs commute.

Proposition 2.3.8 *Let $\Gamma_A = (V, E_A), \Gamma_B = (V, E_B)$ be two vertex transitive graphs. The graphs Γ_A and Γ_B commute, if there exists a relabeling $\sigma \in S_n$ such that for all $i, j \in V$ a common automorphism $\varphi \in Aut(\Gamma_A) \cap Aut(\Gamma_B^\sigma)$ exists with $i^\varphi = j$ and $j^\varphi = i$.*

Proof: We know by Lemma 2.3.5 that Γ_A and Γ_B commute if and only if a relabeling σ of Γ_B exists with

$$|N_A(i) \cap N_B^\sigma(j)| = |N_A(j) \cap N_B^\sigma(i)| \text{ for all } i, j \in V.$$

Since there is a relabeling σ such that for all $i, j \in V$ there exists an automorphism $\varphi \in Aut(\Gamma_A) \cap Aut(\Gamma_B^\sigma)$ with $i^\varphi = j$ and $j^\varphi = i$, it follows for all $i, j \in V$ with suitable automorphism φ:

$$\begin{aligned} |N_A(i) \cap N_B^\sigma(j)| &= |(N_A(j) \cap N_B^\sigma(i))^\varphi| &= |(N_A(i))^\varphi \cap (N_B^\sigma(j))^\varphi| \\ &= |N_A(i^\varphi)) \cap N_B^\sigma(j^\varphi)| &= |N_A(j) \cap N_B^\sigma(i)|. \end{aligned}$$

□

From the view point of centralizer rings the graphs $\Gamma_A = (V, E_A), \Gamma_B^\sigma = (V, E_B^\sigma)$ in the previous proposition are elements of the centralizer ring of $G := Aut(\Gamma_A) \cap Aut(\Gamma_B^\sigma)$. The group G acts on the set V generously transitively (cf. Definition 2.1.19) and since for each pair $(i, j) \in V \times V$ there exist a permutation $\varphi \in G$ such that $(i^\varphi, j^\varphi) = (j, i)$, the 2-orbits of G are symmetric. Symmetric 2-orbits imply symmetric basis matrices of the centralizer ring of G and thus, the centralizer ring of G is commutative by Proposition 2.2.2, in particular, the graphs Γ_A, Γ_B commute.
A further investigation of centralizer rings of permutation groups shows that symmetric basis matrices occur exactly in the case of generously transitive permutation groups:

Proposition 2.3.9 *Let (G, V) be a permutation group. The centralizer ring $\mathcal{V}(G)$ has symmetric basis matrices if and only if (G, V) is transitive and for all pairs $(i, j) \in V \times V$ there exists a permutation $\varphi \in G$ such that $(i^\varphi, j^\varphi) = (j, i)$, i.e., if and only if (G, V) is a generously transitive permutation group.*

Proof: The basis matrices of $\mathcal{V}(G)$ are symmetric if and only if the 2-orbits of (G, V) are symmetric. A 2-orbit is symmetric, if for all elements (i, j) the pair (j, i) is also element of this 2-orbit, i.e., if and only if there exists a permutation $\varphi \in G$ with $(i^\varphi, j^\varphi) = (j, i)$. Since the 2-orbits form a partition of $V \times V$, this condition must be satisfied for all pairs $(i, j) \in V \times V$, i.e., (G, V) is transitive and for all pairs $(i, j) \in V \times V$ it exists a permutation $\varphi \in G$ with $(i^\varphi, j^\varphi) = (j, i)$.

□

A short time before submitting this thesis the author was informed by M. H. Klin that the result of Proposition 2.3.9 is already known (e.g., see [BanI84], Example 2.1). Moreover, in 1980 J. Saxl gave a classification of all generously transitive representations of the symmetric groups S_n, $n > 18$ (in terms of *multiplicity-free representations*, see [Sax81]). The task for $n \leq 18$, where many "sporadic" examples appear, was finally finished by I. E. Pankratova (Kiev) in her Phd-thesis.

The centralizer rings of transitive groups are cells, i.e., all basis graphs are regular. In the following we turn to the case of non-regular graphs, where the groups are intransitive.
It is easy to construct pairs of non-regular, commuting graphs when the graphs are disconnected.

Remark 2.3.10 *Let $C_i, D_i \in M(n_i \times n_i, \mathbb{R}),\ i = 1, \dots, k$. Then the matrices*

$$\begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C_k \end{pmatrix} \text{ and } \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & D_k \end{pmatrix}$$

commute if and only if C_i and D_i commute, $i = 1, \dots, k$.

Proof: We get the assertion immediately by matrix multiplication. □

Hence, if we have pairs of commuting, regular graphs $(\Gamma_i, \Phi_i),\ i = 1, ..., k$, it is easy to join them and to get a pair of commuting regular graphs, one with components $\Gamma_i,\ i = 1, \dots, k$ and one with components $\Phi_i,\ i = 1, \dots, k$.
The more difficult case is to construct pairs of non-regular, connected, commuting graphs. The following lemma gives an idea how to get such a pair of graphs.

Lemma 2.3.11 *([Hei99], Lemma 9)*
Let $C_1, D_1 \in M(n \times n, \mathbb{R})$, $C_2, D_2 \in M(m \times m, \mathbb{R})$ and $X \in M(n \times m, \mathbb{R})$. Then the matrices

$$\begin{pmatrix} C_1 & X \\ X^t & C_2 \end{pmatrix} \text{ and } \begin{pmatrix} D_1 & X \\ X^t & D_2 \end{pmatrix}$$

commute if and only if C_i and D_i commute ($i = 1, 2$) and $(C_1 - D_1)X = X(C_2 - D_2)$.

Proof: Again, we get the assertion immediately by matrix multiplication. □

Now we consider the case when the matrices C_i, D_i are adjacency matrices of commuting graphs $\Gamma_{C_i}, \Gamma_{D_i}, i = 1, 2$. If there exists a nonzero $(0, 1)$-matrix X as described in Lemma 2.3.11, then the two composed matrices in Lemma 2.3.11 commute, they are symmetric and have only entries 0 and 1. Thus, these composed matrices are adjacency matrices of graphs and we have constructed a new pair of commuting graphs.
This reduces the problem of constructing pairs of commuting graphs to the problem of solving the matrix equation $(C_1 - D_1)X = X(C_2 - D_2)$. In matrix theory there are descriptions how to solve such an equation for general matrices (cf. [Gan86]). Here a method for this special case of $(0, 1)$-matrices will be presented.
Denoting $A := C_1 - D_1$ and $B := C_2 - D_2$, we get the matrix equation $AX = XB$ where A, B are matrices with entries 0,1 or -1. The matrices A, B can be considered as adjacency matrices of *weighted graphs* Γ_A, Γ_B with edge weights 1 or -1. Weighted graphs are defined as follows:

Definition 2.3.12 *Take a graph $\Gamma = (V, E)$. The mapping $w : V \times V \to \mathbb{R}$ with*

1. *$(u, v)^w \neq 0$ if $\{u, v\} \in E$ and $(u, v)^w = 0$ if $\{u, v\} \notin E$,*
2. *$(u, v)^w = (v, u)^w$ for all $u, v \in V$.*

is called a **weight function** *and $\Gamma = (V, E, w)$ is called a* **weighted graph**. *For an edge $e = \{u, v\} \in E$ we call $(u, v)^w$ an* **edge weight of** *e.*

For the adjacency matrix $A = (a_{uv})_{u,v \in V}$ of a weighted graph $\Gamma = (V, E, w)$ we define $a_{uv} := (u, v)^w$.

The following proposition gives necessary conditions to solve the matrix equation $AX = XB$ for our special case.

Proposition 2.3.13 *([Hei99], Proposition 11)*
Let $\Gamma_A = (V_A, E_A, w_A), \Gamma_B = (V_B, E_B, w_B)$ be weighted graphs with edge weights 1 or -1 and suppose that $V_A = \{1, ..., n\}, V_B = \{1, ..., m\}$ with $n \leq m$. If there exists a surjective mapping $\varphi : V_B \to V_A$ with

$$\sum_{\substack{k \in N_B(j) \\ k^\varphi = i}} (k, j)^{w_B} = (i, j^\varphi)^{w_A}$$

for all $i \in V_A, j \in V_B$, then the matrix $X = (x_{ij})_{\substack{i \in V_A \\ j \in V_B}}$ with

$$x_{ij} := \begin{cases} 1 & if \quad j^\varphi = i \\ 0 & \quad\quad otherwise \end{cases}$$

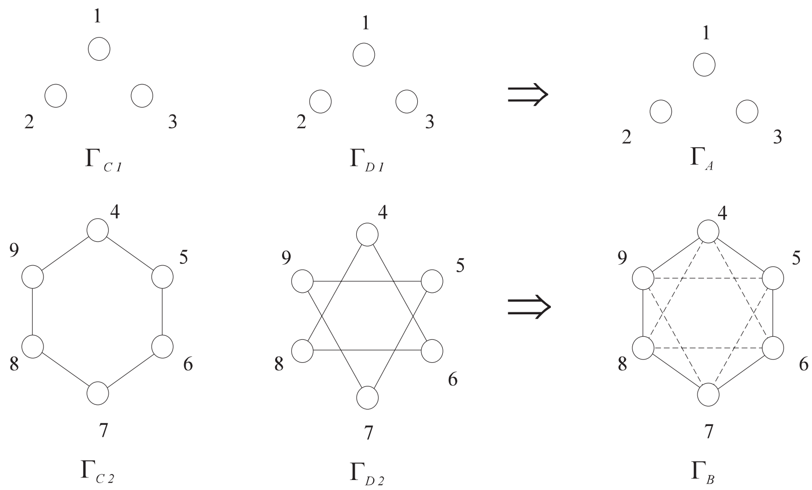
satisfies $AX = XB$.

Proof: We have to show that the equation $AX = XB$ holds, i.e., for all vertices $i \in V_A$, $j \in V_B$ we have $\sum_{l=1}^{n} a_{il} x_{lj} = \sum_{k=1}^{m} x_{ik} b_{kj}$. For all $i \in V_A, j \in V_B$ it holds that

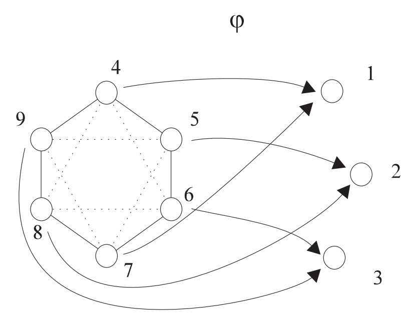
$$\begin{aligned} (i, j^\varphi)^{w_A} = \sum_{\substack{k \in N_B(j) \\ k^\varphi = i}} (k, j)^{w_B} \quad &\Longleftrightarrow \quad a_{i,j^\varphi} = \sum_{\substack{k \in N_B(j) \\ k^\varphi = i}} b_{kj} \\ &\Longleftrightarrow \quad \sum_{\substack{l \in V_A \\ j^\varphi = l}} a_{il} = \sum_{\substack{k \in V_B \\ k^\varphi = i}} b_{kj} \\ &\Longleftrightarrow \quad \sum_{l=1}^{n} a_{il} x_{lj} = \sum_{k=1}^{m} x_{ik} b_{kj}. \end{aligned}$$

□

Example: Constructing commuting graphs
We take two pairs of commuting graphs $\Gamma_{C_1}, \Gamma_{D_1}$ and $\Gamma_{C_2}, \Gamma_{D_2}$. As described we get the weighted graphs Γ_A and Γ_B, where the edges of Γ_{C_2} get weight 1 and the edges of Γ_{D_2} weight -1 (dotted line).



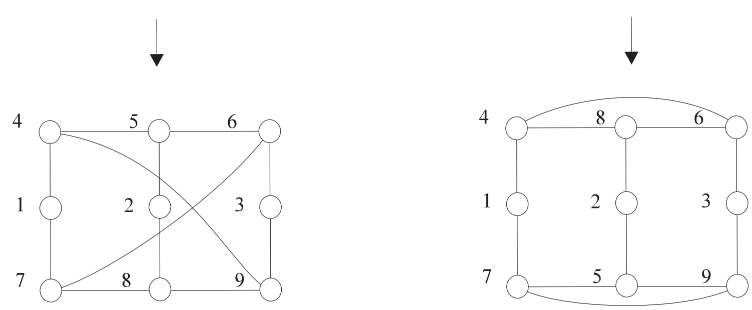
The needed mapping $\varphi : V_B \to V_A$ can be taken as follows:



It is easy to check that the condition of Proposition 2.3.13 is satisfied.
With this mapping φ we get the matrix X and as described in Lemma 2.3.11 we get a new pair of commuting graphs.

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} C_1 & X \\ X^t & C_2 \end{pmatrix} \qquad\qquad \begin{pmatrix} D_1 & X \\ X^t & D_2 \end{pmatrix}$$



In this case we cannot use the argumentation with commutative centralizer rings. Both graphs have identical automorphism groups of order 12. This group is isomorphic to the dihedral group D_6 and has two 1-orbits: $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9\}$. The 2-orbits are

$O_1 := \{(1, 1), (3, 3), (2, 2)\}$,
$O_2 := \{(4, 4), (6, 6), (8, 8), (7, 7), (9, 9), (5, 5)\}$,
$O_3 := \{(1, 2), (3, 1), (2, 3), (1, 3), (2, 1), (3, 2)\}$,
$O_4 := \{(1, 4), (3, 6), (2, 8), (1, 7), (3, 9), (2, 5)\}$,
$O_5 := \{(1, 5), (3, 7), (2, 9), (1, 9), (2, 7), (3, 5), (1, 8), (3, 4), (2, 6), (1, 6), (2, 4), (3, 8)\}$,
$O_6 := \{(4, 1), (6, 3), (8, 2), (7, 1), (9, 3), (5, 2)\}$,
$O_7 := \{(4, 2), (6, 1), (8, 3), (4, 3), (8, 1), (6, 2), (7, 2), (9, 1), (5, 3), (7, 3), (5, 1), (9, 2)\}$,
$O_8 := \{(4, 5), (6, 7), (8, 9), (4, 9), (8, 7), (6, 5), (7, 8), (9, 4), (5, 6), (7, 6), (5, 4), (9, 8)\}$,
$O_9 := \{(4, 6), (6, 8), (8, 4), (4, 8), (8, 6), (6, 4), (7, 9), (9, 5), (5, 7), (7, 5), (5, 9), (9, 7)\}$,
$O_{10} := \{(4, 7), (6, 9), (8, 5), (7, 4), (9, 6), (5, 8)\}$

We can see that the 2-orbits O_1 and O_2 are reflexive, the 2-orbits O_3, O_8, O_9 and O_{10} are symmetric and the pairs (O_4, O_6) and (O_5, O_7) are pairs of antisymmetrical 2-orbits. Thus, the centralizer ring of the automorphism group of the two graphs has non-symmetric 2-orbits and, hence, it is not necessarily commutative. In fact, it is non-commutative, since the corresponding matrices for O_4 and O_8 do not commute.

Chapter 3

Determination of partial difference sets I: Methods and theoretical background

In this chapter we will introduce the notions of strongly regular graphs and partial difference sets. Moreover, we will describe the connection between these two objects and theoretical and computational methods for the determination of partial difference sets.

In the first section necessary definitions and basic results are introduced. For the proofs we refer to the given references.

3.1 Strongly regular graphs and partial difference sets: a survey

3.1.1 Strongly regular graphs

Definition 3.1.1 *A graph* $\Gamma = (V, E)$ *is called a* (v, k, λ, μ)**-strongly regular graph** *if and only if it has* v *vertices, it is regular of valency* k*, each pair of adjacent vertices has exactly* λ *common neighbours and each pair of non-adjacent vertices has exactly* μ *common neighbours.*

A well-known example of a strongly regular graph is the Petersen graph. It is a $(10, 3, 0, 1)$-strongly regular graph. The graph C_5, the cycle with five vertices, is a $(5, 2, 0, 1)$-strongly regular graph.

Remark 3.1.2 *(cf. [vLiW92], p. 231)*
If Γ *is a* (v, k, λ, μ)*-strongly regular graph, then the complement* $\overline{\Gamma}$ *is a* $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$*-strongly regular graph.*

A strongly regular graph Γ is called **trivial** , if Γ or $\overline{\Gamma}$ is disconnected. Otherwise, Γ is called **nontrivial**.
Notice that for a disconnected (v, k, λ, μ)-strongly regular graph Γ follows $\lambda = k - 1$ and $\mu = 0$, i.e., it is a union of copies of complete graphs K_{k+1}. If $v = s(k + 1)$, then Γ consists of s copies of K_{k+1}, i.e., $\Gamma \cong s \circ K_{k+1}$. The nontrivial strongly regular graphs are exactly the primitive strongly regular graphs (cf. Definition 2.1.29).

In this work we will only consider nontrivial strongly regular graphs.

Definition 3.1.3 *(cf. [BroCN89], p. 434)*
Let Γ be a strongly regular graph. Then for a vertex $v \in V$ the subgraph $\Gamma(v) := (V, E(v))$ with $E(v) := \{\{v, w\} \in E | w \in V\}$ is called a **first subconstituent of** Γ*. The subgraph $\Gamma_2(v) := (V, E_2(v))$ with $E_2(v) := \{\{v, w\} | \{v, w\} \notin E\}$ is called a* **second subconstituent of** Γ*.*

For the parameters of a (v, k, λ, μ)-strongly regular graph Γ we have some necessary conditions. Since the parameters for Γ and $\overline{\Gamma}$ are nonnegative and we consider only nontrivial strongly regular graphs, we have

$$v - 1 > k > \mu > 0, \quad v - 2k + \mu - 2 \geq 0, \quad v - 2k + \lambda \geq 0.$$

A simple counting of vertices and edges gives the necessary equality $k(k-1-\lambda) = \mu(v-k-1)$ (cf. [vLiW92], p. 232).
Some other necessary conditions are obtained from the investigation of the adjacency matrix of a strongly regular graph.
The following remark is an easy consequence from the Definition 3.1.1:

Remark 3.1.4 *A graph Γ is a (v, k, λ, μ)-strongly regular graph if and only if for the adjacency matrix A of Γ holds*

$$AJ = kJ_v \text{ and } A^2 + (\mu - \lambda)A + (\mu - k)I_v = \mu J_v,$$

where J_v is the matrix with all entries are 1 and I_v is the unit matrix.

Furthermore, it is not difficult to show the following remark:

Remark 3.1.5 *The matrix ring $\langle I_v, A, \overline{A}\rangle$ over $\mathbb{Z}$ generated by the unit matrix I_v and adjacency matrices A and $\overline{A}$ of a strongly regular graph and its complement is a cellular ring of rank 3.*

The eigenvalues of the adjacency matrix of a strongly regular graph play an important role because they yield some further necessary conditions for the parameters. Since we only consider connected graphs which are regular of valency k, the number k is a simple eigenvalue of A ([ColS57], Satz 2, [CveDS79], Theorem 3.23). In addition to k, there are only two further eigenvalues:

Theorem 3.1.6 *([HesH71], (4.13))*
For a (v, k, λ, μ)-strongly regular graph we have the eigenvalues k, r, s, where

$$r := \frac{1}{2}(\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}) \text{ and } s := \frac{1}{2}(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}).$$

With the spectral properties of strongly regular graphs one can prove the following theorem:

Theorem 3.1.7 *(cf. [HesH71], (4.16))*
For a (v, k, λ, μ)-strongly regular graph the numbers

$$f := \frac{1}{2}\left(v - 1 + \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right) \text{ and } g := \frac{1}{2}\left(v - 1 - \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right)$$

are nonnegative integers.

This condition is called the **integrality condition**. It turns out that f and g are the multiplicities of the two eigenvalues r, s different from k. Moreover, from the results about eigenvalues and their multiplicities we get:

Proposition 3.1.8 *([Hig64], Lemma 7)*
For a (v, k, λ, μ)-strongly regular graph with eigenvalues r, s different from k and corresponding multiplicities f, g one of the following cases holds:

1) $f \neq g$ then $(\mu - \lambda)^2 + 4(k - \mu)$ is a square of integers and the eigenvalues r, s are integers;

2) $f = g$ then $v = 4m + 1$, $k = 2m$, $\lambda = m - 1$, $\mu = m$ for an integer $m \in \mathbb{N}$.

The second case of the previous proposition is the so-called **half case** where we have integer eigenvalues if the number of vertices v of the graph is a square of integers.

A further important condition for the eigenvalues of a strongly regular graph is the **Krein condition** given in the next theorem.

Theorem 3.1.9 *([vLiW92], Theorem 21.3)*
Let A be the adjacency matrix of a (v, k, λ, μ)-strongly regular graph. Then for the eigenvalues k, r, s of A we have

1) $(r + 1)(k + r + 2rs) \leq (k + r)(s + 1)^2$,

2) $(s + 1)(k + s + 2sr) \leq (k + s)(r + 1)^2$.

For the multiplicities of the eigenvalues different from k we have another important property, the so-called **absolute bound**:

Theorem 3.1.10 *([vLiW92], Theorem 21.4)*
Let A be the adjacency matrix of a (v, k, λ, μ)-strongly regular graph. Then for the multiplicities f, g of the eigenvalues $r, s \neq k$ of A we have

$$v \leq \frac{1}{2}f(f+3) \text{ and } v \leq \frac{1}{2}g(g+3).$$

All the above mentioned conditions must be satisfied by all (v, k, λ, μ)-strongly regular graphs. From this point of view, one can call parameters (v, k, λ, μ) satisfying all this conditions **feasible parameters**. In the literature feasible parameters are sometimes defined in different ways. For example, in [vLiW92], Chapter 21, a parameter set is called feasible, if it satisfies the integrality condition. In this case the parameters (28,9,0,4) are feasible, but one can show that there does not exist a (28,9,0,4)-strongly regular graph, because the spectral properties of such strongly regular graph would contradict the Krein condition and the absolute bound.
In Table 4.1 in Section 4.1.1 we give all parameter sets for strongly regular graphs up to 49 vertices which are feasible in the sense of [vLiW92].

3.1.2 Difference sets

Difference sets are well-known in design theory and group theory. Here we will give a short introduction into the main properties. For further details we refer to [Bau71].

Definition 3.1.11 *Let $(H, +)$ be a group of order v. Then a (v, k, λ)-**difference set** is a k-subset $D \subseteq H$ such that each nonzero element $h \in H$ occurs exactly λ times in the multiset $(x - y | x, y \in D)$.*

Sometimes, it is more useful to write difference sets in multiplicative notation. Then for a (v, k, λ)-difference set D in a group $(H, \cdot)$ we have each nonidentity element $h \in H$ exactly λ times in the multiset $(xy^{-1} | x, y \in D)$. In this case the name *quotient set* would be more suitable for D, but due to the tradition we will denote D as difference set though throughout this thesis we will use the multiplicative notation.
A natural way to work with difference sets is the computation in group rings.
With the notation in a group ring ($\underline{T} := \sum_{h \in T} h$, see Definition 2.2.13) one can easily check the following characterization of difference sets:

Lemma 3.1.12 *A subset D in a group H of order v is a (v, k, λ)-difference set in H if and only if*

$$\underline{D}\,\underline{D}^{-1} = k\,\underline{\{e\}} + \lambda\,\underline{H \setminus \{e\}}.$$

Proof: We have $\underline{D}\,\underline{D}^{-1} = \underline{(xy^{-1} | x, y \in D)} = k\,\underline{\{e\}} + \lambda\,\underline{H \setminus \{e\}}$. □

Let H be a group and $S \subseteq H$ be a subset. Then for each $x \in H$ we denote Sx as a **shift** of S by x. An important result is that each shift of a (v, k, λ)-difference set is again a (v, k, λ)-difference set:

Proposition 3.1.13 *Let D be a (v, k, λ)-difference set of a group H. Then for each $x \in H$ the shift Dx is also a (v, k, λ)-difference set.*

Proof: We have $\underline{Dx}(\underline{Dx})^{-1} = \underline{D}xx^{-1}\underline{D}^{-1} = \underline{D}\,\underline{D}^{-1} = k\,\underline{\{e\}} + \lambda\,\underline{H \setminus \{e\}}$. □

A similar result as for the shifts we get for the image of a difference set by a group automorphism:

Proposition 3.1.14 *Let D be a (v, k, λ)-difference set of a group H. Then for each $\alpha \in Aut(H)$ the image $D^\alpha := \{d^\alpha | d \in D\}$ is also a (v, k, λ)-difference set in H.*

Proof: Since each nonidentity element of the group H occurs exactly λ times in the multiset $(xy^{-1}|x, y \in D)$, for $\alpha \in Aut(H)$ each nonidentity element of $H^\alpha = H$ occurs exactly λ times in the multiset $(xy^{-1}|x, y \in D^\alpha)$. □

Propositions 3.1.13 and 3.1.14 enable us to define an equivalence relation on the set of difference sets of a fixed group (see Definition 3.1.35). We will discuss this in Section 3.1.4. Here we will give the definition of multipliers which are of a special interest in the theory of difference sets.

Definition 3.1.15 *(cf. [vLiW92], p. 345)*
Let D be a (v, k, λ)-difference set of a group H. An automorphism $\alpha \in Aut(H)$ is called a **multiplier** *of the difference set D, if an element $h \in H$ exists with $D^\alpha = Dh$, i.e., the difference set D^α is a shift of D.*

Difference sets have great relevance in design theory. One important result is the following theorem:

Theorem 3.1.16 *(cf. [HugP88], Theorem 2.9)*
Let H be a group and $D \subseteq H$ a subset. Then $(H, \{Dh|h \in H\})$ is a symmetric 2-$(v, k, \lambda)-$design if and only if D is a (v, k, λ)-difference set in H.

Furthermore, we like to mention that for each regular subgroup of the automorphism group of a symmetric 2-$(v, k, \lambda)-$design there exists a (v, k, λ)-difference set, which creates the design as described in the above theorem (see [HugP88] for details).

3.1.3 Partial difference sets

In this subsection at first we will introduce Cayley graphs and give some important properties. Then we discuss the notion of partial difference sets.

Definition 3.1.17 *Let H be a group and $S \subseteq H$ be a subset of H with $S = S^{-1}$ and $e \notin S$. Then the graph $Cay(H, S)$ with vertex set H and edge set $E := \{\{h, sh\}|h \in H, s \in S\}$ is called a* **Cayley graph over H with connection set S**.

Here we gave a restricted definition of Cayley graphs, because we are only interested in undirected graphs without loops. In literature one can find more general definitions for Cayley graphs, e.g., directed Cayley graphs, or so-called Cayley objects (cf. [Bab77]).
For Cayley graphs we have several important and well-known observations:

Proposition 3.1.18 *Let H be a group and $S \subseteq H$ be a subset. The Cayley graph $Cay(H, S)$ is connected if and only if S is not contained in a proper subgroup of H.*

The following proposition and lemma is of particular importance for the present work:

Proposition 3.1.19 *(cf. [Big74], Lemma 16.3)*
A graph Γ is isomorphic to a Cayley graph if and only if $Aut(\Gamma)$ has a regular subgroup.

If we know that a graph Γ is isomorphic to a Cayley graph, then by the above proposition we have a regular subgroup H of the automorphism group. This regular subgroup H implies a Cayley graph $Cay(H, S)$ isomorphic to Γ with a suitable connection set S. The following lemma determines this connection set S:

Lemma 3.1.20 *(cf. [Big74], Lemma 16.3)*
Let Γ be a graph whose automorphism group contains a regular subgroup H. Then for a vertex x of Γ we get for $S := \{\omega \in H | x^\omega$ is adjacent to $x\}$ that $e \notin S$, $S^{-1} = S$ and $\Gamma \cong Cay(H, S)$.

Next we will introduce partial difference sets. The notion of partial difference sets is a generalization of the notion of difference sets.

Definition 3.1.21 *Let H be a group of order v. Then a k-subset $D \subseteq H$ is called a $(v, k, \lambda, \mu)-$**partial difference set** if each nonidentity element $d \in D$ occurs exactly λ times in the multiset $(xy^{-1} | x, y \in D)$ and each nonidentity element $h \in H \setminus D$ occurs exactly μ times in the multiset $(xy^{-1} | x, y \in D)$.*

As for difference sets we will always use the multiplicative notation for partial difference sets.

Like for difference sets we have for partial difference sets an alternative description:

Lemma 3.1.22 *A subset D in a group H of order v is a (v, k, λ, μ)-partial difference set in H if and only if*

$$\underline{D}\,\underline{D}^{-1} = k\,\underline{\{e\}} + \lambda\,\underline{D \setminus \{e\}} + \mu\,\underline{(H \setminus D) \setminus \{e\}}.$$

The proof is analogous to the case for difference sets.

With the notion of partial difference sets we can describe an important connection between the concepts of strongly regular graphs, Cayley graphs and partial difference sets:

Theorem 3.1.23 *([Ma94], Proposition 1.1)*
A Cayley graph over a group H with connection set D is a (possibly trivial) (v, k, λ, μ)-strongly regular graph, if and only if D is a (v, k, λ, μ)-partial difference set in H with $D = D^{-1}$ and $e \notin D$.

This theorem is the key for the present work. We determine partial difference sets by strongly regular Cayley graphs.
Notice that there exist partial difference sets D which are connection sets for trivial strongly regular Cayley graphs over a certain group H. One can show that this can only occur, if $D \cup \{e\}$ or $(H \setminus D) \cup \{e\}$ is a subgroup of H (cf. [Ma94], p. 223). For these cases we will call the partial difference sets trivial:

Definition 3.1.24 *A partial difference set D in a group H is called* **trivial**, *if either $D \cup \{e\}$ or $(H \setminus D) \cup \{e\}$ is a subgroup of H. It is called* **reversible**, *if we have $D = D^{-1}$ in H.*
A reversible partial difference set D is called **regular**, *if $e \notin D$.*

For the investigation of partial difference sets and strongly regular graphs by Theorem 3.1.23 it seems to be natural to restrict ourselves to the consideration of nontrivial, regular partial difference set.
The following theorem shows that this "regular condition" is not very restrictive:

Theorem 3.1.25 *(cf. [Ma84], Theorem 2.2, Proposition 3.1)*
If D is a $(v, k, \lambda, \mu)-$partial difference set with $\lambda \neq \mu$, then $D = D^{-1}$.
If D is a reversible partial difference set with $e \in D$, then $D \setminus \{e\}$ is also a reversible partial difference set.

In Theorem 3.1.23 we described the connection between strongly regular graphs and regular partial difference sets which is given by the concept of Cayley graphs. There is also a connection between strongly regular graphs and Schur rings which can be described by Cayley graphs. In Remark 3.1.5 we already mentioned that a strongly regular graph corresponds to a cellular ring W of rank 3. If we consider a strongly regular Cayley graph Γ over a group H, then H corresponds to a regular subgroup of $Aut(\Gamma)$ and by the Galois correspondence in Remark 2.2.8 it follows that W is a cellular subring of the centralizer ring of the regular permutation group (H, H). By Theorem 2.2.18 the cellular ring W corresponds to a Schur ring over H of rank 3. This leads to the following proposition:

Proposition 3.1.26 *([Ma89], Proposition 4.1)*
Let H be a group and D be a subset of H with $D = D^{-1}$ and $e \notin D$. Then the Cayley graph $Cay(H, D)$ is a strongly regular graph if and only if $\langle \underline{\{e\}}, \underline{D}, \underline{(H \setminus D) \setminus \{e\}} \rangle$ is a symmetric, primitive Schur ring of rank 3 over H.

By Theorem 3.1.23 it is clear that the connection set D in Proposition 3.1.26 is a nontrivial, regular partial difference set:

Proposition 3.1.27 *([Ma89], Proposition 5.1)*
Let H be a group and D be a subset of H with $D = D^{-1}$ and $e \notin D$. Then D is a nontrivial, regular partial difference set if and only if $\langle \underline{\{e\}}, \underline{D}, \underline{(H \setminus D) \setminus \{e\}} \rangle$ is a symmetric, primitive Schur ring of rank 3 over H.

Since the notion of partial difference sets is a generalization of difference sets we can consider $(v,k,\lambda)-$difference sets as $(v,k,\lambda,\lambda)-$partial difference sets. However, in this case it can happen that the partial difference sets are not reversible (cf. Theorem 3.1.25). Nevertheless, the following corollary of Proposition 3.1.13 gives an important method to get partial difference sets from difference sets.

Corollary 3.1.28 *Let D be a $(v,k,\lambda)-$difference set in a group H.*
1. Then Dx is a regular $(v,k,\lambda,\lambda)-$partial difference set if and only if $e \notin Dx$ and Dx is reversible.
2. The set $(Dx) \setminus \{e\}$ is a regular $(v,k-1,\lambda-2,\lambda)-$partial difference set if and only if $x^{-1} \in D$ and Dx is a reversible set.

For sake of simplicity we will write partial difference set instead of nontrivial, regular partial difference set in the following chapters, since we are only interested in partial difference sets related to strongly regular graphs.

3.1.4 Equivalent partial difference sets

There are different kinds of equivalence relations for partial difference sets. First of all, we consider the equivalence relation for partial difference sets of a given group which we get from the investigation of Cayley graphs. Considering two Cayley graphs $Cay(H_1,S_1)$ and $Cay(H_2,S_2)$ we can ask when the pairs (H_1,S_1) and (H_2,S_2) should be called equivalent:

Definition 3.1.29 *For i=1,2 let H_i be a group with subset S_i, such that $e \notin S_i$ and $S_i = S_i^{-1}$. The pairs (H_1,S_1) and (H_2,S_2) are called* **equivalent**, *if there exists a group isomorphism $\varphi : H_1 \rightarrow H_2$ which maps S_1 onto S_2.*

Since the groups in Definition 3.1.29 are isomorphic we can modify the situation. Without loss of generality, for the case of equivalence it is sufficient to investigate subsets S_1, S_2 in one group H. In the case of Cayley graphs we call connections sets S_1 and S_2 in H **equivalent**, if there exists a group automorphism $\varphi \in Aut(H)$ such that $S_1^{\varphi} := \{s^{\varphi} | s \in S_1\} = S_2$.
If we have equivalent connection sets S_1, S_2, then it is easy to see that the automorphism φ induces a graph isomorphism between $Cay(H,S_1)$ and $Cay(H,S_2)$. However, in general there exist non-equivalent connection sets which also generate isomorphic Cayley graphs.
Concerned with this problem L. Babai introduced in [Bab77] the notion of a CI-group (CI stands for *Cayley isomorphism property*):

Definition 3.1.30 *(cf. [Bab77], p. 330)*
Let H be a group. The group H is called a **CI-group** *if and only if for all connection sets S_1 and S_2 in H with $Cay(H,S_1) \cong Cay(H,S_2)$ there exists an group automorphism $\varphi \in Aut(H)$ with $S_1^{\varphi} = S_2$.*

In other words: The group H is a CI-group if and only if each pair of non-equivalent connection sets in H generates non-isomorphic Cayley graphs over H. Moreover, all isomorphisms between isomorphic Cayley graphs over H are induced by automorphisms of H.

In 1967 A. Ádám [Ada67] conjectured that all cyclic groups have the Cayley isomorphism property, i.e., all cyclic groups are CI-groups. But it turns out that this is not true. B. Elspas and J. Turner showed in [ElsT70] that $\mathbb{Z}_8$ is not a CI-group. However, since 1967 Ádám's conjecture was proved for several special cases, some are given in the following proposition.

Proposition 3.1.31 *Let p, q be prime numbers and $p \neq q$. Then*

1) $\mathbb{Z}_p$ is a CI-group ([ElsT70], [Djo70]);

2) $\mathbb{Z}_p \times \mathbb{Z}_p$ is a CI-group ([BabF78]);

3) $\mathbb{Z}_{pq}$ is a CI-group ([KliP75],[KliP78],[KliP81]).

For more information about Ádám's conjecture we refer to the survey of P. P. Pálfy [Pál87] which gives an overview over the history of this conjecture over 20 years. In [MuzKP01] M. E. Muzychuk, M. H. Klin and R. Pöschel give a survey of the application of Schur rings in this area.
Regarding partial difference sets we define the following:

Definition 3.1.32 *Let $D_1, D_2 \subseteq H$ be two partial difference sets in a group H. The partial difference sets D_1, D_2 are called* **CI-equivalent** *if and only if there exists a group automorphism $\varphi \in Aut(H)$, such that $D_1^{\varphi} = D_2$.*

For equivalent connection sets of Cayley graphs we have a useful characterization which we can also use for CI-equivalent partial difference sets:

Lemma 3.1.33 *(cf. [Bab77], Lemma 3.1)*
Let Γ be a connected graph whose automorphism group $Aut(\Gamma)$ contains regular subgroups H_1, H_2. For vertices x_1, x_2 of Γ define $S_i := \{\omega \in H_i | x_i^{\omega}$ is adjacent to $x_i\}$, $i = 1, 2$. Then the connection sets S_1, S_2 are equivalent if and only if H_1 is conjugated to H_2 in $Aut(\Gamma)$.

With this lemma we can determine all partial difference sets (up to CI-equivalence) for a strongly regular graph Γ, if we know all regular subgroups of $Aut(\Gamma)$ (up to conjugacy).
Two partial difference sets of a group which are not CI-equivalent may generate isomorphic strongly regular Cayley graphs. Later, we will see some examples.

Sometimes, it is more useful to divide the set of partial difference sets of a group in a coarser partition, e.g., where all partial difference sets which generate isomorphic Cayley graphs are equivalent:

Definition 3.1.34 *Let $D_1, D_2 \subseteq H$ be two partial difference sets in a group H. The partial difference sets D_1, D_2 are called* **srg-equivalent** *if and only if the corresponding Cayley graphs are isomorphic, i.e., $Cay(H, D_1) \cong Cay(H, D_2)$.*

In the special situation when partial difference sets are difference sets, one has further possibilities for equivalence relations.
For difference sets we have an equivalence relation which we can adopt for $(v, k, \lambda, \lambda)-$partial difference sets:

Definition 3.1.35 *(cf. [vLiW92], p. 333)*
Let $D_1, D_2 \subseteq H$ be two $(v, k, \lambda, \lambda)-$partial difference sets in a group H. Then D_1 and D_2 are called **difference equivalent***, if and only if there exists a group automorphism $\varphi \in Aut(H)$ and a group element $h \in H$, such that $D_1^{\varphi} \cdot h = D_2$.*

Comparing this definition with Definition 3.1.32, one can see that the conditions of CI-equivalence are somewhat stronger than for difference equivalence. Later, we give examples of partial difference sets which behave differently for the three kinds of equivalences given above.

Like in the case of partial difference sets and strongly regular graphs we have for difference sets the option to define an equivalence relation by the generated symmetric 2-designs. We mention this definition here for the sake of completeness, it does not play a role in this work (for the definition of isomorphic designs see [HugP88]):

Definition 3.1.36 *Let $D_1, D_2 \subseteq H$ be two $(v, k, \lambda, \lambda)-$partial difference sets in a group H. Then D_1 and D_2 are called* **design equivalent** *if and only if the symmetric 2-designs generated by H and D_1 resp. H and D_2 are isomorphic.*

3.2 Determination of partial difference sets for certain classes of strongly regular graphs

In the previous section we discussed the connection between partial difference sets and strongly regular graphs. In particular, it was shown that the non-CI-equivalent partial difference sets which are connection sets for a strongly regular Cayley graph Γ correspond to the non-conjugated regular subgroups of the automorphism group of Γ (Lemma 3.1.33). In several cases it is possible to determine these non-conjugated regular subgroups of the automorphism group resp. the non-CI-equivalent partial difference sets for the strongly regular graph theoretically. In this section we will present different classes of strongly regular graphs, for which we were able to determine a complete list of partial difference sets or at least some existence or non-existence results. However, in many cases theoretical approaches did not lead to results and, as we present in Section 3.3, computational approaches could be used.

3.2.1 Paley graphs

The discovering of strongly regular graphs that we nowadays call Paley graphs is a consequence of the work of R. Paley on Hadamard matrices (see [Pal33]).

Definition 3.2.1 *Let q be an odd prime power. The graph with vertex set $\mathbb{F}_q$, where two vertices are adjacent if and only if their difference is a nonzero square in $\mathbb{F}_q$, is called* **Paley graph** *$P(q)$.*

Not all graphs of the series of Paley graphs are strongly regular graphs. We can divide the Paley graphs into two classes: on the one hand, strongly regular graphs and on the other hand, the so-called tournaments. A tournament is an orientation of a complete graph for which the vertices can be numbered in such a way that (i, j) is an arc if and only if $i < j$ for vertices $i, j \in \{1, \ldots, n\}$ (see [BroR72]).

Remark 3.2.2 *([FarKM94], p. 130)*
Let q be an odd prime power. For $q \equiv 1 \bmod 4$ the Paley graph $P(q)$ is a strongly regular graph with parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$. For $q \equiv 3 \bmod 4$ the graph $P(q)$ is a tournament.

In [Muz87] M. E. Muzychuk described the automorphism group of the Paley graph $P(q)$, q an odd prime power. Later it turned out that this group can also be obtained by translating a result of L. Carlitz [Car69] in the language of graphs. However, the proofs are based on different ideas.
For $q = p^n$ denote by $H(q)$ the group consisting of all permutations on $\mathbb{F}_q$ of the form $x \mapsto ax^{p^k} + b$, where a is a nonzero square in $\mathbb{F}_q$, $x, b \in \mathbb{F}_q$ and $k \in \{0, \ldots, n-1\}$:

Proposition 3.2.3 *([Muz87], cf. [Car69])*
Let q be an odd prime power. For the automorphism group of the Paley graph $P(q)$ we have $Aut(P(q)) = H(q)$.

From the construction of $H(q)$ it follows immediately that $\{x \mapsto x + b | b \in \mathbb{F}_q\} \cong (\mathbb{F}_q, +)$ is a regular subgroup of $H(q)$. For $q = p^n$ and $p \nmid n$ we can easily get a result for the number of non-conjugated regular subgroups of $H(q)$.

Proposition 3.2.4 *For $q = p^n$, p an odd prime, and $p \nmid n$ there exists exactly one regular subgroup of $Aut(P(q))$ (up to conjugacy). This subgroup is isomorphic to $(\mathbb{F}_q, +)$.*

Proof: The order of $Aut(P(q)) = H(q)$ is $p^n \cdot \frac{p^n-1}{2} \cdot n$. We have $p \nmid \frac{p^n-1}{2}$ and by assumption $p \nmid n$. From this follows that a Sylow p-subgroup of $Aut(P(q))$ has order p^n. By the respective Sylow theorem all Sylow p-subgroups are conjugated. Thus, we have only one subgroup of order $q = p^n$ in $Aut(P(q))$ (up to conjugacy). A representative of this subgroup is $\{x \mapsto x + b | b \in \mathbb{F}_q\} \cong (\mathbb{F}_q, +)$, which obviously acts regularly on $\mathbb{F}_q$. □

In the case of p vertices, p a prime, one can get a more general statement about vertex transitive graphs with p vertices, which does not only hold for Paley graphs.

Proposition 3.2.5 *(cf. [Tur67])*
Let p be a prime. The automorphism group of a connected, vertex transitive graph with p vertices contains exactly one regular subgroup (up to conjugacy) which then is isomorphic to $\mathbb{Z}_p$.

Proof: By [Tur67] a connected, vertex transitive graph Γ with p vertices is isomorphic to a Cayley graph over $\mathbb{Z}_p$ (Γ is *circulant*).
Let H be a regular subgroup of $Aut(\Gamma)$, i.e., $H \cong \mathbb{Z}_p$. Since $H \leq Aut(\Gamma) \leq S_p$ and $p \mid p! = |S_p|$ but $p^2 \nmid p!$, the subgroup H is a Sylow p-subgroup of S_p and, consequently, H is a Sylow $p-$subgroup of $Aut(\Gamma)$. By the respective Sylow theorem all Sylow p-subgroups are conjugated and H is the only regular subgroup of $Aut(\Gamma)$ (up to conjugacy). □

From the definition of Paley graphs and from Proposition 3.2.4 we get a statement about the partial difference sets for Paley graphs:

Proposition 3.2.6 *For $q \equiv 1$ mod 4, $q = p^n$, p an odd prime, and $p \nmid n$ we get exactly one partial difference set for the Paley graph $P(q)$. This partial difference set consists of the nonzero squares of $\mathbb{F}_q$.*

Proof: By Proposition 3.2.4 the automorphism group of $P(q)$ has exactly one regular subgroup (up to conjugacy). Thus, by Lemma 3.1.33 we have exactly one partial difference set. From Definition 3.2.1 we get that this partial difference set consists of the nonzero squares of $\mathbb{F}_q$. □

In addition to this result, it is known that the partial difference sets for Paley graphs given in Proposition 3.2.6 are the only partial difference sets in groups of order p, when p is an odd prime with $p \equiv 1$ mod 4. In [Pös74] R. Pöschel classified all Schur rings over $\mathbb{Z}_p$. From his results and the correspondence between partial difference sets and certain Schur rings (cf. Proposition 3.1.27) we get the following proposition which independently was also formulated by W. G. Bridges and R. A. Mena:

Proposition 3.2.7 *(cf. [Pös74], [BriM79])*
There exists a partial difference set in the group $\mathbb{Z}_p$ if and only if p is a prime with $p \equiv 1$ mod 4. In this case the partial difference set is unique (up to CI-equivalence) and consists of the non-zero squares in $\mathbb{Z}_p$.

Propositions 3.2.5 and 3.2.7 give us the following result about vertex transitive strongly regular graphs with prime number of vertices. (Recall that we only consider nontrivial strongly regular graphs).

Theorem 3.2.8 *Let Γ be a vertex transitive strongly regular graph with p vertices, p an odd prime, $p \equiv 1$ mod 4. Then Γ is isomorphic to the Paley graph $P(p)$.*

Proof: Since by Proposition 3.2.5 all automorphism groups of vertex transitive strongly regular graphs with prime number of vertices have exactly one regular subgroup (up to conjugacy), i.e., each of these graphs yields exactly one partial difference set (up to CI-equivalence, Lemma 3.1.33), and since by Proposition 3.2.7 groups of order p, p an odd prime, $p \equiv 1$ mod 4, have exactly one partial difference set (up to CI-equivalence), which generates the Paley graph, it follows that for all odd primes p with $p \equiv 1$ mod 4 there exists only one vertex transitive strongly regular graph (up to isomorphism) and this is the Paley graph.

□

3.2.2 Triangular graphs

Definition 3.2.9 *(cf. [vLiW92], p. 232)*
Let V be the set of 2-element subsets of a set of cardinality n, $n \geq 5$. The graph $T(n)$ with vertex set V, where two vertices are adjacent if and only if they are not disjoint, is called the **triangular graph** *$T(n)$.*

Remark 3.2.10
1) The triangular graph $T(n)$, $n \geq 5$, is a $(\binom{n}{2}, 2n-4, n-2, 4)$-strongly regular graph.
2) It is easy to see that $T(n) \cong L(K_n), n \geq 5$.

From [Whi32] we know that $\mathrm{Aut}(K_n) \cong \mathrm{Aut}(L(K_n))$, $n \neq 2, 4$. Thus, for the automorphism group of the triangular graphs we have $\mathrm{Aut}(T(n)) \cong \mathrm{Aut}(K_n) \cong S_n$, $n \geq 5$. It follows with $T(n) \cong L(K_n)$ that a regular subgroup of $\mathrm{Aut}(T(n))$ is isomorphic to a subgroup of S_n of order $\frac{1}{2}n(n-1)$ which acts 2-homogeneously but not 2-transitively on $\{1, ..., n\}$.
In [Kan69] W. M. Kantor proved the following proposition and corollary:

Proposition 3.2.11 *([Kan69], Proposition 3.1)*
If H is a transitive permutation group on a finite set V, where $n = |V| > 3$, then the following statements are equivalent:

- *i) H has rank 3 and all orbits of $H_x, x \in V$ have odd lengths.*
- *ii) H is 2-homogeneous but not 2-transitive on V.*
- *iii) n is a prime power, $n \equiv 3 \mod 4$, H is similar to a 2-homogeneous subgroup of $\{x \mapsto x^{\alpha}t + a | \alpha \in Aut(\mathbb{F}_n), t \in \mathbb{F}_n^*, a \in \mathbb{F}_n\}$ the group of all semilinear mappings on $\mathbb{F}_n$.*

If H is represented as in iii), then H contains the set of all translations $\{x \mapsto x + a | a \in \mathbb{F}_n\}$ as a normal subgroup. $H \cap \{x \mapsto xt + a | t \in \mathbb{F}_n^*, a \in \mathbb{F}_n\}$ is a normal Frobenius subgroup of H. If Q is the group of nonzero squares of $\mathbb{F}_n$, then the orbits of H_0 are $\{0\}, Q, -Q$.

Corollary 3.2.12 *([Kan69], Corollary 3.2)*
If H is a sharply 2-homogeneous permutation group of degree n, then H is similar to the group of mappings $x \mapsto xt + a$ on a Dickson nearfield K, where $a \in K$ and t is in the group of nonzero squares of K.

For information about nearfields we refer to [Zas36] or [DixM96], Chapter 7.
From Proposition 3.2.11 follows that there exists a 2-homogeneous but not 2-transitive subgroup H of S_n if and only if n is a prime power, $n > 3$ and $n \equiv 3 \mod 4$.
If there exists a subgroup H which moreover acts regularly on the 2-subsets of $\{1, \ldots, n\}$, then it is sharply 2-homogeneous (Definition 2.1.11) and from Corollary 3.2.12 we get that there is at least one such subgroup (up to similarity), which we get for the (near-)field $\mathbb{F}_n$. If for a prime power n with $n \equiv 3 \mod 4$ there exist more than one nearfield, then there are more options for the existence of sharply 2-homogeneous permutation groups of degree n.

The existence of such a sharply 2-homogeneous permutation group H implies the existence of a regular subgroup of $Aut(T(n))$. In the following we will adopt the result of W. M. Kantor for the determination of regular subgroups of $Aut(T(n))$ and the corresponding partial difference sets.

Lemma 3.2.13 *Let n be a prime power and $n \equiv 3 \bmod 4$ and let K be the subgroup of $\mathbb{F}_n^*$ of order $\frac{1}{2}(n-1)$ consisting of all nonzero squares of $\mathbb{F}_n$. Then for all $t \in K$ we have $-t \notin K$.*

Proof: It is sufficient to show, that $-1 \notin K$.
We know that $n \equiv 3 \bmod 4$, so there exists $s \in \mathbb{N}$ with $n = 4s+3$. Let ω be a primitive element of $\mathbb{F}_n^*$. Then $-1 = \omega^{\frac{n-1}{2}}$. If -1 is a square in $\mathbb{F}_n^*$, then $\frac{n-1}{2}$ is even. But we have $\frac{n-1}{2} = \frac{4s+2}{2} = 2s+1$. Thus, -1 is no square. □

Theorem 3.2.14
Let n be a prime power, $n > 3$ and $n \equiv 3 \bmod 4$. Let $H := \{x \mapsto xt+a | t \in K, a \in \mathbb{F}_n\}$, where K is the subgroup of $\mathbb{F}_n^*$ of order $\frac{1}{2}(n-1)$ consisting of all nonzero squares of $\mathbb{F}_n$, and define $S := \{\sigma_{s,0}, \sigma_{t,-t}, \sigma_{t,1}, \sigma_{s,1-s} | s,t \in K, s \neq 1\}$, where $\sigma_{a,b} \in H$ with $x^{\sigma_{a,b}} := xa+b$, $x \in \mathbb{F}_n$. Then:

- *a) There exist a regular subgroup of $Aut(T(n))$ which is similar to H.*
- *b) The set $S \subset H$ is a partial difference set and the Cayley graph over H with connection set S is isomorphic to the strongly regular graph $T(n)$.*
- *c) If n is not a prime power or $n \not\equiv 3 \bmod 4$, then $Aut(T(n))$ has no regular subgroup.*

Proof: a) The group H acts regularly on the 2-subsets of $\mathbb{F}_n$:
The images of $\{0,1\}$ are of the form $\{a, t+a\}$, $t \in K, a \in \mathbb{F}_n$; they are different for all pairs (a,t), because if there exist $t, t' \in K, a, a' \in \mathbb{F}_n$ with $\{a, t+a\} = \{a', a'+t'\}$, then in the case of $a = a'$ we get $a'+t' = a+t' = a+t \Rightarrow t' = t$ and in the case of $a = a'+t'$ and $a+t = a'$ we get $a' = a-t'$ and $a+t = a-t' \Rightarrow t = -t'$. The last case is impossible, because from Lemma 3.2.13 we know $t \in K \Rightarrow -t \notin K$ for all t. Thus, for the 2-subset $\{0,1\}$ we get all $|K| \cdot |\mathbb{F}_n| = \frac{1}{2}(n-1)n = \binom{n}{2}$ images by the $\binom{n}{2}$ permutations in H and H is similar to a regular subgroup of $\mathrm{Aut}(T(n)) \cong S_n$.
b) Consider the vertex $\{0,1\}$ of $T(n)$. This vertex is adjacent to all vertices of the form $\{0,i\}$ or $\{1,j\}$, $i,j \in \mathbb{F}_n \setminus \{0,1\}$. By Lemma 3.1.20 a partial difference set of H that generates the graph $T(n)$ consists of all permutations of H which map the vertex $\{0,1\}$ onto its neighbours:
The permutations $\sigma_{s,0}$ with $x^{\sigma_{s,0}} = sx$, $s \in K \setminus \{1\}$ maps $\{0,1\}$ onto $\{0,s\}$,
the permutations $\sigma_{t,-t}$ with $x^{\sigma_{t,-t}} = tx-t$, $t \in K$ maps $\{0,1\}$ onto $\{0,-t\}$,
the permutations $\sigma_{s,1-s}$ with $x^{\sigma_{s,1-s}} = sx+1-s$, $s \in K \setminus \{1\}$ maps $\{0,1\}$ onto $\{1, 1-s\}$,
the permutations $\sigma_{t,1}$ with $x^{\sigma_{t,1}} = tx+1$, $t \in K$ maps $\{0,1\}$ onto $\{1, t+1\}$.
Since

$$\mathbb{F}_n \setminus \{0,1\} = \{-t, s | s,t \in K, s \neq 1\} = \{1-s, t+1 | s,t \in K, s \neq 1\}$$

the permutations $\{\sigma_{s,0}, \sigma_{t,-t}, \sigma_{t,1}, \sigma_{s,1-s} | s,t \in K, s \neq 1\}$ map the vertex $\{0,1\}$ onto all of its neighbours $\{\{0,i\}, \{1,j\} | i,j \in \mathbb{F}_n \setminus \{0,1\}\}$ and by Lemma 3.1.20 they form a partial

difference set of H which generates the graph $T(n)$ as a Cayley graph. Consequently, it is a partial difference set with parameters $(\binom{n}{2}, 2n-4, n-2, 4)$.
c) Assume n is not a prime power or $n \not\equiv 3 \bmod 4$ and assume that $\mathrm{Aut}(T(n))$ has a regular subgroup H. Then H is isomorphic to a 2-homogeneous subgroup of S_n that is not 2-transitive. By Proposition 3.2.11 n must be a prime power and $n \equiv 3 \bmod 4$ which contradicts the assumption. □
Notice that the group $H := \{x \mapsto xt + a | t \in K, a \in \mathbb{F}_n\}$ in the previous theorem is the semidirect product $\mathbb{F}_n \rtimes K$, where K is the subgroup of $\mathbb{F}_n^*$ of order $\frac{1}{2}(n-1)$ consisting of all nonzero squares of $\mathbb{F}_n$.

For our purpose it is sufficient to consider prime parameters p for $T(p)$, because the smallest prime power p^m with $p \equiv 3 \bmod 4$ and $m \geq 2$ appears for $p^m = 27$ and $T(27)$ is a graph with 351 vertices. As mentioned in the introduction we consider strongly regular graphs with at most 255 vertices.
If the parameters are restricted to prime parameters we can prove that there exist exactly one regular subgroup of $Aut(T(n))$ (up to conjugacy), i.e., we get exactly one partial difference set (up to CI-equivalence):

Proposition 3.2.15
Let p be a prime, $p > 3$ and $p \equiv 3 \bmod 4$. Let $H := \{x \mapsto xt + a | t \in K, a \in \mathbb{Z}_p\} = \mathbb{Z}_p \rtimes K$, where K is the subgroup of $\mathbb{Z}_p^*$ of order $\frac{1}{2}(p-1)$ consisting of all nonzero squares of $\mathbb{Z}_p$. Then there exists only one regular subgroup in $Aut(T(p))$ (up to conjugacy) and this subgroup is similar to H.

Proof: By Theorem 3.2.14 there exists a regular subgroup of $Aut(T(p))$ which is similar to $H = \mathbb{Z}_p \rtimes K$. The group H has $\mathbb{Z}_p$ as a normal subgroup, therefore H is a subgroup of the holomorph $\mathbb{Z}_p \rtimes Aut(\mathbb{Z}_p)$ of $\mathbb{Z}_p$ (cf. Definition 2.1.21, Remark 2.1.22). Since H is a subgroup of index 2, it is a normal subgroup of the holomorph $\mathbb{Z}_p \rtimes Aut(\mathbb{Z}_p)$ and the holomorph is a subgroup of the normalizer $N_{S_p}(H)$ of H in S_p.
Let $h \in N_{S_p}(H)$, i.e., we have $hHh^{-1} = H$. Since $\mathbb{Z}_p$ is a normal Sylow p-subgroup in H, by the respective Sylow theorem it is unique, and we have $h\mathbb{Z}_p h^{-1} = \mathbb{Z}_p$. Hence, $\mathbb{Z}_p$ is a normal subgroup of $N_{S_p}(H)$ and $N_{S_p}(H)$ is a subgroup of the holomorph $\mathbb{Z}_p \rtimes Aut(\mathbb{Z}_p)$. Consequently, the holomorph $\mathbb{Z}_p \rtimes Aut(\mathbb{Z}_p)$ is the normalizer $N_{S_p}(H)$ of H.
The holomorph has order $p(p-1)$, such that the number of conjugates c_H of H in S_p is

$$c_H = \frac{|S_p|}{|N_{S_p}(H)|} = \frac{p!}{p(p-1)} = (p-2)!.$$

Moreover, we have exactly $(p-1)!$ elements of order p in S_p. Each group similar to H includes exactly $p-1$ elements of order p, because H has exactly one cyclic subgroup of order p. Since different subgroups (similar to H) have different cycles, we have exactly $\frac{(p-1)!}{p-1} = (p-2)!$ options for subgroups in S_p similar to H. These subgroups are the groups in the conjugacy class of H. □

Since each regular subgroup of $Aut(T(n))$ is connected to a nearfield (Corollary 3.2.12), we can get this result also from the complete characterization of nearfields which was obtained by H. Zassenhaus [Zas36].

3.2.3 Partial difference sets for graphs with p^2 vertices

In this section we determine all partial difference sets for groups of order p^2, p a prime.
First of all recall that by Lemma 2.1.7 there exist only two non-isomorphic groups of order p^2, the cyclic group $\mathbb{Z}_{p^2}$ and the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus, it is sufficient to consider these two groups for the determination of all partial difference sets in groups of order p^2.
The next step is to investigate all primitive Schur rings over $\mathbb{Z}_{p^2}$ resp. $\mathbb{Z}_p \times \mathbb{Z}_p$. In the case of $\mathbb{Z}_{p^2}$ we have the following result by Wielandt and Schur:

Theorem 3.2.16 *([Wie64], Theorem 25.4)*
Let H be a finite abelian group which has at least one cyclic Sylow subgroup. There does not exist a nontrivial primitive Schur ring over H unless the order of H is prime.

By the previous theorem there exists no nontrivial primitive Schur ring over $\mathbb{Z}_{p^2}$. Thus, it follows that the strongly regular Cayley graphs of all possible partial difference sets in $\mathbb{Z}_{p^2}$ are trivial (cf. Proposition 3.1.26 and 3.1.27).

For groups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ there exist primitive Schur rings. For the description of symmetric primitive Schur rings over $\mathbb{Z}_p \times \mathbb{Z}_p$ of rank 3 we consider the complete Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$ and all its subrings. Recall that the complete Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$ is the Schur ring $\langle \underline{O}_0, ..., \underline{O}_l \rangle$, where O_i, $0 \leq i \leq l$, are the orbits of $\mathbb{Z}_p^*$ in $\mathbb{Z}_p \times \mathbb{Z}_p$ (Definition 2.2.26).

Lemma 3.2.17 *(M. E. Muzychuk, private communication)*
Let $(H, +)$ be an abelian group and $T \subseteq H$ such that $Cay(H, T)$ has a rational spectrum. Let m be the exponent of the group H. Then for each $\mu \in \mathbb{Z}_m^*$ we have $\mu T = T$.

Proposition 3.2.18 *Each symmetric, primitive Schur ring $\mathcal{S}$ over $\mathbb{Z}_p \times \mathbb{Z}_p$ of rank 3 is a subring of the complete Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$.*

Proof: By Proposition 3.1.8 we know that each strongly regular graph with p^2 vertices has integer eigenvalues and with Proposition 3.1.26 follows that each strongly regular Cayley graph with p^2 vertices corresponds to a nontrivial, symmetric, primitive Schur ring $\mathcal{S}$ of rank 3 over $\mathbb{Z}_p \times \mathbb{Z}_p$. With Lemma 3.2.17 we get for each basis quantity $\underline{T} \in \mathcal{S}$ that T is the union of orbits of $\mathbb{Z}_p^*$ in $\mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, $\mathcal{S}$ is a subring of the Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$.

□

For the previous lemma and proposition we have to add two remarks:
1) Lemma 3.2.17 is of a folklore nature, the author was not able to find a precise formulation or proof for it in literature.

2) In principle, Proposition 3.2.18 is known, as it is mentioned by S. L. Ma in [Ma89] it follows from the results of W. G. Bridges and R. A. Mena in [BriM79], [BriM82].

For the determination of all symmetric primitive Schur rings over $\mathbb{Z}_p \times \mathbb{Z}_p$ of rank 3 by Proposition 3.2.18 it is sufficient to determine all symmetric primitive subrings of rank 3 of the complete Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$. Then, by Proposition 3.1.27, we have all partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$.
We have $p+2$ orbits of $\mathbb{Z}_p^*$ in $\mathbb{Z}_p \times \mathbb{Z}_p$:

$$O_0 = \{(0,0)\},\ O_1 = \{(0,a)|a \in \mathbb{Z}_p^*\}, O_{k+2} = \{(a,ka)|a \in \mathbb{Z}_p^*\}, k \in \mathbb{Z}_p.$$

If we consider the Cayley graph for each orbit O_i, we get $C_i := Cay(\mathbb{Z}_p \times \mathbb{Z}_p, O_i) \cong p \circ K_p$, $1 \leq i \leq p+1$. The vertex sets of the p complete graphs K_p are the cosets of $\mathbb{Z}_p \times \mathbb{Z}_p$ with respect to the subgroup $O_0 \cup O_i$.

As described in [GolIK94], p. 171 these Cayley graphs are associated graphs of parallel classes of an affine plane of order p (this affine plane can be considered as a 2-$(p^2, p, 1)$-design, where the blocks have p points and correspond to the lines of the plane):
For each graph C_i ($\cong p \circ K_p$), $1 \leq i \leq p+1$ each subgraph K_p corresponds to a line and the vertices of K_p are points on this line. For each Cayley graph C_i we get p parallel lines that form a parallel class of lines (in the language of design the parallel lines are the blocks which have no point in common). Altogether, we have $p+1$ parallel classes of lines. All these lines together with the corresponding points define an affine plane.
From this fact follows by Theorem 3.3 in [GolIK94] that $\langle C_0, ..., C_{p+1}\rangle$ resp. $\langle \underline{O}_0, ..., \underline{O}_{p+1}\rangle$ is an amorphic cellular ring, i.e., we know that each partition of the set of basis quantities of this ring corresponds to the basis quantities of a cellular subring (cf. Definition 2.2.7). Hence, from all partitions of the set of basis quantities we get all cellular subrings of the complete Schur ring of traces over $\mathbb{Z}_p \times \mathbb{Z}_p$. In particular, we get all primitive symmetric Schur rings of rank 3 over $\mathbb{Z}_p \times \mathbb{Z}_p$ which correspond to all possible partial difference sets of $\mathbb{Z}_p \times \mathbb{Z}_p$ (cf. Proposition 3.1.27):

Proposition 3.2.19 *Let* $W := \langle \underline{O}_0, \ldots, \underline{O}_{p+1}\rangle$ *be the Schur ring of traces over* $\mathbb{Z}_p \times \mathbb{Z}_p$ *with*

$$O_0 = \{(0,0)\},\ O_1 = \{(0,a)|a \in \mathbb{Z}_p^*\}, O_{k+2} = \{(a,ka)|a \in \mathbb{Z}_p^*\}, k \in \mathbb{Z}_p.$$

Then we get all partial difference sets of $\mathbb{Z}_p \times \mathbb{Z}_p$ *by merging the elements of* i*-subsets of* $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$. *Each partial difference set we get in this way has order* $i(p-1)$ *and generates a strongly regular Cayley graph over* $\mathbb{Z}_p \times \mathbb{Z}_p$ *with valency* $i(p-1)$.

Proof: By Theorem 3.3 in [GolIK94] we know, that W is an amorphic cellular ring. Let D be a union of elements of an i-subset of $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$, i.e., $D = O_{j_1} \cup \ldots \cup O_{j_i}$, where $1 \leq j_1 < \ldots < j_i \leq p+1$. Then $\langle \underline{O_0}, \underline{D}, \underline{((\mathbb{Z}_p \times \mathbb{Z}_p) \setminus D) \setminus O_0}\rangle$ is a Schur ring of rank 3 which is primitive and symmetric. By Proposition 3.1.27 D is a partial difference set of $\mathbb{Z}_p \times \mathbb{Z}_p$ which generates a strongly regular Cayley graph with p^2 vertices and valency $i(p-1)$. □

Notice that the partial difference sets we get in the case $i = 1$ are trivial. For $i > \frac{p+1}{2}$ we have partial difference sets which correspond to partial difference sets we get for $i < \frac{p+1}{2}$ ("complementary cases").

Proposition 3.2.19 yields all partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$. Since we are only interested in partial difference sets up to CI-equivalence, it remains the question which of these partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$ are CI-equivalent. Here we have to investigate the induced action of the automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$ on the i-subsets of $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$.

Lemma 3.2.20 *The automorphism group G of the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is isomorphic to the general linear group $GL(2,p)$. The induced action of G on the i-subsets of $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$ corresponds to the action of $PGL(2,p)$ on the i-sets of one-dimensional subspaces of the vector space $\mathbb{Z}_p \times \mathbb{Z}_p$ over the field $\mathbb{Z}_p$, i.e., the action of $PGL(2,p)$ on $PG(1,p)$.*

Proof: Let G be the automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$. Clearly, every automorphism of the vector space $\mathbb{Z}_p \times \mathbb{Z}_p$ over $\mathbb{Z}_p$ is an element in G. Conversely, we have to take a group automorphism $\varphi \in G$ and have to show, that φ is an automorphism of the vector space. We know that φ is a bijective mapping that respects the operation "+". For $k \in \mathbb{Z}_p$ and $(i,j) \in \mathbb{Z}_p \times \mathbb{Z}_p$ we have:

$$(k(i,j))^{\varphi} = (\underbrace{(i,j) + ... + (i,j)}_{k \text{ times}})^{\varphi} = \underbrace{(i,j)^{\varphi} + ... + (i,j)^{\varphi}}_{k \text{ times}} = k\,(i,j)^{\varphi}.$$

Thus, φ is a bijective vector space homomorphism and G is the automorphism group of the vector space which is isomorphic to $GL(2,p)$.
By definition the sets O_j, $1 \leq j \leq p+1$, are the orbits of $\mathbb{Z}_p^*$ in $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus, $O_j \cup O_0$, $1 \leq j \leq p+1$, are exactly the one-dimensional subspaces of the vector space $\mathbb{Z}_p \times \mathbb{Z}_p$ over $\mathbb{Z}_p$. The group acting on these subspaces induced by $GL(2,p)$ is the group $PGL(2,p)$. Hence, the induced action of G on the i-subsets of $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$ corresponds to the action of $PGL(2,p)$ on the i-sets of one-dimensional subspaces (projective points). □

Now we can characterize the CI-equivalent partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$ by the action of the group $PGL(2,p)$ on $PG(1,p)$:

Lemma 3.2.21 *Two partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$ of order $i(p-1)$, $2 \leq i \leq \frac{p+1}{2}$, are CI-equivalent if and only if the corresponding i-sets of projective points in $PG(1,p)$ are in the same $\{i\}$-orbit of $PGL(2,p)$.*

Proof: Partial difference sets with $i(p-1)$, $2 \leq i \leq \frac{p+1}{2}$ elements are mergings of i-subsets of $\{O_1, ..., O_{p+1}\}$ (cf. Proposition 3.2.19). Two of such partial difference sets are CI-equivalent if and only if the corresponding i-subsets of $\{O_1, ..., O_{p+1}\}$ are in the same $\{i\}$-orbit of the automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$. By Lemma 3.2.20 the action of this automorphism group on the i-subsets corresponds to the action of the group $PGL(2,p)$ on the i-subsets of $PG(1,p)$.
□

The next proposition is a well-known fact in group theory:

Proposition 3.2.22 *(cf. [BigW79], Theorem 2.6.2)*
The group $PGL(2,p)$ acts sharply 3-transitive on the $p+1$ points of $PG(1,p)$.

By this proposition it is clear that we have exactly one $\{2\}$-orbit and exactly one $\{3\}$-orbit. With Lemma 3.2.21 we get immediately:

Corollary 3.2.23 *All partial difference sets of $\mathbb{Z}_p \times \mathbb{Z}_p$ with $2(p-1)$ or $3(p-1)$ elements are CI-equivalent.*

Since each merging of i-subsets of $\{O_1, ..., O_{p+1}\}$ gives CI-equivalent partial difference sets with $i(p-1)$ elements for $i = 2$ and $i = 3$, it remains the question what happens when $i > 3$. This case occurs the first time for $p = 7$, because we have $i \leq \frac{p+1}{2}$. For our purpose we have to consider all prime numbers $p \leq 13$, because we want to investigate groups up to order 255.
We have to examine the partial difference sets of order:
$4(p-1)$ for $p = 7$,
$4(p-1)$, $5(p-1)$, $6(p-1)$ for $p = 11$ and
$4(p-1)$, $5(p-1)$, $6(p-1)$, $7(p-1)$ for $p = 13$.
An easy computation with the computer package GAP [GAP99] (see Section 3.3.3 for information about this program) gives the number of $\{4\}$-orbits, $\{5\}$-orbits, $\{6\}$-orbits and $\{7\}$-orbits of $PGL(2,p)$ on $PG(1,p)$ for all suitable $p = 7, 11, 13$. We have

p	#4-orbits	#5-orbits	#6-orbits	#7-orbits
7	2	-	-	-
11	2	2	4	-
13	3	3	5	5

However, we do not know how the resulting representatives for the $\{i\}$-orbits of the group $PGL(2,p)$, which we get by this computation with GAP, correspond to the concrete $\{i\}$-sets of $\{O_1, ..., O_{p+1}\}$ (the orbits $O_1, ..., O_{p+1}$ are defined in a special way (e.g., see Proposition 3.2.19) and it is not clear that the orbit O_j corresponds to the projective point j in $PG(1,p)$ in GAP). The application of the group $PGL(2,p)$ is very useful for the determination of the number of non-CI-equivalent partial difference sets in $\mathbb{Z}_p \times \mathbb{Z}_p$, but if we want to compute the partial difference sets for a concrete prime number p by GAP, then we have to do all the computations for the action of the automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$ on the i-subsets of $\{O_1, ..., O_{p+1}\}$.
For

$$O_1 = \{(0,a) | a \in \mathbb{Z}_p^*\}, O_{k+2} = \{(a, ka) | a \in \mathbb{Z}_p^*\}, k \in \mathbb{Z}_p$$

we get the following representative non-CI-equivalent partial difference sets $O_{j_1} \cup \ldots \cup O_{j_i}$ in $\mathbb{Z}_p \times \mathbb{Z}_p$, $(4 \leq i \leq \frac{p+1}{2})$:

p	$[j_1, j_2, j_3, j_4]$	$[j_1, j_2, j_3, j_4, j_5]$	$[j_1, j_2, j_3, j_4, j_5, j_6]$	$[j_1, j_2, j_3, j_4, j_5, j_6, j_7]$
7	$[1, 2, 3, 4]$ $[1, 2, 3, 5]$			
11	$[1, 2, 3, 4]$ $[1, 2, 3, 5]$	$[1, 2, 3, 4, 5]$ $[1, 2, 3, 5, 6]$	$[1, 2, 3, 4, 5, 6]$ $[1, 2, 3, 4, 5, 7]$ $[1, 2, 3, 4, 5, 8]$ $[1, 2, 3, 4, 5, 9]$	
13	$[1, 2, 3, 4]$ $[1, 2, 3, 5]$ $[1, 2, 3, 6]$	$[1, 2, 3, 4, 5]$ $[1, 2, 3, 4, 6]$ $[1, 2, 3, 5, 6]$	$[1, 2, 3, 4, 5, 6]$ $[1, 2, 3, 4, 5, 7]$ $[1, 2, 3, 4, 5, 9]$ $[1, 2, 3, 4, 5, 10]$ $[1, 2, 3, 5, 6, 7]$	$[1, 2, 3, 4, 5, 6, 7]$ $[1, 2, 3, 4, 5, 6, 8]$ $[1, 2, 3, 4, 6, 7, 8]$ $[1, 2, 3, 4, 6, 7, 10]$ $[1, 2, 5, 6, 7, 8, 11]$

Each partial difference set generates a strongly regular Cayley graph over $\mathbb{Z}_p \times \mathbb{Z}_p$. In principle, it can happen that one gets isomorphic graphs for non-CI-equivalent partial difference sets. However, since the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is a CI-group (see Proposition 3.1.31), all non-CI-equivalent partial difference sets yield non-isomorphic Cayley graphs over $\mathbb{Z}_p \times \mathbb{Z}_p$.

Example 3.2.24 *Consider the case $p = 7$. We have nine orbits of $\mathbb{Z}_7^*$ on $\mathbb{Z}_7 \times \mathbb{Z}_7$:*

$$\begin{array}{ll} O_0 = \{(0,0)\}, & \\ O_1 = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6)\}, & O_2 = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)\}, \\ O_3 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}, & O_4 = \{(1,2),(2,4),(3,6),(4,1),(5,3),(6,5)\}, \\ O_5 = \{(1,3),(2,6),(3,2),(4,5),(5,1),(6,4)\}, & O_6 = \{(1,4),(2,1),(3,5),(4,2),(5,6),(6,3)\} \\ O_7 = \{(1,5),(2,3),(3,1),(4,6),(5,4),(6,2)\}, & O_8 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}. \end{array}$$

We get non-CI-equivalent partial difference sets, if we merge two, three or four orbits; mergings of five orbits provide partial difference sets which are complementary to the partial difference sets we get by mergings of three orbits.
We get exactly one partial difference set (up to CI-equivalence) by merging two orbits, e.g.,

$$O_1 \cup O_2 = \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0)\}.$$

This partial difference set generates the lattice graph $L_2(7)$ with parameters $(49, 12, 5, 2)$ (cf. Section 3.2.5 for graphs $L_2(n)$).
Merging three orbits provides also exactly one partial difference set, e.g.,

$$\begin{aligned} O_1 \cup O_2 \cup O_3 = \ & \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,0),(2,0),(3,0), \\ & (4,0),(5,0),(6,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}. \end{aligned}$$

This partial difference set generates the graph $L_3(7)$ with parameters $(49, 18, 10, 6)$, one can get from a latin square over $\mathbb{Z}_7$ (cf. Remark 3.2.36 for graphs $L_3(n)$).

Merging four orbits provides two non-CI-equivalent partial difference sets as shown in the table above. Two representatives are

$$O_1 \cup O_2 \cup O_3 \cup O_4 =$$
$$\{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),$$
$$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(2,4),(3,6),(4,1),(5,3),(6,5)\},$$

$$O_1 \cup O_2 \cup O_3 \cup O_5 =$$
$$\{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),$$
$$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,3),(2,6),(3,2),(4,5),(5,1),(6,4)\}.$$

These two partial difference sets generate two non-isomorphic, selfcomplementary graphs with parameters $(49, 24, 11, 12)$ of latin square type $L_4(7)$ (cf. Definition 3.2.34 for latin square type graphs). The first graph is isomorphic to the Paley graph $P(49)$.

3.2.4 Latin square type graphs

The class of latin square type graphs consists of strongly regular graphs one can define by latin squares. In the first part of this section we will give some necessary information about latin squares.

Definition 3.2.25 *(cf. [DenK74], p. 15)*
An $n \times n$-matrix with n different entries, $n \geq 2$, is called a **latin square of order** *n if and only if each entry occurs exactly once in any row and in any column of the matrix.*

An equivalent description of latin squares we find in [vLiW92], p. 157:

Remark 3.2.26 *Let R, C, S be sets of order n, $n \geq 2$, and let $M : R \times C \to S$ be a mapping, such that the equation $M(i,j) = k$ for any two of the three variables $i \in R, j \in C$ and $k \in S$ determines the third one. Then the quadrupel $(R, C, S; M)$ is called a latin square of order n.*

In [GolIK94], p. 182, we find a third equivalent description; it is an abstract combinatorial description of latin squares which is taken from [ArlBGF78]:

Remark 3.2.27 *Let $V := \{1, \ldots, n\}$ be a set of order n and let $L \subset V \times V \times V$ be a set of order n^2. The set L is called a latin square of order n if and only if each of the three sets*

$$L_1 := \{(i,j) | (i,j,k) \in L\}, L_2 := \{(i,k) | (i,j,k) \in L\} \text{ and } L_3 := \{(j,k) | (i,j,k) \in L\}$$

has n^2 distinct elements.

Since we have three equivalent descriptions of latin squares, in the following in each situation we will choose the description which is the most convenient for us.

Next we will give some definitions and statements from the area of latin squares which are of importance for our purpose. For a general introduction into the theory of latin squares we refer to the book of J. Denes and A. D. Keedwell [DenK74].

It is easy to see that the multiplication table of a group (the Cayley table of the group) corresponds to a latin square. More general, one can prove the following theorem:

Theorem 3.2.28 *([DenK74], Theorem 1.1.1)*
The multiplication table of a quasigroup is a latin square.

A quasigroup is a set Q with a binary operation "$\cdot$" such that for all $a, b \in Q$ the equations $a \cdot x = b$ and $y \cdot a = b$ have a unique solution in Q. It follows that the converse of the previous theorem is also true: every latin square can be interpreted as a multiplication table of a quasigroup (cf. [DenK74], p. 23). In [vLiW92] quasigroups are even defined by latin squares.

For every latin square there exist so-called conjugate latin squares. One conjugate is the transpose of a latin square, but we have some more conjugates because we can interchange more than only rows and columns of a latin square:

Proposition 3.2.29 *(cf. [DenK74], p. 125)*
Let $L = (R, C, S; M)$ be a latin square of order n, $n \geq 2$. Then the following objects are also latin squares of order n:

$$\begin{aligned}
&{}^{-1}L = (S, C, R; {}^{-1}M) \text{ with } {}^{-1}M(x, j) = i \Leftrightarrow M(i, j) = x,\\
&L^{-1} = (R, S, C; M^{-1}) \text{ with } M^{-1}(i, x) = j \Leftrightarrow M(i, j) = x,\\
&L^{*} = (C, R, S; M^{*}) \text{ with } M^{*}(j, i) = x \Leftrightarrow M(i, j) = x,\\
&({}^{-1}L)^{*} = (S, R, C; ({}^{-1}M)^{*}) \text{ with } ({}^{-1}M)^{*}(x, i) = j \Leftrightarrow M(i, j) = x,\\
&(L^{-1})^{*} = (C, S, R; (M^{-1})^{*}) \text{ with } (M^{-1})^{*}(j, x) = i \Leftrightarrow M(i, j) = x.
\end{aligned}$$

Proof: The proof follows immediately from the definition of a latin square. □

Definition 3.2.30 *Let $L = (R, C, S; M)$ be a latin square of order n, $n \geq 2$. Then the latin squares defined in Proposition 3.2.29 together with L are called* **conjugates of** *L.*

If we define a latin square L as a set of triples (α, β, γ) as described in Remark 3.2.27, then the concept of conjugates of L corresponds to the permutation of the three coordinates.

The concept of isotopy which is known in the theory of quasigroups can be transformed to isotopy of latin squares:

Definition 3.2.31 *(cf. [vLiW92], p. 158)*
Two latin squares $L = (R, C, S; M), L' = (R', C', S'; M')$ of order n, $n \geq 2$, are called **isotopic** *if and only if we get L' from L by rearranging rows, columns and renaming symbols, i.e., if there exist bijections $\sigma : R \to R'$, $\tau : C \to C'$, $\pi : S \to S'$ such that $M'(i^{\sigma}, j^{\tau}) = (M(i, j))^{\pi}$.*

Isotopic latin squares are also called **equivalent**.
The concepts of isotopy and conjugates provide a partition of the set of all latin squares of order n, $n \geq 2$:

Definition 3.2.32 *([DenK74], p. 126)*
Let Ω_n be the set of all latin squares of order n, $n \geq 2$. A subset of Ω_n that consists of all elements of an isotopy class together with their conjugates is called a **main class**.

There are some more possibilities of partitioning the set Ω_n. For further information see [DenK74].

One of the most famous concepts in the history of the investigation of latin squares is the concept of orthogonality. Already in 1779 L. Euler was concerned with orthogonal latin squares in the case of his "36 officers problem" (see [DenK74], p. 11 or [vLiW92], p. 251).

Definition 3.2.33 *(cf. [DenK74], p. 154, p. 158)*
Let $L_1 := (a_{ij})$ and $L_2 := (b_{ij})$ be two latin squares with n symbols, $n \geq 2$. The latin squares L_1, L_2 are called **orthogonal** *if and only if every ordered pair of symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}), $i, j = 1, \ldots, n$.*
A set of latin squares of order n where each pair of latin squares are orthogonal is called **set of mutually orthogonal latin squares (MOLS)**.

There exists a connection between latin squares and strongly regular graphs. From each system of mutually orthogonal latin squares of order n, $n \geq 2$, one can get a strongly regular graph in the following way:
The vertices are the n^2 items of a square of order n and two vertices are adjacent if and only if the items are in the same row, in the same column or if they have the same symbol in one of the orthogonal latin squares.

Definition 3.2.34 *The graph constructed above from a set of $g-2$ mutually orthogonal latin squares of order n, $n \geq g \geq 2$, is called a* **latin square type graph**. *The class of latin square type graphs will be denoted by $L_g(n)$.*

Proposition 3.2.35 *(cf. [GolIK94], p. 172)*
Every latin square type graph in $L_g(n)$, $n \geq g \geq 2$, is a strongly regular graph with parameters $(n^2, g(n-1), n-2+(g-1)(g-2), g(g-1))$.

Remark 3.2.36 *A special class of the latin square type graphs is the class of so-called* **latin square graphs** *$L_3(n)$, $n \geq 2$. By definition the vertices are the n^2 items of the square and two vertices are adjacent if and only if they are in the same row, in the same column or if they have the same symbol in the latin square.*
If we take the description of latin squares which is given in Remark 3.2.27, then we can describe the latin square graph of a latin square L as follows: The vertices are the triples in L and two vertices are adjacent if and only if they coincide in one coordinate.

If we consider a latin square which corresponds to the Cayley table of a group H of order n, $n \geq 2$, then we will call the associated graph the **latin square graph** $L_3(n)$ **over the group** H and denote it by $L_3(H)$.
In the following we will focus on the case of latin square graphs $L_3(n)$, $n \geq 2$.

Lemma 3.2.37 *Two latin square graphs of conjugated latin squares are isomorphic.*

Proof: Let L and L' be two conjugated latin squares and Γ and Γ' the corresponding latin square graphs. In the notation of Remark 3.2.27 we get L' from L if we permute the three coordinates of all elements in L by a suitable permutation σ simultaneously. This permutation of the coordinates induces a bijective mapping $\tilde{\sigma}$ from the vertex set of Γ to the vertex set of Γ'. Since two triples in L coincide in exactly one coordinate if and only if their images with respect to σ coincide in exactly one coordinate, the mapping $\tilde{\sigma}$ is a graph isomorphism and $\Gamma \cong \Gamma'$. □

With Lemma 3.2.37 we can prove that latin square graphs from latin squares of the same main class are isomorphic.

Proposition 3.2.38 *Let $L = (R, C, S; M), L' = (R', C', S'; M')$ be two latin squares of order n, $n \geq 2$, which are in the same main class. Then the corresponding latin square graphs Γ resp. Γ' are isomorphic.*

Proof: The main classes consist of an isotopy class of latin squares and the conjugates of these latin squares. We know by Lemma 3.2.37 that two conjugate latin squares provide isomorphic latin square graphs. Thus, it is sufficient to show that two latin square graphs of isotopic latin squares are isomorphic.
By Definition 3.2.31 we have bijections $\sigma : R \to R', \tau : C \to C', \pi : S \to S'$ such that $M'(i^\sigma, j^\tau) = (M(i,j))^\pi$. Furthermore, we know that the vertex sets of the graphs are $V(\Gamma) = R \times C$ and $V(\Gamma') = R' \times C'$. Now, we define a mapping $\sigma \times \tau : V(\Gamma) \to V(\Gamma')$ where $(i,j)^{(\sigma\times\tau)} := (i^\sigma, j^\tau)$ for all $(i,j) \in V(\Gamma)$. Clearly, this mapping is a bijection. Let $\{(i,j),(k,l)\}$ be an edge in Γ, i.e., by the construction of Γ from L we have $i = k$ or $j = l$ or $M(i,j) = M(k,l)$. The image of this edge by (σ, τ) is $\{(i^\sigma, j^\tau), (k^\sigma, l^\tau)\}$. Clearly, $i = k$ if and only if $i^\sigma = k^\sigma$, and $j = l$ if and only if $j^\tau = l^\tau$, because σ and τ are bijections. In the case of $M(i,j) = M(k,l)$ we have

$$M(i,j) = M(k,l) \Leftrightarrow (M(i,j))^\pi = (M(k,l))^\pi \Leftrightarrow M'(i^\sigma, j^\tau) = M'(k^\sigma, l^\tau),$$

because the latin squares are isotopic.
Thus, we have a bijection from $V(\Gamma)$ to $V(\Gamma')$ such that two vertices in Γ are adjacent if and only if they are adjacent in Γ', i.e., Γ and Γ' are isomorphic. □

Example 3.2.39 *For $n = 3$ we have only one main class of latin squares (cf. [DenK74], p. 129). One representative is the latin square over the cyclic group with three elements. By Proposition 3.2.38 we have only one latin square graph (up to isomorphism). This strongly*

regular graph is trivial, because its complement is disconnected and has three components each isomorphic to K_3.
For $n = 4$ *we have two main classes of latin squares (cf. [DenK74], p. 129). These are the Cayley tables of the cyclic group with four elements and the Klein four-group. The complements of the corresponding latin square graphs are known as the Shrikhande graph and the lattice graph* $L_2(4)$. *The Shrikhande graph is also called the pseudolattice (see Proposition 3.2.46).*

Now we consider the automorphisms of a latin square graph which we get from the Cayley table of a group of order n, $n \geq 2$.
For this purpose it is convenient to use the description of latin squares as it was given in Remark 3.2.27. In the case when we have the Cayley table of a group H, the corresponding latin square L corresponds to a set of n^2 triples (i, j, k) with $i, j, k \in H$ and $ijk = e$, where e is the identity element in H. As described in Remark 3.2.36 the n^2 elements of L are the vertices of the graph $L_3(H)$ and two vertices are adjacent if and only if they coincide in one coordinate.
It is easy to see that the automorphism group of H induces a subgroup of $Aut(L_3(H))$, because for all $\sigma \in Aut(H)$ and $i, j, k \in H$ with $ijk = e$ we have $(i, j, k)^{\tilde{\sigma}} := (i^\sigma, j^\sigma, k^\sigma)$ and $i^\sigma j^\sigma k^\sigma = (ijk)^\sigma = e^\sigma = e$. Thus, σ induces a bijection $\tilde{\sigma} : L \to L$ and if two triples coincide in one coordinate, then the images also coincide in one coordinate.
Moreover, we can show that $H^3 = H \times H \times H$ corresponds to a subgroup of $Aut(L_3(H))$: For $(\alpha, \beta, \gamma) \in H^3$ and $(i, j, k) \in L$ we define $(i, j, k)^{(\alpha,\beta,\gamma)} := (\alpha i\beta^{-1}, \beta j\gamma^{-1}, \gamma k\alpha^{-1})$. We have $\alpha i\beta^{-1}\beta j\gamma^{-1}\gamma k\alpha^{-1} = \alpha ijk\alpha^{-1} = \alpha e\alpha^{-1} = e$, thus, (α, β, γ) is a bijection $L \to L$ and if two triples coincide in one coordinate, then the images also coincide in one coordinate. However, there are elements of H^3 which have an identical action. It is not difficult to check that these are exactly the elements $N := \{(\alpha, \alpha, \alpha) | \alpha \in Z(H)\}$, where $Z(H)$ is the center of H. Hence, $H^3/_N$ is isomorphic to a subgroup of $Aut(L_3(H))$.
A third subgroup of $Aut(L_3(H))$ is isomorphic to the symmetric group S_3. This subgroup permutes the three coordinates of the elements of the latin square.
In [GolIK94], p. 182, the automorphism group $Aut(L_3(H))$ is given without a proof:

Remark 3.2.40 *([GolIK94], p. 182)*
The automorphism group of a latin square graph $L_3(H)$ *over a group* H *is isomorphic to*

$$H^3 \rtimes (S_3 \times Aut(H))/_{\tilde{N}},$$

where $\tilde{N}$ *is the kernel of the action of* $H^3 \rtimes (S_3 \times Aut(H))$. *If* H *is abelian, then the automorphism group is isomorphic to* $H^2 \rtimes (S_3 \times Aut(H))$.

We already described the action of the three groups H^3, S_3, $Aut(H)$. One can check that the groups S_3 and $Aut(H)$ are acting independently. Moreover, it is not difficult to show that each of these two groups normalizes the group H^3. Thus, we get the semidirect product $H^3 \rtimes (S_3 \times Aut(H))$. The determination of the group $\tilde{N}$ which is the kernel of the action of the group $H^3 \rtimes (S_3 \times Aut(H))$ needs more sophisticated computations which go beyond the scope of this thesis. However, it is clear that the group N (see above) is contained in $\tilde{N}$. Moreover, one can show that $\tilde{N}$ contains a subgroup isomorphic to H.

Proposition 3.2.41 *Let $L_3(H)$ be the latin square graph constructed by a latin square over a group H of order n, $n \geq 2$. Then there exists a regular subgroup in the automorphism group of $L_3(H)$ isomorphic to $H \times H$.*

Proof: Let $L := \{(i,j,k)|i,j,k \in H, ijk = e\}$ be a latin square which corresponds to the Cayley table of the group H. Consider the action of $H \times H$ on L which is defined as following: For $(\alpha,\gamma) \in H \times H$ we have $(i,j,k)^{(\alpha,\gamma)} := (\alpha i, j\gamma^{-1}, \gamma k\alpha^{-1})$. As described above this action is a bijection $L \to L$ because $\alpha ij\gamma^{-1}\gamma k\alpha^{-1} = \alpha ijk\alpha^{-1} = \alpha e\alpha^{-1} = e$ and it induces an automorphism of the graph $L_3(H)$.
It remains the question, if the induced action of $H \times H$ on the set of n^2 triples $(i,j,k) \in L$ is regular. This fact follows immediately because we have n^2 elements $(\alpha,\gamma) \in H \times H$ and each element (α,γ) yields a different image $(\alpha i, j\gamma^{-1}, \gamma k\alpha^{-1})$ of the element $(i,j,k) \in L$ (as one can see at the first and second coordinate). □

Proposition 3.2.42 *Let H be a group of order n, $n \geq 2$. Then the set*

$$D := \{(\alpha,e),(e,\alpha),(\alpha,\alpha)|\alpha \in H, \alpha \neq e\}$$

is a $(n^2, 3(n-1), n, 6)$-partial difference set in $H \times H$ and the Cayley graph $Cay(H \times H, D)$ is isomorphic to the latin square graph $L_3(H)$ over H.

Proof: Since $L_3(H)$ is the latin square graph over the group H, it is a $(n^2, 3(n-1), n, 6)$-strongly regular graph. By Proposition 3.2.41 we know that $H \times H$ is a regular subgroup of $Aut(L_3(H))$. Since (e,e,e) is an element of the latin square, it is a vertex of $L_3(H)$. Thus, by Lemma 3.1.20 we get that for

$$D := \{(\alpha,\gamma) \in H \times H|(e,e,e)^{(\alpha,\gamma)} \text{ is adjacent to } (e,e,e)\}$$

the Cayley graph $Cay(H \times H, D)$ is isomorphic to $L_3(H)$. As described above the action of $H \times H$ on the vertex set of $L_3(H)$ is defined by $(i,j,k)^{(\alpha,\gamma)} := (\alpha i, j\gamma^{-1}, \gamma k\alpha^{-1})$. Since in $L_3(H)$ two vertices are adjacent if and only if they coincide in one coordinate and we have $(e,e,e)^{(\alpha,\gamma)} = (\alpha, \gamma^{-1}, \gamma\alpha^{-1})$ for $(\alpha,\gamma) \in D$, it follows that $\alpha = e, \gamma \neq e$ or $\alpha \neq e, \gamma = e$ or $\alpha = \gamma \neq e$ if the vertex $(\alpha, \gamma^{-1}, \gamma\alpha^{-1})$ is a neighbour of (e,e,e). Hence, we have

$$D := \{(\alpha,e),(e,\alpha),(\alpha,\alpha)|\alpha \in H, \alpha \neq e\}$$

and since $L_3(H)$ is a $(n^2, 3(n-1), n, 6)$-strongly regular graph, by Theorem 3.1.23 the set D is a $(n^2, 3(n-1), n, 6)$-partial difference set in $H \times H$. □

From Proposition 3.2.41 follows that latin square graphs from latin squares over groups are always vertex transitive. However, it is not true that latin square graphs of arbitrary latin squares are vertex transitive. Below we describe an example for a latin square graph which is not vertex transitive. But there are also examples of vertex transitive latin square graphs which are not constructed by a latin square over a group.

Example 3.2.43 *The latin square graph of the latin square L_1 (No. 5.1.2 in [DenK74], p. 130) has an intransitive automorphism group of order 48 with three orbits.*

$$L_1 := \begin{array}{|cccccc} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 3 & 6 & 2 & 4 & 1 \\ 6 & 4 & 2 & 3 & 1 & 5 \end{array} \qquad L_2 := \begin{array}{|cccccc} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 2 & 1 & 3 \\ 5 & 4 & 6 & 3 & 2 & 1 \\ 6 & 5 & 4 & 1 & 3 & 2. \end{array}$$

There are three non-isomorphic latin square graphs with 36 vertices which are Cayley graphs (cf. Section 4.1.2). These graphs correspond to latin squares of different main classes. Since there are only two non-isomorphic groups of order 6 (symmetric group S_3 and cyclic group $\mathbb{Z}_6$), one of these vertex transitive graphs must be a latin square graph over a proper quasigroup. The Cayley table of this quasigroup resp. the latin square is the above given L_2 (No. 3.1.1 in [DenK74], p. 130). The corresponding latin square graph has a transitive automorphism group of order 648.

From Proposition 3.2.35 we can determine the parameters of the strongly regular latin square graphs. All graphs of the class $L_3(n)$, $n \geq 3$, have parameters $(n^2, 3(n-1), n, 6)$. But there exist other strongly regular graphs with this parameter set which are no latin square graphs. For example for $n = 5$ we have two main classes of latin squares of order n. The corresponding graphs are the vertex transitive latin square graph over the cyclic group with five elements and another latin square graph with automorphism group of order 72 which has two orbits. In addition to these two graphs there exist exactly 13 strongly regular graphs with parameter set $(25, 12, 5, 6)$ (cf. [Ros76]). This leads to the following definition:

Definition 3.2.44 *Let Γ be a strongly regular graph with parameter set $(n^2, 3(n-1), n, 6)$. If Γ is not a latin square graph, then it is called* **pseudo latin square graph**.

The definition of pseudo latin square graphs can be generalized to **pseudo latin square type graphs**, i.e., the class of strongly regular graphs with parameters $(n^2, g(n-1), n-2+(g-1)(g-2), g(g-1)), n \geq 3,\ g \geq 2$, that are not latin square type graphs. There exist conditions for n and g when strongly regular graphs with these parameters are latin square type graphs (cf. [GolIK94], p. 172). These results are consequences of R. H. Brucks work on nets (see [Bru51]; [Bru63]; [DenK74], Chapter 9).
Finally, we want to mention that there exist a class of strongly regular graphs with parameters $(n^2, g(n+1), -n-2+(g+1)(g+2), g(g+1)), n \geq 4,\ g \geq 2$. These graphs are called **negative latin square type graphs**.

3.2.5 Square lattice graphs

The square lattice graph (or lattice graph) with n^2 vertices, $n \geq 2$, is a special latin square type graph.
The vertices of this graph are the items of an $n \times n$-square. Two vertices are adjacent if and only if they are in the same row or in the same column. In accordance with Definition

3.2.34 this graph will be denoted by $L_2(n)$. By Proposition 3.2.35 the graph $L_2(n)$ is a strongly regular graph with parameter set $(n, 2(n-1), n-2, 2)$. For $n = 2$ the lattice graph is isomorphic to the cycle with four vertices. This is a trivial strongly regular graph because its complement is disconnected.

Remark 3.2.45 *The lattice graph $L_2(n)$, $n \geq 2$, is isomorphic to the line graph of the complete bipartite graph $K_{n,n}$.*

Proof: Let $V := \{v_1, ..., v_n\}$ and $W := \{w_1, ..., w_n\}$ be two disjoint sets. Consider $V \cup W$ as the vertex set of the bipartite graph $K_{n,n}$, where we have all possible edges between V and W, i.e., $E(K_{n,n}) = \{\{v, w\} | v \in V, w \in W\}$. The line graph has vertex set $E(K_{n,n})$ and two vertices of the line graph are adjacent if and only if the corresponding edges have one common vertex in $K_{n,n}$, i.e., $\{v_i, w_j\}, \{v_k, w_l\} \in E(K_{n,n})$ are adjacent if and only if $i = k$ or $j = l$. Since V and W are disjoint, we can consider $E(K_{n,n})$ as the square $V \times W$ and two vertices $(v_i, w_j), (v_k, w_l) \in V \times W$ are adjacent if and only if they are in the same row or same column. This is just the definition of the lattice graph. □

It is well-known that the lattice graphs $L_2(n)$ for $n \neq 4$ are determined by their parameters:

Proposition 3.2.46 *(cf. [Shr59])*
For $n \geq 2, n \neq 4$, every strongly regular graph with parameters $(n, 2(n-1), n-2, 2)$ is isomorphic to the lattice graph $L_2(n)$.

In the case of $n = 4$ there is exactly one other strongly regular graph with parameters $(16, 6, 2, 2)$. This graph is called *Shrikhande graph* (see Example 3.2.39).

All lattice graphs $L_2(n)$ can be generated as Cayley graphs because for all $n \geq 2$ we have a regular subgroup of $Aut(L_2(n))$:

Proposition 3.2.47 *Let $L_2(n)$, $n \geq 2$, be the lattice graph. Then the automorphism group $Aut(L_2(n))$ has a regular subgroup similar to $H \times K$, where H, K are regular permutation groups of degree n. Moreover, the Cayley graph $Cay(H \times K, D)$ over $H \times K$ with connection set $D := \{(h, e), (e, k) | h \in H, k \in K, h, k \neq e\}$ is isomorphic to $L_2(n)$.*

Proof: From [Whi32] we know that

$$Aut(K_{n,n}) \cong Aut(L(K_{n,n})) \cong Aut(L_2(n)), n \geq 2,$$

i.e., we can consider $Aut(K_{n,n})$ instead of $Aut(L_2(n))$. From the structure of the complete bipartite graph $K_{n,n}$ we know that there is a partition $V \cup W$ of the vertex set, such that the vertices in V resp. in W are not adjacent, but each vertex of V is adjacent to each vertex of W. It follows that any permutation of the vertices of V or of W is an automorphism of $K_{n,n}$. Thus, we can permute each of the disjoint parts of the vertex set independently in an arbitrary way. Hence, for all regular permutation groups H, K of degree n there exists a regular subgroup of $Aut(K_{n,n})$ similar to the group $H \times K$. It is easy to check that this subgroup induces a regular subgroup of $Aut(L_2(n))$.

Since $H \times K$ corresponds to a regular subgroup of the automorphism group of $L_2(n)$, we can construct a Cayley graph with vertex set $H \times K$ which is isomorphic to $L_2(n)$. By Lemma 3.1.20 the connection set for this Cayley graph can be chosen as all elements of $H \times K$ which maps the vertex (e, e) onto its neighbours. Since this mapping is defined by right multiplication, this connection set is $D := \{(h, e), (e, k) | h \in H, k \in K, h, k \neq e\}$, because by definition of the lattice graph the neighbours of (e, e) are all vertices of the form $(h, e), (e, k)$, $h \in H, k \in K, h, k \neq e$.

□

The fact that for groups H, K of order n the set $D := \{(h, e), (e, k) | h \in H, k \in K, h, k \neq e\}$ is a $(n^2, 2(n-1), n-2, 2)$-partial difference set in the group $H \times K$ is also mentioned in [Ma84], Theorem 3.1 (1).

Although it is easy to determine some partial difference sets for $L_2(n)$, it is not easy at all to find all partial difference sets (up to CI-equivalence) which generate a lattice graph. By Lemma 3.1.19 and Lemma 3.1.33 this is equivalent to finding all regular subgroups of $Aut(L_2(n))$ (up to conjugacy). Therefore we need a description of $Aut(L_2(n))$.

Proposition 3.2.48 *(cf. [KliPR88] Example, 4.3.24 c)*
The automorphism group of $L_2(n)$ is the exponentiation $S_n \uparrow S_2$, $n \geq 2$.

Proof: If we take into account that $Aut(K_{n,n}) \cong Aut(L_2(n))$, $n \geq 2$, we can easily check that the two disjoint parts of the vertex set of $K_{n,n}$ can be permuted arbitrarily and independently. Thus, we get all permutations of $S_n \times S_n$. Furthermore, we can interchange these to disjoint subsets and we get $S_2 \wr S_n$ for the automorphism group of $K_{n,n}$. Now, we have to consider the induced action of this group on the edges of $K_{n,n}$. If we consider $\{0, 1\} \times N$, with $N := \{0, \ldots, n-1\}$, as vertex set of this graph, then $N^{\{0,1\}}$ can be taken as the edge set, i.e., as the vertex set of $L(K_{n,n}) \cong L_2(n)$. One can check that the action of $S_2 \wr S_n$ on $\{0, 1\} \times N$ induces exactly the action of $S_n \uparrow S_2$ on $N^{\{0,1\}}$. □

Notice that in general the wreath product $H \wr K$ of two permutation groups is not similar to the exponentiation $K \uparrow H$, but these two groups are isomorphic as abstract groups (cf. [KliPR88], Chapter 1.7).
The description of the automorphism group of $L_2(n)$ reduces the problem of determining all partial difference sets for $L_2(n)$ to a pure group theoretical problem. Nevertheless, it is difficult to solve it in general. If n is a prime number, then it is possible to determine all partial difference sets (see Section 3.2.3). If n is not a prime, then the situation is much more sophisticated; here up to a certain bound for n computational approaches yield results.

3.3 Computational methods for the determination of partial difference sets

There are several methods for the enumeration of partial difference sets which are based on distinct modes of thinking.
The most general approach is to consider the determination of partial difference sets as a special case of the determination of all Schur rings. By Theorem 3.1.27 we know that the existence of partial difference sets in a group H is equivalent to the existence of primitive symmetric Schur rings of rank 3 over H. Thus, the complete determination of all Schur rings over H, in particular, provides all partial difference sets in H. However, in the moment we can use this general approach only for small groups because we are not able to compute complete lists of Schur rings over groups of large order.
A second approach is based on the knowledge of complete lists of other objects. Using complete lists of (v, k, λ)-difference sets in a group we are able to determine complete lists of certain partial difference sets in this group. This method of transforming difference sets works only for partial difference sets with parameters (v, k, λ, λ) and $(v, k-1, \lambda-2, \lambda)$, but in these cases it is very efficient.
Another method is to use complete lists of vertex transitive strongly regular graphs. Here we have to investigate the automorphism groups of these graphs and to determine all regular subgroups which then provides partial difference sets. If we know all vertex transitive strongly regular graphs with parameters (v, k, λ, μ), then by this method we can determine all (v, k, λ, μ)-partial difference sets. However, even if we know all vertex transitive strongly regular graphs with certain parameters, sometimes the automorphism groups are too large for the necessary computations.
Below, we will present these three methods in detail.

3.3.1 Computations in the group ring

According to Theorem 3.1.27 the existence of partial difference sets is equivalent to the existence of primitive symmetric Schur rings of rank 3. Hence, for the complete determination of partial difference sets in a given group we have to discover all primitive symmetric Schur rings of rank 3 over this group.
For this purpose it is possible to use computer programs like the computer package COCO. The computer package COCO (COherent COnfigurations) was created in Moscow in 1990 - 1992 by I. A. Faradžev and M. H. Klin. Its main features are introduced in [FarK91], the algorithm and the methodology are described in [FarKM94], Chapter 2. The computer package COCO has different functions which, in particular, allows to compute all cellular subrings of a given cellular ring. These functions are: the inducing of permutation groups on combinatorial objects, the computation of the colored graph corresponding to the centralizer ring of a permutation group, the computation of the structure constants of a cellular ring, the computation of the cellular subrings of a given cellular ring and the computation of the automorphism group of a cellular ring.

We give a concrete example for the use of COCO:

Example 3.3.1 *We consider the automorphism group of order 432 of one of the vertex transitive strongly regular graphs with parameters (36,14,4,6) (see Section 4.1.6). With COCO we get the information that this group has eight 2-orbits and that there exists an imprimitive symmetric cellular ring resp. Schur ring of rank 8 over this group. The computer package COCO computes all cellular subrings of this cellular ring. Altogether there are 29 symmetrical cellular subrings, three of them are primitive. These three primitive cellular subrings are of rank 3 with parameters (36,10,4), (36,14,4) and (36,15,6). One can check that they correspond to three strongly regular graphs resp. partial difference sets with parameters (36,10,4,2), (36,14,4,6) and (36,15,6,6).*

The computer package COCO is very useful for finding explanations for some computational results given in this thesis. Sometimes it is even possible to make a further, second step and to explain computational results without the aid of computers (so-called *interpretation*, see Section 3.3.6). An example for such an interpretation we have for the Schläfli graph (see Section 4.1.6).
Unfortunately, there are some restrictions for COCO. In its computations COCO determines a set of intermediate objects (so-called "good subsets", see Section 5.2). If this set has more than 1000 elements, then COCO stops the computations. But also if there are less than 1000 "good subsets" the computations sometimes need a very long time and in some cases the computations will practically not terminate because they need weeks or months. The latter case even occurs for groups like $(\mathbb{Z}_2)^5$. A special program for COCO written by F. Fiedler overcame this problem for small groups. With this program F. Fiedler was able to determine all Schur rings over groups of order up to 31 [Fie98]. Recently, C. Pech found a way to manage the case $(\mathbb{Z}_2)^5$.

Besides the "classical" functions of COCO there exist a new function called *srg* (cf. [FieK98]). This new function, written by F. Fiedler, is still in an experimental stage. With this subroutine it is possible to find all strongly regular graphs which are invariant with respect to a prescribed permutation group. The first new strongly regular graph that was discovered with this function was a graph with parameters (512,73,12,10) which is described in [FieK98]. This is the first known strongly regular graph with parameters (512,73,12,10). For this graph there exists an associated partial difference set in the group $(\mathbb{Z}_2)^9$; it is the first partial difference set with this parameters (512,73,12,10).

Apart from the fact that the computer package COCO is useful in general, in special situations for the computations in the group ring it may be a more promising way to write small ad hoc programs. For example, in [Smi95] K. W. Smith found a reversible difference set with parameters (100,45,20) by using the computer after restricting the problem theoretically to a special task. From the existence of this (100,45,20)-difference set follows the existence of a (100,44,18,20)-partial difference set which was recently discovered by L. Jørgensen and M. H. Klin together with some other partial difference sets with 100 elements by switching of edges in strongly regular graphs ([Kli00], M. H. Klin, private communication).

3.3.2 Transformation of difference sets

The transformation of (v,k,λ)-difference sets of a given group H allows us to determine (v,k,λ,λ)- and $(v,k-1,\lambda-2,\lambda)$-partial difference sets in H. If we have a complete list of all (v,k,λ)-difference sets in a group H, then this transformation is a simple and efficient way to determine all partial difference sets in H with parameter sets (v,k,λ,λ) and $(v,k-1,\lambda-2,\lambda)$.

The transformation procedure is based on Proposition 3.1.13 and Corollary 3.1.28. By Proposition 3.1.13 we know that each shifting of a difference set is again a difference set. From Corollary 3.1.28 we get that each shifting D' of a (v,k,λ)-difference set D provides a (v,k,λ,λ)-partial difference set if and only if D' does not contain the identity element and $D'^{-1} = D'$. Furthermore, if the set D' contains the identity element, then $D' \setminus \{e\}$ is a $(v,k-1,\lambda-2,\lambda)$-partial difference set if and only if $D'^{-1} = D'$.
Hence, if we start with a (v,k,λ)-difference set of a group H, then:

1. We construct all shifts Dx, $x \in H$.

2. We select those shifts which are reversible sets in H, i.e., those for which holds $(Dx)^{-1} = Dx$.

3. Each shift Dx which does not contain the identity element is a (v,k,λ,λ)-partial difference set;

4. Each shift which contains the identity element implies a $(v,k-1,\lambda-2,\lambda)$-partial difference set $(Dx) \setminus \{e\}$.

As an example we consider a difference set with 16 elements. In [Kib78] R. E. Kibler gives a complete list of all $(16,6,2)$-difference sets.

Example 3.3.2 *Let H be a regular permutation group of degree and order 16 similar to $\mathbb{Z}_2 \times D_4$ with generators*

$$\begin{aligned} a &:= (1,3,2,4)(5,7,6,8)(9,11,10,12)(13,15,14,16),\\ b &:= (1,5)(2,6)(3,8)(4,7)(9,13)(10,14)(11,16)(12,15),\\ c &:= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16). \end{aligned}$$

We consider the difference set No. E10 from Kibler's list in [Kib78]:

$$D := \{e, a, b, a^2b, c, a^3c\}.$$

The shifts by abc, by a^2bc and by b provide the following sets:

$$\begin{aligned} D_1 &:= Dabc = \{abc, a^2bc, a^3c, ac, ab, b\},\\ D_2 &:= Da^2bc = \{a^2bc, a^3bc, a^2c, c, a^2b, ab\},\\ D_3 &:= Db = \{b, ab, e, a^2, bc, a^3bc\}. \end{aligned}$$

For all sets we have $D_i^{-1} = D_i$, $i = 1, 2, 3$ in the group H. By Corollary 3.1.28 the sets D_1, D_2 and $D_3 \setminus \{e\}$ are partial difference sets in H, where D_1 and D_2 are $(16, 6, 2, 2)$-partial difference sets and $D_3 \setminus \{e\}$ is a $(16, 5, 0, 2)$-partial difference set. If we consider the Cayley graphs over the group H which are generated by these partial difference sets, then we get the Shrikhande graph for D_1, the lattice graph $L_2(4)$ for D_2 and the Clebsch graph for D_3. Notice that in addition to the shift with b resp. abc resp. a^2bc there exist other possibilities to shift the difference set D for getting partial difference sets. Each of these shifts implies one of these three strongly regular graphs as Cayley graph.

3.3.3 Examination of vertex transitive strongly regular graphs and their automorphism groups

The main computational technique that we used to determine partial difference sets is the investigation of the automorphism groups of vertex transitive strongly regular graphs. By Theorem 3.1.23 a strongly regular Cayley graph over a group H is generated by a partial difference set in H and, conversely, each partial difference set in H generates a strongly regular Cayley graph over the group H. Thus, for the determination of all partial difference sets first we have to determine all strongly regular Cayley graphs and then to find all partial difference sets for each graph. Clearly, it is possible that we get more than one partial difference set (up to CI-equivalence) for one strongly regular graph. In the following we will present a procedure, how we can determine all partial difference sets (up to CI-equivalence) for a given strongly regular graph Γ.

1. Check whether the strongly regular graph Γ is vertex transitive.

2. If Γ is vertex transitive, then determine all regular subgroups of its automorphism group (up to conjugacy). By Proposition 3.1.19 we know that Γ is isomorphic to a Cayley graph if and only if there exists a regular subgroup of the automorphism group. From Lemma 3.1.33 it follows that non-conjugated regular subgroups correspond to partial difference sets which are not CI-equivalent.

3. For each regular subgroup from Step 2 determine the partial difference set as described in Lemma 3.1.20. The result is a complete list of non-CI-equivalent partial difference sets for the strongly regular graph Γ.

We will give an example for the above procedure.

Example 3.3.3 *Let Γ be the Schläfli graph on 27 vertices (see Section 4.1.6 for more details about this strongly regular graph). The automorphism group of Γ has order 51840. Using the computer package GAP [GAP99] we get two regular subgroups of the automorphism group (up to conjugacy). Thus, by Lemma 3.1.33 we have two partial difference sets (up to CI-equivalence). Lemma 3.1.20 yields the partial difference sets, here written in abstract notation with group generators a and b:*

group	*GAP*	*partial difference set*
$a^3 = b^3 = 1, bab = (aba)^2$	$(27, 3)$	$\{a, a^2, b, b^2, aba, ab^2a, a^2b^2a^2, a^2ba^2, ab^2a^2b, a^2b^2ab\}$
$a^9 = b^3 = 1, ba = a^4b$	$(27, 4)$	$\{a, a^3, a^6, a^8, ab, ab^2, a^2b^2, a^3b, a^5b, b^2a^6\}$

The column GAP gives the identification number of the group in the GAP catalogue for small groups (see Section 3.3.4 for more information about this group library).

Unfortunately, the knowledge about the existence of strongly regular graphs is still far away from a general classification of these objects. The same situation we have for the strongly regular Cayley graphs. Our knowledge of strongly regular graphs is restricted to some infinite series and a number of exceptional graphs. Only for graphs up to 49 vertices we have a complete catalogue of all strongly regular Cayley graphs. With these graphs we determine a complete list of partial difference sets in groups up to order 49. For strongly regular Cayley graphs with more than 49 vertices a complete determination is given only for some special parameter sets. But up to 255 vertices we have a complete catalogue of all vertex transitive strongly regular graphs with a primitive automorphism group (see Section 4.2). For these graphs we also used the above given procedure to determine almost all corresponding partial difference sets.

The missing knowledge of strongly regular Cayley graphs is one restriction for this approach to determine partial difference sets. There are other restrictions by the power of the computers and the programs.

For our computations we used the computer package GAP [GAP99], which was developed at the Rheinisch-Westfälische Technische Hochschule Aachen, and the share package GRAPE (see [Soi93]). The development of GAP (Groups, Algorithms and Programming) was started in 1986 at the Lehrstuhl D für Mathematik at the RWTH Aachen. The idea for this project came from J. Neubüser who directed the GAP-project until 1997. After his retirement the project management was transferred to the University of St. Andrews. The computer package GAP is a system for computational discrete algebra with particular emphasis on computational group theory. There are several share packages for different applications like codes, monoids and graphs. The share packe GRAPE which was created by L. Soicher is a share package for computations with graphs. It is primarily designed for constructing and analyzing graphs related to groups and finite geometries. For more information about the program GAP we refer to the GAP-websites at the University of St. Andrews (see [GAP99]).

Our task for the system GAP is to determine all regular subgroups (up to conjugacy) of the automorphism group of a given strongly regular graph. Our basis data is a list of strongly regular graphs and their automorphism groups given in GAP-format. For the computation of the regular subgroups of these groups GAP has different commands which are based on different algorithms. However, in many cases the execution of a simple GAP-command is not successful, because the automorphism groups are too large. In such situations we have to use some special tricks or methods which are based on the properties of the groups and regular subgroups. In the following we will present six methods which we used for our task.

Method 1 The simplest way to get all regular subgroups (up to conjugacy) for a given permutation group G with degree n by GAP is to compute the whole subgroup lattice and to check which representatives of the resulting conjugacy classes of subgroups act regularly. This can be done by the GAP-command LatticeSubgroups(G). GAP returns the lattice of subgroups of G from which one can compute a list of representatives of the conjugacy classes of the subgroups of G. From these representatives we only take the groups of order n and check if these act regularly. The result is a list of regular subgroups of G up to conjugacy. The Gap-commands are the following:

```
l:= LatticeSubgroups(G);;
k:=List(ConjugacyClassesSubgroups(l),Representative);;
l:=Filtered(k,x->Size(x)=n);;
regsub:=Filtered(l,IsRegular);;
```

Clearly, this bruteforce algorithm only works for groups with few subgroups. In case of subgroup lattices with too many subgroups, one gets no answer in a reasonable time resp. the system will interrupt.

For the case when G is a solvable group, GAP has the command SubgroupsSolvableGroup(G). With this command the system computes representatives of the conjugacy classes of subgroups of G. The underlying algorithm uses the so-called homomorphism principle and is described in [Hul99]. There is the option to choose some special functions which give conditions for the properties of the subgroups. If one gives an optional function for the order of the subgroups, then GAP computes all the subgroups which have this order (sometimes it can happen that GAP returns in addition some subgroups of other order!). For the determination of the regular subgroups of G with degree n one can use the following GAP-commands:

```
l:=SubgroupsSolvableGroup(G, rec(consider:=SizeConsiderFunction(n)));;
regsub:=Filtered(l,IsRegular);;
```

Method 2 A second method to compute subgroups of a given group G by GAP is the command LatticeByCyclicExtension(G). This command computes the lattice of subgroups of G using the cyclic extension algorithm, i.e., the subgroup lattice is computed by starting from the smallest subgroup and repeated extensions.

A somewhat improved method is to use this GAP-command together with a function f: LatticeByCyclicExtension(G, f). In this case in the cyclic extension algorithm all subgroups will be discarded which do not satisfy the conditions in the function f.

If one chooses a function f that checks whether the computed subgroups are semiregular and of order m, where m divides the degree of G, the program has to do less work than for the first bruteforce algorithm. Starting from the trivial subgroup the algorithm will compute semiregular subgroups of small order and extend them to semiregular subgroups of larger order. The GAP-commands are the following (n is the degree of the automorphism group G):

```
f:= function(H)
  if n mod Size(H) = 0 then return IsSemiRegular(H);
  else return false; fi;
end;
l := LatticeByCyclicExtension(G, f);;
regsub:=Filtered(l,x->Size(x)=n);;
```

The result is the list *regsub* which consists of all regular subgroups of G (up to conjugacy). Nevertheless, the improved algorithm has problems with groups of a large order. If the order of the automorphism groups is too large, one has to check if some other tricks are possible.

Method 3 The third method can be described as "computing regular subgroups with GAP step by step". It is a function which investigates elements of the right cosets of the stabilizer G_a in the permutation group (G, V), where $a \in V$. Since $G = G_a\sigma_1 \cup \ldots \cup G_a\sigma_n$, where $\sigma_1 = e$ is the identity of G and $\sigma_i \in G \setminus G_a$, $i = 2, \ldots, n$, and each coset $G_a\sigma_i$ consists of the permutations which map a onto a^{σ_i}, it is clear that if there exists a regular subgroup of G, it must contain one element of each right coset.
The function starts with computing representatives of the conjugacy classes of the elements of G. From each representative a group is generated and if this group is semiregular and the order of this group divides n, then it is joined to the list l. Now l is a list of semiregular subgroups and each regular group that is contained in G must have a conjugate which has a semiregular subgroup in l. These semiregular groups in the list l will be extended by suitable elements of the right cosets of G_1 in G (we assume that $V := \{1, \ldots, n\}$). Here "suitable" means that for each semiregular group S the new permutation maps the point 1 onto a point x which is not contained in the orbit of 1 with respect to S. The algorithm first determines such a point x and then checks which permutations of the corresponding right coset can be added to S such that S is still semiregular and has an order dividing n. This procedure is repeated until each semiregular group is regular or omitted from the list l because there was no possibility to extend it to a regular subgroup.
The described function in the language of GAP is the following:

```
regsub := function(G)
local cl,clr,s,sl,rc,rcr,l,ll,lll,n,i,j,k,gens,extended,orb,h;
cl := ConjugacyClasses(G); clr := List(cl,Representative);
s := Stabilizer(G,1); sl := AsList(s);
rc := RightCosets(G,s); rcr := List(rc,Representative);
n := LargestMovedPoint(G);
l := Filtered(clr,x->Order(x) > 1 and (n mod Order(x) = 0));
l := List(l,Group); l := Filtered(l,IsSemiRegular);
```

Now l is a list of subgroups, each of them generated by one element, with the following property: Every group in l is semiregular and for every regular subgroup of G there is a conjugate containing at least one of the groups in l.

```
while l <> [ ] and not ForAll(l,h->Size(h) = n) do
h := First(l,h->Size(h) < n);
k := Difference([1..n],Orbit(h,1))[1];
```

```
k := Position(List(rcr,x− >1^ x),k);
ll := List(sl,x− >x * rcr[k]); ll := Filtered(ll,x− > n mod Order(x) = 0);
if ll = [ ] then return [ ]; fi;
```

Now, ll is a list of permutations, each of them is element in a "suitable" right coset. In the following these permutations will be added to the semiregular groups in the list l. If the resulting groups are again semiregular and their orders divide n, then they will be joined to l.

```
lll := [ ];
for i in [1..Length(l)] do
  if Size(l[i]) = n then Add(lll,l[i]);
    else orb := Orbit(l[i],1);
    if 1^ ll[1] in orb then Add(lll,l[i]);
      else gens := GeneratorsOfGroup(l[i]);
      for j in [1..Length(ll)] do
        if n mod Order(ll[j]) = 0 then
        h := Group(Concatenation(gens,[ll[j]]));
        if IsSemiRegular(h) then Add(lll,h); fi; fi; od;
fi; fi; od;
l := lll; od; return l; end;
```

Finally, we have a list l of regular subgroups of G, which may contain groups which are conjugated in G. Hence, we have to check this by the GAP-command IsConjugate(G, H_1, H_2) for all pairs of groups H_1, H_2 in l and to omit the conjugates of each group.

Method 4 The fourth method is a special case of Method 2. In Method 2 semiregular subgroups of a given group G are computed by the cyclic extension algorithm. At the end the semiregular subgroups of order n are the regular subgroups. In the case of large groups or groups which have many semiregular subgroups, this algorithm needs very much time or is interrupted because of lack of working memory. However, for the computational verification that no regular subgroup exists in a given group G, in many cases it is not necessary to compute all semiregular subgroups up to order n. If we assume that in G a regular subgroup H of order n exists, then we know that there also exists a Sylow p-subgroup S of H which has order $p^k \leq n$ for a suitable prime number p and suitable $k \in \mathbb{N}$. The group S must be semiregular, because it is a subgroup of the regular group H. Consequently, if we want to show that no regular subgroup H of G exists, it is sufficient to show that no such semiregular group S of G exists. Since S is a p-group, it is sufficient to check that no such group S exists in the Sylow p-subgroup of G. If the order of S is smaller than the order of H, the cyclic extension algorithm which starts at the smallest group, needs less time for this method than for the other methods. Clearly, this method only makes sense if H is no p-group (because then $H = S$). For a group G of degree $n = up^k$ where $p \nmid u$ and p is a suitable prime number we have the GAP-commands:

```
T:=SylowSubgroup(G,p);;
f:= function(H)
  if p^k mod Size(H) = 0 then return IsSemiRegular(H);
  else return false; fi;
end;
l := LatticeByCyclicExtension(T, f);;
semiregsub:=Filtered(l,x− >Size(x)=p^k);;
```

If the list *semiregsub* is empty, then a regular subgroup of G does not exist.

Method 5 In cases were we have to investigate automorphism groups G of degree p^n, where p is a prime number, we can simplify the computations. In these cases first we compute a Sylow p-subgroup S of G by the GAP-command S:=SylowSubgroup(G, p);;. Notice that all Sylow p-subgroups are conjugated in G (by the Sylow theorems), hence, it is sufficient to consider an arbitrary Sylow p-subgroup. Then we compute all regular subgroups of S with Method 1, Method 2 or Method 3. Afterwards, we have to check, if the regular groups which are non-conjugated in S are conjugated in G. This can be done by the GAP-command IsConjugate(G, H_1, H_2) (where H_1, H_2 are non-conjugated regular subgroups of S).

Method 6 If we have some knowledge about the existence of a normal subgroup N_H of each possible regular subgroup H of a group G, then we know that $H \leq \mathcal{N}_G(N_H)$, i.e., H is a subgroup of the normalizer $\mathcal{N}_G(N_H)$ of the group N_H in G. Hence, it is sufficient to do the computations in the normalizer $\mathcal{N}_G(N_H) \leq G$ which is hopefully smaller than the group G. This method is very useful because in many cases we know that in regular subgroups H of G a normal subgroup N_H exists.
As an example consider a group G with degree 100. Each regular subgroup H of G has order 100. Since $100 = 2^2 \cdot 5^2$ we can use the following lemma:

Lemma 3.3.4 *Let H be a group of order p^2q^2, where p, q are distinct primes. Then a Sylow p-subgroup of H is normal if $p > q^2$.*

Proof: By the respective Sylow theorem a Sylow p-subgroup of H is normal, if the number of Sylow p-subgroups n_p in H is one. Moreover, we know that $n_p \mid q^2$ and $n_p \equiv 1 \mod p$. From these conditions it follows that $n_p \in \{1, q, q^2\}$ and $p \mid (n_p - 1)$. Since $p > q^2$ we get $n_p = 1$, i.e., we have exactly one Sylow p-subgroup of H which, consequently, is normal. □

The lemma grants that a Sylow 5-subgroup (of order 25) of each possible regular subgroup H (of order 100) of G is normal in H. Thus, we can compute a Sylow 5-subgroup S of G and determine all semiregular subgroups $S_1, \ldots, S_k$ of S of order 25 (for example by Method 2). Now, each regular subgroup H of G has a conjugate which contains one of these semiregular groups $S_1, \ldots, S_k$ as normal subgroup. For each of these semiregular groups we compute the normalizer $N_i := \mathcal{N}_G(S_i)$ by the GAP-command N_i:=Normalizer(G, S_i);; and determine all regular subgroups of order 100 for each N_i, $i = 1, ..., k$ (by Method 1, Method 2 or Method 3). Finally, with the GAP-command IsConjugate we have to check if the resulting regular subgroups are conjugated in G.

After the calculation of all non-conjugated regular subgroups of a given group, by Lemma 3.1.20 we can determine all partial difference sets up to CI-equivalence. Here we also used the computer package GAP. A special algorithm which we created for GAP provided one partial difference set for each regular subgroup (see Appendix A). Moreover, with a function IsPds the algorithm checked again if the computed result has the property of a partial difference set. Then it was directly saved in a format which was readable for our LaTex-editor. Thus, all results which appear in this thesis were checked by a second function and we tried to minimize the possibility for mistake by the "human factor".

3.3.4 Presentation of groups

In the present work a complete list of partial difference sets in groups up to order 49 is given. Moreover, partial difference sets for strongly regular graphs with primitive automorphism group and less than 256 vertices are determined. Hence, we have to deal with groups up to order 255 and the question of the presentation of these groups must be discussed.
In general, for the presentation of the groups and for the partial difference sets we give an abstract description in form of generators (like in Example 3.3.3). Nevertheless, in addition to this it is convenient to use the small group library of the computer package GAP [GAP99] which yields group identification numbers. In this library we have all isomorphism classes of groups up to order 1000 (with some exceptions), each isomorphism class is determined by a group identification number. The identification of a group G is possible with the GAP-command $\mathsf{IdGroup}(G)$. GAP returns a pair of numbers $(order, No.)$, where the order of the group is given and the number in the group library of GAP. Conversely, one can get representatives of groups by the GAP-command $\mathsf{SmallGroup}(order, No.)$. For example, $\mathsf{SmallGroup}(9, 2)$ is a group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.
The generators of the groups for which we determined partial difference sets are also determined by a GAP command. For a group G one has to take the command $\mathsf{GeneratorsOfGroup}(G)$. Here we have to mention that this generating set is sometimes much larger than necessary, such that we omitted redundant generators.

3.3.5 Correctness of results

Part of the results in this thesis is obtained with the aid of computers. Hence, the natural question arises whether these results are correct. Certainly, a computational result is not a proof and only a complete theoretical description of all partial difference sets we obtained by our computational approach can yield such a proof. However, the reason for the computer based determination of certain partial difference sets is the fact that, presently, a theoretical description is too tedious. Hopefully, for a number of the computer results a theoretical description will be found in the future. First promising attempts are done (e.g., for the Schläfli graph, see Section 4.1.6) .
A confirmation of a computational result (not a proof!) can be obtained by an independent verification with a computer approach based on different algorithms. This means, if a result

is obtained independently by another computational approach which is based on different algorithms, then we will consider this result as correct. This claim is due to C. W. H. Lam [KolLT90], [Lam91].
For some of our results there are different possibilities for a confirmation. As mentioned before, F. Fiedler has determined all Schur rings over groups of order less than 32 (up to a certain equivalence) [Fie98], i.e., in particular, he has determined partial difference sets in these groups. Due to his approach we can extract from this results all primitive, symmetric Schur rings of rank 3 in groups of order less than 32 up to isomorphism of the corresponding strongly regular graphs, i.e., for each group each such Schur ring corresponds to a different strongly regular graph. With these results we can confirm our computational results for partial difference sets for the Schläfli graph and most of the results for the strongly regular graphs with 16 vertices (all other partial difference set for strongly regular graphs up to 31 vertices are determined without the computer).
As we already mentioned in Section 3.3.2 there is the possibility to determine complete lists of certain partial difference sets by difference sets. We find complete lists of difference sets with parameters (16,6,2) and (36,15,6) in [Kib78]. We can use this information for the confirmation of our results for partial difference sets with parameters (16,6,2,2), (16,5,0,2), (36,15,6,6), (36,14,4,6) which we determined by regular subgroups of the automorphism groups of the associated strongly regular Cayley graphs. Moreover, we can confirm the non-existence result for (40,12,2,4)-partial difference sets by the complete list of (40,13,4)-difference sets. We will describe these cases in detail in the next chapter.

3.3.6 Explanation and interpretation of computational results

As described in the previous section the correctness of mathematical results obtained by computers must be carefully discussed. However, computational results often provide a deeper understanding of certain properties of mathematical objects. The discovering of new properties or even new examples of mathematical objects by the computer gives the possibility to *explain* this computational results afterwards. Sometimes, computational results give new ideas which even provide a complete theoretical description, i.e., an *interpretation* of the results.
In the present thesis partial difference sets are determined with the aid of the computer package GAP. As described before some of the results can also be obtained without computers. However, in many cases a proof of the existence of certain partial difference sets is unknown. The question arises how these "new" partial difference sets resp. the existence of regular subgroups of the automorphism groups of the corresponding strongly regular graphs can be explained or even interpreted. For some cases there are first approaches. Examples are the Schläfli graph (see Section 4.1.6) and the six strongly regular Cayley graphs with parameters (36,14,4,6) and (36,15,6,6) (see Sections 4.1.6 and 4.1.5). For these cases an interpretation resp. explanation of the computational results were obtained by M. H. Klin and the author a short time before submitting this thesis. A detailed description will be prepared in nearest future.

Chapter 4

Determination of partial difference sets II: Results

4.1 Determination of small partial difference sets

4.1.1 A brief survey of all known strongly regular graphs up to 49 vertices

A systematical investigation of strongly regular graphs was started in 1963 in a paper by R. C. Bose [Bos63]. After this starting point strongly regular graphs have become a popular area in algebraic combinatorics during the last decades. As mentioned in the introduction the discovering of unknown strongly regular graphs is one of the most challenging tasks for mathematicians working on this field. A brief survey of known strongly regular graphs can be found in [Bro96]. Notice that since 1996 new graphs were discovered.
The main source for the determination of small partial difference sets in the present work is the complete list of strongly regular graphs from E. Spence. On his website E. Spence gives a list which is complete for strongly regular graphs up to 36 vertices and those with 40 vertices [Spe01]. Moreover, he gives complete lists of strongly regular graphs with 45 and 64 vertices for certain parameter sets. Besides this main source we used some other sources. For example, in the case of strongly regular graphs with 35 and 36 vertices we used information about the determination of these graphs which is presented in [Spe95] and [BusMS81].
In Table 4.1 on Page 80 we list all feasible parameters for strongly regular graphs up to 49 vertices, the number of these graphs (# srg) and of vertex transitive graphs (# vertex trans. srg), and further information. Most of the vertex transitive graphs are introduced in Section 3.2; some graphs are derived from two-graphs (see Section 2.1.3) and in the non-existence case we have conditions described in Section 3.1.1 resp. the connection to the non-existence of a certain conference matrix (see [vLiW92], p. 235 and Chapter 18). For each parameter set there exists a parameter set of the complementary graph (cf. Remark 3.1.2). We dispense with the presentation of these parameters to keep the table as compact as possible.

Table 4.1: General information about strongly regular graphs.

n	k	λ	μ	# srg	# vertex trans. srg	vertex transitive graphs / comments
5	2	0	1	1	1	$P(5)$
9	4	1	2	1	1	$L_2(3) \cong P(9)$
10	3	0	1	1	1	Petersen ($\overline{T(5)}$)
13	6	2	3	1	1	$P(13)$
15	6	1	3	1	1	$T(6)$
16	5	0	2	1	1	Clebsch graph
16	6	2	2	2	2	$L_2(4)$, Shrikhande graph
17	8	3	4	1	1	$P(17)$
21	10	3	6	1	1	$\overline{T(7)}$
21	10	4	5	-	-	Conference
25	8	3	2	1	1	$L_2(5)$
25	12	5	6	15	1	$L_3(5) \cong P(25)$
26	10	3	4	10	-	
27	10	1	5	1	1	Schläfli graph
28	9	0	4	-	-	Krein condition, absolute bound
28	12	6	4	4	1	$T(8)$
29	14	6	7	41	1	$P(29)$
33	16	7	8	-	-	Conference
35	16	6	8	3854	1	from a two-graph
36	10	4	2	1	1	$L_2(6)$
36	14	4	6	180	4	from two-graphs
36	14	7	4	1	1	$T(9)$
36	15	6	6	32548	4	from two-graphs (3 $L_3(6)$-graphs)
37	18	8	9	≥ 6760	1	$P(37)$
40	12	2	4	28	2	no specified name
41	20	9	10	≥ 1	1	$P(41)$
45	12	3	3	78	-†	
45	16	8	4	1	1	$T(10)$
45	22	10	11	≥ 1	-†	
49	12	5	2	1	1	$L_2(7)$
49	18	7	6	≥ 1	$1^\dagger$	$L_3(7)$
49	24	11	12	≥ 2	$2^\dagger$	$L_4(7)$, $P(49)$

† number of strongly regular Cayley graphs.

All strongly regular graphs with 25, 26 and 28 vertices and the orbits of their automorphism groups are given in [Ros76].
In the case of 37, 41, 45, 49 vertices there still does not exist a complete list of strongly regular graphs for each parameter set (for the parameter set $(37, 18, 8, 9)$ B. McKay gives the number of 6760 strongly regular graphs on his website [McK01]). However, by Theorem 3.2.8 we know, that any vertex transitive strongly regular graph with p vertices, p a prime and $p \equiv 1 \mod 4$, is isomorphic to the Paley graph $P(p)$ which has exactly one partial difference set (in $\mathbb{Z}_p$). By Proposition 3.2.7 the partial difference set of the Paley graph $P(p)$ is the only partial difference set in a group of order p. Thus, for 29, 37 and 41 we have exactly one vertex transitive graph. In the case of p^2 vertices, p a prime, we can determine the number of strongly regular Cayley graphs theoretically (see Section 3.2.3). Hence, in the case of 49 vertices we give the number of strongly regular Cayley graphs.
For the graphs with parameters $(45, 12, 3, 3)$ E. Spence announced on his website [Spe01] that he has created a complete catalogue of these graphs. In the case of the strongly regular graphs with parameters $(45, 22, 10, 11)$ we have no information. Nevertheless, in both cases there does not exist any strongly regular Cayley graph: By Proposition 3.1.26 the existence of strongly regular Cayley graphs over a group H is equivalent to the existence of a primitive, symmetric Schur ring over H of rank 3. By Theorem 3.2.16 such a Schur ring does not exist, if H is abelian, H is not of prime order and H has a cyclic Sylow subgroup. It is clear that each group of order 45 has a cyclic Sylow 5-subgroup and that 45 is not a prime. Moreover, we can show that all groups of order 45 are abelian: By the Sylow theorems a group of order 45 has a normal Sylow 5-subgroup and a normal Sylow 3-subgroup which both satisfy the conditions of Proposition 2.1.10. Hence, each group of order 45 is the direct product of its Sylow 5-subgroup and its Sylow 3-subgroup. Since by Lemma 2.1.7 there are two options for the Sylow 3-subgroup, there exist exactly two non-isomorphic groups of order 45, namely $\mathbb{Z}_{45}$ and $\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Both are abelian and, thus, there does not exist a strongly regular Cayley graph with 45 vertices.
The vertex transitive strongly regular graphs with 40 vertices were determined from the complete list of E. Spence [Spe01] using GAP.

In the following we will determine all partial difference sets for each Cayley graph in Table 4.1. Since we restricted the parameters (v, k, λ, μ) of the strongly regular graphs in the table by the condition $k \leq \frac{v}{2}$, i.e., we did not give the complementary graphs (except the case of self-complementary graphs), the determination of (v, k, λ, μ)-partial difference sets is also restricted by this condition. However, it is easy to see that each partial difference set D in a group H corresponds to a partial difference set $(H \setminus D) \setminus \{e\}$, and this is exactly the "complementary" partial difference set (cf. Propositions 3.1.27 and 3.1.26). In the tables of partial difference sets we give in this thesis we do not list the "complementary cases".
The small partial difference sets are determined theoretically if possible. In the other cases the partial difference sets were determined by a computational approach. In these cases we computed the regular subgroups of each automorphism group (up to conjugacy) once with Method 1 and as verification with Method 2. The complete list of partial difference sets is given in Appendix D. The groups are described in Appendix C.

4.1.2 Vertex transitive strongly regular graphs which do not have partial difference sets

As described in Section 3.1.3 vertex transitive graphs cannot be constructed as Cayley graphs in general: By Proposition 3.1.19 we know that only strongly regular graphs whose automorphism group contains a regular subgroup are isomorphic to Cayley graphs and, hence, have a partial difference set.
With the results of Chapter 3 we can easily determine some vertex transitive strongly regular graphs of Table 4.1 which do not have partial difference sets.

Triangular graphs

From Theorem 3.2.14 we get that the automorphism group of the triangular graph $T(n)$, $n \geq 5$, has a regular subgroup if and only if n is a prime power and $n \equiv 3 \bmod 4$. Thus, the graphs $T(5), T(6), T(8), T(9)$ and $T(10)$ have no partial difference set because their automorphism groups have no regular subgroups.

The vertex transitive strongly regular graph with parameters (35,16,6,8)

There is exactly one vertex transitive strongly regular graph with parameters $(35, 16, 6, 8)$. It has an automorphism group of order 40320 (cf. [Spe95]). This graph can be constructed as a descendant from a two-graph (cf. [BusMS81]), however, we can also get this graph by the induced action of the symmetric group S_7 on the 3-subsets of the set $\{1, \ldots, 7\}$, i.e., the action of the permutation group $(S_7, \{{7 \atop 3}\})$. Here we take the 3-subsets as vertices and two vertices are adjacent if and only if they are disjoint or have two common elements (in other words: we consider the centralizer ring $\langle \psi_0, \ldots, \psi_3 \rangle$ of the group $(S_7, \{{7 \atop 3}\})$, where the vertices $A, B \in \{{7 \atop 3}\}$ are adjacent in the basis graph ψ_i if and only if $|A \cap B| = i$, $i = 0, \ldots, 3$. Then the edge set of the strongly regular graph is the union of the edge sets of ψ_0 and ψ_2, see [KliPR88], Chapter 3.4).
The automorphism group of this graph is determined by the following theorem:

Theorem 4.1.1 *(cf. [FarKM94], Theorem 3.2.2 b)*
If $n \geq 3$ and $v = 2n+1$, then there exists only one nontrivial group which contains the group $(S_v, \{{v \atop n}\})$ as subgroup, and this group is isomorphic to S_{v+1}.

Thus, the automorphism group of the strongly regular graph is isomorphic to the symmetric group S_8 which has order 40320.

For this strongly regular graph we can proof in two different ways that there is no partial difference set.
By Proposition 3.1.27 the existence of a partial difference set over a group H is equivalent to the existence of a symmetric, primitive Schur ring over H of rank 3. Since each group H of order 35 is cyclic (Lemma 2.1.8), it is also abelian and with Theorem 3.2.16 one can

show that there exists no primitive, non-trivial Schur ring over H. Thus, we have no partial difference set in this case.
Furthermore, we can prove the following for the automorphism group of the strongly regular graph with parameters (35,16,6,8):

Remark 4.1.2 *The symmetric group S_8 has no subgroup of order 35.*

Proof: Suppose there exists a subgroup H of S_8 with order 35. Then H is cyclic and hence, abelian (Lemma 2.1.8). There exists an element $h \in H$ with order 7 and $h = (a_1, \ldots, a_7)$, $1 \leq a_1 < \ldots < a_7 \leq 8$. Since H is abelian, we have for the centralizer of h: $C_H(h) = H$ and $C_H(h) \leq C_{S_8}(h)$, i.e., $|C_{S_8}(h)| \geq 35$.
On the other hand, we know that

$$|C_{S_8}(h)| = j_1!1^{j_1} \cdot j_2!2^{j_2} \cdot \ldots \cdot j_8!8^{j_8},$$

where j_k is the number of cycles of length k in the permutation h, $1 \leq k \leq 8$ (cf. [KliPR88], Theorem 2.4.6 and 2.4.7). We get $|C_{S_8}(h)| = 1! \cdot 7^1 = 7$ in contradiction to $|C_{S_8}(h)| \geq 35$.
□

From the remark it follows that there does not exist a subgroup of order 35 in the automorphism group of the strongly regular graph. Thus, there is no regular subgroup in the automorphism group and for this strongly regular graph we cannot get a partial difference set.

There are some other vertex transitive graphs in Table 4.1 which are not isomorphic to Cayley graphs. For the verification we took the computer package GAP and got the following results:

Strongly regular graphs with parameters (36,14,4,6)

There are four vertex transitive strongly regular graphs with parameters (36,14,4,6). Three of them are elements in switching classes from latin square type two-graphs, one is from a Steiner type two-graph (cf. [BusMS81] for details). Only the graph of the switching class of the Steiner type two-graph has no regular subgroup. Its automorphism group of order 12096 is isomorphic to the group $P\Gamma U(3,3^2)$ and by GAP no regular subgroup of order 36 exists.

Strongly regular graphs with parameters (36,15,6,6)

Like in the preceding case we have four vertex transitive strongly regular graphs. Two of them are latin square graphs over groups, i.e., by Proposition 3.2.41 their automorphism group contains a regular subgroup and they are isomorphic to Cayley graphs.
Moreover, there is another latin square graph over a quasigroup. A computation with GAP yields the existence of a regular subgroup of its automorphism group.
The fourth vertex transitive strongly regular graph with parameters (36,15,6,6) is not of latin square type. It is from the switching class of a Steiner type two-graph. The automorphism group is isomorphic to the group $PSp(4,3):2$ of order 51840. By GAP this group has no regular subgroup of order 36.

Since the three vertex transitive strongly regular graphs with parameter sets (35,16,6,8), (36,14,4,6) and (36,15,6,6) which are not isomorphic to Cayley graphs are all connected to the same Steiner type two-graph (cf. [BusMS81]), it remains the question, if there is a more unified way to get the described results.

Strongly regular graphs with parameters (40,12,2,4)

There are two vertex transitive graphs with parameters (40,12,2,4). With GAP we get that the corresponding automorphism groups have no regular subgroups. Thus, by Proposition 3.1.19 it follows that all strongly regular graphs with parameters (40,12,2,4) are not isomorphic to Cayley graphs, i.e., there is no partial difference set with parameters (40,12,2,4).
This result can be confirmed in another way: In the complete catalogue of difference sets from Kibler [Kib78] we find exactly one difference set with parameters $(40, 13, 4)$. It is in the group $H := \langle a, b\rangle$ with $a^5 = b^8 = e, ba = a^4 b$. This group has the identification number $(40, 1)$ in the small group library of GAP [GAP99]. If we assume that a $(40, 12, 2, 4)$-partial difference set D in a group K exists, then it is easy to show that $D' := D \cup \{e\}$ is a non-regular $(40, 13, 4, 4)$-partial difference set in K. Since D' as difference set must be equivalent to the $(40, 13, 4)$-difference set in [Kib78], the group K is isomorphic to the group H defined above. Conversely, as described in Section 3.3.2 it is possible to determine all $(40, 12, 2, 4)$-partial difference sets by the known $(40, 13, 4)$-difference set. However, computations with GAP show that we do not get any $(40, 12, 2, 4)$-partial difference set by this difference set.

4.1.3 Computer-free determination of partial difference sets

For several classes of strongly regular graphs it is possible to determine the partial difference sets on a non-computational level.

Paley graphs

In Table 4.1 we have several Paley graphs. By Proposition 3.2.6 the Paley graph $P(p^n)$ with $p^n \equiv 1 \bmod 4$, $p \nmid n$, p a prime, has exactly one partial difference set. This partial difference set consists of the nonzero squares of $\mathbb{F}_q$. Hence, we get all partial difference sets for the Paley graphs in Table 4.1.

Strongly regular graphs with p^2 vertices

In Section 3.2.3 the determination of all partial difference sets for strongly regular graphs with p^2 vertices, p a prime, was discussed. These partial difference sets are always partial difference sets of groups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. They can be represented by mergings of i-subsets of the set $\{O_1, ..., O_{p+1}\}$, $2 \leq i \leq \frac{p+1}{2}$, where

$$O_1 = \{(0, a)|a \in \mathbb{Z}_p^*\}, O_{k+2} = \{(a, ka)|a \in \mathbb{Z}_p^*\}, k \in \mathbb{Z}_p.$$

Moreover, we know by Corollary 3.2.23 that all partial difference sets we get by merging of two resp. three subsets are CI-equivalent. Hence, for each graph in Table 4.1 with 9 resp. 25 vertices we have exactly one partial difference set (up to CI-equivalence). The partial difference sets in the group $\mathbb{Z}_7 \times \mathbb{Z}_7$ were discussed in Example 3.2.24.

Triangular graphs

The partial difference sets generating the triangular graphs $T(n)$ as Cayley graphs were described in Section 3.2.2. There exist partial difference sets if and only if n is a prime power and $n \equiv 3 \bmod 4$. By Proposition 3.2.15 the partial difference sets are unique up to CI-equivalence if n is a prime number. For the triangular graphs in Table 4.1 we have only for $T(7)$ a partial difference set. It is a subset of the group $\mathbb{Z}_7 \rtimes K$, where $K := \{1, 2, 4\}$ is a subgroup of $\mathbb{Z}_7^*$. With Theorem 3.2.14 we get the partial difference set

$$D = \{(1,1),(1,6),(2,0),(2,1),(2,5),(2,6),(4,0),(4,1),(4,3),(4,4)\}$$

for $T(7)$.

4.1.4 Partial difference sets for $L_2(n)$-type graphs

In Section 3.2.3 we gave a complete description of all partial difference sets for the lattice graphs $L_2(n)$ when n is a prime. For each prime number n there exists exactly one partial difference set (up to CI-equivalence) for $L_2(n)$.
However, in the case when n is not a prime the situation is much more difficult. We used the computer package GAP to solve the problem of determining all partial difference sets for the strongly regular graphs $L_2(4), L_2(6)$ and the Shrikhande graph.

The graph $L_2(4)$

The lattice graph $L_2(4)$ has parameters (16,6,2,2) and its automorphism group is isomorphic to the exponentiation $S_4 \uparrow S_2$ of order 1152 (cf. Section 3.2.5). Furthermore, we know by Proposition 3.2.47 that in each of the groups $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$ and $(\mathbb{Z}_2)^2 \times (\mathbb{Z}_2)^2$ exists at least one partial difference set for $L_2(4)$ (these are CI-equivalent to the partial difference sets L_1, L_9, L_{13} in Table 4.2 on Page 86). From the parameter set (16,6,2,2) we immediately see that all partial difference sets from this graph are of special form, because they are also (16,6,2)-difference sets (cf. Section 3.1.3).
Computations with GAP give 13 regular subgroups (up to conjugacy) of the automorphism group of $L_2(4)$. By Lemma 3.1.20 we get 13 partial difference sets associated to $L_2(4)$. In Table 4.2 the 13 regular subgroups of $Aut(L_2(4))$ and the associated partial difference sets are given in an abstract description by generators. The list of 13 partial difference sets was verified by the construction of $(16, 6, 2, 2)$-partial difference sets from $(16, 6, 2)$-difference sets. From [Kib78] we know all $(16, 6, 2)$-difference sets. As described in Section 3.3.2 we can create all $(16, 6, 2, 2)$-partial difference sets from this list. We get exactly 13 partial difference sets (up to CI-equivalence) which generate the lattice graph $L_2(4)$ and which correspond to the partial difference sets in the table.

Table 4.2: Regular subgroups of the automorphism group of the lattice graph $L_2(4)$ (up to conjugacy) and their partial difference sets.

name	abstract description of group	GAP	partial difference sets
L_1	$a^4 = b^4 = 1$ abelian	(16,2)	$\{ab^3, a^2b, a^2b^2, a^2b^3, a^3b, b^2\}$
L_2	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	(16,3)	$\{a^2, ab, a^2c, a^2c^3, a^3b, b\}$
L_3	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	(16,3)	$\{a^2, abc, a^2bc, a^2c, a^3bc, b\}$
L_4	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	(16,3)	$\{ab, abc, a^3b, a^3c, bc, c\}$
L_5	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	(16,3)	$\{a^2, ab, abc, a^2b, a^3b, a^3c\}$
L_6	$a^4 = b^4 = 1,\ ba = a^3b$	(16,4)	$\{ab^2, a^2, a^3b, a^3b^2, a^3b^3, b^2\}$
L_7	$a^8 = 1,\ b^2 = a^2,\ ba = a^5b$	(16,6)	$\{a^3, a^5, ab, a^5b, b^3, b^5\}$
L_8	$a^8 = 1,\ b^2 = a^4,\ ba = a^3b$	(16,8)	$\{ab^3, a^2b, a^2b^3, a^3b, a^3b^2, a^5b^2\}$
L_9	$a^4 = b^2 = c^2 = 1$ abelian	(16,10)	$\{a^2, ac, a^2bc, a^2c, a^3c, b\}$
L_{10}	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	(16,11)	$\{a^2b, a^3b, a^2c, abc, a^2bc, c\}$
L_{11}	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	(16,11)	$\{a^2, abc, a^2b, a^2c, a^3bc, bc\}$
L_{12}	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	(16,11)	$\{a^2, ab, ac, a^2c, a^3c, a^3bc\}$
L_{13}	$a^2 = b^2 = c^2 = d^2 = 1$ abelian	(16,14)	$\{abcd, ac, acd, ad, bd, c\}$

The first column gives a name for each partial difference set. In all tables the column GAP gives the identification number of the groups in the SmallGroup Library of GAP4 [GAP99].

Notice that L_2, L_3, L_4, L_5 are partial difference sets in the same group, but they are not CI-equivalent, because this group has four non-conjugated representations which are regular subgroups of $Aut(L_2(4))$. An analogue situation we have for L_{10}, L_{11}, L_{12}. Moreover, we get the following remark with the aid of GAP:

Remark 4.1.3 *The partial difference sets L_{11} and L_{12} in Table 4.2 are equivalent as difference sets, i.e., difference equivalent. They are clearly srg-equivalent as partial difference sets but not CI-equivalent.*

The different equivalence relations are described in Section 3.1.4.

The Shrikhande graph

As described in Proposition 3.2.46 the Shrikhande graph is an exceptional graph: It is the only graph in the class of $L_2(n)$-type graphs that is not a lattice graph. For this reason it is also called a pseudolattice. The Shrikhande graph is the complement of the latin square graph $L_3(\mathbb{Z}_4)$ (cf. Example 3.2.39). It has an automorphism group of order 192. The group order can be explained by the description of the automorphism group of a latin square graph given in Remark 3.2.40.
Since we can describe the Shrikhande graph as the complement of a latin square graph over the group $\mathbb{Z}_4$, we know by Proposition 3.2.41 that there is at least one partial difference set (this is CI-equivalent to Sh1 in Table 4.3 on Page 87). As for the lattice graph $L_2(4)$ all partial difference sets of the Shrikhande graph are also (16,6,2)-difference sets.

Table 4.3: Regular subgroups of the automorphism group of the Shrikhande graph (up to conjugacy) and their partial difference sets.

name	abstract description of group	GAP	partial difference set
Sh_1	$a^4 = b^4 = 1$ abelian	(16,2)	$\{a, a^3, ab^3, a^3b, b, b^3\}$
Sh_2	$a^8 = 1$, $b^2 = a^2$, $ba = a^5b$	(16,6)	$\{a, a^7, a^3b, a^3b^5, b, b^7\}$
Sh_3	$a^8 = 1$, $b^2 = a^2$, $ba = a^5b$	(16,6)	$\{a, a^2, a^6, a^7, b^3, b^5\}$
Sh_4	$a^8 = 1$, $b^2 = a^4$, $ba = a^3b$	(16,8)	$\{a, a^7, a^3b, a^5b, b, b^3\}$
Sh_5	$a^8 = 1$, $b^2 = a^4$, $ba = a^3b$	(16,8)	$\{a, a^2, a^6, a^7, a^5b, a^7b\}$
Sh_6	$a^4 = b^2 = c^2 = 1$, $bab = a^3$, $ac = ca$, $cb = bc$	(16,11)	$\{a, a^3, a^2bc, a^3b, a^3bc, b\}$

With the aid of GAP we computed all regular subgroups (up to conjugacy) of the automorphism group of the Shrikhande graph and by Lemma 3.1.20 we get the associated partial difference sets. In Table 4.3 the groups and partial difference sets are given in an abstract description by generators. Like for the lattice graph $L_2(4)$ all these partial difference sets where verified by the determination of $(16, 6, 2, 2)$-partial difference sets by $(16, 6, 2)$-difference sets. Notice that the partial difference sets Sh_2, Sh_3 resp. Sh_4, Sh_5 are not CI-equivalent

In Remark 4.1.3 we already gave a first example for two partial difference sets that are not CI-equivalent but srg-equivalent and difference equivalent (see Section 3.1.4 for the definitions). Below, we give some further interesting examples. Since the two partial difference sets in Remark 4.1.3 generate isomorphic Cayley graphs, i.e., they are srg-equivalent, we have in the following remark pairs of partial difference sets which are difference equivalent and generate non-isomorphic Cayley graphs, i.e., they are not srg-equivalent.

Remark 4.1.4 *The partial difference sets of the following groups from Table 4.2 and Table 4.3 are difference equivalent but not srg-equivalent:*

group	*GAP*	*partial difference set*	*KiblerNo.*
L_7	$(16, 6)$	$\{a^3, a^5, ab, a^5b, b^3, b^5\}$	$J23$
Sh_3	$(16, 6)$	$\{a, a^2, a^6, a^7, b^3, b^5\}$	
L_8	$(16, 8)$	$\{ab^3, a^2b, a^2b^3, a^3b, a^3b^2, a^5b^2\}$	$K25$
Sh_4	$(16, 8)$	$\{a, a^7, a^3b, a^5b, b, b^3\}$	
L_{10}	$(16, 11)$	$\{a^2b, a^3b, a^2c, abc, a^2bc, c\}$	$E10$
Sh_6	$(16, 11)$	$\{a, a^3, a^2bc, a^3b, a^3bc, b\}$	

The column Kibler No. gives the number of the corresponding difference set in the catalogue of Kibler [Kib78].

The graph $L_2(6)$

The lattice graph $L_2(6)$ with parameter set (36,10,4,2) has an automorphism group of order 1036800 which is isomorphic to $S_6 \uparrow S_2$. Furthermore, we know by Proposition 3.2.47 that

in each of the groups $\mathbb{Z}_6 \times \mathbb{Z}_6$, $\mathbb{Z}_6 \times S_3$ and $S_3 \times S_3$ exists at least one partial difference set for $L_2(6)$ (these are the partial difference sets labeled with * in Table 4.4).
Like for the Shrikhande graph and the graph $L_2(4)$ we computed all regular subgroups of the automorphism group $Aut(L_2(6))$(up to conjugacy) and determined the corresponding partial difference sets. With the aid of GAP we found 16 regular subgroups (up to conjugacy), thus, we have 16 non-CI-equivalent partial difference sets with parameter set (36,10,4,2) (see Table 4.4). Notice that we can divide these 16 groups in six isomorphism classes.

Table 4.4: Regular subgroups of the automorphism group of the lattice graph $L_2(6)$ (up to conjugacy) and their partial difference sets.

GAP	abstract description of group / partial difference sets
$(36,6)$	$a^3 = b^3 = c^4 = 1, ab = ba, ca = ac, cb = b^2c$
	$\{a^2b^2, ab, c^3, a^2bc^3, ab^2c^3, a^2b, ab^2, c, a^2b^2c, abc\}$
$(36,9)$	$a^3 = b^3 = c^4 = 1, ab = ba, ca = bc, cb = a^2c$
	$\{b, b^2, c^3, ac^3, a^2c^3, a, a^2, c, bc, b^2c\}$
$(36,10)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, cb = bc, ca = a^2c, db = b^2d$
	$\{a, a^2, d, ad, a^2d, b, b^2, cd, b^2cd, bcd\}$
	$\{a, a^2, b^2d, ab^2d, a^2b^2d, a^2b, ab^2, b^2cd, abcd, a^2cd\}$
	$\{a, a^2, c, a^2c, ac, b, b^2, cd, b^2cd, bcd\}$
*	$\{a, a^2, c, a^2c, ac, b, b^2, d, b^2d, bd\}$
	$\{a, a^2, c, a^2c, ac, a^2b, ab^2, cd, ab^2cd, a^2bcd\}$
	$\{a, a^2, bd, abd, a^2bd, b, b^2, c, bc, b^2c\}$
$(36,12)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, db = bd, ca = a^2c, cb = bc$
	$\{d, ab^2, ab^2d, a^2b, a^2bd, b, b^2, c, bc, b^2c\}$
	$\{d, a^2b, a^2bd, ab^2, ab^2d, a, a^2, c, ac, a^2c\}$
	$\{d, b, bd, b^2, b^2d, a, a^2, c, a^2c, ac\}$
	$\{a, a^2, c, a^2c, ac, b, b^2, cd, bcd, b^2cd\}$
*	$\{d, a, ad, a^2, a^2d, b, b^2, c, bc, b^2c\}$
$(36,13)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, db = bd, ca = a^2c, cb = b^2c$
	$\{d, a, ad, a^2, a^2d, b, b^2, c, b^2c, bc\}$
	$\{a, a^2, c, a^2c, ac, b, b^2, cd, b^2cd, bcd\}$
$(36,14)$	$a^3 = b^3 = c^2 = d^2 = 1,$ abelian
*	$\{c, a, ac, a^2, a^2c, d, b, bd, b^2, b^2d\}$

4.1.5 Partial difference sets for other latin square type graphs

Besides the lattice graphs and the pseudolattice graph there are some other latin square type graphs in our list which have partial difference sets. These are the latin square graphs $L_3(5)$, $L_3(6)$ and $L_3(7)$ and the latin square type graphs $L_4(7)$.
Since five and seven are prime numbers, the job for $L_3(5)$, $L_3(7)$ and $L_4(7)$ has already been done in Section 3.2.3: We have exactly one graph for $L_3(5)$ resp. $L_3(7)$ and two graphs of

type $L_4(7)$ (one is $P(49)$). For each graph we get exactly one partial difference set (up to CI-equivalence).
For the case $L_3(6)$ the situation is more sophisticated. As described in Section 4.1.2 we have three vertex transitive strongly regular graphs of type $L_3(6)$ with parameters (36,15,6,6). Two of these graphs are latin square graphs over groups, namely, $L_3(S_3)$ and $L_3(\mathbb{Z}_6)$. For the second graph the order of its automorphism group (= 432) can be easily explained by the description of the automorphism group of latin square graphs in Remark 3.2.40. For the latin square graph over S_3 ($|Aut(L_3(S_3))| = 1296$) the case is a bit more sophisticated, because of the description of the kernel $\tilde{N}$ in Remark 3.2.40.
By Proposition 3.2.41 we know that for each graph there exists at least one partial difference set (in the group $S_3 \times S_3$ for $L_3(S_3)$ resp. in the group $\mathbb{Z}_6 \times \mathbb{Z}_6$ for $L_3(\mathbb{Z}_6)$. These partial difference sets are CI-equivalent to those labeled with * in Table 4.5.

Table 4.5: Regular subgroups of the automorphism group of the latin square graphs of type $L_3(6)$ (up to conjugacy) and their partial difference sets.

graph	GAP	partial difference set
$L_3(S_3)$	$(36,9)$	$\{c, bc, a, a^2, b^2c, c^3, a^2bc^2, a^2c^3, ab, b, a^2b^2, b^2, b^2c^2, ac^3\}$
	$(36,9)$	$\{b^2c^3, bc^3, a, a^2, c^3, c^2, c, abc^2, ac, ab, b, a^2b^2, b^2, a^2b^2c^2, a^2c\}$
*	$(36,10)$	$\{c, a^2c, a, a^2, ac, acd, d, a^2bcd, bd, ab, b, a^2b^2, b^2, b^2cd, b^2d\}$
	$(36,10)$	$\{ad, a^2d, a, a^2, d, c, cd, bc, ab^2cd, b, a^2b, b^2, ab^2, b^2c, a^2bcd\}$
	$(36,12)$	$\{d, a^2bd, ab^2, a^2b, ab^2d, c, ab^2cd, a^2c, abcd, a, b, a^2, b^2, ac, acd\}$
$L_3(\mathbb{Z}_6)$	$(36,6)$	$\{a^2c^2, a, c^2, a^2, ac^2, ac^3, a^2c, a^2b, ab, b^2c^3, b^2c, ab^2, a^2b^2, a^2bc^3, abc\}$
	$(36,6)$	$\{b^2c^2, b, c^2, b^2, bc^2, a^2b^2c, a^2bc^3, ab^2, ab, c, c^3, a^2b, a^2b^2, abc, ab^2c^3\}$
	$(36,11)$	$\{a, a^2bd, c, ac, a^2bcd, a^2, ab^2d, abc, a^2b^2d, d, cd, a^2d, ab^2c, abcd, a^2b^2c\}$
	$(36,13)$	$\{a^2d, a, d, a^2, ad, c, a^2cd, b, a^2b, b^2c, b^2cd, b^2, ab^2, bc, abcd\}$
*	$(36,14)$	$\{a^2c, a, c, a^2, ac, b^2d, ab^2cd, b, a^2b, d, cd, b^2, ab^2, bd, a^2bcd\}$
$L_3(6)$ over	$(36,9)$	$\{a, a^2, ab^2c^2, b^2c^2, a^2b^2c^2, a^2b^2, a^2b, ab, ab^2, c, abc^3, ab^2c, a^2bc, a^2b^2c^3, c^3\}$
quasigroup	$(36,10)$	$\{ab, a^2b^2, b^2cd, a^2bcd, acd, a, b, a^2, b^2, d, bc, ad, b^2c, a^2d, c\}$

For the abstract description of the groups see Table 4.4. Here the group (36,11) occurs the first time. The abstract description is $a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ac, db = bd, cda = ad, cb = bc$.

The third graph of type $L_3(6)$ comes from a latin square which corresponds to a proper quasigroup. The latin square was given in Example 3.2.43, the automorphism group of the associated latin square graph has order 648. This graph is of special interest, since it is the "smallest" latin square graph which is a Cayley graph and which is not a latin square graph over a group (the latin square Cayley graphs with 9, 16 and 25 vertices correspond to groups!). Therefore, it is a task for further research, if this example of a Cayley graph, which was discovered "experimentally" by our computations, can be generalized to an infinite series of Cayley graphs (resp. partial difference sets) or if it is a "sporadic" example.
We computed for each of these three strongly regular graphs all regular subgroups of the automorphism groups (up to conjugacy) and determined the associated partial difference

sets. The results are given in Table 4.5.
For the strongly regular graphs with parameters (36,15,6,6) we have again the situation that all partial difference sets are also (36,15,6)-difference sets. Since we have a complete list of all $(36, 15, 6)$-difference sets in [Kib78], we were able to verify each of the computed $(36, 15, 6, 6)$-partial difference set by the method described in Section 3.3.2.
Like for the other graphs we have partial difference sets in a group which are not CI-equivalent but which are srg-equivalent. Since all $(36, 15, 6, 6)$-partial difference sets are also $(36, 15, 6)$-difference sets it is possible that some of the non-CI-equivalent partial difference sets are equivalent as difference sets. But it turns out, that for these graphs there are no non-CI-equivalent partial difference sets which are equivalent as (36,15,6)-difference sets.

4.1.6 Partial difference sets of other strongly regular graphs

At this point there remain three parameter sets in Table 4.1 for which the partial difference sets are not discussed. These are the parameters for the Clebsch graph, for the Schläfli graph and for the strongly regular graphs with parameter set (36,14,4,6).

The Clebsch graph

The Clebsch graph with parameter set (16,5,0,2) is named after Alfred Clebsch (1833 - 1872), a mathematician from Germany. The Clebsch graph can be defined by the 16 lines of the Clebsch quartic surface, a pair of lines being adjacent if and only if they are skew (cf. [vLiW92], p. 246). Another possibility is, to get the Clebsch graph as a subgraph of the Schläfli graph (see Remark 4.1.5).
In the following we present a third way to create the Clebsch graph based on results of the investigation of the n-dimensional cube (cf. [KliPR88], Chapter 4).

Consider the set $\mathbb{Z}_2^n$. For $n \in \mathbb{N}$ and $i = 0, \ldots, n$ define the graphs $B_i(n)$ of order 2^n with vertex set $V(B_i(n)) := \mathbb{Z}_2^n$ and edge set $E(B_i(n)) := \{\{x, y\} | d(x, y) = i\}$, where $d(x, y)$ denotes the Hamming distance of $x, y \in \mathbb{Z}_2^n$.
The graph $B_1(n)$ is the n-dimensional cube which has automorphism group $S_2 \uparrow S_n$. The centralizer ring of the group $S_2 \uparrow S_n$ is $\langle B_0(n), \ldots, B_n(n)\rangle$. In [KliPR88], Satz 4.1.20 all primitive cellular subrings of this centralizer ring are determined.
Since we are interested in a graph with $16 = 4^2$ vertices, we consider the case $n = 4$ where we have the centralizer ring $W := \langle B_0(4), \ldots, B_4(4)\rangle$. There exist five cellular subrings of W, two of them, the subring $\langle B_0(4), B_1(4) + B_2(4), B_3(4) + B_4(4)\rangle$ and the subring $\langle B_0(4), B_1(4) + B_4(4), B_2(4) + B_3(4)\rangle$, have automorphism groups isomorphic to $(\mathbb{Z}_2)^4 \rtimes S_5$. Their basis graphs $B_1(4) + B_4(4)$ resp. $B_3(4) + B_4(4)$ are isomorphic to the Clebsch graph (here $B_1(4) + B_4(4)$ means the graph whose edge set is the union of the edge sets of the basis graphs $B_1(4), B_4(4)$).
Thus, the Clebsch graph can be created with vertex set $\mathbb{Z}_2^4$ and two vertices are adjacent, if and only if their Hamming distance is 1 or 4 (resp. 3 or 4). The automorphism group of

the Clebsch graph is isomorphic to $(\mathbb{Z}_2)^4 \rtimes S_5$ and has order 1920 (cf. [KliPR88], Beispiel 4.1.19).
Since the automorphism group of the Clebsch graph is isomorphic to $(\mathbb{Z}_2)^4 \rtimes S_5$, one can check that there exists a subgroup isomorphic to $(\mathbb{Z}_2)^4$ which acts regularly on the vertex set $\mathbb{Z}_2^4$. Thus, by Proposition 3.1.19 the Clebsch graph can be generated as a Cayley graph and consequently, at least one partial difference set exists. It is not difficult to see that this partial difference set is $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,1,1)\}$, since two vertices are adjacent if and only if their Hamming distance equals 1 or 4. This partial difference set is the last one in Table 4.6; it can also be described by cyclotomic schemes (see Example 5.3.2 on Page 124).
With the aid of GAP we computed all regular subgroups of the automorphism group (up to conjugacy) and then determined all partial difference sets (up to CI-equivalence). Altogether there are twelve regular subgroups in the automorphism group (up to conjugacy) and, hence, we get twelve non-CI-equivalent partial difference sets. They are listed in Table 4.6.

Table 4.6: Regular subgroups of the automorphism group of the Clebsch graph (up to conjugacy) and their partial difference sets.

GAP	abstract description of group	partial difference set
(16,2)	$a^4 = b^4 = 1$ abelian	$\{a^2b^2, a, a^3, b, b^3\}$
(16,3)	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	$\{a, a^3, a^2b, c, c^3\}$
(16,3)	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	$\{a, a^3, ac, ac^3, a^2b\}$
(16,3)	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	$\{a, a^3, b, ac^3, a^3c\}$
(16,4)	$a^4 = b^4 = 1,\ ba = a^3b$	$\{a, a^3, b, b^3, a^2b^2\}$
(16,6)	$a^8 = 1,\ b^2 = a^2,\ ba = a^5b$	$\{a, a^4, a^7, b, b^7\}$
(16,8)	$a^8 = 1,\ b^2 = a^4,\ ba = a^3b$	$\{a, a^4, a^7, ab^3, a^3b^3\}$
(16,10)	$a^4 = b^2 = c^2 = 1$ abelian	$\{a, a^3, a^2bc, b, c\}$
(16,11)	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	$\{a^2, ab, a^3bc, b, bc\}$
(16,11)	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	$\{ab, a^2c, a^3b, b, bc\}$
(16,11)	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	$\{ac, a^2c, a^3c, b, bc\}$
(16,14)	$a^2 = b^2 = c^2 = d^2 = 1$ abelian	$\{a, b, c, d, abcd\}$

Like for the other graphs here we find non-CI-equivalent partial difference sets in a group (groups No. (16,3), (16,11)).
As described in Section 3.3.2 we can determine partial difference sets for the Clebsch graph by investigating all (16,6,2)-difference sets. If we do this job with the complete list of (16,6,2)-difference sets from Kibler [Kib78], then we get all partial difference sets listed in Table 4.6 as well. Since there exist three partial difference sets (up to CI-equivalence) in the group $(16, 11)$ but only two difference sets (No. E9 and E10 in [Kib78]) we have the situation that the difference set $D := \{e, a, a^2, b, ac, a^2bc\}$ (No. E9 in [Kib78]) provides two non-CI-equivalent partial difference sets: $(Da^2bc) \setminus \{e\}$ and $(Da^3c) \setminus \{e\}$.

The Schläfli graph

The Schläfli graph with parameters (27,10,1,5), named after the swiss mathematician Ludwig Schläfli (1814 - 1895), is well-known as an example for a nontrivial, sharply 4-homogeneous graph (cf. [KliPR88], 4.3.32). For the construction of the Schläfli graph there are several possibilities (cf. [GolK78], [KliPR88], [vLiW92]). One possibility is, to created the graph by a generalized quadrangle $GQ(2,4)$ (cf. [vLiW92], p. 245). A more "local" description we find in [KliPR88], 4.3.33:
Let $V := \{x, y\} \cup V_a \cup V_b \cup V_c \cup V_d$ be the vertex set, where

$$V_a := \{a_i | i \in I\},\ V_b := \{b_i | i \in I\},\ V_c := \{c_i | i \in I\},\ V_d := \{d_{ij} | i, j \in I, i \neq j\},$$

with $I := \{1, 2, 3, 4, 5\}$ and $E := E_1 \cup E_2$ be the edge set, where

$$E_1 := \{\{x, a_i\}, \{x, b_i\}, \{y, a_i\}, \{y, c_i\}, \{a_i, b_i\}, \{a_i, c_i\} | i \in I\},$$
$$E_2 := \{\{b_i, c_j\}, \{a_k, d_{ij}\}, \{b_i, d_{ij}\}, \{c_i, d_{ij}\}, \{d_{ij}, d_{kl}\} | i, j, k, l \in I \text{ pairwise disjoint}\}.$$

Then the graph (V, E) is isomorphic to the Schläfli graph.
As remarked in Section 4.1.6 the Clebsch graph is a subgraph of the Schläfli graph:

Remark 4.1.5 *(cf. [KliPR88], p. 188)*
Let $\Gamma = (V, E)$ be the Schläfli graph and $x \in V$. The subgraph $\overline{\Gamma(x)}$ of Γ which is induced by the vertices $\{y \in V | (\{x, y\} \notin E\}$ is isomorphic to the Clebsch graph.

The Schläfli graph has an automorphism group of order 51840. With the aid of GAP we found two regular subgroups of order 27 (up to conjugacy). In Table 4.7 we give the subgroups and the associated partial difference sets (cf. Example 3.3.3).

Table 4.7: Regular subgroups of the automorphism group of the Schläfli graph (up to conjugacy) and their partial difference sets.

group	GAP	partial difference set
$a^3 = b^3 = 1, bab = (aba)^2$	$(27, 3)$	$\{a, a^2, b, b^2, aba, ab^2a, a^2b^2a^2, a^2ba^2, ab^2a^2b, a^2b^2ab\}$
$a^9 = b^3 = 1, ba = a^4b$	$(27, 4)$	$\{a, a^3, a^6, a^8, ab, ab^2, a^2b^2, a^3b, a^5b, b^2a^6\}$

For the groups we have the following description:
The group (27,3) is isomorphic to the semidirect product $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \langle\alpha\rangle$ with $\alpha \in Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $a^\alpha := ab$, $b^\alpha := b$ for generators a, b of $\mathbb{Z}_3 \times \mathbb{Z}_3$.
The group (27,4) is isomorphic to the semidirect product $\mathbb{Z}_9 \rtimes \langle\beta\rangle$ with $\beta \in Aut(\mathbb{Z}_9)$, where $a^\beta := a^4$ for a generator a of $\mathbb{Z}_9$.

In the following we will describe very briefly a computer-free interpretation of the Schläfli graph. This interpretation covers also the existence of the two non-CI-equivalent partial difference sets. It was obtained post factum by M. H. Klin (M. H. Klin, private communication), i.e., after the computer results given in this thesis were known.

We consider the graph $\Gamma := \overline{3 \circ K_3}$. One can easily check that this graph has the automorphism group $G := S_3 \wr S_3$ of order 1296. Counting the edges of the graph we get exactly 27 edges. Let $\tilde{G}$ be the induced action of G on the 27 edges, i.e., $\tilde{G}$ is a permutation group with degree 27. One can check with the computer package COCO that the Schläfli graph is invariant with respect to $\tilde{G}$. An investigation of the group $\tilde{G}$ yields that there are two subgroups both acting regularly on 27 points, thus, they are acting regularly on the Schläfli graph. These two subgroups coincide with the computational results given above.

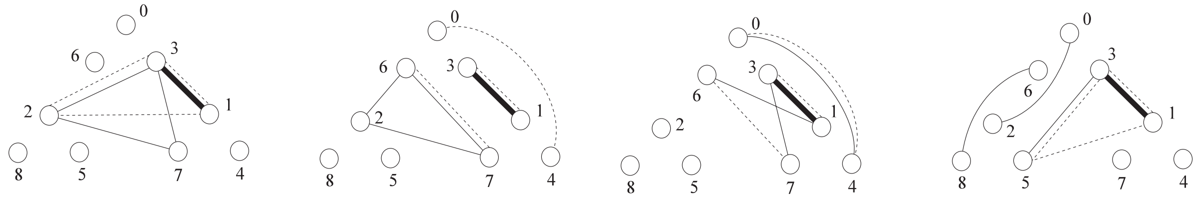
But one can avoid the use of COCO. We consider the following incidence structure $(V, \mathcal{B})$: The points in V are the 27 edges of Γ and the blocks in $\mathcal{B}$ will be represented by sets of three edges of the following two kinds:

1) induced triangles, i.e., from each of the three components $\overline{K_3}$ in Γ we take one vertex and these three vertices will be joined to a triangle;

2) matchings between two components $\overline{K_3}$ in Γ, i.e., each vertex of one component $\overline{K_3}$ will be joined to one vertex of the second component $\overline{K_3}$ such that the three resulting edges do not meet in a vertex.

In this incidence structure we have $3^3 = 27$ induced triangles and $3 \cdot 3! = 18$ matchings, i.e., altogether 45 blocks. This incidence structure represents a generalized quadrangle $GQ(2,4)$, i.e., an incidence structure with 27 points and 45 lines where for each point P and each non-incident line L there exists exactly one line through P which intersects L. The fact that our incidence structure is a $GQ(2,4)$ can be checked as follows:

Since G acts transitively on the edges of Γ it acts transitively on the points of V. Hence, it is sufficient to consider only one point in V, i.e., one edge of Γ. In the picture below this is the edge $\{1,3\}$ (thick line). Now we have to check for each block B_1 in $\mathcal{B}$ which does not contain $\{1,3\}$ if there exists another block $B_2 \in \mathcal{B}$ which contains $\{1,3\}$ and intersects with B_1 in one point. In other words, we have to show that for each induced triangle resp. matching B_1 (see above) which does not contain the edge $\{1,3\}$ there exist an induced triangle resp. matching B_2 which contains $\{1,3\}$ and has one common edge with B_1. Taking into account the action of the group G we can reduce this problem to four cases, which are given in the picture (in each case the block B_1 is represented by the thin lines, the block B_2 is described by the dotted lines, the edge $\{1,3\}$ is the thick line and $\{0,3,6\}, \{1,4,7\}, \{2,5,8\}$ are the vertex sets of the three components $\overline{K_3}$ of Γ):



Thus, as shown in the picture, the incidence structure $(V, \mathcal{B})$ coincides with the structure of the generalized quadrangle $GQ(2,4)$ and it is well-known that the Schläfli graph can be constructed from $GQ(2,4)$ (cf. [vLiW92], p. 245).

The strongly regular graphs with parameters (36,14,4,6)

As described in Section 4.1.2 we have three strongly regular graphs with parameters (36,14,4,6), each of them isomorphic to a Cayley graph. Each of these graphs is in a switching class of a latin square type two-graph with 36 vertices (cf. [BusMS81]). The graphs have automorphism groups of order 144, 432 and 216.
Since the vertex transitive latin square graphs with parameters (36,15,6,6) are in the switching classes of the same two-graphs, we have some connections between these graphs:

No. of two-graph in [Spe95]	(36,14,4,6) #	(36,14,4,6) $\|Aut\|$	(36,15,6,6) #	(36,15,6,6) $\|Aut\|$
184	1	144	1	432
224	1	432	1	1296
227	1	216	1	648

For every line in the table the automorphism group of the strongly regular graph with parameters (36,14,4,6) is a subgroup of the automorphism group of the corresponding latin square graph with parameters (36,15,6,6). Consequently, from the Galois correspondence described in Remark 2.2.8 follows that each of the latin square graphs appears as a basis graph of a cellular subring in the centralizer ring of the automorphism group of one of these (36,14,4,6)-strongly regular graphs (one example is described in Section 3.3.1).
If we compute the non-conjugated regular subgroups for each automorphism group of the (36,14,4,6)-graphs with GAP and determine the associated partial difference sets we get the results of Table 4.8 on Page 95.
Like for the Clebsch graph we can use for these strongly regular graphs the method of determining partial difference sets from difference sets (see Section 3.3.2). In [Kib78] we have a complete list of all (36,15,6)-difference sets. From these we also get all partial difference sets in Table 4.8.
Here we have the situation that there exists only one difference set (No. I. B20 in [Kib78]) for the group $(36, 13)$ which provides the two partial difference sets for the graph with automorphism group of order 144 (see Table 4.8). The group elements for the necessary shiftings of the difference set No. I. B20 are c and abd. For the graph with automorphism group of order 432 there are three difference sets (No. II. D28 - D30 in [Kib78]) which provide the four partial difference sets for the group $(36, 10)$ (see Table 4.8). From the difference set II. D28 we get two partial difference sets by shifting with the group elements d and a^2bc and eliminating the identity element afterwards.

Table 4.8: Regular subgroups of the automorphism group of the strongly regular graphs with parameters (36,14,4,6) (up to conjugacy) and their partial difference sets.

graph	GAP	partial difference set
srg with	$(36, 6)$	$\{c, a^2bc^2, a^2b^2c^3, ab^2c^2, abc^2, abc^3, a, ab^2c, ac^2, a^2, a^2b^2c^2, a^2c^2, a^2bc, c^3\}$
$\lvert Aut\rvert = 144$	$(36, 6)$	$\{c, ab^2c^2, abc^3, a^2bc^2, a^2b^2c^3, abc^2, b, ab^2c, bc^2, b^2, a^2b^2c^2, b^2c^2, a^2bc, c^3\}$
	$(36, 13)$	$\{d, c, cd, a^2b^2c, a^2b^2cd, ac, a, bd, bc, a^2, ab^2c, a^2bc, b^2d, abcd\}$
	$(36, 13)$	$\{c, abd, a^2b^2cd, a^2b^2d, abcd, bd, a, b^2c, ad, a^2, b^2d, a^2d, bc, cd\}$
	$(36, 14)$	$\{c, abcd, abd, a^2b^2cd, a^2b^2d, bcd, a, bc, acd, a^2, b^2cd, a^2cd, b^2c, d\}$
srg with	$(36, 9)$	$\{ac^3, a^2bc, a^2bc^2, ac, c^2, b^2c^2, bc^3, a, bc^2, a^2b^2c^2, a^2b^2c^3, a^2, a^2c^2, b^2c\}$
$\lvert Aut\rvert = 432$	$(36, 9)$	$\{c, c^2, c^3, a^2bc^3, abc^2, a^2bc^2, abc^2, b, bc^2, a^2c^2, a^2b^2c, b^2, ac^2, ab^2c^3\}$
	$(36, 10)$	$\{a^2cd, c, a^2d, ac, ad, ab^2d, ab^2cd, ab^2, abd, a^2bd, bcd, a^2b, a^2b^2d, a^2c\}$
	$(36, 10)$	$\{d, a^2c, a^2cd, c, cd, a^2b^2cd, b^2d, a^2b, abcd, bcd, bd, ab^2, ab^2cd, ac\}$
	$(36, 10)$	$\{c, a^2d, acd, ad, a^2cd, ab^2cd, bc, a^2b, bcd, a^2bcd, b^2c, ab^2, b^2cd, d\}$
	$(36, 10)$	$\{b^2cd, d, a^2bc, a^2d, bc, b^2c, a^2bcd, ab^2, c, a^2c, acd, a^2b, a^2b^2c, ad\}$
	$(36, 12)$	$\{d, a^2c, a^2cd, c, cd, bcd, abd, ab^2, b^2cd, a^2b^2cd, a^2b^2d, a^2b, a^2bcd, ac\}$
	$(36, 12)$	$\{d, b^2cd, b^2c, bc, bcd, a^2bcd, abd, a^2b, abcd, ab^2cd, a^2b^2d, ab^2, a^2b^2cd, c\}$
	$(36, 12)$	$\{c, a^2cd, a^2d, a^2b^2cd, a^2b^2d, ad, ac, a^2b^2, bd, b^2d, a^2c, ab, abd, a^2bcd\}$
srg with	$(36, 9)$	$\{c^2, c, c^3, a^2b^2c^2, a^2b^2, ac^3, bc^2, bc, ac^2, ab, ab^2c^2, a^2bc^2, a^2bc^2, a^2c^3\}$
$\lvert Aut\rvert = 216$	$(36, 10)$	$\{d, c, cd, a^2bd, a^2b, abcd, abd, bc, bd, ab^2, a^2d, ad, b^2ca^2b^2cd\}$
	$(36, 10)$	$\{cd, d, c, a^2bcd, a^2b, b^2c, b^2cd, ad, acd, ab^2, abcd, a^2b^2cd, a^2d, bc\}$

For the abstract description of the groups see Table 4.4.

4.2 Determination of partial difference sets by strongly regular graphs with primitive automorphism group

Besides the results for strongly regular graphs up to 49 vertices we investigated strongly regular graphs with a larger number of vertices. Unfortunately, there does not exist a complete determination for strongly regular graphs with v vertices, where $v > 49$. Only for some special parameter sets the complete number of strongly regular graphs is known. These cases are either exceptional cases where we have exactly one strongly regular graph for a parameter set (e.g., the Hoffman-Singleton graph), infinite series like the triangular graphs (see Section 3.2.2), or cases where all strongly regular graphs where determined by computers like for the parameters $(64, 18, 2, 6)$ (cf. [Spe01]).

It is well-known that the task of determining a complete list of graphs with certain properties is obstructed by the phenomenon of "combinatorial explosion" which means the enormous growth of the number of graphs when the number of vertices increases. This also affects the class of regular graphs and even the strongly regular graphs. Hence, the fact that a complete list of strongly regular graphs does not exist if the number of vertices is over a certain bound is not surprising.

For the present work we are interested in strongly regular Cayley graphs. One approach

to get a complete list of strongly regular Cayley graphs with v vertices is to investigate all transitive permutation groups of degree v and to determine the strongly regular graphs which have these transitive groups as automorphism groups. But here we are again faced with the problem of "combinatorial explosion": the number of transitive permutation groups of degree v explodes if v increases. A. Hulpke has determined all transitive permutation groups with degree up to 31 in his PHD-thesis [Hul86]. To our knowledge this bound is not improved until now. Consequently, by this approach it is only possible to determine all vertex transitive strongly regular graphs up to 31 vertices. But these graphs are already known.
Since the case of transitive permutation groups does not lead to an acceptable result for the determination of strongly regular graphs, it is natural to restrict ourselves to the case of primitive permutation groups.
Since the GAP catalogue [GAP99] on primitive groups contains all primitive groups with degree smaller than 256 (the GAP manual refers to [DixM88], [Sho92] and [The97]), it is possible to determine all strongly regular graphs with primitive automorphism group up to 255 vertices. Using the GAP catalogue of primitive permutation groups C. Pech started to compute all strongly regular graphs which have these primitive groups as automorphism groups. Currently, the catalogue of C. Pech contains all these graphs except the cases with 121 and 169 vertices (C. Pech, private communication). Since these numbers are squares of primes, in both cases we can determine all partial difference sets theoretically (see Section 3.2.3), which completes the catalogue of C. Pech in our sense.

4.2.1 A brief survey of strongly regular graphs up to 255 vertices with primitive automorphism group

In this section we will give a brief survey of the strongly regular graphs which will be investigated in the next sections. Altogether there are 95 strongly regular graphs with primitive automorphism group and v vertices, where $49 < v \leq 255$ (see Table 4.9 starting on Page 97). In the table we give the parameters of the strongly regular graphs and, if known, the name of the graph. Moreover, from the catalogue of C. Pech we took the names of the primitive groups with slight changes to make the table better readable (for some groups no name has been specified). The groups in the GAP library of primitive groups are named like in the thesis of H. Theißen [The97]. The names reflect the cohort structure which is given in [DixM88]. A group "$G\#n.i$" in the table is the i^{th} representation of a permutation group of degree n with socle G in the GAP library. A group $G : H$ is the semidirect product with normal subgroup H and factor group G. The group $G \circ H$ is the central product of matrix groups G and H. The group "G on i-sets.j" is the j^{th} permutation group with degree n and socle G in the GAP library of primitive groups which acts on i-subsets of $\{1, \dots, n\}$. Analogously, the group "G on 1-sets^2.j" is the j^{th} permutation group with degree n and socle G in the GAP library which acts on the elements of $\{1, \dots, n\}^2$. Cyclic groups of order n are simply named by n. The extension of a group G by a group H is denoted by $G.H$. For the groups in the lines 7, 8 and 22 we did not find a description in [The97], we only get the information that the names are "ad-hoc names" which are not necessarily natural for group theorists (A. Hulpke, private communication).

In this thesis we do not give further information about the automorphism groups. Only in some cases when it is necessary we will give details. Most of the groups listed in the table are primitive representations of classical groups. For information about the groups we refer to the Atlas of finite groups [CCNPW85] and the Atlas of finite group representations [BLNPRSTWW01]. Furthermore, the groups can be found in the GAP group library of primitive groups.
The strongly regular graphs which are isomorphic to Cayley graphs will be described in detail below. The strongly regular graphs with 121 and 169 vertices (resp. the associated partial difference sets) were determine theoretically and added to the table afterwards. For the graphs with latin square type parameters we checked with the aid of the GAP-package GRAPE if these graphs are so-called *point graphs of partial linear spaces*. Is this the case then the graphs are in fact latin square type graphs and otherwise they are pseudo latin square type graphs. The underlying idea of this procedure is based on the results of R. C. Bose [Bos63] on strongly regular graphs and partial geometries. An algorithm similar to the GRAPE function was developed by S. Reichard (see [Rei97], [Rei98] for the algorithm and theory).

Table 4.9: Strongly regular graphs with primitive automorphism group

No.	parameters	primitive group	graph
1	50,7,0,1	$PSU(3,5^2):2$	Hoffman-Singleton
2	55,18,9,4	A_{11} on 2-sets.2	$T(11)$
3	56,10,0,2	PSL(3,4)#56.5	Sims-Gewirtz
4	63,30,13,15	PSU(3,3)#63.2	
5	63,30,13,15	PSp(6,2) on projective points.1	
6	64,14,6,2	A_8 on 1-sets^2.4	$L_2(8)$
7	64,18,2,6	$3.A_6.2$ max $GL(3,4).2$	
8	64,21,8,6	$PSL(2,7):2$ max $PSU(3,3):2$ max $Sp(6,2)$	pseudo $L_3(8)$
9	64,21,8,6	$2^6:SL(3,2)\circ SL(2,2)$	$L_3((\mathbb{Z}_2)^3)$
10	64,27,10,12	$2^6:O^{-1}(6,2)$	
11	64,28,12,12	$2^6:O^{+1}(6,2)$	pseudo $L_4(8)$
12	66,20,10,4	A_{12} on 2-sets.2	$T(12)$
13	77,16,0,4	M_{22}#77.2	
14	78,22,11,4	A_{13} on 2-sets.2	$T(13)$
15	81,16,7,2	A_9 on 1-sets^2.4	$L_2(9)$
16	81,20,1,6		
17	81,24,9,6		pseudo $L_3(9)$
18	81,30,9,12		
19	81,32,13,12		$L_4(9)$
20	81,32,13,12		$L_4(9)$
21	81,40,19,20		Paley
22	81,40,19,20	$3^4:N_\Gamma(SL_2(5))_3$	$L_5(9)$
23	85,20,3,5	$PSp(4,4)$ on projective points.2	
24	91,24,12,4	A_{14} on 2-sets.2	$T(14)$

No.	parameters	primitive group	graph
25	100,18,8,2	A_{10} on 1-sets^2.4	$L_2(10)$
26	100,22,0,6	HS#100.2	Higman-Sims
27	100,36,14,12	J_2#100.2	Hall-Janko-Wales
28	105,26,13,4	A_{15} on 2-sets.2	$T(15)$
29	105,32,4,12	$PSL(3,4)$#105.6	
30	112,30,2,10	$PSU(4,3)$#112.8	
31	117,36,15,9	$PSL(4,3)$#117.2	
32	119,54,21,27	$O^{-1}(8,2)$#119.2	
33	120,28,14,4	A_{16} on 2-sets.2	$T(16)$
34	120,42,8,18	$PSL(3,4)$#120.5	
35	120,51,18,24	$PSp(4,4)$#120.2	
36	120,56,28,24	A_7#120.1	
37	120,56,28,24	$O^{+1}(8,2)$#120.2	
38	120,56,28,24	A_{10} on 3-sets.2	
39	121,20,9,2		$L_2(11)$
40	121,30,11,6		$L_3(11)$
41	121,40,15,12		$L_4(11)$
42	121,40,15,12		$L_4(11)$
43	121,50,21,20		$L_5(11)$
44	121,60,29,30		$L_6(11)$
45	121,60,29,30		$L_6(11)$
46	121,60,29,30		Paley
47	125,62,30,31		
48	125,62,30,31		
49	125,62,30,31		
50	125,62,30,31		Paley
51	126,25,8,4	A_{10}#126.2	
52	126,45,12,18	$PSU(4,3)$#126.5	
53	130,48,20,16	$PSL(4,3)$#130.5	
54	135,64,28,32	$O^{+1}(8,2)$#135.2	
55	136,30,15,4	A_{17} on 2-sets.2	$T(17)$
56	136,60,24,28	$PSp(4,4)$#136.2	
57	136,63,30,28	$PSL(2,17)$#136.1	
58	136,63,30,28	$O^{-1}(8,2)$#136.2	
59	144,22,10,2	A_{12} on 1-sets^2.4	$L_2(12)$
60	144,39,6,12	$PSL(3,3)$#144.2	
61	144,55,22,20	M_{12}#144n.1	pseudo $L_5(12)$
62	144,66,30,30	M_{12}#144.2	pseudo $L_6(12)$
63	144,66,30,30	M_{12}#144n.1	pseudo $L_6(12)$
64	153,32,16,4	A_{18} on 2-sets.2	$T(18)$
65	155,42,17,9	$PSL(5,2)$#155.1	
66	156,30,4,6	$PSp(4,5)$ on projective points.2	
67	156,30,4,6	$PSp(4,5)$#156.2	
68	162,56,10,24	$PSU(4,3)$#162.5	
69	165,36,3,9	$PSU(5,2)$ on isotropic projective points.2	
70	169,24,11,2		$L_2(13)$
71	169,36,13,6		$L_3(13)$
72	169,48,17,12		$L_4(13)$
73	169,48,17,12		$L_4(13)$
74	169,48,17,12		$L_4(13)$
75	169,60,23,20		$L_5(13)$

No.	parameters	primitive group	graph
76	169,72,31,30		$L_6(13)$
77	169,72,31,30		$L_6(13)$
78	169,72,31,30		$L_6(13)$
79	169,84,41,42		Paley
80	171,34,17,4	A_{19} on 2-sets.2	$T(19)$
81	175,72,20,36	$PSU(3,5)$#175.2	
82	176,40,12,8	$PSU(5,2)$ on non-isotropic projective points.2	
83	176,70,18,34	M_{22}#176.1	
84	190,36,18,4	A_{20} on 2-sets.2	$T(20)$
85	196,26,12,2	A_{14} on 1-sets^2.4	$L_2(14)$
86	208,75,30,25	$PSU(3,4)$ on non-isotropic projective points.3	
87	210,38,19,4	A_{21} on 2-sets.2	$T(21)$
88	225,28,13,2	A_{15} on 1-sets^2.4	$L_2(15)$
89	231,30,9,3	M_{22}#231.2	
90	231,40,20,4	A_{22} on 2-sets.2	$T(22)$
91	243,22,1,2		
92	243,110,37,60		
93	253,42,21,4	A_{23} on 2-sets.2	$T(23)$
94	253,112,36,60	M_{23}#253a.1	
95	255,126,61,63	$PSp(8,2)$ on projective points.1	

A table with some more details is given in Appendix E.
Most of these graphs belong to special classes of graphs which we already described in Chapter 3. For example, there are eight lattice graphs ($L_2(8),\ldots,L_2(15)$), 13 triangular graphs ($T(11),\ldots,T(23)$), 23 latin square type graphs and seven pseudo latin square type graphs.

Some of the graphs are exceptional graphs in the sense that they are determined by their parameters and they are not members of one of the described infinite series. Here we have:

- the Hoffman-Singleton graph with parameters $(50,7,0,1)$ (see [HofS60], [Jam74]),
- the Sims-Gewirtz graph with parameters $(56,10,0,2)$ (see [Gew69a] [Gew69b]),
- the graph with parameters $(77,16,0,4)$ which is a subconstituent of the Higman-Sims graph (see [Bro83]),
- the graph with parameters $(81,20,1,6)$ (see [vLiS81], [BroH92]),
- the Higman-Sims graph with parameters $(100,22,0,6)$ (see [HigS68]),
- the graph with parameters $(112,30,2,10)$ which is a subconstituent of the McLaughlin graph with parameters $(275,162,105,81)$ (see [CamGS78]),
- the graph with parameters $(162,56,10,24)$ which is also a subconstituent of the McLaughlin graph (see [CamGS78]).

Below we will show that some of these graphs yield partial difference set and others not.

4.2.2 Strongly regular graphs with primitive automorphism group which do not have partial difference sets

The systematical investigation of the strongly regular graphs in Table 4.9 gives 40 strongly regular graphs which are not isomorphic to a Cayley graph, i.e., there does not exist any partial difference set for these graphs. These results were obtained by theoretical and computational methods described in Chapter 3.
We do not give a detailed description of the strongly regular graphs in the non-existence cases. However, some of these graphs play a role for the construction of strongly regular Cayley graphs which we discuss in detail below.

Theoretical results in non-existence cases

As described in the preceding sections there are several results and conditions for the existence of partial difference sets for given strongly regular graphs.
In the case of triangular graphs $T(n)$ by Theorem 3.2.14 we have only partial difference sets, if n is a prime power and $n \equiv 3 \mod 4$. Since in Table 4.9 the triangular graphs $T(11), \ldots, T(23)$ occur, we immediately see that only $T(11), T(19)$ and $T(23)$ yield partial difference sets. For the other graphs $T(12), T(13), \ldots, T(18), T(20), T(21), T(22)$ no partial difference sets exist.

By Theorem 3.2.16 we know that nontrivial, primitive Schur rings do not exist over abelian groups which are not of prime order and which have a cyclic Sylow subgroup. Thus, for all strongly regular graphs with $v = pq$ vertices, where $p > q$ and $q \nmid (p-1)$, we have no partial difference sets. In these cases all groups of order $v = pq$ are cyclic (Lemma 2.1.8) and, consequently, abelian groups, which have a cyclic Sylow subgroup. Therefore, we do not get any partial difference set for the graphs No. 13 with parameters $(77, 16, 0, 4)$, No. 23 with parameters $(85, 20, 3, 5)$ and No. 32 with parameters $(119, 54, 21, 27)$. Moreover, one can check that there exists only one group of order 255 which is cyclic: We have $255 = 3 \cdot 5 \cdot 17$ and from the Sylow theorems and some calculations it follows that all Sylow subgroups satisfy the conditions of Proposition 2.1.10, i.e., each group of order 255 is a direct product of three subgroups of prime order. Since there is only one option for each subgroup (the cyclic group), there exists only one group of order 255 and this is cyclic. Hence, with Theorem 3.2.16 one can show that for the graph No. 95 with parameters $(255, 126, 61, 63)$ there does not exist a partial difference set.

Computational results in non-existence cases

Since we obtain theoretical non-existence results only in 14 cases, the remaining 26 strongly regular graphs of Table 4.9, which do not yield partial difference sets, were investigated by the computer package GAP with the methods described in Section 3.3.3. In Table 4.10 we give all strongly regular graphs of Table 4.9 for which we checked by GAP that no partial difference sets exist. In the column "methods" we give the methods which we used (they are described in Section 3.3.3). If possible we verified a result obtained by one method by a second method.

No.	parameter set	methods	No.	parameter set	methods
1	50,7,0,1	Method 3 & 6	56	136,60,24,28	Method 4
3	56,10,0,2	Method 2 & 3	57	136,63,30,28	Method 2 & 3
4	63,30,13,15	Method 2 & 3	58	136,63,30,28	Method 6
5	63,30,13,15	Method 3 & 6	61	144,55,22,20	Method 3 & 4
29	105,32,4,12	Method 2 & 3	63	144,66,30,30	Method 3 & 4
30	112,30,2,10	Method 6	66	156,30,4,6	Method 3 & 6
31	117,36,15,9	Method 6	67	156,30,4,6	Method 3 & 6
34	120,42,8,18	Method 2 & 3	69	165,36,3,9	Method 6
38	120,56,28,24	Method 4	81	175,72,20,36	Method 3 & 6
51	126,25,8,4	Method 3 & 6	82	176,40,12,8	Method 4
52	126,45,12,18	Method 6	83	176,70,18,34	Method 3 & 4
53	130,48,20,16	Method 6	86	208,75,30,25	Method 3 & 4
54	135,64,28,32	Method 6	89	231,30,9,3	Method 3 & 6

Table 4.10: Non-existence results

4.2.3 Triangular graphs

By Theorem 3.2.14 we have a partial difference set for $T(n)$, if n is a prime power and $n \equiv 3 \bmod 4$. In Table 4.9 this is the case only for $T(11), T(19)$ and $T(23)$. Since the numbers $11, 19$ and 23 are prime numbers, by Proposition 3.2.15 we have exactly one partial difference set D for each of the three graphs. For the graph $T(p)$, $p = 11, 19, 23$, this partial difference set D exists in the group $\mathbb{Z}_p \rtimes K$, where K is the subgroup of $\mathbb{Z}_p^*$ of order $\frac{1}{2}(p-1)$ consisting of all non-zero squares of $\mathbb{Z}_p$, and we have $D := \{\sigma_{s,0}, \sigma_{t,-t}, \sigma_{t,1}, \sigma_{s,1-s} | s, t \in K, s \neq 1\}$, where $\sigma_{a,b} \in \mathbb{Z}_p \rtimes K$ with $x^{\sigma_{a,b}} := xa + b$, $x \in \mathbb{Z}_p$.
(Though the partial difference sets for these triangular graphs are determined theoretically, the partial difference sets for the triangular graphs listed in Appendix G are results of computations by Method 6 in GAP.)

4.2.4 Partial difference sets for lattice graphs $L_2(n)$

In Table 4.9 we find the lattice graphs $L_2(8), \ldots, L_2(15)$. By Proposition 3.2.47 each lattice graph $L_2(n)$ yields a partial difference set D in each group $H \times K$, where H, K are groups of order n and $D = \{(h, e), (e, k) | h \in H, k \in K, h, k \neq e\}$ with identity element e in H and K. In the case of $L_2(p)$, p a prime, by Corollary 3.2.23 we have exactly one partial difference set (up to CI-equivalence) in $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus, for $n = 11, 13$ we have a complete theoretical result.

For the remaining graphs $L_2(n)$, $n = 8, 9, 10, 12, 14, 15$, we used a computational procedure for the investigation of the automorphism groups. It turns out that for the graphs $L_2(8), L_2(12), L_2(14)$ and $L_2(15)$ all our attempts for the calculation of the regular subgroups of the automorphism groups failed. In each of these cases the computer capacity was not sufficient. However, for the graphs $L_2(9)$ and $L_2(10)$ we were successful and got a complete list of 12 resp. 18 partial difference sets (up to CI-equivalence). The regular subgroups of $Aut(L_2(8))$ were computed by Method 3 and those of $Aut(L_2(10))$ by Method 6. The results are listed in Appendix G.

4.2.5 Partial difference sets for the graphs $L_g(p)$, p a prime

The partial difference sets for latin square type graphs $L_g(p)$, $g \geq 3, p$ a prime, are all determined in Section 3.2.3. In the table on Page 58 we gave representatives corresponding to all non-CI-equivalent partial difference sets for $p = 11, 13$. We have ten strongly regular graphs with 121 vertices and each graph yields exactly one partial difference set. An analogue situation we have for the 18 latin square type graphs with 169 vertices, here we have also exactly one partial difference set for each graph. In Appendix G all the partial difference sets are given. Notice that in some cases the corresponding strongly regular graphs have no primitive automorphism group (they are not listed in Table 4.9). Though they are not part of our investigation of graphs with primitive automorphism group we add these partial difference sets in the table in Appendix G, because here we were able to complete the job for all cases.

4.2.6 Partial difference sets for strongly regular graphs with 64 vertices

There exist six strongly regular graphs with 64 vertices and primitive automorphism group. One of these is the lattice graph $L_2(8)$ which was discussed in Section 4.2.4. For all these strongly regular graphs we used computational methods for the determination of partial difference sets. However, in two case we were not successful.

The graph with parameters $(64, 18, 2, 6)$

The strongly regular graph with parameters $(64, 18, 2, 6)$, No. 7 in Table 4.9, has an automorphism group of order 138240. The graph has parameters like a negative latin square type graph. Since we are looking for regular subgroups which have order $64 = 2^6$, it is sufficient to consider the Sylow 2-subgroup of the automorphism group. This subgroup has order 1024. With Method 5 we were able to determine all non-conjugated regular subgroups of the automorphism group of graph No. 7 and hence, to compute all partial difference sets up to CI-equivalence. This graph yields 58 partial difference sets in 30 non-isomorphic groups of order 64. The results are given in Appendix G.

The graphs with parameters $(64, 21, 8, 6)$

There are two graphs with parameters $(64, 21, 8, 6)$ in Table 4.9.
The graph No. 9 in Table 4.9 is the latin square graph over the group $(\mathbb{Z}_2)^3$. The order of its automorphism group is 64512; we can explain this order by the automorphism group of latin square graphs given in Remark 3.2.40. By Proposition 3.2.42 we have one partial difference set $D := \{(a, e), (e, a), (a, a) | a \in (\mathbb{Z}_2)^3, a \neq e\}$, where e is the identity in $(\mathbb{Z}_2)^3$.
Computations with the computer package GAP yield 58 partial difference sets (up to CI-equivalence). The computations were done by Method 5. The 58 partial difference sets are in 30 non-isomorphic groups of order 64. The results are given in Appendix G.
Observing the 58 non-conjugated regular subgroups of the automorphism group of this graph, it turns out that there is a one-to-one correspondence between these groups and those 58 regular subgroups which we found in the automorphism group of the preceding graph No. 7. We have the same number of representative groups in the 30 isomorphism classes. On a first glance it is not clear why this appear; one has to check carefully if there are parallel operations in the construction of these graphs.

For the second graph with parameters $(64, 21, 8, 6)$, the pseudo latin square graph No. 8 in Table 4.9, we calculated 36 non-conjugated regular subgroups of the automorphism group. The computations were done by Method 5. These regular subgroups yield 36 partial difference sets (up to CI-equivalence) in 11 non-isomorphic groups (see Appendix G).

Unsolved cases: The graphs with parameters $(64, 27, 10, 12)$ **and** $(64, 28, 12, 12)$

The two strongly regular graphs No. 10 and No. 11 with parameters $(64, 27, 10, 12)$ and $(64, 28, 12, 12)$ belong to the seven strongly regular graphs in Table 4.9 for which all computations failed. For both graphs for the calculation of all regular subgroups of the automorphism group it is sufficient to consider the Sylow 2-group of the automorphism group, which in both cases has order 8192. Nevertheless, all attempts for the computation of the regular subgroups of these groups failed. In all cases the computer system interrupts the computations after a certain time (hours or days) and reports a lack of workspace.
Using some special functions in GAP we were able to determine at least one regular subgroup in the automorphism group of each graph. Hence, we know that for both graphs partial difference sets exist. In the table in [Ma94] also occur partial difference sets with parameters $(64, 28, 12, 12)$ and $(64, 27, 10, 12)$. But in the first case the partial difference set is derived from a latin square type graph and in the second case the $(64, 27, 10, 12)$-partial difference set corresponds to a difference set. Therefore, these results do not cover our pseudo latin square type graph with parameters $(64, 28, 12, 12)$ and it is not clear if the $(64, 27, 10, 12)$-partial difference sets in our case is covered by the presently known difference sets.

Since we have equal parameters λ and μ for graph No. 11, it is clear that each $(64, 28, 12, 12)$-partial difference set is also a $(64, 28, 12)$-difference set. Hence, as described in Section 3.3.2 it is possible to determine all $(64, 28, 12, 12)$-partial difference sets (up to CI-equivalence), if a complete list of non-equivalent $(64, 28, 12)$-difference sets is available. Moreover, we can determine the $(64, 27, 10, 12)$-partial difference sets with this method. However, unfortunately to our knowledge no such complete list was created until today (James Davis, private communication).

4.2.7 Partial difference sets for strongly regular graphs with 81 vertices

We have eight strongly regular graphs with 81 vertices in Table 4.9. One of these graphs is the lattice graph $L_2(9)$, which was discussed in Section 4.2.4. In the following we will discuss the other strongly regular graphs with 81 vertices and give results about the associated partial difference sets.

The graph with parameters $(81, 20, 1, 6)$

The strongly regular graph with parameters $(81, 20, 1, 6)$, No. 16 in Table 4.9, is one of the exceptional graphs. It is determined by its parameters (cf. [BroH92]). This graph can be constructed by mergings of classes in a cyclotomic scheme. A cyclotomic scheme is an association scheme which can be created by the cosets of a subgroup of $\mathbb{F}_q^*$, where q is a prime power, together with the class $\{0\}$. This means that the classes of this association scheme correspond to a partition of $\mathbb{F}_q$. For details about cyclotomic schemes, in particular, for this graph see Chapter 5 and Example 5.3.12. Regarding the field $\mathbb{F}_{3^4}$ we take the four cosets which we get by the cyclic group $H \leq \mathbb{F}_{3^4}^*$ of order 20 for creating the non-reflexive classes of an association scheme. If we take one class and merge the remaining three classes, then we get an association scheme with two classes which corresponds to the strongly regular graph with parameters $(81, 20, 1, 6)$.
In [vLiS81] the above mentioned construction is described in a simpler way: The vertex set of the graph is $\mathbb{F}_{3^4}$ and two vertices are adjacent, when their difference is a fourth power in $\mathbb{F}_{3^4}$. Besides this construction there exist other descriptions of this graph. In [BroH92] one can find six further descriptions of the unique strongly regular graph with parameters $(81, 20, 1, 6)$. The construction of this graph by the truncated ternary Golay code is analogous to the construction of the strongly regular graph with parameters $(243, 22, 1, 2)$ (see below).

The regular subgroups of the automorphism group were computed by Method 5. Overall we obtained four regular subgroups of the automorphism group (up to conjugacy) which provide four partial difference sets (up to CI-equivalence). For these four groups we can give standard representatives, these are

$$\mathbb{Z}_3 \wr \mathbb{Z}_3,\ (\mathbb{Z}_3)^4,\ \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \langle\alpha\rangle),\ \mathbb{Z}_3 \times (\mathbb{Z}_9 \rtimes \langle\beta\rangle),$$

with $\alpha \in Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\beta \in Aut(\mathbb{Z}_9)$ (the last two groups contain the GAP groups (27,3) and (27,4) as subgroups, see Section 4.1.6, Schläfli graph, for details). The partial difference sets are given in Appendix G.

The graph with parameters $(81, 24, 9, 6)$

We have one strongly regular graph with parameters $(81, 24, 9, 6)$ in Table 4.9. Its automorphism group has order 93312. The parameters of this graph are of latin square type, but this graph is a pseudo latin square graph.
We computed all non-conjugated regular subgroups of the automorphism group by Method 5. We get nine such subgroups (up to conjugacy). Hence, we could determine nine partial difference sets (up to CI-equivalence). These partial difference sets are in five non-isomorphic groups of order 81. For four groups we have standard representatives, these are the same as for the graph with parameters $(81, 20, 1, 6)$. The partial difference sets are listed in Appendix G.

The graph with parameters $(81, 30, 9, 12)$

The strongly regular graph with parameters $(81, 30, 9, 12)$ can also be created by merging of classes of a cyclotomic scheme. We take a cyclotomic scheme over $\mathbb{F}_{3^4}$ created by the eight cosets of a cyclic subgroup of $\mathbb{F}^*_{3^4}$ of order 10. If we merge three "suitable" classes resp. the remaining five classes, then we get an association scheme with three classes which corresponds to this strongly regular graph with parameters $(81, 30, 9, 12)$. Again we refer to Chapter 5 and Example 5.3.12 for details about cyclotomic schemes and this strongly regular graph.
A second possible description of this graph was given by J. H. van Lint and A. Schrijver. They gave a construction of a partial geometry which is called van Lint-Schrijver partial geometry (see [BroCN89], Section 11.5). The point graph of this partial geometry is a strongly regular graph with parameters $(81, 30, 9, 12)$ isomorphic to that in Table 4.9 (cf. [vLiS81] for details).
With the aid of GAP we computed all regular subgroups of the automorphism group (up to conjugacy). By Methods 3 and 5 we calculated seven non-conjugated regular subgroups. These groups yield seven non-CI-equivalent partial difference sets (see Appendix G).

The graphs with parameters $(81, 32, 13, 12)$

Among the strongly regular graphs with parameters $(81, 32, 13, 12)$ there exist two graphs with primitive automorphism group. Their automorphism groups are of order 5184 and 186624. Both graphs belong to the class of latin square type graphs $L_4(9)$.
For each of these strongly regular graphs we computed the non-conjugated regular subgroups of the automorphism group. In both cases we used the Methods 3 and 5. For the first graph (No. 19 in Table 4.9 with automorphism group of order 5184) GAP computed exactly one regular subgroup (up to conjugacy), which is isomorphic to $(\mathbb{Z}_3)^4$. For the second graph (No. 20) there exist six regular subgroups of the automorphism group (up to conjugacy), two of these groups are isomorphic. For four groups we have standard representatives, these are the same like in the case for the strongly regular graph with parameters $(81, 20, 1, 6)$. Finally, we get exactly one partial difference set (up to CI-equivalence) for the first graph (No. 19)

and six partial difference sets (up to CI-equivalence) for the second graph (No. 20). For details see Appendix G.

The graphs with parameters $(81, 40, 19, 20)$

We have two strongly regular graphs with parameters $(81, 40, 19, 20)$ in Table 4.9. Both graphs can be created as subschemes in the cyclotomic scheme mentioned for the strongly regular graph $(81, 20, 1, 6)$ above. For these two graphs we have to merge each time two "suitable" cyclotomic classes (see Chapter 5 and Example 5.3.12). We get two association schemes with two classes which yield these two strongly regular graphs. One of these graphs (No. 21 in Table 4.9) is isomorphic to the Paley graph $P(81)$ with automorphism group of order 12960. The other graph (No. 22) has an automorphism group of order 38880. Both graphs belong to the class of $L_5(9)$-graphs. For the Paley graph $P(81)$ we know by Proposition 3.2.4 that there exists exactly one regular subgroup in the automorphism group $Aut(P(81))$. Hence, we have exactly one partial difference set which by Proposition 3.2.6 consists of all non-zero squares in $\mathbb{F}_{3^4}$.
For the other strongly regular graph we computed by GAP three regular subgroups of its automorphism group (up to conjugacy). For this computation we used Method 2. These three regular subgroups yield three partial difference sets (up to CI-equivalence). Standard representatives of the three groups are $(\mathbb{Z}_3)^4$ and twice the group $\mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes K_1)$, with $K_1 := \langle \alpha \rangle \leq Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$. For the partial difference sets see Appendix G.

4.2.8 Partial difference sets for strongly regular graphs with 100 vertices

There exist exactly three strongly regular graphs with 100 vertices and primitive automorphism group. These are the lattice graph $L_2(10)$ (see Section 4.2.4), the Higman-Sims graph and the Hall-Janko-Wales graph.

The Higman-Sims graph with parameters $(100, 22, 0, 6)$

The Higman-Sims graph is a strongly regular graph which plays an important role in the theory of simple finite groups: Its automorphism group has the Higman-Sims group HS as normal subgroup of index 2. The group HS was discovered in 1968 and it is one of the sporadic simple groups (see [HigS68]). The Higman-Sims graph corresponds to a rank 3 representation of the group HS, i.e., it is a basis graph in the centralizer ring of this representation of HS.
The Higman-Sims graph can also be constructed by the unique Steiner system $S(3, 6, 22)$ which has 22 points and 77 blocks (cf. [BroCN89], Chapter 13.1 B, see Definition 2.1.35 for Steiner systems). As vertex set we take $\{\Omega\} \cup W \cup \mathcal{B}$, where Ω is a symbol, W is the vertex set of the Steiner system and $\mathcal{B}$ is the set of blocks of $S(3, 6, 22)$. Let Ω be adjacent to all elements of W and let $w \in W$ adjacent to $B \in \mathcal{B}$ if and only if w and B are incident. Furthermore, let no two elements of W be adjacent and let $B, B' \in \mathcal{B}$ be adjacent if and

only if $B \cap B' = \emptyset$. This defines a strongly regular graph with parameters $(100, 22, 0, 6)$ isomorphic to the Higman-Sims graph.
Computations by GAP with Method 6 provides four non-conjugated regular subgroups of the automorphism group of the Higman-Sims graph. These correspond to four non-CI-equivalent partial difference sets (see Appendix G for details). Recently, these four partial difference sets were also determined by L. Jørgensen and M. H. Klin [Kli00].

The Hall-Janko-Wales graph with parameters $(100, 36, 14, 12)$

The Hall-Janko-Wales graph is connected to the simple group of Hall-Janko J_2 (see [HalW68], [Jan69]). The automorphism group of the graph is isomorphic to $J_2.2$, i.e. a group which has a subgroup of index 2 isomorphic to J_2. The graph can be constructed as follows: The Janko group J_2 has a subgroup of index 100 which is isomorphic to the group $PSU(3, 3)$ of order 6048. The action of J_2 on the 100 cosets of this subgroup yields a representation of J_2 as a permutation group with degree 100. This representation is of rank 3, i.e., its centralizer ring has two non-reflexive basis graphs. These are the Hall-Janko-Wales graph and its complementary graph.
The automorphism group of the Hall-Janko-Wales graph has exactly two non-conjugated regular subgroups. These groups were determined by Method 2. The subgroups provide two partial difference sets up to CI-equivalence which are listed in Appendix G. Like for the Higman-Sims graph these partial difference sets were recently determined by L. Jørgensen and M. H. Klin [Kli00].

4.2.9 Partial difference sets for strongly regular graphs with 120 vertices

We have six strongly regular graphs with 120 vertices and primitive automorphism group. Three of them are Cayley graphs: one with parameters $(120, 51, 18, 24)$, the other two graphs have parameters $(120, 56, 28, 24)$.

The graph with parameters $(120, 51, 18, 24)$

The strongly regular graph with parameters $(120, 51, 18, 24)$ can be described by a primitive representation of the projective symplectic group $PSp(4, 4)$ (see [FarKM94], Section 3.5). This primitive representation of the group $PSp(4, 4)$ corresponds to a centralizer ring of rank 3. The non-reflexive basis graphs of this centralizer ring are complementary strongly regular graphs with parameters $(120, 51, 18, 24)$ and $(120, 68, 40, 36)$.
With the aid of GAP we computed all regular subgroups of the automorphism group. Since the graph has a large automorphism group, the computations took some time. Finally, by Method 3 we get a result after computations of several hours. We have exactly one regular subgroup in the automorphism group (up to conjugacy). This group is isomorphic to the symmetric group S_5. The single partial difference set (up to CI-equivalence) is listed in Appendix G.

The graphs with parameters $(120, 56, 28, 24)$

In Table 4.9 we have three graphs with parameters $(120, 56, 28, 24)$. One of these graphs is no Cayley graph; this graph (No. 38 in Table 4.9) corresponds to the 3-subsets of a 10-element set where two of these 120 subsets are adjacent, if they have one or three common elements (cf. [FarKM94], Section 3.2). The other two graphs are Cayley graphs.
The first graph (No. 36 in Table 4.9) has an automorphism group of order 5040 which is isomorphic to the symmetric group S_7. By Method 2 we determined all regular subgroups of the automorphism group up to conjugacy. We get two such subgroups, both isomorphic to S_5. This result was confirmed by Method 3. The two corresponding partial difference sets are listed in Appendix G.
A short time before submitting this thesis the author was informed by M. H. Klin that he obtained this result already in 1995. A part of it was presented in the talk [Kli95]. Moreover, based on the idea given in [Kli95] one can also explain the (partial) result for the second graph with parameters $(120, 56, 28, 24)$ (see below).

The second graph (No. 37 in Table 4.9) can be constructed by a quadratic form. A function $Q : \mathbb{F}_q^n \to \mathbb{F}_q$ is called a **quadratic form**, if $(\alpha x)^Q = \alpha^2 x^Q$ for all $\alpha \in \mathbb{F}_q$, $x \in \mathbb{F}_q^n$ and the function $B : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q$ with $(x, y)^B := (x+y)^Q - x^Q - y^Q$ for all $x, y \in \mathbb{F}_q^n$ is a symmetric bilinear form. By [vLiW92], p. 317, a quadratic form can be defined by a homogeneous polynomial of degree 2 in n indeterminates:

$$x^Q = (x_1, \ldots, x_n)^Q = \sum_{i,j=1}^{n} c_{ij} x_i x_j,$$

where $c_{ij} \in \mathbb{F}_q$. Two quadratic forms Q_1 and Q_2 are called **projectively equivalent**, if there exists a non-singular $n \times n$-matrix A over $\mathbb{F}_q$ with $x^{Q_1} = (xA)^{Q_2}$. For a quadratic form Q the least number of indeterminates that occur (with nonzero coefficients) in any projective equivalent quadratic form is called the **rank of** Q. If the rank equals the number of indeterminates of a quadratic form, then it is called **non-degenerate**. A quadratic form of even rank is called **hyperbolic**, if it is projectively equivalent to a quadratic form Q defined by $x^Q = x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n$.
For a quadratic form Q a set of projective points $\{\langle x \rangle \in PG(n, q) | x^Q = 0\}$ is called a **quadric in** $PG(n, q)$. A vector $x \in \mathbb{F}_q^n$ is called **isotropic**, if we have $< x, x >= 0$ for the symmetric inner product in $\mathbb{F}_q^n$. For more details about quadratic forms in combinatorial theory we refer to [vLiW92], Chapter 26.
Now, consider a hyperbolic, non-degenerate quadric in the projective geometry $PG(7, 2)$. The graph consisting of all non-isotropic points of this quadric, where two distinct points are adjacent if they are not orthogonal, is a strongly regular graph with parameters $(120, 56, 28, 24)$ (cf. [BroCN89], 10.3.6; [Bro96]). If we take all points (isotropic and non-isotropic) of this hyperbolic, non-degenerate quadric in the projective geometry $PG(7, 2)$ (by Theorem 26.5 in [vLiW92] these are exactly 135 points), then we can construct a strongly regular graph with parameters $(135, 64, 28, 32)$ (cf. [Bro96]), which has also a primitive automorphism group

isomorphic to the automorphism group of the strongly regular graph with 120 vertices. However, the graph with 135 vertices does not yield any partial difference sets (see Table 4.10 with the non-existence results).
For the automorphism group of the strongly regular graph with 120 vertices we have at least one regular subgroup (up to conjugacy) and, hence, one partial difference set (up to CI-equivalence). The regular subgroup is isomorphic to S_5. In this case we were not successful in the complete determination of all partial difference sets. No computational method led to a result, because the automorphism group is too large. By GAP and the library of small groups in GAP it was possible to check that all groups of order 120 with one exception have a normal Sylow 5-subgroup or a normal Sylow 2-subgroup. Using Method 6 we get the result that there does not exist a regular subgroup of the automorphism group which has a normal Sylow subgroup. Thus, only the mentioned exception, the symmetric group S_5, is a candidate for a regular subgroup and, indeed, in one of the normalizers of a semiregular subgroup which we used in Method 6 such a regular subgroup isomorphic to S_5 was determined. However, we were not able to determine the number of non-conjugated regular subgroups isomorphic to S_5 in the automorphism group.

4.2.10 Partial difference sets for strongly regular graphs with 125 vertices

We have four strongly regular graphs with 125 vertices and primitive automorphism group. Each of these graphs has Paley-type parameters $(125, 62, 30, 31)$. Two of the strongly regular graphs yield one partial difference set respectively, and each of the other two yields three partial difference sets.
The four strongly regular graphs with parameters $(125, 62, 30, 31)$ can be described as descendants of certain regular two-graphs (see Section 2.1.3 for the definition). As described in [BroCN89], Section 1.5 these regular two-graphs correspond to *Taylor graphs*, distance regular graphs with certain parameters which are antipodal double covers of complete graphs (see [BroCN89] for the notions). Each Taylor graph $\Gamma = (V, E)$ is determined by any induced subgraph $\Gamma(\omega)$ where $\omega \in V$. This subgraph $\Gamma(\omega)$ is a strongly regular graph with parameters (v, k, λ, μ) where $2v + 2 = |V|$, $k = 2\mu$ and v is the valency of the distance regular graph Γ. In the case of the four strongly regular graphs with parameters $(125, 62, 30, 31)$ we have $k = 62 = 2 \cdot 31 = 2\mu$. For further details about Taylor graphs we refer to [BroCN89] or to the origin articles of D. E. Taylor [Tay77] and D. E. Taylor & R. Levingston [TayL78].
As mentioned above, two of the strongly regular graphs yield exactly one partial difference set. One of these graphs is the Paley graph $P(125)$ (No. 50 in Table 4.9) which has an automorphism group of order 23250. By Proposition 3.2.4 the group $Aut(P(125))$ has exactly one regular subgroup (up to conjugacy) which corresponds to one partial difference set (up to CI-equivalence). This partial difference set is contained in the group $(\mathbb{Z}_5)^3$. The second graph (No. 47) has an automorphism group of order 3000. In this case we computed all non-conjugated regular subgroups of the automorphism group by Method 2 and 3. There is exactly one such subgroup which corresponds to exactly one partial difference set (up to CI-equivalence). Like for the Paley graph, this partial difference set is contained in a group

isomorphic to $(\mathbb{Z}_5)^3$.
The other two $(125, 62, 30, 31)$-strongly regular graphs (No. 48 and No. 49) have automorphism groups of order 15000. By Method 2 and 3 we found three non-conjugated regular subgroups in each case, which each time yield three partial difference sets (up to CI-equivalence). In both cases we have one group isomorphic to $(\mathbb{Z}_5)^3$ and two isomorphic non-abelian groups of order 125 (see Appendix F for the groups and Appendix G for the partial difference sets).

4.2.11 Partial difference sets for strongly regular graphs with 144 vertices

In Table 4.9 we find five strongly regular graphs with 144 vertices. As we explained in Section 4.2.4 we did not get a complete result for the lattice graph $L_2(12)$, because the automorphism group of this graph was too large. For two strongly regular graphs (No. 61 and No. 63 in Table 4.9) with parameters $(144, 55, 22, 20)$ and $(144, 66, 30, 30)$ there does not exists any partial difference set. The remaining two cases we will describe in the following two subsections.

The graph with parameters $(144, 39, 6, 12)$

The strongly regular graph with parameters $(144, 39, 6, 12)$ in Table 4.9 can be constructed as follows (cf. [FarKM94], pp. 121/122): We consider the induced action of the group $PSL(3, 3)$ on its subgroups of order 13. Since there are 144 cyclic subgroups of order 13 in $PSL(3, 3)$ we obtain a group with degree 144. The centralizer ring of this group has rank 6 and one of the basis graphs is isomorphic to the strongly regular graph with parameters $(144, 39, 6, 12)$ (hence, it follows that this centralizer ring has a cellular subring of rank 3). The automorphism group of this strongly regular graph has order 11232 and contains a subgroup of index 2 isomorphic to $PSL(3, 3)$.
With the aid of GAP we determined by Method 2 and 3 exactly one regular subgroup (up to conjugacy) in the automorphism group of this $(144, 39, 6, 12)$-strongly regular graph. Hence, we have exactly one partial difference set (up to CI-equivalence) which is given in Appendix G.
The existence of a partial difference set for this graph is already mentioned in [Kli00]; it was determined by M. H. Klin and E. K. Lloyd. Here we showed with GAP that there is exactly one partial difference set (up to CI-equivalence).

The graph with parameters $(144, 66, 30, 30)$

In Table 4.9 we have two graphs with parameters $(144, 66, 30, 30)$ (No. 62 and No. 63). Like the $(144, 55, 22, 20)$-strongly regular graph in Table 4.9 they are pseudo latin square type graphs and related to the Mathieu group M_{12}. Only the graph No. 62 is a Cayley graph.
Let $G := Aut(M_{12})$ be the automorphism group of the Mathieu group M_{12}. As described in [ChuI94] the group G has a subgroup H isomorphic to $PGL(2, 11)$ and, hence, we can consider the action of G on the cosets of H. It turns out that there are two orbits O_1 and

O_2 of length 144. Now we consider the action of G on each orbit separately. The action of G on O_1 resp. on O_2 is primitive. Thus, we have two primitive representations G_1, G_2 of G on sets of 144 points; these are not similar. One representation, say G_1, yields a centralizer ring of rank 4 which has cellular subrings of rank 3 corresponding to two strongly regular graphs with parameters $(144, 55, 22, 20)$ and $(144, 66, 30, 30)$ (No. 61 and No. 63 in Table 4.9). The representation G_2 also yields a rank 4 centralizer ring which has only one cellular subring of rank 3 which corresponds to the second strongly regular graph with parameters $(144, 66, 30, 30)$ (No. 62). For more details about this subject we refer to [ChuI94].

Only the graph No. 62 is a Cayley graph. For this graph associated to the group G_2 ($= M_{12}.2$ in Table 4.9) we computed exactly one regular subgroup of G_2 (up to conjugacy) by Method 3. Thus, there exists exactly one partial difference set (up to CI-equivalence) for this graph, which is given in Appendix G.

4.2.12 Partial difference sets for strongly regular graphs with 243 vertices

In Table 4.9 are two strongly regular graphs with 243 vertices. These graphs have isomorphic automorphism groups which have as a one-point-stabilizer a group isomorphic to $M_{11}.2$ (a group containing the Mathieu group M_{11} as normal subgroup of index 2). Both graphs are constructed by a two-weight-code and both provide partial difference sets.

Let $\mathbb{F}_q^n$ be a vector space. A linear (n, k)-code C is a k-dimensional subspace of $\mathbb{F}_q^n$. For $q = 3$ the code C is called a **ternary code**. For a codeword $x = (x_1, \ldots, x_n) \in C$ the number of components x_i with $x_i \neq 0$ is called the **weight of** x. If for all codewords exactly two different weights occur, i.e., $|\{w(x)|x \in C \setminus \{0\}\}| = 2$, then C is called a **two-weight code**. The code $C^\perp := \{y \in \mathbb{F}_q^n | y^t x = 0\}$ is called the **dual code of** C. If the minimum weight of the codewords in $C^\perp$ is greater than 2, then C is called **projective**. A **truncation** of C is a code with codewords of length $n-1$ obtained by deleting a fixed coordinate position. For a more detailed introduction to coding theory we refer to [vLi82].

In [Del72] P. Delsarte proved a one-to-one correspondence between strongly regular Cayley graphs over a group $(\mathbb{Z}_p)^n$ and projective two-weight codes (see also [Ma94], Theorem 8.1 in terms of partial difference sets).

One possibility to construct graphs by codes is the following:

Definition 4.2.1 *Let C be a linear (n, k)-code in $\mathbb{F}_q^n$. Let V be the set of cosets of C in $\mathbb{F}_q^n$. The graph $\Gamma(C)$ with vertex set V, where two vertices are adjacent if and only if representatives of the corresponding cosets exist which differ only in one coordinate (i.e., have Hamming distance one), is called the* **coset graph of** C.

In the following we will give two examples where the coset graph of a code is strongly regular; these are coset graphs of so-called Golay codes. Notice that the strongly regular graph with parameters $(81, 20, 1, 6)$ can also be described as the coset graph of the truncated ternary Golay code. For information about the Golay codes we refer to [Gol49] and [DelG75].

The graph with parameters $(243, 22, 1, 2)$

In Table 4.9 we have one strongly regular graph with parameters $(243, 22, 1, 2)$. This graph can be constructed as the coset graph of the ternary Golay code which is the unique linear $(11, 5)$-code with weights 6 and 9. Since this fact was a result of E. R. Berlekamp, J. H. van Lint and J. J. Seidel [BervLS73], this graph is called the *Berlekamp-van Lint-Seidel graph*.
With the aid of GAP we computed all non-conjugated regular subgroups of the automorphism group. There exist four such subgroups which were determined by Method 5. As mentioned above, by the result of P. Delsarte in [Del72] one group is isomorphic to $(\mathbb{Z}_3)^5$. The other groups and the partial difference sets are given in Appendix F and G.
Strongly regular Cayley graphs with parameters $(243, 22, 1, 2)$ over an abelian group (and their complementary graphs) are in a certain sense exceptional graphs:

Theorem 4.2.2 *(cf. [AraJMP90])*
The following are all possible parameters for a nontrivial (v, k, λ, μ)-strongly regular Cayley graph with $\lambda - \mu = -1$ over an abelian group:

1) $v \equiv 1$ *mod 4 and* $(v, k, \lambda, \mu) = (v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$*;*

2) $(v, k, \lambda, \mu) = (243, 22, 1, 2)$ *or* $(v, k, \lambda, \mu) = (243, 220, 199, 220)$.

The graph with parameters $(243, 110, 37, 60)$

This strongly regular graph with parameters $(243, 110, 37, 60)$ was discovered by P. Delsarte in [Del73]. It can be constructed as a coset graph of a projective ternary $(55, 5)$-code with weights 36 and 45.
With GAP we get by Method 5 a complete list of six non-conjugated regular subgroups of the automorphism group of this strongly regular graph. One group is isomorphic to $(\mathbb{Z}_3)^5$. The other groups and the partial difference sets are given in Appendix F and G.
By [BervLS73], p. 72 both strongly regular graphs with 243 vertices considered here can also be constructed by mergings of classes in a cyclotomic scheme over $\mathbb{F}_3^5$ (see Chapter 5 for definitions).

4.2.13 Partial difference sets for other strongly regular graphs

In this section we discuss the remaining three strongly regular graphs of Table 4.9.

The graph with parameters $(155, 42, 17, 9)$

In the catalogue of primitive groups in GAP [GAP99] we have only one nontrivial primitive permutation group with degree 155. This is the group $PSL(5, 2)$ with its induced action on the lines of the projective geometry $PG(4, 2)$. The points of the projective geometry $PG(4, 2)$ are the 31 non-zero vectors of $(\mathbb{Z}_2)^5$ (which represents the one-dimensional subspaces of $(\mathbb{Z}_2)^5$) and the lines of $PG(4, 2)$ are the 155 distinct two-dimensional subspaces.

If we take the lines of $PG(4,2)$ as vertices and say that two vertices are adjacent if and only if the corresponding lines intersect, then we get a strongly regular graph with parameters $(155, 42, 17, 9)$ (cf. [Hub75], 8.1).
By Method 6 we computed the non-conjugated regular subgroups of the automorphism group. We have exactly one such regular subgroup and this is isomorphic to $\mathbb{Z}_{31} \rtimes \mathbb{Z}_5$. Hence, we get exactly one partial difference set (up to CI-equivalence) which is presented in Appendix G.

The graph with parameters $(162, 56, 10, 24)$

The strongly regular graph with parameters $(162, 56, 10, 24)$ is one of the exceptional graphs, which is determined by its parameters. As mentioned above, it is a subconstituent of the McLaughlin graph with parameters $(275, 112, 30, 56)$. The McLaughlin graph $\Gamma = (V, E)$ can be created by the unique regular two-graph $(X, \mathcal{D})$ with 276 vertices (see [GoeS75] for details about this two-graph): We have $V := X \setminus \{\omega\}$ and $E := \mathcal{D}_\omega$, where ω is a vertex in X and $\mathcal{D}_\omega := \{D \setminus \{\omega\} | \omega \in D \in \mathcal{D}\}$. The McLaughlin graph was first constructed in [McL69]; its automorphism group is $McL.2$, i.e., the group with McL as normal subgroup of index 2 where McL is the sporadic simple group discovered by McLaughlin.
For a vertex $v \in V$ the subconstituents $\Gamma(v)$ and $\Gamma_2(v)$ of the McLaughlin graph Γ are strongly regular graphs with parameters $(112, 30, 2, 10)$ and $(162, 105, 72, 60)$ which are both determined by their parameters (see [CamGS78], Theorems 9.1 and 9.2).
As mentioned in Section 4.2.2 the automorphism group of the strongly regular graph with parameters $(112, 30, 2, 10)$ has no regular subgroups, thus, there exist no partial difference sets for this graph.
For the strongly regular graph with parameters $(162, 105, 72, 60)$ we determined six partial difference sets (up to CI-equivalence) by the six non-conjugated regular subgroups of the automorphism group. These six subgroups were calculated by Method 3. For the partial difference sets see Appendix G.

The graph with parameters $(253, 112, 36, 60)$

In Table 4.9 we have one strongly regular graph with 253 vertices. Its automorphism group is a primitive representation of the Mathieu group M_{23}. The graph can be created by the Steiner system $S(4, 7, 23)$. The vertices are the 253 blocks and two vertices are adjacent, if the blocks intersect in a single point (cf. [Hub75], S. 6).
By Method 6 we computed all regular subgroups of the automorphism group. We get exactly one such group (up to conjugacy). This regular subgroup is isomorphic to the group $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$. The associated partial difference set is given in Appendix G.

Chapter 5

Cyclotomic schemes: A special class of association schemes

The class of cyclotomic schemes is a special class of association schemes which is constructed by finite fields. An important property of cyclotomic schemes is the fact that their intersection numbers p_{ij}^k with $i, j, k \neq 0$ coincide with so-called cyclotomic numbers which are associated to the scheme (see Definition 5.1.4). As described in Section 2.2 (in terms of cellular rings) the intersection numbers (resp. structure constants) of an association scheme determine the multiplication in the associated Bose-Mesner algebra completely. Moreover, as we describe in this chapter, the knowledge of the intersection numbers is sufficient to determine all subschemes of an association scheme.
For an association scheme with d classes we have $(d+1)^3$ intersection numbers p_{ij}^k. Due to the construction of the cyclotomic schemes it is possible to determine the d^3 intersection numbers p_{ij}^k, $i, j, k \neq 0$ by d^2 cyclotomic numbers. Furthermore, the cyclotomic numbers of a cyclotomic scheme over $\mathbb{F}_{p^n}$ are determined by formulas which depend on the parameters p^n, x, y, where $p^n = x^2 + ay^2$ is a certain representation of the prime power p^n. Thus, since the existence of subschemes of an association scheme depends on its intersection numbers, in the special case of cyclotomic schemes the existence of subschemes depends on the parameters p^n, x, y. In principle, it is possible to determine purely theoretically all subschemes of a cyclotomic scheme. However, this approach is restricted by the knowledge of the necessary formulas for the cyclotomic numbers and the extensive calculations one has to do. In this chapter we will present the method for the determination of subschemes of a cyclotomic scheme and we will give results for cyclotomic schemes with three and four classes. In the case of six classes we will give a brief outline.

5.1 Basic definitions and results

In this section we will give the necessary definitions and some basic results about cyclotomic schemes. For a detailed introduction to the theory of cyclotomic schemes we refer to [Sto67].

Throughout this section let p be a prime, $n \in \mathbb{N}$, and as usual $\mathbb{F}_{p^n}$ the field with p^n elements.

Let $\omega \in \mathbb{F}_{p^n}^*$ be a primitive element of the multiplicative group of $\mathbb{F}_{p^n}$. It is easy to check that for $e, f \in \mathbb{N}$ with $p^n = ef + 1$ the set $C_1 := \{\omega^{ek} | k = 0, \dots, f-1\}$ is a cyclic subgroup of $\mathbb{F}_{p^n}^*$. The cosets $C_i := \omega^{i-1}C_1$, $i = 2, \dots, e$, of C_1 together with $C_0 := \{0\}$ correspond to the classes of an association scheme $(\mathbb{F}_{p^n}, \mathcal{R})$, where $R_i \in \mathcal{R}$ is defined by

$$(x, y) \in R_i :\Leftrightarrow x - y \in C_{i-1} \text{ for } i = 1, \dots, e + 1.$$

Definition 5.1.1 *The association scheme $(\mathbb{F}_{p^n}, \mathcal{R})$ constructed above is called a* **cyclotomic scheme***, the cosets C_i, $i = 1, \dots, e$ are called* **cyclotomic classes***.*

A first important observation we find in [ClaG89]:

Lemma 5.1.2 *(cf. [ClaG89], Lemma 4.3.6)*
A cyclotomic scheme $(\mathbb{F}_{p^n}, \mathcal{R})$ with $p^n = ef + 1$ is non-symmetric if and only if e is even and f is odd.

As described in Section 2.2 we have a connection between association schemes and Schur rings:

Remark 5.1.3 *Let $C_1, \dots, C_e$ be the cyclotomic classes of the cyclotomic scheme $(\mathbb{F}_{p^n}, \mathcal{R})$ in Definition 5.1.1 and let $C_0 := \{0\}$. Then the quantities $\underline{C}_0, \dots, \underline{C}_e$ generate a Schur ring over $\mathbb{F}_{p^n}$.*
We have $\underline{C}_i \underline{C}_j = \sum_{k=0}^{e} p_{ij}^k \underline{C}_k$, where p_{ij}^k are the intersection numbers of the cyclotomic scheme.

Notation: In the following for the description of cyclotomic schemes we will switch between the language of Schur rings and the language of association schemes, i.e., we will denote the association scheme that corresponds to $\langle C_0, \dots, C_e \rangle$ as well as the Schur ring $\langle \underline{C}_0, \dots, \underline{C}_e \rangle$ as a cyclotomic scheme and the numbers p_{ij}^k as intersection numbers resp. as structure constants of the cyclotomic scheme. We will always denote $C_0 := \{0\}$. For a cyclotomic scheme over a field $\mathbb{F}_{p^n}$ with e classes we will write $C(p^n, e)$. If it is convenient for us we will describe the cyclotomic classes by their corresponding adjacency matrices or (directed) basis graphs (cf. Section 2.2).

Definition 5.1.4 *Let $\mathbb{F}_{p^n}$ be a field with $p^n = ef + 1$ and primitive element $\omega \in \mathbb{F}_{p^n}^*$. Then for $i, j \in \mathbb{N}$ the numbers*

$$(i, j)_e := |\{(s, t) | \omega^{es+i} + 1 = \omega^{et+j}, 0 \leq s, t \leq f - 1\}|$$

are called **cyclotomic numbers***.*

From the definition it follows that each cyclotomic number $(i, j)_e$ can interpreted as the number of successors of the elements of C_{i+1} in the class C_{j+1}, $(0 \leq i, j \leq e - 1)$.
For a cyclotomic number $(i, j)_e$ we will also write (i, j) if it is clear that it is associated to a pair $(\mathbb{F}_{p^n}, e)$ where $p^n = ef + 1$.

Lemma 5.1.5 *(cf. [Sto67], Lemma 3)*
For a field $\mathbb{F}_{p^n}$ with $p^n = ef+1$ let $C(p^n,e) = \langle \underline{C_0}, \dots, \underline{C_e} \rangle$ be a cyclotomic scheme. For the cyclotomic numbers the following equations hold (with addition modulo e):

1) $(i,j) = (i+ke, j+le)$ *for all* $k,l \in \mathbb{Z}$;

2) $(i,j) = (e-i, j-i) = \begin{cases} (j+\frac{e}{2}, i+\frac{e}{2}) & e \text{ even, } f \text{ odd,} \\ (j,i) & \text{otherwise;} \end{cases}$

3) $\sum_{j=0}^{e-1} (i,j) = f - n_i$, *where* $n_i := \begin{cases} 1 & i \equiv 0 \text{ mod } e \text{ and } f \text{ even,} \\ 1 & i \equiv 0 \text{ mod } e \text{ and } e, f \text{ odd,} \\ 1 & i \equiv \frac{e}{2} \text{ mod } e \text{ and } e \text{ even, } f \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$

Proof: The statement 1 follows from the definition of cyclotomic numbers.
The proof of 2 is given in [Sto67], Lemma 3 for e even or f even. It remains the case for e, f odd: Here $-1 \in \mathbb{F}_{p^n}^*$ equals 1, because $ef+1$ is even and this implies $p=2$. Thus, for a primitive element ω of $\mathbb{F}_{p^n}^*$ we get $\omega^{xe+i}+1 = \omega^{ye+j} \Leftrightarrow \omega^{xe+i} = \omega^{ye+j}+1$ and from this $(i,j) = (j,i)$ follows by the definition of cyclotomic numbers.
The proof of 3 is also given in [Sto67], Lemma 3 except the case e, f odd. In this case we have $p=2$. The sum $\sum_{j=1}^{e}(i,j)$ is the number of successors of the f elements of C_{i+1} in all other classes C_j, i.e., in $\mathbb{F}_{p^n}^*$. Hence, we get f as result if $-1 \notin C_{i+1}$ and $f-1$ if $-1 \in C_{i+1}$ because the successor of -1 is 0 and 0 is not contained in any cyclotomic class. Since $p=2$ implies $-1=1$ we have $-1 \in C_1$ and consequently, we get the result in 3 for e, f odd.

□

Lemma 5.1.6 *(cf. [Man65], p. 91; [Sto67], Lemma 13)*
For $p^n = ef+1$ and $s, t, s+t$ not divisible by e we get for the cyclotomic numbers

$$p^n = \left| \sum_{j=0}^{e-1} exp\left(\frac{2sj\pi i}{e}\right) \sum_{h=0}^{e-1} (j,h)_e \ exp\left(\frac{-(s+t)h\pi i}{e}\right) \right|^2.$$

This lemma gives a relation between the cyclotomic numbers which is very useful. In general, for s, t one chooses small numbers, e.g., $s, t \in \{1,2\}$.
As mentioned above the cyclotomic numbers coincide with certain structure constants of the Schur ring over $\mathbb{F}_{p^n}$ with basis quantities $\underline{C_0}, \underline{C_1}, \dots, \underline{C_e}$:

Lemma 5.1.7 *(cf. [ClaG89], Lemma 4.3.4.2)*
Let $\underline{C_0}, \dots, \underline{C_e}$ be the basis quantities of a Schur ring over $\mathbb{F}_{p^n}$ corresponding to the cyclotomic scheme $C(p^n, e)$. Then we have

$$(j-1, k-1) = p_{1,j}^k, \ j,k \in \{1, \dots, e\}.$$

Proof: For $j \in \{1\dots,e\}$ we have

$$\underline{C_1}\,\underline{C_j} = \sum_{k=0}^{e} p_{1j}^k \underline{C_k}.$$

Each element of $\underline{C_1}$ has the form ω^{xe} and each element of $\underline{C_j}$ has the form ω^{ye+j-1}, where ω is a primitive element of $\mathbb{F}_{p^n}^*$ and $x, y \in \{0,\dots,f-1\}$. The structure constant $p_{1,j}^k$, $j,k \in \{1,\dots,e\}$, equals the number of solutions (x,y,z) of $\omega^{xe} + \omega^{ye+j-1} = \omega^{ze+k-1}$. This equation is equivalent to $1 + \omega^{(y-x)e+j-1} = \omega^{(z-x)e+k-1}$ and the number of solutions of the last equation is $(j-1,k-1)$. □

If we know the cyclotomic numbers of the cyclotomic scheme $\langle \underline{C_0},\dots,\underline{C_e}\rangle$, then it is possible to compute all structure constants p_{ij}^k, $i,j,k \neq 0$ of $\langle \underline{C_0},\dots,\underline{C_e}\rangle$:

Lemma 5.1.8 *([ClaG89], Lemma 4.3.4.3)*
Let $C(p^n,e) = \langle \underline{C_0},\dots,\underline{C_e}\rangle$ be a cyclotomic scheme. Then for $i,j,k \in \{1,\dots,e\}$ and $l \in \mathbb{N}$ we have

$$p_{ij}^k = p_{i+1,j+1}^{k+1} = \ldots = p_{i+l,j+l}^{k+l},$$

where the "+" is defined by $a + b := [(a+b-1) \bmod e] + 1$.

Proof: The sets $C_1,\dots,C_e$ are cosets of $\mathbb{F}_{p^n}^*$ with respect to the subgroup C_1, where $C_i = \omega^{i-1}C_1, i \in \{1,\dots,e\}$. Thus, there is a cyclic transitive action of $\mathbb{F}_{p^n}^*$ on $C_1,\dots,C_e$. Generalizing the idea of the proof of Lemma 5.1.7 we can say that the structure constant p_{ij}^k equals the number of solutions (x,y,z) of the equation $\omega^{xe+i-1} + \omega^{ye+j-1} = \omega^{ze+k-1}$, $i,j,k \in \{1,\dots,e\}$. Since $\mathbb{F}_{p^n}^*$ acts transitively on the cosets by right multiplication we have the same solutions (x,y,z) for the equation $\omega^{xe+i-1+l} + \omega^{ye+j-1+l} = \omega^{ze+k-1+l}$, $l \in \mathbb{N}$. The number of solutions of the last equation equals $p_{i+l,j+l}^{k+l}$ (where "+" is defined as described above). Hence, we have $p_{ij}^k = p_{i+l,j+l}^{k+l}$ for $l \in \mathbb{N}$. □

5.2 Merging of classes in a cyclotomic scheme

In [FarKM94], Section 2.7.5 an algorithm for the computation of all subschemes of a given association scheme is described (Notice that in [FarKM94], Section 2.7.5 the language of cellular rings is used, i.e., the algorithm computes all cellular subrings of a cell.). The computation of subschemes is based on the knowledge of the intersection numbers of the association scheme. The following theorem is essential for this algorithm:

Theorem 5.2.1 *(cf. [FarKM94], p.40)*
Let $W = \langle A_0,\dots,A_d\rangle$ be a cellular ring and let $I = \{0,\dots,d\}$. Let $\{T_0,\dots,T_r\}$ be a partition of the set I and define $B_\alpha := \sum_{k\in T_\alpha} A_k$, $\alpha = 0,\dots,r$. Then $W' = \langle B_0,\dots,B_r\rangle$ is a cellular subring of W if and only if for all $\alpha,\beta,\gamma \in \{0,\dots,r\}$ and for all $k,l \in T_\gamma$ we have

$$\sum_{i\in T_\alpha}\sum_{j\in T_\beta} p_{ij}^k = \sum_{i\in T_\alpha}\sum_{j\in T_\beta} p_{ij}^l,$$

where p_{ij}^k, $i,j,k \in \{0,\dots,d\}$ are the structure constants of W.

The idea of the proof is based on the property of the product of basis matrices B_α, B_β, $\alpha, \beta \in \{0, \dots, r\}$. If W' is a cellular subring of W, we have $B_\alpha B_\beta = \sum_{\gamma=0}^{r} q_{\alpha\beta}^{\gamma} B_\gamma$, where the coefficients $q_{\alpha\beta}^{\gamma}$ are the structure constants of W'. Replacing B_α by $\sum_{k \in T_\alpha} A_k$, $\alpha = 0, \dots, r$, we get the condition for the structure constants p_{ij}^{k} for the cellular ring W given in the theorem.

If we want to determine cellular subrings of a cell, then it is clear that the unit matrix is always a basis matrix. In general, we will denote the unit matrix by A_0, hence, in the above theorem it is sufficient to consider partitions of the set $\{1, \dots, d\}$. Since cyclotomic schemes are association schemes and these correspond to cells (see Section 2.2.2), for the determination of subschemes it is also sufficient to investigate partitions of $\{1, \dots, d\}$.
The algorithm for the determination of all nontrivial subschemes of a given cyclotomic scheme $C(p^n, e)$ investigates all nontrivial partitions of the index set $\{1, \dots, e\}$ of the cyclotomic classes. This investigation is carried out in two steps. At first the so-called "good subsets" are computed (see below) and in a second step all partitions consisting of good subsets will be checked by the condition of Theorem 5.2.1, whether they correspond to a subscheme.
A subset T of $\{1, \dots, e\}$ is called a **good subset**, if it satisfies the following conditions (cf. [FarKM94], p. 83, here we switch to the language of group rings (cf. Definition 2.2.13) for the explanation of the conditions):

1. The set of basis quantities $\mathcal{S}_T := \{\underline{C_i} | i \in T\}$ is either *symmetric* (i.e., in $\mathcal{S}_T$ are only self-inverse basis quantities $\underline{C_i} = \underline{C_i}^{-1}$ or pairs of inverse basis quantities $\underline{C_i}, \underline{C_i}^{-1}$), or $\mathcal{S}_T$ is *antisymmetric* (i.e., there are only basis quantities $\underline{C_i}$ in $\mathcal{S}_T$ for which $\underline{C_i}^{-1} \notin \mathcal{S}_T$).

2. Let $n \in \mathbb{N}$. Then for each $m \in \{1, \dots, n\}$ the coefficients h_k in the decomposition of the m^{th} power $\left(\sum_{i \in T} \underline{C_i}\right)^m = \sum_{k=0}^{e} h_k \underline{C_k}$ are equal for all $k \in T$, e.g., for $m = 2$ we have
$$\left(\sum_{i \in T} \underline{C_i}\right)^2 = \sum_{i,j \in T} \underline{C_i}\,\underline{C_j} = \sum_{i,j \in T} \sum_{k=0}^{e} p_{ij}^{k} \underline{C_k}$$
and for all $k \in T$ the coefficients $h_k := \sum_{i,j \in T} p_{ij}^{k}$ are equal (cf. Theorem 5.2.1 with $\alpha = \beta$).

In [FarKM94], p. 83 it is mentioned that one cannot save much time choosing numbers $n \geq 4$ in the second condition. Experiments with computers have shown that the time for verifying Condition 2 for higher powers is not a sufficient compensation for the slight reduction of the number of good subsets. We will use $n = 2$ in our theoretical determination of subschemes of cyclotomic schemes.
It is easy to check that if a set T, which corresponds to a antisymmetric set of basis quantities $\mathcal{S}_T$, satisfies the second condition, then the set T' formed by the indices of the inverse basis quantities in $\mathcal{S}_{T'} := \{\underline{C_i}^{-1} | i \in T\}$ satisfies also the second condition, i.e., T' is a good subset if and only if T is a good subset. The subsets of order one and the complete set of all indices are trivial good subsets.

The merge $\sum_{i\in T} \underline{C_i} = \underline{\bigcup_{i\in T} C_i}$ of the basis quantities corresponding to a good subset T is a candidate for a basis quantity in a cellular subring.
In the second step of the algorithm all partitions $\{T_1, \ldots, T_r\}$ of $\{1, \ldots, e\}$ consisting of good subsets T_α will be investigated (cf. [FarKM94], pp. 84/85). At first we have to check, if for each good subset T_α, which corresponds to an antisymmetric set $\mathcal{S}_T$, the associated good subset $T'_\alpha := \{j | C_j^{-1} \in \mathcal{S}_T\}$ belongs to the partition (cf. Condition 3 in Definition 2.2.14). If this is the case then for $\underline{B_\alpha} := \sum_{i\in T_\alpha} \underline{C_i}$, $\alpha \in \{1, \ldots, r\}$ all products $\underline{B_\alpha}\, \underline{B_\beta}$ must be computed and checked whether there exist coefficients $q^\gamma_{\alpha\beta}$ such that

$$\underline{B_\alpha}\, \underline{B_\beta} = \sum_{\gamma=1}^{r} q^\gamma_{\alpha\beta} \underline{B_\gamma} + q^0_{\alpha\beta} \underline{B_0}.$$

If these coefficients exist, then by Theorem 5.2.1 this partition corresponds to a subscheme. We will give a small example:

Example 5.2.2 *We consider the trivial cyclotomic scheme $C(5,5)$ over the field $\mathbb{Z}_5$. We have $C_0 := \{0\}$ and the cyclotomic classes are $C_i := \{i\}$, $i = 1, \ldots, 4$. Now, consider the subset $\{1,4\}$ of the index set $\{0, \ldots, 4\}$. This subset is a good subset: (1) It is symmetric, because it consists of two elements whose corresponding basis quantities are inverse to each other ($\underline{C_1}^{-1} = \underline{\{-1\}} = \underline{\{4\}} = \underline{C_4}$), i.e., the elements form a pair of antisymmetric elements and (2) we have*

$$(\underline{C_1 \cup C_4})^2 = (\underline{\{1,4\}})^2 = \underline{\{1+1, 1+4, 4+1, 4+4\}} = 2\underline{\{0\}} + \underline{\{2,3\}},$$

i.e., the coefficients for $\underline{C_1}$ and $\underline{C_4}$ in the decomposition of $(\underline{C_1 \cup C_4})^2$ are equal (to 0). In the same way we get that $\{2,3\}$ is a good subset.
The next step is to check the products $(\underline{C_1 \cup C_4})\,(\underline{C_2 \cup C_3}), \underline{C_0}\,(\underline{C_1 \cup C_4}), \underline{C_0}\,(\underline{C_2 \cup C_3})$. For example, we have

$$(\underline{C_1 \cup C_4})\,(\underline{C_2 \cup C_3}) = \underline{\{1,4\}}\,\underline{\{2,3\}} = \underline{\{1+2, 1+3, 4+2, 4+3\}} = \underline{\{1,4\}} + \underline{\{2,3\}},$$

i.e, the coefficients for $\underline{C_1}$ and $\underline{C_4}$ are equal (to 1) and the coefficients for $\underline{C_2}$ and $\underline{C_3}$ are also equal (to 1). Analogously, we can show that the condition in Step 2 of the algorithm is satisfied for the other two products. Since the one-element subset $\{0\}$ is always a good subset, the partition $\{\{0\}, \{1,4\}, \{2,3\}\}$ corresponds to a subscheme of $C(5,5)$ with classes $C_0, C_1 \cup C_4, C_2 \cup C_3$.

Since it is possible that this algorithm determines all subschemes of an association scheme with d classes by the intersection numbers (structure constants) the advantage in the special case of cyclotomic schemes is clear: Here the necessary d^3 intersection numbers are determined by d^2 cyclotomic numbers and by Lemma 5.1.5 certain cyclotomic numbers are equal. In general, the existence of good subsets and of subschemes of a cyclotomic scheme depends on equations for cyclotomic numbers which only some of the cyclotomic numbers must satisfy. We describe this in the next section.

5.3 Subschemes of cyclotomic schemes

In this section we determine subschemes of cyclotomic schemes by cyclotomic numbers. Therefore, we use the algorithm given in the previous section. We explain the procedure for the first case $C(p^n, 3)$ with $p^n = 3f + 1$, f odd, in detail. For the other cases the necessary steps are presented in a brief manner.
Notice that there exist already results for the investigation of cyclotomic schemes by cyclotomic numbers: In [ClaG89], Sections 4.4 and 4.5., H. L. Claasen and R. W. Goldbach considered 2-class and 3-class cyclotomic schemes. Their aim was to describe the intersection number of the cyclotomic schemes by cyclotomic numbers. The case of so-called uniform cyclotomic numbers is described in [BauMW82]. Here the cyclotomic numbers satisfy the equations $(i, 0) = (0, i) = (i, i) = (0, 1)$ for $i \neq 0$ and $(i, j) = (1, 2)$ for $0 \neq i \neq j \neq 0$. In Section 8 in [BauMW82] results for difference sets are considered. In [vLiS81] and [CalK86] the investigation of cyclotomic schemes was focused on strongly regular graphs and partial difference sets, i.e., the description of certain 2-class cyclotomic schemes.
In this section we make a further step. We want to determine systematically all possible subschemes of certain cyclotomic schemes by the method described in the previous section. Since 2-class association schemes have only trivial subschemes, we start with the case $e = 3$.

5.3.1 The case e=3, f odd

We consider cyclotomic schemes $C(p^n, 3)$ with $p^n = 3f + 1$, f odd. It is easy to check that in this case $p = 2$. By Lemma 5.1.5.2 we have the following equations for the cyclotomic numbers:

$$(0,1) = (1,0) = (2,2), \quad (0,2) = (2,0) = (1,1), \quad (1,2) = (2,1). \tag{5.1}$$

With this equations we get by Lemma 5.1.6:

$$\begin{aligned} p^n &= |(0,0) + 2(1,2) + 3(0,1)\exp(\tfrac{2\pi i}{3}) + 3(0,2)\exp(\tfrac{4\pi i}{3})|^2 \\ &= |(0,0) + 2(1,2) - \tfrac{3}{2}(0,1) - \tfrac{3}{2}(0,2) + i\tfrac{3\sqrt{3}}{2}((0,1) - (0,2))|^2. \end{aligned}$$

Defining $x := 2(0,0) + 4(1,2) - 3(0,1) - 3(0,2)$ and $y := (0,1) - (0,2)$ we have

$$4p^n = x^2 + 27y^2.$$

Lemma 5.1.5.3 yields the equations

$$(0,0) + (0,1) + (0,2) = f - 1, \quad (1,0) + (1,1) + (2,1) = f, \quad (2,0) + (2,1) + (2,2) = f.$$

Using the fact that certain cyclotomic numbers are equal gives the equations

$$(0,0) + (0,1) + (0,2) = f - 1 = \frac{p^n - 4}{3} \text{ and } (0,1) + (0,2) + (1,2) = f = \frac{p^n - 1}{3}.$$

Now we have the situation that each cyclotomic number is equal to one of the four cyclotomic numbers $(0,0), (0,1), (0,2), (1,2)$ which are determined by four equations:

$$\begin{array}{ll} 2(0,0)+4(1,2)-3(0,1)-3(0,2)=x & (0,1)-(0,2)=y \\ (0,0)+(0,1)+(0,2)=\frac{p^n-4}{3} & (0,1)+(0,2)+(1,2)=\frac{p^n-1}{3}. \end{array}$$

Solving this system of equations we get the following formulas for the cyclotomic numbers with variables p^n, x and y (see also [Sto67], Lemma 7):

1. $(0,0) = \frac{1}{9}(x + p^n - 8)$,
2. $(0,1) = (1,0) = (2,2) = \frac{1}{18}(2p^n + 9y - x - 4)$,
3. $(0,2) = (2,0) = (1,1) = \frac{1}{18}(2p^n - 9y - x - 4)$,
4. $(1,2) = (2,1) = \frac{1}{9}(p^n + x + 1)$.

Notice that the parameters x, y, p^n are not independent, because we have $4p^n = x^2 + 27y^2$. Moreover, it is not true that all $x, y \in \mathbb{N}$ with $4p^n = x^2 + 27y^2$ give rise to cyclotomic numbers. By [Sto67], Chapter 6 resp. [ClaG89], Remark 4.5.3 these equations for the cyclotomic numbers only hold, if we have $x \equiv 1 \bmod 3$.

As described in the previous section the existence of subschemes of the cyclotomic scheme depends on certain equalities for the cyclotomic numbers of the scheme. In the following we will determine these conditions and then check, if they contradict the conditions for the numbers x, y and p^n.

We have to check if the following subsets of $\{1,2,3\}$ are good subsets:

$$\{1,2\}, \{1,3\}, \{2,3\}.$$

In the case of three cyclotomic classes all classes are symmetric (Lemma 5.1.2). So, each of these subsets satisfies the first condition for good sets. For the second condition we have to investigate the structure constants in the decomposition

$$(\underline{C_i \cup C_j})^2 = \sum_{k=0}^{3} (p_{ii}^k + p_{ij}^k + p_{ji}^k + p_{jj}^k)\underline{C_k}$$

for $i = 1,2$ and $j = 2,3$. By Lemma 5.1.8 we get for all three subsets the same conditions for the structure constants, because the decompositions of $(\underline{C_1 \cup C_2})^2, (\underline{C_1 \cup C_3})^2, (\underline{C_2 \cup C_3})^2$ provides the equations

$$\begin{array}{rcl} p_{11}^1 + p_{12}^1 + p_{21}^1 + p_{22}^1 &=& p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2, \\ p_{11}^1 + p_{13}^1 + p_{31}^1 + p_{33}^1 &=& p_{11}^3 + p_{13}^3 + p_{31}^3 + p_{33}^3, \\ p_{22}^2 + p_{23}^2 + p_{32}^2 + p_{33}^2 &=& p_{22}^3 + p_{23}^3 + p_{32}^3 + p_{33}^3. \end{array}$$

These three equations are equivalent by Lemma 5.1.8. Thus, it is sufficient to consider only one set, e.g., the set $\{1,2\}$.

From

$$p^1_{11} + p^1_{12} + p^1_{21} + p^1_{22} = p^2_{11} + p^2_{12} + p^2_{21} + p^2_{22}$$

we get by Lemma 5.1.7

$$(0,0) + (1,0) + (2,2) + (0,2) = (0,1) + (1,1) + (2,0) + (0,0)$$

and simplified by the equalities in (5.1), p. 121 we get

$$(0,0) + 2(0,1) + (0,2) = (0,0) + (0,1) + 2(0,2) \Leftrightarrow (0,1) = (0,2).$$

If we substitute the cyclotomic numbers $(0,1)$ and $(0,2)$ by their representations in x, y, p^n we get the equation

$$\frac{1}{18}(2p^n + 9y - x - 4) = \frac{1}{18}(2p^n - 9y - x - 4)$$

which is equivalent to $y = -y$. Consequently, we get $y = 0$ and thus, $4p^n = x^2 + 27y^2 = x^2$. It follows that $x = \pm 2\sqrt{p^n}$ and since $p = 2$ (e, f are odd!) we have $x = \pm 2^{\frac{n+2}{2}}$, i.e., n must be even.

The sets $\{1,2\}, \{1,3\}, \{2,3\}$ are good subsets, if n is even, $x = \pm 2^{\frac{n+2}{2}}$ and, as mentioned above, $x \equiv 1 \bmod 3$.

Now we have to investigate the following partitions (recall that one-element sets are always good subsets):

$$\{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{1\},\{2,3\}\}.$$

Again, by Lemma 5.1.8 the condition that a partition corresponds to a subscheme is equivalent for all of these three partitions. We only consider the first partition $\{\{1,2\},\{3\}\}$:
We have already checked the case $(\underline{C_1 \cup C_2})^2$. For $(\underline{C_1 \cup C_2})\underline{C_3}$ and $\underline{C_3}\underline{C_3}$ we get:

$$\begin{aligned} & p^1_{13} + p^1_{23} = p^2_{13} + p^2_{23} \\ \Leftrightarrow\ & (2,0) + (1,2) = (2,1) + (1,0) \\ \Leftrightarrow\ & (0,1) = (0,2) \Leftrightarrow y = 0. \end{aligned}$$

$$p^1_{33} = p^2_{33} \Leftrightarrow (0,1) = (0,2) \Leftrightarrow y = 0.$$

Altogether, there is no further condition and we get the following theorem:

Theorem 5.3.1 *Let $C(\frac{x^2}{4}, 3) = \langle C_0, C_1, C_2, C_3\rangle$ be a cyclotomic scheme where $x = \pm 2^{\frac{n+2}{2}}$ $x \equiv 1$ mod 3. Then the scheme has the following nontrivial subschemes: $\langle C_0, C_1 \cup C_2, C_3\rangle$, $\langle C_0, C_1, C_2 \cup C_3\rangle$ and $\langle C_0, C_1 \cup C_3, C_2\rangle$.*

Notice that the cyclotomic scheme in the theorem corresponds to an amorphic cellular ring, because we can merge arbitrarily the cyclotomic classes for the construction of subschemes (cf. Definition 2.2.7).

Example 5.3.2 *As an example we consider the case $n = 4$ (n must be even!). We have $2^4 = 16$ and $x = \pm 2^3 = \pm 8$. The condition $x \equiv 1$ mod 3, implies $x = -8$. Merging two classes, e.g., C_1 and C_2, in the cyclotomic scheme gives a subscheme which corresponds to the Clebsch graph. Moreover, all three subschemes correspond to the $(16, 5, 0, 2)$-partial difference set in the group $(\mathbb{Z}_2)^4$ which is unique up to CI-equivalence (cf. Table 4.6).*

In the table we list some cases for small numbers n with the corresponding strongly regular graphs and if possible we give reference to the partial difference sets in the appendix (the computations were done with GAP):

n	p^n	*parameters*	*graph*	*partial difference set*
2	4	(4,1,0,0)	$2 \circ K_2$	
4	16	(16,5,0,2)	Clebsch	App. D 4.1.8.1
6	64	(64,21,8,6)	$L_3((\mathbb{Z}_2)^3)$	App. G 23.2.30.1
8	256	(256,85,24,30)		

5.3.2 The case e=3, f even

In the case $e = 3$, f even, the cyclotomic classes are again symmetric (Lemma 5.1.2). We have to execute the same computations as in the case $e = 3$, f odd. Since we get by Lemma 5.1.5 the same equalities for the cyclotomic numbers, we have the same formulas and the same conditions for the existence of subschemes with one exception: if $p \equiv 1$ mod 3 then the representation $4p^n = x^2 + 27y^2$, $x \equiv 1$ mod 3 only determines the cyclotomic numbers, if $\gcd(p, x) = 1$ (cf. [Sto67], Chapter 6 resp. [ClaG89], Remark 4.5.3).
The fact that the number f is even implies $p \neq 2$. Thus, we have $p^n = \frac{x^2}{4}$ resp. $x = \pm 2\sqrt{p^n}$, where $x \equiv 1$ mod 3. It follows that n and x are even. However, the condition $\gcd(p, x) = 1$ in the case $p \equiv 1$ mod 3 is not satisfied because $\gcd(p, x) = \gcd(p, 2\sqrt{p^n}) = p > 1$. Hence, there are no subschemes for $p \equiv 1$ mod 3 which can be created by merging cyclotomic classes.

Theorem 5.3.3 *Let p be an odd prime with $p \equiv 1$ mod 3 and let $n \in \mathbb{N}$ even. Then no nontrivial subscheme of the cyclotomic scheme $C(p^n, 3)$ can be constructed by merging cyclotomic classes.*

For $p \equiv 2$ mod 3 we do not have this extra condition $\gcd(p, x) = 1$ and, hence, there exists subschemes in these cases. The condition $x \equiv 1$ mod 3 determines the sign of x.

Theorem 5.3.4 *Let p be an odd prime with $p \equiv 2$ mod 3 and let $n \in \mathbb{N}$ even. Then the cyclotomic scheme $C(p^n, 3) = \langle C_0, C_1, C_2, C_3 \rangle$ has the following nontrivial subschemes: $\langle C_0, C_1 \cup C_2, C_3 \rangle$, $\langle C_0, C_1, C_2 \cup C_3 \rangle$ and $\langle C_0, C_1 \cup C_3, C_2 \rangle$.*

For the case $p \equiv 2$ mod 3, p an odd prime and n even we get a cyclotomic scheme $C(p^n, 3)$ which corresponds to an amorphic cellular ring, since arbitrary mergings of its cyclotomic classes yield subschemes.

Example 5.3.5 *Consider the case $p = 5$, $n = 2$. We have the cyclotomic scheme $C(25, 3)$. Merging two cyclotomic classes yields a subscheme which corresponds to the lattice graph $L_2(5)$. Moreover, it corresponds to the (25,8,3,2)-partial difference set in $\mathbb{Z}_5 \times \mathbb{Z}_5$ which is unique up to CI-equivalence (see Corollary 3.2.23).*

Again with GAP we computed some cases for small numbers n with the corresponding strongly regular graphs and if possible we give reference to the partial difference sets in the appendix:

n	p	p^n	*parameters*	*graph*	*partial difference set*
2	*5*	*25*	*(25,8,3,2)*	$L_2(5)$	*App. D 8.1.1.1*
4	*5*	*625*	*(625,208,63,72)*	$NL_8(25)^\dagger$	
2	*11*	*121*	*(121,40,15,12)*	$L_4(11)$	*App. G 39.1.1.1*
2	*17*	*289*	*(289,96,35,30)*	$L_6(17)$	

† *negative latin square type graph (see Page 65)*

In the case of Theorem 5.3.3 we have no subschemes. For example, if we take $p = 7 \equiv 1$ mod 3, $n = 2$, then a subscheme of $C(49, 3)$ would correspond to a strongly regular graph with 49 vertices and valency 16. Feasible parameters are $(49, 16, 3, 6)$ (cf. [Bro96]). In [BusHMW89] a proof is given that such a strongly regular graph does not exist.

5.3.3 The case e=4, f odd

Like in the case $e = 3$ we get for $e = 4, f$ odd, equalities for several cyclotomic numbers by Lemma 5.1.5. And as in the previous case with the representation $p^n = x^2 + 4y^2$ which we get by Lemma 5.1.6 we can determine formulas for the cyclotomic numbers which depend on parameters p^n, x and y. By [Sto67] the additional property $x \equiv 1 \bmod 4$ ensures that x, y, p^n determine the cyclotomic numbers. Notice that $p^n = 4f + 1 \equiv 1 \bmod 4$, i.e., if $p \equiv 3 \bmod 4$, then n must be even.
For the cyclotomic numbers we get by Lemma 5.1.5 and Lemma 5.1.6 the following equations and formulas where $p^n = x^2 + 4y^2$, $x \equiv 1 \bmod 4$ (see also [Sto67], Lemma 19):

1. $(0, 0) = (2, 2) = (2, 0) = \frac{1}{16}(p^n + 2x - 7)$,
2. $(0, 1) = (1, 3) = (3, 2) = \frac{1}{16}(p^n + 2x + 1 - 8y)$,
3. $(0, 2) = \frac{1}{16}(p^n - 6x + 1)$,
4. $(0, 3) = (1, 2) = (3, 1) = \frac{1}{16}(p^n + 2x + 8y + 1)$,
5. $(1, 0) = (2, 1) = (2, 3) = (3, 3) = (3, 0) = (1, 1) = \frac{1}{16}(p^n - 2x - 3)$.

For the determination of subschemes we have to consider the following subsets of $\{1, \ldots, 4\}$:

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}.$$

Notice that by Lemma 5.1.2 the cyclotomic scheme for $e = 4, f$ odd is not symmetric: The cyclotomic classes C_1 and C_3 resp. C_2 and C_4 are antisymmetric pairs. Thus, the 3-subsets of $\{1, \dots, 4\}$ cannot be good subsets, because a 3-subset always contains two indices which are associated to a pair of antisymmetric classes and one index which is associated to a single antisymmetric class. Hence, the first condition for good subsets is not satisfied.
One can check that by Lemma 5.1.8 the conditions for being a good subset are equivalent for the subsets $\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}$ resp. for the subsets $\{1, 3\}, \{2, 4\}$. Thus, we only have to consider the subsets $\{1, 2\}$ and $\{1, 3\}$.
For the second condition for good sets we have to check for $i = 1$, $j = 2, 3$, if the coefficients p_{ij}^k in the expression

$$(\underline{C_i \cup C_j})^2 = \sum_{k=0}^{4} p_{ij}^k \underline{C_k}$$

satisfy the equation

$$p_{ii}^i + p_{ij}^i + p_{ji}^i + p_{jj}^i = p_{ii}^j + p_{ij}^j + p_{ji}^j + p_{jj}^j.$$

Since by Lemma 5.1.8 the equality $p_{11}^1 = p_{22}^2 = p_{33}^3$ holds, the second condition for good subsets simplifies to

$$p_{ij}^i + p_{ji}^i + p_{jj}^i = p_{ii}^j + p_{ij}^j + p_{ji}^j,$$

where $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$.

1. The subset $\{1, 2\}$:
We have

$$p_{12}^1 + p_{21}^1 + p_{22}^1 = p_{11}^2 + p_{12}^2 + p_{21}^2.$$

By Lemma 5.1.7 and Lemma 5.1.8 we obtain:

$$(1, 0) + (3, 3) + (0, 3) = (0, 1) + (1, 1) + (3, 1)$$

and by the above given table of cyclotomic numbers and equations this equation simplifies to $(0, 3) = (0, 1)$. Replacing the cyclotomic numbers by the formulas we get

$$\frac{1}{16}(p^n + 2x + 8y + 1) = \frac{1}{16}(p^n + 2x - 8y + 1)$$

and this implies $y = 0$. Since $p^n = 4f + 1$ and f odd, with $f = 2k + 1$, $k \in \mathbb{N}$ we have $p^n = 8k + 5 \equiv 5 \bmod 8$. With $p^n = x^2 + 4y^2$ and $y = 0$ we have $p^n = x^2$, where $x \equiv 1 \bmod 4$. Since $x = 4l + 1$, $l \in \mathbb{N}$ implies $x^2 = 16l^2 + 8l + 1 \equiv 1 \bmod 8$, it follows that the equation $p^n = x^2$ has no solution. Thus, the subsets $\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}$ are not good subsets.

2. The subset $\{1, 3\}$:
We get

$$p_{13}^1 + p_{31}^1 + p_{33}^1 = p_{11}^3 + p_{13}^3 + p_{31}^3$$

and obtain

$$(2, 0) + (2, 2) + (0, 2) = (0, 2) + (2, 2) + (2, 0).$$

This is always true.
Now we have to consider all possible partitions of $\{1,2,3,4\}$ consisting of good subsets. Notice that for each good subset T in a partition there must be also a good subset $T' := \{j | C_j^{-1} = C_i, i \in T\}$ in the partition. Since $C_1^{-1} = C_3$ and $C_2^{-1} = C_4$ we have to consider the following non-trivial partitions:

$$1)\ \{\{1,3\},\{2\},\{4\}\} \qquad 2)\ \{\{1\},\{2,4\},\{3\}\} \qquad 3)\ \{\{1,3\},\{2,4\}\}$$

Now each of these partitions must be checked, whether it corresponds to a subscheme of the cyclotomic scheme. Again, we can use the fact that by Lemma 5.1.8 for some partitions the conditions are equivalent. This is the case for the Partitions 1 and 2. Thus, we only have to consider the Partitions 1 and 3.
Partition 1:
We have already checked $(\underline{C_1 \cup C_3})^2$. It remains the investigation of the decompositions of $(\underline{C_1 \cup C_3})\underline{C_2}$ and $(\underline{C_2})^2$.

$$\begin{aligned}(\underline{C_1 \cup C_3})\underline{C_2}: \quad & p_{12}^1 + p_{32}^1 = p_{12}^3 + p_{32}^3 \\ & \Leftrightarrow (1,0) + (3,2) = (1,2) + (3,0) \\ & \Leftrightarrow -y = y \Leftrightarrow y = 0.\end{aligned}$$

As we have seen before the condition $y = 0$ yields no solution. Thus, the Partitions 1 and 2 do not correspond to a subscheme.
Partition 3:

$$\begin{aligned}(\underline{C_1 \cup C_3})^2: \quad & p_{11}^2 + p_{13}^2 + p_{31}^2 + p_{33}^2 = p_{11}^4 + p_{13}^4 + p_{31}^4 + p_{33}^4 \\ & \Leftrightarrow (0,1) + (2,1) + (2,3) + (0,3) = (0,3) + (2,3) + (2,1) + (0,1). \\ (\underline{C_2 \cup C_4})^2: \quad & p_{22}^1 + p_{24}^1 + p_{42}^1 + p_{44}^1 = p_{22}^3 + p_{24}^3 + p_{42}^3 + p_{44}^3 \\ & \text{equivalent to first line.} \\ (\underline{C_1 \cup C_3})(\underline{C_2 \cup C_4}): \quad & 1)\ p_{12}^1 + p_{32}^1 + p_{14}^1 + p_{34}^1 = p_{12}^3 + p_{32}^3 + p_{14}^3 + p_{34}^3 \\ & \Leftrightarrow (1,0) + (3,2) + (3,0) + (1,2) = (1,2) + (3,0) + (3,2) + (1,0). \\ & 2)\ p_{12}^2 + p_{32}^2 + p_{14}^2 + p_{34}^2 = p_{12}^4 + p_{32}^4 + p_{14}^4 + p_{34}^4 \\ & \Leftrightarrow (1,1) + (3,3) + (3,1) + (1,3) = (1,3) + (3,1) + (3,3) + (1,1).\end{aligned}$$

For Partition 3 we get no further condition. Thus, in every case the merging of the two antisymmetric pairs of cyclotomic classes will give a subscheme:

Theorem 5.3.6 *Let $p^n = 4f + 1$, where f is odd and let $C(p^n, 4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ be a cyclotomic scheme. Then there exists only one nontrivial subscheme. It can be constructed by merging the classes C_1, C_3 resp. C_2, C_4. This subscheme is symmetric and corresponds to the Paley graph $P(p^n)$.*

The subscheme is symmetric, because $C_1^{-1} = C_3$ and $C_2^{-1} = C_4$. For a primitive element $\omega \in \mathbb{F}_{p^n}$ the sets C_1, C_3 consist of the elements ω^{4s} and ω^{4s+2}, $s = 0, \ldots, f-1$. These elements are exactly the non-zero squares in $\mathbb{F}_{p^n}$ and therefore, the set $C_1 \cup C_3$ is the partial difference set which corresponds to the Paley graph (cf. Proposition 3.2.6).

The fact that the merging of each pair of antisymmetric cyclotomic classes yields the classes of a symmetric subscheme is known as the **symmetrization** or the **symmetric closure** of an association scheme (cf. [BanI84], [ClaG89]). Hence, the result that the merging of C_1, C_3 resp. C_2, C_4 yields a subscheme is not new. However, the important fact in Theorem 5.3.6 is that this subscheme is the only nontrivial subscheme in the mentioned cyclotomic scheme.

5.3.4 The case e=4, f even

For the case $e = 4, f$ even, we have the same representation of p^n by x and y, namely $p^n = x^2 + 4y^2$, where $x \equiv 1 \bmod 4$. For the cyclotomic numbers we get by this representation and by Lemma 5.1.5 the following equalities and formulas (see also [Sto67], Lemma 19):

1. $(0,0) = \frac{1}{16}(p^n - 6x - 11)$,
2. $(0,1) = (3,3) = (1,0) = \frac{1}{16}(p^n + 2x + 8y - 3)$,
3. $(0,2) = (2,2) = (2,0) = \frac{1}{16}(p^n + 2x - 3)$,
4. $(0,3) = (1,1) = (3,0) = \frac{1}{16}(p^n + 2x - 8y - 3)$,
5. $(1,2) = (2,1) = (2,3) = (3,2) = (3,1) = (1,3) = \frac{1}{16}(p^n - 2x + 1)$.

We have to check the same subsets as in the case $e = 4, f$ odd and, in addition, the 3-subsets of $\{1,2,3,4\}$. Here 3-subsets are candidates for good subsets since all cyclotomic classes are symmetric (Lemma 5.1.2). By Lemma 5.1.8 the condition that a 3-subsets is a good subset is the equivalent for all 3-subsets of $\{1,2,3,4\}$.
1. The subset $\{1,2\}$:
We have

$$p^1_{12} + p^1_{21} + p^1_{22} = p^2_{11} + p^2_{12} + p^2_{21}.$$

By Lemma 5.1.7 and Lemma 5.1.8 we obtain:

$$(1,0) + (3,3) + (0,3) = (0,1) + (1,1) + (3,0)$$

which simplifies to $(0,1) = (0,3)$. Replacing the cyclotomic numbers by the corresponding formulas we get $8y = -8y \Leftrightarrow y = 0$ (in this case $y = 0$ is possible).
2. For the subset $\{1,3\}$ we get the same equation as for $e = 4, f$ odd. Thus, this subset is always a good subset.
3. The subset $\{1,2,3\}$:
We have for $(\underline{C_1 \cup C_2 \cup C_3})^2$:

$$\begin{array}{rl}
& p^1_{11} + p^1_{12} + p^1_{13} + p^1_{21} + p^1_{22} + p^1_{23} + p^1_{31} + p^1_{32} + p^1_{33} \\
= & p^2_{11} + p^2_{12} + p^2_{13} + p^2_{21} + p^2_{22} + p^2_{23} + p^2_{31} + p^2_{32} + p^2_{33} \\
= & p^3_{11} + p^3_{12} + p^3_{13} + p^3_{21} + p^3_{22} + p^3_{23} + p^3_{31} + p^3_{32} + p^3_{33} \\
\Leftrightarrow & (0,0) + (1,0) + (2,0) + (3,3) + (0,3) + (1,3) + (2,2) + (3,2) + (0,2) \\
= & (0,1) + (1,1) + (2,1) + (3,0) + (0,0) + (1,0) + (2,3) + (3,3) + (0,3) \\
= & (0,2) + (1,2) + (2,2) + (3,1) + (0,1) + (1,1) + (2,0) + (3,0) + (0,0) \\
\Leftrightarrow & 3(0,2) + (0,1) = 2(0,1) + 2(0,3) = 3(0,2) + (0,3) \\
\Leftrightarrow & 8y = 0 = -8y \Leftrightarrow y = 0.
\end{array}$$

Hence, we get the following non-trivial good subsets:

$$\begin{array}{l} \{1,2\},\{1,4\},\{3,4\},\{2,3\} \text{ if } y=0, \\ \{1,3\},\{2,4\} \text{ for all } x,y, \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} \text{ if } y=0, \end{array}$$

where $x \equiv 1 \pmod 4$ and $p^n = x^2 + 4y^2$.

Now we have to consider all possible partitions of $\{1,2,3,4\}$ consisting of good subsets.

	partition	condition
1	$\{\{1,2\},\{3\},\{4\}\}$	$y=0$
2	$\{\{1,3\},\{2\},\{4\}\}$	
3	$\{\{1,4\},\{2\},\{3\}\}$	$y=0$
4	$\{\{1\},\{2,3\},\{4\}\}$	$y=0$
5	$\{\{1\},\{2,4\},\{3\}\}$	
6	$\{\{1\},\{2\},\{3,4\}\}$	$y=0$
7	$\{\{1,2\},\{3,4\}\}$	$y=0$
8	$\{\{1,3\},\{2,4\}\}$	
9	$\{\{1,4\},\{2,3\}\}$	$y=0$
10	$\{\{1,2,3\},\{4\}\}$	$y=0$
11	$\{\{1,2,4\},\{3\}\}$	$y=0$
12	$\{\{1,3,4\},\{2\}\}$	$y=0$
13	$\{\{2,3,4\},\{1\}\}$	$y=0$

Since the Partitions 1, 3, 4 and 6 resp. 2 and 5 resp. 7 and 9 resp. 10, 11, 12 and 13 yield equivalent conditions, we only have to check the Partitions 1, 2, 7 and 10.
Moreover, the Partition 8 yields the same condition as in the case f odd. Thus, here we have the same result and Theorem 5.3.6 can be reformulated:

Theorem 5.3.7 *Let $p^n \equiv 1$ mod 4 and let $C(p^n,4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ be a cyclotomic scheme. Then there exists a nontrivial subscheme which can be constructed by merging the classes C_1, C_3 resp. C_2, C_4. This subscheme is symmetric and corresponds to the Paley graph $P(p^n)$.*

Now we consider the other partitions.

Partition 1:

$$\begin{aligned}
(\underline{C_1 \cup C_2})\underline{C_3}:\ & p_{13}^1 + p_{23}^1 = p_{13}^2 + p_{23}^2 \\
& \Leftrightarrow (2,0) + (1,3) = (2,1) + (1,0) \\
& \Leftrightarrow (0,2) = (0,1) \Leftrightarrow 8y = 0 \Leftrightarrow y = 0. \\
(\underline{C_1 \cup C_2})\underline{C_4}:\ & p_{14}^1 + p_{24}^1 = p_{14}^2 + p_{24}^2 \\
& \Leftrightarrow (3,0) + (2,3) = (3,1) + (2,0) \\
& \Leftrightarrow (0,3) = (0,2) \Leftrightarrow -8y = 0 \Leftrightarrow y = 0. \\
(\underline{C_3})^2:\ & p_{33}^1 = p_{33}^2 \Leftrightarrow (0,2) = (0,3) \Leftrightarrow y = 0. \\
(\underline{C_4})^2:\ & p_{44}^1 = p_{44}^2 \Leftrightarrow (0,1) = (0,2) \Leftrightarrow y = 0. \\
\underline{C_3}\,\underline{C_4}:\ & p_{34}^1 = p_{34}^2 \Leftrightarrow (1,2) = (1,3) \text{ always true.}
\end{aligned}$$

Hence, for the Partitions 1, 3, 4 and 6 we have the condition $y = 0$. In contrast to the case $e = 4$, f odd, here the condition $y = 0$ does not lead to a contradiction. We have $p^n = 4f + 1$ and with $f = 2k$, $k \in \mathbb{N}$, we have $p^n = 8k + 1 \equiv 1 \bmod 8$. Since we have also $p^n = x^2 \equiv 1 \bmod 8$ (see the case f odd), we have no contradiction. We only get the condition that n is even. Hence, we have the following theorem:

Theorem 5.3.8 *(Partitions 1, 3, 4, 6) Let $p^{2n} \equiv 1$ mod 8 and let $C(p^{2n}, 4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ be a cyclotomic scheme. Then there exist nontrivial subschemes*

$$\langle C_0, C_1 \cup C_2, C_3, C_4 \rangle,\ \langle C_0, C_1, C_4, C_2 \cup C_3 \rangle, \langle C_0, C_1, C_2, C_3 \cup C_4 \rangle \text{ and } \langle C_0, C_1 \cup C_4, C_2, C_3 \rangle.$$

Partition 2:

$$\begin{aligned}
(\underline{C_1 \cup C_3})\underline{C_2}:\ & p_{12}^1 + p_{32}^1 = p_{12}^3 + p_{32}^3 \\
& \Leftrightarrow (1,0) + (3,2) = (1,2) + (3,0) \\
& \Leftrightarrow (0,1) = (0,3) \Leftrightarrow y = -y \Leftrightarrow y = 0. \\
(\underline{C_1 \cup C_3})\underline{C_4}:\ & p_{14}^1 + p_{34}^1 = p_{14}^3 + p_{34}^3 \\
& \text{is equivalent to the first line.} \\
(\underline{C_2})^2:\ & p_{22}^1 = p_{22}^3 \Leftrightarrow (0,3) = (0,1) \Leftrightarrow y = 0. \\
(\underline{C_4})^2:\ & p_{44}^1 = p_{44}^3 \text{ is equivalent to the previous line.} \\
\underline{C_2}\,\underline{C_4}:\ & p_{24}^1 = p_{24}^3 \Leftrightarrow (2,3) = (2,1) \text{ always true.}
\end{aligned}$$

For the Partitions 2 and 5 we have the condition $y = 0$. Hence, we get the following theorem:

Theorem 5.3.9 *(Partitions 2, 5) Let $p^{2n} \equiv 1$ mod 8 and let $C(p^{2n}, 4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ be a cyclotomic scheme. Then there exist nontrivial subschemes $\langle C_0, C_1 \cup C_3, C_2, C_4 \rangle$ and $\langle C_0, C_1, C_3, C_2 \cup C_4 \rangle$.*

Now we consider Partition 7:

$$
\begin{aligned}
(\underline{C_1 \cup C_2})^2: \quad & p^3_{11} + p^3_{12} + p^3_{21} + p^3_{22} = p^4_{11} + p^4_{12} + p^4_{21} + p^4_{22} \\
& \Leftrightarrow (0,2) + (1,2) + (3,1) + (0,1) = (0,3) + (1,3) + (3,2) + (0,2) \\
& \Leftrightarrow (0,1) = (0,3) \Leftrightarrow y = 0. \\
(\underline{C_3 \cup C_4})^2: \quad & p^1_{33} + p^1_{34} + p^1_{43} + p^1_{44} = p^2_{33} + p^2_{34} + p^2_{43} + p^2_{44} \\
& \text{ equivalent to the first line.} \\
(\underline{C_1 \cup C_2})(\underline{C_3 \cup C_4}): \quad & 1)\, p^1_{13} + p^1_{14} + p^1_{23} + p^1_{24} = p^2_{13} + p^2_{14} + p^2_{23} + p^2_{24} \\
& \Leftrightarrow (2,0) + (3,0) + (1,3) + (2,3) = (2,1) + (3,1) + (1,0) + (2,0) \\
& \Leftrightarrow (0,3) = (0,1) \Leftrightarrow y = 0. \\
& 2)\, p^3_{13} + p^3_{14} + p^3_{23} + p^3_{24} = p^4_{13} + p^4_{14} + p^4_{23} + p^4_{24} \\
& \Leftrightarrow (2,2) + (3,2) + (1,1) + (2,1) = (2,3) + (3,3) + (1,2) + (2,2) \\
& \Leftrightarrow (0,3) = (0,1) \Leftrightarrow y = 0.
\end{aligned}
$$

Thus, for Partition 7 and Partition 9 we have the condition $y = 0$.

Theorem 5.3.10 *(Partitions 7, 9) Let* $p^{2n} \equiv 1$ *mod 8 and let* $C(p^{2n}, 4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ *be a cyclotomic scheme. Then there exist nontrivial subschemes* $\langle C_0, C_1 \cup C_2, C_3 \cup C_4 \rangle$ *and* $\langle C_0, C_1 \cup C_4, C_2 \cup C_3 \rangle$.

It remains Partition 10:

$$
\begin{aligned}
(\underline{C_1 \cup C_2 \cup C_3})\underline{C_4} \quad & p^1_{14} + p^1_{24} + p^1_{34} = p^2_{14} + p^2_{24} + p^2_{34} = p^3_{14} + p^3_{24} + p^3_{34} \\
\Leftrightarrow \quad & (3,0) + (2,3) + (1,2) = (3,1) + (2,0) + (1,3) = (3,2) + (2,1) + (1,0) \\
\Leftrightarrow \quad & (0,3) = (0,2) = (0,1) \Leftrightarrow -8y = 0 = 8y \Leftrightarrow y = 0. \\
(\underline{C_4})^2: \quad & p^1_{44} = p^2_{44} = p^3_{44} \\
\Leftrightarrow \quad & (0,1) = (0,2) = (0,3) \Leftrightarrow 8y = 0 = -8y \Leftrightarrow y = 0.
\end{aligned}
$$

We have the condition $y = 0$ for the partitions 10, 11, 12 and 13. Thus, we get the following theorem:

Theorem 5.3.11 *(Partitions 10, 11, 12, 13)*
Let $p^{2n} \equiv 1$ *mod 8 and let* $C(p^{2n}, 4) = \langle C_0, C_1, C_2, C_3, C_4 \rangle$ *be a cyclotomic scheme. Then any merging of three cyclotomic classes yields a nontrivial subscheme.*

We summarize the results in a table. For a cyclotomic scheme $C(p^n, 4)$ with cyclotomic classes C_1, C_2, C_3, C_4 we have the following subschemes:

classes of subscheme	condition	Theorem
$C_0, C_1 \cup C_2, C_3, C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.8
$C_0, C_1 \cup C_3, C_2, C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.9
$C_0, C_1 \cup C_4, C_2, C_3$	$p^n \equiv 1 \bmod 8$, n even	5.3.8
$C_0, C_1, C_2 \cup C_3, C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.8
$C_0, C_1, C_3, C_2 \cup C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.9
$C_0, C_1, C_2, C_3 \cup C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.8
$C_0, C_1 \cup C_2, C_3 \cup C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.10
$C_0, C_1 \cup C_3, C_2 \cup C_4$	$p^n \equiv 1 \bmod 4$	5.3.7
$C_0, C_1 \cup C_4, C_2 \cup C_3$	$p^n \equiv 1 \bmod 8$, n even	5.3.10
$C_0, C_1 \cup C_2 \cup C_3, C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.11
$C_0, C_1 \cup C_2 \cup C_4, C_3$	$p^n \equiv 1 \bmod 8$, n even	5.3.11
$C_0, C_1 \cup C_3 \cup C_4, C_2$	$p^n \equiv 1 \bmod 8$, n even	5.3.11
$C_0, C_1, C_2 \cup C_3 \cup C_4$	$p^n \equiv 1 \bmod 8$, n even	5.3.11

Example 5.3.12 *We consider the cyclotomic scheme $C(3^4, 4)$. We have $3^4 = 81$ and $81 \equiv 1$ mod 8. There are four cyclotomic classes C_1, C_2, C_3, C_4 of $\mathbb{F}_{81}$, each has 20 elements. Now, by the previous theorems for the case $e = 4, f$ even, we get the following types of subschemes:*

1) *A subscheme $\langle C_0, C_1 \cup C_2 \cup C_3, C_4\rangle$ which corresponds to the strongly regular graph $(81, 20, 1, 6)$;*

2) *A subscheme $\langle C_0, C_1 \cup C_2, C_3 \cup C_4\rangle$ which corresponds to a strongly regular graph $(81, 40, 19, 20)$ that is not isomorphic to the Paley graph $P(81)$;*

3) *A subscheme $\langle C_0, C_1 \cup C_3, C_2 \cup C_4\rangle$ which corresponds to the Paley graph $P(81)$ with parameters $(81, 40, 19, 20)$.*

All these graphs appear in Section 4.2.7.
We want to mention that if we consider the cyclotomic scheme $C(3^4, 8)$ we get also the three strongly regular graphs above by merging of classes (since $C(3^4, 4)$ is a subscheme of $C(3^4, 8)$). But, moreover, we get another strongly regular graph with parameters (81,30,9,12) which also is given in Section 4.2.7.

5.3.5 The case e=5

In the case $e = 5, f$ even or odd the determination of the necessary formulas for the cyclotomic numbers for a field $\mathbb{F}_{p^n}$ where $p^n = 5f + 1$ is much more sophisticated. As described in the preceding sections these formulas are derived from the solution of a system of equations we get by the Lemmas 5.1.5 and 5.1.6 (see Section 5.3.1 for this procedure in the case $e = 3, f$ odd). In the case $e = 5$ we do not have enough equations to solve this system of equations and thus, we do not get a description of the cyclotomic numbers which depends only on the parameters p, n, x and y.
To solve this problem we need some further relations between the cyclotomic numbers. However, this is a difficult task, such that we continue with the case $e = 6$.

5.3.6 The case e=6, f odd

The last case which we will consider very briefly is $e = 6, f$ odd. We want to stress that the cyclotomic numbers are determined for further cases, e.g., $e = 6, f$ even and $e = 8,\ f$ odd. However, the computations for $e \geq 6$ are very extensive and it would go beyond the scope of this thesis to perform all these computations and to present them here. For this last case $e = 6, f$ odd, we will restrict ourselves to the determination of good subsets.

By Lemma 5.1.5 we have the following equalities for the 36 cyclotomic numbers.

1. $(0,0) = (3,3) = (3,0)$,
2. $(0,1) = (4,3) = (2,5)$,
3. $(0,2) = (5,3) = (1,4)$,
4. $(0,3)$,
5. $(0,4) = (1,3) = (5,2)$,
6. $(0,5) = (2,3) = (4,1)$,
7. $(1,0) = (2,2) = (5,5) = (3,4) = (3,1) = (4,0)$,
8. $(2,0) = (3,2) = (3,5) = (4,4) = (1,1) = (5,0)$,
9. $(1,2) = (5,1) = (5,4) = (4,2) = (2,4) = (1,5)$,
10. $(2,1) = (4,5)$.

In [Sto67], Theorem 15' (p. 72), formulas for these cyclotomic numbers are given. These formulas are determined by Lemma 5.1.6 for different pairs (s,t). The cyclotomic numbers are given by the representation $p^n = x^2 + 3y^2$, $x \equiv 1 \bmod 3$. We have two consider three different cases which depends on $z \in \{0, \ldots, 5\}$, where $\omega^z = 2 := \omega^0 + \omega^0 \in \mathbb{F}_{p^n}$.

Case 1 $\omega^3 = 2$	Case 2 $\omega^2 = 2$ or $\omega^5 = 2$
$(0,0) = \frac{1}{36}(p^n - 11 - 8x)$,	$(0,0) = \frac{1}{36}(p^n - 11 - 2x)$,
$(0,1) = \frac{1}{36}(p^n + 1 - 2x + 12y)$,	$(0,1) = \frac{1}{36}(p^n + 1 - 2x - 12y)$,
$(0,2) = \frac{1}{36}(p^n + 1 - 2x + 12y)$,	$(0,2) = \frac{1}{36}(p^n + 1 - 8x + 12y)$,
$(0,3) = \frac{1}{36}(p^n + 1 + 16x)$,	$(0,3) = \frac{1}{36}(p^n + 1 + 10x + 12y)$,
$(0,4) = \frac{1}{36}(p^n + 1 - 2x - 12y)$,	$(0,4) = \frac{1}{36}(p^n + 1 - 2x - 12y)$,
$(0,5) = \frac{1}{36}(p^n + 1 - 2x - 12y)$,	$(0,5) = \frac{1}{36}(p^n + 1 + 4x)$,
$(1,0) = \frac{1}{36}(p^n - 5 + 4x + 6y)$,	$(1,0) = \frac{1}{36}(p^n - 5 + 4x + 6y)$,
$(2,0) = \frac{1}{36}(p^n - 5 + 4x - 6y)$,	$(2,0) = \frac{1}{36}(p^n - 5 - 2x - 6y)$,
$(1,2) = \frac{1}{36}(p^n + 1 - 2x)$,	$(1,2) = \frac{1}{36}(p^n + 1 + 4x)$,
$(2,1) = \frac{1}{36}(p^n + 1 - 2x)$.	$(2,1) = \frac{1}{36}(p^n + 1 - 8x + 12y)$.

Case 3 $\omega^1 = 2$ or $\omega^4 = 2$
$(0,0) = \frac{1}{36}(p^n - 11 - 2x)$,
$(0,1) = \frac{1}{36}(p^n + 1 + 4x)$,
$(0,2) = \frac{1}{36}(p^n + 1 - 2x + 12y)$,
$(0,3) = \frac{1}{36}(p^n + 1 + 10x - 12y)$,
$(0,4) = \frac{1}{36}(p^n + 1 - 8x - 12y)$,
$(0,5) = \frac{1}{36}(p^n + 1 - 2x + 12y)$,
$(1,0) = \frac{1}{36}(p^n - 5 - 2x + 6y)$,
$(2,0) = \frac{1}{36}(p^n - 5 + 4x - 6y)$,
$(1,2) = \frac{1}{36}(p^n + 1 + 4x)$,
$(2,1) = \frac{1}{36}(p^n + 1 - 8x - 12y)$.

Now we investigate all possible partitions of $\{1,\dots,6\}$ consisting of good subsets. Since the cyclotomic scheme is not symmetric (Lemma 5.1.2), we must take into account that we have three pairs of antisymmetric cyclotomic classes: $C_1^{-1} = C_4$, $C_2^{-1} = C_5$ and $C_3^{-1} = C_6$.
In the following we will give representatives of subsets which satisfy the first condition for good subsets, i.e. they are symmetric or antisymmetric. These subsets are representatives because we can get all other candidates for good subsets by the cyclic action of $\mathbb{F}_{p^n}^*$ (cf. Lemma 5.1.8):

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,3,5\}, \{1,2,4,5\}.$$

There are no further candidates, especially, no 5-subsets, because we have three pairs of antisymmetric cyclotomic classes and in each good subsets are only indices of symmetric or pairs of antisymmetric classes resp. only indices of antisymmetric, pairwise non-inverse classes.
1. The subset $\{1,2\}$:

$$p_{12}^1 + p_{21}^1 + p_{22}^1 = p_{11}^2 + p_{12}^2 + p_{21}^2 \Leftrightarrow (1,0) + (5,5) + (0,5) = (0,1) + (1,1) + (5,0)$$
$$\Leftrightarrow 2(1,0) + (0,5) = 2(2,0) + (0,1).$$

The last equation must be checked for the formulas in the three tables. In the first case $\omega^3 = 2$ the equation is always true, for the second case $\omega^2 = 2$ or $\omega^5 = 2$ follows $x = -2y$ and in the third case $\omega^1 = 2$ or $\omega^4 = 2$ follows $x = 2y$.
2. The subset $\{1,3\}$:
We get

$$p_{13}^1 + p_{31}^1 + p_{33}^1 = p_{11}^3 + p_{13}^3 + p_{31}^3$$

and obtain

$$2(2,0) + (0,4) = (0,2) + 2(1,0).$$

Here we get for the cases 1, 2 and 3: $y = 0$ resp. $x = -8y$ resp. $x = 8y$.
3. The subset $\{1,4\}$:
We get

$$p_{14}^1 + p_{41}^1 + p_{44}^1 = p_{11}^4 + p_{14}^4 + p_{41}^4$$

and obtain

$$2(3,0)+(0,3)=(0,3)+(3,3)+(3,0).$$

This is always true.
4. The subset $\{1,2,3\}$:
We have for $\underline{(C_1 \cup C_2 \cup C_3)}^2$:

$$\begin{aligned} & p^1_{11}+p^1_{12}+p^1_{13}+p^1_{21}+p^1_{22}+p^1_{23}+p^1_{31}+p^1_{32}+p^1_{33} \\ = & \ p^2_{11}+p^2_{12}+p^2_{13}+p^2_{21}+p^2_{22}+p^2_{23}+p^2_{31}+p^2_{32}+p^2_{33} \\ = & \ p^3_{11}+p^3_{12}+p^3_{13}+p^3_{21}+p^3_{22}+p^3_{23}+p^3_{31}+p^3_{32}+p^3_{33} \\ \Leftrightarrow & \ (0,5)+(0,4)+2(1,2)=(0,1)+(0,5)+2(2,1)=(0,2)+(0,1)+2(1,2) \end{aligned}$$

Here we get in the first case: $y=0$, in the second and third case $x=y$ resp. $x=-y$. The last two cases cannot appear, because $p^n=6f+1=x^2+3y^2=4x^2$ is impossible since $6f+1$ is odd and $4x^2$ even.
5. The subset $\{1,3,5\}$:
We have for $\underline{(C_1 \cup C_3 \cup C_5)}^2$:

$$\begin{aligned} & p^1_{11}+p^1_{13}+p^1_{15}+p^1_{31}+p^1_{33}+p^1_{35}+p^1_{51}+p^1_{53}+p^1_{55} \\ = & \ p^3_{11}+p^3_{13}+p^3_{15}+p^3_{31}+p^3_{33}+p^3_{35}+p^3_{51}+p^3_{53}+p^3_{55} \\ = & \ p^5_{11}+p^5_{13}+p^5_{15}+p^5_{31}+p^5_{33}+p^5_{35}+p^5_{51}+p^5_{53}+p^5_{55} \\ \Leftrightarrow & \ (2,0)+(1,0)+(1,2)=(1,0)+(1,2)=(2,0)=(1,2)=(2,0)+(1,0). \end{aligned}$$

This is always true.
6. The subset $\{1,2,4,5\}$:
Here we only give the result. All equations are equivalent to one equation.
We have for $\underline{(C_1 \cup C_2 \cup C_4 \cup C_5)}^2$:

$$2(1,0)+(0,1)+(0,4)=2(2,0)+(0,2)+(0,5).$$

In the first case we have $y=0$, in the second and third case we have $x=y$ resp. $x=-y$. As shown above the last two cases cannot occur.
We have the following representatives of good subsets (the entry "-" means "impossible" and no entry means no condition):

	set	*condition case* 1	*condition case* 2	*condition case* 3
1	$\{1,2\}$		$x=-2y$	$x=2y$
2	$\{1,3\}$	$y=0$	$x=-8y$	$x=8y$
3	$\{1,4\}$			
4	$\{1,2,3\}$	$y=0$	–	–
5	$\{1,3,5\}$			
6	$\{1,2,4,5\}$	$y=0$	–	–

The next step is to check which partitions of good subsets give rise to a subscheme. We did not perform all the computations. Only for the partitions $\{\{1,3,5\},\{2,4,6\}\}$ and $\{\{1,4\},\{2,5\}\{3,6\}\}$ we want to mention that they always correspond to a subscheme.

List of Symbols

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Appendix A

Algorithms

In this chapter we present the algorithms we used for the determination and further handling of partial difference sets. For each algorithm we give a short description.

A.1 Computation of partial difference sets

The following algorithm computes the partial difference set for a regular subgroup u in the automorphism group of a strongly regular graph g. The resulting partial difference set *pds* will be described by tuples of exponents for the generators of the group u. The generators are given as input in the list m. Furthermore, the algorithm will check by the function IsPds, if *pds* is a partial difference set in the group u. Hence, for all computed partial difference sets *pds* in this thesis we got a confirmation that *pds* is really a partial difference set. The function IsPds is described in the next section.

```
pds:= function(g,u,m)
#g is srg, u is group, m is list of generators
local i,j,o,z,x,y,t,s,w,pds,p,q1,q2,q3,q4;
#determination of the order of the generators
o:=[ ];;z:=1;;
for i in [1..Length(m)] do o[i]:= Order(m[i]);;od;
x:=Maximum(o);;y:=Length(m);;
#determination of all possible tuples (i,j,k,...) for the
#generator products (a^i*b^j*c^k...)
t:=Tuples([0..x-1],y);;
for i in [1..Length(t)] do
  for j in [1..y] do
    if t[i][j]>Order(m[j])-1 then z:=0;; fi;
  od;
  if z=0 then Unbind(t[i]); z:=1; fi;
od;
t:=Set(t);;
```

```
#determination of pds
w:=[ ];;pds:=[ ];;s:=Adjacency(g,1) ;;
#permutations in group u which have representation (a^i*b^j*c^k*...)
for q1 in t do p:=[ ];;
  for i in [1..y] do p[i]:= m[i]^q1[i];;od;
  if 1^Product(p) in s then w[Length(w)+1]:=q1;
    pds[Length(pds)+1]:=Product(p);
    Unbind(s[Position(s,1^Product(p) )]);;s:=Set(s);;
  fi;
od;

#permutations in group u which have representation
#(a^i*b^j*c^k*...)(a^s*b^t*c^u*...)
if Length(s)>0 then
  for q1 in t do
    for q2 in t do p:=[ ];;
      for i in [1..y] do
        p[i+y]:=m[i]^q2[i];; p[i]:= m[i]^q1[i];;
      od;
      if 1^Product(p) in s then
        w[Length(w)+1]:=Concatenation(q1,q2);
        pds[Length(pds)+1]:=Product(p);
        Unbind(s[Position(s,1^Product(p) )]);;s:=Set(s);;
      fi;
  od;od;
fi;

#permutations in group u which have representation
#(a^i*b^j*c^k*...)(a^s*b^t*c^u*...)(...)
if Length(s)>0 then
  for q1 in t do
    for q2 in t do
      for q3 in t do p:=[ ];;
        for i in [1..y] do
          p[i+2*y]:=m[i]^q3[i];;p[i+y]:=m[i]^q2[i];; p[i]:= m[i]^q1[i];;
        od;
        if 1^Product(p) in s then
          w[Length(w)+1]:=Concatenation(q1,q2,q3);
          pds[Length(pds)+1]:=Product(p);
          Unbind(s[Position(s,1^Product(p))]);;s:=Set(s);;
        fi;
  od;od;od;fi;
```

```
#permutations in group u which have representation
#(a^i*b^j*c^k*...)(a^s*b^t*c^u*...)(...)(...)
if Length(s)>0 then
  for q1 in t do
    for q2 in t do
     for q3 in t do
       for q4 in t do p:=[ ];;
         for i in [1..y] do
           p[i+3*y]:=m[i]^q4[i];;p[i+2*y]:=m[i]^q3[i];;
           p[i+y]:=m[i]^q2[i];;p[i]:= m[i]^q1[i];;
         od;;
         if 1^Product(p) in s then
           w[Length(w)+1]:=Concatenation(q1,q2,q3,q4);
           pds[Length(pds)+1]:=Product(p);
           Unbind(s[Position(s,1^Product(p))]);;s:=Set(s);;
         fi;
  od;od;od;od;
fi;

#check, if list pds is really a PDS in group u
Read("ispds.txt");
if IsPds(u,pds) then return w; fi;
Print("fail "); return false;
end;
```

A.2 Verifying partial difference sets

We have two functions for checking if a given subset d of a group g is a partial difference set. These functions differ only in their output: one function returns "true" or "false" and is used in the above given algorithm for the determination of partial difference sets. The second function, which is given below, returns in addition the parameter set of the partial difference set.

```
IsPds:= function(g,d)
local cd,di,h,e,i,s,lambda,mu;

#checking if PDS d is reversible, i.e., d^-1 = d
di:=List(d,x->x^-1); if Set(di)<>Set(d) then return false; fi;
#computing set (g\d)\e, where e is identity element
cd:=Difference(Elements(g),d);; cd:=Difference(cd,[Identity(g)]);;
#compute multiset h:=(xy^-1|xy in d)
h:=[ ];s:=[ ];
```

```
for i in [1..Length(d)] do
  s[i]:=d*d[i]^-1;
  h:=Concatenation(h,s[i]);
od;
k:=Collected(h);
# now k is a tuple which contains pairs (i,n) where i is an
#element in g and n is the multiplicity of i in the multiset
#h=(xy^-1|xy in d). hence, if d is a partial difference set, then
#for all i in d we have (i,lambda) in k, for all j in cd we have
#(j,mu) in k and for e we have (e,Length(d)) in k. This will be
#checked below.
lambda:=0; mu:=0;
for e in k do
  if d[1]=e[1] then lambda:=e[2]; fi;
  if cd[1]=e[1] then mu:=e[2];fi;
od;
for e in k do
  if e[1] = Identity(g) then if e[2]<> Length(d) then return false;fi;fi;
  if e[1] in d then if e[2] <> lambda then return false; fi;fi;
  if e[1] in cd then if e[2]<> mu then return false;fi;fi;
od;
Print("(", Size(g),",",Length(d),",",lambda,",",mu,")-partial difference set, ");
return true;
end;
```

A.3 Transfer of a computed partial difference set to the editor program

After the determination of a partial difference set for a given group in GAP, this partial difference set was saved in a special data format which was useful for the further handling in the editor program we used for preparing the tables.
The input of the function is a partial difference set given as a list t of tuples. Each tuple contains the exponents for a description of the partial difference set by certain known generators $a, b, c, d, ...$ .The output of the function is a list of the form

```
[[a^3,b^2,c^5,d^2], [a^3,b^3,c^2,d^5], [a^4,b^2,c^5,d^4], ...]
```

where for example the first sublist corresponds to the element $a^3b^2c^5d^2$ of the partial difference set in the representation by generators a, b, c, d.

```
pdsprint:=function(t,p)

#t is list of tuples, p is filename

local n,l,i,j,pds,ts,tt,h,k,j1;

#this function works only for groups with at most 8 generators.
n:=Length(t[1]);
if n>8 then return Print("too many generators!");fi;

#n is number of generators
l:=["a","b","c","d","e","f","g","h"];;
ts:=List(t,Length);; tt:=[ ];; tt[1]:=0;pds:=[ ];;
h:=Length(t[Length(t)])/n;;

for k in [1..h] do
  if  Position(ts,(k+1)*n) = fail then
    tt[k+1]:= Length(t);
    else tt[k+1]:=Position(ts,(k+1)*n)-1;
  fi;
  for i in [tt[k]+1..tt[k+1]] do
    pds[Length(pds)+1]:=[ ];
    for j in [1..k*n] do
      if t[i][j]<>0 then
        if j mod n =0 then j1:=n;
          else j1:=j mod n;
        fi;
        if t[i][j]=1 then
          pds[Length(pds)][j]:=[l[j1]];
          else pds[Length(pds)][j]:=[l[j1],"^",t[i][j]];
        fi;
        else pds[Length(pds)][j]:=[ ];
      fi;
    od;
  od;
od;

PrintTo(p,pds);
Print("pds is printed to '",p, "' \n"); return "ready";
end;
```

Appendix B

Table of strongly regular graphs up to 49 vertices

In the following table we give information about the feasible parameters for strongly regular graphs up to 49 vertices. In particular, we give the vertex transitive strongly regular graphs up to 49 vertices: each line in the table corresponds to a vertex transitive strongly regular graph if such a graph exists. In the non-existence case we give a comment.
Moreover, the table contains: (1) the number of known strongly regular graphs for each parameter set (in the column *#srg* in the line where the parameter set appears the first time), (2) all vertex transitive graphs for each parameter set (in the cases labeled with † we have only knowledge about the strongly regular Cayley graphs), (3) the order of the automorphism groups of the vertex transitive graphs, (4) the number of partial difference sets we have determined (in the column *#pds*), (5) how we get these partial difference sets (theoretical approach or by GAP) and (6) in the last column where one can find the partial difference sets in Appendix D.

No.	n	k	λ	μ	# srg	vertex transitive graph Γ /comment	order of Aut(Γ)	#pds	how	No. in App. D
1	5	2	0	1	1	$P(5)$	10	1	theo	1.1
2	9	4	1	2	1	$L_2(3) \cong P(9)$	72	1	theo	2.1
3	10	3	0	1	1	Petersen ($\overline{T(5)}$)	120	-	theo	
4	13	6	2	3	1	$P(13)$	78	1	theo	3.1
5	15	6	1	3	1	$T(6)$	720	-	theo	
6	16	5	0	2	1	Clebsch graph	1920	12	GAP	4.1
7	16	6	2	2	2	Shrikhande graph	192	6	GAP	5.2
8	16	6	2	2		$L_2(4)$	1152	13	GAP	5.1

No.	n	k	λ	μ	# srg	vertex transitive graph Γ /comment	order of Aut(Γ)	#pds	how	No. in App. D
9	17	8	3	4	1	$P(17)$	136	1	theo	6.1
10	21	10	3	6	1	$\overline{T(7)}$	5040	1	theo	7.1
11	21	10	4	5	-	Conference				
12	25	8	3	2	1	$L_2(5)$	28800	1	theo	8.1
13	25	12	5	6	15	$L_3(5) \cong P(25)$	600	1	theo	9.1
14	26	10	3	4	10	no vertex transitive srg				
15	27	10	1	5	1	Schläfli graph	51840	2	GAP	10.1
16	28	9	0	4	-	Krein cond., abs. bound				
17	28	12	6	4	4	$T(8)$	40320	-	theo	
18	29	14	6	7	41	$P(29)$	404	1	theo	11.1
19	33	16	7	8	-	Conference				
20	35	16	6	8	3854	descendant from two-graph	40320	-	theo	
21	36	10	4	2	1	$L_2(6)$	$2^9 3^4 5^2$	16	GAP	12.1
22	36	14	4	6	180	descendant from two-graph	144	5	GAP	13.1
23	36	14	4	6		descendant from two-graph	216	3	GAP	13.3
24	36	14	4	6		descendant from two-graph	432	9	GAP	13.2
25	36	14	4	6		descendant from two-graph	12096	-	GAP	
26	36	14	7	4	1	$T(9)$	362880	-	theo	
27	36	15	6	6	32548	$L_3(\mathbb{Z}_6)$	432	5	GAP	14.2
28	36	15	6	6		$L_3(6)$ over quasigroup	648	2	GAP	14.3
29	36	15	6	6		$L_3(S_3)$	1296	5	GAP	14.1
30	36	15	6	6		descendant from two-graph	51840	-	GAP	
31	37	18	8	9	≥ 6760	$P(37)$	666	1	theo	15.1
32	40	12	2	4	28	no specified name	51840	-	GAP	
33	40	12	2	4		no specified name	51840	-	GAP	
34	41	20	9	10	≥ 1	$P(41)$	810	1	theo	16.1
35	45	12	3	3	78	no Cayley graph†				
36	45	16	8	4	1	$T(10)$	$2^7 3^4 5^2 7^1$	-	theo	
37	45	22	10	11	≥ 1	no Cayley graph†				
38	49	12	5	2	1	$L_2(7)$	$2^9 3^4 5^2 7^2$	1	theo	17.1
39	49	18	7	6	≥ 1	$L_3(7)^\dagger$	1764	1	theo	18.1
40	49	24	11	12	≥ 2	$P(49)$	2352	1	theo	19.1
41	49	24	11	12		$L_4(7)$-type†	3528	1	theo	19.2

Appendix C

Table of all small groups containing partial difference sets

One aim of this work is to give a complete list of partial difference sets in groups of order v, $v \leq 49$. In Appendix D we present a table with all partial difference sets we have determined for these groups. In this section we want to describe the groups for which we discovered the partial difference sets.

In the table on the following page we list all groups up to order 49 which occurred throughout this work. The groups are ordered by their identification number of the small group library of GAP [GAP99] (see Section 3.3.4). For each group we give an abstract description by generators and, if possible, standard representatives.

GAP	description of generators	standard representatives
$(5,1)$	$a^5 = 1$	$\mathbb{Z}_5$
$(9,2)$	$a^3 = b^3 = 1, \text{abelian}$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$(13,1)$	$a^{13} = 1$	$\mathbb{Z}_{13}$
$(16,2)$	$a^4 = b^4 = 1, \text{abelian}$	$\mathbb{Z}_4 \times \mathbb{Z}_4$
$(16,3)$	$a^4 = b^2 = 1,\ ba = ab,\ c^2 = b,\ ca = a^3bc$	
$(16,4)$	$a^4 = b^4 = 1,\ ba = a^3b$	
$(16,6)$	$a^8 = 1,\ b^2 = a^2,\ ba = a^5b$	$M_4(2)$ †1
$(16,8)$	$a^8 = 1,\ b^2 = a^4,\ ba = a^3b$	SD_4 †2
$(16,10)$	$a^4 = b^2 = c^2 = 1, \text{abelian}$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4$
$(16,11)$	$a^4 = b^2 = c^2 = 1,\ bab = a^3,\ ac = ca,\ cb = bc$	$\mathbb{Z}_2 \times D_4$
$(16,14)$	$a^2 = b^2 = c^2 = d^2 = 1, \text{abelian}$	$(\mathbb{Z}_2)^4$
$(17,1)$	$a^{17} = 1$	$\mathbb{Z}_{17}$
$(21,1)$	$a^7 = b^3 = 1, ab = ba^4$	$\mathbb{Z}_7 \rtimes K$ †3
$(25,2)$	$a^5 = b^5 = 1, \text{abelian}$	$\mathbb{Z}_5 \times \mathbb{Z}_5$
$(27,3)$	$a^3 = b^3 = 1, bab = (aba)^2$	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \langle \alpha \rangle$ †4
$(27,4)$	$a^9 = b^3 = 1, ba = a^4b$	$\mathbb{Z}_9 \rtimes \langle \beta \rangle$ †4
$(29,1)$	$a^{29} = 1$	$\mathbb{Z}_{29}$
$(36,6)$	$a^3 = b^3 = c^4 = 1, ab = ba, ca = ac, cb = b^2c$	$\mathbb{Z}_3 \times M$ †5
$(36,9)$	$a^3 = b^3 = c^4 = 1, ab = ba, ca = bc, cb = a^2c$	
$(36,10)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, cb = bc, ca = a^2c, db = b^2d$	$S_3 \times S_3$
$(36,11)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ac, db = bd, cda = ad, cb = bc$	$\mathbb{Z}_3 \times A_4$
$(36,12)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, db = bd, ca = a^2c, cb = bc$	$\mathbb{Z}_6 \times S_3$
$(36,13)$	$a^3 = b^3 = c^2 = d^2 = 1, ab = ba, cd = dc, da = ad, db = bd, ca = a^2c, cb = b^2c$	
$(36,14)$	$a^3 = b^3 = c^2 = d^2 = 1, \text{abelian}$	$\mathbb{Z}_6 \times \mathbb{Z}_6$
$(37,1)$	$a^{37} = 1$	$\mathbb{Z}_{37}$
$(41,1)$	$a^{41} = 1$	$\mathbb{Z}_{41}$
$(49,2)$	$a^7 = b^7 = 1, \text{abelian}$	$\mathbb{Z}_7 \times \mathbb{Z}_7$

†1 The group $M_m(p) := < x, y | x^{p^{m-1}} = y^p = 1, y^{-1}xy = x^{1+2^{m-2}} >$ occurs in [Gor80], p. 190.

†2 The group $SD_m := < x, y | x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{m-2}} >$ is called *semidihedral group* (cf.[Gor80], p. 191).

†3 $K \leq \mathbb{Z}_7^*$, $K = \{1, 2, 4\}$.

†4 $\alpha \in Aut(\mathbb{Z}_3 \times \mathbb{Z}_3)$, $\beta \in Aut(\mathbb{Z}_9)$, see section 4.1.6 for the action of α and β.

†5 The group M is the matrix group generated by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$, where $\epsilon \in \mathbb{C} \setminus \mathbb{R}$ with $\epsilon^3 = 1$.

Appendix D

All partial difference sets in groups of order up to 49

On the following pages we give tables with all determined small partial difference sets, i.e., a complete list of partial difference sets in groups of order up to 49. Notice that for each partial difference set D with parameters (v, k, λ, μ) there exist a $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$−partial difference sets which is the "complement" of D. These "complements" are not contained in the table, consequently, the parameter k is restricted to $k \leq \frac{v}{2}$.

Each partial difference set is coded by four numbers which are ordered hierarchically. The first number represents the parameter set (v, k, λ, μ), the second represents the strongly regular graphs for this parameter set, the third number stands for the non-isomorphic regular subgroups of the automorphism group of the corresponding graph and finally, the fourth number represents the non-CI-equivalent partial difference sets in a regular subgroup.

As an example consider the number 5.2.3.2: the fifth parameter set is $(16, 6, 2, 2)$, the second graph with this parameters in the table is the Shrikhande graph. The automorphism group of this graph has several non-isomorphic regular subgroups, the third subgroup has the GAP-identification number $(16, 8)$ and here we consider the second partial difference set in this group.

The partial difference sets are given in an abstract description by the generators of the corresponding groups. The generators of the groups are described in the table in Appendix C.

Since our table gives a complete list of all partial difference sets for groups of order up to 49, it is clear that some of the partial difference sets are already known. In these cases we give a reference in the last column, however, we are not able to give each time the reference where the respective partial difference set was described the first time.

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	reference to literature
1.1.1.1	5	2	0	1	*Paley*	$(5,1)$	$\{a, a^4\}$	*Prop.* 3.2.6	[*Ma*84], 3.5
2.1.1.1	9	4	1	2	$L_2(3)$	$(9,2)$	$\{a, a^2, b, b^2\}$	*Sec.* 4.1.3	[*Ma*84], 3.4 (1)
3.1.1.1	13	6	2	3	*Paley*	$(13,1)$	$\{a, a^3, a^4, a^9, a^{10}, a^{12}\}$	*Prop.* 3.2.6	[*Ma*84], 3.5
4.1.1.1	16	5	0	2	Clebsch	$(16,2)$	$\{a, a^3, b, b^3, a^2b^2\}$	*Table* 4.6	[*Ma*94], 12.8
2.1	16	5	0	2	Clebsch	$(16,3)$	$\{a, a^3, a^2b, c, c^3\}$	*Table* 4.6	
2.2	16	5	0	2	Clebsch	$(16,3)$	$\{a, a^3, ac, ac^3, a^2b\}$	*Table* 4.6	
2.3	16	5	0	2	Clebsch	$(16,3)$	$\{a, a^3, b, ac^3, a^3c\}$	*Table* 4.6	
3.1	16	5	0	2	Clebsch	$(16,4)$	$\{a, a^3, b, b^3, a^2b^2\}$	*Table* 4.6	
4.1	16	5	0	2	Clebsch	$(16,6)$	$\{a, a^4, a^7, b, b^7\}$	*Table* 4.6	
5.1	16	5	0	2	Clebsch	$(16,8)$	$\{a, a^4, a^7, ab^3, a^3b^3\}$	*Table* 4.6	
6.1	16	5	0	2	Clebsch	$(16,10)$	$\{a, a^3, a^2bc, b, c\}$	*Table* 4.6	[*Ma*94], 12.8
7.1	16	5	0	2	Clebsch	$(16,11)$	$\{a^2, ab, a^3bc, b, bc\}$	*Table* 4.6	
7.2	16	5	0	2	Clebsch	$(16,11)$	$\{ab, a^2c, a^3b, b, bc\}$	*Table* 4.6	
7.3	16	5	0	2	Clebsch	$(16,11)$	$\{ac, a^2c, a^3c, b, bc\}$	*Table* 4.6	
8.1	16	5	0	2	Clebsch	$(16,14)$	$\{a, b, c, d, abcd\}$	*Page* 91	[*Ma*94], 12.8
5.1.1.1	16	6	2	2	$L_2(4)$	$(16,2)$	$\{ab^3, a^2b, a^2b^2, a^2b^3, a^3b, b^2\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)
2.1	16	6	2	2	$L_2(4)$	$(16,3)$	$\{a^2, ab, a^2c, a^2c^3, a^3b, b\}$	*Table* 4.2	
2.2	16	6	2	2	$L_2(4)$	$(16,3)$	$\{a^2, abc, a^2bc, a^2c, a^3bc, b\}$	*Table* 4.2	
2.3	16	6	2	2	$L_2(4)$	$(16,3)$	$\{ab, abc, a^3b, a^3c, bc, c\}$	*Table* 4.2	
2.4	16	6	2	2	$L_2(4)$	$(16,3)$	$\{a^2, ab, abc, a^2b, a^3b, a^3c\}$	*Table* 4.2	
3.1	16	6	2	2	$L_2(4)$	$(16,4)$	$\{ab^2, a^2, a^3b, a^3b^2, a^3b^3, b^2\}$	*Table* 4.2	
4.1	16	6	2	2	$L_2(4)$	$(16,6)$	$\{a^3, a^5, ab, a^5b, b^3, b^5\}$	*Table* 4.2	
5.1	16	6	2	2	$L_2(4)$	$(16,8)$	$\{ab^3, a^2b, a^2b^3, a^3b, a^3b^2, a^5b^2\}$	*Table* 4.2	
6.1	16	6	2	2	$L_2(4)$	$(16,10)$	$\{a^2, ac, a^2bc, a^2c, a^3c, b\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)
7.1	16	6	2	2	$L_2(4)$	$(16,11)$	$\{a^2b, a^3b, a^2c, abc, a^2bc, c\}$	*Table* 4.2	
7.2	16	6	2	2	$L_2(4)$	$(16,11)$	$\{a^2, abc, a^2b, a^2c, a^3bc, bc\}$	*Table* 4.2	
7.3	16	6	2	2	$L_2(4)$	$(16,11)$	$\{a^2, ab, ac, a^2c, a^3c, a^3bc\}$	*Table* 4.2	
8.1	16	6	2	2	$L_2(4)$	$(16,14)$	$\{abcd, ac, acd, ad, bd, c\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	reference to literature
5.2.1.1	16	6	2	2	Shrikhande	(16, 2)	$\{a, a^3, ab^3, a^3b, b, b^3\}$	*Prop.* 3.2.42	[*Ma*84], 3.4 (2)
2.1	16	6	2	2	Shrikhande	(16, 6)	$\{a, a^7, a^3b, a^3b^5, b, b^7\}$	*Table* 4.3	
2.2	16	6	2	2	Shrikhande	(16, 6)	$\{a, a^2, a^6, a^7, b^3, b^5\}$	*Table* 4.3	[*Ma*94], 5.10.2
3.1	16	6	2	2	Shrikhande	(16, 8)	$\{a, a^7, a^3b, a^5b, b, b^3\}$	*Table* 4.3	[*Ma*94], 5.10.1
3.2	16	6	2	2	Shrikhande	(16, 8)	$\{a, a^2, a^6, a^7, a^5b, a^7b\}$	*Table* 4.3	
4.1	16	6	2	2	Shrikhande	(16, 11)	$\{a, a^3, a^2bc, a^3b, a^3bc, b\}$	*Table* 4.3	
6.1.1.1	17	8	3	4	*Paley*	(17, 1)	$\{a, a^2, a^4, a^8, a^9, a^{13}, a^{15}, a^{16}\}$	*Prop.* 3.2.6	[*Ma*84], 3.5
7.1.1.1	21	10	3	6	$\overline{T(7)}$	(21, 1)	$\{a^2, a^3, a^4, a^5, a^2b, a^3b, a^4b, a^2b^2, a^5b^2, a^6b^2\}$	*Prop.* 3.2.15	
8.1.1.1	25	8	3	2	$L_2(5)$	(25, 2)	$\{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$	*Sec.* 3.2.3	[*Ma*84], 3.4 (1)
9.1.1.1	25	12	5	6	$L_3(5)$	(25, 2)	$\{a, a^2, a^3, a^4, b, b^2, b^3, b^4, ab, a^2b^2, a^3b^3, a^4b^4\}$	*Sec.* 3.2.3	[*Ma*84], 3.4 (2)
10.1.1.1	27	10	1	5	Schläfli	(27, 3)	$\{a, a^2, b, b^2, aba, ab^2a, a^2b^2a^2, a^2ba^2, ab^2a^2b, a^2b^2ab\}$	*Table* 4.7	*cf.*[*Ma*94], 14.1
2.1	27	10	1	5	Schläfli	(27, 4)	$\{a, a^3, a^6, a^8, ab, ab^2, a^2b^2, a^3b, a^5b, b^2a^6\}$	*Table* 4.7	*cf.*[*Ma*94], 14.1
11.1.1.1	29	14	6	7	*Paley*	(29, 1)	$\{a, a^4, a^5, a^6, a^7, a^9, a^{13}, a^{16}, a^{20}, a^{22}, a^{23}, a^{24}, a^{25}, a^{28}\}$	*Prop.* 3.2.6	[*Ma*84], 3.5
12.1.1.1	36	10	4	2	$L_2(6)$	(36, 6)	$\{a^2b^2, ab, c^3, a^2bc^3, ab^2c^3, a^2b, ab^2, c, a^2b^2c, abc\}$	*Table* 4.4	
2.1	36	10	4	2	$L_2(6)$	(36, 9)	$\{b, b^2, c^3, ac^3, a^2c^3, a, a^2, c, bc, b^2c\}$	*Table* 4.4	
3.1	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, d, ad, a^2d, b, b^2, cd, b^2cd, bcd\}$	*Table* 4.4	
3.2	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, b^2d, ab^2d, a^2b^2d, a^2b, ab^2, b^2cd, abcd, a^2cd\}$	*Table* 4.4	
3.3	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, c, a^2c, ac, b, b^2, cd, b^2cd, bcd\}$	*Table* 4.4	
3.4	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, c, a^2c, ac, b, b^2, d, b^2d, bd\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)
3.5	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, c, a^2c, ac, a^2b, ab^2, cd, ab^2cd, a^2bcd\}$	*Table* 4.4	
3.6	36	10	4	2	$L_2(6)$	(36, 10)	$\{a, a^2, bd, abd, a^2bd, b, b^2, c, bc, b^2c\}$	*Table* 4.4	
4.1	36	10	4	2	$L_2(6)$	(36, 12)	$\{d, ab^2, ab^2d, a^2b, a^2bd, b, b^2, c, bc, b^2c\}$	*Table* 4.4	
4.2	36	10	4	2	$L_2(6)$	(36, 12)	$\{d, a^2b, a^2bd, ab^2, ab^2d, a, a^2, c, ac, a^2c\}$	*Table* 4.4	
4.3	36	10	4	2	$L_2(6)$	(36, 12)	$\{d, b, bd, b^2, b^2d, a, a^2, c, a^2c, ac\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)
4.4	36	10	4	2	$L_2(6)$	(36, 12)	$\{a, a^2, c, a^2c, ac, b, b^2, cd, bcd, b^2cd\}$	*Table* 4.4	
4.5	36	10	4	2	$L_2(6)$	(36, 12)	$\{d, a, ad, a^2, a^2d, b, b^2, c, bc, b^2c\}$	*Table* 4.4	
5.1	36	10	4	2	$L_2(6)$	(36, 13)	$\{d, a, ad, a^2, a^2d, b, b^2, c, b^2c, bc\}$	*Table* 4.4	
5.2	36	10	4	2	$L_2(6)$	(36, 13)	$\{a, a^2, c, a^2c, ac, b, b^2, cd, b^2cd, bcd\}$	*Table* 4.4	
6.1	36	10	4	2	$L_2(6)$	(36, 14)	$\{c, a, ac, a^2, a^2c, d, b, bd, b^2, b^2d\}$	*Prop.* 3.2.47	[*Ma*84], 3.4 (1)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	reference to literature
13.1.1.1	36	14	4	6	$\|aut\| = 144$	$(36, 6)$	$\{c, a^2bc^2, a^2b^2c^3, ab^2c^2, abc^2, abc^3, a, ab^2c, ac^2, a^2, a^2b^2c^2, a^2c^2, a^2bc, c^3\}$	*Table* 4.8	
1.2	36	14	4	6	$\|aut\| = 144$	$(36, 6)$	$\{c, ab^2c^2, abc^3, a^2bc^2, a^2b^2c^3, abc^2, b, ab^2c, bc^2, b^2, a^2b^2c^2, b^2c^2, a^2bc, c^3\}$	*Table* 4.8	
2.1	36	14	4	6	$\|aut\| = 144$	$(36, 13)$	$\{d, c, cd, a^2b^2c, a^2b^2cd, ac, a, bd, bc, a^2, ab^2c, a^2bc, b^2d, abcd\}$	*Table* 4.8	
2.2	36	14	4	6	$\|aut\| = 144$	$(36, 13)$	$\{c, abd, a^2b^2cd, a^2b^2d, abcd, bd, a, b^2c, ad, a^2, b^2d, a^2d, bc, cd\}$	*Table* 4.8	
3.1	36	14	4	6	$\|aut\| = 144$	$(36, 14)$	$\{c, abcd, abd, a^2b^2cd, a^2b^2d, bcd, a, bc, acd, a^2, b^2cd, a^2cd, b^2c, d\}$	*Table* 4.8	$[Ma94]$, 12.8
13.2.1.1	36	14	4	6	$\|aut\| = 432$	$(36, 9)$	$\{ac^3, a^2bc, a^2bc^2, ac, c^2, b^2c^2, bc^3, a, bc^2, a^2b^2c^2, a^2b^2c^3, a^2, a^2c^2, b^2c\}$	*Table* 4.8	
1.2	36	14	4	6	$\|aut\| = 432$	$(36, 9)$	$\{c, c^2, c^3, a^2bc^3, abc^2, a^2bc^2, abc, b, bc^2, a^2c^2, a^2b^2c, b^2, ac^2, ab^2c^3\}$	*Table* 4.8	
2.1	36	14	4	6	$\|aut\| = 432$	$(36, 10)$	$\{a^2cd, c, a^2d, ac, ad, ab^2d, ab^2cd, ab^2, abd, a^2bd, bcd, a^2b, a^2b^2d, a^2c\}$	*Table* 4.8	
2.2	36	14	4	6	$\|aut\| = 432$	$(36, 10)$	$\{d, a^2c, a^2cd, c, cd, a^2b^2cd, b^2d, a^2b, abcd, bcd, bd, ab^2, ab^2cd, ac\}$	*Table* 4.8	
2.3	36	14	4	6	$\|aut\| = 432$	$(36, 10)$	$\{c, a^2d, acd, ad, a^2cd, ab^2cd, bc, a^2b, bcd, a^2bcd, b^2c, ab^2, b^2cd, d\}$	*Table* 4.8	
2.4	36	14	4	6	$\|aut\| = 432$	$(36, 10)$	$\{b^2cd, d, a^2bc, a^2d, bc, b^2c, a^2bcd, ab^2, c, a^2c, acd, a^2b, a^2b^2c, ad\}$	*Table* 4.8	
3.1	36	14	4	6	$\|aut\| = 432$	$(36, 12)$	$\{d, a^2c, a^2cd, c, cd, bcd, abd, ab^2, b^2cd, a^2b^2cd, a^2b^2d, a^2b, a^2bcd, ac\}$	*Table* 4.8	
3.2	36	14	4	6	$\|aut\| = 432$	$(36, 12)$	$\{d, b^2cd, b^2c, bc, bcd, a^2bcd, abd, a^2b, abcd, ab^2cd, a^2b^2d, ab^2, a^2b^2cd, c\}$	*Table* 4.8	
3.3	36	14	4	6	$\|aut\| = 432$	$(36, 12)$	$\{c, a^2cd, a^2d, a^2b^2cd, a^2b^2d, ad, ac, a^2b^2, bd, b^2d, a^2c, ab, abd, a^2bcd\}$	*Table* 4.8	
13.3.1.1	36	14	4	6	$\|aut\| = 216$	$(36, 9)$	$\{c^2, c, c^3, a^2b^2c^2, a^2b^2, ac^3, bc^2, bc, ac^2, ab, ab^2c^2, a^2bc^2, b^2c, a^2c^3\}$	*Table* 4.8	
2.1	36	14	4	6	$\|aut\| = 216$	$(36, 10)$	$\{d, c, cd, a^2bd, a^2b, abcd, abd, bc, bd, ab^2, a^2d, ad, b^2c, a^2b^2cd\}$	*Table* 4.8	
2.2	36	14	4	6	$\|aut\| = 216$	$(36, 10)$	$\{cd, d, c, a^2bcd, a^2b, b^2c, b^2cd, ad, acd, ab^2, abcd, a^2b^2cd, a^2d, bc\}$	*Table* 4.8	
14.1.1.1	36	15	6	6	$L_3(S_3)$	$(36, 9)$	$\{c, bc, a, a^2, b^2c, ac^2, c^3, a^2bc^2, a^2c^3, ab, b, a^2b^2, b^2, b^2c^2, ac^3\}$	*Table* 4.5	
1.2	36	15	6	6	$L_3(S_3)$	$(36, 9)$	$\{b^2c^3, bc^3, a, a^2, c^3, c^2, c, abc^2, ac, ab, b, a^2b^2, b^2, a^2b^2c^2, a^2c\}$	*Table* 4.5	
2.1	36	15	6	6	$L_3(S_3)$	$(36, 10)$	$\{c, a^2c, a, a^2, ac, acd, d, a^2bcd, bd, ab, b, a^2b^2, b^2, b^2cd, b^2d\}$	*Prop.* 3.2.42	$[Ma84]$, 3.4 (2)
2.2	36	15	6	6	$L_3(S_3)$	$(36, 10)$	$\{ad, a^2d, a, a^2, d, c, cd, bc, ab^2cd, b, a^2b, b^2, ab^2, b^2c, a^2bcd\}$	*Table* 4.5	
3.1	36	15	6	6	$L_3(S_3)$	$(36, 12)$	$\{d, a^2bd, ab^2, a^2b, ab^2d, c, ab^2cd, a^2c, abcd, a, b, a^2, b^2, ac, acd\}$	*Table* 4.5	
14.2.1.1	36	15	6	6	$L_3(\mathbb{Z}_6)$	$(36, 6)$	$\{a^2c^2, a, c^2, a^2, ac^2, ac^3, a^2c, a^2b, ab, b^2c^3, b^2c, ab^2, a^2b^2, a^2bc^3, abc\}$	*Table* 4.5	
1.2	36	15	6	6	$L_3(\mathbb{Z}_6)$	$(36, 6)$	$\{b^2c^2, b, c^2, b^2, bc^2, a^2b^2c, a^2bc^3, ab^2, ab, c, c^3, a^2b, a^2b^2, abc, ab^2c^3\}$	*Table* 4.5	
2.1	36	15	6	6	$L_3(\mathbb{Z}_6)$	$(36, 11)$	$\{a, a^2bd, c, ac, a^2bcd, a^2, ab^2d, abc, a^2b^2d, d, cd, a^2d, ab^2c, abcd, a^2b^2c\}$	*Table* 4.5	
3.1	36	15	6	6	$L_3(\mathbb{Z}_6)$	$(36, 13)$	$\{a^2d, a, d, a^2, ad, c, a^2cd, b, a^2b, b^2c, b^2cd, b^2, ab^2, bc, abcd\}$	*Table* 4.5	
4.1	36	15	6	6	$L_3(\mathbb{Z}_6)$	$(36, 14)$	$\{a^2c, a, c, a^2, ac, b^2d, ab^2cd, b, a^2b, d, cd, b^2, ab^2, bd, a^2bcd\}$	*Prop.* 3.2.42	$[Ma84]$, 3.4 (2)
14.3.1.1	36	15	6	6	$L_3(6)o.Qg$	$(36, 9)$	$\{a, a^2, ab^2c^2, b^2c^2, a^2b^2c^2, a^2b^2, a^2b, ab, ab^2, c, abc^3, ab^2c, a^2bc, a^2b^2c^3, c^3\}$	*Table* 4.5	
2.1	36	15	6	6	$L_3(6)o.Qg$	$(36, 10)$	$\{ab, a^2b^2, b^2cd, a^2bcd, acd, a, b, a^2, b^2, d, bc, ad, b^2c, a^2d, c\}$	*Table* 4.5	
15.1.1.1	37	18	8	9	*Paley*	$(37, 1)$	$\{a, a^3, a^4, a^7, a^9, a^{10}, a^{11}, a^{12}, a^{16}, a^{21}, a^{25}, a^{26}, a^{27}, a^{28}, a^{30}, a^{33}, a^{34}, a^{36}\}$	*Prop.* 3.2.6	$[Ma84]$, 3.5
16.1.1.1	41	20	9	10	*Paley*	$(41, 1)$	$\{a, a^2, a^4, a^5, a^8, a^9, a^{10}, a^{16}, a^{18}, a^{20}, a^{21}, a^{23}, a^{25}, a^{31}, a^{32}, a^{33}, a^{36}, a^{37}, a^{39}, a^{40}\}$	*Prop.* 3.2.6	$[Ma84]$, 3.5
17.1.1.1	49	12	5	2	$L_2(7)$	$(49, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, b, b^2, b^3, b^4, b^5, b^6\}$	*Exa.* 3.2.24	$[Ma84]$,3.4(1)
18.1.1.1	49	18	7	6	$L_3(7)$	$(49, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, b, b^2, b^3, b^4, b^5, b^6, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6\}$	*Exa.* 3.2.24	$[Ma84]$,3.4(2)
19.1.1.1	49	24	11	12	*Paley*	$(49, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, b, b^2, b^3, b^4, b^5, b^6, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, ab^2, a^2b^4,$ $a^3b^6, a^4b, a^5b^3, a^6b^5\}$	*Exa.* 3.2.24	$[Ma84]$,3.5
19.2.1.1	49	24	11	12	$L_4(7)$	$(49, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, b, b^2, b^3, b^4, b^5, b^6, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, ab^3, a^2b^6,$ $a^3b^2, a^4b^5, a^5b, a^6b^4\}$	*Exa.* 3.2.24	$[Ma84]$,3.4(3)

Appendix E

Strongly regular graphs with primitive automorphism group

On the following pages we list the 95 strongly regular graphs with primitive automorphism group and v vertices, where $49 < v \leq 255$ (see Table 4.9). All graphs except those with 121 and 169 vertices are from the catalogue from C. Pech. In the table we give (1) the parameters of the strongly regular graphs, (2) if known, the name of the strongly regular graph, (3) the number of partial difference sets for this graph ($\#pds$), (4) how we get the partial difference sets (by a theoretical approach or by GAP) and (5) where one can find the partial difference sets in Appendix G. Moreover, from the catalogue of C. Pech we took the names of the primitive groups and we computed their order. The groups are named like in [The97]. The names reflect the cohort structure which is given in [DixM88]. A group ”$G\#n.i$” in the table is the i^{th} representation of a permutation group of degree n with socle G in the GAP library. A group $G : H$ is the semidirect product with normal subgroup H and factor group G. The group $G \circ H$ is the central product of matrix groups G and H. The group ”G on i-sets.j” is the j^{th} group of degree n with socle G in the GAP library of primitive groups which acts on the i-subsets of $\{1, \ldots, n\}$. Analogously, the group ”G on 1-sets^2.j” is the j^{th} group of degree n with socle G in the GAP library which acts on the elements of $\{1, \ldots, n\}^2$. Cyclic groups of order n are simply named by n. The extension of a group G by a group H is denoted by $G.H$. For the groups in the lines 7, 8 and 22 we did not find a description in [The97], we only get the information that the names are ”ad-hoc names” which are not necessarily natural for group theorists (A. Hulpke, private communication). For some groups the GAP library of primitive groups specifies no name.

Most of the groups are primitive representations of classical groups. For information about the groups we refer to the Atlas of finite groups [CCNPW85] and the Atlas of finite group representations [BLNPRSTWW01].

No.	parameters	primitive group	order	graph	#pds	how	No. in App. G
1	50,7,0,1	$PSU(3,5^2):2$	252000	Hoffman-Singleton	-	GAP.	
2	55,18,9,4	A_{11} on 2-sets.2	$2^8 3^4 5^2 7^1 11^1$	$T(11)$	1	theo.	20.1
3	56,10,0,2	PSL(3,4)#56.5	80640	Sims-Gewirtz	-	GAP	
4	63,30,13,15	PSU(3,3)#63.2	12096		-	GAP	
5	63,30,13,15	PSp(6,2) on projective points.1	$2^9 3^4 5^1 7^1$		-	GAP	
6	64,14,6,2	A_8 on 1-sets^2.4	$2^{15} 3^4 5^2 7^2$	$L_2(8)$	≥ 1	theo	
7	64,18,2,6	$3.A_6.2$ max $GL(3,4).2$	138240		58	GAP	22.1
8	64,21,8,6	$PSL(2,7):2$ max $PSU(3,3):2$ max $Sp(6,2)$	21504	pseudo $L_3(8)$	36	GAP	23.1
9	64,21,8,6	$2^6:SL(3,2)\circ SL(2,2)$	64512	$L_3((\mathbb{Z}_2)^3)$	58	GAP	23.2
10	64,27,10,12	$2^6:O^{-1}(6,2)$	$2^{13} 3^4 5^1$		≥ 1		
11	64,28,12,12	$2^6:O^{+1}(6,2)$	$2^{13} 3^2 5^1 7^1$	pseudo $L_4(8)$	≥ 1		
12	66,20,10,4	A_{12} on 2-sets.2	$2^{10} 3^5 5^2 7^1 11^1$	$T(12)$	-	theo	
13	77,16,0,4	M_{22}#77.2	887040		-	theo	
14	78,22,11,4	A_{13} on 2-sets.2	$2^{10} 3^5 5^2 7^1 11^1 13^1$	$T(13)$	-	theo	
15	81,16,7,2	A_9 on 1-sets^2.4	$2^{15} 3^8 5^2 7^2$	$L_2(9)$	12	GAP	26.1
16	81,20,1,6		233280		4	GAP	27.1
17	81,24,9,6		93312	pseudo $L_3(9)$	9	GAP	28.1
18	81,30,9,12		116640		7	GAP	29.1
19	81,32,13,12		5184	$L_4(9)$	1	GAP	30.1
20	81,32,13,12		186624	$L_4(9)$	6	GAP	30.2
21	81,40,19,20		12960	Paley	1	theo	31.1
22	81,40,19,20	$3^4:N_\Gamma(SL_2(5))_3$	38880	$L_5(9)$	3	GAP	31.2
23	85,20,3,5	$PSp(4,4)$ on projective points.2	$2^8 3^2 5^2 17^1$		-	theo	
24	91,24,12,4	A_{14} on 2-sets.2	$2^{11} 3^5 5^2 7^2 11^1 13^1$	$T(14)$	-	theo	
25	100,18,8,2	A_{10} on 1-sets^2.4	$2^{17} 3^8 5^4 7^2$	$L_2(10)$	18	GAP	32.1
26	100,22,0,6	HS#100.2	$2^{10} 3^2 5^3 7^1 11^1$	Higman-Sims	4	GAP	33.1
27	100,36,14,12	J_2#100.2	$2^8 3^2 5^2 7^1$	Hall-Janko-Wales	2	GAP	34.1
28	105,26,13,4	A_{15} on 2-sets.2	$2^{11} 3^6 5^3 7^2 11^1 13^1$	$T(15)$	-	theo	
29	105,32,4,12	$PSL(3,4)$#105.6	241920		-	GAP	
30	112,30,2,10	$PSU(4,3)$#112.8	$2^{10} 3^6 5^1 7^1$		-	GAP	
31	117,36,15,9	$PSL(4,3)$#117.2	$2^8 3^6 5^1 13^1$		-	GAP	
32	119,54,21,27	$O^{-1}(8,2)$#119.2	$2^{13} 3^4 5^1 7^1 17^1$		-	theo	
33	120,28,14,4	A_{16} on 2-sets.2	$2^{15} 3^6 5^3 7^2 11^1 13^1$	$T(16)$	-	theo	
34	120,42,8,18	$PSL(3,4)$#120.5	80640		-	GAP	
35	120,51,18,24	$PSp(4,4)$#120.2	$2^9 3^2 5^2 17^1$		1	GAP	35.1
36	120,56,28,24	A_7#120.1	5040		2	GAP	36.1
37	120,56,28,24	$O^{+1}(8,2)$#120.2	$2^{13} 3^5 5^2 7^1$		≥ 1	GAP	36.2

No.	parameters	primitive group	order	graph	#pds	how	No. in App. G
38	120,56,28,24	A_{10} on 3-sets.2	$2^8 3^4 5^2 7^1$		-	GAP	
39	121,20,9,2			$L_2(11)$	1	theo	37.1
40	121,30,11,6			$L_3(11)$	1	theo	38.1
41	121,40,15,12			$L_4(11)$	1	theo	39.1
42	121,40,15,12			$L_4(11)$	1	theo	39.2
43	121,50,21,20			$L_5(11)$	1	theo	40.2
44	121,60,29,30			$L_6(11)$	1	theo	41.1
45	121,60,29,30			$L_6(11)$	1	theo	41.3
46	121,60,29,30			Paley	1	theo	41.4
47	125,62,30,31		3000		1	GAP	42.1
48	125,62,30,31		15000		3	GAP	42.2
49	125,62,30,31		15000		3	GAP	42.3
50	125,62,30,31		23250	Paley	1	theo	42.4
51	126,25,8,4	A_{10}#126.2	$2^8 3^4 5^2 7^1$		-	GAP	
52	126,45,12,18	$PSU(4,3)$#126.5	$2^9 3^6 5^1 7^1$		-	GAP	
53	130,48,20,16	$PSL(4,3)$#130.5	$2^9 3^6 5^1 13^1$		-	GAP	
54	135,64,28,32	$O^{+1}(8,2)$#135.2	$2^{13} 3^5 5^2 7^1$		-	GAP	
55	136,30,15,4	A_{17} on 2-sets.2	$2^{15} 3^6 5^3 7^2 11^1 13^1 17^1$	$T(17)$	-	theo	
56	136,60,24,28	$PSp(4,4)$#136.2	$2^9 3^2 5^2 17^1$		-	GAP	
57	136,63,30,28	$PSL(2,17)$#136.1	2448		-	GAP	
58	136,63,30,28	$O^{-1}(8,2)$#136.2	$2^{13} 3^4 5^1 7^1 17^1$		-	GAP	
59	144,22,10,2	A_{12} on 1-sets^2.4	$2^{21} 3^{10} 5^4 7^2 11^2$	$L_2(12)$	≥ 1	theo	43.1
60	144,39,6,12	$PSL(3,3)$#144.2	11232		1	GAP	44.1
61	144,55,22,20	M_{12}#144n.1	190080	pseudo $L_5(12)$	-	GAP	
62	144,66,30,30	M_{12}#144.2	190080	pseudo $L_6(12)$	1	GAP	45.1
63	144,66,30,30	M_{12}#144n.1	190080	pseudo $L_6(12)$	-	GAP	
64	153,32,16,4	A_{18} on 2-sets.2	$2^{16} 3^8 5^3 7^2 11^1 13^1 17^1$	$T(18)$	-	theo	
65	155,42,17,9	$PSL(5,2)$#155.1	$2^{10} 3^2 5^1 7^1 31^1$		1	GAP	46.1
66	156,30,4,6	$PSp(4,5)$ on projective points.2	$2^7 3^2 5^4 13^1$		-	GAP	
67	156,30,4,6	$PSp(4,5)$#156.2	$2^7 3^2 5^4 13^1$		-	GAP	
68	162,56,10,24	$PSU(4,3)$#162.5	$2^9 3^6 5^1 7^1$		6	GAP	47.1
69	165,36,3,9	$PSU(5,2)$ on isotropic projective points.2	$2^{11} 3^5 5^1 11^1$		-	GAP	
70	169,24,11,2			$L_2(13)$	1	theo	48.1
71	169,36,13,6			$L_3(13)$	1	theo	49.1
72	169,48,17,12			$L_4(13)$	1	theo	50.1

No.	parameters	primitive group	order	graph	#pds	how	No. in App. G
73	169,48,17,12			$L_4(13)$	1	theo	50.2
74	169,48,17,12			$L_4(13)$	1	theo	50.3
75	169,60,23,20			$L_5(13)$	1	theo	51.3
76	169,72,31,30			$L_6(13)$	1	theo	52.1
77	169,72,31,30			$L_6(13)$	1	theo	52.2
78	169,72,31,30			$L_6(13)$	1	theo	52.4
79	169,84,41,42			Paley	1	theo	53.3
80	171,34,17,4	A_{19} on 2-sets.2	$2^{16}3^{8}5^{3}7^{2}11^{1}13^{1}17^{1}19^{1}$	$T(19)$	1	theo	54.1
81	175,72,20,36	$PSU(3,5)$#175.2	252000		-	GAP	
82	176,40,12,8	$PSU(5,2)$ on non-isotropic projective points.2	$2^{11}3^{5}5^{1}11^{1}$		-	GAP	
83	176,70,18,34	M_{22}#176.1	443520		-	GAP	
84	190,36,18,4	A_{20} on 2-sets.2	$2^{18}3^{8}5^{4}7^{2}11^{1}13^{1}17^{1}19^{1}$	$T(20)$	-	theo	
85	196,26,12,2	A_{14} on 1-sets^2.4	$2^{24}3^{10}5^{4}7^{4}11^{2}13^{2}$	$L_2(14)$	≥ 1	theo	55.1
86	208,75,30,25	$PSU(3,4)$ on non-isotropic projective points.3	249600		-	GAP	
87	210,38,19,4	A_{21} on 2-sets.2	$2^{18}3^{9}5^{4}7^{3}11^{1}13^{1}17^{1}19^{1}$	$T(21)$	-	GAP	
88	225,28,13,2	A_{15} on 1-sets^2.4	$2^{24}3^{12}5^{6}7^{4}11^{2}13^{2}$	$L_2(15)$	≥ 1	theo	56.1
89	231,30,9,3	M_{22}#231.2	887040		-	GAP	
90	231,40,20,4	A_{22} on 2-sets.2	$2^{19}3^{9}5^{4}7^{3}11^{2}13^{1}17^{1}19^{1}$	$T(22)$	-	theo	
91	243,22,1,2		$2^{5}3^{7}5^{1}11^{1}$	Berlekamp-van Lint-Seidel	4	GAP	57.1
92	243,110,37,60		$2^{5}3^{7}5^{1}11^{1}$		8	GAP	58.1
93	253,42,21,4	A_{23} on 2-sets.2	$2^{19}3^{9}5^{4}7^{3}11^{2}13^{1}17^{1}19^{1}23^{1}$	$T(23)$	1	theo	59.1
94	253,112,36,60	M_{23}#253a.1	$2^{7}3^{2}5^{1}7^{1}11^{1}23^{1}$		1	GAP	60.1
95	255,126,61,63	$PSp(8,2)$ on projective points.1	$2^{16}3^{5}5^{2}7^{1}17^{1}$		-	theo	

Appendix F

Table of groups for primitive cases

The groups are described by generators. In general, the generating set is not minimal. The first column gives the GAP identification number of the group in the small group library of GAP [GAP99]. The last column gives a standard representative. However, we were not able to give a representative for the groups in all cases.

GAP	abstract description of generators	representative
$(55,1)$	$a^5 = b^{11} = 1, ab = b^3a$	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$
$(64,9)$	$a^4 = b^4 = c^4 = 1, ba = abc, ca = ab^2c, cb = bc^3$	
$(64,18)$	$a^4 = b^4 = c^4 = 1, ba = abc, ca = ac, cb = bc$	
$(64,23)$	$a^4 = b^4 = c^2 = d^2 = 1, ba = abc, ca = acd, da = ad, cb = bc, db = bd, dc = cd$	
$(64,32)$	$a^4 = b^2 = c^2 = d^2 = e^2 = 1, ba = abc, ca = acd, da = ade, ea = ae; b, c, d, e$ commute	$\mathbb{Z}_4 \wr \mathbb{Z}_2$
$(64,33)$	$a^4 = b^4 = c^2 = d^2 = 1, ba = abc, ca = acd, da = ab^2d, cb = bc, db = bd, dc = cd$	
$(64,34)$	$a^4 = b^4 = c^2 = d^2 = 1, ba = abd, ca = acb, da = ab^2d, bc = cb^3, db = bd, dc = cd$	
$(64,35)$	$a^4 = b^4 = c^4 = d^2 = 1, ba = abc, ca = acd, da = ab^2d, b^2 = c^2, bc = cb^3, db = bd, dc = cd$	
$(64,56)$	$a^4 = b^4 = c^2 = d^2 = 1, ba = abd, ca = ac, da = ad, bc = cb, db = bd, dc = cd$	
$(64,60)$	$a^4 = b^4 = c^2 = d^2 = e^2 = 1, ba = abd, ca = ace, da = ad, ea = ae; b, c, d, e$ commute	
$(64,62)$	$a^4 = b^4 = c^2 = d^2 = 1, ba = abd, ca = ab^2c, da = ad, cb = bc, db = bd, dc = cd$	
$(64,67)$	$a^4 = b^4 = c^2 = d^2 = 1, ba^3 = ab^3, ca = acd, da = ad, cb = bc, db = bd, dc = cd$	
$(64,74)$	$a^4 = b^4 = c^2 = d^2 = e^2 = 1, ba = ab^3, ca = acd, da = ad, ea = ae, cb = bce, db = bd, eb = be; c, d, e$ commute	
$(64,88)$	$a^8 = b^2 = c^2 = d^2 = 1, ba = abd, ca = a^5c, da = ad, cb = bc, db = bd, dc = cd$	
$(64,90)$	$a^4 = b^2 = c^2 = d^2 = e^2 = 1, ba = abd, ca = ac, da = ade, ea = ae; b, c, d, e$ commute	
$(64,92)$	$a^8 = b^2 = c^2 = d^2 = 1, ba = abd, ca = ac, da = a^5d, cb = bc, db = bd, dc = cd$	

GAP	abstract description of generators	representative
$(64,102)$	$a^8=b^4=c^2=d^2=1, ba=a^5b, ca=ac, da=adb, cb=bc, db=a^4bd, dc=a^4cd$	
$(64,136)$	$a^4=b^4=c^2=d^2=e^2=1, ba=ab^3, ca=ace, da=adb, ea=ae, cb=a^2bc, bd=a^2db, eb=be, dc=cd,$ $ce=ec, ed=a^2de$	
$(64,138)$	$a^2=b^2=c^2=d^2=e^2=f^2=1, ba=abd, ca=ace, cb=bc, db=bd, eb=bef, fb=bf, dc=cdf,$ $ec=ce, fc=cf; a,d,e,f$ commute	$\mathbb{Z}_2 \wr (\mathbb{Z}_2\times\mathbb{Z}_2)$
$(64,139)$	$a^4=b^2=c^2=d^2=e^2=1, ba=abd, ca=ace, cb=bc, db=bd, eb=a^2be, dc=a^2cd, ec=ce; a,d,e$ commute	
$(64,193)$	$a^4=b^2=c^2=d^2=e^2=1, ba=abe, ca=ac, da=ad, ea=ae; b,c,d,e$ commute	$\mathbb{Z}_2\times\mathbb{Z}_2\times(16,3)$
$(64,199)$	$a^4=b^2=c^2=d^2=e^2=1, ba=abe, ca=ac, da=ade, ea=ae, cb=bce, db=bd, eb=be; c,d,e$ commute	
$(64,202)$	$a^2=b^2=c^2=d^2=e^2=f^2=1, ba=abe, ca=acf, da=ad, ea=ae, fa=af; b,c,d,e,f$ commute	$\mathbb{Z}_2\times((\mathbb{Z}_2\times\mathbb{Z}_2)\wr\mathbb{Z}_2)$
$(64,206)$	$a^4=b^2=c^2=d^2=e^2=1, ab=ba, ac=ca, bd=a^2db, cb=bc, eb=be, cd=dce, ec=ce; a,d,e$ commute	
$(64,215)$	$a^2=b^2=c^2=d^2=e^2=f^2=1, ba=abe, ca=acf, da=ade, ea=ae, fa=af, cb=bce, db=bd, eb=be,$ $fb=bf; c,d,e,f$ commute	
$(64,219)$	$a^4=b^2=c^2=d^2=e^2=1, ba=abe, ca=ace, da=ad, ea=ae, cb=a^2bc, db=bde, eb=be; c,d,e$ commute	
$(64,224)$	$a^4=b^4=c^2=d^2=e^2=1, ba=ab^3, ca=ace, da=ade, ea=ae, bc=cbe, bd=db, be=eb; c,d,e$ commute	
$(64,226)$	$a^4=b^4=c^2=d^2=1, ac=ca^3, bd=db^3, ab=ba, ad=da, bc=cb, cd=dc$	$D_4\times D_4$
$(64,227)$	$a^4=b^2=c^2=d^2=e^2=1, ba=a^3b, ca=ace, da=ad, ea=ae, cb=bc, db=a^2bd, eb=be; c,d,e$ commute	
$(64,232)$	$a^4=b^4=c^2=d^2=1, ba=a^3b, ca=ab^2c, da=ad, cb=bc, db=a^2bd, cd=dc$	
$(64,241)$	$a^2=b^2=c^2=d^2=e^2=f^2=1, ba=abe, ca=acf, da=adef, ea=ae, fa=af, cb=bcef, db=bde, eb=be,$ $fb=bf; c,d,e,f$ commute	
$(64,242)$	$a^4=b^4=c^2=d^2=e^2=1, ba=a^3b, ca=ace, da=a^3de, ea=ae, cb=a^2bce, db=a^2bd, eb=be; c,d,e$ commute	
$(64,264)$	$a^2=b^2=c^2=d^2=e^2=f^2=1, ba=abf, ca=ac, da=adf, ea=ae, fa=af, cb=bcf, db=bd, eb=be, fb=bf;$ c,d,e,f commute	
$(64,267)$	$a^2=b^2=c^2=d^2=e^2=f^2=1;$ abelian	$(\mathbb{Z}_2)^6$
$(81,2)$	$a^9=b^9=1,$ abelian	$\mathbb{Z}_9\times\mathbb{Z}_9$
$(81,3)$	$a^9=b^3=c^3=1, ba=abc, ca=ac, cb=bc$	
$(81,4)$	$a^9=b^9=1, ba=ab^4$	
$(81,7)$	$a^3=b^3=c^3=d^3=1, ab=bac^2, ac=cad^2, ad=da, bc=cb, bd=db, cd=dc$	$\mathbb{Z}_3\wr\mathbb{Z}_3$
$(81,8)$	$a^3=b^9=c^3=1, ab=bac^2, ac=cab^6, bc=cb$	
$(81,9)$	$a^3=b^9=c^3=1, ab=bac^2, ac=cab^3, bc=cb$	
$(81,11)$	$a^9=b^3=c^3=1,$ abelian	$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_9$
$(81,12)$	$a^3=b^3=c^3=1, bab=(aba)^2, ac=ca, bc=cb$	$\mathbb{Z}_3\times(27,3)$
$(81,13)$	$a^9=b^3=c^3=1, ba=a^4b, ac=ca, bc=cb$	$\mathbb{Z}_3\times(27,4)$
$(81,15)$	$a^3=b^3=c^3=d^3=1,$ abelian	$(\mathbb{Z}_3)^4$
$(100,6)$	$a^4=b^5=c^5=1, ba=ab, ca=ac^4, bc=cb$	
$(100,9)$	$a^4=b^5=c^5=1, ba=ab, ca=ac^2, bc=cb$	$\mathbb{Z}_5\times(\mathbb{Z}_5\rtimes\mathbb{Z}_4)$

GAP	abstract description of generators	representative
$(100,10)$	$a^4=b^5=c^5=1, ba=ab^4, ca=ac^3, bc=cb$	
$(100,11)$	$a^4=b^5=c^5=1, ba=ab^2, ca=ac^2, bc=cb$	
$(100,12)$	$a^4=b^5=c^5=1, ba=ab^3, ca=ac^2, bc=cb$	
$(100,13)$	$a^2=b^2=c^5=d^5=1, ba=ab, ca=ac, da=ad^4, cb=bc^4, db=bd, dc=cd$	$D_5\times D_5$
$(100,14)$	$a^2=b^2=c^5=d^5=1, ba=ab, ca=ac, da=ad^4, cb=bc, db=bd, dc=cd$	$\mathbb{Z}_{10}\times D_5$
$(100,15)$	$a^2=b^2=c^5=d^5=1, ba=ab, ca=ac^4, da=ad^4, cb=bc, db=bd, dc=cd$	
$(100,16)$	$a^2=b^2=c^5=d^5=1,$ abelian	$(\mathbb{Z}_2)^2\times(\mathbb{Z}_5)^2$
$(120,34)$	$a^5=b^2=1, ab=(ba^4)^3$	S_5
$(121,2)$	$a^{11}=b^{11}=1,$ abelian	$\mathbb{Z}_{11}\times\mathbb{Z}_{11}$
$(125,3)$	$a^5=b^5=c^5=1, ba=abc, ca=ac, bc=cb$	
$(125,5)$	$a^5=b^5=c^5=1,$ abelian	$(\mathbb{Z}_5)^3$
$(144,182)$	$a^8=b^2=c^3=d^3=1, ab=ba^3, ca=ad, da=acd^2, cb=bcd^2, db=bd^2, cd=dc$	
$(155,1)$	$a^5=b^{31}=1, ab=b^2a$	$\mathbb{Z}_{31}\rtimes\mathbb{Z}_5$
$(162,11)$	$a^2=b^3=c^3=d^3=e^3=1, ba=ab, ca=ac^2, da=ad^2, ea=ae^2, cb=bcd, db=bde, eb=be, dc=cd, ec=ce, de=de$	
$(162,19)$	$a^2=b^3=c^3=d^3=e^3=1, ba=ab^2, ca=ac^2, da=ade^2, ea=ae^2, cb=bcd, db=bde, eb=be, dc=cd, ec=ce, de=de$	
$(162,20)$	$a^2=b^3=c^9=d^3=1, ba=ab^2, ca=ac^8, da=adc^6, cb=bcd, db=bdc^3, dc=cd$	
$(162,36)$	$a^2=b^3=c^3=d^9=1, ba=ab, ca=ac, da=ad^8, cb=bc, db=bd^7, dc=cd$	
$(162,52)$	$a^2=b^3=c^3=d^3=e^3=1, ba=ab, ca=ac, da=ad^2, ea=ae^2, cb=bc, db=bd, eb=be, dc=cd, ec=ce, de=ed$	
$(162,54)$	$a^2=b^3=c^3=d^3=e^3=1, ba=ab^2, ca=ac^2, da=ad^2, ea=ae^2, cb=bc, db=bd, eb=be, dc=cd, ec=ce, de=ed$	
$(169,2)$	$a^{13}=b^{13}=1,$ abelian	$\mathbb{Z}_{13}\times\mathbb{Z}_{13}$
$(171,3)$	$a^9=b^{19}=1, ab=b^5a$	$\mathbb{Z}_{19}\rtimes\mathbb{Z}_9$
$(243,6)$	$a^3=b^9=c^3=d^3=1, ba=abc, ca=acb^6, da=ad, cb=bcd, db=bd, cd=dc$	
$(243,38)$	$a^9=b^3=c^3=d^3=1, ba=a^4b, ca=acd, da=ad, cb=bc, db=bd, cd=dc$	
$(243,51)$	$a^3=b^3=c^3=d^3=e^3=1, ab=bac^2, ac=cad^2, ad=da, ae=ea, bc=cb, bd=db, be=eb, cd=dc, ce=ec, de=ed$	$\mathbb{Z}_3\times(\mathbb{Z}_3\wr\mathbb{Z}_3)$
$(243,56)$	$a^3=b^3=c^3=d^3=e^3=1, ba=abd, ca=ac, da=ade, ae=ea, cb=bce, bd=db, be=eb, cd=dc, ce=ec, de=ed$	
$(243,57)$	$a^9=b^3=c^3=d^3=1, ab=bad, ca=a^4c, da=ad, cb=bc, db=a^3bd, cd=dc$	
$(243,59)$	$a^9=b^9=c^3=d^3=1, a^6=b^3, ba=abd, ca=ac, da=a^4d, cb=a^3bc, db=bd, cd=dc$	
$(243,66)$	$a^9=b^3=c^3=d^3=1, ba=a^4b, ca=ac, da=a^4d, cb=a^3bc, db=bd, cd=dc$	
$(243,67)$	$a^3=b^3=c^3=d^3=e^3=1;$ abelian	$(\mathbb{Z}_3)^5$
$(253,1)$	$a^{11}=b^{23}=1, ab=b^{12}a$	$\mathbb{Z}_{23}\rtimes\mathbb{Z}_{11}$

Appendix G

Partial difference sets for strongly regular graphs up to 255 vertices with primitive automorphism group

On the following pages we give all partial difference sets (up to CI-equivalence) we determined for the strongly regular graphs listed in Appendix E. The table is organized like the table in Appendix D. The groups, given by the GAP identification number of the small group library in GAP [GAP99], are described in Appendix F. In the column *srg* we give the names of the graphs. In cases where we do not know a name we give the number of the graph in the table of Appendix E.

In the last column we give references for the partial difference sets we found in the survey in [Ma94] or in other sources. However, in some case we were not able to recognize if our partial difference sets coincide with those listed in the table in [Ma94], (especially regarding the elementary abelian groups (64,267), (81,15), (125,5), (243,67)).

Notice that the determination of partial difference sets in groups of order 121 and 169 is complete. For these cases we also present the partial difference sets where the associated strongly regular graph has an imprimitive automorphism group and add the comment "not primitive".

As mentioned in Section 4.2 in seven cases we were not successful. However, in all these cases there exist partial difference sets. We add the parameters to this table, but do not give partial results (except No. 36.2.1.i, where all partial difference sets are contained in the group S_5).

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
20.1.1.1	55	18	9	4	$T(11)$	$(55,1)$	$\{b, b^{10}, a, ab^3, ab^4, ab^{10}, a^2, a^2b^4, a^2b^5, a^2b^{10}, a^3, a^3b^8, a^3b^9, a^3b^{10}, a^4, a^4b^2, a^4b^3, a^4b^{10}\}$	*Prop.* 3.2.15	
21.1.1.1	64	14	6	2	$L_2(6)$		no complete determination	*Sec.* 4.2.4	
22.1.1.1	64	18	2	6	7	$(64,9)$	$\{c, c^3, bc, b^2, b^2c^2, b^3c, a, ab, abc^3, ab^2c^3, a^2c^2, a^2b^2, a^2b^3, a^2b^3c^2, a^3, a^3c, a^3bc^3, a^3b^3c^2\}$	*Sec.* 4.2.6	
2.1	64	18	2	6	7	$(64,18)$	$\{b, bc, b^3, b^3c^3, a, ac^2, ac^3, abc^2, ab^2c, ab^3, a^2b^2, a^2b^2c^2, a^3, a^3c, a^3c^2, a^3bc^3, a^3b^2c, a^3b^3c^3\}$	*Sec.* 4.2.6	
3.1	64	18	2	6	7	$(64,23)$	$\{c, cd, b, b^2, b^2d, b^3, a, ab^2c, ab^3d, ab^3cd, a^2, a^2d, a^2bcd, a^2b^3c, a^3, a^3bd, a^3bc, a^3b^2cd\}$	*Sec.* 4.2.6	
3.2	64	18	2	6	7	$(64,23)$	$\{b, bd, bc, b^3, b^3d, b^3c, a, acd, abcd, ab^3, a^2bc, a^2b^2, a^2b^2d, a^2b^3cd, a^3, a^3c, a^3bcd, a^3b^3d\}$	*Sec.* 4.2.6	
3.3	64	18	2	6	7	$(64,23)$	$\{cd, b, bc, b^2cd, b^3, b^3c, a, ab^2d, ab^3, ab^3d, a^2d, a^2c, a^2cd, a^2b^2, a^3, a^3bc, a^3bcd, a^3b^2d\}$	*Sec.* 4.2.6	
3.4	64	18	2	6	7	$(64,23)$	$\{c, cd, bd, b^2c, b^2cd, b^3d, a, ab^2c, ab^3, ab^3c, a^2c, a^2cd, a^2bc, a^2b^3cd, a^3, a^3b, a^3bcd, a^3b^2cd\}$	*Sec.* 4.2.6	
3.5	64	18	2	6	7	$(64,23)$	$\{bd, b^3d, a, ac, abc, ab^3, a^2c, a^2cd, a^2b, a^2b^2, a^2b^2d, a^2b^2c, a^2b^2cd, a^2b^3d, a^3, a^3cd, a^3bcd, a^3b^3\}$	*Sec.* 4.2.6	
3.6	64	18	2	6	7	$(64,23)$	$\{b, b^2, b^2d, b^2c, b^2cd, b^3, a, ac, abcd, ab^3d, a^2c, a^2cd, a^2bd, a^2b^3, a^3, a^3cd, a^3bc, a^3b^3d\}$	*Sec.* 4.2.6	
4.1	64	18	2	6	7	$(64,32)$	$\{b, bd, bc, bcde, a, ae, ace, acd, ab, abde, a^2d, a^2de, a^3, a^3e, a^3c, a^3cd, a^3bce, a^3bcde\}$	*Sec.* 4.2.6	
4.2	64	18	2	6	7	$(64,32)$	$\{ce, cd, b, be, bce, bcde, a, ac, abd, abcd, a^2, a^2e, a^2d, a^2de, a^3, a^3cde, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
4.3	64	18	2	6	7	$(64,32)$	$\{a, ae, ac, acd, abe, abd, a^2d, a^2de, a^2b, a^2bd, a^2bce, a^2bcd, a^3, a^3e, a^3c, a^3cde, a^3bc, a^3bcd\}$	*Sec.* 4.2.6	
4.4	64	18	2	6	7	$(64,32)$	$\{c, ce, b, be, bcd, bcde, a, ade, ab, abd, a^2be, a^2bde, a^2bce, a^2bcd, a^3, a^3d, a^3bc, a^3bcde\}$	*Sec.* 4.2.6	
4.5	64	18	2	6	7	$(64,32)$	$\{c, cd, b, bde, bcd, bcde, a, acde, abd, abce, a^2bc, a^2bce, a^2bcd, a^2bcde, a^3, a^3ce, a^3be, a^3bc\}$		
5.1	64	18	2	6	7	$(64,33)$	$\{b, bd, bc, b^3, b^3d, b^3c, a, ab^2cd, ab^3, ab^3cd, a^2d, a^2bc, a^2bcd, a^2b^2, a^3, a^3b^2c, a^3b^3d, a^3b^3cd\}$	*Sec.* 4.2.6	
5.2	64	18	2	6	7	$(64,33)$	$\{cd, b, bc, b^2cd, b^3, b^3c, a, ad, abd, ab^3, a^2b, a^2bc, a^2bcd, a^2b^3d, a^3, a^3b^2d, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
5.3	64	18	2	6	7	$(64,33)$	$\{c, cd, bc, b^2c, b^2cd, b^3c, acd, ab, abcd, ab^2, a^2d, a^2bc, a^2bcd, a^2b^2, a^3c, a^3bd, a^3bcd, a^3b^2\}$	*Sec.* 4.2.6	
6.1	64	18	2	6	7	$(64,34)$	$\{bc, b^2c, b^2cd, b^3cd, a, ab^2, ab^3, ab^3d, ab^3c, ab^3cd, a^2d, a^2b^2d, a^3, a^3c, a^3b, a^3b^2, a^3b^2cd, a^3b^3d\}$	*Sec.* 4.2.6	
7.1	64	18	2	6	7	$(64,35)$	$\{b, bc^2, a, ac^3, abd, abc^3d, a^2, a^2d, a^2c, a^2cd, a^2c^2, a^2c^2d, a^2bcd, a^2bc^3, a^3, a^3c^3d, a^3bc, a^3bc^2d\}$	*Sec.* 4.2.6	
7.2	64	18	2	6	7	$(64,35)$	$\{d, c, c^2d, c^3, b, bc^2, a, acd, abcd, abc^2, a^2c^2, a^2c^2d, a^2bc, a^2bc^3d, a^3, a^3c^3, a^3bd, a^3bcd\}$	*Sec.* 4.2.6	
7.3	64	18	2	6	7	$(64,35)$	$\{bcd, bc^3d, a, ac^3d, abd, abc, a^2c^2, a^2c^2d, a^2b, a^2bd, a^2bcd, a^2bc^2, a^2bc^2d, a^2bc^3,$ $a^3, a^3c, a^3bc, a^3bc^2\}$	*Sec.* 4.2.6	
7.4	64	18	2	6	7	$(64,35)$	$\{a, ac^2, ac^3, ac^3d, ab, abc^2d, a^2d, a^2c^2d, a^2bc, a^2bc^2, a^2bc^2d, a^2bc^3d, a^3, a^3c, a^3c^2,$ $a^3c^3d, a^3bc^3, a^3bc^3d\}$	*Sec.* 4.2.6	
8.1	64	18	2	6	7	$(64,56)$	$\{d, b, bcd, b^2d, b^3, b^3cd, a, ab^2c, ab^3, ab^3c, a^2d, a^2c, a^2cd, a^2b^2, a^3, a^3bd, a^3bcd, a^3b^2c\}$	*Sec.* 4.2.6	
9.1	64	18	2	6	7	$(64,60)$	$\{d, de, cd, cde, b, bd, ae, ace, ab, abcd, a^2, a^2e, a^2bce, a^2bcd, a^3e, a^3c, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
9.2	64	18	2	6	7	$(64,60)$	$\{b, bd, bce, bcd, a, ae, ade, acde, ab, abc, a^2c, a^2ce, a^3, a^3e, a^3de, a^3cd, a^3bd, a^3bcde\}$	*Sec.* 4.2.6	
9.3	64	18	2	6	7	$(64,60)$	$\{b, bd, a, acd, abe, abce, a^2, a^2e, a^2d, a^2de, a^2c, a^2ce, a^2bc, a^2bcde, a^3, a^3cde, a^3bde, a^3bcd\}$	*Sec.* 4.2.6	
9.4	64	18	2	6	7	$(64,60)$	$\{c, ce, b, be, bd, bde, a, acd, ab, abcde, a^2be, a^2bde, a^2bce, a^2bcd, a^3, a^3cde, a^3bd, a^3bc\}$	*Sec.* 4.2.6	
10.1	64	18	2	6	7	$(64,88)$	$\{d, b, bd, bcd, a, abc, a^2bd, a^2bc, a^3bd, a^3bcd, a^4d, a^4bc, a^5c, a^5b, a^6bd, a^6bc, a^7, a^7c\}$	*Sec.* 4.2.6	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
22.1.11.1	64	18	2	6	7	$(64, 90)$	$\{d, de, b, bde, bc, bcd, ae, acde, ab, abcd, a^2d, a^2de, a^2cd, a^2cde, a^3e, a^3cd, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
11.2	64	18	2	6	7	$(64, 90)$	$\{d, cde, b, bde, bce, bcd, a, ae, ab, abc, a^2de, a^2c, a^2ce, a^2cde, a^3, a^3e, a^3bde, a^3bcde\}$	*Sec.* 4.2.6	
11.3	64	18	2	6	7	$(64, 90)$	$\{c, ce, b, bde, bc, bcd, a, acd, abe, abcde, a^2, a^2e, a^2d, a^2de, a^3, a^3cde, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
11.4	64	18	2	6	7	$(64, 90)$	$\{d, de, a, ace, abd, abcd, a^2e, a^2d, a^2c, a^2cd, a^2b, a^2be, a^2bcd, a^2bcde, a^3, a^3ce, a^3b, a^3bc\}$	*Sec.* 4.2.6	
11.5	64	18	2	6	7	$(64, 90)$	$\{d, de, b, bde, bc, bcd, a, acd, abe, abcde, a^2bd, a^2bde, a^2bcd, a^2bcde, a^3, a^3cde, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
11.6	64	18	2	6	7	$(64, 90)$	$\{c, ce, b, bde, bc, bcd, a, acd, abe, abcde, a^2c, a^2ce, a^2cd, a^2cde, a^3, a^3cde, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
11.7	64	18	2	6	7	$(64, 90)$	$\{cd, cde, b, be, bd, bde, a, ace, ab, abc, a^2e, a^2de, a^2c, a^2cde, a^3, a^3ce, a^3bde, a^3bcde\}$	*Sec.* 4.2.6	
12.1	64	18	2	6	7	$(64, 92)$	$\{c, b, bd, a, ab, abcd, a^2bc, a^3bd, a^4c, a^4b, a^4bd, a^5cd, a^6b, a^6bd, a^6bcd, a^7, a^7cd, a^7bc\}$	*Sec.* 4.2.6	
13.1	64	18	2	6	7	$(64, 102)$	$\{d, c, b^2c, b^3d, a, ad, abd, abcd, ab^2d, ab^3c, a^2cd, a^2b^3cd, a^3d, a^3bd, a^3bc, a^3b^2, a^3b^2cd, a^3b^3d\}$	*Sec.* 4.2.6	
14.1	64	18	2	6	7	$(64, 136)$	$\{d, c, ce, cd, bc, bce, bcde, b^2c, b^2ce, b^3d, ac, ab, abde, ab^2de, ab^2ce, ab^2cd, ab^3, ab^3cde\}$	*Sec.* 4.2.6	
14.2	64	18	2	6	7	$(64, 136)$	$\{d, c, ce, bd, b^2cd, b^3cde, a, ade, ac, abc, abce, abcd, ab^2, ab^2ce, ab^2cde, ab^3de, ab^3c, ab^3ce\}$	*Sec.* 4.2.6	
15.1	64	18	2	6	7	$(64, 138)$	$\{ef, d, c, ce, b, bdf, bc, bcf, bcde, bcdef, a, ad, adf, ade, acf, acef, abef, abde\}$	*Sec.* 4.2.6	
15.2	64	18	2	6	7	$(64, 138)$	$\{ef, d, cf, cef, bf, bd, a, ad, adf, ade, ac, ace, abe, abdef, abce, abcef, abcd, abcdf\}$	*Sec.* 4.2.6	
16.1	64	18	2	6	7	$(64, 139)$	$\{e, d, b, bc, bcde, ad, ace, abe, abde, abce, a^2e, a^2d, a^2c, a^2ce, a^2bd, a^3d, a^3c, a^3bcd\}$	*Sec.* 4.2.6	
16.2	64	18	2	6	7	$(64, 139)$	$\{ce, b, bd, bc, bcde, a, ae, ade, ab, abcd, a^2c, a^3, a^3e, a^3de, a^3cd, a^3cde, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
17.1	64	18	2	6	7	$(64, 193)$	$\{d, de, c, cd, b, be, a, ace, ab, abcde, a^2c, a^2cde, a^2bd, a^2bde, a^3, a^3ce, a^3be, a^3bcd\}$	*Sec.* 4.2.6	
18.1	64	18	2	6	7	$(64, 199)$	$\{d, de, c, cd, b, be, a, ace, ab, abcde, a^2c, a^2cde, a^2bd, a^2bde, a^3, a^3ce, a^3be, a^3bcd\}$	*Sec.* 4.2.6	
20.1	64	18	2	6	7	$(64, 202)$	$\{de, def, c, cf, cd, cdf, b, be, bc, bcef, af, aef, adf, ade, ab, abe, abcd, abcdef\}$	*Sec.* 4.2.6	
20.2	64	18	2	6	7	$(64, 202)$	$\{d, def, cde, cdef, b, be, a, ae, ad, adf, ac, acf, abf, abef, abc, abcef, abcdf, abcde\}$	*Sec.* 4.2.6	
21.1	64	18	2	6	7	$(64, 206)$	$\{d, de, c, ce, b, bc, ac, acd, abde, abce, a^2cd, a^2cde, a^2b, a^2bce, a^3c, a^3cde, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
22.1	64	18	2	6	7	$(64, 215)$	$\{d, de, b, bef, bd, bdf, bcf, bcef, bcd, bcde, a, af, acdf, acde, abde, abdef, abcdf, abcde\}$	*Sec.* 4.2.6	
23.1	64	18	2	6	7	$(64, 219)$	$\{d, de, bd, bde, bc, bcd, ae, acd, abe, abc, a^2b, a^2be, a^2bc, a^2bcde, a^3e, a^3cde, a^3b, a^3bc\}$	*Sec.* 4.2.6	
24.1	64	18	2	6	7	$(64, 224)$	$\{d, de, cd, cde, be, bce, b^2c, b^2ce, b^3e, b^3c, a, ad, ab, abcd, ab^2, ab^2de, ab^3, ab^3cde\}$	*Sec.* 4.2.6	
25.1	64	18	2	6	7	$(64, 226)$	$\{d, c, cd, bc, b^2cd, b^3c, acd, abcd, ab^2c, ab^3d, a^2c, a^2bd, a^2b^2d, a^2b^3d, a^3c, a^3bcd, a^3b^2cd, a^3b^3d\}$	*Sec.* 4.2.6	
26.1	64	18	2	6	7	$(64, 227)$	$\{de, c, ce, b, bc, bcd, acd, ab, abe, abcd, a^2de, a^2be, a^2bc, a^2bcd, a^3cde, a^3bc, a^3bce, a^3bcde\}$	*Sec.* 4.2.6	
27.1	64	18	2	6	7	$(64, 232)$	$\{d, b, bd, bc, b^3, b^3c, a, ab^2d, ab^2cd, ab^3c, a^2d, a^2c, a^2b^2c, a^2b^3d, a^3, a^3cd, a^3bc, a^3b^2d\}$	*Sec.* 4.2.6	
27.2	64	18	2	6	7	$(64, 232)$	$\{d, c, b^2c, b^3cd, a, ac, ab, abd, ab^2d, ab^3, a^2d, a^2bcd, a^3, a^3bc, a^3b^2d, a^3b^2c, a^3b^3d, a^3b^3c\}$	*Sec.* 4.2.6	
28.1	64	18	2	6	7	$(64, 241)$	$\{b, bef, bd, bde, bcde, bcdef, a, af, adf, ade, acd, acde, abdf, abde, abc, abce, abcd, abcdf\}$	*Sec.* 4.2.6	
29.1	64	18	2	6	7	$(64, 242)$	$\{d, ce, cd, be, b^2de, b^2c, b^2cde, b^3e, b^3c, b^3ce, a, ad, ade, ab, ab^2, ab^3, ab^3cd, ab^3cde\}$	*Sec.* 4.2.6	
30.1	64	18	2	6	7	$(64, 264)$	$\{df, def, c, ce, cd, cde, be, bef, bdf, bde, a, aef, ac, acf, abd, abdef, abc, abcf\}$	*Sec.* 4.2.6	
31.1	64	18	2	6	7	$(64, 267)$	$\{f, e, ef, de, cef, cdf, b, bd, bce, bcde, a, adef, ac, acdef, ab, abdf, abcf, abcd\}$	*Sec.* 4.2.6	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
23.1.1.1	64	21	8	6	8	(64, 23)	$\{d, c, cd, b, bd, bc, b^3, b^3d, b^3c, abc, ab^2, ab^2cd, ab^3d, a^2bcd, a^2b^2c, a^2b^2cd, a^2b^3c, a^3bc, a^3b^2, a^3b^2c, a^3b^3\}$	*Sec.* 4.2.6	
1.2	64	21	8	6	8	(64, 23)	$\{d, b, bd, bcd, b^2c, b^2cd, b^3, b^3d, b^3cd, abc, ab^2d, ab^2c, ab^3d, a^2c, a^2cd, a^2bc, a^2b^3cd, a^3bc, a^3b^2d, a^3b^2cd, a^3b^3\}$	*Sec.* 4.2.6	
2.1	64	21	8	6	8	(64, 35)	$\{cd, c^2, c^3d, b, bc^2, ac^2d, ac^3, ab, abc^3d, a^2, a^2d, a^2c, a^2c^2, a^2c^2d, a^2c^3d, a^2b, a^2bd, a^3d, a^3c^3d, a^3bc^2d, a^3bc^3d\}$	*Sec.* 4.2.6	
2.2	64	21	8	6	8	(64, 35)	$\{cd, c^2, c^3d, b, bc^2, ad, ac, ab, abc^3d, a^2c, a^2c^3d, a^2bc, a^2bcd, a^2bc^2, a^2bc^2d, a^2bc^3, a^2bc^3d, a^3cd, a^3c^2d, a^3bc^2d, a^3bc^3d\}$	*Sec.* 4.2.6	
3.1	64	21	8	6	8	(64, 60)	$\{e, d, de, c, ce, bce, bcd, a, ae, ade, ac, abe, abce, a^2b, a^2bd, a^3, a^3e, a^3de, a^3ce, a^3bde, a^3bcd\}$	*Sec.* 4.2.6	
3.2	64	21	8	6	8	(64, 60)	$\{e, c, ce, a, ae, ad, ace, abe, abcde, a^2d, a^2de, a^2be, a^2bde, a^2bce, a^2bcd, a^3, a^3e, a^3d, a^3c, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
3.3	64	21	8	6	8	(64, 60)	$\{e, c, ce, b, be, bd, bde, bc, bcde, ad, ace, abe, abce, a^2, a^2e, a^2be, a^2bde, a^3d, a^3c, a^3bde, a^3bcd\}$	*Sec.* 4.2.6	
4.1	64	21	8	6	8	(64, 62)	$\{d, bcd, b^3cd, a, ad, abd, ab^2c, ab^2cd, ab^3, a^2, a^2cd, a^2b, a^2b^2d, a^2b^2cd, a^2b^3, a^3, a^3d, a^3c, a^3cd, a^3bd, a^3b^3\}$	*Sec.* 4.2.6	
5.1	64	21	8	6	8	(64, 67)	$\{c, bd, bcd, b^2d, b^2c, b^3d, b^3cd, a, ac, ab^2d, ab^2cd, ab^3c, ab^3cd, a^2, a^2d, a^3, a^3cd, a^3b^2d, a^3b^2c, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
5.2	64	21	8	6	8	(64, 67)	$\{bc, b^2d, b^3c, a, acd, ab, abd, ab^2d, ab^2c, a^2, a^2bc, a^2b^2, a^2b^2c, a^2b^2cd, a^2b^3c, a^3, a^3c, a^3b^2d, a^3b^2cd, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
5.3	64	21	8	6	8	(64, 67)	$\{d, bcd, b^3cd, a, ab^2d, ab^3, ab^3d, ab^3c, ab^3cd, a^2, a^2c, a^2bd, a^2b^2d, a^2b^2c, a^2b^3d, a^3, a^3b, a^3bd, a^3b^2d, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
5.4	64	21	8	6	8	(64, 67)	$\{c, bd, bcd, b^2d, b^2c, b^3d, b^3cd, a, ac, ab, abd, ab^2d, ab^2cd, a^2b^2c, a^2b^2cd, a^3, a^3cd, a^3b^2d, a^3b^2c, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
5.5	64	21	8	6	8	(64, 67)	$\{b, bc, b^2, b^2c, b^2cd, b^3, b^3c, ad, ab^2, a^2, a^2bd, a^2bc, a^2b^2d, a^2b^3d, a^2b^3c, a^3d, a^3bc, a^3bcd, a^3b^2, a^3b^3, a^3b^3d\}$	*Sec.* 4.2.6	
5.6	64	21	8	6	8	(64, 67)	$\{cd, b^2d, b^2cd, a, ac, ab, abd, ab^2d, ab^2cd, a^2b, a^2bcd, a^2b^2, a^2b^2d, a^2b^3, a^2b^3cd, a^3, a^3cd, a^3b^2d, a^3b^2c, a^3b^3, a^3b^3d\}$	*Sec.* 4.2.6	
6.1	64	21	8	6	8	(64, 74)	$\{be, bc, bcd, b^2e, b^2d, b^2de, b^3e, b^3ce, b^3cde, ae, ac, acde, ab, abc, abce, ab^2e, ab^2ce, ab^2cd, ab^3, ab^3cd, ab^3cde\}$	*Sec.* 4.2.6	
6.2	64	21	8	6	8	(64, 74)	$\{d, de, c, ce, cd, cde, bd, bc, bcd, b^2e, b^3d, b^3ce, b^3cde, ae, ace, acd, abde, ab^2e, ab^2c, ab^2cde, ab^3de\}$	*Sec.* 4.2.6	
7.1	64	21	8	6	8	(64, 90)	$\{c, cd, cde, a, ade, ac, acde, abe, abc, a^2d, a^2cde, a^2b, a^2be, a^2bcd, a^2bcde, a^3, a^3d, a^3c, a^3cd, a^3bd, a^3bcde\}$	*Sec.* 4.2.6	
7.2	64	21	8	6	8	(64, 90)	$\{c, bcd, bcde, a, ad, ac, acd, abe, abc, a^2d, a^2c, a^2ce, a^2cde, a^2bd, a^2bde, a^3, a^3de, a^3c, a^3cde, a^3bd, a^3bcde\}$	*Sec.* 4.2.6	
7.3	64	21	8	6	8	(64, 90)	$\{c, bcd, bcde, a, ade, ac, acde, abe, abc, a^2, a^2e, a^2d, a^2cde, a^2bcd, a^2bcde, a^3, a^3d, a^3c, a^3cd, a^3bd, a^3bcde\}$	*Sec.* 4.2.6	

$No.$	v	k	λ	μ	srg	GAP	partial difference set	reference	comment/ reference
23.1.7.4	64	21	8	6	8	$(64, 90)$	$\{e, b, be, bd, bde, ade, acde, abe, abc, a^2e, a^2d, a^2ce, a^2cde, a^2b, a^2be, a^2bcd, a^2bcde, a^3d, a^3cd, a^3bd, a^3bcde\}$	$Sec.$ 4.2.6	
7.5	64	21	8	6	8	$(64, 90)$	$\{d, de, ce, bc, bce, bcd, bcde, ad, acd, ab, abde, abce, abcd, a^2e, a^2ce, a^3de, a^3cde, a^3be, a^3bde, a^3bc, a^3bcd\}$	$Sec.$ 4.2.6	
7.6	64	21	8	6	8	$(64, 90)$	$\{e, b, bd, bce, bcd, ade, ac, abe, abcd, a^2, a^2e, a^2d, a^2de, a^2bd, a^2bde, a^2bcd, a^2bcde, a^3d, a^3c, a^3bd, a^3bc\}$	$Sec.$ 4.2.6	
7.7	64	21	8	6	8	$(64, 90)$	$\{e, b, bd, bce, bcd, ade, ac, abe, abcd, a^2c, a^2ce, a^2cd, a^2cde, a^2b, a^2be, a^2bc, a^2bce, a^3d, a^3c, a^3bd, a^3bc\}$	$Sec.$ 4.2.6	
8.1	64	21	8	6	8	$(64, 138)$	$\{def, ce, cef, bde, bdef, bc, bcd, bcdf, bcde, af, ade, ac, ace, acdf, acde, ab, abef, abd, abde, abc, abcdef\}$	$Sec.$ 4.2.6	
8.2	64	21	8	6	8	$(64, 138)$	$\{ef, d, df, cf, cef, bce, bcef, bcde, bcdef, adf, ade, acdf, acde, ab, abe, abd, abdef, abcf, abcef, abcdf, abcde\}$	$Sec.$ 4.2.6	
8.3	64	21	8	6	8	$(64, 138)$	$\{e, ef, de, c, cf, ce, cef, bcf, bcde, aef, adf, acf, acef, acd, acdef, ab, abef, abd, abde, abce, abcd\}$	$Sec.$ 4.2.6	
8.4	64	21	8	6	8	$(64, 138)$	$\{ef, cf, cef, cd, cdf, bde, bdef, bce, bcef, adf, ade, acdf, acde, ab, abe, abd, abdef, abcf, abcef, abcdf, abcde\}$	$Sec.$ 4.2.6	
9.1	64	21	8	6	8	$(64, 193)$	$\{c, bcd, bcde, a, ad, ac, acde, abe, abce, a^2, a^2d, a^2de, a^2ce, a^2bc, a^2bce, a^3, a^3d, a^3c, a^3cde, a^3b, a^3bc\}$	$Sec.$ 4.2.6	
9.2	64	21	8	6	8	$(64, 193)$	$\{e, de, c, b, be, bd, bde, bc, bce, ace, acd, abe, abce, a^2de, a^2ce, a^2bcd, a^2bcde, a^3ce, a^3cd, a^3b, a^3bc\}$	$Sec.$ 4.2.6	
10.1	64	21	8	6	8	$(64, 202)$	$\{ef, d, df, c, cf, cd, cdf, bf, bef, bcf, bce, bcdf, bcde, ad, ade, abdf, abdef, abc, abcef, abcdf, abcde\}$	$Sec.$ 4.2.6	
10.2	64	21	8	6	8	$(64, 202)$	$\{f, df, def, c, cf, cd, cdf, bcdf, bcde, adf, ade, ace, acef, acd, acdf, ab, abe, abdf, abdef, abc, abcef\}$	$Sec.$ 4.2.6	
10.3	64	21	8	6	8	$(64, 202)$	$\{f, c, cf, cde, cdef, bd, bde, bcf, bce, a, aef, adf, adef, ac, acf, acd, acdf, abf, abef, abcf, abce\}$	$Sec.$ 4.2.6	
10.4	64	21	8	6	8	$(64, 202)$	$\{d, df, de, cef, cd, bef, bd, bc, bcdef, ac, acf, acd, acdf, ab, abe, abd, abde, abc, abcef, abcdf, abcde\}$	$Sec.$ 4.2.6	
10.5	64	21	8	6	8	$(64, 202)$	$\{f, e, df, ce, cd, bf, bdef, bcef, bcdf, ad, adf, ade, adef, ac, acf, acd, acdf, ab, abe, abdf, abdef\}$	$Sec.$ 4.2.6	
10.6	64	21	8	6	8	$(64, 202)$	$\{de, c, cf, ce, cdf, bef, bd, bcf, bcde, ade, adef, acde, acdef, ab, abe, abd, abde, abc, abcef, abcdf, abcde\}$	$Sec.$ 4.2.6	
11.1	64	21	8	6	8	$(64, 267)$	$\{e, d, def, c, cf, b, bf, bc, bcf, af, aef, adf, adef, acf, acef, acd, acde, abf, abe, abdf, abde\}$	$Sec.$ 4.2.6	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
23.2.1.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 9)$	$\{c^2, b, b^2c, b^2c^3, b^3, a, ac^3, ab, ab^3c, a^2, a^2b, a^2bc, a^2bc^2, a^2bc^3, a^2b^2c^2, a^2b^3, a^2b^3c^2, a^3, a^3b, a^3bc^3, a^3b^2c\}$	*Sec.* 4.2.6	
2.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 18)$	$\{c^2, b, bc^3, b^2, b^2c^2, b^3, b^3c, a, ac, ac^3, ab, ab^2, ab^3, a^2, a^2c^2, a^3, a^3c, a^3c^3, a^3bc^3, a^3b^2c^2, a^3b^3c\}$	*Sec.* 4.2.6	
3.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, b, bd, bcd, b^2, b^2d, b^3, b^3d, b^3cd, a, acd, abcd, ab^3, a^2, a^2d, a^2bcd, a^2b^3c, a^3, a^3c, a^3bcd, a^3b^3d\}$	*Sec.* 4.2.6	
3.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, c, b, bc, b^2, b^2d, b^2c, b^3, b^3c, a, ab^2d, ab^3, ab^3d, a^2, a^2b^2d, a^2b^2c, a^2b^2cd, a^3, a^3bc, a^3bcd, a^3b^2d\}$	*Sec.* 4.2.6	
3.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, b, b^2, b^2d, b^3, a, ab^2c, ab^3, ab^3c, a^2, a^2d, a^2bcd, a^2b^2, a^2b^2d, a^2b^2c, a^2b^2cd, a^2b^3c, a^3, a^3b, a^3bcd, a^3b^2cd\}$	*Sec.* 4.2.6	
3.4	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, c, cd, b, b^3, a, ab^2c, ab^3, ab^3c, a^2, a^2d, a^2c, a^2cd, a^2bcd, a^2b^2c, a^2b^2cd, a^2b^3c, a^3, a^3b, a^3bcd, a^3b^2cd\}$	*Sec.* 4.2.6	
3.5	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, c, cd, b, b^2, b^2d, b^2c, b^2cd, b^3, a, ac, abc, ab^3, a^2, a^2d, a^2bd, a^2b^3, a^3, a^3cd, a^3bcd, a^3b^3\}$	*Sec.* 4.2.6	
3.6	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 23)$	$\{d, b, b^2c, b^2cd, b^3, a, ab^2c, ab^3, ab^3c, a^2c, a^2cd, a^2bcd, a^2b^2, a^2b^2d, a^2b^2c, a^2b^2cd, a^2b^3c, a^3, a^3b, a^3bcd, a^3b^2cd\}$	*Sec.* 4.2.6	
4.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 32)$	$\{e, d, de, c, cd, b, bde, bcd, bcde, a, acd, ab, abcd, a^2, a^2e, a^2d, a^2de, a^3, a^3c, a^3bde, a^3bcde\}$	*Sec.* 4.2.6	
4.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 32)$	$\{e, d, de, c, cd, b, bde, bcd, bcde, a, acd, ab, abcd, a^2bc, a^2bce, a^2bcd, a^2bcde, a^3, a^3c, a^3bde, a^3bcde\}$	*Sec.* 4.2.6	
4.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 32)$	$\{e, d, de, c, ce, b, be, bc, bce, a, ad, abc, abcde, a^2b, a^2bd, a^2bc, a^2bcde, a^3, a^3de, a^3b, a^3bd\}$	*Sec.* 4.2.6	
4.4	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 32)$	$\{e, d, de, b, bd, bc, bcde, a, ad, acd, acde, ab, abd, a^2d, a^2de, a^3, a^3de, a^3c, a^3ce, a^3bc, a^3bcde\}$	*Sec.* 4.2.6	
4.5	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 32)$	$\{e, d, de, a, ad, acd, acde, abce, abcd, a^2d, a^2de, a^2be, a^2bde, a^2bce, a^2bcd, a^3, a^3de, a^3c, a^3ce, a^3be, a^3bde\}$	*Sec.* 4.2.6	
5.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 33)$	$\{d, c, cd, b, b^2, b^2d, b^2c, b^2cd, b^3, a, ac, abc, ab^3, a^2, a^2d, a^2bd, a^2b^3, a^3, a^3b^2cd, a^3b^3, a^3b^3cd\}$	*Sec.* 4.2.6	
5.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 33)$	$\{d, b, bd, bc, b^2, b^2d, b^3, b^3d, b^3c, a, acd, abd, ab^3c, a^2, a^2b^2d, a^2b^3c, a^2b^3cd, a^3, a^3c, a^3b, a^3b^3c\}$	*Sec.* 4.2.6	
5.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 33)$	$\{d, cd, b, bcd, b^2, b^2d, b^2cd, b^3, b^3cd, a, abc, abcd, ab^2d, a^2bd, a^2bc, a^2bcd, a^2b^3, a^3, a^3d, a^3bd, a^3b^3\}$	*Sec.* 4.2.6	
6.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 34)$	$\{d, c, cd, bc, b^2, b^2d, b^3cd, a, ad, ac, acd, ab, ab^3, a^2d, a^2b^2d, a^3, a^3bd, a^3bc, a^3b^2d, a^3b^3d, a^3b^3cd\}$	*Sec.* 4.2.6	
7.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 35)$	$\{d, c^2, c^2d, b, bc^2, a, ac, abc^2, abc^3, a^2, a^2d, a^2c, a^2cd, a^2c^2, a^2c^2d, a^2bcd, a^2bc^3, a^3, a^3cd, a^3b, a^3bcd\}$	*Sec.* 4.2.6	
7.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 35)$	$\{d, c^2, c^2d, bc, bc^3, a, acd, abd, abc^3, a^2cd, a^2c^3, a^2b, a^2bd, a^2bc, a^2bcd, a^2bc^3, a^2bc^3d, a^3, a^3c^3, a^3b, a^3bc\}$	*Sec.* 4.2.6	
7.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 35)$	$\{c, c^2, c^3, b, bd, bc, bc^2, bc^2d, bc^3, a, ac^3, abd, abc^3d, a^2c^3, a^2c^3d, a^2bc, a^2bc^3d, a^3, a^3c^3d, a^3bc, a^3bc^2d\}$	*Sec.* 4.2.6	
7.4	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 35)$	$\{d, c^2, c^2d, a, ad, ac, ac^3, abcd, abc^3, a^2d, a^2c^2d, a^2b, a^2bd, a^2bcd, a^2bc^3, a^3, a^3cd, a^3c^2d, a^3c^3d, a^3b, a^3bd\}$	*Sec.* 4.2.6	
8.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 56)$	$\{c, cd, b, bc, b^2, b^2c, b^2cd, b^3, b^3c, a, acd, abcd, ab^3, a^2, a^2c, a^2cd, a^2b^2d, a^3, a^3cd, a^3bd, a^3b^3c\}$	*Sec.* 4.2.6	
9.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 60)$	$\{d, c, ce, b, bd, bc, bcde, a, ae, ad, ac, abde, abcd, a^2, a^2d, a^3, a^3e, a^3d, a^3ce, a^3be, a^3bce\}$	*Sec.* 4.2.6	
9.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 60)$	$\{e, d, de, c, ce, b, bd, bc, bcde, a, ac, ab, abc, a^2, a^2e, a^2d, a^2de, a^3, a^3ce, a^3bd, a^3bcde\}$	*Sec.* 4.2.6	
9.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 60)$	$\{e, d, de, c, ce, cd, cde, b, bd, a, ac, ab, abcd, a^2c, a^2ce, a^2bce, a^2bcd, a^3, a^3ce, a^3bd, a^3bce\}$	*Sec.* 4.2.6	
9.4	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 60)$	$\{e, c, ce, cd, cde, b, bd, bc, bcde, a, acde, ab, abcde, a^2, a^2e, a^2c, a^2ce, a^3, a^3cd, a^3bd, a^3bc\}$	*Sec.* 4.2.6	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
23.2.10.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,88)$	$\{c, cd, b, bd, bc, a, ab, a^2b, a^2bc, a^4, a^4c, a^4cd, a^4bcd, a^5cd, a^5bcd, a^6b, a^6bc, a^7, a^7cd, a^7bd, a^7bc\}$	*Sec.* 4.2.6	
11.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, c, ce, cd, cde, b, bde, bc, bcd, a, acd, ab, abcd, a^2bd, a^2bde, a^2bcd, a^2bcde, a^3, a^3cde, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
11.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, de, c, ce, cde, b, bde, bc, bcde, a, ae, ab, abce, a^2, a^2e, a^2d, a^2cde, a^3, a^3e, a^3bde, a^3bcd\}$	*Sec.* 4.2.6	
11.3	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, d, de, c, ce, b, be, bd, bde, a, ac, ab, abce, a^2, a^2d, a^2c, a^2cde, a^3, a^3c, a^3bde, a^3bcd\}$	*Sec.* 4.2.6	
11.4	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, c, ce, cd, cde, a, ac, abd, abcde, a^2, a^2de, a^2c, a^2cd, a^2bd, a^2bde, a^2bc, a^2bce, a^3, a^3c, a^3b, a^3bce\}$	*Sec.* 4.2.6	
11.5	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, d, de, cd, cde, b, bde, bce, bcde, a, acde, ab, abcde, a^2, a^2e, a^2d, a^2de, a^3, a^3cd, a^3bde, a^3bce\}$	*Sec.* 4.2.6	
11.6	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, d, de, cd, cde, b, bde, bce, bcde, a, acde, ab, abcde, a^2c, a^2ce, a^2cd, a^2cde, a^3, a^3cd, a^3bde, a^3bce\}$	*Sec.* 4.2.6	
11.7	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,90)$	$\{e, c, ce, cd, cde, b, bde, bc, bcd, a, acd, ab, abcd, a^2d, a^2de, a^2cd, a^2cde, a^3, a^3cde, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
12.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,92)$	$\{d, cd, b, a, ab, a^2bc, a^2bcd, a^3bd, a^3bc, a^4, a^4d, a^4cd, a^4bd, a^4bc, a^4bcd, a^5cd, a^5bcd, a^6bc, a^6bcd, a^7, a^7cd\}$	*Sec.* 4.2.6	
13.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,102)$	$\{d, b^2, b^3d, a, ad, acd, abd, abc, ab^3d, a^2cd, a^2b, a^2bc, a^2b^3, a^2b^3c, a^2b^3cd, a^3d, a^3bd, a^3b^2, a^3b^2d, a^3b^3c, a^3b^3cd\}$	*Sec.* 4.2.6	
14.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,136)$	$\{e, d, c, cd, bd, b^2, b^2e, b^2ce, b^3cde, ac, ace, acde, ab, ab^2d, ab^2c, ab^2ce, ab^3, ab^3d, ab^3c, ab^3ce, ab^3cd\}$	*Sec.* 4.2.6	
14.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,136)$	$\{e, d, c, ce, cd, bc, bce, b^2, b^2e, b^3d, b^3c, b^3ce, b^3cde, a, ad, ac, acd, abd, abcde, ab^2, ab^2ce\}$	*Sec.* 4.2.6	
15.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,138)$	$\{f, e, ef, d, de, c, cf, ce, cef, b, bd, bc, bcde, a, aef, ad, adf, ab, abd, abce, abcd\}$	*Sec.* 4.2.6	
15.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,138)$	$\{f, e, ef, d, de, b, bd, bcf, bcdef, a, aef, ad, adf, acd, acdf, acde, acdef, ab, abd, abcef, abcdf\}$	*Sec.* 4.2.6	
16.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,139)$	$\{d, c, ce, b, bc, bcde, a, ae, ad, ac, abe, abde, abce, a^2, a^2d, a^2bd, a^3, a^3e, a^3d, a^3ce, a^3bcd\}$	*Sec.* 4.2.6	
16.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,139)$	$\{e, d, de, c, ce, b, bd, bc, a, ac, ab, abc, abcde, a^2, a^2e, a^2d, a^2de, a^2bcde, a^3, a^3ce, a^3bd\}$	*Sec.* 4.2.6	
17.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,193)$	$\{e, d, de, c, cd, b, be, bd, bde, a, ac, ab, abcd, a^2, a^2e, a^2c, a^2cde, a^3, a^3c, a^3be, a^3bcde\}$	*Sec.* 4.2.6	
18.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64,199)$	$\{e, d, de, c, ce, cd, cde, b, bd, a, acd, abd, abcde, a^2, a^2e, a^2be, a^2bd, a^3, a^3cde, a^3bd, a^3bcd\}$	*Sec.* 4.2.6	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
23.2.19.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 202)$	$\{f, e, ef, d, df, b, be, bcd, bcdef, a, ae, ad, adef, ac, acf, acd, acdf,$ $ab, abe, abc, abcef\}$	*Sec.* 4.2.6	
19.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 202)$	$\{f, e, ef, d, de, cd, cdf, b, be, bd, bde, bc, bcef, a, aef, ad, adf, ac, acf, abc, abcef\}$	*Sec.* 4.2.6	
20.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 206)$	$\{e, d, de, c, ce, cd, cde, b, bc, ac, acd, abd, abc, a^2, a^2e, a^2b, a^2bce, a^3c,$ $a^3cde, a^3bde, a^3bc\}$	*Sec.* 4.2.6	
21.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 215)$	$\{f, e, ef, d, de, b, bef, bd, bdf, bc, bce, bcd, bcde, a, af, acd, acdef, abd,$ $abdf, abcdf, abcde\}$	*Sec.* 4.2.6	
22.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 219)$	$\{e, d, de, b, be, bd, bde, bc, bcd, a, acd, ab, abc, a^2, a^2e, a^2bc, a^2bcde, a^3,$ $a^3cde, a^3be, a^3bc\}$	*Sec.* 4.2.6	
23.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 224)$	$\{e, d, de, c, ce, cd, cde, b, bc, b^2, b^2e, b^3, b^3ce, a, ad, ab, abcd, ab^2,$ $ab^2de, ab^3, ab^3cde\}$	*Sec.* 4.2.6	
24.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 226)$	$\{d, c, cd, bd, b^2, b^3d, b^3cd, ac, ab^3d, a^2, a^2b^2, a^2b^2d, a^2b^2c, a^2b^2cd, a^2b^3cd,$ $a^3c, a^3cd, a^3bc, a^3b^2cd, a^3b^3d, a^3b^3c\}$	*Sec.* 4.2.6	
25.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 227)$	$\{e, d, c, ce, b, bc, bcd, acd, ab, abe, abc, abce, abcde, a^2, a^2e, a^2d, a^2be, a^2bc,$ $a^2bcd, a^3cde, a^3bcd\}$	*Sec.* 4.2.6	
26.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 232)$	$\{d, c, bcd, b^2, b^2c, a, ad, ac, ab, abd, abc, ab^3, ab^3c, a^2, a^2d, a^2b^2, a^2b^3cd, a^3, a^3d,$ $a^3b^2c, a^3b^3d\}$	*Sec.* 4.2.6	
26.2	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 232)$	$\{d, c, b, bd, bc, b^2, b^2c, b^3, b^3c, a, ad, acd, ab^3c, a^2, a^2d, a^2b^2, a^2b^3d, a^3,$ $a^3d, a^3bc, a^3b^2cd\}$	*Sec.* 4.2.6	
27.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 241)$	$\{f, e, ef, b, bef, bd, bde, bcd, bcdf, a, af, ad, adef, acd, acde, abdf, abde, abcf,$ $abcef, abcd, abcdf\}$	*Sec.* 4.2.6	
28.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 242)$	$\{e, d, c, cd, b, bc, bce, b^2, b^2e, b^2de, b^2ce, b^2cde, b^3, a, ad, ade, ab, ab^2,$ $ab^3, ab^3cd, ab^3cde\}$	*Sec.* 4.2.6	
29.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 264)$	$\{f, e, ef, d, def, c, cef, cd, cdef, b, bf, bd, bde, a, ae, ac, acf,$ $abd, abde, abc, abcf\}$	*Sec.* 4.2.6	
30.1	64	21	8	6	$L_3(\mathbb{Z}_2)^3$	$(64, 267)$	$\{f, e, ef, d, df, c, ce, cd, cdef, b, bf, bd, bdf, a, ae, ac, ace, ab, abef, abcd, abcdef\}$	*Prop.* 3.2.42	[Ma84], 3.4 (2)
24.1.1.1	64	27	10	12	10		no complete determination	*Sec.* 4.2.6	
25.1.1.1	64	28	12	12	11		no complete determination	*Sec.* 4.2.6	
26.1.1.1	81	16	7	2	$L_2(9)$	$(81, 2)$	$\{b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$	*Prop.* 3.2.47	[Ma84], 3.4(1)
2.1	81	16	7	2	$L_2(9)$	$(81, 3)$	$\{c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$	*Sec.* 4.2.4	
2.2	81	16	7	2	$L_2(9)$	$(81, 3)$	$\{b, b^2, a, a^2, a^3, a^3c, a^3bc, a^3b^2c, a^4, a^5, a^6, a^6c^2, a^6bc^2, a^6b^2c^2, a^7, a^8\}$	*Sec.* 4.2.4	
3.1	81	16	7	2	$L_2(9)$	$(81, 4)$	$\{b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$	*Sec.* 4.2.4	

No.	v	k	λ	μ	srg	GAP	partial difference set	reference	comment/ reference
3.2	81	16	7	2	$L_2(9)$	$(81,4)$	$\{a, ab, a^2, a^2b^5, a^3, a^3b^3, a^4, a^4b^4, a^5, a^5b^8, a^6, a^6b^6, a^7, a^7b^7, a^8, a^8b^2\}$	*Sec.* 4.2.4	
4.1	81	16	7	2	$L_2(9)$	$(81,7)$	$\{d, d^2, cd^2, c^2d, b, bcd^2, bc^2d, b^2, b^2cd^2, b^2c^2d, a, ad, ad^2, a^2, a^2d, a^2d^2\}$	*Sec.* 4.2.4	
4.2	81	16	7	2	$L_2(9)$	$(81,7)$	$\{d, d^2, c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, ab, abd, abd^2, a^2b^2c, a^2b^2cd, a^2b^2cd^2\}$	*Sec.* 4.2.4	
5.1	81	16	7	2	$L_2(9)$	$(81,11)$	$\{c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$	*Prop.* 3.2.47	[*Ma*84], 3.4(1)
6.1	81	16	7	2	$L_2(9)$	$(81,12)$	$\{c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, a^2, bab^2, ba^2b^2, b^2ab, b^2a^2b, aba^2b^2, ab^2a^2b\}$	*Sec.* 4.2.4	
6.2	81	16	7	2	$L_2(9)$	$(81,12)$	$\{c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, a^2, bab^2c, ba^2b^2c^2, b^2abc^2, b^2a^2bc, aba^2b^2c^2, ab^2a^2bc\}$	*Sec.* 4.2.4	
7.1	81	16	7	2	$L_2(9)$	$(81,13)$	$\{c, c^2, b, bc, bc^2, b^2, b^2c, b^2c^2, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$	*Sec.* 4.2.4	
8.1	81	16	7	2	$L_2(9)$	$(81,15)$	$\{d, d^2, c, cd, cd^2, c^2, c^2d, c^2d^2, b, b^2, a, ab, ab^2, a^2, a^2b, a^2b^2\}$	*Prop.* 3.2.47	[*Ma*84], 3.4(1)
27.1.1.1	81	20	1	6	16	$(81,7)$	$\{d, d^2, bd, bcd, bc^2d^2, b^2d^2, b^2cd, b^2c^2d^2, ab, abcd^2, abc^2d^2, ab^2d^2, ab^2cd^2, ab^2c^2d,$ $a^2bd^2, a^2bc, a^2bc^2d^2, a^2b^2d, a^2b^2cd^2, a^2b^2c^2d^2\}$	*Sec.* 4.2.7	
2.1	81	20	1	6	16	$(81,8)$	$\{b, bc^2, b^3, b^4c, b^5c^2, b^6, b^8, b^8c, ab, abc, ab^4c^2, ab^5, ab^8c, ab^8c^2, a^2b, a^2b^5c, a^2b^7c,$ $a^2b^7c^2, a^2b^8, a^2b^8c^2\}$	*Sec.* 4.2.7	
3.1	81	20	1	6	16	$(81,13)$	$\{ab^2c, a^2, a^3, a^4c, a^4c^2, a^4b, a^4bc, a^5c, a^5c^2, a^5bc^2, a^5b^2c, a^6, a^7, a^7bc^2, a^7b^2,$ $a^7b^2c^2, a^8b, a^8bc, a^8b^2, a^8b^2c^2\}$	*Sec.* 4.2.7	
4.1	81	20	1	6	16	$(81,15)$	$\{d, d^2, c, c^2, bcd^2, bc^2d, b^2cd^2, b^2c^2d, acd^2, ac^2d, abcd, abc^2d^2, ab^2cd, ab^2c^2d^2,$ $a^2cd^2, a^2c^2d, a^2bcd, a^2bc^2d^2, a^2b^2cd, a^2b^2c^2d^2\}$	*Exa.* 5.3.12	[*Ma*94], 2.7(2)
28.1.1.1	81	24	9	6	17	$(81,7)$	$\{bd^2, bcd^2, bc^2, b^2d, b^2c, b^2c^2d, ac^2, ac^2d, ac^2d^2, abd^2, abc, abc^2d^2, ab^2d, ab^2cd^2,$ $ab^2c^2d^2, a^2c, a^2cd, a^2cd^2, a^2bd, a^2bc, a^2bc^2, a^2b^2, a^2b^2c, a^2b^2c^2d^2\}$	*Sec.* 4.2.7	
1.2	81	24	9	6	17	$(81,7)$	$\{bc^2, bc^2d, bc^2d^2, b^2c, b^2cd, b^2cd^2, ad^2, acd^2, ac^2d^2, ab, abcd^2, abc^2d, ab^2d^2, ab^2c,$ $ab^2c^2d, a^2d, a^2c, a^2c^2d^2, a^2bd^2, a^2bcd^2, a^2bc^2d^2, a^2b^2d, a^2b^2cd^2, a^2b^2c^2\}$	*Sec.* 4.2.7	
1.3	81	24	9	6	17	$(81,7)$	$\{c, cd, cd^2, c^2, c^2d, c^2d^2, bd, bcd, bc^2d^2, b^2d^2, b^2cd, b^2c^2d^2, abd, abc, abc^2, ab^2d^2,$ $ab^2cd^2, ab^2c^2d, a^2bd^2, a^2bc, a^2bc^2d^2, a^2b^2, a^2b^2cd, a^2b^2c^2d\}$	*Sec.* 4.2.7	
2.1	81	24	9	6	17	$(81,9)$	$\{bc, b^4, b^4c^2, b^5, b^5c, b^8c^2, ac^2, abc, abc^2, ab^3c^2, ab^5, ab^5c^2, ab^6c^2, ab^7, ab^8c, a^2c,$ $a^2bc^2, a^2b^3c, a^2b^4, a^2b^4c, a^2b^5c, a^2b^5c^2, a^2b^6c, a^2b^8\}$	*Sec.* 4.2.7	
2.2	81	24	9	6	17	$(81,9)$	$\{c, c^2, b^2, b^2c, b^3c, b^3c^2, b^4c, b^5c^2, b^6c, b^6c^2, b^7, b^7c^2, ab, abc, ab^2, ab^7c^2, ab^8c, ab^8c^2,$ $a^2b, a^2b^2c, a^2b^4c, a^2b^4c^2, a^2b^8, a^2b^8c^2\}$	*Sec.* 4.2.7	
3.1	81	24	9	6	17	$(81,12)$	$\{bc, b^2c^2, ac^2, abc, a^2c, a^2b^2, ba, bab, babc, babc^2, bab^2, ba^2bc, ba^2b^2, b^2abc, b^2ab^2c^2,$ $b^2a^2c^2, b^2a^2bc^2, b^2a^2b^2, b^2a^2b^2c, b^2a^2b^2c^2, aba^2c, aba^2bc^2, ab^2a^2c^2, ab^2a^2b^2c\}$	*Sec.* 4.2.7	
4.1	81	24	9	6	17	$(81,13)$	$\{c, c^2, ac, ac^2, a^2, a^2bc^2, a^3c, a^3c^2, a^4bc^2, a^4b^2c, a^5b^2, a^5b^2c^2, a^6c, a^6c^2, a^7, a^7b,$ $a^7bc, a^7b^2, a^7b^2c^2, a^8c, a^8c^2, a^8b, a^8bc, a^8b^2c\}$	*Sec.* 4.2.7	
4.2	81	24	9	6	17	$(81,13)$	$\{bc, b^2c^2, abc, ab^2c^2, a^2c, a^2c^2, a^2b, a^2bc^2, a^2b^2c^2, a^3bc, a^3b^2c^2, a^4, a^4b, a^4bc^2,$ $a^4b^2, a^4b^2c, a^5, a^5bc, a^6bc, a^6b^2c^2, a^7c, a^7c^2, a^8b^2, a^8b^2c\}$	*Sec.* 4.2.7	
5.1	81	24	9	6	17	$(81,15)$	$\{bc^2, bc^2d, bc^2d^2, b^2c, b^2cd, b^2cd^2, ad, acd, ac^2d, abd, abcd^2, abc^2, ab^2d, ab^2c,$ $ab^2c^2d^2, a^2d^2, a^2cd^2, a^2c^2d^2, a^2bd^2, a^2bcd, a^2bc^2, a^2b^2d^2, a^2b^2c, a^2b^2c^2d\}$	*Sec.* 4.2.7	
29.1.1.1	81	30	9	12	18	$(81,7)$	$\{b, bc, bc^2d, b^2, b^2cd^2, b^2c^2, a, ad, ad^2, ac, acd, acd^2, ab, abcd, abc^2, ab^2d, ab^2cd^2, a^2,$ $ab^2c^2d^2, a^2d, a^2d^2, a^2c^2, a^2c^2d, a^2c^2d^2, a^2bd, a^2bc, a^2bc^2, a^2b^2d^2, a^2b^2cd^2, a^2b^2c^2d\}$	*Sec.* 4.2.7	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
1.2	81	30	9	12	18	$(81,7)$	$\{c, cd, cd^2, c^2, c^2d, c^2d^2, b, bc, bc^2d, b^2, b^2cd^2, b^2c^2, ac^2, ac^2d, ac^2d^2, abd^2, abc, abc^2d^2, ab^2, ab^2cd, ab^2c^2d, a^2c, a^2cd, a^2cd^2, a^2bd^2, a^2bcd, a^2bc^2d, a^2b^2, a^2b^2c, a^2b^2c^2d^2\}$	*Sec.* 4.2.7	
2.1	81	30	9	12	18	$(81,8)$	$\{c, c^2, b, bc^2, b^3c, b^3c^2, b^4c, b^5c^2, b^6c, b^6c^2, b^8, b^8c, ac^2, abc, abc^2, ab^3c^2, ab^4, ab^5c, ab^6c^2, ab^8, ab^8c^2, a^2c, a^2b, a^2bc, a^2b^2c, a^2b^2c^2, a^2b^3c, a^2b^4c^2, a^2b^6c, a^2b^8\}$	*Sec.* 4.2.7	
2.2	81	30	9	12	18	$(81,8)$	$\{b, bc^2, b^4c, b^5c^2, b^8, b^8c, ac, ac^2, abc, ab^2c^2, ab^3c, ab^3c^2, ab^5, ab^5c, ab^6c, ab^6c^2, ab^7, ab^7c^2, a^2c, a^2c^2, a^2bc, a^2b^3c, a^2b^3c^2, a^2b^5c^2, a^2b^6c, a^2b^6c^2, a^2b^7, a^2b^7c^2, a^2b^8, a^2b^8c\}$	*Sec.* 4.2.7	
3.1	81	30	9	12	18	$(81,13)$	$\{c, c^2, bc, b^2c^2, a, abc, ab^2, ab^2c, a^2c, a^2c^2, a^2bc, a^2b^2c^2, a^3c, a^3c^2, a^3bc, a^3b^2c^2, a^4b^2c^2, a^5b, a^5bc^2, a^5b^2, a^5b^2c, a^6c, a^6c^2, a^6bc, a^6b^2c^2, a^7c, a^7c^2, a^7b, a^7bc^2, a^8\}$	*Sec.* 4.2.7	
3.2	81	30	9	12	18	$(81,13)$	$\{b, bc^2, b^2, b^2c, a, ac, ab, ab^2c, a^2b^2, a^3b, a^3bc^2, a^3b^2, a^3b^2c, a^4c^2, a^5c, a^5bc^2, a^5b^2c, a^5b^2c^2, a^6b, a^6bc^2, a^6b^2, a^6b^2c, a^7bc, a^7bc^2, a^7b^2, a^7b^2c^2, a^8, a^8c^2, a^8b, a^8bc\}$	*Sec.* 4.2.7	
4.1	81	30	9	12	18	$(81,15)$	$\{d, d^2, c, c^2, b, b^2, a, ad, ac, acd, abd^2, abcd^2, abc^2, abc^2d, ab^2d^2, ab^2cd^2, ab^2c^2, ab^2c^2d, a^2, a^2d^2, a^2c^2, a^2c^2d^2, a^2bd, a^2bc, a^2bcd^2, a^2bc^2d, a^2b^2d, a^2b^2c, a^2b^2cd^2, a^2b^2c^2d\}$	*Sec.* 4.2.7	[*Ma*94], 10.6(1)
30.1.1.1	81	32	13	12	$L_4(9)$	$(81,15)$	$\{c, c^2, bd^2, bcd, bc^2d^2, b^2d, b^2cd, b^2c^2d^2, ac, acd^2, ac^2d, abd, abd^2, abcd^2, abc^2, abc^2d, abc^2d^2, ab^2d^2, ab^2c^2, ab^2c^2d^2, a^2cd^2, a^2c^2, a^2c^2d, a^2bd, a^2bc, a^2bcd, a^2b^2d, a^2b^2d^2, a^2b^2c, a^2b^2cd, a^2b^2cd^2, a^2b^2c^2d\}$	*Sec.* 4.2.7	
30.2.1.1	81	32	13	12	$L_4(9)$	$(81,7)$	$\{d, d^2, c, cd, cd^2, c^2, c^2d, c^2d^2, b, bd, bd^2, b^2, b^2d, b^2d^2, a, ac, ac^2, abd, abc, abc^2d^2, ab^2, ab^2cd, ab^2c^2d^2, a^2, a^2cd^2, a^2c^2d, a^2bd, a^2bcd, a^2bc^2d, a^2b^2, a^2b^2cd, a^2b^2c^2d^2\}$	*Sec.* 4.2.7	
1.2	81	32	13	12	$L_4(9)$	$(81,7)$	$\{d, d^2, b, bc, bc^2d, b^2, b^2cd^2, b^2c^2, a, ad, ad^2, ac^2, ac^2d, ac^2d^2, ab, abcd^2, abc^2d^2, ab^2, ab^2c, ab^2c^2d^2, a^2, a^2d, a^2d^2, a^2c, a^2cd, a^2cd^2, a^2bd, a^2bcd^2, a^2bc^2d, a^2b^2d, a^2b^2cd^2, a^2b^2c^2d^2\}$	*Sec.* 4.2.7	
2.1	81	32	13	12	$L_4(9)$	$(81,9)$	$\{b, bc^2, b^2c^2, b^3, b^6, b^7c, b^8, b^8c, a, ac, ab^2c, ab^3, ab^3c, ab^4, ab^6, ab^6c, ab^7c, ab^7c^2, ab^8, ab^8c^2, a^2, a^2c^2, a^2b, a^2bc, a^2b^2, a^2b^3, a^2b^3c^2, a^2b^6, a^2b^6c^2, a^2b^7c^2, a^2b^8c, a^2b^8c^2\}$	*Sec.* 4.2.7	
3.1	81	32	13	12	$L_4(9)$	$(81,12)$	$\{c, c^2, a, abc^2, ab^2c^2, a^2, a^2b, a^2b^2, ba, babc, bab^2, ba^2c, ba^2bc^2, ba^2b^2, b^2a, b^2ab, b^2ab^2c, b^2a^2c, b^2a^2b, b^2a^2b^2c^2, aba^2, aba^2c, aba^2c^2, aba^2b^2, aba^2b^2c, aba^2b^2c^2, ab^2a^2, ab^2a^2c, ab^2a^2c^2, ab^2a^2b, ab^2a^2bc, ab^2a^2bc^2\}$	*Sec.* 4.2.7	
4.1	81	32	13	12	$L_4(9)$	$(81,13)$	$\{b, bc^2, b^2, b^2c, a, ab, abc^2, ab^2, ab^2c, a^2b^2, a^2b^2c, a^3, a^3b, a^3bc^2, a^3b^2, a^3b^2c, a^4c, a^4c^2, a^5c, a^5c^2, a^5b, a^5bc^2, a^5b^2c^2, a^6, a^6b, a^6bc^2, a^6b^2, a^6b^2c, a^7bc, a^7b^2c^2, a^8, a^8bc\}$	*Sec.* 4.2.7	
5.1	81	32	13	12	$L_4(9)$	$(81,15)$	$\{d, d^2, b, bd, bd^2, bc, bcd, bcd^2, b^2, b^2d, b^2d^2, b^2c^2, b^2c^2d, b^2c^2d^2, a, acd^2, ac^2d^2, ab, abc, abc^2d, ab^2, ab^2cd, ab^2c^2, a^2, a^2cd, a^2c^2d, a^2b, a^2bc, a^2bc^2d^2, a^2b^2, a^2b^2cd^2, a^2b^2c^2\}$	*Sec.* 4.2.7	
31.1.1.1	81	40	19	20	*Paley*	$(81,15)$	$\{c, cd^2, c^2, c^2d, b, bc, bcd^2, b^2, b^2c^2, b^2c^2d, a, ad^2, ac, acd, acd^2, ac^2d^2, abd^2, abc, abcd, abc^2, abcd^2, d, a^2c^2d^2, a^2bd^2, a^2bcd^2, a^2bc^2, a^2b^2d, a^2b^2c, a^2b^2cd^2, a^2b^2c^2, a^2b^2c^2d, a^2b^2c^2d^2\}$	*Prop.* 3.2.6	[*Ma*84], 3.5
31.2.1.1	81	40	19	20	$L_5(9)$	$(81,12)$	$\{b, bc, b^2, b^2c^2, abc, ab^2, ab^2c^2, a^2bc, a^2bc^2, a^2b^2, ba, bab, babc, bab^2, bab^2c, bab^2c^2, ba^2, ba^2c, ba^2bc, ba^2b^2, ba^2b^2c, ba^2b^2c^2, b^2ac, b^2ac^2, b^2ab^2c^2, b^2a^2c^2, b^2a^2b^2, b^2a^2b^2c^2, aba^2, aba^2c, aba^2b, aba^2bc^2, aba^2b^2, aba^2b^2c^2, ab^2a^2, ab^2a^2c^2, ab^2a^2b, ab^2a^2bc, ab^2a^2b^2, ab^2a^2b^2c\}$	*Sec.* 4.2.7	
1.2	81	40	19	20	$L_5(9)$	$(81,12)$	$\{c, c^2, bc, b^2c^2, a, ac, ac^2, abc, ab^2, ab^2c^2, a^2, a^2c, a^2c^2, a^2b, a^2bc^2, a^2b^2c, bac^2, babc, babc^2, ba^2, ba^2c, ba^2b, b^2a, b^2ac, b^2ab, b^2abc, b^2abc^2, b^2ab^2, b^2a^2c^2, b^2a^2b, b^2a^2bc, b^2a^2bc^2, b^2a^2b^2c, b^2a^2b^2c^2, aba^2c, aba^2bc^2, aba^2b^2c, ab^2a^2c^2, ab^2a^2bc^2, ab^2a^2b^2c\}$	*Sec.* 4.2.7	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
2.1	81	40	19	20	$L_5(9)$	$(81,15)$	$\{b, bd, bc, bcd^2, bc^2d^2, b^2, b^2d^2, b^2cd, b^2c^2, b^2c^2d, a, ad^2, ac, acd, acd^2, ab, abd, abcd, abcd^2, abc^2, ab^2, ab^2d^2, ab^2cd^2, ab^2c^2, ab^2c^2d^2, a^2, a^2d, a^2c^2, a^2c^2d, a^2c^2d^2, a^2b, a^2bd, a^2bc, a^2bcd, a^2bc^2d, a^2b^2, a^2b^2d^2, a^2b^2c, a^2b^2c^2d, a^2b^2c^2d^2\}$	$Exa.5.3.12$	
32.1.1.1	100	18	8	2	$L_2(10)$	$(100,6)$	$\{bc, bc^4, b^2c^2, b^2c^3, b^3c^2, b^3c^3, b^4c, b^4c^4, a, abc^4, ab^2c^3, ab^3c^2, ab^4c, a^3, a^3bc, a^3b^2c^2, a^3b^3c^3, a^3b^4c^4\}$	$Sec.$ 4.2.4	
2.1	100	18	8	2	$L_2(10)$	$(100,9)$	$\{bc^2, bc^4, b^2c^3, b^2c^4, b^3c, b^3c^2, b^4c, b^4c^3, a, abc^4, ab^2c^3, ab^3c^2, ab^4c, a^3, a^3bc^2, a^3b^2c^4, a^3b^3c, a^3b^4c^3\}$	$Sec.$ 4.2.4	
3.1	100	18	8	2	$L_2(10)$	$(100,10)$	$\{bc^3, bc^4, b^2c, b^2c^3, b^3c^2, b^3c^4, b^4c, b^4c^2, a, abc^3, ab^2c, ab^3c^4, ab^4c^2, a^3, a^3bc^4, a^3b^2c^3, a^3b^3c^2, a^3b^4c\}$	$Sec.$ 4.2.4	
4.1	100	18	8	2	$L_2(10)$	$(100,12)$	$\{bc, bc^4, b^2c^2, b^2c^3, b^3c^2, b^3c^3, b^4c, b^4c^4, a, abc^4, ab^2c^3, ab^3c^2, ab^4c, a^3, a^3bc, a^3b^2c^2, a^3b^3c^3, a^3b^4c^4\}$	$Sec.$ 4.2.4	
5.1	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, b, bd, bd^2, bd^3, bd^4, a, ac, ac^2, ac^3, ac^4\}$	$Sec.$ 4.2.4	
5.2	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, b, bc, bc^2, bc^3, bc^4, a, ad, ad^2, ad^3, ad^4\}$	$Prop.$ 3.2.47	$[Ma84]$, 3.4(1)
5.3	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, a, ad, ad^2, ad^3, ad^4, ab, abc, abc^2, abc^3, abc^4\}$	$Sec.$ 4.2.4	
5.4	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, cd^4, c^2d^3, c^3d^2, c^4d, a, ad, ad^2, ad^3, ad^4, ab, abcd^4, abc^2d^3, abc^3d^2, abc^4d\}$	$Sec.$ 4.2.4	
5.5	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, cd^4, c^2d^3, c^3d^2, c^4d, bc^2, bc^2d, bc^2d^2, bc^2d^3, bc^2d^4, ab, abcd^4, abc^2d^3, abc^3d^2, abc^4d\}$	$Sec.$ 4.2.4	
5.6	100	18	8	2	$L_2(10)$	$(100,13)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, a, ac, ac^2, ac^3, ac^4, ab, abd, abd^2, abd^3, abd^4\}$	$Sec.$ 4.2.4	
6.1	100	18	8	2	$L_2(10)$	$(100,14)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, b, bc, bc^2, bc^3, bc^4, a, ad, ad^2, ad^3, ad^4\}$	$Prop.$ 3.2.47	$[Ma84]$, 3.4(1)
6.2	100	18	8	2	$L_2(10)$	$(100,14)$	$\{d, d^2, d^3, d^4, cd^3, c^2d, c^3d^4, c^4d^2, b, bcd^3, bc^2d, bc^3d^4, bc^4d^2, a, ad, ad^2, ad^3, ad^4\}$	$Sec.$ 4.2.4	
6.3	100	18	8	2	$L_2(10)$	$(100,14)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, b, bd, bd^2, bd^3, bd^4, a, ac, ac^2, ac^3, ac^4\}$	$Sec.$ 4.2.4	
6.4	100	18	8	2	$L_2(10)$	$(100,14)$	$\{c, cd^4, c^2, c^2d^3, c^3, c^3d^2, c^4, c^4d, b, bcd^4, bc^2d^3, bc^3d^2, bc^4d, abd^3, abcd^3, abc^2d^3, abc^3d^3, abc^4d^3\}$	$Sec.$ 4.2.4	
6.5	100	18	8	2	$L_2(10)$	$(100,14)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, a, ac, ac^2, ac^3, ac^4, ab, abd, abd^2, abd^3, abd^4\}$	$Sec.$ 4.2.4	
7.1	100	18	8	2	$L_2(10)$	$(100,15)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, a, ac, ac^2, ac^3, ac^4, abc^3, abc^3d, abc^3d^2, abc^3d^3, abc^3d^4\}$	$Sec.$ 4.2.4	
7.2	100	18	8	2	$L_2(10)$	$(100,15)$	$\{d, d^2, d^3, d^4, c, c^2, c^3, c^4, b, bd, bd^2, bd^3, bd^4, a, ac, ac^2, ac^3, ac^4\}$	$Sec.$ 4.2.4	
8.1	100	18	8	2	$L_2(10)$	$(100,16)$	$\{c, cd^9, c^2, c^2d^8, c^3, c^3d^7, c^4, c^4d^6, c^5, c^5d^5, c^6, c^6d^4, c^7, c^7d^3, c^8, c^8d^2, c^9, c^9d\}$	$Prop.$ 3.2.47	$[Ma84]$, 3.4(1)
33.1.1.1	100	22	0	6	*Higman–Sims*	$(100,9)$	$\{c^2, c^3, ac^3, abc, ab^2, ab^3, ab^4c, a^2c, a^2c^2, a^2bc^3, a^2bc^4, a^2b^2, a^2b^2c^4, a^2b^3, a^2b^3c^4, a^2b^4c^3, a^2b^4c^4, a^3c, a^3bc^2, a^3b^2, a^3b^3, a^3b^4c^2\}$	$Sec.$ 4.2.8	$[Kli00]$
2.1	100	22	0	6	*Higman–Sims*	$(100,10)$	$\{c^2, c^3, ac^3, ab, ab^2c^4, ab^3, ab^4c^3, a^2c^3, a^2c^4, a^2b, a^2bc, a^2b^2c, a^2b^2c^2, a^2b^3c, a^2b^3c^2, a^2b^4, a^2b^4c, a^3c^4, a^3b, a^3b^2c^2, a^3b^3, a^3b^4c^4\}$	$Sec.$ 4.2.8	$[Kli00]$
3.1	100	22	0	6	*Higman–Sims*	$(100,11)$	$\{bc^2, b^4c^3, ac^2, ac^4, ab^2c^2, ab^4c^3, ab^4c^4, a^2c^4, a^2b, a^2bc^2, a^2bc^3, a^2bc^4, a^2b^2c^3, a^2b^3c, a^2b^3c^4, a^2b^4, a^2b^4c^4, a^3c^3, a^3c^4, a^3b^3c, a^3b^3c^3, a^3b^4c^4\}$	$Sec.$ 4.2.8	$[Kli00]$
4.1	100	22	0	6	*Higman–Sims*	$(100,12)$	$\{b^2, b^3, abc^3, ab^3, ab^3c, ab^4c^2, ab^4c^4, a^2c, a^2c^2, a^2bc^4, a^2b^2c^4, a^2b^3, a^2b^3c^3, a^2b^4, a^2b^4c, a^2b^4c^2, a^2b^4c^3, a^3b^2c^3, a^3b^2c^4, a^3b^3c, a^3b^4, a^3b^4c^2\}$	$Sec.$ 4.2.8	$[Kli00]$

No.	v	k	λ	μ	srg	GAP	partial difference set	reference	comment/ reference
34.1.1.1	100	36	14	12	$Hall-Janko-Wales$	$(100,11)$	$\{c^2, c^3, b, bc^3, b^4, b^4c^2, a, ac^3, abc, ab^2, ab^3, ab^3c^2, ab^4, ab^4c^2, ab^4c^3, ab^4c^4, a^2c, a^2c^3, a^2c^4, a^2bc, a^2b^2, a^2b^2c, a^2b^3c^4, a^2b^4, a^2b^4c^2, a^2b^4c^4, a^3, a^3c, a^3b, a^3bc^4, a^3b^2c^2, a^3b^3, a^3b^3c, a^3b^3c^3, a^3b^3c^4, a^3b^4\}$	$Sec.$ 4.2.8	$[Kli00]$
2.1	100	36	14	12	$Hall-Janko-Wales$	$(100,12)$	$\{c, c^4, bc^2, bc^3, b^4c^2, b^4c^3, a, ac^2, ac^3, ac^4, abc, abc^2, ab^2, ab^3, ab^4c, ab^4c^2, a^2c^2, a^2c^4, a^2bc^2, a^2b^2, a^2b^2c, a^2b^2c^4, a^2b^3, a^2b^3c, a^2b^3c^4, a^2b^4c^2, a^3, a^3c, a^3c^3, a^3c^4, a^3b, a^3b^2c^2, a^3b^2c^4, a^3b^3c^2, a^3b^3c^4, a^3b^4\}$	$Sec.$ 4.2.8	$[Kli00]$
35.1.1.1	120	51	18	24	35	$(120,34)$	$\{a^2, a^3, bab, ba^4b, abab, aba^2, aba^3, a^2ba, a^2bab, a^2ba^3b, a^2ba^4, a^3ba, a^3ba^2b, a^3ba^4, a^3ba^4b, a^4ba^2, a^4ba^3, a^4ba^4b, baba^2, baba^2b, baba^3b, ba^2bab, ba^2ba^4b, ba^3bab, ba^3ba^4b, ba^4ba^2b, ba^4ba^3, ba^4ba^3b, aba^2ba^2, aba^2ba^2b, aba^3ba, aba^4ba^2, aba^4ba^3, aba^4ba^3b, a^2ba^2ba, a^2ba^2bab, a^2ba^4ba, a^2ba^4ba^2, a^2ba^4ba^3, a^3ba^4ba, a^3ba^4bab, a^3ba^4ba^2, baba^2ba^2, baba^2ba^2b, baba^4ba^3, baba^4ba^3b, ba^2ba^2ba, ba^2ba^2bab, ba^3ba^4ba, ba^3ba^4bab, aba^2ba^2bab\}$	$Sec.$ 4.2.9	
36.1.1.1	120	56	28	24	36	$(120,34)$	$\{b, ab, a^2, a^3, a^3b, bab, ba^2, ba^4, ba^4b, abab, aba^2, aba^3, aba^4, a^2bab, a^2ba^3, a^2ba^3b, a^2ba^4, a^2ba^4b, a^3bab, a^3ba^2, a^3ba^2b, a^3ba^4, a^4ba, a^4bab, a^4ba^2b, a^4ba^3b, baba^3, baba^3b, baba^4b, ba^2ba, ba^2bab, ba^2ba^4b, ba^3ba, ba^3bab, ba^4ba, ba^4bab, ba^4ba^2, ba^4ba^2b, ba^4ba^3, ba^4ba^3b, aba^2bab, aba^2ba^4b, aba^4bab, aba^4ba^2, aba^4ba^2b, a^2ba^2ba, a^2ba^2ba^2b, a^2ba^4ba, a^2ba^4ba^3, a^3ba^4ba, a^3ba^4bab, a^3ba^4ba^2, a^3ba^4ba^3b, baba^4ba^3b, ba^3ba^4ba, aba^2ba^2ba\}$	$Sec.$ 4.2.9	$cf.$ $[Kli95]$
1.2	120	56	28	24	36	$(120,34)$	$\{ab, a^2, a^3, a^3b, a^4b, ba, bab, ba^2, ba^4, ba^4b, aba, aba^3, aba^4b, a^2ba, a^2bab, a^2ba^2b, a^2ba^3, a^2ba^4, a^2ba^4b, a^3ba, a^3ba^4b, a^4ba^2, a^4ba^3, a^4ba^4, a^4ba^4b, baba^2, baba^2b, baba^3, baba^3b, baba^4, ba^2ba^4b, ba^3ba^3, ba^3ba^4b, ba^4bab, ba^4ba^3, aba^2bab, aba^2ba^2, aba^2ba^4, aba^3ba, aba^3bab, aba^3ba^4, aba^3ba^4b, aba^4ba, aba^4ba^2, a^2ba^2ba, a^2ba^2ba^4b, a^2ba^4ba^2, a^2ba^4ba^2b, a^2ba^4ba^3, a^2ba^4ba^3b, a^3ba^4bab, a^3ba^4ba^2b, a^3ba^4ba^3, baba^2ba^2b, ba^3ba^4ba, aba^2ba^2ba\}$	$Sec.$ 4.2.9	$cf.$ $[Kli95]$
36.2.1.1	120	56	28	24	37	$(120,34)$	$\{ab, a^2, a^2b, a^3, a^3b, a^4b, ba, bab, ba^2, ba^3, ba^4, ba^4b, aba^2, aba^3b, aba^4b, a^2ba, a^2bab, a^2ba^2b, a^2ba^4b, a^3bab, a^3ba^3b, a^3ba^4, a^3ba^4b, a^4bab, a^4ba^2b, a^4ba^3, baba^2, baba^2b, baba^3, baba^4, ba^2bab, ba^2ba^2, ba^2ba^4, ba^3ba, ba^3ba^3, ba^3ba^4b, ba^4ba, ba^4ba^2, ba^4ba^3, ba^4ba^3b, aba^2ba, aba^2ba^4b, aba^3ba^3b, aba^3ba^4b, aba^4ba, aba^4ba^2, aba^4ba^2b, a^2ba^2ba^2, a^2ba^2ba^4b, a^2ba^4ba, a^2ba^4bab, a^2ba^4ba^3, a^2ba^4ba^3b, a^3ba^4ba^2, a^3ba^4ba^2b, a^3ba^4ba^3\}$	$Sec.$ 4.2.9	M. H. Klin private comm.
1.i	120	56	28	24	37	$(120,34)$	there may be more than one partial difference set in group (120,34)	$Sec.$ 4.2.9	
37.1.1.1	121	20	9	2	$L_2(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(1)
38.1.1.1	121	30	11	2	$L_3(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(2)
39.1.1.1	121	40	15	12	$L_4(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
39.2.1.1	121	40	15	12	$L_4(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
40.1.1.1	121	50	21	20	$L_5(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8\}$	*Sec.* 3.2.3	not primitive $[Ma84]$, 3.4(3)
40.2.1.1	121	50	21	20	$L_5(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8, ab^4, a^2b^8, a^3b, a^4b^5, a^5b^9, a^6b^2, a^7b^6, a^8b^{10}, a^9b^3, a^{10}b^7\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
41.1.1.1	121	60	29	30	$L_6(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8, ab^4, a^2b^8, a^3b, a^4b^5, a^5b^9, a^6b^2, a^7b^6, a^8b^{10}, a^9b^3, a^{10}b^7\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
41.2.1.1	121	60	29	30	$L_6(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8, ab^5, a^2b^{10}, a^3b^4, a^4b^9, a^5b^3, a^6b^8, a^7b^2, a^8b^7, a^9b, a^{10}b^6\}$	*Sec.* 3.2.3	not primitive $[Ma84]$, 3.4(3)
41.3.1.1	121	60	29	30	$L_6(11)$	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8, ab^6, a^2b, a^3b^7, a^4b^2, a^5b^8, a^6b^3, a^7b^9, a^8b^4, a^9b^{10}, a^{10}b^5\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
41.4.1.1	121	60	29	30	*Paley*	$(121,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b, a^7b^3, a^8b^5, a^9b^7, a^{10}b^9, ab^3, a^2b^6, a^3b^9, a^4b, a^5b^4, a^6b^7, a^7b^{10}, a^8b^2, a^9b^5, a^{10}b^8, ab^7, a^2b^3, a^3b^{10}, a^4b^6, a^5b^2, a^6b^9, a^7b^5, a^8b, a^9b^8, a^{10}b^4\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
42.1.1.1	125	62	30	31	47	$(125,5)$	$\{c, c^4, b, bc, bc^4, b^2c, b^2c^4, b^3c, b^3c^4, b^4, b^4c, b^4c^4, a, ac, ac^2, ac^3, ac^4, ab, abc, abc^4, ab^2c^2, ab^2c^3, ab^3c^2, ab^3c^3, ab^4, ab^4c, ab^4c^4, a^2b, a^2bc^2, a^2bc^3, a^2b^2c, a^2b^2c^4, a^2b^3c, a^2b^3c^4, a^2b^4, a^2b^4c^2, a^2b^4c^3, a^3b, a^3bc^2, a^3bc^3, a^3b^2c, a^3b^2c^4, a^3b^3c, a^3b^3c^4, a^3b^4, a^3b^4c^2, a^3b^4c^3, a^4, a^4c, a^4c^2, a^4c^3, a^4c^4, a^4b, a^4bc, a^4bc^4, a^4b^2c^2, a^4b^2c^3, a^4b^3c^2, a^4b^3c^3, a^4b^4, a^4b^4c, a^4b^4c^4\}$	*Sec.* 4.2.10	
42.2.1.1	125	62	30	31	48	$(125,3)$	$\{c, c^4, b, bc^2, bc^4, b^2c^3, b^2c^4, b^3c, b^3c^2, b^4, b^4c, b^4c^3, a, ac, ac^2, ac^3, ac^4, ab, abc^2, abc^4, ab^2, ab^2c^4, ab^3c, ab^3c^2, ab^4c, ab^4c^2, ab^4c^4, a^2b, a^2bc^2, a^2bc^3, a^2b^2, a^2b^2c^4, a^2b^3c^2, a^2b^3c^3, a^2b^4, a^2b^4c^2, a^2b^4c^4, a^3bc, a^3bc^3, a^3bc^4, a^3b^2c^3, a^3b^2c^4, a^3b^3, a^3b^3c^4, a^3b^4, a^3b^4c^2, a^3b^4c^4, a^4, a^4c, a^4c^2, a^4c^3, a^4c^4, a^4b, a^4bc^2, a^4bc^3, a^4b^2c, a^4b^2c^2, a^4b^3c^2, a^4b^3c^3, a^4b^4c, a^4b^4c^2, a^4b^4c^4\}$	*Sec.* 4.2.10	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
1.2	125	62	30	31	48	$(125,3)$	$\{c^2, c^3, b, bc, bc^2, b^2, b^2c^2, b^3, b^3c^3, b^4, b^4c^3, b^4c^4, a, ac, ac^2, ac^3, ac^4, ab, abc^3, abc^4, ab^2, ab^2c^2, ab^3c, ab^3c^4, ab^4c, ab^4c^2, ab^4c^3, a^2b, a^2bc, a^2bc^2, a^2b^2c^2, a^2b^2c^4, a^2b^3, a^2b^3c^3, a^2b^4, a^2b^4c, a^2b^4c^2, a^3bc, a^3bc^2, a^3bc^3, a^3b^2c, a^3b^2c^3, a^3b^3, a^3b^3c^2, a^3b^4, a^3b^4c, a^3b^4c^2, a^4, a^4c, a^4c^2, a^4c^3, a^4c^4, a^4bc, a^4bc^2, a^4bc^3, a^4b^2c^2, a^4b^2c^4, a^4b^3, a^4b^3c^2, a^4b^4c, a^4b^4c^2, a^4b^4c^3\}$	*Sec.* 4.2.10	
2.1	125	62	30	31	48	$(125,5)$	$\{c, c^4, b, bc, bc^2, bc^4, b^2c, b^3c^4, b^4, b^4c, b^4c^3, b^4c^4, a, ac^3, ac^4, abc^2, ab^2c, ab^2c^4, ab^3c^3, ab^4c, ab^4c^3, ab^4c^4, a^2c^2, a^2c^4, a^2b, a^2bc^2, a^2bc^3, a^2bc^4, a^2b^2, a^2b^2c, a^2b^2c^2, a^2b^2c^3, a^2b^3, a^2b^3c^4, a^2b^4, a^2b^4c, a^2b^4c^4, a^3c, a^3c^3, a^3b, a^3bc, a^3bc^4, a^3b^2, a^3b^2c, a^3b^3, a^3b^3c^2, a^3b^3c^3, a^3b^3c^4, a^3b^4, a^3b^4c, a^3b^4c^2, a^3b^4c^3, a^4, a^4c, a^4c^2, a^4bc, a^4bc^2, a^4bc^4, a^4b^2c^2, a^4b^3c, a^4b^3c^4, a^4b^4c^3\}$	*Sec.* 4.2.10	
42.3.1.1	125	62	30	31	49	$(125,3)$	$\{c, c^4, bc, bc^2, bc^3, b^2, b^2c^2, b^3, b^3c^3, b^4c^2, b^4c^3, b^4c^4, a, ac^2, ab, abc, abc^2, ab^2, ab^2c, ab^2c^2, ab^2c^3, ab^2c^4, ab^3, ab^3c^3, ab^3c^4, ab^4, ab^4c^3, a^2, a^2c, a^2c^2, a^2bc^2, a^2bc^4, a^2b^2, a^2b^2c^2, a^2b^3c^2, a^2b^3c^3, a^2b^3c^4, a^3, a^3c^3, a^3c^4, a^3b^2c^2, a^3b^2c^3, a^3b^2c^4, a^3b^3c^2, a^3b^3c^4, a^3b^4, a^3b^4c^3, a^4, a^4c^3, a^4bc, a^4bc^4, a^4b^2, a^4b^2c^3, a^4b^2c^4, a^4b^3, a^4b^3c, a^4b^3c^2, a^4b^3c^3, a^4b^3c^4, a^4b^4, a^4b^4c, a^4b^4c^4\}$	*Sec.* 4.2.10	
1.2	125	62	30	31	49	$(125,3)$	$\{c, c^4, bc^2, bc^3, bc^4, b^2c^2, b^2c^4, b^3c, b^3c^3, b^4c, b^4c^2, b^4c^3, a, ac^3, ab, abc, abc^4, ab^3, ab^3c, ab^3c^4, ab^4, ab^4c^2, a^2c, a^2c^2, a^2c^3, a^2bc, a^2bc^4, a^2b^2, a^2b^2c^3, a^2b^3c, a^2b^3c^2, a^2b^3c^3, a^2b^4, a^2b^4c, a^2b^4c^2, a^2b^4c^3, a^2b^4c^4, a^3c^2, a^3c^3, a^3c^4, a^3b, a^3bc, a^3bc^2, a^3bc^3, a^3bc^4, a^3b^2, a^3b^2c^3, a^3b^2c^4, a^3b^3c, a^3b^3c^4, a^3b^4c, a^3b^4c^3, a^4, a^4c^2, a^4bc^2, a^4bc^4, a^4b^2c^2, a^4b^2c^3, a^4b^2c^4, a^4b^4, a^4b^4c, a^4b^4c^2\}$	*Sec.* 4.2.10	
2.1	125	62	30	31	49	$(125,5)$	$\{c, c^4, b, bc, bc^2, bc^4, b^2c, b^3c^4, b^4, b^4c, b^4c^3, b^4c^4, a, ac, ac^2, ab, abc^3, abc^4, ab^2c^4, ab^3c^2, ab^3c^4, ab^4, a^2c, a^2c^3, a^2b, a^2bc, a^2bc^2, a^2b^2c^2, a^2b^2c^4, a^2b^3c, a^2b^3c^2, a^2b^3c^3, a^2b^3c^4, a^2b^4, a^2b^4c, a^2b^4c^2, a^2b^4c^4, a^3c^2, a^3c^4, a^3b, a^3bc, a^3bc^3, a^3bc^4, a^3b^2c, a^3b^2c^2, a^3b^2c^3, a^3b^2c^4, a^3b^3c, a^3b^3c^3, a^3b^4, a^3b^4c^3, a^3b^4c^4, a^4, a^4c^3, a^4c^4, a^4b, a^4b^2c, a^4b^2c^3, a^4b^3c, a^4b^4, a^4b^4c, a^4b^4c^2\}$	*Sec.* 4.2.10	
42.4.1.1	125	62	30	31	*Paley*	$(125,5)$	$\{c, c^4, b, bc, bc^2, bc^4, b^2c, b^3c^4, b^4, b^4c, b^4c^3, b^4c^4, a, ab, abc^2, abc^4, ab^2c, ab^2c^2, ab^2c^4, ab^3, ab^4c^3, ab^4c^4, a^2c, a^2c^2, a^2c^3, a^2c^4, a^2bc, a^2bc^2, a^2bc^3, a^2bc^4, a^2b^2c, a^2b^2c^2, a^2b^3, a^2b^3c^2, a^2b^3c^4, a^2b^4, a^2b^4c, a^3c, a^3c^2, a^3c^3, a^3c^4, a^3b, a^3bc^4, a^3b^2, a^3b^2c, a^3b^2c^3, a^3b^3c^3, a^3b^3c^4, a^3b^4c, a^3b^4c^2, a^3b^4c^3, a^3b^4c^4, a^4, a^4bc, a^4bc^2, a^4b^2, a^4b^3c, a^4b^3c^3, a^4b^3c^4, a^4b^4, a^4b^4c, a^4b^4c^3\}$	*Prop.* 3.2.6	$[Ma84]$, 3.5
43.1.1.1	144	22	10	2	$L_(12)$		no complete determination	*Sec.* 4.2.4	
44.1.1.1	144	39	6	12	60	$(144,182)$	$\{b, bd, bcd, bcd^2, bc^2, bc^2d^2, a, acd, ac^2, abc, abc^2, abc^2d, a^2d, a^2cd, a^2c^2, a^2bc, a^2bcd^2, a^2bc^2, a^3bd, a^3bd^2, a^3bcd, a^4, a^4d, a^4c, a^4b, a^4bc^2, a^4bc^2d, a^5bd, a^5bd^2, a^5bcd, a^6d, a^6c^2d, a^6c^2d^2, a^7, a^7cd, a^7cd^2, a^7bc, a^7bcd^2, a^7bc^2d\}$	*Sec.* 4.2.11	*cf.* $[Kli00]$

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
45.1.1.1	144	66	30	30	62	(144, 182)	$\{c, cd, cd^2, c^2, c^2d, c^2d^2, b, bd^2, bc, bcd^2, bc^2, bc^2d, ad^2, ac, ac^2d, abd, abc^2, abc^2d^2, a^2d, a^2c, a^2c^2d, a^2b, a^2bd, a^2bd^2, a^2bcd, a^2bc^2d, a^2bc^2d^2, a^3d^2, a^3cd, a^3cd^2, a^3c^2, a^3c^2d, a^3c^2d^2, a^3b, a^3bc, a^3bc^2d^2, a^4, a^4d, a^4cd, a^4cd^2, a^4c^2, a^4c^2d, a^4bc, a^4bcd, a^4bcd^2, a^5d^2, a^5c, a^5cd, a^5c^2, a^5c^2d, a^5c^2d^2, a^5bd, a^5bc, a^5bc^2d, a^6c, a^6cd^2, a^6c^2d^2, a^6b, a^6bc, a^6bc^2, a^7d, a^7c, a^7c^2d^2, a^7b, a^7bc^2, a^7bc^2d^2\}$	*Sec.* 4.2.11	
46.1.1.1	155	42	17	9	65	(155, 1)	$\{b, b^{13}, b^{14}, b^{17}, b^{18}, b^{30}, a, ab, ab^8, ab^{14}, ab^{15}, ab^{17}, ab^{21}, ab^{22}, ab^{30}, a^2b, a^2b^6, a^2b^7, a^2b^{12}, a^2b^{14}, a^2b^{15}, a^2b^{24}, a^2b^{25}, a^2b^{26}, a^3b^2, a^3b^3, a^3b^6, a^3b^7, a^3b^{14}, a^3b^{20}, a^3b^{24}, a^3b^{27}, a^3b^{28}, a^4, a^4b, a^4b^2, a^4b^3, a^4b^{15}, a^4b^{18}, a^4b^{20}, a^4b^{28}, a^4b^{29}\}$	*Sec.* 4.2.13	
47.1.1.1	162	56	10	24	68	(162, 11)	$\{e, e^2, ce^2, cde^2, cd^2, c^2e, c^2d, c^2d^2e, bce, bcde^2, bcd^2e, bc^2e, bc^2de^2, bc^2d^2e^2, b^2ce, b^2cd, b^2cd^2, b^2c^2e, b^2c^2de, b^2c^2d^2, ae, ae^2, ad, ade, ad^2e, ad^2e^2, ace^2, acd, acd^2e, ac^2e, ac^2d, aac^2d^2e, b, abe^2, abd, abde, abd^2, abd^2e, abce, abcde, abcd^2e, abc^2e^2, abc^2de, abc^2d^2e^2, ab^2, ab^2e^2, ab^2d, ab^2de^2, ab^2d^2e, ab^2d^2e^2, ab^2ce, ab^2cd, ab^2cd^2e^2, ab^2c^2e^2, ab^2c^2de, ab^2c^2d^2e^2\}$	*Sec.* 4.2.13	
2.1	162	56	10	24	68	(162, 19)	$\{e, e^2, c, cd, cd^2e, c^2, c^2de^2, c^2d^2, bce^2, bcde, bcd^2e, bc^2, bc^2d, bc^2d^2e^2, b^2ce, b^2cde^2, b^2cd^2e, b^2c^2e^2, b^2c^2d, b^2c^2d^2, a, ae, ad, ade, ad^2e, ad^2e^2, ace^2, acde^2, acd^2, ac^2e^2, ac^2d, ac^2d^2e, abe, abe^2, abd, abde, abd^2, abd^2e, abc, abcd, abcd^2e, abc^2e^2, abc^2de^2, abc^2d^2e^2, ab^2, ab^2e, ab^2de, ab^2de^2, ab^2d^2, ab^2d^2e, ab^2ce^2, ab^2cde^2, ab^2cd^2, ab^2c^2e, ab^2c^2d, ab^2c^2d^2e^2\}$	*Sec.* 4.2.13	
3.1	162	56	10	24	68	(162, 20)	$\{c^2, c^3, c^4d, c^4d^2, c^5d, c^5d^2, c^6, c^7, bc, bcd^2, bc^2d, bc^4d, bc^5, bc^5d^2, b^2c^2d^2, b^2c^4, b^2c^4d, b^2c^5, b^2c^5d, b^2c^7d^2, ad^2, acd, ac^2d, ac^3, ac^4, ac^4d^2, ac^5, ac^5d, ac^5d^2, ac^6d, ac^8, ac^8d^2, abcd^2, abc^2, abc^2d, abc^2d^2, abc^3, abc^3d, abc^3d^2, abc^4, abc^4d, abc^5, abc^5d^2, abc^8d, ab^2d^2, ab^2c, ab^2c^2, ab^2c^2d, ab^2c^2d^2, ab^2c^3d, ab^2c^4d, ab^2c^4d^2, ab^2c^5, ab^2c^5d^2, ab^2c^6, ab^2c^8d\}$	*Sec.* 4.2.13	
4.1	162	56	10	24	68	(162, 36)	$\{d^3, d^4, d^5, d^6, cd^2, cd^4, c^2d^5, c^2d^7, bd^5, bd^7, bcd^4, bcd^5, bc^2d^2, bc^2d^4, b^2d^7, b^2d^8, b^2cd, b^2cd^2, b^2c^2d^2, b^2c^2d^7, a, ad, ad^2, ad^4, ac, acd, acd^7, acd^8, ac^2, ac^2d, ac^2d^7, ac^2d^8, ab, abd, abd^2, abd^7, abcd, abcd^3, abcd^7, abcd^8, abc^2d, abc^2d^4, abc^2d^6, abc^2d^8, ab^2, ab^2d, ab^2d^4, ab^2d^8, ab^2cd^4, ab^2cd^5, ab^2cd^6, ab^2cd^7, ab^2c^2d, ab^2c^2d^3, ab^2c^2d^4, ab^2c^2d^5\}$	*Sec.* 4.2.13	
5.1	162	56	10	24	68	(162, 52)	$\{e, e^2, de^2, d^2e, cde, cd^2e^2, c^2de, c^2d^2e^2, bd, bd^2, bcd, bcd^2, bc^2de, bc^2d^2e^2, b^2d, b^2d^2, b^2cde, b^2cd^2e^2, b^2c^2d, b^2c^2d^2, ae^2, ade, ad^2e, ad^2e^2, ace, acde, acd^2, acd^2e, ac^2e, ac^2de, ac^2d^2, ac^2d^2e, abe^2, abd, abd^2, abd^2e, abc, abcde, abcd^2, abcd^2e^2, abc^2e^2, abc^2de^2, abc^2d^2, abc^2d^2e^2, ab^2e^2, ab^2d, ab^2d^2, ab^2d^2e, ab^2ce^2, ab^2cde^2, ab^2cd^2, ab^2cd^2e^2, ab^2c^2, ab^2c^2de, ab^2c^2d^2, ab^2c^2d^2e^2\}$	*Sec.* 4.2.13	
6.1	162	56	10	24	68	(162, 54)	$\{de, d^2e^2, ce, cd, cde, c^2e^2, c^2d^2, c^2d^2e^2, be^2, bde, bde^2, bce, bce^2, bcde, b^2e, b^2d^2e, b^2d^2e^2, b^2c^2e, b^2c^2e^2, b^2c^2d^2e^2, ad^2, ad^2e, ad^2e^2, ace^2, acd, acde, acde^2, acd^2, acd^2e^2, ac^2e, ac^2d^2e^2, ac^2e^2, abe, abd, abde, abde^2, abd^2e, abd^2e^2, abc, abcd, abcde^2, abc^2e^2, abc^2de^2, abc^2d^2e^2, ab^2e, ab^2e^2, ab^2d^2e^2, ab^2ce, ab^2cd, ab^2cd^2e^2, ab^2c^2, ab^2c^2e, ab^2c^2e^2, ab^2c^2d, ab^2c^2de^2, ab^2c^2d^2e^2\}$	*Sec.* 4.2.13	

$No.$	v	k	λ	μ	srg	GAP	partial difference set	reference	comment/ reference
48.1.1.1	169	24	11	2	$L_2(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(1)
49.1.1.1	169	36	13	6	$L_3(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(2)
50.1.1.1	169	48	17	12	$L_4(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^2,a^2b^4,a^3b^6,a^4b^8,a^5b^{10},a^6b^{12},a^7b,a^8b^3,a^9b^5,a^{10}b^7,$ $a^{11}b^9,a^{12}b^{11}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)
50.2.1.1	169	48	17	12	$L_4(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^3,a^2b^6,a^3b^9,a^4b^{12},a^5b^2,a^6b^5,a^7b^8,a^8b^{11},a^9b,a^{10}b^4,$ $a^{11}b^7,a^{12}b^{10}\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)
50.3.1.1	169	48	17	12	$L_4(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^4,a^2b^8,a^3b^{12},a^4b^3,a^5b^7,a^6b^{11},a^7b^2,a^8b^6,a^9b^{10},a^{10}b,$ $a^{11}b^5,a^{12}b^9\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)
51.1.1.1	169	60	23	20	$L_5(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^2,a^2b^4,a^3b^6,a^4b^8,a^5b^{10},a^6b^{12},a^7b,a^8b^3,a^9b^5,a^{10}b^7,$ $a^{11}b^9,a^{12}b^{11},ab^3,a^2b^6,a^3b^9,a^4b^{12},a^5b^2,a^6b^5,a^7b^8,a^8b^{11},a^9b,a^{10}b^4,a^{11}b^7,a^{12}b^{10}\}$	$Sec.$ 3.2.3	not primitive $[Ma84]$, 3.4(3)
51.2.1.1	169	60	23	20	$L_5(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^2,a^2b^4,a^3b^6,a^4b^8,a^5b^{10},a^6b^{12},a^7b,a^8b^3,a^9b^5,a^{10}b^7,$ $a^{11}b^9,a^{12}b^{11},ab^4,a^2b^8,a^3b^{12},a^4b^3,a^5b^7,a^6b^{11},a^7b^2,a^8b^6,a^9b^{10},a^{10}b,a^{11}b^5,a^{12}b^9\}$	$Sec.$ 3.2.3	not primitive $[Ma84]$, 3.4(3)
51.3.1.1	169	60	23	20	$L_5(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^3,a^2b^6,a^3b^9,a^4b^{12},a^5b^2,a^6b^5,a^7b^8,a^8b^{11},a^9b,a^{10}b^4,$ $a^{11}b^7,a^{12}b^{10},ab^4,a^2b^8,a^3b^{12},a^4b^3,a^5b^7,a^6b^{11},a^7b^2,a^8b^6,a^9b^{10},a^{10}b,a^{11}b^5,a^{12}b^9\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)
52.1.1.1	169	72	31	30	$L_6(13)$	$(169,2)$	$\{a,a^2,a^3,a^4,a^5,a^6,a^7,a^8,a^9,a^{10},a^{11},a^{12},b,b^2,b^3,b^4,b^5,b^6,b^7,$ $b^8,b^9,b^{10},b^{11},b^{12},ab,a^2b^2,a^3b^3,a^4b^4,a^5b^5,a^6b^6,a^7b^7,a^8b^8,a^9b^9,$ $a^{10}b^{10},a^{11}b^{11},a^{12}b^{12},ab^2,a^2b^4,a^3b^6,a^4b^8,a^5b^{10},a^6b^{12},a^7b,a^8b^3,a^9b^5,a^{10}b^7,$ $a^{11}b^9,a^{12}b^{11},ab^3,a^2b^6,a^3b^9,a^4b^{12},a^5b^2,a^6b^5,a^7b^8,a^8b^{11},a^9b,a^{10}b^4,$ $a^{11}b^7,a^{12}b^{10},ab^4,a^2b^8,a^3b^{12},a^4b^3,a^5b^7,a^6b^{11},a^7b^2,a^8b^6,a^9b^{10},a^{10}b,$ $a^{11}b^5,a^{12}b^9\}$	$Sec.$ 3.2.3	$[Ma84]$, 3.4(3)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
52.2.1.1	169	72	31	30	$L_6(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8\}$	*Sec.* 3.2.3	[*Ma*84], 3.4(3)
52.3.1.1	169	72	31	30	$L_6(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^7, a^2b, a^3b^8, a^4b^2, a^5b^9, a^6b^3, a^7b^{10}, a^8b^4, a^9b^{11}, a^{10}b^5, a^{11}b^{12}, a^{12}b^6\}$	*Sec.* 3.2.3	not primitive [*Ma*84], 3.4(3)
52.4.1.1	169	72	31	30	$L_6(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^8, a^2b^3, a^3b^{11}, a^4b^6, a^5b, a^6b^9, a^7b^4, a^8b^{12}, a^9b^7, a^{10}b^2, a^{11}b^{10}, a^{12}b^5\}$	*Sec.* 3.2.3	[*Ma*84], 3.4(3)
52.5.1.1	169	72	31	30	$L_6(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8\}$	*Sec.* 3.2.3	not primitive [*Ma*84], 3.4(3)
53.1.1.1	169	84	41	42	$L_7(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8\}$	*Sec.* 3.2.3	not primitive [*Ma*84], 3.4(3)
53.2.1.1	169	84	41	42	$L_7(13)$	$(169, 2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^6, a^2b^{12}, a^3b^5, a^4b^{11}, a^5b^4, a^6b^{10}, a^7b^3, a^8b^9, a^9b^2, a^{10}b^8, a^{11}b, a^{12}b^7\}$	*Sec.* 3.2.3	not primitive [*Ma*84], 3.4(3)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
53.3.1.1	169	84	41	42	*Paley*	$(169,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8, ab^6, a^2b^{12}, a^3b^5, a^4b^{11}, a^5b^4, a^6b^{10}, a^7b^3, a^8b^9, a^9b^2, a^{10}b^8, a^{11}b, a^{12}b^7\}$	*Sec.* 3.2.3	$[Ma84]$, 3.4(3)
53.4.1.1	169	84	41	42	$L_7(13)$	$(169,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab, a^2b^2, a^3b^3, a^4b^4, a^5b^5, a^6b^6, a^7b^7, a^8b^8, a^9b^9, a^{10}b^{10}, a^{11}b^{11}, a^{12}b^{12}, ab^2, a^2b^4, a^3b^6, a^4b^8, a^5b^{10}, a^6b^{12}, a^7b, a^8b^3, a^9b^5, a^{10}b^7, a^{11}b^9, a^{12}b^{11}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8, ab^8, a^2b^3, a^3b^{11}, a^4b^6, a^5b, a^6b^9, a^7b^4, a^8b^{12}, a^9b^7, a^{10}b^2, a^{11}b^{10}, a^{12}b^5\}$	*Sec.* 3.2.3	not primitive $[Ma84]$, 3.4(3)
53.5.1.1	169	84	41	42	$L_7(13)$	$(169,2)$	$\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, b^{10}, b^{11}, b^{12}, ab^3, a^2b^6, a^3b^9, a^4b^{12}, a^5b^2, a^6b^5, a^7b^8, a^8b^{11}, a^9b, a^{10}b^4, a^{11}b^7, a^{12}b^{10}, ab^4, a^2b^8, a^3b^{12}, a^4b^3, a^5b^7, a^6b^{11}, a^7b^2, a^8b^6, a^9b^{10}, a^{10}b, a^{11}b^5, a^{12}b^9, ab^5, a^2b^{10}, a^3b^2, a^4b^7, a^5b^{12}, a^6b^4, a^7b^9, a^8b, a^9b^6, a^{10}b^{11}, a^{11}b^3, a^{12}b^8, ab^6, a^2b^{12}, a^3b^5, a^4b^{11}, a^5b^4, a^6b^{10}, a^7b^3, a^8b^9, a^9b^2, a^{10}b^8, a^{11}b, a^{12}b^7, ab^9, a^2b^5, a^3b, a^4b^{10}, a^5b^6, a^6b^2, a^7b^{11}, a^8b^7, a^9b^3, a^{10}b^{12}, a^{11}b^8, a^{12}b^4\}$	*Sec.* 3.2.3	not primitive $[Ma84]$, 3.4(3)
54.1.1.1	171	34	17	4	$T(19)$	$(171,3)$	$\{b, b^{18}, a, ab^{14}, ab^{15}, ab^{18}, a^2b^{14}, a^2b^{15}, a^2b^{17}, a^2b^{18}, a^3b^{10}, a^3b^{11}, a^3b^{17}, a^3b^{18}, a^4b, a^4b^2, a^4b^{10}, a^4b^{11}, a^5b, a^5b^2, a^5b^3, a^5b^4, a^6b^3, a^6b^4, a^6b^{11}, a^6b^{12}, a^7b^5, a^7b^6, a^7b^{11}, a^7b^{12}, a^8, a^8b, a^8b^5, a^8b^6\}$	*Prop.* 3.2.15	
55.1.1.1	196	26	12	2	$L_2(14)$		no complete determination	*Sec.* 4.2.4	
56.1.1.1	225	28	13	2	$L_2(15)$		no complete determination	*Sec.* 4.2.4	
57.1.1.1	243	22	1	2	91	$(243,6)$	$\{b, b^3d, b^3d^2, b^4cd^2, b^4c^2d, b^5cd, b^5c^2d^2, b^6d, b^6d^2, b^8, ab^2cd, ab^4c, ab^7, ab^7c^2, ab^8d, ab^8c^2d, a^2b, a^2bcd, a^2b^5, a^2b^5cd^2, a^2b^7c^2d^2, a^2b^8c^2d\}$	*Sec.* 4.2.12	
2.1	243	22	1	2	91	$(243,38)$	$\{d, d^2, ad, ab, ab^2cd^2, ab^2c^2, a^2cd, a^2c^2d^2, a^2bd, a^2b^2, a^3d^2, a^4b^2d^2, a^5bcd^2, a^5bc^2d^2, a^5b^2c, a^5b^2c^2d^2, a^6d, a^7cd^2, a^7c^2d, a^7bcd^2, a^7bc^2d^2, a^8d^2\}$	*Sec.* 4.2.12	
3.1	243	22	1	2	91	$(243,51)$	$\{e, e^2, de^2, d^2e, bd^2, bcd^2, bc^2, b^2d, b^2c, b^2c^2d, abe^2, abcd^2, abc^2d^2e, ab^2, ab^2ce^2, ab^2c^2d^2e, a^2bde^2, a^2bcd^2e, a^2bc^2d, a^2b^2d, a^2b^2cd^2e, a^2b^2c^2d^2e^2\}$	*Sec.* 4.2.12	
4.1	243	22	1	2	91	$(243,67)$	$\{ce^2, c^2e, bc, bcde, bc^2, bc^2de^2, b^2c, b^2cd^2e, b^2c^2, b^2c^2d^2e^2, ade, ade^2, ac^2e, ac^2d^2e, abce^2, ab^2cd^2e^2, a^2d^2e, a^2d^2e^2, a^2ce^2, a^2cde^2, a^2bc^2de, a^2b^2c^2e\}$	*Sec.* 4.2.12	$[Ma94]$, 8.3(2)

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
58.1.1.1	243	110	37	70	92	(243, 38)	$\{bcd, bcd^2, bc^2, bc^2d, b^2c, b^2cd^2, b^2c^2d, b^2c^2d^2, ac, abcd, ab^2, ab^2c, ab^2cd, ab^2c^2d, a^2, a^2cd^2, a^2c^2, a^2c^2d^2, a^2bc^2d^2, a^2b^2c^2, a^3, a^3c, a^3cd, a^3cd^2, a^3c^2, a^3c^2d, a^3c^2d^2, a^3bc, a^3bcd^2, a^3bc^2, a^3bc^2d^2, a^3b^2cd, a^3b^2cd^2, a^3b^2c^2, a^3b^2c^2d^2, a^4d, a^4d^2, a^4c, a^4cd, a^4cd^2, a^4c^2d, a^4c^2d^2, a^4bd, a^4bd^2, a^4bc, a^4bcd, a^4bcd^2, a^4bc^2, a^4bc^2d, a^4b^2cd^2, a^5d, a^5d^2, a^5c, a^5cd, a^5c^2, a^5c^2d, a^5c^2d^2, a^5b, a^5bcd, a^5bc^2, a^5bc^2d, a^5b^2, a^5b^2c, a^5b^2c^2d, a^5b^2c^2d^2, a^6, a^6c, a^6cd, a^6cd^2, a^6c^2, a^6c^2d, a^6c^2d^2, a^6bc, a^6bcd, a^6bc^2d, a^6bc^2d^2, a^6b^2c, a^6b^2cd, a^6b^2c^2, a^6b^2c^2d, a^7, a^7cd, a^7cd^2, a^7c^2, a^7b, a^7bc, a^7bcd^2, a^7bc^2d^2, a^7b^2d, a^7b^2d^2, a^7b^2c, a^7b^2cd, a^7b^2cd^2, a^7b^2c^2, a^7b^2c^2d^2, a^8c^2d, a^8bd, a^8bd^2, a^8bc, a^8bcd^2, a^8bc^2, a^8bc^2d, a^8bc^2d^2, a^8b^2d, a^8b^2d^2, a^8b^2cd, a^8b^2cd^2, a^8b^2c^2, a^8b^2c^2d, a^8b^2c^2d^2\}$	*Sec.* 4.2.12	
2.1	243	110	37	70	92	(243, 51)	$\{d, d^2, be^2, bd^2, bd^2e, bce^2, bcd^2, bcd^2e, bc^2, bc^2e, bc^2de^2, b^2e, b^2d, b^2de^2, b^2c, b^2ce^2, b^2cd^2e, b^2c^2e, b^2c^2d, b^2c^2de^2, ae, ae^2, ad, ade^2, ad^2, ad^2e, acd, acde, acde^2, acd^2, acd^2e, acd^2e^2, ac^2, ac^2e^2, ac^2de, ac^2de^2, ac^2d^2, ac^2d^2e, ab, abe, abe^2, abd, abde^2, abd^2e, abce, abce^2, abcd, abcd^2, abcd^2e, abcd^2e^2, abc^2, abc^2e, abc^2de^2, abc^2d^2, abc^2d^2e, abc^2d^2e^2, ab^2e, ab^2d, ab^2de^2, ab^2ce^2, ab^2cd, ab^2cde, ab^2c^2e, ab^2c^2e^2, ab^2c^2d^2, a^2e, a^2e^2, a^2d, a^2de^2, a^2d^2, a^2d^2e, a^2c, a^2ce^2, a^2cde, a^2cde^2, a^2cd^2, a^2cd^2e, a^2c^2, a^2c^2e, a^2c^2e^2, a^2c^2d^2, a^2c^2d^2e, a^2c^2d^2e^2, a^2be, a^2be^2, a^2bd, a^2bcd, a^2bcde^2, a^2bcd^2e, a^2bc^2, a^2bc^2e, a^2bc^2de^2, a^2b^2e, a^2b^2e^2, a^2b^2d, a^2b^2de, a^2b^2de^2, a^2b^2d^2, a^2b^2ce^2, a^2b^2cd, a^2b^2cde, a^2b^2cd^2, a^2b^2cd^2e, a^2b^2cd^2e^2, a^2b^2c^2e, a^2b^2c^2d, a^2b^2c^2de^2, a^2b^2c^2d^2, a^2b^2c^2d^2e, a^2b^2c^2d^2e^2\}$	*Sec.* 4.2.12	
2.2	243	110	37	70	92	(243, 51)	$\{d, d^2, b, be, be^2, bd, bde^2, bd^2e, bc, bce, bce^2, bcd, bcde^2, bcd^2e, bc^2e, bc^2d, bc^2de, bc^2de^2, bc^2d^2, bc^2d^2e^2, b^2, b^2e, b^2e^2, b^2de^2, b^2d^2, b^2d^2e, b^2ce^2, b^2cd, b^2cde, b^2cd^2, b^2cd^2e, b^2cd^2e^2, b^2c^2, b^2c^2e, b^2c^2e^2, b^2c^2de^2, b^2c^2d^2, b^2c^2d^2e, ae, ae^2, ad, ade^2, ad^2, ad^2e, ac, ace^2, acde, acde^2, acd^2, acd^2e, ac^2, ac^2e, ac^2e^2, ac^2d, ac^2de, ac^2de^2, abe^2, abd^2, abd^2e, abce, abce^2, abcd, abc^2e, abc^2d^2, abc^2d^2e^2, ab^2e, ab^2d, ab^2de^2, ab^2cd, ab^2cd^2e, ab^2cd^2e^2, ab^2c^2de^2, ab^2c^2d^2, ab^2c^2d^2e, a^2e, a^2e^2, a^2d, a^2de^2, a^2d^2, a^2d^2e, a^2cd, a^2cde, a^2cde^2, a^2cd^2, a^2cd^2e, a^2cd^2e^2, a^2c^2e, a^2c^2e^2, a^2c^2d, a^2c^2de, a^2c^2d^2, a^2c^2d^2e^2, a^2bd, a^2bde^2, a^2bd^2e, a^2bce, a^2bce^2, a^2bcd, a^2bc^2, a^2bc^2e, a^2bc^2de^2, a^2b^2e, a^2b^2e^2, a^2b^2d^2, a^2b^2c, a^2b^2ce^2, a^2b^2cd^2e, a^2b^2c^2de^2, a^2b^2c^2d^2, a^2b^2c^2d^2e\}$	*Sec.* 4.2.12	

$No.$	v	k	λ	μ	srg	GAP	partial difference set	reference	comment/ reference
58.1.3.1	243	110	37	70	92	$(243, 56)$	$\{e, e^2, b, be, bd, bde, bd^2, bd^2e, bc, bce, bcde, bcde^2, bcd^2, bcd^2e^2, bc^2e, bc^2e^2, bc^2d, bc^2de,$ $bc^2d^2, bc^2d^2e^2, b^2, b^2e^2, b^2d, b^2de^2, b^2d^2, b^2d^2e^2, b^2c, b^2ce, b^2cd, b^2cde^2,$ $b^2cd^2e, b^2cd^2e^2, b^2c^2, b^2c^2e, b^2c^2de, b^2c^2de^2, b^2c^2d^2, b^2c^2d^2e^2, a, ae^2, ad, ade^2, ad^2e, ad^2e^2,$ $ac, ace, acde, acde^2, acd^2e, acd^2e^2, ac^2e, ac^2e^2, ac^2d, ac^2de, ac^2d^2e, ac^2d^2e^2, abe, abde,$ $abd^2e^2, abce, abcde^2, abcd^2e, abc^2, abc^2de^2, abc^2d^2e^2, ab^2e, ab^2d, ab^2d^2, ab^2ce^2, ab^2cde,$ $ab^2cd^2e, ab^2c^2e, ab^2c^2d, ab^2c^2d^2, a^2, a^2e, a^2d, a^2de, a^2d^2e, a^2d^2e^2, a^2ce, a^2ce^2, a^2cd, a^2cde,$ $a^2cd^2, a^2cd^2e, a^2c^2, a^2c^2e^2, a^2c^2d, a^2c^2de, a^2c^2d^2, a^2c^2d^2e^2, a^2b, a^2bde^2, a^2bd^2, a^2bce,$ $a^2bcd, a^2bcd^2e, a^2bc^2e, a^2bc^2d, a^2bc^2d^2e, a^2b^2e^2, a^2b^2de, a^2b^2d^2e^2, a^2b^2c, a^2b^2cde,$ $a^2b^2cd^2e, a^2b^2c^2e^2, a^2b^2c^2de^2, a^2b^2c^2d^2e\}$	$Sec.$ 4.2.12	
4.1	243	110	37	70	92	$(243, 57)$	$\{b, bd, bc, bcd, bcd^2, bc^2d, bc^2d^2, b^2, b^2d, b^2c^2, b^2c^2d, ad^2, ac, acd, ac^2d^2, ab, abd, abcd, abcd^2,$ $ab^2c^2, a^2, a^2d^2, a^2c, a^2cd, a^2cd^2, a^2c^2, a^2c^2d^2, a^2b, a^2bcd^2, a^2bc^2,$ $a^2bc^2d^2, a^2b^2d^2, a^2b^2c, a^2b^2c^2, a^2b^2c^2d^2, a^3, a^3b, a^3bd^2, a^3bc^2, a^3bc^2d^2, a^3b^2d, a^3b^2d^2,$ $a^3b^2c, a^3b^2cd, a^3b^2cd^2, a^3b^2c^2, a^3b^2c^2d^2, a^4, a^4d, a^4d^2, a^4cd^2, a^4c^2, a^4c^2d, a^4c^2d^2, a^4bd^2,$ $a^4bc, a^4bc^2, a^4bc^2d^2, a^4b^2, a^4b^2d, a^4b^2c, a^4b^2cd^2, a^5, a^5d, a^5d^2, a^5cd, a^5c^2, a^5c^2d, a^5c^2d^2,$ $a^5bd, a^5bd^2, a^5bc, a^5bcd, a^5b^2, a^5b^2d, a^5b^2cd, a^5b^2cd^2, a^6, a^6bd, a^6bd^2, a^6bc, a^6bcd, a^6bcd^2,$ $a^6bc^2, a^6bc^2d, a^6b^2, a^6b^2d^2, a^6b^2c, a^6b^2cd, a^6b^2cd^2, a^6b^2c^2d, a^6b^2c^2d^2, a^7,$ $a^7d, a^7c, a^7cd, a^7cd^2, a^7c^2, a^7c^2d, a^7bc^2d, a^7b^2d^2, a^7b^2cd, a^7b^2c^2d, a^7b^2c^2d^2, a^8d, a^8c,$ $a^8cd^2, a^8c^2d, a^8bc^2d, a^8b^2c^2d\}$	$Sec.$ 4.2.12	
5.1	243	110	37	70	92	$(243, 59)$	$\{bc^2d, b^2, b^2cd^2, b^2c^2, b^2c^2d^2, b^3, b^4d, b^4d^2, b^4c, b^4cd, b^5d, b^5d^2, b^5c, b^5cd, b^6, b^7,$ $b^7cd^2, b^7c^2, b^7c^2d^2, b^8c^2d, ac, abd, abd^2, abc, abcd, abc^2, abc^2d, abc^2d^2, ab^2,$ $ab^2d, ab^2d^2, ab^2cd, ab^2c^2, ab^2c^2d, ab^2c^2d^2, ab^3, ab^3cd, ab^3cd^2, ab^3c^2, ab^4, ab^4d, ab^4cd, ab^4cd^2,$ $ab^5d, ab^5d^2, ab^5c, ab^5cd, ab^5cd^2, ab^5c^2, ab^5c^2d, ab^6d, ab^6d^2, ab^6c^2d, ab^6c^2d^2, ab^7, ab^7d^2,$ $ab^7c, ab^7cd^2, ab^7c^2, ab^7c^2d, ab^7c^2d^2, ab^8, ab^8c, ab^8cd^2, ab^8c^2d^2, a^2, a^2d, a^2c, a^2cd, a^2b, a^2bd,$ $a^2bd^2, a^2bc, a^2bcd, a^2bcd^2, a^2bc^2, a^2b^2, a^2b^2d, a^2b^2d^2, a^2b^2cd, a^2b^2cd^2, a^2b^2c^2d, a^2b^2c^2d^2,$ $a^2b^3d^2, a^2b^3cd^2, a^2b^3c^2, a^2b^3c^2d, a^2b^4d, a^2b^4cd^2, a^2b^4c^2d, a^2b^4c^2d^2, a^2b^5c,$ $a^2b^5cd^2, a^2b^5c^2, a^2b^5c^2d, a^2b^6c^2d^2, a^2b^7, a^2b^7d^2, a^2b^7c, a^2b^7cd, a^2b^7c^2, a^2b^7c^2d,$ $a^2b^7c^2d^2, a^2b^8, a^2b^8d, a^2b^8d^2, a^2b^8c, a^2b^8cd, a^2b^8c^2, a^2b^8c^2d^2\}$	$Sec.$ 4.2.12	
6.1	243	110	37	70	92	$(243, 66)$	$\{c, cd^2, c^2, c^2d, bc^2d, bc^2d^2, b^2c, b^2cd^2, b^2c^2, b^2c^2d, b^2c^2d^2, ac, acd, acd^2, ac^2d^2, abd,$ $abd^2, abc, abc^2d, ab^2, ab^2d^2, ab^2c, ab^2cd, ab^2cd^2, ab^2c^2d, ab^2c^2d^2, a^2, a^2d,$ $a^2c^2d^2, a^2bd, a^2bd^2, a^2bc, a^2bcd^2, a^2bc^2, a^2bc^2d, a^2b^2cd, a^2b^2cd^2, a^2b^2c^2d^2, a^3, a^3cd, a^3cd^2,$ $a^3c^2, a^3c^2d^2, a^3bc, a^3bcd, a^3bcd^2, a^3bc^2, a^3bc^2d^2, a^3b^2c, a^3b^2cd, a^4d^2, a^4c, a^4c^2, a^4c^2d,$ $a^4bcd, a^4bcd^2, a^4bc^2, a^4bc^2d^2, a^4b^2c, a^5d^2, a^5c, a^5cd, a^5c^2, a^5c^2d, a^5c^2d^2, a^5b, a^5bcd, a^5bc^2d^2,$ $a^5b^2, a^5b^2d^2, a^5b^2c, a^5b^2c^2, a^5b^2c^2d, a^5b^2c^2d^2, a^6, a^6c, a^6cd, a^6c^2d, a^6c^2d^2, a^6bc, a^6bcd,$ $a^6bcd^2, a^6bc^2, a^6bc^2d, a^6b^2cd, a^6b^2cd^2, a^6b^2c^2, a^6b^2c^2d, a^6b^2c^2d^2, a^7, a^7d,$ $a^7cd, a^7cd^2, a^7b, a^7bc, a^7bcd, a^7bcd^2, a^7b^2d, a^7b^2cd, a^7b^2cd^2, a^7b^2c^2, a^8cd^2, a^8c^2, a^8c^2d,$ $a^8bc^2, a^8bc^2d, a^8bc^2d^2, a^8b^2d, a^8b^2c^2, a^8b^2c^2d\}$	$Sec.$ 4.2.12	

No.	v	k	λ	μ	*srg*	GAP	partial difference set	reference	comment/ reference
58.1.7.1	243	110	37	70	92	$(243, 67)$	$\{de, d^2e^2, c, ce^2, cd, cde, cde^2, cd^2, c^2, c^2e, c^2d, c^2d^2, c^2d^2e, c^2d^2e^2, be, bd, bde, bce^2, bcd^2,$ $bcd^2e^2, bc^2d, bc^2de, bc^2de^2, bc^2d^2, bc^2d^2e, bc^2d^2e^2, b^2e^2, b^2d^2, b^2d^2e^2, b^2cd, b^2cde, b^2cde^2,$ $b^2cd^2, b^2cd^2e, b^2cd^2e^2, b^2c^2e, b^2c^2d, b^2c^2de, ae^2, ad, ade, ade^2, ad^2, ad^2e^2, ac^2e,$ $ac^2d, ac^2de, ac^2d^2, ac^2d^2e, ac^2d^2e^2, abe, abe^2, abde, abde^2, abd^2e, abd^2e^2, abcd, abcd^2,$ $abcd^2e^2, abc^2d, abc^2de, abc^2d^2, ab^2de, ab^2d^2, ab^2d^2e, ab^2cd, ab^2cde^2, ab^2cd^2e^2, ab^2c^2e,$ $ab^2c^2e^2, ab^2c^2d, ab^2c^2de, ab^2c^2d^2, ab^2c^2d^2e^2, a^2e, a^2d, a^2de, a^2d^2, a^2d^2e, a^2d^2e^2, a^2ce^2,$ $a^2cd, a^2cde, a^2cde^2, a^2cd^2, a^2cd^2e^2, a^2bd, a^2bde^2, a^2bd^2e^2, a^2bce, a^2bce^2, a^2bcd, a^2bcde,$ $a^2bcd^2, a^2bcd^2e^2, a^2bc^2de, a^2bc^2d^2, a^2bc^2d^2e, a^2b^2e, a^2b^2e^2, a^2b^2de, a^2b^2de^2, a^2b^2d^2e,$ $a^2b^2d^2e^2, a^2b^2cd, a^2b^2cd^2, a^2b^2cd^2e^2, a^2b^2c^2d, a^2b^2c^2de, a^2b^2c^2d^2\}$	*Sec.* 4.2.12	
59.1.1.1	253	42	21	4	$T(23)$	$(253, 1)$	$\{b, b^{22}, a, ab^{20}, ab^{21}, ab^{22}, a^2b^{16}, a^2b^{17}, a^2b^{20}, a^2b^{21}, a^3b^8, a^3b^9,$ $a^3b^{16}, a^3b^{17}, a^4b^8, a^4b^9, a^4b^{15}, a^4b^{16}, a^5b^6, a^5b^7, a^5b^{15}, a^5b^{16},$ $a^6b^6, a^6b^7, a^6b^{11}, a^6b^{12}, a^7b^{11}, a^7b^{12}, a^7b^{21}, a^7b^{22}, a^8b^{18}, a^8b^{19}, a^8b^{21}, a^8b^{22},$ $a^9b^{12}, a^9b^{13}, a^9b^{18}, a^9b^{19}, a^{10}, a^{10}b, a^{10}b^{12}, a^{10}b^{13}\}$	*Prop.* 3.2.15	
60.1.1.1	253	112	36	60	94	$(253, 1)$	$\{b, b^2, b^5, b^7, b^8, b^{11}, b^{12}, b^{15}, b^{16}, b^{18}, b^{21}, b^{22}, ab, ab^4, ab^7, ab^8, ab^{10},$ $ab^{12}, ab^{13}, ab^{16}, ab^{18}, ab^{21}, a^2b^2, a^2b^5, a^2b^6, a^2b^8, a^2b^{12}, a^2b^{15},$ $a^2b^{16}, a^2b^{18}, a^2b^{19}, a^2b^{22}, a^3b^3, a^3b^4, a^3b^6, a^3b^7, a^3b^9, a^3b^{12}, a^3b^{13}, a^3b^{15}, a^3b^{16},$ $a^3b^{18}, a^4, a^4b^4, a^4b^5, a^4b^6, a^4b^8, a^4b^9, a^4b^{10}, a^4b^{12}, a^4b^{14}, a^4b^{18}, a^5,$ $a^5b^2, a^5b^4, a^5b^6, a^5b^{10}, a^5b^{12}, a^5b^{14}, a^5b^{15}, a^5b^{16}, a^5b^{19}, a^6, a^6b, a^6b^3, a^6b^4, a^6b^6,$ $a^6b^7, a^6b^{10}, a^6b^{11}, a^6b^{14}, a^6b^{20}, a^7, a^7b^2, a^7b^4, a^7b^5, a^7b^8, a^7b^{11}, a^7b^{14},$ $a^7b^{17}, a^7b^{19}, a^7b^{21}, a^8b, a^8b^2, a^8b^5, a^8b^7, a^8b^{10}, a^8b^{11}, a^8b^{14}, a^8b^{15}, a^8b^{19},$ $a^8b^{21}, a^9b, a^9b^2, a^9b^6, a^9b^7, a^9b^{10}, a^9b^{11}, a^9b^{16}, a^9b^{19}, a^9b^{20}, a^9b^{21}, a^{10}b,$ $a^{10}b^5, a^{10}b^8, a^{10}b^{11}, a^{10}b^{14}, a^{10}b^{15}, a^{10}b^{17}, a^{10}b^{18}, a^{10}b^{19}, a^{10}b^{21}\}$	*Sec.* 4.2.13	

Lebenslauf

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Schulausbildung

1978 – 1982:	Kath. Grundschule IV Langholt
1982 – 1984:	Orientierungsstufe Rhauderfehn
1984 – 1991:	Gymnasium Papenburg, Abschluss mit Abitur

Studium

WiSe 1991/92 - SoSe 1993:	Grundstudium an der Carl von Ossietzky-Universität Oldenburg im Studiengang Lehramt Gymnasium für die Fächer Mathematik und Chemie. Abschluss mit der Zwischenprüfung, Note ”sehr gut”
WiSe 1993/94 - WiSe 1996/97	Hauptstudium an der Carl von Ossietzky-Universität Oldenburg Examensarbeit im Fach Mathematik mit dem Thema ”Kantenmorphismen von Graphen und Hypergraphen” Abschluss mit dem 1. Staatsexamen, Note: ”sehr gut”

Promotion

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01.04.1997 - 28.02.1999	wissenschaftliche Hilfskraft am Fachbereich Mathematik
seit 01.03.1999	wissenschaftlicher Mitarbeiter am Fachbereich Mathematik
01.02.2000 - 01.04.2000	Forschungsaufenthalt bei Prof. Dr. Mikhail Klin, Ben-Gurion-University of the Negev, Beer-Sheva, Israel
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