CPPI strategies and the problem of long-term guarantees

Von der Fakultät für Mathematik und Naturwissenschaften der Carl von Ossietzky Universität Oldenburg zur Erlangung des Grades und Titels eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

angenommene Dissertation

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geboren am 9. Juli 1984 in Chiayi, Taiwan

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Tag der Disputation: 15.03.2016

Abstract

Due to the incomplete downside-risk protection possessed by portfolio insurance strategies, we are particularly interested in one of the most classical ones: **C**onstant **P**roportion **P**ortfolio **I**nsurance (CPPI), and the consequential risk its issuers confront with based on the unfulfilled guarantee.

In the thesis we propose a new model for the risky asset dynamic concerning the gap risk, and further loosen the traditional restriction in the CPPI strategies with regard to the non-risky asset, which is constantly assumed to evolve with riskfree rate. The cushion dynamic is under the new framework driven by a bivariate Lévy process, the solution to the stochastic differential equation is a generalized Ornstein-Uhlenbeck process. Hence we are able to derive explicitly the risk measures through stochastic integration.

Empirical results are also provided in the end of the thesis. We compare the simulation outcome from the new model with Kou and Merton models, along with their performances inside the CPPI portfolio.

Zusammenfassung

Verschiedene Portfolio Insurance Strategien weisen eine unvollständige Absicherung gegenüber dem Downside-Risiko auf. Hier wird eine der klassischen Strategien untersucht, die **C**onstant **P**roportion **P**ortfolio **I**nsurance-(CPPI-)Strategie, vor allem im Hinblick auf das resultierende Risiko für die Emittenten, das aus nicht erfüllten Garantien entsteht.

In dieser Dissertation stellen wir ein neues Modell für die riskante Dynamik von Assets vor, welches auch Gap-Risiko berücksichtigt und außerdem lockern wir die klassische Einschränkung der CPPI-Strategien für nicht-riskante Assets, grundsätzlich eine Asset-Entwicklung mit risikolosem Zinssatz anzunehmen. Nach diesem neuen Ansatz wird die Cushion-Dynamik von einem bivariaten Lévy-Prozess gesteuert. Die Lösung der zugehörigen stochastischen Differentialgleichung ist ein verallgemeinerter Ornstein-Uhlenbeck-Prozess. Daher können wir die Risikomaße mit stochastischer Integration herleiten.

Am Ende dieser Dissertation werden empirische Ergebnisse dargestellt. In einer Simulation vergleichen wir das neue Modell mit den Modellen von Kou und Merton und ihre Performance im CPPI-Portfolio.

Acknowledgements

The completion of this thesis could not have been possible without the support from my adviser, Prof. Dr. Angelika May. I appreciate the opportunity to work and study under her guidance, and all the wise advices and encouragement she had kindly given me. The admiration to my second adviser, Prof. Dr. Dietmar Pfeifer, is beyond words. His intelligence and the passion for knowledge pushed me forward along the way. My sincere gratitude to erudite Prof. Dr. Peter Ruckdeschel, not only for taking part in the committee, but also for the fruitful suggestions that he had given me, either for my thesis or for my future career.

Many thanks are dedicated to my colleagues who have accompanied me through highs and lows: Andreas and Hervé, those coffee and tea we drank together these years have finally turned into this work, I am truly grateful. The discussion with Kornelius at the final stage of my thesis in the matter of improvement was of special help to me.

Without my dearest friends here I would not have pulled my life as a PhD student through: Lena, my hero, in her I see a true friend and I look up to her. I cannot thank her enough in my life for her generosity. Jörg, my dear friend who showed me the kindness of Germans during my first visit, makes me fall in love ever since with this beautiful country, Germany. Yu-Chuan, my savior, her existence in a foreign country is already an enormous comfort for me. Mensa-Gang: Melanie, Sinje, Sven and Thomas, the support I gain from them makes me strong enough to cope with every situation I confront with. My deepest gratitude to these friends, who are practically my families here.

Last but not least, I thank my families, Yun-Fu and friends in Taiwan genuinely, who make me realize where my root is. Without their love and support I would not have come this far.

Contents

1 | Introduction

European Central Bank reduced its benchmark interest rate on June 2014 to -0.1%, and up to December 2015 the interest rate has been further cut to -0.3% ([ECB, 2015]). Particular interest is therefore drawn to investment strategies since the riskfree investments will not contribute much to the capital for investors. For instance, capital protection strategies provide the investors an occasion to participate in the long term return provided by risky investments, and at the same time limit the risk of loss.

CPPI, as the most basic form of portfolio insurance strategy, is the approach on which we lay our eyes in the thesis. The aim of the strategy is to protect the investors from market risk through guaranteeing them a certain percentage of the initial capital investment. Portfolio value is contributed from two sorts of investments, one in the risky asset equal to the product of a constant multiplier and the cushion, which is the difference between the current portfolio value and the guaranteed amount; the other in the non-risky asset. Leverage is therefore created from this multiplier so as to increase the position in the risky asset (See Chapter 3 for further detail). Thus, the setup results in a *buy high sell low* strategy. It can be further customized to meet investors' needs with respect to the individual risk preference, e.g. the level of leverage and the guarantee level.

CPPI was introduced by [Black and Jones, 1987] as an alternative method to allocate investments dynamically between risky and non-risky assets over a time horizon. Since then, several comparisons between CPPI and other portfolio insurance strategies have been investigated. For example, the comparison between CPPI and other dynamic strategies: Buy-and-hold, constant mix and **O**ption **B**ased **P**ortfolio **I**nsurance (OBPI) was given in [Perold and Sharpe, 1988] with respect to the payoff and exposure diagrams. Later, [Bertrand and Prigent, 2005] analyzed OBPI and CPPI in accordance to their payoffs at maturity, stochastic dominance of their returns. Dynamic hedging properties were also examined, in particular classical delta hedging. [Lin and Shyu, 2008] and [Joossens and Schoutens, 2008] provided overviews on the differences between time invariant portfolio protection, **C**onstant **P**roportion **D**ebt **O**bligation (CPDO) and CPPI, respectively. More recently, the performance of the CPPI and OBPI strategies was analyzed by [Bertrand and Prigent, 2011] using Omega measure, which was first introduced by [Keating and Shadwick, 2002], under which the CPPI method outperformed the OBPI.

The properties of continuous-time CPPI strategies have also been studied extensively in the literature, cf. [Black and Perold, 1992] investigated the effect of transaction costs and borrowing constraints on the strategy. An extension for CPPI was provided in [Prigent and Tahar, 2005] with an additional insurance on the cushion. Furthermore, a discrete-time version of the continuous-time CPPI strategy was proposed by [Balder et al., 2009], in which the trading was restricted to discrete time. The result was further extended by [Weng, 2014], in which a double-sided Laplace inversion method was developed to compute the Omega measure of a CPPI portfolio.

Despite its name, the insured portfolio in practice is not typically literally insured, which is the reason encouraging us to assess the risk encountered. On one hand, risky asset in the strategy has often been assumed under the scheme of [Black and Scholes, 1973], which had been shown that CPPI strategies under such a model setup never result in violation of the guarantee; on the other hand, heavy-tailed returns of the risky asset are widely recognized and empirically observed in the financial market, e.g. [Borak et al., 2010]. These facts rendered a spur to research on jump-diffusion models, which was firstly introduced by [Merton, 1976].

An amount of empirical and theoretical research proved the existence of jumps and their substantial impact on financial management. In the case of pricing and hedging, since [Merton, 1976] derived an option pricing formula when the underlying stock returns are generated by a mixture of both continuous and jump processes, research on alternative models due to various distributed jumps has boosted, cf. [Ramezani and Zeng, 1998] proposed the Pareto-Beta jump-diffusion model, assuming that good and bad news are generated by two independent Poisson processes and jump magnitudes are drawn from the Pareto and Beta distributions. [Kou, 2002] proposed another model whose jump is characterized by an asymmetric double exponential distribution, and later demonstrated the result of option pricing with regard to American options in [Kou and Wang, 2004]. For European option pricing, see e.g. [Escobar et al., 2011]. [Cai and Kou, 2011] extended the analytical tractability of the Black-Scholes model to alternative models with arbitrary jump size distributions, such as Gamma, Pareto, and Weibull.

In the case of portfolio and risk management, [Cont and Tankov, 2009] studied the behavior of CPPI concerning the price jumps and derive various associated risk measures in the context of a jump-diffusion price process, along with the problem of downside-risk hedging by using options. Later for another structured credit derivative CPDO, [Cont and Jessen, 2012] gave a thorough risk analysis of the strategy by using a top-down approach and obtained the numerical results by Monte Carlo method. The capital requirement for a long-tern guarantee under the framework of Solvency II with respect to the different risk measurements was discussed in [Devolder, 2011]. [Weng, 2013] investigated the CPPI portfolio under some popular Lévy models from Merton, Kou, variance gamma and normal inverse Gaussian models. The OBPI strategy which minimizes the **V**alue-**a**t-**R**isk (VaR) of the hedged position in a continuous time, regime-switching jump-diffusion market was investigated by [Ramponi, 2013]. [Pézier and Scheller, 2013] concluded CPPI strategies still outperformed OBPI ones under the consideration of gap risk.

In order to implement the theoretical models into practice, parameter calibration for the empirical data is the next aim. **M**aximum **L**ikelihood **E**stimation method (MLE) is one of the most widely implemented estimation methods, which produces the most efficient parameter estimates. This is, however, only possible when the likelihood function is in a tractable form. Alternative techniques are proposed so as to cope with such difficulties arising from the likelihood function.

Due to the one-to-one correspondence between the distribution functions and the characteristic functions, **E**mpirical **C**haracteristic **F**unction method (ECF) is one of the desirable methods while estimating. The original idea of this estimation method was initialized by [Parzen, 1962], and later obtained major theoretical support in the works from [Feuerverger and Mureika, 1977], [Feuerverger and McDunnough, 1981] and [Feuerverger, 1990] with asymptotic efficiency results and block approach for stationary time-series. The basic idea behind the method is to minimize the weighted difference between the empirical and theoretical characteristic functions. Various weighted functions have been investigated due to the research from [Feuerverger and Mureika, 1977], cf. [Jiang and Knight, 2002], Chacko and Viceira [2003].

Moreover, [Carrasco and Florens, 2000] concluded that continuous ECF estimator can be seen as a special case of the generalized method of moments with a continuum of moment conditions. An overview of the ECF method and its application on affine jump-diffusion models was provided by [Yu, 2004]. More recently, [Levin and Khramtsov, 2015] considered the method for estimating parameters of affine jump-diffusions with unobserved stochastic volatility.

The aim of this thesis is to construct a stochastic model for the self-financing CPPI strategy under the continuous-time framework. Furthermore, the gap risk and its resulting effect on the strategy are taken into consideration. In the light of the above objectives, the thesis is outlined as follows.

In Chapter 2 an overview on the prerequisite tools in the field of stochastic analysis which we implement throughout the thesis is provided. Particularly in this chapter we refer to [Sato, 2005], [Protter, 2005], [Applebaum, 2009] and references therein for general theory on Lévy processes and stochastic integration, and [Cont and Tankov, 2004] and [Pascucci, 2011] for their applications in finance modeling.

The following chapter is dedicated to the CPPI strategy under the classical framework - the Black-Scholes model. We first clarify the mechanism behind the strategy, in which the risky asset is described as a geometric Brownian motion. From the closed-form expression of the value process, it is foreseeable the model is not adequate for risk assessment.

In order to deal with the impractical problem we face in the former chapter, we include the gap risk in the dynamic of the risky asset in Section 4.1. We start from reviewing the features of well-known Merton and Kou models, respectively. According to the result of data fitting we propose a new model with a different jump characteristic in order to capture more accurately the sudden jumps of market prices and to get a better characterization under the mathematical framework.

Apart from the modification of the price dynamic in risky asset, in Section 4.2 we further relax the restriction of the CPPI strategy on the non-risky investment, which does not necessarily evolve from the riskfree rate anymore, yet it can be chosen from a pool of financial products with higher yields. Based on this new setup in the CPPI strategy, we derive a closed-form solution to the cushion in the end of this chapter, as well as the solution to the portfolio value.

Chapter 5 focuses on the effect of modified CPPI strategies on statistical evaluation and various risk measures, which allows the portfolio insurance issuers to further assess the extent of risk they confront with. The results from Section 5.1 and 5.1 are derived based on the the former chapter and the moments of Doléans-Dade exponential. In terms of VaR and **C**onditional VaR (CVaR) we use the inverse Fourier transform to retrieve the distribution function.

Empirical data from various assets in major markets are implemented in Chapter 6. Section 6.1 illustrates the estimation results from Chapter 4 with regard to the jump-diffusion models. Parameters are estimated from daily log-returns with ECF and MLE methods, which are reviewed in Section 6.1. Additionally, we combine the two estimation methods to calibrate the parameter. Two sources of initial values are also provided. Overall, parameter from each asset is estimated in 6 different ways for each model. With the help of Q-Q plots and **A**kaike **I**nformation **C**riterion (AIC) we compare the performances among Kou, Merton and our new models with respect to different estimation methods in Section 6.2.

After modifying the dynamic of the risky asset we are interested to see how it unfolds and if it further possesses the forecasting ability. The analysis is done by out-of-sample testing with respect to the performance of the CPPI strategy. The result and the comparison to other models are demonstrated in Section 6.3.

2 | Preliminaries

In this chapter a few preliminaries in stochastic analysis will be introduced. A complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be given throughout the thesis. Furthermore this probability space will be provided with a *filtration* $\{\mathcal{F}_t\}_{t\geq 0}$, which is an increasing sequence of σ-algebras with $\mathcal{F}_t \in \mathcal{F}$, $\forall t \geq 0$, to obtain a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

In addition, the filtered probability space is said to fulfill the *usual hypotheses* if it is *complete*, i.e. $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets, and right continuous, i.e. $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$ is equal to $\mathcal{F}_t$ for all $t \geq 0$. Note that we assume the usual hypotheses are satisfied in this thesis.

Section 2.1 will give a brief overview to stochastic processes, and then the focus will be set on a special class of stochastic processes called "*semimartingales*", which is the key to the topic *stochastic integration* in the next section. Attention will move on to Lévy processes in Section 2.3. From Lévy-Itô decomposition we are able to see the relation between Lévy processes and semimartingales.

2.1 Stochastic Processes and Semimartingales

A *stochastic process* X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of real-valued random variables $\{X_t\}_{t\geq 0}$. X is said to be an *adapted* process if $X_t \in \mathcal{F}_t$, $\forall t \geq 0$, which means $\{X_t\}_{t\geq 0}$ is non-anticipating with respect to the information structure $\{\mathcal{F}_t\}_{t\geq 0}$. The filtration which is generated by the past values of the stochastic process X is called a *natural filtration* $\{\mathcal{F}_t^0\}_{t\geq 0}$. That is, $\{\mathcal{F}_t^0\}_{t\geq 0}$ is the smallest filtration that makes X adapted. One can obtain the so-called *augmented natural filtration* by extending the natural filtration such that it satisfies the usual hypothesis.

Moreover, a stochastic process is said to be *càdlàg* if it almost surely (a.s.) has sample paths which are right continuous (**c**ontinue **à** **d**roite), with left limits (**l**imite **à** **g**auche). Similarly, a stochastic process X is said to be càglàd if it a.s. has sample paths which are left continuous, with right limits.

An adapted stochastic process $\{M_t\}_{t\geq 0}$ is said to be a *martingale* with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, if $\mathrm{E}\left[|X_t|\right] < \infty$ for all $t \geq 0$, and $\mathrm{E}\left[X_t|\mathcal{F}_s\right] = X_s$ a.s. for $0 \leq s \leq t$.

Next we introduce *stopping time* as a random variable $\tau : \Omega \to [0, \infty)$ which fulfills the following condition: Event $\{\tau \leq t\} \in \mathcal{F}_t$, $\forall t > 0$, i.e., event $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable, for each t. And then we define the "good integrators" on an appropriate class of adapted process, that is, semimartingale, which later will be used in our model setup.

Definition 2.1.1. ***(Local Martingale)*** *An adapted stochastic process X is a local martingale if there is a sequence of increasing stopping times $\{\tau_n\}_{n>0}$ with $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that $\{X_{t\wedge\tau_n}\}_{t\geq 0}$ is a martingale for each n.*

Definition 2.1.2. ***(Semimartingale)*** *An adapted stochastic process X is called a semimartingale, if it can be written as the following form*

$$X_t = X_0 + M_t + A_t,$$

where M is a local martingale and A is a process that has finite-variation.

2.2 Stochastic Integration

In the process of deriving the explicit solutions to risk measures in Chapter 5, some stochastic integrals driven by semimartingales need to be overcome. Therefore we will give an introduction regarding the stochastic integration, which was initially developed by Itô with respect to the standard Brownian motion.

We introduced in the last section the “good integrators”, now we wish to know the processes we can consider as integrands.

Definition 2.2.1. ***(Simple predictable process)*** *A stochastic process H is called a simple predictable process if it can be represented as*

$$H_t = H_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t),$$

where $\tau_0 = 0 < \tau_1 < \tau_2 < ... < \tau_n < \tau_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{\tau_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$.

The stochastic integral of the simple predictable process H with respect to a stochastic process X is defined as

$$I^X(H) = \int_0^t H dX = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{\tau_{i+1}\wedge t} - X_{\tau_i \wedge t}),$$

where $I^X : \mathcal{S} \to \mathcal{L}^0$ is a mapping. And $\mathcal{S}$ represents the space of simple predictable processes, whereas the space of finite-valued random variables is denoted by $\mathcal{L}^0$.

We should notice here the mapping I^X has to hold for bounded convergence in probability. That is, if $\{H^n\}_{n\geq 0}$ is a sequence of predictable processes converging to a process H and uniformly bounded, then the stochastic integral converges in probability.

$$\sup_{s\leq t} |I_s^X(H^n) - I_s^X(H)| \to 0$$

Otherwise, a very small difference in the integrand can cause a large change in the resulting integral. It is therefore preferable to have this property, especially in the aspect of the implementation in finance. This class of stochastic process X is the mentioned “good integrators”

- semimartingales. Moreover, we can analogously enlarge the space of possible integrands, e.g. locally bounded predictable integrands (For details please refer to [Protter, 2005]). In addition, the stochastic integral of an adapted process H with respect to a semimartingale is, by Definition 2.1.2, the sum of two integrals, one with respect to the local martingale and the other with respect to the finite variation process, which can be calculated path by path as the Stieltjes integral.

In the rest of this section we give some important extensions of stochastic integrals which will be used frequently throughout this thesis.

Definition 2.2.2. ***(Quadratic covariation)*** *Given two semimartingales X_t and Z_t, the quadratic covariation process $\{[X,Z]_t\}_{t\geq 0}$ is the semimartingale defined by*

$$[X,Z]_t = X_t Z_t - X_0 Z_0 - \int_0^t X_{s^-} dZ_s - \int_0^t Z_{s^-} dX_s$$

The quadratic covariation is alo called the *bracket process*. Its definition leads us to the *stochastic integration by parts* formula. Next we present the Itô-Döblin theorem for a special case: semimartingales. It explains the structure of a process $f(X)$ given a “nice” function f.

Theorem 2.2.1. ***(Itô-Döblin theorem for semimartingales)*** *Given a semimartingale X and f be a C^2 function of X. Then $f(X)$ is also a martingale and written in the following form*

$$\begin{aligned} f(X_t) =& f(X_0) + \int_0^t f'\left(X_{s^-}\right) dX_s + \frac{1}{2}\int_0^t f''\left(X_{s^-}\right) d[X,X]_s^c \\ &+ \sum_{0<s\leq t} \left\{ f\left(X_s\right) - f\left(X_{s^-}\right) - f'\left(X_{s^-}\right)\Delta X_s \right\} \end{aligned}$$

Proof See [Protter, 2005], Theorem II.32. □

One of the applications from Itô-Döblin Theorem is the derivation of the solution to

$$dZ = Z_- dX$$

If X is a deterministic process, e.g. $X_t = t$, the solution is given as a exponential function. However, if X and Z are two semimartingales, the solution to it is given in the following theorem.

Theorem 2.2.2. ***(Doléans-Dade exponential)*** *Let X be a semimartingale. There exists an unique semimartingale $\{Z_t\}_{t\geq 0}$ such that*

$$\frac{dZ_t}{Z_t^-} = dX_t, \quad Z_0 = 1$$

which is called stochastic or Doléans-Dade exponential of X, denoted by $\mathcal{E}(X)_t$ and written explicitly as

$$Z_t = \mathcal{E}(X)_t = e^{X_t - X_0 - \frac{1}{2}[X,X]_t^c} \prod_{0<s\leq t} \left(1 + \Delta X_s\right) e^{-\Delta X_s}$$

Proof See [Cont and Tankov, 2004], Proposition 8.21. □

2.3 Lévy process

Another popular class of stochastic processes will be presented in this section, the Lévy process. Again, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is given and the usual hypothesis is fulfilled.

Definition 2.3.1. ***(Lévy process)*** *A càdlàg stochastic process $\{X_t\}_{t\geq 0}$ with $X_0 = 0$ a.s. is called a Lévy process if it possesses the following properties:*

1. *Independent increments: for every increasing sequence of times $t_0, t_1, ..., t_n$, the random variables $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, ... , $X_{t_n} - X_{t_{n-1}}$ are independent*
2. *Stationary increments: $\forall h > 0$, the distribution of $X_{t+h} - X_t$ does not depend on t*
3. *Continuity in probability: $\forall \epsilon > 0$, $\lim_{h\to 0} \mathbb{P}\left(|X_{t+h} - X_t| > \epsilon\right) = 0$*

Further information can be found in [Protter, 2005] and [Cont and Tankov, 2004].

The most elementary and well-known jump process is the *homogeneous Poisson process*. Before any further detail is given, we should first have some basic concept of the *counting process*, denoted by $\{N_t\}_{t\geq 0}$. The value of the process is given by

$$\begin{aligned} N_t &= \sum_{k=1}^{\infty} \mathbb{1}_{[T_k,\infty)}(t), t \in \mathbb{R}^+ \\ \Delta N_{T_k} &= N_{T_k} - N_{T_k^-} = 1, \end{aligned}$$

where $\{T_k\}_{k\in\mathbb{N}}$ represents the time when jumps occur. If the random times $\{T_k\}_{k\in\mathbb{N}}$ are partial sums of i.i.d. exponential random variables, then $\{N_t\}_{t\geq 0}$ is a homogenous Poisson process if it satisfies the following properties:

1. Independent increments: for all $0 < t_0 < t_1 < ... < t_n$, $n \in \mathbb{N}$, the random variables $N_{t_1} - N_{t_0}$, $N_{t_2} - N_{t_1}$, ... , $N_{t_n} - N_{t_{n-1}}$ are independent
2. Stationary increments: for all $h > 0$, $0 \leq u < t$, $N_{t+h} - N_{u+h}$ has the same distribution as $N_t - N_u$

Proposition 2.3.1. *The homogeneous Poisson process is a Lévy process*

The homogeneous Poisson process defined above counts events that occur at a constant rate. Moreover, $N_t - N_u$ follows the *Poisson distribution* with parameter $\lambda(t-u)$. Therefore, the expected value of N_t is

$$\mathrm{E}[N_t] = \lambda t$$

Obviously the homogeneous Poisson process has its own restrictions to describe the behavior of the asset price jumps. Thereby we introduce another more general process: *compound Poisson Process* so as to mimic the asset price jumps better.

Define $\{J_k\}_{k\in\mathbb{N}}$ as a sequence of i.i.d. random variables with probability distribution π and independent of the Poisson process $\{N_t\}_{t\geq 0}$ whose intensity rate is λ. The following definition is obtained.

Definition 2.3.2. *The process $\{Y_t\}_{t\geq 0}$ which is given as*

$$Y_t = \sum_{k=1}^{N_t} J_k$$

is called a compound Poisson process

Proposition 2.3.2. *The compound Poisson process is a Lévy process*

By construction, $\{Y_t\}_{t\geq 0}$ has paths that are constant apart from a finite number of jumps in any finite time interval. In comparison to the counting process $\{N_t\}_{t\geq 0}$, the jumps of $\{Y_t\}_{t\geq 0}$ occur at the same time as the jumps of $\{N_t\}_{t\geq 0}$. The only difference is the jumps of $\{N_t\}_{t\geq 0}$ are always of size 1, whereas the jumps of $\{Y_t\}_{t\geq 0}$ are of random size.

Proposition 2.3.3. *The characteristic function of a compound Poisson process $\{Y_t\}_{t\geq 0}$ with intensity λ is*

$$\mathrm{E}\left[e^{iuY_t}\right] = e^{\lambda t \int_{-\infty}^{\infty}\left(e^{iuy}-1\right)\pi(dh)}$$

The first and second moments are therefore derived as follows:

$$\mathrm{E}\left[Y_t\right] = \lambda t \mathrm{E}\left[J_1\right] \;\; \textit{and} \;\; \mathrm{Var}\left[Y_t\right] = \lambda t \mathrm{E}\left[{J_1}^2\right]$$

Proof The moments can be attained by Proposition 2.4.1. Or we can prove it either by using the characteristic function or by the law of total expectation. The characteristic function can also be derived easily

$$\begin{aligned}
\mathrm{E}\left[e^{iuY_t}\right] &= \sum_{n=0}^{\infty} \mathrm{E}\left(e^{iu\sum\limits_{k=1}^{n} J_k}\right)\mathbb{P}\left(N_t = n\right) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\left(\lambda t\right)^n}{n!}\mathrm{E}\left(e^{iu\sum\limits_{k=1}^{n} J_k}\right) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}\left(\lambda t\right)^n}{n!}\left[\mathrm{E}\left(e^{iuJ_1}\right)\right]^n \\
&= e^{-\lambda t}\sum_{n=0}^{\infty} \frac{\left[\lambda t \mathrm{E}\left(e^{iuJ_1}\right)\right]^n}{n!} \\
&= e^{\lambda t\left[\mathrm{E}\left(e^{iuJ_1}\right)-1\right]} \\
&= e^{\lambda t \int_{-\infty}^{\infty}\left(e^{iuy}-1\right)\pi(dy)}
\end{aligned}$$

For the expectation, we have

$$\begin{aligned}
\mathrm{E}\left[Y_t\right] &= \frac{1}{i}\left(\frac{d}{du}\mathrm{E}\left[e^{iuY_t}\right]|_{u=0}\right) \\
&= \lambda t \int_{-\infty}^{\infty} y\pi(dy) \\
&= \lambda t \mathrm{E}\left[J_1\right]
\end{aligned}$$

The variance can hereby be derived. $\square$

Definition 2.3.3. *The process $\{M_t\}_{t\geq 0}$ with $M_t := N_t - \lambda t$ is called a compensated Poisson process, where $\{N_t\}_{t\geq 0}$ is a counting process and λ is its intensity.*

Now that we have reviewed the commonly used processes which describe pure jumps, we would like to focus on the Lévy processes which possess jumps, and how it can be characterized. To each càdlàg stochastic process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}$, one can associate jump measure $\bar{J}^X$ with $\{X_t\}_{t\geq 0}$ as follows:

$$\bar{J}^X(B) = \#\{(\Delta X_t, t) \in B\}, \forall B \in \mathcal{B}(\mathbb{R} \times [0, \infty))$$

where $\Delta X_t = X_t - X_{t^-}$.

In a period of time, e.g. $[t_1, t_2]$, the jump measure $\bar{J}^X(A \times [t_1, t_2])$ counts the number of jumps of X whose jump sizes are in $A \in \mathcal{B}(\mathbb{R})$ between the time t_1 and t_2. If X_t is a compound Poisson process with intensity λ and jump size distribution π, then the jump measure of which is a Poisson random measure with intensity measure $\lambda\pi(dx)dt = \nu(dx)dt$, where ν is the Lévy measure which will be shown in the next definition.

From the interpretation of the intensity measure of a compound Poisson process, it is clear to see that its Lévy measure can be seen as the average number of jumps per unit of time. And this holds for all Lévy processes.

Definition 2.3.4. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by*

$$\nu(A) = \mathrm{E}\left[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}\right], A \in \mathcal{B}(\mathbb{R})$$

is called the Lévy measure of X: $\nu(A)$ is the expected number, per unit time, of jumps where sizes belong to A.

If a Lévy process has only a finite number of jumps in any bounded time interval (e.g. compound Poisson process) we say that it is a finite activity Lévy process. Otherwise we say that it has infinite activity, which means that singularities, i.e. infinitely many jumps, can occur around the origin. Moreover, the jump mass away from the origin of a finite activity Lévy process is bounded, i.e. only a finite number of big jumps can occur. One of most well known models that deals with finite activity Lévy processes is the Merton jump-diffusion model [Merton, 1976], which is the independent sum of a Brownian motion with drift and a compound Poisson process.

In order to cope with the convergence problem when it comes to infinite activity, every Lévy process can be represented in the following form.

Theorem 2.3.1. ***(Lévy-Itô decomposition)*** *Let $\{X_t\}_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with jump measure J and Lévy measure ν fulfilling*

$$\int_{|x|>1} \nu(dx) < \infty \quad \text{and} \quad \int_{|x|\leq 1} x^2 \nu(dx) < \infty$$

For any $R > 0$ we can write X_t into the sum of independent Lévy process $X_t^{(1)}$, $X_t^{(2)}$ and $X_t^{(3)}$ where

1. *$X_t^{(1)} = \mu_R t + \sigma W_t$, which is a Brownian motion with drift, where $\mu_S = \mu_R - \int_{S<|x|\leq R} x\nu(dx)$*

2. $X_t^{(2)} = \int_0^t \int_{|x|\geq R} xJ(dx, ds) = \sum\limits_{0<s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\geq R\}}$, *which is a compound Poisson process that is responsible for the large jumps*

3. $X_t^{(3)} = \int_0^t \int_{\epsilon\leq|x|<R} x\{\tilde{J}(dx, ds)\} = \sum\limits_{\substack{0<s\leq t\\ \epsilon\leq|x|<R}} \Delta X_s - t\mathrm{E}[\Delta X_1 \mathbb{1}_{\{\epsilon\leq|\Delta X_s|<R\}}]$, *which is a L^2-martingale that deals with the small jumps and $\epsilon \to 0^+$*

Definition 2.3.5. *(μ_R, σ^2, ν) is called the Lévy R-triplet of a Lévy process X.*

In most of the literature it can be found that the choice of $R = 1$ is common. And if the process has finite activity, we no longer need to truncate the small jumps, the jump part can actually separated from the continuous part of the process by setting R go to zero. The Lévy triplet for finite activity Lévy process is (μ_0, σ^2, ν), where

$$\mu_0 = \mu_1 - \int_{0<|x|\leq 1} x\nu\,(dx)\,,$$

which has an intrinsic interpretation as the continuous part of the process, whereas μ_1 depends on the truncation function.

In the next section we can observe that, according to the Lévy-Itô decomposition it follows that all Lévy processes are semimartingales with respect to the augmented natural filtration.

Theorem 2.3.2. ***(Lévy-Khintchine representation)*** *Let $\{X_t\}_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with Lévy triplet (μ_1, σ^2, ν) Then*

$$\mathrm{E}\left[e^{i\theta X_t}\right] = e^{t\psi(\theta)}, \quad \theta \in \mathbb{R}$$

where the Lévy exponent

$$\psi\,(\theta) = i\mu_1\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{\{0<|x|\leq 1\}}\right)\nu(dx) \tag{2.1}$$

Note that the conditions on the Lévy measure are sufficient to ensure that the integral in (2.1) converges since the integrand is $O(1)$ for $|x| > 1$ and $O(x^2)$ for $|x| \leq 1$.

In the case of finite activity the equation (2.1) can be simplified as

$$\psi\,(\theta) = i\mu_0\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1\right)\nu\,(dx)$$

2.4 Cumulants

The Lévy-Khintchine formula allows us to compute easily the cumulants of a Lévy process. In the thesis, cumulants are also used in deriving the closed-form solutions to risk measures. Furthermore, we also use them to locate a set of reasonable initial values for parameters estimation.

Definition 2.4.1. *The characteristic function of the random variables X, with values in $\mathbb{R}$, is the function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined by*

$$\begin{aligned}\phi_X(u) &= \mathrm{E}[e^{iux}], u \in \mathbb{R}\\ &= \int_{\mathbb{R}} e^{iux} P_X(dx), u \in \mathbb{R}\end{aligned}$$

ϕ_X is simply the Fourier transform of the distribution P_X of X. In particular, if P_X has a density f, then we write $\phi_X = F(f)$. In addition, if $X \in L^p$, then

$$m_p^{'} = \mathrm{E}[X^p] = \frac{1}{i^p}\frac{d^p}{du^p}\phi_X(u)|_{u=0}$$

is called the *p-th moment* of X, whereas

$$m_p = \mathrm{E}[(X - \mathrm{E}[X])^p]$$

is called the *p-th central moment* of X.

Since $\phi_X(0) = 1$ and ϕ_X is a continuous function, it can be proved in Lemma 7.6 in [Sato, 2005] that there exists a unique continuous function $\tilde{\psi}_X$ such that $\phi_X(u) = e^{\tilde{\psi}_X(u)}$ and $\tilde{\psi}_X(0) = 0$. The function $\tilde{\psi}_X$ is called the cumulant generating function of X, and along with it is the cumulants of X defined as follows:

$$c_n(X) = \frac{1}{i^n}\frac{d^n}{du^n}\tilde{\psi}_X(u)|_{u=0}$$

Differentiating the cumulant generating function, for instance, results in

$$\begin{aligned}c_1(X) &= \mathrm{E}[X]\\ c_2(X) &= \mathrm{Var}(X)\end{aligned}$$

The relation between cumulants and central moments is the key to the **C**umulant **M**atching **M**ethod (CMM), which will be further discussed in Section 6.1.3.

Proposition 2.4.1. *Let X be a Lévy process on $\mathbb{R}$ generated by Lévy triplet (μ_1, σ^2, ν). The n-absolute moment $\mathrm{E}[|X_t|^n]$ is finite if and only if*

$$\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$$

In particular, we have

$$c_n(X_t) = tc_n(X_1), n \geq 1$$

Proof See [Pascucci, 2011], Proposition 13.45 □

Proposition 2.4.2. *Let X be a Lévy process on $\mathbb{R}$. If $\mathrm{E}[|X_1|] < \infty$, then*

$$X_t - \mathrm{E}\left[X_t\right]$$

is a martingale.

Proof Since $\mathrm{E}[X_t] = t\mathrm{E}[X_1]$ from Proposition 2.4.1, and if $\mathrm{E}[|X_1|] < \infty$, then $X_t - \mathrm{E}[X_t]$ is integrable. Moreover, by the independence of increments we have

$$\begin{aligned}\mathrm{E}[(X_t - \mathrm{E}[X_t]) - (X_s - \mathrm{E}[X_s]) \mid \mathcal{F}_s] &= \mathrm{E}[X_t - X_s \mid \mathcal{F}_s] - E[X_1](t-s) \\ &= \mathrm{E}[X_{t-s}] - E[X_1](t-s) \\ &= 0\end{aligned}$$

and therefore $X_t - \mathrm{E}[X_t]$ is a martingale. □

Corollary 2.4.1. *The compensated Poisson process followed by Definition 2.3.3 is a càdlàg martingale with respect to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

3 | Stochastic Model without Gap Risk Assumption

The theoretical background introduced in the preceding chapter enables us to describe the dynamic of the CPPI portfolio by using continuous-time stochastic processes. In this chapter we introduce the classical setup of the portfolio, in which the gap risk is omitted.

CPPI is a dynamic portfolio insurance strategy which provides downside protection for the portfolio by setting up a threshold to the portfolio value. The protection is obtained by reallocating the exposure to the risky asset based on the surplus to the discounted guarantee. Throughout the thesis we assume a self-financing CPPI portfolio in a frictionless market.

The portfolio constructed under the CPPI framework consists of two parts of investments. One part in non-risky asset, say B, which evolves with a riskfree rate r; and the other in risky asset, S, a geometric Brownian motion with μ and σ as the expected rate of return and the volatility of S, respectively. μ, $\sigma \in \mathbb{R}$.

The classical dynamics of B and S are given as follows

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t dt + \sigma S_t dW_t \end{aligned}$$

The *Guarantee* of the CPPI portfolio is assumed to be G. It is the least payment the investor should receive at maturity T. Set P_t to be the present value of G, discounted by the riskfree rate r, which forms the *Floor*.

In other words, once the *Floor* is violated, the *Guarantee* will be unfulfilled at maturity. Consequently, the insurance issuer is obligated to cover the difference between the *Guarantee* and the final portfolio value. Mathematically we describe the *Floor* in the following way.

$$\begin{aligned} P_t &= Ge^{-r(T-t)} \\ dP_t &= rP_t dt \end{aligned}$$

Consider a CPPI portfolio whose *Value* is written as V, which contains B and S, we write

$$dV_t = \alpha_t \frac{dS_t}{S_t} + (V_t - \alpha_t)\frac{dB_t}{B_t}, \alpha_t \in \mathbb{R} \tag{3.1}$$

The amount of money which is invested in the risky asset is denoted by $\alpha_t = mC_t$, where $C_t = V_t - P_t$ is the *Cushion*, and m is the *Multiplier*, which is a finite positive number. One can rewrite (3.1).

$$dV_t = [rV_t + m(V_t - P_t)(\mu - r)]\, dt + m(V_t - P_t)\sigma dW_t \tag{3.2}$$

Rearrange (3.2), the equation becomes

$$dV_t = rP_t dt + (V_t - P_t)\{[m(\mu - r) + r]\, dt + m\sigma dW_t\}$$

It is clear to see that the cushion process $\{C_t\}_{t\in[0,T]}$ is driven by the process $\{D_t\}_{t\in[0,T]}$

$$dD_t = [m(\mu - r) + r]\, dt + m\sigma dW_t$$

Hence we get

$$\begin{cases} dV_t = C_t dD_t + rP_t dt \\ V_0 = v \end{cases}$$

or

$$\begin{cases} dC_t = C_t dD_t \\ C_0 = v - P_0 = v - Ge^{-rT} \end{cases}$$

Since the risk premium is bounded, the above differential equation has an unique solution.

Theorem 3.0.1. *The value of the cushion follows the following process*

$$C_t = C_0 \cdot \mathcal{E}(D)_t = C_0 e^{\left[m(\mu-r)+r-\frac{m^2\sigma^2}{2}\right]t + m\sigma W_t} \tag{3.3}$$

Proof A direct result from Theorem 2.2.2. $\square$

4 | Stochastic Model Concerned with Gap Risk

From the formula (3.3), the value of the continuously traded portfolio apparently will not drop under the floor, since C_t is always positive for every $t \in [0, T]$, see for instance Figure 4.1, where the asset price is simulated from the Black-Scholes model.

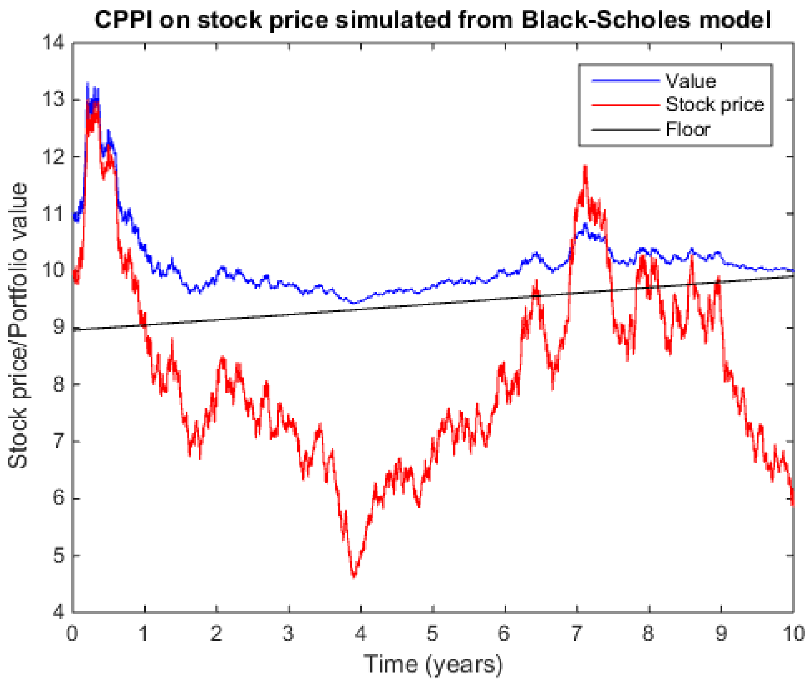

Figure 4.1: CPPI on stock price simulated from Black-Scholes model

However in the real world, jumps of the asset prices would occur and are widely recognized. In order to characterize the gap risk, jump process, is therefore added to our model setup. The different asset price dynamics can be seen in Figure 4.2.

Firstly in Section 4.1 the attention is placed on the risky investment in the CPPI strategy, the theoretical backgrounds of well-known models, e.g. Merton and Kou models are being investigated, and a different setup of asset dynamic is proposed by using different way to interpret the jump. In the next Section 4.2, focus is set on relaxing the traditional restriction

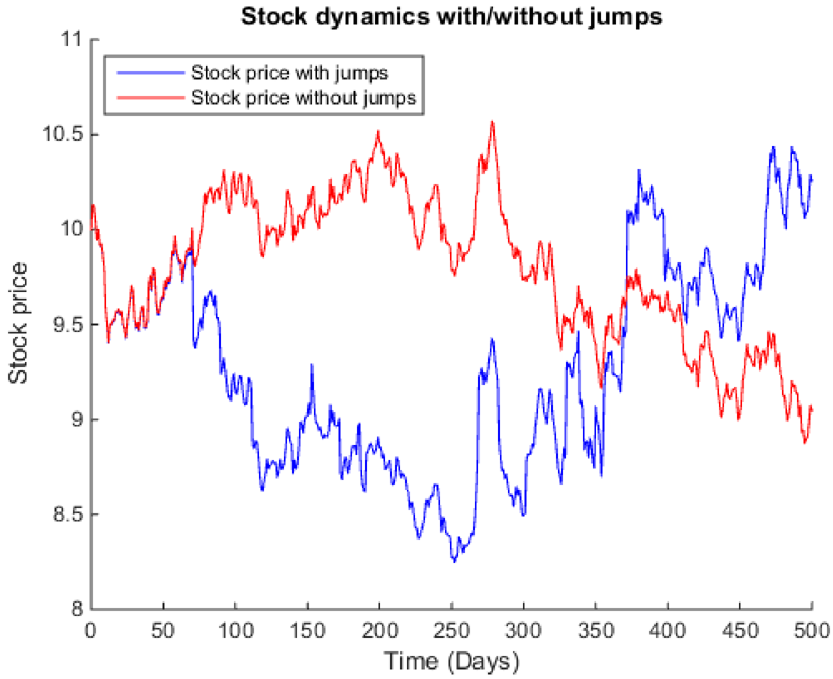

Figure 4.2: The effect of jumps on asset price

on the non-risky investment in the CPPI strategy. Consequently our new model is born.

4.1 Model Setup for the Risky Asset

In this section we focus on a special case of Lévy process, which is also the assumption of risky asset dynamic in the thesis - *jump-diffusion process*. It is a combination of Lévy Processes, including a geometric Brownian motion and a jump described by a compound Poisson process with random jump sizes.

$$X_t = \mu t + \sigma W_t + Y_t = X_t^c + Y_t, \tag{4.1}$$

where X_t^c is the continuous part of the process X, and Y is the compound Poisson process. The differential of a function of the above process can be seen as a special case of Theorem 2.2.1. Due to the finite activity Y has, Theorem 2.2.1 for the jump-diffusion process (4.1) can then be simplified as

$$\begin{aligned} f(X_t) =& f(X_0) + \int_0^t f'(X_{s^-}) dX_s^c + \frac{\sigma^2}{2} \int_0^t f''(X_{s^-}) ds + \\ & \sum_{0<s\leq t} (f(X_{s^-} + \Delta X_s) - f(X_{s^-})), \end{aligned}$$

where $dX_s^c = dX_s - \Delta X_s$.

Similarly we can derive Theorem 2.2.2 for the case of Lévy process with finite activity as a special case. With all conditions the same, only an extra condition: $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ added, then we have,

$$\mathcal{E}(X)_t = e^{X_t^c - \frac{\sigma^2 t}{2}} \prod_{0<s\leq t} (1 + \Delta X_s) \tag{4.2}$$

In view of the above, we continue to introduce dynamics of assets with practical adjustments and look at its effect on the whole CPPI strategy. The setup of the risky asset dynamic S is described as Equation (4.3) with the participation of jumps which follow a compound Poisson process $Y_t = \sum_{k=1}^{N_t} (J_k - 1)$, whereas J is a sequence of i.i.d. random variables with probability distribution π and independent of the Brownian motion W and the Poisson process N whose intensity rate is λ.

$$\frac{dS_t}{S_t^-} = \mu dt + \sigma dW_t + dY_t \tag{4.3}$$

The difference between Merton and Kou models is how they describe the jumps using different distribution π. Under this setup (4.3) Merton introduced his model in 1976 with jumps following log-normal distribution [Merton, 1976]; Later in 2002, Kou proposed another model by introducing double exponentially distributed jumps [Kou, 2002].

- Merton: $X = \log \frac{S}{S_-} = \log J \sim \mathcal{N}(\mu_j, \sigma_j^2)$
- Kou: $X = \log \frac{S}{S_-} = \log J$ has an asymmetric double exponential distribution, and the density for X is

$$f_X(x) = p\frac{1}{\eta_-}e^{-\frac{|x|}{\eta_-}}\mathbb{1}_{\{x<0\}} + (1-p)\frac{1}{\eta_+}e^{-\frac{x}{\eta_+}}\mathbb{1}_{\{x>0\}}$$

  with η_-, $\eta_+ > 0$, where $p \geq 0$ represent the probability downward jumps occur.

In comparison with Merton model, in which only one random variable reflects downward and upward jumps, Kou model has better economical interpretation with asymmetric double exponential distributed jumps, which could capture better the leptokurtic feature of the empirical log return distribution, but when it comes to parameter estimation, an extra parameter can also bring inevitably computing burden.

Moreover, we modify the jumps to follow log-Gumbel distribution due to the fact that the returns of our underlyings are mostly right-skewed, and a phenomenon we observe after simulating from Merton and Kou models. The number of parameters in our new model remains the same as in Merton.

- new: $X = \log \frac{S}{S_-} = \log J \sim \mathcal{G}(\alpha, \beta)$

The characteristics that different types of jump possess are offered in Figure 4.3. The left side figure presents different distributions with the same mean and variance. Either exponential or Gumbel distributed jump has fatter right tail than the normal distributed one. If we only focus on the 5-parameter models, i.e. Merton and new model, the difference between each other is evident based on the right side figure.

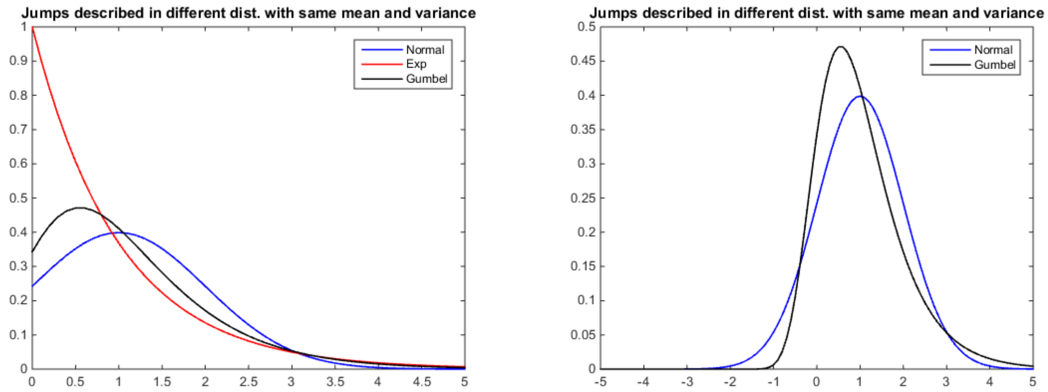

Figure 4.3: Comparison between different jumps in different distributions with same mean and variance

Note that for the sake of convergence when the process has infinite activity, it is useful to compensate the jump process when one is applied into the dynamic of the asset. But in this special case (finite activity) of the Lévy-Itô decomposition, we do not need to compensate the compound Poisson process. In other words, we separate the continuous part of the process from the jump part. Assume the jumps occur in $\{T_i\}_{i\in\mathbb{N}}$, by the construction we have the jump in S at T_i represented as:

$$S_{T_i} - S_{T_{i-}} = S_{T_{i-}}(Y_{T_i} - Y_{T_{i-}}) = S_{T_{i-}}(J_i - 1)$$

Hence $S_{T_i} = S_{T_{i-}} J_i$. This reveals that the J_i are the ratios of the asset price after and before a jump, which indicates the jumps are multiplicative. This also explains why $J_i - 1$ is used in this thesis rather than simply J_i. The solution to the stochastic differential equation (SDE) (4.3) according to the special case (4.2) of Doléans-Dade exponential is then

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{k=1}^{N_t} J_k,$$

or rewrite it in the form of log-return

$$\log \frac{S_t}{S_0} = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{k=1}^{N_t} \log J_k$$

In the meanwhile, the dynamic of the non-risky asset B remains unchanged. Therefore for the cushion process, we have

$$\frac{dC_t}{C_t^-} = [r + m(\mu - r)]\, dt + m\sigma dW_t + mdY_t \tag{4.4}$$

Analogously, the solution to the Equation (4.4) can be written as

$$C_t = C_0 e^{[r + m(\mu - r) - \frac{1}{2}m^2\sigma^2]t + m\sigma W_t} \prod_{k=1}^{N_t} (1 + m(J_k - 1))$$

4.2 Model Setup for the CPPI Strategy

In the previous section we focus on the cushion whose dynamic is driven by a Lévy process. In order to move onto the main focus of this thesis, we will first introduce how the idea of this topic was born in the beginning of this section. And then a new model is built by loosing the setup of our current assumption, which will lead to a bivariate-Lévy-process-driven cushion.

4.2.1 When Yield Does Not Coincide with Money Market Return

Most of the former models concerned with CPPI strategy consider only the riskfree rate when it comes to the investment in the non-risky asset. Nevertheless, according to Solvency II framework, sovereign bonds are free of capital charges under the standard formula. In other words, the sovereign bonds are still seen as riskfree asset, regardless the credit risk. Therefore we discard the assumption of the non-risky asset being savings in a bank, which is to follow the riskfree rate r.

Hence when the insurance company invests in the non-risky asset while applying the CPPI strategy, there are actually plenty of products to select from in the financial market besides those at the riskfree rate r. Which of the above will lead to a question: what if the yield, y, from the non-risky asset does not coincide with the riskfree rate, r? The answer to this question is what we are seeking for in this section, and then further we study the effect of the CPPI strategy in such setup on risk measures.

The relation between y and r can be rationally assumed to be $y > r$, for the surplus can be seen as a risk premium. Along with the assumption the setup of CPPI strategy is consequently modified.

$$
\begin{aligned}
dB_t &= yB_{t^-}dt \\
dS_t &= \mu S_{t^-}dt + \sigma S_{t^-}dW_t + S_{t^-}dY_t \\
dP_t &= rP_{t^-}dt
\end{aligned}
$$

It is trivial to see the processes of the money market account and the non-risky asset are actually both continuous from the setup, the dynamic of the portfolio and the cushion are hereby described as

$$
\begin{aligned}
dV_t &= yP_t dt + (V_{t^-} - P_t)\left\{[m(\mu - y) + y]\,dt + m\sigma dW_t + mdY_t\right\} \\
dC_t &= (y - r)\,P_t dt + C_{t^-}\left\{[m(\mu - y) + y]\,dt + m\sigma dW_t + mdY_t\right\}
\end{aligned}
$$

And the differential equation of the cushion becomes

$$
\begin{cases}
dC_t - C_{t^-}d\tilde{D}_t = (y - r)\,P_t dt = d\tilde{P}_t \\
C_0 = v - P_0 = v - Ge^{-rT}
\end{cases}
$$

where

$$
d\tilde{D}_t = [m(\mu - y) + y]\,dt + m\sigma dW_t + mdY_t
$$

The strategy of the CPPI indicates, once the violation occurs, all the money should be withdrawn from the risky asset and immediately invested in the non-risky asset, i.e. the cushion process stops at the time when the floor is broken through. A loss occurs, if for some $t \in \mathbb{R}^+$, $V_t < P_t$, which is equivalent to the event $C_t < 0$. Set $\tau := inf\{t > 0 : C_t < 0\}$. In this case we can adjust the above equation according to the characteristics of CPPI strategy as follows.

$$\begin{cases} dC_t - C_{t^-} d\tilde{D}_t = (y - r) P_t dt = d\tilde{P}_t, & if \ t < \tau \\ C_t = C_\tau, & if \ t \geq \tau \\ C_0 = v - P_0 = v - Ge^{-rT} \end{cases}$$

Or equivalently in terms of stochastic integrals,

$$\begin{cases} C_t - C_0 = \int_0^t d\tilde{P}_s + \int_0^t C_{s^-} d\tilde{D}_s, & if \ t < \tau \\ C_t = C_\tau, & if \ t \geq \tau \\ C_0 = v - P_0 = v - Ge^{-rT} \end{cases} \tag{4.5}$$

We can see from SDE (4.5) that the cushion is driven by a bivariate Lévy Process $\{\tilde{D}_t, \tilde{P}_t\}_{t \geq 0}$. In order to solve the SDE (4.5), we will come across quadratic covariation between different processes. Therefore, preliminary knowledge before the proof is hereby given as follows. The solution to the SDE is derived in Theorem 4.2.1.

The result of the quadratic covariation of each process can be heuristically derived and is shown in the table below [Etheridge, 2002].

$\times$	dt	dW_t	dN_t
dt	0	0	0
dW_t	0	dt	0
dN_t	0	0	dN_t

Table 4.1: Quadratic covariation for time, Brownian motion, and counting process

Next we provide the solution to our SDE (4.5) with the assistance of the above preliminaries.

Theorem 4.2.1. *$\tilde{D}$ and $\tilde{P}$ followed from Equation* (4.5) *are both Lévy processes. Define the stopping time as*

$$\tau := \inf\{t > 0 : C_t \leq 0\}$$

and

$$\tau^* := min\{\tau, T\}$$

then the unique solution to equation (4.5) *is given by*

$$C_t = C_t^a \mathbb{1}_{(0,\tau^*]}(t) + C_t^b \mathbb{1}_{(\tau^*,T]}(t), \quad t > 0, \tag{4.6}$$

where

$$
\begin{aligned}
C_t^a &= \mathcal{E}(\tilde{D})_t \left(C_0 + \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \right) \\
&= \mathcal{E}(\tilde{D})_t \left(C_0 + G(y-r)e^{-rT} \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} e^{rs} ds \right) \\
C_t^b &= \mathcal{E}(\tilde{D})_\tau \left(C_0 + \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \right) \\
&= \mathcal{E}(\tilde{D})_\tau \left(C_0 + G(y-r)e^{-rT} \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} e^{rs} ds \right)
\end{aligned}
$$

Proof Under the structure of CPPI, once the value of the portfolio at time t drops under the floor, the investment in the risky asset is then terminated. In this case of $t > \tau$, the value of the cushion shall remain to be $C_t = C_\tau$. Therefore in this manner we need to focus on the case when $t \leq \tau$ solely. It is hence to prove the following equation satisfying the SDE (4.5).

$$
C_t = \mathcal{E}(\tilde{D})_t \left(C_0 + \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \right) \tag{4.7}
$$

Hereby we write $C_t = C_t^1 C_t^2$, where C_t^1 and C_t^2 are both semimartingales w.r.t. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$

$$
C_t^1 = \mathcal{E}(\tilde{D})_t \;\text{ and }\; C_t^2 = C_0 + \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s
$$

Using the definition of stochastic integration by parts, and the fact that $d\mathcal{E}(\tilde{D})_s = \mathcal{E}(\tilde{D})_{s^-} d\tilde{D}_s$, we have following equations

$$
\begin{aligned}
C_t - C_0 &= \int_0^t C_{s^-}^1 dC_s^2 + \int_0^t C_{s^-}^2 dC_s^1 + \left[C^1, C^2 \right]_t \\
&= \int_0^t \mathcal{E}(\tilde{D})_{s^-} \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \\
&\quad + \int_0^t \left(C_0 + \int_0^s \left[\mathcal{E}(\tilde{D})_{h^-} \right]^{-1} d\tilde{P}_h \right) \mathcal{E}(\tilde{D})_{s^-} d\tilde{D}_s \\
&\quad + \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\left(\left[\mathcal{E}(\tilde{D}), \tilde{P} \right]_s \right) \\
&= \int_0^t d\tilde{P}_s + \int_0^t C_{s^-}^1 C_{s^-}^2 d\tilde{D}_s \\
&= \int_0^t d\tilde{P}_s + \int_0^t C_{s^-} d\tilde{D}_s
\end{aligned}
$$

This proves that (4.7) is a solution to (4.5). And the uniqueness can be proven by Theorem V.7 in [Protter, 2005].

□

We can also prove it by using the result as pointed out in [Maller et al., 2009], a SDE in term of the following form

$$dV_t = V_t dU_t + dL_t, \quad t \geq 0, \tag{4.8}$$

where $\{U, L\}$ is a bivariate Lévy process which is constructed from another pair of Lévy processes $\{\xi, \eta\}$ by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -\xi_t + \sum\limits_{0<s\leq t} (e^{-\Delta\xi_s} - 1 + \Delta\xi_s) + t\sigma_\xi^2/2 \\ \eta_t + \sum\limits_{0<s\leq t} (e^{-\Delta\xi_s} - 1)\Delta\eta_s - t\sigma_{\xi,\eta} \end{pmatrix}, \quad t \geq 0,$$

and $\{\Delta\xi_t, \Delta\eta_t\} = \{\xi_t - \xi_{t-}, \eta_t - \eta_{t-}\}$ represents the jump processes of $\{\xi, \eta\}$ at time t. In addition, there exists an unique solution to the SDE (4.8) which is given as

$$V_t = e^{-\xi_t} \left(V_0 + \int_0^t e^{\xi_{s-}} d\eta_s \right), \quad t \geq 0. \tag{4.9}$$

Equation (4.9) is also called as a *generalized Ornstein-Uhlenbeck process* $\{V_t\}_{t\geq 0}$ driven by the bivariate Lévy process $\{\xi_t, \eta_t\}_{t\geq 0}$. If $\{\eta_t\}_{t\geq 0}$ is a Brownian motion, then we obtain the classical Ornstein-Uhlenbeck process.

By definition of Doléans-Dade exponential, we know it is equivalent to write $\mathcal{E}(U)_t = e^{-\xi_t}$. Hence Equation (4.9) can be written in terms of Doléans-Dade exponential.

$$V_t = \mathcal{E}(U)_t \left(V_0 + \int_0^t [\mathcal{E}(U)_s]^{-1} d\eta_s \right), \quad t \geq 0. \tag{4.10}$$

Here we need to be careful with the modification in (4.10). Doléans-Dade exponential by definition can take non-positive values, but on the contrary $e^{-\xi_t}$ can only take positive values. Therefore the Lévy measure ν oof U should have no mass on $(-\infty, -1]$, i.e. $\nu_U((-\infty, -1]) = 0$.

The above condition is exactly fulfilled by the setup of the CPPI strategy, since cushion dynamics stops evolving once the cushion value becomes non-positive.

Corollary 4.2.1. *Stopping time τ can also be written as*

$$\tau = \inf\{t > 0 : 1 + \Delta\tilde{D}_t \leq 0\}$$

Moreover, by assumption, $C_t = C_\tau$ when $t > \tau$, i.e. the value of cushion stops varying after time τ. In other words, the single jump at time τ that is less or equal to $1 - \frac{1}{m}$ is crucial to the whole risk assessment. The value of the portfolio is additionally presented in the next corollary.

Corollary 4.2.2. *The value of the portfolio is*

$$V_t = V_t^a \mathbb{1}_{(0,\tau^*]}(t) + V_t^b \mathbb{1}_{(\tau^*,T]}(t), \quad t > 0,$$

where

$$V_t^a = P_t + \mathcal{E}(\tilde{D})_t \left(C_0 + \int_0^t \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \right)$$

$$V_t^b = e^{y(t-\tau)} \left[P_\tau + \mathcal{E}(\tilde{D})_\tau \left(C_0 + \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s^-} \right]^{-1} d\tilde{P}_s \right) \right]$$

It is clear to see from either Theorem 4.2.1 or Corollary 4.2.2, unlike the model constructed in the end of Chapter 3, the solution to the cushion process in the Section 4 is likely to be negative, i.e. the floor could be violated in the new set up, which is more realistic than the model proposed in the Chapter 3.

5 | Measuring the Gap Risk

Under the new framework of the model proposed in Section 4.2, CPPI strategy still maintains its capital protection, but not anymore in an absolute sense due to the gap risk. According to the violation of the floor we have the following two scenarios. See for instance: Figure 5.1 and 5.2.

The following risk measure will be discussed respectively: probability, expectation and the variance of loss in the aspect of cushion in Section 5.1, 5.2 and 5.3. In Section 5.4, VaR and CVaR will be investigated by using the *inverse Fourier transform* to acquire the probability density function.

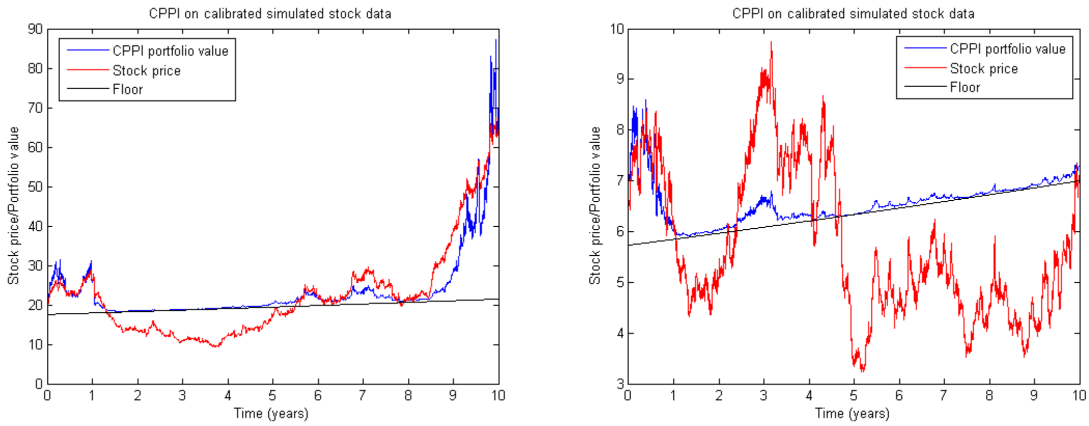

Figure 5.1: Floor not violated: Different stock dynamics (left: up; right: fluctuate) and the performance of CPPI strategy

5.1 Probability of Loss

Corollary 4.2.1 implies that $C_t \leq 0$ if and only if $J_{N_t} \leq 1-\frac{1}{m}$. The jumps of the Lévy process $\tilde{D}$ follows a compound Poisson process with the intensity described by the Lévy measure in time period $(0, T]$. Hence, the probability of loss for the cushion can be interpreted in the proposition below, where the jump size of $1-\frac{1}{m}$ is excluded for general case.

However, the exclusion of $1-\frac{1}{m}$ with respect to the models mentioned in Section 4.1 is not necessary, since the jump size follows continuous probability distribution.

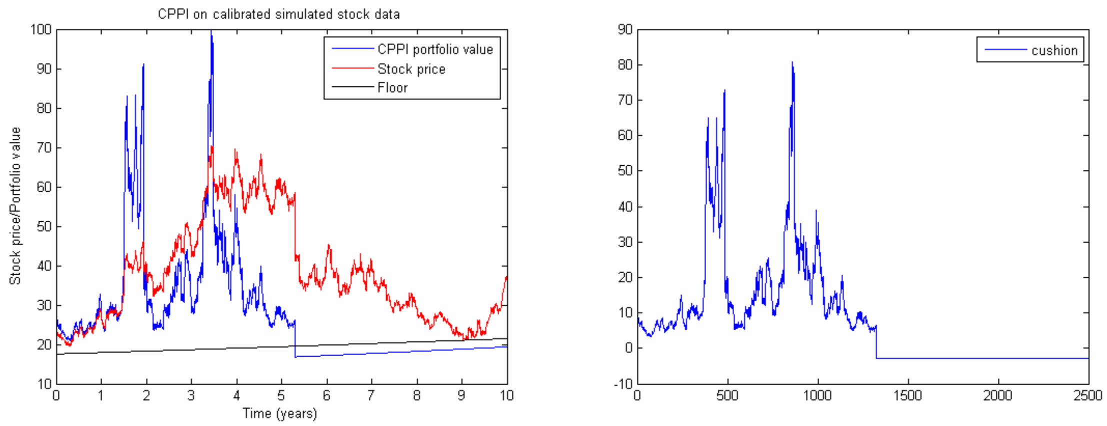

Figure 5.2: Floor violated: Dynamics of portfolio value (left) and cushion value when the floor is broken

Proposition 5.1.1. *The cushion-driven-dynamic $\tilde{D}$ is a Lévy process with Lévy measure ν, where the jump process is independent of the continuous one. The probability of loss for the cushion is*

$$\mathbb{P}\left(C_t < 0, t \in (0, T]\right) = 1 - e^{-T\lambda \mathbb{P}\left(J_{N_t} < 1 - \frac{1}{m}\right)}$$

Proof The following proof will be given based on the two equivalent events:

$$C_t < 0 \quad \text{if and only if} \quad \Delta \tilde{D}_t < -1$$

Let ω be the first hitting time of $C_t < 0$, $t \in (0, T]$, and the cumulative density function of ω is $F_W(\omega)$. Moreover, V is a counting process which counts the number of times of the event $\{\Delta \tilde{D}_t < -1\}$ occurring in time period $(0, T]$, and V $\sim$ Poisson$(T\nu\left((-\infty, -1)\right))$.

$$\begin{aligned}
F_W(\omega) &= F\left(W < \omega\right) \\
&= 1 - \mathbb{P}\left(W \geq \omega\right) \\
&= 1 - \mathbb{P}\left(V = 0\right) \\
&= 1 - e^{-T\nu((-\infty,-1))} \\
&= 1 - e^{-T\lambda \mathbb{P}\left(J_{N_t} < 1 - \frac{1}{m}\right)}
\end{aligned}$$

$\square$

Note that the probability of loss of the portfolio is the same with the probability of loss of cushion only when $y = r$. Since y is set to be greater than r, it is possible that the following event occurs: the portfolio value falls under the floor, nevertheless the loss is so little that at maturity the yield from the non-risky investment "saves" the portfolio, see for instance Figure 5.3. In this case, the probability of loss of the portfolio should be considered by using another floor which is discounted by y as a threshold (brown line in Figure 5.3), i.e. $\mathbb{P}\left(C_t < B_t - P_t, t \in (0, T]\right)$.

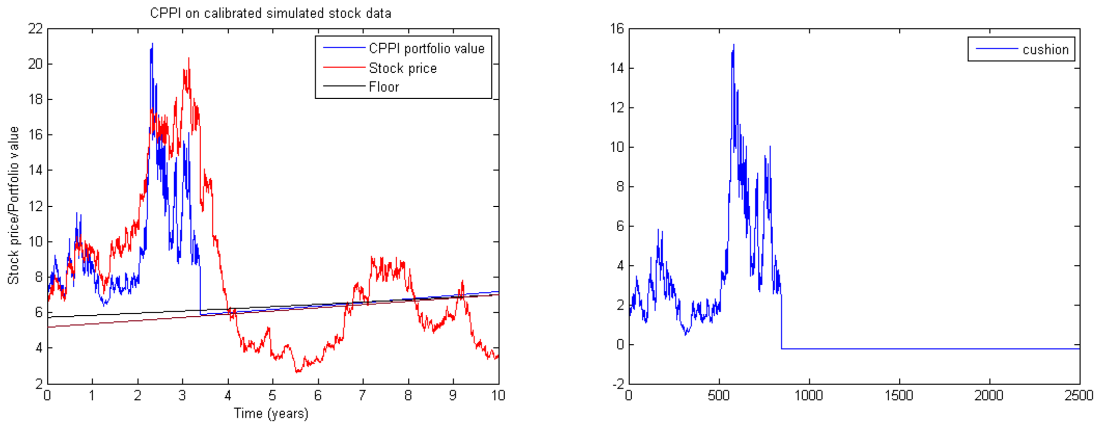

Figure 5.3: Floor violated: Dynamics of portfolio value (left) and cushion value when the floor is broken

However, under the framework of $y \neq r$, $\{B_t - P_t\}$ is negative, $\forall t$, which indicates that every jump size could be the cause for a negative cushion depending on the previous cushion value. Nevertheless, the probability in Proposition 5.1.1 is still an upper bound for the probability of loss of the portfolio.

5.2 Expectation of Loss

For the computation of the expectation of loss, we make use of the important fact that $\tilde{D}_t - t\mathrm{E}[\tilde{D}_1]$ is a martingale by Proposition 2.4.2 and the preliminaries from Chapter 2. Before the explicit solution is derived, we present two required lemmas.

Lemma 5.2.1. *Let $\{X_s\}_{s\geq 0}$ be a Lévy process and $\{Y_s\}_{s\geq 0}$ an adapted, càdlàg process. If $\mathrm{E}[|X_1|] < \infty$ and $\mathrm{E}\left[\sup_{0<s\leq 1} |Y_s|\right] < \infty$, then for $t > 0$ we have*

$$\mathrm{E}\left[\int_0^t Y_{s^-} dX_s\right] = \mathrm{E}[X_1] \int_0^t \mathrm{E}[Y_{s^-}] ds$$

Proof Proposition 2.4.2 shows that $X_s - s\mathrm{E}[X_1]$ is a martingale. Let Z represents the stochastic integral

$$Z_t := \int_0^t Y_{s^-}(dX_s - s\mathrm{E}[X_1])$$

Since $\mathrm{E}\left[\sup_{0<s\leq 1} |Y_s|\right] < \infty$, Z is also a martingale by dominated convergence theorem. According to the definition of martingale we know $\mathrm{E}[|Z_t|] < \infty$, $\forall t \geq 0$, which indicates that $\{Z_t\}_{t\geq 0}$ is uniformly integrable. Thus we have,

$$\mathrm{E}\left[\int_0^t Y_{s^-} dX_s\right] = \mathrm{E}[X_1]\mathrm{E}\left[\int_0^t Y_{s^-} ds\right]$$

By applying Fubini's theorem we finish the proof. □

In the following proposition we explore the relation between the stochastic integral and the ordinary one.

Proposition 5.2.1. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process with Lévy triplet $\left(\mu, \sigma^2, \nu\right)$ and $Z_t = \mathcal{E}(X)_t$. If $Z > 0$ a.s. then there exists a Lévy process $\{\tilde{X}_t\}_{t\geq 0}$ such that $Z_t = \mathcal{E}(X)_t = e^{\tilde{X}_t}$ where*

$$\tilde{X}_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0\leq s\leq t} \left[\ln|1+\Delta X_s|) - \Delta X_s\right]$$

Its Lévy triplet $\left(\mu, \sigma^2, \nu\right)$ is given by

$$\begin{aligned}
&\tilde{\sigma}^2 = \sigma^2 \\
&\tilde{\nu}(A) = \nu\left(\{x : \ln|1+x| \in A\}\right) = \int \mathbb{1}_A\left(\ln|1+x|\right)\nu\left(dx\right) \\
&\tilde{\mu} = \mu - \frac{\sigma^2}{2} + \int \{\ln|1+x|\mathbb{1}_{[-1,1]}\left(\ln|1+x|\right) - x\mathbb{1}_{[-1,1]}\left(x\right)\}\nu\left(dx\right)
\end{aligned}$$

Proof See [Cont and Tankov, 2004], Proposition 8.22.

Next proposition and remark we move forward with respect to the last lemma into the characteristics of Doléans-Dade exponential, in which we discuss its first two moments.

Proposition 5.2.2. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process generated by Lévy triplet (μ, σ^2, ν) with $\nu(\{\Delta X \leq -1\}) = 0$ and $\gamma \in \mathbb{N}$. Then $\mathrm{E}[\mathcal{E}(X)_t^\gamma] < \infty$ if and only if $\mathrm{E}[|X_1|^\gamma] < \infty$.*

Proof Based on Proposition 5.2.1, there exists a Lévy process such that $\mathcal{E}(X)_t^\gamma = e^{\gamma\tilde{X}_t}$, where $\left(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\nu}\right)$ is the Lévy triplet for $\tilde{X}$. By Proposition 1 [Eberlein, 2009],

$$\mathrm{E}[e^{\gamma\tilde{X}_t}] < \infty \text{ if and only if } \int_{|x|>1} e^{\gamma x}\tilde{\nu}(dx) < \infty$$

Thus we need to prove $\int_{|x|>1} e^{\gamma x}\tilde{\nu}(dx) < \infty$ is a necessary and sufficient condition of $\mathrm{E}[|X_1|^\gamma] < \infty$. From Proposition 5.2.1, we know $\int_{|x|>1} e^{\gamma x}\tilde{\nu}(dx)$ is equivalent to $\int_A |1+x|^\gamma \nu\left(dx\right)$, where $A = \{x : |\ln|1+x|| > 1\}$. According to Example 25.12 [Sato, 2005], the later integral is finite if and only if X_t^γ has finite mean, $\forall t > 0$. And by Proposition 2.4.2, the proof is complete. □

Remark 5.2.1. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process generated by Lévy triplet (μ, σ^2, ν) with $\nu(\{\Delta X \leq -1\}) = 0$. The first and second moments of its Doléans-Dade exponential are written as follows.*

$$\mathrm{E}[\mathcal{E}(X)_t] = e^{t\mathrm{E}[X_1]} \tag{5.1}$$

$$\mathrm{E}[\mathcal{E}(X)_t)^2] = e^{t(\mathrm{var}(X_1)+2\mathrm{E}[X_1])} \tag{5.2}$$

Proof Applying Lemma 5.2.1 on the definition of Doléans-Dade exponential implies

$$\mathrm{E}[\mathcal{E}(X)_t] = 1 + \mathrm{E}[X_1] \int_0^t \mathrm{E}[\mathcal{E}(X)_s] ds \tag{5.3}$$

After differentiating both sides with respect to t, we have

$$\frac{d\mathrm{E}[\mathcal{E}(X)_t]}{dt} = E[X_1]\mathrm{E}[\mathcal{E}(X)_t]$$

Since by definition, $\mathcal{E}(X)_0 = 1$, a.s., The first moment (5.1) is proved.
The second moment will be proved by using stochastic integration by parts and the associativity between stochastic integrals.

$$\begin{aligned}
\mathcal{E}(X)_t^2 &= 1 + 2\int_0^t \mathcal{E}(X)_{s^-} dX_s + \left[\int_0^\cdot \mathcal{E}(X)_{s^-} dX_s, \int_0^\cdot \mathcal{E}(X)_{s^-} dX_s\right]_t \\
&\quad + 2\int_0^t \left(\int_0^s \mathcal{E}(X)_{u^-} dX_u\right) d\left(\int_0^s \mathcal{E}(X)_{u^-} dX_u\right) \\
&= 1 + 2\int_0^t \mathcal{E}(X)_{s^-} dX_s + \int_0^t (\mathcal{E}(X)_{s^-})^2 \, d[X,X]_s \\
&\quad + 2\int_0^t (\mathcal{E}(X)_{s^-} - 1)\, \mathcal{E}(X)_{s^-} dX_s \\
&= 1 + \int_0^t (\mathcal{E}(X)_{s^-})^2 \, d[X,X]_s + 2\int_0^t (\mathcal{E}(X)_{s^-})^2 \, dX_s
\end{aligned}$$

Again by applying Lemma 5.2.1, we have,

$$\mathrm{E}[(\mathcal{E}(X)_t)^2] = 1 + (\mathrm{E}\,[X,X]_1 + 2\mathrm{E}[X_1]) \int_0^t \mathrm{E}\left[(\mathcal{E}(X)_{s^-})^2\right] ds$$

Thus by differentiation on the both side of the above equation implies

$$\frac{d\mathrm{E}[(\mathcal{E}(X)_t)^2]}{dt} = (\mathrm{E}\,[X,X]_1 + 2\mathrm{E}[X_1])\, \mathrm{E}[(\mathcal{E}(X)_t)^2]$$

Analogous to the first moment, the second moment is written as

$$\mathrm{E}[(\mathcal{E}(X)_t)^2] = e^{t(\mathrm{E}[X,X]_1 + 2\mathrm{E}[X_1])}$$

Since by integration by parts $\mathrm{E}\,[X,X]_1 = \mathrm{E}[X_1^2] - 2\mathrm{E}\left[\int_0^1 X_{s^-} dX_s\right]$. And by Proposition 2.4.1, $\mathrm{E}\,[X,X]_1 = \mathrm{E}[X_1^2] - 2\mathrm{E}\,[X_1] \int_0^1 s\mathrm{E}\,[X_1]\, ds = \mathrm{var}(X_1)$. Hence Equation (5.2) is obtained. □

Continue with the notation in Theorem 4.2.1, the expectation of loss is derived as follows.

Proposition 5.2.3. *The expectation of loss of the cushion is*

$$\begin{aligned}
\mathrm{E}\left[C_T \mathbb{1}_{\{\tau \le T\}}\right] = \lambda^* \left(1 + \mathrm{E}[\Delta \tilde{D}_\tau]\right) \Bigg(& C_0 \frac{e^{(\mathrm{E}[\tilde{D}_1] - \lambda^*)T} - 1}{\mathrm{E}[\tilde{D}_1] - \lambda^*} + G(y-r) \times \\
& \frac{(\mathrm{E}[\tilde{D}_1] - \lambda^*) e^{(r-\lambda^*)T} - (r - \lambda^*) e^{(\mathrm{E}[\tilde{D}_1] - \lambda^*)T} + (r - \mathrm{E}[\tilde{D}_1])}{e^{rT}(r - \mathrm{E}[\tilde{D}_1])(r - \lambda^*)(\mathrm{E}[\tilde{D}_1] - \lambda^*)} \Bigg),
\end{aligned}$$

where λ^* *is the intensity of jump size below* -1*, i.e.* $\lambda^* := \nu\left((-\infty,-1)\right) = \lambda \int_{-\infty}^{1-\frac{1}{m^-}} \pi(dh)$*. And the expected jump of cushion-driven-dynamic given in crucial timing* τ *is denoted by* $\mathrm{E}[\Delta \tilde{D}_\tau] = \int_{-\infty}^{1-\frac{1}{m^-}} m(h-1)\pi(dh)$

Proof Recall C_T in Equation (4.6). It can be further formulated as

$$\begin{aligned} C_T \mathbb{1}_{\{\tau \le T\}} &= \mathcal{E}(\tilde{D})_\tau \left(C_0 + \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s-}\right]^{-1} d\tilde{P}_s \right) \mathbb{1}_{\{\tau \le T\}} \\ &= \mathcal{E}(\tilde{D})_{\tau-} \left(C_0 + \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s-}\right]^{-1} d\tilde{P}_s \right) (1+\Delta\tilde{D}_\tau)\mathbb{1}_{\{\tau \le T\}} \end{aligned}$$

Notice that the event $\{\tau \le T\}$ is equivalent to the event $\{C_t < 0, t \in (0,T]\}$, and according to Proposition 5.2.2 the expectation of loss is

$$\begin{aligned} \mathrm{E}[C_T \mathbb{1}_{\{\tau \le T\}}] =& \mathrm{E}\left[\mathcal{E}(\tilde{D})_{\tau-} \left(C_0 + \int_0^\tau \left[\mathcal{E}(\tilde{D})_{s-}\right]^{-1} d\tilde{P}_s \right) \mathbb{1}_{\{\tau \le T\}}\right] \times \\ & \mathrm{E}[1+\Delta\tilde{D}_\tau] \\ =& \left(1+\mathrm{E}[\Delta\tilde{D}_\tau]\right)\Bigg(C_0 \mathrm{E}[\mathcal{E}(\tilde{D})_{\tau-}\mathbb{1}_{\{\tau \le T\}}] + \\ & G(y-r)e^{-rT}\mathrm{E}\left[\mathcal{E}(\tilde{D})_{\tau-}\int_0^\tau \left[\mathcal{E}(\tilde{D})_{s-}\right]^{-1} e^{rs} ds \mathbb{1}_{\{\tau \le T\}}\right]\Bigg) \\ =& \left(1+\mathrm{E}[\Delta\tilde{D}_\tau]\right)\Bigg(C_0 \int_0^T \mathrm{E}[\mathcal{E}(\tilde{D})_{t-}]\lambda^* e^{-\lambda^* t} dt + \\ & G(y-r)e^{-rT}\int_0^T \mathrm{E}\left[\mathcal{E}(\tilde{D})_{t-}\int_0^t \left[\mathcal{E}(\tilde{D})_{s-}\right]^{-1} e^{rs} ds\right]\lambda^* e^{-\lambda^* t} dt\Bigg) \end{aligned}$$

The first integral above is easy to tackle, but for the calculation of the second integral, we need to handle it with the help of Lemma 5.2.1, Theorem 2.2.2 and a few knowledge of integration equations [Polyanin and Manzhirov, 1998]. First we rewrite the expected value in the following integral:

$$X_t := \mathrm{E}\left[\mathcal{E}(\tilde{D}_{t-})\int_0^t \left[\mathcal{E}(\tilde{D}_{s-})\right]^{-1} e^{rs} ds\right] = \frac{e^{rt}-1}{r} + \mathrm{E}[\tilde{D}_1]\int_0^t X_s ds$$

Solving the integral equation which has a form of *Volterra integral equations of the second kind*, we have,

$$X_t = \frac{e^{rt}-1}{r} + \mathrm{E}[\tilde{D}_1]\int_0^t e^{\mathrm{E}[\tilde{D}_1](t-s)}\left(\frac{e^{rs}-1}{r}\right) ds$$

Combining the results together, the expectation of loss of the cushion is attained as follows.

$$\begin{aligned}
\mathrm{E}[C_T \mathbb{1}_{\{\tau\leq T\}}] &= \lambda^* \left(1+\mathrm{E}[\Delta \tilde{D}_\tau]\right) \left(C_0 \int_0^T e^{(\mathrm{E}[\tilde{D}_1]-\lambda^*)t} dt + \right. \\
&\qquad \left. G(y-r)e^{-rT} \int_0^T X_t e^{-\lambda^* t} dt \right) \\
&= \lambda^* \left(1+\mathrm{E}[\Delta \tilde{D}_\tau]\right) \left(C_0 \frac{e^{(\mathrm{E}[\tilde{D}_1]-\lambda^*)T}-1}{\mathrm{E}[\tilde{D}_1]-\lambda^*} + G(y-r)\times \right. \\
&\qquad \left. \frac{(\mathrm{E}[\tilde{D}_1]-\lambda^*)e^{(r-\lambda^*)T} - (r-\lambda^*)e^{(\mathrm{E}[\tilde{D}_1]-\lambda^*)T} + (r-\mathrm{E}[\tilde{D}_1])}{e^{rT}(r-\mathrm{E}[\tilde{D}_1])(r-\lambda^*)(\mathrm{E}[\tilde{D}_1]-\lambda^*)} \right)
\end{aligned}$$

□

Corollary 5.2.1. *The conditional expectation for the value of the cushion given that the floor is broken through is*

$$\begin{aligned}
\mathrm{E}\left[C_T \mid \tau \leq T\right] = \frac{\lambda^* \left(1+\mathrm{E}[\Delta \tilde{D}_\tau]\right)}{1-e^{-T\lambda \int_{-\infty}^{1-\frac{1}{m^-}} \pi(dh)}} &\left(C_0 \frac{e^{(\mathrm{E}[\tilde{D}_1]-\lambda^*)T}-1}{\mathrm{E}[\tilde{D}_1]-\lambda^*} + G(y-r)\times \right. \\
&\left. \frac{(\mathrm{E}[\tilde{D}_1]-\lambda^*)e^{(r-\lambda^*)T} - (r-\lambda^*)e^{(\mathrm{E}[\tilde{D}_1]-\lambda^*)T} + (r-\mathrm{E}[\tilde{D}_1])}{e^{rT}(r-\mathrm{E}[\tilde{D}_1])(r-\lambda^*)(\mathrm{E}[\tilde{D}_1]-\lambda^*)} \right)
\end{aligned}$$

Proof It is a direct consequence from Proposition 5.2.3 and 5.1.1. □

5.3 Variance of Loss

Computing the variance involves dealing with the square term, which will be handled analogously as the procedure of deriving the expected value, but yet more complex. Hence, the previous section can be seen as preliminaries for the derivation in this section. We continue with the same notation as before, and start with a few required theorems before giving out the variance of the loss.

Theorem 5.3.1. *Let X and Y be two semimartingales with $X_0 = 0$ and $Y_0 = 0$. Then the product of their own Doléans-Dade exponentials is*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y+[X,Y])$$

Proof See [Protter, 2005], Theorem II.38. □

Secondly, we investigate the characteristics of the inverse of Doléans-Dade exponential as an extension of Proposition 5.2.2 and Remark 5.2.1.

Proposition 5.3.1. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process generated by Lévy triplet (μ, σ^2, ν) with $\nu(\{\Delta X \leq -1\}) = 0$. Suppose that $U_t = -X_t + [X,X]_t^c + \sum_{0<s\leq t} (1+\Delta X_s)^{-1}(\Delta X_s)^2$, then*

(a) $\mathcal{E}(U)_t = (\mathcal{E}(X)_t)^{-1}$

(b) The first moment of $\mathcal{E}(U)_t$ *follows Remark 5.2.1, where*

$$\mathrm{E}[U_1] = -\mu + \sigma^2 + \int_{[-1,1]} \frac{x^2}{1+x}\nu(dx) - \int_{|x|>1} \frac{x}{1+x}\nu(dx)$$

Proof

(a) Denote the jump of U_t as $J_t = \sum\limits_{0<s\leq t} (1+\Delta X_s)^{-1}(\Delta X_s)^2$.

Let $A = \mathcal{E}(U)$, $B = \mathcal{E}(X)$, then $A_0 = B_0 = 1$. We will prove that $AB = 1$. According to stochastic integration by parts,

$$\begin{aligned} d(AB) =& A_- dB + B_- dA + d[A,B] \\ =& A_- B_- dX + B_- A_- dU + A_- B_- d[X,U] \\ =& A_- B_- dX + B_- A_- (-dX + d[X,X]^c + dJ) \\ &+ A_- B_- (-d[X,X] + d[X,[X,X]^c] + d[J,X]) \\ =& A_- B_- (d[X,X]^c + dJ - d[X,X] + d[J,X]) \end{aligned}$$

By definition of quadratic covariation, we have

$$\begin{aligned} d[X,X] =& [X,X]^c + \sum_{0<s\leq t} (\Delta X_s)^2 \\ d[J,X] =& \sum_{0<s\leq t} \Delta J_s \Delta X_s = \sum_{0<s\leq t} (1+\Delta X_s)^{-1}(\Delta X_s)^3 \end{aligned}$$

We proved that $d(AB) = 0$, $\forall t \geq 0$. With the initial condition the result of (a) follows.

(b) Similarly as the proof in [Cont and Tankov, 2004], Proposition 8.22, we are able to write $\Delta U = \dfrac{-\Delta X}{1+\Delta X}$, which fulfills the condition in Remark 5.2.1 since $\nu(\Delta X \leq -1) = 0$. Therefore, Equation (5.1) and (5.2) also apply to U.

Next, X and U are separated into jump part and continuous part by the Lévy-Itô decomposition, and then we insert them into the assumption, in which both parts should coincide with each other. Since the Brownian motion parts for U and $-X$ are the same, for the drift part we obtain,

$$\begin{aligned} \mu_u t &= -\mu t + \sigma^2 t + t\int_{|x|\leq 1} x\nu(dx) + t\int_{|x|\leq 1} x\nu_u(dx) \\ &= -\mu t + \sigma^2 t + t\int_{\mathbb{R}} \left(x\mathbb{1}_{\{|x|\leq 1\}} - \frac{x}{1+x}\mathbb{1}_{\{x\geq -\frac{1}{2}\}} \right) \nu(dx) \end{aligned}$$

Let $t = 1$. Since $\mathrm{E}[U_1] = \mu_u + \int_{|x|>1} x\nu_u(dx) = \mu_u - \int_{x<-\frac{1}{2}} \frac{x}{1+x}\nu(dx)$, we have

$$\mathrm{E}[U_1] = -\mu + \sigma^2 + \int_{\{|x|\leq 1\}} \frac{x^2}{1+x}\nu(dx) - \int_{|x|>1} \frac{x}{1+x}\nu(dx)$$

□

Now we are ready to derive the variance of the loss with help of the above preliminary.

Proposition 5.3.2. *The variance of the loss of the cushion is*

$$\mathrm{Var}(C_T \mathbb{1}_{\tau<T}) = \mathrm{E}\left[(C_T \mathbb{1}_{\tau<T})^2\right] - \mathrm{E}\left[C_T \mathbb{1}_{\{\tau<T\}}\right]^2,$$

where

$$\begin{aligned}
\mathrm{E}\left[(C_T \mathbb{1}_{\tau<T})^2\right] =& \lambda^* \left(1 + 2\mathrm{E}[\Delta\tilde{D}_\tau] + \mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right) \left\{ \frac{C_0^2 \left(e^{(\mathrm{E}[X_1]-\lambda^*)T} - 1\right)}{(\mathrm{E}[X_1]-\lambda^*)} + \right. \\
& \frac{2C_0 G e^{-rT}(y-r)}{\mathrm{E}[V_1]-\mathrm{E}[X_1]} \left[\frac{e^{(\mathrm{E}[V_1]-\lambda^*)T} - 1}{\mathrm{E}[V_1]-\lambda^*} - \frac{e^{(\mathrm{E}[X_1]-\lambda^*)T} - 1}{\mathrm{E}[X_1]-\lambda^*} \right] + \\
& G^2 e^{-2rT} \left[\frac{(2a - \mathrm{E}[X_1])\left(e^{(a-\lambda^*)T} - 1\right)}{a(a-\mathrm{E}[V_1])(a-\mathrm{E}[X_1])(a-\lambda^*)} - \right. \\
& \frac{(2\mathrm{E}[V_1] - \mathrm{E}[X_1])\left(e^{(\mathrm{E}[V_1]-\lambda^*)T} - 1\right)}{\mathrm{E}[V_1](a-\mathrm{E}[V_1])(\mathrm{E}[V_1]-\mathrm{E}[X_1])(\mathrm{E}[V_1]-\lambda^*)} + \\
& \left.\left. \frac{1 - e^{-\lambda^* T}}{a\lambda^* \mathrm{E}[V_1]} + \frac{e^{(\mathrm{E}[X_1]-\lambda^*)T} - 1}{(a-\mathrm{E}[X_1])(\mathrm{E}[V_1]-\mathrm{E}[X_1])(\mathrm{E}[X_1]-\lambda^*)} \right] \right\}
\end{aligned}$$

$\mathrm{E}\left[C_T \mathbb{1}_{\{\tau<T\}}\right]$ *is as given in Proposition 5.2.3, where the notation in the above formula is presented as follows:*

$$\begin{aligned}
X_t &= 2\tilde{D}_t + [\tilde{D}, \tilde{D}]_t \\
U_t &= -\tilde{D}_t + \sigma^2 t + \sum_{0<s\le t} \frac{(\Delta\tilde{D}_t)^2}{1+\Delta\tilde{D}_s} \\
V_t &= X_t + U_t + [X, U]_t + rt \\
a &= \mathrm{E}[V_1] + \mathrm{E}[U_1] + \mathrm{E}([V, U]_1) + r
\end{aligned}$$

Proof With the result of Proposition 5.2.3, the variance of the loss can be obtained by computing $\mathrm{E}\left[\left(C_T \mathbb{1}_{\{\tau<T\}}\right)^2\right]$ additionally.

$$\begin{aligned}
\left(C_T \mathbb{1}_{\{\tau<T\}}\right)^2 =& \left[\mathcal{E}(\tilde{D})_\tau \left(C_0 + (y-r) \int_0^\tau \left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1} P_s ds \right) \mathbb{1}_{\{\tau<T\}} \right]^2 \\
=& C_0^2 \left(\mathcal{E}(\tilde{D})_\tau\right)^2 \mathbb{1}_{\{\tau<T\}} + \\
& 2C_0(y-r) \left(\mathcal{E}(\tilde{D})_\tau\right)^2 \int_0^\tau \left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1} P_s ds \mathbb{1}_{\{\tau<T\}} + \\
& \left(\mathcal{E}(\tilde{D})_\tau\right)^2 \left(\int_0^\tau \left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1} P_s ds \right)^2 \mathbb{1}_{\{\tau<T\}} \qquad (5.4)
\end{aligned}$$

For convenience, the expected value of $\left(C_T \mathbb{1}_{\{\tau<T\}}\right)^2$ will hereby be calculated step by step

by separating itself into three parts as (5.4). First of all, we deal with

$$\begin{aligned}\mathrm{E}\left[C_0^2\left(\mathcal{E}(\tilde{D})_\tau\right)^2 \mathbb{1}_{\{\tau<T\}}\right] =& C_0^2\mathrm{E}\left[\left(\mathcal{E}(\tilde{D})_{\tau^-}\right)^2\left(1+\Delta\tilde{D}_\tau\right)^2 \mathbb{1}_{\{\tau<T\}}\right]\\ =&\lambda^* C_0^2\mathrm{E}\left[1+2\Delta\tilde{D}_\tau+(\Delta\tilde{D}_\tau)^2\right]\int_0^T e^{t(\mathrm{E}[X_1]-\lambda^*)}dt\\ =&\lambda^* C_0^2\mathrm{E}\left[1+2\Delta\tilde{D}_\tau+(\Delta\tilde{D}_\tau)^2\right]\frac{e^{(\mathrm{E}[X_1]-\lambda^*)T}-1}{\mathrm{E}[X_1]-\lambda^*},\end{aligned}$$

Secondly,

$$\begin{aligned}&\mathrm{E}\left[2C_0(y-r)(\mathcal{E}(\tilde{D})_\tau)^2\int_0^\tau\left(\mathcal{E}(\tilde{D})_{s^-}\right)^{-1}P_s ds\mathbb{1}_{\{\tau<T\}}\right]\\ =&2C_0(y-r)\left(1+2\mathrm{E}[\Delta\tilde{D}_\tau]+\mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right)\times\\ &\mathrm{E}\left[(\mathcal{E}(\tilde{D})_{\tau^-})^2\int_0^\tau\left(\mathcal{E}(\tilde{D})_{s^-}\right)^{-1}P_s ds\mathbb{1}_{\{\tau<T\}}\right]\\ =&2C_0Ge^{-rT}(y-r)\left(1+2\mathrm{E}[\Delta\tilde{D}_\tau]+\mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right)\times\\ &\int_0^T\mathrm{E}\left[(\mathcal{E}(\tilde{D})_{t^-})^2\int_0^t\left(\mathcal{E}(\tilde{D})_{s^-}\right)^{-1}e^{rs}ds\right]\lambda^*e^{-\lambda^*t}dt\end{aligned}$$

From Theorem 5.3.1 we can rewrite the expected value of the above integral as

$$Y_t := \mathrm{E}\left[\mathcal{E}(X)_{t^-}\int_0^t\left(\mathcal{E}(\tilde{D})_{s^-}\right)^{-1}e^{rs}ds\right],$$

Y_t can be furthermore formulated into

$$\begin{aligned}Y_t =& \int_0^t e^{s\mathrm{E}[V_1]}ds+\mathrm{E}[X_1]\int_0^t Y_s ds\\ =&\frac{e^{t\mathrm{E}[V_1]}-1}{\mathrm{E}[V_1]}+\mathrm{E}[X_1]\int_0^t Y_s ds\end{aligned}$$

The calculation of the integral equation will be handled analogously as the procedure when deriving the expected value in Proposition 5.2.3. Hence,

$$Y_t = \frac{e^{t\mathrm{E}[V_1]}-1}{\mathrm{E}[V_1]}+\mathrm{E}[X_1]\int_0^t e^{\mathrm{E}[X_1](t-s)}\left(\frac{e^{s\mathrm{E}[V_1]}-1}{\mathrm{E}[V_1]}\right)ds$$

Therefore we have,

$$\begin{aligned}&\mathrm{E}\left[2C_0(y-r)\left(\mathcal{E}(\tilde{D})_\tau\right)^2\int_0^\tau\left(\mathcal{E}(\tilde{D})_{s^-}\right)^{-1}P_s ds\mathbb{1}_{\{\tau<T\}}\right]\\ =&\frac{2\lambda^*C_0Ge^{-rT}(y-r)\left(1+2\mathrm{E}[\Delta\tilde{D}_\tau]+\mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right)}{\mathrm{E}[V_1]-\mathrm{E}[X_1]}\times\\ &\left(\frac{e^{(\mathrm{E}[V_1]-\lambda^*)T}-1}{\mathrm{E}[V_1]-\lambda^*}-\frac{e^{(\mathrm{E}[X_1]-\lambda^*)T}-1}{\mathrm{E}[X_1]-\lambda^*}\right)\end{aligned}$$

And similarly,

$$\begin{aligned}
&\mathrm{E}\left[\left(\mathcal{E}(\tilde{D})_\tau\right)^2\left(\int_0^\tau\left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1}P_s ds\right)^2 \mathbb{1}_{\{\tau<T\}}\right]\\
=&\lambda^* G^2 e^{-2rT}\left(1+2\mathrm{E}[\Delta\tilde{D}_\tau]+\mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right)\times\\
&\int_0^T \mathrm{E}\left[\mathcal{E}(X)_{t-}\left(\int_0^t\left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1}e^{rs}ds\right)^2\right]e^{-\lambda^* t}dt
\end{aligned}$$

Let $Z_t := \mathrm{E}\left[\mathcal{E}(X)_{t-}\left(\int_0^t\left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1}e^{rs}ds\right)^2\right]$, we have

$$\begin{aligned}
Z_t =&\mathrm{E}\left[2\int_0^t \mathcal{E}(X)_{s-}\int_0^s\left(\mathcal{E}(\tilde{D})_{u-}\right)^{-1}e^{ru}du\left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1}e^{rs}ds+\right.\\
&\left.\int_0^t\left(\int_0^s\left(\mathcal{E}(\tilde{D})_{u-}\right)^{-1}e^{ru}du\right)^2\mathcal{E}(X)_{s-}dX_s\right]\\
=&2\int_0^t \mathrm{E}\left[\mathcal{E}(X)_{s-}\left(\mathcal{E}(\tilde{D})_{s-}\right)^{-1}e^{rs}\int_0^s\left(\mathcal{E}(\tilde{D})_{u-}\right)^{-1}e^{ru}du\right]ds+\\
&\mathrm{E}[X_1]\int_0^t Z_s ds\\
=&2\int_0^t \mathrm{E}\left[\mathcal{E}(V)_{s-}\int_0^s\left(\mathcal{E}(\tilde{D})_{u-}\right)^{-1}e^{ru}du\right]ds+\mathrm{E}[X_1]\int_0^t Z_s ds
\end{aligned} \tag{5.5}$$

The expectation in the first integral is exactly the same type as Y_t, therefore the integral of it is

$$\begin{aligned}
\frac{f(t)}{2} :=&\frac{e^{\mathrm{E}[V_1]t}}{\mathrm{E}[V_1]+\mathrm{E}[U_1]+\mathrm{E}([V,U]_1)+r}\times\\
&\left(\frac{e^{(\mathrm{E}[U_1]+\mathrm{E}([V,U]_1)+r)t}-1}{\mathrm{E}[U_1]+\mathrm{E}([V,U]_1)+r}+\frac{e^{-\mathrm{E}[V_1]t}-1}{\mathrm{E}[V_1]}\right)
\end{aligned}$$

Hence, Equation (5.5) is a integral equation as written below:

$$Z_t = f(t)+\mathrm{E}[X_1]\int_0^t Z_s ds$$

Z_t has again the form of Volterra integral equations of the second kind, with which we handle similarly as above, the solution is presented as follows

$$\begin{aligned}
Z_t =&f(t)+\mathrm{E}[X_1]\int_0^t e^{\mathrm{E}[X_1](t-s)}f(s)ds\\
=&\frac{e^{at}(2a-\mathrm{E}[X_1])-(a-\mathrm{E}[X_1])-ae^{\mathrm{E}[X_1]t}}{a(a-\mathrm{E}[V_1])(a-\mathrm{E}[X_1])}-\\
&\frac{e^{\mathrm{E}[V_1]t}(2\mathrm{E}[V_1]-\mathrm{E}[X_1])-(\mathrm{E}[V_1]-\mathrm{E}[X_1])-\mathrm{E}[V_1]e^{\mathrm{E}[X_1]t}}{\mathrm{E}[V_1](a-\mathrm{E}[V_1])(\mathrm{E}[V_1]-\mathrm{E}[X_1])},
\end{aligned}$$

With the result of above,

$$
\begin{aligned}
&\mathrm{E}\left[\left(\mathcal{E}(\tilde{D}_\tau)\right)^2\left(\int_0^\tau\left[\mathcal{E}(\tilde{D}_{s^-})\right]^{-1}P_s ds\right)^2\mathbb{1}_{\{\tau<T\}}\right]\\
=&\lambda^* G^2 e^{-2rT}\left(1+2\mathrm{E}[\Delta\tilde{D}_\tau]+\mathrm{E}[(\Delta\tilde{D}_\tau)^2]\right)\\
&\left[\frac{(2a-\mathrm{E}[X_1])\left(e^{(a-\lambda^*)T}-1\right)}{a(a-\mathrm{E}[V_1])(a-\mathrm{E}[X_1])(a-\lambda^*)}-\right.\\
&\frac{(2\mathrm{E}[V_1]-\mathrm{E}[X_1])\left(e^{(\mathrm{E}[V_1]-\lambda^*)T}-1\right)}{\mathrm{E}[V_1](a-\mathrm{E}[V_1])(\mathrm{E}[V_1]-\mathrm{E}[X_1])(\mathrm{E}[V_1]-\lambda^*)}+\\
&\left.\frac{1-e^{-\lambda^* T}}{a\lambda^*\mathrm{E}[V_1]}+\frac{e^{(\mathrm{E}[X_1]-\lambda^*)T}-1}{(a-\mathrm{E}[X_1])(\mathrm{E}[V_1]-\mathrm{E}[X_1])(\mathrm{E}[X_1]-\lambda^*)}\right]
\end{aligned}
$$

Thus the formula is obtained. □

5.4 Value at Risk and Conditional Value at Risk

Despite the fact that VaR is not a coherent risk measure as it does not possess the sub-additivity property, it is still an important tool for risk management since it summarizes the downside-risk by a quantile. By the definition, VaR of our CPPI portfolio can be written in the following equation:

Given a confidence level $\alpha \in (0,1)$

$$
\begin{aligned}
VaR_\alpha &= \inf\{x\in\mathbb{R} \mid \mathbb{P}(G-V_T>x)\le 1-\alpha\}\\
&= \inf\{x\in\mathbb{R} \mid \mathbb{P}(-C_T>x)\le 1-\alpha\}\\
&= \inf\{x\in\mathbb{R} \mid \mathbb{P}(L_T>x)\le 1-\alpha\} \qquad (5.6)
\end{aligned}
$$

From Equation (5.6), it is clear to see that in order to calculate the VaR, we need to first find the distribution function of either $\{C_T\}_{T\ge0}$, or $\{L_T\}_{T\ge0}$ which represents $\{-C_T\}_{T\ge0}$. Although the random variable of $\{C_T\}_{T\ge0}$ may not have an analytical expression for its distribution function, its characteristic function always exists. The one to one relationship with probability density functions is one of the basic properties of characteristic functions. In the following section, we will discuss how we can use the connection between the two, and then find the distribution function and probability density function which are linked to the characteristic function.

CVaR is another concept which generates from VaR. It is an alternative to VaR which is more sensitive to the shape of the loss distribution in the tail. Mathematically speaking, CVaR is derived by taking a weighted average between the VaR and losses exceeding the VaR. Moreover, unlike VaR, CVaR is actually a coherent risk measure, which means it satisfies properties of monotonicity, sub-additivity, homogeneity, and translational invariance. For the CVaR of the portfolio which follows the strategy of CPPI, we have the following formulation:

$$\begin{aligned} CVaR_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_x dx \\ &= \frac{\displaystyle\int_{x>VaR_\alpha} x f_L(x)dx}{1-\alpha} \\ &= VaR_\alpha + \frac{\displaystyle\int_{\mathbb{R}} [x-VaR_\alpha]^+ f_L(x)dx}{1-\alpha} \end{aligned} \tag{5.7}$$

where $f_L(x)$ can be replaced by $f_C(-x)$ since L is written as the random variable $-C$.

Let g be a complex-valued integrable function on $\mathbb{R}$. The Fourier transform of g is the function ϕ from $\mathbb{R}$ to $\mathbb{C}$ given as follows.

Definition 5.4.1. ***(Fourier transform)*** *If $g \in L^1$, then*

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} g(x)dx, \tag{5.8}$$

and ϕ is called the Fourier transform of g.

If g is a probability density function, then ϕ is its characteristic function as already defined in Definition 2.4.1.

The inversion formula was firstly demonstrated by [Lévy, 1925]:

$$F(x) - F(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-iux}}{iu}\phi(u)du,$$

where $F(x)$ is a distribution function. Later [Gurland, 1948] and [Gil-Pelaez, 1951] develop another expressions of the inversion theorem. Hereby we review briefly the Fourier inversion theorem, and then move on to the particular form of the Gil-Pelaez inversion integral, with which we find the characteristic function for the loss.

Theorem 5.4.1. ***(Fourier inversion theorem)*** *If g and $\phi \in L^1$, and ϕ is written as*

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} g(x)dx,$$

then

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)du$$

Proof See [Gut, 2005], Theorem 1.4. □

Theorem 5.4.2. *If $F(x)$ is a one-dimension distribution function, its characteristic function, ϕ, is written as* (5.8)*, and $\phi \in L^1$, then,*

$$F(x) = \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iux}\phi(u)}{iu}du$$

Proof Recall Dirichlet integral:

$$\int_0^\infty \frac{\sin t}{t} dt = \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \tag{5.9}$$

Observe that $\sin t/t$ is an even function, we can extend (5.9) further to

$$sgn(z) = \frac{2}{\pi} \int_0^\infty \frac{\sin tz}{t} dt,$$

where

$$sgn(z) = \begin{cases} 1, & if \;\; z > 0 \\ 0, & if \;\; z = 0 \\ -1, & if \;\; z < 0 \end{cases}$$

Moreover,

$$\int_{-\infty}^\infty sgn(z - x) dF(z) = 1 - 2F(x)$$

With knowledge of Fourier transform, Euler's formula and Fubini's theorem, we are able to write

$$\begin{aligned}
\int_{-\infty}^\infty \frac{e^{-iux}\phi(u)}{iu} du &= \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{e^{-iux}\phi(u) - e^{iux}\phi(-u)}{iu} du \\
&= 2 \lim_{\epsilon \to 0} \int_\epsilon^\infty \int_{-\infty}^\infty \frac{\sin u(z-x)}{u} dF(z) du \\
&= 2 \lim_{\epsilon \to 0} \int_{-\infty}^\infty \int_\epsilon^\infty \frac{\sin u(z-x)}{u} du dF(z) \\
&= \pi \int_{-\infty}^\infty sgn(z - x) dF(z) \\
&= \pi(1 - 2F(x)),
\end{aligned}$$

which completes the proof. □

With Theorem 5.4.2, we are able to calculate the VaR and CVaR for the CPPI portfolio via its characteristic function.

Proposition 5.4.1. *ϕ_C is the characteristic function for C_T and $\phi_C \in L^1$, the distribution function for L_T is then represented as*

$$\mathbb{P}(L_T \leq x) = F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-iux}\phi_C(-u)}{iu} du \tag{5.10}$$

Therefore, the VaR_α can be reformulated in

$$\inf \left\{ x \in \mathbb{R} \;\middle|\; \int_{-\infty}^\infty \frac{e^{-iux}\phi_C(-u)}{iu} du \leq \pi(1 - 2\alpha) \right\} \tag{5.11}$$

Furthermore, $CVaR_\alpha$ is written as

$$VaR_\alpha - \frac{\frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iuVaR_\alpha}}{u^2}\phi_C(-u)\right)du}{1-\alpha},$$

where $\Re(z)$ represents the real part of a complex number z.

Proof Based on Theorem 5.4.2 and the fact that $\phi_L(u) = \phi_C(-u)$ lead to Equation (5.10). VaR_α in (5.11) is a direct result followed by Equation (5.6) and (5.10). Here we focus on formulating the CVaR of the CPPI portfolio.

With the VaR already being calculated, we only need to find the integral in Equation (5.7).

$$\begin{aligned}
\int_{-\infty}^{\infty} [x - VaR_\alpha]^+ f_L(x)dx &= \int_{VaR_\alpha}^{\infty} (x - VaR_\alpha)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi_C(-u)dudx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{VaR_\alpha}^{\infty}(x - VaR_\alpha)e^{-iux}dx\right)\phi_C(-u)du \\
&= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iuVaR_\alpha}}{u^2}\phi_C(-u)du \\
&= \frac{-1}{\pi}\int_0^{\infty}\Re\left(\frac{e^{-iuVaR_\alpha}}{u^2}\phi_C(-u)\right)du
\end{aligned}$$

Hence the result follows. □

6 | Parameter Calibration and Simulation

ECF and MLE are the two methods applied in this thesis to calibrate parameters for the three models concerning normal (Merton), asymmetric double exponential (Kou) and Gumbel (new) distributed jumps. The risky assets are selected from major market indices, global ETFs and top 10 most-weighted components from German DAX and U.S. Dow Jones, respectively, which possess time period of 10 years up to 31.December.2014. The simulation and parameters estimation are performed in a computer with four computing cores running at 3.00 GHz with 4GB RAM.

Parameters are estimated according to the methods given in Section 6.1. The numerical results with the market data are presented in Section 6.2. Furthermore, in the end of the chapter we also present the forecasting ability of these three models with parameters retrieved from the data.

6.1 Estimation Methods

In each model we mentioned in Section 4.1, there are two parameters for the continuous part of the stock dynamic: drift μ and volatility σ for the geometric Brownian motion; As for the jump part, apart from the intensity parameter λ of the Poisson process, both Merton and new models we present require two parameters, respectively, for normal distributed and Gumbel distributed jumps.

Kou model has instead three parameters in order to describe the asymmetric exponentially distributed jumps: probability p for downside jumps with mean of η_-, and probability $1-p$ for upside jumps with mean of η_+.

6.1.1 Maximum Likelihood Estimation

Assume we have a vector of observation (x_1, x_2,..., x_n) from a unknown population, and the density function of this observing data vector given the parameter θ is written as

$$l(\theta) := f_\theta(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n|\theta),$$

which is defined as likelihood function $l(\theta)$. If the observations are stochastically independent, then the above representation can be written in a product of individual density.

The purpose of MLE is to find the parameter θ^* which maximizes the likelihood of observing the given data, in other words, θ^* is the value that makes the observed data the most probable to have been generated from the population. Traditionally the MLE approach is widely favored in financial applications due to its generality and asymptotic efficiency.

Based on the assumption of mutual independence among Brownian motion, Poisson process and the jump size, the characteristic function of observed log-returns $X_{\Delta t}$ between t and $t + \Delta t$ under new model is

$$\begin{aligned}\phi_\theta^G(u) &= \mathrm{E}\left[e^{iu(\mu-\frac{1}{2}\sigma^2)\Delta t}\right] \times \mathrm{E}\left[e^{iu\sigma W_{\Delta t}}\right] \times \mathrm{E}\left[e^{iu\sum\limits_{k=1}^{N_{\Delta t}}\log J_k}\right] \\ &= e^{\Delta t\left[iu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}+\lambda\left[\Gamma(1-i\beta u)e^{i\alpha u}-1\right]\right]}\end{aligned}$$

The density function of log-returns $X_{\Delta t}$ can therefore be derived from Theorem 5.4.1, which has the following representation.

$$\begin{aligned}f_{X_{\Delta t}}^G(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi_\theta^G(u)du \\ &= \frac{1}{\pi}\int_0^{\infty}\Re\left[e^{-iux}e^{\Delta t\left[iu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}+\lambda\left[\Gamma(1-i\beta u)e^{i\alpha u}-1\right]\right]}\right]du\end{aligned}$$

The procedure applied in Merton and Kou models resemble the above one. Hence, the characteristic functions of $X_{\Delta t}$ under Merton and Kou models are respectively as follows:

$$\begin{aligned}\phi_\theta^M(u) &= e^{\Delta t\left[iu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}+\lambda\left(e^{i\mu_j u-\frac{\sigma_j^2u^2}{2}}-1\right)\right]} \\ \phi_\theta^K(u) &= e^{\Delta t\left[iu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}+\frac{\lambda p}{1+iu\eta_-}+\frac{\lambda(1-p)}{1-iu\eta_+}-\lambda\right]}\end{aligned}$$

Once again from Theorem 5.4.1 we derive each density function of log-returns, respectively. It is worth mentioning that Merton model has one attractive characteristic, that is, the density function has a closed-form formulation. Unfortunately both Kou and new models do not acquire the same nice property.

$$\begin{aligned}f_{X_{\Delta t}}^M(x) &= \frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\left(\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\frac{e^{-\frac{1}{2}\frac{\left[x-\left(\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+k\mu_j\right)\right]^2}{\sigma^2\Delta t+k\sigma_j^2}}}{\sqrt{\sigma^2\Delta t+k\sigma_j^2}}\right) \\ f_{X_{\Delta t}}^K(x) &= \frac{1}{\pi}\int_0^{\infty}\Re\left[e^{-iux}e^{\Delta t\left[iu\left(\mu-\frac{\sigma^2}{2}\right)-\frac{\sigma^2u^2}{2}+\frac{\lambda p}{1+iu\eta_-}+\frac{\lambda(1-p)}{1-iu\eta_+}-\lambda\right]}\right]du\end{aligned}$$

6.1.2 Empirical Characteristic Function Method

The advantage of using ECF method is that one can avoid difficulties inherent in calculating or maximizing the likelihood function, e.g. the likelihood function can be unbounded, but its Fourier transform, characteristic function, is always bounded. Moreover, while the likelihood function is not tractable or has no closed form, the Fourier transform can have a closed-form expression. Hence, we also exploit ECF method to estimate the parameters. For example, to calibrate parameters from Kou model requires much computing power when applying MLE due to lack of availability of closed-form density, but on account of the attainability of explicit closed-form characteristic function of Kou model, the estimating process is much more efficient. Furthermore, ECF estimator is shown to be consistent and asymptotically normal under regularity conditions. Some theoretical background for the estimation procedure has been introduced by [Paulson et al., 1975]. [Yu, 2004] presents the applications of ECF method to fit time series models. The basic idea for this procedure, which is mathematically interpreted as below, is to minimize the weighted distance between the theoretical characteristic function $\phi(u)$ and empirical characteristic function $\hat{\phi}_n(u)$.

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \int_{-\infty}^{\infty} w(u)|\hat{\phi}_n(u) - \phi_\theta(u)|^2 du$$

where θ is the set of parameters, and $\hat{\phi}_n(u)$ is defined as:

$$\hat{\phi}_n(u) = \frac{\sum\limits_{k=1}^{n} e^{iuX_k}}{n} \tag{6.1}$$

The weight function $w(u)$ is optimal in the sense that the estimator attains maximum likelihood estimator efficiency. And there is optimal weight function $w^*(u)$, if the likelihood function has closed-form expression.

$$w^*(u) = \frac{1}{2\pi} \int \frac{\partial \log f_\theta(x)}{\partial x} e^{-iux} dx, \tag{6.2}$$

Otherwise the optimal weight remains unknown [Feuerverger and McDunnough, 1981]. As a consequence, in an actual implementation, an arbitrary weight function should be used. The widely chosen weight functions are, for example, normal density function [Rockinger and Semenova, 2005], exponential density function [Yu, 2004], and equally weighted function [Levin and Khramtsov, 2015]. Exponential weight function has often been used due to the computational convenience. It brings along the possibility to calculate the integral by Hermitian quadrature, and also it gives more importance to the characteristic function in a neighborhood of zero [Heathcote, 1977]. Normal density weight function has one more attracting feature than the exponential one: the variance of the sample is being taken into account.

On the basis of the advantage from normal weight function, our first intention is, inspired by [Cont and Tankov, 2009], to take the inverse of variance of empirical characteristic function $\hat{\phi}_n$ as our weight function.

$$\begin{aligned} w(u) &= \frac{1}{\mathrm{E}\left[\left(\hat{\phi}_n(u) - \mathrm{E}\left[\hat{\phi}_n(u)\right]\right)^2\right]} \\ &\approx \frac{1}{\mathrm{E}\left[\left(\hat{\phi}_n(u) - \phi_{\theta^*}(u)\right)^2\right]} \\ &= \frac{n}{1 - e^{-\sigma^2 u^2}}, \end{aligned} \tag{6.3}$$

where θ^* is the true set of parameters. Without loss of generality, we can remove n in Equation (6.3). In accordance with the divergence of this weight function when it reaches to origin, instead of Equation (6.3) we use

$$\begin{aligned} w(u) &= 1 - (1 - e^{-\sigma^2 u^2}) \\ &= e^{-\sigma^2 u^2} \end{aligned} \tag{6.4}$$

to secure convergence. As a result in the thesis we define Equation (6.4) as our weight function. On one hand this weight function is a variation of exponential function; on the other hand it takes the effect of variance into account. These are the advantages of both exponential and normal density weight functions respectively.

6.1.3 Cumulant Matching Method

Another problem one needs to be aware of is the choice of initial values, since maximizing the likelihood function could lead to numerical computational problem. In addition, the existence of multiple extrema also stresses on the importance on "good" initial values. Therefore, the initial values in this thesis are obtained not only from reasonable presumption, but also from CMM. CMM is a variant of the method of moments, by which model parameters are expressed by its cumulants, which are written as a combination of sample central moments calculated from Matlab.

Based on the preliminaries discussed in Chapter 2. The cumulants for our new model are derived from differentiation as follows:

$$\begin{aligned} c_1 &= \Delta t \left[\mu - \frac{\sigma^2}{2} + \lambda \left(\alpha - \beta \int_0^\infty e^{-t} \ln t dt \right) \right] \\ c_2 &= \Delta t \left[\sigma^2 + \lambda \left(\alpha^2 - 2\alpha\beta \int_0^\infty e^{-t} \ln t dt + \beta^2 \int_0^\infty e^{-t} (\ln t)^2 dt \right) \right] \\ c_n &= \lambda \Delta t \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^{n-k} \beta^k \int_0^\infty e^{-t} (\ln t)^k dt, n \geq 3 \end{aligned}$$

Since the cumulant c_n, $n > 1$ is a polynomial in the first n central moments m_n. Here the first four expressions are listed. See, e.g., Lemma A.86 [Pascucci, 2011].

$$\begin{aligned} c_1 &= m_1' \\ c_2 &= m_2 \\ c_3 &= m_3 \\ c_4 &= -3m_2^2 + m_4 \end{aligned} \tag{6.5}$$

c_1 is by definition the expected value. Note that we use the sample mean m_1' so as to distinguish it from the first central moment m_1. For convenience, we assume the mean of log-jump size is 0, i.e. $\alpha + \beta\gamma = 0$, where γ is the Euler's constant. Therefore initial values for our new model can be expressed in term of central moments as follows.

$$\beta = \frac{3m_2^2 - m_4}{m_3} \times \frac{\gamma^3 + 3\gamma^2\Gamma_1 + 3\gamma\Gamma_2 + \Gamma_3}{\gamma^4 + 4\gamma^3\Gamma_1 + 6\gamma^2\Gamma_2 + 4\gamma\Gamma_3 + \Gamma_4}$$

$$\alpha = -\beta\gamma$$

$$\lambda = \frac{m_4 - 3m_2^2}{\Delta t} \times \frac{1}{\beta^4(\gamma^4 + 4\gamma^3\Gamma_1 + 6\gamma^2\Gamma_2 + 4\gamma\Gamma_3 + \Gamma_4)}$$

$$\sigma = \sqrt{\frac{m_2}{\Delta t} - \beta^2\lambda(\gamma^2 + 2\gamma\Gamma_1 + \Gamma_2)}$$

$$\mu = \frac{m_1'}{\Delta t} + \frac{\sigma^2}{2} + \lambda\beta(\gamma + \Gamma_1)$$

where

$$\Gamma_1 = \int_0^\infty e^{-t}\ln t dt, \quad \Gamma_2 = \int_0^\infty e^{-t}(\ln t)^2 dt$$
$$\Gamma_3 = \int_0^\infty e^{-t}(\ln t)^3 dt, \quad \Gamma_4 = \int_0^\infty e^{-t}(\ln t)^4 dt$$

For Merton model we have the first six cumulants derived as follows:

$$c_1 = \Delta t\left(\mu - \frac{\sigma^2}{2} + \lambda\mu_j\right)$$
$$c_2 = \Delta t\left[\sigma^2 + \lambda\left(\mu_j^2 + \sigma_j^2\right)\right]$$
$$c_3 = \lambda\Delta t\left(\mu_j^3 + 3\mu_j\sigma_j^2\right)$$
$$c_4 = \lambda\Delta t\left(\mu_j^4 + 6\mu_j^2\sigma_j^2 + 3\sigma_j^4\right)$$
$$c_5 = \lambda\Delta t\left(\mu_j^5 + 10\mu_j^3\sigma_j^2 + 15\mu_j\sigma_j^4\right)$$
$$c_6 = \lambda\Delta t\left(\mu_j^6 + 15\mu_j^4\sigma_j^2 + 45\mu_j^2\sigma_j^4 + 15\sigma_j^6\right)$$

Following the same assumption we have $\mu_j = 0$. The initial values for Merton model can be written in the following manner:

$$\mu_j = 0$$

$$\sigma_j = \sqrt{\frac{c_6}{5c_4}}$$

$$\lambda = \frac{25c_4^3}{3c_6^2\Delta t}$$

$$\sigma = \sqrt{\frac{1}{\Delta t}\left(c_2 - \frac{5c_4^2}{3c_6}\right)}$$

$$\mu = \frac{1}{\Delta t}\left[c_1 + \frac{1}{2}\left(c_2 - \frac{5c_4^2}{3c_6}\right)\right]$$

with extra two cumulants expressed in central moments:

$$
\begin{aligned}
c_5 &= -10m_2m_3 + m_5 \\
c_6 &= 30m_2^3 - 15m_2m_4 - 10m_3^2 + m_6
\end{aligned}
$$

The above initial values for Merton model can therefore be obtained as functions of central moments.

Next, cumulants for Kou model are calculated in the same manner, and shown as follows:

$$
\begin{aligned}
c_1 &= \Delta t\left[\mu - \frac{\sigma^2}{2} - \lambda\left(p\eta_- - (1-p)\,\eta_+\right)\right] \\
c_2 &= \Delta t\left[\sigma^2 + 2\lambda\left(p\eta_-^2 + (1-p)\,\eta_+^2\right)\right] \\
c_n &= n!\lambda\Delta t\left[(1-p)\,\eta_+^n + (-1)^n p\eta_-^n\right], \quad n \geq 3
\end{aligned} \tag{6.6}
$$

Follow the assumption used in former models, we have the following equation

$$
\mu_j = -p\eta_- + (1-p)\eta_+ = 0
$$

Eliminate η_- in (6.6) using the relation shown above and rearrange the cumulants by central moments, we can write η_+ as a function of $x = \frac{1-p}{p}$ in two following ways

$$
\eta_+ = \frac{c_4}{4c_3} \times \frac{1-x}{1-x+x^2} = \frac{c_5}{5c_4} \times \frac{1-x+x^2}{(1-x)(1+x^2)} \tag{6.7}
$$

The above equation in (6.7) ends up in a polynomial of degree 4. Solve the equation and use the appropriate real root to obtain six initial parameters as follows

$$
\begin{aligned}
p &= \frac{1}{1+x} \\
\eta_+ &= \frac{c_4}{4c_3} \times \frac{1-x}{1-x+x^2} \\
\eta_- &= \eta_+ x \\
\lambda &= \frac{c_3}{6\Delta t(1-p)\eta_+^3(1-x^2)} \\
\sigma &= \sqrt{\frac{c_2}{\Delta t} - \frac{c_3}{3\Delta t\eta_+(1-x)}} \\
\mu &= \frac{m_1'}{\Delta t} + \frac{1}{2}\left(\frac{c_2}{\Delta t} - \frac{c_3}{3\Delta t\eta_+(1-x)}\right)
\end{aligned} \tag{6.8}
$$

From the result of the numerical implementation in the next section one can see that reasonable presumed initial values not only are sufficient compared to those which are estimated from CMM, but also outperform the other in the aspect of elapsed time and likelihood.

6.2 Numerical Implementation

In the beginning of this section parameters from each model are being estimated, the procedure will be performed not only with each method mentioned in the last section, but also the combination of them. See Table 6.1 for further details. Apart from the aspect of models, the goodness of fit in underlying coming from different categories is also investigated. Next we carry on reviewing the performance of these models by using AIC as a criterion.

The estimation is performed on 30 risky assets individually, as shown in Table A.1 in the

Model	# of Param.	Initial Values	Method
Kou	6 $[\mu, \sigma, \lambda, p, \eta_+, \eta_-]$	presumed values∗/ CMM values	ECF/MLE/ ECFMLE
Merton	5 $[\mu, \sigma, \lambda, \mu_j, \sigma_j]$	presumed values/ CMM values	ECF/MLE/ ECFMLE
new	5 $[\mu, \sigma, \lambda, \alpha, \beta]$	presumed values/ CMM values	ECF/MLE/ ECFMLE

presumed values= $[0, 0.15, 50, 0, 0.05]$
presumed values*= $[0, 0.15, 50, 0, 0.02, 0.02]$

Table 6.1: Parameters-estimating methods and initial values applied in different models

Appendices. An overlook of the general results is as well provided in Table 6.2 (See Table A.2 - A.19 in the Appendices for detailed results). From the results we observe that, on one hand, the efficiency applying the ECF method comparing to any involved with MLE method is much improved, especially in the cases of new and Kou models. This is one of the drawbacks applying MLE method due to the lack of closed-form likelihood function: each iteration involves going through the improper integral, which requires much computing power. Therefore, methods which involve with MLE take averagely 20-50 minutes for estimating parameters from new and Kou models. On the contrary, it takes 1-2 minutes in average for a model with a closed-form likelihood function like Merton one.

On the other hand, it is however undeniable that parameters estimated from any method which includes MLE method is outperforming with respect to the AIC.

In particular from Table 6.2 we observe that initial values obtained from CMM method not only do not bring out evident effects on better fit as expected, but also slow down the procedure of estimation in most cases. Another interesting finding has been discovered after applying the combination of ECF and MLE methods, that one can use ECF method as an auxiliary aid to locate a better set of initial values for the MLE method, and consequently improve distinctly the efficiency by saving $30 - 50\%$ of elapsed time. In addition to such a combination, the parameters found are no worse than the ones estimated directly from MLE method.

Although Kou model dominates the other models in most cases with an extra parameter describing jumps. It is worth mentioning that the new model has the best performance in

Method	Kou	Merton	new
ECF	-13957.4 (14.7)	-13951.6 (11.1)	-13952.2 (7.8)
CMMECF	-13929.1 (32.8)	-13911.9 (10.9)	-13950.7 (10.5)
MLE	-13993.1 (3348.6)	-13964.6 (103.3)	-13965.8 (2925.2)
CMMMLE	-13979.2 (2983.1)	-13964.6 (74.2)	-13965.6 (3265.4)
ECFMLE	-13994.9 (2052.3)	-13964.6 (69.9)	-13965.8 (1416.6)
CMMECFMLE	-13994.8 (2585.3)	-13964.6 (76.9)	-13965.8 (1450.0)

Table 6.2: Average AIC and elapsed time (in seconds) with respect to different models

the case of parameters estimated from ECF method with CMM calibrated initial values, and also out perform Merton model in all other cases. Moreover, both models in the case of ECF and CMMECF have approximately 7-10 seconds as duration of estimation.

The results can also be compared from another perspective according to different cate-

Method	Kou	Merton	new
ECF	-14787.9 (15.3)	-14792.7 (11.9)	-14794.4 (7.9)
CMMECF	-14698 (53.6)	-14716.2 (10.6)	-14789.7 (12.9)
MLE	-14840.2 (4024.4)	-14804.5 (106.6)	-14813.4 (3251)
CMMMLE	-14797.7 (3659.5)	-14804.5 (73.4)	-14812.7 (3898.5)
ECFMLE	-14844.3 (2324.5)	-14804.5 (66.4)	-14813.4 (1621.1)
CMMECFMLE	-14844.2 (4384.1)	-14804.5 (70.5)	-14813.4 (1610.3)

Table 6.3: Average AIC and elapsed time (in seconds) for indices only with respect to different models

gories, e.g. indices and stocks, whereas ETFs are classified here in the category of indices. See Table 6.3 and Table 6.4. The new model compared to Merton one provides a better fit with respect to all estimation methods when indices, rather than stocks, are taken as underlying. Nevertheless, Merton model clearly shows the dominating applicability to the others with its time-efficiency while applying MLE method.

Apart from the assistance of AIC for our model selection (in a "relative" sense), we present the Q-Q plots in Figure 6.1-6.4 as an auxiliary to see how good the models really fit with 3 different undelyings as examples: DAX (right-skewed), iShares MSCI Emerging Markets (right-skewed) and IBM (left-skewed), whose parameters are estimated from CMMECF and ECFMLE. For respective parameters please refer to Table 6.5 or the Appendices. Although jump parts seem different from the first glance, however we can observe that the jump size and intensity rate λ compensate each other. Moreover, the volatility of jump size and λ has the same effect, too. Another finding on λ is, in Kou model it is at most time larger than in the others. The cause of this phenomenon is due to the fact that exponential distributed variable is more centered to the origin, which means that it produces more small jumps than in the normal and Gumbel distribution, see Figure 4.3. Hence, a larger intensity is

Method	Kou	Merton	new
ECF	-13542.1 (14.4)	-13531.1 (10.7)	-13531.1 (7.8)
CMMECF	-13544.6 (22.3)	-13509.8 (11.1)	-13531.2 (9.3)
MLE	-13569.6 (3010.6)	-13544.6 (101.7)	-13542 (2762.3)
CMMMLE	-13570 (2644.9)	-13544.6 (74.6)	-13542 (2948.8)
ECFMLE	-13570.2 (1916.2)	-13544.6 (71.7)	-13542 (1314.4)
CMMECFMLE	-13570.2 (1685.9)	-13544.6 (80.2)	-13542 (1369.8)

Table 6.4: Average AIC and elapsed time (in seconds) for stocks only with respect to different models

demanded in Kou model.

Obviously in Figure 6.1 we can see that empirical returns have fatter tails than the normal distribution. Furthermore, the Q-Q plots reveal the insufficiency of Merton and Kou models on capturing the right tail of returns, which is one of the reasons we introduce log-Gumbel jumps into the new model.

After parameters estimating we present next the result of simulation from CMMECF and ECFMLE with respect to the chosen three underlyings in Table 6.6. Each result consists of 10000 times of simulation followed by the general settings: $m = 5$, $r = 1\%$, $y = 3\%$, $G = 0.9V_0$ and $V_0 = 1.1S_0$. Those which have greater jump intensity λ consume more time while simulating. The results are evidently distinct due to parameters and characteristics of each model, however there are still some consistencies, e.g. default occurs more in the Kou model. That is because it has not only greater jump intensity but also more extreme jumps, which can be seen as a nice quality for its conservative. Although log-Gumbel jumps might capture better the characteristics of returns, however from the Q-Q plots it also shows in the case of iShares MSCI Emerging Markets that the new model, in return of capturing the right tail, produces a relatively more optimistic results for the left tail, which might be a drawback from the point of view of risk assessment. A similar finding can also be seen from the results of simulation.

Ticker	Model	CMMECF	ECFMLE
DAX	Kou	(0.4478, 0.1590, 50.0997, 0.8219, 0.0230, 0.0137)	(0.4203, 0.0688, 387.4424, 0.5426, 0.0071, 0.0076)
	Merton	(0.1914, 0.1683, 20.0096, -0.0063, 0.0314)	(0.2572, 0.1439, 52.6966, -0.0031, 0.0227)
	new	(0.4594, 0.1394, 48.7904, -0.018, 0.0182)	(0.6067, 0.1093, 136.0029, -0.0109, 0.0124)
EEM	Kou	(0.3646, 0.2106, 49.9683, 0.6480, 0.0279, 0.0241)	(0.5880, 0.1814, 94.3161, 0.7030, 0.0245, 0.0178)
	Merton	(0.2246, 0.2201, 27.8195, -0.0064, 0.0446)	(0.2674, 0.2000, 42.6912, -0.0040, 0.0401)
	new	(0.4719, 0.1931, 49.2803, -0.0226, 0.0286)	(0.4373, 0.1931, 51.2686, -0.0233, 0.0293)
IBM	Kou	(0.3468, 0.1260, 111.8565, 0.6180, 0.0122, 0.0115)	(-0.0054, 0.1220, 135.5647, 0.4011, 0.0095, 0.0127)
	Merton	(0.1680, 0.1613, 24.3340, -0.0041, 0.0293)	(0.1553, 0.1391, 50.6664, -0.0016, 0.0233)
	new	(0.4276, 0.1382, 49.9331, -0.0169, 0.0178)	(0.2539, 0.1433, 43.7309, -0.0155, 0.0205)

Table 6.5: Parameter with respect to target underlyings estimated from CMMECF and ECFMLE

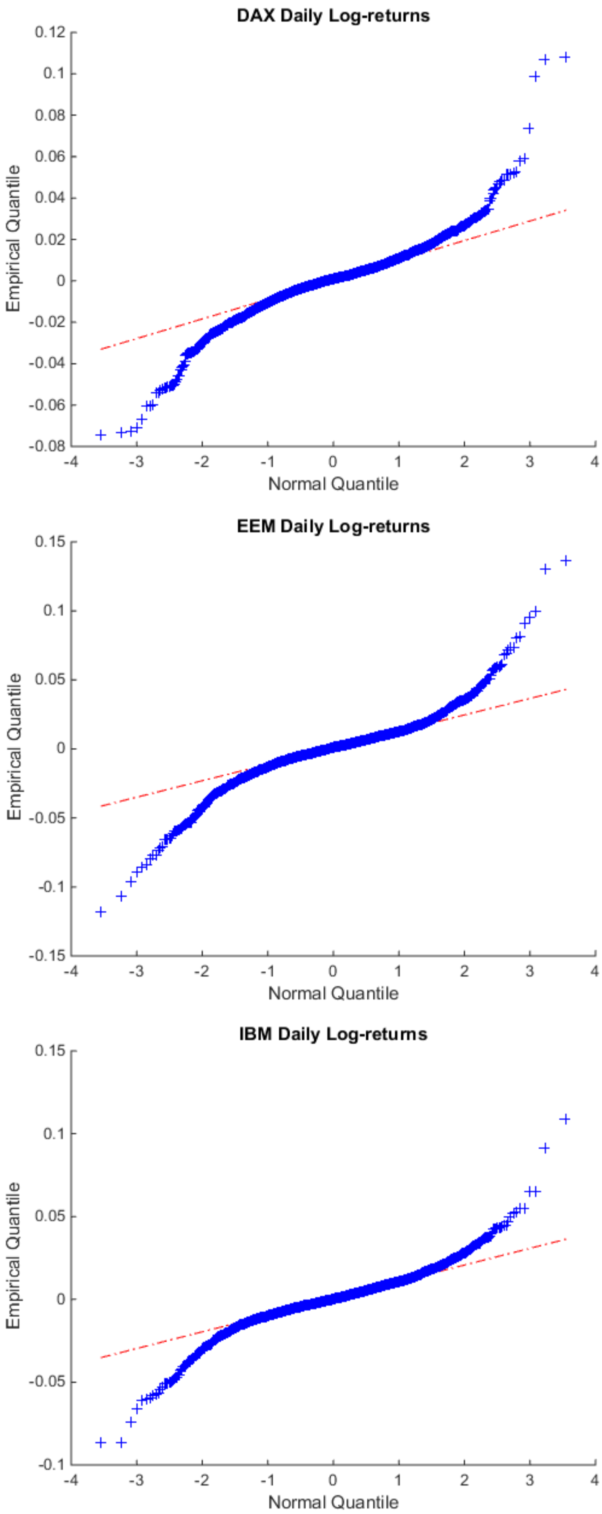

Figure 6.1: Q-Q plots of DAX, iShares MSCI Emerging Markets and IBM empirical quantiles versus theoretical quantiles from a normal distribution

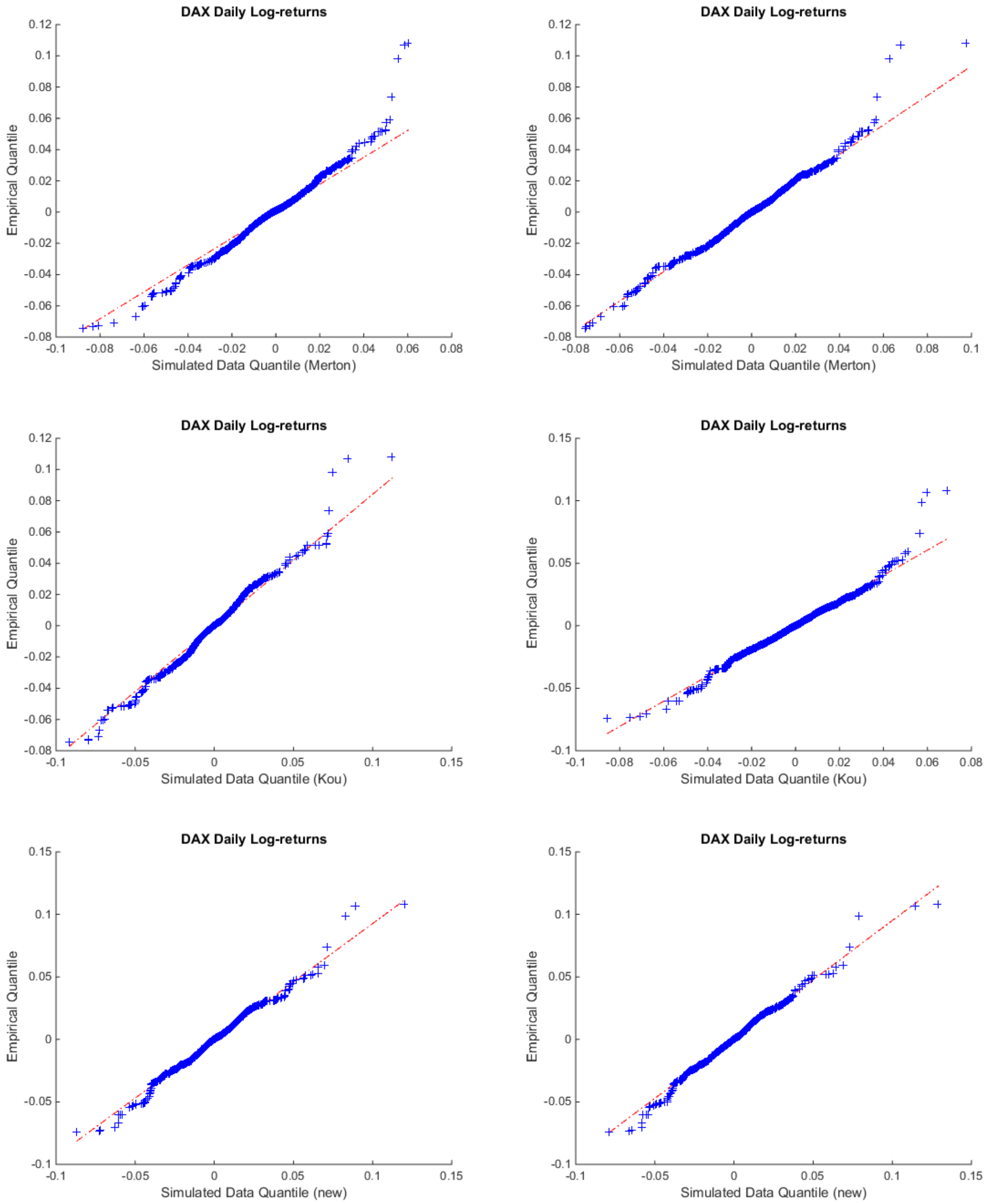

Figure 6.2: Q-Q plots of DAX empirical quantiles versus theoretical quantiles from each model whose parameters estimated from CMMECF(left) and ECFMLE, respectively

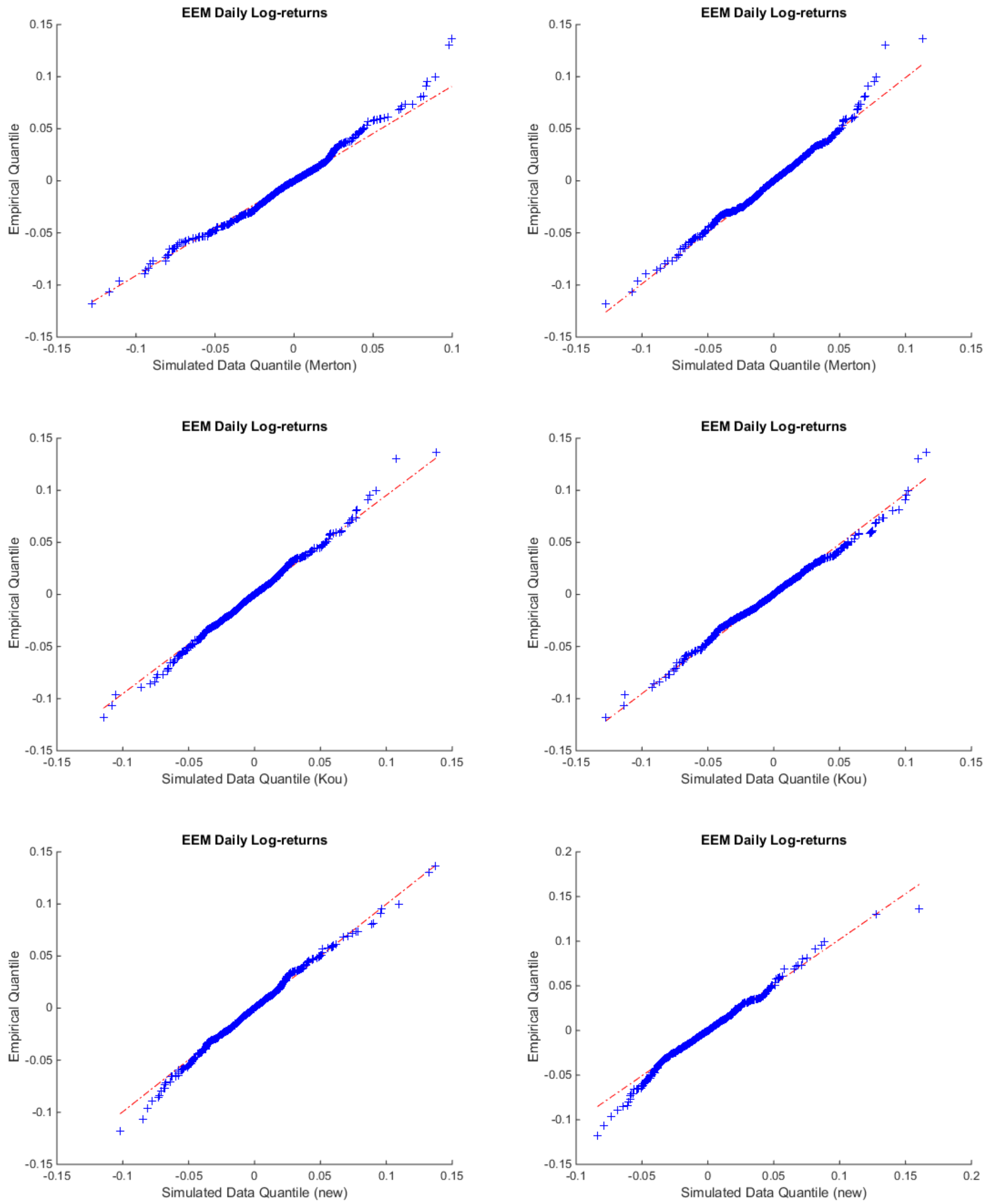

Figure 6.3: Q-Q plots of iShares MSCI Emerging Markets empirical quantiles versus theoretical quantiles from each model whose parameters estimated from CMMECF(left) and ECFMLE, respectively

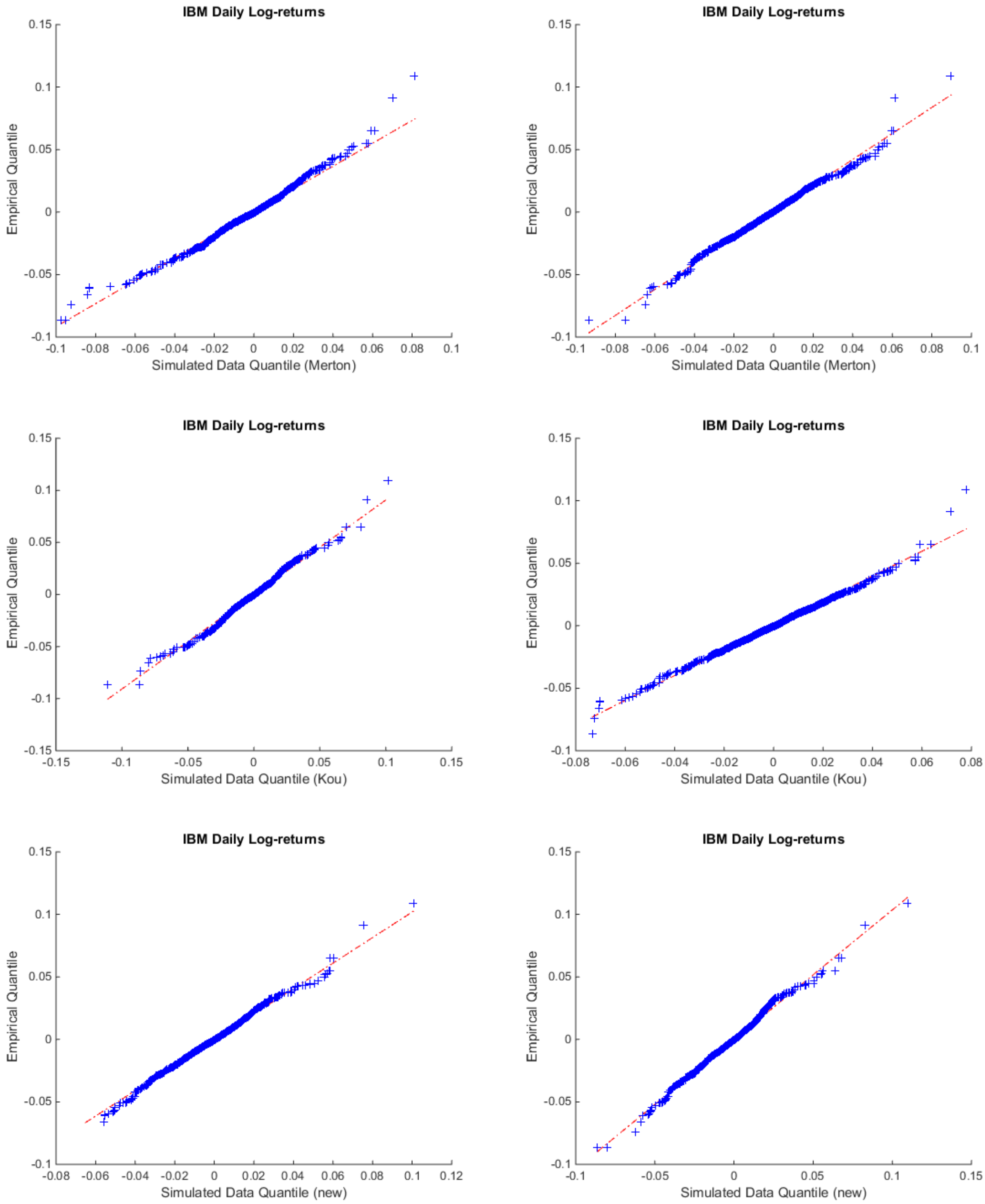

Figure 6.4: Q-Q plots of IBM empirical quantiles versus theoretical quantiles from each model whose parameters estimated from CMMECF(left) and ECFMLE, respectively

			Average Ratio (V_T/V_0)	Stdev. (V_T/V_0)	Default Rate	Conditional Average Loss	Average Loss	Stdev. of Loss Ratio	Elapsed Time
Kou	DAX	CMMECF	8.64	306.15	0	0	0	0	132.36
		ECFMLE	3.65	104.65	0	0	0	0	541.94
	EEM	CMMECF	1.08	1.94	0.0182	0.4392	0.0080	1.2785	130.98
		ECFMLE	2.25	66.17	0.0049	3.3653	0.0165	18.4343	217.69
	IBM	CMMECF	2.32	34.8	0.0001	0.2289	0.0001	0	244.41
		ECFMLE	3.03	48.73	0	0	0	0	289.75
Merton	DAX	CMMECF	2.73	25.42	0	0	0	0	128.07
		ECFMLE	9.61	112.81	0	0	0	0	186.21
	EEM	CMMECF	1.48	18.42	0.0015	0.0751	0.0002	0.1083	145.81
		ECFMLE	3.91	56.16	0.0005	0.2011	0.0002	0.2288	164.06
	IBM	CMMECF	4.25	50.21	0	0	0	0	137.24
		ECFMLE	5.31	70.25	0	0	0	0	192.66
New	DAX	CMMECF	14.39	189.66	0	0	0	0	195.43
		ECFMLE	855.92	8860.42	0	0	0	0	335.43
	EEM	CMMECF	21.75	389.67	0	0	0	0	193.09
		ECFMLE	4.32	92.90	0	0	0	0	195.74
	IBM	CMMECF	14.19	174.85	0	0	0	0	194.37
		ECFMLE	6.32	80.49	0	0	0	0	168.32

Table 6.6: Simulation results with respect to target underlyings estimated from CMMECF and ECFMLE

6.3 Forecasting Ability

In the former sections the discussion has been restricted to in-sample parameter estimation, and according to the result of goodness of the fit of each model with respect to the the estimated parameters, we are interested in how well these models can predict the future stock price dynamic, and furthermore affect the performance of the CPPI strategy. In current section we investigate the out-of-sample forecast by using three years data between 2004 and 2014 as in-sample data, and the following year from 2007 to 2015 as the out-of sample data. For example, daily data between 2004 and 2006 are considered as in-sample data and its parameters are estimated by CMMECF, and then applied in the simulation of 1-year data for 10000 times so that the performance of the investment using CPPI strategy can be observed. Next we compare it with the empirical result of out-of-sample data in 2007. The above estimation and simulation are performed under the same conditions and setup as in the former sections

Once again we take the index: DAX, ECF: EEM and stock: IBM as the underlyings. Table 6.7-6.9 shows the results of Kou, Merton and new models. The last column of each table represents the empirical portfolio values. We mark the greater value between simulated and empirical ratios in bold, so that it is convenient to compare.

From the empirical results, we can see that none of these risky investments result in floor violation. Comparing these tables we find that, these three models are generally more optimistic than the empirical world if we only look at the average ratio. However, Kou model is still relatively conservative compared to the others, whereas the new model is the most optimistic on the basis of default rate. If the times when simulated average ratio is greater than the empirical one are compared, we can see from Table 6.8 and Table 6.9 that for Merton model it occurs even more times than for the new model.

In Table 6.7 it is worth noticing that in 2008, simulated IBM data has a uncommon average ratio, which is due to the in-sample parameters estimation $(\mu, \sigma, \lambda, p, \eta_+, \eta_-) = (0.48, 0.24, 35.43, 0.94, 0.62, 0.01)$. Although the initial values CMM provides seem fit, we obtain a large probability for downside jumps, and as a compensation, the average size of upward jumps is unusually huge, which leads to the necessity on restricting the parameter field.

In-Sample	Out-of-Sample	Underlyings	Average V_T/V_0	Stdev. V_T/V_0	Default Rate	Conditional Average Loss	Stdev. Loss	Empirical V_T/V_0
2004-2006	2007	DAX	**1.18**	0.22	0	0	0	0.97
		EEM	**1.32**	0.75	0	0	0	1.06
		IBM	1.04	0.12	0.0032	1.18	1.41	**1.05**
2005-2007	2008	DAX	**1.14**	0.23	0	0	0	0.91
		EEM	**1.35**	1.26	0.0309	1.62	3.49	0.91
		IBM	**1.06**	0.17	0.0001	1.9	0	0.91
2006-2008	2009	DAX	0.98	0.17	0.0001	13.22	0	**1.11**
		EEM	0.98	0.59	0.1489	0.46	3.12	**1.29**
		IBM	1.02	0.19	0.0008	0.41	0.32	**1.53**
2007-2009	2010	DAX	0.98	0.26	0.0002	6.5	4.85	**1.18**
		EEM	1.00	1.19	0.3894	2.27	25.84	**1.03**
		IBM	**1.10**	0.61	0	0	0	1.06
2008-2010	2011	DAX	**1.02**	0.34	0	0	0	0.94
		EEM	**1.01**	2.25	0.148	1.32	16.02	0.92
		IBM	**600.81**	31273.2	0.0001	179.94	0	1.13
2009-2011	2012	DAX	1.12	0.44	0.0007	68.56	87.01	**1.15**
		EEM	**1.16**	1.88	0.0002	0.04	0.02	1.08
		IBM	**1.39**	0.82	0	0	0	1.01
2010-2012	2013	DAX	1.11	0.3	0.0001	61.44	0	**1.16**
		EEM	**1.05**	0.32	0.0048	2.68	9.7	0.98
		IBM	**1.14**	0.28	0	0	0	0.98
2011-2013	2014	DAX	**1.09**	0.29	0	0	0	1.01
		EEM	**1.01**	0.2	0.0003	0.06	0.07	0.98
		IBM	**1.09**	0.23	0.0005	4.76	6.65	0.95
2012-2014	2015	DAX	**1.10**	0.3	0	0	0	1.01
		EEM	**1.01**	0.24	0.0004	1.21	1.99	0.94
		IBM	**1.09**	0.23	0.0005	1.78	1.31	0.95

Table 6.7: Out-of-sample forecast with respect to Kou model estimated from CMMECF

In-Sample	Out-of-Sample	Underlyings	Average V_T/V_0	Stdev. V_T/V_0	Default Rate	Conditional Average Loss	Stdev. Loss	Empirical V_T/V_0
2004-2006	2007	DAX	**1.19**	0.22	0	0	0	0.97
		EEM	**1.38**	0.69	0	0	0	1.06
		IBM	1.04	0.12	0	0	0	**1.05**
2005-2007	2008	DAX	**1.14**	0.22	0	0	0	0.91
		EEM	**1.58**	1.26	0	0	0	0.91
		IBM	**1.06**	0.16	0	0	0	0.91
2006-2008	2009	DAX	0.99	0.14	0.0001	30.52	0	**1.11**
		EEM	0.99	0.58	0.0402	0.36	1.17	**1.29**
		IBM	1.04	0.22	0	0	0	**1.53**
2007-2009	2010	DAX	1.00	0.19	0.0001	7.6	0	**1.18**
		EEM	**1.09**	2.45	0.0414	0.47	1.09	1.03
		IBM	**1.14**	0.53	0	0	0	1.06
2008-2010	2011	DAX	**1.04**	0.27	0.0001	0.86	0	0.94
		EEM	**1.04**	1.47	0.0384	0.77	2.23	0.92
		IBM	1.13	0.46	0	0	0	1.13
2009-2011	2012	DAX	1.15	0.42	0	0	0	1.15
		EEM	**1.25**	1	0	0	0	1.08
		IBM	**1.40**	0.76	0	0	0	1.01
2010-2012	2013	DAX	1.13	0.33	0	0	0	**1.16**
		EEM	**1.11**	0.37	0	0	0	0.98
		IBM	**1.15**	0.29	0	0	0	0.98
2011-2013	2014	DAX	**1.12**	0.29	0	0	0	1.01
		EEM	**1.03**	0.19	0	0	0	0.98
		IBM	**1.10**	0.23	0	0	0	0.95
2012-2014	2015	DAX	**1.16**	0.13	0	0	0	1.01
		EEM	**1.05**	0.15	0	0	0	0.94
		IBM	**1.02**	0.12	0	0	0	0.95

Table 6.8: Out-of-sample forecast with respect to Merton model estimated from CMMECF

In-Sample	Out-of-Sample	Underlyings	Average V_T/V_0	Stdev. V_T/V_0	Default Rate	Conditional Average Loss	Stdev. Loss	Empirical V_T/V_0
2004-2006	2007	DAX	**1.22**	0.24	0	0	0	0.97
		EEM	**1.40**	0.68	0	0	0	1.06
		IBM	1.05	0.13	0	0	0	1.05
2005-2007	2008	DAX	**1.17**	0.23	0	0	0	0.91
		EEM	**1.46**	1.12	0	0	0	0.91
		IBM	**1.09**	0.18	0	0	0	0.91
2006-2008	2009	DAX	0.99	0.17	0	0	0	**1.11**
		EEM	1.03	0.92	0.0005	0.29	0.61	**1.29**
		IBM	1.05	0.26	0	0	0	**1.53**
2007-2009	2010	DAX	1.00	0.21	0	0	0	**1.18**
		EEM	**1.11**	1.41	0.0001	0.05	0	1.03
		IBM	**1.14**	0.54	0	0	0	1.06
2008-2010	2011	DAX	**1.04**	0.34	0	0	0	0.94
		EEM	**1.10**	1.67	0	0	0	0.92
		IBM	**1.14**	0.48	0	0	0	1.13
2009-2011	2012	DAX	**1.17**	0.48	0	0	0	1.15
		EEM	**1.26**	1.02	0	0	0	1.08
		IBM	**1.40**	0.75	0	0	0	1.01
2010-2012	2013	DAX	1.15	0.36	0	0	0	**1.16**
		EEM	**1.09**	0.33	0	0	0	0.98
		IBM	**1.17**	0.29	0	0	0	0.98
2011-2013	2014	DAX	**1.13**	0.32	0	0	0	1.01
		EEM	**1.05**	0.22	0	0	0	0.98
		IBM	**1.13**	0.26	0	0	0	0.95
2012-2014	2015	DAX	**1.16**	0.23	0	0	0	1.01
		EEM	**1.06**	0.16	0	0	0	0.94
		IBM	**1.04**	0.14	0	0	0	0.95

Table 6.9: Out-of-sample forecast with respect to new model estimated from CMMECF

7 | Conclusion

The protection from the CPPI strategies is incomplete due to the well-recognized gap risk. For those portfolios whose guarantees are failed to be fulfilled at maturity, the issuers need to compensate the investors for the gap between the guarantee and the final portfolio value, which might further result in violations of regulations, e.g. Solvency II.

In view of above, this work gives contribution to not only the evaluation of the risk measure under the consideration of gap risk in CPPI portfolios, but also the generalization of the CPPI strategies.

On the perspective of the risky asset dynamic, two well-known models, Merton and Kou, are examined. We observed their insufficiency on capturing the right tail. Therefore, a modification is launched by introducing a Gumbel variable to characterize the size of jumps. From the result of AIC it is shown that this modification outperformed Merton model in terms of 5-parameter models.

We generalized the setup of CPPI strategies by loosening the traditional restriction, in which the non-risky asset follows the riskfree rate. In the thesis, this restriction is relaxed by allowing the non-risky asset evolving with another yield, which is reasonably assumed to be greater than the riskfree rate. Accordingly, we generalized the CPPI strategy by providing a bigger pool of possible non-risky investments to select from. For the generalized self-financing CPPI strategy, a continuous-time stochastic model is constructed. Therein we derived a closed-form solution to the SDE for the cushion dynamic, which can be explicitly written as a generalized Ornstein-Uhlenbeck process. The solution for the portfolio value is obtained according to the mechanics of the insurance strategy after its cushion dynamic being solved.

An interesting extension with regard to CPPI strategies is to set the riskfree rate or the yield to follow another arbitrary Lévy process, instead of a constant. Moreover, now that we have loosed the restriction on the yield of the non-risky asset, how to hedge under the new setup is also another problem that is worth further investigation.

Another focus is set on the problem of the long-term guarantee in the CPPI strategies and how to describe it mathematically. We provided the statistical evaluation of probability, expectation and variance of loss with the help of the derived closed-form expression. Based on the existence of the characteristic function we also used the Fourier inversion theorem to retrieve the distribution function in order to derive the VaR and CVaR.

Particular interest is attracted to the numerical performance of the new model setup with respect to the risky investment and the CPPI portfolio. In addition, the comparison among three models are being investigated empirically. We selected various types of risky assets from global markets, including Asia, Europe and the United States. Parameters are estimated from three methods, each of the three started with two different types of initial values. According to the estimation results, we observed that initial values estimated from CMM method not only did not bring out evident effect on behalf of AIC, but also did not improve the time-efficiency.

In the aspect of estimation methods, we discovered that on one hand, parameters estimated from MLE had the lowest values from AIC, but cost almost one hour in Kou and new models; on the other hand, ECF had the best time-efficiency when estimating, yet the likelihood with respect to the parameters was relatively small. Nevertheless, the combination of ECF and MLE exhibited remarkable improvements by saving 30-50% of elapsed time, the performance of the parameters is not a bit compromised by its time-efficiency.

Given that Kou has one more parameter than the other two models, Kou model performed better in most of the cases, except for the case when parameters are estimated from the CMMECF. In this very case, the modified new model outperformed the other two. Thus, we provided Q-Q plots for our target underlyings with their parameters estimated from CMMECF and ECFMLE with respect to each of the three models. From the plots we found that right tails of each underlying are better captured in the new model. However, one should be cautious while applying the new model, since the improvement on fitting the right tail could have compromised the ability of capturing the left tail.

In this thesis we proposed another weight function for the application in the ECF method. This weight function possesses both the advantages from exponential and normal weighted functions, respectively. Since the focus is to assess the risk exposure, assigning more weight on the left tail is worthwhile being further investigated.

We examined the forecasting ability of each model from 2004 to 2014 by taking three years as the in-sample period, and the following year as the out-of-sample period. The result showed that all three models were in general too optimistic. The Kendall's τ of each asset between simulated and empirical ratios (V_T/V_0) were very low ($[-0.3, 0.2]$) with p-Values greater than 0.4. In this manner, we concluded all three models are lack of the ability to forecast. Nevertheless, from the results we observed evidently the downside-risk protection provided by the CPPI strategies.

In this thesis we have generalized the CPPI strategies and proposed another dynamic for the risky asset, which can be seen as a start to reexamine and compare other portfolio insurance strategies based on the similar manner. Especially for those strategies that consist of non-risky investments, e.g. time-invariant portfolio protection strategies, which are variation of the CPPI strategies, see [Mantilla-García, 2014]. Moreover, discrete-time trading as well as trading costs can also be taken into account for further research.

Appendices

Type	Symbol	Name
Index	ˆDJI	Dow Jones Industrial Average 30
	ˆFCHI	CAC 40
	ˆFTSE	FTSE 100
	ˆGDAXI	DAX 30
	ˆGSPC	S&P 500
	ˆHSI	Hang Seng Index
	ˆN225	Nikkei 225
	ˆTWII	TSEC Weighted Index
ETF	EEM	iShares MSCI Emerging Markets
	EZU	iShares MSCI Eurozone
Stock	AAPL	Apple Inc.
	BA	The Boeing Company
	DIS	The Walt Disney Company
	GS	The Goldman Sachs Group, Inc.
	HD	The Home Depot, Inc.
	IBM	International Business Machines Corporation
	MMM	3M Company
	NKE	NIKE, Inc.
	TRV	The Travelers Companies, Inc.
	UNH	UnitedHealth Group Incorporated
	ALV.DE	Allianz SE
	BAS.DE	BASF SE
	BAYN.DE	Bayer AG
	BMW.DE	Bayerische Motoren Werke AG
	DAI.DE	Daimler AG
	DBK.DE	Deutsche Bank AG
	DTE.DE	Deutsche Telekom AG
	LIN.DE	Linde AG
	SAP.DE	SAP SE
	SIE.DE	Siemens AG

Table A.1: Analyzed undelyings from U.S. Dow Jones (July, 2015), German DAX (June, 2015), indices and ETFs

Symbol	μ	σ	λ	p	η_+	η_-	AIC
ˆDJI	0.1515	0.1148	49.9959	0.5466	0.0141	0.0157	-16057.1
ˆFTSE	0.2392	0.1288	49.9932	0.6908	0.0167	0.0133	-15717.5
ˆGDAXI	0.3877	0.157	50.0104	0.7546	0.0196	0.0145	-14878.4
ˆGSPC	0.2623	0.1251	49.9987	0.6429	0.017	0.0158	-15687
ˆHSI	0.4253	0.1729	49.9854	0.8053	0.0276	0.0153	-14369.2
ˆN225	0.1598	0.1783	50.0003	0.4635	0.0134	0.0204	-14280.2
ˆTWII	0.4482	0.1525	49.9867	0.8642	0.0175	0.0118	-15341.6
ˆFCHI	0.1944	0.1605	50.0035	0.6316	0.0179	0.0166	-14662.6
EEM	0.3599	0.2104	49.9943	0.6441	0.0278	0.0242	-13138.7
EZU	0.3212	0.1869	50.0195	0.6386	0.0244	0.023	-13747
ALV.DE	0.3207	0.2047	49.871	0.6535	0.0315	0.0247	-13163.5
BAS.DE	0.1836	0.2042	50.0033	0.5268	0.0214	0.0211	-13476.2
BAYN.DE	0.26	0.2038	50.0043	0.5413	0.0192	0.0198	-13565.3
BMW.DE	0.2402	0.2384	50.0021	0.5905	0.0253	0.0203	-12839.6
DAI.DE	0.2806	0.2473	49.9988	0.5818	0.0276	0.0256	-12500.8
DBK.DE	0.169	0.25	55.7234	0.6031	0.036	0.0294	-12035.3
DTE.DE	0.2492	0.1726	50.0066	0.7327	0.0228	0.0136	-14445.7
LIN.DE	0.3453	0.1769	50.0049	0.6629	0.0215	0.0173	-14206.6
SAP.DE	0.4452	0.1695	50.0027	0.8046	0.0266	0.0145	-14449.7
SIE.DE	-0.0068	0.2167	35.3925	0.3506	0.021	0.032	-13392.9
AAPL	0.47	0.2662	49.9925	0.5572	0.025	0.0233	-12334.7
BA	0.2921	0.2098	50.0077	0.5896	0.0214	0.0213	-13435.9
DIS	0.2569	0.1857	49.9987	0.5918	0.0222	0.0191	-13904.8
GS	0.2115	0.2371	49.9977	0.5425	0.029	0.0304	-12429.9
HD	0.2029	0.2041	50.0032	0.6195	0.0228	0.0163	-13675.5
IBM	0.295	0.155	50.0037	0.6758	0.017	0.0147	-14972.5
MMM	0.3732	0.1539	49.9957	0.692	0.0185	0.016	-14903.6
NKE	0.2266	0.179	50.0048	0.5891	0.025	0.0189	-13954.1
TRV	0.309	0.1712	49.9782	0.6455	0.0269	0.0196	-14071.2
UNH	0.0539	0.2148	49.9909	0.4201	0.0218	0.0279	-13083.5

Table A.2: Parameters of Kou model estimated by ECF using presumed initial values

Symbol	μ	σ	λ	p	η_+	η_-	AIC
ˆDJI	0.3238	0.1188	49.9888	0.7559	0.0183	0.013	-16039.7
ˆFTSE	0.1124	0.113	74.3454	0.52	0.0122	0.0135	-15741.4
ˆGDAXI	0.4478	0.159	50.0997	0.8219	0.023	0.0137	-14873.6
ˆGSPC	0.2742	0.1248	50.6269	0.6547	0.0172	0.0156	-15688
ˆHSI	0.3161	0.169	49.9795	0.6935	0.0217	0.0171	-14379.9
ˆN225	-0.0229	0.1877	50.0967	0.2342	0.01	0.03	-14261.5
ˆTWII	0.2843	0.1507	49.9994	0.6462	0.0117	0.0137	-15347.7
ˆFCHI	0.3417	0.1651	49.53	0.7947	0.0237	0.0144	-14660.4
EEM	0.3646	0.2106	49.9683	0.648	0.0279	0.0241	-13138.5
EZU	0.2857	0.1704	27.3319	0.5176	0.8957	0.6149	-12849.6
ALV.DE	0.3883	0.1875	72.2725	0.6547	0.0263	0.0209	-13182.8
BAS.DE	0.1837	0.2042	49.9891	0.5269	0.0214	0.0211	-13476.2
BAYN.DE	0.26	0.2039	49.9911	0.5414	0.0192	0.0198	-13565.3
BMW.DE	0.4671	0.2057	121.652	0.656	0.019	0.0139	-12866.7
DAI.DE	0.367	0.2467	57.0994	0.637	0.0264	0.0217	-12503.5
DBK.DE	0.169	0.25	55.7152	0.6031	0.036	0.0294	-12035.3
DTE.DE	0.2072	0.1749	43.8382	0.7041	0.0229	0.0146	-14444.6
LIN.DE	0.3453	0.1769	49.9928	0.6629	0.0215	0.0173	-14206.6
SAP.DE	0.3573	0.1774	34.8366	0.7824	0.0298	0.0169	-14448.4
SIE.DE	-0.0851	0.2081	49.8768	0.3123	0.0177	0.0287	-13400.3
AAPL	0.5727	0.2275	133.8625	0.5697	0.0172	0.0156	-12362.3
BA	0.8393	0.174	151.244	0.7574	0.0169	0.0117	-13451.2
DIS	0.2611	0.1839	52.7572	0.5921	0.0217	0.0187	-13907.3
GS	0.2319	0.2515	33.5552	0.5812	0.0385	0.0362	-12410.8
HD	0.2225	0.2012	55.85	0.6297	0.022	0.0156	-13682.1
IBM	0.3468	0.126	111.8565	0.618	0.0122	0.0115	-15003.1
MMM	0.4185	0.1295	94.1993	0.6385	0.014	0.0132	-14938.5
NKE	0.2265	0.179	49.9948	0.5891	0.025	0.0189	-13954.1
TRV	0.1927	0.2004	17.3608	0.585	0.0442	0.0353	-13970.8
UNH	0.0665	0.2176	46.0218	0.4265	0.0227	0.0289	-13082.1

Table A.3: Parameters of Kou model estimated by ECF using CMM initial values

Symbol	μ	σ	λ	p	η_+	η_-	AIC
ˆDJI	0.3334	0.0721	196.9474	0.5926	0.0085	0.0083	-16138.5
ˆFTSE	0.0012	0.0886	196.4289	0.4307	0.0074	0.0096	-15758.5
ˆGDAXI	0.0091	0.0644	425.5452	0.415	0.0061	0.0082	-14925.3
ˆGSPC	-0.0029	0.0797	193.6681	0.4071	0.0077	0.011	-15744.9
ˆHSI	-0.0057	0.047	472.2244	0.437	0.0069	0.0086	-14454.5
ˆN225	0.2693	0.1736	64.0486	0.5667	0.0138	0.0164	-14282.6
ˆTWII	0.2505	0.0842	276.7152	0.4721	0.0064	0.0087	-15449.3
ˆFCHI	0.3886	0.1291	152.0699	0.637	0.0114	0.0104	-14682.3
EEM	0.5878	0.1814	94.2968	0.7029	0.0245	0.0178	-13159.6
EZU	0.6291	0.1335	162.5869	0.6665	0.0153	0.0131	-13806.7
ALV.DE	0.3548	0.1586	127.2283	0.6042	0.02	0.0166	-13198.1
BAS.DE	0.3607	0.1675	129.8124	0.5847	0.0157	0.0141	-13496.5
BAYN.DE	0.1817	0.1662	137.8326	0.4968	0.0137	0.0138	-13579.2
BMW.DE	-0.0095	0.1498	282.9089	0.471	0.0117	0.0121	-12881
DAI.DE	0.3286	0.1846	192.9819	0.5493	0.0155	0.0146	-12518.1
DBK.DE	0.135	0.2122	108.5138	0.5584	0.0254	0.0226	-12054
DTE.DE	-0.1414	0.1321	163.0684	0.4204	0.0103	0.0116	-14477
LIN.DE	0.4087	0.1474	116.7132	0.6281	0.0151	0.0126	-14224.7
SAP.DE	0.1308	0.1604	66.5168	0.4733	0.0145	0.0171	-14467.5
SIE.DE	-0.0008	0.1881	81.692	0.4191	0.0169	0.0203	-13408.5
AAPL	0.0076	0.1767	279.3681	0.4009	0.0116	0.0144	-12371.2
BA	0.3177	0.1714	144.0375	0.543	0.0134	0.0137	-13459.9
DIS	0.2356	0.1571	108.3727	0.5501	0.0158	0.0144	-13925.5
GS	0.2482	0.2053	89.7066	0.552	0.0252	0.0236	-12447.2
HD	0.0036	0.1232	260.2979	0.4919	0.0109	0.0104	-13747.4
IBM	-0.0015	0.1221	135.2217	0.4031	0.0095	0.0127	-15014
MMM	0.0037	0.111	175.2258	0.3821	0.0086	0.0127	-14946
NKE	0.3516	0.1449	130.3912	0.6065	0.0162	0.0128	-13976.8
TRV	0.1665	0.1401	94.9343	0.5127	0.018	0.0177	-14108.1
UNH	-0.0011	0.1915	84.0531	0.4573	0.0197	0.0219	-13091.3

Table A.4: Parameters of Kou model estimated by MLE using presumed initial values

Symbol	μ	σ	λ	p	η_+	η_-	AIC
ˆDJI	0.3333	0.0721	196.8636	0.5926	0.0085	0.0083	-16138.5
ˆFTSE	0.1194	0.0906	181.2644	0.4947	0.008	0.0093	-15763
ˆGDAXI	0.323	0.0719	376.269	0.5146	0.007	0.0078	-14934
ˆGSPC	0.3813	0.0799	175.7985	0.6028	0.0098	0.0095	-15769.3
ˆHSI	-0.2046	0.051	462.2152	0.3856	0.0066	0.0092	-14448.3
ˆN225	0.2692	0.1736	64.0384	0.5667	0.0138	0.0164	-14282.6
ˆTWII	0.3853	0.1878	70.551	0.9999	0.0001	0.0032	-14999.6
ˆFCHI	-0.0113	0.1161	212.1964	0.4391	0.0084	0.0107	-14675.5
EEM	0.5878	0.1814	94.279	0.7028	0.0245	0.0179	-13159.6
EZU	0.6292	0.1335	162.581	0.6665	0.0153	0.0131	-13806.7
ALV.DE	0.3548	0.1586	127.2282	0.6042	0.02	0.0166	-13198.1
BAS.DE	0.3604	0.1675	129.9117	0.5846	0.0157	0.0141	-13496.5
BAYN.DE	0.1817	0.1662	137.8418	0.4968	0.0137	0.0138	-13579.2
BMW.DE	0.2569	0.149	285.598	0.5374	0.0123	0.0114	-12882.2
DAI.DE	0.3286	0.1846	193.0212	0.5493	0.0155	0.0146	-12518.1
DBK.DE	0.0013	0.2126	107.2625	0.519	0.0245	0.0235	-12053.3
DTE.DE	-0.1414	0.1321	163.0685	0.4204	0.0103	0.0116	-14477
LIN.DE	0.1908	0.1494	109.0629	0.5227	0.0139	0.014	-14221.7
SAP.DE	0.1308	0.1604	66.5168	0.4733	0.0145	0.0171	-14467.5
SIE.DE	-0.0997	0.1869	85.8004	0.3746	0.0161	0.0209	-13409
AAPL	0.2445	0.1817	255.932	0.4536	0.0125	0.0142	-12371.8
BA	0.2655	0.1706	146.506	0.5204	0.013	0.0139	-13459.8
DIS	0.2356	0.1571	108.3727	0.5501	0.0158	0.0144	-13925.5
GS	0.2482	0.2053	89.7065	0.552	0.0252	0.0236	-12447.2
HD	0.2169	0.1242	259.7438	0.5561	0.0116	0.0099	-13748.8
IBM	-0.0054	0.122	135.5647	0.4011	0.0095	0.0127	-15014
MMM	0.203	0.1143	151.2892	0.4772	0.0098	0.0122	-14948.8
NKE	0.3516	0.1449	130.3723	0.6065	0.0162	0.0128	-13976.8
TRV	0.1665	0.1401	94.932	0.5127	0.018	0.0177	-14108.1
UNH	0.2962	0.1881	89.66	0.5787	0.0214	0.0192	-13095.9

Table A.5: Parameters of Kou model estimated by MLE using CMM initial values

Symbol	μ	σ	λ	p	η_+	η_-	AIC
^DJI	0.3334	0.0721	196.9475	0.5926	0.0085	0.0083	-16138.5
^FTSE	0.2525	0.0919	172.5455	0.5676	0.0088	0.0089	-15764.9
^GDAXI	0.4203	0.0688	387.4424	0.5426	0.0071	0.0076	-14934.7
^GSPC	0.3813	0.0799	175.7961	0.6028	0.0098	0.0095	-15769.3
^HSI	0.0351	0.0479	462.51	0.4484	0.007	0.0085	-14454.7
^N225	0.2693	0.1736	64.0491	0.5667	0.0138	0.0164	-14282.6
^TWII	0.2506	0.0842	276.9188	0.4721	0.0064	0.0087	-15449.3
^FCHI	0.3889	0.129	152.2731	0.637	0.0114	0.0104	-14682.3
EEM	0.588	0.1814	94.3161	0.703	0.0245	0.0178	-13159.6
EZU	0.6293	0.1335	162.5646	0.6665	0.0153	0.0131	-13806.7
ALV.DE	0.3548	0.1586	127.2283	0.6042	0.02	0.0166	-13198.1
BAS.DE	0.3606	0.1675	129.8421	0.5847	0.0157	0.0141	-13496.5
BAYN.DE	0.1817	0.1662	137.8383	0.4968	0.0137	0.0138	-13579.2
BMW.DE	0.2569	0.149	285.5978	0.5374	0.0123	0.0114	-12882.2
DAI.DE	0.3286	0.1846	193.0213	0.5493	0.0155	0.0146	-12518.1
DBK.DE	0.135	0.2122	108.5136	0.5584	0.0254	0.0226	-12054
DTE.DE	-0.1414	0.1321	163.0683	0.4204	0.0103	0.0116	-14477
LIN.DE	0.4088	0.1475	116.6847	0.6281	0.0151	0.0126	-14224.7
SAP.DE	0.1308	0.1604	66.5168	0.4733	0.0145	0.0171	-14467.5
SIE.DE	-0.0997	0.1869	85.8005	0.3746	0.0161	0.0209	-13409
AAPL	0.2461	0.1819	255.3139	0.4539	0.0125	0.0142	-12371.8
BA	0.3177	0.1714	144.0375	0.543	0.0134	0.0137	-13459.9
DIS	0.2356	0.1571	108.3727	0.5501	0.0158	0.0144	-13925.5
GS	0.2482	0.2053	89.7066	0.552	0.0252	0.0236	-12447.2
HD	0.2169	0.1242	259.768	0.5561	0.0116	0.0099	-13748.8
IBM	-0.0054	0.122	135.5647	0.4011	0.0095	0.0127	-15014
MMM	0.2045	0.1144	150.871	0.4775	0.0098	0.0122	-14948.8
NKE	0.3515	0.1449	130.342	0.6065	0.0162	0.0128	-13976.8
TRV	0.1665	0.1401	94.9341	0.5127	0.018	0.0177	-14108.1
UNH	0.294	0.1884	89.1463	0.5781	0.0215	0.0192	-13095.9

Table A.6: Parameters of Kou model estimated by ECF and MLE using presumed initial values

Symbol	μ	σ	λ	p	η_+	η_-	AIC
ˆDJI	0.3334	0.0722	196.8558	0.5926	0.0085	0.0083	-16138.5
ˆFTSE	0.2524	0.0919	172.437	0.5676	0.0088	0.0089	-15764.9
ˆGDAXI	0.4203	0.0688	387.2564	0.5427	0.0071	0.0076	-14934.7
ˆGSPC	0.3812	0.0799	175.8698	0.6027	0.0098	0.0095	-15769.3
ˆHSI	0.0438	0.0499	457.778	0.4489	0.007	0.0086	-14454.5
ˆN225	0.2693	0.1736	64.0409	0.5667	0.0138	0.0164	-14282.6
ˆTWII	0.2507	0.0842	276.8754	0.4721	0.0064	0.0087	-15449.3
ˆFCHI	0.3886	0.1291	152.07	0.637	0.0114	0.0104	-14682.3
EEM	0.5878	0.1814	94.2968	0.7029	0.0245	0.0178	-13159.6
EZU	0.6292	0.1335	162.5817	0.6665	0.0153	0.0131	-13806.7
ALV.DE	0.3548	0.1586	127.2284	0.6042	0.02	0.0166	-13198.1
BAS.DE	0.3606	0.1675	129.8421	0.5847	0.0157	0.0141	-13496.5
BAYN.DE	0.1817	0.1662	137.8422	0.4968	0.0137	0.0138	-13579.2
BMW.DE	0.2569	0.149	285.5979	0.5374	0.0123	0.0114	-12882.2
DAI.DE	0.3286	0.1846	192.982	0.5493	0.0155	0.0146	-12518.1
DBK.DE	0.135	0.2122	108.5136	0.5584	0.0254	0.0226	-12054
DTE.DE	-0.1414	0.1321	163.0689	0.4204	0.0103	0.0116	-14477
LIN.DE	0.4087	0.1474	116.7129	0.6281	0.0151	0.0126	-14224.7
SAP.DE	0.1308	0.1604	66.5168	0.4733	0.0145	0.0171	-14467.5
SIE.DE	-0.0998	0.1869	85.8207	0.3746	0.0161	0.0209	-13409
AAPL	0.2444	0.1817	255.9394	0.4536	0.0125	0.0142	-12371.8
BA	0.3177	0.1714	144.0374	0.543	0.0134	0.0137	-13459.9
DIS	0.2356	0.1571	108.3727	0.5501	0.0158	0.0144	-13925.5
GS	0.2482	0.2053	89.7066	0.552	0.0252	0.0236	-12447.2
HD	0.2169	0.1242	259.7675	0.5561	0.0116	0.0099	-13748.8
IBM	-0.0054	0.122	135.5647	0.4011	0.0095	0.0127	-15014
MMM	0.2045	0.1144	150.8712	0.4775	0.0098	0.0122	-14948.8
NKE	0.3516	0.1449	130.3867	0.6065	0.0162	0.0128	-13976.8
TRV	0.1666	0.1401	94.9245	0.5128	0.018	0.0177	-14108.1
UNH	0.2939	0.1884	89.118	0.5781	0.0215	0.0193	-13095.9

Table A.7: Parameters of Kou model estimated by ECF and MLE using CMM initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
^DJI	0.1628	0.0965	49.9778	-0.0024	0.022	-16061.9
^FTSE	0.1198	0.1091	49.9713	-0.0018	0.0221	-15711.9
^GDAXI	0.2325	0.1403	49.9823	-0.0033	0.0236	-14875.5
^GSPC	0.1861	0.1031	49.9766	-0.0026	0.0243	-15698.8
^HSI	0.1879	0.1528	49.984	-0.003	0.0267	-14383
^N225	0.2545	0.1605	49.9804	-0.0036	0.0253	-14248.6
^TWII	0.2714	0.1375	49.9711	-0.0044	0.0195	-15402.9
^FCHI	0.1317	0.1459	49.9842	-0.0028	0.0248	-14651.8
EEM	0.2481	0.1934	50.0021	-0.0038	0.0353	-13124.5
EZU	0.2594	0.1728	49.9951	-0.005	0.0314	-13767.8
ALV.DE	0.1805	0.1861	50.001	-0.0033	0.0371	-13160.3
BAS.DE	0.1637	0.1882	50.0016	-0.0007	0.0306	-13469.8
BAYN.DE	0.2443	0.1887	49.9959	-0.0016	0.0285	-13542.7
BMW.DE	0.1416	0.2234	50.0057	-0.0002	0.0323	-12849.4
DAI.DE	0.2258	0.2362	50.0102	-0.0026	0.035	-12491.6
DBK.DE	0.0379	0.2384	50.0062	-0.0023	0.0451	-12024.2
DTE.DE	0.0496	0.1541	49.9927	-0.0005	0.0248	-14445.3
LIN.DE	0.2224	0.1592	49.9874	-0.0021	0.0276	-14201.3
SAP.DE	0.2069	0.1495	49.9842	-0.0024	0.0257	-14425
SIE.DE	0.1022	0.1901	49.9994	0.0004	0.0308	-13371
AAPL	0.4184	0.2523	50.0117	-0.0011	0.0342	-12344.9
BA	0.2691	0.1998	50.0005	-0.0035	0.0287	-13448.6
DIS	0.177	0.1683	49.9929	-0.001	0.0296	-13905.8
GS	0.1751	0.2608	20.031	-0.0067	0.0627	-12386.5
HD	0.0746	0.1879	49.9993	0.0005	0.028	-13710.3
IBM	0.1919	0.138	49.9715	-0.0024	0.0234	-14997.6
MMM	0.2605	0.1355	49.9846	-0.0032	0.025	-14926.4
NKE	0.1207	0.1596	49.996	0.0005	0.0309	-13939.3
TRV	0.166	0.1505	49.9922	-0.0011	0.0317	-14060.6
UNH	0.1477	0.1976	49.9985	-0.0003	0.0344	-13049.2

Table A.8: Parameters of Merton model estimated by ECF using presumed initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
ˆDJI	0.1266	0.136	15.5154	-0.0055	0.032	-15957.4
ˆFTSE	0.1	0.1371	22.2441	-0.0031	0.0284	-15683.3
ˆGDAXI	0.1914	0.1683	20.0096	-0.0063	0.0314	-14855.3
ˆGSPC	0.1564	0.139	20.684	-0.0051	0.0323	-15618.7
ˆHSI	0.1675	0.1716	30.9204	-0.0043	0.0312	-14363.5
ˆN225	0.1986	0.1967	12.955	-0.0112	0.04	-14265.5
ˆTWII	0.6403	0.0871	254.3634	-0.0023	0.0105	-14915.4
ˆFCHI	0.1085	0.1663	27.9882	-0.0042	0.0297	-14648.9
EEM	0.2246	0.2201	27.8195	-0.0064	0.0446	-13113.1
EZU	0.2339	0.1916	33.2504	-0.0069	0.0362	-13741.3
ALV.DE	0.1697	0.2031	36.7402	-0.0042	0.042	-13155.9
BAS.DE	0.1615	0.232	11.4894	-0.0026	0.0567	-13423.3
BAYN.DE	0.2393	0.1995	37.3982	-0.002	0.0315	-13547.6
BMW.DE	0.1428	0.241	30.8547	-0.0003	0.0385	-12837.2
DAI.DE	0.2115	0.2568	29.2921	-0.004	0.0431	-12490.2
DBK.DE	0.0288	0.2641	30.988	-0.0035	0.0565	-12016.5
DTE.DE	0.0453	0.1946	9.5679	-0.0013	0.0461	-14398.5
LIN.DE	0.1919	0.203	11.3028	-0.0073	0.0484	-14149.6
SAP.DE	0.1778	0.1798	19.5716	-0.0048	0.0354	-14447.2
SIE.DE	0.1198	0.2279	15.339	0.0005	0.0499	-13370.1
AAPL	0.4044	0.293	9.3209	-0.0044	0.1747	-12179
BA	0.3287	0.147	123.2947	-0.0019	0.0218	-13355.5
DIS	0.1707	0.2007	20.4466	-0.0019	0.0414	-13878.1
GS	0.1751	0.2608	20.0427	-0.0067	0.0626	-12386.6
HD	0.0825	0.2103	25.1404	0.001	0.0356	-13666.3
IBM	0.168	0.1613	24.334	-0.0041	0.0293	-14966.8
MMM	0.2604	0.1355	49.922	-0.0032	0.0251	-14926.4
NKE	0.1245	0.1737	37.4945	0.0007	0.0343	-13944.8
TRV	0.158	0.1929	18.2324	-0.0022	0.0479	-13998.8
UNH	0.1538	0.2313	21.3682	-0.001	0.0489	-13057.6

Table A.9: Parameters of Merton model estimated by ECF using CMM initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
ˆDJI	0.204	0.0971	66.6324	-0.0022	0.0189	-16081.5
ˆFTSE	0.1752	0.1134	59.3609	-0.0023	0.0197	-15727.4
ˆGDAXI	0.2572	0.1439	52.6966	-0.0031	0.0227	-14878.5
ˆGSPC	0.2418	0.1042	64.0499	-0.0029	0.0212	-15715.7
ˆHSI	0.2538	0.1479	60.5474	-0.0032	0.0253	-14386
ˆN225	0.2184	0.1865	23.9427	-0.0066	0.0325	-14273.3
ˆTWII	0.3411	0.1201	66.7075	-0.0043	0.0187	-15424
ˆFCHI	0.1864	0.1541	46.8014	-0.0036	0.0246	-14657.6
EEM	0.2674	0.2	42.6912	-0.004	0.0401	-13129.4
EZU	0.2884	0.1622	60.9128	-0.004	0.0298	-13772.2
ALV.DE	0.1369	0.1901	48.1391	-0.001	0.0395	-13163.6
BAS.DE	0.235	0.1944	44.9435	-0.0022	0.0327	-13472.3
BAYN.DE	0.2073	0.2027	34.5214	-0.0004	0.0347	-13550.5
BMW.DE	0.1415	0.2152	57.5933	0.0002	0.0315	-12850.9
DAI.DE	0.2002	0.2435	41.5244	-0.0013	0.0393	-12494.6
DBK.DE	0.0139	0.2422	46.5493	-0.0005	0.0493	-12027.4
DTE.DE	0.03	0.162	44.9109	0.0003	0.0262	-14448.9
LIN.DE	0.2098	0.1696	41.3354	-0.0015	0.0298	-14205.4
SAP.DE	0.1754	0.1728	27.8741	-0.0027	0.0327	-14451.4
SIE.DE	0.0987	0.2086	31.2722	-0.0001	0.0397	-13384.1
AAPL	0.4442	0.2423	56.1273	-0.0016	0.0342	-12347
BA	0.2772	0.1948	51.5347	-0.0028	0.0292	-13450
DIS	0.1601	0.1748	45.6433	-0.0002	0.0309	-13907.8
GS	0.1673	0.2291	41.1033	-0.0017	0.0492	-12412.2
HD	0.0664	0.1717	64.2284	0.001	0.0266	-13719.5
IBM	0.1553	0.1391	50.6664	-0.0016	0.0233	-14998.5
MMM	0.2622	0.1369	52.6817	-0.003	0.0242	-14927.5
NKE	0.1678	0.1723	44.6095	0.0001	0.032	-13947.2
TRV	0.1644	0.1569	45.9636	-0.0005	0.0355	-14066.7
UNH	0.1519	0.2164	31.8868	-0.0011	0.0454	-13066.4

Table A.10: Parameters of Merton model estimated by MLE using presumed initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
ˆDJI	0.204	0.0971	66.6324	-0.0022	0.0189	-16081.5
ˆFTSE	0.1752	0.1134	59.3609	-0.0023	0.0197	-15727.4
ˆGDAXI	0.2572	0.1439	52.6966	-0.0031	0.0227	-14878.5
ˆGSPC	0.2418	0.1042	64.0499	-0.0029	0.0212	-15715.7
ˆHSI	0.2538	0.1479	60.5474	-0.0032	0.0253	-14386
ˆN225	0.2184	0.1865	23.9427	-0.0066	0.0325	-14273.3
ˆTWII	0.3411	0.1201	66.7075	-0.0043	0.0187	-15424
ˆFCHI	0.1864	0.1541	46.8015	-0.0036	0.0246	-14657.6
EEM	0.2674	0.2	42.6912	-0.004	0.0401	-13129.4
EZU	0.2884	0.1622	60.9129	-0.004	0.0298	-13772.2
ALV.DE	0.1369	0.1901	48.1391	-0.001	0.0395	-13163.6
BAS.DE	0.235	0.1944	44.9435	-0.0022	0.0327	-13472.3
BAYN.DE	0.2073	0.2027	34.5214	-0.0004	0.0347	-13550.5
BMW.DE	0.1415	0.2152	57.5933	0.0002	0.0315	-12850.9
DAI.DE	0.2002	0.2435	41.5244	-0.0013	0.0393	-12494.6
DBK.DE	0.0139	0.2422	46.5493	-0.0005	0.0493	-12027.4
DTE.DE	0.03	0.162	44.9109	0.0003	0.0262	-14448.9
LIN.DE	0.2098	0.1696	41.3354	-0.0015	0.0298	-14205.4
SAP.DE	0.1754	0.1728	27.8741	-0.0027	0.0327	-14451.4
SIE.DE	0.0987	0.2086	31.2722	-0.0001	0.0397	-13384.1
AAPL	0.4442	0.2423	56.1273	-0.0016	0.0342	-12347
BA	0.2772	0.1948	51.5346	-0.0028	0.0292	-13450
DIS	0.1601	0.1748	45.6433	-0.0002	0.0309	-13907.8
GS	0.1673	0.2291	41.1033	-0.0017	0.0492	-12412.2
HD	0.0664	0.1717	64.2284	0.001	0.0266	-13719.5
IBM	0.1553	0.1391	50.6664	-0.0016	0.0233	-14998.5
MMM	0.2622	0.1369	52.6817	-0.003	0.0242	-14927.5
NKE	0.1678	0.1723	44.6095	0.0001	0.032	-13947.2
TRV	0.1644	0.1569	45.9636	-0.0005	0.0355	-14066.7
UNH	0.1519	0.2164	31.8868	-0.0011	0.0454	-13066.4

Table A.11: Parameters of Merton model estimated by MLE using CMM initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
ˆDJI	0.204	0.0971	66.6323	-0.0022	0.0189	-16081.5
ˆFTSE	0.1752	0.1134	59.3609	-0.0023	0.0197	-15727.4
ˆGDAXI	0.2572	0.1439	52.6966	-0.0031	0.0227	-14878.5
ˆGSPC	0.2418	0.1042	64.0499	-0.0029	0.0212	-15715.7
ˆHSI	0.2538	0.1479	60.5474	-0.0032	0.0253	-14386
ˆN225	0.2184	0.1865	23.9427	-0.0066	0.0325	-14273.3
ˆTWII	0.3411	0.1201	66.7075	-0.0043	0.0187	-15424
ˆFCHI	0.1864	0.1541	46.8015	-0.0036	0.0246	-14657.6
EEM	0.2674	0.2	42.6912	-0.004	0.0401	-13129.4
EZU	0.2884	0.1622	60.9128	-0.004	0.0298	-13772.2
ALV.DE	0.1369	0.1901	48.1391	-0.001	0.0395	-13163.6
BAS.DE	0.235	0.1944	44.9435	-0.0022	0.0327	-13472.3
BAYN.DE	0.2073	0.2027	34.5214	-0.0004	0.0347	-13550.5
BMW.DE	0.1415	0.2152	57.5933	0.0002	0.0315	-12850.9
DAI.DE	0.2002	0.2435	41.5244	-0.0013	0.0393	-12494.6
DBK.DE	0.0139	0.2422	46.5493	-0.0005	0.0493	-12027.4
DTE.DE	0.03	0.162	44.9109	0.0003	0.0262	-14448.9
LIN.DE	0.2098	0.1696	41.3354	-0.0015	0.0298	-14205.4
SAP.DE	0.1754	0.1728	27.8741	-0.0027	0.0327	-14451.4
SIE.DE	0.0987	0.2086	31.2722	-0.0001	0.0397	-13384.1
AAPL	0.4442	0.2423	56.1273	-0.0016	0.0342	-12347
BA	0.2772	0.1948	51.5347	-0.0028	0.0292	-13450
DIS	0.1601	0.1748	45.6433	-0.0002	0.0309	-13907.8
GS	0.1673	0.2291	41.1033	-0.0017	0.0492	-12412.2
HD	0.0664	0.1717	64.2284	0.001	0.0266	-13719.5
IBM	0.1553	0.1391	50.6664	-0.0016	0.0233	-14998.5
MMM	0.2622	0.1369	52.6818	-0.003	0.0242	-14927.5
NKE	0.1678	0.1723	44.6095	0.0001	0.032	-13947.2
TRV	0.1644	0.1569	45.9636	-0.0005	0.0355	-14066.7
UNH	0.1519	0.2164	31.8868	-0.0011	0.0454	-13066.4

Table A.12: Parameters of Merton model estimated by ECF and MLE using presumed initial values

Symbol	μ	σ	λ	μ_j	σ_j	AIC
ˆDJI	0.204	0.0971	66.6324	-0.0022	0.0189	-16081.5
ˆFTSE	0.1752	0.1134	59.3609	-0.0023	0.0197	-15727.4
ˆGDAXI	0.2572	0.1439	52.6966	-0.0031	0.0227	-14878.5
ˆGSPC	0.2418	0.1042	64.0499	-0.0029	0.0212	-15715.7
ˆHSI	0.2538	0.1479	60.5475	-0.0032	0.0253	-14386
ˆN225	0.2184	0.1865	23.9427	-0.0066	0.0325	-14273.3
ˆTWII	0.3411	0.1201	66.7075	-0.0043	0.0187	-15424
ˆFCHI	0.1864	0.1541	46.8015	-0.0036	0.0246	-14657.6
EEM	0.2674	0.2	42.6912	-0.004	0.0401	-13129.4
EZU	0.2884	0.1622	60.9128	-0.004	0.0298	-13772.2
ALV.DE	0.1369	0.1901	48.1391	-0.001	0.0395	-13163.6
BAS.DE	0.235	0.1944	44.9435	-0.0022	0.0327	-13472.3
BAYN.DE	0.2073	0.2027	34.5214	-0.0004	0.0347	-13550.5
BMW.DE	0.1415	0.2152	57.5933	0.0002	0.0315	-12850.9
DAI.DE	0.2002	0.2435	41.5244	-0.0013	0.0393	-12494.6
DBK.DE	0.0139	0.2422	46.5493	-0.0005	0.0493	-12027.4
DTE.DE	0.03	0.162	44.9109	0.0003	0.0262	-14448.9
LIN.DE	0.2098	0.1696	41.3355	-0.0015	0.0298	-14205.4
SAP.DE	0.1754	0.1728	27.8741	-0.0027	0.0327	-14451.4
SIE.DE	0.0987	0.2086	31.2722	-0.0001	0.0397	-13384.1
AAPL	0.4442	0.2423	56.1273	-0.0016	0.0342	-12347
BA	0.2772	0.1948	51.5347	-0.0028	0.0292	-13450
DIS	0.1601	0.1748	45.6433	-0.0002	0.0309	-13907.8
GS	0.1673	0.2291	41.1033	-0.0017	0.0492	-12412.2
HD	0.0664	0.1717	64.2284	0.001	0.0266	-13719.5
IBM	0.1553	0.1391	50.6664	-0.0016	0.0233	-14998.5
MMM	0.2622	0.1369	52.6817	-0.003	0.0242	-14927.5
NKE	0.1678	0.1723	44.6095	0.0001	0.032	-13947.2
TRV	0.1644	0.1569	45.9636	-0.0005	0.0355	-14066.7
UNH	0.1519	0.2164	31.8868	-0.0011	0.0454	-13066.4

Table A.13: Parameters of Merton model estimated by ECF and MLE using CMM initial values

Symbol	μ	σ	λ	α	β	AIC
ˆDJI	0.3924	0.0961	49.9453	-0.0162	0.0168	-16046.4
ˆFTSE	0.3513	0.1102	49.9516	-0.0156	0.0167	-15714.7
ˆGDAXI	0.4662	0.1384	49.9574	-0.0178	0.0181	-14878.7
ˆGSPC	0.4202	0.103	49.9647	-0.0174	0.0186	-15687.9
ˆHSI	0.4234	0.1516	49.969	-0.0188	0.0208	-14399.3
ˆN225	0.4898	0.1593	49.9738	-0.0188	0.0194	-14236.3
ˆTWII	0.4923	0.1338	49.9489	-0.017	0.0146	-15412.9
ˆFCHI	0.3672	0.1449	49.9679	-0.0179	0.0191	-14666.1
EEM	0.4765	0.1923	49.9977	-0.0225	0.0284	-13136
EZU	0.49	0.1698	49.9888	-0.0224	0.0249	-13766
ALV.DE	0.4028	0.1853	49.9925	-0.0224	0.0302	-13158.9
BAS.DE	0.3996	0.1909	49.9884	-0.0179	0.0242	-13480
BAYN.DE	0.4849	0.1903	49.9862	-0.0181	0.0223	-13550.3
BMW.DE	0.3805	0.2258	49.998	-0.0182	0.0257	-12847
DAI.DE	0.4642	0.2363	50.0007	-0.0215	0.0281	-12498
DBK.DE	0.1174	0.2697	26.8069	-0.029	0.0524	-12003.3
DTE.DE	0.2845	0.1567	49.9733	-0.0155	0.0192	-14441.5
LIN.DE	0.4596	0.16	49.9777	-0.0182	0.0216	-14199.4
SAP.DE	0.4427	0.1491	49.9685	-0.0179	0.02	-14408.5
SIE.DE	0.3364	0.1946	49.9909	-0.0167	0.0242	-13351.5
AAPL	0.664	0.2541	50.0159	-0.0199	0.0272	-12333.8
BA	0.5091	0.1985	50.0005	-0.0202	0.0225	-13451.1
DIS	0.4117	0.1706	49.9871	-0.0178	0.0233	-13898.5
GS	0.2112	0.2669	15.9436	-0.0349	0.0627	-12368.5
HD	0.3109	0.1914	49.9902	-0.0158	0.022	-13703.4
IBM	0.4278	0.1382	49.9565	-0.0169	0.0178	-14965.2
MMM	0.496	0.1344	49.9612	-0.0183	0.0193	-14902.7
NKE	0.349	0.1637	49.9861	-0.0167	0.0246	-13952.4
TRV	0.3952	0.1525	49.9847	-0.0185	0.0254	-14058.8
UNH	0.3808	0.201	49.9958	-0.0187	0.0274	-13049.7

Table A.14: Parameters of new model estimated by ECF using presumed initial values

Symbol	μ	σ	λ	α	β	AIC
ˆDJI	0.382	0.0981	47.7773	-0.0164	0.017	-16047.8
ˆFTSE	0.3506	0.1103	49.8413	-0.0156	0.0168	-15714.8
ˆGDAXI	0.4594	0.1394	48.7904	-0.018	0.0182	-14878.8
ˆGSPC	0.4195	0.1032	49.8493	-0.0174	0.0186	-15688.1
ˆHSI	0.42	0.1521	49.3949	-0.0189	0.0209	-14399
ˆN225	0.4927	0.1596	49.9642	-0.0188	0.0193	-14237.1
ˆTWII	0.8298	0.095	116.0681	-0.0132	0.0114	-15363.5
ˆFCHI	0.3621	0.1456	49.0643	-0.0179	0.0193	-14666.6
EEM	0.4719	0.1931	49.2803	-0.0226	0.0286	-13136
EZU	0.482	0.1711	48.7862	-0.0225	0.0251	-13765.2
ALV.DE	0.4028	0.1853	49.9859	-0.0225	0.0302	-13158.9
BAS.DE	0.3993	0.1909	49.935	-0.0179	0.0242	-13479.9
BAYN.DE	0.4848	0.1903	49.978	-0.0181	0.0223	-13550.3
BMW.DE	0.3802	0.2258	49.9502	-0.0182	0.0257	-12847
DAI.DE	0.4639	0.2364	49.9467	-0.0215	0.0281	-12498
DBK.DE	0.1173	0.2697	26.7917	-0.029	0.0524	-12003.2
DTE.DE	0.2836	0.1569	49.7985	-0.0155	0.0192	-14441.6
LIN.DE	0.4585	0.1602	49.792	-0.0182	0.0216	-14199.4
SAP.DE	0.4415	0.1492	49.753	-0.0179	0.02	-14408.9
SIE.DE	0.332	0.1954	49.0939	-0.0168	0.0244	-13351.8
AAPL	0.6639	0.2541	50.0027	-0.0199	0.0272	-12333.8
BA	0.4989	0.1998	48.3685	-0.0203	0.0228	-13451.2
DIS	0.4078	0.1713	49.2536	-0.0178	0.0234	-13898.6
GS	0.2112	0.2669	15.9421	-0.0349	0.0627	-12368.5
HD	0.3109	0.1914	49.9932	-0.0158	0.022	-13703.4
IBM	0.4276	0.1382	49.9331	-0.0169	0.0178	-14965.2
MMM	0.4965	0.1345	49.927	-0.0184	0.0192	-14902.6
NKE	0.3458	0.1644	49.3104	-0.0167	0.0248	-13952.4
TRV	0.391	0.1534	49.2116	-0.0186	0.0255	-14059.1
UNH	0.3707	0.2028	48.074	-0.0189	0.0278	-13049.8

Table A.15: Parameters of new model estimated by ECF using CMM initial values

Symbol	μ	σ	λ	α	β	AIC
^DJI	0.3534	0.0874	96.5037	-0.0103	0.0128	-16093.3
^FTSE	0.3228	0.1068	78.5033	-0.0115	0.0138	-15736.2
^GDAXI	0.6067	0.1093	136.0062	-0.0109	0.0124	-14901.2
^GSPC	0.3814	0.0962	86.0285	-0.0121	0.0148	-15721.4
^HSI	0.4585	0.1379	80.553	-0.015	0.0178	-14407.8
^N225	0.2934	0.1868	22.983	-0.0247	0.026	-14262.3
^TWII	0.4505	0.1217	63.0695	-0.0149	0.0152	-15425.2
^FCHI	0.3442	0.1498	54.858	-0.0162	0.018	-14672.4
EEM	0.4373	0.1931	51.2686	-0.0233	0.0293	-13137.7
EZU	0.5066	0.1512	80.8166	-0.0177	0.021	-13776.2
ALV.DE	0.2808	0.1881	52.4162	-0.0211	0.0312	-13163.4
BAS.DE	0.4349	0.1864	58.0473	-0.0184	0.0231	-13481.9
BAYN.DE	0.3826	0.1944	47.695	-0.0177	0.0242	-13552.9
BMW.DE	0.5149	0.1881	111.4163	-0.0144	0.0195	-12859
DAI.DE	0.4869	0.2274	63.6776	-0.0204	0.0263	-12499.2
DBK.DE	0.2057	0.2361	54.2855	-0.0251	0.0376	-12023.3
DTE.DE	0.1842	0.1583	54.5025	-0.0136	0.0195	-14446.3
LIN.DE	0.4195	0.1562	65.7207	-0.0152	0.0194	-14204.1
SAP.DE	0.243	0.1769	22.8471	-0.0224	0.0291	-14443.3
SIE.DE	0.2057	0.2129	27.7015	-0.0231	0.0349	-13363
AAPL	0.7842	0.2274	82.6479	-0.0186	0.0235	-12341.9
BA	0.4308	0.1981	46.7593	-0.0202	0.0244	-13452.7
DIS	0.3205	0.1691	56.5238	-0.0159	0.023	-13901.8
GS	0.3399	0.2219	50.0576	-0.0251	0.036	-12408
HD	0.3666	0.1509	116.3077	-0.0117	0.0169	-13730.6
IBM	0.2539	0.1433	43.7311	-0.0155	0.0205	-14977.2
MMM	0.3591	0.1405	47.0574	-0.017	0.0208	-14911.9
NKE	0.3459	0.1613	63.0072	-0.0154	0.0222	-13956.2
TRV	0.2712	0.1545	51.1499	-0.0179	0.0275	-14065.7
UNH	0.2951	0.2104	38.6419	-0.0236	0.0339	-13057.4

Table A.16: Parameters of new model estimated by MLE using presumed initial values

Symbol	μ	σ	λ	α	β	AIC
ˆDJI	0.2347	0.0924	80.2538	-0.0107	0.0141	-16086.8
ˆFTSE	0.3228	0.1068	78.5032	-0.0115	0.0138	-15736.2
ˆGDAXI	0.5674	0.111	130.0557	-0.011	0.0126	-14900.9
ˆGSPC	0.3813	0.0961	86.1469	-0.0121	0.0148	-15721.4
ˆHSI	0.4585	0.1379	80.5526	-0.015	0.0178	-14407.8
ˆN225	0.2934	0.1868	22.983	-0.0247	0.026	-14262.3
ˆTWII	0.4505	0.1217	63.0695	-0.0149	0.0152	-15425.2
ˆFCHI	0.3442	0.1498	54.858	-0.0162	0.018	-14672.4
EEM	0.4373	0.1931	51.2686	-0.0233	0.0293	-13137.7
EZU	0.5066	0.1512	80.8166	-0.0177	0.021	-13776.2
ALV.DE	0.2807	0.1881	52.4003	-0.0211	0.0312	-13163.4
BAS.DE	0.4349	0.1864	58.0473	-0.0184	0.0231	-13481.9
BAYN.DE	0.3826	0.1944	47.6759	-0.0177	0.0242	-13552.9
BMW.DE	0.5149	0.1881	111.4069	-0.0144	0.0195	-12859
DAI.DE	0.4872	0.2274	63.7636	-0.0204	0.0263	-12499.2
DBK.DE	0.2057	0.2361	54.2855	-0.0251	0.0376	-12023.3
DTE.DE	0.1842	0.1583	54.5026	-0.0136	0.0195	-14446.3
LIN.DE	0.4195	0.1562	65.7203	-0.0152	0.0194	-14204.1
SAP.DE	0.243	0.1769	22.8471	-0.0224	0.0291	-14443.3
SIE.DE	0.2452	0.2126	28.1962	-0.0233	0.0345	-13362.7
AAPL	0.7842	0.2274	82.6513	-0.0186	0.0235	-12341.9
BA	0.4308	0.1981	46.7593	-0.0202	0.0244	-13452.7
DIS	0.3205	0.1691	56.5237	-0.0159	0.023	-13901.8
GS	0.3399	0.2219	50.0575	-0.0251	0.036	-12408
HD	0.3668	0.1509	116.3763	-0.0117	0.0169	-13730.6
IBM	0.2539	0.1433	43.7311	-0.0155	0.0205	-14977.2
MMM	0.3591	0.1405	47.0574	-0.017	0.0208	-14911.9
NKE	0.3459	0.1613	63.0072	-0.0154	0.0222	-13956.2
TRV	0.2712	0.1545	51.1501	-0.0179	0.0275	-14065.7
UNH	0.2951	0.2104	38.642	-0.0236	0.0339	-13057.4

Table A.17: Parameters of new model estimated by MLE using CMM initial values

Symbol	μ	σ	λ	α	β	AIC
ˆDJI	0.3535	0.0874	96.4994	-0.0103	0.0128	-16093.3
ˆFTSE	0.3228	0.1068	78.5032	-0.0115	0.0138	-15736.2
ˆGDAXI	0.6067	0.1093	136.0029	-0.0109	0.0124	-14901.2
ˆGSPC	0.3808	0.0962	85.8644	-0.0121	0.0148	-15721.4
ˆHSI	0.4585	0.1379	80.5629	-0.015	0.0178	-14407.8
ˆN225	0.2934	0.1868	22.9829	-0.0247	0.026	-14262.3
ˆTWII	0.4505	0.1217	63.0694	-0.0149	0.0152	-15425.2
ˆFCHI	0.3442	0.1498	54.858	-0.0162	0.018	-14672.4
EEM	0.4373	0.1931	51.2686	-0.0233	0.0293	-13137.7
EZU	0.5066	0.1512	80.8166	-0.0177	0.021	-13776.2
ALV.DE	0.2807	0.1881	52.4003	-0.0211	0.0312	-13163.4
BAS.DE	0.4349	0.1864	58.0473	-0.0184	0.0231	-13481.9
BAYN.DE	0.3828	0.1944	47.7187	-0.0177	0.0242	-13552.9
BMW.DE	0.5149	0.1881	111.407	-0.0144	0.0195	-12859
DAI.DE	0.487	0.2274	63.6722	-0.0204	0.0263	-12499.2
DBK.DE	0.2057	0.2361	54.2855	-0.0251	0.0376	-12023.3
DTE.DE	0.1842	0.1583	54.5025	-0.0136	0.0195	-14446.3
LIN.DE	0.4196	0.1562	65.741	-0.0152	0.0194	-14204.1
SAP.DE	0.243	0.1769	22.8471	-0.0224	0.0291	-14443.3
SIE.DE	0.2057	0.2129	27.7015	-0.0231	0.0349	-13363
AAPL	0.7842	0.2274	82.6512	-0.0186	0.0235	-12341.9
BA	0.4308	0.1981	46.7593	-0.0202	0.0244	-13452.7
DIS	0.3205	0.1691	56.5237	-0.0159	0.023	-13901.8
GS	0.3399	0.2219	50.0575	-0.0251	0.036	-12408
HD	0.3668	0.1509	116.3763	-0.0117	0.0169	-13730.6
IBM	0.2539	0.1433	43.7309	-0.0155	0.0205	-14977.2
MMM	0.3591	0.1405	47.0574	-0.017	0.0208	-14911.9
NKE	0.3459	0.1613	63.0071	-0.0154	0.0222	-13956.2
TRV	0.2712	0.1545	51.15	-0.0179	0.0275	-14065.7
UNH	0.2951	0.2104	38.6419	-0.0236	0.0339	-13057.4

Table A.18: Parameters of new model estimated by ECF and MLE using presumed initial values

Symbol	μ	σ	λ	α	β	AIC
ˆDJI	0.3535	0.0874	96.5662	-0.0103	0.0127	-16093.3
ˆFTSE	0.3228	0.1068	78.5032	-0.0115	0.0138	-15736.2
ˆGDAXI	0.6067	0.1093	136.0063	-0.0109	0.0124	-14901.2
ˆGSPC	0.3813	0.0961	86.1167	-0.0121	0.0148	-15721.4
ˆHSI	0.4585	0.1379	80.5522	-0.015	0.0178	-14407.8
ˆN225	0.2934	0.1868	22.983	-0.0247	0.026	-14262.3
ˆTWII	0.4505	0.1217	63.0695	-0.0149	0.0152	-15425.2
ˆFCHI	0.3442	0.1498	54.8579	-0.0162	0.018	-14672.4
EEM	0.4373	0.1931	51.2686	-0.0233	0.0293	-13137.7
EZU	0.5066	0.1512	80.8166	-0.0177	0.021	-13776.2
ALV.DE	0.2807	0.1881	52.4003	-0.0211	0.0312	-13163.4
BAS.DE	0.4349	0.1864	58.0597	-0.0184	0.0231	-13481.9
BAYN.DE	0.3828	0.1944	47.7182	-0.0177	0.0242	-13552.9
BMW.DE	0.5149	0.1881	111.4017	-0.0144	0.0195	-12859
DAI.DE	0.4872	0.2274	63.7636	-0.0204	0.0263	-12499.2
DBK.DE	0.2057	0.2361	54.2855	-0.0251	0.0376	-12023.3
DTE.DE	0.1842	0.1583	54.5025	-0.0136	0.0195	-14446.3
LIN.DE	0.4196	0.1562	65.741	-0.0152	0.0194	-14204.1
SAP.DE	0.243	0.1769	22.8471	-0.0224	0.0291	-14443.3
SIE.DE	0.2057	0.2129	27.7015	-0.0231	0.0349	-13363
AAPL	0.7842	0.2274	82.6484	-0.0186	0.0235	-12341.9
BA	0.4308	0.1981	46.7593	-0.0202	0.0244	-13452.7
DIS	0.3205	0.1691	56.5237	-0.0159	0.023	-13901.8
GS	0.3399	0.2219	50.0575	-0.0251	0.036	-12408
HD	0.3668	0.1509	116.3763	-0.0117	0.0169	-13730.6
IBM	0.2539	0.1433	43.7317	-0.0155	0.0205	-14977.2
MMM	0.3591	0.1405	47.0574	-0.017	0.0208	-14911.9
NKE	0.3459	0.1613	63.0071	-0.0154	0.0222	-13956.2
TRV	0.2712	0.1545	51.15	-0.0179	0.0275	-14065.7
UNH	0.2951	0.2104	38.6419	-0.0236	0.0339	-13057.4

Table A.19: Parameters of new model estimated by ECF and MLE using CMM initial values

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Abbreviation

CPPI: constant proportion portfolio insurance.

OBPI: option based portfolio insurance.

CPDO: constant proportion debt obligation.

càdlàg: right continuous, with left limits.

VaR: value at risk.

CVaR: conditional value at risk.

CMM: cumulant matching method.

MLE: maximum likelihood method.

ECF: empirical characteristic function method.

AIC: Akaike information criterion.

Preliminaries

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space.

$\{\mathcal{F}_t\}_{t\geq 0}$: filtration, which is an increasing sequence of σ-algebras with $\mathcal{F}_t \in \mathcal{F}$, $\forall t \geq 0$.

τ: stopping time.

$\wedge$: infix operator which outputs the minimum of two real scalar arguments.

$\mathcal{E}(X)$: Doléans-Dade exponential of a semimartingale X.

$[X_1, X_2]$: quadratic covariation of two semimartingales X_1 and X_2.

$\{J_t\}_{t\geq 0}$: sequence of i.i.d. random variable which generates the jump size.

$\{N_t\}_{t\geq 0}$: counting process.

$\{Y_t\}_{t\geq 0}$: compound Poisson process which is constructed by N and J.

(μ_R, σ^2, ν): Lévy R-triplet.

$\tilde{\psi}_Z$: cumulant generating function of a random variable Z.

$c_i(Z)$: i-th cumulant of a random variable Z.

CPPI strategy

G: guarantee.

m: multiplier, which offers leverage.

r: riskfree rate.

T: maturity.

y: yield from non-risky asset.

σ: volatility.

B_t: non-risky asset price at time t.

C_t: cushion value at time t.

P_t: discounted guarantee at time t, which forms the *floor*.

S_t: risky asset price at time t.

V_t: portfolio value at time t.

W_t: Brownian motion at time t.

α_t: the amount of risky investment at time t.

Risk measure

$\mathrm{E}[X|A]$: condtional expectation of X given A.

VaR_α: value at risk of the portfolio at the confidence level α.

CVaR_α: conditional value at risk of the portfolio at the confidence level α.

sgn: sign function.

$\Re$: operator which acquires the real part from a complex number

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Eidesstattliche Erklärung

Ich erkläre hiermit, dass ich diese Dissertation selbstständig und nur unter Zuhilfenahme der angegebenen Hilfsmittel und Quellen angefertigt habe.

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