Unconditionality in spaces of holomorphic functions

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Abstract

Let $\mathcal{P}({}^{\Lambda}X)$ be the set of all polynomials spanned by the monomials z^α, $\alpha \in \Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ on a Banach sequence space X, e.g. $X = \ell_p$. The unconditional basis constant of the monomials in $\mathcal{P}({}^{\Lambda}X)$ is the best constant $c \geq 1$ such that

$$\sum_{\alpha \in \Lambda} |c_\alpha x^\alpha| \leq c \sup_{\xi \in \mathrm{B}_X} \Big| \sum_{\alpha \in \Lambda} c_\alpha \xi^\alpha \Big|$$

for any $x \in \mathrm{B}_X$ and any $P = \sum_\alpha c_\alpha z^\alpha \in \mathcal{P}({}^{\Lambda}X)$.

We establish upper and lower bounds for the unconditional basis constant in terms of the cardinality of the index set Λ and study inequalities of this type for spaces of holomorphic functions on Reinhardt domains in a Banach sequence space X.

In particular, we prove that the unconditional basis constant of the monomials in $\mathcal{P}({}^{\Lambda(x)}\ell_p)$ (where $\Lambda(x) := \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \,|\, \mathfrak{p}^\alpha = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} \cdots \leq x\}$ and $\mathfrak{p}$ denoting the sequence of primes) is bounded by

$$x^{1-\frac{1}{\min\{p,2\}}} \exp\Big(\big(-\sqrt{2}\,\big(1 - \tfrac{1}{\min\{p,2\}}\big) + o(1)\big)\sqrt{\log x \log\log x}\Big).$$

For $p = \infty$ this resembles a deep result proved in a series of papers by Konyagin and Queffélec [42]; de la Bretèche [23]; and Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip [27]. We are able to generalize this result not only to the case of other Banach sequence spaces, but also to the case of other index sets with similar structural properties.

Finally, this enables us to investigate the domains of monomial convergence for the set $H_\infty(\mathrm{B}_X)$ of all bounded holomorphic functions on B_X. This is the set of all sequences $x \in \mathbb{C}^{\mathbb{N}}$ such that the power series expansion of any $f \in H_\infty(\mathrm{B}_X)$ converges absolutely in x. Moreover, we introduce the concept of ℓ_1–multipliers for sets of Dirichlet series and translate the results obtained for domains of monomial convergence to this setting.

Kurzzusammenfassung

Sei $\mathcal{P}(^{\Lambda}X)$ der Raum aller Polynome aufgespannt von den Monomen z^α, $\alpha \in \Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ auf einem Banach Folgenraum X, z.B. $X = \ell_p$. Die unbedingte Basiskonstante der Monome in $\mathcal{P}(^{\Lambda}X)$ ist die kleinste Konstante $c \geq 1$, so dass

$$\sum_{\alpha \in \Lambda} |c_\alpha x^\alpha| \leq c \sup_{\xi \in \mathrm{B}_X} \Big| \sum_{\alpha \in \Lambda} c_\alpha \xi^\alpha \Big|$$

für jedes $x \in \mathrm{B}_X$ und jedes $P = \sum_\alpha c_\alpha z^\alpha \in \mathcal{P}(^{\Lambda}X)$ gilt.

Für die unbedingte Basiskonstante der Monome in $\mathcal{P}(^{\Lambda}X)$ zeigen wir obere und untere Schranken in Abhängigkeit der Kardinalität der Indexmenge Λ und studieren Ungleichungen der obigen Art für Räume holomorpher Funktionen auf Reinhardt-Gebieten in einem Banach Folgenraum X.

Wir zeigen insbesondere, dass die unbedingte Basiskonstante der Monome in $\mathcal{P}(^{\Lambda(x)}\ell_p)$, wobei $\Lambda(x) := \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid \boldsymbol{p}^\alpha = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}7^{\alpha_4}\cdots \leq x\}$ und $\boldsymbol{p}$ die Folge der Primzahlen bezeichne, beschränkt ist durch

$$x^{1-\frac{1}{\min\{p,2\}}} \exp\Big(\big(-\sqrt{2}\,\big(1-\tfrac{1}{\min\{p,2\}}\big)+o(1)\big)\sqrt{\log x \log\log x}\Big).$$

Für $p = \infty$ stellt dies ein tiefgehendes Resultat, bewiesen von Konyagin, Queffélec [42], de la Bretèche [23], Defant, Frerick, Ortega-Cerdà, Ounaïes und Seip [27], dar. Wir konnten dieses Resultat nicht nur auf weitere Banach Folgenräume verallgemeinern, sondern auch auf weitere Indexmengen mit ähnlichen strukturellen Eigenschaften.

Schließlich können wir mittels dieser Resultate die Gebiete der absoluten Konvergenz der Potenzreihenentwicklungen für $H_\infty(\mathrm{B}_X)$, der Raum aller beschränkten holomorphen Funktionen auf B_X, untersuchen. Dies sind die Mengen aller Folgen $x \in \mathbb{C}^{\mathbb{N}}$, in denen die Potenzreihenentwicklung einer jeden Funktion $f \in H_\infty(\mathrm{B}_X)$ absolut konvergiert. Des Weiteren führen wir das Konzept der ℓ_1-Multiplikatoren für Räume von Dirichletschen Reihen ein und übersetzen die Resultate über die Gebiete der absoluten Konvergenz der Potenzreihenentwicklungen in diese Sprache.

An dieser Stelle möchte ich allen, die zu dem Gelingen dieser Dissertation beigetragen haben, herzlich danken.

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Contents

Chapter 1.

Introduction

For an infinite dimensional BANACH space E a natural question is whether it possesses an unconditional SCHAUDER basis $(b_k)_k$. The unconditional basis constant of a basic sequence $(b_k)_k$ in a BANACH space E is defined as the best constant $c \geq 1$ such that for any $x = \sum_k x_k b_k \in \overline{\operatorname{span}\{b_k \mid k \in \mathbb{N}\}}^E$ and any choice of $(\theta_k)_k \in \mathbb{T}^{\mathbb{N}}$ (where $\mathbb{T}$ denotes the set of complex numbers with absolute value one)

$$\Big\| \sum_k \theta_k x_k b_k \Big\|_E \leq c \Big\| \sum_k x_k b_k \Big\|_E .$$

In the following, we denote the unconditional basis constant of a basic sequence $(b_k)_k$ in E by $\chi\big((b_k)_k; E\big)$. The existence of such a constant is equivalent to the unconditional convergence of the representing series $x = \sum_k x_k b_k$.

In the finite dimensional case any basis is unconditional. However, also in this case it is of great interest to determine the unconditional basis constant.

In particular, we will investigate the unconditional basis constant in spaces of polynomials on BANACH sequence spaces X (for now, one may think of X as ℓ_p with $1 \leq p \leq \infty$). The monomials are prototypical polynomials: For a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $\mathbb{N}_0^{(\mathbb{N})} := \big\{\alpha \in \mathbb{N}_0^{\mathbb{N}} \,\big|\, |\{k \mid \alpha_k \neq 0\}| < \infty\big\}$ we define the monomial $z^\alpha : X \to \mathbb{C}$ by $x \mapsto x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. In the BANACH space of m–homogeneous polynomials on X endowed with the supremum norm over the unit ball of X (denoted by $\mathcal{P}(^m X)$), however, the monomials constitute in general not even a basis.

For a finite set $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ consider the closed subspace $\mathcal{P}(^\Lambda X) := \operatorname{span}\{z^\alpha \mid \alpha \in \Lambda\}$ of all polynomials on X. In this setting the monomials clearly define a basis of $\mathcal{P}(^\Lambda X)$

and we may consider the unconditional basis constant

$$\chi\big((z^\alpha)_{\alpha\in\Lambda}; \mathcal{P}(^\Lambda X)\big)\,.$$

In this Part I of thesis we establish upper and lower bounds for the unconditional basis constant in terms of the cardinality and structure of the index set Λ and in terms of the underlying Banach sequence space.

It turns out that the unconditional basis constant of the monomials is closely linked to several other fields of research. We shed light on these connections in Part II of the thesis at hand.

1.1. Connection to Dirichlet series

The domains of convergence of an ordinary Dirichlet series $D(s) = \sum_n a_n n^{-s}$ are given by half-planes $[\mathrm{Re} > \sigma] := \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \sigma\}$. Whenever a Dirichlet series converges in $\sigma + it \in \mathbb{C}$, it also converges on the half-plane $[\mathrm{Re} > \sigma]$. For a Dirichlet series D the abscissa of convergence (denoted by $\sigma_c(D)$) is defined as the infimum over all $\sigma \in \mathbb{R}$ such that D converges on $[\mathrm{Re} > \sigma]$. The abscissae of absolute and uniform convergence are defined analogously and denoted by $\sigma_a(D)$ and $\sigma_u(D)$ respectively. We clearly have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$ for any Dirichlet series D.

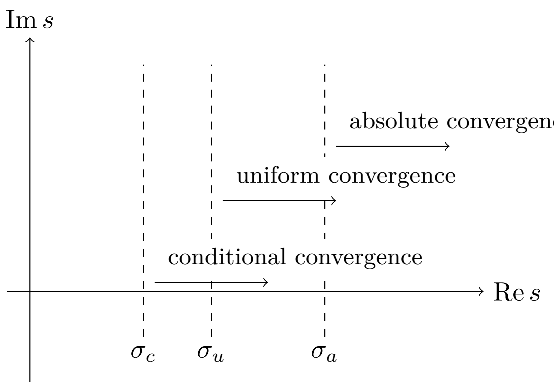

Figure 1.1.: Abscissae of convergence.

In 1913, BOHR asked in his paper [18] what the maximal difference between $\sigma_u(D)$ and $\sigma_a(D)$ might be. He was already able to show that

$$S := \sup\bigl\{\sigma_a(D) - \sigma_u(D) \bigm| D \text{ a Dirichlet series}\bigr\} \leq \tfrac{1}{2}$$

and asked for equality. As a consequence of his result,

$$\sum_{n=1}^{\infty} |a_n| \frac{1}{n^{\frac{1}{2}+\varepsilon}} < \infty \tag{1 · A}$$

for any $\varepsilon > 0$ and any DIRICHLET series $D = \sum_n a_n n^{-s} \in \mathcal{H}_\infty$ (where $\mathcal{H}_\infty$ denotes the set of all DIRICHLET series defining a bounded holomorphic function on $[\mathrm{Re} > 0]$).

The theory of DIRICHLET series constituted a glamorous topic at that time, so this question went down in history as "BOHR'S absolute convergence problem". 18 years later, BOHNENBLUST and HILLE [17] answered the question in the positive using a fairly unbiased approach. Their result implies that the exponent $\frac{1}{2}$ in $(1 \cdot \mathrm{A})$ is optimal.

However, the interest in BOHR'S absolute convergence problem didn't disappear completely. In recent times a series of improvements to the results of BOHR, BOHNENBLUST and HILLE where made. KONYAGIN and QUEFFÉLEC [42]; DE LA BRETÈCHE [23]; and finally DEFANT, FRERICK, ORTEGA-CERDÀ, OUNAÏES, and SEIP [27] proved the following proposition. For a thorough proof see also the recently published book [51].

Proposition 1.1. *Let $x \in (2, \infty)$. The best constant $c \geq 1$ such that*

$$\sum_{n \leq x} |a_n| \leq c \sup_{t \in \mathbb{R}} \Bigl| \sum_{n \leq x} a_n n^{-it} \Bigr|$$

for every choice of scalars $(a_n)_n \in \mathbb{C}^{\mathbb{N}}$ is given by

$$x^{\frac{1}{2}} \exp\Bigl(\bigl(-\tfrac{1}{\sqrt{2}} + o(1)\bigr) \sqrt{\log x \log\log x}\Bigr).$$

The proposition gives us furthermore a fine-grained version of $(1 \cdot \mathrm{A})$: The supremum over all $c \geq 0$ such that

$$\sum_{n=1}^{\infty} |a_n| \frac{\mathrm{e}^{c\sqrt{\log x \log\log x}}}{n^{\frac{1}{2}}} < \infty$$

for all $D \in \mathcal{H}_\infty$ is given by $\frac{1}{\sqrt{2}}$.

Astonishingly, the question of the convergence of DIRICHLET series is closely related to the question of convergence of power series on the unit ball of the space of all scalar sequences converging to zero, denoted by c_0. This crucial connection is due to a brilliant observation BOHR made in his paper [18]: By the fundamental theorem of arithmetics, every $n \in \mathbb{N}$ has a unique prime number decomposition; in other words there exists a unique multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ such that $n = \mathfrak{p}^\alpha = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \cdots$ (where $\mathfrak{p}$ denotes the sequence of primes). Then the so-called BOHR transform is the algebra homomorphism

$$\mathfrak{B} : \mathfrak{P} \to \mathfrak{D}, \qquad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \mapsto \sum_{n \in \mathbb{N}} a_n n^{-s} \quad \text{where } a_{\mathfrak{p}^\alpha} := c_\alpha$$

between the algebra $\mathfrak{P}$ of all (formal) power series and the algebra $\mathfrak{D}$ of all DIRICHLET series. It turns out (see [18], [41] or for an alternative proof the upcoming book [31]) that $\mathfrak{B}$ induces an isometric isomorphism $H_\infty(\mathrm{B}_{c_0}) \to \mathcal{H}_\infty$ between the BANACH space of all bounded holomorphic functions on the unit ball of c_0 endowed with the supremum norm

$$\|f\|_{\mathrm{B}_{c_0}} := \sup_{x \in \mathrm{B}_{c_0}} |f(x)| \quad \text{for } f \in H_\infty(\mathrm{B}_{c_0})$$

and $\mathcal{H}_\infty$ endowed with the supremum norm

$$\|D\|_{\mathcal{H}_\infty} := \sup_{s \in [\mathrm{Re}>0]} |f(s)| \quad \text{for } D \in \mathcal{H}_\infty,$$

where $f : [\mathrm{Re} > 0] \to \mathbb{C}$ denotes the bounded holomorphic function defined by D.

Any statement about absolute convergence of a DIRICHLET series in $\mathcal{H}_\infty$ now translates into a statement about absolute convergence of the power series expansion of a bounded holomorphic function in $H_\infty(\mathrm{B}_{c_0})$. We will come back to this connection in Chapter 8.

With the knowledge of the BOHR transform we can translate Proposition 1.1 into a statement about polynomials: With $\Lambda(x) := \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid \mathfrak{p}^\alpha \le x\}$

$$\sum_{\alpha \in \Lambda(x)} |c_\alpha| \le x^{\frac{1}{2}} \exp\Big(\big(-\tfrac{1}{\sqrt{2}} + o(1)\big)\sqrt{\log x \log\log x}\Big) \sup_{\xi \in \mathrm{B}_{c_0}} \Big|\sum_{\alpha \in \Lambda(x)} c_\alpha \xi^\alpha\Big| \qquad (1 \cdot \mathrm{B})$$

for every choice of scalars $(c_\alpha)_\alpha \in \mathbb{C}^{\Lambda(x)}$. Moreover, this inequality is sharp. We conclude easily that

$$\chi\big((z^\alpha)_\alpha; \mathcal{P}(^{\Lambda(x)}\ell_\infty)\big) = x^{\frac{1}{2}} \exp\Big(\big(-\tfrac{1}{\sqrt{2}} + o(1)\big)\sqrt{\log x \log\log x}\Big).$$

1.2. The origin of our research

At this point our research, this thesis is based on, began: *How does the unconditional basis constant change if we consider polynomials on other sequence spaces (for example ℓ_p with $1 \leq p \leq \infty$)?* One crucial step in the proof of Proposition 1.1 is given by the BOHNENBLUST-HILLE inequality. For $1 \leq p < \infty$ however, it is not applicable: It turns out that unconditional basis constants of spaces of m–homogeneous polynomials are the natural replacement at this point. Such an estimate is established in Theorem 4.1 which finally enables us to prove the following result.

Theorem (cf. Theorem 6.10)**.** *Let $1 \leq p \leq \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. Then for any choice of scalars $(c_\alpha)_\alpha \in \mathbb{C}^{\Lambda(x)}$ and any $\xi \in \mathrm{B}_{\ell_p}$*

$$\sum_{\alpha \in \Lambda(x)} |c_\alpha \xi^\alpha| \leq x^\sigma \exp\Big(\big(-\sqrt{2}\sigma + o(1)\big)\sqrt{\log x \log\log x}\Big) \sup_{\zeta \in \mathrm{B}_{\ell_p}} \Big|\sum_{\alpha \in \Lambda(x)} c_\alpha \zeta^\alpha\Big| .$$

The abstract tool given by Theorem 4.1 was eventually the answer to another question: *How does the unconditional basis constant change if we replace $\Lambda(x)$ by another set of multi-indices?* Depending on the structure of the set in question different results are obtained. As we will demonstrate in Section 3.1, the general case doesn't permit a precise estimate. For index sets with structural properties similar to those of $\Lambda(x)$ we obtain Theorem 6.4, which gives an estimate in the case that the index set is generated by an increasing sequence different from the sequence of primes. To be more precise, we consider the set of indices $\Lambda_{\boldsymbol{q}}(x) := \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \,|\, \boldsymbol{q}^\alpha \leq x\}$ where $\boldsymbol{q} := \big(k\,(\log(k+2))^\theta\big)_k$ for some $\theta \in (\frac{1}{2}, 1]$. We get the following result:

Theorem (cf. Theorem 6.4)**.** *Let $1 \leq p \leq \infty$, $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and $\boldsymbol{q}$ as defined before. Then for any choice of scalars $(c_\alpha)_\alpha \in \mathbb{C}^{\Lambda(x)}$ and any $\xi \in \mathrm{B}_{\ell_p}$ we have*

$$\sum_{\alpha \in \Lambda_{\boldsymbol{q}}(x)} |c_\alpha \xi^\alpha| \leq x^\sigma \exp\Big(\big(-2\sigma\sqrt{\theta - \tfrac{1}{2}} + o(1)\big)\sqrt{\log x \log\log x}\Big) \sup_{\zeta \in \mathrm{B}_{\ell_p}} \Big|\sum_{\alpha \in \Lambda_{\boldsymbol{q}}(x)} c_\alpha \zeta^\alpha\Big| .$$

Note that this result perfectly fits with the result for the sequence of primes: For $\theta = 1$ the sequence $\boldsymbol{q}$ is asymptotically equivalent to the sequence of primes and the constants in the respective inequalities equal.

1.3. Interfaces — domains of monomial convergence, multipliers for Dirichlet series, and Bohr radii

Part II discusses further questions in related areas. In Chapter 7 we investigate for which $x \in \mathbb{C}^{\mathbb{N}}$ the power series expansion of a holomorphic function converges absolutely.

By $H_\infty(R)$ we denote the set of all bounded holomorphic functions $f : R \to \mathbb{C}$ on a Reinhardt domain $R \subset X$. For a subset $\mathcal{F}(R) \subset H_\infty(R)$ we consider the domain of monomial convergence defined by

$$\operatorname{mon} \mathcal{F}(R) := \{x \in \mathbb{C}^{\mathbb{N}} \,|\, \forall f \in \mathcal{F}(R) : \sum_\alpha |c_\alpha(f)\, x^\alpha| < \infty\}\,.$$

Different from the finite dimensional case the set of monomial convergence in general doesn't match the entire domain of holomorphy. First attempts to study the domain of monomial convergence were made, although in a different fashion, by Bohr [18]. In order to prove that

$$S := \sup\bigl\{\sigma_a(D) - \sigma_u(D) \,\big|\, D \text{ a Dirichlet series}\bigr\} \le \tfrac{1}{2}$$

he showed (stated in our notation) that

$$M := \sup\bigl\{r \ge 1 \,\big|\, \ell_p \cap \mathrm{B}_{c_0} \subset \operatorname{mon} H_\infty(\mathrm{B}_{c_0})\bigr\} \ge 2$$

and established the equality $S = \frac{1}{M}$. In 1999, Lempert [45] gave a precise characterization for $p = 1$; namely

$$\operatorname{mon} H_\infty(\mathrm{B}_{\ell_1}) = \mathrm{B}_{\ell_1}\,. \qquad (1 \cdot \mathrm{C})$$

Furthermore, in [32] it is shown that for $1 < p < \infty$ and $1 \le q \le 2$ such that $\frac{1}{q} = \frac{1}{2} + \frac{1}{\max\{p,2\}}$

$$\mathrm{B}_{\ell_p} \cap \ell_q \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) \subset \mathrm{B}_{\ell_p} \cap \ell_{q+\varepsilon}$$

for any $\varepsilon > 0$. The question whether $\varepsilon = 0$ is possible remained open.

Using the results of Chapter 6 we find Theorem 7.10, which gives an approximation in the other cases:

Theorem (cf. Theorem 7.10)**.** *Let* $1 \leq p \leq \infty$ *and set* $\sigma := 1 - \frac{1}{\min\{p,2\}}$*. Then*

$$\left(\frac{1}{k^{\sigma}(\log(k+2))^{\theta\sigma}}\right)_k \cdot \mathrm{B}_{\ell_p} \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$$

for every $\theta > \frac{1}{2}$ *if* $1 \leq p \leq 2$ *or every* $\theta > 0$ *if* $2 \leq p \leq \infty$*. On the other hand, if*

$$\left(\frac{1}{k^{\sigma+\frac{1}{p}}(\log(k+2))^{\beta}}\right)_k \in \operatorname{mon} H_\infty(B_{\ell_p}) \,,$$

then $\beta \geq \frac{1}{p}$*.*

As a consequence of the theorem we obtain the negative answer to the question whether $\varepsilon = 0$ is possible:

Theorem (cf. Theorem 7.11)**.** *Let* $1 < p < \infty$ *and set* $\frac{1}{q} := \frac{1}{2} + \frac{1}{\max\{p,2\}}$*. Then*

$$\mathrm{B}_{\ell_p} \cap \ell_q \subsetneq \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) \,.$$

Another interesting case is $\mathcal{F}(R)$ denoting the space of m–homogeneous polynomials on ℓ_p, i.e. $\mathcal{F}(R) = \mathcal{P}(^m\ell_p)$. Also for this set of holomorphic functions a complete characterization was known in the cases $p = 1$ and $p = \infty$. For the cases $1 < p < \infty$ we obtain approximations which are stated in Theorem 7.6. If we replace ℓ_p by the LORENTZ space $\ell_{p,\infty}$ with $2 \leq p < \infty$, we even get a complete characterization (cf. Corollary 7.7).

We are furthermore able to establish results (for instance Theorem 7.5) which provide tools to tackle the general case of bounded holomorphic functions on any REINHARDT domain R (in particular the unit ball of BANACH sequence spaces X).

In Chapter 8 we go back to the setting of DIRICHLET series and try to translate our new results back to this setting. We start with the study of so-called ℓ_1–multipliers for certain sets of DIRICHLET series: We call a sequence $(b_n)_n \in \mathbb{C}^{\mathbb{N}}$ an ℓ_1–multiplier for a set $\mathcal{D}$ of DIRICHLET series if

$$\sum_{n=1}^{\infty} |a_n b_n| < \infty$$

for every $D = \sum_n a_n n^{-s} \in \mathcal{D}$. The set $\mathcal{H}_\infty$ will play a particular role as it is via the Bohr transform isometrically isomorphic to the set of all bounded holomorphic functions on B_{c_0}.

It turns out that the multiplicative ℓ_1–multipliers for $\mathcal{D}$ (those for which $b_{nm} = b_n\, b_m$ for every $n, m \in \mathbb{N}$) are exactly those for which $(b_{p_k})_k$ lies in the domain of monomial convergence for $\mathfrak{B}^{-1}(\mathcal{D})$.

In Chapter 9 we point out the connection of the unconditional basis constant with the so-called Bohr radii. We define the n^{th} Bohr radius by

$$K_n := \sup\Big\{0 \le r \le 1 \,\Big|\, \forall f \in H_\infty(\mathrm{B}_{\ell_\infty^n}) : \sup_{x \in r\mathrm{B}_{\ell_\infty^n}} \sum_{\alpha \in \mathbb{N}_0^n} \big|c_\alpha(f)\, x^\alpha\big| \le \|f\|_{\mathrm{B}_{\ell_\infty^n}}\Big\}.$$

Bohr introduced this concept originally to tackle the convergence of power series. He was already able to prove that $K_1 = \frac{1}{3}$ and recently it was proved by Bayart, Pellegrino, and Seoane-Sepúlveda [11], using the method of Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip [27], that

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1\,.$$

Our research enables us to give lower bounds of an even more general definition of Bohr radii: For an index set $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ and a Reinhardt domain $R \subset X$ define

$$K(R;\Lambda) := \sup\Big\{0 \le r \le 1 \,\Big|\, \forall f \in H_\infty(R) : \sup_{x \in rR} \sum_{\alpha \in \Lambda} \big|c_\alpha(f)\, x^\alpha\big| \le \|f\|_R\Big\}.$$

We obtain the following lower estimate. In the case that X is ℓ_p this lower estimate was already proven by Defant and Frerick [26]; by a result of Boas [15] we know that in the case $X = \ell_p$ this lower bound is optimal:

Theorem (cf. Theorem 9.2)**.** *Let $1 \le p \le \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. There exists a constant $c \ge 1$ such that for any p–concave* Banach *sequence space X*

$$c^{-1}\left(\frac{\log n}{n}\right)^{\sigma} \le K\big(\mathrm{B}_{X_n}; \mathbb{N}_0^{(\mathbb{N})}\big)\,.$$

Finally we close this thesis with Chapter 10, which gives a brief overview of the open questions remaining.

Some results presented in this thesis arose from a joint work with Bayart, Defant, and Schlüters [10] (submitted). This pertains Theorem 4.1, the Theorems 6.4 and 6.10, and the Theorems 7.6 and 7.10.

Chapter 2.

Preliminaries — notations and the objects of our study

We use throughout this thesis standard notation from BANACH space theory as used for example in [46] or [35]. The reader is expected to be familiar with the basic results of calculus, function theory, and BANACH space theory.

As usual $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the natural numbers, non-negative integers, integers, real numbers, and the complex numbers respectively. By $\mathbb{T}$, called torus, we want to denote the set of all $x \in \mathbb{C}$ with $|x| = 1$. We say that a function $f : \mathbb{T} \to \mathbb{C}$ is LEBÉSGUE measurable if the mapping $t \mapsto f(\mathrm{e}^{it})$ is LEBÉSGUE measurable on $[0, 2\pi]$. Analogously a function f on the torus is said to be LEBÉSGUE integrable if $t \mapsto f(\mathrm{e}^{it})$ is integrable on $[0, 2\pi]$; in this case we set

$$\int_{\mathbb{T}} f(x)\,\mathrm{d}m_1(x) := \int\limits_0^{2\pi} f(\mathrm{e}^{it})\,\frac{\mathrm{d}t}{2\pi} = \frac{1}{2\pi i} \oint\limits_{|\xi|=1} \frac{f(\xi)}{\xi}\,\mathrm{d}\xi$$

where the latter integral denotes the contour integral along the boundary of the unit disc in $\mathbb{C}$. We refer to m_1 as the normalized LEBÉSGUE measure on $\mathbb{T}$ and by m_n we denote the respective product measure on $\mathbb{T}^n$. The functions $\xi \mapsto \xi^k$, $k \in \mathbb{Z}$ are easily seen to be an orthonormal system in $L_2(\mathbb{T})$; that is

$$\int_{\mathbb{T}} \xi^k\,\mathrm{d}m_1(\xi) = \begin{cases} 1 & \text{if } k = 0 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For a BANACH space E we denote its norm by $\|\cdot\|_E$; we may omit the subscript where it is clear from the context which space is meant. The open unit ball in E will be denoted by B_E. We consider only complex BANACH spaces.

The space of all bounded linear operators from E into another BANACH space F is denoted by $\mathcal{L}(E;F)$; by $E' := \mathcal{L}(E;\mathbb{C})$ we denote the dual space of E. The pairing of a functional $x' \in E'$ and an element $x \in E$ will sometimes denoted by $\langle x', x\rangle := x'(x)$.

Recall that a sequence $(b_k)_k$ in a BANACH space E is called SCHAUDER basis if for every $x \in E$ there exists a unique sequence $(x_k)_k \in \mathbb{C}^{\mathbb{N}}$ such that $x = \sum_k x_k b_k$. Note that in the description of a SCHAUDER basis the order of the sequence is not negligible. We call a sequence $(b_k)_k$ SCHAUDER basic sequence in E if $(b_k)_k$ is a SCHAUDER basis of $\overline{\operatorname{span}\{b_k \mid k \in \mathbb{N}\}}^E$, the closure of the linear span of $(b_k)_k$ in E.

As we will consider only SCHAUDER bases in infinite dimensional spaces, we shall speak merely of bases.

Proposition 2.1 (cf. Proposition 1.a.3 in [46])**.** *A sequence $(b_k)_k$ in a* BANACH *space E is a basic sequence if and only if the following two conditions hold true:*

(i) $b_k \neq 0$ for any $k \in \mathbb{N}$.

(ii) There exists $c \geq 1$ such that for all $n, N \in \mathbb{N}$ with $N \geq n$ and $(a_k)_k \in \mathbb{C}^{\mathbb{N}}$

$$\Big\|\sum_{k=1}^{n} a_k b_k\Big\|_E \leq c \Big\|\sum_{k=1}^{N} a_k b_k\Big\|_E .$$

For $1 \leq p < \infty$ we denote by ℓ_p the BANACH space of sequences $x = (x_k)_k \in \mathbb{C}^{\mathbb{N}}$ whose p^{th} power is summable; i.e. those for which $\|x\|_{\ell_p} := (\sum_k |x_k|^p)^{\frac{1}{p}} < \infty$. The space ℓ_∞ is the BANACH space of all bounded sequences $x = (x_k)_k$ equipped with the usual supremum norm $\|x\|_{\ell_\infty} := \sup_k |x_k|$. As usual we denote for $1 \leq p \leq \infty$ by p' the conjugate exponent; that is $1 \leq p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention $\frac{1}{\infty} := 0$.

The closed subspace of all sequences converging to 0 is denoted by c_0. For $n \in \mathbb{N}$ and $1 \leq p \leq \infty$, ℓ_p^n denotes the linear space $\mathbb{C}^n$ equipped with the respective norm. It will be convenient to think of ℓ_p^n as a subspace of ℓ_p and of ℓ_∞^n as a subspace of c_0 or ℓ_∞.

For sequences $x = (x_k)_k \in \ell_\infty$ we denote by $|x|$ the sequence $(|x_k|)_k$ and by x^* the sequence $(x_k^*)_k$ defined by

$$x_k^* := \inf\Big\{\sup\big\{|x_j| \,\big|\, j \in \mathbb{N} \setminus A\big\} \,\Big|\, A \subset \mathbb{N}, |A| < k\Big\}.$$

x^* is called the non-increasing rearrangement of $|x|$. If $x \in c_0$, then there exists a permutation σ of the natural numbers such that $x_k^* = x_{\sigma(k)}$.

Notions such as $|y| \le |x|$, x^σ, or $x \cdot y$ for sequences $x, y \in \ell_\infty$ and $\sigma \in \mathbb{R}$ are meant to be understood pointwise; for instance $|y| \le |x|$ if and only if $|y_k| \le |x_k|$ for all $k \in \mathbb{N}$ or $x \cdot y := (x_k y_k)_k$. For $\omega \in \ell_\infty$ we define

$$D_\omega : \ell_\infty \to \ell_\infty, \quad x \mapsto \omega \cdot x .$$

For $1 \le p, q \le \infty$ we define the Lorentz spaces $\ell_{p,q}$ as the space of all $x \in \ell_\infty$ for which $\big(k^{\frac{1}{p}-\frac{1}{q}} x_k^*\big)_k \in \ell_q$. Define

$$\rho_{p,q}(x) := \left(\sum_{k=1}^{n} \big(k^{\frac{1}{p}-\frac{1}{q}} x_k^*\big)^q\right)^{\frac{1}{q}} .$$

In the case $q = \infty$ the definition of $\rho_{p,q}(\,\cdot\,)$ is modified in the usual way, i.e.

$$\rho_{p,\infty}(x) := \sup_{k \in \mathbb{N}} k^{\frac{1}{p}} x_k^* .$$

In general, $\rho_{p,q}(\,\cdot\,)$ does not define a norm on $\ell_{p,q}$ but rather a complete quasi-norm; i.e. the triangle inequality holds with a constant $c > 1$: $\rho_{p,q}(x+y) \le c\,\big(\rho_{p,q}(x) + \rho_{p,q}(y)\big)$. It is easy to see that $\ell_{p,p} = \ell_p$ and that $\ell_{p,q} \subset \ell_{\tilde{p},\tilde{q}}$ whenever $(p,q) \le (\tilde{p},\tilde{q})$ lexicographically.

Where applicable, we denote by e_k, $k \in \mathbb{N}$ the canonical sequences $(0, \ldots, 0, \overset{\substack{k^{\text{th}}\text{ position}\\ \downarrow}}{1}, 0 \ldots)$. It is well known that these sequences form a Schauder basis of ℓ_p for $1 \le p < \infty$ and of c_0. The biorthogonal functionals are denoted by e_k', $k \in \mathbb{N}$; i.e. $e_k' : \ell_\infty \to \mathbb{C}$ such that $e_k'(e_l) = 1$ if $k = l$ and $e_k'(e_l) = 0$ otherwise. Note that the $e_k', k \in \mathbb{N}$ not necessarily define a basis (in the sense of a Schauder basis) of the dual space.

We call a linear subspace $X \subset \ell_\infty$ equipped with a complete norm $\|\cdot\|_X$ a Banach sequence space if $x \in X$ and $y \in \ell_\infty$ with $|y| \le |x|$ implies $y \in X$ and $\|y\|_X \le \|x\|_X$. Without loss of generality we may assume that $\{e_k \,|\, k \in \mathbb{N}, k \le \dim X\} \subset X$ and that

$\|e_k\|_X = 1$ for every $k \in \mathbb{N}$ with $k \leq \dim X$. A BANACH sequence space X is called symmetric if $x^* \in X$ if and only if $x \in X$ and in this case $\|x^*\|_X = \|x\|_X$.

For $n \in \mathbb{N}$ we define the n–dimensional section of X by $X_n := \operatorname{span}\{e_k \,|\, k \leq n\}$. A norm one projection onto X_n is given by

$$P_n : X \to X, \quad x \mapsto \sum_{k=1}^{n} \langle e'_k, x \rangle e_k.$$

2.1. Unconditionality

There exist several notations of summability in BANACH spaces. Those important for our research will be introduced in this section. For a more general study we refer the reader to DIESTEL, JARCHOW, and TONGE [35] or LINDENSTRAUSS and TZAFRIRI [46].

A sequence $(x_k)_k$ in a BANACH space E is called summable if the series $\sum_k x_k$ is convergent. $(x_k)_k$ is called unconditionally summable if $(x_{\pi(k)})_k$ is summable for every permutation π of the natural numbers.

$(x_k)_k$ is called absolutely summable if $(\|x_k\|_E)_k$ as a sequence in $\mathbb{R}$ is summable. In this case define $\|(x_k)_k\|_1 := \sum_k \|x_k\|_E$. We obtain by a straightforward argument:

Proposition 2.2. *Let $(x_k)_k$ be a sequence in a* BANACH *space E. Absolute summability of $(x_k)_k$ implies unconditional summability of $(x_k)_k$ and unconditional summability of $(x_k)_k$ implies summability of $(x_k)_k$.*

Proposition 2.3 (cf. Theorem 1.9 in [35])**.** *Let $(x_k)_k$ be a sequence in a* BANACH *space E. The following are equivalent:*

(i) *$(x_k)_k$ is unconditionally summable.*

(ii) *$(x_k)_k$ is sign summable, i.e. the sequence $(\theta_k x_k)_k$ is summable for every choice $(\theta_k)_k \in \{-1, 1\}^{\mathbb{N}}$ of signs.*

(iii) *$(x_k)_k$ is complex sign summable, i.e. the sequence $(\theta_k x_k)_k$ is summable for every choice $(\theta_k)_k \in \mathbb{T}^{\mathbb{N}}$.*

(iv) *For every $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that $\left\|\sum_{k \in A} x_k\right\| < \varepsilon$ whenever $A \subset \mathbb{N}$ finite with* $\min A \geq N$.

In this case the limit $\sum_k x_{\pi(k)}$ does not depend on the permutation $\pi : \mathbb{N} \to \mathbb{N}$.

Let $(b_k)_k$ be a basic sequence in a Banach space E. Every $x \in \overline{\operatorname{span}\{b_k \mid k\}}^E$ then has a unique representation as a series $x = \sum_k x_k b_k$. The basic sequence is called unconditional if for every x the representing series converges unconditionally. By Proposition 2.3 this is equivalent to the convergence of $\sum_k \theta_k x_k b_k$ for every choice of $(\theta_k)_k \in \mathbb{T}^{\mathbb{N}}$.

By a closed graph argument we have that the operator

$$M_\theta : \overline{\operatorname{span}\{b_k \mid k\}}^E \to \overline{\operatorname{span}\{b_k \mid k\}}^E, \quad \sum_k x_k b_k \mapsto \sum_k \theta_k x_k b_k$$

is continuous for every $\theta = (\theta_k)_k \in \mathbb{T}^{\mathbb{N}}$ and, again by a closed graph theorem, that

$$\chi\big((b_k)_k; E\big) := \sup\big\{\|M_\theta\| \,\big|\, \theta = (\theta_k)_k \in \mathbb{T}^{\mathbb{N}}\big\}$$

is finite. $\chi\big((b_k)_k; E\big)$ is called the unconditional basis constant of the basic sequence $(b_k)_k$ in E; if a basic sequence $(b_k)_k$ is not unconditional we write $\chi\big((b_k)_k; E\big) := \infty$. The following observation is immediate:

Proposition 2.4. *Let $(b_k)_k$ be an unconditional basic sequence in a* Banach *space E. Then for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ is $(b_{\pi(k)})_k$ an unconditional basic sequence with $\chi\big((b_k)_k; E\big) = \chi\big((b_{\pi(k)})_k; E\big)$.*

By a continuity argument it clearly suffices to check the inequalities involved only for finite linear combinations. To be more precise, we have:

Lemma 2.5. *Let $(b_k)_k$ be a basic sequence in a* Banach *space E. $(b_k)_k$ is unconditional if and only if there exists a constant $c \geq 1$ such that for any $n \in \mathbb{N}$, any $(x_k)_k \in \mathbb{C}^n$, and any $(\theta_k)_k \in \mathbb{T}^n$*

$$\Big\|\sum_{k=1}^n \theta_k x_k b_k\Big\|_E \leq c \,\Big\|\sum_{k=1}^n x_k b_k\Big\|_E .$$

In this case $\chi\big((b_k)_k; E\big)$ equals the infimum of all $c \geq 1$ fulfilling the above inequality.

The unconditional basis constant of a Banach space E is the infimum of $\chi\big((b_k)_k; E\big)$ over all possible bases $(b_k)_k$ of E. We denote the unconditional basis constant of E by $\chi(E)$; if E does not posses an unconditional basis we set $\chi(E) := \infty$.

Proposition 2.6. *Let X be a* Banach *sequence space. Then the canonical sequences $(e_k)_{k=1}^{\dim X}$ define an unconditional basic sequence with $\chi\big((e_k)_k; X\big) = 1$.*

Proof. If X is finite dimensional, the canonical sequences clearly define a basis. Otherwise, we have by definition for every $n, N \in \mathbb{N}$ with $n \le N$ that

$$\Big\|\sum_{k=1}^{n} x_k e_k\Big\|_X \le \Big\|\sum_{k=1}^{N} x_k e_k\Big\|_X$$

for any choice of $(x_k)_k \in \mathbb{C}^N$. This proves, by Proposition 2.1, that the $(e_k)_k$ define a basic sequence. Moreover, $\|\theta \cdot x\|_X \le \|x\|_X$ for any $\theta \in \mathbb{T}^{\mathbb{N}}$ and any sequence x, since $|\theta \cdot x| \le |x|$. Therefore, $\chi\big((e_k)_k; X\big) = 1$. $\square$

It is worth noting, that if a Banach space E possesses an unconditional basis, the dual space E' doesn't need to have an unconditional basis. An easy example is given by the Banach space ℓ_1. The canonical vectors are easily seen to be an 1–unconditional basis, but the dual space $\ell_1' = \ell_\infty$ doesn't have a basis at all. However, we have the following theorem:

Theorem 2.7. *Let E be a* Banach *space with a basis $(b_k)_k$ assume that the biorthogonal functionals $(b_k')_k$ define a basis of E'. Then $(b_k)_k$ is an unconditional basis if and only if $(b_k')_k$ is an unconditional basis. In this case*

$$\chi\big((b_k')_k; E'\big) = \chi\big((b_k)_k; E\big).$$

Proof. Let $\theta \in \mathbb{T}^{\mathbb{N}}$. Then for each $x = \sum_k \mu_k b_k \in E$ and $x' = \sum_k \eta_k b_k' \in E'$

$$\langle x', M_\theta x\rangle = \Big\langle \sum_k \eta_k b_k', \sum_l \theta_l \mu_l b_l \Big\rangle = \sum_k \theta_k \eta_k \mu_k = \Big\langle \sum_k \theta_k \eta_k b_k', \sum_l \mu_l b_l \Big\rangle = \langle M_\theta x', x\rangle .$$

Hence, for any $\theta \in \mathbb{T}^{\mathbb{N}}$

$$\|M_\theta : E \to E\| = \sup_{x\in E}\sup_{x'\in E'} |\langle x', M_\theta x\rangle| = \sup_{x'\in E'}\sup_{x\in E} |\langle M_\theta x', x\rangle| = \|M_\theta : E' \to E'\| . \quad \square$$

Sometimes the unconditional basis constant is defined as

$$\sup\big\{\|M_\theta\| \,\big|\, \theta = (\theta_k)_k \in \{-1,1\}^{\mathbb{N}}\big\} .$$

Up to a constant this definition is equivalent to the one we gave above. To be more precise, we have (see e.g. Proposition 1.c.7 in [46])

$$\chi\big((b_k)_k; E\big) \leq 2 \cdot \sup\Big\{\|M_\theta\| \,\Big|\, \theta = (\theta_k)_k \in \{-1,1\}^{\mathbb{N}}\Big\}$$

and obviously

$$\sup\Big\{\|M_\theta\| \,\Big|\, \theta = (\theta_k)_k \in \{-1,1\}^{\mathbb{N}}\Big\} \leq \chi\big((b_k)_k; E\big)\,.$$

For the sake of completeness, let us present an alternative characterization:

Theorem 2.8. *Let $(b_k)_k$ be a sequence in E. The following are equivalent:*

(i) *$(b_k)_k$ is an unconditional basic sequence.*

(ii) *(1)* *$b_k \neq 0$ for every k.*

(2) *There exists $c \geq 1$ such that for every finite $I \subset \mathbb{N}$, every $J \subset I$, every $(\varepsilon_k)_k \in \mathbb{T}^J$, and every $(x_k)_k \in \mathbb{C}^I$*

$$\Big\|\sum_{k\in J} \varepsilon_k x_k b_k\Big\|_E \leq c\,\Big\|\sum_{k\in I} x_k b_k\Big\|_E\,. \tag{2 · A}$$

In this case $\chi\big((b_k)_k; E\big) = \inf\{c \geq 1 \,|\, c$ *fulfills* $(2\cdot\mathrm{A})\}$.

Proof. Let $(b_k)_k$ be an unconditional basic sequence. Obviously $b_k \neq 0$ for any $k \in \mathbb{N}$. Let $I \subset \mathbb{N}$ finite, $J \subset I$, $(\varepsilon_k)_k \in \mathbb{T}^J$, and $(x_k)_k \in \mathbb{C}^I$. Then

$$\begin{aligned}\Big\|\sum_{k\in J} \varepsilon_k x_k b_k\Big\|_E &\leq \sup_{x'\in \mathrm{B}_{E'}} \sum_{k\in I} \big|x_k x'(b_k)\big| = \sup_{(\varepsilon_k)_k\in\mathbb{T}^I}\ \sup_{x'\in \mathrm{B}_{E'}} \Big|\sum_{k\in I} \varepsilon_k x_k x'(b_k)\Big| \\ &= \sup_{(\varepsilon_k)_k\in\mathbb{T}^I} \Big\|\sum_{k\in I} \varepsilon_k x_k b_k\Big\|_E \leq \chi\big((b_k)_k; E\big)\,\Big\|\sum_{k\in I} x_k b_k\Big\|_E\,.\end{aligned}$$

Thus $(2\cdot\mathrm{A})$ holds true with a constant less than or equal $\chi\big((b_k)_k; E\big)$.

Let now the second statement hold true. By Proposition 2.1 and the fact that $(2\cdot\mathrm{A})$ holds true for $\varepsilon_k = 1$ we have that $(b_k)_k$ is a basic sequence. Furthermore, $(2\cdot\mathrm{A})$ implies for any $n \in \mathbb{N}$, any $(\varepsilon_k)_k \in \mathbb{T}^n$, and any $(x_k)_k \in \mathbb{C}^n$ that

$$\Big\|\sum_{k=1}^{n} \varepsilon_k x_k b_k\Big\|_E \leq c\,\Big\|\sum_{k=1}^{n} x_k b_k\Big\|_E\,.$$

By Lemma 2.5 this proves the claim. □

2.2. Multilinerar forms, polynomials and holomorphic functions

In this section we want to introduce the main objects of our studies. We will restrict our attention to those essentials needed in our considerations. For a deeper study of the topic and proofs of some of the presented results we refer the reader to Dineen [37].

Let in what follows $m \in \mathbb{N}$ and $E, E^{(1)}, \ldots, E^{(m)}$ be Banach spaces. A mapping $L : E^{(1)} \times \cdots \times E^{(m)} \to \mathbb{C}$ is called m–linear if L is linear in each variable while having the other $m-1$ variables fixed. The linear space of all continuous m–linear mappings from $E^{(1)} \times \cdots \times E^{(m)}$ into $\mathbb{C}$ will be denoted by $\mathcal{L}(E^{(1)}, \ldots, E^{(m)}; \mathbb{C})$. Equipped with the norm

$$\|L\| := \sup_{\substack{x^{(k)} \in \mathrm{B}_{E^{(k)}} \\ k=1,\ldots,m}} |L(x^{(1)}, \ldots, x^{(m)})|$$

this defines a Banach space. In the case $E^{(1)} = \cdots = E^{(m)} = E$ we write $\mathcal{L}({}^m E; \mathbb{C})$ for short.

A mapping $P : E \to \mathbb{C}$ is called m–homogeneous polynomial if there exists an m–linear mapping $L : E^m \to \mathbb{C}$ such that $P = L \circ \Delta_m$ where $\Delta_m : E \ni x \mapsto (x, \ldots, x) \in E^m$. In this case we say the m–linear form L is associated to P. By $\mathcal{P}({}^m E)$ we denote the space of all continuous m–homogeneous polynomials on E; this space endowed with the supremum norm

$$\|P\|_{\mathrm{B}_E} := \sup_{x \in \mathrm{B}_E} |P(x)| \tag{2·B}$$

is a Banach space as well.

We call a mapping $P : E \to \mathbb{C}$ polynomial if P is the sum of finitely many homogeneous polynomials, i.e. $P = \sum_{m=1}^{M} P_m$ with $P_m \in \mathcal{P}({}^m E)$. For obvious reasons, P_m is called the m–homogeneous part of P. The space of all polynomials endowed with the norm defined as in (2 · B) is a Banach space and will be denoted by $\mathcal{P}(E)$.

Let now $U \subset E$ open. A function $f : U \to \mathbb{C}$ is said to be holomorphic if f is Fréchet differentiable on U, i.e. for every $x \in U$ there exists a functional $x' \in E'$ such that

$$\lim_{\substack{h \in E \\ h \to 0}} \frac{f(x+h) - f(x) - \langle x', h \rangle}{\|h\|} = 0\,.$$

We denote the linear space of all holomorphic functions or bounded holomorphic functions $f : U \to \mathbb{C}$ on U by $H(U)$ and $H_\infty(U)$ respectively; endowed with the supremum norm

$$\|f\|_U := \sup_{x \in U} |f(x)|$$

$H_\infty(U)$ is a BANACH space. A straightforward calculation shows that every m–homogeous polynomial on E is holomorphic on B_E:

Proposition 2.9. *Let E denote a* BANACH *space. Then $\mathcal{P}(^m E)$ and $\mathcal{P}(E)$ are closed subspaces of $H_\infty(\mathrm{B}_E)$.*

2.2.1. Polarization — connecting polynomials and multilinear forms

Let Σ_m denote the set of all permutations of $\{1, 2, \ldots, m\}$. We call an m–linear mapping $L \in \mathcal{L}(^m E; \mathbb{C})$ symmetric if for every permutation $\sigma \in \Sigma_m$ and any choice of $x^{(1)}, \ldots, x^{(m)} \in E$

$$L(x^{(1)}, \ldots, x^{(m)}) = L(x^{(\sigma(1))}, \ldots, x^{(\sigma(m))}).$$

The subspace of $\mathcal{L}(^m E; \mathbb{C})$ of all symmetric m–linear forms is denoted by $\mathcal{L}_s(^m E; \mathbb{C})$. Furthermore, the symmetrization operator $\mathcal{S} : \mathcal{L}(^m E; \mathbb{C}) \to \mathcal{L}(^m E; \mathbb{C})$ is defined by

$$\mathcal{S}L(x^{(1)}, \ldots, x^{(m)}) := \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L(x^{(\sigma(1))}, \ldots, x^{(\sigma(m))}).$$

We check at once that $\mathcal{S}$ defines a projection onto $\mathcal{L}_s(^m E; \mathbb{C})$ with $\|\mathcal{S}\| = 1$. Furthermore, we see immediately that $L \circ \Delta_m = \mathcal{S}L \circ \Delta_m$ for $L \in \mathcal{L}(^m E; \mathbb{C})$. Hence, for every $P \in \mathcal{P}(^m E)$ there exists a symmetric m–linear form L associated to P. We get that

$$\mathcal{L}_s(^m E; \mathbb{C}) \to \mathcal{P}(^m E), \quad L \mapsto L \circ \Delta_m$$

is a surjection with norm 1. Moreover, the polarization formula (see e.g. Proposition 1.5, Corollary 1.6 and Proposition 1.8 in [37]) shows that we have in fact an isomorphism:

Proposition 2.10 (cf. Corollary 1.6 in [37])**.** *Let E be a complex* BANACH *space. For $P \in \mathcal{P}(^m E)$ and $L \in \mathcal{L}_s(^m E; \mathbb{C})$ with $L \circ \Delta_m = P$ then*

$$L\big(x^{(1)}, \ldots, x^{(m)}\big) = \frac{1}{2^m m!} \sum_{\varepsilon_k = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\Big(\sum_{k=1}^m \varepsilon_k x^{(k)}\Big)$$

for every choice of $x^{(1)}, \ldots, x^{(m)} \in E$. In particular, $\|P\|_{\mathrm{B}_E} \le \|L\|_{\mathrm{B}_E} \le \frac{m^m}{m!} \|P\|_{\mathrm{B}_E}$.

In some cases the following result provides a substantially better estimate. This result is particularly useful in the case $k = 1$ were we get a constant independent of m. The result is due to HARRIS and can be found as Theorem 1 in [40].

Proposition 2.11 (cf. Theorem 1 in [40])**.** *Let $P \in \mathcal{P}(^m E)$, E a complex* BANACH *space, and let L denote the symmetric m–linear form associated to P. Then for any $k \in \{0, 1, \dots, m\}$*

$$\sup_{x,y\in \mathrm{B}_E} |L(\overbrace{x, \dots, x}^{(m-k)\text{–}times}, \underbrace{y, \dots, y}_{k\text{–}times})| \le \frac{(m-k)!\, k!\, m^m}{(m-k)^{m-k}\, k^k\, m!} \|P\|_{\mathrm{B}_E} .$$

In particular,

$$\sup_{x,y\in \mathrm{B}_E} |L(x, \dots, x, y)| \le \Big(1 + \frac{1}{m-1}\Big)^{m-1} \|P\|_{\mathrm{B}_E} \le \mathrm{e}\, \|P\|_{\mathrm{B}_E} .$$

We have seen that we can reconstruct the symmetric m–linear form L associated to a polynomial $P \in \mathcal{P}(^m E)$ and that $\|L\| \le \frac{m^m}{m!} \|P\|$ by Proposition 2.10.

In the case that $L \in \mathcal{L}(^m E; \mathbb{C}) \setminus \mathcal{L}_s(^m E; \mathbb{C})$ and $P = L \circ \Delta_m$ we do in general not have an norm estimate like in Proposition 2.10. Let for example $L : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{C}$ be defined by $L(x, y) := x_1 y_2 - x_2 y_1$. Then $P = L \circ \Delta_2 = 0$, but $\|L\| = 2$.

Stating certain restrictions on L we can overcome this shortcoming. We introduce the required theory and the result in Section 5.4.4.

2.2.2. Monomials — prototypical polynomials

By $\mathbb{N}_0^{(\mathbb{N})}$ we denote the set of those sequences α of non-negative integers such that $\alpha_k = 0$ for all but finitely many $k \in \mathbb{N}$. We call the elements $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ multi-index and define for such a multi-index $|\alpha| := \alpha_1 + \alpha_2 + \cdots$ and $\alpha! := \alpha_1!\alpha_2! \cdots$. Moreover, for an element $x \in \ell_\infty$ we define $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and with this the monomial $z^\alpha : \ell_\infty \to \mathbb{C}$ by $x \mapsto x^\alpha$. If $|\alpha| = m$ it is clear that $z^\alpha \in \mathcal{P}(^m X)$.

Monomials are in some sort prototypical polynomials. However, the m–homogeneous polynomials define in general not a basis of the full space of m–homogeneous polynomials: For a BANACH sequence space X and $m \in \mathbb{N}$ define the space of approximable polynomials by

$$\mathcal{P}_{\mathrm{app}}(^m X) := \overline{\left\{ \sum_{k=1}^{n} c_k \left(x'_k(\,\cdot\,) \right)^m \,\middle|\, n \in \mathbb{N}, c_k \in \mathbb{C}, x'_k \in X' \right\}}^{\mathcal{P}(^m X)} .$$

Obviously, $z^\alpha \in \mathcal{P}_{\mathrm{app}}(^m X) \subset \mathcal{P}(^m X)$ and thus necessary conditions for the monomials to form a basis of $\mathcal{P}(^m X)$ are

(i) the monomials form a basis of $\mathcal{P}_{\mathrm{app}}(^m X)$ and

(ii) $\mathcal{P}(^m X) = \mathcal{P}_{\mathrm{app}}(^m X)$.

The following proposition now gives a criterion for the monomials to be a basis of $\mathcal{P}_{\mathrm{app}}(^m X)$. Stated originally by Ryan [52] the proof of this result contained a flaw which was corrected in [36].

Proposition 2.12 (cf. Proposition 4.4 in [37])**.** *Let X be a* Banach *sequence space such that the $(e'_k)_k$ define a basis of X'. Then under an appropriate order the monomials $z^\alpha, \alpha \in \Lambda := \left\{ \alpha \in \mathbb{N}_0^{(\mathbb{N})} \,\middle|\, |\alpha| = m \right\}$ form a basis of $\mathcal{P}_{\mathrm{app}}(^m X)$.*

Moreover, Alencar [2] proved under the assumption that X is a Banach sequence space with the approximation property that the second condition holds true if and only if $\mathcal{P}(^m X)$ is reflexive. To be more precise, he proved:

Proposition 2.13 (cf. [2])**.** *Let E be a* Banach *space with the approximation property and let $m \in \mathbb{N}$. Then*

$$\mathcal{P}(^m E) = \mathcal{P}_{\mathrm{app}}(^m E)$$

if and only if $\mathcal{P}(^m E)$ is reflexive.

Pełczyński [48] proved that $\mathcal{P}(^m \ell_p)$ is not reflexive if and only if $m \geq p$. Summarizing, these results show that the monomials in general do not define a basis of $\mathcal{P}(^m X)$.

Since every subsequence of a basic sequence is again a basic sequence we obtain easily the following corollary of Proposition 2.12:

Corollary 2.14. *Let X be a* Banach *sequence space such that the $(e'_k)_k$ define a basis of X' and let $\Lambda \subset \left\{ \alpha \in \mathbb{N}_0^{(\mathbb{N})} \,\middle|\, |\alpha| = m \right\}$. Then the monomials $(z^\alpha)_{\alpha \in \Lambda}$ under an appropriate order form a basic sequence in $\mathcal{P}(^m X)$.*

A suitable order is given by the so-called square order of the monomials (see e.g. Grecu and Ryan [39] or Prengel [50]): For $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ define $\operatorname{len}\alpha := \max\{k \mid \alpha_k \neq 0\}$ and $\alpha^* := (\alpha_1, \dots, \alpha_{n-1}, \alpha_n - 1, 0, \dots)$ if $n = \operatorname{len}\alpha$. For $\alpha, \beta \in \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid |\alpha| = m\}$ we say $\alpha \tilde{\leq}_{\mathrm{sq}} \beta$ if $\alpha = \beta$, or

$$\operatorname{len}\alpha < \operatorname{len}\beta\,,$$

or

$$\operatorname{len}\alpha = \operatorname{len}\beta \quad \text{and} \quad \alpha^* \tilde{\leq}_{\mathrm{sq}} \beta^*\,.$$

This defines an linear order on the set $\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid |\alpha| = m\}$. Let us extend this order to the whole set $\mathbb{N}_0^{(\mathbb{N})}$. We write $\alpha \leq_{\mathrm{sq}} \beta$ for $\alpha, \beta \in \mathbb{N}_0^{(\mathbb{N})}$ if $\alpha = \beta$, or

$$\operatorname{len}\alpha + \sqrt{2}\,|\alpha| < \operatorname{len}\beta + \sqrt{2}\,|\beta|\,,$$

or

$$\operatorname{len}\alpha + \sqrt{2}\,|\alpha| = \operatorname{len}\beta + \sqrt{2}\,|\beta| \quad \text{and} \quad \alpha^* \leq_{\mathrm{sq}} \beta^*\,.$$

This order clearly extends the previously defined one, since for $\alpha, \beta \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = |\beta|$ we have $\alpha \leq_{\mathrm{sq}} \beta$ if and only if $\alpha \tilde{\leq}_{\mathrm{sq}} \beta$. Furthermore, for any $\gamma \in \mathbb{N}_0^{(\mathbb{N})}$ there exist only finitely many $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ such that $\alpha \leq_{\mathrm{sq}} \gamma$, since there exist only finitely many $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ such that $\operatorname{len}\alpha + \sqrt{2}\,|\alpha| \leq \operatorname{len}\gamma + \sqrt{2}\,|\gamma|$.

Theorem 2.15. *The (extended) square order defines a linear order on $\mathbb{N}_0^{(\mathbb{N})}$; that is $\leq_{\mathrm{sq}}$ is reflexive, antisymmetric, transitive, and total.*

Proof. The *reflexivity* is given by definition. To prove the remaining three properties we proceed inductively. For $n \in \mathbb{N}$ define $\Lambda_n := \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid N(\alpha) \leq n\}$ where we write $N(\alpha) := \operatorname{len}\alpha + \sqrt{2}\,|\alpha|$ to keep the proof lucid. Obviously $\leq_{\mathrm{sq}}$ defines a linear order on $\Lambda_1 = \{(0, \dots)\}$. Now let $n > 1$ and assume that $\leq_{\mathrm{sq}}$ defines a linear order on Λ_{n-1}.

We begin by proving the *antisymmetry*. Let $\alpha, \beta \in \Lambda_n$ with $\alpha \leq_{\mathrm{sq}} \beta$ and $\beta \leq_{\mathrm{sq}} \alpha$. Then obviously $N(\alpha) = N(\beta)$, $\alpha^* \leq_{\mathrm{sq}} \beta^*$, and $\beta^* \leq_{\mathrm{sq}} \alpha^*$. Clearly, $\alpha^*, \beta^* \in \Lambda_{n-1}$ and thus $\alpha^* = \beta^*$ by the induction hypothesis. Now $\operatorname{len}\alpha = \operatorname{len}\beta$ and $|\alpha| = |\beta|$, since 1 and $\sqrt{2}$ are linearly independent over the rationals. This implies $\alpha = \beta$.

To prove *transitivity* take $\alpha, \beta, \gamma \in \Lambda_n$ such that $\alpha \leq_{\mathrm{sq}} \beta \leq_{\mathrm{sq}} \gamma$. If $N(\alpha) < N(\gamma)$ we have immediately $\alpha \leq_{\mathrm{sq}} \gamma$. Otherwise $N(\alpha) = N(\beta) = N(\gamma)$ and $\alpha^* \leq_{\mathrm{sq}} \beta^* \leq_{\mathrm{sq}} \gamma^*$.

Since $\alpha^*, \beta^*, \gamma^* \in \Lambda_{n-1}$, we have by the induction hypothesis $\alpha^* \leq_{\mathrm{sq}} \gamma^*$. Thus $\alpha \leq_{\mathrm{sq}} \beta$.

It remains to show that $\leq_{\mathrm{sq}}$ is *total*. Take $\alpha, \beta \in \Lambda_n$. If $N(\alpha) \neq N(\beta)$, then we have either $N(\alpha) < N(\beta)$ or $N(\beta) < N(\alpha)$ and thus $\alpha \leq_{\mathrm{sq}} \beta$ or $\beta \leq_{\mathrm{sq}} \alpha$ respectively. Otherwise, if $N(\alpha) = N(\beta)$, we have by the induction hypothesis $\alpha^* \leq_{\mathrm{sq}} \beta^*$ or $\beta^* \leq_{\mathrm{sq}} \alpha^*$ and thus $\alpha \leq_{\mathrm{sq}} \beta$ or $\beta \leq_{\mathrm{sq}} \alpha$ respectively. $\square$

Whenever applicable, we will implicitly use the square order when referring to the monomials as a basic sequence.

2.2.3. Power series expansion of holomorphic functions

Throughout this section we consider holomorphic functions on Reinhardt domains R in Banach sequence spaces X. We call an open subset $R \subset X$ a Reinhardt domain if $y \in R$ whenever there exists $x \in R$ with $|y| \leq |x|$. In particular the unit ball of any Banach sequence space is a Reinhardt domain. We want to point out that the setting of holomorphic functions especially includes the case of (homogeneous) polynomials.

Proposition 2.16 (cf. Proposition 3.2 in [37]). *Let X be a Banach sequence space and $R \subset X$ a Reinhardt domain. Then for any $f \in H_\infty(R)$ there exists a unique sequence $(P_m)_m$ of homogeneous polynomials (with $P_m \in \mathcal{P}(^m X)$) such that*

$$f(x) = \sum_{m=0}^{\infty} P_m(x)$$

for every $x \in R$ and

$$P_m(x) = \frac{1}{2\pi i} \oint_{|\xi|=1} f(\xi x)\, \xi^{-(m+1)} \, \mathrm{d}\xi = \int_{\mathbb{T}} f(\xi x)\, \xi^{-m} \, \mathrm{d}m_1(\xi)$$

for every $x \in R$ and $m \in \mathbb{N}$. In particular, $\|P_m\|_R \leq \|f\|_R$ for any $m \in \mathbb{N}$.

Let X be a Banach sequence space, $R \subset X$ a Reinhardt domain, and $f \in H_\infty(R)$. From Chapter 3 of [37] we know that on each finite dimensional section $R_n := R \cap X_n$

of R the function f has a power series expansion (uniform convergence on compact subsets of R_n)

$$f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha^{(n)} z^\alpha .$$

The coefficient $c_\alpha^{(n)}(f)$ is given by

$$\begin{aligned} c_\alpha^{(n)}(f) &= \frac{1}{(2\pi i)^n} \oint\limits_{|\xi_n|=r_n} \cdots \oint\limits_{|\xi_1|=r_1} \frac{f(\xi)}{\xi^\alpha \cdot \xi_1 \cdots \xi_n} \,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n \\ &= \int_{\mathbb{T}^n} f(r \cdot \xi)\,(r \cdot \xi)^{-\alpha} \,\mathrm{d}m_n(\xi) \end{aligned} \tag{2$\cdot$C}$$

where $r = (r_k)_k \in R \cap (0, \infty)^n$. This formula, being referred to as the Cauchy integral formula, is well known to be independent of the choice of r. Furthermore, we check easily that $c_\alpha^{(n)}(f) = c_\alpha^{(n+1)}(f)$ for any $\alpha \in \mathbb{N}_0^n \subset \mathbb{N}_0^{n+1}$. Hence, we obtain a unique family $\big(c_\alpha(f)\big)_\alpha$ of complex numbers such that for any $n \in \mathbb{N}$ and any $x \in R_n$

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha x^\alpha .$$

Note that the series expansion on the right-hand side does not necessarily converge for every $x \in R$. We will investigate this circumstance in Chapter 7.

From the Cauchy integral formula we get the following three essential results:

Lemma 2.17. *Let X be a* Banach *sequence space, $R \subset X$ a* Reinhardt *domain, and $f \in H_\infty(R)$. Let $\alpha \in \mathbb{N}_0^{(\mathbb{N})} \setminus \{0\}$ and fix $k \in \mathbb{N}$ such that $\alpha_k \neq 0$. Assume that $t \mapsto f(x + te_k)$ is constant for every $x \in R$. Then $c_\alpha(f) = 0$.*

Proof. Let $n \in \mathbb{N}$ such that $\alpha \in \mathbb{N}_0^n$ and fix $r \in R \cap (0, \infty)^n$. For $\xi \in \mathbb{T}^n$ and α write $\xi' := (\xi_1, \ldots, \xi_{k-1}, 0, \xi_{k+1}, \ldots, \xi_n)$ and analogously α'. By Fubini's theorem then

$$\begin{aligned} c_\alpha(f) &= \int_{\mathbb{T}^n} f(r \cdot \xi)\,(r \cdot \xi)^{-\alpha} \,\mathrm{d}m_n(\xi) \\ &= \int_{\mathbb{T}^{n-1}} f(r \cdot \xi')\,(r \cdot \xi')^{-\alpha'} \,\mathrm{d}m_{n-1}(\xi') \cdot \int_{\mathbb{T}} (r_k \xi_k)^{-\alpha_k} \,\mathrm{d}m_1(\xi_k) , \end{aligned}$$

since $t \mapsto f(\xi + te_k)$ is constant. The latter integral now vanishes as $(z^k)_k$ defines an orthonormal system in $L_2(\mathbb{T})$. $\square$

Proposition 2.18. *Let X be a* Banach *sequence space, $R \subset X$ a* Reinhardt *domain, and $f \in H_\infty(R)$ with $\|f\|_R \le 1$. Then for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$*

$$\big|c_\alpha(f)\big| \le \inf_{r\in R} \frac{1}{|r^\alpha|} .$$

Proof. Fix $n \in \mathbb{N}$ such that $\alpha_k = 0$ for $k > n$. With $r \in R \cap (0,\infty)^n$ the Cauchy integral formula $(2\cdot\mathrm{C})$ shows

$$\big|c_\alpha(f)\big| = \Big|\int_{\mathbb{T}^n} f(r\cdot\xi)\,(r\cdot\xi)^{-\alpha}\,\mathrm{d}m_n(\xi)\Big| \le r^{-\alpha}\int_{\mathbb{T}^n}\big|f(r\cdot\xi)\big|\,\mathrm{d}m_n(\xi) \le r^{-\alpha} ,$$

since $\big|f(r\cdot\xi)\big| \le \|f\|_R \le 1$ for any $\xi \in \mathbb{T}^n$. This yields the claim as r was arbitrary. □

Corollary 2.19. *Let $1 \le p \le \infty$ and $f \in H_\infty(\mathrm{B}_{\ell_p})$ with $\|f\|_{\mathrm{B}_{\ell_p}} \le 1$. Then for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = m$*

$$\big|c_\alpha(f)\big| \le \Big(\frac{m^m}{\alpha^\alpha}\Big)^{\frac{1}{p}} \le \mathrm{e}^{\frac{m}{p}}\Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}} \le \big(\mathrm{e}^m m!\big)^{\frac{1}{p}} .$$

Proof. By the previous proposition we obtain with $\xi := m^{-\frac{1}{p}}\,(\alpha_1{}^{\frac{1}{p}}, \alpha_2{}^{\frac{1}{p}}, \dots) \in \mathrm{B}_{\ell_p}$

$$\big|c_\alpha(f)\big| \le \inf_{r\in\mathrm{B}_{\ell_p}} \frac{1}{|r^\alpha|} \le \frac{1}{\xi^\alpha} = \Big(\frac{m^m}{\alpha^\alpha}\Big)^{\frac{1}{p}} .$$

The second and third inequality now follow by a straightforward calculation. □

We even get this estimate for the full range of Lorentz spaces:

Corollary 2.20. *Let $1 \le p,q \le \infty$ and $f \in H_\infty(\mathrm{B}_{\ell_{p,q}})$ with $\|f\|_{\mathrm{B}_{\ell_{p,q}}} \le 1$. Then for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = m$*

$$\big|c_\alpha(f)\big| \le \Big(\frac{m^m}{\alpha^\alpha}\Big)^{\frac{1}{p}} \le \mathrm{e}^{\frac{m}{p}}\Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}} \le \big(\mathrm{e}^m m!\big)^{\frac{1}{p}} .$$

Proof. At first notice that $\mathrm{B}_{\ell_{p-\varepsilon}} \subset \mathrm{B}_{\ell_{p,q}}$ for any $\varepsilon > 0$. Thus by Proposition 2.18 and the proof of the previous corollary

$$\big\|c_\alpha : \mathcal{P}({}^\Lambda\ell_{p,q}) \to \mathbb{C}\big\| \le \inf_{x\in\mathrm{B}_{\ell_{p,q}}} \frac{1}{|x^\alpha|} \le \inf_{x\in\mathrm{B}_{\ell_{p-\varepsilon}}} \frac{1}{|x^\alpha|} \le \mathrm{e}^{\frac{m}{p-\varepsilon}}\Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p-\varepsilon}} .$$

Letting $\varepsilon \to 0$ we obtain the claim. □

The inequality of the foregoing proposition is sharp: For $f := \left(\sup_{r\in R}|r^\alpha|\right)^{-1} z^\alpha$ obviously $\|f\|_R = 1$ and $c_\alpha(f) = \inf_{r\in R}|r^\alpha|^{-1}$. With Lemma 1.38 in [37] we have

$$\|c_\alpha : H_\infty(\mathrm{B}_{\ell_p}) \to \mathbb{C}\| = \left(\frac{m^m}{\alpha^\alpha}\right)^{\frac{1}{p}}.$$

Lemma 2.21. *Let X be a* Banach *sequence space, $R \subset X$ a* Reinhardt *domain, and $f : R \to \mathbb{C}$ a holomorphic function. Let furthermore $\omega \in \ell_\infty$ with $\|\omega\|_{\ell_\infty} \le 1$. Then is $f \circ D_\omega : R \to \mathbb{C}$ a holomorphic function as well and $c_\alpha(f \circ D_\omega) = \omega^\alpha\, c_\alpha(f)$ for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$.*

Proof. Having the definitions of Reinhardt domains and Banach sequence spaces in mind, we check easily that D_ω maps R linearly into R with $\|D_\omega\| \le \|\omega\|_{\ell_\infty}$ and thus $f \circ D_\omega$ is holomorphic on R.

Let now $f \in H_\infty(R)$, $\alpha \in \mathbb{N}_0^n \subset \mathbb{N}_0^{(\mathbb{N})}$, and $r \in R_n \cap (0,\infty)^n$. By Lemma 2.17 we may assume that $\omega_k \ne 0$ for all k with $\alpha_k \ne 0$ as otherwise the claim is trivial. Therefore,

$$\begin{aligned} c_\alpha(f \circ D_\omega) &= \int_{\mathbb{T}^n} f \circ D_\omega(r\cdot\xi)\,(r\cdot\xi)^{-\alpha}\,\mathrm{d}m_n(\xi) \\ &= \omega^\alpha \int_{\mathbb{T}^n} f\big((\omega\cdot r)\cdot\xi\big)\,\big((\omega\cdot r)\cdot\xi\big)^{-\alpha}\,\mathrm{d}m_n(\xi) \\ &= \omega^\alpha\, c_\alpha(f). \end{aligned}$$

□

2.2.4. Specific spaces of polynomials

Let now X denote a Banach sequence space. We are interested in specific subspaces of $H_\infty(\mathrm{B}_X)$. For $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ define

$$\begin{aligned} \mathcal{P}({}^\Lambda X) &:= \left\{ f \in H_\infty(\mathrm{B}_X) \,\middle|\, \forall \alpha \in \mathbb{N}_0^{(\mathbb{N})} \setminus \Lambda : c_\alpha(f) = 0 \right\}, \\ \mathcal{P}_{\mathrm{fin}}({}^\Lambda X) &:= \operatorname{span}\left\{ z^\alpha \,\middle|\, \alpha \in \Lambda \right\}, \end{aligned}$$

and

$$\mathcal{P}_{\mathrm{mon}}({}^\Lambda X) := \overline{\operatorname{span}\{z^\alpha \mid \alpha \in \Lambda\}}^{\mathcal{P}({}^\Lambda X)}.$$

Note that by Proposition 2.12 the monomials form a basis of $\mathcal{P}_{\mathrm{app}}({}^m X)$ if the coefficent functionals $(e_k')_k$ form a basis of X'; in this case we have have $\mathcal{P}_{\mathrm{app}}({}^m X) = \mathcal{P}_{\mathrm{mon}}({}^\Lambda X)$ where Λ denotes the set of all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = m$.

If Λ is a finite set of indices we clearly have that the monomials define a basis of $\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X) = \mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X) = \mathcal{P}({}^{\Lambda}X)$. However, in the case that Λ is not finite it is vital to note that the spaces $\mathcal{P}({}^{\Lambda}X)$ and $\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)$ in general do not coincide and thus the monomials define in general not a basis of $\mathcal{P}({}^{\Lambda}X)$.

Given $n, m \in \mathbb{N}$ we consider throughout this thesis the following index sets

$$\begin{aligned} \boldsymbol{\Lambda}(n,m) &:= \left\{\alpha \in \mathbb{N}_0^n \,\middle|\, |\alpha| = m\right\}, \\ \boldsymbol{\Lambda}(\infty,m) &:= \bigcup_n \boldsymbol{\Lambda}(n,m), \end{aligned} \tag{2 $\cdot$ D}$$

and

$$\boldsymbol{\Lambda}(n,\,\cdot\,) := \bigcup_m \boldsymbol{\Lambda}(n,m). \tag{2 $\cdot$ E}$$

It follows easily that $\mathcal{P}({}^{m}X) = \mathcal{P}({}^{\boldsymbol{\Lambda}(\infty,m)}X)$ for any $m \in \mathbb{N}$ and $\mathcal{P}({}^{\Lambda}X) = \mathcal{P}({}^{\Lambda}X_n)$ if $\Lambda \subset \boldsymbol{\Lambda}(n,\,\cdot\,)$. Moreover, if Λ is finite, then $\mathcal{P}({}^{\Lambda}X) = \operatorname{span}\{z^\alpha \,|\, \alpha \in \Lambda\}$ and there exists some $n \in \mathbb{N}$ such that $\Lambda \subset \boldsymbol{\Lambda}(n,\,\cdot\,)$.

Most of the time it will be convenient to use a second index notation. Let again $n, m \in \mathbb{N}$. We define the index sets

$$\mathcal{I}(n,m) := \left\{\boldsymbol{i} = (i_1,\ldots,i_m) \in \mathbb{N}^m \,\middle|\, \forall k : i_k \le n\right\} = \{1,2,\ldots,n\}^m$$

and

$$\mathcal{J}(n,m) := \left\{\boldsymbol{j} = (j_1,\ldots,j_m) \in \mathcal{I}(n,m) \,\middle|\, j_1 \le j_2 \le \cdots \le j_m\right\}.$$

$\mathcal{I}(\infty,m)$, $\mathcal{I}(n,\,\cdot\,)$, $\mathcal{J}(\infty,m)$, and $\mathcal{J}(n,\,\cdot\,)$ are defined analogous to the definitions (2 $\cdot$ D) and (2 $\cdot$ E). For indices $\boldsymbol{i} = (i_1,\ldots,i_m), \boldsymbol{j} = (j_1,\ldots,j_{\tilde{m}}) \in \mathcal{I}(\infty,\,\cdot\,)$ we write $(\boldsymbol{i},\boldsymbol{j})$ for the concatenation of the two, i.e. $(\boldsymbol{i},\boldsymbol{j}) = (i_1,\ldots,i_m,j_1,\ldots,j_{\tilde{m}})$. In expressions such as $(\boldsymbol{i},k)$ with $\boldsymbol{i} \in \mathcal{I}(\infty,\,\cdot\,)$ and $k \in \mathbb{N}$ we interpret k as an index in $\mathcal{J}(\infty,1)$.

There is a bijective relation between $\boldsymbol{\Lambda}(n,m)$ and $\mathcal{J}(n,m)$. Given $\boldsymbol{j} \in \mathcal{J}(n,m)$ define $\alpha = \alpha(\boldsymbol{j})$ by $\alpha_l := \big|\{k \,|\, j_k = l\}\big|$. Vice versa set $\boldsymbol{j} = \boldsymbol{j}(\alpha) := (1, \overset{\alpha_1}{\ldots}, 1, 2, \overset{\alpha_2}{\ldots}, 2, \ldots)$ for any $\alpha \in \boldsymbol{\Lambda}(n,m)$.

Using this identification, the notation $\mathcal{P}({}^{J}X)$ with $J \subset \mathcal{J}(\infty,\,\cdot\,)$ is well-defined and we have for the monomials $z_{\boldsymbol{j}} : X \ni x \mapsto x_{\boldsymbol{j}} := x_{j_1} \cdots x_{j_m}$, $\boldsymbol{j} \in \mathcal{J}(\infty,\,\cdot\,)$ that $z_{\boldsymbol{j}} = z^\alpha$ whenever $\boldsymbol{j} = \boldsymbol{j}(\alpha)$.

On $\mathcal{I}(n,m)$ we define an equivalence relation as follows: $\boldsymbol{i} \sim \boldsymbol{j}$ if $\boldsymbol{i}$ is a rearrangement of $\boldsymbol{j}$, i.e. if there exists a permutation $\sigma \in \Sigma_m$ such that $\boldsymbol{j} = \sigma(\boldsymbol{i}) := (i_{\sigma(1)}, \ldots, i_{\sigma(m)})$. The equivalence class of $\boldsymbol{i} \in \mathcal{I}(n,m)$ with respect to this equivalence relation is denoted by $[\boldsymbol{i}]$. Note that for every $\boldsymbol{i} \in \mathcal{I}(n,m)$ there exits a unique $\boldsymbol{j} \in \mathcal{J}(n,m)$ so that $\boldsymbol{i} \sim \boldsymbol{j}$. We will denote this unique representative by $\boldsymbol{i}^*$. Using the identification above we obtain by a straightforward combinatorial argument that $\big|[\boldsymbol{j}]\big| = \frac{m!}{\alpha!}$ for $\boldsymbol{j} = \boldsymbol{j}(\alpha)$.

Homogeneous polynomials are, by definition, the restriction of multilinear forms to the diagonal. Those multilinear forms defining the monomials are of a certain form. Let $\boldsymbol{i} \in \mathcal{I}(n,m)$ and define

$$e'_{\boldsymbol{i}} : X^m \to \mathbb{C}, \quad \left(x^{(1)}, \ldots, x^{(m)}\right) \mapsto e'_{i_1}\left(x^{(1)}\right) \cdots e'_{i_m}\left(x^{(m)}\right).$$

For $\boldsymbol{j} \in \mathcal{J}(n,m)$ we see at once that $e'_{\boldsymbol{j}} \circ \Delta_m = z_{\boldsymbol{j}}$ and that $e'_{\boldsymbol{i}} \circ \Delta_m = e'_{\boldsymbol{j}} \circ \Delta_m = z_{\boldsymbol{j}}$ if $\boldsymbol{i} \sim \boldsymbol{j}$. On the other hand, these mappings define a basis of $\mathcal{L}({}^m X_n; \mathbb{C})$. Indeed, for every $L \in \mathcal{L}({}^m X_n; \mathbb{C})$

$$L = \sum_{\boldsymbol{i} \in \mathcal{I}(n,m)} c_{\boldsymbol{i}}(L)\, e'_{\boldsymbol{i}}$$

where $c_{\boldsymbol{i}}(L) := L(e_{i_1}, \ldots, e_{i_m})$. We check at once that L is symmetric if and only if $c_{\boldsymbol{i}}(L) = c_{\boldsymbol{j}}(L)$ whenever $\boldsymbol{i} \sim \boldsymbol{j}$. In the case that L is symmetric, we have for $P = L \circ \Delta_m$ and $\boldsymbol{j} \in \mathcal{J}(n,m)$

$$c_{\boldsymbol{j}}(P) = \big|[\boldsymbol{j}]\big|\, c_{\boldsymbol{j}}(L) \tag{2 $\cdot$ F}$$

since

$$\begin{aligned} L \circ \Delta_m &= \Bigg(\sum_{\boldsymbol{i} \in \mathcal{I}(n,m)} c_{\boldsymbol{i}}(L)\, e'_{\boldsymbol{i}} \Bigg) \circ \Delta_m \\ &= \sum_{\boldsymbol{j} \in \mathcal{J}(n,m)} \sum_{\boldsymbol{i} \in [\boldsymbol{j}]} c_{\boldsymbol{i}}(L)\, e'_{\boldsymbol{i}} \circ \Delta_m = \sum_{\boldsymbol{j} \in \mathcal{J}(n,m)} \big|[\boldsymbol{j}]\big|\, c_{\boldsymbol{j}}(L)\, z_{\boldsymbol{j}}\,. \end{aligned}$$

Part I.

Unconditional basis constants of spaces of polynomials

Chapter 3.

Introduction and first results

Let Λ be a set of indices and X a Banach sequence space. Assume for the moment that the monomials define (extended square order) a basic sequence in $\mathcal{P}({}^{\Lambda}X)$. We have seen that the unconditionality of the monomials is equivalent to the finiteness of the unconditional basis constant (see the definition in Section 2.1)

$$\chi\big((z^{\alpha})_{\alpha};\mathcal{P}({}^{\Lambda}X)\big)=\sup\Big\{\|M_{\theta}\|\;\Big|\;\theta=(\theta_{\alpha})_{\alpha}\in\mathbb{T}^{\Lambda}\Big\}$$

where $M_{\theta}:\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)\to\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)$ is defined by $z^{\alpha}\mapsto\theta_{\alpha}z^{\alpha}$. Thus, by Lemma 2.5, the monomials $(z^{\alpha})_{\alpha}$ are an unconditional basic sequence if and only if there exists $c\geq 1$ such that for any $\gamma\in\mathbb{N}_0^{(\mathbb{N})}$, any choices of $(\theta_{\alpha})_{\alpha}\in\mathbb{T}^{\Lambda}$, and any $(c_{\alpha})_{\alpha}\in\mathbb{C}^{\Lambda}$

$$\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}\theta_{\alpha}c_{\alpha}z^{\alpha}\Big\|_{\mathrm{B}_X}\leq c\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}c_{\alpha}z^{\alpha}\Big\|_{\mathrm{B}_X}. \tag{3$\cdot$A}$$

On the other hand we have for any $\gamma\in\mathbb{N}_0^{(\mathbb{N})}$ and every choice of $(c_{\alpha})_{\alpha}\in\mathbb{C}^{\Lambda}$ that

$$\begin{aligned}\sup_{(\theta_{\alpha})_{\alpha}\in\mathbb{T}^{\Lambda}}\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}\theta_{\alpha}c_{\alpha}z^{\alpha}\Big\|_{\mathrm{B}_X}&=\sup_{(\theta_{\alpha})_{\alpha}\in\mathbb{T}^{\Lambda}}\sup_{x\in\mathrm{B}_X}\Big|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}\theta_{\alpha}c_{\alpha}x^{\alpha}\Big|\\ &=\sup_{x\in\mathrm{B}_X}\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}\big|c_{\alpha}x^{\alpha}\big|=\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}|c_{\alpha}|\,z^{\alpha}\Big\|_{\mathrm{B}_X}.\end{aligned} \tag{3$\cdot$B}$$

Therefore, the existence of a constant $c\geq 1$ fulfilling (3 · A) implies for any $\beta,\gamma\in\mathbb{N}_0^{(\mathbb{N})}$ with $\beta\leq_{\mathrm{sq}}\gamma$ and any choice of $(c_{\alpha})_{\alpha}\in\mathbb{C}^{\Lambda}$ that

$$\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\beta}}c_{\alpha}z^{\alpha}\Big\|_{\mathrm{B}_X}\leq\sup_{x\in\mathrm{B}_X}\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\beta}}\big|c_{\alpha}x^{\alpha}\big|\leq\sup_{x\in\mathrm{B}_X}\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}\big|c_{\alpha}x^{\alpha}\big|\leq c\Big\|\sum_{\substack{\alpha\in\Lambda\\ \alpha\leq_{\mathrm{sq}}\gamma}}c_{\alpha}z^{\alpha}\Big\|_{\mathrm{B}_X}.$$

By Proposition 2.1 this implies that the monomials define a basic sequence in $\mathcal{P}({}^{\Lambda}X)$ which, moreover, is unconditional. We collect these results in the following theorem:

Theorem 3.1. *Let X be a* BANACH *sequence space, $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$, and $c \geq 1$. Then the following are equivalant:*

(i) $(z^\alpha)_\alpha$ is an unconditional basic sequence in $\mathcal{P}({}^{\Lambda}X)$ with $\chi\big((z^\alpha)_\alpha; \mathcal{P}({}^{\Lambda}X)\big) \leq c$.

(ii) $(z^\alpha)_\alpha$ is a basic sequence in $\mathcal{P}({}^{\Lambda}X)$ and for any $P \in \mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)$ and any choice of $(\theta_\alpha)_\alpha \in \mathbb{T}^\Lambda$

$$\Big\| \sum_{\alpha \in \Lambda} \theta_\alpha c_\alpha(P)\, z^\alpha \Big\|_{\mathrm{B}_X} \leq c\, \|P\|_{\mathrm{B}_X} .$$

(iii) For any $P \in \mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X)$ and any choice of $(\theta_\alpha)_\alpha \in \mathbb{T}^\Lambda$

$$\Big\| \sum_{\alpha \in \Lambda} \theta_\alpha c_\alpha(P)\, z^\alpha \Big\|_{\mathrm{B}_X} \leq c\, \|P\|_{\mathrm{B}_X} .$$

We denote now

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) := \inf\big\{ c \geq 1 \,\big|\, c \text{ fulfilling (iii) of Theorem 3.1 } \big\} .$$

In the case that there does not exist a constant fulfilling (iii) of Theorem 3.1, we write $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) = \infty$.

From $(3 \cdot \mathrm{B})$ we obtain further characterizations: $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big)$ denotes the best constant $c > 1$ satisfying any of the three inequalities

$$\Big\| \sum_{\alpha \in \Lambda} \theta_\alpha c_\alpha(P)\, z^\alpha \Big\| \leq c\, \|P\|_{\mathrm{B}_X} \qquad \text{for all } P \in \mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X) \text{ and } (\theta_\alpha)_\alpha \in \mathbb{T}^\Lambda,$$

$$\sum_{\alpha \in \Lambda} \big| c_\alpha(P)\, x^\alpha \big| \leq c\, \|P\|_{\mathrm{B}_X} \qquad \text{for all } P \in \mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X) \text{ and } x \in \mathrm{B}_X,$$

or

$$\Big\| \sum_{\alpha \in \Lambda} \big|c_\alpha(P)\big|\, z^\alpha \Big\| \leq c\, \|P\|_{\mathrm{B}_X} \qquad \text{for all } P \in \mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X).$$

At this point it may seem to be a constraint to consider only the monomials as a (potentially unconditional) basic sequence in $\mathcal{P}({}^{\Lambda}X)$. We will see in Chapter 5, Theorem 5.2, that this is in fact not a restriction: For any set of indices $\Lambda \subset \mathbf{\Lambda}(\infty, m)$ we have

$$\chi\big(\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)\big) \leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) \leq 2^m \cdot \chi\big(\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X)\big) .$$

In other words: $\mathcal{P}_{\text{mon}}({}^{\Lambda}X)$ possesses an unconditional basis if and only if the monomials define an unconditional basis of $\mathcal{P}_{\text{mon}}({}^{\Lambda}X)$. And in this case the unconditional basis constants differ only up to a factor 2^m.

3.1. A general estimate and extreme examples

We may now introduce a general estimate for the unconditional basis constant. The result is in most cases not optimal, but can't be improved in general. We will provide extreme examples which indicate this.

Theorem 3.2. *Let Λ be a finite set of indices and X a* Banach *sequence space. Then*

$$\chi_{\text{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) \le |\Lambda|^{\frac{1}{2}}.$$

The proof of this general estimate uses only elementary methods. The following result is essential in this and forthcoming proofs.

Lemma 3.3. *Let $\alpha, \beta \in \mathbb{N}_0^n$. Then*

$$\int_{\mathbb{T}^n} \xi^{\alpha-\beta} \,\mathrm{d}m_n(\xi) = \begin{cases} 1 & \text{if } \alpha = \beta \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition we have

$$\int_{\mathbb{T}^n} \xi^{\alpha-\beta} \,\mathrm{d}m_n(\xi) = \prod_{k=1}^{n} \int_{\mathbb{T}} \xi^{\alpha_k} \xi^{-\beta_k} \,\mathrm{d}m_1(\xi) = \prod_{k=1}^{n} \int_{\mathbb{T}} \xi^{\alpha_k} \overline{\xi^{\beta_k}} \,\mathrm{d}m_1(\xi)\,.$$

$(z^k)_{k\in\mathbb{Z}}$ defines an orthonormal system in $L_2(\mathbb{T})$; therefore, the product vanishes whenever $\alpha_k - \beta_k \neq 0$ for some k; otherwise each factor evaluates to 1. $\square$

Lemma 3.4. *Let $\Lambda \subset \mathbf{\Lambda}(n,\,\cdot\,)$ be a finite set of indices and let $P \in \mathcal{P}({}^{\Lambda}\ell_\infty)$. Then*

$$\Big(\sum_{\alpha\in\Lambda} |c_\alpha(P)|^2\Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{T}^n} |P(\xi)|^2 \,\mathrm{d}m_n(\xi)\Big)^{\frac{1}{2}}.$$

In particular,

$$\Big(\sum_{\alpha\in\Lambda} |c_\alpha(P)|^2\Big)^{\frac{1}{2}} \le \sup_{\xi\in\mathbb{T}^n} |P(\xi)| \le \|P\|_{\mathrm{B}_{\ell_\infty}}\,.$$

Proof. Let $P \in \mathcal{P}({}^{\Lambda}\ell_\infty)$. Then

$$\int_{\mathbb{T}^n} |P(\xi)|^2 \,\mathrm{d}m_n(\xi) = \int_{\mathbb{T}^n} \Big(\sum_{\alpha\in\Lambda} c_\alpha(P)\, \xi^\alpha \Big) \Big(\sum_{\beta\in\Lambda} \overline{c_\beta(P)}\, \xi^{-\beta} \Big) \,\mathrm{d}m_n(\xi)$$
$$= \sum_{\alpha,\beta\in\Lambda} c_\alpha(P)\, \overline{c_\beta(P)} \int_{\mathbb{T}^n} \xi^{\alpha-\beta} \,\mathrm{d}m_n(\xi)$$

which, by Lemma 3.3, evaluates to

$$= \sum_{\alpha\in\Lambda} \big|c_\alpha(P)\big|^2 . \qquad \square$$

Proof of Theorem 3.2. Since Λ is finite, we can choose $n \in \mathbb{N}$ such that $\Lambda \subset \mathbf{\Lambda}(n, \cdot\,)$. Let us at first investigate the case $X = \ell_\infty^n$. Let $P \in \mathcal{P}({}^{\Lambda}\ell_\infty^n)$ and $x \in \mathrm{B}_{\ell_\infty^n}$. By CAUCHY-SCHWARZ inequality and Lemma 3.4, we have

$$\sum_{\alpha\in\Lambda} \big|c_\alpha(P) x^\alpha\big| \le \sum_{\alpha\in\Lambda} \big|c_\alpha(P)\big| \le |\Lambda|^{\frac12} \Big(\sum_{\alpha\in\Lambda} \big|c_\alpha(P)\big|^2 \Big)^{\frac12} \le |\Lambda|^{\frac12} \, \|P\|_{\mathrm{B}_{\ell_\infty}} . \tag{3·C}$$

We now consider any BANACH sequence space X. Let $P \in \mathcal{P}({}^{\Lambda}X_n)$ and $x \in \mathrm{B}_{X_n}$. Choose $\omega \in \mathrm{B}_{\ell_\infty^n}$ and $r \in \mathrm{B}_X$ such that $x = r \cdot \omega$; for example $r = (1+\varepsilon)x$ and $\omega = (\frac{1}{1+\varepsilon})_k$ with $\varepsilon > 0$ sufficiently small. Then, by (3 · C),

$$\sum_{\alpha} \big|c_\alpha(P)\, x^\alpha\big| = \sum_{\alpha} \big|\big(c_\alpha(P)\, r^\alpha\big)\, \omega^\alpha\big| \le |\Lambda|^{\frac12} \, \Big\| \sum_{\alpha} \big(c_\alpha(P)\, r^\alpha\big)\, z^\alpha \Big\|_{\mathrm{B}_{\ell_\infty}} .$$

On the other hand, we have by the definition of BANACH sequence spaces $r \cdot \mathrm{B}_{\ell_\infty^n} \subset \mathrm{B}_{X_n}$ and thus

$$\Big\| \sum_{\alpha} \big(c_\alpha(P)\, r^\alpha\big)\, z^\alpha \Big\|_{\mathrm{B}_{\ell_\infty}} = \sup_{\omega\in r\cdot \mathrm{B}_{\ell_\infty^n}} \Big| \sum_{\alpha} c_\alpha(P)\, \omega^\alpha \Big| \le \|P\|_{\mathrm{B}_X} . \qquad \square$$

The estimate of Theorem 3.2 can, in general, not be improved. Indeed, we have the following extreme example:

Proposition 3.5. *Let $m := 2^l$ with $l \in \mathbb{N}$. For $\Lambda := \{1, 2, \dots, m\} \subset \mathbf{\Lambda}(1, \cdot\,) = \mathbb{N}_0^1$ then*

$$\tfrac{1}{\sqrt{2}} \, |\Lambda|^{\frac12} \le \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty^1)\big) \le |\Lambda|^{\frac12} .$$

The construction used in this proof is a variant of a construction of Shapiro and Rudin, due to Brillhart and Carlitz [20].

Proof. Define recursively $P_0(z) := z$ and $P_k(z) := P_{k-1}(z^2) + \frac{1}{z} P_{k-1}(-z^2)$ for $k \in \mathbb{N}$. For $x \in \mathbb{T}$ we obtain by the parallelogram identity that

$$\begin{aligned} \left|P_k(x)\right|^2 + \left|P_k(-x)\right|^2 &= \tfrac{1}{2}\Big(\left|P_k(x) + P_k(-x)\right|^2 + \left|P_k(x) - P_k(-x)\right|^2\Big) \\ &= \tfrac{1}{2}\Big(\left|2P_{k-1}(x^2)\right|^2 + \left|\tfrac{2}{x} P_{k-1}(-x^2)\right|^2\Big) \\ &= 2\Big(\left|P_{k-1}(x^2)\right|^2 + \left|P_{k-1}(-x^2)\right|^2\Big)\,. \end{aligned}$$

We now proceed by induction and check for every $k \in \mathbb{N}_0$ that

$$P_k(z) = \sum_{j=1}^{2^k} \pm z^j$$

and

$$\left|P_k(x)\right|^2 + \left|P_k(-x)\right|^2 = 2^{k+1}\,.$$

We have constructed a polynomial $P_l \in \mathcal{P}({}^{\Lambda}\ell_\infty^1)$ such that

$$\begin{aligned} 2^l = \sup_{x \in \mathrm{B}_{\ell_\infty^1}} \sum_{j=1}^{2^l} |\pm x^j| &\le \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty^1)\big) \sup_{\xi \in \mathrm{B}_{\ell_\infty^1}} \left|P_l(\xi)\right| \\ &= \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty^1)\big) \sup_{\xi \in \mathbb{T}} \left|P_l(\xi)\right| \le \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty^1)\big) \sqrt{2^{l+1}}\,, \end{aligned}$$

since, by the maximum modulus principle, the supremum is attained on the boundary $\mathbb{T}$ of $\mathrm{B}_{\ell_\infty^1}$. Hence, we have

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty^1)\big) \ge \tfrac{1}{\sqrt{2}} \cdot \sqrt{2^l} = \tfrac{1}{\sqrt{2}} \cdot |\Lambda|^{\frac{1}{2}}\,. \qquad \square$$

Even for the generality of Banach sequence spaces this proposition yields an extreme example. Using the following lemma we can transfer this example to any Banach sequence space.

Lemma 3.6. *Let $\Lambda \subset \mathbf{\Lambda}(1, \cdot)$. Then for any* Banach *sequence space X*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) = \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty)\big)\,.$$

Proof. This proof uses the idea that $\mathcal{P}({}^{\Lambda}X)$ equals $\mathcal{P}({}^{\Lambda}X_1)$ and that $\mathcal{P}({}^{\Lambda}\ell_\infty)$ equals $\mathcal{P}({}^{\Lambda}\ell_\infty^1)$ as a consequence of the special structure of Λ; furthermore, it uses that X_1 is isometrically isomorphic to ℓ_∞^1 for every Banach sequence space X.

Hence, for any polynomial $P \in \mathcal{P}({}^{\Lambda}X)$

$$\|P\|_{\mathrm{B}_{\ell_\infty}} = \|P\|_{\mathrm{B}_{\ell_\infty^1}} = \|P\|_{\mathrm{B}_{X_1}} = \|P\|_{\mathrm{B}_X} . \qquad \square$$

However, there exist sets of indices and Banach sequence spaces for which the estimate is extremely imprecise.

Proposition 3.7. *Let $n, m \in \mathbb{N}$ and set*

$$\Lambda := \big\{(m, 0, \dots, 0), (0, m, 0, \dots, 0), \dots, (0, \dots, 0, m)\big\} \subset \mathbf{\Lambda}(n, m) .$$

Then

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) = 1 .$$

This proposition is an direct consequence of Theorem 1.3 of Aron and Globevnik [4]. For the readers convenience, however, we want to give a short proof.

Proof. Let us denote by $\alpha^{(k)}$ the multi-index in Λ with m at the k^{th} position and notice that $x^{\alpha^{(k)}} = x_k^m$ for any $x \in X$.

For given $P \in \mathcal{P}({}^{\Lambda}X)$ and $x \in \mathrm{B}_X$ choose $(\omega_k)_{k=1}^n \in \mathbb{T}^n$ such that $\omega_k^m x_k^m c_{\alpha^{(k)}}(P) \geq 0$ for every k. With this

$$\begin{aligned} \sum_{\alpha\in\Lambda} \big|c_\alpha(P)\, x^\alpha\big| &= \sum_{k=1}^{n} \big|c_{\alpha^{(k)}}(P)\, x_k^m\big| \\ &= \sum_{k=1}^{n} c_{\alpha^{(k)}}(P)\, \omega_k^m x_k^m = \Big|\sum_{\alpha\in\Lambda} c_\alpha(P)\, (\omega\cdot x)^\alpha\Big| \leq \|P\|_{\mathrm{B}_X} , \end{aligned}$$

since $\|\omega \cdot x\|_X = \|x\|_X < 1$. $\square$

Using the general idea of this proof, we can obtain easily further examples:

Corollary 3.8. *Let X be a Banach sequence space and $\Lambda \subset \mathbf{\Lambda}(n, \cdot)$. Assume that either $\alpha = \beta$ or $\alpha_k \cdot \beta_k = 0$ for all k, whenever $\alpha, \beta \in \Lambda$. Then*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) = 1 .$$

We see that the structure of an index set Λ dominantly influences the magnitude of the unconditional basis constant; the cardinality of Λ and the BANACH sequence space play a minor part.

3.2. Uniform bounds of the unconditional basis constant

Proposition 3.9. *Let X be a* BANACH *sequence space, $\Lambda \subset \mathbf{\Lambda}(\infty,\cdot)$, and $\Gamma \subset \Lambda$. Then*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Gamma}X)\big) \leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big)\,.$$

Proof. Clearly, $\mathcal{P}_{\mathrm{fin}}({}^{\Gamma}X)$ is a subspace of $\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X)$. Hence, for any $P \in \mathcal{P}_{\mathrm{mon}}({}^{\Gamma}X)$ and any $x \in \mathrm{B}_X$

$$\sum_{\alpha\in\Gamma}|c_\alpha(P)\,x^\alpha| = \sum_{\alpha\in\Lambda}|c_\alpha(P)\,x^\alpha| \leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big)\,\|P\|_{\mathrm{B}_X}$$

as $c_\alpha(P) \neq 0$ only if $\alpha \in \Gamma$. □

Having this proposition we may ask if the knowledge about unconditional bases of certain subspaces in $\mathcal{P}({}^{\Lambda}X)$ enables us to conclude the existence of an unconditional basis in $\mathcal{P}({}^{\Lambda}X)$. We get the following theorem:

Theorem 3.10. *Let X be a* BANACH *sequence space and $\Lambda \subset \mathbf{\Lambda}(\infty,m)$. For $n \in \mathbb{N}$ define $\Lambda(n) := \Lambda \cap \mathbf{\Lambda}(n,m)$ and assume that there exists $c \geq 1$ such that*

$$\sup_{n\in\mathbb{N}} \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda(n)}X)\big) \leq c\,.$$

Then $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) \leq c$.

Note that this result doesn't give the existence of an unconditional basis. In fact, we obtain that the monomials $z^\alpha, \alpha \in \Lambda$ define an unconditional basic sequence in $\mathcal{P}({}^{\Lambda}X)$. However, in general we have $\mathcal{P}_{\mathrm{mon}}({}^{\Lambda}X) \neq \mathcal{P}({}^{\Lambda}X)$.

The case $X = \ell_1$ and $\Lambda = \mathbf{\Lambda}(\infty,m)$ gives an counterexample: For $m \in \mathbb{N}$, Theorem 3 of [24] shows the existence of a constant $c \geq 1$ such that

$$\sup_{n\in\mathbb{N}} \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\mathbf{\Lambda}(n,m)}\ell_1)\big) \leq c\,.$$

However, $\mathcal{P}({}^m\ell_1)$ doesn't have a basis at all. In the case $m = 1$ this is easily seen, since $\mathcal{P}({}^1\ell_1) = \ell_1' = \ell_\infty$. For $m > 1$ fix $\varphi_0 \in \mathcal{P}({}^1\ell_1)$ with $\|\varphi_0\| = 1$ and consider the embedding

$$\mathcal{P}({}^1\ell_1) \hookrightarrow \mathcal{P}({}^m\ell_1), \quad \varphi \mapsto \varphi \cdot \varphi_0^{m-1}.$$

We check at once that this defines an isometric embedding of ℓ_∞ into $\mathcal{P}({}^m\ell_1)$. The existence of a basis of $\mathcal{P}({}^m\ell_1)$ would now imply separability of ℓ_∞, a contradiction.

Proof of Theorem 3.10. Let $P \in \mathcal{P}_{\text{fin}}({}^\Lambda X)$ and pick $n \in \mathbb{N}$ such that $P \in \mathcal{P}_{\text{fin}}({}^{\Lambda(n)}X)$. Then for any $x \in \mathrm{B}_X$

$$\sum_{\alpha\in\Lambda} |c_\alpha(P)\, x^\alpha| \le \chi_{\text{mon}}\big(\mathcal{P}({}^{\Lambda(n)}X)\big)\, \|P\|_{\mathrm{B}_X} \le c\, \|P\|_{\mathrm{B}_X}. \qquad \square$$

3.3. The trick — extending known results

In the proof of Theorem 3.2 we used a simple trick to transfer a known result from the case $X = \ell_\infty^n$ to the case of any Banach sequence space. This idea goes back to the work of Boas and Khavinson [16].

At first, let us explain this trick in more detail; having understood the trick, we can then use this technique in an even more elaborate way. By definition, we can exhaust the unit ball of every Banach sequence space X by dilated unit balls of ℓ_∞; i.e. there exists a set $\mathcal{R} \subset [0,\infty)^{\mathbb{N}}$ (a suitable choice of $\mathcal{R}$ is given by $\mathrm{B}_X \cap [0,\infty)^{\mathbb{N}}$) such that

$$\mathrm{B}_X = \bigcup_{r\in\mathcal{R}} r \cdot \mathrm{B}_{\ell_\infty} = \bigcup_{r\in\mathcal{R}} \big\{(r_k\omega_k)_k \,\big|\, \omega = (\omega_k)_k \in \mathrm{B}_{\ell_\infty}\big\}.$$

With this we can easily transfer results obtained for ℓ_∞ to the case of any Banach sequence space:

Theorem 3.11. *Let X be a Banach sequence space and Λ be a set of indices. Then*

$$\chi_{\text{mon}}\big(\mathcal{P}({}^\Lambda X)\big) \le \chi_{\text{mon}}\big(\mathcal{P}({}^\Lambda \ell_\infty)\big).$$

Proof. Let $P \in \mathcal{P}_{\text{fin}}({}^\Lambda X)$ and $x \in \mathrm{B}_X$. Choose $\omega \in \mathrm{B}_{\ell_\infty}$ and $r \in \mathrm{B}_X \cap [0,\infty)^{\mathbb{N}}$ such that $x = r \cdot \omega$. Now, since $r \cdot \mathrm{B}_{\ell_\infty} \subset \mathrm{B}_X$,

$$\Big|\sum_\alpha |c_\alpha(P)|\, x^\alpha\Big| = \Big|\sum_\alpha |c_\alpha(P)\, r^\alpha|\, \omega^\alpha\Big|$$

$$\begin{aligned}&\leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty)\big) \sup_{\omega\in \mathrm{B}_{\ell_\infty}} \Big|\sum_{\alpha\in\Lambda} \big(c_\alpha(P)\, r^\alpha\big)\, \omega^\alpha\Big| \\ &\leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty)\big) \sup_{\xi\in r\cdot \mathrm{B}_{\ell_\infty}} \Big|\sum_{\alpha\in\Lambda} c_\alpha(P)\, \xi^\alpha\Big| \\ &\leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_\infty)\big)\, \|P\|_{\mathrm{B}_X}\,. \end{aligned}$$

□

3.3.1. The concept of p–exhaustibility

Let X be a Banach sequence space and $R\subset X$ a Reinhardt domain. We say that a Reinhardt domain R is p–exhaustible for $1\leq p\leq\infty$ if there exists a set $\mathcal{R}\subset[0,\infty)^{\mathbb{N}}$ such that

$$R=\mathcal{R}\cdot \mathrm{B}_{\ell_p}=\bigcup_{r\in\mathcal{R}} r\cdot \mathrm{B}_{\ell_p}\,.$$

It is evident that the unit ball of any Banach sequence space is a Reinhardt domain and that every Reinhardt domain is ∞–exhaustible. We have the following generalization of the theorem above:

Theorem 3.12. *Let $1\leq p\leq\infty$ and X be a Banach sequence space with p–exhaustible unit ball. Let furthermore $\Lambda\subset\mathbf{\Lambda}(\infty,\,\cdot\,)$ be a set of indices.*

Let $\theta=(\theta_\alpha)_{\alpha\in\Lambda}\in\mathbb{C}^\Lambda$ and assume that

$$M_\theta:\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}\ell_p)\to\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}\ell_p),\qquad z^\alpha\mapsto\theta_\alpha z^\alpha$$

defines a bounded linear operator. Then M_θ is as well a bounded operator on $\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X)$ with

$$\big\|M_\theta:\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X)\to\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}X)\big\|\leq\big\|M_\theta:\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}\ell_p)\to\mathcal{P}_{\mathrm{fin}}({}^{\Lambda}\ell_p)\big\|\,.$$

In particular, by taking $\theta\in\mathbb{T}^\Lambda$,

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big)\leq\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}\ell_p)\big)\,.$$

This theorem enables us to transfer many of the results for $\mathcal{P}({}^{\Lambda}\ell_p)$ to the space $\mathcal{P}({}^{\Lambda}X)$. Philosophically speaking: "If an inequality is true for $\mathcal{P}({}^{\Lambda}\ell_p)$ and is X a Banach sequence space with p–exhaustible unit ball, then the inequality is most likely also true for $\mathcal{P}({}^{\Lambda}X)$."

Proof. Let $P \in \mathcal{P}_{\text{fin}}({}^{\Lambda}X)$. For $x \in \mathrm{B}_X$ choose $r \in [0,\infty)^{\mathbb{N}}$ such that $x \in r \cdot \mathrm{B}_{\ell_p} \subset \mathrm{B}_X$ and write $x = r \cdot \omega$ with $\omega \in \mathrm{B}_{\ell_p}$. Then

$$\begin{aligned} |M_\theta P(x)| &= \Big| \sum_{\alpha \in \Lambda} \big(c_\alpha(P)\, \theta_\alpha\big)\, x^\alpha \Big| \\ &= \Big| \sum_{\alpha \in \Lambda} \big(c_\alpha(P)\, \theta_\alpha r^\alpha\big)\, \omega^\alpha \Big| \\ &\leq \big\| M_\theta : \mathcal{P}_{\text{fin}}({}^{\Lambda}\ell_p) \to \mathcal{P}_{\text{fin}}({}^{\Lambda}\ell_p) \big\| \sup_{\omega \in \mathrm{B}_{\ell_p}} \Big| \sum_{\alpha \in \Lambda} \big(c_\alpha(P)\, r^\alpha\big)\, \omega^\alpha \Big| \\ &\leq \big\| M_\theta : \mathcal{P}_{\text{fin}}({}^{\Lambda}\ell_p) \to \mathcal{P}_{\text{fin}}({}^{\Lambda}\ell_p) \big\| \, \|P\|_{\mathrm{B}_X} . \end{aligned}$$ ☐

It remains to investigate under what circumstances the unit ball of a Banach sequence space is p–exhaustible. Using Hölders inequality we obtain easily that $\mathrm{B}_{\ell_q} = \mathrm{B}_{\ell_p} \cdot \mathrm{B}_{\ell_s}$ if $\frac{1}{q} = \frac{1}{p} + \frac{1}{s}$. Thus:

Proposition 3.13. *Let $1 \leq p \leq \infty$. Then B_{ℓ_p} is q–exhaustible for every $q \geq p$.*

3.3.2. Characterization of Banach sequence spaces with p–exhaustible unit ball

In this section we aim on giving a description of those Banach sequence spaces with p–exhaustible unit ball. In Theorem 3.15 we give a necessary condition and in Theorem 3.22 we give a sufficient condition for a Banach sequence space X to have p–exhaustible unit ball.

Proposition 3.14. *Let X be a Banach sequence space with p–exhaustible unit ball and let $q \geq p$. Then X has q–exhaustible unit ball.*

Proof. By assumption there exists $\mathcal{R} \subset [0,\infty)^{\mathbb{N}}$ such that $\mathrm{B}_X = \mathcal{R} \cdot \mathrm{B}_{\ell_p}$. Proposition 3.13 shows the existence of $\mathcal{R}_p \subset [0,\infty)^{\mathbb{N}}$ such that $\mathrm{B}_{\ell_p} = \mathcal{R}_p \cdot \mathrm{B}_{\ell_q}$. Hence $\mathrm{B}_X = \tilde{\mathcal{R}} \cdot \mathrm{B}_{\ell_q}$ for $\tilde{\mathcal{R}} := \mathcal{R} \cdot \mathcal{R}_p$. ☐

Theorem 3.15. *Let X be a Banach sequence space and $R \subset \mathrm{B}_X$ a p–exhaustible Reinhardt domain. Then $R \subset \mathrm{B}_{\ell_p}$. In particular, if the unit ball of X is p–exhaustible, then $\mathrm{B}_X \subset \mathrm{B}_{\ell_p}$.*

Proof. Assume that $R \not\subset \mathrm{B}_{\ell_p}$. Then there exists $x \in R$ such that $\|x\|_{\ell_p} \geq 1$. Choose $r \in [0,\infty)^{\mathbb{N}}$ such that $x \in r \cdot \mathrm{B}_{\ell_p} \subset R$. Since $\overline{\mathrm{B}_{\ell_\infty}} \cdot \mathrm{B}_{\ell_p} \subset \mathrm{B}_{\ell_p}$ and $x \notin \mathrm{B}_{\ell_p}$ there exists $k \in \mathbb{N}$ such that $r_k > 1$. Hence $\|(1-\varepsilon)r_k e_k\|_X = (1-\varepsilon)r_k > 1$ for $\varepsilon > 0$ sufficiently small, but $(1-\varepsilon)r_k e_k \in r \cdot \mathrm{B}_{\ell_p} \subset R \subset \mathrm{B}_X$ for every $\varepsilon > 0$; a contradiction. □

As a consequence we immediately get the following corollary:

Corollary 3.16. *Let X be a* Banach *sequence space with p–exhaustible unit ball. Then $\|x\|_{\ell_p} \leq \|x\|_X$ for any $x \in X$ and hence $\|\mathrm{id} : X \hookrightarrow \ell_p\| \leq 1$.*

To obtain a sufficient condition we have to introduce the notion of p–convexity and p–concavity. We will touch only a few aspect of the theory and omit the proofs as the theory is well documented. For the proofs see for instance Section 1.d. in [47].

A Banach sequence space X is called p–convex if there exists a constant $c \geq 1$ such that for any choice of $x^{(1)}, \ldots, x^{(n)} \in X$

$$\Big\| \Big(\sum_k |x^{(k)}|^p \Big)^{\frac{1}{p}} \Big\|_X \leq c \Big(\sum_k \|x^{(k)}\|_X^p \Big)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ and

$$\Big\| \sup_k |x^{(k)}| \Big\|_X \leq c \sup_k \|x^{(k)}\|_X$$

if $p = \infty$. The best constant in these inequalities is called p–convexity constant of X and denoted by $M^{(p)}(X)$. The Banach sequence space X is called q–concave if there exists a constant $c \geq 1$ such that for any choice of $x^{(1)}, \ldots, x^{(n)} \in X$

$$\Big(\sum_k \|x^{(k)}\|_X^q \Big)^{\frac{1}{q}} \leq c \Big\| \Big(\sum_k |x^{(k)}|^q \Big)^{\frac{1}{q}} \Big\|_X$$

if $1 \leq q < \infty$ and

$$\sup_k \|x^{(k)}\|_X \leq c \Big\| \sup_k |x^{(k)}| \Big\|_X$$

if $q = \infty$. The best constant in these inequalities is called q–concavity constant of X and denoted by $M_{(q)}(X)$.

We have the following two properties:

Lemma 3.17 (cf. Proposition 1.d.4. in [47])**.** *Let X be a* Banach *sequence space and let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then X is p–convex (p–concave) if and only if the dual space X' is q–concave with $M^{(p)}(X) = M_{(q)}(X')$ (respectively q–convex with $M_{(p)}(X) = M^{(q)}(X')$).*

Lemma 3.18 (cf. Proposition 1.d.8. in [47])**.** *Let X be a* Banach *sequence space. Assume that X is r–convex and s–concave for $1 \leq r \leq s \leq \infty$. Then there exists a norm $\|\!|\cdot|\!\|$ on X, equivalent to $\|\cdot\|_X$, such that X endowed with $\|\!|\cdot|\!\|$ has r–convexity and s–concavity constants equal to one and that the order is preserved (that means $\|\!|y|\!\| \leq \|\!|x|\!\|$ whenever $|y| \leq |x|$).*

We now consider products of Banach sequence spaces. A good reference here is given by Schep [53]; for the proofs of the lemmas below see [53]. Note that Schep considers Banach function spaces, a wider class of Banach spaces, and that in our setting of Banach sequence spaces (that are Banach function spaces over the measure space $(\mathbb{N}, \Omega, \mu)$, Ω being the power set of $\mathbb{N}$, and μ the counting measure) many of the assumptions are automatically fulfilled.

For Banach sequence spaces X and Y define the so-called product space

$$X \cdot Y := \big\{ x \cdot y \,\big|\, x \in X, y \in Y \big\}.$$

A not necessarily complete norm on $X \cdot Y$ is then given by

$$\|\xi\|_{X \cdot Y} := \inf\Big\{ \sum_{k=1}^{N} \|x^{(k)}\|_X \|y^{(k)}\|_Y \,\Big|\, |\xi| \leq \sum_{k=1}^{N} x^{(k)} y^{(k)}, 0 \leq x^{(k)} \in X, 0 \leq y^{(k)} \in Y \Big\}$$

for $\xi \in X \cdot Y$.

For Banach sequence spaces X and Z we define furthermore the space of multipliers from X into Z by

$$\mathcal{M}(X; Z) := \big\{ \xi \in \ell_\infty \,\big|\, \xi \cdot x \in Z \text{ for every } x \in X \big\}.$$

When endowed with the norm $\|\xi\| := \sup\big\{ \|\xi x\|_Z \,\big|\, x \in \mathrm{B}_X \big\}$ this space is, as a closed subspace of all bounded operators from X into Z, again a Banach space.

Lemma 3.19 (cf. Proposition 1.4 in [53])**.** *Let X and Y be* Banach *sequence spaces. Then*

$$(X \cdot Y)' = \mathcal{M}(X; Y') = \mathcal{M}(Y; X')$$

isometrically.

The norm $\|\cdot\|_X$ of a BANACH sequence space is said to have the FATOU property if $0 \leq x^{(n)} \in X$ with $x^{(n)} \nearrow x \in \ell_\infty$ implies $\|x^{(n)}\|_X \nearrow \|x\|_X$ if $x \in X$ and $\|x^{(n)}\|_X \nearrow \infty$ otherwise.

Lemma 3.20 (cf. Theorem 2.4 in [53])**.** *Let X and Y be* BANACH *sequence spaces with the* FATOU *property and assume that $X \cdot Y$ is a* BANACH *sequence space. For $0 \leq \xi \in X \cdot Y$ then exist $0 \leq x \in X$ and $0 \leq y \in Y$ so that $\xi = x \cdot y$ and $\|\xi\|_{X \cdot Y} = \|x\|_X \|y\|_Y$.*

Lemma 3.21 (cf. Theorem 3.8 in [53])**.** *Let X be a p–concave* BANACH *sequence space with $M_{(p)}(X) = 1$. Then is $\ell_p \cdot \mathcal{M}(\ell_p; X)$ a* BANACH *sequence space and we have $\ell_p \cdot \mathcal{M}(\ell_p; X) = X$ isometrically.*

Finally we can state our sufficient condition.

Theorem 3.22. *Let X be a* BANACH *sequence space with the* FATOU *property and assume that X is p–concave with $M_{(p)}(X) = 1$. Then X has p–exhaustible unit ball.*

Proof. Our proof starts with the observation that, by Lemma 3.21, $X = \ell_p \cdot \mathcal{M}(\ell_p; X)$. On account of Lemma 3.20, for every $x \in \mathrm{B}_X$ there exist $0 \leq r \in \mathcal{M}(\ell_p; X)$ and $0 \leq \omega \in \ell_p$ such that $|x| = r \cdot \omega$ and $\|x\|_X = \|r\|_{\mathcal{M}(\ell_p;X)} \cdot \|\omega\|_{\ell_p}$.

We may assume without loss of generality that $\|\omega\|_{\ell_p} = \|x\|_X$ and $\|r\|_{\mathcal{M}(\ell_p;X)} = 1$; we have found $r \in [0, \infty)^{\mathbb{N}}$, such that

$$x \in r \cdot \mathrm{B}_{\ell_p} \subset \mathrm{B}_X\,. \qquad \square$$

3.4. Lower estimates

To give an idea of the optimality of our results we have to construct polynomials with small norm. The main tool for this is given in the following proposition, which is due to BAYART [8].

Proposition 3.23 (cf. Theorem 3.1 in [8])**.** *Let X be a* BANACH *sequence space, let $1 \leq q \leq 2$, and let $n, m \in \mathbb{N}$ with $m \geq 2$. For any family $(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(n,m)}$ of complex numbers there exist a choice of signs $(\varepsilon_{\boldsymbol{j}})_{\boldsymbol{j}} \in \mathbb{T}^{\mathcal{J}(n,m)}$ such that*

$$\Big\| \sum_{\boldsymbol{j} \in \mathcal{J}(n,m)} \varepsilon_{\boldsymbol{j}} c_{\boldsymbol{j}} z_{\boldsymbol{j}} \Big\|_{\mathrm{B}_X} \leq c(m,n;q) \sup_{\boldsymbol{j} \in \mathcal{J}(n,m)} |c_{\boldsymbol{j}}| \, \big|[\boldsymbol{j}]\big|^{\frac{1}{q}} \sup_{x \in \mathrm{B}_X} \left(\sum_{k=1}^{n} |x_k|^q \right)^{\frac{m}{q}}$$

with

$$c(m,n;q) = \begin{cases} c\,(\log n + \log\log m) & \textit{if } q = 1 \textit{ and} \\ c\,(n\log m)^{1-\frac{1}{q}} & \textit{otherwise} \end{cases}$$

where $c \geq 1$ is independent of m and n.

The proof of this proposition uses a probabilistic argument to show that the probability for the Rademacher variables $\big(\varepsilon_{\boldsymbol{j}}(\omega)\big)_{\boldsymbol{j}}$ to fail the inequality in question is less than one.

We will apply the proposition several times in the following fashion:

Corollary 3.24. *Let X be an Banach sequence space, $n, m \in \mathbb{N}$ with $m \geq 2$, and $1 < q \leq 2$. For $\xi \in \mathrm{B}_X$ assume that there exists a constant $\tilde{c} \geq 1$ such that*

$$\sum_{\boldsymbol{j}\in\mathcal{J}(n,m)} |c_{\boldsymbol{j}}(P)\,\xi_{\boldsymbol{j}}| \leq \tilde{c}\,\|P\|_{\mathrm{B}_{X_n}}$$

for any polynomial $P \in \mathcal{P}(^m X_n)$. Then

$$\Big(\sum_{k=1}^{n} |\xi_k|\Big)^m \leq \tilde{c}\cdot c(m,n;q)\, m^{m(1-\frac{1}{q})}\, \big\|\mathrm{id} : X_n \hookrightarrow \ell_q^n\big\|^m$$

with $c(m,n;q)$ denoting the constant of the proposition.

Proof. At first notice that we have for any choice of $(\varepsilon_{\boldsymbol{j}})_{\boldsymbol{j}} \in \mathbb{T}^{\mathcal{J}(n,m)}$

$$\Big(\sum_{k=1}^{n} |\xi_k|\Big)^m = \sum_{\boldsymbol{j}\in\mathcal{J}(n,m)} \Big|\varepsilon_{\boldsymbol{j}}\,|[\boldsymbol{j}]|\,\xi_{\boldsymbol{j}}\Big| \leq \tilde{c}\,\Big\|\sum_{\boldsymbol{j}\in\mathcal{J}(n,m)} \varepsilon_{\boldsymbol{j}}\,|[\boldsymbol{j}]|\,z_{\boldsymbol{j}}\Big\|_{\mathrm{B}_X}.$$

By the proposition we may now chose $(\varepsilon_{\boldsymbol{j}})_{\boldsymbol{j}} \in \mathbb{T}^{\mathcal{J}(n,m)}$ such that

$$\begin{aligned}\Big\|\sum_{\boldsymbol{j}\in\mathcal{J}(n,m)} \varepsilon_{\boldsymbol{j}}\,|[\boldsymbol{j}]|\,z_{\boldsymbol{j}}\Big\|_{\mathrm{B}_X} &\leq c(m,n;p) \sup_{\boldsymbol{j}\in\mathcal{J}(n,m)} |c_{\boldsymbol{j}}|\,\big|[\boldsymbol{j}]\big|^{\frac{1}{p}} \sup_{x\in\mathrm{B}_X} \Big(\sum_{k=1}^{n} |x_k|^p\Big)^{\frac{m}{p}} \\ &= c(m,n;p) \sup_{\boldsymbol{j}\in\mathcal{J}(n,m)} \big|[\boldsymbol{j}]\big|^{1-\frac{1}{p}}\, \big\|\mathrm{id} : X \hookrightarrow \ell_p\big\|^m.\end{aligned}$$

To conclude the claim, it now suffices to notice that $|[\boldsymbol{j}]| \leq m^m$ for any $\boldsymbol{j} \in \mathcal{J}(n,m)$. $\square$

Let us introduce another reformulation in the proposition above:

Corollary 3.25. *Let X be an* Banach *sequence space, let $n, m \in \mathbb{N}$ with $m \geq 2$, and let $1 < q \leq 2$. Then*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^m X_n)\big) \geq \big(c(m,n;q)\big)^{-1} m^{-m(1-\frac{1}{q})} \big\|\mathrm{id} : X_n \hookrightarrow \ell_1^n\big\|^m \big\|\mathrm{id} : X_n \hookrightarrow \ell_q^n\big\|^{-m}$$

with $c(m,n;q)$ denoting the constant in the proposition.

Proof. With $\tilde{c} := \chi_{\mathrm{mon}}\big(\mathcal{P}(^m X_n)\big)$ we can apply the previous corollary and obtain for any $\xi \in \mathrm{B}_X$

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^m X_n)\big) \geq \big(c(m,n;q)\big)^{-1} m^{-m(1-\frac{1}{q})} \big\|\mathrm{id} : X_n \hookrightarrow \ell_q^n\big\|^{-m} \Big(\sum_{k=1}^n |\xi_k|\Big)^m$$

with $c(m,n;q)$ denoting the constant of the proposition. By taking the supremum over all $\xi \in \mathrm{B}_X$ we obtain the claim. □

Chapter 4.

Better estimates in the $\boldsymbol{\ell_p}$ case

In this chapter we establish an upper bound for the unconditional basis constant of the monomials in $\mathcal{P}({}^J\ell_p)$ where $J \subset \mathcal{J}(n,m)$. Depending on the structure of J this gives much better results than the general estimate established in the previous chapter.

The result and its proof arose in a joint work with Bayart, Defant, and Schlüters [10] and is inspired by a more abstract approach which we will present in the next chapter.

For an index set $J \subset \mathcal{J}(n,m)$ denote in the following by J^* the *reduced index set*

$$\begin{aligned} J^* &:= \left\{ \boldsymbol{j} \in \mathcal{J}(n,m-1) \,\middle|\, \exists k \in \mathbb{N} : (\boldsymbol{j},k) \in J \right\} \\ &= \left\{ (j_1,\dots,j_{m-1}) \,\middle|\, (j_1,\dots,j_m) \in J \right\}. \end{aligned} \tag{4 $\cdot$ A}$$

Theorem 4.1. *Let $1 \le p \le \infty$, set $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and let $J \subset \mathcal{J}(n,m)$. Then there exists a constant $c(p,m) \ge 1$ such that for every $P \in \mathcal{P}({}^m\ell_p^n)$ and every $x \in \ell_p^n$*

$$\sum_{\boldsymbol{j}\in J} |c_{\boldsymbol{j}}(P)x_{\boldsymbol{j}}| \le c(p,m)\, |J^*|^{\sigma}\, \|x\|_{\ell_p}^m\, \|P\|_{\mathrm{B}_{\ell_p^n}} \tag{4 $\cdot$ B}$$

where

$$c(p,m) \le \begin{cases} \mathrm{e}m\mathrm{e}^{\frac{m-1}{p}} & \text{if } p \le 2 \text{ and} \\ \mathrm{e}m2^{\frac{m-1}{2}} & \text{if } p \ge 2. \end{cases}$$

Note that in both cases $c(p,m) \le \mathrm{e}^{2m}$. In particular, we obtain an estimate for the unconditional basis constant:

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^J\ell_p^n)\big) \le c(p,m)\, |J^*|^{\sigma}. \tag{4 $\cdot$ C}$$

Note that (4 · B) gives insofar a stronger result as (4 · C) gives the statement of (4 · B) only for polynomials $P \in \mathcal{P}({}^J\ell_p^n)$.

From Theorem 3.12 we have the following corollary:

Corollary 4.2. *Let $1 \le p \le \infty$, $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and $J \subset \mathcal{J}(n,m)$. Then for any* BANACH *sequence space with p–exhaustible unit ball*

$$\chi_{\text{mon}}\big(\mathcal{P}({}^J X)\big) \le c(p,m)\,|J^*|^\sigma$$

where $c(p,m)$ denotes the constant of the previous theorem.

The proof of the theorem differs in the two cases $p \le 2$ and $p \ge 2$; we begin with the case $p \le 2$.

4.1. Proof for $p \le 2$

We need two lemmas for the proof.

Lemma 4.3. *Let $1 \le p \le 2$ and $2 \le q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let furthermore*

$$Q := \sum_{k=1}^{n} \Bigg(\sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} b_{(\boldsymbol{j},k)} z_{\boldsymbol{j}} \Bigg) e'_k(\,\cdot\,) \in \mathcal{L}\big(\ell_p^n, \mathcal{P}({}^{m-1}\ell_p^n)\big)\,.$$

Then for any $\boldsymbol{j} \in \mathcal{J}(n,m-1)$

$$\Big(\sum_{k=1}^{n} |b_{(\boldsymbol{j},k)}|^q\Big)^{\frac{1}{q}} \le \mathrm{e}^{\frac{m-1}{p}} \big|[\boldsymbol{j}]\big|^{\frac{1}{p}}\, \|Q\|$$

where $\|Q\|$ denotes the operator norm in $\mathcal{L}\big(\ell_p^n, \mathcal{P}({}^{m-1}\ell_p^n)\big)$.

Proof. Fix $\omega \in \mathrm{B}_{\ell_p^n}$. Then $Q\omega \in \mathcal{P}({}^{m-1}\ell_p^n)$. By Corollary 2.19,

$$\Big|\sum_{k=1}^{n} b_{(\boldsymbol{j},k)}\omega_k\Big| = \big|c_{\boldsymbol{j}}(Q\omega)\big| \le \mathrm{e}^{\frac{m-1}{p}} \big|[\boldsymbol{j}]\big|^{\frac{1}{p}} \sup_{x \in \mathrm{B}_{\ell_p^n}} \big|Q\omega(x)\big| \le \mathrm{e}^{\frac{m-1}{p}} \big|[\boldsymbol{j}]\big|^{\frac{1}{p}}\, \|Q\|\,.$$

Taking the supremum over $\omega \in \mathrm{B}_{\ell_p^n}$ we see that this provides the claim as $(\ell_p^n)' = \ell_q^n$. $\square$

Lemma 4.4. *Let $1 \leq p \leq 2$ and let q denote its conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Fix $P \in \mathcal{P}({}^m\ell_p^n)$. Then for any $\boldsymbol{j} \in \mathcal{J}(n, m-1)$*

$$\Big(\sum_{k=j_{m-1}}^{n} \big| c_{(\boldsymbol{j},k)}(P) \big|^q \Big)^{\frac{1}{q}} \leq m \mathrm{e}^{1+\frac{m-1}{p}} \big| [\boldsymbol{j}] \big|^{\frac{1}{p}} \, \|P\|_{\mathrm{B}_{\ell_p^n}} .$$

Proof. Let L denote the unique symmetric m–linear form associated to P and define $Q \in \mathcal{L}\big(\ell_p^n, \mathcal{P}({}^{m-1}\ell_p^n)\big)$ by $Q\omega(x) := L(x, \ldots, x, \omega)$. Then

$$Q\omega(x) = L(x, \ldots, x, \omega) = \sum_{\boldsymbol{i} \in \mathcal{I}(n,m-1)} \sum_{k=1}^{n} c_{(\boldsymbol{i},k)}(L) \, e'_{\boldsymbol{i}} \circ \Delta_{m-1}(x) \, e'_k(\omega)$$

and thus

$$\begin{aligned}
Q &= \sum_{\boldsymbol{i} \in \mathcal{I}(n,m-1)} \sum_{k=1}^{n} c_{(\boldsymbol{i},k)}(L) \, \underbrace{\big(e'_{\boldsymbol{i}} \circ \Delta_{m-1}\big)}_{= z_{\boldsymbol{i}^*}} \, e'_k(\,\cdot\,) \\
&= \sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} \sum_{\boldsymbol{i} \in [\boldsymbol{j}]} \sum_{k=1}^{n} c_{(\boldsymbol{i},k)}(L) \, z_{\boldsymbol{j}} \, e'_k(\,\cdot\,) \\
&= \sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} \sum_{k=1}^{n} \Big(\big|[\boldsymbol{j}]\big| \, c_{(\boldsymbol{j},k)^*}(L) \Big) \, z_{\boldsymbol{j}} \, e'_k(\,\cdot\,) \\
&= \sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} \sum_{k=1}^{n} \underbrace{\left(\big|[\boldsymbol{j}]\big| \, \frac{c_{(\boldsymbol{j},k)^*}(P)}{\big|[(\boldsymbol{j},k)]\big|} \right)}_{=: \, b_{(\boldsymbol{j},k)}} \, z_{\boldsymbol{j}} \, e'_k(\,\cdot\,) ,
\end{aligned}$$

since, by $(2 \cdot \mathrm{F})$, $c_{(\boldsymbol{j},k)^*}(P) = \big|[(\boldsymbol{j},k)]\big| \, c_{(\boldsymbol{j},k)^*}(L)$. For $\boldsymbol{j} \in \mathcal{J}(n, m-1)$ and $k \geq j_{m-1}$ we have now

$$c_{(\boldsymbol{j},k)}(P) = \frac{|[(\boldsymbol{j},k)]|}{|[\boldsymbol{j}]|} \, b_{(\boldsymbol{j},k)} \leq m \, b_{(\boldsymbol{j},k)} \, ;$$

therefore, by the previous lemma,

$$\Big(\sum_{k=j_{m-1}}^{n} \big| c_{(\boldsymbol{j},k)}(P) \big|^q \Big)^{\frac{1}{q}} \leq m \Big(\sum_{k=j_{m-1}}^{n} \big| b_{(\boldsymbol{j},k)} \big|^q \Big)^{\frac{1}{q}} \leq m \mathrm{e}^{\frac{m-1}{p}} \big| [\boldsymbol{j}] \big|^{\frac{1}{p}} \, \|Q\|$$

for every $\boldsymbol{j} \in J(n, m-1)$. By HARRIS' inequality (Proposition 2.11), $\|Q\| \leq \mathrm{e} \, \|P\|_{\mathrm{B}_{\ell_p^n}}$, which completes the proof. □

We are now able to prove Theorem 4.1 in the case $p \leq 2$.

Proof of Theorem 4.1, $1 \le p \le 2$. For $P \in \mathcal{P}(^m\ell_p^n)$ and $x \in \ell_p^n$ we have

$$\begin{aligned}\sum_{\boldsymbol{j}\in J} \left|c_{\boldsymbol{j}}(P) x_{\boldsymbol{j}}\right| &= \sum_{\boldsymbol{j}\in J^*} \sum_{\substack{k\\(\boldsymbol{j},k)\in J}} \left|c_{(\boldsymbol{j},k)}(P)\, x_{(\boldsymbol{j},k)}\right| \\ &= \sum_{\boldsymbol{j}\in J^*} |x_{\boldsymbol{j}}| \sum_{\substack{k\\(\boldsymbol{j},k)\in J}} \left|c_{(\boldsymbol{j},k)}(P)\right| |x_k| \,.\end{aligned}$$

By Hölders inequality ($\frac{1}{p} + \frac{1}{q} = 1$), this is

$$\le \sum_{\boldsymbol{j}\in J^*} |x_{\boldsymbol{j}}| \Big(\sum_{\substack{k\\(\boldsymbol{j},k)\in J}} \left|c_{(\boldsymbol{j},k)}(P)\right|^q \Big)^{\frac{1}{q}} \Big(\sum_{\substack{k\\(\boldsymbol{j},k)\in J}} |x_k|^p \Big)^{\frac{1}{p}} .$$

The last sum is obviously bounded by $\|x\|_{\ell_p}$ and Lemma 4.4 provides an estimate for the middle factor. We obtain

$$\le m e^{1+\frac{m-1}{p}} \sum_{\boldsymbol{j}\in J^*} \left|[\boldsymbol{j}]\right|^{\frac{1}{p}} |x_{\boldsymbol{j}}| \, \|x\| \, \|P\|_{\mathrm{B}_{\ell_p^n}} .$$

Application of Hölders inequality now provides the claim:

$$\begin{aligned}&\le m e^{1+\frac{m-1}{p}} \Big(\sum_{\boldsymbol{j}\in J^*} \left|[\boldsymbol{j}]\right| \cdot |x_{\boldsymbol{j}}|^p \Big)^{\frac{1}{p}} \Big(\sum_{\boldsymbol{j}\in J^*} 1 \Big)^{\frac{1}{q}} \|x\| \, \|P\|_{\mathrm{B}_{\ell_p^n}} \\ &\le m e^{1+\frac{m-1}{p}} |J^*|^{1-\frac{1}{p}} \|x\|^m \, \|P\|_{\mathrm{B}_{\ell_p^n}} ,\end{aligned}$$

since

$$\Big(\sum_{\boldsymbol{j}\in J^*} \left|[\boldsymbol{j}]\right| |x_{\boldsymbol{j}}|^p \Big)^{\frac{1}{p}} \le \Big(\sum_{\boldsymbol{j}\in \mathcal{J}(n,m-1)} \left|[\boldsymbol{j}]\right| |x_{\boldsymbol{j}}|^p \Big)^{\frac{1}{p}} = \bigg(\Big(\sum_{k=1}^n |x_k|^p \Big)^{\frac{1}{p}} \bigg)^{m-1} . \qquad \square$$

4.2. Proof for $p \ge 2$

We prove the claim for $p = \infty$. Using the trick introduced in Theorem 3.12 we will then be able to prove the theorem for the full range $2 \le p < \infty$. Again, we need an auxiliary result. This lemma was already proven in a more general fashion by Bayart, Defant, Frerick, Maestre, and Sevilla-Peris [9] and in a different fashion it also appeared in [34] and [27]. However, we want to give the condensed proof for our setting.

Lemma 4.5. *Let $P \in \mathcal{P}(^m\ell_\infty^n)$. Then*

$$\sum_{k=1}^{n} \Big(\sum_{\substack{\boldsymbol{j} \in \mathcal{J}(n,m-1) \\ j_{m-1} \leq k}} \big|c_{(\boldsymbol{j},k)}(P)\big|^2 \Big)^{\frac{1}{2}} \leq \mathrm{e}m2^{\frac{m-1}{2}} \, \|P\|_{\mathrm{B}_{\ell_\infty^n}} .$$

The proof utilizes the following Khintchine-Steinhaus type inequality, which relates different integral norms of polynomials. It is stated as follows:

Proposition 4.6 (cf. Theorem 9 in [7])**.** *Given $1 \leq p \leq q < \infty$ there exists a constant $0 < c(p,q) \leq \sqrt{\frac{q}{p}}$ such that for every polynomial $P \in \mathcal{P}(^m\ell_\infty^n)$ we have*

$$\Big(\int_{\mathbb{T}^n} \big|P(x)\big|^q \, \mathrm{d}m_n(x) \Big)^{\frac{1}{q}} \leq c(p,q)^m \Big(\int_{\mathbb{T}^n} \big|P(x)\big|^p \, \mathrm{d}m_n(x) \Big)^{\frac{1}{p}} .$$

A recent result of Defant and Mastyło [33] moreover shows that the constants $\sqrt{\frac{q}{p}}$ can't be improved significantly.

Already in 1980, Weissler [59] proved the following result, which is something of the kind of an 1–dimensional ancestor: For $P \in \mathcal{P}(\ell_\infty^1)$ we have

$$\Big(\int_{\mathbb{T}} \big|P\big(\sqrt{\tfrac{p}{q}}x\big)\big|^q \, \mathrm{d}m_1(x) \Big)^{\frac{1}{q}} \leq \Big(\int_{\mathbb{T}^n} \big|P(x)\big|^p \, \mathrm{d}m_1(x) \Big)^{\frac{1}{p}} .$$

Using Fubini's theorem it is possible to generalize this result to the the n–dimensional case, which provides the proposition above; this is done in [7]. The case relevant for us ($p = 1$ and $q = 2$) was, to our best knowledge, first proved by Beauzamy, Bombieri, Enflo, and Montgomery [12] in 1990, however, with a slightly larger constant.

Proof of Lemma 4.5. Let $P \in \mathcal{P}(^m\ell_\infty^n)$ and let $L \in \mathcal{L}_s(^m\ell_\infty^n;\mathbb{C})$ denote the unique symmetric m–linear form associated to P. By $(2\cdot\mathrm{F})$, $c_{\boldsymbol{j}}(P) = \big|[\boldsymbol{j}]\big| \, c_{\boldsymbol{j}}(L)$ for every $\boldsymbol{j} \in \mathcal{J}(n,m)$.

Therefore, for $k \in \mathbb{N}$

$$\sum_{\substack{\boldsymbol{j} \in \mathcal{J}(n,m-1) \\ j_{m-1} \leq k}} \big|c_{(\boldsymbol{j},k)}(P)\big|^2 = \sum_{\substack{\boldsymbol{j} \in \mathcal{J}(n,m-1) \\ j_{m-1} \leq k}} \Big(\big|[(\boldsymbol{j},k)]\big| \, |c_{(\boldsymbol{j},k)}(L)| \Big)^2 \leq m^2 \sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} \Big(\big|[\boldsymbol{j}]\big| \, |c_{(\boldsymbol{j},k)}(L)| \Big)^2 . \tag{4$\cdot$D}$$

By Lemma 3.4, the latter sum can be written as an integral

$$\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)} \Big(\big|[\boldsymbol{j}]\big|\,|c_{(\boldsymbol{j},k)}(L)|\Big)^2 = \int_{\mathbb{T}^n}\Big|\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)} \big|[\boldsymbol{j}]\big|\, c_{(\boldsymbol{j},k)}(L)\,\omega_{\boldsymbol{j}}\Big|^2 \mathrm{d}m_n(\omega)$$

which is, by Proposition 4.6, bounded by

$$\leq 2^{m-1}\Bigg(\int_{\mathbb{T}^n}\Big|\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)} |[\boldsymbol{j}]|\, c_{(\boldsymbol{j},k)}(L)\,\omega_{\boldsymbol{j}}\Big| \mathrm{d}m_n(\omega)\Bigg)^2.$$

Together with $(4\cdot\mathrm{D})$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\Big(\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)}\big|c_{(\boldsymbol{j},k)}(P)\big|^2\Big)^{\frac{1}{2}}\\
&\qquad\leq m2^{\frac{m-1}{2}}\int_{\mathbb{T}^n}\sum_{k=1}^{n}\Big|\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)}\big|[\boldsymbol{j}]\big|\,c_{(\boldsymbol{j},k)}(L)\,\omega_{\boldsymbol{j}}\Big|\mathrm{d}m_n(\omega)\\
&\qquad= m2^{\frac{m-1}{2}}\int_{\mathbb{T}^n}\sup_{\xi\in\mathrm{B}_{\ell_\infty^n}}\Big|\sum_{k=1}^{n}\sum_{\boldsymbol{j}\in\mathcal{J}(n,m-1)}\big|[\boldsymbol{j}]\big|\,c_{(\boldsymbol{j},k)}(L)\,\omega_{\boldsymbol{j}}\,\xi_k\Big|\mathrm{d}m_n(\omega)\\
&\qquad= m2^{\frac{m-1}{2}}\int_{\mathbb{T}^n}\sup_{\xi\in\mathrm{B}_{\ell_\infty^n}}\Big|\sum_{\boldsymbol{i}\in\mathcal{I}(n,m)}c_{\boldsymbol{i}}(L)\,\omega_{i_1}\cdots\omega_{i_{m-1}}\,\xi_{i_m}\Big|\mathrm{d}m_n(\omega)\\
&\qquad\leq m2^{\frac{m-1}{2}}\sup_{\omega\in\mathbb{T}^n}\sup_{\xi\in\mathrm{B}_{\ell_\infty^n}}|L(\omega,\ldots,\omega,\xi)|\\
&\qquad= m2^{\frac{m-1}{2}}\sup_{\omega,\xi\in\mathrm{B}_{\ell_\infty^n}}|L(\omega,\ldots,\omega,\xi)|\,.
\end{aligned}$$

Harris' inequality (Proposition 2.11) completes the proof. □

Proof of Theorem 4.1, $p=\infty$. For $P\in\mathcal{P}({}^m\ell_\infty^n)$ and $x\in\mathrm{B}_{\ell_\infty^n}$

$$\sum_{\boldsymbol{j}\in J}|c_{\boldsymbol{j}}(P)\,x_{\boldsymbol{j}}|\leq\sum_{k=1}^{n}\Bigg(\sum_{\substack{\boldsymbol{j}\in J^*\\(\boldsymbol{j},k)\in J}}|c_{\boldsymbol{j}}(P)|\Bigg)$$

and, by the Cauchy-Schwarz inequality,

$$\leq\sum_{k=1}^{n}\Bigg(\sum_{\substack{\boldsymbol{j}\in J^*\\(\boldsymbol{j},k)\in J}}|c_{\boldsymbol{j}}(P)|^2\Bigg)^{\frac{1}{2}}\big|\{\boldsymbol{j}\in J^*\mid(\boldsymbol{j},k)\in J\}\big|^{\frac{1}{2}}$$

$$\leq \sum_{k=1}^{n} \Bigg(\sum_{\substack{\boldsymbol{j} \in \mathcal{J}(n,m-1) \\ \boldsymbol{j}_{m-1} \leq k}} |c_{\boldsymbol{j}}(P)|^2 \Bigg)^{\frac{1}{2}} |J^*|^{\frac{1}{2}} .$$

The previous lemma now provides the claim. □

Proof of Theorem 4.1, $2 \leq p \leq \infty$. In the previous proof we have shown that for any $P \in \mathcal{P}({}^m\ell_\infty^n)$

$$\sup_{x \in \mathrm{B}_{\ell_\infty^n}} \sum_{\boldsymbol{j} \in J} |c_{\boldsymbol{j}}(P)\, x_{\boldsymbol{j}}| \leq \mathrm{e} m 2^{\frac{m-1}{2}} \, |J^*|^{\frac{1}{2}} \, \|P\|_{\mathrm{B}_{\ell_\infty^n}} . \tag{4·E}$$

Recall (3 · B) and note that (4 · E) is equivalent to

$$\left\| M_\theta : \mathcal{P}({}^m\ell_\infty^n) \to \mathcal{P}({}^m\ell_\infty^n) \right\| \leq \mathrm{e} m 2^{\frac{m-1}{2}} \, |J^*|^{\frac{1}{2}}$$

for any $\theta \in \mathbb{C}^{\mathcal{J}(n,m)}$ with $\theta_{\boldsymbol{j}} \in \mathbb{T}$ if $\boldsymbol{j} \in J$ and $\theta_{\boldsymbol{j}} = 0$ otherwise. By Theorem 3.12, we obtain the desired inequality for every $2 \leq p \leq \infty$. □

4.3. The unconditional basis constant for the full space of m–homogeneous polynomials

We want introduce the following estimate as a first application of our result:

Theorem 4.7. *Let $1 \leq p \leq \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. There exists a constant $c \geq 1$ such that for any $n, m \in \mathbb{N}$*

$$\frac{1}{c(m \log m)^\sigma} \left(\frac{n}{m} \right)^{(m-1)\,\sigma} \leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^m\ell_p^n)\big) \leq \mathrm{e}^{3m} \left(1 + \frac{n}{m-1} \right)^{(m-1)\,\sigma} .$$

Let us outhouse a technical result to keep the proof of the theorem lucid:

Lemma 4.8. *Let $n, m \in \mathbb{N}$. Then*

$$\binom{n+m-2}{m-1} \leq \mathrm{e}^{m-1} \left(1 + \frac{n}{m-1} \right)^{m-1} .$$

Proof. Using the series expansion of the exponential function we have

$$\begin{aligned}\mathrm{e}^{m-1} &\geq \frac{(m-1)^{m-1}}{(m-1)!} \geq \frac{(m-1)^{m-1}}{(m-1)!} \cdot \frac{n}{n+m-1} \cdots \frac{n+m-2}{n+m-1}\\ &= \frac{(m-1)^{m-1}(n+m-2)!}{(n-1)!(n+m-1)^{m-1}(m-1)!} = \left(\frac{m-1}{n+m-1}\right)^{m-1} \binom{n+m-2}{m-1}.\end{aligned}$$

Thus, after multiplying both sides by $\left(\frac{m-1}{n+m-1}\right)^{-(m-1)} = \left(1+\frac{n}{m-1}\right)^{m-1}$, we obtain the inequality to prove. $\square$

Proof of Theorem 4.7. We start with the upper estimate. By Theorem 4.1, we have that

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^m\ell_p^n)\big) \leq \mathrm{e}^{2m}|\mathcal{J}(n,m)^*|^{\sigma}\,.$$

We check easily that $\mathcal{J}(n,m)^* = \mathcal{J}(n,m-1)$; therefore, by a combinatorial argument and the previous lemma,

$$|\mathcal{J}(n,m)^*| = |\mathcal{J}(n,m-1)| = \binom{n+m-2}{m-1} \leq \mathrm{e}^{m-1}\left(1+\frac{n}{m-1}\right)^{m-1},$$

which provides the upper estimate.

For $p=1$ the lower estimate is trivially true as in this case $\sigma = 0$. Let $p>1$. The lower estimate makes use of Corollary 3.25: we get with a universal constant $c \geq 1$

$$\begin{aligned}\chi_{\mathrm{mon}}\big(\mathcal{P}({}^m X_n)\big) &\geq \big(c\,(n\log m)^{\sigma}\big)^{-1} m^{-m\sigma}\,\big\|\mathrm{id}:\ell_p^n \hookrightarrow \ell_1^n\big\|^m \big\|\mathrm{id}:\ell_p^n \hookrightarrow \ell_{\min\{p,2\}}^n\big\|^{-m}\\ &= \big(c\,(n\log m)^{\sigma}\big)^{-1} m^{-m\sigma}\,n^{m\sigma} = \frac{1}{c(m\log m)^{\sigma}}\left(\frac{n}{m}\right)^{(m-1)\,\sigma}.\end{aligned}$$

$\square$

Chapter 5.

Abstract viewpoint on unconditional basis constants

In the previous chapter we saw an elementary approach to an estimate of the unconditional basis constant of the monomials in $\mathcal{P}({}^J\ell_p)$ where $J \subset \mathcal{J}(n,m)$. At this point we want to introduce an abstract and more general approach to estimate the unconditional basis constant: For one thing, this abstract proof helped pave the way for the elegant proof in the previous chapter, for another thing, this abstract approach gives independently interesting results about the Gordon-Lewis constant and the projection constant of the discussed Banach spaces.

The main theorem of this chapter is heavily inspired by the work of Defant and Frerick [26] and reads as follows:

Theorem 5.1. *Let X be a* Banach *sequence space and $J \subset \mathcal{J}(m,n)$ a set of indices. Given $J^\dagger \subset \mathcal{J}(m-1,n)$ and operators $R : \mathcal{P}({}^J X) \to \mathcal{L}\big(X; \mathcal{P}({}^{J^\dagger} X)\big)$ and $S : \mathcal{L}\big(X; \mathcal{P}({}^{J^\dagger} X)\big) \to \mathcal{P}({}^J X)$ such that* $\mathrm{id}_{\mathcal{P}({}^J X)} = SR$, *we have*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^J X)\big) \leq 2^m \|R\| \|S\| \, \lambda\big(\mathcal{P}({}^{J^\dagger} X)\big) .$$

The expression $\lambda\big(\mathcal{P}({}^{J^\dagger} X)\big)$ denotes the projection constant of the space $\mathcal{P}({}^{J^\dagger} X)$. For a precise definition see Section 5.1.4.

In the proof of the main theorem of this chapter we will see that the existence of any unconditional basis in $\mathcal{P}_{\mathrm{mon}}({}^J X)$ is equivalent to the unconditionality of the monomials and that the unconditional basis constant differ at most by a factor of 2^m. To be more precise, we have the following theorem:

Theorem 5.2. *Let X be a* Banach *sequence space and $J \subset \mathcal{J}(\infty, m)$. Then the monomials define an unconditional basis of $\mathcal{P}_{\mathrm{mon}}({}^J X)$ if and only if there exists some unconditional basis of $\mathcal{P}_{\mathrm{mon}}({}^J X)$. Moreover,*

$$\chi\big(\mathcal{P}_{\mathrm{mon}}({}^J X)\big) \leq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^J X)\big) \leq 2^m \chi\big(\mathcal{P}_{\mathrm{mon}}({}^J X)\big) .$$

A weaker version of the upper estimate (with a constant $c(m) \geq 1$) in this theorem was already proven by Defant, Díaz, García, and Maestre [24], though only for the full index set $J = \mathcal{J}(\infty, m)$; the constant 2^m was then established by Defant and Frerick [26], likewise only for the full index set $J = \mathcal{J}(\infty, m)$.

The theory needed to state the proofs of Theorem 5.1 and Theorem 5.2 will be introduced in the first section of this chapter. Subsequently we give the proofs of these theorems. In the latter sections we will show applications of the main theorem; we present possible choices of R, S, and $J^\dagger$ respectively.

5.1. Preliminaries of the proof

The concepts presented in this section are valid for any Banach space; we are not restricted to Banach sequence spaces or spaces of polynomials. As the theory is already well understood, we will touch only a few aspects. A good reference here is given by Diestel, Jarchow, and Tonge [35]; for some of the proofs we refer the reader to the literature.

5.1.1. Summing operators

We have already seen the definition of absolute summability. The following definition generalizes this strong notion of summability. For $p = 1$ we obtain the notion of absolute summability.

A sequence $(x_k)_k$ in E is called strongly p–summable if $\big(\|x_k\|\big)_k$ is an element of ℓ_p; we denote the space of all strongly p–summable sequences in E by $\ell_p(E)$ and define $\|(x_k)_k\|_p := \big\|\big(\|x_k\|\big)_k\big\|_{\ell_p}$ for $(x_k)_k \in \ell_p(E)$.

We call a sequence $(x_k)_k$ in E weakly p–summable if for every $x' \in E'$ the sequence $\big(\langle x', x_k\rangle\big)_k$ belongs to ℓ_p. By $\ell_p^w(E)$ we denote the space of all weakly p–summable sequences; a norm on $\ell_p^w(E)$ is given by

$$w_p\big((x_k)_k\big) := \sup\Big\{ \big\|\big(\langle x', x_k\rangle\big)_k\big\|_{\ell_p} \,\Big|\, x' \in \mathrm{B}_{E'} \Big\}$$

for any $(x_k)_k \in \ell_p^w(E)$.

Proposition 5.3 (cf. Chapter 2 of [35])**.** *Let E be a* Banach *space and $1 \le p \le \infty$. Then $\big(\ell_p(E), \|\cdot\|_p\big)$ and $\big(\ell_p^w(E), w_p\big)$ are* Banach *spaces.*

As the name suggests, strong p–summability implies weak p–summability:

Proposition 5.4 (cf. Chapter 2 of [35])**.** *Let E be a* Banach *space and $1 \le p \le \infty$. Then $\ell_p(E) \subset \ell_p^w(E)$ and $\|\mathrm{id} : \ell_p(E) \hookrightarrow \ell_p^w(E)\| \le 1$.*

Let $T : E \to F$ be a bounded operator between Banach spaces and define

$$\hat{T} : (x_k)_k \mapsto (Tx_k)_k\,.$$

We check easily that $\hat{T}$ always defines a bounded operator $\ell_p^w(E) \to \ell_p^w(F)$ and a bounded operator $\ell_p(E) \to \ell_p(F)$:

Proposition 5.5 (cf. Chapter 2 of [35])**.** *Let $T : E \to F$ be a bounded operator between* Banach *spaces. $\hat{T}$ then defines a bounded operator $\ell_p^w(E) \to \ell_p^w(F)$ and likewise a bounded operator $\ell_p(E) \to \ell_p(F)$. In both cases we have $\|\hat{T}\| \le \|T\|$.*

For some $T : E \to F$, $\hat{T}$ even induces a bounded operator $\ell_p^w(E) \to \ell_p(F)$. Those operators are of peculiar interest.

Let $1 \le p \le \infty$. An operator $T : E \to F$ is called p–summing if $\hat{T}$ induces a bounded operator $\ell_p^w(E) \to \ell_p(F)$. The set of all p–summing operators mapping E into F is denoted by $\Pi_p(E; F)$ and the p–summing norm of an operator T is defined by

$$\pi_p(T) := \big\|\hat{T} : \ell_p^w(E) \to \ell_p(F)\big\|\,.$$

To check whether an operator $T : E \to F$ is p–summing it suffices to consider only finite sequences. In other words:

Lemma 5.6. *The following are equivalent* *(i) T is p–summing.*

(ii) There exists a constant $c > 0$ such that

$$\big\|(Tx_k)_k\big\|_p \le c\, w_p\big((x_k)_k\big) \tag{5·A}$$

for every $n \in \mathbb{N}$ and any choice of $(x_k)_{k=1}^n \subset E$.

In this case $\pi_p(T) = \inf\{c \,|\, c$ fulfills $(5\cdot\mathrm{A})\}$.

Proof. It is clear that (i) implies (ii). Conversely take $(x_k)_k \in \ell_p^w(E)$. Then for every $n \in \mathbb{N}$

$$\Big(\sum_{k=1}^n \|Tx_k\|_F^p\Big)^{\frac{1}{p}} \le c \sup_{x' \in \mathrm{B}_{E'}} \Big(\sum_{k=1}^n |\langle x', x_k\rangle|^p\Big)^{\frac{1}{p}} \le c\, w_p\big((x_k)_k\big)\,.$$

Letting $n \to \infty$ we see that T is p–summing with $\pi_p(T) \le c$. ☐

Proposition 5.7 (cf. 2.6 Proposition and 2.4 Ideal Property in [35])**.** *Let E and F be* Banach *spaces and $1 \le p \le \infty$. Then $\big(\Pi_p(E;F), \pi_p\big)$ is a* Banach *space.*

Moreover, let G and H be Banach *spaces, $T \in \Pi_p(E;F)$, and $R : G \to E$, $S : F \to H$ be bounded operators. Then $STR : G \to H$ is p–summing with*

$$\pi_p(STR) \le \|S\|\, \pi_p(T)\, \|R\|\,.$$

5.1.2. Factorable operators

For a Banach space F define

$$\kappa_F : F \to F'', \quad x \mapsto \langle\,\cdot\,, x\rangle\,.$$

It is well known that κ_F defines an isometric embedding of F into its bidual space F''.

Let $1 \le p \le \infty$. An operator $T : E \to F$ between Banach spaces is called p–factorable if there exists a measure space (Ω, Σ, μ) and operators $R : E \to L_p(\mu)$ and $S : L_p(\mu) \to F''$ such that $\kappa_F \circ T = SR$. In this case we define

$$\gamma_p(T) := \inf \|R\|\,\|S\|$$

where the infimum is taken over all possible factorizations of $\kappa_F \circ T$ through some L_p space. The space of all p–factorable operators $T : E \to F$ is denoted by $\Gamma_p(E;F)$.

Proposition 5.8 (cf. 7.1 Theorem in [35]). *Let E and F be* Banach *spaces and let $1 \leq p \leq \infty$. Then $\big(\Gamma_p(E;F), \gamma_p\big)$ is a* Banach *space.*

Moreover, for G and H Banach *spaces, $T \in \Gamma_p(E;F)$, and $R : G \to E$, $S : F \to H$ bounded operators $STR : G \to H$ is p–factorable with*

$$\gamma_p(STR) \leq \|S\| \, \gamma_p(T) \, \|R\| \, .$$

The following proposition will enable us to give a much simpler characterization if E and F are finite dimensional.

Proposition 5.9 (cf. 3.2 Theorem in [35]). *Let (Ω, Σ, μ) be a measure space and let $1 \leq p \leq \infty$. Then for any $\varepsilon > 0$ and any finite dimensional subspace $X \subset L_p(\mu)$ there exists a finite dimensional subspace $Y \subset L_p(\mu)$ and an isomorphism $U : F \to \ell_p^{\dim Y}$ such that $X \subset Y$ and $\|U\| \, \|U^{-1}\| \leq 1 + \varepsilon$.*

For finite dimensional Banach spaces E and F we have thus an alternative characterization of γ_p; namely

$$\gamma_p(T) = \inf\big\{ \|R\| \, \|S\| \; \big| \; n \in \mathbb{N}, R : E \to \ell_p^n, S : \ell_p^n \to F, T = SR \big\}$$

for any p–factorable operator $T : E \to F$.

Proposition 5.10 (cf. 7.2 Proposition in [35]). *Let E and F be* Banach *spaces and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For a bounded operator $T : E \to F$ the following statements are equivalent:*

(i) T is p–factorable.

(ii) The dual operator $T' : F' \to E'$, defined by $T'y' = y' \circ T$, is q–factorable.

In this case, $\gamma_p(T) = \gamma_q(T')$.

5.1.3. The Gordon-Lewis property

A Banach space E is said to have the Gordon-Lewis property if every 1–summing operator $T : E \to \ell_2$ is 1–factorable. This is equivalent to the existence of a constant $c > 0$ such that for every $T \in \Pi_1(E; \ell_2)$

$$\gamma_1(T) \leq c \, \pi_1(T) \, . \qquad (5 \cdot \mathrm{B})$$

The GORDON-LEWIS constant of E is then defined as

$$\mathrm{gl}(E) := \sup\bigl\{\gamma_1(T) \bigm| T \in \Pi_1(E;\ell_2), \pi_1(T) \leq 1\bigr\} = \inf\bigl\{c \bigm| c \text{ fulfills } (5\cdot\mathrm{B})\bigr\}\,.$$

Let us introduce some fundamental properties of the GORDON-LEWIS constant. For a deeper discussion of this topic see Chapter 17 in [35].

Theorem 5.11. *Let E and F be* BANACH *spaces and suppose that F has the* GORDON-LEWIS *property. Let $R : E \to F$ and $S : F \to E$ be bounded operators such that* $\mathrm{id}_E = SR$. *Then E as well has the* GORDON-LEWIS *property and*

$$\mathrm{gl}(E) \leq \|R\| \cdot \|S\|\ \mathrm{gl}(F)\,.$$

Proof. Let $T : E \to \ell_2$ be an 1–summing operator. Then $TS : F \to \ell_2$ is also 1–summing and, since F has the GORDON-LEWIS property, TS is 1–factorable. By Proposition 5.8, $T = TSR$ is then as well 1–factorable.

Moreover, by Proposition 5.8 and 5.7,

$$\gamma_1(T) \leq \gamma_1(TS)\,\|R\| \leq \mathrm{gl}(F)\,\pi_1(TS)\,\|R\| \leq \|S\|\,\|R\|\ \mathrm{gl}(F)\,\pi_1(T)\,. \qquad \square$$

To use the GORDON-LEWIS constant in our investigations we need to know how the unconditional basis constant and GORDON-LEWIS constant are related. The following two theorems are essential.

Theorem 5.12. *Let E be a* BANACH *space with a basis $(b_k)_k$. Then*

$$\mathrm{gl}(E) \leq \chi\bigl((b_k)_k; E\bigr).$$

Moreover, $\mathrm{gl}(E) \leq \chi(E)$.

Proof. Let $T \in \Pi_1(E;\ell_2)$ and define $R : E \to \ell_1$ by $b_k \mapsto \|Tb_k\|_{\ell_2}\, e_k$ and $S : \ell_1 \to \ell_2$ by $e_k \mapsto \bigl(\|Tb_k\|_{\ell_2}\bigr)^{-1}\, Tb_k$. We check at once that R and S are well-defined (note that T is 1–summing). We obtain a factorization of T through ℓ_1.

For $x = \sum_k x_k b_k \in E$ we have

$$\|Rx\|_{\ell_1} = \sum_k \|Tb_k\|_{\ell_2} \cdot |x_k| = \sum_k \|T(x_k b_k)\|_{\ell_2} \leq \pi_1(T)\, w_1\bigl((x_k b_k)_k\bigr)$$

and

$$
\begin{aligned}
w_1\big((x_k b_k)_k\big) &\le \sup_{x' \in \mathrm{B}_{E'}} \sum_k \big|x'(x_k b_k)\big| \\
&= \sup_{x' \in \mathrm{B}_{E'}} \sup_{(\varepsilon_k)_k \in \mathbb{T}^{\mathbb{N}}} \Big|\sum_k \varepsilon_k x'(x_k e_k)\Big| = \sup_{(\varepsilon_k)_k \in \mathbb{T}^{\mathbb{N}}} \Big\|\sum_k \varepsilon_k x_k b_k\Big\|_E \le \chi\big((b_k)_k; E\big) \, \|x\| \,.
\end{aligned}
$$

For $y = (y_k)_k \in \ell_1$ we have

$$
\|Sy\|_{\ell_2} = \Big\|\sum_k y_k \cdot \big(\|Tb_k\|_{\ell_2}\big)^{-1} \, Tb_k\Big\|_{\ell_2} \le \sum_k |y_k| = \|y\|_{\ell_1} \,.
$$

From these inequalities, we have $\gamma_1(T) \le \|R\| \, \|S\| \, \pi_1(T) \le \chi\big((b_k)_k; E\big) \, \pi_1(T)$. To deduce $\mathrm{gl}(E) \le \chi(E)$ from this, simply take the infimum over all possible bases $(b_k)_k$.
$\square$

We now have that the Gordon-Lewis constant is always bounded by the unconditional basis constant. In certain situations we have the opposite inequality. A result similar to the following one was, to our best knowledge, first proved independently by Pisier [49] and Schütt [54]. In [24] the result can be found as stated.

Theorem 5.13 (cf. Lemma 2 in [24])**.** *Let E be a finite dimensional* Banach *space with a basis $(b_k)_{k=1}^n$ and coefficient functionals $(b'_k)_{k=1}^n$. Suppose that there exist constants $c_1, c_2 \ge 1$ such that for every choice of $(\lambda_k)_k, (\mu_k)_k \in \mathbb{C}^n$*

$$
\begin{aligned}
D_\lambda : E &\to \ell_2^n \\
\sum_k x_k b_k &\mapsto (\lambda_k x_k)_k
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
D_\mu : E' &\to \ell_2^n \\
\sum_k y_k b'_k &\mapsto (\mu_k y_k)_k
\end{aligned}
$$

satisfy

$$
\pi_1(D_\lambda) \le c_1 \, \Big\|\sum_k \lambda_k b'_k\Big\|_{E'} \qquad \text{and} \qquad \pi_1(D_\mu) \le c_2 \, \Big\|\sum_k \mu_k b_k\Big\|_E \,.
$$

Then

$$
\chi\big((b_k)_k; E\big) \le c_1 c_2 \; \mathrm{gl}(E) \,.
$$

In the general infinite dimensional case an upper bound of the unconditional basis constant in terms of the Gordon-Lewis constant does not exist. However in certain cases the theorem above allows suitable bounds.

Although the theorem was already proved by Defant, Díaz, García, and Maestre [24], we give the proof for the readers convenience. The proof of the theorem makes use of the following lemma. In [35] this result can be found as a combination of 5.16 Theorem and 6.14 Lemma.

Lemma 5.14. *Let E and F be finite dimensional* Banach *spaces and $R : E \to F$ and $S : F \to E$ operators. Then*

$$\big|\mathrm{tr}(SR)\big| \leq \pi_1(S)\,\gamma_\infty(R)\,.$$

Proof. Choose any factorization $R : E \overset{K}{\to} \ell_\infty^n \overset{L}{\to} F$. There is no loss of generality in assuming $\|K\| \leq 1$. As finite rank operators we can write K and L as $K = \sum_k x'_k \otimes e_k$ with $(x'_k)_{k=1}^n \subset E'$ respectively $L = \sum_k e'_k \otimes y_k$ with $(y_k)_{k=1}^n \subset F$. Then

$$1 \geq \|K\| = \sup_{x \in \mathrm{B}_E} \Big\|\sum_k x'_k(x) e_k\Big\|_{\ell_\infty} = \sup_{x\in\mathrm{B}_E} \sup_k |x'_k(x)| = \sup_k \|x'_k\|$$

and

$$\begin{aligned}\|L\| = \sup_{\xi \in \mathrm{B}_{\ell_\infty^n}} \Big\|\sum_k e'_k(\xi) y_k\Big\|_F &= \sup_{\xi \in \mathrm{B}_{\ell_\infty^n}} \sup_{y' \in \mathrm{B}_{F'}} \Big|\sum_k \xi_k y'(y_k)\Big| \\ &= \sup_{y'\in\mathrm{B}_{F'}} \sum_k \big|\langle y', y_k\rangle\big| = w_1\big((y_k)_k\big)\,.\end{aligned}$$

Now,

$$\begin{aligned}\big|\mathrm{tr}(SR)\big| = \Big|\mathrm{tr}\,\Big(S \circ \sum_k x'_k \otimes y_k\Big)\Big| &= \Big|\mathrm{tr}\,\Big(\sum_k x'_k \otimes Sy_k\Big)\Big| = \Big|\sum_k x'_k(Sy_k)\Big| \\ &\leq \sum_k \|Sy_k\| \leq \pi_1(S)\, w_1\big((y_k)_k\big) \leq \pi_1(S)\,\|K\|\,\|L\|\,.\end{aligned}$$

Taking the infimum over all possible factorizations of R through ℓ_∞^n, $n \in \mathbb{N}$ now yields the claim. $\square$

Proof of Theorem 5.13. Let $(\theta_k)_k \in \mathbb{T}^n$ and take $(\lambda_k)_k, (\mu_k)_k \in \mathbb{C}^n$. Then

$$\begin{aligned}\Big|\langle \sum_k \theta_k \mu_k b_k, \sum_j \lambda_j b'_j\rangle\Big| &= \Big|\sum_k \theta_k\mu_k\lambda_k\Big| \\ &= \Big|\mathrm{tr}\,\Big(\ell_2^n \xrightarrow{D'_\lambda} E' \xrightarrow{D_\mu} \ell_2^n \xrightarrow{D_\theta} \ell_2^n\Big)\Big|\end{aligned}$$

which is by Lemma 5.14

$$\begin{aligned} &\le \gamma_\infty(D'_\lambda)\,\pi_1(D_\theta \circ D_\mu) \\ &\le \gamma_1(D_\lambda)\,\pi_1(D_\mu)\,, \end{aligned}$$

since $\|D_\theta\| \le 1$. Hence,

$$\begin{aligned} &\le \mathrm{gl}(E)\,\pi_1(D_\lambda)\pi_1(D_\mu) \\ &\le \mathrm{gl}(E)\,c_1c_2\,\Big\|\sum_k \lambda_k b'_k\Big\|_{E'}\Big\|\sum_k \mu_k b_k\Big\|_E \end{aligned}$$

by assumption. By our considerations above, we have now

$$\begin{aligned} \Big\|\sum_k \theta_k\mu_k b_k\Big\| &= \sup_{x'\in \mathrm{B}_{E'}} \Big|\Big\langle \sum_k \theta_k\mu_k b_k, x'\Big\rangle\Big| \\ &\le \sup_{x'\in \mathrm{B}_{E'}} \mathrm{gl}(E)\,c_1c_2\,\|x'\|_{E'}\Big\|\sum_k \mu_k b_k\Big\|_E \\ &= c_1c_2\ \mathrm{gl}(E)\,\Big\|\sum_k \mu_k b_k\Big\|_E\,. \end{aligned}$$

Thus $\chi\big((b_k)_k; E\big) \le c_1c_2\ \mathrm{gl}(E)$. □

5.1.4. The projection constant

For a Banach space F and a closed subspace $E \subset F$ the relative projection constant of E in F is defined as

$$\lambda(E,F) := \inf\big\{\|\mathfrak{P}\| \,\big|\, \mathfrak{P} : F \to F \text{ bounded projection onto } E\big\}\,.$$

If E is not complemented in F, i.e. if there doesn't exist a bounded projection $\mathfrak{P} : F \to F$ onto E, we set $\lambda(E,F) = \infty$. The projection constant of a Banach space E defined as

$$\lambda(E) := \sup\big\{\lambda\big(i(E),F\big) \,\big|\, F \text{ Banach space, } i : E \hookrightarrow F \text{ isometric embedding}\big\}\,.$$

The projection constant is of particular interest in different fields of mathematics. However, this topic exceeds the scope of this thesis. For a deeper discussion of this topic and proofs we refer the reader to Tomczak-Jaegermann [58].

For a finite dimensional BANACH space E it is easy to see that $\lambda(E) < \infty$ and to obtain a rough estimate: Let $x_1, \dots, x_n$ be a basis of $i(E) \subset F$ and let $x_1', \dots, x_n'$ denote the biorthogonal functionals in F'. Then $\mathfrak{P} := \sum_k x_k' \otimes x_k$ defines a bounded projection onto $i(E)$ with $\|\mathfrak{P}\| \leq n = \dim E$.

A better estimate on the projection constant is given in the following theorem. This estimate was first established by KADEC and SNOBAR in 1971.

Theorem 5.15 (cf. Proposition 9.12 in [58])**.** *Let E be a finite dimensional* BANACH *space. Then*

$$\lambda(E) \leq \sqrt{\dim E}\,.$$

To obtain even better estimates the following proposition will be useful:

Proposition 5.16 (cf. Proposition 32.1 in [58])**.** *Let E be a* BANACH *space. Then*

$$\lambda(E) = \gamma_\infty(\mathrm{id}_E)\,.$$

This proposition now enables us to prove the following estimate:

Theorem 5.17. *Let $1 \leq p < 2$ and let X be a* BANACH *sequence space with p–exhaustible unit ball. For $\Lambda \subset \mathbf{\Lambda}(n,m)$ then*

$$\lambda\big(\mathcal{P}(^\Lambda X)\big) \leq \left(\sup_{\alpha\in\Lambda} \Big(\frac{\alpha!}{m!}\Big)^{\frac{1}{p}} \big\|c_\alpha : \mathcal{P}(^\Lambda X) \to \mathbb{C}\big\|\right) \cdot |\Lambda|^{1-\frac{1}{p}}\,.$$

Proof. We aim to use the previous proposition; we have to construct a suitable factorization. Let $\Gamma := \mathrm{B}_X$ and consider the isometric embedding

$$\mathcal{P}(^\Lambda X) \hookrightarrow \ell_\infty(\Gamma), \quad P \mapsto \big(P(x)\big)_{x\in\Gamma}\,.$$

We claim that there exists a projection $\mathfrak{P} : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ onto $\mathcal{P}(^\Lambda X)$ such that

$$\|\mathfrak{P}\| \leq \left(\sup_{\alpha\in\Lambda} \Big(\frac{\alpha!}{m!}\Big)^{\frac{1}{p}} \big\|c_\alpha : \mathcal{P}(^\Lambda X) \to \mathbb{C}\big\|\right) \cdot |\Lambda|^{1-\frac{1}{p}}\,.$$

By HAHN-BANACH'S theorem, the coefficient functionals $c_\alpha : \mathcal{P}(^\Lambda X) \to \mathbb{C}$ extend to functionals $\pi_\alpha : \ell_\infty(\Gamma) \to \mathbb{C}$ with $\|\pi_\alpha\| = \|c_\alpha\|$ and $\pi_\alpha\big|_{\mathcal{P}(^\Lambda X)} = c_\alpha$. With these functionals define

$$\mathfrak{P} : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma), \quad f \mapsto \sum_{\alpha\in\Lambda} \pi_\alpha(f)\, z^\alpha\,.$$

$\mathfrak{P}$ is indeed a projection onto $\mathcal{P}({}^{\Lambda}X)$; it remains to estimate its norm. For any $f \in \ell_\infty(\Gamma)$ with $\|f\| \le 1$ we obtain

$$\begin{aligned}
\|\mathfrak{P}f\| &= \sup_{x\in \mathrm{B}_X} \Big|\sum_{\alpha\in\Lambda} \pi_\alpha(f)\, x^\alpha\Big| \\
&\le \sup_{x\in \mathrm{B}_X} \sum_{\alpha\in\Lambda} \|\pi_\alpha\|\, |x^\alpha| \\
&\le \sup_{\alpha\in\Lambda} \Big(\frac{\alpha!}{m!}\Big)^{\frac{1}{p}} \|\pi_\alpha\| \cdot \sup_{x\in \mathrm{B}_X} \sum_{\alpha\in\Lambda} \Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}} |x^\alpha| \,.
\end{aligned}$$

Using Hölders inequality, we can estimate the latter supremum and obtain

$$\begin{aligned}
\sup_{x\in \mathrm{B}_X} \sum_{\alpha\in\Lambda} \Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}} |x^\alpha| &\le \sup_{x\in \mathrm{B}_X} \Big(\sum_{\alpha\in\Lambda} 1\Big)^{1-\frac{1}{p}} \Big(\sum_{\alpha\in\Lambda} \frac{m!}{\alpha!} |x^\alpha|^p\Big)^{\frac{1}{p}} \\
&\le \sup_{x\in \mathrm{B}_X} |\Lambda|^{1-\frac{1}{p}} \Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \big(|x_1|^p, \dots, |x_n|^p\big)^\alpha\Big)^{\frac{1}{p}} \\
&= \sup_{x\in \mathrm{B}_X} |\Lambda|^{1-\frac{1}{p}} \bigg(\Big(\sum_{k=1}^n |x_k|^p\Big)^{\frac{1}{p}}\bigg)^m \,.
\end{aligned}$$

Here the last factor is a power of $\|\mathrm{id} : X \hookrightarrow \ell_p\|$; therefore, by Corollary 3.16,

$$\le |\Lambda|^{1-\frac{1}{p}} \,.$$

Hence,

$$\|\mathfrak{P}\| \le \sup_{\alpha\in\Lambda} \Big(\frac{\alpha!}{m!}\Big)^{\frac{1}{p}} \|\pi_\alpha\| \cdot |\Lambda|^{1-\frac{1}{p}} \,. \qquad \square$$

For specific choices of X we obtain concrete estimates. From Corollary 2.19 and Corollary 2.20 we have:

Corollary 5.18. *Let $n, m \in \mathbb{N}$, $\Lambda \subset \mathbf{\Lambda}(n,m)$, $1 \le p < 2$, and $1 \le q \le \infty$. Then*

$$\lambda\big(\mathcal{P}({}^{\Lambda}\ell_{p,q})\big) \le \mathrm{e}^{\frac{m}{p}} |\Lambda|^{1-\frac{1}{p}} \,.$$

In particular,

$$\lambda\big(\mathcal{P}({}^{\Lambda}\ell_{p})\big) \le \mathrm{e}^{\frac{m}{p}} |\Lambda|^{1-\frac{1}{p}} \,.$$

5.2. Proof of the theorems

We are now able to prove our Theorems 5.1 and 5.2. The first step in our proof consists of an upper bound of the unconditional basis constant in terms of the GORDON-LEWIS constant of our space of polynomials.

Theorem 5.19. *Let X be a* BANACH *sequence space and $J \subset \mathcal{J}(n,m)$. Then*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^J X)\big) \leq 2^m \,\mathrm{gl}\left(\mathcal{P}(^J X)\right).$$

We have divided the proof into a sequence of lemmas.

Lemma 5.20. *Let $J \subset \mathcal{J}(n,m)$ and define $\Phi : \mathcal{P}(^J \ell_\infty^n) \to \ell_2(J)$ by $P \mapsto \big(c_{\boldsymbol{j}}(P)\big)_{\boldsymbol{j}}$. Then Φ is 1–summing with $\pi_1(\Phi) \leq \sqrt{2}^m$.*

Proof. For every family $(P_k)_k \subset \mathcal{P}(^J \ell_\infty^n)$

$$\sum_k \|\Phi P_k\|_{\ell_2} \leq \sum_k \sqrt{2}^m \int_{\mathbb{T}^n} |P_k(x)| \,\mathrm{d}m_n(x)$$

by Lemma 3.4 and Proposition 4.6. We proceed on estimating the right-hand side:

$$\begin{aligned}
&\leq \sqrt{2}^m \sup_{x \in \ell_\infty^n} \sum_k |P_k(x)| \\
&\leq \sqrt{2}^m \sup_{\substack{P' \in \mathcal{P}(^J \ell_\infty^n)' \\ \|P'\| \leq 1}} \sum_k \big|\langle P', P_k\rangle\big| \\
&= w_1\big((P_k)_k\big). \qquad \square
\end{aligned}$$

Lemma 5.21. *Let $J \subset J(n,m)$ and let $z'_{\boldsymbol{j}}$ denote a basis of $\mathcal{P}(^J X)'$ biorthogonal to the monomials in $\mathcal{P}(^J X)$. Define*

$$\Phi_\lambda : \mathcal{P}(^J X) \to \mathcal{P}(^J \ell_\infty^n), \quad z_{\boldsymbol{j}} \mapsto \lambda_{\boldsymbol{j}} z_{\boldsymbol{j}}$$

and

$$\Phi_\mu : \mathcal{P}(^J X)' \to \mathcal{P}(^J \ell_\infty^n), \quad z'_{\boldsymbol{j}} \mapsto \mu_{\boldsymbol{j}} z_{\boldsymbol{j}}.$$

for $(\lambda_{\boldsymbol{j}})_{\boldsymbol{j}}, (\mu_{\boldsymbol{j}})_{\boldsymbol{j}} \in \mathbb{C}^J$. Then $\|\Phi_\lambda\| \leq \big\|\sum_{\boldsymbol{j}} \lambda_{\boldsymbol{j}} z'_{\boldsymbol{j}}\big\|_{\mathcal{P}(^J X)'}$ and $\|\Phi_\mu\| \leq \big\|\sum_{\boldsymbol{j}} \mu_{\boldsymbol{j}} z_{\boldsymbol{j}}\big\|_{\mathrm{B}_X}$.

Proof. For $x \in \mathrm{B}_{\ell_\infty^n}$ and $P \in \mathcal{P}({}^J X)$

$$\begin{aligned} \big|\Phi_\lambda P(x)\big| &= \big|\sum_{\boldsymbol{j}} \lambda_{\boldsymbol{j}} c_{\boldsymbol{j}}(P)\, x_{\boldsymbol{j}}\big| \\ &= \big|\big\langle \sum_{\boldsymbol{j}} c_{\boldsymbol{j}}(P)\, x_{\boldsymbol{j}} z_{\boldsymbol{j}}, \sum_{\boldsymbol{j}} \lambda_{\boldsymbol{j}} z'_{\boldsymbol{j}} \big\rangle\big| \leq \|P\|_{\mathrm{B}_X} \big\|\sum_{\boldsymbol{j}} \lambda_{\boldsymbol{j}} z'_{\boldsymbol{j}}\big\|_{\mathcal{P}({}^J X)'} , \end{aligned} \tag{5 $\cdot$ C}$$

since

$$\Big\|\sum_{\boldsymbol{j}} c_{\boldsymbol{j}}(P)\, x_{\boldsymbol{j}} z_{\boldsymbol{j}}\Big\|_{\mathrm{B}_X} = \sup_{\omega \in \mathrm{B}_X} |P(x \cdot \omega)| \leq \sup_{\xi \in \mathrm{B}_X} |P(\xi)| = \|P\|_{\mathrm{B}_X} .$$

Analogously, for $x \in \mathrm{B}_{\ell_\infty^n}$ and $P' = \sum_{\boldsymbol{j}} c_{\boldsymbol{j}} z'_{\boldsymbol{j}} \in \mathcal{P}({}^J X)'$

$$\big|\Phi_\mu P'(x)\big| = \big|\sum_{\boldsymbol{j}} \mu_{\boldsymbol{j}} c_{\boldsymbol{j}} x_{\boldsymbol{j}}\big| = \big|\big\langle \sum_{\boldsymbol{j}} \mu_{\boldsymbol{j}}\, x_{\boldsymbol{j}} z_{\boldsymbol{j}}, \sum_{\boldsymbol{j}} c_{\boldsymbol{j}} z'_{\boldsymbol{j}} \big\rangle\big| \leq \big\|\sum_{\boldsymbol{j}} \mu_{\boldsymbol{j}} z_{\boldsymbol{j}}\big\|_{\mathrm{B}_X} \|P'\|_{\mathcal{P}({}^J X)'} . \tag{5 $\cdot$ D}$$

Taking the supremum over $x \in \mathrm{B}_{\ell_\infty^n}$ in both inequalities (5 · C) and (5 · D) yields the claim. □

Proof of Theorem 5.19. We consider the monomials $(z_{\boldsymbol{j}})_{\boldsymbol{j}}$ as a basis of $\mathcal{P}({}^J X)$ (note that we assumed J to be finite) and denote by $(z'_{\boldsymbol{j}})_{\boldsymbol{j}}$ its biorthogonal basis of $\mathcal{P}({}^J X)'$.

As we check at once, we have for D_λ and D_μ as defined in Theorem 5.13 that $D_\lambda = \Phi \circ \Phi_\lambda$ and $D_\mu = \Phi \circ \Phi_\mu$. Furthermore, by the previous lemmas and Proposition 5.7,

$$\pi_1(D_\lambda) \leq \|\Phi_\lambda\| \pi_1(I) \leq \sqrt{2}^m \big\|\sum_{\boldsymbol{j}} \lambda_{\boldsymbol{j}} z'_{\boldsymbol{j}}\big\|_{\mathcal{P}({}^J X)'}$$

and

$$\pi_1(D_\mu) \leq \|\Phi_\mu\| \pi_1(I) \leq \sqrt{2}^m \big\|\sum_{\boldsymbol{j}} \mu_{\boldsymbol{j}} z_{\boldsymbol{j}}\big\|_{\mathrm{B}_X} .$$

The premises of Theorem 5.13 are fulfilled and we obtain the inequality in question. □

We proceed with the second step in the proof of our main theorem.

Theorem 5.22. *Let X be a* Banach *sequence space and $J \subset \mathcal{J}(n, m-1)$. Then*

$$\mathrm{gl}\Big(\mathcal{L}\big(X_n; \mathcal{P}({}^J X)\big)\Big) \leq \lambda\big(\mathcal{P}({}^J X)\big) .$$

Proof. The proof is divided into two steps. For the first step let $\mathrm{id}_{\mathcal{P}(^J X)} = SR$ be any factorization with $R \in \mathcal{L}\big(\mathcal{P}(^J X); \ell_\infty^d\big)$ and $S \in \mathcal{L}\big(\ell_\infty^d; \mathcal{P}(^J X)\big)$. We naturally obtain a factorization $\mathrm{id}_{\mathcal{L}(X_n;\,\mathcal{P}(^J X))} = VU$ with

$$U : \mathcal{L}\big(X_n; \mathcal{P}(^J X)\big) \to \mathcal{L}(X_n; \ell_\infty^d), \quad T \mapsto RT$$

and

$$V : \mathcal{L}(X_n; \ell_\infty^d) \to \mathcal{L}\big(X_n; \mathcal{P}(^J X)\big), \quad L \mapsto SL\,.$$

Obviously $\|U\| \le \|R\|$ and $\|V\| \le \|S\|$. Hence, by Theorem 5.11,

$$\mathrm{gl}\left(\mathcal{L}\big(X_n; \mathcal{P}(^J X)\big)\right) \le \|R\|\,\|S\|\; \mathrm{gl}\left(\mathcal{L}(X_n; \ell_\infty^d)\right).$$

By Proposition 5.16, we have after taking the infimum over all possible factorizations $\mathrm{id}_{\mathcal{P}(^J X)} = SR$

$$\mathrm{gl}\left(\mathcal{L}\big(X_n; \mathcal{P}(^J X)\big)\right) \le \lambda\big(\mathcal{P}(^J X)\big) \cdot \mathrm{gl}\left(\mathcal{L}(X_n; \ell_\infty^d)\right). \tag{5$\cdot$E}$$

In the second step we show that the GORDON-LEWIS constant on the right-hand side of (5 · E) equals one. The coefficient functionals $(e_k')_{k=1}^n$ define a basis of X_n' and the canonical sequences $(e_l)_{l=1}^d$ define a basis of ℓ_∞^d. Hence $(e_k' \otimes e_l)_{k,l}$ is a basis of $\mathcal{L}(X_n; \ell_\infty^d)$.

For any choice of $(y_j')_j \subset X'$ an easy calculation shows

$$\Big\|\sum_j y_j' \otimes e_j\Big\|_{\mathcal{L}(X;\ell_\infty^d)} = \sup_{x \in \mathrm{B}_X} \Big\|\sum_j y_j'(x) e_j\Big\|_{\ell_\infty} = \sup_j \sup_{x \in \mathrm{B}_X} \big|y_j'(x)\big| = \sup_j \big\|y_j'\big\|_{X'}\,.$$

Thus, for $(c_{k,j})_{k,j} \in \mathbb{C}^{n \times d}$ and $(\varepsilon_{k,j})_{k,j} \in \mathbb{T}^{n \times d}$ we have

$$\begin{aligned}
\Big\|\sum_{k,j} \varepsilon_{k,j} c_{k,j} x_k' \otimes e_j\Big\|_{\mathcal{L}(X;\ell_\infty^d)} &= \Big\|\sum_j \Big(\sum_k \varepsilon_{k,j} c_{k,j} x_k'\Big) \otimes e_j\Big\|_{\mathcal{L}(X;\ell_\infty^d)} \\
&= \sup_j \Big\|\sum_k \varepsilon_{k,j} c_{k,j} x_k'\Big\|_{X'} \\
&\le \sup_j \Big\|\sum_k c_{k,j} x_k'\Big\|_{X'}\,,
\end{aligned}$$

since the coefficient functionals form an 1–unconditional basis of X_n'. By the same calculation as before this is

$$= \Big\|\sum_{k,j} c_{k,j} x_k' \otimes e_j\Big\|_{\mathcal{L}(X;\ell_\infty^d)}\,.$$

We have shown that the unconditional basis constant of $\mathcal{L}(X_n;\ell_\infty^d)$ equals one. Theorem 5.12 now completes the proof. $\square$

Proof of Theorem 5.1. Recall the assumptions and note that $\mathcal{P}({}^J X) = \mathcal{P}({}^J X_n)$, since $J \subset \mathcal{J}(n,m)$. By Theorem 5.19 and Theorem 5.11,

$$\begin{aligned}\chi_{\mathrm{mon}}\big(\mathcal{P}({}^J X_n)\big) &\le 2^m \,\mathrm{gl}\left(\mathcal{P}({}^J X_n)\right)\\ &\le 2^m \|R\|\,\|S\|\,\mathrm{gl}\Big(\mathcal{L}\big(X_n;\mathcal{P}({}^J X_n)\big)\Big)\,.\end{aligned}$$

Theorem 5.22 then completes the proof. We obtain the upper bound

$$\le 2^m \|R\|\,\|S\|\,\lambda\big(\mathcal{P}({}^J X_n)\big)\,. \qquad \square$$

The preparations made to prove Theorem 5.1 also enable us to prove Theorem 5.2.

Proof of Theorem 5.2. Assume that there exists an unconditional basis of $\mathcal{P}_{\mathrm{mon}}({}^J X)$. We then have by Theorem 5.12 that $\mathrm{gl}\left(\mathcal{P}_{\mathrm{mon}}({}^J X)\right) \le \chi\big(\mathcal{P}_{\mathrm{mon}}({}^J X)\big)$.

Let $n \in \mathbb{N}$ and define $J(n) := J \cap \mathcal{J}(n,m)$. For $P \in \mathcal{P}_{\mathrm{mon}}({}^J X)$ then

$$P\big|_{X_n} = \sum_{\boldsymbol{j}\in J(n)} c_{\boldsymbol{j}}(P)\, z_{\boldsymbol{j}} \in \mathcal{P}_{\mathrm{mon}}({}^{J(n)} X)\,.$$

Define

$$\mathfrak{P} : \mathcal{P}_{\mathrm{mon}}({}^J X) \to \mathcal{P}_{\mathrm{mon}}({}^{J(n)} X), \quad P \mapsto P\big|_{X_n}\,.$$

We check at once that $\|\mathfrak{P}P\| = \|P\big|_{X_n}\| \le \|P\|$ and thus

$$\begin{aligned}\mathrm{gl}\left(\mathcal{P}({}^{J(n)} X)\right) &\le \big\|\mathrm{id} : \mathcal{P}({}^{J(n)} X) \hookrightarrow \mathcal{P}_{\mathrm{mon}}({}^J X)\big\|\,\big\|\mathfrak{P}\big\|\,\mathrm{gl}\left(\mathcal{P}_{\mathrm{mon}}({}^J X)\right)\\ &= \mathrm{gl}\left(\mathcal{P}_{\mathrm{mon}}({}^J X)\right)\,.\end{aligned}$$

By Theorem 5.19, we have for any $n \in \mathbb{N}$

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(n)} X)\big) \le 2^m\,\mathrm{gl}\left(\mathcal{P}({}^{J(n)} X)\right) \le 2^m\,\mathrm{gl}\left(\mathcal{P}_{\mathrm{mon}}({}^J X)\right) \le 2^m \chi\big(\mathcal{P}_{\mathrm{mon}}({}^J X)\big)\,.$$

This yields a uniform bound on the unconditional basis constants $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(n)} X)\big)$. Therefore, by Theorem 3.10,

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^J X)\big) \le 2^m \chi\big(\mathcal{P}_{\mathrm{mon}}({}^J X)\big)\,.$$

The other implication and inequality is trivial. $\square$

5.3. Symmetric reduction method

In this and the following section we present two applications of Theorem 5.1. We have seen estimates of the projection constant $\lambda\big(\mathcal{P}({}^{J^\dagger}X)\big)$ in terms of the cardinality of $J^\dagger$; therefore, we may choose $J^\dagger \subset \mathcal{J}(n, m-1)$ significantly smaller than J. For this reason we call the selection of suitable operators and index set $J^\dagger$ *reduction*. The idea of the symmetric reduction method, which we will explain in short, is originally due to Defant and Frerick [26].

The idea of this reduction is the following: For a polynomial $P \in \mathcal{P}({}^J X) \subset \mathcal{P}({}^m X)$ choose the unique symmetric multilinear form $L \in \mathcal{L}_s({}^m X; \mathbb{C})$ associated to P. By fixing one variable we obtain an $m-1$–linear form, which itself defines an $m-1$–homogeneous polynomial in $\mathcal{P}({}^{J^\dagger}X)$.

Let X be a Banach sequence space and $J \subset \mathcal{J}(n, m)$. For a polynomial $P \in \mathcal{P}({}^J X_n)$ let $L \in \mathcal{L}_s({}^m X_n; \mathbb{C})$ be its associated symmetric multilinear form. Then

$$\begin{aligned} L(\Delta_{m-1}x, \omega) &= \sum_{\boldsymbol{i} \in \mathcal{I}(n,m)} \frac{c_{\boldsymbol{i}^*}(P)}{\big|[\boldsymbol{i}^*]\big|} x_{i_1} \cdots x_{i_{m-1}} \omega_{i_m} \\ &= \sum_{\boldsymbol{i} \in \mathcal{I}(n,m-1)} \sum_{k=1}^{n} \frac{c_{(\boldsymbol{i},k)^*}(P)}{\big|[(\boldsymbol{i},k)^*]\big|} x_{\boldsymbol{i}^*} \omega_k \\ &= \sum_{\boldsymbol{j} \in \mathcal{J}(n,m-1)} \sum_{k=1}^{n} \frac{c_{(\boldsymbol{j},k)^*}(P)}{\big|[(\boldsymbol{j},k)^*]\big|} x_{\boldsymbol{j}} \omega_k \,, \end{aligned}$$

where $\boldsymbol{i}^*$ denotes the unique index $\boldsymbol{j} \in \mathcal{J}(n, m)$ such that $\boldsymbol{i} \sim \boldsymbol{j}$. $(\boldsymbol{i}, k)^*$ and $(\boldsymbol{j}, k)^*$ are to be understood accordingly. Now $c_{(\boldsymbol{j},k)^*}(P) \neq 0$ only if $(\boldsymbol{j}, k)^* \in J$. The remaining summands are

$$= \sum_{\boldsymbol{j} \in J^\dagger} \sum_{\substack{k=1 \\ (\boldsymbol{j},k)^* \in J}}^{n} \frac{c_{(\boldsymbol{j},k)^*}(P)}{\big|[(\boldsymbol{j},k)^*]\big|} x_{\boldsymbol{j}} \omega_k$$

where

$$J^\dagger := \big\{ \boldsymbol{j} \in \mathcal{J}(n, m-1) \,\big|\, \exists k \in \mathbb{N} : (\boldsymbol{j}, k)^* \in J \big\} \,.$$

Define furthermore

$$R : \mathcal{P}({}^J X_n) \to \mathcal{L}\big(X_n; \mathcal{P}({}^{J^\dagger} X_n)\big), \quad P \mapsto \sum_{k=1}^{n} \Bigg(\sum_{\substack{\boldsymbol{j} \in J^\dagger \\ (\boldsymbol{j},k)^* \in J}} \frac{c_{(\boldsymbol{j},k)^*}(P)}{\big|[(\boldsymbol{j},k)^*]\big|} z_{\boldsymbol{j}} \Bigg) \otimes e'_k$$

and $S := \mathfrak{P}\tilde{S}$ where

$$\tilde{S} : \mathcal{L}\big(X_n; \mathcal{P}({}^{J^\dagger} X_n)\big) \to \mathcal{P}({}^m X_n), \quad \sum_{k=1}^{n} \Bigg(\sum_{\boldsymbol{j} \in J^\dagger} c_{(\boldsymbol{j},k)} z_{\boldsymbol{j}} \Bigg) \otimes e'_k \mapsto \sum_{k=1}^{n} \sum_{\boldsymbol{j} \in J^\dagger} c_{(\boldsymbol{j},k)} z_{(\boldsymbol{j},k)^*}$$

and $\mathfrak{P} : \mathcal{P}({}^m X_n) \to \mathcal{P}({}^m X_n)$ denotes the projection onto $\mathcal{P}({}^J X_n)$.

Theorem 5.23. *Let X be a* Banach *sequence space and $J \subset \mathcal{J}(n,m)$. Define R, S and $J^\dagger$ as before. Then R and S are well-defined, $\mathrm{id}_{\mathcal{P}({}^J X)} = SR$, $\|R\| \leq \mathrm{e}$, and $\|S\| \leq \|\mathfrak{P}\|$.*

Proof. A straightforward verification shows that R and S are well-defined and that $\mathrm{id}_{\mathcal{P}({}^J X)} = SR$. To check the first norm estimate recall the construction of R and apply Harris' inequality (Proposition 2.11), we get

$$\|RP\| = \sup_{\omega \in \mathrm{B}_X} \sup_{x \in \mathrm{B}_X} |L(\Delta_{m-1} x, \omega)| \leq \mathrm{e}\, \|P\|$$

for every polynomial $P \in \mathcal{P}({}^J X)$ and $L \in \mathcal{L}_s({}^m X; \mathbb{C})$ denoting its associated symmetric multilinear form. The second norm estimate, the estimate of $\|\tilde{S}\|$, is trivial, since

$$\|P\| = \sup_{x \in \mathrm{B}_X} |Q(\Delta_m x)| \leq \sup_{\omega \in \mathrm{B}_X} \sup_{x \in \mathrm{B}_X} |Q(\Delta_{m-1} x, \omega)|$$

for any multilinear form $Q \in \mathcal{L}({}^m X; \mathbb{C})$ with $P = Q \circ \Delta_m$. $\square$

Using the symmetric reduction method we can now obtain easily Theorem 1.2 of [26]:

Theorem 5.24. *There exists a constant $c \geq 1$ such that for each $1 \leq p \leq \infty$ and all $n, m \in \mathbb{N}$*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^m \ell_p^n)\big) \leq c^m \binom{n+m-2}{m-1}^{1-\frac{1}{\min\{p,2\}}} \leq c^m \left(1 + \frac{n}{m-1}\right)^{(m-1)\left(1-\frac{1}{\min\{p,2\}}\right)}.$$

Proof. We are in the case $J = \mathcal{J}(n,m)$ and hence $\mathfrak{P} = \mathrm{id}_{\mathcal{P}({}^J \ell_p^n)}$. By our considerations above, Theorem 5.1, and the estimates of the projection constant,

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^m \ell_p^n)\big) \leq 2^m \mathrm{e} \lambda\big(\mathcal{P}({}^{J^\dagger} \ell_p^n)\big) \leq c_0^m |J^\dagger|^{1-\frac{1}{\min\{p,2\}}}$$

where $c_0 = 2\,\mathrm{e}^{\frac{1}{m}}$ in the case $p \geq 2$ and $c_0 = 2\,\mathrm{e}^{\frac{1}{m}+\frac{1}{p}}$ if $p < 2$. It remains to estimate the cardinality of $J^\dagger$. Indeed $J^\dagger = \mathcal{J}(n, m-1)$; therefore, by simple combinatorial analysis

$$|J^\dagger| = \binom{n+m-2}{m-1} \leq \mathrm{e}^m \big(1 + \tfrac{n}{m-1}\big)^{m-1}$$

where last inequality follows from Lemma 4.8. □

5.4. Asymmetric reduction method

The symmetric reduction method described in the previous section doesn't yield optimal results in every case. Therefore, we want to introduce another reduction method which is very similar, but in certain cases much more efficient.

Let again X be a Banach sequence space and $J \subset \mathcal{J}(n, m)$. For $P \in \mathcal{P}({}^J X_n)$ define $L \in \mathcal{L}({}^m X_n; \mathbb{C})$ by

$$L := \sum_{\boldsymbol{j} \in J} c_{\boldsymbol{j}}(P)\, e'_{\boldsymbol{j}}\,.$$

We check at once that $L \circ \Delta_m = P$, but L is in general not symmetric. We fix again the last variable and obtain

$$L(\Delta_{m-1}x, \omega) = \sum_{\boldsymbol{j} \in J} c_{\boldsymbol{j}}(P)\, x_{j_1} \cdots x_{j_{m-1}} \omega_{j_m} = \sum_{\boldsymbol{j} \in J^*} \sum_{\substack{k=1 \\ (\boldsymbol{j},k) \in J}}^{n} c_{(\boldsymbol{j},k)}(P)\, x_{\boldsymbol{j}} \omega_k$$

where

$$J^* := \big\{ \boldsymbol{j} \in \mathcal{J}(n, m-1) \,\big|\, \exists k \in \mathbb{N} : (\boldsymbol{j}, k) \in J \big\}\,.$$

Note that this definition coincides with the definition $(4 \cdot \mathrm{A})$ of J^* in the previous chapter. Moreover, define

$$R : \mathcal{P}({}^J X_n) \to \mathcal{L}\big(X_n; \mathcal{P}({}^{J^*} X_n)\big), \quad P \mapsto \sum_{k=1}^{n} \Bigg(\sum_{\substack{\boldsymbol{j} \in J^* \\ (\boldsymbol{j},k) \in J}} c_{(\boldsymbol{j},k)}(P)\, z_{\boldsymbol{j}} \Bigg) \otimes e'_k$$

and $S := \mathfrak{P}\tilde{S}$ with

$$\tilde{S} : \mathcal{L}\big(X_n; \mathcal{P}({}^{J^*} X_n)\big) \to \mathcal{P}({}^m X_n), \quad \sum_{k=1}^{n} \Bigg(\sum_{\boldsymbol{j} \in J^*} c_{(\boldsymbol{j},k)} z_{\boldsymbol{j}} \Bigg) \otimes e'_k \mapsto \sum_{k=1}^{n} \sum_{\boldsymbol{j} \in J^*} c_{(\boldsymbol{j},k)} z_{(\boldsymbol{j},k)^*}$$

and $\mathfrak{P}$ denoting the projection $\mathcal{P}(^m X_n) \to \mathcal{P}(^m X_n)$ onto $\mathcal{P}(^J X_n)$ as in the symmetric case.

By the same argument we used in the previous section R and S are well-defined with $\mathrm{id}_{\mathcal{P}(^J X)} = SR$ and we get the identical estimate for the norm of S. However, it is extremely difficult to obtain a reasonable estimate of $\|R\|$ as Harris' inequality is no longer applicable due to the fact that we defined R using a non-symmetric linear form. We have the following theorem.

Theorem 5.25. *Let X be a* Banach *sequence space and $J \subset \mathcal{J}(n,m)$. Define R, S and J^* as before. Then R and S are well-defined, $\mathrm{id}_{\mathcal{P}(^J X)} = SR$, and $\|S\| \leq \|\mathfrak{P}\|$.*

Moreover, there exists a constant $c \geq 1$ independent of n and m so that

$$\|R\| \leq \begin{cases} c^m & \text{if } X \text{ is } p\text{–concave for } 1 \leq p < 2 \text{ and} \\ c^m (\log n)^{m-1} & \text{otherwise.} \end{cases}$$

To give a proof of the theorem we have to introduce the theory of Schur multipliers. The proof is then given at the end of Section 5.4.4.

5.4.1. Schur multipliers

One may call Schur multiplication the dream of every student in his first semester; Schur multiplication is just the entrywise multiplication of matrices. Nevertheless, this leads to an interesting theoretical structure.

At first we give the usual definition of Schur multipliers and cite important results for our following investigations. Afterwards we generalize this definition to higher dimensions and find a way to use existing results. In the latter subsection we use this theory to estimate the operator norm of the inverse of a certain symmetrization operator.

For a recent account of the theory we refer the reader to Bennett [14] and Sukochev and Tomskova [56, 55].

Let in what follows $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ be two infinite matrices with complex entries. The Schur product of A and B is defined as $A * B := (a_{ij} b_{ij})_{i,j}$. For $k, l \in \mathbb{N}$ define furthermore $c_{kl} : \mathbb{C}^{\mathbb{N} \times \mathbb{N}} \to \mathbb{C}$ by $(a_{ij})_{i,j} \mapsto a_{kl}$.

In particular, we will be interested in matrices A which preserve certain matrix classes under SCHUR multiplication. Recall that every bounded linear operator $L : X \to Y$ between BANACH sequence spaces X and Y defines an infinite matrix via

$$\mathcal{L}(X;Y) \to \mathbb{C}^{\mathbb{N}\times\mathbb{N}}, \quad L \mapsto \big(\langle Le_j, e_i' \rangle\big)_{i,j} \tag{5·F}$$

and vice versa every infinite matrix $A = (a_{ij})_{i,j} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ defines a mapping rule via

$$(x_k)_k \mapsto \Big(\sum_j a_{ij} x_j\Big)_i .$$

The SCHUR product $A * L \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ where A is a matrix and L is a linear mapping is well-defined.

Let X and Y be BANACH sequence spaces. We call an infinite matrix $A \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ an (X,Y)–multiplier (SCHUR multiplier) if $A * L$ defines a bounded linear operator from X into Y whenever L is a bounded linear operator from X into Y.

The set of all (X,Y)–multipliers is denoted by $\mathcal{M}(X,Y)$. For $A \in \mathcal{M}(X,Y)$ we define the SCHUR norm of A by $\mu_{X,Y}(A) := \big\|m_A : \mathcal{L}(X;Y) \to \mathcal{L}(X;Y)\big\|$ where

$$m_A : \mathcal{L}(X;Y) \to \mathcal{L}(X;Y), \quad L \mapsto A * L .$$

From the definition we immediately get the following rules of computing:

Lemma 5.26. *Let X, Y be* BANACH *sequence spaces and $A, B \in \mathcal{M}(X;Y)$. Then*

$$\mu_{X,Y}(A * B) \le \mu_{X,Y}(A) \cdot \mu_{X,Y}(B)$$

and

$$\mu_{X,Y}(A + B) \le \mu_{X,Y}(A) + \mu_{X,Y}(B) .$$

A majority of the theory discusses the special case $X = \ell_p$ and $Y = \ell_q$. Such a multiplier is then called (p,q)–multiplier with the SCHUR norm denoted by $\mu_{p,q}$ and the set of all (p,q)–multipliers denoted by $\mathcal{M}(p,q)$.

The more general definition above was introduced by SUKOCHEV and TOMSKOVA [55]. Although SUKOCHEV and TOMSKOVA restrict themselves to symmetric BANACH sequence spaces, a careful analysis of the proofs reveals that this restriction is not needed for the results we use.

Proposition 5.27 (cf. Theorem 4.6 in [56]). *Let X and Y be* Banach *sequence spaces. Then $\big(\mathcal{M}(X,Y),\mu_{X,Y}\big)$ is a* Banach *space.*

It is possible to transfer many results about (p,q)–multipliers to the case of Banach sequence spaces. First results in [56] show:

Proposition 5.28 (cf. Theorem 4.14 in [56]). *Let X and Y be* Banach *sequence spaces. Then*

$$\mathcal{M}(\infty,1)\subset\mathcal{M}(X,Y)\subset\mathcal{M}(1,\infty).$$

We even have $\frac{1}{c}\,\mu_{1,\infty}(A)\le\mu_{X,Y}(A)\le c\,\mu_{\infty,1}(A)$ for $A\in\mathcal{M}(\infty,1)$ where $c>0$ is a constant only depending on X and Y:

Proposition 5.29. *Let X_0, Y_0, X, and Y be* Banach *sequence spaces such that*

$$\mathcal{M}(X_0,Y_0)\subset\mathcal{M}(X,Y)\,.$$

Then there exists $c>0$ such that $\mu_{X,Y}(A)\le c\,\mu_{X_0,Y_0}(A)$ for every $A\in\mathcal{M}(X_0,Y_0)$.

Proof. We check easily that $\mu_{X,Y}(A)\ge|a_{ij}|$ for every i,j. Therefore, convergence with respect to $\mu_{X,Y}$ implies entrywise convergence; a closed graph argument yields then that

$$\mathrm{id}:\big(\mathcal{M}(X_0,Y_0),\mu_{X_0,Y_0}\big)\to\big(\mathcal{M}(X,Y),\mu_{X,Y}\big)$$

is continuous. □

Together with the following proposition we are able to transfer known results about (p,q)–multipliers to a variety of Banach sequence spaces.

Proposition 5.30 (cf. Theorem 4.13 in [56]). *Let $1\le p,q\le\infty$. Suppose that X_0,Y_0,X,Y are* Banach *spaces with the* Fatou *property such that X_0 is p–convex, Y_0 is q–concave, X is p–concave, and Y is q–convex. Then $\mathcal{M}(X_0,Y_0)\subset\mathcal{M}(X,Y)$.*

Since ℓ_p is p–convex and p–concave, we obtain easily as a corollary:

Corollary 5.31. *Let $1\le p,q\le\infty$. Suppose that X,Y are* Banach *sequence spaces such that X is p–concave and Y is q–convex. Then $\mathcal{M}(p,q)\subset\mathcal{M}(X,Y)$.*

5.4.2. The main triangle projection and other examples of SCHUR multipliers

Three important SCHUR multipliers are given by the matrices $\mathbf{1}, \mathbf{1}_n$, and $\mathbf{1}^{kl}$ defined as

$$c_{ij}(\mathbf{1}) := 1\,,$$
$$c_{ij}(\mathbf{1}_n) := \begin{cases} 1 & \text{if } i \le n \text{ and } j \le n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_{ij}(\mathbf{1}^{kl}) := \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.32. *Let X and Y be* BANACH *sequence spaces. The three matrices defined above are (X,Y)–multipliers with $\mu_{X,Y}(\mathbf{1}) = \mu_{X,Y}(\mathbf{1}_n) = \mu_{X,Y}(\mathbf{1}^{kl}) = 1$.*

Proof. We show that $\|A * L\| \le \|L\|$ for every $L \in \mathcal{L}(X,Y)$ where $A = \mathbf{1}$, $A = \mathbf{1}_n$, respectively $A = \mathbf{1}^{kl}$. This is trivial for $A = \mathbf{1}$, since $\mathbf{1} * L = L$.

Let now $L \in \mathcal{L}(X,Y)$ and $A = \mathbf{1}_n$. Write $x = \sum_j x_j e_j$ for $x \in X$. An easy calculation gives

$$\begin{aligned} \|\mathbf{1}_n * L\| &= \sup_{x \in \mathrm{B}_X} \Big\| \Big(\sum_{j=1}^{\infty} c_{ij}(\mathbf{1}_n)\, \langle Le_j, e_i' \rangle\, x_j \Big)_{i=1}^{\infty} \Big\|_Y = \sup_{x \in \mathrm{B}_X} \Big\| \Big(\sum_{j=1}^{n} \langle Le_j, e_i' \rangle\, x_j \Big)_{i=1}^{n} \Big\|_Y \\ &= \sup_{x \in \mathrm{B}_X} \Big\| \Big(\Big\langle L\Big(\sum_{j=1}^{n} x_j e_j \Big), e_i' \Big\rangle \Big)_{i=1}^{n} \Big\|_Y \le \sup_{x \in \mathrm{B}_X} \|L\| \Big\| \sum_{j=1}^{n} x_j e_j \Big\|_X \le \|L\|\,. \end{aligned}$$

The case $A = \mathbf{1}^{kl}$ follows analogously. $\square$

Let us introduce two more sophisticated examples. Let D and T denote the matrices defined by

$$c_{ij}(D) := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_{ij}(T) := \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

The projection $L \mapsto T * L$ induced by T is called the main triangle projection and was studied by many (among others KWAPIEŃ and PEŁCZYŃSKI [43]; BENNETT [13, 14]; and SUKOCHEV and TOMSKOVA [56]). T is in general not an (X, Y)–multiplier whereas D defines an (X, Y)–multiplier for every choice of BANACH sequence spaces X and Y. We have the following results:

Proposition 5.33. *Let $1 \leq p, q \leq \infty$. Then D is a (p, q)–multiplier with $\mu_{p,q}(D) = 1$.*

A useful tool to check whether a given matrix defines an (p, q)–multiplier is given in Theorem 4.3 of [14]. We give the statement without proof in the following lemma and use this to prove our proposition.

Lemma 5.34 (cf. Theorem 4.3 in [14])**.** *A is a (p, q)–multiplier if and only if for each $\omega \in \ell_p$ the mapping $A \circ D_\omega : \ell_p' = \ell_{p'} \to \ell_\infty$ is q–summing. In this case*

$$\mu_{p,q}(A) = \sup_{\omega \in \mathrm{B}_{\ell_p}} \pi_q\big(A \circ D_\omega\big).$$

Proof of Proposition 5.33. We see at once that $D \circ D_\omega = D_\omega$. It remains to check whether $D_\omega : \ell_{p'} \to \ell_\infty$ is q–summing with $\pi_q(D_\omega) \leq 1$ for every $\omega \in \mathrm{B}_{\ell_p}$.

We use Lemma 5.6 and verify for $(x^{(k)})_{k=1}^n \subset \ell_{p'}$

$$\begin{aligned} \sup_{\omega \in \mathrm{B}_{\ell_p}} \big\| \big(D_\omega\, x^{(k)}\big)_k \big\|_q &= \sup_{\omega \in \mathrm{B}_{\ell_p}} \Big(\sum_{k=1}^n \big\| D_\omega\, x^{(k)} \big\|_{\ell_\infty}^q \Big)^{\frac{1}{q}} \\ &\leq \sup_{\omega \in \mathrm{B}_{\ell_p}} \Big(\sum_{k=1}^n \big| \big\langle \omega, x^{(k)} \big\rangle \big|^q \Big)^{\frac{1}{q}} = w_q\big((x^{(k)})_k\big). \end{aligned}$$ □

For the main triangle projection we have the following characterization by KWAPIEŃ and PEŁCZYŃSKI [43] and BENNETT [13]:

Proposition 5.35 (cf. Proposition 1.2 in [43], Theorem 5.1 in [13])**.** *Let $1 \leq p, q \leq \infty$.*

(i) *If* $1 \le p < q \le \infty$, *then is* T *a* (p,q)*–multiplier.*

(ii) *If* $p \neq 1$, $q \neq \infty$ *and* $q \le p$, *then is* T *not a* (p,q)*–multiplier. However,* $\mathbf{1}_n * T$ *is for any* $n \in \mathbb{N}$ *a* (p,q)*–multiplier with*

$$c^{-1} \log n \le \mu_{p,q}(\mathbf{1}_n * T) \le c \log n$$

where $c = c(p,q) > 1$ *is independent of* n.

Using Corollary 5.31 we obtain:

Corollary 5.36. *Let* $1 \le p, q \le \infty$ *and let* X *and* Y *be* BANACH *sequence spaces such that* X *is* p*–concave and* Y *is* q*–convex. Then*

(i) *If* $1 \le p < q \le \infty$, *then* T *is a* (X,Y)*–multiplier.*

(ii) *If* $p \neq 1$, $q \neq \infty$ *and* $q \le p$, *then* $\mathbf{1}_n * T$ *is for any* $n \in \mathbb{N}$ *a* (X,Y)*–multiplier with*

$$\mu_{X,Y}(\mathbf{1}_n * T) \le c \log n$$

where $c = c(X,Y) > 1$ *is independent of* n.

5.4.3. SCHUR multipliers acting on multilinear forms

Any bounded bilinear form $L : X \times Y \to \mathbb{C}$ might be seen as a bounded operator $X \to Y'$ via the identification

$$\mathcal{L}(X,Y;\mathbb{C}) \to \mathcal{L}(X;Y'), \quad L \mapsto \left[x \mapsto L(x,\,\cdot\,)\right]. \tag{5·G}$$

It is easy to see that this is in fact an isometry. Therefore, we may think of SCHUR multipliers acting on bilinear forms. This indicates that the following generalization is somewhat natural.

Let $m \in \mathbb{N}$. We call a family $A = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)} \in \mathbb{C}^{\mathcal{I}(\infty,m)}$ of complex numbers an m–dimensional cube or m–cube. Clearly, 2–cubes coincide with infinite matrices. For m–cubes $A = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)}$ and $B = (b_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)}$ we define the SCHUR product as in the two dimensional case, that is $A * B := (a_{\boldsymbol{i}} b_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)}$. Furthermore define for $\boldsymbol{j} \in \mathcal{I}(\infty,m)$ the mapping $c_{\boldsymbol{j}} : \mathbb{C}^{\mathcal{I}(\infty,m)} \to \mathbb{C}$ by $(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)} \mapsto a_{\boldsymbol{j}}$.

There is a natural correspondence between m–linear forms $L \in \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C})$ on BANACH sequence spaces $X^{(1)}, \dots, X^{(m)}$ and m–dimensional cubes via

$$\mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C}) \to \mathbb{C}^{\mathcal{I}(\infty,m)}, \quad L \mapsto \big(L(e_{i_1}, \dots, e_{i_m})\big)_{\boldsymbol{i} \in \mathcal{I}(\infty,m)}$$

and vice versa every m–cube $(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{I}(\infty,m)}$ defines a mapping rule

$$\big(x^{(1)}, \dots, x^{(m)}\big) \mapsto \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} a_{\boldsymbol{i}} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m} \,.$$

The expression $A * L$ for A an m–cube and $L \in \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C})$ is hence well-defined. We say that an m–dimensional cube A is an $(X^{(1)} \times \cdots \times X^{(m)})$–multiplier or SCHUR multiplier if $A * L \in \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C})$ whenever $L \in \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C})$. In this case, by a closed graph argument,

$$m_A : \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C}) \to \mathcal{L}(X^{(1)}, \dots, X^{(m)}; \mathbb{C}), \quad L \mapsto A * L$$

defines a bounded operator. The SCHUR norm of A is then defined as the operator norm of m_A and denoted by $\mu_{X^{(1)} \times \cdots \times X^{(m)}}(A)$. In the case $X^{(1)} = \cdots = X^{(m)} = X$ we write $\mu_{X^m}(A)$ for short and call A an X^m–multiplier.

With the identification $(5 \cdot \mathrm{G})$ it is clear that this definition extends the usual one. The only difference is that $(X \times Y)$–multipliers in this setting coincide with (X, Y')–multipliers as defined in the previous subsection.

Again, we immediately get the following rules of computing:

Lemma 5.37. *Let $X^{(1)}, \dots, X^{(m)}$ be* BANACH *sequence spaces and let $A, B \in \mathbb{C}^{\mathcal{I}(\infty,m)}$ be $(X^{(1)} \times \cdots \times X^{(m)})$–multipliers. Then*

$$\mu_{X^{(1)} \times \cdots \times X^{(m)}}(A * B) \leq \mu_{X^{(1)} \times \cdots \times X^{(m)}}(A) \cdot \mu_{X^{(1)} \times \cdots \times X^{(m)}}(B)$$

and

$$\mu_{X^{(1)} \times \cdots \times X^{(m)}}(A + B) \leq \mu_{X^{(1)} \times \cdots \times X^{(m)}}(A) + \mu_{X^{(1)} \times \cdots \times X^{(m)}}(B) \,.$$

Analogous to the definition in the preceding section we define $\mathbf{1}, \mathbf{1}_n$, and $\mathbf{1}^{\boldsymbol{j}}$ by

$$c_{\boldsymbol{i}}(\mathbf{1}) := 1 \,,$$

$$c_{\boldsymbol{i}}(\mathbf{1}_n) := \begin{cases} 1 & \text{if } i_k \leq n \text{ for all } k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_{\boldsymbol{i}}(\mathbf{1}^{\boldsymbol{j}}) := \begin{cases} 1 & \text{if } \boldsymbol{i} = \boldsymbol{j}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.38. *Let $X^{(1)}, \ldots, X^{(m)}$ be* Banach *sequence spaces. Then these three m–dimensional cubes are $(X^{(1)} \times \cdots \times X^{(m)})$–multipliers with*

$$\mu_{X^{(1)} \times \cdots \times X^{(m)}}(\mathbf{1}) = \mu_{X^{(1)} \times \cdots \times X^{(m)}}(\mathbf{1}_n) = \mu_{X^{(1)} \times \cdots \times X^{(m)}}(\mathbf{1}^{\boldsymbol{j}}) = 1.$$

Proof. The proof follows analogously to that of Proposition 5.32. We have for instance

$$\begin{aligned} \|\mathbf{1}^{\boldsymbol{j}} * L\| &= \sup_{\substack{x^{(k)} \in \mathrm{B}_{X^{(k)}} \\ k=1,\ldots,m}} |c_{\boldsymbol{j}}(L)\, x^{(1)}_{j_1} \cdots x^{(m)}_{j_m}| \\ &= \sup_{\substack{x^{(k)} \in \mathrm{B}_{X^{(k)}} \\ k=1,\ldots,m}} |L(x^{(1)}_{j_1} e_{j_1}, \ldots, x^{(m)}_{j_m} e_{j_m})| \leq \|L\|\,. \end{aligned}$$

□

To get further examples we can "lift" known results from the two dimensional to the m–dimensional case.

Proposition 5.39. *Let $X^{(1)}, \ldots, X^{(m)}$ be* Banach *sequence spaces and $\tilde{A} = (\tilde{a}_{ij})_{i,j}$ be an $(X^{(1)} \times X^{(2)})$–multiplier. Define $A = (a_{\boldsymbol{i}})_{\boldsymbol{i}} \in \mathbb{C}^{\mathcal{I}(\infty,m)}$ by $a_{\boldsymbol{i}} := \tilde{a}_{i_1 i_2}$. Then A is a $(X^{(1)} \times \cdots \times X^{(m)})$–multiplier with $\mu_{X^{(1)} \times \cdots \times X^{(m)}}(A) \leq \mu_{X^{(1)} \times X^{(2)}}(\tilde{A})$.*

Proof. Let $L \in \mathcal{L}(X^{(1)}, \ldots, X^{(m)}; \mathbb{C})$ and for $\xi = (\xi^{(3)}, \ldots, \xi^{(m)}) \in \mathrm{B}_{X^{(3)}} \times \cdots \times \mathrm{B}_{X^{(m)}}$ define

$$\phi_\xi : X^{(1)} \times X^{(2)} \to X^{(1)} \times \cdots X^{(m)}, \quad (x, y) \mapsto (x, y, \xi^{(3)}, \ldots, \xi^{(m)})\,.$$

For the bilinear form $\tilde{L}_\xi = L \circ \phi_\xi$ we have now $(L * A) \circ \phi_\xi = \tilde{L}_\xi * \tilde{A}$. We obtain

$$\begin{aligned} \|L * A\| &= \sup_{\xi^{(1)}, \ldots, \xi^{(m)}} |L * A(\xi_1, \ldots, \xi_m)| = \sup_{\xi^{(3)}, \ldots, \xi^{(m)}} \|\tilde{L}_\xi * \tilde{A}\| \\ &\leq \sup_{\xi^{(3)}, \ldots, \xi^{(m)}} \mu_{X^{(1)} \times X^{(2)}}(\tilde{A})\, \|\tilde{L}_\xi\| = \mu_{X^{(1)} \times X^{(2)}}(\tilde{A})\, \|L\| \end{aligned}$$

where the supremum is taken over $\xi^{(k)} \in \mathrm{B}_{X^{(k)}}$, $k = 1, \ldots, m$ respectively. □

We may now lift the SCHUR multipliers D and T defined in the previous section. Let D^k and T^k define the m–dimensional cubes defined by

$$c_{\boldsymbol{i}}(D^k) := \begin{cases} 1 & \text{if } i_k = i_m \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_{\boldsymbol{i}}(T^k) := \begin{cases} 1 & \text{if } i_k \leq i_m \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5.40. *Let X be a* BANACH *sequence space and $1 \leq k \leq m$. Then D^k is an X^m–multiplier. In the case that $X = \ell_p$ with $1 \leq p \leq \infty$ we additionally have that $\mu_{(\ell_p)^m}(D^k) = 1$.*

Corollary 5.41. *Let X be a* BANACH *sequence space and $1 \leq k \leq m$. Assume that X is p–concave for some $1 \leq p \leq \infty$.*

(i) If $1 \leq p < 2$, then T^k is a X^m–multiplier.

(ii) If $p \geq 2$, then there exists a constant $c = c(X) > 1$ such that for any $n \in \mathbb{N}$

$$\mu_{p,q}(\mathbf{1}_n * T) \leq c \log n\,.$$

Both corollaries are an immediate consequence of Proposition 5.39. Regarding the second corollary it is vital to note that every BANACH sequence space is ∞–concave.

5.4.4. Symmetrization of partly symmetric multilinear mappings

Consider an asymmetric m–linear form L. In general we do not have estimates of the form $\|L\| \leq c\,\|\mathcal{S}L\|$ with an universal constant $c > 0$. There even exist multilinear forms $L \neq 0$ such that $\mathcal{S}L = 0$ (see Section 2.2.1). However, if we pose some restrictions on our multilinear form L we can obtain such estimates.

Theorem 5.42. *Let X a* BANACH *sequence space and $L \in \mathcal{L}(^m X; \mathbb{C})$ such that the following two conditions hold true:*

(i) For the coefficients of L

$$c_{\boldsymbol{i}}(L) = L(e_{i_1}, \dots, e_{i_m}) \neq 0$$

only if $\boldsymbol{i} \in \mathcal{I}(n,m)$ with $i_m \geq i_k$ for all k. In particular, L only lives on $\big(X_n\big)^m$.

(ii) *L is symmetric in the first $m-1$ variables; that means $c_{\boldsymbol{i}}(L) = c_{\sigma(\boldsymbol{i})}(L)$ for all permutations $\sigma \in \Sigma_m$ with $\sigma(m) = m$.*

Then there exists a universal constant $c = c(X)$ independent of n and m such that

$$\|L\| \leq \left(\prod_{k=1}^{m-1} \mu_{X^m}\left(\mathbf{1}_n * T^k\right) \right) m \left(1 + 2c\right)^m \|\mathcal{S}L\| .$$

With Corollary 5.41 we obtain:

Corollary 5.43. *Let X be an p–concave* Banach *sequence space for some $1 \leq p < 2$. There exists a constant $c > 0$ such that for every m–linear form $L : X^m \to \mathbb{C}$ with the properties as in Theorem 5.42*

$$\|L\| \leq c^m \|\mathcal{S}L\| .$$

Proof of Theorem 5.42. Let us take a look at the coefficients of the symmetrization $\mathcal{S}L$ of L. We have for $\boldsymbol{i} \in \mathcal{I}(n,m)$

$$\begin{aligned} c_{\boldsymbol{i}}(\mathcal{S}L) &= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} c_{\sigma(\boldsymbol{i})}(L) \\ &= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{(m-1)!} \sum_{\substack{\sigma \in \Sigma_m \\ \sigma(m)=k}} c_{\sigma(\boldsymbol{i})}(L) . \end{aligned}$$

Using the second assumption, i.e. that L is symmetric in its first $m-1$ variables, and that we have $(m-1)!$ summands in the latter sum this is

$$= \frac{1}{m} \sum_{k=1}^{m} c_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m,i_k)}(L) .$$

We proceed to write L as the Schur product of an m–cube $\mathfrak{A} = (\mathfrak{a}_{\boldsymbol{i}})_{\boldsymbol{i}}$ and $\mathcal{S}L$. We distinguish two cases.

For $\boldsymbol{i}$ such that $i_m < i_k$ for some k we have $c_{\boldsymbol{i}}(L) = 0$ by the first assumption. Therefore, we set in this case $\mathfrak{a}_{\boldsymbol{i}} = 0$ and have $c_{\boldsymbol{i}}(L) = \mathfrak{a}_{\boldsymbol{i}}\, c_{\boldsymbol{i}}(\mathcal{S}L)$.

Otherwise, if $i_m \geq i_k$ for all k we have by our considerations above

$$c_{\boldsymbol{i}}(\mathcal{S}L) = \frac{1}{m} \sum_{k=1}^{m} c_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m,i_k)}(L)$$

$$= \frac{1}{m} \sum_{\substack{k=1 \\ i_k=i_m}}^{m} c_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m,i_k)}(L)\,,$$

since our first assumption assures $c_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m,i_k)}(L) = 0$ if $i_k < i_m$. Using that $i_k = i_m$ for every summand and that L is symmetric in its first $m-1$ variables this evaluates to

$$= \frac{1}{m} \sum_{\substack{k=1 \\ i_k=i_m}}^{m} c_{\boldsymbol{i}}(L)$$

$$= \frac{|\{k \,|\, i_k = i_m\}|}{m}\, c_{\boldsymbol{i}}(L)\,.$$

Hence, we set $\mathfrak{a}_{\boldsymbol{i}} := \frac{m}{|\{k \,|\, i_k=i_m\}|}$ in this case.

With $\mathfrak{A} = (\mathfrak{a}_{\boldsymbol{i}})_{\boldsymbol{i}}$ defined by

$$\mathfrak{a}_{\boldsymbol{i}} := \begin{cases} \frac{m}{|\{k \,|\, i_k=i_m\}|} & \text{if } \boldsymbol{i} \in \mathcal{I}(n,m) \text{ with } i_k \leq i_m \text{ for every } k \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

we have $\mathfrak{A} * \mathcal{S}L = L$.

To estimate the Schur norm of $\mathfrak{A}$ we will now write $\mathfrak{A}$ as a composition of simpler m–cubes and extensively use Lemma 5.37. ■

We interrupt the proof to outhouse the decomposition of the matrix $\mathfrak{A}$ into a separate lemma:

Lemma 5.44. *Let $\mathfrak{A} = (\mathfrak{a}_{\boldsymbol{i}})_{\boldsymbol{i}}$ be the m–cube defined by*

$$\mathfrak{a}_{\boldsymbol{i}} := \begin{cases} \frac{m}{|\{k \,|\, i_k=i_m\}|} & \textit{if } \boldsymbol{i} \in \mathcal{I}(n,m) \textit{ with } i_k \leq i_m \textit{ for every } k \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\mathfrak{A} = \Big(\mathop{*}_{k=1}^{m-1} \big(\mathbf{1}_n * T^k\big)\Big) * \Big(\sum_{l=1}^{m} \frac{m}{l} \cdot A^l\Big) \tag{5 · H}$$

with

$$A^l := \sum_{\substack{Q \subset \{1,\ldots,m\} \\ |Q|=l}} \Big(\mathop{*}_{q\in Q} D^q\Big) * \Big(\mathop{*}_{q\in Q^c} \big(\mathbf{1} - D^q\big)\Big)$$

where Q^c denotes the complement of Q, i.e. $Q^c := \{1,\ldots,m\} \setminus Q$.

Proof. Let us denote the m–dimensional cube on the right-hand side of $(5 \cdot \mathrm{H})$ by A. Fix $\boldsymbol{i} \in \mathcal{I}(\infty, m)$. We distinguish three cases.

In case that $\boldsymbol{i} \notin \mathcal{I}(n,m)$ we have $c_{\boldsymbol{i}}(\mathbf{1}_n) = 0$. If $\boldsymbol{i} \in \mathcal{I}(n,m)$ with $i_k > i_m$ for some k, then $c_{\boldsymbol{i}}(T^k) = 0$. In both cases $c_{\boldsymbol{i}}(A) = 0 = c_{\boldsymbol{i}}(\mathfrak{A})$.

It remains to show equality in the third case: $\boldsymbol{i} \in \mathcal{I}(n,m)$ with $i_k \le i_m$ for all k. In this case $c_{\boldsymbol{i}}(T^k) = 1$ for every k and thus

$$c_{\boldsymbol{i}}\Big(\mathop{*}_{k=1}^{m-1}\big(\mathbf{1}_n * T^k\big)\Big) = 1\,.$$

Let $\tilde{Q} = \{k \,|\, i_k = i_m\}$ and $\tilde{Q}^c$ its complement with respect to $\{1,\dots,m\}$. We claim that $c_{\boldsymbol{i}}(A^l) = 1$ only if $l = |\tilde{Q}|$ and $c_{\boldsymbol{i}}(A^l) = 0$ otherwise. By definition of D^q we have $c_{\boldsymbol{i}}(D^q) = 1$ only if $q \in \tilde{Q}$ and $c_{\boldsymbol{i}}(D^q) = 0$ otherwise. Therefore, $c_{\boldsymbol{i}}(\mathbf{1} - D^q) = 1$ only if $q \in \tilde{Q}^c$ and $c_{\boldsymbol{i}}(\mathbf{1} - D^q) = 0$ otherwise. Together this shows

$$c_{\boldsymbol{i}}\Big(\Big(\mathop{*}_{q\in Q} D^q\Big) * \Big(\mathop{*}_{q\in Q^c}\big(\mathbf{1} - D^q\big)\Big)\Big) = \begin{cases} 1 & \text{if } Q = \tilde{Q} \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_{\boldsymbol{i}}(A^l) = \begin{cases} 1 & \text{if } l = |\tilde{Q}| \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Altogether we obtain $c_{\boldsymbol{i}}(A) = \frac{m}{|\{k \,|\, i_k = i_m\}|}$ and hence $\mathfrak{A} = A$. □

Having this decomposition of the m–cube $\mathfrak{A}$ in mind we can proceed with the proof of Theorem 5.42.

Proof of Theorem 5.42 (continuation). It remains to estimate the Schur norm of $\mathfrak{A}$. Thereby we prove the claim, since $\|L\| = \|\mathfrak{A} * \mathcal{S}L\| \le \mu_{X^m}(\mathfrak{A})\,\|\mathcal{S}L\|$.

From Lemma 5.37 we obtain for the Schur norm $\mu_{X^m}(A^l)$

$$\begin{aligned} \mu_{X^m}(A^l) &= \mu_{X^m}\Bigg(\sum_{\substack{Q\subset\{1,\dots,m\}\\ |Q|=l}} \Big(\mathop{*}_{q\in Q} D^q\Big) * \Big(\mathop{*}_{q\in Q^c}\big(\mathbf{1} - D^q\big)\Big)\Bigg) \\ &\le \sum_{\substack{Q\subset\{1,\dots,m\}\\ |Q|=l}} \mu_{X^m}\Big(\Big(\mathop{*}_{q\in Q} D^q\Big)\Big)\,\mu_{X^m}\Big(\Big(\mathop{*}_{q\in Q^c}\big(\mathbf{1} - D^q\big)\Big)\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\substack{Q\subset\{1,\dots,m\}\\|Q|=l}} \prod_{q\in Q} \mu_{X^m}(D^q) \prod_{q\in Q^c} \big(1+\mu_{X^m}(D^q)\big) \\
&\leq \binom{m}{k} c^l (1+c)^{m-l}
\end{aligned}
$$

with a universal constant $c = c(X) > 1$. Hence, for $\mathfrak{A}$

$$
\begin{aligned}
\mu_{X^m}(\mathfrak{A}) &= \mu_{X^m}\Bigg(\Big(\mathop{*}_{k=1}^{m} \big(\mathbf{1}_n * T^k\big)\Big) * \Big(\sum_{k=1}^{m} \frac{m}{k}\cdot A^k\Big)\Bigg) \\
&\leq \Bigg(\prod_{k=1}^{m-1} \mu_{X^m}\big(\mathbf{1}_n * T^k\big)\Bigg)\Bigg(\sum_{k=1}^{m} \frac{m}{k}\mu_X\big(A^k\big)\Bigg) \\
&\leq \Bigg(\prod_{k=1}^{m-1} \mu_{X^m}\big(\mathbf{1}_n * T^k\big)\Bigg)\Bigg(\sum_{k=1}^{m} \frac{m}{k}\binom{m}{k} c^k (1+c)^{m-k}\Bigg) \\
&\leq \Bigg(\prod_{k=1}^{m-1} \mu_{X^m}\big(\mathbf{1}_n * T^k\big)\Bigg) m\,\big(1+2c\big)^m . \qquad \square
\end{aligned}
$$

We are finally able to give the proof of our theorem:

Proof of Theorem 5.25 (estimate of the norm of R). Recall the definition of R. We have for $P \in \mathcal{P}(^J X)$

$$
RP = \sum_{k=1}^{n}\Bigg(\sum_{\substack{\boldsymbol{j}\in J^*\\(\boldsymbol{j},k)\in J}} c_{(\boldsymbol{j},k)}(P)\, z_{\boldsymbol{j}}\Bigg)\otimes e'_k = \sum_{k=1}^{n}\Bigg(\sum_{\substack{\boldsymbol{j}\in J^*\\(\boldsymbol{j},k)\in J}} \sum_{\boldsymbol{i}\in[\boldsymbol{j}]} \frac{c_{(\boldsymbol{j},k)}(P)}{|[\boldsymbol{j}]|}\, z_{\boldsymbol{i}}\Bigg)\otimes e'_k .
$$

Set

$$
Q = \sum_{k=1}^{n} \sum_{\substack{\boldsymbol{j}\in J^*\\(\boldsymbol{j},k)\in J}} \sum_{\boldsymbol{i}\in[\boldsymbol{j}]} \frac{c_{(\boldsymbol{j},k)}(P)}{|[\boldsymbol{j}]|}\, e'_{(\boldsymbol{i},k)} \in \mathcal{L}(^m X_n; \mathbb{C}) .
$$

We check at once that Q is a multilinear form such that

(i) $Q \circ \Delta_m = P$ and with this $\|\mathcal{S}Q\| \leq \mathrm{e}^m \|P\|$;

(ii) $RP\omega(x) = Q(\Delta_{m-1}x, \omega)$ for every $x, \omega \in \mathrm{B}_X$ and thus $\|RP\| \leq \|Q\|$; and

(iii) Q fulfills the premises of Theorem 5.42.

Corollary 5.43 proves immediately the claim if X is p–concave for some $1 \leq p < 2$. In the remaining cases Corollary 5.41, (ii) together with Theorem 5.42 does the trick. $\square$

5.5. Comparison of the elementary approach and the abstract viewpoint

To conclude this chapter we want to compare a few aspects of this chapter and the previous one. The previous chapter essentially consists of Theorem 4.1. This theorem gives an estimate of the unconditional basis constant $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big)$ for any index set $\Lambda \subset \mathbf{\Lambda}(n,m)$.

The results of this chapter on the other hand establish a mesh of results relating the unconditional basis constant with the GORDON-LEWIS constant and the projection constant. Theorems such as Theorem 5.2 are not provable with the techniques of the previous chapter. We are able to relate several BANACH space invariants; among other things we obtain an lower bound of the projection constant.

In the next subsection we want to discuss the consequences for the applicability in the next chapter. Afterwards we present an application of this abstract viewpoint which is not possible to obtain using the elementary approach of the previous chapter.

5.5.1. Practical consequences

In the case of a p–concave BANACH sequence space (with $1 \leq p < 2$) we obtain from Theorem 5.1 and 5.25 the following:

Theorem 5.45. *Let $1 \leq p < 2$ and X be a p–concave* BANACH *sequence space. There exists a constant $c \geq 1$ such that for any $n, m \in \mathbb{N}$ and $\Lambda \subset \mathbf{\Lambda}(n,m)$*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{\Lambda}X)\big) \leq c^m \lambda\big(\mathcal{P}({}^{\Lambda^*}X)\big)\, \|\mathfrak{P}_\Lambda\|$$

where $\mathfrak{P}_\Lambda$ denotes the projection $\mathcal{P}({}^m X_n) \to \mathcal{P}({}^m X_n)$ onto $\mathcal{P}({}^{\Lambda}X)$.

In the next chapter we present an application of the results obtained in this and the previous chapter. There we have to estimate the unconditional basis constant for certain index sets: Let $2 < y < x$ and $m \in \mathbb{N}$. Choose $n := \pi(x)$ and define

$$\begin{aligned} \Lambda^+(x,y;m) &:= \big\{\alpha \in \mathbf{\Lambda}(n,m) \,\big|\, p^\alpha = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}\cdots \leq x \text{ and } \alpha_k = 0 \text{ for all } k \leq \pi(y)\big\} \\ &= \mathbf{\Lambda}(n,m) \cap \underbrace{\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid \alpha_k = 0 \text{ for all } k \leq \pi(y)\}}_{=:\,\Lambda^+(y)} \cap \underbrace{\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid p^\alpha \leq x\}}_{=:\,\Lambda(x)} \end{aligned}$$

where p denotes the sequence of primes and π is the prime-counting function. We are now interested in an upper bound of

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^{\Lambda^+(x,y;m)}X)\big)$$

especially for $X = \ell_p$ with $1 \le p < 2$. To make use of Theorem 5.45 we have to estimate the norm of the occurring projection and the projection constant of the space $\mathcal{P}(^{\Lambda^+(x,y;m)^*}X)$. We begin by estimating the norm of the projection.

Let Λ be either $\Lambda^+(x,y;m)$, $\Lambda^+(y)$, or $\Lambda(x)$ and define

$$\mathfrak{P}_\Lambda : \mathcal{P}(^m X_n) \to \mathcal{P}(^m X_n), \quad P \mapsto \sum_{\alpha\in\Lambda} c_\alpha(P)\, z^\alpha\,.$$

We have then $\mathfrak{P}_{\Lambda^+(x,y;m)} = \mathfrak{P}_{\Lambda^+(y)} \circ \mathfrak{P}_{\Lambda(x)}$. By a straightforward calculation we obtain the following lemma:

Lemma 5.46. *Let X be a* Banach *sequence space, $n, m \in \mathbb{N}$, and $y > 2$. Then*

$$\big\|\mathfrak{P}_{\Lambda^+(y)} : \mathcal{P}(^m X_n) \to \mathcal{P}(^m X_n)\big\| \le 1\,.$$

Proof. Let $P \in \mathcal{P}(^m X_n)$. Then

$$\Big\|\sum_{\alpha\in\Lambda^+(y)} c_\alpha(P)\, z^\alpha\Big\|_{\mathrm{B}_{X_n}} = \sup\big\{|P(x)| \,\big|\, x \in \mathrm{B}_{X_n} \text{ with } x_k = 0 \text{ for } k \le \pi(y)\big\} \le \|P\|_{\mathrm{B}_{X_n}}\,.$$

□

Using the following result of Balasubramanian, Calado, and Queffélec [5] we are able to determine the norm of $\mathfrak{P}_{\Lambda(x)}$.

Lemma 5.47 (cf. Lemma 1.1 in [5])**.** *There exits a constant $c \ge 1$ such that for any $x > 2$ and any* Dirichlet *series $D = \sum_n a_n n^{-s} \in \mathcal{H}_\infty$ which converges on the half-plane* $[\mathrm{Re} > 0]$ *to a bounded holomorphic function f*

$$\Big\|\sum_{n\le x} a_n n^{-s}\Big\|_{\mathcal{H}_\infty} \le c \log x\, \|f\|_\infty\,.$$

With Kronecker's theorem we are able to transfer this result to polynomials. Let p denote the sequence of primes. We check at once that $1, \log p_1, \log p_2, \ldots, \log p_n$ are $\mathbb{Q}$–linearly independent reals so that we can use Theorem 7.9 of [3] (Kronecker's theorem) in the following fashion:

Lemma 5.48 (cf. Theorem 7.9 in [3]). *Let $\mathfrak{p}$ denote the sequence of primes and let $n \in \mathbb{N}$. Then*

$$\left\{(\mathfrak{p}_1^{-it}, \ldots, \mathfrak{p}_n^{-it}) \,\middle|\, t \in \mathbb{R}\right\} = \left\{(\mathrm{e}^{-it\log \mathfrak{p}_1}, \ldots, \mathrm{e}^{-it\log \mathfrak{p}_n}) \,\middle|\, t \in \mathbb{R}\right\}$$

is a dense subset of $\mathbb{T}^n$.

Corollary 5.49. *There exists a constant $c \geq 1$ such that for $n, m \in \mathbb{N}$, $x > 2$, and $P \in \mathcal{P}(^m\ell_\infty^n)$*

$$\Big\| \sum_{\substack{\alpha \in \mathbf{\Lambda}(n,m) \\ \mathfrak{p}^\alpha \leq x}} c_\alpha(P)\, z^\alpha \Big\|_{\mathrm{B}_{\ell_\infty^n}} \leq c \log x \, \|P\|_{\mathrm{B}_{\ell_\infty^n}} .$$

In other words,

$$\big\| \mathfrak{P}_{\Lambda(x)} : \mathcal{P}(^m\ell_\infty^n) \to \mathcal{P}(^m\ell_\infty^n) \big\| \leq c \log x \, .$$

Proof. From the maximum modulus principle, we see that any polynomial on ℓ_∞^n attains its maximum on the combined boundary $\mathbb{T}^n$ of $\mathrm{B}_{\ell_\infty^n}$. Hence, by Lemma 5.48,

$$\begin{aligned}
\Big\| \sum_{\substack{\alpha \in \mathbf{\Lambda}(n,m) \\ \mathfrak{p}^\alpha \leq x}} c_\alpha(P)\, z^\alpha \Big\|_{\mathrm{B}_{\ell_\infty^n}} &= \sup_{x \in \mathbb{T}^n} \Big| \sum_{\substack{\alpha \in \mathbf{\Lambda}(n,m) \\ \mathfrak{p}^\alpha \leq x}} c_\alpha(P)\, x^\alpha \Big| \\
&= \sup_{t \in \mathbb{R}} \Big| \sum_{\substack{\alpha \in \mathbf{\Lambda}(n,m) \\ \mathfrak{p}^\alpha \leq x}} c_\alpha(P)\, (\mathfrak{p}_1^{-it}, \ldots, \mathfrak{p}_n^{-it})^\alpha \Big| = \Big\| \sum_{\substack{\alpha \in \mathbf{\Lambda}(n,m) \\ \mathfrak{p}^\alpha \leq x}} c_\alpha(P)\, (\mathfrak{p}^\alpha)^{-s} \Big\|_{\mathcal{H}_\infty}
\end{aligned}$$

for any $P \in \mathcal{P}(^m\ell_\infty^n)$. We leave it to the reader to apply Lemma 5.47 and to do the same calculation in the opposite direction to complete the proof. □

Using the trick presented in Section 3.3 we transfer this result to the case of any Banach sequence space. We check at once that $\mathfrak{P}_{\Lambda(x)}$ is of the required form and obtain:

Proposition 5.50. *Let X be a Banach sequence space. There exits a constant $c \geq 1$ such that for any $n, m \in \mathbb{N}$ and $x > 2$*

$$\big\| \mathfrak{P}_{\Lambda(x)} : \mathcal{P}(^m X_n) \to \mathcal{P}(^m X_n) \big\| \leq c \log x \, .$$

Lemma 5.46 together with Proposition 5.50 now provide

$$\big\| \mathfrak{P}_{\Lambda^+(x,y;m)} : \mathcal{P}(^m X_n) \to \mathcal{P}(^m X_n) \big\| \leq c \log x \, .$$

To apply Theorem 5.45 it remains to estimate the projection constant. For the Banach sequence space $X = \ell_{p,q}$ where $1 \leq p < 2$ and $1 \leq q \leq \infty$ we have from Corollary 5.18

$$\lambda\big(\mathcal{P}(^{\Lambda^+(x,y;m)}\ell_{p,q})\big) \leq \mathrm{e}^{\frac{m}{p}} \, |\Lambda^+(x,y;m)^*|^{1-\frac{1}{p}} \, .$$

Let us collect our results so far:

Theorem 5.51. *Let $X = \ell_{p,q}$ for $1 \leq p < 2$ and $1 \leq q \leq \infty$. Then there exists a constant $c \geq 1$ such that for any $x > y > 2$ and any $m \in \mathbb{N}$*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^{\Lambda^+(x,y;m)^*}X)\big) \leq c^m \log x \, |\Lambda^+(x,y;m)^*|^{1-\frac{1}{p}} \, .$$

In comparison to Theorem 4.1 this result has some drawbacks: for $X = \ell_p$ with $p \geq 2$ it doesn't yield an estimate; it doesn't provide an estimate such as $(4 \cdot \mathrm{B})$; and it has the additional factor $\log x$.

For the application in the next chapter these drawbacks are acceptable. The trick of Section 3.3 provides an workaround for $p \geq 2$ and the additional factor $\log x$ gets imbibed by an $o(1)$ term.

Furthermore, this abstract approach has, compared to the elementary approach, the advantage that we are not restricted to the ℓ_p case.

5.5.2. Spaces of polynomials without an unconditional basis

In the preceding sections and chapter we saw several estimates on the unconditional basis constant of spaces of polynomials. We now want to present examples of spaces which do not have an unconditional basis. Theorem 5.53, the aim of this section, is due to Defant and Kalton [30]; however, we want to present the idea as it demonstrates the power of the abstract theory developed in this chapter.

Theorem 5.52. *Let X be a Banach sequence space and assume that X contains uniformly complemented copies of ℓ_p^n, $n \in \mathbb{N}$ for some $1 < p \leq \infty$. In other words, there exists a constant $c \geq 1$ and operators $R_n : \ell_p^n \to X$ and $S_n : X \to \ell_p^n$ such that $\mathrm{id}_{\ell_p^n} = S_n R_n$ and $\|R_n\| \, \|S_n\| \leq c$ for every $n \in \mathbb{N}$.*

Then $\mathcal{P}(^m X)$ does not have an unconditional basis.

Proof. Let $n, m \in \mathbb{N}$ and define

$$U_n : \mathcal{P}(^m\ell_p^n) \to \mathcal{P}(^mX), \quad P \mapsto P \circ S_n$$

and

$$V_n : \mathcal{P}(^mX) \to \mathcal{P}(^m\ell_p^n), \quad P \mapsto P \circ R_n\,.$$

Then $\mathrm{id}_{\mathcal{P}(^m\ell_p^n)} = V_nU_n$ and we have $\|U_n\| \leq \|S_n\|^m$, since

$$\left\|P \circ S_n\right\|_{\mathcal{P}(^mX)} \leq \sup_{y \in \|S_n\| \cdot \mathrm{B}_{\ell_p^n}} |P(y)| = \|S_n\|^m \, \|P\|_{\mathcal{P}(^m\ell_p^n)}\,.$$

Analogously, we get $\|V_n\| \leq \|R_n\|^m$.

Under the assumption that $\mathcal{P}(^mX)$ possesses an unconditional basis we have by Theorem 5.19, Theorem 5.11, and Theorem 5.12

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^m\ell_p^n)\big) \leq 2^m \ \mathrm{gl}\left(\mathcal{P}(^m\ell_p^n)\right) \leq (2\,c)^m \ \mathrm{gl}\left(\mathcal{P}(^mX)\right) \leq (2\,c)^m \, \chi\big(\mathcal{P}(^mX)\big)$$

as $\|V_n\| \, \|U_n\| \leq \|R_n\|^m \|S_n\|^m \leq c^m$. By Theorem 4.7 the left-hand side tends to infinity for $n \to \infty$, which contradicts the finiteness of the right-hand side. $\square$

Now, we can prove the following theorem which resembles one implication of Theorem 1.1 by Defant and Kalton [30].

Theorem 5.53 (cf. Theorem 1.1 in [30])**.** *Let X be an infinite dimensional* Banach *sequence space and $m \geq 2$. Then $\mathcal{P}(^mX)$ does not have an unconditional basis.*

Proof. Assume that $\mathcal{P}(^mX)$ has a basis. This implies that $\mathcal{P}(^mX)$ is separable; therefore, by Proposition 3.2 of [30], X contains uniformly complemented copies of ℓ_p^n, $n \in \mathbb{N}$ for either $p = 2$ or $p = \infty$.

The preceding theorem then proves that $\mathcal{P}(^mX)$ doesn't have an unconditional basis. $\square$

Chapter 6.

Index sets generated by increasing sequences

The preceding chapters where dedicated to the study of the unconditional basis constant of the monomials in subspaces of the m–homogeneous polynomials. In this chapter we use the established results to investigate the unconditional basis constant $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J}\ell_p)\big)$ where $1 \le p \le \infty$ for non-homogeneous index sets J with a special structure.

Throughout this chapter we denote by $\boldsymbol{q} = (q_k)_{k\in\mathbb{N}}$ a strictly increasing sequence with $q_1 > 1$ and $q_k \to \infty$ for $k \to \infty$. As usual, set $q_{\boldsymbol{j}} := q_{j_1} \cdots q_{j_k}$ for $\boldsymbol{j} = (j_1, \dots, j_k) \in \mathbb{N}^k$. For technical reasons we denote furthermore by $\vartheta := (\ )$ the index of zero length and set $q_\vartheta := 1$ as well as $(\vartheta, \boldsymbol{j}) = (\boldsymbol{j}, \vartheta) = \boldsymbol{j}$ for any index $\boldsymbol{j}$.

For $x > 2$ we define now

$$J(x) := \big\{\boldsymbol{j} \in \mathcal{J}(\infty, \cdot\,) \,\big|\, q_{\boldsymbol{j}} \le x\big\} \cup \{\vartheta\}$$

and for $m \in \mathbb{N}$

$$J(x,m) := J(x) \cap \mathcal{J}(\infty, m)\,.$$

Our purpose is to get upper estimates of $\chi_{\mathrm{mon}}\mathcal{P}({}^{J(x)}\ell_p)$ in terms of $x \in (2,\infty)$. At first we want to introduce a general technique suitable to tackle this problem. In the second and third section we analyze the involved sets of indices for specific sequences $\boldsymbol{q}$ and subsequently apply the introduced technique. The last section finally discusses the optimality of the results obtained.

The technique we want to introduce is due to KONYAGIN and QUEFFÉLEC. In their ingenious paper [42] they used this technique to prove that there do not exist RUDIN-SHAPIRO like DIRICHLET polynomials (cf. Section 4 of their paper). To be more precise, they proved that there exist constants $\alpha, \beta > 0$ such that

$$\sum_{n \le x} |a_n| \le \alpha \sqrt{x} \mathrm{e}^{-\beta \sqrt{\log x \log \log x}} \sup_{t \in \mathbb{R}} \Big| \sum_{n \le x} a_n n^{-it} \Big|$$

for any DIRICHLET polynomial $A = \sum_{n \le x} a_n n^{-s}$. Using BOHR'S transform (which we will explain in detail in Chapter 8) we see that this is equivalent to

$$\chi_{\mathrm{mon}} \mathcal{P}(^{J(x)} \ell_\infty) \le \alpha \sqrt{x} \mathrm{e}^{-\beta \sqrt{\log x \log \log x}}$$

where $J(x)$ is generated by the sequence of primes.

The very general idea of their proof (see the mentioned paper [42] for more details) is the following: Split the sum $\sum_{n \le x} |a_n|$ into three sums, i.e.

$$\sum_{n \le x} |a_n| = \sum_u \sum_m \sum_v |a_{u \cdot v}|$$

where the first sum runs over integers with small prime factors and the third sum runs over all integers with exactly m great prime factors. This splitting technique then reduces the problem now to a m–homogeneous one.

6.1. The KONYAGIN-QUEFFÉLEC method

We may now introduce the technique in great detail for our abstract setting of an arbitrary generating sequence $\boldsymbol{q}$.

Let $2 < y < x$ and set $\pi_{\boldsymbol{q}}(y) := \max\{k \in \mathbb{N} \,|\, \boldsymbol{q}_k \le y\}$. We define

$$J^-(x,y) := \big\{ \boldsymbol{j} = (j_1, \ldots, j_k) \in J(x) \,\big|\, j_k \le \pi_{\boldsymbol{q}}(y) \big\} \cup \{\vartheta\} \tag{6 $\cdot$ A}$$

and for any $\boldsymbol{i} \in J^-(x,y)$, $m \in \mathbb{N}$

$$\begin{aligned} J^+_{\boldsymbol{i}}(x,y;m) &:= \big\{ \boldsymbol{j} \in J(x,m) \,\big|\, \pi_{\boldsymbol{q}}(y) < j_1, (\boldsymbol{i}, \boldsymbol{j}) \in J(x) \big\}, \\ J^+(x,y;m) &:= \big\{ \boldsymbol{j} \in J(x,m) \,\big|\, \pi_{\boldsymbol{q}}(y) < j_1 \big\}. \end{aligned} \tag{6 $\cdot$ B}$$

For $m = 0$ we set

$$J^+_{\boldsymbol{i}}(x, y; 0) := J^+(x, y; 0) := \{\vartheta\}\,.$$

From the general construction of these sets we can already say something about their size:

Lemma 6.1. *Let* $2 < y < x$. *For any* $\boldsymbol{j} = (j_1, \dots, j_k) \in J(x)$ *we have* $k \leq \frac{\log x}{\log q_1}$ *and thus*

(i) $|J^-(x, y)| \leq \left(1 + \frac{\log x}{\log q_1}\right)^{\pi_q(y)}$ *and*

(ii) $J^+(x, y; m) = \emptyset$ *for any* $m > \frac{\log x}{\log q_1}$.

Proof. Let $\boldsymbol{j} = (j_1, \dots, j_k) \in J(x) \supset J^-(x, y) \cup J^+(x, y; m)$. Then

$$\log x \geq \log q_{\boldsymbol{j}} = \log(q_{j_1} \cdots q_{j_k}) \geq \log q_1^k = k \log q_1$$

and with this $k \leq \frac{\log x}{\log q_1}$. Thus, $J^+(x, y; m) = \emptyset$ for any $m > \frac{\log x}{\log q_1}$ and furthermore

$$J^-(x, y) \subset \left\{(j_1, \dots, j_k) \,\middle|\, k \in \mathbb{N}, k \leq \frac{\log x}{\log q_1}, j_1 \leq \dots \leq j_k \leq \pi_q(y)\right\}.$$

Now $\boldsymbol{j} \mapsto \alpha(\boldsymbol{j})$ defines a bijection between this superset of $J^-(x, y)$ and

$$\left\{\alpha \in \mathbb{N}_0^{\pi_q(y)} \,\middle|\, \forall k : \alpha_k \leq \frac{\log x}{\log q_1}\right\}.$$

Obviously, the cardinality of the latter set is given by $\left(1 + \frac{\log x}{\log q_1}\right)^{\pi_q(y)}$. $\square$

Recall (see Chapter 4, $(4 \cdot \mathrm{A})$) that for $J \subset \mathcal{J}(n, m)$ the reduced index J^* was defined by $J^* := \left\{\boldsymbol{j} \in \mathcal{J}(n, m-1) \,\middle|\, \exists k \in \mathbb{N} : (\boldsymbol{j}, k) \in J\right\}$.

Lemma 6.2. *We have for the reduced index sets*

$$J^+(x, y; m)^* \subset J^+\left(x^{\frac{m-1}{m}}, y; m-1\right)$$

and

$$J(x, m)^* \subset J\left(x^{\frac{m-1}{m}}, m-1\right).$$

Proof. Let J be either $J^+(x,y;m)$ or $J(x,m)$ and let $\boldsymbol{j} = (j_1,\dots,j_{m-1}) \in J^*$. Then there exists $k \geq j_{m-1}$ such that $(\boldsymbol{j},k) \in J$ and hence $q_{\boldsymbol{j}} \cdot q_k = q_{(\boldsymbol{j},k)} \leq x$. Since $q_k \geq q_{j_{m-1}}$ this implies either $q_k > x^{\frac{1}{m}}$ or $q_{j_1} \leq \ldots \leq q_{j_{m-1}} \leq q_k \leq x^{\frac{1}{m}}$. In both cases $q_{\boldsymbol{j}} = q_{j_1} \cdots q_{j_{m-1}} \leq x^{\frac{m-1}{m}}$. □

Let now $1 \leq p \leq \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. Choose $n \in \mathbb{N}$ such that $J(x) \subset \mathcal{J}(n,\,\cdot\,)$ and fix $f \in H_\infty(\mathrm{B}_{\ell_p})$. Fix furthermore $\xi \in \mathrm{B}_{\ell_p}$ and write $\xi = \xi^- + \xi^+$ where $\xi_k^- = 0$ whenever $k > \pi_q(y)$ and $\xi_k^+ = 0$ if $k \leq \pi_q(y)$. This implies $\|\xi\|^p = \|\xi^-\|^p + \|\xi^+\|^p$ and $\xi^- + x \in \mathrm{B}_{\ell_p}$ whenever $\|x\| \leq \|\xi^+\|$ and $x_k = 0$ for $k \leq \pi_q(y)$.

Any index $\boldsymbol{k} \in J(x)$ may now be decomposed as $\boldsymbol{k} = (\boldsymbol{i},\boldsymbol{j})$ with $\boldsymbol{i} \in J^-(x;y)$ and $\boldsymbol{j} \in J^+(x,m;y)$ for some $m \in \mathbb{N}_0$. For such $\boldsymbol{i}$ and $\boldsymbol{j}$ we have $\xi_{\boldsymbol{i}} = \xi_{\boldsymbol{i}}^-$, $\xi_{\boldsymbol{j}} = \xi_{\boldsymbol{j}}^+$ and hence $\xi_{\boldsymbol{k}} = \xi_{(\boldsymbol{i},\boldsymbol{j})} = \xi_{\boldsymbol{i}}\,\xi_{\boldsymbol{j}} = \xi_{\boldsymbol{i}}^-\,\xi_{\boldsymbol{j}}^+$.

Therefore,

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in J(x)} |c_{\boldsymbol{k}}(f)\,\xi_{\boldsymbol{k}}| \\
&\qquad = \sum_{\boldsymbol{i}\in J^-(x,y)} \sum_{m\in\mathbb{N}_0} \sum_{\boldsymbol{j}\in J^+(x,y;m)} |c_{(\boldsymbol{i},\boldsymbol{j})}(f)\,\xi_{(\boldsymbol{i},\boldsymbol{j})}| \\
&\qquad = \sum_{\boldsymbol{i}\in J^-(x,y)} \sum_{m\in\mathbb{N}_0} |\xi_{\boldsymbol{i}}^-| \sum_{\boldsymbol{j}\in J^+(x,y;m)} |c_{(\boldsymbol{i},\boldsymbol{j})}(f)\,\xi_{\boldsymbol{j}}^+|\,.
\end{aligned}$$

From Theorem 4.1 we get that this is

$$\leq \sum_{\boldsymbol{i}\in J^-(x,y)} \sum_{m\in\mathbb{N}_0} |\xi_{\boldsymbol{i}}^-|\ \mathrm{e}^{2m} |J^+(x,y;m)^*|^\sigma \sup_{\|x\|\leq\|\xi^+\|} \Big| \sum_{\substack{\boldsymbol{j}\in\mathcal{J}(n,m)\\ j_1>\pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(f)\,x_{\boldsymbol{j}} \Big|\,.$$

We may assume, since $j_1 > \pi_q(y)$ for all $\boldsymbol{j} \in J^+(x,y;m)$, that the supremum is taken over all $x \in \ell_p$ such that $\|x\| \leq \|\xi^+\|$ and $x_k = 0$ for $k \leq \pi_q(y)$. Therefore, $\|\xi^- + x\| \leq 1$ by the considerations above. Thus

$$\begin{aligned}
&= \sum_{\boldsymbol{i}\in J^-(x,y)} \sum_{m\in\mathbb{N}_0} \mathrm{e}^{2m} |J^+(x,y;m)^*|^\sigma \sup_{\|x\|\leq\|\xi^+\|} \Big| \sum_{\substack{\boldsymbol{j}\in\mathcal{J}(n,m)\\ j_1>\pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(f)\,\xi_{\boldsymbol{i}}^-\,x_{\boldsymbol{j}} \Big| \\
&\leq \sum_{\boldsymbol{i}\in J^-(x,y)} \sum_{m\in\mathbb{N}_0} \mathrm{e}^{2m} |J^+(x,y;m)^*|^\sigma\, \Big\| \sum_{\substack{\boldsymbol{j}\in\mathcal{J}(n,m)\\ j_1>\pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(f)\,z_{(\boldsymbol{i},\boldsymbol{j})} \Big\|_{\mathrm{B}_{\ell_p}}\,.
\end{aligned}$$

To estimate the latter norm we use the following lemma. For any $\boldsymbol{i} \in \mathcal{J}(\infty, k)$ with $i_k \leq \pi_q(y)$ and any $P \in \mathcal{P}_{\text{fin}}({}^{\mathcal{J}(\infty,k+m)}\ell_p)$ we define

$$P_{\boldsymbol{i}} := \sum_{\substack{\boldsymbol{j} \in \mathcal{J}(\infty,m) \\ j_1 > \pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(P)\, z_{(\boldsymbol{i},\boldsymbol{j})}\,.$$

Lemma 6.3. *Let $P \in \mathcal{P}_{\text{fin}}({}^{\mathcal{J}(\infty,k+m)}\ell_p)$ and let $\boldsymbol{i} = (i_1, \dots, i_k) \in \mathcal{J}(\infty, k)$ with $i_k \leq \pi_q(y)$. Then for any $x \in \ell_p$*

$$P_{\boldsymbol{i}}(x) = \int_{\mathbb{T}^{\pi_q(y)}} P(\tilde{\zeta} \cdot x)\, \zeta_{\boldsymbol{i}}^{-1}\, \mathrm{d}m_{\pi_q(y)}(\zeta)$$

where $\tilde{\zeta} := (\zeta_1, \dots, \zeta_{\pi_q(y)}, 1, \dots)$. As a consequence $\|P_{\boldsymbol{i}}\|_{\mathrm{B}_{\ell_p}} \leq \|P\|_{\mathrm{B}_{\ell_p}}$.

Proof. By a straightforward calculation,

$$\int_{\mathbb{T}^{\pi_q(y)}} P(\tilde{\zeta} \cdot x)\, \zeta_{\boldsymbol{i}}^{-1}\, \mathrm{d}m_{\pi_q(y)}(\zeta) = \sum_{\boldsymbol{j} \in \mathcal{J}(\infty,k+m)} c_{\boldsymbol{j}}(P)\, x_{\boldsymbol{j}} \cdot \int_{\mathbb{T}^{\pi_q(y)}} \tilde{\zeta}_{\boldsymbol{j}}\, \zeta_{\boldsymbol{i}}^{-1}\, \mathrm{d}m_{\pi_q(y)}(\zeta)\,.$$

By Lemma 3.3, the integral on the right-hand side evaluates to 1 if $(j_1, \dots, j_k) = \boldsymbol{i}$ and $j_k \leq \pi_q(y) < j_{k+1}$. Otherwise the integral vanishes. We obtain

$$\begin{aligned} &= \sum_{\substack{\boldsymbol{j} \in \mathcal{J}(\infty,m) \\ j_1 > \pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(P)\, x_{(\boldsymbol{i},\boldsymbol{j})} \\ &= P_{\boldsymbol{i}}(x)\,. \end{aligned}$$

□

Hence, for any $\boldsymbol{i} = (j_1, \dots, j_k) \in J^-(x, y)$

$$\Big\| \sum_{\substack{\boldsymbol{j} \in \mathcal{J}(n,m) \\ j_1 > \pi_q(y)}} c_{(\boldsymbol{i},\boldsymbol{j})}(f)\, z_{(\boldsymbol{i},\boldsymbol{j})} \Big\|_{\mathrm{B}_{\ell_p}} \leq \Big\| \sum_{\boldsymbol{j} \in \mathcal{J}(n,k+m)} c_{\boldsymbol{j}}(f)\, z_{\boldsymbol{j}} \Big\|_{\mathrm{B}_{\ell_p}} \leq \|f\|_{\mathrm{B}_{\ell_p}}$$

by Proposition 2.16. We have proven so far

$$\begin{aligned} \sum_{\boldsymbol{j} \in J(x)} |c_{\boldsymbol{j}}(f)\, \xi_{\boldsymbol{j}}| &\leq \sum_{\boldsymbol{i} \in J^-(x,y)} \sum_{m \in \mathbb{N}_0} \mathrm{e}^{2m} \big| J^+(x, y; m)^* \big|^{\sigma}\, \|f\|_{\mathrm{B}_{\ell_p}} \\ &= \big| J^-(x, y) \big| \sum_{m \in \mathbb{N}_0} \mathrm{e}^{2m} \big| J^+(x, y; m)^* \big|^{\sigma}\, \|f\|_{\mathrm{B}_{\ell_p}}\,. \end{aligned}$$

From Lemma 6.1 and 6.2 we have upper bounds on $\left|J^-(x,y)\right|$ and $\left|J^+(x,y;m)^*\right|$. Furthermore, we have that $J^+(x^{\frac{m-1}{m}}, y; m-1) = \emptyset$ whenever $m-1 > \frac{\log x}{\log q_1}$. Hence, this is

$$\leq \Big(1 + \frac{\log x}{\log q_1}\Big)^{\pi_q(y)+1} \sup_{m \in \mathbb{N}_0} \mathrm{e}^{2m} \left|J^+(x^{\frac{m-1}{m}}, m-1)\right|^{\sigma} \|f\|_{\mathrm{B}_{\ell_p}} .$$

Without specific knowledge about q we can't go any further. In the following section we will investigate this expression for a specific choice of q.

6.2. Specific choices of the generating sequence

Let us now discuss two specific choices of the generating sequence; namely

$$q := \Big(k\,\big(\log(k+2)\big)^{\theta}\Big)_k \tag{6 · C}$$

for some $\theta \in (\frac{1}{2}, 1]$ and

$$p := \big(2, 3, 5, 7, 11, \dots\big)\,, \tag{6 · D}$$

the sequence of primes. The aim of this section is to prove for these choices Theorem 6.4 and Theorem 6.10 respectively.

Theorem 6.4. *Let $1 \leq p \leq \infty$, $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and q as defined in (6 · C) with $\theta \in (\frac{1}{2}, 1]$. Then for $f \in H_\infty(\mathrm{B}_{\ell_p})$ and $\xi \in \mathrm{B}_{\ell_p}$*

$$\sum_{\boldsymbol{j} \in J(x)} |c_{\boldsymbol{j}}(f)\,\xi_{\boldsymbol{j}}| \leq x^{\sigma} \exp\Big(\Big(-2\sigma\sqrt{\theta - \tfrac{1}{2}} + o(1)\Big)\sqrt{\log x \log\log x}\Big)\,\|f\|_{\mathrm{B}_{\ell_p}}$$

where the $o(1)$–term only depends on p and θ. In particular,

$$\chi_{\mathrm{mon}}\big(\mathcal{P}(^{J(x)}\ell_p)\big) \leq x^{\sigma} \exp\Big(\Big(-2\sigma\sqrt{\theta - \tfrac{1}{2}} + o(1)\Big)\sqrt{\log x \log\log x}\Big) .$$

For the proof we need to analyze the specific properties of the sequence q.

Proposition 6.5. *Let q denote the sequence defined in (6 · C). For $x \geq q_2$ we have*

$$\frac{1}{2^{1+\theta}}\,\frac{x}{(\log x)^{\theta}} \leq \pi_q(x) \leq 2^{\theta}\,\frac{x}{(\log x)^{\theta}} .$$

Proof. Our proof starts with the observation that $\pi_q(x) \geq 2$ for $x \geq q_2$ and

$$\pi_q(x)\Big(\log\big(\pi_q(x)+2\big)\Big)^\theta \leq x < \big(\pi_q(x)+1\big)\Big(\log\big(\pi_q(x)+3\big)\Big)^\theta \leq \big(\pi_q(x)+2\big)^2$$

by the definition of $\pi_q(x)$. Hence, for every $x \geq q_2$

$$\frac{x}{(\log x)^\theta} \geq \frac{\pi_q(x)\Big(\log\big(\pi_q(x)+2\big)\Big)^\theta}{\Big(\log\big((\pi_q(x)+2)^2\big)\Big)^\theta} = 2^{-\theta}\,\pi_q(x)$$

and

$$\frac{x}{(\log x)^\theta} \leq \frac{\big(\pi_q(x)+1\big)\Big(\log\big(\pi_q(x)+3\big)\Big)^\theta}{\big(\log \pi_q(x)\big)^\theta} \leq 2^{1+\theta}\,\pi_q(x)\,. \qquad \square$$

The following considerations differ slightly in the cases $\theta = 1$ and $\theta < 1$. To simplify notation we set

$$g_\theta(x) := \begin{cases} \frac{(\log x)^{1-\theta}}{1-\theta} & \text{if } \theta < 1 \text{ and} \\ \log\log x & \text{if } \theta = 1. \end{cases}$$

Lemma 6.6. *Let q denote the sequence defined in* $(6\cdot\mathrm{C})$*. Then for $x > 2$*

$$\sum_{k \leq x} \frac{1}{q_k} \leq g_\theta(x) + c_q$$

where $c_q := q_1^{-1} + q_2^{-1} + q_3^{-1}$.

Proof. Obviously $q_k = k\big(\log(k+2)\big)^\theta \geq k(\log k)^\theta$ and $\frac{1}{k(\log k)^\theta}$ is monotonically decreasing; therefore,

$$\sum_{3<k\leq x} \frac{1}{q_k} \leq \sum_{3<k\leq x} \frac{1}{k(\log k)^\theta} \leq \int_3^x \frac{1}{t(\log t)^\theta}\,\mathrm{d}t = \int_{\log 3}^{\log x} \frac{1}{s^\theta}\,\mathrm{d}s = g_\theta(x) - g_\theta(3)\,. \qquad \square$$

Using this result we are finally able to estimate the cardinality of $J^+(x, y; m)$.

Proposition 6.7. *Let $J^+(x, y; m)$ be the respective set of indices generated by q as defined in* $(6\cdot\mathrm{C})$*. Then*

$$\big|J^+(x, y; m)\big| \leq xy^{-m}\exp\Big(y\,\big(g_\theta(x) + c_q\big)\Big)\,.$$

Proof. From the definition of q it is easily seen that

$$q_{l+k} - q_l \geq q_k \,. \tag{6 · E}$$

We introduce a completely multiplicative function and use what is sometimes called Rankin's trick:

$$\begin{aligned}
|J^+(x,y;m)| &= \sum_{\boldsymbol{j}\in J^+(x,y;m)} 1 \\
&\leq \frac{x}{y^m} \sum_{\boldsymbol{j}\in J^+(x,y;m)} \frac{y}{q_{j_1}} \cdots \frac{y}{q_{j_m}} \\
&\leq \frac{x}{y^m} \prod_{\pi_q(y)<k<x} \left(\sum_{\nu=0}^{\infty} \left(\frac{y}{q_k}\right)^{\nu} \right) \\
&= \frac{x}{y^m} \exp\left(-\sum_{\pi_q(y)<k<x} \log\left(1-\frac{y}{q_k}\right)\right).
\end{aligned}$$

Using the series expansion of the logarithm around 1, we obtain for the exponent

$$\begin{aligned}
-\sum_{\pi_q(y)<k<x} \log\left(1-\frac{y}{q_k}\right) &= \sum_{\pi_q(y)<k<x} \sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{y}{q_k}\right)^{\nu} \\
&\leq \sum_{\pi_q(y)<k<x} \frac{y}{q_k} \frac{1}{1-\frac{y}{q_k}} \\
&= y \sum_{\pi_q(y)<k<x} \frac{1}{q_k - y} \,.
\end{aligned}$$

By (6 · E), $q_k - y > q_k - q_{\pi_q(y)} \geq q_{k-\pi_q(y)}$; therefore, this is

$$\leq y \sum_{\pi_q(y)<k<x} \frac{1}{q_{k-\pi_q(y)}} \,.$$

Lemma 6.6 now completes the proof. □

We proceed with the proof of Theorem 6.4.

Proof of Theorem 6.4. Having the considerations of the previous section in mind it suffices to estimate

$$\left(\frac{\log x}{\log q_1}\right)^{\pi_q(y)+1} \sup_{m\in\mathbb{N}_0} \mathrm{e}^{2m} \left|J^+(x^{\frac{m-1}{m}}, y; m-1)\right|^{\sigma} .$$

By Proposition 6.7,

$$\begin{aligned}\Big(\frac{\log x}{\log q_1}\Big)^{\pi_q(y)+1} &\sup_{m\in\mathbb{N}_0} \mathrm{e}^{2m} \left|J^+(x^{\frac{m-1}{m}}, y; m-1)\right|^\sigma \\ &\leq \Big(\frac{\log x}{\log q_1}\Big)^{\pi_q(y)+1} \sup_{m\in\mathbb{N}_0} \left(\mathrm{e}^{2m} x^{\frac{m-1}{m}} y^{-m} \exp\Big(y\cdot\big(g_\theta(x)+c_q\big)\Big)\right)^\sigma .\end{aligned}$$

Choosing $y := \frac{(\log x)^{\theta-\frac{1}{2}}}{\log\log x}$, this is

$$= x^\sigma \exp\Big(o(1)\sqrt{\log x \log\log x}\Big) \sup_m \big(\overbrace{\mathrm{e}^{2m} x^{-\frac{1}{m}} y^{-m}}^{=:\,\exp h_{x,y}(m)}\big)^\sigma$$

as

$$y\, g_\theta(x) \leq 2\sqrt{\log x} = o(1)\sqrt{\log x \log\log x}$$

and

$$\pi_q(y)\,\log\log x \leq 2^\theta\,\frac{y\log\log x}{(\log y)^\theta} = o(1)\sqrt{\log x\log\log x}$$

by Proposition 6.5. After differentiating

$$h_{x,y}(m) = 2m - \frac{1}{m}\,\log x - m\log y$$

we see that it attains its maximum at

$$M = \sqrt{\frac{\log x}{\log y - 2}} \geq \sqrt{\frac{\log x}{\log y}}\,;$$

therefore,

$$\begin{aligned} h_{x,y}(m) &\leq h_{x,y}(M) \\ &= \underbrace{\sqrt{\frac{\log x}{\log y-2}}}_{=o(1)\sqrt{\log x\log\log x}} -2\sqrt{\log x\log y} \\ &= \Big(-2\sqrt{\theta-\tfrac{1}{2}}+o(1)\Big)\sqrt{\log x\log\log x}\,, \end{aligned}$$

which proves the claim. □

We now turn to the sequence of primes; i.e. $\boldsymbol{p}$ as defined in $(6\cdot\mathrm{D})$. We check at once that in this case the set $J(x)$ is via $\boldsymbol{j}\mapsto\alpha(\boldsymbol{j})$ in bijection with

$$\Lambda(x) := \big\{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\;\big|\; p^\alpha = p_1^{\alpha_1}p_2^{\alpha_2}\cdots\leq x\big\}\,.$$

We see that every index $\boldsymbol{j} \in J(x)$, $\alpha \in \Lambda(x)$ respectively represents the prime number decomposition of a natural number less or equal x. Since in this case π_p denotes the prime counting function, the indices in $J^-(x,y)$ and $J^+(x,y;m)$ represent accordingly the prime number decomposition of natural numbers whose prime factors are all less or equal y, respectively those numbers with exactly m prime factors which are all strictly greater than y. We have as a substitute of Proposition 6.5 the well-known prime number theorem:

Proposition 6.8 (cf. Chapter I.1 in [57])**.** *For x to infinity we have the asymptotical equivalence*

$$|\{k \mid p_k \le x\}| = \pi_p(x) \sim \frac{x}{\log x}\,.$$

As an replacement for Proposition 6.7 we have the following proposition, whose proof is due to Balazard [6].

Proposition 6.9 (cf. Corollaire 1 in [6])**.** *Let $J^+(x,y;m)$ be the respective set of indices generated by the sequence of primes. Then there exits a universal constant $c > 0$ such that*

$$\big|J^+(x,y;m)\big| \le xy^{-m}\exp\Big(y\,\big(\log\log x + c\big)\Big) = xy^{-m}\exp\Big(y\,\big(g_1(x)+c\big)\Big)\,.$$

Going through the proof of Theorem 6.4 (while having $\theta = 1$ in mind) we obtain analogously the following theorem.

Theorem 6.10. *Let $1 \le p \le \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. For $f \in H_\infty(\mathrm{B}_{\ell_p})$ and $\xi \in \mathrm{B}_{\ell_p}$*

$$\sum_{\boldsymbol{j} \in J(x)} |c_{\boldsymbol{j}}(f)\,\xi_{\boldsymbol{j}}| \le x^\sigma \exp\Big(\big(-\sqrt{2}\sigma + o(1)\big)\sqrt{\log x \log\log x}\Big)\,\|f\|_{\mathrm{B}_{\ell_p}}$$

where the $o(1)$–term only depends on p. In particular,

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x)}\ell_p)\big) \le x^\sigma \exp\Big(\big(-\sqrt{2}\sigma + o(1)\big)\sqrt{\log x \log\log x}\Big)\,.$$

For $p = \infty$ this theorem, although stated in a different fashion, was proven in a series of papers by Konyagin and Queffélec [42]; de la Bretèche [23]; and Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip [27]. The Bohnenblust-Hille inequality seemed to be the central building block. However, the proof at hand shows that Lemma 4.5 (which is one of the ingredients of the Bohnenblust-Hille inequality) and thus Theorem 4.1 yields the critical estimate.

6.3. Optimality

The results of Section 3.4 enable us to give an idea of the optimality of the results of this chapter. We can prove that the exponents of x in the Theorems 6.4 and 6.10 are optimal in the following sense:

Theorem 6.11. *Let* $1 \leq p \leq \infty$ *and set* $\sigma := 1 - \frac{1}{\min\{p,2\}}$. *Let furthermore* $x \in (2, \infty)$ *and* $m \in \mathbb{N}$ *such that* $m \geq 2$. *Let* $J(x, m)$ *be the respective index set generated by either the sequence defined in* $(6 \cdot \mathrm{C})$ *or the sequence of primes. Then there exists an constant* $c(m) > 0$ *independent of* x *such that*

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x,m)}\ell_p)\big) \geq c(m)\, \frac{x^{\sigma\, \frac{m-1}{m}}}{(\log x)^{\sigma\, \theta\, (m-1)}}\,.$$

Proof. For $p = 1$ the claim is trivial, since in this case $\sigma = 0$. Therefore, we may assume $p > 1$. We make use of Proposition 3.23; more precisely Corollary 3.25 with $q := \min\{p, 2\}$. Let $n := \pi_q\big(x^{\frac{1}{m}}\big) \in \mathbb{N}$. With this obviously $\mathcal{J}(n, m) \subset J(x, m)$ and hence

$$\begin{aligned}\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x,m)}\ell_p^n)\big) &\geq \chi_{\mathrm{mon}}\big(\mathcal{P}({}^{m}\ell_p^n)\big)\\ &\geq \big(c(m,n;q)\big)^{-1} m^{-m(1-\frac{1}{q})} \big\|\mathrm{id} : \ell_p^n \hookrightarrow \ell_1^n\big\|^m \big\|\mathrm{id} : \ell_p^n \hookrightarrow \ell_q^n\big\|^{-m}\end{aligned}$$

with $c(m, n; q)$ denoting the constant in Proposition 3.23. A well known result gives now $\big\|\mathrm{id} : \ell_p^n \hookrightarrow \ell_1^n\big\| = n^{1-\frac{1}{p}}$ and $\big\|\mathrm{id} : \ell_p^n \hookrightarrow \ell_q^n\big\| = n^{\frac{1}{q}-\frac{1}{p}}$. Therefore, we have an universal constant $c \geq 1$ such that

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x,m)}\ell_p)\big) \geq c^{-1}\, (\log m\, m^m)^{-\sigma}\, n^{(m-1)\sigma} \geq c(m)\, \frac{x^{\sigma\, \frac{m-1}{m}}}{(\log x)^{\sigma\, \theta\, (m-1)}}$$

by Proposition 6.5 and Proposition 6.8 respectively. □

In the special case $p = \infty$ and $J(x)$ generated by the sequence of primes the investigations of Konyagin and Queffélec [42], de la Bretèche [23] and Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip [27] show that the result is optimal, i.e.

$$\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x)}\ell_\infty)\big) = x^{\frac{1}{2}} \exp\Big(\big(-\tfrac{1}{\sqrt{2}} + o(1)\big)\sqrt{\log x \log\log x}\Big)\,.$$

Part II.

Domains of convergence and other viewpoints

Chapter 7.

Sets of monomial convergence

When talking about holomorphy in finite dimensions, it is well known that the approaches of Cauchy and Weierstrass are equivalent; a function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is Fréchet or complex differentiable in x (cf. definition of holomorphy in Section 2.2) if and only if f can be expressed as a power series in a neighborhood of x. In infinite dimensions these approaches do not coincide.

Let X be a Banach sequence space, R be a Reinhardt domain in X, and $f : R \to \mathbb{C}$ a holomorphic function. We saw in Section 2.2 that such a function has a (formal) power series expansion

$$f = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f)\, z^\alpha\,.$$

Contrary to what happens in the finite dimensional case, this power series does not necessarily converge at every point $x \in R$. A classical result of Toeplitz shows that there exists a 2–homogeneous polynomial $P : c_0 \to \mathbb{C}$ (thus P is holomorphic on c_0) such that

$$\forall \varepsilon > 0 \exists x \in \ell_{4+\varepsilon} : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(P)\, x^\alpha| = \infty\,.$$

This motivates the following definition: Let $\mathcal{F}(R)$ be a closed subset of $\big(H_\infty(R), \|\cdot\|_R\big)$ (of the space of all bounded holomorphic functions on the Reinhardt domain R). We call

$$\operatorname{mon} \mathcal{F}(R) := \Big\{ x \in \mathbb{C}^{\mathbb{N}} \,\Big|\, \forall f \in \mathcal{F}(R) : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)\, x^\alpha| < \infty \Big\}$$

the domain of monomial convergence for $\mathcal{F}(R)$.

Our aim in this chapter is to describe the sets of monomial convergence for $\mathcal{P}(^m X)$ and $H_\infty(\mathrm{B}_X)$ with X denoting a BANACH sequence space.

In [22], DAVIE and GAMELIN showed that every function $f \in H_\infty(\mathrm{B}_{c_0})$ can be extended to a function $f \in H_\infty(\mathrm{B}_{\ell_\infty})$ with equal norm. From this we get (see Remark 6.4 in [32] for a proof)

$$\operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty}) = \operatorname{mon} H_\infty(\mathrm{B}_{c_0})$$

and

$$\operatorname{mon} \mathcal{P}(^m \ell_\infty) = \operatorname{mon} \mathcal{P}(^m c_0)\,.$$

7.1. Preliminaries and essential results

Proposition 7.1. *Let X be a* BANACH *sequence space, $R \subset X$ a* REINHARDT *domain, and $\mathcal{F}(R)$ a closed subset of $H_\infty(R)$ with $\mathcal{P}(^m R) := \{P\big|_R \,|\, P \in \mathcal{P}(^m X)\} \subset \mathcal{F}(R)$ for some $m \geq 2$. Then* $\operatorname{mon} \mathcal{F}(R) \subset c_0$.

Proof. Since we assumed $\mathcal{P}(^m R) \subset \mathcal{F}(R)$, we have $\operatorname{mon} \mathcal{F}(R) \subset \operatorname{mon} \mathcal{P}(^m R)$. It remains to show that $\operatorname{mon} \mathcal{P}(^m R) \subset c_0$. Let $x \in \operatorname{mon} \mathcal{P}(^m R)$; we may assume without loss of generality that $\|x\|_X < 1$. By a closed graph argument, we find a constant $\tilde{c} \geq 1$ such that

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(P)\, x^\alpha| \leq \tilde{c}\, \|P\|_R$$

for every $P \in \mathcal{P}(^m R)$. Assume that $x \notin c_0$; that means there is $\delta > 0$ and a strictly increasing sequence $(k_j)_j$ of natural numbers such that $|x_{k_j}| \geq \delta$ for every $j \in \mathbb{N}$.

Set $Y_n := \operatorname{span}\{e_{k_j} \,\big|\, j \in \{1, \ldots, n\}\}$ and define $\xi := \sum_j x_{k_j} e_{k_j} \in \mathrm{B}_{Y_n}$. We apply Corollary 3.24 with $q = 2$ to this setting and obtain a universal constant $c \geq 1$ such that

$$(n\delta)^m \leq \Big(\sum_{j=1}^n |x_{k_j}|\Big)^m \leq c\, (m^m \,\log m)^{\frac{1}{2}}\, n^{\frac{1}{2} + \frac{m}{2}}$$

since $\big\|\mathrm{id} : Y_n \hookrightarrow \ell_2^n\big\| \leq \big\|\mathrm{id} : \ell_\infty^n \hookrightarrow \ell_2^n\big\| = n^{\frac{1}{2}}$. For n to infinity and fixed $m \geq 2$ this is clearly a contradiction. $\square$

We call a Reinhardt domain R symmetric if $x \in R$ if and only if $x^* \in R$; i.e. if the decreasing rearrangement of x is an element of R if and only if $x \in R$. For a holomorphic function $f : R \to \mathbb{C}$ on such a Reinhardt domain and a permutation σ of the natural numbers we define

$$f_\sigma : R \to \mathbb{C}, \quad (x_k)_k \mapsto f\big((x_{\sigma(k)})_k\big) .$$

Let R be a symmetric Reinhardt domain. We call a set $\mathcal{F}(R) \subset H_\infty(R)$ symmetric if $f_\sigma \in \mathcal{F}(R)$ for every $f \in \mathcal{F}(R)$ and every permutation σ of the natural numbers. We check at once that $H_\infty(R)$ and $\mathcal{P}(^m R)$ are symmetric for any symmetric Reinhardt domain R.

Many of the proofs show that for a sequence x the decreasing rearrangement is contained in the domain of monomial convergence. Hence the following result will be beneficial in our investigations.

Proposition 7.2. *Let X be a* Banach *sequence space, $R \subset X$ be a symmetric* Reinhardt *domain, and $\mathcal{F}(R) \subset H_\infty(R)$. Assume $\operatorname{mon} \mathcal{F}(R) \subset c_0$. Then $x \in \operatorname{mon} \mathcal{F}(R)$ if and only if $x^* \in \operatorname{mon} \mathcal{F}(R)$.*

Proof. Let $x \in \operatorname{mon} \mathcal{F}(R) \subset c_0$. Then there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $x_k^* = x_{\sigma(k)}$ for all $k \in \mathbb{N}$. By assumption, $f_\sigma \in \mathcal{F}(R)$ and $f_{\sigma^{-1}} \in \mathcal{F}(R)$ for every $f \in \mathcal{F}(R)$. Hence $x^* \in \operatorname{mon} \mathcal{F}(R)$.

Analogously we have $x \in \operatorname{mon} \mathcal{F}(R)$ for every $x \in c_0$ with $x^* \in \operatorname{mon} \mathcal{F}(R) \subset c_0$. $\square$

As most of the proofs show that a certain sequence is contained in the monomial domain, the following results provide a method to classify these results. For the proof of the following theorem we refer the reader to Lemma 2 in [29].

Theorem 7.3 (cf. Lemma 2 in [29])**.** *Let $x \in \mathrm{B}_{\ell_\infty}$ and assume that there exists some $u \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty})$ such that $|x_k| \leq |u_k|$ for all but finitely many $k \in \mathbb{N}$.*

Then $x \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty})$.

For homogeneous polynomials we obtain a much stronger result.

Theorem 7.4. *Let X be a* Banach *sequence space, $m \in \mathbb{N}$, and $\mathcal{F}(R) \subset \mathcal{P}(^m X)$. Let $x \in \ell_\infty$ and assume there exists $u \in \operatorname{mon} \mathcal{F}(R)$ and $c > 0$ such that $|x_k| \leq c\,|u_k|$ for every $k \in \mathbb{N}$. Then $x \in \operatorname{mon} \mathcal{F}(R)$.*

Proof. For any $f \in \mathcal{F}(R)$ we have

$$\sum_{\alpha} |c_\alpha(f)\, x^\alpha| \le \sum_{\alpha} |c_\alpha(f)\,(c\,u)^\alpha| = c^m \sum_{\alpha} |c_\alpha(f)\, u^\alpha| < \infty\,,$$

since $\mathcal{F}(R) \subset \mathcal{P}({}^m X)$ and therefore $c_\alpha(f) \neq 0$ only if $|\alpha| = m$. $\square$

The following theorem is inspired by the trick introduced in Section 3.3. For the statement we have to introduce a new notation: Let X be a BANACH sequence space, $R \subset X$ a p–exhaustible REINHARDT domain, and $\mathcal{F}(R)$ a closed set of bounded holomorphic functions on R. We define

$$\big[\mathcal{F}(R)\big]_p := \big\{ f \circ D_r \,\big|\, r \in \mathcal{R}, f \in \mathcal{F}(R) \big\}$$

where $\mathcal{R} \subset [0,\infty)^{\mathbb{N}}$ such that $R = \mathcal{R} \cdot \mathrm{B}_{\ell_p}$. $\big[\mathcal{F}(R)\big]_p$ is then a set of holomorphic functions on B_{ℓ_p} and we have the following result:

Theorem 7.5. *Let X be a* BANACH *sequence space, $R \subset X$ a p–exhaustible* REINHARDT *domain with $R = \mathcal{R} \cdot \mathrm{B}_{\ell_p}$, and $\mathcal{F}(R) \subset H_\infty(R)$. Then*

$$\mathcal{R} \cdot \operatorname{mon} \big[\mathcal{F}(R)\big]_p \subset \operatorname{mon} \mathcal{F}(R)\,.$$

Proof. Let $f \in \mathcal{F}(R)$ and $x = r \cdot \omega \in \mathcal{R} \cdot \operatorname{mon} \big[\mathcal{F}(R)\big]_p$. $f \circ D_r$ is then a holomorphic function on B_{ℓ_p} with $c_\alpha(f \circ D_r) = r^\alpha\, c_\alpha(f)$ for every $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ by Lemma 2.21.

Since $\omega \in \operatorname{mon} \big[\mathcal{F}(R)\big]_p$,

$$\sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} \big|c_\alpha(f)\, x^\alpha\big| = \sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} \big|c_\alpha(f)\, r^\alpha \omega^\alpha\big| = \sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} \big|c_\alpha(f \circ D_r)\, \omega^\alpha\big| < \infty\,. \qquad \square$$

7.2. Homogeneous polynomials

As we can extend any m–homogeneous polynomial defined on the unit ball B_X to a m–homogeneous polynomial on X, it is clear how to interpret $\mathcal{P}({}^m X)$ as a closed subspace of $\big(H_\infty(\mathrm{B}_X), \|\cdot\|_{\mathrm{B}_X}\big)$.

For the BANACH sequence spaces ℓ_p with $1 \le p \le \infty$ various results are already known. In the extreme cases $p = 1$ and $p = \infty$ we have a complete characterization:

$$\operatorname{mon} \mathcal{P}({}^m \ell_1) = \ell_1 \tag{7 · A}$$

and

$$\operatorname{mon} \mathcal{P}({}^m\ell_\infty) = \ell_{\frac{2m}{m-1},\infty}\,. \tag{7 · B}$$

Proof of (7 · A). In order to prove equality we have to show that for any $P \in \mathcal{P}({}^m\ell_1)$ and any $x \in \ell_1$

$$\sum_\alpha |c_\alpha(P)\, x^\alpha| < \infty\,.$$

By Corollary 2.19,

$$\sum_\alpha |c_\alpha(P)\, x^\alpha| \leq \mathrm{e}^m m!\, \|P\|_{\mathrm{B}_{\ell_1}} \sum_\alpha |x^\alpha|$$

which is finite, since

$$\sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} |x^\alpha| = \lim_{N\to\infty} \sum_{\alpha \in \mathbb{N}_0^N} |x^\alpha| = \lim_{N\to\infty} \prod_{n=1}^N \sum_{k=0}^\infty |x_n|^k = \lim_{N\to\infty} \prod_{n=1}^N \frac{1}{1-|x_n|} < \infty\,. \qquad \square$$

The proof of (7 · B) can be found in [9]. For $1 < p < \infty$ only approximations are known. The following theorem represents the state of the art.

Theorem 7.6. *For* $1 < p < 2$ *and every* $\varepsilon > 0$ *we have*

$$\ell_{(mp')'-\varepsilon,\infty} \subset \operatorname{mon} \mathcal{P}({}^m\ell_p) \subset \ell_{(mp')',\infty}\,, \tag{7 · C}$$

and for $2 \leq p < \infty$

$$\ell_{\frac{2m}{m-1},\infty} \cdot \ell_p \subset \operatorname{mon} \mathcal{P}({}^m\ell_p) \subset \ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1},\infty}\,. \tag{7 · D}$$

The upper inclusion of (7 · C) is due to Defant, Maestre, and Prengel [32] and the lower inclusion yields an partial solution of a conjecture in [32]. In what follows we want to give a proof of this theorem. We begin by proving the upper inclusions in (7 · C) and (7 · D).

Proof of the upper inclusions. Let $1 < p < \infty$, set $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and fix a sequence $x \in \operatorname{mon} \mathcal{P}({}^m\ell_p)$. By definition, $\sum_\alpha |c_\alpha(P)\, x^\alpha| < \infty$ for every $P \in \mathcal{P}({}^m\ell_p)$. By a closed graph argument, there exists a constant $\tilde{c} \geq 1$ such that

$$\sum_\alpha |c_\alpha(P)\, x^\alpha| \leq \tilde{c}\, \|P\|_{\mathrm{B}_{\ell_p}}$$

for every $P \in \mathcal{P}({}^m\ell_p)$.

By Proposition 7.2, it suffices to show that the decreasing rearrangement x^* of x is contained in the right-hand side of $(7\cdot\mathrm{C})$ respectively $(7\cdot\mathrm{D})$ as the sequence spaces are clearly symmetric. We may therefore assume $x = x^* \geq 0$ in the following. By Corollary 3.24 (applied with $q := \min\{p,2\}$) then for any $n \in \mathbb{N}$

$$\Big(\sum_{k=1}^{n} x_k\Big)^m \leq c\,(n\log m)^\sigma m^{m\sigma} \left\|\mathrm{id} : \ell_p^n \hookrightarrow \ell_{\min\{p,2\}}^n\right\|^m = c(m)\, n^\sigma n^{\left(\frac{1}{2}-\frac{1}{\max\{p,2\}}\right)m} \tag{7$\cdot$E}$$

with $c(m) \geq 1$ not depending on n. Taking the m^{th} root and diving by n we obtain

$$\frac{1}{n}\sum_{k=1}^{n} x_k \leq c(m)^{\frac{1}{m}}\, n^{\frac{\sigma}{m}-\frac{1}{2}-\frac{1}{\max\{p,2\}}} = c(m)^{\frac{1}{m}}\, n^{-\frac{1}{q_m}} \tag{7$\cdot$F}$$

with $-\frac{1}{q_m} := \frac{\sigma}{m} - \frac{1}{2} - \frac{1}{\max\{p,2\}}$. Since

$$\sup_{n\in\mathbb{N}} n^{\frac{1}{q_m}}\, x_n \leq \sup_{n\in\mathbb{N}} n^{\frac{1}{q_m}} \cdot \frac{1}{n}\sum_{k=1}^{n} x_k \leq c(m)^{\frac{1}{m}}\,,$$

we get $x \in \ell_{q_m,\infty}$. We leave it to the reader to verify that $q_m = (mp')'$ for $p \leq 2$ and $q_m = \left(\frac{m-1}{2m} + \frac{1}{p}\right)^{-1}$ for $p \geq 2$. $\square$

Proof of the lower inclusion of $(7\cdot\mathrm{D})$. We prove a more general result: Let X be any Banach sequence space. Then B_X is a Reinhardt domain and ∞–exhaustible with $\mathrm{B}_X = \mathrm{B}_X \cdot \mathrm{B}_{\ell_\infty}$. Thus

$$\left[\mathcal{P}({}^m X)\right]_\infty \subset \mathcal{P}({}^m\ell_\infty)\,.$$

From Theorem 7.5 we obtain by the natural extension of a bounded polynomial on B_{ℓ_∞} to a continuous polynomial on ℓ_∞

$$X \cdot \ell_{\frac{2m}{m-1},\infty} = \mathrm{B}_X \cdot \mathrm{mon}\,\mathcal{P}({}^m\ell_\infty) \subset \mathrm{B}_X \cdot \mathrm{mon}\left[\mathcal{P}({}^m X)\right]_\infty \subset \mathrm{mon}\,\mathcal{P}({}^m X)\,. \qquad \square$$

A careful analysis of the preceding two proofs reveal that we obtain a precise characterization in the case of Lorentz spaces:

Corollary 7.7. *Let* $2 \leq p \leq \infty$. *Then*

$$\mathrm{mon}\,\mathcal{P}({}^m\ell_{p,\infty}) = \ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1},\infty}\,.$$

Proof. For the upper inclusion it suffices to notice that $(7 \cdot \mathrm{E})$ remains true if we replace ℓ_p by $\ell_{p,\infty}$. Indeed,

$$\big\|\mathrm{id} : \ell_{p,\infty}^n \hookrightarrow \ell_2^n\big\| = \sup\Big\{ \Big(\sum_{k=1}^{n} |x_k|^2\Big)^{\frac{1}{2}} \,\Big|\, x \in \mathbb{C}^n,\ \sup_{k=1,\dots,n} k^{\frac{1}{p}} x_k^* \le 1 \Big\} \le \frac{\pi}{\sqrt{6}}\, n^{\frac{1}{2}-\frac{1}{p}} \,.$$

In the proof of the lower inclusion we use Theorem 7.5 and obtain (with $X = \ell_{p,\infty}$)

$$\ell_{p,\infty} \cdot \ell_{\frac{2m}{m-1},\infty} \subset \operatorname{mon} \mathcal{P}({}^m \ell_{p,\infty}) \,.$$

Let now $x \in \ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1},\infty}$. By Proposition 7.2, we may assume $x = x^*$ and hence have

$$\infty > \sup_k x_k k^{\frac{m-1}{2m}+\frac{1}{p}} = \sup_k \,(x_k k^{\frac{1}{p}})\, k^{\frac{m-1}{2m}} \,.$$

Therefore, we have $(x_k)_k = (x_k k^{\frac{1}{p}} k^{-\frac{1}{p}})_k \in \ell_{p,\infty} \cdot \ell_{\frac{2m}{m-1},\infty}$, since $(k^{-\frac{1}{p}})_k \in \ell_{p,\infty}$. □

Theorem 7.8. *Let* $1 \le p \le 2$. *Let* $\mathfrak{p}$ *denote the sequence of primes and define for* $\varepsilon > \frac{1}{p}$ *the sequence* $(\xi_k)_k$ *by*

$$\xi_k := \mathfrak{p}_k^{\frac{m-1}{m}\left(1-\frac{1}{p}\right)} \left(\log(\mathfrak{p}_k + \mathrm{e}^2)\right)^{\varepsilon} .$$

Then

$$\tfrac{1}{\xi} \cdot \ell_p \subset \operatorname{mon} \mathcal{P}({}^m \ell_p) \,.$$

For the proof of this theorem we use the following well-known result of Landau [44]. For a proof see §56 of [44] or Chaper II.6 of [57].

Lemma 7.9 (cf. §56 of [44])**.** *Let* $m \in \mathbb{N}$ *and let* $\mathfrak{p}$ *denote the sequence of primes. Recall the definition*

$$J(x,m) := \big\{ \boldsymbol{j} \in \mathcal{J}(\infty,m) \,\big|\, \mathfrak{p}_{\boldsymbol{j}} \le x \big\} \,.$$

Then

$$\big|J(x,m)\big| \sim \frac{x}{\log x} \, \frac{(\log\log x)^{m-1}}{(m-1)!}$$

asymptotically for x *to infinity.*

Therefore, by Lemma 6.2, for the reduced index set $J(x,m)^*$ with $x := \mathrm{e}^N$

$$\big|J(\mathrm{e}^N,m)^*\big| \le c(m)\,\frac{\mathrm{e}^{N\frac{m-1}{m}}}{N}\,\big(\log N\big)^{m-1} \tag{7·G}$$

with a constant $c(m) \ge 1$ independent of N.

Proof of Theorem 7.8. Notice at first that for any index $\boldsymbol{j} \in J(x,m)$

$$\prod_{k=1}^m \log\big(p_{j_k}+\mathrm{e}^2\big) \ge \sum_{k=1}^m \log\big(p_{j_k}+\mathrm{e}^2\big) = \log\bigg(\prod_{k=1}^m \big(p_{j_k}+\mathrm{e}^2\big)\bigg) \ge \log\big(p_{\boldsymbol{j}}+\mathrm{e}^2\big)\,;$$

therefore,

$$\xi_{\boldsymbol{j}} = p_{\boldsymbol{j}}^{\frac{m-1}{m}\left(1-\frac{1}{p}\right)}\,\Big(\log\big(p_{j_1}+\mathrm{e}^2\big)\cdots\log\big(p_{j_m}+\mathrm{e}^2\big)\Big)^{\varepsilon} \ge q_{\boldsymbol{j}}^{\frac{m-1}{m}\left(1-\frac{1}{p}\right)}\,\Big(\log\big(p_{\boldsymbol{j}}+\mathrm{e}^2\big)\Big)^{\varepsilon}.$$

For $x \in \ell_p$ and every $P \in \mathcal{P}({}^m\ell_p)$ we have

$$\begin{aligned}
&\sum_{\boldsymbol{j}\in J(\infty,m)} \big|c_\alpha(P)\,\xi_{\boldsymbol{j}}^{-1}x_{\boldsymbol{j}}\big| \\
&\qquad= \sum_{N=1}^\infty \sum_{\substack{\boldsymbol{j}\in J(\mathrm{e}^N,m)\\ \mathrm{e}^{N-1}<p_{\boldsymbol{j}}}} \big|c_\alpha(P)\,\xi_{\boldsymbol{j}}^{-1}x_{\boldsymbol{j}}\big| \\
&\qquad\le \sum_{N=1}^\infty \mathrm{e}^{-N\frac{m-1}{m}\left(1-\frac{1}{p}\right)}\,N^{-\varepsilon} \sum_{\boldsymbol{j}\in J(\mathrm{e}^N,m)} \big|c_\alpha(P)\,x_{\boldsymbol{j}}\big|\,.
\end{aligned}$$

By Theorem 4.1 and (7 · G),

$$\begin{aligned}
&\le \sum_{N=1}^\infty \mathrm{e}^{-N\frac{m-1}{m}\left(1-\frac{1}{p}\right)}\,N^{-\varepsilon}\,\big|J(\mathrm{e}^N,m)^*\big|^{1-\frac{1}{p}}\,\|P\|_{\mathrm{B}_{\ell_p}}\,\|x\|_{\ell_p}^m \\
&\le \sum_{N=1}^\infty \mathrm{e}^{-N\frac{m-1}{m}\left(1-\frac{1}{p}\right)}\,N^{-\varepsilon}\left(\frac{\mathrm{e}^{N\frac{m-1}{m}}}{N}\big(\log N\big)^m\right)^{1-\frac{1}{p}}\,\|P\|_{\mathrm{B}_{\ell_p}}\,\|x\|_{\ell_p}^m \\
&= \sum_{N=1}^\infty \frac{\big(\log N\big)^{m\left(1-\frac{1}{p}\right)}}{N^{1-\frac{1}{p}+\varepsilon}}\,\|P\|_{\mathrm{B}_{\ell_p}}\,\|x\|_{\ell_p}^m \\
&< \infty\,.
\end{aligned}$$

□

Proof of the lower inclusion of (7 · C). Fix any $\varepsilon > 0$ and choose $\delta > 0$ such that $\frac{1}{(mp')'-\varepsilon} = \frac{1}{(mp')'} + \delta$. Moreover, let $x \in \ell_{(mp')'-\varepsilon,\infty}$. By Proposition 7.2 we may assume that x is positive and non-increasing, i.e. $x = x^*$.

By definition, there exists a constant $c > 0$ such that $x_k\, k^{\frac{1}{(mp')'}+\delta} \leq c$ for every $k \in \mathbb{N}$. Hence,

$$x_k\, k^{\frac{m-1}{m}\left(1-\frac{1}{p}\right)+\frac{\delta}{2}} = x_k\, k^{1-\frac{1}{m}\left(1-\frac{1}{p}\right)-\frac{1}{p}+\frac{\delta}{2}} = x_k\, k^{\frac{1}{(mp')'}+\delta-\frac{1}{p}-\frac{\delta}{2}} \leq c\, k^{-\frac{1}{p}-\frac{\delta}{2}}$$

and thus

$$\left(x_k\, k^{\frac{m-1}{m}\left(1-\frac{1}{p}\right)+\frac{\delta}{2}}\right)_k \in \ell_p\,,$$

which implies $x \in \frac{1}{\xi} \cdot \ell_p$. Theorem 7.8 concludes the argument. $\square$

7.3. Holomorphic functions

We now turn to the case of bounded holomorphic functions on the unit ball of ℓ_p, that is $\mathcal{F}(R) = H_\infty(\mathrm{B}_{\ell_p})$. As in the polynomial case, various results are already known: In the extreme cases $p = 1$ and $p = \infty$ we have

$$\operatorname{mon} H_\infty(\mathrm{B}_{\ell_1}) = \mathrm{B}_{\ell_1} \tag{7 · H}$$

and

$$B \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty}) \subset \overline{B} \tag{7 · I}$$

where

$$B := \left\{x \in \mathrm{B}_{\ell_\infty} \;\middle|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |x_k^*|^2 < 1\right\}$$

and

$$\overline{B} := \left\{x \in \mathrm{B}_{\ell_\infty} \;\middle|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |x_k^*|^2 \leq 1\right\}.$$

The characterization (7 · H) is due to Lempert (see e.g. [45] or [32]), whereas the characterization (7 · I) is proven by Bayart, Defant, Frerick, Maestre, and Sevilla-Peris [9].

In [32] it was furthermore shown that for $1 < p < \infty$, q defined by $\frac{1}{q} = \frac{1}{2} + \frac{1}{\max\{p,2\}}$ and every $\varepsilon > 0$

$$\mathrm{B}_{\ell_p} \cap \ell_q \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) \subset \mathrm{B}_{\ell_p} \cap \ell_{q+\varepsilon}\,. \tag{7·J}$$

In the following we improve the lower inclusion and show in particular that $\varepsilon = 0$ is not possible. More precisely, we give necessary and sufficient conditions on $\alpha, \beta \in [0, \infty)$ so that

$$\left(\frac{1}{k^\alpha(\log(k+2))^\beta}\right)_k \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})\,. \tag{7·K}$$

Note that, by (7 · I), a sufficient condition in the case $p = \infty$ is given by $\alpha \geq \frac{1}{2}$ and $\beta > 0$. However, we do not know whether $\alpha = \frac{1}{2}$ and $\beta = 0$ is possible in this situation. Moreover, by (7 · H),

$$\left(\frac{1}{k(\log(k+2))^\beta}\right)_k \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_1})$$

if and only if $\beta > 1$. The following theorem collects results for the remaining cases:

Theorem 7.10. *Let $1 \leq p \leq \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$.*

(i) If $1 \leq p \leq 2$, then

$$\left(\frac{1}{k^\sigma(\log(k+2))^{\theta\sigma}}\right)_k \cdot \mathrm{B}_{\ell_p} \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$$

for every $\theta > \frac{1}{2}$. In particular, (7 · K) holds true for $\alpha = \sigma + \frac{1}{p} = 1$ and any $\beta > \frac{1}{2}\left(1 + \frac{1}{p}\right)$.

(ii) If $2 \leq p \leq \infty$, then

$$\left(\frac{1}{k^\sigma(\log(k+2))^{\theta\sigma}}\right)_k \cdot \mathrm{B}_{\ell_p} \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$$

for every $\theta > 0$. In particular, (7 · K) holds true for $\alpha = \sigma + \frac{1}{p} = \frac{1}{2} + \frac{1}{p}$ and any $\beta > \frac{1}{p}$.

(iii) If

$$\left(\frac{1}{k^{\sigma+\frac{1}{p}}(\log(k+2))^\beta}\right)_k \in \operatorname{mon} H_\infty(B_{\ell_p})\,,$$

then $\beta \geq \frac{1}{p}$.

Proof. We start with the proof of (i). Let $1 \le p \le 2$ and $\theta > \frac{1}{2}$. Recall that the sequence q was defined by $q_k := k\big(\log(k+2)\big)^{\theta}$.

By Theorem 6.4 for any $f \in H_\infty(\mathrm{B}_{\ell_p})$ and $u \in \mathrm{B}_{\ell_p}$

$$
\begin{aligned}
&\sum_{\boldsymbol{j}} |c_{\boldsymbol{j}}(f)\,(q^{-\sigma}u)_{\boldsymbol{j}}| \\
&\qquad = \sum_{N=1}^{\infty} \sum_{\substack{\boldsymbol{j}\in J(\mathrm{e}^N)\\ \mathrm{e}^{N-1}<q_{\boldsymbol{j}}}} \frac{1}{q_{\boldsymbol{j}}^{\sigma}}\,|c_{\boldsymbol{j}}(f)\,u_{\boldsymbol{j}}| \\
&\qquad \le \sum_{N=1}^{\infty} \frac{1}{e^{(N-1)\sigma}} \sum_{\boldsymbol{j}\in J(\mathrm{e}^N)} |c_{\boldsymbol{j}}(f)\,u_{\boldsymbol{j}}| \\
&\qquad \le \sum_{N=1}^{\infty} \frac{1}{e^{(N-1)\sigma}}\,\mathrm{e}^{N\sigma} \exp\Big(\big(-2\sigma\sqrt{\theta-\tfrac{1}{2}}+o(1)\big)\sqrt{N\log N}\Big)\,\|f\|_{\mathrm{B}_{\ell_p}} \\
&\qquad < \infty\,.
\end{aligned}
$$

Hence, $q^{-\sigma}\cdot u \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$ for every $u \in \mathrm{B}_{\ell_p}$, which had to be demonstrated.

We proceed to prove (ii). Let $2 \le p \le \infty$ and recall that $\mathrm{B}_{\ell_p} = \mathrm{B}_{\ell_p}\cdot \mathrm{B}_{\ell_\infty}$. By Theorem 7.5,

$$
\operatorname{mon}\big[H_\infty(\mathrm{B}_{\ell_p})\big]_\infty \cdot \mathrm{B}_{\ell_p} \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})\,.
$$

We see at once that $\big[H_\infty(\mathrm{B}_{\ell_p})\big]_\infty = H_\infty(\mathrm{B}_{\ell_\infty})$. Therefore, it suffices to check that

$$
(\xi_k)_k = \left(\frac{1}{k^{\frac{1}{2}}(\log(k+2))^{\theta}}\right)_k \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty})
$$

for every $\theta > 0$. We have

$$
\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k(\log(k+2))^{2\theta}} \le \frac{1}{\log n}\left(\int_3^n \frac{1}{t(\log t)^{2\theta}}\,\mathrm{d}t + c\right) \le \frac{(\log n)^{1-2\theta}+c}{\log n}
$$

with a universal constant $c > 0$; therefore, by $(7\cdot\mathrm{I})$, $\xi \in B \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_\infty})$.

Finally, we get to (iii). In the case $p = 1$ the claim follows directly from the already known result $(7\cdot\mathrm{H})$. The remaining cases $1 < p \le 2$ and $2 \le p < \infty$ will be proved separately. In the following denote by x the sequence in question (i.e. $x_k = k^{-\sigma-\frac{1}{p}}\big(\log(k+2)\big)^{-\beta}$) and assume $x \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$.

Let $1 < p \leq 2$. By a closed graph argument we find a constant $\tilde{c} \geq 1$ such that for every $f \in H_\infty(\mathrm{B}_{\ell_p})$

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)\, x^\alpha| \leq \tilde{c}\, \|f\|_{\mathrm{B}_{\ell_p}} .$$

From Corollary 3.24, we obtain a constant $c \geq 1$ such that for any $n, m \in \mathbb{N}$

$$\Big(\sum_{k=1}^{n} |x_k| \Big)^m \leq c\, (n \log m)^{1-\frac{1}{p}}\, m^{m(1-\frac{1}{p})} .$$

Taking the m^{th} root, we get

$$\sum_{k=1}^{n} \frac{1}{k(\log(k+2))^\beta} \leq c^{\frac{1}{m}}\, (n \log m)^{\frac{1}{m}(1-\frac{1}{p})}\, m^{1-\frac{1}{p}}$$

for every $n, m \in \mathbb{N}$ with a universal constant $c \geq 1$. The left-hand side of this equation is now asymptotically equivalent to $(\log n)^{1-\beta}$ and with $m := \lfloor \log n \rfloor$ the right-hand side is asymptotically equivalent to $(\log n)^{1-\frac{1}{p}}$ as $n \to \infty$. Hence, $\beta \geq \frac{1}{p}$.

Now, let $p \geq 2$. Define $\xi := \big(k^{-\frac{1}{q}} (\log(k+2))^{-\frac{1}{q}-\varepsilon} \big)_k$ for some $\varepsilon > 0$ where q is determined by $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Consider $f \in H_\infty(\mathrm{B}_{\ell_2})$ and set $g := f \circ D_\xi$. By Hölder's inequality, D_ξ defines a bounded operator $\ell_p \to \ell_2$. Therefore, $g \in H_\infty(\mathrm{B}_{\ell_p})$ and thus

$$\begin{aligned} \sum_{\boldsymbol{j}} |c_{\boldsymbol{j}}(f)| \frac{1}{j_1 (\log(j_1+2))^{\frac{1}{q}+\beta+\varepsilon}} \cdots \frac{1}{j_m (\log(j_m+2))^{\frac{1}{q}+\beta+\varepsilon}} \\ = \sum_{\boldsymbol{j}} \big| \big(c_{\boldsymbol{j}}(f)\, \xi_{\boldsymbol{j}} \big)\, x_{\boldsymbol{j}} \big| = \sum_{\boldsymbol{j}} \big| c_{\boldsymbol{j}}(g)\, x_{\boldsymbol{j}} \big| < \infty \end{aligned}$$

as we assumed $x \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$. Hence, we have

$$\Big(k \big(\log(k+2) \big)^{\frac{1}{q}+\beta+\varepsilon} \Big)_k \in \operatorname{mon} H_\infty(B_{\ell_2}) .$$

From what was already proven (the case $p = 2$), we obtain that $\frac{1}{q} + \beta + \varepsilon \geq \frac{1}{2}$ and thus $\beta + \varepsilon \geq \frac{1}{p}$ for every $\varepsilon > 0$. $\square$

We are finally able to give an answer to our previously stated question: The upper inclusion $(7 \cdot \mathrm{J})$ holds *not* true for $\varepsilon = 0$.

Theorem 7.11. *Let $1 < p < \infty$ and set $\frac{1}{q} := \frac{1}{2} + \frac{1}{\max\{p,2\}}$. Then*

$$\mathrm{B}_{\ell_p} \cap \ell_q \neq \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) .$$

Proof. Assume equality and let $\mathfrak{q} := \big(k\log(k+2)\big)_k$. By Theorem 7.10, this implies that the diagonal operator $\ell_p \to \ell_q$ induced by the sequence $\mathfrak{q}^{-\sigma}$ where $\sigma := 1 - \frac{1}{\min\{p,2\}}$ is well-defined and, by a closed graph argument, bounded. Hence,

$$\Big(\sum_{k=1}^{\infty} \big|\mathfrak{q}_k^{-\sigma}\big|^r\Big)^{\frac{1}{r}} = \sup_{x\in \mathrm{B}_{\ell_p}} \Big(\sum_{k=1}^{\infty} \big|x_k\, \mathfrak{q}_k^{-\sigma}\big|^q\Big)^{\frac{1}{q}} = \|D_{\mathfrak{q}^{-\sigma}}\| < \infty$$

where $\frac{1}{q} := \frac{1}{p} + \frac{1}{r}$. Therefore, we have $\mathfrak{q}^{-\sigma} \in \ell_r$, but

$$\sum_{k=1}^{\infty} \mathfrak{q}_k^{-\sigma r} = \sum_{k=1}^{\infty} \frac{1}{k\log(k+2)} = \infty\,,$$

a contradiction. □

Theorem 7.12. *Let $1 \le p \le \infty$, set $\sigma := 1 - \frac{1}{\min\{p,2\}}$, and let $\mathfrak{p}$ denote the sequence of primes. Then*

$$\mathfrak{p}^{-\sigma} \cdot \mathrm{B}_{\ell_p} \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$$

and the exponent σ is optimal.

Proof. We proceed analogously to the proof of Theorem 7.10, (i). By Theorem 6.10, for any $f \in H_\infty(\mathrm{B}_{\ell_p})$ and every $u \in \mathrm{B}_{\ell_p}$

$$\begin{aligned}
&\sum_{\boldsymbol{j}} \big|c_{\boldsymbol{j}}(f)\,(\mathfrak{p}^{-\sigma}u)_{\boldsymbol{j}}\big| \\
&\qquad = \sum_{N=1}^{\infty} \sum_{\substack{\boldsymbol{j}\in J(\mathrm{e}^N)\\ \mathrm{e}^{N-1}<\mathfrak{p}_{\boldsymbol{j}}}} \frac{1}{\mathfrak{p}_{\boldsymbol{j}}^{\sigma}}\,|c_{\boldsymbol{j}}(f)\,u_{\boldsymbol{j}}| \\
&\qquad \le \sum_{N=1}^{\infty} \frac{1}{e^{(N-1)\sigma}} \sum_{\boldsymbol{j}\in J(\mathrm{e}^N)} |c_{\boldsymbol{j}}(f)\,u_{\boldsymbol{j}}| \\
&\qquad \le \sum_{N=1}^{\infty} \frac{1}{e^{(N-1)\sigma}}\,\mathrm{e}^{N\sigma} \exp\Big(\big(-\sqrt{2}\sigma + o(1)\big)\sqrt{N\log N}\Big)\,\|f\|_{\mathrm{B}_{\ell_p}} \\
&\qquad < \infty\,.
\end{aligned}$$

Hence, $\mathfrak{p}^{-\sigma}\cdot u \in \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$ for every $u \in \mathrm{B}_{\ell_p}$, which had to be demonstrated. □

Analogously to the result $(7\cdot\mathrm{I})$ for $p=\infty$ a plausible conjecture could be

$$B_p \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) \subset \overline{B}_p \qquad (7\cdot\mathrm{L})$$

with B_p and $\overline{B}_p$ defined by

$$B_p = \Big\{ x \in \mathrm{B}_{\ell_\infty} \;\Big|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} \big|x_k^*\big|^q < 1 \Big\}$$

$$\overline{B}_p = \Big\{ x \in \mathrm{B}_{\ell_\infty} \;\Big|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} \big|x_k^*\big|^q \leq 1 \Big\}$$

where $\frac{1}{q} := \frac{1}{2} + \frac{1}{\max\{p,2\}}$. This conjecture (at least the lower inclusion) is false. Indeed, for sufficiently small $\beta > 0$

$$\xi := \left(\frac{1}{k^{\frac{1}{p}+\sigma} (\log(k+2))^{\beta}} \right)_k \notin \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$$

by Theorem 7.10; but

$$\frac{1}{\log n} \sum_{k=1}^{n} |\xi_k^*|^q = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k(\log(k+2))^{\beta q}} \sim (\log n)^{-\beta q} \to 0$$

as $n \to \infty$.

Chapter 8.

Interfaces with DIRICHLET series

The investigations of this thesis are closely linked to the theory of DIRICHLET series as we already mentioned in our introduction. In this chapter we want to point out this connection in more detail. An ordinary DIRICHLET series is a series of the form

$$D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$$

with complex coefficients $(a_n)_n$ and a complex variable s. Such a series is conditional, uniform, and absolute convergent on half-planes

$$[\mathrm{Re} > \sigma] := \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \sigma \} \,.$$

For a DIRICHLET series D, we define the abscissa of conditional convergence $\sigma_c(D)$ as the infimum over all $\sigma \in \mathbb{R}$ such that D converges conditionally on $[\mathrm{Re} > \sigma]$. The abscissae of uniform and absolute convergence are defined analogously and denoted by $\sigma_a(D)$ and $\sigma_u(D)$ respectively. Clearly we have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$ for any DIRICHLET series D.

On its half-plane of uniform convergence any DIRICHLET series D converges to a holomorphic function $f : [\mathrm{Re} > \sigma_u(D)] \to \mathbb{C}$. By $\sigma_b(D)$ we want to denote the abscissa of boundedness, which is defined as the infimum over all $\sigma \in \mathbb{R}$ such that f can be extended to a bounded holomorphic function on $[Re > \sigma]$. An outstanding result of BOHR [18] shows that $\sigma_u(D) = \sigma_b(D)$ for any DIRICHLET series D.

By $\mathcal{H}_\infty$ we want to denote the linear space of all DIRICHLET series converging to a bounded holomorphic function on the half-plane $[\mathrm{Re} > 0]$; $\mathcal{H}_\infty$ forms a BANACH space when endowed with the supremum norm on $[\mathrm{Re} > 0]$.

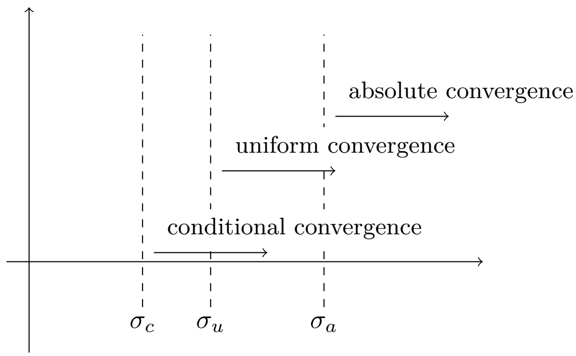

Figure 8.1.: Abscissae of convergence.

8.1. The BOHR transform — connecting DIRICHLET series and power series

In his paper [18], BOHR introduced an algebra isomorphism between the set of formal power series in infinitely many variables and the set of all ordinary DIRICHLET series. By the fundamental theorem of arithmetics we have a correspondence between the natural numbers and the set of all multi-indices $\mathbb{N}_0^{(\mathbb{N})}$: $n = p^\alpha$ where n and α determine each other uniquely.

What we call today the BOHR transform is then the algebra homomorphism

$$\mathfrak{B} : \mathfrak{P} \to \mathfrak{D}, \qquad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \mapsto \sum_{n \in \mathbb{N}} a_n n^{-s} \quad \text{where } a_{p^\alpha} := c_\alpha .$$

A natural question might be: Which spaces on the side of power series correspond to which spaces on the side of ordinary DIRICHLET series? Do we have isomorphisms or even isometries?

HEDENMALM, LINDQVIST, and SEIP [41] first proved that the BOHR transform defines an isometry between $H_\infty(\mathrm{B}_{c_0})$ and $\mathcal{H}_\infty$. For an alternative proof see the upcoming book [31].

Proposition 8.1 (cf. Section 2.2 of [41])**.** *The* BOHR *transform defines a bijective isometry*

$$\mathfrak{B} : H_\infty(\mathrm{B}_{c_0}) \to \mathcal{H}_\infty .$$

In addition to the space $\mathcal{H}_\infty$ as the image of $H_\infty(\mathrm{B}_{c_0})$ under the Bohr transform we can construct further examples of spaces of Dirichlet series. For $m \in \mathbb{N}$ define

$$\mathcal{H}_\infty^m := \mathfrak{B}\big(\mathcal{P}(^m c_0)\big)\,.$$

We easily check for $D = \sum_n n^{-s} \in \mathcal{H}_\infty^m$ that $a_n \neq 0$ only if n has exactly m prime factors (counting with multiplicity). Such Dirichlet series are called m–homogeneous.

Let $1 \leq p \leq \infty$ and denote by m the normalized product measure on the infinite dimensional polytorus $\mathbb{T}^\infty$. For a function $f \in L_p(\mathbb{T}^\infty)$ we define the Fourier coefficient $\hat{f}(\alpha)$ with $\alpha \in \mathbb{Z}^{(\mathbb{N})}$ by

$$\hat{f}(\alpha) := \int_{\mathbb{T}^n} f(\omega)\,\omega^{-\alpha}\,\mathrm{d}m(\omega) = \langle f, z^\alpha\rangle_{L_p(\mathbb{T}^\infty),L_{p'}(\mathbb{T}^\infty)}\,.$$

The so-called Hardy spaces are then defined as

$$H_p(\mathbb{T}^\infty) := \big\{ f \in L_p(\mathbb{T}^\infty) \,\big|\, \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} : \hat{f}(\alpha) = 0 \big\}\,.$$

It is well known that these are Banach spaces when endowed with the L_p norm. For $m \in \mathbb{N}$ define furthermore

$$H_p^m(\mathbb{T}^\infty) := \big\{ f \in H_p(\mathbb{T}^\infty) \,\big|\, \hat{f}(\alpha) = 0 \text{ if } |\alpha| \neq m \big\}\,.$$

From [21] we know that $H_p^m(\mathbb{T}^\infty)$ is the completion of the m–homogeneous trigonometric polynomials in $H_p(\mathbb{T}^\infty)$. By means of the Bohr transform, applied on the Fourier series expansion, we define now the Banach spaces (transferring the respective topology)

$$\mathcal{H}_p := \mathfrak{B}\big(H_p(\mathbb{T}^\infty)\big)$$

and

$$\mathcal{H}_p^m := \mathfrak{B}\big(H_p^m(\mathbb{T}^\infty)\big)\,.$$

For $X = \ell_p$ where $1 \leq p < \infty$ and $X = c_0$ define moreover

$$\mathcal{H}_\infty[X] := \mathfrak{B}\big(H_\infty(\mathrm{B}_X)\big)$$

and

$$\mathcal{H}_\infty^m[X] := \mathfrak{B}\big(H_\infty(\mathcal{P}(^m X)\big)\,.$$

With this obviously $\mathcal{H}_\infty[c_0] = \mathcal{H}_\infty$ and $\mathcal{H}_\infty^m[c_0] = \mathcal{H}_\infty^m$.

8.2. Multipliers on spaces of DIRICHLET series

Let $\mathcal{D}$ denote a set of DIRICHLET series. We call a sequence $(b_n)_n$ of complex numbers an ℓ_p–multiplier for $\mathcal{D}$ if

$$\left\|(a_n b_n)_n\right\|_{\ell_p} = \Big(\sum_{n=1}^{n} |a_n b_n|^p\Big)^{\frac{1}{p}} < \infty$$

for all $\sum_n a_n n^{-s} \in \mathcal{D}$. In [9], BAYART, DEFANT, FRERICK, MAESTRE, and SEVILLA-PERIS conduct an profound research about the set of ℓ_1–multipliers for $\mathcal{H}_\infty$, $\mathcal{H}_\infty^m$, and the spaces $\mathcal{H}_p$, $\mathcal{H}_p^m$. Using results presented in Chapter 7 and the fact that the BOHR transform defines an bijective isometry $H_\infty(\mathrm{B}_{c_0}) \to \mathcal{H}_\infty$ they find (among others):

Theorem 8.2 (cf. Theorem 4.2 in [9]). *Let $(b_n)_n$ be a completely multiplicative sequence of complex numbers, that is $b_{nm} = b_n b_m$ for any $n, m \in \mathbb{N}$. Then:*

(i) If $\left|b_{p_k}\right| < 1$ for every $k \in \mathbb{N}$ and

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(b_{p_k}^*\right)^2 < 1\,,$$

then is $(b_n)_n$ an ℓ_1–multiplier for $\mathcal{H}_\infty$.

(ii) If $(b_n)_n$ is an ℓ_1–multiplier for $\mathcal{H}_\infty$, then $\left|b_{p_k}\right| < 1$ for all $k \in \mathbb{N}$ and

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(b_{p_k}^*\right)^2 \leq 1\,.$$

In particular, $(n^{-\frac{1}{2}})_n$ is an ℓ_1–multiplier for $\mathcal{H}_\infty$ and $(n^{-\frac{1}{2}+\varepsilon})_n$ is not *an ℓ_1–multiplier for $\mathcal{H}_\infty$ for every $\varepsilon > 0$.*

In their proof they use the following evident connection, which follows directly from the definition.

Lemma 8.3. *Let X a BANACH sequence space and $b = (b_n)_n$ be a completely multiplicative sequence with $\left|b_{p_k}\right| < 1$ for every $k \in \mathbb{N}$. Then:*

(i) b is an ℓ_1–multiplier for $\mathcal{H}_\infty[X]$ if and only if $(b_{p_k})_k \in \operatorname{mon} H_\infty(\mathrm{B}_X)$.

(ii) b is an ℓ_1–multiplier for $\mathcal{H}_\infty^m[X]$ if and only if $(b_{p_k})_k \in \operatorname{mon} \mathcal{P}(^m X)$.

From Theorem 7.6, (7 · J), and the preceding lemma we get the following characterization for the ℓ_1–multipliers for $\mathcal{H}_\infty[\ell_p]$ and $\mathcal{H}_\infty^m[\ell_p]$:

Theorem 8.4. *Let $b = (b_n)_n$ be a completely multiplicative sequence of complex numbers with $\left|b_{p_k}\right| < 1$ for every $k \in \mathbb{N}$ and let $1 \le p < \infty$. Then:*

(i) b is an ℓ_1–multiplier for $\mathcal{H}_\infty[\ell_1]$ if and only if $(b_{p_k})_k \in \ell_1$.

(ii) In the case $1 < p < \infty$ and q defined by $\frac{1}{q} = \frac{1}{2} + \frac{1}{\max\{p,2\}}$:

(1) If $(b_{p_k})_k \in \mathrm{B}_{\ell_p} \cap \ell_q$, then is b is an ℓ_1–multiplier for $\mathcal{H}_\infty[\ell_p]$.

Conversely:

(2) If b is an ℓ_1–multiplier for $\mathcal{H}_\infty[\ell_p]$, then $(b_{p_k})_k \in \mathrm{B}_{\ell_p} \cap \ell_{q+\varepsilon}$ for every $\varepsilon > 0$.

Furthermore, for $m \in \mathbb{N}$ we have:

(iii) b is an ℓ_1–multiplier for $\mathcal{H}_\infty^m[\ell_1]$ if and only if $(b_{p_k})_k \in \ell_1$.

(iv) In the case that $1 < p < 2$:

(1) If $(b_{p_k})_k \in \ell_{(mp')'-\varepsilon,\infty}$ for some $\varepsilon > 0$, then is b an ℓ_1–multiplier for $\mathcal{H}_\infty^m[\ell_p]$.

Conversely:

(2) If b is an ℓ_1–multiplier for $\mathcal{H}_\infty^m[\ell_p]$, then $(b_{p_k})_k \in \ell_{(mp')',\infty}$.

(v) In the case $2 \le p < \infty$:

(1) If $(b_{p_k})_k \in \ell_{\frac{2m}{m-1},\infty} \cdot \ell_p$, then is b an ℓ_1–multiplier for $\mathcal{H}_\infty^m[\ell_p]$.

Conversely:

(2) If b is an ℓ_1–multiplier for $\mathcal{H}_\infty^m[\ell_p]$, then $(b_{p_k})_k \in \ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1},\infty}$.

The analysis of the underlying results of this summarizing theorem, in particular Theorem 6.10, brings another interesting finding to light: Let $(a_n)_n$ be a sequence of complex numbers. We verify easily that

$$\sup_{(b_n)_n \in \mathrm{B}_{\ell_p}} \sum_{n=1}^{N} |a_n b_n| = \sup_{(b_n)_n \in \mathrm{B}_{\ell_p}} \left| \sum_{n=1}^{N} a_n b_n \right| \tag{8 · A}$$

for every $N \in \mathbb{N}$. Furthermore, a quick calculation shows that for any completely multiplicative sequence $(b_n)_n$ of complex numbers

$$(b_n)_n \in \ell_p \Leftrightarrow (b_{p_k})_k \in \ell_p \text{ and } \forall k : \big|b_{p_k}\big| < 1 \,. \tag{8·B}$$

Having (8·A) and (8·B) in mind it is peculiar that there exist sequences $(a_n)_n$ of complex numbers such that

$$\sup_{\substack{(b_n)_n \text{ mult.} \\ (b_{p_k})_k \in \mathrm{B}_{\ell_p}}} \sum_{n=1}^{N} |a_n b_n| > \sup_{\substack{(b_n)_n \text{ mult.} \\ (b_{p_k})_k \in \mathrm{B}_{\ell_p}}} \Big| \sum_{n=1}^{N} a_n b_n \Big| \,.$$

Indeed, in the proof of Proposition 3.5 we constructed a polynomial $P = \sum_{k=1}^{N} c_k z^k$ such that $c_k = \pm 1$ and $|P(x)| \leq \sqrt{2N}$ for every $x \in \mathbb{T}$. With $a_{2^k} = c_k$ for $k = 1, \ldots, N$ and $a_n = 0$ otherwise the right-hand side is bounded by $\sqrt{2N}$ whereas the left-hand side evaluates to N.

From Theorem 6.10 we obtain the following curious inequality:

Theorem 8.5. *Let $(a_n)_n$ be a sequence of complex numbers. For $1 \leq p \leq \infty$ set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. For any $N \in \mathbb{N}$ then*

$$\sup_{\substack{(b_n)_n \text{ mult.} \\ (b_{p_k})_k \in \mathrm{B}_{\ell_p}}} \sum_{n=1}^{N} |a_n b_n| \leq N^{\sigma} \exp\Big(\big(-\sqrt{2}\sigma + o(1)\big) \sqrt{\log N \log\log N} \Big) \sup_{\substack{(b_n)_n \text{ mult.} \\ (b_{p_k})_k \in \mathrm{B}_{\ell_p}}} \Big| \sum_{n=1}^{N} a_n b_n \Big| \,.$$

We conclude this chapter by interpreting the results of BOHR and BOHNENBLUST and HILLE in this new fashion. The result of BOHR, namely

$$S := \sup\big\{ \sigma_a(D) - \sigma_u(D) \,\big|\, D \text{ a DIRICHLET series} \big\} \leq \tfrac{1}{2} \,,$$

is equivalent to the fact that $(n^{-\frac{1}{2}-\varepsilon})_n$ is for every $\varepsilon > 0$ an ℓ_1–multiplier for $\mathcal{H}_\infty$. Conversely, $S \geq \frac{1}{2}$, which was proved by BOHNENBLUST and HILLE, is equivalent to the fact that $(n^{-\frac{1}{2}+\varepsilon})_n$ is for any $\varepsilon > 0$ not an ℓ_1–multiplier for $\mathcal{H}_\infty$. Both statements can hence be concluded from Theorem 8.2.

BOHNENBLUST and HILLE showed in their proof of the lower bound $S \geq \frac{1}{2}$ that

$$S_m := \sup\big\{ \sigma_a(D) - \sigma_u(D) \,\big|\, D \text{ a } m\text{–homogeneous DIRICHLET series} \big\} = \tfrac{m-1}{2m} \,.$$

This is equivalent to two of the statements in Theorem 8.4: $(n^{-\frac{m-1}{2m}-\varepsilon})_n$ is for every $\varepsilon > 0$ an ℓ_1–multiplier for $\mathcal{H}_\infty^m$ and $(n^{-\frac{m-1}{2m}+\varepsilon})_n$ is for every $\varepsilon > 0$ not an ℓ_1–multiplier for $\mathcal{H}_\infty^m$.

Chapter 9.

Bohr radii

Already in 1913, Bohr was aware that the absolute convergence of Dirichlet series is closely related to the absolute convergence of power series in infinitely many variables; we introduced the Bohr transform, which relates these facts, in the previous chapter.

A reasonable strategy to tackle the convergence of power series is to consider finite dimensional sections. We define the n^{th} Bohr radius as

$$K_n := \sup\Big\{0 \le r \le 1 \,\Big|\, \forall f \in H_\infty(\mathrm{B}_{\ell_\infty^n}) : \sup_{x \in r\mathrm{B}_{\ell_\infty^n}} \sum_{\alpha \in \mathbb{N}_0^n} \big|c_\alpha(f)\, x^\alpha\big| \le \|f\|_{\mathrm{B}_{\ell_\infty^n}}\Big\}\,.$$

Bohr's power series theorem states that $K_1 = \frac{1}{3}$ and Bayart, Pellegrino, and Seoane-Sepúlveda [11] recently proved, using ideas of [27], that

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1\,.$$

We introduce a more general definition: For a Reinhardt domain $R \subset \ell_\infty$ and an index set $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ define

$$K(R;\Lambda) := \sup\Big\{0 \le r \le 1 \,\Big|\, \forall f \in H_\infty(R) : \sup_{x \in rR} \sum_{\alpha \in \Lambda} \big|c_\alpha(f)\, x^\alpha\big| \le \|f\|_R\Big\}\,.$$

We call $K(R;\Lambda)$ the Bohr radius of the Reinhardt domain R with respect to Λ. With this we have clearly $K(\mathrm{B}_{\ell_\infty^n};\mathbb{N}_0^{(\mathbb{N})}) = K_n$. In the ℓ_p case we have by results of Dineen and Timoney [38]; Boas and Khavinson [16]; Aizenberg [1]; Boas [15]; and Defant and Frerick [25, 26] the following theorem:

Theorem 9.1 (cf. Theorem 3 in BOAS [15] and Theorem 1.1 in [26])**.** *Let $1 \le p \le \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. There exists a constants $c \ge 1$ such that*

$$c^{-1}\left(\frac{\log n}{n}\right)^{\sigma} \le K\big(\mathrm{B}_{\ell_p^n}; \mathbb{N}_0^{(\mathbb{N})}\big) \le c\left(\frac{\log n}{n}\right)^{\sigma}$$

for every $n \in \mathbb{N}$.

The upper estimate is due to BOAS [15] (see also [28]); the proof uses a probabilistic argument. In [26] a proof of the lower estimate can be found which uses local BANACH space theory and symmetric tensor products. Using Theorem 4.1, or rather its corollary, we want to give a simplified proof of the lower estimate, which moreover covers a wider range of BANACH sequence spaces:

Theorem 9.2. *Let $1 \le p \le \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. For any* BANACH *sequence space with p–exhaustible unit ball there exists a constants $c \ge 1$ such that*

$$c^{-1}\left(\frac{\log n}{n}\right)^{\sigma} \le K\big(\mathrm{B}_{X_n}; \mathbb{N}_0^{(\mathbb{N})}\big)$$

for every $n \in \mathbb{N}$.

The proof is based on the following lemma:

Lemma 9.3. *Let $1 \le p \le \infty$ and set $\sigma := 1 - \frac{1}{\min\{p,2\}}$. There exists a constants $c \ge 1$ such that for any index set Λ and every $n \in \mathbb{N}$*

$$K(\mathrm{B}_{X_n}; \Lambda) \ge \frac{c}{\sup\limits_m |\Lambda(n,m)^*|^{\frac{\sigma}{m}}}$$

where $\Lambda(n,m) := \Lambda \cap \mathbf{\Lambda}(n,m)$. Moreover, we have $c \ge \frac{1}{3\mathrm{e}^2}$.

We will at first give the proof of Theorem 9.2. Afterwards, we give the proof of this lemma.

Proof of Theorem 9.2. Take the full index set $\Lambda = \mathbb{N}_0^{(\mathbb{N})}$. Obviously $\Lambda(n,m) = \mathbf{\Lambda}(n,m)$ and $\Lambda(n,m)^* = \mathbf{\Lambda}(n,m-1)$; thus, by Lemma 4.8,

$$|\Lambda(n,m)^*| = |\mathbf{\Lambda}(n,m-1)| = \binom{(m-1)+n-1}{m-1} \le \mathrm{e}^{m-1}\left(1 + \frac{n}{m-1}\right)^{m-1}.$$

Distinguishing the two cases $n \leq m-1$ and $n \geq m$ we have now

$$|\Lambda(n,m)^*| \leq \mathrm{e}^{m-1}\left(1+\frac{n}{m-1}\right)^{m-1} \leq \begin{cases} 2\mathrm{e} & \text{if } n \leq m-1 \text{ and} \\ \big(2\mathrm{e}\,\frac{n}{m-1}\big)^{m-1} & \text{if } n \geq m. \end{cases}$$

From Lemma 9.3, we hence obtain

$$K(\mathrm{B}_{X_n};\Lambda) \geq \frac{c}{\sup\limits_m |\Lambda(n,m)^*|^{\frac{\sigma}{m}}} \geq c\,\min\left\{\big(2\mathrm{e}\big)^{-\sigma}, \inf_m \big(2\mathrm{e}\,\tfrac{n}{m-1}\big)^{-\frac{m-1}{m}\sigma}\right\}.$$

It remains to find a lower bound of the infimum. Let $h_n(m) := \frac{m-1}{m}\big(\log n - \log(m-1)\big)$. By differentiation we find that h_n attains its maximum at $M = W(\frac{n}{\mathrm{e}}) + 1$ where W denotes the LAMBERT W function; that is the inverse function of $x \mapsto x\mathrm{e}^x$ on $(0,\infty)$. Therefore, with an absolute constant $c \geq 1$

$$\left(\frac{n}{m-1}\right)^{\frac{m-1}{m}} \exp h_n(m) \leq \exp h_n(M) = \left(\frac{n}{W(\frac{n}{\mathrm{e}})}\right)^{\frac{W(\frac{n}{\mathrm{e}})}{W(\frac{n}{\mathrm{e}})+1}} \leq c\,\frac{n}{\log n}$$

for any $m \in \mathbb{N}$ as $W(x) = \log x - \log\log x + o(1)$. Together, we obtain

$$K(\mathrm{B}_{X_n};\Lambda) \geq c\left(\frac{\log n}{n}\right)^{\sigma}. \qquad \square$$

We proceed with the proof of Lemma 9.3. For this purpose we need adaptations of Lemma 2.1 and Theorem 2.2 in [28]:

Proposition 9.4 (cf. Lemma 2.1 in [28])**.** *For each* BANACH *sequence space X, any set of indices Λ, and any $n, m \in \mathbb{N}$*

$$K\big(\mathrm{B}_{X_n};\Lambda(n,m)\big) = \frac{1}{\sqrt[m]{\chi_{\mathrm{mon}}\big(\mathcal{P}(^{\Lambda(n,m)}X_n)\big)}}.$$

Proof. Let $P \in \mathcal{P}(^{\Lambda(n,m)}X_n)$ and $(\theta_\alpha)_\alpha \in \mathbb{T}^{\Lambda(n,m)}$. Then with $k_n := K\big(\mathrm{B}_{X_n};\Lambda(n,m)\big)$

$$\begin{aligned}\Big\|\sum_{\alpha\in\Lambda(n,m)} \theta_\alpha c_\alpha(P)\,z^\alpha\Big\|_{\mathrm{B}_{X_n}} &\leq \sup_{x\in\mathrm{B}_{X_n}} \sum_{\alpha\in\Lambda(n,m)} |c_\alpha(P)\,x^\alpha| \\ &= \sup_{x\in k_n\cdot\mathrm{B}_{X_n}} \sum_{\alpha\in\Lambda(n,m)} |c_\alpha(P)\,\big(\tfrac{x}{k_n}\big)^\alpha| \leq \frac{1}{k_n^m}\|P\|_{\mathrm{B}_{X_n}}.\end{aligned}$$

This yields the upper estimate of $K\big(\mathrm{B}_{X_n}; \Lambda(n,m)\big)$. On the other hand,

$$\sup_{x\in \mathrm{B}_{X_n}} \sum_{\alpha\in\Lambda(n,m)} |c_\alpha(P)\, x^\alpha| = \Big\| \sum_{\alpha\in\Lambda(n,m)} |c_\alpha(P)|\, z^\alpha \Big\|_{\mathrm{B}_{X_n}} \le \chi_{\mathrm{mon}}\big(\mathcal{P}\big(^{\Lambda(n,m)}X_n\big)\big)\, \|P\|_{\mathrm{B}_{X_n}}$$

and thus

$$\sum_{\alpha\in\Lambda(n,m)} |c_\alpha(P)\, x^\alpha| \le \|P\|_{\mathrm{B}_{X_n}}$$

for any $x \in X_n$ with $\|x\|^m \le \Big(\chi_{\mathrm{mon}}\big(\mathcal{P}\big(^{\Lambda(n,m)}X_n\big)\big)\Big)^{-1}$. □

Proposition 9.5 (cf. Theorem 2.2 in [28])**.** *Let X be a* BANACH *sequence space and $R \subset X$ a* REINHARDT *domain. Then for any set of indices Λ and any $n \in \mathbb{N}$*

$$K\big(R_n; \Lambda\big) \ge \frac{1}{3} \inf_m K\big(R_n; \Lambda(n,m)\big)\,.$$

Proof. For simplicity we write $k_n := \inf_m K\big(R_n; \Lambda(n,m)\big)$ for $n \in \mathbb{N}$. Let $f \in H_\infty(R_n)$ with $\|f\|_{R_n} \le 1$, fix $x \in R_n$, and define

$$g : \big\{\xi \in \mathbb{C} \,\big|\, |\xi| \le 1\big\} \to \mathbb{C}, \quad \xi \mapsto \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha(f)\, \xi^{|\alpha|} x^\alpha\,.$$

Clearly $|g(\xi)| \le 1$ for $|\xi| \le 1$ and thus $\mathrm{Re}(1-\mathrm{e}^{i\theta} g) \ge 1$ for θ so that $\mathrm{e}^{i\theta} c_0(f) = |c_0(f)|$. By CARATHÉODORY'S inequality for any $m \ge 1$

$$\Big| \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=m}} c_\alpha(f)\, x^\alpha \Big| \le 2\,\mathrm{Re}\,\big(1 - \mathrm{e}^{i\theta} c_0(f)\big) = 2\big(1 - |c_0(f)|\big)\,.$$

Hence, for any $m \in \mathbb{N}$

$$\sum_{\alpha\in\Lambda(n,m)} \big|c_\alpha(f)\, \big(x\, \tfrac{k_n}{3}\big)^\alpha\big| \le 2\big(1 - |c_0(f)|\big)\, \frac{1}{3^m}$$

and thus

$$\sum_{\alpha\in\Lambda} |c_\alpha(f)\, x^\alpha| = |c_0(f)| + \sum_{m=1}^{\infty} \sum_{\alpha\in\Lambda(n,m)} |c_\alpha(f)\, x^\alpha| \le |c_0(f)| + 2\big(1 - |c_0(f)|\big) \sum_{m=1}^{\infty} \frac{1}{3^m} = 1$$

for any $x \in R_n$ with $\|x\|_X \le \frac{k_n}{3}$.

Altogether, we have

$$K\big(R_n; \Lambda\big) \ge \frac{k_n}{3} = \frac{1}{3} \inf_m K\big(R_n; \Lambda(n,m)\big)\,.$$ □

Proof of Lemma 9.3. We now simply have to combine the previous propositions:

$$\begin{aligned} K(\mathrm{B}_{X_n};\Lambda) &\geq \frac{1}{3}\,\inf_m K\big(\mathrm{B}_{X_n};\Lambda(n,m)\big) \\ &= \inf_m \frac{1}{3}\,\frac{1}{\sqrt[m]{\chi_{\mathrm{mon}}\big(\mathcal{P}\big({}^{\Lambda(n,m)}X_n\big)\big)}} \end{aligned}$$

which is, by Corollary 4.2,

$$\geq \inf_m \frac{1}{3\mathrm{e}^2}\,\frac{1}{|\Lambda(n,m)^*|^{\frac{\sigma}{m}}}\,. \qquad \square$$

Chapter 10.

Outlook — where to continue

To conclude this thesis we want give a short outlook and draw the readers attention to some questions remaining unanswered.

Regarding Part I: In Part I we investigated the unconditional basis constant of the monomials in certain spaces of polynomials. Although we were able to establish an abstract inequality to get upper bounds, we presented only one vigorous application; namely the case that the index set is generated by an increasing sequence.

In this setting the asymmetric reduction method displays its full strength. However, in the setting presented in Theorem 5.24 the asymmetric reduction has no advantage over the symmetric reduction method. *Are there other relevant examples of index sets, different from index sets generated by increasing sequences, for which the asymmetric reduction method shows its full potential?*

In Theorem 6.11 we tried to give an idea of the optimality of Theorem 6.4 and 6.10. However, we were only able to prove that the exponent of x is optimal; different from $\chi_{\mathrm{mon}}\big(\mathcal{P}({}^{J(x)}\ell_\infty)\big)$ with $J(x)$ generated by the sequence of primes no precise lower bound are known. *Are the estimates in Theorem 6.4 and 6.10 optimal?*

Regarding Part II: In the latter part we used Theorem 6.4 to investigate the sets of monomial convergence and ℓ_1–multipliers for sets of Dirichlet series. It seems that only index sets generated by increasing sequences yield results useful in this context. *Is there another choice of index sets more reasonable for this purpose?*

Regarding domains of monomial convergence: Theorem 7.3 yields a useful tool to prove that a certain sequence lies in the domain of monomial convergence for $\mathcal{P}({}^m\ell_\infty)$ or $H_\infty(\mathrm{B}_{\ell_\infty})$. It is perfectly reasonable to expect that this result holds (possibly with additional assumptions) also true for ℓ_p (or even any Banach sequence space) instead of ℓ_∞. However, the issues one stumbles upon trying to adapt the proof in the ℓ_∞ case seem indissoluble. Attempts to use the trick of Theorem 7.5 result in contrasting assumptions preventing success. *It remains open to prove (or disprove) Theorem 7.3 for any* Banach *sequence space instead of ℓ_∞.*

In (7 · L) we presented a plausible conjecture for an approximation of $\operatorname{mon} H_\infty(\mathrm{B}_{\ell_p})$. Unfortunately we were instantly able to reveal a flaw. If we modify the sets B_p and $\overline{B}_p$ a little we can bypass this shortcoming. We conjecture for $p \geq 2$

$$B_p \subset \operatorname{mon} H_\infty(\mathrm{B}_{\ell_p}) \subset \overline{B}_p$$

with B_p and $\overline{B}_p$ defined by

$$B_p := \Big\{x \in \mathrm{B}_{\ell_\infty} \;\Big|\; \limsup_n \frac{1}{(\log n)^{\frac{p}{p+2}}} \sum_{k=1}^n |x_k^*|^q < 1\Big\}$$
$$\overline{B}_p := \Big\{x \in \mathrm{B}_{\ell_\infty} \;\Big|\; \limsup_n \frac{1}{(\log n)^{\frac{p}{p+2}}} \sum_{k=1}^n |x_k^*|^q \leq 1\Big\}$$

where $\frac{1}{q} := \frac{1}{2} + \frac{1}{p}$.

For ℓ_1, Lempert [45] proved that the domain of monomial convergence for $H_\infty(\mathrm{B}_{\ell_1})$ coincides with the whole domain of holomorphy, i.e. $\operatorname{mon} H_\infty(\mathrm{B}_{\ell_1}) = \mathrm{B}_{\ell_1}$. It is unclear if there exist other Banach sequence spaces X for which this is the case. *Prove or disprove:*

$$\operatorname{mon} H_\infty(\mathrm{B}_X) = \mathrm{B}_X \quad \Rightarrow \quad X = \ell_1\,.$$

This is equivalent to the implication

$$\inf_n K(\mathrm{B}_{X_n}; \mathbb{N}_0^{(\mathbb{N})}) > 0 \quad \Rightarrow \quad X = \ell_1\,.$$

By Proposition 9.4, this is on the other hand equivalent to the implication

$$\exists c \geq 1 \forall m, n \in \mathbb{N} : \chi_{\mathrm{mon}}\big(\mathcal{P}({}^m X_n)\big) \leq c^m \quad \Rightarrow \quad X = \ell_1\,.$$

Regarding DIRICHLET *series:* Theorem 6.10 in the case $p = \infty$ reads in the setting of DIRICHLET series as: For any DIRICHLET polynomial $D = \sum_{n \le x} a_n n^{-s} \in \mathcal{H}_\infty$

$$\sum_{n \le x} |a_n| \le \sqrt{x} \exp\Big(\big(-\tfrac{1}{\sqrt{2}} + o(1) \big) \sqrt{\log x \log \log x} \Big) \sup_{t \in \mathbb{R}} \Big| \sum_{n \le x} a_n n^{-it} \Big| .$$

Here a reasonable question might be: *Can we obtain an analogous result for* DIRICHLET *series in* $\mathcal{H}_p$*?*

In Chapter 8, we investigated ℓ_p–multipliers of sets of DIRICHLET series and obtained results for multiplicative ℓ_1–multipliers. It remains open to investigate on one hand non-multiplicative ℓ_1–multipliers and on the other hand to identify ℓ_p–multipliers at all.

We furthermore defined the spaces $\mathcal{H}_\infty[\ell_p]$ as the image of $H_\infty(\mathrm{B}_{\ell_p})$ under the BOHR transform and proved conditions on multiplicative sequences $(b_n)_n$ to be ℓ_1–multipliers. However, we didn't investigate how the DIRICHLET series in $\mathcal{H}_\infty[\ell_p]$ and $\mathcal{H}_\infty^m[\ell_p]$ look like.

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Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

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