TAUTNESS AND KOBAYASHI HYPERBOLICITY

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INTRODUCTION

During the last fifty years there was a great and rapid development in the field of complex analysis. Nevertheless, quite lots of basic problems are not exactly studied yet. Especially, the following basic problems related to holomorphically contractible pseudodistances (which are now called 'invariant pseudodistances') are still unsolved. Given a domain $G \subset \mathbb{C}^n$ and an invariant pseudodistance d_G, one asks:

- does d_G separate points of G, i.e. is d_G a distance?
- does d_G generate the initial topology of G?
- is (G, d_G) complete whenever d_G is a distance?

Partial answers to these questions may be found in [Jar-Pfl 93] and its references.

The purpose of this thesis is to study the 'hyperbolicity' and 'completeness' of a given domain in $\mathbb{C}^n$ with respect to a holomorphically contractible function (which is from now on called 'invariant function'). Another important and related subject is the notion of taut domains, introduced by H. Wu ([Wu 67]) in 1967. It may be considered as a generalization of Montel's theory of normal families. The following relationship between the above two subjects with respect to the Kobayashi pseudodistance k and tautness (see e.g. [Roy 71], [Jar-Pfl 93]) is well known:

(a) $$k\text{-complete} \implies \text{taut} \implies k\text{-hyperbolic}$$

For more detailed historical remarks related to the above subjects we refer to ([Aba 89], pp. 148-150, [Kim-Kra 99], [Gau 99]).

Let us explain the main interest in this thesis in more detail:

- Hyperbolicity with respect to the Lempert function $\tilde{k}$ (shortly $\tilde{k}$-hyperbolicity);
- Comparison between k-, $\tilde{k}$-hyperbolicity and Brody hyperbolicity;
- Examination of hyperbolicities, tautness, and completeness in the class of special domains (e.g. Hartogs type domains, balanced domain).

The first problem, which attracts our attention, is about hyperbolicity with respect to invariant functions on a given domain. In particular, we want to discuss the relations between $\tilde{k}$-hyperbolicity which has not been treated so far and the other well-known hyperbolicities (e.g. k-hyperbolicity, Brody hyperbolicity). Obviously, one has:

(b) $$k\text{-hyperbolic} \implies \tilde{k}\text{-hyperbolic} \implies \text{Brody hyperbolic.}$$

We are interested in studying the following topics:

1) Characterization of k-, $\tilde{k}$-hyperbolicity, and Brody hyperbolicity;
2) Examination of the difference between the three hyperbolicities in (b);
3) Finding counterexamples to the converse implications in (b).

Notice that counterexamples to many problems including hyperbolicities and completeness with respect to invariant functions could be derived for Hartogs type domains (see e.g. [Sib 81], [Azu 83], [Jar-Pfl 93], etc). To seek the answer to 3), we will investigate mainly Hartogs type domains.

Related to tautness, there is an invariant function $k_G^{(2)}$ defined by

$$k_G^{(2)}(z,w) := \inf\{p(0,\alpha)+p(0,\beta) : \varphi,\psi \in \mathcal{O}(E,G),\ \alpha,\beta\in E,\\ \varphi(0)=z,\ \varphi(\alpha)=\psi(0),\ \psi(\beta)=w\},\quad z,w\in G.$$

Using it, in 1971 H. Royden got the following criterion for domains in $\mathbb{C}^n$ to be taut (Proposition 1.4.4).

[A]. ([Roy 71]) *Let $G\subset\mathbb{C}^n$ be a domain. Then G is taut iff the set $\{w\in G : k_G^{(2)}(z,w)<R\}$ is relatively compact in G for any $R>0$ and $z\in G$.*

Using **[A]**, he also showed that a given domain, finitely compact with respect to the Kobayashi distance, is taut. Unfortunately, we could not find any other paper in which Royden's criterion was intensely studied. Some of the well-known invariant functions satisfy the triangle inequality and have the product-property. In particular, the (invariant) Lempert function $\tilde{k}$ which is not a pseudodistance has the product-property. But the invariant function $k^{(2)}$ for which the inequality $k\le k^{(2)}\le\tilde{k}$ holds, does not satisfy in general the triangle inequality nor the product-property. So we guess that Royden's criterion did not attract the attention of mathematicians, because of the weak properties of the function $k^{(2)}$.

Before we go to give applications of Royden's criterion, we will present some properties of $k^{(2)}$ (cf. §1.3, §1.4):

- In general, $\tilde{k}\ne k^{(2)}\ne k$ (see [Jar-Pfl 93], Exercise 3.1).
- If G is a convex domain in $\mathbb{C}^n$ or a pseudoconvex Reinhardt domain in $(\mathbb{C}_*)^n$, then $\tilde{k}_G=k_G^{(2)}=k_G$ ([Lem 82], [Zwo 00a]).
- $k^{(2)}$ is, in general, not continuous. Moreover, the following result holds:

Proposition 1.4.2. *For a taut domain G in $\mathbb{C}^n$ the function $k_G^{(2)}$ is continuous.*

Concerning tautness the two following results are known:

[B]. ([Jar-Pfl-Zwo 00]) *Let $\Omega=\{(z,w)\in G\times\mathbb{C}^m : H(z,w)<1\}$ be a bounded pseudoconvex Hartogs domain over a domain $G\subset\mathbb{C}^n$ with m-dimensional balanced fibers, where H is upper semicontinuous in $G\times\mathbb{C}^m$ and $H(z,\lambda w)=|\lambda|H(z,w)$, $\lambda\in\mathbb{C}$, $z\in G$, $w\in\mathbb{C}^m$. Then Ω is taut iff G is taut and H is continuous on $G\times\mathbb{C}^m$.*

[C]. ([Tha-Duc 00]) *Let $\Omega=\{(z,\lambda)\in G\times\mathbb{C} : |\lambda|e^{u(z)}<1\}$ be a Hartogs domain over a domain $G\subset\mathbb{C}^n$ with 1-dimensional balanced fibers, where u is plurisubharmonic in G. Then Ω is taut iff G is taut and u is continuous on G.*

Let us discuss the difference between the proof of statements **[B]** and **[C]**. In both cases, there is no difference in proving the necessity. In case **[B]** it is not difficult to prove the sufficiency by using Montel's theorem. Observe that the Hartogs domain Ω is assumed to be bounded. On the other hand, in case **[C]**, even though $m=1$, we can not use the method used in the proof from **[B]** directly. In fact, the authors proved the sufficiency using the following result **[D]** (Proposition 3.1.8) given by D. D. Thai and N. L. Huong and a result **[E]** ((2) in Theorem 1.5.11) by N. Sibony:

[D]. ([Tha-Huo 93]) *A holomorphic fiber bundle is taut iff both the fiber and the base are taut*

[E]. ([Sib 81]) *Any domain admitting a bounded plurisubharmonic exhaustion function is taut.*

In [Tha-Huo 93], the method of proving the statement **[D]** is based on Zorn's lemma and the proof is not elementary. So we tried to find an easier way to prove the statement **[C]**. We will give a new proof using Royden's criterion. Moreover, we also can re-prove the statements **[B]**, **[C]** and **[E]** and some other known results, namely:

[F]. ([Bar 83]) *A balanced domain in $\mathbb{C}^n$ is taut iff it is bounded and the associated Minkowski function is continuous plurisubharmonic in $\mathbb{C}^n$.*

[G]. ([Ker-Ros 81]) *If a bounded domain in $\mathbb{C}^n$ is locally taut, then it is taut.*

Looking at **[G]** the following question seems to be natural:

(c) "Is any unbounded locally taut domain taut?"

A sufficient condition for an unbounded domain to be taut is given by F. Berteloot:

[H]. ([Ber 94]) *Let $G \subset \mathbb{C}^n$ be a domain and let $p_0 \in \partial G$. If G admits a local plurisubharmonic peak function at p_0, then:*

- (Localization) *there exist $s, r \in (0,1)$ such that*
$$\forall\, g \in \mathcal{O}(E,G) \ : \ g(0) \in \mathbb{B}_n(p_0,s) \implies g(rE) \subset G \cap \mathbb{B}_n(p_0,r)$$
where $\mathbb{B}_n(z,R)$ is the Euclidean open ball with center $z \in \mathbb{C}^n$ and radius $R > 0$;
- *if, moreover, there exists a sequence $(\varphi_j)_{j\geq 1} \subset \mathrm{Aut}(G)$ with $\lim_{j\to\infty} \varphi_j(z_0) = p_0$ for some $z_0 \in G$, then there exists a subsequence $(\varphi_{j_\nu})_{\nu\geq 1}$ of $(\varphi_j)_{j\geq 1}$ such that $\varphi_{j_\nu} \overset{\mathrm{K}}{\Longrightarrow} p_0$ on G as $\nu \to \infty$ (so G is k-hyperbolic). Moreover, G is taut iff it is locally taut at $p_0 \in \partial G$.*

Considering the above 'localization property of analytic disks', H. Gaussier ([Gau 99]) has given some sufficient conditions regarding 'an effective localization argument at infinity'(see the comments after Theorem 1.5.15) for a given unbounded domain to be k-hyperbolic, taut, or k-complete. For this, he introduced the new notion of a 'local plurisubharmonic (anti-)peak function at infinity'. The precise result given in his paper is the following one (Theorem 1.5.15, Proposition 3.1.11):

[I]. ([Gau 99]) *Let $G \subset \mathbb{C}^n$ be a domain. Suppose that G admits local plurisubharmonic peak and antipeak functions at infinity. Then G is k-hyperbolic. Moreover, if G is locally taut, then G is taut.*

The statement **[I]** can be considered as a generalization of **[G]**. Notice that to prove tautness in **[I]**, H. Gaussier first proved that G is k-hyperbolic and then, using hyperbolicity, he proved that G is taut.

Thus the answer to (c) is positive for some subclass of domains that are k-hyperbolic and locally taut. In Theorem 3.1.12, we give a new partial positive answer to (c). For this, we introduce a new notion of a 'local plurisubharmonic weak-peak function at infinity'; more explicitly, we say that an unbounded domain $G \subset \mathbb{C}^n$ has a *local plurisubharmonic weak-peak function φ at infinity* if there is a constant $R > 0$ such that $\varphi \in PSH(G\cap U) \cap \mathcal{C}(\bar{G} \cap U)$ and
$$\lim_{G\ni z\to\infty} \varphi(z) = 0 > \varphi(z), \quad z \in G\cap U,$$
where $U := \mathbb{C}^n \setminus \overline{\mathbb{B}_n(0,R)}$. Then our result is the following one.

Theorem 3.1.12. *Let $G \subset \mathbb{C}^n$ be a locally taut domain. Suppose that $\mathcal{O}(E,G)$ is equicontinuous with respect to the Euclidean distance $\|\cdot\|$. If G has a local plurisubharmonic weak-peak function φ at infinity, then G is taut.*

Notice that if G has a local plurisubharmonic peak function at infinity, then G has also a local plurisubharmonic weak-peak function at infinity (cf. [**I**]). Recall that k-hyperbolicity is necessary for a domain to be taut (see (a)). If $\mathcal{O}(E,G)$ is equicontinuous with respect to $\|\cdot\|$, then G is k-hyperbolic, but the converse does not hold in general. Moreover, in general, tautness of a domain $G \subset \mathbb{C}^n$ does not imply that $\mathcal{O}(E,G)$ is equicontinuous with respect to $\|\cdot\|$ ((2) in Remark 1.5.8). Hence, the converse of Theorem 3.1.12 does not hold in general.

On the other hand, for any bounded domain $G \subset \mathbb{C}^n$ the family $\mathcal{O}(E,G)$ is equicontinuous with respect to $\|\cdot\|$ ((3) in Remark 1.5.8). However, there is an unbounded domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E,G)$ is equicontinuous with respect to $\|\cdot\|$. As a simple consequence of this result, we have the following:

Example 1.5.9 & Remark 1.5.10. *For any $n \geq 2$ there exists an unbounded pseudoconvex non-taut domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E,G)$ is equicontinuous with respect to $\|\cdot\|$.*

In this point of view, Theorem 3.1.12 could be considered as a generalization of the statement [**G**] and a positive answer to (c).

Next, let us point out one more fact. H. Gaussier did not mention in his paper ([Gau 99]) whether the existence of the local plurisubharmonic peak function at infinity in statement [**I**] is necessary for the tautness. In fact, the answer is negative as the following result may show:

Example 2.2.18 & Remark 2.2.19. *For any $n \geq 4$ there exists an unbounded k-complete pseudoconvex Reinhardt domain in $\mathbb{C}^n$ which admits a local plurisubharmonic antipeak function at infinity, but which does not admit a local plurisubharmonic peak function at infinity.*

Notice that we do not know yet the existence of a domain which has a local plurisubharmonic (weak-)peak function at infinity, but which does not have a local plurisubharmonic antipeak function at infinity.

Now, let us discuss more about [**E**] and [**H**]. For this, we first recall the following result due to N. Sibony ((1) in Theorem 1.5.11):

[**J**]**.** ([Sib 81]) *If a domain $G \subset \mathbb{C}^n$ has a 'bounded plurisubharmonic function' that is $\mathcal{C}^2$ and strictly plurisubharmonic near a point $z_0 \in G$, then*

$$\exists_{C>0,\, V=V(z_0)\subset G} \;:\; S_G(z;X) \geq C\|X\|, \quad z \in V,\, X \in \mathbb{C}^n,$$

where S_G is the Sibony pseudometric for G (see e.g. Chapter 4 in [Jar-Pfl 93]).

In [Sib 81], it is mentioned that the following statement could be obtained by modifying some part of the proof of [**J**]:

[E-0]. ([Sib 81]) *Any domain admitting a bounded plurisubharmonic exhaustion function is k-hyperbolic.*

The original proof of [**E**] is based on [**E-0**]. On the other hand, the proof of the localization in [**H**] is also based on the idea of the proof of [**J**]. In Theorem 1.5.11, for the sake of completeness, we will give a detailed proof of [**E-0**]; moreover, we also present a new proof of [**E**] using Royden's criterion.

Next, let us also think about the hyperbolicities, tautness, and completeness of Hartogs type domains.

Let u, v be upper semicontinuous functions on a domain $G \subset \mathbb{C}^n$ (shortly $u, v \in \mathcal{C}^{\uparrow}(G)$) and let $h \in \mathcal{C}^{\uparrow}(\mathbb{C}^m)$ such that $h(\lambda w) = |\lambda| h(w)$, $\lambda \in \mathbb{C}$, $w \in \mathbb{C}^m$. Put

$$\Omega = \Omega_{u,h}(G) := \{(z, w) \in G \times \mathbb{C}^n : H(z, w) := h(w) e^{u(z)} < 1\},$$
$$\Sigma = \Sigma_{u,v}(G) := \{(z, \lambda) \in G \times \mathbb{C} : e^{v(z)} < |\lambda| < e^{-u(z)}\}.$$

We say that Ω (resp. Σ) is a Hartogs domain over a base G with m-dimensional balanced fibers (resp. a Hartogs-Laurent domain over a base G) (cf. [**B**], [**C**]).

From now on, we are interested in studying the differences between the notions of hyperbolicity for the above domains Ω and Σ.

For the case $m = 1$, the k-hyperbolicity of the Hartogs domain Ω was investigated in [Zai 83], [Tha-Tho 98], [Tha-Duc 00], and [Die-Tha 90]. Based on these results, we give the following property of k-hyperbolicity of Ω for the case $m \geq 1$:

Proposition 2.1.4. *Denote $D = D_h := \{w \in \mathbb{C}^m : h(w) < 1\}$. Then one has*

$$\Omega \text{ is } k\text{-hyperbolic} \iff \begin{cases} G \text{ is } k\text{-hyperbolic}, \\ D \text{ is bounded in } \mathbb{C}^m, \\ u \text{ is locally bounded on } G. \end{cases}$$

The question whether it is possible to characterize the c-hyperbolicity of Ω seems to be very difficult. For example,

[K]. ([Sib 81]) *There exists a Hartogs domain $\Omega = \Omega_{u,|\cdot|}(E) \subset \mathbb{C}^2$ over E with an 1-dimensional balanced fiber such that $u \in (\mathcal{C} \cap SH)(E)$ and Ω is k-complete, but not c-hyperbolic.*

For more examples with respect to c-hyperbolicity, see Remark 2.1.8 and Example 2.1.9. We see that there is a great difference between the notions of k-hyperbolicity and c-hyperbolicity for the Hartogs domain Ω. On the other hand, several sufficient conditions for Ω to be k-complete can be found in [Die-Tha 00]. In §3.5, we prove the following statement:

Theorem 3.5.2. *Let $\Omega \subset \mathbb{C}^{n+m}$ be a Hartogs domain over a k-complete domain $G \subset \mathbb{C}^n$. Suppose that for any $(z, w) \in \partial\Omega$ with $z \in G$, there are an open neighborhood $V = V(z) \subset G$ and a mapping $f \in \mathcal{O}(\Omega', U)$, where $\Omega' = \Omega \cap (V \times \mathbb{C}^m)$ and U is a k-complete domain, such that the sequence $(f(z_\nu, w_\nu))_{\nu \geq 1}$ is not relatively compact in U for any sequence $((z_\nu, w_\nu))_{\nu \geq 1}$ converging to the point (z, w). Then Ω is k-complete.*

Notice that Theorem 3.5.2 can be regarded as a generalized statement of the example situation in [**K**].

Additionally, there is the following result with respect to $\tilde{k}$-hyperbolicity:

Theorem 2.1.14. *Suppose that $u \in PSH(G, \mathbb{R})$ and h is a plurisubharmonic quasinorm in $\mathbb{C}^m$ with $h^{-1}(0) = \{0\}$. If one of the following conditions is satisfied:*
- *G is taut;*
- *$G \Subset \mathbb{C}^m$;*
- *G is $\tilde{k}$-hyperbolic and u is bounded from above,*

then Ω is $\tilde{k}$-hyperbolic.

Observe that any balanced domain $D = D_h \subset \mathbb{C}^m$ that is either bounded or convex has the associated Minkowski function h which is a quasinorm on $\mathbb{C}^m$ (see Lemma 1.1.6).

Using Proposition 2.1.4 and Royden's criterion we can prove the following complete characterization of Ω to be taut:

Proposition 3.1.3. *The domain Ω is taut iff the sets G and $D = D_h$ are taut and $u \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$.*

As a simple consequence, we get a sufficient condition for Σ to be taut:

Corollary 3.1.6. *If G is taut and $u, v \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, then Σ is taut.*

However, the converse of Corollary 3.1.6 does not hold in general. For more details, see Example 2.2.10 and Remark 2.2.11.

The following three results make a comparison between tautness and hyperconvexity of the domain Ω and Σ. Here, the latter notion was introduced by J.-L. Stehl'e ([Ste 73/74]). The following implication is due to N. Kerzman and J.-P. Rosay ([Ker-Ros 81]):

$$\text{bounded hyperconvex} \quad \Longrightarrow \quad \text{taut.}$$

The next result is a characterization of bounded hyperconvex Hartogs domains with m-dimensional balanced fibers ((1) of Proposition 3.4.1):

[L]. ([Jar-Pfl-Zwo 00]) *Suppose that Ω is bounded in $\mathbb{C}^{n+m}$. Then Ω is hyperconvex iff G is hyperconvex and $H \in (\mathcal{C} \cap PSH)(G \times \mathbb{C}^m, \mathbb{R})$* (cf. Proposition 3.1.3).

Moreover, in §3.4 we give a sufficient condition for Σ to be hyperconvex, namely:

Proposition 3.4.1. *Suppose that Σ is bounded in $\mathbb{C}^{n+1}$. If G is hyperconvex and $u, v \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, then Σ is hyperconvex* (cf. Corollary 3.1.6).

Let us return to discuss the notion of hyperbolicities. Since $G \times \{0\} \subset \Omega$, it is clear that

(d) if Ω is hyperbolic (resp. taut, complete), so is G.

Since $\Sigma = \Sigma_{u,v}(G) \subset \Omega_{u,|\cdot|} =: \Omega'$, we could often get hyperbolicities (tautness, completeness) of the domain Σ from the corresponding characteristics of Ω'. So it is natural to ask whether (d) remains true for a Hartogs-Laurent domain Σ, i.e.

(e) "If Σ is hyperbolic (resp. taut, complete), so is G?"

In §2.2, we are interested in studying the question (e) for hyperbolicities. In general, the answer to (e) is negative for all hyperbolicities. To give a negative answer, we first prove a sufficient condition for the Hartogs-Laurent domain Σ to be Brody hyperbolic:

Lemma 2.2.9. *Let $G \subset \mathbb{C}^n$ be a domain and let $u \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$ be nonconstant and bounded from below on G. Suppose that G is not Brody hyperbolic and $u \circ \varphi$ is not a constant for any nonconstant $\varphi \in \mathcal{O}(\mathbb{C}, G)$. Then the domain $\Sigma = \Sigma_{u,-\infty}(G)$ is Brody hyperbolic.*

Using Lemma 2.2.9 and the fact that all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains (see Theorem 1.5.21), we show that:

Example 2.2.10. *There is a pseudoconvex Reinhardt Hartogs-Laurent domain Σ which is hyperbolic, but its base G is not hyperbolic.*

However, there is a certain significant subclass of Hartogs-Laurent domains for which the answer to (e) is always positive, namely:

Theorem 2.2.15. *Let Σ be a pseudoconvex Reinhardt domain with $u \not\equiv -\infty$ and $v \not\equiv -\infty$. Then Σ is hyperbolic iff G is hyperbolic and* $\max\{u, v\} > -\infty$.

The proof of Theorem 2.2.15 is based on a result (Theorem 1.5.21) of S. Fu [Fu 94] (cf. [Zwo 99]).

On the other hand, it is also well-known that there is an example of a pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^2$ which is Brody hyperbolic but not k-hyperbolic (see [Azu 83]; [Jar-Pfl 93], Example 7.1.4). Inspired by this example, A. Kodama ([Kod 82]) showed that the notions of 'bounded' and 'k-hyperbolic' coincide for balanced domains. Moreover, J. Siciak proved the following:

[M]. ([Sic 85]) *A pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^2$ is Brody hyperbolic iff $h^{-1}(0) = \{0\}$.*

From this, we know that there is a difference between the notions of k-hyperbolicity and Brody hyperbolicity even in the class of pseudoconvex balanced domains in $\mathbb{C}^2$. From this point of view, it is very interesting to know whether there is a difference between the notions of '$\tilde{k}$-hyperbolicity', 'k-hyperbolicity', and 'Brody hyperbolicity' in the class of pseudoconvex balanced domains in $\mathbb{C}^n$. Let us check in case $n = 2$. There is a good candidate, namely, the balanced domain $D = D_h \subset \mathbb{C}^n$ due to K. Azukawa. Let us recall that D is Brody hyperbolic. However, so far it is not clear whether it is $\tilde{k}$-hyperbolic. Unfortunately, we have only a partial answer, namely (Remark 4.1.4, Example 4.1.5):

$$\tilde{k}_D((a, z), (b, w)) > 0 \quad \text{whenever} \quad a \neq b \ \text{ or } \ [\, a = b \neq 0 \ \& \ z \neq w \,].$$

For $n \geq 3$ we have the following example:

Example 4.1.7. *Let $a \in \mathbb{C} \setminus \{0\}, b, c \in \mathbb{C}$ with $b \neq c$, and take $M_1 > 0$ and $M_2 > |a|$. Then there exists a Brody hyperbolic, pseudoconvex, balanced domain D in $\mathbb{C}^3$ such that $D \subset (M_1 E) \times (M_2 E) \times \mathbb{C}$ and $\tilde{k}_D((0, a, b), (0, a, c)) = 0$.*

From this example, it turns out that for any $n \geq 3$ there is a pseudoconvex balanced domain in $\mathbb{C}^n$ which is Brody hyperbolic but not $\tilde{k}$-hyperbolic. Therefore, for $n \geq 3$, in general, Brody hyperbolicity of a balanced domain in $\mathbb{C}^n$ does not imply $\tilde{k}$-hyperbolicity.

Now we turn to study the Kobayashi completeness. In view of (a) and [**E**], the following question was suggested by P. Pflug ([Sic 85]).

(f) " Is any bounded pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^n$ with a continuous Minkowski function h complete with respect to k_D?"

For $n \geq 3$, it is known that the answer is negative; moreover, there is only one counterexample to (f) due to M. Jarnicki and P. Pflug ([Jar-Pfl 91b]). In fact, the construction of this example is based on the idea of the proof of the following result due to N. Sibony:

[**N**]. *There is a pseudoconvex non-k-complete domain $G \Subset \mathbb{C}^2$ with a $\mathcal{C}^\infty$-boundary except of one point.*

Details are published in Theorem 7.5.9 of [Jar-Pfl 93].

In Theorem 4.2.1, we give a new counterexample to (f), which is based on the method used in [Jar-Pfl 91c]. Different from [Jar-Pfl 91c], we use a new analytic chain with better properties. Also, as a consequence from both constructions of two counterexamples, we obtain the following result (cf. Corollary 3.5.3, Corollary 3.5.4, and [**K**]):

Corollary 4.2.4. *There exists a pseudoconvex Hartogs domain $\Omega = \Omega_H(G)$ over $G \subset \mathbb{C}^2$ with m-dimensional balanced fibers such that the domain G is k-complete and H is continuous on $G \times \mathbb{C}^m$, but Ω is not k-complete.*

On the other hand, using the idea of Sibony's original proof for [**N**], we present a new example as in [**N**], namely:

Theorem 4.2.5. *There is a pseudoconvex non-k-complete domain $G \Subset \mathbb{B}_2(0,2)$ given as a connected component of $\{z \in \mathbb{B}_2(0,4)\ ;\ u(z) < 1\}$, where $u \in (\mathcal{C} \cap PSH)(\mathbb{B}_2(0,4)) \cap \mathcal{C}^\infty(\mathbb{B}_2(0,4) \setminus \{0\})$ and $\operatorname{grad} u(z) \neq 0$ if $z \neq 0$, and $u(0) = 1$.*

Let us recall a result obtained by N. Q. Dieu and D. D. Thai (Proposition 3.5.7):

[**O**]. ([Die-Tha 00]) *Let $G \subset \mathbb{C}$ be a k-complete domain and let $u \in (\mathcal{C}^2 \cap SH)(G)$. Put $\Omega := \Omega_{u,|\cdot|}(G)$ and suppose that*

$$\forall_{z_0 \in G},\ \exists_{N \geq 4,\, U = U(z_0) \subset G}\ :\ \begin{cases} u \text{ is of class } \mathcal{C}^N \text{ in } U; \\ \exists_{1 \leq \alpha \leq \beta - 1 \leq N-1}\ :\ \dfrac{\partial^\beta u}{\partial z^{\beta-\alpha} \partial \bar{z}^\alpha} \neq 0 \text{ on } U. \end{cases}$$

Then for any $(z_0, \lambda_0) \in \partial\Omega$ with $z_0 \in G$, there is an open neighborhood V of (z_0, λ_0) such that (z_0, λ_0) is a local peak point for $\mathcal{O}(\Omega \cap V)$. Moreover, Ω is a k-complete hyperbolic domain.

Using [**O**], in §4.3 we prove a sufficient condition for balanced domains in $\mathbb{C}^2$ to be k-complete:

Proposition 4.3.3. *Let $D = D_h \Subset \mathbb{C}^2$ be a pseudoconvex balanced domain and denote $\Omega := \Omega_{\log h(\cdot,1),|\cdot|}(\pi_1(D'))$, where $D' := D \cap (\mathbb{C} \times \mathbb{C}_*)$ and $\pi_1(z) := z_1$. Suppose* *that h is continuous on $\mathbb{C}^2 \setminus \{(0,0)\}$ and for any $p := (a,b) \in \partial D$ with $b \neq 0$ it holds that:*

$$\exists_{N \geq 4,\, U = U(a/b) \subset \pi_1(D')} \quad : \quad \begin{cases} \log h(\cdot, 1) \textit{ is of class } \mathcal{C}^N \textit{ in } U; \\ \exists_{1 \leq \alpha \leq \beta - 1 \leq N-1} \quad : \quad \dfrac{\partial^\beta \log h(\cdot,1)}{\partial z^{\beta-\alpha} \partial \bar{z}^\alpha} \neq 0 \textit{ on } U. \end{cases}$$

Then D is locally c-finitely compact and so (globally) k-complete.

Each chapter begins with a short summary including a general outline of the material. Sometimes we use notions, well-known results, and some standard symbols without explanation. Readers may find them in the 'Appendix' and the 'List of Symbols'.

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CHAPTER 1. PRELIMINARIES

Summary. This chapter is organized as follows.

In §1.1, we introduce some elementary domains (e.g. Hartogs type domains, balanced domain, etc.) and present some of their properties. For example, in Lemma 1.1.6, we prove that the associated Minkowski function h for a bounded balanced domain $D = D_h \subset \mathbb{C}^n$ is a quasinorm on $\mathbb{C}^n$, which will be used in Chapter 2.

In §1.2, we recall the definitions of invariant functions which satisfy the so-called 'holomorphically contractible property' and treat some relationships between them.

In §1.3, we study some properties of the family $k^{(\mu)} := (k_G^{(\mu)})_{G\in\mathcal{G}}$ of invariant function introduced in §1.2.

In §1.4, we start with defining the concept of a 'taut domain' introduced by H. Wu. In Proposition 1.4.2, we present some properties of $k^{(\mu)}$ in the class of taut domains. Also, we introduce Royden's criterion for taut domains (cf. Proposition 1.4.4). By using Lemma 1.4.6 obtained from this criterion, we give another proof of the characterization of tautness for balanced domains due to T. J. Barth. Finally, we define hyperconvexity and present some related results.

In §1.5, we consider hyperbolicity and completeness with respect to invariant functions. From Proposition 1.5.3 and Remark 1.5.4, we obtain a motivation to study $\tilde{k}$-hyperbolicity of given domains, which are not investigated so far. Later, in Chapter 2 and 4, we will study $\tilde{k}$-hyperbolicity in the class of Hartogs type domains and balanced domains. In Remark 1.5.8, we observe a relationship between the k-hyperbolicity of a given domain $G \subset \mathbb{C}^n$ and the equicontinuity of the family $\mathcal{O}(E, G)$; in particular, in Example 1.5.9, we give an example of an unbounded domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E, G)$ is equicontinuous with respect to the Euclidean distance $\|\cdot\|$. As a consequence, we have that for any $n \geq 2$ there exists an unbounded pseudoconvex non-taut domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$ (Remark 1.5.10). In Theorem 1.5.11, we state a well-known sufficient condition for a domain to be taut due to N. Sibony. We present a new proof of this assertion by using Royden's criterion. In Definition 1.5.12, we introduce the concepts of 'local plurisubharmonic peak and antipeak functions at infinity' dealt by H. Gaussier. In Remark 1.5.13 and Example 1.5.14, we give some examples related to them. Using these concepts we present some sufficient conditions for k-hyperbolicity and k-completeness of a given domain, which are due to H. Gaussier. We point out that, in general, these conditions are not necessary.

§1.1. Definitions for domains and some remarks.

Let E be the unit disk in the complex plane. For domains $G_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, let us denote by $\mathcal{O}(G_1, G_2)$ the set of all holomorphic maps from G_1 to G_2 with the compact-open topology, $\mathcal{O}(G_1) := \mathcal{O}(G_1, \mathbb{C})$, and by $PSH(G_1)$ the set of all plurisubharmonic functions on G_1. We will shortly write 'psh' instead of the notion 'plurisubharmonic'.

Now we define the sets which are basic for this paper.

Definition 1.1.1. A set $S \subset \mathbb{C}^m$ is called:
- *circled* if $\lambda w \in S$ for any $w \in S$, $\lambda \in \partial E$;
- *Reinhardt* (or *m-circled*) if $\{(\lambda_1 w_1, \cdots, \lambda_m w_m) : (w_1, \cdots, w_m) \in S,\ \lambda_j \in \partial E,\ 1 \leq j \leq m\} \subset S$;

- *balanced* if $\lambda w \in S$ for any $w \in S$, $\lambda \in \bar{E}$;
- *complete Reinhardt* (or *complete m-circled*) if $\{(\lambda_1 w_1, \cdots, \lambda_n w_m) : (w_1, \cdots, w_m) \in S,\ \lambda_j \in \bar{E},\ 1 \leq j \leq m\} \subset S$.

We recall some basic properties of balanced domains and introduce some associated notions.

Remark 1.1.2. Let $D \subset \mathbb{C}^m$ be a balanced domain. We define a function $h \equiv h_D : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ by

$$h_D(w) := \inf \{\alpha > 0 : w/\alpha \in D\}, \quad w \in \mathbb{C}^m.$$

Then $D = \{w \in \mathbb{C}^m : h(z) < 1\} =: D_h$; h_D is called the *Minkowski function of* D. It is easy to check that h is *absolutely homogeneous* (i.e. $h(\lambda w) = |\lambda| h(w)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$), upper semicontinuous on $\mathbb{C}^m$; $h \equiv 0$ iff $D_h = \mathbb{C}^m$. Furthermore, if $h(\lambda_1 w_1, \cdots, \lambda_m w_m) \leq h(w)$ for $\lambda_1, \cdots, \lambda_m \in \bar{E}$, $w \in \mathbb{C}^m$, then h is continuous on $\mathbb{C}^m$ (cf. [Jak-Jar 01], Lemma 1.6.1).

The next statement is a well-known characterization for the pseudoconvexity of Reinhardt and balanced domains.

Proposition 1.1.3. *(1) Let $G \subset \mathbb{C}^n$ be a Reinhardt domain. Then G is pseudoconvex iff the following two conditions are satisfied:*
- $\log G := \{x \in \mathbb{R}^n : (e^{x_1}, \cdots, e^{x_n}) \in G\}$ *is convex,*
- *for every $1 \leq j \leq n$, if $G \cap (\mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}) \neq \emptyset$, then*

$$\{(z', \lambda z_j, z'') \in \mathbb{C}^{j-1} \times \mathbb{C} \times \mathbb{C}^{n-j} : (z', z_j, z'') \in G,\ \lambda \in \bar{E}\} \subset G.$$

(2) For a balanced domain $D \subset \mathbb{C}^m$ it holds that:

$$h_D \in PSH(\mathbb{C}^m) \iff \log h_D \in PSH(\mathbb{C}^m) \iff D \text{ is pseudoconvex.}$$

For a proof, we refer to ([Jar-Pfl 00], Proposition 1.9.19, Proposition 2.1.29).

Remark 1.1.2 explains that the notions $D = D_h$ and h establish a one-to-one correspondence between the balanced domains D in $\mathbb{C}^m$ and the Minkowski functions h on $\mathbb{C}^m$. Now we recall some relations between a balanced domain $D = D_h$ and the associated Minkowski function h.

Remark 1.1.4. Let $D = D_h \subset \mathbb{C}^m$ be a balanced domain in $\mathbb{C}^m$. Then:

(1) D is bounded iff there is a $C > 0$ such that $h(w) \geq C\|w\|$ for any $w \in \mathbb{C}^m$;

(2) D does not contain a complex line through 0 iff h is *positive definite* on $\mathbb{C}^m$, which means that $h(w) > 0$ for any $w \in (\mathbb{C}^m)_*$;

(3) D is convex iff h is *subadditive* on $\mathbb{C}^m$, i.e. $h(w_1 + w_2) \leq h(w_1) + h(w_2)$, $w_1, w_2 \in \mathbb{C}^m$;

(4) If $h \in \mathcal{C}(\mathbb{C}^m)$ (iff $\partial D = \{h = 1\}$) and $h > 0$ on $\mathbb{C}^m \setminus \{0\}$, then D is bounded;

(5) For any complete Reinhardt or convex balanced domain of $\mathbb{C}^m$ the associated Minkowski function is continuous on $\mathbb{C}^m$.

The above properties are obvious or could be found in the following references, e.g. [Jar-Pfl 93], [Azu 86], [Khr 89].

The following examples show that the converses of (4) and (5) in Remark 1.1.4 do not hold in general.

Example 1.1.5. (1) For $m \geq 2$ there exists an absolutely homogeneous psh function $\tilde{h}$ on $\mathbb{C}^m$, $\tilde{h} \not\equiv$ constant, such that $\tilde{h}^{-1}(0)$ is a dense proper subset of $\mathbb{C}^m$. It is due to J. Siciak ([Sic 85]). Fix such a $\tilde{h}$ and put $h := \tilde{h} + \|\cdot\|$ on $\mathbb{C}^m$. Then $D = D_h$ is a bounded pseudoconvex balanced domain in $\mathbb{C}^m$ with $h^{-1}(0) = \{0\}$ and $h \notin \mathcal{C}(\mathbb{C}^m)$. For more details, see Example 3.1.12 in [Jar-Pfl 93].

(2) There is a balanced domain $D = D_h$ with continuous Minkowski function h such that D is neither Reinhardt nor convex. For example:

a. (unbounded case) If we put $h(z) := |z_1^2 + z_2^2|^{1/2}$ for $z \in \mathbb{C}^2$, then $D = D_h$ is clearly unbounded, $h \in \mathcal{C}(\mathbb{C}^2)$, $h(1,i) = h(i,1) = 0$, and $h(\frac{1+i}{2}, \frac{1+i}{2}) = 1$, where $i^2 = -1$, so D is not convex;

b. (bounded case) If we put $h(w) := \frac{1}{2}(|w_1 + w_2| + |w_1^2 + w_2^2|^{1/2})$, $w \in \mathbb{C}^2$, then $h \in \mathcal{C}(\mathbb{C}^2)$, $h^{-1}(0) = \{(0,0)\}$, and so $D = D_h$ is bounded by (4) in Remark 1.1.4; moreover, it is not convex because $(1,i), (i,1) \in D$, $h(\frac{1+i}{2}, \frac{1+i}{2}) > 1$, where $i^2 = -1$.

A function $f : \mathbb{C}^m \to \mathbb{R}_{\geq 0}$ is called a *quasinorm* on $\mathbb{C}^m$ if f is absolutely homogeneous and there is a constant $C \geq 1$ such that $f(w' + w'') \leq C(f(w') + f(w''))$ for any $w', w'' \in \mathbb{C}^m$ (cf. e.g. [Din 89], p 83; [Kal 86]). For example, if a balanced domain $D = D_h \subset \mathbb{C}^m$ is convex, it follows from (3) of Remark 1.1.4 that h is a quasinorm on $\mathbb{C}^m$. Note that a bounded balanced domain is, in general, not convex (e.g. (2-b) in Example 1.1.5). But the following property is true:

Lemma 1.1.6. *If a balanced domain $D = D_h \subset \mathbb{C}^m$ is bounded, then h is a quasinorm on $\mathbb{C}^m$.*

Proof. Since 0 is an interior point of $D = D_h$, we can take an $r > 0$ such that $\mathbb{B}_m(0,r) := \{w \in \mathbb{C}^m : \|w\| < r\} \Subset D$. Also since $D \Subset \mathbb{C}^m$, there is a constant $R > 0$ such that $Rh(w) \geq \|w\|$ for any $w \in \mathbb{C}^m$. Obviously, $D \subset \mathbb{B}_m(0,R)$ implies that $R/r \geq 1$. Let $w', w'' \in \mathbb{C}^m$ be not all zero. Clearly, $h(w') \geq \frac{\|w'\|}{R}$, $h(w'') \geq \frac{\|w''\|}{R}$, and $r\frac{w'+w''}{\|w'\|+\|w''\|} \in D$. Hence,

$$1 > \frac{r}{\|w'\| + \|w''\|} h(w' + w'') = \frac{r}{R} \frac{h(w' + w'')}{\frac{\|w'\|}{R} + \frac{\|w''\|}{R}} \geq \frac{r}{R} \frac{h(w' + w'')}{h(w') + h(w'')},$$

that is, $h(w' + w'') \leq \frac{R}{r}(h(w') + h(w''))$ and we are done. $\square$

Hartogs type domains are basic objects for studying complex analysis as well as the domains presented in Definition 1.1.1. Moreover, they are very useful for finding counterexamples to certain properties. Now we define some necessary notions of Hartogs type domains to investigate 'hyperbolicity and completeness with respect to invariant functions', which will be defined in §1.4 and §1.5.

Definition 1.1.7. Let $m, n \geq 1$ and $\Omega \subset \mathbb{C}^{n+m}$ be a domain, and let $G := \pi(\Omega) \subset \mathbb{C}^n$, where

$$\mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m \ni (z,w) \stackrel{\pi}{\longmapsto} z \in \mathbb{C}^n.$$

For each $z \in G$, we define the *m-dimensional fiber* of z by $\Omega_z := \{w \in \mathbb{C}^m : (z,w) \in \Omega\}$. A domain Ω is called:

- *a Hartogs domain over G with m-circled fibers* if Ω_z is m-circled for any $z \in G$ (In that case, in general, Ω_z is not connected);

- *a Hartogs domain over G with m-dimensional balanced fibers* if Ω_z is balanced for any $z \in G$;
- *a Hartogs domain over G with complete m-circled fibers* if Ω_z is complete m-circled for any $z \in G$;
- *a Hartogs-Laurent domain over G* if Ω_z is an annulus for any $z \in G$, that is, $\Omega_z = \{w \in \mathbb{C} : r_1(z) < |w| < r_2(z)\}$ for some $0 \leq r_1(z) < r_2(z) \leq +\infty$, $z \in G$.

Like in the case of a balanced domain (Remark 1.1.2), any Hartogs domain with m-dimensional balanced fibers can be expressed by a corresponding function. The next remark establishes some relationships between the Hartogs domains of this type and the corresponding functions.

Remark 1.1.8. Let Ω be a Hartogs domain over a domain $G \subset \mathbb{C}^n$ with m-dimensional balanced fibers. We define $H(z,w) := h_{\Omega_z}(w)$ for $(z,w) \in G \times \mathbb{C}^m$, where h_{Ω_z} is the Minkowski function of Ω_z, $z \in G$. Then we have

(H1) $$\Omega = \big\{(z,w) \in G \times \mathbb{C}^m : H(z,w) < 1\big\} =: \Omega_H(G),$$

(H2) $$H(z,\lambda w) = |\lambda| H(z,w), \quad \lambda \in \mathbb{C},\ (z,w) \in G \times \mathbb{C}^m,$$

(H3) $$H \text{ is upper semicontinuous on } G \times \mathbb{C}^m.$$

Conversely, if a function $H = H_G : G \times \mathbb{C}^m \to \mathbb{R}_{\geq 0}$ satisfies the above properties (H2) and (H3), then the set $\Omega_H(G)$ defined in (H1) is a Hartogs domain over G with m-dimensional balanced fibers. Moreover, for domains $G_1, G_2 \subset \mathbb{C}^n$ one has

$$G_1 \subset G_2 \iff H_{G_1}(z,w) \geq H_{G_2}(z,w),\ \ (z,w) \in G_1 \times \mathbb{C}^m.$$

For a Hartogs domain $\Omega = \Omega_H(G)$ over a domain $G \subset \mathbb{C}^n$ the following properties are true:

(1) If Ω is convex, then G is convex and H is subadditive in the second variable. But, in general, the converse does not hold.

(2) $\Omega \equiv \Omega_H(G) \Subset \mathbb{C}^{n+m}$ iff $G \Subset \mathbb{C}^n$, $\exists_{C>0} : H(z,w) \geq C\|w\|$, $(z,w) \in G \times \mathbb{C}^m$.

(3) If $z \in G$ with $D_{h_{\Omega_z}} \Subset \mathbb{C}^m$, then

$$\exists_{C=C(z)\geq 1} \ : \ H(z, w_1 + w_2) \leq C\left[H(z,w_1) + H(z,w_2)\right], \quad w_1, w_2 \in \mathbb{C}^m.$$

In particular, if $H(z,w) := h(w)e^{u(z)}$, $(z,w) \in G \times \mathbb{C}^m$, where $u : G \to [-\infty, \infty)$ is upper semicontinuous (shortly $u \in \mathcal{C}^{\uparrow}(G)$) and h is a quasinorm on $\mathbb{C}^m$, we can take a constant $C > 0$, which satisfies the previous inequality and is independent of the choice of the point $z \in G$,

(4) If Ω is a Hartogs domain over G with complete m-circled fibers, then one has

$$H(z, \lambda_1 w_1, \cdots, \lambda_m w_m) \leq H(z,w), \quad \lambda_1, \cdots, \lambda_m \in \bar{E},\ z \in \Omega,\ w \in \mathbb{C}^m.$$

In particular, for $m = 1$ one has that Ω is a Hartogs domain over G with complete 1-circled fibers iff

$$\Omega \equiv \Omega_{u,|\cdot|}(G) := \{(z,w) \in G \times \mathbb{C}^1 : |w| < e^{-u(z)}\},$$

for some $u \in \mathcal{C}^{\uparrow}(G)$.

The above properties may be found in ([Jak-Jar 01], Remark 1.6.4) or could be easily proved (cf. Remark 1.1.4, Lemma 1.1.6).

The following lemma is a characterization of Hartogs-Laurent domains.

Lemma 1.1.9. *Let $\Omega \subset \mathbb{C}^{n+1}$, $G \subset \mathbb{C}^n$ be domains. Then Ω is a Hartogs-Laurent domain over G iff*

$$\Omega = \{(z,w) \in G \times \mathbb{C}^1 : e^{v(z)} < |w| < e^{-u(z)}\} =: \Sigma_{u,v}(G) \equiv \Sigma,$$

where $u, v \in \mathcal{C}^{\uparrow}(G)$ with $u + v < 0$ on G. Moreover,
(1) $\Sigma = \Sigma_{u,v}(G) \Subset \mathbb{C}^{n+1}$ iff $G \Subset \mathbb{C}^n$ and u is bounded from below on G;
(2) Σ is Reinhardt iff G is Reinhardt and

$$u(z) = u(|z_1|, \cdots, |z_n|), \quad v(z) = v(|z_1|, \cdots, |z_n|), \quad z \in G.$$

Proof. Let $r_1, r_2 : G \to [0, \infty]$ be functions such that $r_1 < r_2$ and $\Omega_z = \{w \in \mathbb{C} : r_1(z) < |w| < r_2(z)\}$, $z \in G$. Then it is easy to check that r_1, $1/r_2 \in \mathcal{C}^{\uparrow}(G)$, that is, $u := -\log r_2$, $v := \log r_1 \in \mathcal{C}^{\uparrow}(G)$. From which we obtain the first assertion. (1) is obvious. For the necessity of (2), assume that $u \not\equiv -\infty$ in G and there are $z \in G$ and $\theta := (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n$ such that $u(z) \neq u(z_1 e^{i\theta_1}, \cdots, z_n e^{i\theta_n}) =: u_\theta(z)$. Without loss of generality, we may assume that $u(z) < u_\theta(z)$. Choose a number $\lambda_\theta \in \mathbb{C}$ so that $\max\{e^{v(z)}, e^{-u_\theta(z)}\} < |\lambda_\theta| < e^{-u(z)}$. Then $(z, \lambda_\theta) \in \Sigma$ and because Σ is Reinhardt, one has $e^{v_\theta(z)} < |\lambda_\theta e^{ix}| < e^{-u_\theta(z)}$ for any $x \in \mathbb{R}$, which is a contradiction. By a similar way, we also see that $v(z) = v(|z_1|, \cdots, |z_n|)$, $z = (z_1, \cdots, z_n) \in G$. $\square$

We conclude this section by recalling the characterization of pseudoconvex Hartogs type domains.

Proposition 1.1.10. *Let $\Omega \subset \mathbb{C}^{n+m}$, $G \subset \mathbb{C}^n$ be domains.*
(1) A Hartogs domain $\Omega = \Omega_H(G)$ over G with m-dimensional balanced fibers is pseudoconvex iff G is pseudoconvex and $\log H \in PSH(G \times \mathbb{C}^m)$.
In particular, let $H(z,w) := h(w)e^{u(z)}$, $z \in G$, $w \in \mathbb{C}^m$, where $u \in \mathcal{C}^{\uparrow}(G)$ and $h \in \mathcal{C}^{\uparrow}(\mathbb{C}^m)$, $h \not\equiv 0$, and $h(\lambda w) = |\lambda| h(w)$, $\lambda \in \mathbb{C}$, $w \in \mathbb{C}^m$. Denote $\Omega_H(G)$ by $\Omega_{u,h}(G)$. Then

$$\Omega \equiv \Omega_{u,h}(G) \text{ is pseudoconvex} \quad \Longleftrightarrow \quad \begin{cases} G \text{ is pseudoconvex,} \\ u \in PSH(G), \\ \log h \in PSH(\mathbb{C}^m). \end{cases}$$

(2) A Hartogs-Laurent domain $\Sigma = \Sigma_{u,v}(G)$ over G, where $u, v \in \mathcal{C}^{\uparrow}(G)$, $u + v < 0$ on G, is pseudoconvex iff G is pseudoconvex and $u, v \in PSH(G)$.

For (1), we refer e.g. to Proposition 2.2.22 in [Jar-Pfl 00]. For 'if' and 'only if' in (2), see e.g. pp.130-132 in [Vla 66] and Corollary 3.1.10 in [Jar-Pfl 00], respectively.

§1.2. Definitions of invariant functions and some remarks.

Let S be a nonempty set. A function $d : S \times S \to \mathbb{R}_{\geq 0}$ is called a *pseudodistance on S* if
- $d(x,x) = 0$, $x \in S$;
- d is symmetric, i.e. $d(x,y) = d(y,x)$, $x, y \in S$;
- d satisfies the triangle inequality, i.e. $d(x,y) \leq d(x,z) + d(z,y)$, $x, y, z \in S$.

A pseudodistance d on S is called a *distance on* S if the following property is satisfied:
- for any $x, y \in S$, $d(x,y) = 0 \iff x = y$.

Now we shall recall the definitions of some invariant functions. For this, let $\mathcal{G}$ denote the set of all domains in all $\mathbb{C}^n$'s and let p be the *Poincaré* (or *hyperbolic*) distance on E, i.e.

$$p(\lambda, \zeta) := \tanh^{-1}\left(|\lambda - \zeta|/|1 - \overline{\lambda}\zeta|\right), \quad \lambda, \zeta \in E.$$

Let us recall

Classical Schwarz Lemma. *For any $f \in \mathcal{O}(E, E)$ it holds*

$$p(f(\lambda), f(\zeta)) \leq p(\lambda, \zeta), \quad \lambda, \zeta \in E.$$

Moreover, if the equality holds for some $\lambda \neq \zeta$, then f is an automorphism of E.

By considering the above phenomena, we can define the following: A family $\underline{d} := (d_G)_{G \in \mathcal{G}}$ of functions $d_G : G \times G \to \mathbb{R}_{\geq 0}$ is called *holomorphically contractible* whenever:

(a) $\underline{d}$ is *normalized*, i.e. $d_E = p$;
(b) $\underline{d}$ satisfies the *decreasing property*, i.e. for any $G, D \in \mathcal{G}$ one has

$$d_D(f(z), f(w)) \leq d_G(z, w), \quad f \in \mathcal{O}(G, D),\ z, w \in G.$$

The last property may be interpreted as a generalization of the classical Schwarz Lemma. The condition (b) implies that the family d is invariant with respect to biholomorphic mappings. Sometimes, to be short, we call $d_G \in \underline{d}$ ($G \in \mathcal{G}$) an *invariant function*. Furthermore, (b) also implies that for $G_j \in \mathcal{G}$, $z'_j, z''_j \in G_j, j \in \{1, 2\}$ one has

$$(*) \qquad d_{G_1 \times G_2}((z'_1, z'_2), (z''_1, z''_2)) \geq \max\{d_{G_1}(z'_1, z''_1),\ d_{G_2}(z'_2, z''_2)\}$$

We say that a family $\underline{d}$ of invariant functions has the *product-property* if the equality in $(*)$ always holds for any $G_j \in \mathcal{G}$, $z'_j, z''_j \in G_j$, $j \in \{1, 2\}$.

Remark 1.2.1. Let $G \in \mathcal{G}$ and $z, w \in G$. Then there is a map $\varphi \in \mathcal{O}(E, G)$ whose range contains both z and w (see e.g. [Jar-Pfl 93], Remark 3.1.1; [Din 89], p. 49). For any $f \in \mathcal{O}(G, E)$, one has $f \circ \varphi \in \mathcal{O}(E, E)$ and by the above Schwarz lemma one has

$$p(f(z), f(w)) = p((f \circ \varphi)(\lambda), (f \circ \varphi)(\zeta)) \leq p(\lambda, \zeta)$$

where $\lambda, \zeta \in E$ with $\varphi(\lambda) = z$ and $\varphi(\zeta) = w$.

From the above observation, we can define two holomorphically contractible families by considering maps from G into E and maps from E into G, respectively, as follows: For points z and w in $G \in \mathcal{G}$ we set

$$c_G(z, w) := \sup_{f \in \mathcal{O}(G, E)} p(f(z), f(w)), \qquad \tilde{k}_G(z, w) := \inf_{\substack{\varphi \in \mathcal{O}(E, G) \\ \varphi(\lambda) = z, \varphi(\zeta) = w}} p(\lambda, \zeta).$$

By Remark 1.2.1, the function $\tilde{k}_G$ is nonnegative real-valued and so is c_G. It is easily seen that $\underline{c} := (c_G)_{G\in\mathcal{G}}$ and $\underline{\tilde{k}} := (\tilde{k}_G)_{G\in\mathcal{G}}$ are holomorphically contractible systems of functions. Note that for any $G \in \mathcal{G}$ the function c_G is a pseudodistance and $\tilde{k}_G$ is symmetric but, in general, it does not satisfy the triangle inequality. Moreover, if $(d_G)_{G\in\mathcal{G}}$ is any holomorphically contractible system of functions, then

$$c_G \leq d_G \leq \tilde{k}_G, \quad G \in \mathcal{G}.$$

In this sense we says that $(c_G)_{G\in\mathcal{G}}$ and $(\tilde{k}_G)_{G\in\mathcal{G}}$ are the *smallest* and *largest* holomorphically contractible family of functions, respectively.

We are now in a position to ask whether there exists the largest holomorphically contractible family of pseudodistances. To find such a system, we want to modify the system $(\tilde{k}_G)_{G\in\mathcal{G}}$, and from which we can obtain a new holomorphically contractible family $(k_G)_{G\in\mathcal{G}}$ of pseudodistances as follows:

$$k_G := \text{ the greatest pseudodistance on } G \text{ below of } \tilde{k}_G.$$

It is easy to see that $\underline{k} := (k_G)_{G\in\mathcal{G}}$ is the largest holomorphically contractible family of pseudodistances, i.e.

$$c_G \leq d_G \leq k_G \leq \tilde{k}_G, \;\; G \in \mathcal{G}$$

where d_G is any holomorphically contractible pseudodistance on $G \in \mathcal{G}$. In particular, for $\mu \in \mathbb{N}$ and $z, w \in G \in \mathcal{G}$ we put

$$k_G^{(\mu)}(z,w) := \inf\left\{\sum_{j=1}^{\mu} \tilde{k}_G(p_{j-1}, p_j) : z = p_0,\, w = p_\mu,\, (p_j)_{j=0}^{\mu} \subset G\right\}.$$

Then

$$\tilde{k}_G = k_G^{(1)} \geq k_G^{(\mu)} \geq k_G^{(\mu+1)} \geq \lim_{\nu\to\infty} k_G^{(\nu)} =: k_G^{(\infty)} \equiv k_G.$$

Note. Since the end of nineteen century several mathematicians including H. Poincaré in France, S. Bergman in Poland, C. Carathéodory in Germany and S. Kobayashi in Japan have defined several (pseudo-)distances which play an important role in the research of holomorphic functions. In fact, the idea for constructing the invariant distances for suitable maps, was conceived already by B. Riemann in the eighteenth century in Germany. To study automorphism mappings, H. Poincaré constructed the hyperbolic distance p as a model of a non-Euclidean geometry ([Poi 1881]). After that, the pseudodistance c_G was introduced by C. Carathéodory in 1927, who was mainly interested in the class of bounded domains ([Car 27]). By considering an antipodal situation to the definition of c_G, in 1967, S. Kobayashi introduced the pseudodistance k_G in the form

$$\begin{aligned} k_G(z,w) = \inf\Bigg\{\sum_{j=0}^{\mu} p(\lambda_j, \zeta_j) :\; & \mu \in \mathbb{N},\, \{z = p_0, p_1, \cdots, p_{\mu-1}, p_\mu = w\} \subset G, \\ & ((\lambda_j, \zeta_j))_{j=1}^{\mu} \subset E \times E,\, (\varphi_j)_{j=1}^{\mu} \subset \mathcal{O}(E, G), \\ & \varphi_j(\lambda_j) = p_{j-1},\, \varphi_j(\zeta_j) = p_j, 1 \leq j \leq \mu\Bigg\}, \;\; z, w \in G. \end{aligned}$$

([Kob 67a, 67b]). Later on, in [Lem 81, 82, 84], L. Lempert studied intensively the pseudodistance k_G in convex domains. In particular, he was mainly concerned with the function $\tilde{k}_G$ defined by taking only a single disk instead of chains of discs as in the definition of k_G. He proved that both c_G and $\tilde{k}_G$ coincide on convex domains.

In this view, for $G \in \mathcal{G}$, we say that c_G, k_G, and $\tilde{k}_G$ is the *Carathéodory, Kobayashi pseudodistance*, and the *Lempert function* for G, respectively.

Also, in 1979, Harris introduced the concept of so-called *'Schwarz-Pick system'* which is a very convenient tool for the comparison between the different invariant functions. In this thesis, this system is called a 'holomorphically contractible family' as in [Jar-Pfl 93]. On the other hand, the product-property for $\underline{d} \in \{\underline{\tilde{k}}, \underline{k}\}$ is due to S. Kobayashi (see e.g. §3.7 in [Jar-Pfl 93]). In [Kob 76], firstly, he claimed the same property for $\underline{d} = \underline{c}$ without proof. This fact is proved at first by M. Jarnicki and P. Pflug in [Jar-Pfl 89].

Now we finish this section by introducing a holomorphically contractible family of functions introduced by M. Klimek ([Kli 85]). For this, let $G \in \mathcal{G}$ and define a function

$$g_G(a, z) := \sup_{u \in \mathcal{K}_G(a)} u(z), \quad a, z \in G,$$

where $\mathcal{K}_G(a) := \{u : G \to [0, 1) : \log u \in PSH(G), \exists_{C,r>0} : u(w) \leq C\|w - a\|, w \in B(a, r) \subset G\}$. If $\Omega \in \mathcal{G}$, and if $f \in \mathcal{O}(G, \Omega)$ and $u \in \mathcal{K}_\Omega(f(a))$, it is easy to check that $u \circ f \in \mathcal{K}_G(a)$, which implies that $g_\Omega(f(a), f(z)) \leq g_G(a, z)$. Recall

Schwarz Lemma for subharmonic function. *Let* $\log u \in SH(E)$*. If* $u(\lambda)/|\lambda|$ *is bounded near zero and if* $\limsup_{|\lambda|\to 1^-} u(\lambda) \leq 1$*, then* $u(\lambda) \leq |\lambda|$ *for any* $\lambda \in E$.

This gives us that $\tanh^{-1} g_E$ coincides with the Poincaré distance p on E. Therefore, $(\tanh^{-1} g_G)_{G\in\mathcal{G}}$ is a holomorphically contractible family of functions and also

$$\tanh c_G \leq g_G \leq \tanh \tilde{k}_G, \quad G \in \mathcal{G}.$$

Moreover, g_G is upper semicontinuous, but not symmetric (see e.g. Remark of Proposition 4.4.2 in [Jar-Pfl 93]). On the other hand, if $G \subset \mathbb{C}^1$, then $-\log g_G(a, \cdot)$ coincides with the *classical Green function* for G with pole at a, and in this case $g_G^2(a, \cdot)$ is of class $\mathcal{C}^2$ near a. For this reason, for any $G \in \mathcal{G}$ we say that $\log g_G$ is the *pluricomplex Green function* for G. Notice that the product-property for $(\tanh^{-1} g_G)_{G\in\mathcal{G}}$ is firstly studied by M. Jarnicki and P. Pflug ([Jar-Pfl 91a]) and is completely proved by A. Edigarian in [Edi 97].

For the sake of more information for invariant functions we refer to [Jar-Pfl 93], [Kli 91], [Din 89], [Kob 98].

§1.3. Basic properties for $k^{(\mu)}$.

In case that $\mu = 1$ or ∞, the basic properties for $k^{(\mu)}$ are well-known (see e.g. [Jar-Pfl 93]). Furthermore, in case $\mu \geq 2$, the basic properties for $k^{(\mu)}$ are almost the same as those for $k^{(1)}$ and the ideas of those proofs are essentially the same.

Let $\mu \in \mathbb{N} \cup \{\infty\}$. Since $(\tilde{k}_G)_{G\in\mathcal{G}}$ is a holomorphically contractible family of functions, so is $\underline{k}^{(\mu)} := (k_G^{(\mu)})_{G\in\mathcal{G}}$; moreover, the family $\underline{k}^{(\mu)}$ satisfies $(*)$ in §2.1. Notice that, in general, $k_G^{(\mu)} \not\equiv k_G^{(\mu+1)}$ (see. [Jar-Pfl 93], Exercises 3.1).

Let us recall the following theorem on holomorphic covering, due to S. Kobayashi (cf. [Kob 98], Theorem 3.2.8):

Theorem 1.3.1. *Let $\pi : \tilde{G} \to G$ be a holomorphic coverings and $\mu \in \mathbb{N} \cup \{\infty\}$. If $x, y \in G$ and $\tilde{x} \in \tilde{G}$ with $\pi(\tilde{x}) = x$, then*

$$k_G^{(\mu)}(x,y) = \inf_{\tilde{y}\in\tilde{G},\, \pi(\tilde{y})=y} k_{\tilde{G}}^{(\mu)}(\tilde{x},\tilde{y}).$$

In the above theorem, the infimum may, in general, not be attained. An example is due to W. Zwonek ([Zwo 98]). Note that Theorem 1.3.1 was used in [Pfl-Zwo 98], [Zwo 99], etc., as an important tool for calculating some effective formulae of invariant functions.

Remark 1.3.2. (1) Let $G \subset \mathbb{C}$ be a domain and let $\mu \in \mathbb{N}$. In view of the uniformization theorem, $k_G^{(\mu)} = k_G$. Hence $k_G^{(\mu)}$ is a pseudodistance. Moreover, $k_{\mathbb{C}} = k_{\mathbb{C}_*} = 0 \neq k_{\mathbb{C}\setminus\{0,1\}}$, but $c_{\mathbb{C}} = c_{\mathbb{C}_*} = c_{\mathbb{C}\setminus\{0,1\}} = 0$.

(2) As a deep result of L. Lempert ([Lem 81, 82, 84]), we have that $k_G^{(\mu)} = k_G$ for any convex domain $G \subset \mathbb{C}^n$ and $\mu \in \mathbb{N}$. For more details, see e.g. Chapter 8 in [Jar-Pfl 93].

(3) Recall that a domain $\omega \subset \mathbb{R}^n$ is convex iff the tube domain $\omega + i\mathbb{R}^n$ is convex, where $i^2 = -1$ (see e.g. [Kra 92], Theorem 3.5.1). So by (1) of Proposition 1.1.3, the above property (2), and Theorem 1.3.1, it is easy to check that if $G \subset (\mathbb{C}_*)^n$ is a pseudoconvex Reinhardt domain, then $k_G^{(\mu)} = k_G$ for any $\mu \in \mathbb{N}$ ([Zwo 00a], Lemma 3).

Remark 1.3.3. For a domain $G \subset \mathbb{C}^n$, one has $k_G \in \mathcal{C}(G\times G)$ and $k_G^{(\mu)} \in \mathcal{C}^{\uparrow}(G\times G)$ for $\mu \in \mathbb{N}$, but $k_G^{(\mu)}$ is, in general, not continuous for $\mu \geq 2$ (see e.g. [Jar-Pfl 93], Proposition 3.1.9, Proposition 3.1.13).

§1.4. Tautness and hyperconvexity.

The study of a normal family of holomorphic mappings between complex manifolds in the general setting of complex manifolds was studied by H. Grauert and H. Reckziegel in 1965 ([Gra-Rec 65]). The following concept of taut domain was introduced by H. Wu ([Wu 67]).

Definition 1.4.1. A domain $G \subset \mathbb{C}^n$ is called *taut* if $\mathcal{O}(E,G)$ is a *normal family*, which means that for every sequence $(f_j)_{j\geq 1} \subset \mathcal{O}(E,G)$ there is a subsequence $(f_{j_\nu})_{\nu\geq 1}$ which is either

- *normally convergent in* $\mathcal{O}(E,G)$, i.e. it converges uniformly on compact subsets to a map $f \in \mathcal{O}(E,G)$ (briefly, $f_{j_\nu} \overset{\mathrm{K}}{\Longrightarrow} f$), or
- *compactly divergent*, i.e. for every compact sets $K \subset E$, $L \subset G$, the set $f_{j_\nu}(K) \cap L$ is empty for all large enough ν.

In some literature, authors call a domain G taut when $\mathcal{O}(\Omega, G)$ is normal for every domain $\Omega \in \mathcal{G}$. In fact, this is the original definition introduced in [Wu 67]. Later T. J. Barth ([Bar 70]) proved that if a domain G is taut in the sense of Definition 1.4.1, then it is also taut in the sense of Wu.

In complex analysis, the concept of tautness is one of the most important tools. It is studied not only for itself but also as an important tool for another research problems which deal with families of functions. For more information, see e.g. ([Aba 89], pp. 148-159; [Gau 99]; [Kim-Kra 99]).

In contrast to Remark 1.3.3, there is a significant class of domains $G \in \mathcal{G}$ for which the function $k_G^{(\mu)}$ is continuous for $\mu \in \mathbb{N}$.

Proposition 1.4.2. *Let $G \subset \mathbb{C}^n$ be a taut domain. Then $k_G^{(\mu)}$ is continuous for any $\mu \in \mathbb{N}$. Moreover, for any $z, w \in G$, $\mu \in \mathbb{N}$ there exists an extremal family for $k_G^{(\mu)}(z,w)$, i.e. there are two finite sequences $(\varphi_j)_{j=1}^{\mu} \subset \mathcal{O}(E,G)$ and $(\alpha_j)_{j=1}^{\mu}$ such that $\varphi_1(0) = z$, $\varphi_j(\alpha_j) = \varphi_{j+1}(0)\,(2 \le j \le \mu - 1)$, $\varphi_\mu(\alpha_\mu) = w$, and $k_G^{(\mu)}(z,w) = \sum_{j=1}^{\mu} p(0,\alpha_j)$.*

Note that, in general, the above extremal family for $k_G^{(\mu)}(z,w)$ is not unique. For example, let $\mu \in \mathbb{N}$, $G := E^2$, and put $z := (0,0)$, $w := (a,0)$ for some $a \in E_*$. Then it follows from (2) of Remark 1.3.2 and the product property of $\tilde{k}$ that $k_G^{(\mu)}(z,w) = \tilde{k}_G(z,w) = p(0,a)$. On the other hand, if for any $g \in \mathcal{O}(E,E)$ with $g(0) = g(a) = 0$, we define $\varphi_\mu(\lambda) := (\lambda, g(\lambda))$, $\lambda \in E$, and put $\varphi_j := (0,0)$, $\alpha_j = 0$ for $1 \le j \le \mu - 1$, then $(\varphi_j)_{j=1}^{\mu}$ is an extremal family for $k_G^{(\mu)}(z,w)$.

Proof of Proposition 1.4.2. We will show the first assertion by an indirect proof. For this, let us assume the contrary. In view of Remark 1.3.3, we can take a point $(z_0, w_0) \in G \times G$ such that $k_G^{(\mu)}$ is not lower semicontinuous at (z_0, w_0). Take a sequence $((z_j, w_j))_{j \in \mathbb{N}} \subset G \times G$ such that $z_j \overset{j\to\infty}{\longrightarrow} z_0$, $w_j \overset{j\to\infty}{\longrightarrow} w_0$, and

$$k_G^{(\mu)}(z_j, w_j) \le k_G^{(\mu)}(z_0, w_0) - \epsilon =: M \in (0, \infty)$$

for a suitable $\epsilon > 0$. Then for any $j \in \mathbb{N}$ there are finite sequences $(\varphi_j^m)_{m=1}^{\mu} \subset \mathcal{O}(E,G)$, $(\alpha_j^m)_{m=1}^{\mu} \subset [0,1)$ such that $\varphi_j^1(0) = z_j$, $\varphi_j^m(\alpha_j^m) = \varphi_j^{m+1}(0)\,(2 \le m \le \mu - 1)$, $\varphi_j^{\mu}(\alpha_j^{\mu}) = w_j$, and

$$\sum_{m=1}^{\mu} p(0, \alpha_j^m) < k_G^{(\mu)}(z_j, w_j) + \frac{\epsilon}{2} \le k_G^{(\mu)}(z_0, w_0) - \frac{\epsilon}{2}.$$

In particular, since $M < \infty$ we may assume that $\alpha_j^m \overset{j\to\infty}{\longrightarrow} {}^\exists\alpha_0^m \in [0,1)$ for every $m \in \{1, \cdots, \mu\}$. Because of the tautness of G and the fact that $z_j \overset{j\to\infty}{\longrightarrow} z_0$, we can take a sequence $(\varphi_{1j}^1)_{j \in \mathbb{N}} \subset (\varphi_j^1)_{j \in \mathbb{N}}$ such that $\varphi_{1j}^1 \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi_0^1 \in \mathcal{O}(E,G)$ as $j \to \infty$. In particular,

$$\lim_{j\to\infty} \varphi_{1j}^2(0) = \lim_{j\to\infty} \varphi_{1j}^1(\alpha_{1j}^1) = \varphi_0^1(\alpha_0^1) \in G.$$

From the tautness of G, there exists a sequence $(\varphi_{2j}^2)_{j \in \mathbb{N}} \subset (\varphi_{1j}^2)_{j \in \mathbb{N}}$ such that $\varphi_{2j}^2 \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi_0^2 \in \mathcal{O}(E,G)$ as $j \to \infty$. In particular,

$$\varphi_0^1(\alpha_0^1) = \lim_{j\to\infty} \varphi_{2j}^1(\alpha_{2j}^1) = \lim_{j\to\infty} \varphi_{2j}^2(0) = \varphi_0^2(0),$$
$$\lim_{j\to\infty} \varphi_{2j}^3(0) = \lim_{j\to\infty} \varphi_{2j}^2(\alpha_{2j}^2) = \varphi_0^2(\alpha_0^2) \in G.$$

By induction we get for $m \in \{3, \cdots, \mu - 1\}$ that there exists a sequence $(\varphi^m_{mj})_{j\in\mathbb{N}} \subset (\varphi^m_{m-1\,j})_{j\in\mathbb{N}}$ such that $\varphi^m_{mj} \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi^m_0 \in \mathcal{O}(E, G)$ as $j \to \infty$; moreover,

$$\varphi^{m-1}_0(\alpha^{m-1}_0) = \lim_{j\to\infty} \varphi^{m-1}_{mj}(\alpha^{m-1}_{mj}) = \lim_{j\to\infty} \varphi^m_{mj}(0) = \varphi^m_0(0),$$
$$\lim_{j\to\infty} \varphi^{m+1}_{mj}(0) = \lim_{j\to\infty} \varphi^m_{mj}(\alpha^m_{mj}) = \varphi^m_0(\alpha^m_0) \in G.$$

Finally, we obtain a sequence $(\varphi^\mu_{\mu j})_{j\in\mathbb{N}} \subset (\varphi^\mu_{\mu-1\,j})_{j\in\mathbb{N}}$ such that $\varphi^\mu_{\mu j} \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi^\mu_0 \in \mathcal{O}(E, G)$ as $j \to \infty$ and also

$$\varphi^{\mu-1}_0(\alpha^{\mu-1}_0) = \lim_{j\to\infty} \varphi^{\mu-1}_{\mu j}(\alpha^{\mu-1}_{\mu j}) = \lim_{j\to\infty} \varphi^\mu_{\mu j}(0) = \varphi^\mu_0(0), \quad \varphi^\mu_0(\alpha^\mu_0) = w_0.$$

Consequently, we have two sequences $(\varphi^m_0)^\mu_{m=1} \subset \mathcal{O}(E, G)$ and $(\alpha^m_0)^\mu_{m=1} \subset [0, 1)$ satisfying:

$$\varphi^1_0(0) = z_0, \;\; \varphi^m_0(\alpha^m_0) = \varphi^{m+1}_0(0) \;(1 \le m \le \mu - 1), \;\; \varphi^\mu_0(\alpha^\mu_0) = w_0.$$

Thus it follows that

$$\begin{aligned} k^{(\mu)}_G(z_0, w_0) \le \sum_{m=1}^{\mu} p(0, \alpha^m_0) &= \sum_{m=1}^{\mu} p(0, \lim_{j\to\infty} \alpha^m_j) \\ &= \sum_{m=1}^{\mu} \left(\lim_{j\to\infty} p(0, \alpha^m_j) \right) \\ &= \lim_{j\to\infty} \sum_{m=1}^{\mu} p(0, \alpha^m_j) \le k^{(\mu)}_G(z_0, w_0) - \frac{\epsilon}{2}, \end{aligned}$$

which leads to a contradiction and we are done. □

Let us recall some results related to the notion of tautness. H. Wu ([Wu 67]) proved, by means of the so-called Kontinuitätssatz, that any taut domain in $\mathbb{C}^n$ is pseudoconvex. But the converse does not hold in general. N. Kerzman & J.-P. Rosay ([Ker 81], [Ker-Ros 81]) have found an example of a bounded pseudoconvex Hartogs domain over E with 1-dimensional balanced fibers which is not taut. Another one could be obtained in the class of balanced domains by using the following result:

Theorem 1.4.3. *Let $D = D_h \subset \mathbb{C}^m$ be a balanced domain with the Minkowski function h. Then D is taut iff D is bounded in $\mathbb{C}^m$ and $h \in (\mathcal{C} \cap PSH)(\mathbb{C}^m)$.*

This characterization is due to T. J. Barth ([Bar 83], Theorem 1).

On the other hand, in 1971, H. J. Royden studied carefully the properties of the family $k^{(\mu)}$ to obtain a characterization of taut domains. First observe that

$$\mathbb{B}_{\tilde{k}_G}(z, R) \subset \mathbb{B}_{k^{(\mu)}_G}(z, R) \subset \mathbb{B}_{k_G}(z, R), \quad z \in G,\; R > 0,\; \mu \in \mathbb{N},$$

where $\mathbb{B}_{d_G}(z, R) := \{w \in G : d_G(z, w) < R\}$ for a function d_G on $G \times G$. Now we recall that a characterization of taut domains ([Roy 71], Proposition 6):

Proposition 1.4.4. (Royden's criterion for taut domains) *Let $G \subset \mathbb{C}^n$ be a domain. Then the following properties are equivalent:*

(a) G is taut;

(b) $\mathbb{B}_{k_G^{(\mu)}}(z, R) \Subset G$ for any $\mu \in \mathbb{N}, R > 0$, and $z \in G$;

(c) $\mathbb{B}_{k_G^{(2)}}(z, R) \Subset G$ for any $R > 0, z \in G$.

For a proof, see ([Jar-Pfl 93], Proposition 3.2.1).

As a simple consequence of this criterion we have the following:

Remark 1.4.5. (1) Let d_G be an invariant pseudodistance on a domain $G \subset \mathbb{C}^n$. If all d_G-balls with finite radii are relatively compact (with respect to the Euclidean topology of G) inside of G, then G is taut.

(2) Let $\pi : \tilde{G} \to G$ be a holomorphic covering between domains in $\mathbb{C}^n$. If $\tilde{G}$ is taut, then, by Theorem 1.3.1, for $\mu \in \mathbb{N}$, $x, y \in G$, and $\tilde{x} \in \tilde{G}$ with $\pi(\tilde{x}) = x$, there exists $\tilde{y} \in \tilde{G}$ with $\pi(\tilde{y}) = y$ such that $k_G^{(\mu)}(x, y) = k_{\tilde{G}}^{(\mu)}(\tilde{x}, \tilde{y})$ (cf. [Jar-Pfl 93], (a) of Remark 3.3.8).

Also, in virtue of Proposition 1.4.4 we have the following result:

Lemma 1.4.6. *Let $G \subset \mathbb{C}^n$ be a domain. If G is not taut, then there exist an $R > 0$, sequences $(z_j)_{j\geq 0} \subset G$, $(f_j)_{j\geq 1}$, $(g_j)_{j\geq 1} \in \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \in [0, 1)$, such that for any $j \geq 1$:*

$$k_G^{(2)}(z_0, z_j) < R, \tag{†1}$$

$$f_j(0) = z_0 \in G, \tag{†2}$$

$$f_j(\alpha_j) = g_j(0), \tag{†3}$$

$$g_j(\beta_j) = z_j, \quad z_j \overset{j\to\infty}{\longrightarrow} {}^\exists\hat{z}_0 \in \partial G \ \text{ or } \ \|z_j\| \overset{j\to\infty}{\longrightarrow} \infty, \tag{†4}$$

$$\alpha_j \overset{j\to\infty}{\longrightarrow} \alpha_0, \ \ \beta_j \overset{j\to\infty}{\longrightarrow} \beta_0. \tag{†5}$$

Remark 1.4.7. Lemma 1.4.6 will be a very useful tool for showing the tautness of a given domain by an indirect proof.

Now we are going to give a new proof of Theorem 1.4.3 based on Lemma 1.4.6.

Alternative proof for the sufficiency in Theorem 1.4.3. Suppose that $D = D_h$ is not taut. By Lemma 1.4.6 with $G := D$, we may take sequences $z_0 \in D$, $(z_j)_{j\geq 0} \subset D \setminus \{0\}$, $(f_j)_{j\geq 1}, (g_j)_{j\geq 1} \subset \mathcal{O}(E, D)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \subset [0, 1)$ satisfying the properties (†2) $\sim$ (†5). Note that $\|z_j\| h(z_j/\|z_j\|) = h(z_j) < 1$ for $j \geq 1$. Since $h \in \mathcal{C}(\mathbb{C}^n)$, it is clear that $\|z_j\| \not\to \infty$ as $j \to \infty$, which implies that there exists a subsequence $(z_{1j})_{j\geq 1}$ of $(z_j)_{j\geq 1}$ such that $z_{1j} \overset{j\to\infty}{\longrightarrow} {}^\exists z_* \in \partial D$. On the other hand, since D is bounded in $\mathbb{C}^m$, in view of Montel's theorem, we may take sequences $(f_{2j})_{j\geq 1} \subset (f_{1j})_{j\geq 1}$, $(g_{2j})_{j\geq 1} \subset (g_{1j})_{j\geq 1}$, and $f, g \in \mathcal{O}(E, \bar{D})$ such that $f_{2j} \overset{\mathrm{K}}{\Longrightarrow} f$ and $g_{2j} \overset{\mathrm{K}}{\Longrightarrow} g$. By (†4) and (†5) it follows that

$$\partial D \ni z_* = \lim_{j\to\infty} g_{2j}(\beta_{2j}) = g(\beta_0),$$

and, therefore, using the continuity of h we have

$$\lim_{j\to\infty} h(g_{2j}(\beta_{2j})) = h\big(\lim_{j\to\infty} g_{2j}(\beta_{2j})\big) = h(g(\beta_0)) = 1.$$

Here, in the last equality, we used (4) of Remark 1.1.4. Notice that $\beta_0 \in E$ and $h \circ g \in SH(E)$. Therefore, in view of the maximum principle for the subharmonic function $h \circ g$, we obtain $h \circ g \equiv 1$ on E. Similarly, one has

$$h(f(\alpha_0)) = \lim_{j\to\infty} h(f_{2j}(\alpha_{2j})) = h\big(\lim_{j\to\infty} f_{2j}(\alpha_{2j})\big) = h\big(\lim_{j\to\infty} g_{2j}(0)\big) = h(g(0)) = 1.$$

Here in the first (resp. third) equality we have used the property (†5) (resp. (†3)). Since $\alpha_0 \in E$ and $h\circ f \in SH(E)$, it follows from the maximum principle that $h\circ f \equiv 1$ on E, but the condition (†2) implies that $h(f(0)) = h(z_0) < 1$; a contradiction. $\square$

Now we will recall a notion which is connected with the one of tautness. We define the hyperconvexity of a domain, which was introduced by J.-L. Stehlé ([Ste 73/74]).

Definition 1.4.8. A domain $G \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a continuous *bounded psh* function u on G, which means that $u \in (\mathcal{C} \cap PSH)(G, [-\infty, 0))$, such that $\{z \in G : u(z) < \alpha\} \Subset G$ for any $\alpha \in \mathbb{R}_{<0}$. Any function u enjoying the last property is called an *exhaustion function* of G.

Notice that the hyperconvexity of domains is invariant with respect to biholomorphic mappings. It is known that:

- any domain $G \Subset \mathbb{C}$ is hyperconvex iff it is regular with respect to the Dirichlet problem;
- every pseudoconvex domain in $\mathbb{C}^n$ is the union of an increasing sequence of bounded hyperconvex subdomains;
- any balanced taut domain is also hyperconvex (cf. Theorem 1.4.3).

On the other hand, the following notions were introduced by N. Kerzman and J.-P. Rosay in 1981 ([Ker-Ros 81]; cf. [Kob 98]).

Definition 1.4.9. We say that a domain $G \subset \mathbb{C}^n$ is:

- *locally taut* (resp. *locally hyperconvex*) *at a point* $p \in \partial G$ if there is a constant $r > 0$ such that any connected component of $G \cap \mathbb{B}_n(p, r)$ is taut (resp. hyperconvex);
- *locally taut* (resp. *locally hyperconvex*) if it is locally taut (resp. locally hyperconvex) at every point of ∂G.

Obviously, any taut (resp. hyperconvex) domain is also locally taut (resp. locally hyperconvex), but the converse does not hold in general; e.g. the Hartogs domain $\{z \in \mathbb{C}^2 : |z_1| < |z_2|\}$ (resp. $\{z \in \mathbb{C}^2 : |z_1 z_2| < 1\}$). Observe that both domains are not bounded (cf. Theorem 1.5.23). The following results are due to N. Kerzman and J.-P. Rosay ([Ker-Ros 81]; [Kob 98], pp. 251-255). Sometimes we will say that the result (3) below is the *Kerzman-Rosay Theorem*.

Theorem 1.4.10. *Let $G \subset \mathbb{C}^n$ be a bounded domain. Then:*
(1) if G is locally hyperconvex, then it is hyperconvex;
(2) if G is (locally) hyperconvex, then it is (locally) taut;
(3) if G is locally taut, then it is taut.

Using it, they also proved that

Corollary 1.4.11. *Any bounded pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^1$-boundary is hyperconvex, so also taut.*

Another proof for the last assertion may be found in ([Kra 92], Exercises 10, p.476). Note that there is a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^1$-boundary but not pseudoconvex (so not taut). Also, there is a bounded Reinhardt pseudoconvex domain which is taut but not hyperconvex (see e.g. (1) in Example 1.5.24). Therefore, hyperconvexity is a stronger condition than tautness. On the other hand,

Theorem 1.4.12. *Any bounded pseudoconvex domain $G \subset \mathbb{C}^n$ with a Lipschitz boundary,* which means that ∂G is locally defined by a Lipschitz function, *is taut.*

This is a generalization of Corollary 1.4.11. It is due to J. P. Demailly ([Dem 87]).

In Chapter 3, we will deal with some generalized versions of (3) in Theorem 1.4.10 and of Theorem 1.4.12 in the case of unbounded domains in $\mathbb{C}^n$.

§1.5. Hyperbolicity and completeness with respect to invariant functions.

In this section we discuss the relations between geometrical properties of domains and hyperbolicities with respect to invariant functions of those domains.

Definition 1.5.1. Let $(d_G)_{G\in\mathcal{G}}$ be a family of invariant functions. We say that a domain $G \subset \mathbb{C}^n$ is *d-hyperbolic* if $d_G(z,w) > 0$ for any $z, w \in G$ with $z \neq w$; if $d_G \in \mathcal{C}^{\uparrow}(G \times G)$, we will denote top$d_G$ (resp. top$G$) the topology generated by the subbasis consisting of all d_G-balls (resp. the Euclidean topology of G).

Note that top$c_G \subset$ top$k_G \subset$ top$\tilde{k}_G$. Obviously, a domain G is hyperbolic with respect to an invariant function d_G whenever top$d_G =$ top$G$. It is of interest to know whether the converse holds. That is,

(T) Does the d-hyperbolicity of G imply that top$d_G =$ top$G$?

First let us remark that

Remark 1.5.2. Let d_G be a holomorphically contractible upper semicontinuous function on a domain $G \subset \mathbb{C}^n$. If d_G satisfies the triangle inequality, then topd_G coincides with the topology generated by the basis consisting of all d_G-balls. Moreover, if d_G is continuous, then top$d_G \subset$ top$G$.

In general, the answer to (T) in case that $d = c$ is negative. For $n \geq 3$, there exists a c-hyperbolic domain $G \subset \mathbb{C}^n$ with top$c_G \subsetneq$ top$G$, which is due to M. Jarnicki, P. Pflug, and J.-P. Vigué ([Jar-Pfl-Vig 91]). However, the answer of (T) in case that $d = k$ is always positive; see Theorem 1.5.7 below.

Let us recall the following elementary properties:

Proposition 1.5.3. *Let $G \subset \mathbb{C}^n$ be a domain and consider:*
(1) G is bounded.
(2) G is biholomorphic to a bounded domain in $\mathbb{C}^n$.
(3) top$G =$ topc_G.
(4) G is c-hyperbolic.
(5) G is k-hyperbolic.

(6) G is $\tilde{k}$-hyperbolic.
(7) G is Brody hyperbolic, i.e. every $f \in \mathcal{O}(\mathbb{C}, G)$ is constant.
(8) G does not contain any affine complex line.
(9) No complex line through 0 stays inside G.
Then, obviously,

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (6) \Longrightarrow (7) \Longrightarrow (8) \Longrightarrow (9).$$

In particular, the following properties are true:
(a) If $n = 1$, then (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ $H^\infty(G) \ncong \mathbb{C}$ ([Jar-Pfl 91b], [Sib 75]).
(b) If G is convex, then (2) $\Longleftrightarrow$ (8)*; and if, moreover, G contains the origin, then* (2) $\Longleftrightarrow$ (9) ([Bar 80]).
(c) If G is balanced, then (1) $\Longleftrightarrow$ (5) ([Kod 82]; cf. §5.1).
(d) If $G = G_h$ is balanced with a continuous Minkowski function h (e.g. any complete n-circled domains), then (1) $\Longleftrightarrow$ (9) (cf. Remark 1.1.4).
(e) If $G = G_h$ is balanced pseudoconvex, then (8) $\Longleftrightarrow$ (9) ([Azu 83]).
(f) If $\pi : \tilde{G} \to G$ is a holomorphic covering, then $\tilde{G}$ is k-hyperbolic iff G is k-hyperbolic ([Kob 70]).

In §4.1, we will discuss more details about the hyperbolicity of balanced domains.

Remark 1.5.4. (i) In Proposition 1.5.3 it is known that there exist examples of domains satisfying $(j+1)$ but not (j), except for the cases $j = 5, 6$, but including the case (7) $\not\Longrightarrow$ (5), see e.g. [Azu 83], [Suz 83], [Jar-Pfl-Vig 91], or [Jar-Pfl 93]. We would like to point out that a large part of such counterexamples was found in the class of Hartogs type domains.

(ii) In fact, the notion of $\tilde{k}$-hyperbolicity was studied by W. Zwonek in [Zwo 99, 00a, 00b]. He has shown that the above properties (2) and (7) are equivalent for any pseudoconvex Reinhardt domain in $\mathbb{C}^n$ (see Theorem 1.5.21). But he did not discuss the $\tilde{k}$-hyperbolicity for other domains. So far we do not know whether there are essential differences between the $\tilde{k}$-hyperbolicity and the other hyperbolicities presented in Proposition 1.5.3. Therefore, it is interesting to know whether there exist examples of domains satisfying $(j+1)$ but not (j) for $j = 5, 6$. Observe that those domains are non-convex and n-dimensional with $n \geq 2$ by (1) and (2) in Remark 1.3.2. In Chapter 2, we will study the $\tilde{k}$-hyperbolicity of some Hartogs type domains.

(iii) The following was recently shown by W. Jarnicki and N. Nikolov ([Jar-Nik 02]): *If F is a convex closed set in $\mathbb{C}^n$ $(n \geq 2)$ containing at most $(n-1)$-dimensional complex hyperplane, then $\tilde{k}_{\mathbb{C}^n \setminus F} \equiv 0$.* Hence, in contrast to (b) in Proposition 1.5.3, any invariant function of $\mathbb{C}^n \setminus F$ is trivial; so $\mathbb{C}^n \setminus F$ is not $\tilde{k}$-hyperbolic.

Before we go further, let us recall some notions and their basic properties related to the k-hyperbolicity of domains.

Definition 1.5.5. Let $D \subset \mathbb{C}^m$ be a domain and let (G, d_G) be a metric space in the usual sense, i.e. d_G is a usual distance on $G \subset \mathbb{C}^n$ inducing the topology top$G$ of G. We say that a family $\mathcal{F} \subset \mathcal{O}(D, G)$ is *equicontinuous* with respect to a distance d_G if for any $\epsilon > 0$ and $z_0 \in D$, there exists an open neighborhood $U \equiv U(z_0) \subset D$ of z_0 such that $d_G(f(z_0), f(z)) < \epsilon$ for any $z \in U$, $f \in \mathcal{F}$.

Remark 1.5.6. Equicontinuity depends on the choice of the distance (cf. (1) and (2) in Remark 1.5.8).

Now let us recall the following characterization of k-hyperbolicity (cf. [Jar-Pfl 93]: Theorem 7.2.2).

Theorem 1.5.7. *For a domain $G \subset \mathbb{C}^n$ the following properties are equivalent:*
(a) G is k-hyperbolic;
(b) $\mathrm{top}G = \mathrm{top}k_G$;
(c) for any $z_0 \in G$ and any neighborhood $U = U(z_0) \subset G$, there exist neighborhoods $V = V(0) \subset E$ of 0, $W = W(z_0) \subset U$ of z_0 such that $f(V) \subset U$ for any $f \in \mathcal{O}(E, G)$ with $f(0) \in W$;
(d) for any $z_0 \in G$ there exist a constant $C = C(z_0) > 0$ and a neighborhood $U = U(z_0) \subset G$ of z_0 such that $k_G(z, w) \geq C\|z - w\|$ for any $z, w \in U$;
(e) for any $z_0 \in G$ there is an open neighborhood $U = U(z_0) \subset G$ and a constant $C > 0$ such that $\kappa_G(z; X) \geq C\|X\|$, $z \in U$, $X \in \mathbb{C}^n$, where

$$\kappa_G(z; X) := \inf\{\alpha > 0 : \exists\, \varphi \in \mathcal{O}(E, G),\ \varphi(0) = z,\ \varphi'(0) = X/\alpha\}$$

(In that case, the domain G is called *κ-hyperbolic*).

This is a combination of results expressed in [Gra-Rec 65], [Kie 70], [Roy 71], and [Har 79].

Remark 1.5.8. (1) Let $(d_G)_{G\in\mathcal{G}}$ be a holomorphically contractible family of pseudodistances. If a domain $G \subset \mathbb{C}^n$ is d-hyperbolic (i.e. d_G is a distance on G), then $\mathcal{O}(E, G)$ is equicontinuous with respect to k_G because (G, k_G) is a metric space with $\mathrm{top}G = \mathrm{top}k_G$ and $k_G(f(\lambda), f(\zeta)) \leq p(\lambda, \zeta)$ for any $\lambda, \zeta \in E$ and $f \in \mathcal{O}(E, G)$.

(2) Using (c) in Theorem 1.5.7, it is easy to see that if for a domain $G \subset \mathbb{C}^n$ the family $\mathcal{O}(E, G)$ is equicontinuous with respect to d_G, where (G, d_G) is a metric space in the usual sense, then G is k-hyperbolic. However, in general, neither tautness nor k-hyperbolicity of a domain $G \subset \mathbb{C}^n$ implies that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$.

(3) Let $G \Subset \mathbb{C}^n$ be a domain. Then there is a constant $R > 0$ such that $G \Subset \mathbb{B}_n(z, R)$ for any $z \in G$, so

$$k_G(z, w) \geq k_{\mathbb{B}_n(z,R)}(z, w) = p(0, \frac{\|z - w\|}{R}) \geq \frac{\|z - w\|}{R}, \quad z, w \in G.$$

Thus the domain G is k-hyperbolic (cf. the above (b)) and hence $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$.

(4) Using (c) in Theorem 1.5.7, it is easy to see that any taut domain in $\mathbb{C}^n$ is k-hyperbolic; but its converse does not hold in general. For example, any bounded domain is k-hyperbolic but it is not taut if it is not pseudoconvex, e.g. any bounded balanced domain $D = D_h \subset \mathbb{C}^m$ with $h \notin PSH(\mathbb{C}^m)$ is such a case (by (2) of Proposition 1.1.3, Theorem 1.4.3).

Observe that the family $\mathcal{O}(E, G)$ is not equicontinuous with respect to $\|\cdot\|$ whenever $G \subset \mathbb{C}^n$ is an unbounded non-k-hyperbolic domain. Now we can ask whether there is an example of an unbounded domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$. Fortunately, the answer is positive and we will give such an example, which was proposed by Professor W. Zwonek, as follows:

Example 1.5.9. Let $G := \{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \pi/2\}$. Now we will verify that $\mathcal{O}(E,G)$ is equicontinuous with respect to $|\cdot|$. For this, it is enough to see that for any $\epsilon > 0$ there is a small constant $\delta = \delta(\epsilon) > 0$ such that

$$(1.5.9a) \qquad |\zeta| < \delta,\ f \in \mathcal{O}(E,G) \quad \Longrightarrow \quad |f(\zeta) - f(0)| < \epsilon.$$

In fact, let $\epsilon > 0$, $a \in E$ and put $\varphi_a(\zeta) := \frac{\zeta - a}{1 - \bar{a}\zeta}$, $\zeta \in E$. Then φ_a is an automorphism of E with $\varphi_a^{-1} = \varphi_{-a}$ and $\varphi_{-a}(0) = a$. Observe that

$$(1.5.9b) \qquad |f(\lambda) - f(a)| = |(f \circ \varphi_{-a})(\varphi_a(\lambda)) - (f \circ \varphi_{-a})(0)|, \quad f \in \mathcal{O}(E,G),$$

Let $\delta = \delta(\epsilon) > 0$ be as in (1.5.9a). Obviously, we can take a constant $\delta' > 0$ so small that

$$|\lambda - a| < \delta' \quad \Longrightarrow \quad |\varphi_a(\lambda)| < \delta.$$

Hence, the required assertion follows directly from (1.5.9a) and (1.5.9b).

Now, to show (1.5.9a), let $f \in \mathcal{O}(E,G)$. Clearly,

$$\{e^\lambda : \lambda \in G\} \subset \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta > 0\} =: H_+.$$

Since

$$k_{H_+}(\lambda, \zeta) = \tanh^{-1}\left(\left|\frac{\lambda - \zeta}{\lambda + \bar{\zeta}}\right|\right), \quad \lambda, \zeta \in H_+$$

(e.g. Exercise 2.2 in [Jar-Pfl 93]), the decreasing property of the Kobayashi pseudodistance implies that for any $\lambda \in E$ one has

$$\begin{aligned}(1.5.9c) \qquad |\lambda| &\geq \tanh k_G(f(\lambda), f(0)) \\ &\geq \tanh k_{H_+}(e^{f(\lambda)}, e^{f(0)}) \\ &\geq \frac{|e^{f(\lambda)} - e^{f(0)}|}{|e^{f(\lambda)}| + |e^{f(0)}|} = \frac{|e^{f(\lambda)-f(0)} - 1|}{e^{\mathrm{Re}(f(\lambda)-f(0))} + 1} \\ &\geq \frac{||e^{f(\lambda)-f(0))}| - 1|}{e^{\mathrm{Re}(f(\lambda)-f(0))} + 1} = \frac{|e^{\mathrm{Re}(f(\lambda)-f(0))} - 1|}{e^{\mathrm{Re}(f(\lambda)-f(0))} + 1}.\end{aligned}$$

Since the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) := \frac{e^x - 1}{e^x + 1}$ for $x \in \mathbb{R}$, is increasing and $g(0) = 0$, (1.5.9c) implies that

$$(1.5.9d) \qquad \lim_{|\lambda| \to 0} \mathrm{Re}(f(\lambda) - f(0)) = 0.$$

Moreover, we can assume that $|e^{\mathrm{Re}(f(\lambda)-f(0))}| < 2$ for $|\lambda| \ll 1$, and for such a $\lambda \in E$, one has

$$(1.5.9e) \qquad |\lambda| \geq \frac{|e^{f(\lambda)-f(0)} - 1|}{e^{\mathrm{Re}(f(\lambda)-f(0))} + 1} \geq \frac{1}{3}|e^{f(\lambda)-f(0)} - 1|.$$

Here, in the first inequality, we used the third inequality in (1.5.9c). Observe that

$$|e^\zeta - 1|^2 = (-1 + e^{\zeta_1}\cos\zeta_2)^2 + (e^{\zeta_1}\sin\zeta_2)^2$$

for any $\zeta \in \mathbb{C}$, $\zeta_1 := \mathrm{Re}\,\zeta$, $\zeta_2 := \mathrm{Im}\,\zeta$. Hence, by (1.5.9e) and (1.5.9d), we have

$$\lim_{|\lambda| \to 0} \sin(\mathrm{Im}(f(\lambda) - f(0))) = 0, \quad \lim_{|\lambda| \to 0} \cos(\mathrm{Im}(f(\lambda) - f(0))) = 1.$$

Because of $|\mathrm{Im}\, f(\lambda)| < \frac{\pi}{2}$ for any $\lambda \in E$, it holds that

$$(1.5.9f) \qquad \lim_{|\lambda| \to 0} \mathrm{Im}(f(\lambda) - f(0)) = 0.$$

Consequently, (1.5.9a) follows from (1.5.9d) and (1.5.9f).

As a simple consequence of Example 1.5.9, we have the following results:

Remark 1.5.10. For any $n \geq 1$ there is an unbounded pseudoconvex domain G in $\mathbb{C}^n$ such that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$. In particular, in case $n \geq 2$, we can find such a domain which is not taut.

In view of (3) in Remark 1.5.8, our interest in the notion of k-hyperbolicity is restricted only to unbounded domains. The following results are due to N. Sibony ([Sib 81]):

Theorem 1.5.11. *(1) Any domain $G \subset \mathbb{C}^n$ having a bounded psh function that is $\mathcal{C}^2$ and strictly psh near a point $z_0 \in G$ satisfies the condition (e) in Theorem 1.5.7 for z_0.*

(2) Any domain $G \subset \mathbb{C}^n$ admitting a bounded psh exhaustion function is k-hyperbolic. In fact, this domain is taut.

Recall that any pseudoconvex domain has an unbounded strictly psh exhaustion function of class $\mathcal{C}^\infty$ (see e.g. [Jak-Pfl 00], Proposition 2.2.6). Note that any bounded domain having the property (2) above is also hyperconvex (so pseudoconvex) (cf. [Jak-Jar 01], Proposition 3.4.33). In [Sib 81], it is mentioned that the first assertion in (2) could be obtained by modifying some part of proof of the property (1) (cf. [Jar-Pfl 93], Exercise 7.4). For the sake of completeness we now give the proof; moreover, we also present a new proof of the second assertion in (2) by using Royden's criterion.

Alternative proof of (2) in Theorem 1.5.11. (k*-hyperbolicity*) Let $u \in PSH(G, [0, 1))$ be an exhaustion function for G and let $\epsilon > 0$, $0 \leq \alpha < 1$, and $G_\alpha := G \cap \{u < \alpha\}$. Put $\Omega_\epsilon := G \cap \{\delta_G > \epsilon\}$, where $\delta_G(z) := \sup\{r > 0 : \mathbb{B}_n(z, r) \subset G\}$. Fix $\alpha' > 0$ with $\alpha < \alpha' < 1$. Note that $\tilde{G}_\alpha := \{z \in G : u(z) \leq \alpha\} \subset G_{\alpha'}$. Since G is pseudoconvex, we may take a sequence $(v_j)_{j \geq 1} \subset (\mathcal{C}^\infty \cap PSH)(G)$ such that $v_j \searrow u$ pointwise on G as $j \to \infty$ (cf. '26' in Appendix). Moreover, we may take a strictly decreasing sequence $(u_\epsilon)_\epsilon \subset \mathcal{C}^\infty(G)$ of strictly psh functions such that $u_\epsilon \searrow u$ pointwise on G as $\epsilon \searrow 0$ (cf. '25' in Appendix). From this, for any $\alpha \in [0, 1)$ we can take an $\epsilon_\alpha > 0$ and an open set U_α such that $G_{\alpha'} \Subset U_\alpha \Subset \Omega_{\epsilon_\alpha}$ and $u < u_{\epsilon_\alpha}$ on $\bar{U}_\alpha$. Put

$$A_\alpha := \inf_{z \in \partial U_\alpha} u(z), \quad M_\alpha := \max_{z \in \partial U_\alpha} u_{\epsilon_\alpha}(z).$$

Note that $A_\alpha > \alpha$ because $G_{\alpha'} \cap \partial U_\alpha = \emptyset$. Let $F_\alpha : \mathbb{R} \to \mathbb{R}$ be a convex increasing continuous function such that $F_\alpha(x) = x$ if $x < \alpha + \frac{A_\alpha - \alpha}{2}$ and $F_\alpha(A_\alpha) > M_\alpha$. Define a function $\varphi_\alpha : G \to [-\infty, +\infty)$ by

$$\varphi_\alpha(z) := \begin{cases} \max\{u_{\epsilon_\alpha}(z),\, F_\alpha(u(z))\} & \text{if} \quad z \in U_\alpha, \\ F_\alpha(u(z)) & \text{if} \quad z \in G \setminus U_\alpha. \end{cases}$$

Then

$$\limsup_{U_\alpha \ni z \to w} u_{\epsilon_\alpha}(z) \leq M_\alpha < F_\alpha(A_\alpha) \leq F_\alpha(u(w)), \quad w \in \partial U_\alpha.$$

Here in the last inequality we used the fact that F_α is increasing. Moreover, $u_{\epsilon_\alpha} \in PSH(U_\alpha)$, $F_\alpha \circ u \in PSH(G)$, and so in view of the gluing lemma for psh functions we have $\varphi_\alpha \in PSH(G)$. In particular, $0 \leq \varphi_\alpha \leq \max\{C, F_\alpha(1)\}$, where $C := \max_{w \in \bar{U}_\alpha} u_{\epsilon_\alpha}(w) < \infty$ because of the continuity of u_{ϵ_α}. On the other hand, there is an open neighborhood V_α of $\tilde{G}_\alpha$ such that $V_\alpha \Subset G_{\alpha'}$ and $F_\alpha(u(z)) = u(z)$, $z \in V_\alpha$, so

$\varphi_\alpha = u_{\epsilon_\alpha}$ on V_α. Therefore φ_α is $\mathcal{C}^\infty$ strictly psh on V_α and hence the property (1) gives us that the domain G satisfies the condition (e) in Theorem 1.5.7 for every point $z_0 \in \tilde{G}_\alpha$. But since $(G_\alpha)_{\alpha\in[0,1)}$ is an exhaustion of G we have that G is κ-hyperbolic and also k-hyperbolic.

(*Tautness*) Suppose the contrary. By Lemma 1.4.6, we may take sequences $z_0 \in G$, $(z_j)_{j\geq 0} \subset G$, $(f_j)_{j\geq 1}, (g_j)_{j\geq 1} \subset \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \subset [0, 1)$ satisfying the properties $(\dagger 2) \sim (\dagger 5)$. For a sequence $(g_{j_\nu})_{\nu\geq 1} \subset (g_j)_{j\geq 1}$ we put $\tilde{u} = \tilde{u}_{(g_{j_\nu})_{\nu\geq 1}} := \limsup_{\nu\to\infty} u \circ g_{j_\nu}$. Since $u < 1$ on G, one has $\tilde{u}^* \in SH(E)$ where '$*$' is the upper semicontinuous regularization; moreover, $\limsup_{\nu\to\infty} \tilde{u}^*(\beta_{j_\nu}) \leq \tilde{u}^*(\beta_0)$.

Now we are going to see that $\tilde{u}^*(\beta_0) = \tilde{u}^*_{(g_{j_\nu})_{\nu\geq 1}}(\beta_0) = 1$. Assume that $\tilde{u}^*(\beta_0) < 1$ and take $C > 0$, so that $\tilde{u}^*(\beta_0) < C < 1$. Then there exists $\mu_1 \in \mathbb{N}$ such that $\limsup_{\nu\to\infty} u(g_{j_\nu}(\beta_{j_\mu})) \leq C$ for $\mu \geq \mu_1$. Hence it follows from $(\dagger 4)$, $(\dagger 5)$, and the Hartogs Lemma for subharmonic functions that there are $\mu_2 \geq \mu_1$ and $\nu_1 \in \mathbb{N}$ such that $u(g_{j_\nu}(\beta_{j_\mu})) < \frac{1+C}{2}$ for $\nu \geq \nu_1$, $\mu \geq \mu_2$. This implies that

$$u(z_{j_\nu}) = u(g_{j_\nu}(\beta_{j_\nu})) < \frac{1+C}{2} < 1, \quad \nu \geq \nu_0 := \max\{\nu_1, \mu_2\},$$

and also $(z_{j_\nu})_{\nu\geq\nu_0} \subset \{z \in G : u(z) < \frac{1+C}{2}\} \Subset G$ because u is an exhaustion for G. Thus $(z_{j_\nu})_{\nu\geq\nu_0}$ has a subsequence converging to a point in G; a contradiction to $(\dagger 4)$. Thus we then get the required assertion and also it follows from the maximum principle for subharmonic functions that

$$\text{(1.5.11a)} \qquad \big(\limsup_{\nu\to\infty} u \circ g_{j_\nu}\big)^* \equiv 1 \text{ on } E \text{ for any sequence } (g_{j_\nu})_{\nu\geq 1} \subset (g_j)_{j\geq 1}.$$

Next we will show that either $g_j(0) \overset{j\to\infty}{\longrightarrow} \partial G$ or $\|g_j(0)\| \overset{j\to\infty}{\longrightarrow} \infty$.

To see this, suppose the contrary, i.e. there exists a sequence $S := (g_{1j})_{j\geq 1} \subset (g_j)_{j\geq 1}$ such that $g_{1j}(0)$ converges to a point $a \in G$ and take a constant $0 < C' < 1$ so that $u(a) < C'$. Then we can choose $\nu_0 \geq 1$ and an open neighborhood $W = W(a) \subset G$ of a such that $u < C'$ on W and $(g_{1j}(0))_{j\geq\nu_0} \subset W$. Since G is k-hyperbolic, it follows from (c) of Theorem 1.5.7 that there exists an open neighborhood $V = V(0) \subset E$ such that $g_{1j}(\lambda) \in W$ for any $\lambda \in V$ and any $j \geq \nu_0$. This implies that $u(g_{1j}(\lambda)) < C'$ for any $\lambda \in V$ and any $j \geq \nu_0$. Hence $\tilde{u}_S(\lambda) \leq C'$ for any $\lambda \in V$. Then we have that $\tilde{u}^*_S(0) \leq C' < 1$; a contradiction to (1.5.11a) and thus the required assertion holds.

From the previous result and $(\dagger 3)$, we have either $f_j(\alpha_j) \overset{j\to\infty}{\longrightarrow} \partial G$ or $\|f_j(\alpha_j)\| \overset{j\to\infty}{\longrightarrow} \infty$, which is a similar phenomenon as in $(\dagger 4)$ for the sequence $(g_j(\beta_j))_{j\geq 1}$. Hence we can repeatedly carry out the above procedures by replacing $(g_j(\beta_j))_{j\geq 1}$ with $(f_j(\alpha_j))_{j\geq 1}$ to obtain that $f_j(0) \overset{j\to\infty}{\longrightarrow} \partial G$ or $\|f_j(0)\| \overset{j\to\infty}{\longrightarrow} \infty$; a contradiction to $(\dagger 2)$ and we are done. $\square$

To recall the next theorem, we need the following notions.

Definition 1.5.12. We say that a domain $G \subset \mathbb{C}^n$ has:

- a *local psh peak function φ at infinity* whenever there exists a $R > 0$ such that $\varphi \in \mathcal{C}(\bar{G} \cap U_R(\infty)) \cap PSH(G \cap U_R(\infty))$ and

$$\lim_{G\ni w\to\infty} \varphi(w) = 0 > \varphi(z), \quad z \in \bar{G} \cap U_R(\infty),$$

where $U_R(\infty) = U_R^n(\infty) := \mathbb{C}^n \setminus \overline{\mathbb{B}_n(0,R)}$;

- a *local psh antipeak function φ at infinity* whenever there exists a $R > 0$ such that $\varphi \in \mathcal{C}(\bar{G} \cap U_R(\infty)) \cap PSH(G \cap U_R(\infty))$ and

$$\lim_{G \ni w \to \infty} \varphi(w) = -\infty < \varphi(z), \quad z \in \bar{G} \cap U_R(\infty).$$

These notions were introduced by H. Gaussier in 1999.

Remark 1.5.13. (1) Assume that a domain $G \subset \mathbb{C}^n$ has a *local holomorphic peak function f at infinity*, which means that there exists a $R > 0$ such that $f \in \mathcal{C}(\bar{G} \cap U_R(\infty)) \cap \mathcal{O}(G \cap U_R(\infty))$ and

$$\lim_{G \ni w \to \infty} |f(w)| = 1 > |f(z)|, \quad z \in \bar{G} \cap U_R(\infty).$$

Then it is easy to check that G also has local psh peak and antipeak functions at infinity. Such domains can be found in ([Gau 99], Example 3.2.1, Example 3.2.2).

(2) Let $G' \Subset \mathbb{C}^n$. Then it is easy to check that any unbounded subdomain G of $G' \times \mathbb{C}$ has a local psh antipeak function at infinity. For example, for any sequence $(z^\nu)_{\nu \geq 1} \subset G$ with $\lim_{\nu \to \infty} \|z^\nu\| = \infty$, where $z^\nu = (\tilde{z}^\nu, z^\nu_{n+1}) \in G' \times \mathbb{C}$, one has $\lim_{\nu \to \infty} |z^\nu_{n+1}| = \infty$. Put $\varphi(z) := -\log |z_{n+1}|$ for $z \in G$. Then $\varphi \in (\mathcal{C} \cap PSH)(\mathbb{C}^n \times \mathbb{C}_*, \mathbb{R})$ with $\lim_{G \ni z \to \infty} \varphi(z) = -\infty$, that is, φ is a local psh antipeak function at infinity.

(3) As a simple consequence from (2), any unbounded Hartogs domain over a bounded domain G with 1-dimensional balanced fibers has a local psh antipeak function at infinity.

Now we will show that for any $n \geq 3$, there exists an unbounded domains in $\mathbb{C}^n$ which have no local psh peak and antipeak functions at infinity.

Example 1.5.14. (1) If a domain $G \subset \mathbb{C}^n$ has no local psh peak (resp. antipeak) function at infinity, then any domain in $\mathbb{C}^n$ containing G does not have local psh peak (resp. antipeak) functions at infinity.

(2) For $j = 1, 2$, let $D_j \subset \mathbb{C}^{m_j}$ be a balanced domain with the associated Minkowski function h_j. Assume that there exists a point $z^0 \in \mathbb{C}^{m_1} \setminus \{0\}$ such that $h_1(z^0) = 0$. Then the unbounded domain $\Omega = \Omega_{\log h_1, h_2}(D_{h_1})$ has no local psh peak functions at infinity. To check this assertion, suppose the contrary. Let φ be a local psh peak function at infinity, that is, there exists a constant $R > 0$ such that $\varphi \in \mathcal{C}(\bar{\Omega} \cap U) \cap PSH(\Omega \cap U)$ and

$$\lim_{G \ni w \to \infty} \varphi(w) = 0 > \varphi(z), \quad z \in \bar{\Omega} \cup U, \tag{1.5.14a}$$

where $U := U_R^{m_1+m_2}(\infty)$. Fix a point $w^0 \in \mathbb{C}^{m_2}$ with $\|w_0\| = 2R$. By our assumption, it is clear that $h_1 = 0$ on $\mathbb{C}z^0$ and also $\varphi(\lambda z^0, w^0) < 0$ for any $\lambda \in \mathbb{C}$. Thus it follows from the Liouville type theorem for subharmonic functions that $\varphi(\lambda z^0, w^0) \equiv$ constant $=: 2C < 0$ for any $\lambda \in \mathbb{C}$, so we have $\lim_{|\lambda| \to \infty} \varphi(\lambda z^0, w^0) \leq C < 0$; a contradiction to (1.5.14a).

(3) By (1) and (2) the domain $G_n := \{z \in \mathbb{C}^n : |z_1 \cdots z_n| < 1\}$, $n \geq 3$, has no local psh peak function at infinity; moreover, this domain has also no local psh antipeak

function at infinity. To check this, fix $n \geq 3$ and let ψ be a function defined on $W := G_n \cap U_R(\infty)$ for some $R > 0$. Suppose that $\psi \in PSH(W)$ and $\psi|_W > -\infty = \lim_{W \ni z \to \infty} \psi(z)$. Fix $a \in \mathbb{C}$ with $|a| = 2R$. Then $\psi(\cdot, a) \in PSH(\tilde{W})$ where $\tilde{W} := \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} : |z_1 \cdots z_{n-1}| < 1/(2R)\}$, and so $\psi_a := \psi(\cdot, 0, \cdots, 0, a) \in SH(\mathbb{C})$. Our assumption gives us that $\lim_{|\lambda| \to \infty} \psi_a(\lambda) = -\infty$. Therefore, it follows from the maximum principle for subharmonic function that $\psi_a \equiv -\infty$, which is a contradiction to the fact that $\psi_a > -\infty$ on $\mathbb{C}$.

Let us give a sufficient condition for k-hyperbolicity of unbounded domains.

Theorem 1.5.15. *Any unbounded domain $G \subset \mathbb{C}^n$ having local psh peak and antipeak functions at infinity is k-hyperbolic.*

This result is due to H. Gaussier ([Gau 99]). At first he showed the following localization lemma under the given assumption:

$$\forall_{R>0}, \quad \exists_{R'>0} \ : \ \forall_{g \in \mathcal{O}(E,G)}, \quad g(0) \in U_{R'}(\infty) \implies g(\frac{1}{2}E) \subset U_R(\infty)$$

(cf. (c) in Theorem 1.5.7); and then he proved that the condition (e) in Theorem 1.5.7 is satisfied.

Remark 1.5.16. The converse of Theorem 1.5.15, in general, does not hold. In fact, there exists an unbounded k-hyperbolic domain which has a local psh antipeak function at infinity but no local psh peak function at infinity. For examples, see e.g. Example 2.2.18 and Remark 2.2.19.

Next, let us recall the notions of completeness with respect to invariant distances d ($= c$ or k).

Definition 1.5.17. Let $G \subset \mathbb{C}^n$ be a domain with an invariant distance d_G. Then we say that G is:

- *d_G-complete* if any d_G-Cauchy sequence converges to a point in G (with respect to top$G$);
- *d_G-finitely compact* if any d_G-ball with finite radius is relatively compact (with respect to top$G$) inside G;
- *H^∞-sequentially convex* if for any $z_0 \in G$ and any sequence $(z_\nu)_{\nu \geq 1} \subset G$ without accumulation points in G, there exists a function $f \in \mathcal{O}(G, E)$ such that $f(z_0) = 0$ and $\sup_{\nu \geq 1} |f(z_\nu)| = 1$.

The last notion was introduced by P. Pflug ([Pfl 87]). By the decreasing property of invariant distances d (=c or k) the following implications are true:

$$c\text{-complete} \implies k\text{-complete},$$
$$c\text{-finitely compact} \implies k\text{-finitely compact} \implies \text{taut}.$$

Here, the last implication follows directly from Royden's criterion ((1) in Remark 1.4.5). For any invariant distance d, the following implication also holds:

$$d\text{-finitely compact} \implies d\text{-complete}.$$

Moreover, the following results are well-known:

Theorem 1.5.18. *(a) k-finitely compact $\Longleftrightarrow$ k-complete* ([Rin 61]).

(b) c-finitely compact $\Longleftrightarrow$ H^∞-sequentially convex ([Jar-Pfl 93]; cf. [Pfl 84], [Che 74]).

(c) In $\mathbb{C}$, c-finitely compact $\Longleftrightarrow$ c-complete ([Sel 74]), ([Sib 75]).

(d) If $\pi : D \to G$ denotes a holomorphic covering between domains in $\mathbb{C}^n$, then D is k-complete iff so is G ([Kob 67a]).

(e) (Eastwood Theorem) *Let $G \Subset \mathbb{C}^n$ be a domain. If for any $z_0 \in \partial G$, there is an open neighborhood $U = U(z_0) \Subset \mathbb{C}^n$ of z_0 such that any connected component of $G \cap U$ is k-complete, then G is k-complete* ([Eas 75]).

Remark 1.5.19. (1) By the above (b), any bounded domain $G \subset \mathbb{C}^n$ is c-finitely compact if any boundary point of G has a holomorphic peak function. For example,

- any bounded strongly pseudoconvex domain in $\mathbb{C}^n$,
- any bounded pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary ([Bed-For 78]), and
- any bounded domains of finite type in $\mathbb{C}^2$ ([For-Sib 89])

are of such a type.

(2) In virtue of (b) in Theorem 1.5.18 (or above (1)) and Eastwood's theorem, a bounded domain $G \subset \mathbb{C}^n$, for which every point of ∂G has a local holomorphic peak function is k-complete. But, in that case, G is in general not c-complete. That is, c-completeness is not a local property, e.g. see (2) in Example 1.5.24.

(3) Recall that $\log(\tanh c_G) \in (\mathcal{C} \cap PSH)(G \times G, \mathbb{R}_{<0})$ for any domain $G \subset \mathbb{C}^n$ (cf. [Jar-Pfl 93], Proposition 2.4.1, Proposition 2.5.1). Therefore any c-finitely compact domain is hyperconvex (cf. [Berg 79]). But, in general, hyperconvex $\not\Longrightarrow$ k-complete. For example, E_* is k-complete but not regular with respect to the Dirichlet problem; or, see Theorem 4.2.1 below.

The following result is obtained by H. Gaussier ([Gau 99], Theorem 1).

Theorem 1.5.20. *Let $G \subset \mathbb{C}^n$ be a domain. If there exists a local holomorphic peak function at any point of $\partial G \cup \{\infty\}$, then G is k-complete.*

Note that we can consider this theorem as a generalization of the first statement in (2) of Remark 1.5.19.

In view of (1) of Remark 1.5.13 and Theorem 1.5.15, such a domain G in Theorem 1.5.20 is always k-hyperbolic. Note that, in general, the converse of the previous theorem does not hold, see e.g. Example 2.2.18 and Remark 2.2.19.

On the other hand, the completeness of invariant distances on pseudoconvex Reinhardt domain was studied by P. Pflug ([Pfl 84]). He proved that:

- any bounded pseudoconvex complete Reinhardt domain is c-finitely compact.

And then, S. Fu ([Fu 94]) proved that:

- any bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$ is k-complete;
- any bounded pseudoconvex Reinhardt domain $G \subset \mathbb{C}^n$ satisfying the following condition:

$$(\clubsuit) \qquad \text{if } \bar{G} \cap V_j \neq \emptyset, \text{ then } G \cap V_j \neq \emptyset,$$

  where $V_j := \{z \in \mathbb{C}^n : z_j = 0\}$, is c-finitely compact.

For this, he used a similar method as in [Pfl 84] and applied a localization principle for the Kobayashi pseudodistance. Note that every bounded Reinhardt domain with

$\mathcal{C}^1$-boundary satisfies the condition (♣). For details, see ([Fu 94], Lemma 3). Finally, the following full characterization of hyperbolic Reinhardt domains was obtained by W. Zwonek ([Zwo 99, 00b]).

Theorem 1.5.21. *For a pseudoconvex Reinhardt domain $G \subset \mathbb{C}^n$ the following properties are equivalent:*
(a) G is c-hyperbolic;
(b) G is $\tilde{k}$-hyperbolic;
(c) G is Brody hyperbolic;
(d) G is biholomorphic to a bounded Reinhardt domain;
(e) G is k-complete;
(f) G is taut.

Remark 1.5.22. (1) In Theorem 1.5.21, the condition (e) is a consequence of Fu's theorem via (d). However, W. Zwonek proved (e) using the effective formulas for invariant functions on so-called *elementary Reinhardt domains*, which were calculated by P. Pflug and W. Zwonek ([Pfl-Zwo 98]).

(2) In view of Theorem 1.5.21, all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains. Therefore, we will speak only on hyperbolic pseudoconvex Reinhardt domains.

It turns out that there is the following complete characterization of Carathéodory completeness in hyperbolic Reinhardt domains, due to W. Zwonek ([Zwo 00a, 00b]):

Theorem 1.5.23. *Let $G \subset \mathbb{C}^n$ be a hyperbolic pseudoconvex Reinhardt domain. Then the following properties are equivalent:*
(a) G is c-finitely compact;
(b) G is c-complete;
(c) G is bounded and satisfies the so-called 'Fu-condition' (♣)*;*
(d) There is no boundary sequences $(z_j)_{j\geq 1} \subset G$ with $\sum_{j=1}^{\infty} g_G(z_j, z_{j+1}) < \infty$;
(e) For any $z_0 \in G$, it holds that $g_G(z_0, z) \to 1$ as $z \to \partial G \cup \{\infty\}$;
(f) G is hyperconvex.

The implications $(c) \iff (f)$ was obtained by M. Carlehed, U. Cegrell, F. Wikström ([Car-Ceg-Wik 99]) and P. Pflug, W. Zwonek ([Zwo 00a], Remark 11)

Finally, we give two well-known examples, frequently used as 'counterexamples' to some properties of the above mentioned notions.

Example 1.5.24. (1) The Hartogs Triangle $\triangle_H := \{z \in \mathbb{C}^2 : |z_2| < |z_1| < 1\}$ is a bounded pseudoconvex Reinhardt domain without $\mathcal{C}^1$-boundary. It is hyperbolic and k-complete, but neither c-complete nor hyperconvex (because it does not satisfy the Fu-condition). In particular, $\triangle_H = \Omega_{\log(1/|\cdot|),|\cdot|}(E_*) = \Sigma_{0,\log|\cdot|}(E) \cong E \times E_*$.

(2) Let $(a_\nu)_{\nu\geq 1}$ be a discrete sequence of points in E such that every boundary point of E is a nontangential limit of a subsequence and α_ν's are positive numbers with $\sum_{\nu\geq 1} \alpha_\nu \log(|a_\nu|/2) > -\infty$. Put

$$u(\lambda) := \exp\left(\sum_{\nu=1}^{\infty} \alpha_\nu \log \frac{|\lambda - a_\nu|}{2}\right), \quad \lambda \in E.$$

The Hartogs domain $\Omega = \Omega_{u,|\cdot|}(E)$ was firstly studied by N. Sibony ([Sib 75], [Eas 75]; cf. [Jar-Pfl 93], [Jar-Pfl 00]). It is easy to check that u is continuous on E, so the (non-Reinhardt) bounded domain Ω is hyperconvex. Moreover, it is locally c-complete (so k-complete by (e) of Theorem 1.5.18), but not c-complete. For details, see e.g. Example 7.4.8 in [Jar-Pfl 93].

Note that some sufficient conditions for such a Hartogs type domain to be k-complete are studied by N. Q. Dieu and D. D. Thai ([Die-Tha 00]) (cf. §3.5).

Final Remark: In fact, most of facts introduced in Chapter 1 can be found in [Jar-Pfl 93] and [Pfl 00]. These references contain also many open problems. Specially, [Pfl 00] contains recent results related to the hyperbolicity and completeness with respect to invariant distances.

CHAPTER 2. HYPERBOLICITY OF SOME HARTOGS TYPE DOMAINS

Summary. In this section, we study the hyperbolicities (mainly k-, $\tilde{k}$-, Brody hyperbolicity) of some Hartogs type domains.

Section 2.1 is devoted to study the hyperbolicities of a Hartogs domain $\Omega = \Omega_{u,h}(G)$ over a domain $G \subset \mathbb{C}^n$ with m-dimensional balanced fibers. In Proposition 2.1.4, k-hyperbolicity of Ω is completely characterized. The remaining part of this section is devoted to study mostly $\tilde{k}$-hyperbolicity of the Hartogs domain Ω. In Theorem 2.1.14 we present some sufficient conditions for Ω to be $\tilde{k}$-hyperbolic. For this, the following properties are used:

- Normality of a special subfamily of $\mathcal{O}(E, G)$;
- Maximum principle of plurisubharmonic functions;
- The fact that h is a quasinorm on $\mathbb{C}^m$.

Theorem 2.1.14 shows that there is a difference between k- and $\tilde{k}$-hyperbolicity of Ω. Moreover, from Proposition 2.1.4 and Theorem 2.1.14, we could obtain the example which were already announced in (ii) of Remark 1.5.4. That is, there is an example of a Hartogs domain that is $\tilde{k}$-hyperbolic, but not k-hyperbolic. In Proposition 2.1.18, we study shortly Brody hyperbolicity of Ω. In Remark 2.1.19, we compare g_Ω with $\tanh \tilde{k}_\Omega$ at some points of Ω.

Section 2.2 starts with studying the hyperbolicities of Hartogs-Laurent domains $\Sigma = \Sigma_{u,v}(G)$ over a domain $G \subset \mathbb{C}^n$. Because of $\Sigma_{u,v}(G) \subset \Omega_{u,|\cdot|}(G)$, the hyperbolicities of Ω imply those of Σ in most cases. From this point of view, our aim is to find the differences between the hyperbolicities of those domains. First of all, in Lemma 2.2.2, we point out that the property "$\max\{u, v\}$ is locally bounded" is necessary for Σ to be k-hyperbolic. However, we do not know whether the converse also holds. Proposition 2.2.4 and Proposition 2.2.5 are obtained by applying the results for the Hartogs domains $\Omega' := \Omega_{u,|\cdot|}(G)$ and $\Omega'' := \Omega_{v,|\cdot|}(G)$ (cf. Lemma 2.1.13/Theorem 2.1.14) which were already shown in section 2.1.

By Theorem 2.1.14, it is known that the hyperbolicities of Hartogs domains Ω' and Ω'' imply those of these base G. So it is natural to ask whether Σ has the same property as Ω' and Ω'', i.e. whether the following implication

$$\text{(Q)} \qquad \Sigma : \text{hyperbolic} \quad \Longrightarrow \quad G : \text{hyperbolic}$$

is true. The remaining part of this section is devoted to give an answer to (Q). In general, the answer is 'NO' for all hyperbolicities. In Lemma 2.2.9, we give a sufficient condition for Σ to be Brody hyperbolic. From this fact and Theorem 1.5.21, as a negative answer to (Q), we find examples of pseudoconvex Hartogs domains for all hyperbolicities (Example 2.2.10).

Example 2.2.18 and Remark 2.2.19 show that for any $n \geq 4$ there exists an example of a domain in $\mathbb{C}^n$ which has a local psh antipeak function at infinity, but which does not have a local psh peak function at infinity. To see it, we use Oka's theorem and a technical lemma (Lemma 2.2.17). In Lemma 2.2.13, we show that the continuity of u, v are necessary conditions for Σ to be taut. Using it, in Theorem 2.2.15, we prove that: Let Σ be a pseudoconvex Reinhardt domain and let $u \not\equiv -\infty$, $v \not\equiv -\infty$. Then Σ is hyperbolic iff G is hyperbolic and $\max\{u, v\} > -\infty$ on G. In this case, the answer to (Q) is always positive for all notions of hyperbolicity (cf. Theorem 1.5.21). Moreover, in Example 2.2.16, we give an example of a non-Reinhardt domain for which (Q) is true for $\tilde{k}$-hyperbolicity.

§2.1. Hyperbolicity of certain Hartogs domains with balanced fibers.

We will investigate the hyperbolicity of Hartogs domains Ω over G with m-dimensional balanced fibers. In this section, unless otherwise stated, we shall use the following notations: Let $G \subset \mathbb{C}^n$ be a domain, $u \in \mathcal{C}^{\uparrow}(G)$, $D = D_h \subset \mathbb{C}^m$ be a balanced domain with the associated Minkowski function h, $h \not\equiv 0$, $H(z,w) := h(w)e^{u(z)}$, $z \in G$, $w \in \mathbb{C}^m$. We write $\Omega = \Omega_H(G) = \Omega_{u,h}(G)$.

First let us check the following elementary properties.

Lemma 2.1.1. *For any $(z,w) \in \Omega$ one has*

$$\tilde{k}_\Omega((z,0),(z,w)) \le p(0,H(z,w)). \tag{2.1.1a}$$

Moreover, if $H \in PSH(G \times \mathbb{C}^m)$, then

$$\tilde{k}_\Omega((z,0),(z,w)) = p(0,H(z,w)), \quad (z,w) \in \Omega. \tag{2.1.1b}$$

Note that this lemma holds also for a (general) function H defining a Hartogs domain $\Omega = \Omega_H(G)$ over G with m-dimensional balanced fibers (cf. Remark 1.1.8).

Proof. Fix $(z,w) \in \Omega$. First suppose $H(z,w) \neq 0$ and define a holomorphic function $\varphi : E \to \mathbb{C}^n \times \mathbb{C}^m$ by

$$\varphi(\lambda) := (z, \frac{\lambda w}{H(z,w)}), \quad \lambda \in E.$$

Then $\varphi \in \mathcal{O}(E,\Omega)$, $\varphi(0) = (z,0)$, and $\varphi(H(z,w)) = (z,w)$. So (2.1.1a) follows directly by the decreasing property of $\tilde{k}$.

Now suppose $H(z,w) = 0$ and define a family $(\varphi_t)_{t>1}$ of analytic discs φ_t by $\varphi_t(\lambda) := (z, \lambda t w)$ for any $\lambda \in E$. Clearly $(\varphi_t)_{t>1} \subset \mathcal{O}(E,\Omega), \varphi_t(0) = (z,0)$, and $\varphi_t(1/t) = (z,w)$ for any $t > 1$. Therefore, (2.1.1a) follows immediately by the decreasing property of $\tilde{k}$.

Next, to show (2.1.1b), let $\varphi \equiv (\varphi_1,\varphi_2) \in \mathcal{O}(E,\Omega)$ such that $\varphi_1(0) = \varphi_1(\sigma) = z$, $\varphi_2(0) = 0$, and $\varphi_2(\sigma) = w$ for some $\sigma \in [0,1)$, where $\varphi_1 \in \mathcal{O}(E,G)$ and $\varphi_2 \in \mathcal{C}(E,\mathbb{C}^m)$. Then φ_2 can be written in the form $\varphi_2(\lambda) = \lambda\tilde{\varphi}_2(\lambda)$ with $\tilde{\varphi}_2 \in \mathcal{O}(E,\mathbb{C}^m)$. Note that $H(\varphi(\lambda)) = |\lambda| H(\varphi_1(\lambda),\tilde{\varphi}_2(\lambda)) < 1$ for any $\lambda \in E$. Since $H \in PSH(G \times \mathbb{C}^n)$, it follows from the maximum principle for subharmonic functions that $H(\varphi_1,\tilde{\varphi}_2) \le 1$, which implies that

$$H(z,w) = H(\varphi(\sigma)) = \sigma H(\varphi_1(\sigma),\tilde{\varphi}_2(\sigma)) \le \sigma.$$

Therefore, we get that $p(0,H(z,w)) \le \tilde{k}_\Omega((z,0),(z,w))$. Since $(z,w) \in \Omega$ is arbitrary, we are done. $\square$

In Lemma 2.1.1, if we replace $\tilde{k}_\Omega$ by c_Ω, then, in general, (2.1.1b) does not hold, even if $H \in PSH(G \times \mathbb{C}^m)$. For example, if $u := \alpha \in \mathbb{R}$ and $h_\alpha := e^\alpha h$, then $\Omega = G \times D_{h_\alpha}$ and also

$$\tanh c_\Omega((z,0),(z,w)) = \tanh c_{D_{h_\alpha}}(0,w) \le h_\alpha(w) = H(z,w), \quad (z,w) \in \Omega.$$

Here, in the first equality, we used the fact that the family $\underline{c}$ has the product-property. If h is not a seminorm, then there exists a point $w_0 \in \mathbb{C}^m$ such that $\tanh c_{D_{h_\alpha}}(0,w_0) \lneqq h_\alpha(w_0)$ (cf. [Jar-Pfl 93], (c) in Proposition 2.2.1; (3) in Remark 1.1.2). On the other hand, in the left side of (2.1.1a), if the first coordinates do not coincide, then (2.1.1a) does not hold in general. For example,

Example 2.1.2. Let $G := \{\lambda \in \mathbb{C} : 0 < |\lambda+1| < 1\}$ and put $H(\lambda,\zeta) := \frac{|\zeta|}{|\lambda+1|}$, $(\lambda,\zeta) \in G \times \mathbb{C}$. Then

$$c^*_\Omega((a,0),(\lambda,\zeta)) = \max\left\{\frac{|a-\lambda|}{|1-(\bar{a}+1)(\lambda+1)|},\ H(\lambda,\zeta)\right\} \geq H(\lambda,\zeta).$$

In particular, $c^*_\Omega((-\frac{1}{2},0),(-\frac{1}{3},\zeta)) > H(-\frac{1}{3},\zeta)$ if $|\zeta| < \frac{1}{6}$.

Proof. Consider two maps

$$\begin{aligned}
&\varphi : \Omega \to \mathbb{C}^2,\ \ \varphi(\lambda,\zeta) := (\lambda+1,\zeta), \quad (\lambda,\zeta) \in \Omega,\\
&\psi : \mathbb{C}_* \times \mathbb{C} \to \mathbb{C}^2,\ \ \psi(\lambda,\zeta) := (\lambda,\zeta/\lambda), \quad (\lambda,\zeta) \in \mathbb{C}_* \times \mathbb{C}.
\end{aligned}$$

Clearly, $\Omega \overset{\varphi}{\cong} \triangle_H \overset{\psi}{\cong} E_* \times E$, where $\triangle_H$ denotes the Hartogs triangle. Thus for any $(a,0),(\lambda,\zeta) \in \Omega$

$$\begin{aligned}
c^*_\Omega((a,0),(\lambda,\zeta)) &= c^*_{E_*\times E}((\psi\circ\varphi)(a,0),(\psi\circ\varphi)(\lambda,\zeta))\\
&= \max\left\{c^*_{E_*}(a+1,\lambda+1), c^*_E(0,\frac{\zeta}{\lambda+1})\right\}\\
&= \max\left\{\frac{|a-\lambda|}{|1-\overline{(a+1)}(\lambda+1)|}, \frac{|\zeta|}{|\lambda+1|}\right\}.
\end{aligned}$$

□

Now we will give a characterization for the k-hyperbolicity of Ω. First note that:

Remark 2.1.3. Observe that $G \times \{0\} \subset \Omega$. Therefore, if $u(z_0) = -\infty$ for some $z_0 \in G$, then $\{z_0\} \times \mathbb{C}^m \subset \Omega$. Hence if Ω is hyperbolic with respect to an invariant function d_Ω, then G is also d-hyperbolic and u is real-valued.

Moreover, we have:

Proposition 2.1.4. *If $\Omega = \Omega_{u,h}(G)$ is d-hyperbolic where d_Ω is a continuous invariant function, then G is d-hyperbolic, $D \Subset \mathbb{C}^m$, and u is locally bounded on G. In the case $d = k$, it holds that*

$$\Omega \text{ is } k\text{-hyperbolic} \quad \Longleftrightarrow \quad \begin{cases} G \text{ is } k\text{-hyperbolic},\\ D \text{ is bounded in } \mathbb{C}^m,\\ u \text{ is locally bounded on } G. \end{cases}$$

Remark 2.1.5. In case that $m = 1$, $h(\lambda) := |\lambda|$, $\lambda \in \mathbb{C}$, and $d = k$, the above result is obtained by N. Q. Dieu & D. D. Thai ([Die-Tha 00]); see also ([Tha-Tho 98]), ([Zai 83]).

To verify the sufficiency of the second assertion in Proposition 2.1.4, we need the following statement which is due to A. Eastwood ([Eas 75]).

Theorem 2.1.6. *Let $\pi : \Omega_1 \to \Omega_2$ be a holomorphic map of domains. If Ω_2 is (k-complete) k-hyperbolic and has an open covering (U_j) such that $\pi^{-1}(U_j)$ is (k-complete) k-hyperbolic, then Ω_1 is also (k-complete) k-hyperbolic.*

In general, in case that $\pi^{-1}(z)$ is k-hyperbolic for any $z \in \Omega_2$, the domain Ω_1 may not be k-hyperbolic. For example, see (1) in Example 2.1.12 below. For a proof of Theorem 2.1.6, see e.g. Theorem 3.2.15 in [Kob 98].

Proof of Proposition 2.1.4. Suppose that d is continuous and Ω is d-hyperbolic.

(i) *u is locally bounded on G:* Suppose not. Then there exist a point $z_0 \in G$ and a sequence $(z_j)_{j\geq 1} \subset G$ such that $\lim_{j\to\infty} z_j = z_0$ and $\lim_{j\to\infty} u(z_j) = -\infty$. Now we take a point $w_0 \in (\mathbb{C}^m)_*$ with $(z_0, w_0) \in \Omega$. Without loss of generality, we may assume that $\{(z_j, w_0)\}_{j\geq 1} \subset \Omega$. So it follows from Lemma 2.1.1 that

$$0 \leq d_\Omega((z_j, 0), (z_j, w_0)) \leq p(0, H(z_j, w_0)), \quad j \geq 1.$$

But,

$$\lim_{j\to\infty} H(z_j, w_0) = h(w_0) \exp\left(\lim_{j\to\infty} u(z_j)\right) = 0.$$

Thus, by the continuity of d_Ω, we then have $d_\Omega((z_0, 0), (z_0, w_0)) = 0$, which is a contradiction to the fact that $w_0 \neq 0$.

(ii) *$D = D_h$ is bounded in $\mathbb{C}^m$:* Suppose not. Let $R > 0$ be so small that $\mathbb{B}_m(0, R) \Subset D$, and choose a sequence $(w_j)_{j\geq 1} \subset D$ such that $\max\{R, 1\} < \|w_j\| \to \infty$ as $j \to \infty$. Fix a point $z^0 \in G$. By (i) it is clear that $u(z^0) > -\infty$. Observe that

$$(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|}) \in \Omega, \quad j \geq 1.$$

By Lemma 2.1.1, one has

$$0 \leq d_\Omega((z_0, 0), (z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|})) \leq p(0, H(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|})), \quad j \geq 1.$$

However,

$$0 \leq H(z_0, Re^{-u(z_0)} \frac{w_j}{\|w_j\|}) = \frac{R}{\|w_j\|} h(w_j) < \frac{R}{\|w_j\|} \overset{j\to\infty}{\longrightarrow} 0.$$

On the other hand, by the compactness of $\{w \in \mathbb{C}^m : \|w\| = 1\}$, we may assume that there exists a point $w_0 \in \mathbb{C}^m$ with $\|w_0\| = 1$ such that $\lim_{j\to\infty}(w_j/\|w_j\|) = w_0$. In particular, $(z_0, Re^{-u(z_0)} w_0) \in \Omega$ by the choice of R. The continuity of d_Ω gives us that

$$\begin{aligned} 0 &\leq d_\Omega((z_0, 0), (z_0, Re^{-u(z_0)} w_0)) \\ &= \lim_{j\to\infty} d_\Omega((z_0, 0), (z_0, \frac{Rw_j/\|w_j\|}{e^{u(z_j)}})) \leq \lim_{j\to\infty} p(0, H(z_0, \frac{Rw_j/\|w_j\|}{e^{u(z_j)}})) = 0, \end{aligned}$$

which is a contradiction to the fact that $e^{-u(z_0)} w_0 \neq 0$.

Now we will verify that the converse of the first assertion, in case $d = k$, is true. Obviously, in view of (b) in Theorem 1.5.7, for every $z \in G$ we may choose a k-hyperbolic open neighborhood $U(z)$ of z in G, so that $(U(z))_{z\in G}$ is an open covering of G. Since

u is locally bounded, we can take $U(z)$ so small that $R(z) := \inf_{z' \in U(z)} u(z') > -\infty$. Hence,

$$\pi^{-1}(U(z)) = \Omega_{u,h}(U(z)) \subset U(z) \times \{w \in \mathbb{C}^m : h(w) < e^{-R(z)}\}$$

where $\pi : \Omega \to G$ is defined by $\pi(z, w) := z$ for $(z, w) \in (G \times \mathbb{C}^m) \cap \Omega$. Since $D = D_h \Subset \mathbb{C}^m$, it follows from (1) of Remark 1.1.4 that there is a $C > 0$ such that $h(w) \geq C\|w\|$ for any $w \in \mathbb{C}^m$. From which we have $V_z := \{w \in \mathbb{C}^m : h(w) < e^{-R(z)}\} \subset \mathbb{B}_m(0, e^{-R(z)}/C)$, i.e. V_z is bounded in $\mathbb{C}^m$. So $\pi^{-1}(U(z))$ is k-hyperbolic, because of the decreasing property of $\underline{k}$. Thus, the required assertion follows directly from Theorem 2.1.6. $\square$

Remark 2.1.7. In the case $m = 1, h(\lambda) = |\lambda|$, there are some differences between the previous proof and the proof in [Die-Tha 00]. For the latter, to show the fact that u is locally bounded on G, the authors did not use Lemma 2.1.1. In this case, there is another simpler direct proof by Professor P. Pflug. More explicitly: Suppose the contrary and keep the same notations as in the previous proof. For any $j \geq 1$ we define a map $\varphi_j : E \to \Omega$ by $\varphi_j(\zeta) := (z_j, e^{-u(z_j)}\zeta)$ for $\zeta \in E$. Then $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(E, \Omega)$ and since $((z_j, w_0))_{j\gg 1} \subset \Omega$, it follows from the continuity of d_Ω and the decreasing property of $\underline{d}$ that

$$0 \leq d_\Omega((z_0, 0), (z_0, w_0)) = \lim_{j\to\infty} d_\Omega((z_j, 0), (z_j, w_0)) \leq \lim_{j\to\infty} d_E(0, e^{u(z_j)}w_0) = 0,$$

for any $j \geq 1$, which is a contradiction to the fact that $w_0 \neq 0$.

Remark 2.1.8. Even if Ω is pseudoconvex with $u \in PSH(G, \mathbb{R})$, the continuity of u, in general, does not imply the c-hyperbolicity of Ω. In particular, there exists a k-complete (of course, k-hyperbolic) Hartogs domain over E with 1-dimensional balanced fibers which is not c-hyperbolic, which is due to N. Sibony (see below (4) in Example 2.1.9).

Example 2.1.9. (1) If $u \in \mathcal{C}^\uparrow(\mathbb{C})$ with $(1/\nu)_{\nu\geq 1} \subset u^{-1}(-\infty)$, then $\Omega = \Omega_{u,|\cdot|}(\mathbb{C})$ is not c-hyperbolic. In fact, $c_\Omega \equiv 0$ because $H^\infty(\Omega) \cong \mathbb{C}$, i.e. all bounded holomorphic functions on Ω are constant. For this, let $f \in H^\infty(\Omega)$. Then the function f can be represented by the Hartogs series

$$f(\lambda, \zeta) = \sum_{j=0}^{\infty} f_j(\lambda)\zeta^j, \quad (\lambda, \zeta) \in \Omega,$$

where $f_j \in \mathcal{O}(\mathbb{C})$ for any $j \geq 0$ (cf. Proposition 1.6.5 in [Jak-Jar 01]). Since f is bounded on Ω, it follows from Liouville's theorem that $f(1/\nu, \cdot)$ is a constant on $\mathbb{C}$ for any $\nu \in \mathbb{N}$. Let $j \geq 1$. By the Cauchy inequality one has $f_j(1/\nu) = 0$ for any $\nu \in \mathbb{N}$. Hence the continuity of f_j yields that $f_j(0) = 0$. Since 0 is the limit point of the sequence $(1/\nu)_{\nu\geq 1}$, the identity theorem implies that $f_j \equiv 0$ for any $j \geq 1$. Thus $f(\lambda, \zeta) = f_0(\lambda)$. Since f_0 is a bounded entire function, it is a constant (by Liouville's theorem), so is f.

(2) If $u(\lambda) := \log|\lambda - 1|$, $\lambda \in E$, then $u \in (\mathcal{C} \cap SH)(E)$ and the domain $\Omega = \Omega_{u,|\cdot|}(E)$ is unbounded pseudoconvex c-hyperbolic (e.g. consider the map $f(z) = (z_1 - 1)z_2$ for $z \in \Omega$).

(3) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $u_\alpha(\lambda) := \alpha \log |\lambda|$, $\lambda \in \mathbb{C}$, then $u_\alpha \in (\mathcal{C} \cap SH)(\mathbb{C})$ and the pseudoconvex domain $\Omega = \Omega_{u_\alpha, |\cdot|}(\mathbb{C})$ is not c-hyperbolic. In fact, $c_\Omega \equiv 0$ because $H^\infty(\Omega) \cong \mathbb{C}$. More explicitly: First note that $\Omega = \{z \in \mathbb{C}^2 : |z_1|^\alpha |z_2| < 1\}$. Let $f \in H^\infty(\Omega)$ with $\|f\|_\infty \leq 1$. Then f can be represented by

$$f(z) = \sum_{\nu=(\nu_1,\nu_2)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}} a_\nu z_1^{\nu_1} z_2^{\nu_2}, \quad z \in \Omega$$

where $(a_\nu)_{\nu\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}}$ is a sequence in $\mathbb{C}$ (cf. Proposition 2.6.2 in [Jak-Jar 01]). Let $(t_1, t_2) \in (\partial\Omega) \cap (\mathbb{R}_{>0})^2$. The Cauchy inequality implies that

$$|a_\nu| \leq t_1^{-\nu_1} t_2^{-\nu_2} = t_1^{\alpha\nu_2 - \nu_1}, \quad \nu = (\nu_1, \nu_2) \in (\mathbb{Z}_{\geq 0})^2.$$

Moreover, either $\alpha\nu_2 - \nu_1 \neq 0$ or $\nu_1 = \nu_2 = 0$. Hence, by letting $t_1 \to 0$ (or $t_1 \to \infty$), we get that $|a_\nu| = 0$ for all $\nu \neq (0,0)$, which implies that $f(z) = a_{(0,0)} \in \mathbb{C}$, so f is a constant.

(4) ([Sib 81], III. An example, pp. 366-369) Let $(a_\nu)_{\nu\geq 1} \subset E$ be a sequence such that $a_\nu \neq a_\mu$ for $\nu \neq \mu$ and every boundary point of E is the nontangential limit of a subsequence of $(a_\nu)_{\nu\geq 1}$. Choose two sequences $(m_\nu)_{\nu\geq 1}$ and $(n_\nu)_{\nu\geq 1}$ of natural numbers satisfying:

- $m_\nu \geq n_\nu$ for $\nu \geq 1$;
- $\sum_{\nu=1}^\infty \frac{1}{n_\nu} \log \frac{|a_\nu|}{2} > -\infty$;
- $\mathbb{B}_1(a_\nu, 3e^{-\nu m_\nu}) \cap \mathbb{B}_1(a_\mu, 3e^{-\mu m_\mu}) = \emptyset$ for $\nu \neq \mu$;
- $\mathbb{B}_1(a_\nu, 3e^{-\nu m_\mu}) \subset E$ for $j \geq 1$.

Define

$$u(\lambda) := \sum_{\nu=1}^\infty \frac{1}{n_\nu} \max\left\{-\nu m_\nu,\, \log\frac{|\lambda - a_\nu|}{2}\right\}, \quad \lambda \in E.$$

Then $u \in (\mathcal{C} \cap SH)(E)$ and the domain $\Omega \equiv \Omega_{u,|\cdot|}(E)$ is k-hyperbolic and k-complete (cf. Corollary 3.5.3 below). However, for any $f \in H^\infty(\Omega)$, there exists $g \in \mathcal{O}(E)$ such that $f(z) = g(z_1)$ for any $z = (z_1, z_2) \in \Omega$, which implies that $H^\infty(\Omega)$ does not separate points in Ω. Hence Ω is not c-hyperbolic.

From now on, we will deal with the $\tilde{k}$-hyperbolicity of a Hartogs domain $\Omega = \Omega_{u,h}(G)$. First, we will try to find a necessary condition for Ω to be $\tilde{k}$-hyperbolic.

Proposition 2.1.10. *Assume that* $\lim_{z\not\to z_0} u(z) = -\infty$ *for some* $z_0 \in G$. *Then* $\tilde{k}_\Omega = 0$ *on* $(\{z_0\} \times \mathbb{C}^m) \cap \Omega$. *In particular,* Ω *is not* $\tilde{k}$*-hyperbolic.*

The fact that u is real-valued on G is a necessary condition for Ω to be $\tilde{k}$-hyperbolic (Remark 2.1.3). If either $u \in PSH(G, \mathbb{R})$ or Ω is $\tilde{k}$-hyperbolic, then $\lim_{z\to z_0} u(z) \neq -\infty$ for any $z_0 \in G$.

Proof. For this, let $w_1, w_2 \in \mathbb{C}^m$ be not all zero with $\max\{h(w_1), h(w_2)\} < e^{-u(z_0)}$. For any $j \geq 1$, put $M_j := \max_{\|w\|\leq r_j} h(w)$ where $r_j := (1+j)\|w_1\| + j\|w_2\|$. Since $h \not\equiv 0$, there exists $j_0 \geq 1$ such that $0 < M_j < \infty$ for $j \gg j_0$. By the assumption, for any $j \geq j_0$, we may take $\delta_j > 0$ such that:

$$0 < \|z - z_0\| < \delta_j \implies z \in G,\ u(z) < -\log M_j.$$

Fix $j \geq j_0$ and choose $\alpha_j > 0$ so small that $\alpha_j < \frac{\delta_j}{(j^2+j)\sqrt{n}}$. Define two mappings $f_j : E \to \mathbb{C}^n$ and $g_j : E \to \mathbb{C}^m$ by

$$f_j(\lambda) := z_0 + \alpha_j(1 - j\lambda)j\lambda\mathbb{I}, \quad g_j(\lambda) := (1 - j\lambda)w_1 + j\lambda w_2, \quad \lambda \in E,$$

where $\mathbb{I} := (1, \cdots, 1) \in \mathbb{C}^n$. Observe that $f_j(0) = f_j(\frac{1}{j}) = z_0$, $g_j(0) = w_1$, $g_j(\frac{1}{j}) = w_2$, and

$$0 < \|f_j(\lambda) - z_0\| < \alpha_j(j^2 + j)\sqrt{n} < \delta_j, \quad \|g_j(\lambda)\| < r_j, \quad \lambda \in E \setminus \{0, \frac{1}{j}\}.$$

Therefore,

$$H(f_j(\lambda), g_j(\lambda)) = h(g_j(\lambda))e^{u(f_j(\lambda))} < M_j e^{-\log M_j} = 1, \quad \lambda \in E.$$

Here we used the fact that $f_j(0) = f_j(1/j) = z_0$, $g_j(0) = w_1$, and $g_j(1/j) = w_2$. Thus, $\Psi_j := (f_j, g_j) \in \mathcal{O}(E, \Omega)$ with $\Psi_j(0) = (z_0, w_1)$, $\Psi_j(\frac{1}{j}) = (z_0, w_2)$, which implies that

$$0 \leq \tilde{k}_\Omega((z_0, w_1), (z_0, w_2)) = \tilde{k}_\Omega(\Psi_j(0), \Psi_j(\frac{1}{j})) \leq p(0, \frac{1}{j}) \overset{j\to\infty}{\longrightarrow} 0. \qquad \square$$

Remark 2.1.11. As a simple consequence of Proposition 2.1.10, if $\Omega = \Omega_{u_\alpha,|\cdot|}$, $\alpha > 0$, is as in (3) of Example 2.1.9, then Ω is not $\tilde{k}$-hyperbolic.

Example 2.1.12. (1) Let $G := E$ and define $u(\lambda) := \log|\lambda|$ for $\lambda \in E \setminus \{0\}$ and $u(0) = 0$. We will say that $\Omega = \Omega_{u,|\cdot|}(E)$ is the *Eisenman-Taylor Domain*. This domain was firstly studied by D. Eisenman and L. Taylor. They showed that Ω is Brody hyperbolic but not k-hyperbolic (cf. [Kob 70], p. 130). If $\pi : \Omega \to G$, $\pi(\lambda, \zeta) := \lambda, (\lambda, \zeta) \in \Omega$, then for any $\lambda \in E$, $\pi^{-1}(\lambda)$ is bounded, so k-hyperbolic (cf. Theorem 2.1.6). On the other hand, by Proposition 2.1.10, it is clear that $\tilde{k}_\Omega = 0$ on $(\{0\} \times \mathbb{C}) \cap \Omega$. Thus the Eisenman-Taylor Domain, itself, is a counterexample for the converse implication of (6) $\Longrightarrow$ (7) in Proposition 1.5.3.

(2) If $D \not\Subset \mathbb{C}^m$, then Ω is, in general, not $\tilde{k}$-hyperbolic. For example:

a. Define $G := \mathbb{C} \setminus \bar{E}$, $u(\lambda) := \log|\lambda|$ $(\lambda \in \mathbb{C})$, and $h(z) = |z_2|$, $z \in \mathbb{C}^2$. Then G is $\tilde{k}$-hyperbolic, $u \in H(G, \mathbb{R})$, $D = D_h \not\Subset \mathbb{C}^2$. Moreover,

$$\Omega = \Omega_{u,h}(G) = \{z \in \mathbb{C}^3 : |z_1| > 1, |z_1 z_2| < 1\}$$

is a pseudoconvex Reinhardt domain in $\mathbb{C}^3$ with $\Omega \cap (\mathbb{C} \times \{0\} \times \mathbb{C}) = G \times \{0\} \times \mathbb{C}$, which implies that Ω is not $\tilde{k}$-hyperbolic (so it is not k-hyperbolic);

b. Let $m \geq 2$, $u \in PSH(G)$ and let $\tilde{h}$ be as in (1) of Example 1.1.5. Obviously, $D = D_{\tilde{h}} \not\Subset \mathbb{C}^m$. By Lemma 2.1.1, it holds that $\tilde{k}_\Omega((z, 0), (z, w)) = p(0, \tilde{h}(w)e^{u(z)})$ for any $(z, w) \in \Omega = \Omega_{u,\tilde{h}}(G)$. This implies that Ω is not $\tilde{k}$-hyperbolic.

Recall that $H \in PSH(G \times \mathbb{C}^m)$ iff $u \in PSH(G)$ and $h \in PSH(\mathbb{C}^m)$. Moreover, any balanced domain $D = D_h$, either bounded or convex, has an associated Minkowski function h that is a quasinorm on $\mathbb{C}^m$.

Lemma 2.1.13. *Suppose that G is Brody hyperbolic, $u \in PSH(G,\mathbb{R})$, $h \in PSH(\mathbb{C}^m)$ is a positive definite quasinorm on $\mathbb{C}^m$. Moreover, assume that*

$(\star)$: *any sequence $(f_\nu)_{\nu\geq 1}$ of holomorphic functions $f_\nu \in \mathcal{O}(r_\nu E, G)$ with $f_1(0) = f_\nu(0) = f_\nu(1)$ for $\nu \geq 1$, where $(r_\nu)_{\nu\geq 1}$ is a sequence in $\mathbb{R}_{>0}$ with $1 < r_\nu < r_{\nu+1} \nearrow \infty$ as $\nu \to \infty$, has a subsequence $(f_{\nu_j})_{j\geq 1}$ converging to an $f \in \mathcal{O}(\mathbb{C}, G)$ uniformly on every compact subset of $\mathbb{C}$.*

Then for any $(a,z),(a,w) \in \Omega \cap (G \times \mathbb{C}^m)$ it holds that

$$(\star\star) \qquad \tilde{k}_\Omega((a,z),(a,w)) = 0 \quad \Longleftrightarrow \quad z = w.$$

Proof. Fix $(a,z),(a,w) \in \Omega \cap (G \times \mathbb{C}^m)$ and assume that $\tilde{k}_\Omega((a,z),(a,w)) = 0$. Then there are two sequences $(r_j)_{j\geq 1} \subset \mathbb{R}$ and $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(r_j E, \Omega)$ such that $\varphi_j(0) = (a,z)$, $\varphi_j(1) = (a,w)$, and $1 < r_j < r_{j+1} \nearrow \infty$ as $j \to \infty$. Let $j \geq 1$ and put $\varphi_j := (f_j, g_j)$, where $f_j \in \mathcal{O}(r_j E, \mathbb{C}^n)$ and $g_j \in \mathcal{O}(r_j E, \mathbb{C}^m)$. Note that the mapping g_j can be written in the form $g_j(\lambda) = z + \lambda \tilde{g}_j(\lambda)$ for some $\tilde{g}_j \in \mathcal{O}(r_j E, \mathbb{C}^m)$. Because of $\varphi_j(r_j E) \subset \Omega$, one has $H(f_j(\lambda), g_j(\lambda)) < 1$ for any $\lambda \in r_j E$. The fact that h is a quasinorm on $\mathbb{C}^m$ gives us that there exists a constant $C > 0$ such that for any $\lambda \in r_j E$

$$|\lambda| H(f_j(\lambda), \tilde{g}_j(\lambda)) = H(f_j(\lambda), \lambda\tilde{g}_j(\lambda)) \leq C\left(H(f_j(\lambda), g_j(\lambda)) + H(f_j(\lambda), z)\right).$$

Now, put $\epsilon := \frac{1}{2}\mathrm{dist}(a, \partial G) > 0$. Obviously, $\mathbb{B}_n(a,\epsilon) \Subset G$ and also

$$\log M := \max_{\|\zeta - a\| \leq \epsilon} u(\zeta) < \infty,$$

because of the upper semicontinuity of u. By the condition $(\star)$ and the fact that G is Brody hyperbolic, without loss of generality we may assume that $f_j \overset{\mathrm{K}}{\Longrightarrow} a$ on $\mathbb{C}$. Thus, for $R > 1$, we may choose $j_R \geq 1$ such that $r_{j_R} > R$ and $f_j(\lambda) \in \mathbb{B}_n(a,\epsilon)$ for $|\lambda| < R$ and $j \geq j_R$. This implies that

$$|\lambda| H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq C(1 + h(z)M), \quad |\lambda| \leq R,\ j \geq j_R.$$

It follows from the maximum principle for the subharmonic function $H(f_j, \tilde{g}_j)$ that

$$H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq \frac{C}{R}(1 + Mh(z)), \quad |\lambda| \leq R,\ j \geq j_R.$$

On the other hand, $f_j(1) = a$, $\tilde{g}_j(1) = g_j(1) - z = w - z$ for any $j \geq 1$, so the previous inequality tells us that $h(w-z)e^{u(a)} \leq \frac{C}{R}(1 + Mh(z))$. Since h is nonnegative and $R > 1$ is arbitrary, we then get that $h(w-z)e^{u(a)} = 0$ by letting $R \to \infty$. Because u is real-valued, it must be $h(w-z) = 0$. The fact that $h^{-1}(0) = \{0\}$ gives us that $z - w = 0$. $\square$

Now we shall give some sufficient conditions for Ω to be $\tilde{k}$-hyperbolic. In particular, by considering the remark noted just below in Proposition 2.1.10, our interest is restricted only to the case where u is psh on G.

Theorem 2.1.14. *Let $u \in PSH(G,\mathbb{R})$, and let $D = D_h \subset \mathbb{C}^m$, whose associated Minkowski function h a positive definite psh quasinorm. If one of the following conditions is satisfied:*
(a) G is taut;
(b) $G \Subset \mathbb{C}^n$;
(c) G is $\tilde{k}$-hyperbolic and u is bounded from above,
then Ω is $\tilde{k}$-hyperbolic.

Proof. Under our hypotheses the base G is always $\tilde{k}$-hyperbolic. So, it is enough to verify that the condition $(\star\star)$ is satisfied. For this, assume that $\tilde{k}_\Omega((a,z),(a,w)) = 0$ for some $(a,z),(a,w) \in \Omega \cap (G \times \mathbb{C}^m)$. Then there exist three sequences $(r_j)_{j\geq 1} \subset \mathbb{R}$, $(f_j)_{j\geq 1} \subset \mathcal{O}(E_j, G)$, and $(g_j)_{j\geq 1} \subset \mathcal{O}(E_j, \mathbb{C}^m)$, where $E_j := r_j E$, such that $(f_j, g_j) \in \mathcal{O}(E_j, \Omega)$, $f_j(0) = f_j(1) = a$, $g_j(0) = z$, $g_j(1) = w$ for $j \geq 1$, and $1 < r_j < r_{j+1} \to \infty$ as $j \to \infty$.

The case (a). Clearly, $\mathcal{F}_1 := \{f_j\big|_{E_1} : j \geq 1\} \subset \mathcal{O}(E_1, G)$. Then the tautness of G and the fact that $f_j(0) = a \in G$ tell us that we can take a sequence $(f_{1j}\big|_{E_1})_{j\geq 1} \subset \mathcal{F}_1$ such that $f_{1j}\big|_{E_1} \overset{\mathrm{K}}{\Longrightarrow} F_1 \in \mathcal{O}(E_1, G)$. Next consider the family $\mathcal{F}_2 := \{f_{1j}\big|_{E_2} : j \geq 1\} \subset \mathcal{O}(E_2, G)$. By the same reasoning, we can obtain a sequence $(f_{2j})_{j\geq 1} \subset \mathcal{F}_2$ such that $f_{2j}\big|_{E_2} \overset{\mathrm{K}}{\Longrightarrow} F_2 \in \mathcal{O}(E_2, G)$. Note that $F_2\big|_{E_1} \equiv F_1$. Continuing this process, for any $\nu \geq 2$ we may extract a sequence $(f_{\nu j}\big|_{E_\nu})_{j\geq 1} \subset \mathcal{F}_\nu := \{f_{\nu-1\, j}\big|_{E_{\nu-1}} : j \geq 1\}$ such that $f_{\nu j}\big|_{E_\nu} \overset{\mathrm{K}}{\Longrightarrow} F_\nu \in \mathcal{O}(E_\nu, G)$; moreover, $F_\nu\big|_{E_{\nu-1}} \equiv F_{\nu-1}$. Therefore, the diagonal sequence $(f_{jj})_{j\geq 1}$ converges uniformly to a map $F \in \mathcal{O}(\mathbb{C}, G)$ with $F\big|_{E_j} \equiv F_j$ for any $j \geq 1$. Hence the condition $(\star)$ holds, so does $(\star\star)$.

The case (b). In view of Montel's theorem, the family $\mathcal{T}_1 := \{f_\ell\big|_{E_1} : \ell \geq 1\}$ is normally convergent in $\mathcal{O}(E_1, \bar{G})$, i.e. there exists a sequence $(f_{1\ell}\big|_{E_1})_{\ell\geq 1} \subset \mathcal{T}_1$ such that $f_{1\ell}\big|_{E_1} \overset{\mathrm{K}}{\Longrightarrow} T_1 \in \mathcal{O}(E_1, \bar{G})$. In particular, $T_1(0) = a$. Applying Montel's theorem to the corresponding family $\mathcal{T}_2 := \{f_{1\ell}\big|_{E_2} : \ell \geq 1\}$, we get, as in the case (a), a subsequence of $\mathcal{T}_2$ which converges to a map $T_2 \in \mathcal{O}(E_2, \bar{G})$. By induction, we may take a subsequence of $(f_j)_{j\geq 1}$ which converges to a map $T \in \mathcal{O}(\mathbb{C}, \bar{G})$ with $T\big|_{E_j} \equiv T_j$ for any $j \geq 1$. On the other hand, since G is bounded in $\mathbb{C}^n$, we can take $s > 0$ so large that $G \Subset \mathbb{B}_n(0,s)$. Therefore, $T \in \mathcal{O}(\mathbb{C}, \mathbb{B}_n(0,s))$. Since every bounded Euclidean ball in $\mathbb{C}^n$ is Brody hyperbolic and $T(0) = a$, we get that $T \equiv \text{constant} = a$ and thus the condition $(\star)$ holds, so does $(\star\star)$.

The case (c). For this, we will use a similar argument as in the proof of Lemma 2.1.13. Take $N > 0$ so large that

$$\sup_{|\lambda|<r_j} u(f_j(\lambda)) \leq \log N, \quad j \geq 1.$$

Let $j \geq 1$. Observe that the mapping g_j can be written in the form $g_j(\lambda) = z + \lambda \tilde{g}_j(\lambda)$ for some $\tilde{g}_j \in \mathcal{O}(E_j, \mathbb{C}^m)$. By our assumption for h, we may take a constant $C > 0$ so large that

$$|\lambda| H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq C\left(H(f_j(\lambda), g_j(\lambda)) + H(f_j(\lambda), z)\right) \leq C(1 + Nh(z)), \quad \lambda \in E_j.$$

Thus the maximum principle for the subharmonic function $H(f_j, \tilde{g}_j)$ implies that

$$H(f_j(\lambda), \tilde{g}_j(\lambda)) \leq \frac{C}{r_j}(1 + Nh(z)), \quad \lambda \in E_j.$$

In particular, the right hand side of the previous inequality tends to 0 as $j \to \infty$, because of the nonnegativity of h. Since u is real-valued, h is positive definite, and

$$H(f_j(1), \tilde{g}_j(1)) = h(g_j(1) - z)e^{u(a)} = h(z - w)e^{u(a)}, \quad j \geq 1,$$

it follows that $w - z = 0$. Thus the required result $(\star\star)$ is obtained. $\square$

Example 2.1.15. Assume that $u \in PSH(G, \mathbb{R})$ and that $D = D_h \subset \mathbb{C}^m$ has the associated Minkowski function h which is a psh quasinorm with $h^{-1}(0) = \{0\}$. Then:

(1) As a simple consequence of Theorem 2.1.14, one has that the domain $\Omega = \Omega_{u,h}(G)$ is $\tilde{k}$-hyperbolic if G is a $\tilde{k}$-hyperbolic domain satisfying one of the following conditions:

- $n = 1$, i.e. $G \subset \mathbb{C} \setminus \{\text{two points}\}$ (cf. (1) in Remark 1.3.2);
- $G \subset \mathbb{C}^n$ is a pseudoconvex Reinhardt domain (cf. Theorem 1.5.21);

(2) Other examples of a $\tilde{k}$-hyperbolic domain Ω can be found in Example 3.1.5 below. Those examples do not satisfy any of the conditions (a), (b), (c) in Theorem 2.1.14. Nevertheless they satisfy the condition $(\star)$ in Lemma 2.1.13;

(3) Let G be a domain. Suppose that $\mathcal{O}(E, G)$ is equicontinuous w.r.t $\|\cdot\|$ and any sequence $(f_\nu)_{\nu\geq 1}$ of holomorphic functions $f_\nu \in \mathcal{O}(r_\nu E, G)$ with $f_1(0) = f_\nu(0) = f_\nu(1)$ for $\nu \geq 1$ converges pointwise to a map in $\mathcal{C}(\mathbb{C}, G)$. Here, $(r_\nu)_{\nu\geq 1}$ is a sequence in $\mathbb{R}_{>0}$ with $1 < r_\nu < r_{\nu+1} \nearrow \infty$ as $\nu \to \infty$. Then Ω is $\tilde{k}$-hyperbolic. For this, in view of Lemma 2.1.13, it is enough to see that the condition $(\star)$ is satisfied. Since $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$, so is $\mathcal{F} := \{f_\nu : \nu \geq 1\}$. By the Arzelà-Ascoli theorem the family $\mathcal{F}$ is relatively compact in $\mathcal{O}(\mathbb{C}, G)$, which means that $\mathcal{F}$ contains a subsequence converging to a map in $\mathcal{O}(\mathbb{C}, G)$ uniformly on every compact subset of $\mathbb{C}$. Thus the desired condition $(\star)$ is obtained.

Remark 2.1.16. Let $\Omega = \Omega_{u,|\cdot|}(E)$ be the Eisenman-Taylor Domain. Recall that Ω is not $\tilde{k}$-hyperbolic; moreover, u is not psh. However, by using the Montel's theorem, it is easy to check that

$$\tilde{k}_\Omega((a,z),(a,w)) > 0 \quad \text{whenever} \quad (a,z),(a,w) \in \Omega \cap (\mathbb{C}_* \times \mathbb{C}),\ z \neq w.$$

Now we will give some concrete examples of $\tilde{k}$-hyperbolic but not k-hyperbolic domains; such an example is a counterexample for the converse implication of (5) $\Rightarrow$ (6) in Proposition 1.5.3.

Example 2.1.17. (1) As a generalization of the big Picard theorem, it is well-known that: Let $G \subset \mathbb{C}^n$ be a k-hyperbolic domain. If $f \in \mathcal{O}(E_*, G)$ has the following property:

$$\text{(2.1.17a)} \qquad \begin{cases} \text{there is a sequence } (\lambda_\nu)_{\nu\geq 1} \subset E_* \text{ converging to 0 such that} \\ \text{the sequence } (f(\lambda_\nu))_{\nu\geq 1} \text{ converges to a point } z_0 \in G. \end{cases}$$

Then f extends to a map $\tilde{f} \in \mathcal{O}(E, G)$. This is due to M. H. Kwack ([Kwa 69]).

We say that a domain $G \subset \mathbb{C}^n$ has the *E_*-extension property* if for every $f \in \mathcal{O}(E_*, G)$ there exists an $\tilde{f} \in \mathcal{O}(E, G)$ with $\tilde{f}\big|_{E_*} = f$. This notion was firstly introduced in [Tha 91].

In 1998, D. D. Thai and P. Thomas have constructed the following example: For $\nu \geq 1$, choose $\alpha_\nu, \beta_\nu \in (0,1)$, $\lambda_\nu \in E_*$, such that

$$\lim_{\nu\to\infty} \lambda_\nu = 0, \quad \lim_{\nu\to\infty} \beta_\nu^{\alpha_\nu} = 0, \quad \sum_{\nu=1}^{\infty} \alpha_\nu \log|\lambda_\nu| > -\infty$$

(e.g. $\lambda_\nu := 2^{-\nu}$, $\alpha_\nu := 2^{-\nu-1}$, $\beta_\nu := \nu^{-1/\alpha_\nu}$). If we define a function $u : E \to [-\infty, \infty)$ by

$$u(\lambda) := \sum_{\nu=1}^{\infty} \alpha_\nu \log(\beta_\nu^2 + |\lambda - \lambda_\nu|^2), \quad \lambda \in E.$$

Obviously, $u \in SH(E,\mathbb{R}) \cap \mathcal{C}^\infty(E_*)$ and $\liminf_{\lambda \not\to 0} u(\lambda) = -\infty$. Hence $\Omega = \Omega_{u,|\cdot|}(E) \subset \mathbb{C}^2$ is a pseudoconvex domain which has the E_*-extension property (see [Tha-Tho 98]; pp. 1128-1129). It is Brody hyperbolic but not k-hyperbolic by Proposition 2.1.4. Moreover, according to Theorem 2.1.14, the domain Ω is $\tilde{k}$-hyperbolic. Sometimes we will say that Ω is the *Thai-Thomas Domain*.

Now we can ask whether any $\tilde{k}$-hyperbolic domain has the E_*-extension property. Unfortunately, the answer is negative. Recall that any domain which has the E_*-extension property is Brody hyperbolic and pseudoconvex (cf. [Tha 91], p. 21, Corollary 1.10). Even though any $\tilde{k}$-hyperbolic domain is Brody hyperbolic (Proposition 1.5.3), it does not need to be pseudoconvex. In addition, the k-completeness of a domain Ω, in general, does not imply the fact that Ω have the E_*-extension property. For example, take $\Omega := E_*$, $f := \mathrm{id}_{E_*}$, which does not have the property (2.1.17a).

Next, we shall consider the following condition for a domain $G \subset \mathbb{C}^n$:

$$\text{(2.1.17b)} \qquad \begin{cases} \text{every sequence } (f_j)_{j\geq 1} \subset \mathcal{O}(E_*, G) \text{ has a subsequence } (f_{j_\nu})_{\nu \geq 1} \\ \text{such that } f_{j_\nu} \overset{\mathrm{K}}{\rightrightarrows} {}^\exists f_0 \in \mathcal{O}(E_*, G) \text{ as } \nu \to \infty. \end{cases}$$

Clearly, any domain in $\mathbb{C}^n$ satisfying the condition (2.1.17b) is taut. Using the Kontinuitätssatz, we can verify that any domain in $\mathbb{C}^n$ satisfying the condition (2.1.17b) is pseudoconvex. On the other hand, it is easy to check that the Thai-Thomas Domain Ω does not satisfy the condition (2.1.17b). In fact, any domain in $\mathbb{C}^n$ which has the E_*-extension property and satisfies the condition (2.1.17b) is taut (so k-hyperbolic).

(2) The next example was constructed by K. Diederich and N. Sibony ([Die-Sib 79]; or [Jar-Pfl 93], Remark 3.5.11). It gives an example of a domain which has, in some sense, a very strange complex structure: Define

$$u(\lambda) := \sum_{k=2}^{\infty} \frac{1}{k^2} \max\Big\{ -k^3, \log \frac{|\lambda - \frac{1}{k}|}{2} \Big\}, \ \lambda \in E.$$

Put $G := \Big\{ z \in E \times \mathbb{C} : |z_2| e^{\|z\|^2 + u(z_1)} < 1 \Big\}$. Then $u \in SH(E,\mathbb{R})$ and the pseudoconvex domain G is Brody hyperbolic, but not k-hyperbolic because $k_G((0,0),(0,w)) = 0$ for any $(0,w) \in G$ (cf. [Jar-Pfl 93], Example 3.5.11). However, as a simple consequence of Theorem 2.1.14, we can check that G is $\tilde{k}$-hyperbolic. In detail: Since $|z_2| e^{u(z_1)} < e^{-\|z\|^2} \leq 1$ for any $z \in G$, one has $G \subset \Omega_{u,|\cdot|}(E) \equiv \Omega$. By Theorem 2.1.14 Ω is $\tilde{k}$-hyperbolic. Therefore G is also $\tilde{k}$-hyperbolic. Here we used the decreasing property of $\tilde{k}$.

Proposition 2.1.18. *If G, D are Brody hyperbolic and $u > -\infty$ on G, then also Ω is Brody hyperbolic. Conversely, if Ω is Brody hyperbolic, so is G.*

Similarly, if G, D contain no affine complex lines and $u > -\infty$ on G, then Ω contains no affine complex lines, and conversely, if G contains an affine complex line, so does Ω.

Proof. Let $(f, g) \in \mathcal{O}(\mathbb{C}, (G \times \mathbb{C}^m) \cap \Omega)$. Suppose that G, D are Brody hyperbolic. Then $f \equiv$ constant $=: z_0 \in G$, and so

$$h(e^{u(z_0)}g(\lambda)) = h(g(\lambda))e^{u(z_0)} < 1, \quad \lambda \in \mathbb{C}.$$

Thus the mapping $\tilde{g} := e^{u(z_0)}g$ is in $\mathcal{O}(\mathbb{C}, D)$. Using that D is Brody hyperbolic, we conclude that $\tilde{g}$ is a constant. On the other hand, assume that G is not Brody hyperbolic and let $\varphi \in \mathcal{O}(\mathbb{C}, G)$. Then $H(\varphi, 0) \equiv 0$ on $\mathbb{C}$, which implies that Ω is not Brody hyperbolic. Similarly, the other two assertions are true. $\square$

Finally, we will finish this section by considering a relationship between the pluricomplex Green function g_Ω and the Lempert function $\tilde{k}_\Omega$. Since $\hat{g}_\Omega := \tanh^{-1} g_\Omega \leq \tilde{k}_\Omega$, the $\hat{g}$-hyperbolicity of Ω implies its $\tilde{k}$-hyperbolicity.

Remark 2.1.19. By Lemma 2.1.1 one has

$$g_\Omega((a, 0), (a, z)) \leq H(a, z), \quad (a, z) \in \Omega. \tag{2.1.19a}$$

Now we will show that if $\log H \in PSH(G \times \mathbb{C}^m)$ and u is locally bounded on G, then

$$g_\Omega((a, 0), (a, z)) \geq H(a, z), \quad (a, z) \in \Omega. \tag{2.1.19b}$$

For this, fix $(a, z) \in \Omega$. By the assumption of u we may find an open neighborhood $U = U(a) \Subset G$ so small that $u \leq \log M_2$ on U for some $M_2 > 0$. Since $h \in \mathcal{C}^\uparrow(\mathbb{C}^m)$, one has $M_1 := \sup_{\|\zeta\| \leq 1} h(\zeta) < \infty$. Hence one has $h(w) \leq M_1\|(b - a, w)\|$ for $b \in U$ and $w \in \mathbb{C}^m$, and also

$$H(b, w) = h(w)e^{u(b)} \leq M_1\|(b - a, w)\|e^{u(b)} \leq M_1M_2\|(b - a, w)\|$$

for any $(b, w) \in U \times \mathbb{C}^m$. This tells us that the function $\log H$ has a logarithmic pole at $(a, 0)$. Since $\log H \in PSH(\Omega, [-\infty, 0))$ and $H(a, 0) = 0$, we then get the required inequality (2.1.19b). Moreover, the property (2.1.19a) gives us that

$$g_\Omega((a, 0), (a, z)) = H(a, z), \quad (a, z) \in \Omega.$$

Therefore, by Lemma 2.1.1,

$$\tanh c_\Omega((a, 0), (a, z)) \leq H(a, z) = g_\Omega((a, 0), (a, z)) = \tanh \tilde{k}_\Omega((a, 0), (a, z))$$

for any $(a, z) \in \Omega$.

§2.2. Hyperbolicity of Hartogs-Laurent domains.

In this section, unless otherwise stated, we will keep the following notations: Let $G \subset \mathbb{C}^n$ be a domain, $u, v \in \mathcal{C}^{\uparrow}(G)$ with $u + v < 0$ on G, and $\Sigma \equiv \Sigma_{u,v}(G)$. Denote $\Sigma' = \Sigma'_{u,v}(G) := \{(z, w) \in G \times \mathbb{C}_* : (z, 1/w) \in \Sigma\}$.

First let us observe that:

Remark 2.2.1. Let $\underline{d}$ be a family of invariant functions. By the decreasing property of $\underline{d}$, the following properties are true:

(1) Clearly, $\{z_0\} \times \mathbb{C}_* \subset \Sigma$ whenever $u(z_0) = v(z_0) = -\infty$ for some $z_0 \in G$. Therefore the d-hyperbolicity of Σ implies that $\max\{u, v\} > -\infty$ on G. But, in general, the converse does not hold.

(2) Obviously, Σ is biholomorphic to Σ'. Moreover Σ is d-hyperbolic iff Σ' is d-hyperbolic.

(3) If $\Omega_{u,|\cdot|}(G)$ or $\Omega_{v,|\cdot|}(G)$ is d-hyperbolic, then $\Sigma_{u,v}(G)$ is d-hyperbolic (and also $\Sigma'_{u,v}(G)$).

Moreover, we have the following properties:

Lemma 2.2.2. *(1) If an invariant function d_Σ is continuous and Σ is d-hyperbolic, then* $\max\{u, v\}$ *is locally bounded on G.*

(2) If G is k-hyperbolic and u (or v) is locally bounded on G, then Σ is also k-hyperbolic.

Proof. (1) Suppose the contrary. Since $u, v \in \mathcal{C}^{\uparrow}(G)$, there exist a point $z_0 \in G$ and a sequence $(z_j)_{j\geq 1} \subset G$ converging to z_0 such that

$$\max\{u(z_j), v(z_j)\} \in \mathbb{R}_{<0} \ (j \geq 1), \qquad \lim_{j\to\infty} u(z_j) = \lim_{j\to\infty} v(z_j) = -\infty.$$

Now we may choose[1] a sequence $(\alpha_j)_{j\geq 1} \subset (0, 1]$ such that

$$v(z_j) \leq \alpha_j u(z_j) < 0 \ (j \geq 1), \qquad \lim_{j\to\infty} \alpha_j u(z_j) = -\infty.$$

Fix a point $\lambda_0 \in \mathbb{C}_*$ so that $(z_0, \lambda_0) \in \Sigma$ and $\lambda_0 = e^{\zeta_0}$ for some $\zeta_0 \in \mathbb{R}$.

(i) The case $\lambda_0 \leq 1$. Without loss of generality, we may assume that $\alpha_j u(z_j) - \zeta_0 < 0$ for any $j \geq 1$. Let $j \geq 1$. We define a holomorphic mapping $\varphi_j : E \to G \times \mathbb{C}$ by

$$\varphi_j(\lambda) := \left(z_j,\ \lambda_0 e^{(\alpha_j u(z_j) - \zeta_0)\lambda}\right), \quad \lambda \in E.$$

Observe that

$$\begin{aligned} e^{v(z_j)} \leq e^{\alpha_j u(z_j)} = \lambda_0 e^{\alpha_j u(z_j) - \zeta_0} &< e^{\alpha_j u(z_j) \operatorname{Re}\lambda} \\ &= \left| e^{\alpha_j u(z_j)\lambda} \right| < e^{-\alpha_j u(z_j)} < e^{-u(z_j)} \end{aligned}$$

for any $\lambda \in E$, so $\varphi_j \in \mathcal{O}(E, \Sigma)$; moreover $\varphi_j(0) = (z_j, \lambda_0) \in \Sigma$. Note that $\Sigma_0 := \{\lambda \in \mathbb{C} : e^{v(z_0)} < |\lambda| < e^{-u(z_0)}\}$ is a nonempty open subset of $\mathbb{C}$. Take $w_0 \in \mathbb{C}_*$ with $v(z_0) - \zeta_0 < \operatorname{Re} w_0 < -u(z_0) - \zeta_0$. Then $\lambda_0 e^{w_0} \in \Sigma_0$. Since $\zeta_j := \alpha_j u(z_j) - \zeta_0 \to -\infty$

[1] e.g. if $v(z_j) \leq u(z_j)$, then put $\alpha_j := 1$; if $v(z_j) > u(z_j)$, then put $\alpha_j := \frac{v(z_j)}{2u(z_j)}$.

as $j \to \infty$, it is clear that $w_0/\zeta_j \in E$ and $\varphi_j(w_0/\zeta_j) = (z_j, \lambda_0 e^{w_0})$ for $j \gg 1$. Therefore the continuity of d_Σ and the decreasing property of $\underline{d}$ give us that

$$
\begin{aligned}
0 \le d_\Sigma((z_0,\lambda_0),\,(z_0,\lambda_0 e^{w_0})) &= \lim_{j\to\infty} d_\Sigma((z_j,\lambda_0),\,(z_j,\lambda_0 e^{w_0})) \\
&= \lim_{j\to\infty} d_\Sigma(\varphi_j(0),\varphi(\frac{w_0}{\zeta_j})) \le \lim_{j\to\infty} p(0,\frac{w_0}{\zeta_j}) = 0,
\end{aligned}
$$

which is a contradiction to the d-hyperbolicity of Σ.

(ii) The case $\lambda_0 > 1$. Recall that the function $\Phi = (\Phi_1, \Phi_2) : \Sigma \to \Sigma'$ defined by $\Phi(z,\lambda) := (z, \frac{1}{\lambda})$ for $(z,\lambda) \in G \times \mathbb{C}$ is biholomorphic. Put $\Phi_2(z_0,\lambda_0) = \frac{1}{\lambda_0} =: \lambda_0' \in E_*$. By applying the case $\lambda_0' \le 1$ to the first case we obtain that Σ' is not d-hyperbolic, but this is a contradiction to (2) of Remark 2.2.1.

(2) It follows directly from Proposition 2.1.4 and the two properties (2) and (3) of Remark 2.2.1. $\square$

Proposition 2.2.3. *If* $\lim_{z \not\to z_0} \max\{u(z), v(z)\} = -\infty$ *for some* $z_0 \in G$*, then* $\tilde{k}_\Sigma = 0$ *on* $(\{z_0\} \times \mathbb{C}) \cap \Sigma$*. In particular,* Σ *is not* $\tilde{k}$*-hyperbolic.*

The fact that $\max\{u, v\}$ is real-valued on G is necessary for Ω to be $\tilde{k}$-hyperbolic ((1) in Remark 2.2.1). If either $\max\{u,v\} \in PSH(G,\mathbb{R})$ or Ω is $\tilde{k}$-hyperbolic, then $\lim_{z\to z_0} \max\{u(z), v(z)\} \ne -\infty$ for any $z_0 \in G$.

Proof. For this it suffices that

$$
\tilde{k}_\Sigma((z_0,w'),(z_0,w'')) = 0, \quad (z_0,w'),(z_0,w'') \in \Sigma,\ w' \in \mathbb{R}.
$$

To show this, fix two point $(z_0, e^\alpha), (z_0, e^{\beta+i\theta}) \in \Sigma$, $\alpha, \beta \in \mathbb{R}$, $0 \le \theta < 2\pi$, where $i^2 = -1$. For $j \ge 1$ put $r_j := \alpha - j(|\alpha - \beta| + 2\pi)$ and $R_j := |\alpha| + j(|\alpha| + |\beta| + 2\pi)$. By the hypothesis, we may take $j_0 \ge 1$ so large that for any $j \ge j_0$, there exists $\delta_j > 0$ such that:

(2.2.3a) $$0 < \|z_0 - z\| < \delta_j \implies z \in G, \quad \max\{u(z), v(z)\} < \min\{r_j, -R_j\}.$$

Fix $j \ge j_0$ and choose $C_j > 0$ so small that $C_j < \frac{\delta_j}{(j^2+j)\sqrt{n}}$. Define two analytic disks $f_j : E \to \mathbb{C}^n$ and $g_j : E \to \mathbb{C}$ by

$$
f_j(\lambda) := z_0 + C_j(1 - j\lambda)j\lambda\mathbb{I}, \quad g_j(\lambda) := e^{(1-j\lambda)\alpha + j\lambda(\beta+i\theta)}, \quad \lambda \in E,
$$

where $\mathbb{I} := (1, \cdots, 1) \in \mathbb{C}^n$. Observe that $f_j(0) = f_j(\frac{1}{j}) = z_0$, $g_j(0) = e^\alpha =: w'$, $g_j(\frac{1}{j}) = e^{\beta+i\theta} =: w''$, and $0 < \|f_j(\lambda) - z_0\| < C_j(j^2 + j)\sqrt{n} < \delta_j$ for any $\lambda \in E \setminus \{0, \frac{1}{j}\}$. Hence it follows from (2.2.3a) that for any $\lambda \in E$,

$$
\begin{aligned}
e^{v(f_j(\lambda))} < e^{r_j} &\le e^{\alpha - j(\alpha-\beta)\mathrm{Re}\,\lambda - j\theta \mathrm{Re}(i\lambda)} \\
&= e^{\mathrm{Re}((1-j\lambda)\alpha + j\lambda(\beta+i\theta))} = |g_j(\lambda)| \le e^{R_j} < e^{-u(f_j(\lambda))}.
\end{aligned}
$$

Here we used the fact that $f_j(0) = f_j(1/j) = z_0$, $g_j(0) = w'$, $g_j(1/j) = w''$, and $(z_0, w'), (w_0, w'') \in \Sigma$. Thus, $\Psi_j := (f_j, g_j) \in \mathcal{O}(E, \Sigma)$ with $\Psi_j(0) = (z_0, w')$, $\Psi_j(\frac{1}{j}) = (z_0, w'')$. Then we get that

$$
0 \le \tilde{k}_\Sigma((z_0,w'),(z_0,w'')) = \tilde{k}_\Sigma(\Psi_j(0), \Psi_j(\frac{1}{j})) \le p(0, \frac{1}{j}).
$$

Therefore, by setting $j \to \infty$, we obtain that $\tilde{k}_\Sigma((z_0,w'),(z_0,w'')) = 0$. $\square$

The next statement follows immediately from Theorem 2.1.14 and (3) of Remark 2.2.1.

Proposition 2.2.4. *Suppose that* $u \in PSH(G, \mathbb{R})$ *(resp.* $v \in PSH(G, \mathbb{R})$*). If one of the following conditions is satisfied:*
(a) G *is taut;*
(b) $G \Subset \mathbb{C}^n$*;*
(c) G *is* $\tilde{k}$*-hyperbolic and* u *(resp.* v*) is bounded from above on* G*,*
then Σ *is* $\tilde{k}$*-hyperbolic.*

If a domain G is taut or bounded, then the condition $(\star)$ for G holds. Compare the proof of Theorem 2.1.14. In the same case, to show that a domain $\Omega_{u,|\cdot|}$ (resp. $\Omega_{v,|\cdot|}(G)$) satisfies the condition $(\star\star)$ in Lemma 2.1.13, it suffices that the function u (resp. v) is psh and real-valued on G, such as in Lemma 2.1.13. Hence, in view of (2) and (3) in Remark 2.2.1, to show that the domain $\Sigma_{u,v}(G)$ with $u, v \in PSH(G)$ is $\tilde{k}$-hyperbolic, we only need the fact that $u(z) > -\infty$ or $v(z) > -\infty$ for any $z \in G$. Thus we have also the following properties:

Proposition 2.2.5. *If one of the following conditions is satisfied:*
(a) G *is taut and* $u, v \in PSH(G)$ *with* $\max\{u, v\} > -\infty$ *on* G*;*
(b) $G \Subset \mathbb{C}^n$*,* $u, v \in PSH(G)$ *with* $\max\{u, v\} > -\infty$ *on* G*,*
then Σ *is* $\tilde{k}$*-hyperbolic.*

So far we have discussed the k- and $\tilde{k}$-hyperbolicity of Σ. Now we shall give a sufficient condition for Σ to be Brody hyperbolic.

Proposition 2.2.6. *If* G *is Brody hyperbolic and* $\max\{u, v\} > -\infty$ *on* G*, then* Σ *is Brody hyperbolic.*

Proof. Suppose the contrary. Then there exists a nonconstant holomorphic mapping $(f, g) \in \mathcal{O}(\mathbb{C}, \Sigma)$, $f \in \mathcal{O}(\mathbb{C}, G)$, $g \in \mathcal{O}(\mathbb{C})$. Clearly, the map f must be a constant because G is Brody hyperbolic, i.e. $f = z_0$ for some $z_0 \in G$. Thus g is not a constant. Moreover, $g(\mathbb{C}) \subset \{\lambda \in \mathbb{C} : e^{v(z_0)} < |\lambda| < e^{-u(z_0)}\}$. Hence it follows from the little Picard theorem that $u(z_0) = v(z_0) = -\infty$, which is a contradiction. $\square$

After this, we like to study the differences between the hyperbolicities of $\Sigma_{u,v}(G)$ and $\Omega_{u,|\cdot|}(G)$.

Remark 2.2.7. As above, the hyperbolicity of $\Sigma_{u,v}(G)$ is similar to that of $\Omega_{u,|\cdot|}(G)$ (or $\Omega_{v,|\cdot|}(G)$), which was already observed in §2.1. Recall that if $\Omega = \Omega_{u,\|\cdot\|}(G)$ is d-hyperbolic, so is G (Remark 2.1.3). In view of Lemma 2.1.13, it seems that Σ is $\tilde{k}$-hyperbolic whenever $u \in PSH(G)$ and G is Brody hyperbolic though it is not $\tilde{k}$-hyperbolic. From that, we can ask the following question:

(Q) "Does the hyperbolicity of $\Sigma_{u,v}(G)$ imply the hyperbolicity of G ?"

From now on we shall mainly deal with the above mentioned question (Q). We start with observing the following examples.

Example 2.2.8. (1) There exists a Brody hyperbolic domain Σ with $\max\{u, v\} > -\infty$ on G, such that G is not Brody hyperbolic. For example, put $G := \mathbb{C}$, $u(\lambda) := -\log(|\lambda| + 1), v(\lambda) := \log|\lambda|$, $\lambda \in \mathbb{C}$. Then $\Sigma = \Sigma_{u,v}(G)$ is Brody hyperbolic but not pseudoconvex. That is, in the case where Σ is not pseudoconvex, the answer to (Q) with respect to the Brody hyperbolicity is, in general, negative.

(2) There exists a Brody hyperbolic domain $\Sigma \subset \mathbb{C}^3$ that is not $\tilde{k}$-hyperbolic. For example, let G be the Eisenman-Taylor Domain. Define

$$u(z) = v(z) := \log\left(\frac{1 + |z_1(z_1 + z_2)|}{3}\right), \quad z = (z_1, z_2) \in G.$$

Clearly, $\max\{u, v\} > -\infty$ on G and $G \times \{1\} \subset \Sigma_{u,v}(G) = \Sigma$. Then it holds that

$$0 \leq \tilde{k}_\Sigma(((0, \lambda), 1), ((0, 0), 1)) \leq \tilde{k}_G((0, \lambda), (0, 0)), \quad \lambda \in E.$$

Thus, by (1) of Example 2.1.12, the domain Σ is not $\tilde{k}$-hyperbolic.

Next, to discuss the problem to (Q) in other cases, we need the following auxiliary lemma:

Lemma 2.2.9. *Let $G \subset \mathbb{C}^n$ be a domain and let $u \in PSH(G, \mathbb{R})$ be nonconstant and bounded from below on G. Suppose that the domain G is not Brody hyperbolic and that $u \circ \varphi$ is not a constant for any nonconstant $\varphi \in \mathcal{O}(\mathbb{C}, G)$. Then the domain $\Sigma := \Sigma_{u,-\infty}(G)$ is Brody hyperbolic.*

Proof. Suppose the contrary. Then there exists a nonconstant mapping $\psi := (\psi_1, \psi_2) \in \mathcal{O}(\mathbb{C}, \Sigma)$, where $\psi_1 \in \mathcal{O}(\mathbb{C}, G)$ and $\psi_2 \in \mathcal{O}(\mathbb{C}, \mathbb{C})$. By our assumption, we can choose a constant $M > 0$ so large that $u > -\log M$ on G, which implies that $|\psi_2(\lambda)| < M$ for any $\lambda \in \mathbb{C}$. Then Liouville's theorem implies that $\psi_2 \equiv$ constant $=: A \in \mathbb{C}_*$. On the other hand, our assumption gives us that $u \circ \psi_1$ is not a constant on $\mathbb{C}$. Hence, it follows from the Liouville type theorem for subharmonic functions that there exists a sequence $(\lambda_\nu)_{\nu \geq 1} \subset \mathbb{C}$ such that $u(\psi_1(\lambda_\nu)) \to \infty$ as $\nu \to \infty$. Therefore, we can take a $\nu_0 \in \mathbb{N}$ such that $0 < e^{-u(\psi_1(\lambda_\nu))} < |A|$ for any $\nu \geq \nu_0$, which is a contradiction to the fact that $\psi(\mathbb{C}) \subset \Sigma$. $\square$

Now we give an example of a hyperbolic pseudoconvex Reinhardt domain Σ such that its base G is not hyperbolic.

Example 2.2.10. Let $n \geq 2$ and let $G := \{z \in \mathbb{C}^n : |z_1 \cdots z_n| < 1\}$. Define $u(z) := \max_{1 \leq j \leq n} |z_j|$ for $z \in G$. It is easy to check that $u \in PSH(G)$ and G is not Brody hyperbolic. In view of the little Picard theorem, $u \circ \psi = \max_{1 \leq j \leq n} |\psi_j|$ is not a constant for any nonconstant mapping $\psi := (\psi_1, \cdots, \psi_n) \in \mathcal{O}(\mathbb{C}, G)$, where $\psi_j \in \mathcal{O}(\mathbb{C})$, $j = 1, \cdots, n$. Thus Lemma 2.2.9 implies that the pseudoconvex Reinhardt domain $\Sigma = \Sigma_{u,-\infty}(G)$ is Brody hyperbolic.

Moreover, Σ has a local psh antipeak function φ, defined by

$$\varphi(w', w_{n+1}) := \log |w_{n+1}|, \quad (w', w_{n+1}) \in \mathbb{C}^n \times \mathbb{C},$$

at infinity, because $0 < |w_{n+1}| \to 0$ as $\Sigma \ni (w', w_{n+1}) \to \infty$. However, in case $n \geq 3$, the domain Σ has no local psh peak function at infinity. For details, see Example 2.2.18 below.

Remark 2.2.11. In view of Theorem 1.5.21, Example 2.2.10 example gives us a negative answer to (Q) in terms of all hyperbolicities.

It is also natural to ask whether a similar phenomenon as in Example 2.2.10 happens in the class of all (pseudoconvex) Reinhardt Hartogs-Laurent domain $\Sigma_{u,v}(G)$ with the condition $v \equiv -\infty$. However, the phenomenon as in Example 2.2.10 does not always happen. For example:

Example 2.2.12. Suppose that G is not Brody hyperbolic. If either u is bounded from above or $u \le \log|g|$ on G for some $g \in \mathcal{O}(G, \mathbb{C}_*)$, then $\Sigma = \Sigma_{u,-\infty}(G)$ is not Brody hyperbolic, even if Σ is pseudoconvex Reinhardt.

Next, to discuss a positive case to the previous question, we need the following necessary condition for the domain Σ to be taut:

Lemma 2.2.13. *If Σ is taut, then u and v are continuous on G.*

Proof. Let us suppose the contrary. Since tautness is invariant with respect to biholomorphic mappings, without loss of generality, we may assume that $u \notin \mathcal{C}(G)$. Choose a constant $A \in \mathbb{R}$ and a sequence $(z_j)_{j\ge 0} \subset G$ such that $z_j \to z_0$ as $j \in \mathbb{N}$ and $-u(z_0) < -A < -u(z_j)$ for any $j \in \mathbb{N}$. Note that $u(z_0) \ne -\infty$. Since $u(z_0) + v(z_0) < 0$, we may take an $\tilde{\alpha} \in \mathbb{R}$ such that $v(z_0) < -\tilde{\alpha} < -u(z_0)$. Because of the upper semicontinuity of v, we may assume that $v(z_j) < -\tilde{\alpha}$ for any $j \in \mathbb{N}$. Now, put $C := \frac{1}{2}\min\{-u(z_0) + \tilde{\alpha}, -A + u(z_0)\} > 0$ and $\breve{\Sigma} := \Sigma_{\breve{u},\breve{v}}(G)$, where

$$\breve{v} := v + u(z_0) + \frac{C}{2}, \qquad \breve{u} := u - u(z_0) - \frac{C}{2}.$$

Clearly, the mapping

$$\Sigma \ni (z, w) \longmapsto (z, w\exp(u(z_0) + \frac{C}{2})) \in \breve{\Sigma}$$

is well-defined and biholomorphic, so $\breve{\Sigma}$ is a taut domain. Moreover, if we put

$$\breve{A} := -u(z_0) - \frac{C}{2} + A, \qquad \breve{\alpha} := -u(z_0) - \frac{C}{2} + \tilde{\alpha},$$

then $\breve{v}(z_j) < -\breve{\alpha}$ for any $j \ge 1$. Hence, it holds that

$$\max\{\breve{v}(z_0), \breve{v}(z_j)\} < -\breve{\alpha} < -C < 0 < -\breve{u}(z_0) < C < -\breve{A} < -\breve{u}(z_j) \tag{2.2.13a}$$

for any $j \ge 1$. For $j \in \mathbb{N}$ we define $f_j(\lambda) := (z_j, e^{C\lambda})$ for any $\lambda \in E$. Then

$$e^{\breve{v}(z_j)} < e^{-C} < |e^{C\lambda}| < e^{C} < e^{-\breve{u}(z_j)}, \quad j \ge 1,\ \lambda \in E,$$

and so $(f_j)_{j\in\mathbb{N}} \subset \mathcal{O}(E, \breve{\Sigma})$. Moreover, $f_j(0) = (z_j, e^0) = (z_j, 1) \overset{j\to\infty}{\longrightarrow} (z_0, 1) \in \breve{\Sigma}$, because $e^{\breve{v}(z_0)} < e^{-C} < e^0 < e^{-\breve{u}(z_0)}$. Therefore the tautness of $\breve{\Sigma}$ gives us that $f_j \overset{\mathrm{K}}{\Longrightarrow} (z_0, e^{C\lambda}) \in \mathcal{O}(E, \breve{\Sigma})$ as $j \to \infty$, which implies that $e^{\breve{v}(z_0)} < e^{C\mathrm{Re}\lambda} < e^{-\breve{u}(z_0)}$ for any $\lambda \in E$. Consequently, we obtain a contradiction to (2.2.13a) by setting $E \ni \lambda \to 1$. $\square$

Remark 2.2.14. In general, the tautness of Σ does not imply the tautness of G (cf. Remark 2.2.11). However, if G is taut and $u, v \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, then Σ is taut. For more details, see Corollary 3.1.6 below.

Next, we are going to give a class of Hartogs-Laurent domains $\Sigma = \Sigma_{u,v}(G)$ for which the answer to (Q) is always positive.

Theorem 2.2.15. *If Σ is a pseudoconvex Reinhardt domain with $u \not\equiv -\infty$ and $v \not\equiv -\infty$, then:*

$$\Sigma \textit{ is hyperbolic iff } G \textit{ is hyperbolic and } \max\{u, v\} > -\infty.$$

Recall that all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains ((2) in Remark 1.5.22, (3) in Lemma 1.1.9).

Proof. In view of Theorem 1.5.21 and Proposition 2.2.6, it is enough to verify the necessity. Assume that Σ is hyperbolic. In view of Theorem 1.5.21 and (1) of Lemma 2.2.2, the function $\max\{u, v\}$ is locally bounded on G. Seeking for a contradiction, suppose that G is not Brody hyperbolic. Then there is a nonconstant $\varphi \in \mathcal{O}(\mathbb{C}, G)$. Note that $(u + v) \circ \varphi < 0$ on $\mathbb{C}$. By the Liouville type theorem for subharmonic function, one has that $u \circ \varphi + v \circ \varphi = \text{constant} =: \alpha \in [-\infty, 0)$.

(i) The case $-\infty < \alpha < 0$: Note that $u\circ\varphi = -v\circ\varphi+\alpha$. Since $u\circ\varphi, v\circ\varphi \in SH(\mathbb{C})$, one has $u \circ \varphi, v \circ \varphi \in H(\mathbb{C})$, and so $v \circ \varphi = \operatorname{Re} F$ for some $F \in \mathcal{O}(\mathbb{C})$. Take a number $\beta \in \mathbb{R}$ such that $1 < \beta < e^{-\alpha}$. Define $\Psi = \Psi_{\varphi,F,\beta} : \mathbb{C} \to \mathbb{C}^{n+1}$ by

$$\Psi_{\varphi,F,\beta}(\lambda) := (\varphi(\lambda), \beta e^{F(\lambda)}), \quad \lambda \in \mathbb{C}.$$

Observe that

$$e^{v(\varphi(\lambda))} = e^{\operatorname{Re} F(\lambda)} < \beta|e^{F(\lambda)}| < e^{-\alpha}|e^{F(\lambda)}| = e^{-\alpha+v(\varphi(\lambda))} = e^{-u(\varphi(\lambda))}, \quad \lambda \in \mathbb{C},$$

which implies that Ψ is nonconstant holomorphic with $\Psi(\mathbb{C}) \subset \Sigma$; a contradiction.

(ii) The case $\alpha = -\infty$: Since the hyperbolicity is an invariant property under biholomorphic mappings (cf. (2) in Remark 2.2.1), without loss of generality, we may assume that $u(\varphi(\lambda_0)) > -\infty$, $v(\varphi(\lambda_0)) = -\infty$ for some $\lambda_0 \in \mathbb{C}$. In view of Theorem 1.5.21, Proposition 2.2.6, and Lemma 2.2.13, one has $u \circ \varphi \in \mathcal{C}(\mathbb{C})$, and so we may take an open neighborhood $W = W(\lambda_0) \subset \mathbb{C}$ such that $u \circ \varphi > -\infty$ on W. It follows from the integrability theorem that $v \circ \varphi = -\infty$ on W, i.e. the Lebesgue measure of $(v \circ \varphi)^{-1}(-\infty)$ is nonzero, and so $v \circ \varphi \equiv -\infty$ on $\mathbb{C}$. Since $\varphi(\mathbb{C})$ is a nonempty open subset of G (use the open mapping theorem for analytic functions), the Lebesgue measure of $v^{-1}(-\infty) \cap G$ is also nonzero. Therefore, the integrability theorem gives us that $v \equiv -\infty$ on G; a contradiction to our assumption. $\square$

What happens with the question (Q) in the class of non-Reinhardt Hartogs-Laurent domains? In the next example, we will give a $\tilde{k}$-hyperbolic non-Reinhardt domain $\Sigma = \Sigma_{u,-\infty}(G) \subset \mathbb{C}^3$ such that G is Brody hyperbolic, but not $\tilde{k}$-hyperbolic.

Example 2.2.16. Let $G := \{z \in \mathbb{C}^2 : |z_1 z_2| < 1\}$ and put $u(z) := \max\{|z_1|, |z_2|\}$, $z \in G$. Consider the following domains:

$$G_7 := \text{the Eisenman-Taylor Domain,}$$
$$G_8 := \{z \in \mathbb{C}^2 : |z_1 z_2| < 1\} \setminus \{(0, \lambda), (\lambda, 0) : |\lambda| \geq 1\},$$

Let $j = 7, 8$. It is easy to show that G_j is an example of a domain that satisfies (j) but not $(j - 1)$ in Proposition 1.5.3 (cf. Example 2.1.12). Since $G_j \subset G$, one has $\Sigma_j := \Sigma_{u,-\infty}(G_j) \subset \Sigma_{u,-\infty}(G) = \Sigma$. Then the decreasing property of $\underline{\tilde{k}}$ implies that

$$\max\{\tilde{k}_{\Sigma_7}, \tilde{k}_{\Sigma_8}\} \geq \tilde{k}_{\Sigma}.$$

Since the pseudoconvex Reinhardt domain Σ is hyperbolic (by Example 2.2.10), so is Σ_j for $j = 7, 8$.

We will use the remaining part of this section to find the counterexample for the converses of Theorem 1.5.15 and Theorem 1.5.20. For this, we need the following auxiliary lemma.

Lemma 2.2.17. *Let $\mathcal{M} = \{z \in \mathbb{C}^2 : |z_1 z_2| < 1\}$. Then $u \in PSH(\mathcal{M})$ is bounded from above if and only if there exists $\tilde{u} \in SH(E)$ that is bounded from above on E such that $u(z) = \tilde{u}(z_1 z_2)$ for $z \in \mathcal{M}$.*

Proof. To prove the necessity, we may assume, without loss of generality, that $u < 0$ on $\mathcal{M}$. Let $\mathcal{M}_1 := (\mathbb{C}_* \times \mathbb{C}) \cap \mathcal{M}$. Observe that the mapping $\varphi : \mathbb{C}_* \times E \to \mathcal{M}_1$, defined by $\varphi(\lambda, \zeta) := (\lambda, \zeta/\lambda)$ for $(\lambda, \zeta) \in \mathbb{C}_* \times E$, is holomorphic. Then $u \circ \varphi \in PSH(\mathbb{C}_* \times E, [-\infty, 0))$. Also $(u \circ \varphi)(\cdot, \zeta_0) \in SH(\mathbb{C}_*)$ for every $\zeta_0 \in E$. Because u is bounded from above on $\mathcal{M}$, it follows from the removable singularity theorem that for any $\zeta_0 \in E$ the function $\widetilde{u \circ \varphi}$ on $\mathbb{C} \times E$ defined by

$$(\widetilde{u \circ \varphi})(\lambda, \zeta_0) := \begin{cases} (u \circ \varphi)(\lambda, \zeta_0), & (\lambda \neq 0), \\ \limsup\limits_{\lambda' \not\to 0} (u \circ \varphi)(\lambda', \zeta_0) & (\lambda = 0), \end{cases}$$

is subharmonic on $\mathbb{C}$. By the Liouville type theorem, $(\widetilde{u \circ \varphi})(\cdot, \zeta_0) \equiv \mathrm{const}_{\zeta_0} =: \tilde{u}(\zeta_0)$; moreover, $\tilde{u} \in SH(E)$.

On the other hand, for any $(\lambda, \zeta) \in \mathcal{M}_1$ it holds that

$$u(\lambda, \zeta) = u(\lambda, \frac{\lambda\zeta}{\lambda}) = (u \circ \varphi)(\lambda, \lambda\zeta) = (\widetilde{u \circ \varphi})(\lambda, \lambda\zeta) = \tilde{u}(\lambda\zeta).$$

Note that the function $f : \mathcal{M} \ni (\lambda, \zeta) \longmapsto \log|\lambda\zeta|$ is psh. Observe that $\mathcal{M} \setminus \mathcal{M}_1$ is a closed subset of $\mathcal{M}$ and $\mathcal{M} \setminus \mathcal{M}_1 \subset f^{-1}(-\infty)$, that is, $\mathcal{M} \setminus \mathcal{M}_1$ is a closed pluripolar subset of $\mathcal{M}$. Thus, by the removable singularity theorem, there exists $v \in PSH(\mathcal{M})$ such that $v\big|_{\mathcal{M}_1} \equiv u\big|_{\mathcal{M}_1}$ and so the weak identity principle implies that $v \equiv u$ on $\mathcal{M}$. Furthermore, for any $\zeta_0 \in \mathbb{C}$ with $(0, \zeta_0) \in \mathcal{M}$, one has

$$u(0, \zeta_0) = v(0, \zeta_0) = \limsup_{\lambda \not\to 0} u(\lambda, \zeta_0) = \limsup_{\lambda \not\to 0} \tilde{u}(\lambda\zeta_0) = \limsup_{\lambda' \not\to 0} \tilde{u}(\lambda') = \tilde{u}(0).$$

The converse is obvious. $\square$

Example 2.2.18. Let $G := \{z \in \mathbb{C}^3 : |z_1 z_2 z_3| < 1\}$ and put $u(z) := \max_{j=1,2,3} |z_j|$ for $z \in G$. Consider the domain

$$\Sigma \equiv \Sigma_{u,-\infty}(G) = \left\{(z, \lambda) \in G \times \mathbb{C} : 0 < |\lambda| < e^{-u(z)}\right\}.$$

Observe that the pseudoconvex Reinhardt domain Σ is k-complete and has a local psh antipeak function at infinity (Example 2.2.10, Theorem 1.5.21). Now we are going to show that the domain Σ has no local psh peak function at infinity. For this, we assume that there are a constant $R > 0$ and a function $\varphi \in \mathcal{C}(\bar{\Sigma} \cap U) \cap PSH(\Sigma \cap U)$, where $U := U_R^4(\infty)$, such that

(2.2.18a) $$\varphi(z, \lambda) < 0, \quad (z, \lambda) \in \bar{\Sigma} \cap U,$$

(2.2.18b) $$\lim_{\Sigma \ni (z,\lambda) \to \infty} \varphi(z, \lambda) = 0.$$

Fix a point $a \in \mathbb{C}$ with $|a| = 2R$. Let $G_a := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < \frac{1}{2R}\}$, $u_a := u(\cdot, a)$ on G_a, and put

$$\Omega_a := \Omega_{u_a,|\cdot|}(G_a) = \left\{((z_1, z_2), \lambda) \in G_a \times \mathbb{C} : |\lambda| < e^{-u_a(z_1,z_2)}\right\}.$$

Now we define a function $\varphi_a : \Omega_a \to [-\infty, +\infty)$ by $\varphi_a(z_1, z_2, \lambda) := \varphi(z_1, z_2, a, \lambda)$ for any $(z_1, z_2, \lambda) \in \Omega_a$. Observe that $\{(z_1, z_2, a, \lambda) : (z_1, z_2, \lambda) \in (\mathbb{C}^2 \times \mathbb{C}_*) \cap \Omega_a\} \subset \Sigma$ and $\varphi_a(z_1, z_2, 0) = \lim_{|\zeta| \not\to 0} \varphi(z_1, z_2, a, \zeta)$. Moreover, $\left\{(z_1, z_2, a, \lambda) : (z_1, z_2, \lambda) \in \overline{\Omega_a}\right\} \subset \bar{\Sigma}$ and $\varphi_a \le 0$ on $\overline{\Omega_a}$. Then $\varphi_a \in PSH(\Omega_a)$ and also $\varphi_a(\cdot, 0) \in PSH(G_a)$. Now, in virtue of Lemma 2.2.17, we may take a function $\tilde{\varphi}_{(a,0)} \in SH(\frac{1}{2R}E)$ such that

$$\varphi_a(z_1, z_2, 0) = \tilde{\varphi}_{(a,0)}(z_1 z_2), \quad (z_1, z_2) \in G_a.$$

Moreover, from Oka's theorem it follows that

$$\tilde{\varphi}_{(a,0)}(0) = \limsup_{\mathbb{R} \ni t \to \infty} \tilde{\varphi}_{(a,0)}(\frac{1}{2Rt}) = \limsup_{\mathbb{R} \ni t \to \infty} \varphi_a(t, \frac{1}{2Rt^2}, 0) =: -C. \tag{2.2.18c}$$

Obviously, $C \ge 0$. If $C = 0$, the maximum principle for the subharmonic function $\tilde{\varphi}_{(a,0)}$ implies that $\tilde{\varphi}_{(a,0)} = 0$ on $\frac{1}{2R}E$ and so $\varphi(\cdot, a, 0) = 0$ on G_a; a contradiction to our assumption (2.2.18a). Hence, $C > 0$.

In view of (2.2.18c), we may take a constant $M' \gg 1$ so large that

$$\varphi_a(t, \frac{1}{2Rt^2}, 0) < -\frac{3}{4}C, \quad t \in \mathbb{R},\ t > M'.$$

Let $t \in \mathbb{R}$ with $t > M'' := \max\{M', |a|\}$. Then

$$\varphi_a(t, \frac{1}{2Rt^2}, 0) = \lim_{|\lambda| \not\to 0} \varphi(t, \frac{1}{2Rt^2}, a, \lambda) < -\frac{3}{4}C.$$

Hence we may take $M_t > t$ so large that

$$\varphi(t, \frac{1}{2Rt^2}, a, \lambda) < -\frac{1}{2}C, \quad 0 < |\lambda| < e^{-M_t}. \tag{2.2.18d}$$

Notice that $u(t, \frac{1}{2Rt^2}, a) = t$ for any $t \in \mathbb{R}$ with $t > M''$ and $\lim_{\mathbb{R} \ni t \to \infty} M_t = \infty$. Consequently, we may choose a sequence $(t_j, \frac{1}{2Rt_j^2}, a, \lambda_j) \in (\mathbb{R} \times \mathbb{R} \times \{a\} \times \mathbb{C}) \cap \Sigma$ such that $t_j > M''$, $0 < |\lambda_j| < e^{-M_{t_j}} \le e^{-t_j}$, and $\lim_{j\to\infty} t_j = \infty$. From (2.2.18d) it follows that

$$\lim_{j \to \infty} \varphi(t_j, \frac{1}{2Rt_j^2}, a, \lambda_j) \le -\frac{1}{4}C < 0,$$

which is a contradiction to our assumption (2.2.18b). Thus the domain Σ has no the function φ as above, so we obtain the required assertion.

As a simple consequence of Example 2.2.18, we have the following one:

Remark 2.2.19. For any $n \ge 4$ there exists an unbounded k-complete pseudoconvex Reinhardt domain in $\mathbb{C}^n$ which admits a local psh antipeak function at infinity, but which does not admit a local psh peak function at infinity.

Finally, we mention that:

Remark 2.2.20. It is *not known* whether there exists a domain which has a local plurisubharmonic (weak-)peak function at infinity, but no local psh antipeak function at infinity.

CHAPTER 3. A NEW APPLICATION OF ROYDEN'S CRITERION FOR TAUT DOMAINS

Summary. In this chapter, new applications of Royden's criterion on taut domain are given.

First of all, in Proposition 3.1.1 and 3.1.2, we recall two characterizations for the tautness of certain Hartogs type domains with balanced fibers ([Jar-Pfl-Zwo 00], [Tha-Duc 00]). In Proposition 3.1.3, using Royden's criterion, we give a complete characterization of tautness in the class of Hartogs domains $\Omega = \Omega_{u,h}(G)$. Using it, we find the example which was announced in (2) of Example 2.1.15 (Example 3.1.5). Moreover, we present a necessary condition for a Hartogs-Laurent domain $\Sigma = \Sigma_{u,v}(G)$ to be taut (Corollary 3.1.6).

In Proposition 3.1.8, we give a new proof of the following result due to D. D. Thai and N. L. Huong ([Tha-Huo 93]), namely: A holomorphic fiber bundle is taut iff both the fiber and the base are taut.

Moreover, we study a sufficient condition for an unbounded domain to be taut. For this, we first recall:

- a known sufficient condition for an unbounded domain to be taut due to F. Barteloot ([Bar 94]);
- a generalized version of the Kerzman-Rosay Theorem ((3) of Theorem 1.4.10) due to H. Gaussier ([Gau 99]).

Using Royden's criterion, we find a relationship between (global) tautness and local tautness (Theorem 3.1.12), which may be considered as a generalization of the Kerzman-Rosay Theorem.

In §3.4 we study the hyperconvexity of bounded Hartogs type domains.

In §3.5 we deal with the k-completeness of a Hartogs domain with m-dimensional balanced fibers.

§3.1. Main results.

The following characterization for the tautness of a bounded Hartogs domain with m-dimensional balanced fibers can be found in ([Jar-Pfl-Zwo 00], Proposition 3.8):

Proposition 3.1.1. *Let $G \subset \mathbb{C}^n$ be a domain. Let $H \in \mathcal{C}^{\uparrow}(G \times \mathbb{C}^m)$, $H(z, \lambda w) = |\lambda| H(z, w)$, $\lambda \in \mathbb{C}$, $z \in G$, $w \in \mathbb{C}^m$. Suppose that $\Omega = \Omega_H(G)$ is bounded pseudconvex. Then Ω is taut iff G is taut and H is continuous on $G \times \mathbb{C}^m$.*

The case $m = 1$ without boundedness was also studied in ([Tha-Duc 00], Theorem B). For details:

Proposition 3.1.2. *Let $G \subset \mathbb{C}^n$ be a domain and let $u \in PSH(G)$. Then $\Omega = \Omega_{u,|\cdot|(G)}$ is taut iff G is taut and u is continuous on G.*

Let us discuss the difference between the proofs of Proposition 3.1.1 and Proposition 3.1.2. In both cases, there is no difference in proving the necessity. In Proposition 3.1.1, it is not difficult to prove the sufficiency by using Montel's theorem. Observe that the Hartogs domain Ω is assumed to be bounded. On the other hand, in Proposition 3.1.2, even though $m = 1$, we can not use the method used in the proof from

[Jar-Pfl-Zwo 00] directly. In fact, the authors proved the sufficiency using (2) in Theorem 1.5.11 and a result given by D. D. Thai and N. L. Huong (see Proposition 3.1.8 below). In our work, applying Royden's criterion, but not these two results, we will give the following characterization of a Hartogs domain (without the assumption of boundedness) with m-dimensional balanced fibers.

Proposition 3.1.3. *The domain $\Omega = \Omega_{u,h}(G)$ is taut if and only if $G, D = D_h$ are taut, $u \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$.*

Remark 3.1.4. (1) By (1) of Proposition 1.1.10, the plurisubharmonicity of u and h and the pseudoconvexity of G are necessary for $\Omega_{u,h}(G)$ to be pseudoconvex.

(2) In view of Proposition 2.1.4 and (4) of Remark 1.5.8, the boundedness of $D = D_h$ in Proposition 3.1.3 is necessary for the domain G to be taut.

(3) In [Jar-Pfl-Zwo 00], to prove the necessity of Proposition 3.1.1, the fact that Ω is bounded was not used. Therefore, the same proof for the necessity in Proposition 3.1.3, except the boundedness of D_h, remains valid.

Now we are able to give some examples of domains which were mentioned in (2) of Example 2.1.15.

Example 3.1.5. (1) *There exists a Hartogs domain $\tilde{\Omega} := \Omega_{\tilde{\varphi},|\cdot|}(G)$ such that*

- *G is unbounded $\tilde{k}$-hyperbolic which is pseudoconvex but not taut;*
- *$\tilde{\Omega}$ is pseudoconvex but not taut, and if $\tilde{\varphi} \in PSH(G)$ then G is $\tilde{k}$-hyperbolic. Here, it is possible that $\tilde{\varphi}$ is not bounded from above.*

To find such an example, let $\Omega = \Omega_{u,|\cdot|}(E)$ be the Thai-Thomas Domain and let $(\alpha_\nu)_{\nu \geq 1}$ be as in (1) of Example 2.1.17. Put $A := (\log 5) \sum_{\nu=1}^{\infty} \alpha_\nu < \infty$ and take an $\alpha \in (0, 1)$ so that $(\log \alpha) + A < 0$. Define

$$\varphi(z_1, z_2) := u(z_1), \quad \psi(z_1, z_2) := \max\{\log \alpha, \log |z_2|\}, \quad (z_1, z_2) \in \Omega.$$

Consider the domain

$$\Sigma \equiv \Sigma_{\varphi,\psi}(\Omega) = \left\{((z_1, z_2), z_3) \in \Omega \times \mathbb{C} : e^{\psi(z_1,z_2)} < |z_3| < e^{-\varphi(z_1,z_2)}\right\}.$$

Clearly, $\varphi + \psi < 0$ on Ω. Thus the domain Σ is well-defined. Since Ω is $\tilde{k}$-hyperbolic (cf. (1) of Example 2.1.17) and φ is bounded from above on Ω, Σ is $\tilde{k}$-hyperbolic (by (c) in Proposition 2.2.4). But since φ is not continuous at the point $(0, z_2) \in \Omega$, Σ is not taut (by Lemma 2.2.13). Therefore, for any $\tilde{\varphi} \in PSH(\Sigma)$, the domain

$$\tilde{\Omega} := \Omega_{\tilde{\varphi},|\cdot|}(\Sigma) = \left\{(z', z_4) \in \Sigma \times \mathbb{C} : |z_4| e^{\tilde{\varphi}(z')} < 1\right\}$$

is unbounded, pseudoconvex, and $\tilde{k}$-hyperbolic (for a proof, see §3.2), but not taut (by Proposition 3.1.3). In particular, Lemma 1.1.9 and Proposition 1.1.10 imply that Σ is unbounded pseudoconvex.

(2) *There exists a Hartogs domain $\Omega = \Omega_{u,|\cdot|}(G)$ such that*

- *G is k-hyperbolic but not taut;*
- *Ω is $\tilde{k}$-hyperbolic, not k-hyperbolic; also, it is not pseudoconvex in general.*

To find such an example, let $u_1 \in \mathcal{C}^{\uparrow}(E, \mathbb{R})$ and put $D_1 := \Omega_{u_1,|\cdot|}(E)$. For $j = 2, 3$, define $D_j := \Omega_{u_j,|\cdot|}(D_{j-1})$ where $u_j \in \mathcal{C}^{\uparrow}(D_{j-1}, \mathbb{R})$. If u_j ($j = 1, 2, 3$) are not continuous, u_j $(j = 1, 2)$ are locally bounded, and u_3 is not locally bounded, then D_1, D_2 are k-hyperbolic (by Proposition 2.1.4), but not taut (by Proposition 3.1.3). If, moreover, $u_3 \in PSH(D_2)$, then D_3 is $\tilde{k}$-hyperbolic (for a proof, see §3.2), but not k-hyperbolic (by Proposition 2.1.4) (hence it is not taut by (4) in Remark 1.5.8).

As an easy consequence of Proposition 3.1.1, we get a sufficient condition for a Hartogs-Laurent domain to be taut, namely:

Corollary 3.1.6. *If G is taut and $u, v \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, then $\Sigma = \Sigma_{u,v}(G)$ is taut.*

Remark 3.1.7. (1) By (2) of Proposition 1.1.10, the plurisubharmonicity of u, v and the pseudoconvexity of G are necessary conditions for the Hartogs-Laurent domain $\Sigma_{u,v}(G)$ to be pseudoconvex.

(2) By Lemma 2.2.13, the continuity of u, v is necessary for the domain G to be taut.

(3) In general, the converse of Corollary 3.1.6 does not hold. For example, see Example 2.2.10 and Remark 2.2.11.

In 1993, D. D. Thai and N. L. Huong proved that a holomorphic fiber bundle is taut iff both the fiber and the base are taut. Their proof is based on Zorn's Lemma. Now we will give a new proof of the following statement for a domain in $\mathbb{C}^n$, using Royden's criterion.

Proposition 3.1.8. *Let $G \subset \mathbb{C}^n$, $\Omega \subset \mathbb{C}^m$ be domains and let $\pi : G \to \Omega$ be a holomorphic mapping. Suppose that for any $p \in \Omega$ there exists an open neighborhood $U = U(p)$ of p in Ω such that $\pi^{-1}(U)$ is taut. If Ω is taut, then G is also taut.*

Remark 3.1.9. (1) S. Kobayashi ([Kob 98], Theorem 5.1.8) has shown Proposition 3.1.8 under an additional assumption that π is a proper map.

(2) As mentioned above, in [Tha-Duc 00] (in case $m = 1$), to show Proposition 3.1.3, the authors used Proposition 3.1.8 and (2) in Theorem 1.5.11.

(3) As a consequence of Proposition 3.1.8, we may get a similar result as the one of S. Kobayashi (resp. A. Eastwood) for k-completeness (resp. k-hyperbolicity). It is due to D. D. Thai and N. L. Huong ([Tha-Huo 93]): *If $\pi : D \to G$ denotes a holomorphic covering between domains in $\mathbb{C}^n$, then D is taut if and only if so is G.*

From the Kerzman-Rosay Theorem ((3) in Theorem 1.4.10), it is natural to ask the following:

(K-R) "Is any unbounded locally taut domain taut?"

Now let us recall a sufficient condition for an unbounded domain to be taut, due to F. Berteloot ([Ber 94]), namely:

Theorem 3.1.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $p_0 \in \partial\Omega$. If Ω admits a local psh peak function at p_0, then:*

(1) there are $C > 0$, $\beta \in (0, 1)$ such that

$$\kappa_\Omega(z; X) \geq C\|X\|, \quad z \in \Omega \cap \mathbb{B}_n(p_0, \beta),\ X \in \mathbb{C}^n;$$

(2) (Localization) *there exist* $s, r \in (0,1)$ *such that*

$$\forall\, g \in \mathcal{O}(E, G) \ \ : \ \ g(0) \in \mathbb{B}_n(p_0, s) \implies g(rE) \subset \Omega \cap \mathbb{B}_n(p_0, r);$$

(3) if, moreover, there exists a sequence $(\varphi_j)_{j\geq 1} \subset \mathrm{Aut}(G)$ *with* $\lim_{j\to\infty} \varphi_j(z_0) = p_0$ *for some* $z_0 \in G$*, then there exists a subsequence* $(\varphi_{j_\nu})_{\nu\geq 1}$ *of* $(\varphi_j)_{j\geq 1}$ *such that* $\varphi_{j_\nu} \overset{\mathrm{K}}{\Longrightarrow} p_0$ *on* G *as* $\nu \to \infty$ *(so* G *is k-hyperbolic). Moreover,* G *is taut iff it is locally taut at* $p_0 \in \partial G$*.*

This result was based on the idea of (1) in Theorem 1.5.11.

By modifying the above localization, recently, H. Gaussier showed the following result ([Gau 99], Proposition 2):

Proposition 3.1.11. *Let* G *be a domain in* $\mathbb{C}^n$*. Assume that* G *is locally taut at each point of* ∂G *and that there are local peak and antipeak psh functions at infinity. Then* G *is taut.*

To prove tautness of G, H. Gaussier used that G is k-hyperbolic (Theorem 1.5.15). Proposition 3.1.11 is a positive answer to (K-R). Moreover, it is also considered as a generalized version of the Kerzman-Rosay Theorem.

Now, using Royden's criterion, we will give a new partial answer to (K-R). For this, we introduce a new notion of a 'local psh weak-peak function at infinity':

Theorem 3.1.12. *Let* $G \subset \mathbb{C}^n$ *be a domain. Suppose that* G *is locally taut and that* $\mathcal{O}(E, G)$ *is equicontinuous with respect to* $\|\cdot\|$*. If* G *has a 'local psh weak-peak function* φ *at infinity', i.e. there is a constant* $R > 0$ *such that* $\varphi \in PSH(G \cap U_R(\infty)) \cap \mathcal{C}(\bar{G} \cap U_R(\infty))$ *and*

$$\varphi(z) < 0, \quad z \in G \cap U_R(\infty), \tag{3.1.12a}$$

$$\lim_{G \ni z \to \infty} \varphi(z) \ \text{ exists and is } 0. \tag{3.1.12b}$$

Then G *is taut.*

Remark 3.1.13. (1) In view of (3) in Remark 1.5.8 and Remark 1.5.10, Theorem 3.1.12 is a generalized version of the Kerzman-Rosay Theorem.

(2) Notice that any domain in $\mathbb{C}^n$, admitting a local psh peak function at infinity, also has a local psh weak-peak function at infinity.

(3) Recall that for any $n \geq 2$ there is an unbounded non-taut domain $G \subset \mathbb{C}^n$ such that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$ (Remark 1.5.10). Therefore, in general, although the equicontinuity of $\mathcal{O}(E, G)$, Theorem 3.1.12 does not hold.

(4) There exists a locally taut domain which is not Brody hyperbolic (so not k-hyperbolic and not taut). For example, the pseudoconvex Reinhardt Hartogs domain $\Omega = \Omega_{\log(1/|\cdot|),|\cdot|}(\mathbb{C}_*)$, which is biholomorphic to $\mathbb{C}_* \times E$, is locally taut, but not k-hyperbolic (so not taut). On the other hand, Ω has a local psh antipeak function φ at infinity, defined by $\varphi(z) := -\log|z_1|$ for $z \in \mathbb{C} \times \mathbb{C}_*$.

§3.2. Proofs of propositions, corollary, and example.

Proof of Proposition 3.1.3. In view of Proposition 2.1.4 and (3) of Remark 3.1.4, it suffices to verify the sufficiency of this statement. For seeking a contradiction, suppose the contrary, i.e. Ω is not taut. By Lemma 1.4.6, we may choose an $R > 0$, sequences $(z_j)_{j\geq 0} \subset \Omega$, $(f_j)_{j\geq 1}$, $(g_j)_{j\geq 1} \in \mathcal{O}(E,\Omega)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \in [0,1)$ having the properties (†1) $\sim$ (†5). Since $k^{(2)}$ satisfies the decreasing property, it holds that $k^{(2)}_\Omega(z_0, z_j) \geq k^{(2)}_G(z^1_0, z^1_j)$, where $z_j =: (z^1_j, z^2_j) \in G \times \mathbb{C}^m$, $j \geq 0$, and so the property (†1) implies that $(z^1_j)_{j\geq 1} \subset \mathbb{B}_{k^{(2)}_G}(z^1_0, R)$. But since G is taut, we may assume, in view of Royden's criterion, that $z^1_j \overset{j\to\infty}{\longrightarrow} {}^\exists a^1_0 \in G$. For any $j \geq 1$, denote $f_j =: (f^1_j, f^2_j), g_j =: (g^1_j, g^2_j) \in \mathcal{O}(E,G) \times \mathcal{O}(E,\mathbb{C}^m)$. Because of the tautness of G and the property (†2), we may extract a sequence $(f^1_{1j})_{j\geq 1} \subset (f^1_j)_{j\geq 1}$ such that $f^1_{1j} \overset{\mathrm{K}}{\Longrightarrow} {}^\exists f^1_0 \in \mathcal{O}(E,G)$ as $j \to \infty$. Hence, the properties (†3) and (†5) yield that

$$\lim_{j\to\infty} g^1_{1j}(0) = \lim_{j\to\infty} f^1_{1j}(\alpha_{1j}) = f^1_0(\alpha_0) \in G.$$

So, the tautness of G implies that there is a sequence $(g^1_{2j})_{j\geq 1} \subset (g^1_{1j})_{j\geq 1}$ such that $g^1_{2j} \overset{\mathrm{K}}{\Longrightarrow} {}^\exists g^1_0 \in \mathcal{O}(E,G)$ as $j \to \infty$. In particular,

$$g^1_0(\beta_0) = \lim_{j\to\infty} g^1_{2j}(\beta_{2j}) = \lim_{j\to\infty} z^1_{2j} = a^1_0.$$

On the other hand, since D is taut, Theorem 1.4.3 implies that $D \Subset \mathbb{C}^m$. By (1) of Remark 1.1.4, there is a constant $C > 0$ such that $h(w) \geq C\|w\|$, $\forall\, w \in \mathbb{C}^m$. Since u is real-valued, one has $h(z^2_j) \leq \exp(-u(z^1_j))$ for $j \geq 1$. The continuity of u gives us that

$$\limsup_{j\to\infty} \|z^2_j\| \leq \frac{1}{C} \limsup_{j\to\infty} h(z^2_j) \leq \frac{1}{C} \exp(-u(a^1_0)) < \infty,$$

which implies that $z^2_j \not\to \infty$ as $j \to \infty$. Thus, in view of (†4), we may take a point $a^2_0 \in \mathbb{C}^m$ so that

$$\lim_{j\to\infty} z_j = (a^1_0, a^2_0) = \hat{z}_0 \in \partial\Omega.$$

Step I. Choose $c_2 \in (0,1)$ so that $\beta_j \in c_2 E$ for $j \geq 0$. For any $j \geq 1$, we define a map $\tilde{g}_j : \frac{1}{c_2}E \to \mathbb{C}^n \times \mathbb{C}^m$ by

$$\tilde{g}_j(\lambda) = (\tilde{g}^1_j(\lambda), \tilde{g}^2_j(\lambda)) := g_j(\beta_j\lambda), \quad \lambda \in \frac{1}{c_2}E =: E_2.$$

Clearly, it is well-defined and $(\tilde{g}_j)_{j\geq 1} \subset \mathcal{O}(E_2,\Omega)$. Now we shall show that $(\tilde{g}^2_{2j})_{j\geq 1}$ is bounded on E_2. Let $F_2 := \bigcup_{j\geq 0}(\beta_j E_2)$. Using (†5), it is easy to check that $F_2 \Subset E$. Let $L := g^1_0(\bar{F}_2)$. Obviously, $L \Subset G$ and so $\delta := \frac{1}{3}\mathrm{dist}(L,\partial G) > 0$. Since g^1_{2j} converges uniformly on $\bar{F}_2$ to g^1_0 as $j \to \infty$, we may take $j_0 \in \mathbb{N}$ such that

$$\|g^1_{2j}(\lambda) - g^1_0(\lambda)\| < \delta, \quad \lambda \in \bar{F}_2,\ j \geq j_0.$$

Hence,

$$\|g^1_{2j}(\lambda) - \hat{v}_0\| \geq \|g^1_0(\lambda) - \hat{v}_0\| - \|g^1_{2j}(\lambda) - g^1_0(\lambda)\| \geq \mathrm{dist}(L,\partial G) - \delta \geq 2\delta$$

for $j \geq j_0$, $\lambda \in \bar{F}_2$, $\hat{v}_0 \in \partial G$. That is, $\mathrm{dist}(g^1_{2j}(\bar{F}_2), \partial G) \geq 2\delta > 0$ for $j \geq j_0$, which gives us that $K := g^1_0(\bar{F}_2) \bigcup (\cup_{j \geq j_0} g^1_{2j}(\bar{F}_2)) \Subset G$. In particular,

$$K' := \left\{ g^1_{2j}(\beta_{2j}\lambda),\, g^1_0(\beta_0\lambda) : \lambda \in E_2,\, j \geq j_0 \right\} \subset K.$$

Since u is uniformly continuous on $\bar{K}$, we may take a constant $C' > 0$ so that $|u(x) - u(y)| < C'$ for $x, y \in K$. Therefore, for any $j \geq j_0$ and $\lambda \in E_2$ it holds that

$$C\|\tilde{g}^2_{2j}(\lambda)\| \leq h(\tilde{g}^2_{2j}(\lambda)) < e^{-u(\tilde{g}^1_{2j}(\lambda))} \leq e^{-u(g^1_0(\beta_0\lambda))+C'} \leq e^{-\inf_{x \in \bar{K}} u(x)+C'} < \infty.$$

Here, in the third inequality, we used the fact that $K' \subset K$. So, the family $(\tilde{g}^2_{2j})_{j\geq 1}$ is uniformly bounded on E_2. In view of Montel's theorem, we may choose a sequence $(\tilde{g}^2_{3j})_{j \geq 1} \subset (\tilde{g}^2_{2j})_{j\geq 1}$ such that $\tilde{g}^2_{3j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\tilde{g}^2_0 \in \mathcal{O}(E_2, \mathbb{C}^m)$ as $j \to \infty$. In particular,

$$\tilde{g}^2_0(1) = \lim_{j\to\infty} \tilde{g}^2_{3j}(1) = \lim_{j\to\infty} g^2_{3j}(\beta_{3j}) = \lim_{j\to\infty} z^2_{3j} = a^2_0.$$

For $j \geq 1$ we put $\varphi_{3j} := H \circ \tilde{g}_{3j}$ on E_2. Since $\varphi_{3j} < 1$ on E_2 for any $j \geq 1$, one has $\varphi_0 := H \circ \tilde{g}_0 \leq 1$ on E_2, where $\tilde{g}_0 := (\tilde{g}^1_0, \tilde{g}^2_0)$, $\tilde{g}^1_0(\lambda) := g^1_0(\beta_0\lambda)$, $\lambda \in E_2$. In particular, $\varphi_0(1) = H(\hat{z}_0) = 1$. Hence, the maximum principle for $\varphi_0 \in SH(E_2)$ implies that $\varphi_0 \equiv 1$ on E_2, and also

$$\tilde{g}_0(0) = (\tilde{g}^1_0(0), \tilde{g}^2_0(0)) \in \partial\Omega. \tag{3.1.3a}$$

Step II. From now on, we are going to apply the same argument as in the step I to $(f_j)_{j\geq 1}$ and $(\alpha_j)_{j \geq 0}$. Choose $c_1 \in (0, 1)$ so that $\alpha_j \in c_1 E$ for $j \geq 0$. Define a holomorphic mapping $\tilde{f}_j : \frac{1}{c_1}E \to \Omega$ by

$$\tilde{f}_j(\lambda) = (\tilde{f}^1_j(\lambda), \tilde{f}^2_j(\lambda)) := f_j(\alpha_j\lambda), \quad \forall\, \lambda \in \frac{1}{c_1}E =: E_1.$$

Then we may verify, as in step I, that $(\tilde{f}^2_{3j})_{j\geq 1}$ is bounded on E_1. Again, applying Montel's theorem, we may choose a sequence $(\tilde{f}^2_{4j})_{j\geq 1} \subset (\tilde{f}^2_{3j})_{j \geq 1}$ such that $\tilde{f}^2_{4j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\tilde{f}^2_0 \in \mathcal{O}(E_1, \mathbb{C}^m)$ as $j \to \infty$. From which and (†3), we obtain that

$$\tilde{g}_0(0) = \lim_{j\to\infty} \tilde{g}_j(0) = \lim_{j\to\infty} g_{4j}(0) = \lim_{j\to\infty} f_{4j}(\alpha_{4j}) = \lim_{j\to\infty} \tilde{f}_{4j}(1) = \tilde{f}_0(1), \tag{3.1.3b}$$

where $\tilde{f}_0 := (\tilde{f}^1_0, \tilde{f}^2_0)$, $\tilde{f}^1_0(\lambda) := f^1_0(\alpha_0\lambda)$, $\lambda \in E_1$. Observe that $\psi_0(1) = H(\tilde{f}_0(1)) = 1$ by (3.1.3a) and (3.1.3b). But since $\psi_0 := H \circ \tilde{f}^2_0 \leq 1$ on $\lambda \in E_1$ as above, it follows from the maximum principle for $\psi_0 \in SH(E_1)$ that $\psi_0 \equiv 1$ on E_2, which implies that

$$\partial\Omega \ni \tilde{f}_0(0) = \lim_{j\to\infty} \tilde{f}_{4j}(0) = \lim_{j\to\infty} f_{4j}(0) = z_0.$$

This is a contradiction to (†2) and we are done. □

Proof of Example 3.1.5. To see that $\tilde{\Omega}$ (resp. D_2) is $\tilde{k}$-hyperbolic, it suffices to show that, as Σ (resp. D_2) replaces G in Lemma 2.1.13, the condition $(\star)$ for Σ (resp. D_2)

holds, because Σ (resp. D_2) is $\tilde{k}$-hyperbolic. To verify them, let $a = (a_1, a_2, a_3) \in \Sigma, b = (b_1, b_2, b_3) \in D_2$. Take two sequences $f_\nu \in \mathcal{O}(E_\nu, \Sigma)$, $g_\nu \in \mathcal{O}(E_\nu, D_2)$, $\nu \geq 1$, such that $f_\nu(0) = f_\nu(1) = a$, $g_\nu(0) = g_\nu(1) = b$, where $E_\nu := r_\nu E$ and $1 < r_\nu \nearrow \infty$ as $\nu \to \infty$.

(1) For $j = 1, 2, 3$, define $\pi_j : \mathbb{C}^3 \to \mathbb{C}$ by $\pi_j(z_1, z_2, z_3) := z_j$, and denote $f_\nu := (f_\nu^1, f_\nu^2, f_\nu^3)$, where $f_\nu^j \in \mathcal{O}(E_\nu, \mathbb{C})$. Since $\pi_1(\Sigma) \subset \pi_1(\Omega) \subset E$, the Montel theorem gives us that there exists a sequence $(f_{1\nu}^1)_{\nu\geq 1} \subset (f_\nu^1)_{\nu \geq 1}$ such that $f_{1\nu}^1 \overset{\mathrm{K}}{\Longrightarrow} a_1$ as $\nu \to \infty$. Since $\psi\big|_\Omega \geq \log\alpha$, one has $\bigcup_{\nu=1}^\infty f_\nu^3(E_\nu) \subset \pi_3(\Omega) \subset \mathbb{C} \setminus (\alpha E)$. So, by the tautness of $\mathbb{C}\setminus(\alpha E)$, we can take a sequence $(f_{2\nu}^3)_{\nu\geq 1} \subset (f_{1\nu}^3)_{\nu\geq 1}$ such that $f_{2\nu}^3 \overset{\mathrm{K}}{\Longrightarrow} a_3$. On the other hand, since $f_{2\nu}(E_{2\nu}) \subset \Sigma$ for any $\nu \geq 1$, it holds that

$$|f_{2\nu}^2(\lambda)| = e^{\log|f_{2\nu}^2(\lambda)|} \leq e^{\psi(f_{2\nu}^1(\lambda), f_{2\nu}^2(\lambda))} < |f_{2\nu}^3(\lambda)|, \quad \forall\, \lambda \in E_{2\nu}.$$

Thus there is $\nu_0 \in \mathbb{N}$ such that $|f_{2\nu}^2(\lambda)| \leq |a_3| + 1$ for any $\lambda \in E_{2\nu}$ and $\nu \geq \nu_0$. In view of Montel's theorem, we can choose a sequence $(f_{3\nu}^2)_{\nu\geq 1} \subset (f_{2\nu}^2)_{\nu\geq 1}$ such that $f_{3\nu}^2 \overset{\mathrm{K}}{\Longrightarrow} a_2$. Consequently, $f_{3\nu} \overset{\mathrm{K}}{\Longrightarrow} a$ as $\nu \to \infty$.

(2) Let $\nu \geq 1$. Put $g_\nu := (g_\nu^1, g_\nu^2) \in \mathcal{O}(E_\nu, D_2)$ and $g_\nu^1 := (\varphi_\nu^1, \varphi_\nu^2) \in \mathcal{O}(E_\nu, D_1)$, where $\varphi_\nu^1 \in \mathcal{O}(E_\nu, E)$. In view of Montel's theorem, we can extract a sequence $(\varphi_{1\nu}^1)_{\nu\geq 1} \subset (\varphi_\nu^1)_{\nu\geq 1}$ such that $\varphi_{1\nu}^1 \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi^1 \in \mathcal{O}(\mathbb{C}, \bar{E})$ as $\nu \to \infty$, and it follows from the Liouville theorem that $\varphi^1 \equiv$ constant $= \varphi_{11}^1(0) = b_1 \in E$. Now, put $\epsilon := \frac{1}{2}\mathrm{dist}(b_1, \partial E) > 0$ and fix $0 < s < 1$. Then we may choose $\nu_s \geq 1$ such that $\varphi_{1\nu}^1(\lambda) \in \mathbb{B}_1(b_1, \epsilon)$ for any $\nu \geq \nu_s$ and any $\lambda \in s\bar{E}$. Because of $\mathbb{B}_1(b_1, \epsilon) \Subset E$, it follows that

$$\left|\varphi_{1\nu}^2(\lambda)\right| \leq \exp\big(\max_{|\zeta - b_1|\leq\epsilon} u_1(\zeta)\big) =: \alpha < \infty, \quad \lambda \in s\bar{E},\, \nu \geq \nu_s.$$

Here, in the last inequality, we used the fact that u_1 is locally bounded on E. But since s is arbitrary, the family $(\varphi_{1\nu}^2)_{\nu\gg 1}$ is locally bounded. So by Montel's theorem, we can choose a sequence $(\varphi_{2\nu}^2)_{\nu\geq 1} \subset (\varphi_{1\nu}^2)_{\nu\geq 1}$ such that $\varphi_{2\nu}^2 \overset{\mathrm{K}}{\Longrightarrow} {}^\exists\varphi^2 \in \mathcal{O}(\mathbb{C}, \alpha\bar{E})$ as $\nu \to \infty$. By applying Liouville's theorem to the entire function φ^2, we then get that $\varphi^2 \equiv$ constant $= \varphi_{21}^2(0) = b_2$. Hence, $g_{2\nu}^1 \overset{\mathrm{K}}{\Longrightarrow} (b_1, b_2) \in D_1$ as $\nu \to \infty$. Applying the same method to the family $(g_{2\nu})_{\nu\geq 1}$, we can obtain a sequence $(g_{3\nu})_{\nu\geq 1} \subset (g_{2\nu})_{\nu\geq 1}$ such that $g_{3\nu} \overset{\mathrm{K}}{\Longrightarrow} b \in D_2$ as $\nu \to \infty$. □

To show Corollary 3.1.6, we will use Proposition 3.1.3 as already mentioned.

Proof of Corollary 3.1.6. Let $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(E, \Sigma)$ be a sequence. Observe that $\Sigma = \Sigma_{u,v}(G) \subset \Omega_{u,|\cdot|}(G) =: \Omega$. By Proposition 3.1.3 the domain Ω is taut, so $(\varphi_j)_{j\geq 1}$ is a normal subfamily of $\mathcal{O}(E, \Omega)$. That is, there is a sequence $(\varphi_{1j})_{j\geq 1} \subset (\varphi_j)_{j\geq 1}$ which is either normally convergent in $\mathcal{O}(E, \Omega)$ or compactly divergent. In the latter case, the sequence $(\varphi_{1j})_{j\geq 1}$, as a subfamily of $\mathcal{O}(E, \Sigma)$, diverges compactly.

For $j \geq 1$ we put $\varphi_j := (f_j, g_j)$, where $(f_j)_{j\geq 1} \subset \mathcal{O}(E, G)$ and $(g_j)_{j\geq 1} \subset \mathcal{O}(E)$.

From now on, we only suppose that $(\varphi_{1j})_{j\geq 1}$ is normally convergent in $\mathcal{O}(E, \Omega)$. Take a function $\varphi := (f, g) \in \mathcal{O}(E, \Omega)$, where $f \in \mathcal{O}(E, G)$, $g \in \mathcal{O}(E)$, such that $f_{1j} \overset{\mathrm{K}}{\Longrightarrow} f$ and $g_{1j} \overset{\mathrm{K}}{\Longrightarrow} g$ as $j \to \infty$. Note that

$$e^{(v\circ f_j)(\lambda)} < |g_j(\lambda)| < e^{-(u\circ f_j)(\lambda)}, \quad \lambda \in E,\, j \geq 1,$$
$$|g(\lambda)| < e^{-(u\circ f)(\lambda)}, \quad \lambda \in E.$$

Since $g_j^{-1}(0) = \emptyset$ for any $j \geq 1$, it follows from Hurwitz's Theorem that either $g \equiv 0$ or g never vanishes. In the former case, it is clear that $\varphi(E) \subset \partial\Sigma$, which implies that $(\varphi_{1j})_{j\geq 1}$, as a subfamily of $\mathcal{O}(E, \Sigma)$, is compactly divergent. Now we assume that $g \not\equiv 0$ and define

$$\tilde{u}(\lambda) := |g(\lambda)| e^{(u\circ f)(\lambda)}, \quad \tilde{v}(\lambda) := \frac{1}{|g(\lambda)|} e^{(v\circ f)(\lambda)}, \quad \lambda \in E.$$

Observe that $\tilde{u}, \tilde{v} \in SH(E)$ and $\max\{\tilde{u}, \tilde{v}\} \leq 1$ on E. Then the maximum principle for the subharmonic function $\tilde{u}$ (resp. $\tilde{v}$) implies that either $\tilde{u}\big|_E \equiv 1$ or $\tilde{u}\big|_E < 1$ (resp. either $\tilde{v}\big|_E \equiv 1$ or $\tilde{v}\big|_E < 1$). These properties yield that either $\varphi(E) \subset \partial\Sigma$ or $\varphi(E) \subset \Sigma$. Consequently, the sequence $(\varphi_{1j})_{j\geq 1}$ is either normally convergent in $\mathcal{O}(E, \Sigma)$ or compactly divergent. □

Proof of Proposition 3.1.8. Suppose the contrary. Then by Lemma 1.4.6, we may take sequences $(z_j)_{j\geq 0} \subset G$, $f_j, g_j \in \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq 0}, (\beta_j)_{j\geq 0} \subset [0, 1)$ satisfying $(\dagger 2) \sim (\dagger 5)$. Because Ω is taut and the family $(\pi \circ f_j)_{j\geq 1} \subset \mathcal{O}(E, \Omega)$ satisfies

$$\lim_{j\to\infty} (\pi \circ f_j)(0) = \lim_{j\to\infty} \pi(f_j(0)) = \pi(z_0) \in \Omega,$$

there exists a sequence $(f_{1j})_{j\geq 1} \subset (f_j)_{j\geq 1}$ such that

$$\pi \circ f_{1j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\varphi_1 \in \mathcal{O}(E, \Omega) \quad \text{as} \quad j \to \infty. \tag{3.1.8a}$$

In particular, by $(\dagger 3)$

$$\lim_{j\to\infty} (\pi \circ g_{1j})(0) = \lim_{j\to\infty} (\pi \circ f_{1j})(\alpha_{1j}) = \varphi_1(\alpha_0) \in \Omega.$$

That is, the sequence $(\pi \circ g_{1j})_{j\geq 1} \subset \mathcal{O}(E, \Omega)$ does not diverge compactly on G, and because of the tautness of Ω, we may extract a sequence $(g_{2j})_{j\geq 1} \subset (g_{1j})_{j\geq 1}$ such that

$$\pi \circ g_{2j} \overset{\mathrm{K}}{\Longrightarrow} {}^{\exists}\varphi_2 \in \mathcal{O}(E, \Omega) \quad \text{as} \quad j \to \infty. \tag{3.1.8b}$$

Step I. For any $\lambda \in E$, there exist open neighborhoods $V_\lambda \Subset E$ of λ, $U_{\varphi_2(\lambda)} \subset \Omega$ of $\varphi_2(\lambda)$, and $j_\lambda \in \mathbb{N}$, such that $\pi^{-1}(U_{\varphi_2(\lambda)})$ is taut and $g_{2j}(V_\lambda) \Subset \pi^{-1}(U_{\varphi_2(\lambda)}) \subset G$ for any $j \geq j_\lambda$.

Subproof. Fix $\lambda \in E$. Clearly, $\varphi_2(\lambda) \in \Omega$ and by our assumption, we may take an open neighborhood $U_{\varphi_2(\lambda)} \subset \Omega$ of $\varphi_2(\lambda)$ such that $\pi^{-1}(U_{\varphi_2(\lambda)})$ is taut. Take $r_\lambda := r(\lambda) > 0$ so that $\mathbb{B}(\varphi_2(\lambda), 3r_\lambda) \subset U_{\varphi_2(\lambda)}$. Because of the continuity of φ_2, the set $B_\lambda := \varphi_2^{-1}(\mathbb{B}(\varphi_2(\lambda), r_\lambda)) \subset E$ is open containing the point λ, and also, we may take an open neighborhood $V_\lambda = V(\lambda) \Subset B_\lambda$ of λ so small that $\|\varphi_2(\zeta) - \varphi_2(\lambda)\| < r_\lambda$ for any $\zeta \in \bar{V}_\lambda$. Now, in view of (3.1.8b) we may choose $j_\lambda \in \mathbb{N}$ so large that $\|(\pi \circ g_{2j})(\zeta) - \varphi_2(\zeta)\| < r_\lambda$ for any $\zeta \in \bar{V}_\lambda$ and $j \geq j_\lambda$. Hence

$$\|(\pi \circ g_{2j})(\zeta) - \varphi_2(\lambda)\| \leq \|(\pi \circ g_{2j})(\zeta) - \varphi_2(\zeta)\| + \|\varphi_2(\zeta) - \varphi_2(\lambda)\| < 2r_\lambda$$

for any $\zeta \in \bar{V}_\lambda$ and $j \geq j_\lambda$. Thus we get that

$$(\pi \circ g_{2j})(V_\lambda) \subset (\pi \circ g_{2j})(\bar{V}_\lambda) \subset \mathbb{B}(\varphi_2(\lambda), 3r_\lambda) \subset U_{\varphi_2(\lambda)}$$

for any $j \geq j_\lambda$. $\square$

Step II. Take a point $0 < s < 1$ so that $[-s, \beta_0]$ is compact in E, we may take a finite set $\{x_\mu : \mu = 1, \cdots, m\} \subset [-s, \beta_0]$ such that $[-s, \beta_0] \subset \bigcup_{\mu=1}^m V_{x_\mu}$ and

$$\forall_{\mu \in \{1,\cdots,m\}}, \quad \exists_{\nu \in \{1,\cdots,m\}\setminus\{\mu\}} \ : \ V_{x_\nu} \cap V_{x_\mu} \neq \emptyset,$$

and moreover, after a rearrangement, we may assume that

$$\beta_0 \in V_m, \qquad V_{x_\mu} \cap V_{x_{\mu+1}} \neq \emptyset, \quad \forall\, \mu \in \{1, \cdots, m-1\}.$$

Now, we will consider the case $\lambda = \beta_0$. Suppose that there exists a subsequence $(g_{3j})_{j\geq1} \subset (g_{2j})_{j\geq1}$ such that $g_{3j} \overset{\mathrm{K}}{\Longrightarrow} {}^\exists g_{\beta_0} \in \mathcal{O}(V_{\beta_0}, \pi^{-1}(U_{\varphi_2(\beta_0)}))$ as $j \to \infty$. By the first property of (†4), it holds that

$$\lim_{j\to\infty} z_{3j} = \lim_{j\to\infty} g_{3j}(\beta_{3j}) = g_{\beta_0}(\beta_0) \in G,$$

which is a contradiction to the divergence of the sequence $(z_j)_j$ in (†4). Hence, in view of Step I, the sequence $(g_{2j})_j$ diverges compactly on V_{β_0}. But since $\beta_0 \in V_{x_m} \cap V_{\beta_0}$, in view of Step I, we may extract a sequence $(g_{4j})_{j\geq1} \subset (g_{2j})_{j\geq1}$ such that $(g_{4j})_{j\geq1}$ diverges compactly on V_{x_m}. But since $V_{x_{m-1}} \cap V_{x_m} \neq \emptyset$, in view of Step I, we may extract a sequence $(g_{5j})_{j\geq1} \subset (g_{4j})_{j\geq1}$ such that $(g_{5j})_{j\geq1}$ diverges compactly on $V_{x_{m-1}}$. Of course, we can proceed to $m-2$ and so on. Thus, in this manner, we may get $\mu_0 \in \{1, \cdots, m\}$ with $0 \in V_{x_{\mu_0}}$ and a sequence $(g_{6j})_{j\geq1} \subset (g_{5j})_{j\geq1}$ such that $(g_{6j})_{j\geq1}$ diverges compactly on $V_{x_{\mu_0}}$.

Thus, the result of Step II gives us, in view of (†3), that

$$\text{(3.1.8c)} \qquad \text{either} \quad f_{6j}(\alpha_{6j}) \overset{j\to\infty}{\longrightarrow} {}^\exists \hat{a}_0 \in \partial G, \quad \text{or} \quad \|f_{6j}(\alpha_{6j})\| \overset{j\to\infty}{\longrightarrow} \infty.$$

This is a similar situation as in (†4) for the sequence $(g_j(\beta_j))_j$. Hence we can repeatedly carry out the procedures of Step I (using (3.1.8a) and our assumption) and Step II (using the condition (3.1.8c)) to the sequence $(f_{6j}(\alpha_{6j}))_j$, so we may obtain a subsequence $(f_{7j})_{j\in\mathbb{N}}$ of $(f_{6j})_{j\in\mathbb{N}}$ such that $(f_{7j}(0))_{j\in\mathbb{N}}$ does not converge to a point in G; a contradiction to (†2). $\square$

§3.3. Proof of Theorem 3.1.12.

To prove Theorem 3.1.12, we need the following auxiliary lemma:

Lemma 3.3.1. *Let $G \subset \mathbb{C}^n$ be a locally taut domain. Suppose that $\mathcal{O}(E, G)$ is equicontinuous with respect to $\|\cdot\|$. If there are sequences $(z_j)_{j\geq1} \subset G$, $(\psi_j)_{j\geq1} \subset \mathcal{O}(E, G)$, and $(\alpha_j)_{j\geq1} \subset [0, 1)$ such that $\psi_j(\alpha_j) = z_j \overset{j\to\infty}{\longrightarrow} {}^\exists \hat{z}_0 \in \partial G$ and $\alpha_j \overset{j\to\infty}{\longrightarrow} {}^\exists \alpha_0 \in [0, 1)$, then there exists a subsequence $(\psi_{j_\nu})_{\nu\geq1}$ of $(\psi_j)_{j\geq1}$ such that $\psi_{j_\nu}(0) \overset{\nu\to\infty}{\longrightarrow} {}^\exists \hat{v}_0 \in \partial G$.*

Proof. Because of the equicontinuity of the family $\mathcal{O}(E, G)$, we may choose an open covering $(B_x)_{x\in[0,\alpha_0]}$ of the closed interval $[0, \alpha_0] \subset E$ such that

$$\text{(3.3.1a)} \qquad \|\psi_j(\lambda) - \psi_j(x)\| < 1, \quad \forall\, j \in \mathbb{N}, \forall\, \lambda \in B_x := \mathbb{B}_1(x, r_x) \Subset E,$$

where $(r_x)_{x\in[0,\alpha_0]}$ is a suitable family of positive real numbers. By the compactness of $[0,\alpha_0]$, we can extract a finite subcover $(B_{x_\nu})_{\nu=1}^N \subset (B_x)_{x\in[0,\alpha_0]}$, so that $[0,\alpha_0] \subset B := \bigcup_{\nu=1}^N B_{x_\nu}$. In particular, B is connected and we may assume that

$$\text{(3.3.1b)} \quad \alpha_0 \in B_{x_N},\ \ B_{x_\nu} \not\subset B_{x_\mu}\ \ (\nu \neq \mu),\ \ B_{x_\nu} \cap B_{x_{\nu+1}} \neq \emptyset\ \ (\forall\, \nu = 1, \cdots, N-1).$$

Since $\psi_j(\alpha_j) = z_j \to \hat{z}_0 \in \partial G$ as $j \to \infty$, we may take $j' \in \mathbb{N}$ so large that

$$\text{(3.3.1c)} \qquad \alpha_j \in B_{x_N}, \quad \|\psi_j(\alpha_j) - \hat{z}_0\| < 1, \quad j \geq j'.$$

If $\zeta \in B$, then $\exists_{N_\zeta \in \{1,\cdots,N\}}$ with $\zeta \in B_{x_{N_\zeta}}$, and in view of (3.3.1b), we may take $\lambda_\nu \in B_{x_\nu} \cap B_{x_{\nu+1}}$ $(\nu = N_\zeta, \cdots, N-1)$ and also

$$\begin{aligned}
&\|\psi_j(\zeta) - \hat{z}_0\| \\
&\quad \leq \|\psi_j(\zeta) - \psi_j(x_{N_\zeta})\| + \|\psi_j(x_{N_\zeta}) - \psi_j(\lambda_{N_\zeta})\| + \|\psi_j(\lambda_{N_\zeta}) - \psi_j(x_{N_\zeta+1})\| \\
&\qquad + \|\psi_j(x_{N_\zeta+1}) - \psi_j(\lambda_{N_\zeta+1})\| + \|\psi_j(\lambda_{N_\zeta+1}) - \psi_j(x_{N_\zeta+2})\| + \cdots \\
&\qquad + \|\psi_j(\lambda_{N-2}) - \psi_j(x_{N-1})\| + \|\psi_j(x_{N-1}) - \psi_j(\lambda_{N-1})\| \\
&\qquad + \|\psi_j(\lambda_{N-1}) - \psi_j(x_N)\| + \|\psi_j(x_N) - \psi_j(\alpha_j)\| + \|\psi_j(\alpha_j) - \hat{z}_0\|
\end{aligned}$$

and by (3.3.1a) and (3.3.1c), one has $\|\psi_j(\zeta) - \hat{z}_0\| < 2N - 2N_\zeta + 3$ for any $j \geq j'$. Therefore, we then have that $\|\psi_j(\zeta)\| < 2N + 3 + \|\hat{z}_0\| < \infty$ for $\forall\, j \geq j'$, $\forall\, \zeta \in B$. Hence, in view of Montel's theorem, we may extract a sequence $(\psi_{1j})_{j\in\mathbb{N}} \subset (\psi_j)_{j\in\mathbb{N}}$ such that

$$\text{(3.3.1d)} \qquad \psi_{1j} \overset{\mathrm{K}}{\rightrightarrows} {}^{\exists}\psi_0 \in \mathcal{O}(B, \bar{G}) \quad \text{as} \quad j \to \infty.$$

In particular,

$$\text{(3.3.1e)} \qquad \psi_0(\alpha_0) = \lim_{j\to\infty} \psi_{1j}(\alpha_{1j}) = \hat{z}_0 \in \partial G.$$

Now, put $K := \{\lambda \in B : \psi_0(\lambda) \in \partial G\}$. Clearly, the set K is nonempty, relative closed in B; moreover, it is open. Because: If $\lambda_0 \in K$, then $\hat{p}_0 := \psi_0(\lambda_0) \in \partial G$, and by our assumption, we may take a constant $c_0 = c_0(\hat{p}_0) > 0$, so that any connected component of $G \cap \mathbb{B}_n(\hat{p}_0, c_0)$ is taut. From the continuity of ψ_0,

$$\exists_{\gamma_0=\gamma(\lambda_0,c_0)>0} \ \ : \ \ \|\psi_0(\lambda) - \psi_0(\lambda_0)\| < \frac{c_0}{3}, \quad \forall\, \lambda \in \mathbb{B}_1(\lambda_0, \gamma_0) \Subset B.$$

In view of (3.3.1d),

$$\exists_{j_0=j(\gamma_0)\in\mathbb{N}} \ \ : \ \ \|\psi_{1j}(\lambda) - \psi_0(\lambda)\| < \frac{c_0}{3}, \quad \forall\, \lambda \in \mathbb{B}_1(\lambda_0, \gamma_0), \forall\, j \geq j_0.$$

Therefore, for any $j \geq j_0$ and $\lambda \in \mathbb{B}_1(\lambda_0, \gamma_0)$, it holds that

$$\|\psi_{1j}(\lambda) - \psi_0(\lambda_0)\| \leq \|\psi_{1j}(\lambda) - \psi_0(\lambda)\| + \|\psi_0(\lambda) - \psi_0(\lambda_0)\| < \frac{2}{3}c_0.$$

Thus, $\psi_{1j}(\mathbb{B}_1(\lambda_0,\gamma_0)) \subset G \cap \mathbb{B}_n(\hat{p}_0, c_0)$ for any $j \geq j_0$, and so, in view of (3.3.1e), no sequence in $(\psi_{1j})_{j\geq j_0}$ can be uniformly convergent on every compact subset of $\mathbb{B}_1(\lambda_0,\gamma_0)$. Moreover, the boundedness of $\mathbb{B}_n(\hat{p}_0, c_0)$ and (3.3.1d) give us that

$$\psi_0(\mathbb{B}_1(\lambda_0,\gamma_0)) \subset \partial G.$$

That is, $\mathbb{B}_1(\lambda_0,\gamma_0) \subset K$, i.e. λ_0 is an interior point of K. But since λ_0 is arbitrary, the set K is open, and so the connectedness of B gives us that $K = B$. Since $0 \in B$, we have $\psi_{1j}(0) \overset{j\to\infty}{\longrightarrow} {}^{\exists}\hat{v}_0 \in \partial G$. □

Proof of Theorem 3.1.12. Suppose the contrary. By Lemma 1.4.6, we may choose sequences $(z_j)_{j\geq 0} \subset G$, $(f_j)_{j\geq 1}, (g_j)_{j\geq 1} \in \mathcal{O}(E,G)$, and $(\alpha_j)_{j\geq 1}, (\beta_j)_{j\geq 1} \subset [0,1)$ satisfying (†2) $\sim$ (†5).

First suppose that $g_j(\beta_j) = z_j \overset{j\to\infty}{\longrightarrow} \hat{z}_0 \in \partial G$. Then in view of Lemma 3.3.1, we may extract a subsequence $(g_{1j})_{j\in\mathbb{N}}$ of $(g_j)_{j\in\mathbb{N}}$ such that $f_{1j}(\alpha_{1j}) = g_{1j}(0) \to {}^{\exists}\hat{w}_0 \in \partial G$ as $j \to \infty$. Note that $\alpha_{1j} \to \alpha_0$ as $j \to \infty$. Now, we apply again Lemma 3.3.1 for the sequence $(f_{1j}(\alpha_{1j}))_j$. Then we may choose a subsequence $(f_{2j})_{j\in\mathbb{N}}$ of $(f_{1j})_{j\in\mathbb{N}}$ such that $f_{2j}(0) \to {}^{\exists}\hat{v}_0 \in \partial G$ as $j \to \infty$, which is a contradiction to the property (†2). Thus it follows that

$$\|g_j(\beta_j)\| = \|z_j\| \to \infty \quad \text{as} \quad j \to \infty. \tag{3.1.12c}$$

Observe that, as in the proof of Lemma 3.3.1, using the equicontinuity of $\mathcal{O}(E,G)$ we may take an open covering $(B_y)_{y\in[0,\beta_0]}$ of the closed interval $[0,\beta_0] \subset E$ such that

$$\|g_j(\lambda) - g_j(y)\| < 1, \quad j \in \mathbb{N},\, \lambda \in B_y := \mathbb{B}_1(y, r_y) \Subset E, \tag{3.1.12d}$$

where $(r_y)_{y\in[0,\beta_0]}$ is a suitable family of positive real numbers. By the compactness of $[0,\beta_0]$, we can extract a finite subcover $(B_{y_\nu})_{\nu=1}^M \subset (B_y)_{y\in[0,\beta_0]}$ so that $[0,\beta_0] \subset B := \bigcup_{\nu=1}^M B_{y_\nu}$. In particular, B is connected and we may assume that

$$\beta_0 \in B_{y_M},\ \ B_{y_\nu} \not\subset B_{y_\mu}\ \ (\nu \neq \mu),\ \ B_{y_\nu} \cap B_{y_{\nu+1}} \neq \emptyset\ \ (\forall\, \nu = 1, \cdots, M-1). \tag{3.1.12e}$$

By (3.1.12c), we can take $j_R \in \mathbb{N}$ so large that

$$\beta_j \in B_{y_M}, \quad \|g_j(\beta_j)\| > 2R, \quad \forall\, j \geq j_R. \tag{3.1.12f}$$

Let $\zeta \in B$. Then $\zeta \in B_{y_{M_\zeta}}$ for some $\exists_{M_\zeta \in \{1,\cdots,M\}}$. In virtue of (3.1.12d) and (3.1.12e), it holds that for any $\lambda_\mu \in B_{y_\mu} \cap B_{y_{\mu+1}}$ ($\mu = M_\zeta, \cdots, M-1$),

$$\begin{aligned}
\|g_j(\zeta) - g_j(\beta_j)\| \leq\ & \|g_j(\zeta) - g_j(y_{M_\zeta})\| + \|g_j(y_{M_\zeta}) - g_j(\lambda_{M_\zeta})\| \\
& + \|g_j(\lambda_{M_\zeta}) - g_j(y_{M_\zeta+1})\| + \|g_j(y_{M_\zeta+1}) - g_j(\lambda_{M_\zeta+1})\| + \cdots \\
& + \|g_j(\lambda_{M-2}) - g_j(y_{M-1})\| + \|g_j(y_{M-1}) - g_j(\lambda_{M-1})\| \\
& + \|g_j(\lambda_{M-1}) - g_j(y_M)\| + \|g_j(y_M) - g_j(\beta_j)\| \\
<\ & 2(M - M_\zeta + 1) < 2(M+1), \qquad \forall\, j \in \mathbb{N}.
\end{aligned}$$

Then, taking (3.1.12f) into account,

$$\|g_j(\zeta)\| + 2(M+1) > \|g_j(\beta_j)\| > 2R, \quad \forall\, j \geq j_R, \forall\, \zeta \in B,$$

which shows that there exists $j_1 \in \mathbb{N}$ with $j_1 \geq j_R$ such that $g_j(B) \subset G \cap U$ for any $j \geq j_1$. Now, $\varphi \circ g_j$ is well-defined on B for any $j \geq j_1$. Put $\tilde{\varphi} := \limsup_{\nu\to\infty} \varphi \circ g_\nu$ on B. In view of (3.1.12a) and (3.1.12b), $(\varphi \circ g_j)_{j \geq j_1}$ is bounded from above on B, and so $\tilde{\varphi}^* \in SH(B)$ and $\tilde{\varphi}^* \leq 0$, where '$*$' denotes the upper semicontinuous regularization. Now, we will show that $\tilde{\varphi}^*(\beta_0) = 0$. For seeking a contradiction, assume that $\tilde{\varphi}^*(\beta_0) < 0$. Choose a constant $s > 0$ so small that $\tilde{\varphi}^*(\beta_0) < -3s < 0$. By the definition of $\tilde{\varphi}^*$, we may take an open neighborhood $V = V(\beta_0) \subset B$ of β_0 such that

$$\left(\limsup_{\nu\to\infty} \varphi \circ g_\nu\right)(\lambda) < -2s, \quad \forall\, \lambda \in V.$$

So, in view of Hartogs' lemma for psh functions,

$$\forall_{\text{open } W \Subset V,\, \beta_0 \in W}, \quad \exists_{j'_W \geq j_1} \;:\; \sup_{\lambda \in W} (\varphi \circ g_j)(\lambda) < -s, \quad \forall\, j \geq j'_W.$$

Fix such a W. By (†5) we may take $j''_W \in \mathbb{N}$ with $j''_W \geq j'_W$ so large that $\beta_j \in W$ for any $j \geq j''_W$. Then one has

$$\varphi(g_j(\beta_j)) = (\varphi \circ g_j)(\beta_j) < -s, \quad \forall\, j \geq j''_W,$$

which is a contradiction to (3.1.12b). We then get the required property $\tilde{\varphi}^*(\beta_0) = 0$, and so the maximum principle for subharmonic functions gives us that $\tilde{\varphi}^* \equiv 0$ on B. On the other hand, $\tilde{\varphi} = \tilde{\varphi}^*$ almost everywhere on B. Let $m_0 \in \mathbb{N}$ be so large that $\frac{1}{m_0}E \Subset B$. For any $m \in \mathbb{N}$ with $m \geq m_0$, we can extract a point $\lambda_m \in \frac{1}{m}E$ so that $\tilde{\varphi}(\lambda_m) = \tilde{\varphi}^*(\lambda_m) = 0$, i.e. $\limsup_{j\to\infty} \varphi \circ g_j(\lambda_m) = 0$. Hence for any $m \in \mathbb{N}$ with $m \geq m_0$, we may take a sequence $(m_j)_{j\in\mathbb{N}} \subset (j)_{j \geq j_1}$ such that $\lim_{j\to\infty} \varphi(g_{m_j}(\lambda_m)) = 0$. From which, we may choose an increasing sequence $(\mu_m)_{m \geq m_0} \subset \mathbb{N}$ satisfying $-\frac{1}{m} < \varphi(g_{\mu_m}(\lambda_m)) < 0$ for any $m \geq m_0$. Therefore, (3.1.12a), (3.1.12b), and the continuity of φ imply that there exists a subsequence $(\mu_{1\ell})_{\ell\in\mathbb{N}}$ of $(\mu_m)_{m\in\mathbb{N}}$ such that

$$\text{(3.1.12g)} \qquad \text{either} \quad g_{\mu_{1\ell}}(\lambda_{1\ell}) \overset{\ell\to\infty}{\longrightarrow} {}^{\exists}\hat{v}_0 \in \partial G, \quad g_{\mu_{1\ell}}(\lambda_{1\ell}) \in U\,(\forall\, \ell \in \mathbb{N}),$$

$$\text{(3.1.12h)} \qquad \text{or} \quad \|g_{\mu_{1\ell}}(\lambda_{1\ell})\| \overset{\ell\to\infty}{\longrightarrow} \infty, \quad g_{\mu_{1\ell}}(\lambda_{1\ell}) \in U\,(\forall\, \ell \in \mathbb{N}).$$

In case (3.1.12g), we obtain, using the fact that $\lambda_{1\ell} \overset{\ell\to\infty}{\longrightarrow} 0$ and $\mathcal{O}(E, G)$ is equicontinuous, that

$$\text{(3.1.12i)} \qquad g_{\mu_{1\ell}}(0) \longrightarrow \hat{v}_0 \in \partial G \;\text{ as }\; \ell \to \infty.$$

On the other hand, we may assume that $0 \in B_{x_1}$ for some x_1. In view of (3.1.12d) and the fact that $\lambda_{1\ell} \overset{\ell\to\infty}{\longrightarrow} 0$, we may assume that for any $\ell \gg 1$ with $\lambda_{1\ell} \in B_{x_1}$, it holds that $\|g_{\mu_{1\ell}}(\lambda_{1\ell})\| < 2 + \|g_{\mu_{1\ell}}(0)\|$ for any $\ell \gg 1$. So the case (3.1.12h) gives us that

$$\text{(3.1.12j)} \qquad \|g_{\mu_{1\ell}}(0)\| \longrightarrow \infty \quad \text{as} \quad \ell \to \infty.$$

Thus it follows from (†3) that either $f_{\mu_{1\ell}}(\alpha_{\mu_{1\ell}}) \overset{\ell\to\infty}{\longrightarrow} \partial G$, or $\|f_{\mu_{1\ell}}(\alpha_{\mu_{1\ell}})\| \overset{\ell\to\infty}{\longrightarrow} \infty$. In the former, as in the first paragraph of this proof, using Lemma 3.3.1 we get a contradiction to the property (†2). In the latter, it is also a similar situation as in (3.1.12c). So, after we replace $(g_j)_j$ (resp. $(\beta_j)_j$) by $(f_{\mu_{1\ell}})_\ell$ (resp. $(\alpha_{\mu_{1\ell}})_\ell$), we can repeat the previous argument. By carrying out this procedure, we may extract a subsequence $(f_{0j})_{j\in\mathbb{N}}$ of $(f_{\mu_{1\ell}})_{\ell\in\mathbb{N}}$ such that $(f_{0j}(0))_{j\in\mathbb{N}}$ does not converge to a point in G; compare (3.1.12i) and (3.1.12j). This is a contradiction to the property (†2). Consequently, in each case we are led to the desired contradiction. Thus the proof is complete. □

§3.4. Hyperconvexity of Hartogs type domains.

In this section we will only deal with bounded hyperconvex domains. So, in this section, we will always assume that any domain is bounded. Note that the boundedness of domains is not an invariant property under biholomorphic mappings. Recall that any hyperconvex domain is taut. In view of Proposition 3.1.1, Lemma 2.2.13, and Proposition 1.1.10, the continuity and plurisubharmonicity of u, h (resp. u, v) are necessary for the Hartogs domain $\Omega = \Omega_{u,h}(G)$ (resp. $\Sigma = \Sigma_{u,v}(G)$) to be hyperconvex. Moreover, we have the following statements:

Proposition 3.4.1. *(1) $\Omega_H(G)$ is hyperconvex iff G is hyperconvex and $H \in (\mathcal{C} \cap PSH)(G \times \mathbb{C}^m, \mathbb{R})$. Here H is any function defining a Hartogs domain $\Omega_H(G)$ with m-dimensional balanced fibers.*

(2) If G is hyperconvex and $u, v \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, then $\Sigma_{u,v}(G)$ is hyperconvex.

Proof. For the proof of (1), we refer to Proposition 3.8 in [Jar-Pfl-Zwo 00] (cf. Proposition 3.1.1).

(2) Because of the hyperconvexity of G, there exists an exhaustion $\varphi \in (\mathcal{C} \cap PSH)(G, \mathbb{R}_{<0})$ of G. Define a function $\Phi : G \times \mathbb{C}_* \to [-\infty, \infty)$ by $\Phi(z, \lambda) := \max\{\varphi(z), \psi(z, \lambda)\}$ for $(z, \lambda) \in G \times \mathbb{C}_*$, where

$$\psi(z, \lambda) := \max\{u(z) + \log|\lambda|,\, v(z) - \log|\lambda|\}, \quad (z, \lambda) \in G \times \mathbb{C}_*.$$

Since $u, v \in (\mathcal{C} \cap PSH)(G)$, clearly $\psi \in (\mathcal{C} \cap PSH)(G \times \mathbb{C}_*)$. Therefore, $\Phi \in (\mathcal{C} \cap PSH)(G \times \mathbb{C}_*)$; moreover, $\Phi \in (\mathcal{C} \cap PSH)(\Sigma, \mathbb{R}_{<0})$ and it is an exhaustion of Σ. Thus Σ has a bounded continuous psh exhaustion function Φ. □

Remark 3.4.2. Assume that Σ is a pseudoconvex Reinhardt domain. By Lemma 1.1.9 and (c) in Theorem 1.5.23, one has: if Σ is hyperconvex, then G is hyperconvex and u is bounded from below on G. However, its converse, in general, does not hold. For example, consider the Hartogs triangle $\triangle_H = \Sigma_{0,\log|\cdot|}(E)$ (cf. Example 2.1.2, Example 1.5.24).

§3.5. k-completeness of Hartogs type domains.

In this section we shall recall some results for k-completeness of Hartogs type domains. Those results for a Hartogs type domain with 1-dimensional balanced fibers are mainly obtained by N. Q. Dieu and D. D. Thai ([Die-Tha 00]).

Remark 3.5.1. Obviously, if a Hartogs domain over $G \subset \mathbb{C}^n$ with m-dimensional balanced fibers is k-complete, so is the base G. In general, however, in case of Hartogs-Laurent domains, the same property does not hold (see Example 2.2.18 and Remark 2.2.19). Moreover, for any $n \geq 1$ there exists an example of a pseudoconvex non-k-complete Hartogs domain $\Omega = \Omega_H(G)$ such that the base G is k-complete and $H \in \mathcal{C}(G \times \mathbb{C}^n)$. For more details, see Corollary 4.2.4 below.

The following result can be regarded as a generalized statement of the example situation of (4) in Example 2.1.9.

Theorem 3.5.2. *Let $\Omega \subset \mathbb{C}^{n+m}$ be a Hartogs domain over a k-complete domain $G \subset \mathbb{C}^n$. Suppose that for any $(z,w) \in \partial\Omega$ with $z \in G$, there are an open neighborhood $V = V(z) \subset G$ and a mapping $f \in \mathcal{O}(\Omega', U)$, where $\Omega' = \Omega \cap (V \times \mathbb{C}^m)$ and U is a k-complete domain, such that the sequence $(f(z_\nu, w_\nu))_{\nu \geq 1}$ is not relatively compact in U for any sequence $((z_\nu, w_\nu))_{\nu \geq 1}$ converging to the point (z,w). Then Ω is k-complete.*

In [Die-Tha 00], the authors stated and proved a special version of Theorem 3.5.2. However, in fact, their proof guarantees that Theorem 3.5.2 is also true. For the reader's convenience we will give the proof. For this, we need the following lemma.

Localization Lemma. *Let $\omega \subset \mathbb{C}^n$ be a domain and let $\epsilon, r > 0$, and $p_0 \in \omega$. Then we may take a constant $C > 1$ such that*

$$k_{\mathbb{B}_{k_\omega}(p_0, 3r+\epsilon)}(p,q) \leq C k_\omega(p,q), \quad p,q \in \mathbb{B}_{k_\omega}(p_0, r)$$

For a proof, we refer to e.g. Proposition 3.1.19 in [Kob 98].

Proof of Theorem 3.5.2. Let $(p_j)_{j \geq 1}$ be a k_Ω-Cauchy sequence in Ω, and denote $p_j := (z_j, w_j) \in G \times \mathbb{C}^m$ for $j \geq 1$. Then $p_j \longrightarrow {}^{\exists}p_0 := (z_0, w_0) \in \bar{\Omega}$. We claim that $p_0 \in \Omega$. By considering the decreasing property of $\underline{k}$, it is easy to check that $(z_j)_{j \geq 1}$ is a k_G-Cauchy sequence in G. So the k-completeness of G implies that $z_j \longrightarrow {}^{\exists}z_0' \in G$ as $j \to \infty$. In particular, $z_0 = z_0'$.

Suppose that $p_0 \in \partial\Omega$. By our hypotheses, we may take an open neighborhood $V = V(z_0) \subset G$ of z_0 and a mapping $f \in \mathcal{O}(\Omega', U)$ where $\Omega' := \Omega \cap (V \times \mathbb{C}^m)$ and U is a k-complete domain, such that

$$\{f(p_j) : j \geq 1\} \not\Subset U. \tag{3.5.2a}$$

Take a constant $r > 0$ with

$$5r \leq \inf\{k_G(z_0, w) : w \in G \setminus V\} =: k_G(z_0, G \setminus V). \tag{3.5.2b}$$

Here, we used that G is k-hyperbolic and k_G is continuous (Remark 1.3.3). Choose $j_0 \in \mathbb{N}$ so large that

$$\{p_j : j \geq j_0\} \subset \mathbb{B}_{k_\Omega}(p_{j_0}, r). \tag{3.5.2c}$$

Now we will see that

$$\mathbb{B}_{k_\Omega}(p_{j_0}, 4r) \subset \Omega'. \tag{3.5.2d}$$

For this, let $q \in \mathbb{B}_{k_\Omega}(p_{j_0}, 4r)$ and set $\pi(z,w) := z$ for $(z,w) \in \Omega \cap (G \times \mathbb{C}^m)$. By the continuity of k_G, it holds that

$$k_G(\pi(p_{j_0}), z_0) = k_G(z_{j_0}, z_0) = \lim_{j\to\infty} k_G(z_{j_0}, z_j) \le \lim_{j\to\infty} k_\Omega(p_{j_0}, p_j) \le r.$$

Here, in the third (resp. fourth) inequality, we used the decreasing property of $\underline{k}$ (resp. the condition (3.5.2c)). Therefore,

$$k_G(\pi(q), z_0) \le k_G(\pi(q), \pi(p_{j_0})) + k_G(\pi(p_{j_0}), z_0) \le k_\Omega(q, p_{j_0}) + r \lneqq 5r \le k_G(z_0, G\setminus V),$$

and so (3.5.2b) implies that $\pi(q) \subset V$, i.e. $q \in \pi^{-1}(V) = \Omega'$. Hence, we get the required property (3.5.2d). Thus, by the above Localization Lemma and (3.5.2c), there is a constant $C > 0$ such that

$$k_{\mathbb{B}_{k_\Omega}(p_{j_0},4r)}(p_{j_0}, p_j) \le C k_\Omega(p_{j_0}, p_j), \quad j \ge j_0,$$

and also the decreasing property of $\underline{k}$ and (3.5.2d) give us that

$$k_U(f(p_{j_0}), f(p_j)) \le k_{\Omega'}(p_{j_0}, p_j) \le k_{\mathbb{B}_{k_\Omega}(p_{j_0},4r)}(p_{j_0}, p_j) \le Cr =: R, \quad j \ge j_0.$$

Since U is k-complete, it follows from (a) of Theorem 1.5.18 that

$$\{f(p_j) : j \ge j_0\} \subset \mathbb{B}_{k_\Omega}(f(p_{j_0}), R) \subset \mathbb{B}_{k_U}(f(p_{j_0}), R) \Subset U,$$

which is a contradiction to (3.5.2a). Thus $p_0 \in \Omega$ and we are done. □

The following corollary is the exact case studied in (4) of Example 2.1.9. For more details, see pp. 366-369 in [Sib 81].

Corollary 3.5.3. *Let $\Omega \subset \mathbb{C}^{n+m}$ be a Hartogs domain over a k-complete domain $G \subset \mathbb{C}^n$. Suppose that for any $(z,w) \in \partial\Omega$ with $z \in G$, there are an open neighborhood $V = V(z) \subset G$ and a mapping $f \in \mathcal{O}(\Omega')$, where $\Omega' := \Omega \cap (V \times \mathbb{C}^m)$, such that*

$$\lim_{\Omega \ni \eta \to (z,w)} |f(\eta)| = 1 > |f(\zeta)|, \quad \zeta \in \Omega'.$$

Then Ω is k-complete.

The following two statements may be also found in [Die-Tha 00]. In this paper, the authors studied only the case that $m = 1$ and $h(\lambda) := |\lambda|$, but we can also apply the same argument to prove our assertion in the case $h(w) = \|w\|$, $w \in \mathbb{C}^m$.

As a simple consequence of Corollary 3.5.3, we have the following:

Corollary 3.5.4. *Suppose that G is a k-complete domain, $u \in (\mathcal{C} \cap PSH)(G, \mathbb{R})$, and that for any $z \in G$, there exist an open neighborhood $U = U(z) \subset G$ and a function $f_z \in \mathcal{O}(U)$ such that*

$$u(w) \ge \log|f_z(w)| \ (w \in U), \qquad u(z) = \log|f_z(z)|.$$

Then $\Omega_{u,\|\cdot\|}(G)$ is k-complete.

As a particular case of Corollary 3.5.4 we have

Corollary 3.5.5. *If G is a k-complete domain and u is strictly psh on G, then $\Omega_{u,\|\cdot\|}(G)$ is k-complete.*

The proof follows directly from considering the Taylor expansion of u.

In [Die-Tha 00], the authors pointed out that the assumption of Corollary 3.5.4 is rarely satisfied. For example:

Remark 3.5.6. (1) Let $u \in PSH(G)$ and $z \in G$. Suppose that $u(z) \neq -\infty$ and that u satisfies the assumption of Corollary 3.5.4. Then $G_u := \{(z, \lambda) \in G \times \mathbb{C} : \mathrm{Re}\lambda + u(z) < 0\}$ has a holomorphic support graph at z in the following sense: there exist an open neighborhood $U = U(z) \subset G$ and a function $\varphi \in \mathcal{O}(U)$ with $\varphi(z) = u(z)$, but

$$G_u \cap \{(z', \varphi(z')) : z' \in U\} = \emptyset.$$

(2) Let us recall an example of a domain found in [Sib 91]. For this, let $k \geq 3$ and $|t| \leq \frac{k^2}{2k-1}$. Define $u_t(\lambda) := |\lambda|^{2k} + t|\lambda|^2\mathrm{Re}(\lambda^{2k-2})$, $\lambda \in \mathbb{C}$. Then u_t is real analytic subharmonic on $\mathbb{C}$. If we put $G := \mathbb{C}_{u_t}$ as in (1), then G is a pseudoconvex domain with a smooth boundary. But since G does not have a supporting analytic set at $\lambda = 0$, the function u_t does not satisfy the assumption of Corollary 3.5.4 at $\lambda = 0$.

At this point, it would be interesting whether there is a Hartogs domain $\Omega = \Omega_{u,\|\cdot\|}(G)$ such that Ω is k-complete and $u \in (\mathcal{C} \cap PSH)(G)$, but u does not satisfy the assumption of Corollary 3.5.4. The following positive answer of the question can be found in [Die-Tha 00].

Proposition 3.5.7. *Let $G \subset \mathbb{C}$ be a k-complete domain and let $u \in (\mathcal{C}^2 \cap SH)(G)$. Put $\Omega \equiv \Omega_{u,|\cdot|}(G)$ and suppose that*

$$\forall_{z_0 \in G}, \quad \exists_{N \geq 4,\, U \equiv U(z_0) \subset G} \quad : \begin{cases} u \text{ is of class } \mathcal{C}^N \text{ in } U, \\ \exists_{1 \leq \alpha \leq \beta - 1 \leq N-1} \quad : \quad \dfrac{\partial^\beta u}{\partial z^{\beta-\alpha} \partial \bar{z}^\alpha} \neq 0 \text{ on } U. \end{cases}$$

Then for any $(z_0, \lambda_0) \in \partial\Omega$ with $z_0 \in G$, there is an open neighborhood V of (z_0, λ_0) such that (z_0, λ_0) is a local peak point for $\mathcal{O}(\Omega \cap V)$. Moreover, Ω is a k-complete hyperbolic domain.

In [Die-Tha 00] the proof of Proposition 3.5.7 was based on the following result of E. Bedford and J. E. Foraeness:

Theorem 3.5.8. ([Bed-For 78]) *Let $P_{2m}(\lambda) := \sum_{j=0}^{2m} a_j \lambda^j \bar{\lambda}^{2m-j}$ ($\lambda \in \mathbb{C}$) be a homogenous subharmonic polynomial of degree $2m$ which is not harmonic. Then for $\epsilon > 0$ small enough, $(0, 0) \in \partial\mathcal{D}_\epsilon$ is a peak point for $\mathcal{O}(\mathcal{D}_\epsilon) \cap \mathcal{C}(\overline{\mathcal{D}_\epsilon})$, where*

$$\mathcal{D}_\epsilon := \left\{(\lambda, \zeta) \in \mathbb{C}^2 : \mathrm{Re}\,\zeta + P_{2m}(\lambda) < \epsilon(|\zeta| + |\lambda|^{2m})\right\}.$$

In §4.3, using Proposition 3.5.7, we are going to give a sufficient condition for balanced domains in $\mathbb{C}^2$ to be k-complete.

CHAPTER 4. $\tilde{k}$-HYPERBOLICITY AND COMPLETENESS OF BALANCED DOMAINS

Summary. This chapter is devoted to study $\tilde{k}$-hyperbolicity and completeness of balanced domains.

In §4.1, we give a partial answer to the following question: Is any unbounded pseudoconvex balanced domain $D = D_h$ with $h^{-1}(0) = \{0\}$ $\tilde{k}$-hyperbolic ? This question arose from Kodama's theorem for k-hyperbolicity and Siciak's theorem for Brody hyperbolicity.

In case $n = 2$, we are interested in an example of a balanced domain $D = D_h \subset \mathbb{C}^2$ introduced by K. Azukawa (see Example 4.1.1). By modifying the proof of Lemma 6.3 in [Azu 83] we prove that for any $z, w \in D$,

$$\tilde{k}_D(z,w) > 0 \quad \text{if} \quad z_1 \neq w_1 \quad \text{or} \quad [\, z_1 = w_1 \neq 0 \;\&\; z_2 \neq w_2 \,]$$

(Remark 4.1.4, Example 4.1.5).

In Example 4.1.7 (and Remark 4.1.9), we show that for any $n \geq 3$ there is an unbounded pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^n$ that is not $\tilde{k}$-hyperbolic, although it is Brody hyperbolic and $h^{-1}(0) = \{0\}$.

§4.2 is devoted to give another example of the following result obtained by M. Jarnicki and P. Pflug: For $n \geq 3$ there is a bounded pseudoconvex balanced domain $D = D_h$ in $\mathbb{C}^n$ with continuous Minkowski function h that is not k-complete. Moreover, we get the counterexample announced in Remark 3.5.1, namely: There is a pseudoconvex Hartogs domain $\Omega = \Omega_H(G)$ over $G \subset \mathbb{C}^2$ with m-dimensional balanced fibers such that the domain G is k-complete and $H \in \mathcal{C}(G \times \mathbb{C}^m)$ but Ω is not k-complete (Corollary 4.2.4).

It is well-known that there is a pseudoconvex non-k-complete domain $G \Subset \mathbb{C}^2$ which has a $\mathcal{C}^\infty$-boundary except of one point. This example is given by N. Sibony in 1991; a detailed proof can be found in [Jar-Pfl 93]. Here, in Theorem 4.2.5, we present a new example of that type.

Finally, in §4.3, as an application of Proposition 3.5.7, we give a sufficient condition for balanced domains in $\mathbb{C}^2$ to be k-complete.

§4.1. $\tilde{k}$-hyperbolicity of balanced domains.

Modifying some examples of A. Sadullaev and T. Barth ([Sad 80], [Bar 80]), K. Azukawa ([Azu 83]) found the following example:

Example 4.1.1. Define a function $h : \mathbb{C}^2 \to \mathbb{R}$ by

$$h(z) = \begin{cases} |z_2| e^{\varphi(\frac{z_1}{z_2})} & (z_2 \neq 0), \\ |z_1| & (z_2 = 0), \end{cases}$$

where $\varphi : \mathbb{C} \to [-\infty, \infty)$ is defined by

$$\varphi(\lambda) := \max\left\{ \log|\lambda|,\ \sum_{k=2}^{\infty} \frac{1}{k^2} \log\left|\lambda - \frac{1}{k}\right| \right\},\ \ \lambda \in \mathbb{C}.$$

Then h is positive definite on $\mathbb{C}^2$ and the balanced domain $D = D_h := \{z \in \mathbb{C}^2 : h(z) < 1\}$ is pseudoconvex and Brody hyperbolic (Lemma 6.3 in [Azu 83]; cf. Theorem 4.1.2 and Remark 4.1.4 below) but not k-hyperbolic (cf. Proposition 2.1.4). Sometimes we will say that D is the *Azukawa Domain.*

In this spirit, A. Kodama obtained the following result.

Kodama's Theorem. *(cf. (c) in Proposition 1.5.3)*
Any k-hyperbolic balanced domain in $\mathbb{C}^n$ is bounded.

In general, a k-hyperbolic Reinhardt domain, even if it is pseudoconvex, is not bounded (cf. Theorem 1.5.21).

On the other hand, there is the following characterization of the Brody hyperbolicity for pseudoconvex balanced domains in $\mathbb{C}^2$ due to J. Siciak ([Sic 85]; [Jar-Pfl 93], Theorem 7.1.3):

Theorem 4.1.2. *For a pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^2$ the following are equivalent:*
(a) D is Brody hyperbolic;
(b) No complex lines through 0 stays inside D;
(c) h is positive definite.

Here, the implications $(a) \Longrightarrow (b) \Longrightarrow (c)$ are trivial.

We are now in a position to ask what about the $\tilde{k}$-hyperbolicity of pseudoconvex balanced domains?

Observe that the condition $h^{-1} = \{0\}$ is a necessary condition for a balanced domain $D = D_h \subset \mathbb{C}^n$ to be Brody hyperbolic. At first, we would like to know whether the Azukawa Domain $D = D_h \subset \mathbb{C}^2$ is $\tilde{k}$-hyperbolic.

Before answering these questions, let us first mention the following example:

Example 4.1.3. Let $g : \mathbb{C}^n \to [0, \infty)$ be upper semicontinuous such that

$$\ell := \lim_{\|z\| \to \infty} \frac{g(z)}{\|z\|} \quad \text{exists and is finite.} \tag{4.1.3a}$$

Define $h : \mathbb{C}^n \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ by

$$h(z, z_{n+1}) = h_g(z, z_{n+1}) := \begin{cases} |z_{n+1}| g(\frac{z}{z_{n+1}}) & (z_{n+1} \neq 0), \\ \ell\|z\| & (z_{n+1} = 0). \end{cases} \tag{4.1.3b}$$

For $(z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C}$, put $\pi_1(z, z_{n+1}) := z$ and $\pi_2(z, z_{n+1}) := z_{n+1}$. Clearly, h is absolutely homogeneous and upper semicontinuous on $\mathbb{C}^{n+1}$. Let $D := D_h$. In the following we shall consider the condition:

$$\exists_{C>0} \ : \ h(z, 1) \geq C\|z\|, \quad z \in \mathbb{C}^n. \tag{4.1.3c}$$

Then the following properties are true:
(1) if $D \Subset \mathbb{C}^{n+1}$, then:
a. $g^{-1}(0) = \emptyset$,
b. there exists a $C > 0$ such that $g(z) > C\|z\|$ for any $z \in \mathbb{C}^n$,

c. $h^{-1}(0) = \{0\}$ and h satisfies (4.1.3c);

(2) if h is positive definite and satisfies (4.1.3c), then:

a. $\pi_1(D) \Subset \mathbb{C}^n$,

b. $\pi_2(D) \neq \mathbb{C}$ iff $D \Subset \mathbb{C}^{n+1}$,

c. if, moreover, $h \in PSH(\mathbb{C}^{n+1})$ and $\{z\} \times \mathbb{C} \not\subset D$ for any $z \in \pi_1(D)$, then D is Brody hyperbolic.

Proof. (1) Since $D = D_h \Subset \mathbb{C}^{n+1}$, there is a $C > 0$ such that

$$h(z, z_{n+1}) \geq C\|(z, z_{n+1})\| = C\big(\|z\|^2 + |z_{n+1}|^2\big)^{1/2}, \quad (z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C},$$

which implies that $g(z) \geq C\|(z,1)\| > C\|z\|$ for $z \in \mathbb{C}^n$, and also $g^{-1}(0) = \emptyset$. The last property is trivial.

(2) (a) Note that $\ell \geq 0$. If $\ell = 0$, then $h\big|_{\mathbb{C}^n \times \{0\}} \equiv 0$, which is a contradiction to $h^{-1}(0) = \{0\}$ and so $\ell > 0$. Observe that $h(z,1) = g(z)$, $\forall\, z \in \mathbb{C}^n$. Let $(z, z_{n+1}) \in D$. If $z_{n+1} = 0$, then $\|z\| < \frac{1}{\ell}$; if $z_{n+1} \neq 0$, then

$$1 \geq |z_{n+1}| h(\frac{z}{z_{n+1}}, 1) \geq |z_{n+1}| \left(C\frac{\|z\|}{|z_{n+1}|}\right) = C\|z\|,$$

that is, $\|z\| < \frac{1}{C}$. Hence one has $\|z\| < \max\{\frac{1}{C}, \frac{1}{\ell}\} =: \alpha$. But since z is arbitrary, $\pi(D) \subset \mathbb{B}_n(0,\alpha) \Subset \mathbb{C}^n$.

(b) Let $a \notin \pi_2(D)$. Suppose that there exists $b \in \pi_2(D)$ with $a \neq b$ such that $|b| \geq |a|$. Then $h(z_b, b) < 1$ for some $z_b \in \mathbb{C}^n$. Take $\theta \in [0, 2\pi)$ and $0 < \beta \leq 1$ so that $\arg a = \theta + \arg b$ and $|a| = \beta|b|$. Then

$$h(\beta e^{i\theta} z_b, a) = |\beta||e^{i\theta}| h(z_b, b) = |\beta| h(z_b, b) \leq h(z_b, b) < 1.$$

That is, $(\beta e^{i\theta} z_b, a) \in D$ and so $a \in \pi_2(D)$, which is a contradiction. Therefore, $\pi_2(D) \subset \mathbb{B}_1(0, |a|) \Subset \mathbb{C}$ and thus, the required result is obtained by (2-a).

(c) Suppose the contrary. There exists a map $\varphi := (f, g) \in \mathcal{O}(\mathbb{C}, D), \varphi \neq$ constant, where $f \in \mathcal{O}(\mathbb{C}, \pi_1(D))$ and $g \in \mathcal{O}(\mathbb{C}, \pi_2(D))$. Since $\pi_1(D) \Subset \mathbb{C}^n$ by (2-a), the Liouville type theorem gives us that the mapping f must be a constant, set $f\big|_{\mathbb{C}} \equiv {}^{\exists}a \in \pi_1(D)$. Since φ is not a constant, one has $\pi_2(D) = \mathbb{C}$ by (2-a) and so the little Picard theorem yields that $g(\mathbb{C}) \supset \mathbb{C} \setminus \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$, and because of $h \in PSH(\mathbb{C}^{n+1})$, it follows from the removable singularity theorem and Liouville's theorem that $h(a, \cdot) =$ constant $=: M < 1$ on $\mathbb{C}$, which implies that $\{a\} \times \mathbb{C} \subset D$; a contradiction. $\square$

Remark 4.1.4. Let $D = D_h \subset \mathbb{C}^2$ be the Azukawa Domain. Note that $h^{-1}(0) = \{0\}$ and since

$$|\lambda| \leq \exp\varphi(\lambda) \leq \max\big\{|\lambda|, (|\lambda|+1)^{(\frac{\pi^2}{6}-1)}\big\}, \quad \lambda \in \mathbb{C},$$

we have $\lim_{|\lambda|\to\infty} \frac{\exp(\varphi(\lambda))}{|\lambda|} = 1$ and $h(\lambda, 1) = \exp(\varphi(\lambda)) \geq \exp(\log|\lambda|) = |\lambda|$ for any $\lambda \in \mathbb{C}$. By (2-a) of Example 4.1.3, the set $\{z_1 \in \mathbb{C} : z \in D\}$ is bounded in $\mathbb{C}$, in fact, $\{z_1 \in \mathbb{C} : z \in D\} = E$. Then the decreasing property of $\tilde{k}$ gives us that

$$\tilde{k}_D(z, w) > 0, \quad z, w \in D \text{ with } z_1 \neq w_1. \qquad \text{(4a)}$$

Since D is not bounded in $\mathbb{C}^2$, one has $\{z_2 \in \mathbb{C} : z \in D\} = \mathbb{C}$ by (2-b) of Example 4.1.3. On the other hand, because of $h \in PSH(\mathbb{C}^2)$, it follows from Lemma 2.1.1 that $\tilde{k}_D(0, z) = p(0, h(z))$ for any $z \in D$. In particular,

$$\tilde{k}_D(0, z) > 0, \quad z \in D \text{ with } z \neq 0 \tag{4b}$$

(Observe there is no $\lambda_0 \in E$ such that $\{\lambda_0\} \times \mathbb{C} \subset D$, so (2-c) of Example 4.1.3 gives us that D is Brody hyperbolic: cf. the proof of Lemma 6.3 in [Azu 83]).

Now, we shall study the $\tilde{k}$-hyperbolicity of the Azukawa Domain.

Example 4.1.5. Let $D = D_h \subset \mathbb{C}^2$ be the Azukawa Domain. Then

$$\tilde{k}_D((a, z_2), (a, w_2)) > 0, \quad (a, z_2), (a, w_2) \in D \cap (\mathbb{C}_* \times \mathbb{C}),\ z_2 \neq w_2. \tag{4c}$$

To verify this, we will use some ideas of the proof of Lemma 6.3 in [Azu 83].

Proof. Let $a \in E_*$. Suppose that there are two points $z_2, w_2 \in \mathbb{C}$ with $(a, z_2), (a, w_2) \in D$ such that $\tilde{k}_D((a, z_2), (a, w_2)) = 0$. Then we may take sequences $(s_\nu)_{\nu \geq 1} \subset \mathbb{R}$ and $(\varphi_\nu)_{\nu \geq 1} \subset \mathcal{O}(s_\nu E, D)$ such that

$$\varphi_\nu := (f_\nu, g_\nu),\ \ f_\nu, g_\nu \in \mathcal{O}(s_\nu E, \mathbb{C}), \quad \nu \geq 1, \tag{4.1.5a}$$

$$f_\nu(0) = f_\nu(1) = a, \quad \nu \geq 1, \tag{4.1.5b}$$

$$g_\nu(0) = z_2,\ \ g_\nu(1) = w_2, \quad \nu \geq 1, \tag{4.1.5c}$$

$$1 < s_\nu \nearrow \infty \quad \text{as} \quad \nu \to \infty. \tag{4.1.5d}$$

Fix a constant $\epsilon > 0$ so small that $\epsilon \ll \frac{1}{2}\min\{|a|, 1 - |a|\}$. Put

$$D_a := D \cap (\mathbb{B}_1(a, \epsilon) \times \mathbb{C}).$$

Since $f_\nu(s_\nu E) \subset E$ for any $\nu \geq 1$, in virtue of Montel's theorem, Liouville's theorem, and (4.1.5b), we may assume, without loss of generality, that $f_\nu(s_\nu E) \subset \mathbb{B}_1(a, \epsilon)$ for any $\nu \geq 1$.

Now take a sequence $(r_j)_{j \geq 2} \subset \mathbb{R}_{>0}$ (e.g. $r_j := \frac{1}{2j^3}$) such that:

$$r_j \searrow 0 \quad \text{as} \quad j \to \infty, \tag{4.1.5e}$$

$$r_j + r_{j+1} < \frac{1}{j(j+1)}, \quad j \geq 2, \tag{4.1.5f}$$

$$\alpha := \sum_{k=2}^{\infty} \frac{\log r_k}{k^2} > -\infty. \tag{4.1.5g}$$

We define

$$\Omega_j := \bigcup_{|a - x| < \epsilon} (\{x\} \times A_j^x),$$

where $A_j^x := \left\{\zeta \in \mathbb{C} : \left|\frac{x}{\zeta} - \frac{1}{j}\right| < r_j\right\}$ for $j \geq 2$. Clearly, $\Omega_j \subset \mathbb{C}^2$ is open for $j \geq 2$. Observe that

$$\Omega_j \cap \Omega_k = \emptyset \quad \text{whenever} \quad j, k \geq 2,\ j \neq k, \tag{4.1.5h}$$

(cf. (4.1.5f)). We claim that

$$D_a \subset \Omega_0 \cup \Big(\bigcup_{j=2}^{\infty} \Omega_j\Big), \tag{4.1.5i}$$

where $\Omega_0 := \mathbb{B}_1(a,\epsilon) \times \mathbb{B}_1(0, e^{-\alpha})$. For this, it is enough to check that

$$(D_a \cap (\mathbb{C} \times \mathbb{C}_*)) \setminus \Big(\bigcup_{j\ge 2} \Omega_j\Big) \subset \mathbb{B}_1(a,\epsilon) \times \mathbb{B}_1(0, e^{-\alpha}).$$

More explicitly, let $(x,\lambda) \in (D_a \cap (\mathbb{C}\times\mathbb{C})) \setminus \big(\bigcup_{j\ge 2}\Omega_j\big)$ with $\lambda \neq 0$. Then $(x,\lambda) \in D$, $x \in \mathbb{B}_1(a,\epsilon)$, and $\lambda \notin \big(\bigcup_{j\ge 2} A_j^x\big)$, that is, $\left|\frac{x}{\lambda} - \frac{1}{j}\right| \ge r_j$ for any $j \ge 2$. Moreover, one has

$$1 > |\lambda| \exp(\varphi(\frac{x}{\lambda})) \ge |\lambda| \exp\left(\sum_{k=2}^{\infty} \frac{1}{k^2} \log\left|\frac{x}{\lambda} - \frac{1}{k}\right|\right) \ge |\lambda| \exp\left(\sum_{k=2}^{\infty} \frac{1}{k^2}\log r_k\right),$$

which implies that

$$|\lambda| < \exp\left(-\sum_{k=2}^{\infty}\frac{1}{k^2}\log r_k\right) = \exp(-\alpha).$$

Hence (4.1.5i) is true.

On the other hand, using (4.1.5f) it is easy to check that

$$A_j^x \subset \left\{\zeta \in \mathbb{C} : \frac{|x|}{\frac{1}{j} + r_j} < |\zeta| < \frac{|x|}{\frac{1}{j} - r_j}\right\}, \quad j \ge 2,\ x \in \mathbb{B}_1(a,\epsilon). \tag{4.1.5j}$$

Hence, by (4.1.5e), (4.1.5h), and (4.1.5j), there exists a $j_0 \ge 0$ with $j_0 \neq 1$ such that

$$\Omega_0 \cap \Omega_j \neq \emptyset \ \ (j \le j_0), \qquad \Omega_0 \cap \Omega_j = \emptyset \ \ (j > j_0), \tag{4.1.5k}$$

and

$$\left\{\zeta \in \mathbb{C} :\ \exists_{x\in\mathbb{C}},\ (x,\zeta) \in \underbrace{\Omega_0 \cup \Big(\bigcup_{j=2}^{j_0}\Omega_j\Big)}_{=:\tilde{\Omega}_0}\right\} \Subset \mathbb{C}. \tag{4.1.5l}$$

Moreover,

$$\tilde{\Omega}_0 \cap \Big(\bigcup_{j>j_0}\Omega_j\Big) = \emptyset, \qquad D_a \subset \tilde{\Omega}_0 \cup \Big(\bigcup_{j>j_0}\Omega_j\Big). \tag{4.1.5m}$$

Now, since $\varphi_\nu(s_\nu E)$ is connected in D_a for any $\nu \ge 1$, it follows from (4.1.5c), (4.1.5k), and (4.1.5m) that:

$$\text{either} \quad \{(a,z_2),(a,w_2)\} \subset \bigcup_{\nu\ge 1}\varphi_\nu(s_\nu E) \subset \tilde{\Omega}_0$$

$$\text{or} \qquad \{(a,z_2),(a,w_2)\} \subset \bigcup_{\nu\ge 1}\varphi_\nu(s_\nu E) \subset \Omega_{j_a} \quad \text{for some} \quad j_a \in \mathbb{N},\ j_a > j_0.$$

Therefore, in view of (4.1.5j) and (4.1.5l), we can choose an $R > 0$ so large that $g_\nu(s_\nu E) \subset \mathbb{B}(0,R)$ for $\nu \ge 1$. So, using Montel's theorem, Liouville's theorem, (4.1.5b), and (4.1.5c), we get that $z_2 = w_2$. Consequently,

$$\tilde{k}_D((a,z_2),(a,w_2)) > 0 \quad \text{whenever} \quad a \neq 0,\ z_2 \neq w_2. \qquad \square$$

At this point, the following question naturally arises:

Remark 4.1.6. Let $(0,z_2),(0,w_2) \in D \cap (\mathbb{C} \times \mathbb{C}_*)$ with $z_2 \neq w_2$. It would be interesting to know whether $\tilde{k}_D((0,z_2),(0,w_2))$ is positive or zero. This remains still an open problem.

Next, the following example shows that for $n \geq 3$, in general, a Brody hyperbolic pseudoconvex balanced domain $D = D_h \subset \mathbb{C}^n$ need not to be $\tilde{k}$-hyperbolic.

Example 4.1.7. Let $a \in \mathbb{C}\setminus\{0\}$, $b,c \in \mathbb{C}$ with $b \neq c$, and take $M_1 > 0$ and $M_2 > |a|$. Then there exists a Brody hyperbolic, pseudoconvex, balanced domain D in $\mathbb{C}^3$ such that $D \subset (M_1 E) \times (M_2 E) \times \mathbb{C}$ and

$$\tilde{k}_D((0,a,b),(0,a,c)) = 0. \tag{4.1.7a}$$

In particular, D is not $\tilde{k}$-hyperbolic.

The idea for this example was proposed by Professor M. Jarnicki and Professor P. Pflug.

Proof. Let $(r_j)_{j\geq 1}$ be a sequence in $\mathbb{R}$ such that $1 < r_j \nearrow \infty$ as $j \to \infty$ and $\lim_{j\to\infty} \frac{\log(\log(r_j^2+r_j))}{\log j} \in \mathbb{R}$ (e.g. $r_j := \frac{1}{2}e^{j/2}$). Take a sequence $(s_j)_{j\geq 1} \subset (0,1)$ so that $r_j^2 + r_j < 1/s_j < 2(r_j^2 + r_j)$, $(j \geq 1)$. Moreover, we define a sequence $(\varphi_j)_{j\geq 1} \subset \mathcal{O}(E_j, \mathbb{C}^3)$ of mappings $\varphi_j =: (\varphi_j^1, \varphi_j^2, \varphi_j^3)$, where $E_j := r_j E$, by

$$\varphi_j^1(\lambda) := s_j\lambda(\lambda-1), \quad \varphi_j^2(\lambda) := a, \quad \varphi_j^3(\lambda) := (c-b)\lambda + b, \quad \lambda \in E_j$$

and set

$$Q_j(z) := z_1 z_2 - \frac{a s_j}{(c-b)^2}\big(z_3 - \frac{b}{a}z_2\big)\big(z_3 - \frac{c}{a}z_2\big), \quad z = (z_1,z_2,z_3) \in \mathbb{C}^3.$$

For any $j \geq 1$, put $t_j := \sqrt{j}/s_j$ and $\epsilon_j := 2^{-j-1}$. Obviously, $\eta_j := t_j s_j < \eta_{j+1} \to \infty$ as $j \to \infty$ and $\sum_{j=1}^\infty \epsilon_j = \frac{1}{2}$. Moreover, it is easy to see that

$$\sum_{j=1}^\infty \epsilon_j \log\frac{1}{\eta_j} > -\infty, \quad \sum_{j=1}^\infty \epsilon_j \log\frac{1}{t_j} > -\infty \tag{4.1.7b}$$

(use e.g. the Log-test for series). Now we define a function $h : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ by

$$h(z) := \max\left\{\frac{|z_1|}{M_1}, \frac{|z_2|}{M_2}, h_0(z)\right\}, \quad z = (z_1,z_2,z_3) \in \mathbb{C}^3,$$

where

$$h_0(z) := \prod_{j=1}^\infty \left(\frac{|Q_j(z)|}{\eta_j}\right)^{\epsilon_j} = \exp\left(\sum_{j=1}^\infty \epsilon_j \log\frac{|Q_j(z)|}{\eta_j}\right), \quad z \in \mathbb{C}^3.$$

Now, we are going to show that $D = D_h := \{z \in \mathbb{C}^3 : h(z) < 1\}$ is an unbounded, Brody hyperbolic, pseudoconvex balanced domain, and that $(4.1.7a)$ is satisfied.

$1°$. *h_0 is alsolutely homogeneous on $\mathbb{C}^3$, so is h; moreover, h is positive definite.*

Subproof. Fix $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ and fix $\lambda \in \mathbb{C}$. Then it is clear that $|Q_j(\lambda z)| = |\lambda|^2 |Q_j(z)|$ for any $j \geq 1$, and also

$$\sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(\lambda z)|}{\eta_j} = \sum_{j=1}^{\infty} \epsilon_j \log |\lambda|^2 + \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(z)|}{\eta_j}$$
$$= 2(\log|\lambda|) \sum_{j=1}^{\infty} \epsilon_j + \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(z)|}{\eta_j} = \log|\lambda| + \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(z)|}{\eta_j}.$$

This implies that $h_0(\lambda z) = |\lambda| h_0(z)$. Since $z \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ are arbitrary, the function h_0 is absolutely homogeneous, so is h.

On the other hand, it is clear that $h^{-1}(0) \subset \{0\} \times \{0\} \times \mathbb{C}$. Let $\lambda \in \mathbb{C}$. Observe that

$$|Q_j(0,0,\lambda)| = \frac{|a| s_j}{|c-b|^2} |\lambda|^2, \quad j \geq 1.$$

Then

$$\sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(0,0,\lambda)|}{\eta_j} = \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a||\lambda|^2}{|c-b|^2},$$

which implies that $h_0(0,0,\lambda) = 0$ iff $\lambda = 0$, so h is positive definite. $\square$

2°. *h_0 is psh in $\mathbb{C}^3$, so is h; in particular, D is pseudoconvex.*

Subproof. For this, fix $\alpha > 0$. Then for any $z \in (\alpha E)^3$, it holds that

$$|Q_j(z)| \leq |z_1||z_2| + \frac{|a| s_j}{|c-b|^2} \left(|z_3|^2 + \frac{|b+c|}{|a|} |z_3||z_2| + \frac{|bc|}{|a|^2} |z_2|^2 \right)$$
$$\leq \alpha^2 + s_j \underbrace{\frac{|a|\alpha^2}{|c-b|^2} \left(1 + \frac{|b+c|}{|a|} + \frac{|bc|}{|a|^2} \right)}_{=: M_\alpha = M_\alpha(a,b,c) > 0}, \quad j \geq 1.$$

Since $\lim_{j\to\infty} \eta_j = \infty$ and $0 < s_j < 1$, we have

$$\exists_{j_\alpha \in \mathbb{N}} \; : \; \frac{|Q_j(z)|}{\eta_j} \leq \frac{1 + M_\alpha s_j}{\eta_j} \leq \frac{1 + M_\alpha}{\eta_j} < 1, \quad z \in \mathbb{B}_3(0,\alpha), \, j \geq j_\alpha.$$

This implies that $h_0 \in PSH(\mathbb{B}_3(0,\alpha))$. Since $\alpha > 0$ is arbitrary, one has $h_0 \in PSH(\mathbb{C}^3)$ and also $h \in PSH(\mathbb{C}^3)$. Moreover, the pseudoconvexity of D follows directly from (2) of Proposition 1.1.3. $\square$

3°. *$\tilde{k}_D((0,a,b),(0,a,c)) = 0$, and so D is not $\tilde{k}$-hyperbolic; in particular, D is unbounded and not k-hyperbolic.*

Subproof. It is easy to check that

$$(Q_j \circ \varphi_j)(\lambda) = 0, \quad \lambda \in r_j E, \quad j \geq 1.$$

This implies that $\varphi_j(r_j E) \subset D$, i.e. $\varphi_j \in \mathcal{O}(r_j E, D)$ for any $j \geq 1$. In particular, $\varphi_j(0) = (0,a,b)$ and $\varphi_j(1) = (0,a,c)$. $\square$

4°. *D is Brody hyperbolic.*

Subproof. Let $f := (f_1, f_2, f_3) \in \mathcal{O}(\mathbb{C}, D)$, where $f_j \in \mathcal{O}(\mathbb{C}), j = 1, 2, 3$. Since $D \subset (M_1 E) \times (M_2 E) \times \mathbb{C}$, it follows from Liouville's theorem that $f_1 =$ constant $=: \zeta_1$ and $f_2 =$ constant $=: \zeta_2$. Suppose that f is not a constant, i.e. f_3 is not a constant. Then, in view of the Little Picard Theorem, one has $f_3(\mathbb{C}) \supset \mathbb{C} \setminus \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$, which implies that $h(\zeta_1, \zeta_2, \lambda) < 1$ for any $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$. Hence, $h(\zeta_1, \zeta_2, \cdot) < 1$ on $\mathbb{C}$; in particular, $h_0(\zeta_1, \zeta_2, \cdot) < 1$ on $\mathbb{C}$. Thus, by the Liouville type theorem, we conclude that $h_0(\zeta_1, \zeta_2, \cdot) \equiv$ constant on $\mathbb{C}$. Observe that

$$h_0(\zeta_1, \zeta_2, \lambda) = 0$$

for any $\lambda \in \mathbb{C}$ such that $Q_j(\zeta_1, \zeta_2, \lambda) = 0$ for some $j \geq 1$. Therefore, in order to get a contradiction, it is enough to verify that

$$\forall_{(\zeta_1,\zeta_2)\in(M_1E)\times(M_2E)}, \quad \exists_{\lambda\in\mathbb{C}} \ : \ \log h_0(\zeta_1, \zeta_2, \lambda) > -\infty. \tag{4.1.7c}$$

To show this, fix a point $(\zeta_1, \zeta_2) \in (M_1 E) \times (M_2 E)$. Now we shall discuss the following four cases.

(i) *the case* $\zeta_2 = 0$: For any $\lambda \neq 0$, one has

$$Q_j(\zeta_1, 0, \lambda) = -\frac{a s_j \lambda^2}{(c-b)^2}, \quad \forall j \geq 1.$$

Therefore,

$$\begin{aligned} \log h_0(\zeta_1, 0, \lambda) &= \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(\zeta_1, 0, \lambda)|}{\eta_j} \\ &= \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a||\lambda|^2}{|c-b|^2} > -\infty. \end{aligned}$$

Here, in the last inequality, we used (4.1.7b).

(ii) *the case* $\zeta_1 = 0$ *and* $\zeta_2 \neq 0$: Take a point $\lambda \in \mathbb{C} \setminus \{0, b\zeta_2/a, c\zeta_2/a\}$. Then

$$Q_j(0, \zeta_2, \lambda) = -\frac{a s_j (\lambda - \frac{b}{a}\zeta_2)(\lambda - \frac{c}{a}\zeta_2)}{(c-b)^2 a^2} \neq 0, \quad j \geq 1.$$

This implies that

$$\begin{aligned} \log h_0(0, \zeta_2, 0) &= \sum_{j=1}^{\infty} \epsilon_j \log \frac{|Q_j(0, \zeta_2, 0)|}{\eta_j} \\ &= \sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j} + \frac{1}{2} \log \frac{|a|\left|(\lambda - \frac{b}{a}\zeta_2)(\lambda - \frac{c}{a}\zeta_2)\right|}{|c-b|^2} > -\infty. \end{aligned}$$

Here, in the last inequality, we used (4.1.7b) .

(iii) *the case* $\zeta_1\zeta_2 \neq 0$ *and* $\frac{1}{s_j} \neq \frac{bc\zeta_2}{a(c-b)^2\zeta_1}$ $(j \geq 1)$: Note that

$$|Q_j(\zeta_1, \zeta_2, 0)| = \left| \zeta_1\zeta_2 - \frac{bc\zeta_2^2 s_j}{(c-b)^2 a} \right| \geq |\zeta_1\zeta_2| - s_j \frac{|bc\zeta_2^2|}{|a||c-b|^2}, \quad \forall j \geq 1.$$

Since $\lim_{j\to\infty} s_j = 0$,

$$\exists_{j_0\in\mathbb{N}} \;:\; |Q_j(\zeta_1,\zeta_2,0)| \geq \frac{1}{2}|\zeta_1\zeta_2| > 0, \quad j \geq j_0.$$

By our assumption,

$$Q_j(\zeta_1,\zeta_2,0) = \zeta_1\zeta_2 - \frac{bc\zeta_2^2 s_j}{(c-b)^2 a} = \zeta_2\left(\zeta_1 - \frac{bc\zeta_2 s_j}{(c-b)^2 a}\right) \neq 0, \quad j \geq 1,$$

which implies that

$$C := \sum_{j=1}^{j_0-1} \epsilon_j \log\frac{|Q_j(\zeta_1,\zeta_2,0)|}{\eta_j} \in \mathbb{R}.$$

Note that $\sum_{j=1}^{j_0-1}\epsilon_j\log(1/\eta_j) > -\infty$. Therefore, we have

$$\begin{aligned}\log h_0(\zeta_1,\zeta_2,0) &= \sum_{j=1}^{\infty}\epsilon_j\log\frac{|Q_j(\zeta_1,\zeta_2,0)|}{\eta_j}\\ &\geq \sum_{j=1}^{j_0-1}\epsilon_j\log\frac{|Q_j(\zeta_1,\zeta_2,0)|}{\eta_j} + \sum_{j=j_0}^{\infty}\epsilon_j\log\frac{|\zeta_1\zeta_2|}{2\eta_j}\\ &= C + \left(\sum_{j=j_0}^{\infty}\epsilon_j\right)\log\frac{|\zeta_1\zeta_2|}{2} + \sum_{j=j_0}^{\infty}\epsilon_j\log\frac{1}{\eta_j} > -\infty.\end{aligned}$$

Here, in the last inequality, we used (4.1.7b).

(iv) *the case* $\zeta_1\zeta_2 \neq 0$ *and* $\frac{1}{s_j} = \frac{bc\zeta_2}{a(c-b)^2\zeta_1}$ *for some* $j \geq 1$: Note that, in this case, $bc \neq 0$. Take a point $\lambda_0 \in \mathbb{C}\setminus\{0\}$ such that

$$|\zeta_1\zeta_2| > \frac{|\zeta_2|^2|(\lambda_0-b)(\lambda_0-c)|}{|a||c-b|^2}.$$

Put $\lambda := \frac{\zeta_2}{a}\lambda_0$. Since $0 < s_j < 1$ for any $j \geq 1$, it is easy to check that $Q_j(\zeta_1,\zeta_2,\lambda) \neq 0$ for any $j \geq 1$. In particular,

$$|Q_j(\zeta_1,\zeta_2,\lambda)| \geq \frac{|\zeta_1\zeta_2|}{2}, \quad \forall j \geq 1.$$

By the similar argument as in (iii), we may get that $\log h_0(\zeta_1,\zeta_2,\lambda) > -\infty$. □

So the proof is finished. □

In the previous example, the value of $\lim_{|\lambda|\to 0}\frac{h(z/\lambda,1)}{\|z/\lambda\|}$ depends on the choices of $z \in \mathbb{C}^2$ (cf. (4.1.3a)). For more details:

Remark 4.1.8. Keep the same notations as in the proof of Example 4.1.7. Clearly, $|Q_j(\zeta,0,1)| = \frac{|a|s_j}{|c-b|^2}$ for $\zeta \in \mathbb{C}$ and $j \geq 1$. Then for any $z_1 \in \mathbb{C}\setminus\{0\}$, one has

$$\lim_{|\lambda|\not\to 0} \frac{h_0(z_1/\lambda,0,1)}{|z_1/\lambda|} = \lim_{|\lambda|\not\to 0} |\lambda| \frac{\exp\left(\sum_{j=1}^{\infty} \epsilon_j \log \frac{|a|}{t_j|b-c|^2}\right)}{|z_1|} = 0,$$

and also $\ell' := \lim_{|\lambda|\not\to 0} \frac{h(z_1/\lambda,0,1)}{|z_1/\lambda|} = 1/M_1$. On the other hand, it is easy to check that

$$h_0(0,\frac{z_2}{\lambda},1) = \frac{1}{|\lambda|} \exp\big(\sum_{j=1} \epsilon_j \log \frac{|a|\left|(\lambda - \frac{b}{a}z_2)(\lambda - \frac{c}{a}z_2)\right|}{t_j|c-b|^2}\big), \quad z_2 \neq 0,\ \lambda \neq 0,$$

and also

$$\lim_{|\lambda|\not\to 0} \frac{h_0(0,z_2/\lambda,1)}{|z_2/\lambda|} = \frac{\sqrt{|bc|}}{\sqrt{|a|}|c-b|} \exp\big(\sum_{j=1}^{\infty} \epsilon_j \log \frac{1}{t_j}\big) =: M_3 \geq 0.$$

Here, M_3 is positive whenever $bc \neq 0$. Therefore, one has $\ell'' := \lim_{|\lambda|\to 0} \frac{h(0,z_2/\lambda,1)}{|z_2/\lambda|} = \max\{1/M_2, M_3\}$, so we have, in general, $\ell' \neq \ell''$.

Remark 4.1.9. As a consequence of Example 4.1.7, for any $n \geq 3$ there is a pseudoconvex balanced domain in $\mathbb{C}^n$ which is Brody hyperbolic but not $\tilde{k}$-hyperbolic.

§4.2. Some counterexamples for k-completeness.

The following result is due to M. Jarnicki and P. Pflug ([Jar-Pfl 91c]): For $n \geq 3$, there is a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with continuous Minkowski function that is not k-complete. The main aim of this section is to give another example belonging to the same category. For this, we shall apply the method used in [Jar-Pfl 91c] (or [Jar-Pfl 93], Theorem 7.5.7). Different from [Jar-Pfl 91c], we use a new analytic chain X with better properties.

Let us give the main theorem.

Theorem 4.2.1. *There exists a bounded balanced pseudoconvex domain $G = \{z \in \mathbb{C}^3 : h(z) < 1\}$ in $\mathbb{C}^3$ with continuous Minkowski function h that is not k_G-finitely compact.*

Also, we get the following by taking $\Omega := E^{n-3} \times G$, $n \geq 3$.

Corollary 4.2.2. *For $n \geq 3$, there is a bounded balanced domain of holomorphy in $\mathbb{C}^n$ with continuous Minkowski function that is not Kobayashi complete.*

For the proof of Theorem 4.2.1, we need the following lemma:

Lemma 4.2.3. *There are a connected set $X \subset \mathbb{B}_2(0,2)\setminus\{0\}$ and a sequence $(a_j)_{j\in\mathbb{N}} \subset X$, $a_j \to 0 \in \partial X$ as $j \to \infty$, with the following properties:*

(i) *For each j there are $\lambda_j, \zeta_j \in E$ and a mapping $\varphi_j \in \mathcal{O}(E, X)$ such that $\varphi_j(\lambda_j) = a_j$, $\varphi_j(\zeta_j) = a_{j+1}$, and $\sum_{j=1}^{\infty} p(\lambda_j, \zeta_j) < \infty$.*

(ii) *There is a continuous logarithmically psh* (shortly, *log-psh*) *function $F : \mathbb{C}^2 \to \mathbb{R}_{>0}$ such that:*
- *$F(0) = 1$, $F\big|_X < 1$;*
- *$F(z) \le 4\|z\|^{1/2}$ for $\|z\| \ge 1$;*
- *the open set $\{z \in \mathbb{C}^2 : \|z\| < 2,\ F(z) < 1\}$ has a connected component G' such that $X \subset G'$, $0 \in \partial X \cap \partial(G')$, and G' is not $k_{G'}$-finitely compact.*

Using the previous lemma, we also obtain the following statement which was mentioned in Remark 3.5.1.

Corollary 4.2.4. *There exists a pseudoconvex Hartogs domain $\Omega = \Omega_H(G)$ over $G \subset \mathbb{C}^2$ with m-dimensional balanced fibers such that G is k-complete and H is continuous on $G \times \mathbb{C}^m$, but Ω is not k-complete.*

On the other hand, N. Sibony proved that

(S) There is a pseudoconvex non-k-complete domain $G \Subset \mathbb{B}_2(0,1)$ with a $\mathcal{C}^\infty$-boundary except of one point.

It was published in detail in Theorem 7.5.9 of [Jar-Pfl 93] later. Note that the original construction of Theorem 4.2.1 by M. Jarnicki and P. Pflug is based on the idea of the proof of (S).

Using some part of the following construction of Theorem 4.2.1 and using the idea of Sibony's original proof, we may also get a new example in the same category, namely:

Theorem 4.2.5. *There is a pseudoconvex non-k-complete domain $G \Subset \mathbb{B}_2(0,2)$ given as a connected component of $\{z \in \mathbb{B}_2(0,4)\ ;\ u(z) < 1\}$, where $u \in (\mathcal{C} \cap PSH)(\mathbb{B}_2(0,4)) \cap \mathcal{C}^\infty(\mathbb{B}_2(0,4) \setminus \{0\})$ and* $\operatorname{grad} u(z) \neq 0$ *if $z \neq 0$, and $u(0) = 1$.*

Now we shall verify Lemma 4.2.3. Its proof will be based on a series of steps.

Proof of Lemma 4.2.3. We will start defining a family of analytic sets as follows:

Step I. *Let $k \in \mathbb{N}$ and define $X := \bigcup_{j=1}^{\infty} X_j$, where*

$$\begin{aligned} X_{2k-1} &:= \left\{(z_1, z_2) \in \mathbb{C}^2 : 2^{2k-1} z_1 + 2^{2k} z_2 = 1,\ |z_1| \le 2^{-2k+1} + 2^{(-2k+1)/2}\right\}, \\ X_{2k} &:= \left\{(z_1, z_2) \in \mathbb{C}^2 : 2^{2k+1} z_1 + 2^{2k} z_2 = 1,\ |z_2| \le 2^{-2k} + 2^{-k}\right\}. \end{aligned}$$

Put $a_{2k} := (0, 2^{-2k}) \in X_{2k-1} \cap X_{2k}$ and $a_{2k+1} := (2^{-2k-1}, 0) \in X_{2k} \cap X_{2k+1}$.

Note that the X_j's are connected, so is X. Also, $X_{2k-1} \subsetneq (\frac{3}{2}E) \times E$ and $X_{2k} \subsetneq E^2$. In particular, even if the X_j's are compact, X is not compact; moreover, $X \setminus \mathbb{B}_2(0,R)$ is compact for every $0 < R < 1$.

Step 2. *For $k \in \mathbb{N}$ we define holomorphic mappings $\varphi_k : E \to X_k \subset X$ by*

$$\varphi_{2k-1}(\lambda) := \Big(\frac{1}{2^{(2k-1)/2}}\big(\frac{1}{2^{(2k-1)/2}} - \lambda\big),\ \frac{1}{2^{(2k+1)/2}}\lambda\Big),$$
$$\varphi_{2k}(\lambda) \quad := \Big(\frac{1}{2^{k+1}}\lambda,\ \frac{1}{2^k}\big(\frac{1}{2^k} - \lambda\big)\Big)$$

for all $\lambda \in E$. Then

$$\varphi_{2k-1}\big(\frac{1}{2^{(2k-1)/2}}\big) = a_{2k} = \varphi_{2k}(0) \quad \textit{and} \quad \varphi_{2k}\big(\frac{1}{2^{2k/2}}\big) = a_{2k+1} = \varphi_{2k+1}(0)$$

Moreover, $\sum_{k=1}^{\infty} p(0, 2^{-k/2}) =: \Theta < \infty$, where p is the Poincaré distance.

Proof. It suffices to check that the last assertion is satisfied. If

$$\log\big(\frac{1+x}{1-x}\big) \leq 4x, \quad 0 < x \leq \frac{1}{2}, \tag{2a}$$

then
$$\sum_{k=2}^{\infty} p(0, \frac{1}{2^{k/2}}) = \frac{1}{2}\sum_{k=2}^{\infty} \log\frac{2^{k/2}+1}{2^{k/2}-1} \leq 2\sum_{k=2}^{\infty}\frac{1}{2^{k/2}} = 2 + \sqrt{2},$$

which implies that $\Theta < \infty$. Thus it is enough to see that $(2a)$ holds. For this, define $\varphi(x) := (1-x)\exp(4x) - x - 1$ for $x \in \mathbb{R}$. A direct computation yields that $\varphi'(x) = (3-4x)\exp(4x) - 1$ for every $x \in \mathbb{R}$. But since $\varphi' > 0$ on $(0, 1/2]$ and $\varphi(0) = 0$, we have $\varphi \geq 0$ on $(0, 1/2]$ and so $(2a)$ is true. □

To continue with the next steps, we need the following notations: Let $k \in \mathbb{N}$, $\alpha > 2$, $0 < \epsilon_0 < 1$, and choose $\theta_0 > 0$ with $\sigma := \theta_0/(1-\epsilon_0) < 1$. Put $\epsilon_k := \theta_0\epsilon_0^{k-1}$ and $r_k := 2^{k+2}$. Choose two constants $A, c \in (0,1)$ such that $\log A = -2(\log 2)\sum_{j=1}^{\infty}\epsilon_j(j+1)$ and $\log c < (1/\sigma)\log(1/\alpha)$. Next, we define a function $f : \mathbb{C}^2 \to (-\infty, +\infty)$ by $f(z) := (1-\sigma)\log A + \sigma\max\big\{g(z),\ \log(Ac)\big\}$ for $z = (z_1, z_2) \in \mathbb{C}^2$, where

$$g(z) := \sum_{j=1}^{\infty}\epsilon_j \log\frac{|P_j(z)|}{r_j}; \qquad P_j(z) := \frac{1}{2^j} - \frac{3+(-1)^j}{2}z_1 - \frac{3-(-1)^j}{2}z_2.$$

Let $\chi : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ be a $\mathcal{C}^1$-function on $\mathbb{C}^2$ with the following properties:

$$\operatorname{supp}\chi \subset \overline{E}^2,\ \chi(z) = \chi(|z_1|, |z_2|), \text{ and } \int_{\mathbb{C}^2}\chi(\lambda)dV_4(\lambda) = 1$$

where dV_4 is the Lebesgue measure on $\mathbb{R}^4$. For $\eta > 0$, denote

$$f_\eta(z) = (f * \chi_\eta)(z) := \int_{\mathbb{C}^2} f(w)\chi_\eta(z-w)\,dV_4(w), \quad z \in \mathbb{C}^2,$$

where $\chi_\eta(z) = (1/\eta^4)\chi(z/\eta)$ for $z \in \mathbb{C}^2$. Here "$*$" denotes the convolution operator.

Now we will show some basic properties that will be used in the following steps.

Step 3. *We have the following properties:*
(1) $\sum_{j=1}^{\infty} \epsilon_j (j+1) < \infty$.
(2) $A \nearrow 1$ *as* $\theta_0 \searrow 0^+$;
(3) *The function* g *is psh on* $\mathbb{C}^2$, *and so is* f.

Proof. Since
$$\frac{\epsilon_0^j (j+1)}{\epsilon_0^{j-1} j} = \epsilon_0 \frac{j+1}{j} \longrightarrow \epsilon_0 < 1 \quad \text{as} \quad j \to +\infty,$$
by the Ratio test, the series $\sum_{j=1}^{\infty} \epsilon_0^{j-1} j =: T_0$ is convergent and so is $\sum_{j=1}^{\infty} \epsilon_j (j+1)$. Moreover, $\log A = -2\theta_0 (\log 2)\big(T_0 + 1/(1-\epsilon_0)\big)$, and hence we get (1) and (2). For (3) it suffices to verify that
$$\forall_{m \in \mathbb{N}}, \quad \exists_{k \equiv k(m) \in \mathbb{N}} \quad \text{s.t. } u_{\langle k \rangle} := \sum_{j=k}^{\infty} \epsilon_j \log \frac{|P_j|}{r_j} \in PSH(\mathbb{B}_2(0,m)).$$

Fix $m_0 \in \mathbb{N}$. Note that $2\big(\max_{\|z\| \le m_0} |P_k(z)|\big) \le 1 + 6m_0$. Put $k_0 := \min\big\{k \in \mathbb{N} : 1 + 6m_0 \le 2^{k+2}\big\}$. Then $|P_k(z)| \le r_k$ for $\|z\| \le m_0$ and $k \ge k_0$. Therefore, if $\|z\| \le m_0$ and $k \ge k_0$, one has $\log\big(|P_k(z)|/r_k\big) < 0$, and also
$$u_i(z) := \sum_{k=k_0}^{k_0+i} \epsilon_k \log \frac{|P_k(z)|}{r_k} \searrow u_{\langle k_0 \rangle}(z) \quad \text{as} \quad i \to \infty.$$
Consequently, $u_{\langle k_0 \rangle} \in PSH(\mathbb{B}_2(0,m_0))$ because of $u_i \in PSH(\mathbb{B}_2(0,m_0))$. Hence $g \in PSH(\mathbb{B}_2(0,m_0))$ and so $g \in PSH(\mathbb{C}^2)$. □

Step 4. *Let* $c_0 := \max\big\{3/4,\ Ac\big\}$, $\tau_0 := \max\{2\sigma^2, \sigma\}$. *Then* $0 < c_0 < 1$ *and the following properties are fulfilled:*

$$f(0) = \log A, \tag{4}$$
$$f\big|_X \equiv \log A + \sigma \log c \le f(z) \quad \textit{for} \quad z \in \mathbb{C}^2, \tag{5}$$
$$f(z) < 0 \quad \textit{for} \quad \|z\| < 1, \tag{6}$$
$$f(z) \le \tau_0 \log \|z\| \quad \textit{for} \quad \|z\| \ge c_0. \tag{7}$$

In particular, f *is not continuous at* $z = 0 \in \partial X$, *but*

$$f \ \textit{is a constant on a neighborhood of } X. \tag{8}$$

Proof. For $k \in \mathbb{N}$ it holds that
$$|P_k(0)| = 2^{-k},$$
$$P_k\big|_{X_k} \equiv 0,$$
$$|P_{2k-1}(z)|/r_{2k-1} < 2^{-2k-1}\big(2^{-2k+1} + 3\|z\|\big) \le 1/2^4 + (3/2^3)\|z\|,$$
$$|P_{2k}(z)|/r_{2k} < 2^{-2k-2}\big(2^{-2k} + 3|z|\big) \le 1/2^6 + (3/2^4)\|z\|,$$
$$|P_k(z)|/r_k < 1 \quad \text{for} \quad \|z\| < 1,$$
$$|P_k(z)|/r_k < \|z\|^2 \quad \text{for} \quad \|z\| \ge c_0,$$

and so

$$g(0) = \sum_{j=1}^{\infty} \epsilon_j \log(1/2^{2j+2}) = \log A,$$
$$g\big|_X \equiv -\infty,$$
$$g(z) < 0 \quad \text{for} \quad \|z\| < 1,$$
$$g(z) \leq \sum_{j=1}^{\infty} \epsilon_j (\log \|z\|^2) = 2\sigma \log \|z\| \quad \text{for} \quad \|z\| \geq c_0.$$

If we put $h(z) := \sigma \max\big\{g(z),\ \log(Ac)\big\}$ for $z \in \mathbb{C}^2$, then it holds that

$$h(0) = \sigma \log A,$$
$$h\big|_X \equiv \sigma \log(Ac),$$
$$h(z) < 0 \quad \text{for} \quad \|z\| < 1,$$
$$h(z) \leq \sigma \max\big\{2\sigma \log \|z\|,\ \log(Ac)\big\} \leq \tau_0 \log \|z\| \quad \text{for} \quad \|z\| \geq c_0.$$

Therefore, we get directly the required properties (4)~(7). Here, to get (6) and (7), we used the two conditions $0 < \sigma < 1$ and $0 < A < 1$.

The discontinuity of f at the origin follows directly from (4) and (5). To show (8), fix $k \in \mathbb{N}$. Note that $|P_k(z + w)| \leq |P_k(z)| + 3\|w\|$ for all $z, w \in \mathbb{C}^2$. In particular, $|P_k(z + w)| \leq 3\|w\|$, if $z \in X_k$. Fix $z_0 \in X_k$ and choose $\delta_k \in \mathbb{R}$ with $0 < \delta_k < \frac{1}{2} \min_{w \in X_k} \|w\|$, so that $\epsilon_k \log\big(3\delta_k/r_k\big) \leq \log(Ac)$. Then we have that

$$g(z_0 + z) \leq \epsilon_k \log \frac{|P_k(z_0 + z)|}{r_k} \leq \epsilon_k \log \frac{3\|z\|}{r_k} \leq \log(Ac) \quad \text{for} \quad \|z\| \leq \delta_k,$$

Here, in the first inequality, we used the fact that $|P_j(z_0 + z)|/r_j < 1$ for $\|z\| \leq \delta_k$ and $j \in \mathbb{N}$. Hence, $h(w) = h(z_0)$ for $w \in \overline{B}(z_0, \delta_k)$. Note that $0 \notin \overline{B}(z_0, \delta_k)$. But since k is arbitrary, there is an open neighborhood $V(\not\ni 0)$ of the set X such that $h\big|_V \equiv \sigma \log(Ac)$. Thus, the function f is a constant on a neighborhood of X. $\square$

Step 5. *For $\eta > 0$, the sequence (f_η) of psh $\mathcal{C}^1$-functions on $\mathbb{C}^2$ satisfies the following properties:*

(9) $$f_\eta \searrow f \quad as \quad \eta \searrow 0,$$

(10) $$f_\eta \longrightarrow f \ \ uniformly\ on\ a\ compact\ subset\ of\ X \ \ as\ \ \eta \searrow 0,$$

(11) $$f_\eta(0) \geq f(0) = \log A,$$

(12) $$\exists_{\eta_0 \in (0,1)} \ : \ f_\eta(z) < 0 \quad if \quad \|z\| \leq 1 \quad and \quad 0 < \eta \leq \eta_0,$$

(13) $$f_\eta(z) \leq \tau_0 \log(2\|z\|) \quad if \quad 0 < \eta \leq (1 - c_0)/\sqrt{2} \quad and \quad \|z\| \geq 1.$$

In particular,

(14) $$\forall_{\text{compact set } K \subset X},\ \exists_{\eta_K \in (0,1)} \ : \ f_\eta = f \quad on \quad K \quad for \quad 0 < \eta \leq \eta_K.$$

Proof. Since χ is of class $\mathcal{C}^1$, so is $f * \chi_\eta$. Also, the plurisubharmonicity of f_η and (9) are well-known, and so (11) is trivial. For (10), let $K \subset X$ be compact, $z \in K$. Then

$$|f_\eta(z) - f(z)| \leq \int_{\mathbb{C}^2} \chi_\eta(z-\lambda)\ |f(\lambda) - f(z)|\ dV_4(\lambda) \leq \sup_{w\in K} \|f - f(w)\|_{w+(\eta\overline{E})^2},$$

so the property (10) follows from (8) and the compactness of the set $K \subset X$. Moreover, the preceding inequalities also give us the property (14), because the function f is a constant on a neighborhood of K.

For (12), fix $z \in \mathbb{C}^2$ with $\|z\| \leq 1$. By the properties (6), (9), and the continuity of f_η, we may find an open neighborhood U_z of z and a small $\eta_z \in (0,1)$ so that $f_\eta(w) < 0$ for $w \in U_z$ and $0 < \eta \leq \eta_z$. Hence, the property (12) follows directly from the compactness of $\overline{\mathbb{B}_2(0,1)}$.

In order to show (13), let $0 < \eta \leq (1-c_0)/\sqrt{2}$ and let $z \in \mathbb{C}^2$ with $\|z\| \geq 1$. If $\lambda \in \overline{E}^2$, then

$$\|z - \eta\lambda\| \geq \|z\| - \eta\|\lambda\| \geq 1 - (1-c_0) = c_0,$$
$$\|z - \eta\lambda\| \leq \|z\| + \eta\|\lambda\| \leq \|z\| + (1-c_0) < 2\|z\|.$$

But since $\operatorname{supp}\chi \subset \overline{E}^2$ and $\int_{\mathbb{C}^2} \chi(\lambda) dV_4(\lambda) = 1$, it follows from (7) that

$$\int_{\overline{E}^2} f(z-\eta\lambda)\,\chi(\lambda)\,dV_4(\lambda) \leq \int_{\overline{E}^2} \tau_0 \log\|z-\eta\lambda\|\,\chi(\lambda)\,dV_4(\lambda) \leq \tau_0\log(2\|z\|). \qquad \square$$

Step 6. *For $0 < \eta \leq \min\{\eta_0,\ (1-c_0)/\sqrt{2}\}$, define a map $F_\eta : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ by $F_\eta(z) := [\exp f_\eta(z)]/[\exp f_\eta(0)]$ for $z \in \mathbb{C}^2$. Then the following properties are fulfilled:*

$$F_\eta \text{ is a } \mathcal{C}^1 \text{ (locally bounded), log-psh function on } \mathbb{C}^2, \tag{15}$$
$$F_\eta(0) = 1, \tag{16}$$
$$F_\eta(z) \leq 1/A \quad \text{for} \quad \|z\| \leq 1, \tag{17}$$
$$F_\eta(z) \leq (2\|z\|)^{\tau_0}/A \quad \text{for} \quad \|z\| \geq 1. \tag{18}$$

In particular, $F_\eta(z) \neq 0$ for all $z \in \mathbb{C}^2$.

Proof. Clearly, (15) and (16) are true. The property (17) follows directly from (11) and the definition of F_η. Since $0 < \eta \leq \min\{\eta_0,\ (1-c_0)/\sqrt{2}\}$, it follows from (13) that

$$F_\eta(z) \leq \frac{1}{A}\exp f_\eta(z) \leq \frac{1}{A}\exp\left[\tau_0\log(2\|z\|)\right] = \frac{(2\|z\|)^{\tau_0}}{A}, \quad \|z\| \leq 1.$$

Thus, the property (18) holds. In particular, the positivity of F_η follows immediately from (5) and (9). $\square$

To construct the peak function which is required in Lemma 4.2.3, we shall apply the so-called *Bishop's construction of peak functions* (see e.g. [Gam 84]). For this, we need the following preparation:

Step 7. *Let $\beta > 1$. Then there are two sequences $(U_k)_{k\geq 1}$, $(\eta_k)_{k\geq 1}$ satisfying the following properties:*

$$U_k \supset U_{k+1} \ni 0, \quad \textit{for} \quad k = 1, 2, \cdots, \tag{19}$$

$$\eta_k > \eta_{k+1} > 0 \quad \textit{for} \quad k = 1, 2, \cdots, \tag{20}$$

$$F_{\eta_j} < \frac{1}{\alpha} \quad \textit{on} \quad X \setminus U_j, \;\; j = 1, 2 \cdots, \tag{21}$$

$$F_{\eta_\nu} < 1 + \frac{1}{\alpha \cdot \beta^j} \quad \textit{on} \quad U_j, \;\; 1 < j, \;\; 1 \leq \nu < j. \tag{22}$$

Proof. To obtain the required sequences, we shall use induction with respect to k. First put $U_1 := \mathbb{B}_2(0, 1/4)$. Since $X \setminus U_1$ is compact, combining (5) and (14) we may choose a small η_1 with $0 < \eta_1 \leq \min\{\eta_0, (1-c_0)/\sqrt{2}\}$ so that $f_\eta = \log A + \sigma \log c$ on $X \setminus U_1$ for all $0 < \eta \leq \eta_1$. Hence, using (11), it is easy to see that $F_\eta < 1/\alpha$ on $X \setminus U_1$ for all $0 < \eta \leq \eta_1$, because $\log c < (1/\sigma)\log(1/\alpha)$. Suppose we have already constructed two finite sequences $(U_j)_{1\leq j\leq k}$ and $(\eta_j)_{1\leq j\leq k}$ satisfying the properties (19) $\sim$ (22). We define the next set

$$U_{k+1} := \left\{ z \in U_k : F_{\eta_j}(z) < 1 + \frac{1}{\alpha \cdot \beta^{k+1}}, \;\; 1 \leq j \leq k \right\}.$$

Since $X \setminus U_{k+1}$ is compact, by the same method as in the case $k = 1$, we may choose $\eta_{k+1} \in (0, \eta_k)$ so that $F_\eta < 1/\alpha$ on $X \setminus U_{k+1}$ for all $0 < \eta \leq \eta_{k+1}$. Thus we have obtained two sequences $(U_k)_{k\in\mathbb{N}}$ and $(\eta_k)_{k\in\mathbb{N}}$ having the desired properties. □

Applying these sequences to Bishop's construction, we get the following:

Step 8. *We find a continuous log-psh function $F : \mathbb{C}^2 \to \mathbb{R}_{>0}$ such that $F(0) = 1$, $F\big|_X < 1$, and also $F(z) \leq C\|z\|^{\tau_0}$ for $\|z\| \geq 1$, where $C := 2^{\tau_0}/A$.*

Proof. Using the sequence $(F_j)_{j\geq 1}$ obtained in Step 7, we define a function $F : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ by

$$F(z) := (\beta - 1)\sum_{j=1}^{\infty} \frac{1}{\beta^j} F_j(z) \quad \text{for} \quad z \in \mathbb{C}^2.$$

Combining (12) and (13), it is easy to see that this series is locally uniformly convergent in $\mathbb{C}^2$. So the function F is continuous, logarithmically psh in $\mathbb{C}^2$ with $F(0) = 1$. In particular, by (18) (or (13)), one has $F(z) \leq (2^{\tau_0}/A)\|z\|^{\tau_0}$ for $\|z\| \geq 1$. On the other hand, the property (19) says that $X \setminus U_1 \subset X \setminus U_j$ for $j \in \mathbb{N}$. Hence, by (21) we get $F_j < 1/\alpha$ on $X \setminus U_1$ for $j \in \mathbb{N}$, which implies that $F < 1/\alpha < 1$ on $X \setminus U_1$. Next, to show that $F < 1$ on X, let $k \in \mathbb{N}$ and fix $z \in X \cap (U_k \setminus U_{k+1})$. Since

$$\frac{1}{\beta - 1}F(z) = \sum_{j=1}^{k-1} \frac{1}{\beta^j}F_j(z) + \frac{1}{\beta^k}F_k(z) + \sum_{j=k+1}^{\infty} \frac{1}{\beta^j}F_j(z),$$

it follows from (17), (19), (21), and (22) that

$$\begin{aligned} \frac{1}{\beta - 1}F(z) &\leq (1 + \frac{1}{\alpha \cdot \beta^k}) \cdot \frac{(1/\beta)\left\{1 - (1/\beta^{k-1})\right\}}{1 - (1/\beta)} + \frac{1}{\beta^k} \cdot \frac{1}{A} + \frac{1/\beta^{k+1}}{1 - (1/\beta)} \cdot \frac{1}{\alpha} \\ &= \frac{1}{\beta - 1}(1 - \frac{1}{\beta^{k-1}} + \frac{1}{\alpha \cdot \beta^k} - \frac{\beta}{\alpha \cdot \beta^{2k}}) + \frac{1}{\beta^k \cdot A} + \frac{1}{\alpha(\beta - 1) \cdot \beta^k}, \end{aligned}$$

that is,

$$F(z) \le 1 + \frac{1}{\beta^k}(\frac{2}{\alpha} + \frac{\beta-1}{A} - \beta - \frac{\beta}{\alpha\cdot\beta^k}).$$

Since $\alpha > 2$, by (2) we may choose $A > 0$ so that $A > \frac{\beta-1}{\beta-(2/\alpha)}$, i.e. $\beta - (2/\alpha) > (\beta-1)/A$, and we are done. □

Remark 4.2.6. Before going to the next step, we consider the following special case in Step 8. Let $\alpha = \beta := 4, \epsilon_0 := 1/2$, and choose θ_0 so that $2^{12\theta_0} < 7/6$. Clearly, $\theta_0 < 1/4$ and so $0 < \sigma = \theta_0/(1-\epsilon_0) < 1/2$. This implies that $\tau_0 < 1/2$. In particular,

$$\log A = -2\theta_0(\log 2)\Big(\sum_{j=1}^{\infty} \epsilon_0^{j-1} j + \frac{1}{1-\epsilon_0}\Big) = -12\theta_0 \log 2 > \log\frac{6}{7} = \log\frac{\beta-1}{\beta-(2/\alpha)}.$$

Here, in the second equality, we used the fact that $\sum_{j=1}^{\infty}\frac{j}{2^j} = 2$. Hence, $6/7 < A < 1$, $C < 2/(1/2) = 4$, and also, $F(z) \le 4\|z\|^{1/2}$ for $\|z\| \ge 1$.

As the final consequence, we obtain the following result:

Step 9. *Let α, β, ϵ_0, and θ_0 be as in Remark 4.2.6. Set $G_F := \{z \in \mathbb{C}^2 : F(z) < 1\}$. Then $\lim_{\|z\|\to\infty} F(z)/\|z\| = 0$ and the open set $\tilde{G} := G_F \cap \mathbb{B}_2(0,2)$ has a connected component G' that is not $k_{G'}$-finitely compact.*

Proof. Since $F(z) \le 4\|z\|^{1/2}$ for $\|z\| \ge 1$, one has

$$0 \le \lim_{\|z\|\to\infty}\frac{F(z)}{\|z\|} \le \lim_{\|z\|\to\infty}\frac{4}{\|z\|^{1/2}} = 0.$$

Note that $X \subset \tilde{G}$ and $0 \in (\partial X)\cap(\partial\tilde{G})$. Let G' be a connected component of $\tilde{G}$ with $X \subset G'$, and let $(a_k)_{k\in\mathbb{N}}$ be the sequence as in Step 1. Then, by Step 2 it holds that $k_{G'}(a_1, a_k) \le \Theta < \infty$ for all $k \in \mathbb{N}$, where Θ is as in Step 2. Since $G' \ni a_k \to 0 \in \partial G'$ as $k \to \infty$, we then obtain that G' is not $k_{G'}$-finitely compact. □

Thus the proof of Lemma 4.2.3 is completed. □

For the proof of Theorem 4.2.1, we will use the same notations as in the proof of Lemma 4.2.3.

Proof of Theorem 4.2.1. Define a function $h_0 : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{R}_{\ge 0}$ by

$$h_0(z, z_3) := \begin{cases} |z_3|F(z/z_3) & (z_3 \ne 0) \\ 0 & (z_3 = 0) \end{cases} \qquad (z, z_3) \in \mathbb{C}^2 \times \mathbb{C}.$$

Clearly, h_0 is absolutely homogeneous on $\mathbb{C}^3$ and continuous log-psh in $\mathbb{C}^2 \times \mathbb{C}_*$. On the other hand, observe that

$$\lim_{\mathbb{C}_*\ni z_3\to 0} h_0(z, z_3) = \lim_{\|w\|\to\infty}\frac{\|z\|}{\|w\|}F(w) = 0 = h_0(z, 0), \quad z \in \mathbb{C}^2$$

so $h_0 \in \mathcal{C}(\mathbb{C}^3)$. Since F is locally bounded from above in $\mathbb{C}^2$, the removable singularity theorem implies that $h_0 \in PSH(\mathbb{C}^3)$. Now, let $h : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ be defined as

$$h(z, z_3) := \max\left\{h_0(z, z_3),\ \frac{1}{\sqrt{5}}\|(z, z_3)\|\right\}, \quad (z, z_3) \in \mathbb{C}^2 \times \mathbb{C}.$$

Then h is absolutely homogeneous, continuous in $\mathbb{C}^3$ with $h^{-1}(0) = \{0\}$. Moreover, for any $M \geq \sqrt{5}$, one has

$$Mh(z, z_3) \geq \frac{M}{\sqrt{5}}\|(z, z_3)\| \geq \|(z, z_3)\|, \quad (z, z_3) \in \mathbb{C}^2 \times \mathbb{C}.$$

Therefore, $G := \{(z, z_3) \in \mathbb{C}^3 : h(z, z_3) < 1\}$ is a bounded pseudoconvex balanced domain. Since $G' \times \{1\}$ is a connected component of $G \cap (\mathbb{C}^n \times \{1\}) = \tilde{G} \times \{1\}$, and since $G \ni (a_j, 1) \longrightarrow (0, 1) \in \partial G$ as $j \to \infty$, the domain G is not k_G-finitely compact. By the decreasing property of the Kobayashi distance, we have

$$k_G((a_1, 1), (a_j, 1)) \leq k_{G'}(a_1, a_j) \leq \Theta < \infty \quad \text{for} \quad j \in \mathbb{N}$$

and the desired assertion is proved. □

As a consequence of Theorem 4.2.1, we get the following proof.

Proof of Corollary 4.2.2. Let h be as in the proof of Theorem 4.2.1. Fix $n \geq 3$ and put $\Omega := E^{n-3} \times G$. If we define a mapping $\mathfrak{h} : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ by $\mathfrak{h}(w, w') := \max\{|w_1|, \cdots, |w_{n-3}|, h(w')\}$ for $(w, w') \in \mathbb{C}^{n-3} \times \mathbb{C}^3$, $w = (w_1, \cdots, w_{n-3})$, then $\mathfrak{h}$ is homogeneous continuous psh in $\mathbb{C}^n$. In particular, $\Omega = \{(w, w') \in \mathbb{C}^{n-3} \times \mathbb{C}^3 : \mathfrak{h}(w, w') < 1\}$ and also,

$$\Omega \ni (\mathfrak{o}, (a_j, 1)) := (\underbrace{0, \cdots, 0}_{(n-3)\text{-times}}, (a_j, 1)) \overset{j\to\infty}{\longrightarrow} (\underbrace{0, \cdots, 0}_{(n-3)\text{-times}}, (0, 1)) \in \partial\Omega.$$

But,

$$\begin{aligned} k_\Omega((\mathfrak{o}, (a_1, 1)), (\mathfrak{o}, (a_j, 1))) &= \max\{k_{E^{n-3}}(\mathfrak{o}, \mathfrak{o}),\ k_G((a_1, 1), (a_j, 1))\} \\ &= k_G((a_1, 1), (a_j, 1)) \leq \Theta < \infty \qquad j \geq 1. \end{aligned}$$

Thus the proof is completed. □

Finally, the following proof follows from Lemma 4.2.3.

Proof of Corollary 4.2.4. Let X, F, and $(a_j)_{j\geq 1}$ be as in Lemma 4.2.3. Recall that $\log F \in (\mathcal{C} \cap PSH)(\mathbb{C}^2, \mathbb{R})$. Let $G \subset \mathbb{C}^2$ be a pseudoconvex domain containing an open neighborhood of the origin in $\mathbb{C}^2$. Then it follows from Lemma 4.2.3 that the pseudoconvex Hartogs domain $\Omega = \Omega_{\log F, \|\cdot\|}(G)$ is not k-complete because

$$\Omega \ni (a_j, (1, 0, \cdots, 0)) \to (0, (1, 0, \cdots, 0)) \in \partial\Omega \quad \text{as} \quad j \to \infty.$$

In particular, if G is a ball (or a polydisk), then G is k-complete. □

§4.3. A sufficiency for balanced domains in $\mathbb{C}^2$ to be k-complete.

As mentioned in the last part of §3.5, the aim of this section is to give a sufficient condition for balanced domains $D = D_h \subset \mathbb{C}^2$ to be k-complete. For this, we need the following two lemmas:

Lemma 4.3.1. *Let $D = D_h \Subset \mathbb{C}^2$ be a balanced domain. If h is continuous on $\mathbb{C}^2 \setminus \{(0,0)\}$, then the function $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(z) := h(1,0)z_1$, $z \in \mathbb{C}^2$, is a local holomorphic peak function at every point of $(\partial D) \cap (\mathbb{C} \times \{0\})$*

Recall that the boundedness of $D = D_h$ implies that h is positive definite on $\mathbb{C}^2$.

Proof. By the continuity of h, one has

$$h(1,0) = \lim_{|\zeta|\to\infty} h(1, \frac{1}{\zeta}) = \lim_{|\zeta|\to\infty} \frac{h(\zeta,1)}{|\zeta|} \in [0,\infty); \tag{4.3.1a}$$

in particular, $h(1,0) > 0$ because of $D \Subset \mathbb{C}^2$. It is easy to check that $(\frac{1}{h(1,0)}, 0) \in \partial D \cap (\mathbb{C}_* \times \{0\})$. Let $(z_1, z_2) \in D \cap (\mathbb{C}_* \times \mathbb{C})$. Observe that $h(1, z_2/z_1) \neq 0$, and

$$0 < h(z) = |z_1| h(1, \frac{z_2}{z_1}) < 1 = \frac{1}{h(1,0)} h(1,0).$$

By (4.3.1a), it follows that

$$|z_1| h(1,0) < \frac{h(1,0)}{h(1, z_2/z_1)} \longrightarrow 1 \quad \text{as} \quad z_2 \to 0.$$

In particular, the left side of the previous inequality does not depend on the choice of z_2, so we are done. $\square$

Lemma 4.3.2. *Let $D = D_h \Subset \mathbb{C}^2$ be a pseudoconvex balanced domain and put $\Omega := \Omega_{\log h(\cdot,1),|\cdot|}(\pi_1(D'))$, where $D' := D \cap (\mathbb{C} \times \mathbb{C}_*)$ and $\pi_1(z) := z_1$. Suppose that h is continuous on $\mathbb{C}^2 \setminus \{(0,0)\}$. Moreover, assume for any $p := (a,b) \in \partial D$ with $b \neq 0$ that:*

$$\exists_{N\geq 4,\, U=U(a/b)\subset\pi_1(D')} \quad : \quad \begin{cases} \log h(\cdot,1) \text{ is of class } \mathcal{C}^N \text{ in } U; \\ \exists_{1\leq\alpha\leq\beta-1\leq N-1} \quad : \quad \dfrac{\partial^\beta \log h(\cdot,1)}{\partial z^{\beta-\alpha}\partial\bar{z}^\alpha} \neq 0 \text{ on } U. \end{cases}$$

Then there are open neighborhoods $V = V(a/b, b) \subset \mathbb{C}^2$ and $W = W(p) \subset \mathbb{C}^2$ of $(a/b, b)$ and p, respectively, such that $(a/b, b)$ and p are local peak points for $\mathcal{O}(\Omega \cap V)$ and $\mathcal{O}(D \cap W)$, respectively.

Recall that every bounded domain in $\mathbb{C}$ is k-complete and that $D = D_h \subset \mathbb{C}^m$ is pseudoconvex iff $\log h \in PSH(\mathbb{C}^m)$.

Proof. Fix a point $p = (a,b) \in \partial D$ with $b \neq 0$. Under our hypotheses, the neighborhood $V = V(a/b, b)$ having the desired property is obtained directly from Proposition 3.5.7. Take a function $f_0 \in \mathcal{O}(\Omega \cap V)$ such that

$$|f_0(a/b, b)| = 1 > |f_0(z)|, \quad z \in \Omega \cap V.$$

Define a mapping $\varphi : D' \to \Omega$ by $\varphi(z) := (z_1/z_2, z_2)$ for $z \in D'$. Then it is well-defined and $\varphi\big|_{\partial(D')}$ is one-to-one. In particular, $\varphi(\partial(D')) = \partial\Omega$. Hence we can take an open neighborhood $W = W(p) \subset \mathbb{C}^2$ of p such that $\varphi(D' \cap W) \subset \Omega_{\log h(\cdot,1),|\cdot|}(V)$ and also

$$|(f_0 \circ \varphi)(p)| = 1 > |(f_0 \circ \varphi)(z)|, \quad z \in D' \cap W.$$

Because of $b \neq 0$, we can choose the open neighborhood $W \equiv W(p)$ so small that $D' \cap W = D \cap W$ and thus we are done. □

As an immediate consequence of Lemma 4.3.1, Lemma 4.3.2, and (e) of Theorem 1.5.18, we obtain the following:

Proposition 4.3.3. *Every balanced pseudoconvex domain $D = D_h \Subset \mathbb{C}^2$ satisfying all assumptions of Lemma 4.3.2 is locally c-finitely compact and so (globally) k-complete.*

APPENDIX

The following material was used in this thesis. Most of it is well-known. For the convenience of the reader, we collect them in this section. Details can be found in e.g. [Vla 66], [Rud 74], [Kob 98], [Jak-Jar 01].

A *partial order* on a set S is a binary relation "$\sim$" that is reflexive, antisymmetric and transitive. A partial order '$\sim$' on a set S is called a *total order* on S if '$a \sim b$' or '$b \sim a$' for any $a, b \in S$. A set $(S, \sim)$ with a partial (resp. total) order "$\sim$" on S is called a *partially* (resp. *totally*) *ordered set.*

1. [Zorn Lemma] *Let S be a nonempty partial ordered set. If every total ordered set $A \subset S$ has an upper bound, then the set S has a maximal element.*

2. [Fatou Theorem] *If $f \in H^\infty(E)$, then for almost all $\theta \in \partial E$ the function f has a nontangential limit at θ, i.e. the limit $\lim_{\Gamma_\alpha(\theta)\ni\lambda\to\theta} f(\lambda)$ exists and is independent of $\alpha > 1$, where $\Gamma_\alpha(\theta) := \{\lambda \in E : |\lambda - \theta| < \alpha(1 - |\lambda|)\}$.*

3. [Identity Theorem] *Let $U \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(U)$. If the set $U \cap f^{-1}(0)$ has an accumulation point in U, then $f \equiv 0$ on U.*

4. [Liouville Theorem] *If $f \in \mathcal{O}(\mathbb{C})$ is bounded, then $f \equiv$* constant.

5. [Little Picard Theorem] *If f is a nonconstant entire function, then the image of f contains all complex numbers except at most one.*

6. [Big Picard Theorem] *If $f \in \mathcal{O}(E_*, \mathbb{C})$ has an essential singularity at 0, then $f(E_*)$ contains the whole $\mathbb{C}$ except at most one point.*

7. [Cauchy Inequality] *If $f \in \mathcal{O}(P(a,r)) \cap \mathcal{C}(\overline{P(a,r)})$ for some $a \in \mathbb{C}^n$, $r = (r_1, \cdots, r_n) \in (\mathbb{R}_{>0})^n$, then*

$$\left| \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} f(a) \right| \le \frac{\alpha_1! \cdots \alpha_n!}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}} \sup_{z \in \partial_0 P(a,r)} |f(z)|$$

for any $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{Z}_{>0})^n$.

8. [Open Mapping Theorem] *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $f \in \mathcal{O}(\Omega)$ with $f \not\equiv$* constant, *then f is an open mapping.*

9. [Hurwitz Theorem] *Let $(f_j)_{j\ge1}$ be a sequence of nowhere-vanishing holomorphic functions on a domain Ω in $\mathbb{C}$. If $(f_j)_{j\ge1}$ converges uniformly on every compact subset of Ω to a function $f \in \mathcal{O}(\Omega)$, then either $f \equiv 0$ or f never vanishes.*

We say that a domain $G \Subset \mathbb{C}^n$ has the *Lipschitz boundary* if for any $z_0 \in \partial G$ there is an open neighborhood $U = U(z_0) \subset \mathbb{C}^n$ of z_0 and a function $\varrho : U \to \mathbb{R}$ such that $|\varrho(z) - \varrho(w)| \le \|z - w\|$ for $z, w \in U$, $U \cap G = \{z \in U : \varrho(z) < 0\}$, and $U \cap \partial G = \{z \in U : \varrho(z) = 0\}$.

A *holomorphic covering* is a holomorphic map $\pi : M \to N$, M, N connected complex manifolds, the following property satisfying: any point $y \in N$ has a neighborhood $W = W(y)$ such that $\pi^{-1}(W)$ is the union of pairwise disjoint open subsets V_j of M, such that $\pi\big|_{V_i} : V_i \to W$ is biholomorphic, $i \in I$.

10. [Uniformization Theorem] *For any domain $G \subset \mathbb{C}$ there is a holomorphic covering $\pi : M \to G$, where $M := \begin{cases} E & \text{if } \#(\mathbb{C} \setminus G) \geq 2 \\ \mathbb{C} & \text{otherwise} \end{cases}$, such that for any other holomorphic covering $\tilde{\pi} : \tilde{M} \to G$ and for any points $z' \in M, \tilde{z}' \in \tilde{M}$ with $\pi(z') = \tilde{\pi}(\tilde{z}')$, there is an unique holomorphic mapping $\varphi : M \to \tilde{M}$ satisfying $\tilde{\pi} \circ \varphi = \pi$ and $\varphi(z') = \tilde{z}'$.*

Let X, Y be two topological spaces. By $\mathcal{C}(X, Y)$ we denote the set of all continuous maps from X to Y with the compact-open topology. If Y is a metric space, then the compact-open topology coincides with the topology of uniform convergence on compact sets.

11. [Arzelà-Ascoli Theorem] *Let X be a locally compact, separable topological space and Y a locally compact metric space with distance function d_Y. Then a family $\mathcal{F} \subset \mathcal{C}(X, Y)$ is normally convergent in $\mathcal{C}(X, Y)$ if and only if*
- *$\mathcal{F}$ is equicontinuous at every point $x \in X$;*
- *for every $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is relatively compact in Y.*

Let X and Y be locally compact, separable spaces with pseudodistances d_X and d_Y, respectively. We put

$$\mathcal{D}(X, Y) := \{f \in \mathcal{C}(X, Y) : d_Y(f(x'), f(x'')) \leq d_X(x', x'') \text{ for any } x', x'' \in X\}.$$

Then $\mathcal{D}(X, Y)$ is closed in $\mathcal{C}(X, Y)$. If Y is a metric space with distance function d_Y, then $\mathcal{D}(X, Y)$ is equicontinuous, so is every subfamily of it.

12. [Montel Theorem] *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\mathcal{F} \subset \mathcal{O}(\Omega, \mathbb{C})$. If $\mathcal{F}$ is locally uniformly bounded in Ω, then $\mathcal{F}$ is normally convergent in $\mathcal{O}(\Omega)$.*

We denote by $H(G)$ the set of all harmonic functions on a domain G in $\mathbb{C}$.

13. [Maximum Principle for Harmonic Functions] *Let $G \subset \mathbb{C}$ be a domain and let $h \in H(G)$ with $h \not\equiv$ const. Then h has no local maximum in G. If, moreover, G is bounded, then*

$$h(\lambda) < \sup_{\zeta \in \partial G} \big(\limsup_{G \ni \eta \to \zeta} u(\eta) \big), \quad \lambda \in G.$$

Let $G \subset \mathbb{C}$ be a bounded domain and let $\varphi \in \mathcal{C}(\partial\Omega, \mathbb{R})$. The *Dirichlet problem* asks to find a function $h \in \mathcal{C}(\bar{G}) \cap H(G)$ such that $h = \varphi$ on ∂G. By the previous maximum principle, it follows that if such a solution h exists, then it is unique. h is called the *solution of the Dirichlet problem for G with boundary data φ.* If the Dirichlet problem for G has a solution for any boundary data, then we say that G is *regular with respect to the Dirichlet problem.*

It is well-known that if each connected component of the boundary of G contains more than one point (e.g. G is a disk or an annulus in $\mathbb{C}$), then G is regular with respect to the Dirichlet problem. Note that the Dirichlet problem on the punctured disk E_* has no solution for the boundary data $\varphi : \partial(E_*) \to \mathbb{R}$ defined by

$$\varphi(\lambda) := \begin{cases} 0 & (|\lambda| = 1), \\ 1 & (|\lambda| = 0). \end{cases}$$

Let $G \subset \mathbb{C}$ be a domain and fix $a \in G$. The classical Green function of G with pole at a is a function $\mathfrak{g}_G(a, \cdot) : G \setminus \{a\} \to \mathbb{R}$ satisfying the following properties:

- $\mathfrak{g}_G(a,\cdot) \in H(G \setminus \{a\})$;
- $\lim_{G\setminus\{a\}\ni\lambda\to a} [\mathfrak{g}_G(a,\lambda) + \log|\lambda - a|]$ exists and is finite;
- there is a polar set $M \subset \partial G$ such that:
  - if $\lambda \in (\partial G) \setminus M$, then $\lim_{G\ni\zeta\to\lambda} \mathfrak{g}_G(a,\zeta) = 0$,
  - if $\lambda \in M$ or $\lambda = \infty$, then $\mathfrak{g}_G(a,\cdot)$ is bounded near λ.

14. *Let $G \subset \mathbb{C}$ be a domain. If ∂G is polar, then $\mathfrak{g}_G(a,\cdot) \equiv 0$. If ∂G is not polar, then for every $a \in G$ the function $\mathfrak{g}_G(a,\cdot)$ exists and is unique. Moreover $g_G(a,\cdot) =$* $\exp(-\mathfrak{g}_G(a,\cdot))$*, where g_G denotes the 'pluricomplex Green function' of G* (cf. §1.2).

15. [Weak Identity Principle] *Let $G \subset \mathbb{C}$ be open and let $u, v \in SH(G)$. If $u = v$ almost everywhere on G, then $u \equiv v$ on G.*

16. [Integrability Theorem] *Let $G \subset \mathbb{C}$ be a domain. If $u \in SH(G)$ with $u \not\equiv -\infty$ on G, then u is locally integrable on G, i.e. $\int_K |u| dA$ for each compact set $K \subset G$, where dA denotes the two-dimensional Lebesgue measure.*

17. [Liouville Type Theorem] *If $u \in SH(\mathbb{C})$ is bounded from above, then $u \equiv$* constant

18. [Oka Theorem] *Let $G \subset \mathbb{C}$ be open. Then for a $u \in SH(G)$ and a curve $\gamma :$* $[0,1] \to G$ *the following is true:*

$$u(\gamma(0)) = \limsup_{t\to 0^+} u(\gamma(t)).$$

19. [Upper Semicontinuous Regularization] Let G be a domain in $\mathbb{C}^n$. Let $u : G \to \mathbb{R}$ be locally bounded from above. We define the *upper semicontinuous regularization* u^* of u by

$$u^*(z) := \limsup_{w \not\to z} u(w) = \inf\{\varphi(z) : \varphi \in \mathcal{C}(G, \mathbb{R}),\, u \le \varphi\}, \quad z \in G.$$

Obviously, $u^* \ge u$ on G. Moreover, the following properties are true:
- *if $v \in \mathcal{C}^{\uparrow}(G)$ with $u \le v$ on G, then $u^* \le v$ on G;*
- *if a sequence $(u_j)_{j\ge 1} \in PSH(G)$ is locally bounded from above, the upper semicontinuous regularization φ^* of $\varphi := \limsup_{j\to\infty} u_j$ is plurisubharmonic on G, and $\varphi = \varphi^*$ almost everywhere in G.*

20. [Gluing Lemma for Plurisubharmonic Functions] *Let $G \subset \Omega$ be open subsets of $\mathbb{C}^n$ and let $u \in PSH(\Omega)$, $v \in PSH(G)$. Assume that*

$$\limsup_{G\ni z\to z_0} v(z) \le u(z_0), \quad z_0 \in (\partial G) \cap \Omega.$$

Let

$$\tilde{u}(z) := \begin{cases} \max\{v(z), u(z)\} & (z \in G), \\ u(z) & (z \in \Omega \setminus G). \end{cases}$$

Then $\tilde{u} \in PSH(\Omega)$.

21. [Maximum Principle for Plurisubharmonic Functions] *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $u \in PSH(\Omega)$ with $u \not\equiv$ const. Then u does not attain its global maximum in Ω. If, moreover, Ω is bounded, then*

$$u(z) < \sup_{p \in \partial\Omega} \big(\limsup_{\Omega \ni w \to p} u(w) \big), \quad z \in \Omega.$$

A set $M \subset \mathbb{C}^n$ is called *pluripolar* if for every $a \in M$ there are a connected open neighborhood $U_a \subset \Omega$ and a function $v_a \in PSH(U_a)$ with $v_a \not\equiv -\infty$ such that $M \cap U_a \subset v_a^{-1}(-\infty)$.

22. [Removable Singularities of Plurisubharmonic Functions] *Let $\Omega \subset \mathbb{C}^n$ be open and let $M \subset \Omega$ be a closed pluripolar subset of Ω. Let $u \in PSH(\Omega \setminus M)$ be locally bounded from above in Ω. Then the function*

$$\tilde{u}(z) := \begin{cases} \limsup\limits_{\Omega \setminus M \ni w \to z} u(w) & (z \in M), \\ u(z) & (z \in \Omega \setminus M) \end{cases}$$

is psh on Ω. If Ω is connected, so is $\Omega \setminus M$. In particular, if $M \subset \Omega$ is a closed pluripolar set, then for any $u \in PSH(G)$

$$u(z) = \limsup_{\Omega \setminus M \ni w \to z} u(w), \quad z \in \Omega.$$

23. [Hartogs Lemma] *Let $\Omega \subset \mathbb{C}^n$ be open. If $(u_j)_{j \geq 1}$ is locally bounded from above and*

$$\limsup_{j \to \infty} u_j \leq M \quad \text{for some} \quad M \in \mathbb{R},$$

then for any compact $K \subset \Omega$ and any $\epsilon > 0$ there exists $j_0 = j_0(K, \epsilon) \in \mathbb{N}$ such that

$$\max_{z \in K} u_j(z) \leq M + \epsilon, \quad j \geq j_0.$$

A domain $G \subset \mathbb{C}^n$ is said to be *pseudoconvex* if the function $-\log \operatorname{dist}(\cdot, \partial G)$ is psh on G, where $\operatorname{dist}(z, \partial G) := \inf_{w \notin G} \|z - w\|$ for $z \in G$.

24. [Kontinuitätssatz] *A domain $G \subset \mathbb{C}^n$ is pseudoconvex iff:*

$$\forall_{(\varphi_\alpha)_{\alpha \in I} \subset \mathcal{C}(\bar{E}, G) \cap \mathcal{O}(E, G)} \ : \ \bigcup_{\alpha \in I} \varphi_\alpha(\partial E) \Subset G \quad \Longrightarrow \quad \bigcup_{\alpha \in I} \varphi_\alpha(\bar{E}) \Subset G.$$

A function $u \in \mathcal{C}(\Omega, \mathbb{R})$ is said to be *strictly psh* on Ω if for any open set $\tilde{\Omega} \Subset \Omega$ there exists an $\epsilon = \epsilon(\tilde{\Omega}) > 0$ such that the function $\tilde{\Omega} \ni z \longmapsto u(z) - \epsilon \|z\|^2$ is psh. It is easy to see that a function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ is strictly psh iff

$$\sum_{\nu, \mu = 1}^{n} \frac{\partial^2 u}{\partial z_\nu \partial \bar{z}_\mu}(a) X_\nu \bar{X}_\mu > 0, \quad a \in G, \, X \in \mathbb{C}^n \setminus \{0\}.$$

25. [Richberg Theorem] *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $u \in PSH(\Omega, \mathbb{R})$, $\tau \in \mathcal{C}(\Omega, \mathbb{R}_{>0})$. Then the following results are true:*

- *if u is strictly psh, then there is a strictly psh function $v \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ such that $u \leq v \leq u + \tau$ on Ω;*
- *if u is continuous, then there is a strictly psh function $v \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ such that $u \leq v \leq u + \tau$ on Ω.*

26. [Fornaess-Narasimhan Theorem] *If Ω is a pseudoconvex domain, then for any $u \in PSH(\Omega)$ there is a sequence $(u_j)_{j \geq 1} \subset (\mathcal{C}^\infty \cap PSH)(\Omega)$ such that $u_j \searrow u$ pointwise on Ω.*

LIST OF SYMBOLS

$A_{>0} := \{x \in A : x > 0\}$, where $A \subset \mathbb{R}$;
$A_{<0} := \{x \in A : x < 0\}$, where $A \subset \mathbb{R}$;
$A_{\geq 0} := \{x \in A : x \geq 0\}$, where $A \subset \mathbb{R}$;
$A_{\leq 0} := \{x \in A : x \leq 0\}$, where $A \subset \mathbb{R}$;
$U_* := U \setminus \{0\}$, where $0 \in U \subset \mathbb{C}^n$;
$S^n := S \times \cdots \times S$ (n-times);
$S \Subset T$ means that S is relatively compact in T;
$\partial S :=$ the boundary of S in the topology of X, $S \subset X$;
$\forall_{x \in S}$ means that 'for every $x \in S$';
$^{\exists}x \in S$ means that there is an $x \in S$;
$\mathbb{Z} :=$ the set of all integers;
$\mathbb{R} :=$ the field of real numbers;
$\mathbb{C} :=$ the field of complex numbers;
$\mathrm{Re}z :=$ the real part of $z \in \mathbb{C}$;
$\mathrm{Im}z :=$ the imaginary part of $z \in \mathbb{C}$;
$|\cdot| \equiv \|\cdot\|_{\mathbb{C}} :=$ the Euclidean norm in $\mathbb{C}$;
$\|\cdot\| \equiv \|\cdot\|_{\mathbb{C}^n} :=$ the Euclidean norm in $\mathbb{C}^n$;
$\mathbb{B}_{d_G}(z, R) := \{w \in G : d_G(z, w) < R\}$, $z \in G$, $r > 0$, d is a function on $G \times G$;
$\mathbb{B}_1(\lambda, r) := \mathbb{B}_{|\cdot|}(\lambda, r)$, $\lambda \in \mathbb{C}$, $r > 0$;
$\mathbb{B}_n(z, r) := \mathbb{B}_{\|\cdot\|}(z, r)$, $z \in \mathbb{C}^n$, $r > 0$;
$E := \mathbb{B}_1(0, 1) =$ the unit disk in the complex plane;
$U_R(\infty) = U_R^n(\infty) := \mathbb{C}^n \setminus \overline{\mathbb{B}_n(0, R)}$, $R > 0$;
$P(a, r) := \mathbb{B}_1(a_1, r_1) \times \cdots \times \mathbb{B}_1(a_n, r_n)$, $a \in \mathbb{C}^n$, $r \in (\mathbb{R}_{>0})^n$;
$\partial_0 P(a, r) := \partial\mathbb{B}_1(a_1, r_1) \times \cdots \times \partial\mathbb{B}_1(a_n, r_n)$, $a \in \mathbb{C}^n$, $r \in (\mathbb{R}_{>0})^n$;
$\mathcal{C}^{\uparrow}(G) :=$ the set of all upper semicontinuous functions $f : G \to [-\infty, +\infty)$;
$\mathcal{C}(G, G') :=$ the set of all continuous functions from G to G';
$\mathcal{C}(G) := \mathcal{C}(G, \mathbb{C})$;
$\mathcal{C}^{\mu}(G) :=$ the set of all $\mathcal{C}^{\mu}$-functions $f : G \to \mathbb{C}$, $\mu \in \mathbb{N} \cup \{\infty\}$;
$\mathcal{O}(G, G') :=$ the set of all holomorphic maps from G to G';
$\mathcal{O}(G) := (G, \mathbb{C})$;
$H^{\infty}(G) :=$ the set of all bounded holomorphic functions on G;
$H^{\infty}(G) \cong \mathbb{C}$ means that all bounded holomorphic function G are constant;
$H(G) :=$ the set of all harmonic functions on G;
$SH(G) :=$ the set of all subharmonic functions on G;
psh := plurisubharmonic;
$PSH(G) :=$ the set of all plurisubharmonic functions on G;
$\mathcal{G} :=$ the set of all domains in all $\mathbb{C}^n$'s;
p - the Poincaré (hyperbolic) distance;
c_G - the Carathéodory pseudodistance of G;
$\tilde{k}_G$ - the Lempert function of G;

$k_G^{(\mu)}(z,w) := \inf\{\sum_{j=1}^{\mu} \tilde{k}_G(p_{j-1},p_j) : z = p_0,\ w = p_\mu,\ (p_j)_{j=0}^{\mu} \subset G\}$, $\mu \in \mathbb{N}$;
k_G - the Kobayashi pseudodistance of G;
κ_G - the Kobayashi pseudometric of G;
S_G - the Sibony pseudometric of G;
$\log g_G$ - the pluricomplex Green function of G;
$d := (d_G)_{G\in\mathcal{G}}$ - a family of invariant functions;
$d_G^* := \tanh d_G$;
topd_G:= the topology generated by the subbasis consisting of all d_G-balls, $G \in \mathcal{G}$;
top$G$:=the Euclidean topology of G, $G \in \mathcal{G}$;
h_D - the associated Minkowski function of a balanced domain $D \subset \mathbb{C}^m$;
$D_h := \{z \in \mathbb{C}^m : h(z) < 1\}$ - the balanced domain with Minkowski function h;
$\Omega_z := \{w \in \mathbb{C}^m : (z,w) \in \Omega\}$, $\Omega \subset \mathbb{C}^{n+m}$, $z \in \pi(\Omega)$,
where $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ is the projection of Ω onto $\mathbb{C}^m$,
$\Omega_H(G) := \{(z,w) \in G \times \mathbb{C}^m : H(z,w) < 1\}$, H is a function on $G \times \mathbb{C}^m$;
$\Omega_{u,h}(G) := \{(z,w) \in G \times \mathbb{C}^m : h(w)e^{u(z)} < 1\}$;
$\Sigma_{u,v}(G) := \{(z,\lambda) \in G \times \mathbb{C} : e^{v(z)} < |\lambda| < e^{-u(z)}\}$, $u, v \in \mathbb{C}^{\uparrow}(G), u + v < 0$ on G;
$\overset{\mathrm{K}}{\Longrightarrow}$ - locally uniform convergence;
$\operatorname{grad} u(z)$:= the gradient of u at z.

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