Topology and Spectrum in Quantum Layers

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von Frau Diala Yacoub
geboren am 01. September 1980 in Latakia, Syrien.

Gutachter:	Prof. Dr. Daniel Grieser
Zweitgutachter:	Prof.Dr. Hannes Uecker
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Abstract

Given a complete non-compact surface Σ embedded in $\mathbb{R}^3$, we consider the Dirichlet Laplacian, $-\Delta_D^\Omega$, on the quantum layer Ω, that is defined as a tubular neighborhood of constant width about Σ. Recently, sufficient geometrical and topological conditions have been found which guarantee the existence of discrete spectrum, that is isolated points of the spectrum which are eigenvalues with finite multiplicity.

The purpose of this thesis is to identify relationships between the topology of Σ and the spectrum of the Dirichlet Laplacian on the quantum layer Ω. More explicitly, we find a lower bound on the number of eigenvalues of $-\Delta_D^\Omega$ in terms of the genus of Σ.

We consider two classes of surfaces Σ: Firstly, Σ is a Euclidean plane with handles, whose distance to each other is greater than or equal to a constant. Secondly, Σ is a Euclidean plane outside a compact set with nontrivial topology.

The first result depends on the theorem of Carron, Exner and Krejcirik on quantum layers around surfaces whose curvatures vanish at infinity. The second result is based on the papers of Wachsmuth, Teufel and Lampart on approximating the eigenvalues of the Dirichlet Laplacian on a thin quantum layer and on a paper of Grigor'yan and Netrusov on estimating the number of the negative eigenvalues of Schrödinger operator.

Zusammenfassung

Für eine vollständige nicht kompakte Fläche Σ, die im $\mathbb{R}^3$ eingebettet ist, betrachten wir den Dirichlet Laplace Operator, $-\Delta_D^\Omega$, auf einer Tubenumgebung Ω von Σ mit konstanter Breite. In letzter Zeit sind hinreichende geometrische und topologische Bedingungen gefunden worden, die die Existenz von diskretem Spektrum garantieren, das heißt von isolierten Punkten des Spektrums, die Eigenwerten mit endlicher Vielfachheit sind.

Das Ziel dieser Arbeit ist es, Beziehungen zwischen der Topologie von Σ und dem Spektrum von $-\Delta_D^\Omega$ zu finden. Genauer finden wir eine untere Schranke für die Anzahl der Eigenwerte von $-\Delta_D^\Omega$ in Termen des Geschlechts von Σ.

Wir betrachten zwei Klassen von Flächen Σ: Zum einen ist Σ eine Euklidische Ebene mit Henkeln, deren Abstand zueinander größer oder gleich einer Konstanten ist. Zweitens ist Σ eine Euklidische Ebene außerhalb einer kompakten Menge mit nicht trivialer Topologie.

Das erste Ergebnis shützt sich auf einen Satz von Carron, Exner und Krejcirik für Tubenumgebungen von Flächen, deren Krümmungen im Unendlichen verschwinden. Das zweite Ergebnis beruht auf Veröffentlichungen von Wachsmuth, Teufel und Lampart über die Approximation der Eigenwerte des Dirichlet Laplace Operators auf einer dünnen Tubenumgebung einer Fläche und auf einer Veröffentlichung von Grigor'yan und Netrusov über die Abschätzung der Anzahl der negativen Eigenwerte des Schrödinger-Operators.

Contents

Introduction

The spectrum of the Laplacian on a manifold is a classic research domain within geometric analysis. One of its interesting areas is the spectrum of the Dirichlet Laplacian on non-compact Riemannian manifolds which are much less understood than their compact counterparts. In particular, it is often not even known whether such a manifold has any discrete spectrum. It has shown recently that a certain type of a non-compact manifold, called the quantum layer, has a non-empty discrete spectrum by assuming certain geometrical and topological conditions.

Furthermore, the existence of the discrete spectrum of the Laplacian on a manifold is an interesting phenomenon in both mathematics and physics. The bound states as a physics terminology are the normalized eigenfunctions that correspond to each point in the discrete spectrum and the ground state is the normalized eigenfunction that corresponds to the lowest eigenvalue in the discrete spectrum.

Review of the Literature and Research Goal

The general mathematical problem considered in this thesis and its related work is the existence of the discrete spectrum, σ_{disc}, below the essential spectrum, σ_{ess}, of the Dirichlet Laplacian on quantum layers.

A quantum layer Ω is defined in Chapter 1 as a tubular neighbourhood of radius a about an orientable complete non-compact surface Σ embedded in $\mathbb{R}^3$. If the surface is a locally deformed plane, the existence of the discrete spectrum below the essential spectrum of the Dirichlet Laplacian on the quantum layer Ω, $-\Delta_D^\Omega$, was demonstrated in [DEK00]. A more general situation was treated in [DEK01] by assuming that Σ has asymptotically vanishing curvatures and possesses a pole (i.e., the exponential map is a diffeomorphism) and several conditions are fulfilled. The extension of these conditions for the existence of the discrete spectrum without assuming the existence of poles on the surface Σ and without making any other topological and geometrical assumptions was demonstrated in [CEK04].

To state the main theorem of [CEK04], we need to mention the following assumptions

(H_1) The Gauss curvature K of Σ is integrable, i.e., $\int_\Sigma |K| d\Sigma < \infty$.

(H_2) The radius a of the layer is less than the inverse of the maximum principal curvatures k_1, k_2 of Σ, i.e., $a < \rho_m = \big(\max\{\|k_1\|_\infty \|k_2\|_\infty\}\big)^{-1}$.

Theorem (Carron, Exner and Krejcirik, 2004)**.** *Let Σ be a complete non-compact connected surface embedded in $\mathbb{R}^3$ and satisfying (H_1). Let the quantum layer Ω be defined as a tubular neighborhood of radius a around Σ satisfying (H_2).*

i) If the Gauss curvature K and the mean curvature M of Σ vanish at infinity, then

$$\inf \sigma_{ess}(-\Delta_D^\Omega) = \kappa_1^2 := \left(\frac{\pi}{2a}\right)^2.$$

ii) If the surface Σ is not a plane, then any of the conditions $a) - d)$ below is sufficient to guarantee that

$$\inf \sigma_{disc}(-\Delta_D^\Omega) < \kappa_1^2.$$

The conditions are:

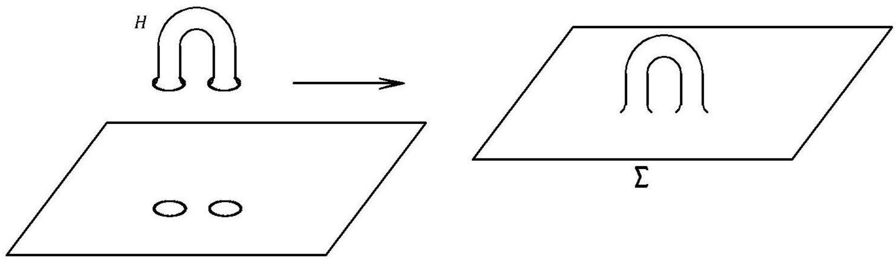
a) The total Gauss curvature is non-positive, i.e., $\mathcal{K} = \int_\Sigma K d\Sigma \leq 0$,

b) a is small enough and $\nabla M \in L^2_{loc}(\Sigma)$,

c) The total mean curvature is infinite, i.e., $\mathcal{M}^2 = \int_\Sigma M^2 d\Sigma = \infty$, and $\nabla M \in L^2(\Sigma)$,

d) Σ contains a cylindrically symmetric end with positive total Gauss curvature.

In our research, we are interested in the spectral results for a quantum layer Ω around the surface Σ which is a Euclidian plane with finite number of handles attached. Moreover, adding a handle H to a surface increases its genus by one, so the genus of Σ is equal to the number of handles.



Euclidean plane with one handle is constructed by smoothly attaching to it a curved cylindrical surface H.

Then, for such Σ the total Gauss curvature of Σ is given by the generalization of Gauss-Bonnet theorem due to (Huber 1957) and (Hartmann 1964)

$$\mathcal{K} = \int_\Sigma K d\Sigma = -4\pi g,$$

Therefore, if there is at least one handles then our surface Σ has a negative total Gauss curvature.

Based on that, the **main goal** of this thesis is to find a lower bound on the number of the eigenvalues of the Dirichlet Laplacian on this quantum layer Ω in terms of the genus of Σ. Achieving this main goal passes through writing two theorems. The first main theorem is as follows:

Main Theorem 1. *There is a constant $\mathcal{C}$, so that for all $m \in \mathbb{N}$ the following is true: If the reference surface Σ is a Euclidean plane with m-handles $H_1, ..., H_m$ whose distance to each other is at least C, then there are at least m eigenvalues of the Dirichlet Laplacian $-\Delta_D$ on the quantum layer Ω that are less than κ_1^2.*

It is important to mention here that the radius of the quantum layer Ω only satisfies (H_2) and no additional assumption.

A different approach to quantum layers is to consider the asymptotic behavior for the thickness of the radius a going to zero. The asymptotic behavior of the spectrum can be described using an effective Hamiltonian on Σ. Following this approach led us formulate our second theorem that depends on sufficiently small a.

The derivation of effective Hamiltonians for constrained quantum layers has been considered many times in the literature with different motivations and applications, see for example [WT09] and [WTL10]. By applying the results of there papers to our case, we construct an effective Hamiltonian H^a_{eff} on $L^2(\Sigma)$ which is unitary equivalent to the Hamiltonian on the thin quantum layer Ω, $H^a = -a^2\Delta_\Omega$ on $L^2(\Omega)$ up to errors of order a^3. Therefore, the eigenvalues of the effective Hamiltonian H^a_{eff} are close to those of the Hamiltonian H^a, so that we can write all eigenvalues of the Hamiltonian H^a such that

$$\lambda = E^a + O(a^3)$$

where E^a represents the eigenvalues of the H^a_{eff}, which is given by

$$H^a_{\text{eff}} = -a^2\Delta_\Sigma - a^2(M^2 - K) + \left(\frac{\pi}{2}\right)^2 + O(a^3).$$

After this result, we found a lower bound on the number of negative eigenvalues of the Schrödinger operator

$$L = -\Delta_\Sigma - (M^2 - K),$$

to estimate a lower bound on the eigenvalues of the Dirichlet Laplacian on the thin quantum layer Ω. Therefore, to get our second main theorem, we use some information from [GNY04] and take more assumptions on our surface Σ as in the following theorem.

Main Theorem 2. *Let Σ be a complete, connected and non-compact surface embedded in $\mathbb{R}^3$ with integrable Gauss curvature. Assume that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 have a common boundary, such that:*

i) $\Sigma_1 = \mathbb{R}^2 \setminus B_R(0)$, *for $R > 0$, with the Euclidean metric.*

ii) Σ_2 *is a compact surface with boundary, it has bounded diameter, volume and bounded Gauss curvature from below,*

$$\begin{aligned} diam(\Sigma_2) &\leq D \\ \mathrm{vol}(\Sigma_2) &\leq \vartheta \\ K &\geq -\kappa^2, \end{aligned}$$

where $D, \vartheta > 0$ and $\kappa \in \mathbb{R}$.

Denote the number of the negative eigenvalue of the Schrödinger operator L by $Neg(L)$. Let $r' = 8\max\{D-R, R\}$, then there is a constant $C_{(\kappa,\vartheta,r')} > 0$ only depending on κ, ϑ, r', such that we have:

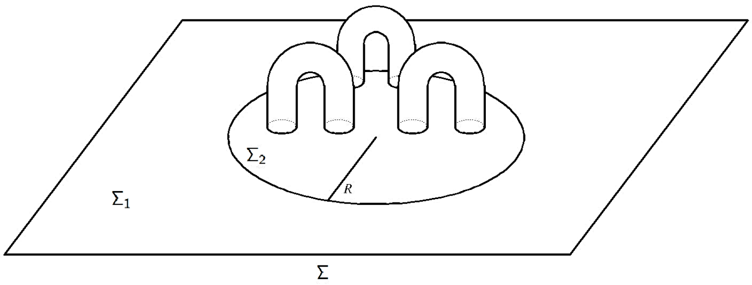
$$Neg(L) \geq \lfloor C_{(\kappa,\vartheta,r')} \int_{\Sigma} (M^2 - K) d\Sigma \rfloor.$$

In particular,

$$Neg(L) \geq \lfloor C_{(\kappa,\vartheta,r')} g \rfloor, \tag{0.0.1}$$

where g is the genus of Σ.

A very simple example of such a surface is shown in the figure below.



$\Sigma = \Sigma_1 \cup \Sigma_2$.

We found in Main Theorem 1 that the number of eigenvalues of the Dirichlet Laplacian on the quantum layer Ω around our first class surface Σ is greater than or equal to the number of handles. As from Main Theorem 2, the number of eigenvalues of Ω around our second class of surfaces depends on the genus g and a constant C.

Thesis Structure

The contents of this thesis are divided as follows:

In the first chapter, we will introduce some geometrical and topological notations on a surface and we will present some assumptions on quantum layers. We suppose that our reference surface Σ is complete, connected, orientable, non-compact and embedded in $\mathbb{R}^3$, and that its Gauss curvature K is integrable. The quantum layer Ω of width $2a$ around Σ, which we are going to study in this work, is parameterized by the map

$$\begin{aligned} \mathcal{L} : \widetilde{\Omega} = \Sigma \times (-a, a) &\to \Omega \subseteq \mathbb{R}^3 \\ (x, u) &\mapsto \mathcal{L}(x, u) = x + un(x), \end{aligned}$$

where $\mathcal{L}$ is a diffeomorphism. Then the metric of $\widetilde{\Omega}$ is defined as the pull-back of the Euclidean metric via $\mathcal{L}$. Finally, we will consider the Hamiltonian on Ω as the Dirichlet Laplacian $H := -\Delta_D^{\Omega}$ on $L^2(\Omega)$.
The assumptions presented in Chapter 1 form the basic foundations for the rest of this

thesis.

In the second chapter we refer to the References [RS80] and [RS78] to get more background information on the spectrum of self-adjoint operators on Hilbert spaces and their properties. In particular, we are interested in the discrete spectrum (isolated points of the spectrum which are eigenvalues with finite multiplicity) and its complement in the total spectrum, the essential spectrum. We will use the notations $\sigma(-\Delta)$, $\sigma_{disc}(-\Delta)$, $\sigma_{ess}(-\Delta)$ to denote the spectrum, discrete spectrum and essential spectrum, respectively. In Fact, the spectrum of the Dirichlet Laplacian on a layer of width $2a$ around the plane has a purely essential spectrum and coincides with the interval $\big[(\frac{\pi}{2a})^2, \infty\big)$.

In the third chapter we give some background information on topological properties of surfaces using [Mas89], where we refer to the classification theorem for a compact surface with or without boundary. At the end of this chapter, we give some topologically notations on non-compact surfaces and we will apply these to our reference surface Σ.

In Chapter 4, we estimate the first eigenvalue of the essential spectrum of the Dirichlet Laplacian under the assumption that the surface is asymptotically planar in the sense that its Gauss and mean curvatures vanish at infinity. We find that this part of the spectrum is bounded from below by $\kappa_1^2 = (\frac{\pi}{2a})^2$.

We prove in Chapter 5 that there exist three conditions (adopted from [CEK04]) that guarantee the existence of the discrete spectrum below κ_1^2 for the Dirichlet Laplacian on the quantum layer Ω around the surface Σ which is asymptotically planar. We substantially employ the consequence of [CEK04] that, if the Gauss curvature is integrable then there always exists a sequence of functions on Σ having the properties of the mollifiers mentioned in Lemma 5.0.6.

The first main theorem in this thesis is presented in Chapter 6 and states, if the reference surface Σ is a Euclidian plane with m-handles, that are pairwise at distance greater than or equal to a constant C, then there are at least m eigenvalues of the $-\Delta_D^\Omega$ below κ_1^2. (This constant C will be calculated in the proof of the theorem). Moreover, the number of eigenvalues depends only on the genus g of the surface and the radius a of the layer satisfying the hypothesis (H_2).

In Chapter 7, we want to approximate the eigenvalues of the Dirichlet Laplacian on a thin quantum layer Ω, $H^a = -a^2\Delta^\Omega$, by constructing the effective Hamiltonian H^a_{eff} on $L^2(\Sigma)$. This is done by applying the expression of the effective Hamiltonian in Theorem 3.1 in [WTL10] to our case. Based on that, we can write the eigenvalues λ of $-\Delta_D^\Omega$ as follows

$$\lambda = \Big(\frac{\pi}{2a}\Big)^2 + E + O(a) \quad \text{as} \quad a \to o,$$

where E are the corresponding eigenvalues of the Schrödinger operator

$$L = -\Delta_\Sigma - (M^2 - K).$$

In Chapter 8, we make more assumptions on our surface Σ to find a lower bound on the negative eigenvalues of the Schrödinger operator L, where this is our second main theorem. Then, we can estimate the lower bound on the eigenvalues of the Dirichlet Laplacian of the corresponding thin quantum layer. Moreover, this lower bound depends on the genus g and a constant.

Chapter 1

Quantum Layers

The major goal behind this chapter is to give the geometrical and topological definitions of a surface concept that will be treated here, that of a complete non-compact surface Σ embedded in $\mathbb{R}^3$. This concept is investigated here as the starting point to define the wider concept of the quantum layer Ω around this surface. Metric, volume form, Hamiltonian operator and its quadratic form are defined for the quantum layer Ω.

This chapter starts with general definitions concerning the geometry and topology of the surface Σ. Then, we look of the the geometrical properties of the quantum layer Ω. Based on the latter, a metric and volume properties of the quantum layer are specified. Quadratic forms and self-adjoint operators are introduced in this chapter as well. Finally, we consider a Hamiltonian operator on the quantum layer Ω which in this case is the Dirichlet Laplacian and show how it is used on the quantum layer to derive the spectrum of the Dirichlet Laplacian that will be explained in detail in the next chapter.

An introduction to the geometry and topology of the surface Σ and the corresponding quantum layer Ω can be found in the article [CEK04]. For more details about general definitions on the surface Σ see [Gri09] and [Küh02]. The basic definitions and properties of the quadratic forms and self-adjoint operators can be found in [CL90], [RS80], and [RS78].

1.1 The Geometry and Topology of the Reference Surface

Let Σ be a smooth surface embedded in $\mathbb{R}^3$ without boundary. We will always make the following assumptions on Σ, as for example that are made in the reference [CEK04]:

(1) Σ is connected, i.e., between any two points in Σ there is a continuous path in Σ.

(2) Σ is complete, i.e., Σ is a connected surface in which for every point $p \in \Sigma$, and for any parameterized geodesic $\gamma : [0, \varepsilon) \to \Sigma$, starting from $p = \gamma(0)$, this geodesic may be extended to a parameterized geodesic $\widetilde{\gamma} : \mathbb{R} \to \Sigma$, defined on the entire line $\mathbb{R}$.

(3) Σ is orientable, i.e., there is a globally defined unit normal vector field $n : \Sigma \to \mathbb{R}^3$.

(4) Σ is non-compact.

Before going into details, it is necessary to recall some basic facts about fundamental forms and curvatures of Σ, for more details see, [Küh02] and [Gri09] :

Definition 1.1.1. *Let $\Sigma \subset \mathbb{R}^3$. For any point $x \in \Sigma$ denoted by $T_x\Sigma$ the tangent space on Σ at x, and define*

$$\begin{aligned} g_x : T_x\Sigma \times T_x\Sigma &\to \mathbb{R} \\ g_x(X,Y) &\mapsto \langle X, Y\rangle_{\mathbb{R}^3} \end{aligned}$$

for $X, Y \in T_x\Sigma$.
g_x is called the first fundamental form (I) or the induced Riemannian metric of Σ at x.

Definition 1.1.2. *Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface and n a unit normal vector field for Σ. For any $x \in \Sigma$, the Weingarten map is the linear map given by*

$$\begin{aligned} L_x : T_x\Sigma &\to T_x\Sigma \\ X &\mapsto -dn_x(X). \end{aligned}$$

The second fundamental form of Σ at x is

$$II_x(X,Y) := g_x(L_xX, Y),$$

for $X, Y \in T_x\Sigma$.

Definition 1.1.3. *Let $\Sigma \subset \mathbb{R}^3$ be a surface, $x \in \Sigma$ and $X \in T_x\Sigma$ with $\|X\| = 1$. We call $II_x(X,X)$ the normal curvature K_n of Σ at x in the direction of X.*

Definition 1.1.4. *Let Σ be a surface, $x \in \Sigma$. The principal curvatures of Σ at x are the minimum and the maximum value of $II(X,X)$ over all $X \in T_x\Sigma$ with $\|X\| = 1$. They are denoted by $k_1 = k_1(x)$, $k_2 = k_2(x)$. If $k_1 \neq k_2$, then the corresponding X are called the principal curvature directions.*

Proposition 1.1.5. *The principal curvatures at x are the eigenvalues of the Weingarten map L_x. If $k_1 \neq k_2$ then the principal curvature directions are the corresponding eigenvectors are orthogonal to each other.*

Definition 1.1.6. *Let Σ be a surface, $x \in \Sigma$, k_1, k_2 the principal curvatures at x with respect to an unit normal vector field n.*

(i) the Gaussian curvature of Σ at x is

$$K = \det(L_x) = \kappa_1\kappa_2.$$

(ii) the mean curvature of Σ at x is

$$M = \frac{1}{2}tr(L_x) = \frac{1}{2}(k_1 + k_2).$$

Definition 1.1.7. *Let $x = (x_1, x_2)$ be local coordinates of Σ. The 2×2-matrix (g_{ij}) is the matrix of the first fundamental form of the surface Σ corresponding to these coordinates.*

i) The volume element form of the surface Σ *is given as follows*

$$d\Sigma = \sqrt{\det g} dx.$$

ii) The total Gauss curvature $\mathcal{K}$ *and the total mean curvature* $\mathcal{M}$ *are given by the integrals*

$$\mathcal{K} := \int_{\Sigma} K d\Sigma, \quad \mathcal{M}^2 := \int_{\Sigma} M^2 d\Sigma. \tag{1.1.1}$$

Remark 1.1.8. *The total mean curvature always exists (although it may be* $+\infty$ *), and in this work we assume that the Gauss curvature of* Σ *is integrable, i.e.,*

$$K \in L^1(\Sigma). \tag{1.1.2}$$

We are also interested in surfaces which have finite genus and a finite number of ends, which they are defined as follows

Definition 1.1.9. *An open set* $E \subseteq \Sigma$ *is called an end of* Σ *if it is connected, unbounded and if its boundary* ∂E *is compact (see Figure. 1.1.1); its total curvatures are defined by means of* (1.1.1) *with the domain of integration being the subset* E *only.*

Definition 1.1.10. *A manifold embedded in* $\mathbb{R}^3$ *is cylindrically symmetric if it is invariant under rotations around a fixed axis in* $\mathbb{R}^3$.

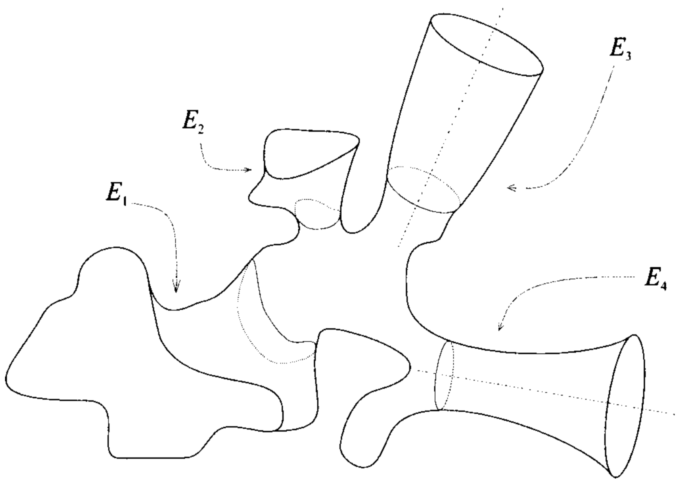


Figure 1.1.1: Surface with four ends $E_1, \ldots, E_4$, with E_3, E_4 cylindrically symmetric.

Definition 1.1.11. *The genus of a surface* Σ *is the maximum number of non-intersecting closed curves which can be drawn on* Σ *without disconnecting the surface.*

It can be proved that any surface of finite genus is homomorphic to plane or sphere with finitely many handles attached (see Figure 1.1.2). Then the genus is equal to the number of handles.

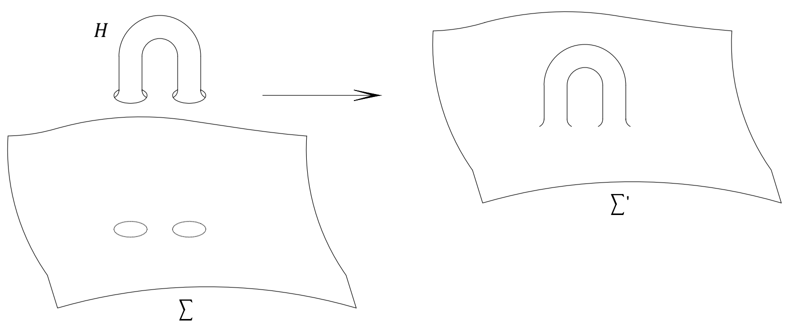


Figure 1.1.2: Surface with a handle Σ' is constructed from Σ by smoothly attaching a curved cylindrical surface H.

1.2 The Quantum Layer Geometry

First, we introduce the quantum layer geometry and formulate some basic assumptions.

Definition 1.2.1. *A quantum layer Ω is defined as a tubular neighborhood of radius $a > 0$ about an orientable complete non-compact surface Σ,i.e.,*

$$\Omega := \{z \in \mathbb{R}^3 \mid \mathrm{dist}(z, \Sigma) < a\}. \tag{1.2.1}$$

We will assume that

a) The principal curvatures k_1, k_2 of the surface Σ are bounded and

$$a < \rho_m := \Big(max\big\{\|k_1\|_\infty, \|k_2\|_\infty\big\}\Big)^{-1}, \tag{1.2.2}$$

the number ρ_m is naturally interpreted as the minimal normal curvature radius of Σ (the radius of the normal curvature of a surface at a given point is the radius of a circle that best fits a normal section, the intersection of the surface with a plane containing the normal to the surface at a particular point). For plane surfaces one can put $\rho_m := \infty$.

b) Ω does not have self intersection, in the sense that the map $\mathcal{L}$ defined below in 1.2.3 is injective.

Also, we may define the quantum layer Ω of width $2a$ as the image of the mapping

$$\begin{array}{rccc} \mathcal{L}: & \Sigma \times (-a, a) & \to & \mathbb{R}^3 \\ & (x, u) & \mapsto & x + un(x). \end{array} \tag{1.2.3}$$

Taking into account that the assumption $a)$ and $b)$ above are always true, then $\mathcal{L}$ is injective. Also, we will prove below in Lemma 1.2.5 that $\mathcal{L}$ is an immersion, and thus an embedding.

An embedding is a diffeomorphism onto its image, and in particular the image of an embedding is a submanifold, Therefore, $\mathcal{L} : \Sigma \times (-a, a) \to \Omega$ is a diffeomorphism and Ω is the submanifold of $\mathbb{R}^3$ of points located between two parallel surfaces at the distance a from Σ (see Figure. 1.2.1), i.e., if Σ has empty boundary the definitions of Ω via (1.2.1) and (1.2.3) are equivalent.

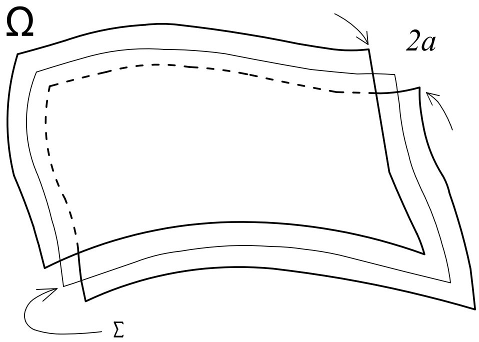


Figure 1.2.1: Quantum layer

1.2.1 Metric Properties of the Layer

First, recall what the concept of pull-pack is: If $F : M \to N$ is a smooth map between manifolds and w is a k-form on N then F^*w is a k-contravariant tensor on M defined by: For $p \in M$ and $v_1, .., v_k \in T_pM$, set

$$(F^*w)_p(v_1, .., v_k) := w_{F(p)}\big(dF_{|p}(v_1), .., dF_{|p}(v_k)\big).$$

Writing the metric g_{eucl} on Ω with respect to the parameterization $\mathcal{L}$ in 1.2.3 means taking the pull-back

$$G = \mathcal{L}^* g_{eucl},$$

Now we compute G

$$G : T_{(x,u)}\big(\Sigma \times (-a, a)\big) \times T_{(x,u)}\big(\Sigma \times (-a, a)\big) \to \mathbb{R}.$$

with

$$G(X, Y) = \big\langle d\mathcal{L}(X), d\mathcal{L}(Y)\big\rangle_{\mathbb{R}^3}$$

Then,

$$\mathcal{L} : \big(\Sigma \times (-a, a), G\big) \to \big(\Omega, g_{eukl}\big)$$

is an isometry, and since

$$X, Y \in T_{(x,u)}\big(\Sigma \times (-a, a)\big) = T_x\Sigma \times T_u(-a, a) = T_x\Sigma \times \mathbb{R},$$

we may write

$$X = (X_\Sigma, X') \quad \text{and} \quad Y = (Y_\Sigma, Y'),$$

and decompose $d\mathcal{L}(X)$ as

$$\begin{aligned} d\mathcal{L}(X) &= d\mathcal{L}(X_\Sigma, X') \\ &= d\mathcal{L}(X_\Sigma, 0) + d\mathcal{L}(0, X') \end{aligned}$$

and similarly

$$d\mathcal{L}(Y) = d\mathcal{L}(Y_\Sigma, 0) + d\mathcal{L}(0, Y').$$

We will fix u in (1.2.3), we can write

$$\begin{aligned} \mathcal{L}_u : \Sigma &\to \mathbb{R}^3 \\ x &\mapsto \mathcal{L}(x, u) \end{aligned}$$

then we obtain :

$$\mathcal{L}_u = id + un.$$

Then:

$$\begin{aligned} d\mathcal{L}(X_\Sigma, 0) = d\mathcal{L}_u(X_\Sigma) &= X_\Sigma + udn(X_\Sigma) \\ &= X_\Sigma - uL(X_\Sigma) \\ &= (I - uL)X_\Sigma \end{aligned}$$

and

$$d\mathcal{L}(Y_\Sigma, 0) = (I - uL)Y_\Sigma,$$

where $I = I_x$ denotes the identity map on $T_x\Sigma$ and $L = L_x$ the Weingarten map of Σ.

Moreover, fix x in (1.2.3)

$$\mathcal{L}_x = x + (id)n(x)$$

and it follows that

$$\begin{aligned} d\mathcal{L}(0, X') &= d\mathcal{L}_x(X') = n(x)X', \\ d\mathcal{L}(0, Y') &= d\mathcal{L}_x(Y') = n(x)Y'. \end{aligned}$$

By using the all above equalities, we get

$$\begin{aligned} G(X, Y) &= \big\langle d\mathcal{L}(X), d\mathcal{L}(Y) \big\rangle_{\mathbb{R}^3} \\ &= \big\langle (I - uL)X_\Sigma + n(x) \cdot X', (I - uL)Y_\Sigma + n(x) \cdot Y' \big\rangle_{\mathbb{R}^3} \\ &= \big\langle (I - uL)X_\Sigma, (I - uL)Y_\Sigma \big\rangle_g + 0 + 0 + \big\langle X', Y' \big\rangle_{\mathbb{R}} \\ &= g\big((I - uL)X_\Sigma, (I - uL)Y_\Sigma\big) + X'Y'. \end{aligned}$$

Thus, the metric of $\Sigma \times (-a, a)$ has a block matrix, such that

$$G_{ij} = \begin{pmatrix} (I - uL)^2 g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2.4}$$

1.2.2 The Volume Form of the Layer

The volume form of $\Sigma \times (-a, a)$ is by definition the pull-back of the volume form of Ω under the map $\mathcal{L}$, which is given by the following Lemma .

Lemma 1.2.2.

$$d\big(\Sigma \times (-a, a)\big) = (1 - 2Mu + Ku^2)d\Sigma du. \tag{1.2.5}$$

Where M, K denote the mean curvature and the Gauss curvature of Σ, respectively.

Proof. Let us apply the definition of pull-back to our situation, but first for a fixed u. For brevity, we will write

$$\begin{aligned} F = \mathcal{L}_u : \Sigma \times \{u\} &\to \Sigma_u \\ (., u) &\mapsto id + un \end{aligned}$$

where $\Sigma_u = \mathcal{L}_u(\Sigma \times \{u\})$.

So we want to calculate $F^*(d\Sigma_u)$, where $d\Sigma_u$ denotes the volume form of Σ_u. Recall the definition of it: Since Σ_u is 2-dimensional, $d\Sigma_u$ is a 2-form and for $q \in \Sigma_u$ and $V, W \in T_q\Sigma_u$, where the pair V, W is positively oriented, then we have by definition $(d\Sigma_u)_q(V, W) =$ the area of the parallelogram spanned by V and W.

Here, the area is measured with respect to the metric on $T_q\Sigma_u$, the metric is just the Euclidean metric on $\mathbb{R}^3$ restricted to the subspace $T_q\Sigma_u$, and the usual formula area $= |V \times W|$ applies. However, we will not use this formula.

Now, let $x \in \Sigma$ and $X, Y \in T_x\Sigma$. Of course X, Y also may be considered tangent vectors to $\Sigma \times \{u\}$ at the point $p = (x, u)$. Then, with $A = dF_{|p} = d\mathcal{L}_u(X, 0) = (I - uL)X$,

$$\begin{aligned} \big(F^*(d\Sigma_u)\big)_p(X, Y) &= (d\Sigma_u)_{F(p)}(AX, AY) \\ &= \text{the area of the parallelogram spanned by } AX, AY. \end{aligned}$$

To evaluate this, we first need to understand the meaning of the equation $d\mathcal{L}_u(X, 0) = (I - uL)X$, which in our current notation says $dF_p = I - uL$. This has a very important geometric meaning (besides the concrete formula): Since both I and L map the tangent space $T_x\Sigma$ to itself, it shows that dF_p also maps $T_x\Sigma$ to itself. Of course dF_p also maps $T_x\Sigma$ to $T_{F(p)}\Sigma_u$, so we obtain that, the tangent space of Σ_u at $F(p) = (x, u)$ is the same as the tangent space of Σ at x.
(By the way, this also shows the normal spaces are the same. This will be needed later in this work.)

Thus, we have $A = I - uL$, and it maps $T_x\Sigma$ to itself. So by well-known properties of the area of parallelograms, for $X, Y \in T_x\Sigma$ the area of the parallelogram spanned by AX, AY is $|\det A|$ times the area of the parallelogram spanned by X, Y.

So, the area of the latter parallelogram is $d\Sigma(X, Y)$, where $d\Sigma$ is the volume element of Σ, by the same reason as above for Σ_u.

Putting things together, we get

$$\begin{aligned}\big(F^*(d\Sigma_u)\big)_p(X,Y) &= (d\Sigma_u)_{F(p)}(AX,AY)\\ &=\mid \det A \mid d\Sigma(X,Y),\end{aligned}$$

and becouse this is true for all X, Y, we obtain

$$\big(F^*(d\Sigma_u)\big) =\mid \det A \mid d\Sigma.$$

Since $\det A > 0$ for $|u| < a$ (see Lemma 1.2.4),

$$\big(F^*(d\Sigma_u)\big) = (\det A)d\Sigma.$$

For small $a > 0$ we have

$$\begin{aligned}\mathrm{dvol}\big(\Sigma\times(-a,a)\big) &= \big(F^*(d\Sigma_u)\big)du\\ &= (\det A)d\Sigma du.\end{aligned}$$

Since the matrix corresponding to the Weingarten map L in local coordinates has the principle curvatures k_1, k_2 as eigenvalues, we get

$$\det A = \det(I - uL)$$

$$= (1-uk_1)(1-uk_2) = 1 - u(k_1+k_2) + u^2(k_1k_2),$$

and by using the definitions of the Gauss curvature and the mean curvature, we get

$$\det A = (1 - 2Mu + Ku^2)$$

which shows that

$$\mathrm{dvol}\big(\Sigma\times(-a,a)\big) = (1-2Mu+Ku^2)d\Sigma du,$$

therefore

$$d\big(\Sigma\times(-a,a)\big) = (1-2Mu+Ku^2)d\Sigma du. \quad \square$$

Remark 1.2.3. *Since*

$$d\big(\Sigma\times(-a,a)\big) = \sqrt{\det G}dxdu$$

and

$$d\Sigma = \sqrt{\det g}dx,$$

we have

$$\det G = (1-2Mu+Ku^2)^2 \det g.$$

Lemma 1.2.4. *Assume that $|u| < a$. Then the metric G of the $\Sigma\times(-a,a)$ can be estimated by the metric g of the surface Σ as follows:*

$$C_-\sqrt{\det g} \le \sqrt{\det G} \le C_+\sqrt{\det g}, \tag{1.2.6}$$

where $C_\pm := (1 \pm a\rho_m^{-1})^2$.

Proof. We have

$$|u| < a < \rho_m := \Big(max\{\|k_1\|_\infty, \|k_2\|_\infty\}\Big)^{-1},$$

$$\rho_m^{-1} = \max\big\{\|k_1\|_\infty, \|k_2\|_\infty\big\}.$$

And

$$k_1 \leq \|k_1\|_\infty \leq \rho_m^{-1},$$

(If Σ is a plane we can set $\rho_m = \infty$, $\rho_m^{-1} = 0$).

Thus

$$|uk_1| \leq a\rho_m^{-1} \Leftrightarrow -a\rho_m^{-1} \leq uk_1 \leq a\rho_m^{-1}$$

$$\Leftrightarrow 1 - a\rho_m^{-1} \leq 1 - uk_1 \leq 1 + a\rho_m^{-1}$$

and similarly

$$1 - a\rho_m^{-1} \leq 1 - uk_2 \leq 1 + a\rho_m^{-1}.$$

Not that all the elements are positive because $a\rho_m^{-1} < 1$.

This implies

$$(1 - a\rho_m^{-1})^2 \leq (1 - uk_1)(1 - uk_2) \leq (1 + a\rho_m^{-1})^2$$

$$(1 - a\rho_m^{-1})^2 \leq (1 - 2uM + u^2K) \leq (1 + a\rho_m^{-1})^2. \tag{1.2.7}$$

Since g is a Riemannian metric, it is positive definite, so the proof is completed by using the remark above. $\square$

Finally, we can prove the following Lemma:

Lemma 1.2.5. *The map $\mathcal{L} : \Sigma \times (-a, a) \to \mathbb{R}^3$ is an immersion.*

Proof. Recall that $\mathcal{L}$ is an immersion if for each point $(x, u) \in \Sigma \times (-a, a)$ the map $d\mathcal{L}_{(x,u)} : T_{(x,u)}\big(\Sigma \times (-a, a)\big) \to T_{\mathcal{L}(x,u)}(\Omega) = \mathbb{R}^3$ is injective.

This follows from the fact that $\det G \neq 0$, where it is proved in Lemma 1.2.4. $\square$

1.3 Quadratic Forms and Self-Adjoint Operators

In this section we will recall some definitions and important propositions about quadratic forms and self-adjoint operators. (For more details see [CL90], [RS80]and [RS78]).

Definition 1.3.1. *A quadratic form is a map $q : D(q) \times D(q) \to \mathbb{C}$, where $D(q)$ is a linear subset of a Hilbert space $\mathcal{H}$ called the domain of the form q, such that $q(., \psi)$ is conjugate linear and $q(\varphi, .)$ is linear for all $\varphi, \psi \in D(q)$.*
q is said to be:

a) Densely defined if $\overline{D(q)} = \mathcal{H}$.

b) Symmetric if $q(\varphi, \psi) = \overline{q(\psi, \varphi)}$.

c) non-negative if $q(\varphi, \varphi) \geq 0$ *for all* $\varphi \in D(q)$.

d) Semibounded (from below) if there exists $M \in \mathbb{R}$*, such that* $q(\varphi, \varphi) \geq -M\|\varphi\|^2$ *for all* $\varphi \in D(q)$.

Definition 1.3.2. *Let* q *be a semibounded quadratic form.* q *is called closed if the vector space* $D(q)$ *is complete with respect to the norm*

$$\|\varphi\|_{(q)} = \left[q(\varphi, \varphi) + (M+1)\|\varphi\|^2\right]^{\frac{1}{2}}.$$

Definition 1.3.3. *Let* A *be a non-negative, self-adjoint operator. Then the quadratic form* q *defined by* $D(q) = D(A^{\frac{1}{2}})$ *and*

$$q(\varphi, \psi) = \langle A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\psi\rangle \quad \textit{for} \quad \varphi, \psi \in D(A^{\frac{1}{2}})$$

is a closed non-negative form, and we say that it is the quadratic form of the operator A.

Remark 1.3.4. *The right hand side of the last equation will very often be denoted* $\langle A\varphi, \psi\rangle$ *even though* φ *may not belong to the domain of* A.

Theorem 1.3.5. *If* q *is a closed and semibounded quadratic form, then* q *is the quadratic form of a unique self-adjoint operator.*

Proof. See [RS80]

Moreover, let us recall the definition of Sobolev spaces and some related notations.

Definition 1.3.6. *Let* $\Lambda \subset \mathbb{R}^n$ *be an open set and let* $m \in \mathbb{N}_0$. $H^m(\Lambda)$ *is the set of functions* $f \in L^2(\Lambda)$ *whose distributional derivatives* $D^\alpha f$ *are in* $L^2(\Lambda)$ *for all* α *with* $|\alpha| \leq m$. $H^m(\Lambda)$ *is a Hilbert space with respect to the norm*

$$\|f\|_m = \Big(\sum_{|\alpha| \leq m} \|D^\alpha f\|_2^2\Big)^{1/2}.$$

$H_0^m(\Lambda)$ *is defined to be the completion of* $\mathcal{C}_0^\infty(\Lambda)$ *in the norm* $\|\cdot\|_m$. $H^m(\Lambda)$ *and* $H_0^m(\Lambda)$ *are called* **Sobolev spaces**.

In general, $H_0^m(\Lambda)$ is proper subset of $H^m(\Lambda)$.

Proposition 1.3.7. *Let* Λ *be an open set and suppose that* $m \geq j \geq 0$. *Then:*

a) $H_0^m(\Lambda) \subset H^m(\Lambda)$.

b) $H^m(\Lambda) \subset H^j(\Lambda)$.

c) $H_0^m(\Lambda) \subset H_0^j(\Lambda)$.

Proof. (a), (b), and (c) follow immediately from the definitions.

Two important examples of self-adjoint operators are the Dirichlet and Neumann Laplacian which we define now.

Definition 1.3.8. *Let Λ be an open subset of $\mathbb{R}^n$.*

i) The Dirichlet Laplacian for Λ, $-\Delta_D^\Lambda$, is the unique self-adjoint operator on $L^2(\Lambda)$ corresponding to the quadratic form

$$q(\psi,\varphi) := \int_\Lambda \langle \nabla\psi, \nabla\varphi\rangle d\Lambda \quad \textit{with domain} \quad D(q) := H_0^1(\Lambda).$$

ii) The Neumann Laplacian for Λ, $-\Delta_N^\Lambda$, is the unique self-adjoint operator on $L^2(\Lambda)$ whose quadratic form is given by

$$q(\psi,\varphi) := \int_\Lambda \langle \nabla\psi, \nabla\varphi\rangle d\Lambda \quad \textit{on the domain} \quad D(q) := H^1(\Lambda).$$

An important property of the Dirichlet and Neumann boundary conditions is that they decouple the space in the sense as in the following Proposition 1.3.10. To be precise about the decoupling we make some preliminary remarks about direct sums of self-adjoint operators.

Definition 1.3.9. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a direct sum of Hilbert spaces, and let A_1 be a self-adjoint operator on $\mathcal{H}_1$, A_2 a self-adjoint operator on $\mathcal{H}_2$. Let A be the operator with domain*

$$D(A) = \big\{(\varphi,\psi) \mid \varphi \in D(A_1), \psi \in D(A_2)\big\}$$

defined by

$$A(\varphi,\psi) = (A_1\varphi, A_2\psi).$$

We shall write $A = A_1 \oplus A_2$.

The properties of $A_1 \oplus A_2$ include:

1) $A_1 \oplus A_2$ is self-adjoint.

2) $q(A_1 \oplus A_2) = q(A_1) \oplus q(A_2)$.

3) If $(\varphi,\psi) \in D(A_1) \oplus D(A_2)$, then

$$\big((\varphi,\psi), (A_1 \oplus A_2)(\varphi,\psi)\big) = (\varphi, A_1\varphi) + (\psi, A_2\psi)$$

Proposition 1.3.10. *Let Λ_1 and Λ_2 be disjoint open sets. Then*

$$L^2(\Lambda_1 \cup \Lambda_2) = L^2(\Lambda_1) \oplus L^2(\Lambda_2).$$

Under this decomposition, it holds

$$\begin{aligned} -\Delta_D^{\Lambda_1\cup\Lambda_2} &= -\Delta_D^{\Lambda_1} \oplus -\Delta_D^{\Lambda_2} \\ -\Delta_N^{\Lambda_1\cup\Lambda_2} &= -\Delta_N^{\Lambda_1} \oplus -\Delta_N^{\Lambda_2} \end{aligned}$$

Proof. (See [RS78] Proposition 3, Page 269)

Another important property is that the Laplace operator with Dirichlet boundary condition is larger than the Laplace operator with Neumann boundary condition. But to make this fact rigorous we need some sort of order on operators or quadratic forms. More precisely:

Definition 1.3.11. *Let A and B be self-adjoint operators on $\mathcal{H}$ bounded from below. We say that $A \leq B$ if and only if:*

i) $D(B) \subset D(A)$,

ii) for all $\psi \in D(B)$, one has $(A\psi, \psi) \leq (B\psi, \psi)$.

The typical examples include the Dirichlet and Neumann Laplacians for bounded domains. In fact, from the definitions immediately get:

Proposition 1.3.12. *If Λ and Λ' are bounded domains such that $\Lambda \subset \Lambda'$ then one has:*

$$0 \leq -\Delta_D^{\Lambda'} \leq -\Delta_D^{\Lambda}$$

and for each bounded domain Λ:

$$0 \leq -\Delta_N^{\Lambda} \leq -\Delta_D^{\Lambda}$$

which goes under the name of Dirichlet- Neumann bracketing.

Proof. (See [RS78] Proposition 4, Page 270)

1.4 The Hamiltonian on the Layer

After these preliminaries let us define the Hamiltonian H of our model.
The Hamiltonian H is the Dirichlet Laplacian

$$H := -\Delta_D^{\Omega} \quad \text{on} \quad L^2(\Omega), \tag{1.4.1}$$

which is defined for an open set $\Omega \subset \mathbb{R}^3$, whose the associated quadratic form is

$$q(\psi, \varphi) := \int_{\Omega} \langle \nabla\psi, \nabla\varphi \rangle d\Omega \quad \text{with domain} \quad D(q) := H_0^1(\Omega).$$

Here ∇ is the gradient corresponding to the metric g_{eucl} and $\langle ., . \rangle$ denotes the inner product in the manifold Ω induced by g_{eucl}, the associated norm will be denoted by $|.|$. Similarly, the inner product and the norm in the Hilbert space $L^2(\Omega)$ will be benoted by $(., .)$ and $\|.\|$, respectively. We shall sometimes abuse the notation slightly by writing $(., .) = \int_{\Omega} |.| d\Omega$.

A natural way to investigate the operator (1.4.1) is to pass to the coordinates (x, u) on $\widetilde{\Omega} = \Sigma \times (-a, a)$ in which it acquires the Laplace-Beltrami form

$$\widetilde{H} := -\Delta_D^{\widetilde{\Omega}} = -\frac{1}{\sqrt{\det G}} \Big[\sum \partial_i \big(\sqrt{\det G} G^{ij} \partial_j \big) \Big]$$

on $L^2\big(U \times (-a,a), \sqrt{\det G}dxdu\big)$, where we denote by $(x_\mu) \equiv (x_1, x_2)$ local coordinates for $U \subset \Sigma$ and by G^{ij} the coefficients of the inverse of the matrix G_{ij} (which is given in the equation 1.2.4) in the coordinates $(x_i) \equiv (x_\mu, u)$ for $\widetilde{\Omega}$. Then $-\Delta_D^{\widetilde{\Omega}}$ splits into a sum of two parts

$$-\Delta_D^{\widetilde{\Omega}} = -\frac{1}{\sqrt{\det G}}\Big[\sum \partial_\mu\big(\sqrt{\det G}G^{\mu\nu}\partial_\nu\big) + \partial_u\big(\sqrt{\det G}G^{uu}\partial_u\big)\Big].$$

We will denoted by

$$-\Delta_\Sigma = -\frac{1}{\sqrt{\det G}}\sum \partial_\mu\big(\sqrt{\det G}G^{\mu\nu}\partial_\nu\big)$$

$$-\Delta_u = -\frac{1}{\sqrt{\det G}}\partial_u\big(\sqrt{\det G}G^{uu}\partial_u\big)$$

$$= -\frac{1}{(1-2Mu+Ku^2)\sqrt{\det g}}\Big[(-2M+2uK)\sqrt{\det g}\partial_u + (1-2uM+u^2K)\sqrt{\det g}\partial_u^2\Big]$$

by using the fact that $\sqrt{\det g}$ does not depend on u, then

$$-\Delta_u = 2M_u\partial_u - \partial_u^2$$

where

$$M_u := \frac{M-Ku}{1-2Mu+Ku^2}.$$

Then, we can write

$$-\Delta_D^{\widetilde{\Omega}} = -\Delta_\Sigma + 2M_u\partial_u - \partial_u^2. \tag{1.4.2}$$

The above coordinate change is nothing else than the unitary transformation

$$\widetilde{U} : L^2(\Omega, d\Omega) \to L^2\big(\Sigma \times (-a,a), \sqrt{\det G}dxdu\big)$$
$$\psi \mapsto \widetilde{U}\psi := \psi \circ \mathcal{L},$$

which relates the two operators by $\widetilde{H} = \widetilde{U}(-\Delta_D^\Omega)\widetilde{U}^{-1}$.

Lemma 1.4.1. *The mean curvature and the Gauss curvature of the parallel surface $\mathcal{L}(\Sigma \times \{u\})$ are given respectively:*

$$M_u = \frac{M-Ku}{1-2Mu+Ku^2}, \tag{1.4.3}$$

$$K_u = \frac{K}{1-2Mu+Ku^2}. \tag{1.4.4}$$

Proof. Fix u, denote for the moment by $F = \mathcal{L}_u = id + un$ the map that sends a point on Σ to the corresponding point on the parallel surface $\Sigma_u = \mathcal{L}(\Sigma \times \{u\})$.
Now the normal vector to Σ_u is the same as the normal to Σ at the corresponding point (as we see in the Proof of the Lemma 1.2.2). In other words, if we denote for the moment the normal vector field to Σ_u by n_u, then for all x we have

$$n_u\big(F(x)\big) = n(x).$$

Now differentiate this expression with respect to x to obtain

$$d\big(n_u \circ F\big) = dn, \quad \big(dn_{u_{|F}}\big)\big(dF\big) = dn \qquad \text{(chain rule)}, \quad \big(dn_{u_{|F}}\big)\big(d\mathcal{L}_u\big) = dn.$$

Using the definition of the Weingarten map L_u of Σ_u, we have

$$L_u = -(dn_{u_{|F}}),$$

and we know that

$$(I - uL) = (d\mathcal{L}_u),$$

thus we get

$$-L_u(I - uL) = -L \Leftrightarrow L_u(I - uL) = L$$

$$\Leftrightarrow L_u = L(I - uL)^{-1}.$$

Since the matrix of the Weingarten map L has the principle curvatures k_1, k_2 as eigenvalues, then we can write in a basis of eigenvalues

$$(I - uL)^{-1} = \begin{pmatrix} 1 - uk_1 & 0 \\ 0 & 1 - uk_2 \end{pmatrix}^{-1}$$

$$= \frac{1}{\det(I - uL)} \begin{pmatrix} 1 - uk_2 & 0 \\ 0 & 1 - uk_1 \end{pmatrix}$$

and we obtain

$$L_u = \frac{1}{\det(I - uL)} \left[\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} 1 - uk_2 & 0 \\ 0 & 1 - uk_1 \end{pmatrix} \right]$$

$$= \frac{1}{1 - 2Mu + Ku^2} \left[\begin{pmatrix} k_1 - uk_1k_2 & 0 \\ 0 & k_2 - uk_1k_2 \end{pmatrix} \right].$$

Moreover, denoted by M_u' the mean curvature of Σ_u has by definition

$$M_u' = \frac{1}{2} tr L_u$$

and thus

$$M_u' = \frac{1}{2(1 - 2Mu + Ku^2)} (k_1 + k_2 - 2uk_1k_2)$$

$$= \frac{2M - 2uK}{2(1 - 2Mu + Ku^2)}$$

$$= \frac{M - uK}{1 - 2Mu + Ku^2} = M_u.$$

Also, the Gauss curvature of the parallel surface $\mathcal{L}(\Sigma \times \{u\})$ is given by

$$\begin{aligned} K_u &= \det L_u \\ &= \frac{1}{(1-2Mu+Ku^2)^2}\big(k_1k_2 - uk_1k_2(k_1+k_2) + u^2k_1^2k_2^2\big) \\ &= \frac{K(1-2Mu+Ku^2)}{(1-2Mu+Ku^2)^2} \end{aligned}$$

it follows that

$$K_u = \frac{K}{1-2Mu+Ku^2}. \quad \square$$

Chapter 2

The Spectrum

Relations between the geometry of a quantum layer Ω in $\mathbb{R}^d$, boundary conditions at $\partial\Omega$, and spectral properties of the corresponding Laplacian are one of the vintage problems of mathematical physics. Recent years brought new motivations and focused attention to aspects of the problem which attracted little attention earlier.

We consider one of the most attractive and important objects from the theory of functional analysis: the spectra for self-adjoint operators on Hilbert space. The spectrum of an operator of a finite dimensional vector space is precisely the set of eigenvalues. However an operator on an infinite dimensional space may have additional elements in its spectrum, and may have no eigenvalues.

In this chapter we start by recalling the definitions of the spectrum of a self-adjoint operator and its subsets, which are essential spectrum and discrete spectrum. Then we display some important properties of the essential and discrete spectrum of self-adjoint operators which we need them in our work. Finally, we calculate the essential spectrum of the Dirichlet Laplacian on the planer layer $\Omega_0 := \mathbb{R}^2 \times (-a, a)$.

For more details about the spectrum of self-adjoint operators on Hilbert spaces and its properties see [RS80] and [RS78].

2.1 The Spectrum of Self-Adjoint Operators

We will mainly be interested in studying self-adjoint operators, whose spectrum is always a non-empty subset of the real numbers.

Definition 2.1.1. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. A real number λ is said to be in the resolvent set $\rho(A)$ of A if $\lambda I - A$ is a bijection with a bounded inverse. $R_\lambda(A) = (\lambda I - A)^{-1}$ is called the resolvent of A at λ. If $\lambda \notin \rho(A)$, then λ is said to be in the spectrum of A, denoted by $\sigma(A)$.*

Remark 2.1.2. *Let $A : D \subset \mathcal{H} \to \mathcal{H}$ be a linear operator. Then the operator $\lambda I - A : D \to \mathcal{H}$ has bounded inverse, if there exists a bounded operator $S : \mathcal{H} \to \mathcal{H}$ such that:*

$$S(\lambda I - A) = I_D, \qquad (\lambda I - A)S = I_{\mathcal{H}}.$$

Remark 2.1.3. *If λ is an eigenvalue of A, then the operator $\lambda I - A$ is not injective, and therefore its inverse $(\lambda I - A)^{-1}$ is not defined. However, the converse statement is not true: the operator $\lambda I - A$ may not have an inverse and it is injective but not surjective, even if λ is not an eigenvalue. Thus the spectrum of an operator always contain all its eigenvalues, but is not limited to them.*

The spectrum of a self-adjoint operator decomposes into two necessarily disjoint subsets as follows:

Definition 2.1.4. *Let A be a self-adjoint operator.*

a) The discrete spectrum of A, $\sigma_{disc}(A)$, is defined to be those eigenvalues λ of A which are of finite multiplicity, i.e., $\{\psi | A\psi = \lambda\psi\}$ is finite dimensional, and are isolated points of the spectrum.

b) The essential spectrum of A, $\sigma_{ess}(A)$, is the complement the $\sigma_{disc}(A)$ in $\sigma(A)$, $\sigma_{ess} = \sigma(A) \setminus \sigma_{disc}(A)$.

Remark 2.1.5. *If $\sigma_{ess}(A) = \emptyset$, then A is said to have purely discrete spectrum. If $\sigma_{disc}(A) = \emptyset$, then A is said to have purely essential spectrum.*

The following Theorem is called the Weyl's criterion theorem, which give the necessary and sufficient condition for the λ in spectrum:

Theorem 2.1.6. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then:*

$$\lambda \in \sigma(A) \Leftrightarrow \exists \{\psi_n\}_{n=1}^{\infty} \quad \text{so that} \quad \|\psi_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(A - \lambda)\psi_n\| = 0.$$

$$\Leftrightarrow \exists \psi_n \quad \text{and} \quad C > 0 \quad \text{so that} \quad \|\psi_n\| \geq C, \forall n \quad \text{and} \quad \lim_{n\to\infty} \|(A - \lambda)\psi_n\| = 0.$$

2.2 The Discrete Spectrum of Self-Adjoint Operators

In this section we deal with self-adjoint operators which are bounded from below. The min-max principle is an important and useful tool for studying and comparing discrete spectrum of these operators, see [RS78].

Theorem 2.2.1. *(min-max principle, operator form) Let A be a self-adjoint operator that is bounded from below, i.e., $A \geq cI$ for some c. Define*

$$\mu_n(A) = \sup_{\varphi_1, .., \varphi_{n-1}} U_A(\varphi_1, .., \varphi_{n-1})$$

where

$$U_A(\varphi_1, .., \varphi_{n-1}) = \inf_{\substack{\psi \in D(A); \|\psi\|=1 \\ \psi \in [\varphi_1, .., \varphi_{n-1}]^{\perp}}} (\psi, A\psi)$$

$[\varphi_1, .., \varphi_{n-1}]^{\perp}$ is shorthand for $\{\psi | (\psi, \varphi_i) = 0, i = 1, ..., n-1\}$. Note that the φ_i are not necessarily independent.

Then, for each fixed n, either:

a) *there are n eigenvalues (counting degenerate eigenvalues a number of times equal to their multiplicity) below the bottom of the essential spectrum, and* $\mu_n(A)$ *is the nth eigenvalue counting multiplicity;*

or

b) μ_n *is the bottom of the essential spectrum, i.e.,* $\mu_n = \inf\{\lambda | \lambda \in \sigma_{ess}(A)\}$ *and in that case* $\mu_n = \mu_{n+1} = \mu_{n+2} = \cdots$ *and there are at most* $n-1$ *eigenvalues (counting multiplicity) below* μ_n.

Proof. See [RS78], Sec. XIII. 1.

From the definition of the $\mu_n(A)$, we immediately have:

Lemma 2.2.2. *Let A and B be self-adjoint operators on* $\mathcal{H}$ *that are bounded from below. If* $A \leq B$ *then* $\mu_n(A) \leq \mu_n(B)$*, for all* $n \geq 1$.

Proof. From the definition of $A \leq B$, we have

$$(\psi, A\psi) \leq (\psi, B\psi) \qquad \forall \psi \in D(B).$$

Because $D(B) \subseteq D(A)$, then we get

$$\inf_{\substack{\psi\in D(A)\\ \|\psi\|=1}} (\psi, A\psi) \leq \inf_{\substack{\psi\in D(B)\\ \|\psi\|=1}} (\psi, A\psi) \leq \inf_{\substack{\psi\in D(B)\\ \|\psi\|=1}} (\psi, B\psi),$$

and so

$$\sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{\psi\in D(A), \|\psi\|=1\\ \psi\in[\varphi_1,..,\varphi_{n-1}]^\perp}} (\psi, A\psi) \leq \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{\psi\in D(B), \|\psi\|=1\\ \psi\in[\varphi_1,..,\varphi_{n-1}]^\perp}} (\psi, B\psi),$$

which implies

$$\mu_n(A) \leq \mu_n(B) \qquad \text{for all} \quad n \geq 1. \quad \square$$

We complete this section by giving criteria that guarantee that a semibounded self-adjoint operator has purely discrete spectrum.

Theorem 2.2.3. *Let* Λ *be a bounded open set in* $\mathbb{R}^n$*. The Dirichlet and Neumann Laplacian* $-\Delta_D^\Lambda, -\Delta_N^\Lambda$ *has compact resolvent,*

Proof. see [RS78], Sec.XIII.14.

Corollary 2.2.4. *If the Dirichlet or Neumann Lablacian has compact resolvent, then it has purely discrete spectrum and a complete set of eigenfunctions.*

2.3 The Essential Spectrum of Self-Adjoint Operators

In this section we will recall some methods of determining the essential spectrum for a self-adjoint operator A and display important properties that we will need later.

Theorem 2.3.1. *Let A be a self-adjoint operator. Then* $\lambda \in \sigma_{ess}(A)$ *if and only if there exists* $\{\psi_n\}_{n=1}^\infty$ *so that* $\|\psi_n\| = 1$, $\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0$ *and* $\{\psi_n\}$ *can be chosen so that* $\psi_n \to 0$ *weakly.*

A sequence as in Theorem 2.3.1 is called a Weyl sequence for A at λ.

Remark 2.3.2. *Weak convergence of a sequence $\psi_n \in \mathcal{H}$ to an element ψ means that*

$$\int\limits_{\mathbb{R}^d} \psi_n f d\mu \to \int\limits_{\mathbb{R}^d} \psi f d\mu$$

for all function $f \in L^2$ (or, more typically, for all f in a dense subset of L^2 such as a space of Schwarz functions, if the sequence $\{\psi_n\}$ is bounded).

Another theorem that determiner the essential spectrum for self-adjoint operators is given in [Don81] as follows:

Theorem 2.3.3. *A necessary and sufficient condition for $\lambda \in \sigma_{ess}(A)$ of the self-adjoint operator A is that, for all $\epsilon > 0$ there exist an infinite dimensional subspace $S \subset D(A)$, for which:*

$$\|(A - \lambda I)f\| \leq \varepsilon \|f\|, \quad f \in S.$$

Now, to state the classical Weyl theorem which we will use in Section 4, we need the following definition:

Definition 2.3.4. *Let A be a self-adjoint operator and $A \geq 0$. An operator B with $D(A) \subset D(B)$ is called relatively compact with respect to A if and only if $B(A+1)^{-1}$ is compact.*

Theorem 2.3.5. *(classical Weyl theorem) Let A be a self-adjoint operator and let B be a relatively compact operator with respect to A. then*

$$\sigma_{ess}(A) = \sigma_{ess}(A + B).$$

Proof. See [RS78], Sec. XIII. 4.

Remark 2.3.6. *If A is self-adjoint and B is compact then $\sigma_{ess}(A) = \sigma_{ess}(A + B)$. This holds because B is automatically relatively compact.*

Example 2.3.7. *Let $f \in C_0^\infty(\mathbb{R}^n)$, then M_f the multiplication operator is relatively compact with respect to $-\Delta$.*

The proof of the Example (see [RS78]).

Finally, by using the Weyl sequence, we get the following proposition:

Proposition 2.3.8. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and let A_1 be a self-adjoint operator on $\mathcal{H}_1$, A_2 a self-adjoint operator on $\mathcal{H}_2$. Let $A = A_1 \oplus A_2$. Then:*

$$\sigma_{ess}(A) = \sigma_{ess}(A_1) \cup \sigma_{ess}(A_2).$$

Proof. $'\subseteq'$, by the theorem 2.3.1 if $\lambda \in \sigma_{ess}(A)$ then there is a Weyl sequence $\{\psi_n\}$ for A at λ, where $\psi_n = (\psi_{1,n}, \psi_{2,n}) \in D(A)$ with $\psi_{i,n} \in D(A_i), i = 1, 2$.

So that

$$\lim_{n\to\infty} \left\|(A-\lambda)\psi_n\right\|^2 = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \left\|\big((A_1 \oplus A_2) - \lambda\big)\big(\psi_{1,n}, \psi_{2,n}\big)\right\|^2 = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \left\|(A_1-\lambda)\psi_{1,n}, (A_2-\lambda)\psi_{2,n}\right\|^2 = 0$$

$$\Leftrightarrow \lim_{n\to\infty} \left\|(A_1-\lambda)\psi_{1,n}\right\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \left\|(A_2-\lambda)\psi_{2,n}\right\|^2 = 0.$$

Since for each n, $\|\psi_{1,n}\|^2 + \|\psi_{2,n}\|^2 = 1$, therefore $\|\psi_{1,n}\|^2 \geq \frac{1}{2}$ or $\|\psi_{2,n}\|^2 \geq \frac{1}{2}$.

Now, let

$$S_1 = \big\{m_1 : \|\psi_{1,m_1}\|^2 \geq \frac{1}{2}\big\}$$
$$S_2 = \big\{m_2 : \|\psi_{2,m_2}\|^2 \geq \frac{1}{2}\big\}$$

then $S_1 \cup S_2 = \mathbb{N}$ therefore, S_1 or S_2 must be infinite. Then it follows from theorem 2.1.6 that $\lambda \in \sigma(A_1)$ or $\lambda \in \sigma(A_2)$.

Now, without loss of generality, assume that S_1 is infinite that follows $\lambda \in \sigma(A_1)$ and there exists $\psi_{1,n}$ satisfying the Weyl's criterion. Since λ also in $\sigma_{ess}(A)$ then, for all $f = (f_1, f_2) \in \mathcal{H}$, we have

$$\langle f, \psi_n\rangle \xrightarrow[n\to\infty]{} 0.$$

Let $f = (f_1, 0)$, we get

$$\big\langle (f_1, 0), (\psi_{1,n}, \psi_{2,n})\big\rangle \xrightarrow[n\to\infty]{} 0 \Leftrightarrow \langle f_1, \psi_{1,n}\rangle \xrightarrow[n\to\infty]{} 0.$$

It follows that $\lambda \in \sigma_{ess}(A_1)$.
Similarly, by assume that $\lambda \in \sigma(A_2)$ we get as above that $\lambda \in \sigma_{ess}(A_2)$.

$'\supseteq'$, let $\lambda \in \sigma_{ess}(A_1) \cup \sigma_{ess}(A_2)$ then,
either, $\lambda \in \sigma_{ess}(A_1)$ by theorem 2.3.1 is equivalent to the existence of a Weyl sequence $\{\psi_{1,n}\} \in \mathcal{H}_1$ for A_1 at λ. Then it is easy to check that $\{\psi_n\} = \{(\psi_{1,n}, 0)\} \in \mathcal{H}$ is a Weyl sequence for A at λ. so we get $\lambda \in \sigma_{ess}(A)$.
or, $\lambda \in \sigma_{ess}(A_2)$, similarly as above we get $\lambda \in \sigma_{ess}(A)$. $\square$

2.4 The Essential Spectrum of the Dirichlet Laplacian on the Planar Layer $\Omega_0 = \mathbb{R}^2 \times (-a, a)$

First, we compute the essential spectrum of the Laplacian $-\Delta$ on $\mathbb{R}^d$.

Lemma 2.4.1. *Let $-\Delta$ be the Laplacian considered as self-adjoint operator in $L^2(\mathbb{R}^d)$ with domain the Sobolev space $H^2(\mathbb{R}^d)$. Then $\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0, \infty)$.*

Proof. We first consider $\nu \geq 0$. The functions $w(x) = e^{ix\xi}$ are the solutions to

$$-\Delta w = \nu w \quad \text{for} \quad \nu = |\xi|^2.$$

For fixed $\nu \in [0, \infty)$, we take

$$\psi_n(x) = \frac{1}{n^{\frac{d}{2}}}\zeta\Big(\frac{x}{n}\Big)w(x) = \frac{1}{n^{\frac{d}{2}}}\zeta\Big(\frac{x}{n}\Big)e^{ix\xi},$$

where $\zeta(x)$ is a smooth function supported on a cube $[1, 2]^d$ and

$$\int\limits_{\mathbb{R}^d} \big|\zeta(x)\big|^2 dx = 1,$$

then $\zeta\big(\frac{x}{n}\big)$ is supported on $[n, 2n]^d$ and

$$\frac{1}{n^d}\int\limits_{\mathbb{R}^d} \Big|\zeta\Big(\frac{x}{n}\Big)\Big|^2 dx = 1.$$

Because $\psi_n(x) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we have $\psi_n(x) \in H^2(\mathbb{R}^d)$.

Now, we have

$$\big\|\psi_n(x)\big\|^2 = \frac{1}{n^d}\int\limits_{\mathbb{R}^d} \Big|\zeta\Big(\frac{x}{n}\Big)e^{ix\xi}\Big|^2 dx$$

$$= \frac{1}{n^d}\int\limits_{\mathbb{R}^d} \Big|\zeta\Big(\frac{x}{n}\Big)\Big|^2\Big|e^{ix\xi}\Big|^2 dx,$$

where $|e^{ix\xi}|^2 = 1$ and with the properties of ζ it follows

$$\big\|\psi_n(x)\big\|^2 = \frac{1}{n^d}\underbrace{\int\cdots\int}_{d-times}\Big|\zeta\Big(\frac{x}{n}\Big)\Big|^2 dx_1\dots dx_d = 1.$$

Moreover,

$$\Big\|(-\Delta - \nu)\psi_n(x)\Big\|^2 = \frac{1}{n^d}\int\limits_{\mathbb{R}^d}\Big| -\Delta\Big(\zeta\Big(\frac{x}{n}\Big)\Big)e^{ix\xi} - 2i\xi\nabla\Big(\zeta\Big(\frac{x}{n}\Big)\Big)e^{ix\xi} + \nu\zeta\Big(\frac{x}{n}\Big)e^{ix\xi} - \nu\zeta\Big(\frac{x}{n}\Big)e^{ix\xi}\Big|^2 dx$$

$$= \frac{1}{n^d}\int\limits_{\mathbb{R}^d}\Big|\frac{1}{n^2}\Big(-\Delta\zeta\Big)\Big(\frac{x}{n}\Big) - \frac{2i\xi}{n}\Big(\nabla\zeta\Big)\Big(\frac{x}{n}\Big)\Big|^2\Big|e^{ix\xi}\Big|^2 dx$$

$$= \int\limits_{\mathbb{R}^d}\Big|\frac{1}{n^2}\big(-\Delta\zeta\big)(y) - \frac{2i\xi}{n}\big(\nabla\zeta\big)(y)\Big|^2 dy,$$

hence

$$\lim_{n\to\infty} \left\|(-\Delta-\nu)\psi_n(x)\right\|^2 = 0,$$

thus $\nu \in \sigma(-\Delta)$.

Now we show that $\psi_n \to 0$ weakly. Indeed, for any function $f \in \mathcal{S}(\mathbb{R}^d)$, the set of Schwarz functions, then there are constants M_d such that

$$\left|f(x)\right| \leq M_d\left|x\right|^{-d} \quad \text{as } x \to \infty.$$

Then, by using the Schwarz inequality and polar coordinates, we get

$$\begin{aligned}
\left|\int\limits_{\mathbb{R}^d} \psi_n f dx\right| &= \left|\int\limits_{|x|\in[n,2n]} \psi_n f dx\right| \\
&\leq \Bigg(\int\limits_{|x|\in[n,2n]} \left|\psi_n\right|^2 dx\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{|x|\in[n,2n]} \left|f\right|^2 dx\Bigg)^{\frac{1}{2}} \\
&\leq M_d \Bigg(\int\limits_{|x|\in[n,2n]} \left|x\right|^{-2d} dx\Bigg)^{\frac{1}{2}} \\
&= M_d \Bigg(\int\limits_{n}^{2n} r^{-2d} r^{d-1} dr\Bigg)^{\frac{1}{2}}
\end{aligned}$$

and an easy computation shows that

$$\int\limits_{n}^{2n} r^{-d-1} dr = -\frac{1}{d}\left[\frac{1-2^d}{(2n)^d}\right]$$

which tends to 0 as $n \to \infty$. This implies $\nu \in \sigma_{ess}(-\Delta)$, and so

$$[0,\infty) \subset \sigma_{ess}(-\Delta).$$

Moreover, since $-\Delta$ is self-adjoint, it holds $\sigma(-\Delta) \subset \mathbb{R}$, and because of $-\Delta \geq 0$, i.e., $(-\Delta\phi, \phi) \geq 0$, we have $\sigma(-\Delta) \subset [0,\infty)$. Then

$$\sigma_{\mathrm{ess}}(-\Delta) \subset [0,\infty).$$

From the above it now follows that

$$\sigma_{\mathrm{ess}}(-\Delta) = [0,\infty). \quad \square$$

Finally, we will use the argument above in combination with some basic tools from the theory of partial differential equations to prove the following theorem:

Lemma 2.4.2. *The spectrum of* $-\Delta$ *on the planar layer* $\Omega_0 := \mathbb{R}^2 \times (-a,a)$, *with Dirichlet conditions is purely essential and coincides with the interval* $\left[(\frac{\pi}{2a})^2, \infty\right)$.

Proof. In order to show this, we first look for solutions to

$$\begin{cases} -\Delta\phi(x,u) = \lambda\phi(x,u), & \text{in} \quad \mathbb{R}^2 \times (-a,a) \\ \phi(x,u) = 0, & \text{on} \quad \partial\big(\mathbb{R}^2 \times (-a,a)\big) \end{cases} \tag{2.4.1}$$

which have the multiplicative form

$$\phi(x,u) = \eta(x)\chi(u),$$

where $\phi(x,u)$ have separated variables $x \in \mathbb{R}^2$ and $u \in (-a,a)$.

Then, we have

$$-\Delta\phi(x,u) = -\Delta\eta(x)\chi(u) - \eta(x)\chi''(u),$$

hence,

$$-\Delta\eta(x)\chi(u) - \eta(x)\chi''(u) = \lambda\eta(x)\chi(u).$$

This holds if and only if

$$-\frac{\Delta\eta(x)}{\eta(x)} - \lambda = \frac{\chi''(u)}{\chi(u)} \tag{2.4.2}$$

for all $x \in \mathbb{R}^2$ and $u \in (-a,a)$ such that $\eta(x), \chi(u) \neq 0$.

Now observe that the left hand side of (2.4.2) depends only on x and the right hand side depends only on u. This is impossible unless each side is constant, say

$$-\frac{\Delta\eta(x)}{\eta(x)} - \lambda = -\mu = \frac{\chi''(u)}{\chi(u)} \quad \text{for all } x \in \mathbb{R}^2 \text{ and } u \in (-a,a).$$

Then

$$\Delta\eta(x) + (\lambda - \mu)\eta(x) = 0 \tag{2.4.3}$$

$$\chi''(u) + \mu\chi(u) = 0. \tag{2.4.4}$$

We must solve these equations for η, χ and μ.

Notice that Dirichlet conditions for $\phi(x,u)$ at $u = \pm a$ imply

$$\chi(a) = \chi(-a) = 0,$$

thus a solution of (2.4.4) is of the form

$$\chi_k(u) = \sqrt{\frac{1}{a}} \sin\Big((u+a)\frac{\pi k}{2a}\Big),$$

for $k = 1, 2, ...,$ (see [Cha84]).

Using (2.4.4) to compute μ_k, we arrive at

$$-\Big(\frac{\pi k}{2a}\Big)^2 \sqrt{\frac{1}{a}} \sin\Big((u+a)\frac{\pi k}{2a}\Big) + \mu_k \cdot \sqrt{\frac{1}{a}} \sin\Big((u+a)\frac{\pi k}{2a}\Big) = 0,$$

hence

$$\mu_k = \Big(\frac{\pi k}{2a}\Big)^2. \tag{2.4.5}$$

We now divide the rest of the proof into two parts:

1. $\big[(\frac{\pi}{2a})^2, \infty\big) \subseteq \sigma_{ess}(-\Delta_D^{\Omega_0})$
2. $\sigma_{ess}(-\Delta_D^{\Omega_0}) \subseteq \big[(\frac{\pi}{2a})^2, \infty\big)$

1. If $\lambda \in \big[(\frac{\pi}{2a})^2, \infty\big)$ and following $\lambda \geq \mu_1$ then from the Lemma 2.4.1 there is a Weyl sequence ψ_n for $-\Delta$ at $\lambda - \mu_1$ on $\mathbb{R}^2$.

   We now seek a Weyl sequence $\{\Psi_N\}$ for $-\Delta$ at λ. Let

$$\Psi_N(x,u) = \psi_N(x)\chi_1(u)$$

   where

$$\begin{aligned}\chi_1(u) &= \sqrt{\frac{1}{a}}\sin\Big(\frac{\pi}{2a}u + \frac{\pi}{2}\Big)\\ &= \sqrt{\frac{1}{a}}\cos\Big(\frac{\pi}{2a}u\Big).\end{aligned}$$

   We see that,

$$\big\|\Psi_n(x)\big\|^2 = \int_{\mathbb{R}^2}\int_{-a}^{a} \big|\psi_n(x)\chi_1(u)\big|^2 dxdu$$

$$= \int_{\mathbb{R}^2} \big|\psi_n(x)\big|^2 dx \int_{-a}^{a} \big|\chi_1(u)\big|^2 du = 1$$

   Moreover,

$$\big\|(-\Delta-\lambda)\Psi_n(x,u)\big\|^2 = \int_{\mathbb{R}^2}\int_{-a}^{a} \Big|\big(-\Delta\psi_n(x)\big)\chi_1(u) + \psi_n(x)\big(-\Delta\chi_1(u)\big) - \lambda\Psi_n(x,u)\Big|^2 dxdu$$

$$= \int_{\mathbb{R}^2}\int_{-a}^{a} \Big|\big(-\Delta\psi_n(x)\big)\chi_1(u) + \mu_1\psi_n(x)\chi_1(u) - \lambda\Psi_n(x,u)\Big|^2 dxdu$$

$$= \int_{\mathbb{R}^2} \Big|\big(-\Delta-(\lambda-\mu_1)\big)\psi_n(x)\Big|^2 dx \int_{-a}^{a} \big|\chi_1(u)\big|^2 du$$

   since $\psi_n(x)$ is a weyl sequence of $\lambda - \mu$, and

$$\int_{-a}^{a} \big|\chi_1(u)\big|^2 du = 1.$$

   Then

$$\lim_{n\to\infty} \Big\|\big(-\Delta-(\lambda-\mu)\big)\psi_n(x,u)\Big\| = 0.$$

and it follows that

$$\lim_{n\to\infty}\Big\|\big(-\Delta-\lambda\big)\Psi_n(x,u)\Big\|=0.$$

Thus $\lambda\in\sigma(-\Delta_D^{\Omega_0})$ which shows that

$$\Big[\big(\frac{\pi}{2a}\big)^2,\infty\Big)\subseteq\sigma(-\Delta_D^{\Omega_0}) \tag{I}$$

The proof of the first part is completed by showing that Ψ_n tends weakly to 0. Let $f(x,u)\in\mathcal{S}\big(\mathbb{R}^n\times(-a,a)\big)$. Similarly as above

$$\Big|\int\limits_{\mathbb{R}^2}\int\limits_{-a}^{a}\overline{\Psi_n(x,u)}f(x,u)\Big|\xrightarrow[n\to\infty]{}0.$$

This implies that $\lambda-\mu\in\sigma_{ess}(-\Delta)$ and therefore,

$$\Big[\Big(\frac{\pi}{2a}\Big)^2,\infty\Big)\subseteq\sigma_{ess}(-\Delta_D^{\Omega_0}). \tag{II}$$

2. To prove the second part of the theorem, we want to show that $\inf\sigma(-\Delta_D^{\Omega_0})\geq(\frac{\pi}{2a})^2$. We have:

$$\big(-\Delta\phi(x,u),\phi(x,u)\big)=\int\limits_{\mathbb{R}^2}\int\limits_{-a}^{a}\big|\nabla\phi(x,u)\big|^2dxdu$$

$$=\int\limits_{\mathbb{R}^2}\int\limits_{-a}^{a}\Big[\big|\nabla_{\mathbb{R}^2}\phi(x,u)\big|^2+\big|\nabla_u\phi(x,u)\big|^2\Big]dxdu \tag{*}$$

Since

$$\inf\sigma_{ess}\Big(-\frac{d^2}{du^2}\Big)_D=\inf_{\substack{\psi:\psi\neq0\\ \psi(.,\pm a)=0}}\frac{\int_{-a}^{a}\big|\nabla_u\phi(x,u)\big|^2du}{\int_{-a}^{a}\big|\phi(x,u)\big|^2du}=\Big(\frac{\pi}{2a}\Big)^2,$$

then, for any ϕ, $\phi(.,\pm a)=0$

$$\int\limits_{-a}^{a}\big|\nabla_u\phi(x,u)\big|^2du\geq\Big(\frac{\pi}{2a}\Big)^2\int\limits_{-a}^{a}\big|\phi(x,u)\big|^2du,$$

then

$$\int\limits_{\mathbb{R}^2}\int\limits_{-a}^{a}\big|\nabla_u\phi(x,u)\big|^2dxdu\geq\Big(\frac{\pi}{2a}\Big)^2\int\limits_{\mathbb{R}^2}\int\limits_{-a}^{a}\big|\phi(x,u)\big|^2dxdu=\Big(\frac{\pi}{2a}\Big)^2\big\|\phi(x,u)\big\|^2,$$

and the first term of $(\star)$ is non-negative, we obtain

$$(-\Delta\phi,\phi)\geq\Big(\frac{\pi}{2a}\Big)^2\|\phi\|^2$$

i.e., $\inf\sigma(-\Delta_D^{\Omega_0})\geq\big(\frac{\pi}{2a}\big)^2$, and hence

$$\sigma(-\Delta_D^{\Omega_0})\subseteq\Big[\Big(\frac{\pi}{2a}\Big)^2,\infty\Big). \tag{III}$$

From (I), (II) and (III) it follows that

$$\sigma_{ess}(-\Delta_D^{\Omega_0}) = \left[\left(\frac{\pi}{2a}\right)^2, \infty\right)$$

and so the spectrum of the planar layer $\Omega_0 := \mathbb{R}^2 \times (-a, a)$ is purely essential and coincides with the interval $\left[\left(\frac{\pi}{2a}\right)^2, \infty\right)$.

Definition 2.4.3. *The spectral threshold κ_1^2 of the planar layer of width $2a$ is the first eigenvalue of the Dirichlet Laplacian, i.e., $\kappa_1^2 := \left(\frac{\pi}{2a}\right)^2$.*

In what follows we will use the corresponding normalized eigenfunction given explicitly by

$$\chi_1(u) := \sqrt{\frac{1}{a}} \cos \kappa_1 u.$$

Chapter 3

The Topology of Surfaces

In mathematics, specifically, in topology, a surface is a 2-dimensional topological manifold such that each point has a neighbourhood homeomorphic to the Euclidean plane.

In the field of topology, one common problem is to find a classification of various topological spaces. For compact 2-manifolds there is a classification theorem , giving a simple procedure for obtaining all possible compact 2-manifolds up to homeomorphy. Moreover, by using the Euler characteristic for surfaces, this theorem enables us to decide whether or not any two compact 2-manifolds are homeomorphic. This may be considered an ideal theorem.

In this chapter, we define the Euler characteristic of a topological space, then we calculate it in the special case of a triangulated compact surface, where this surface is defined below. After that we display the classification theorem for compact connected surfaces and for compact , connected surfaces with boundary. Finally, we give some remarks on non-compact surfaces and use these tools to study some important topological properties of our reference surface. Recall that this is a complete, connected and non-compact surface with a finite total Gauss curvature.

For more details about the topology of surfaces, see [Mas89]. There all theorems given below are proved (unless stated otherwise) and it is this monograph the figures were taken from.

3.1 The Euler Characteristic of a Topological Space

The most familiar definition of the Euler characteristic of a topological space is as follows

Definition 3.1.1. *The Euler characteristic of a topological space X, denoted by $\chi(X)$, is defined as the alternating sum of Betti numbers*

$$\chi(X) = \sum_{k\geq 0}(-1)^k b_k(X),$$

where the k-th Betti number $b_k(X)$ is the dimension of the homology group $H_k(X)$. $\chi(X)$ is only defined for spaces X which satisfy $b_k(X) < \infty$, for all k, and $b_k(X) = 0$ for $k \geq k_0$ for some $k_0 \in \mathbb{N}$.

For a surface, $b_i = 0$ for $i > 2$, therefore its Euler characteristic is $\chi = b_0 - b_1 + b_2$. An informal interpretation of the first three Betti numbers is:

. b_0 is the number of connected components

. b_1 is the number of two-dimensional or "circular" holes

. b_2 is the number of three-dimensional holes or "voids".

The calculation of Betti numbers for connected surfaces is studied in [Spr81], we only want to give the following theorem since we will need it in our work:

Theorem 3.1.2. *For any connected surface , $b_0 = 1$. For any compact orientable surface, $b_2 = 1$. For any non-compact orientable surface, $b_2 = 0$.*

Proof. See [Spr81], chapter 5.

3.2 The Euler Characteristic of a Triangulated Compact Surface

In this section we assume that the given surface is triangulated, i.e., divided up into triangles which fit together nicely. Such a subdivision is very useful in the study of compact surfaces in general.

Definition 3.2.1. *A triangulation of a compact surface S consists of a finite family of closed subsets $\{T_1, T_2, \ldots, T_n\}$ that cover S, and a family of homeomorphisms $\varphi_i : T_i' \to T_i, i = 1, \ldots, n$, where each T_i' is a triangle in the plane $\mathbb{R}^2$ (i.e., a compact subset of $\mathbb{R}^2$ bounded by three distinct straight lines). The T_i are called "triangles". The subsets of the T_i that are the images of the vertices and edges of the triangle T_i' under φ_i are also called vertices and edges, respectively. Finally, it is required that any two distinct triangles, T_i and T_j, either be disjoint, have a single vertex in common, or have one entire edge in common.*

Perhaps the conditions in the definition are clarified by Figure 3.2.1, which shows three unallowable types of intersection of triangles.

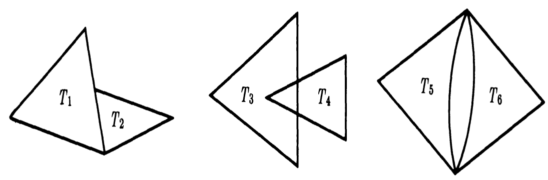


Figure 3.2.1: Some types of intersection forbidden in a triangulation.

Theorem 3.2.2. *(T.Rado 1925) For any compact surface S, there exists a triangulation of S.*

Now, we calculate the Euler characteristic of a triangulated surface.

Theorem 3.2.3. *Let S be a compact surface with triangulation $\{T_1, \ldots, T_n\}$. Let*

$$v = \text{total number of vertices of } S,$$

$$e = \text{total number of edges of } S,$$

$$t = \text{total number of triangles (in this case, } t = n).$$

Then the Euler characteristic of S is

$$\chi(S) = v - e + t \tag{3.2.1}$$

Corollary 3.2.4. *The number $v - e + t$ depends only on S, not on the triangulation chosen.*

Definition 3.2.5. *The quotient space of the 2-sphere S^2 obtained by identifying every pair of diametrically opposite points is called the projective plane. We shall also refer to any space homeomorphic to this quotient space as the projective plane.*

Example 3.2.6. *Figure 3.2.2 suggests uniform methods of triangulating the sphere, tours, and projective plane so that we may make the number of triangles as large as we please. Using such triangulations, we see that the Euler characteristics of the sphere, torus, and projective plane are $2, 0$, and 1, respectively. It also shows that the Euler characteristics are independent of the number of vertical and horizontal dividing lines in the diagrams for the sphere and torus, and of the number of radial lines or concentric circles in the case of the diagram for the projective plane.*

3.3 Statement of Classification theorem of Compact, Connected Surfaces

Our present goal is to find an easy method for distinguishing between compact surfaces. In order to state our main result we first need to define the connected sum of two such surfaces.

Definition 3.3.1. *Let S_1 and S_2 be disjoint compact surfaces. The connected sum of S_1 and S_2, denoted $S_1 \# S_2$, is constructed by removing a disk from each one, and then joining them along the boundaries of the holes.*

To be precise see figure 3.3.1 for a schematic display of the connected sum of two tori.

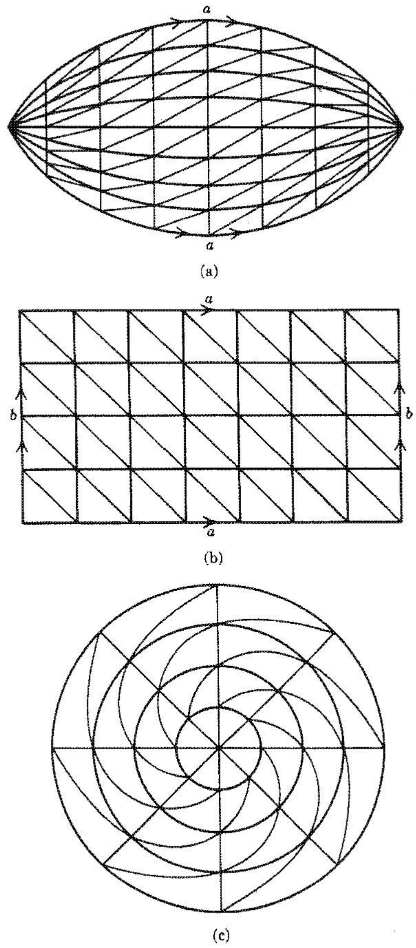


Figure 3.2.2: Computing the Euler characteristic from a triangulation. (a) Sphere. (b) Torus. (c) Projective plane.

Proposition 3.3.2. *Let S_1 and S_2 be compact surfaces. The Euler characteristic of S_1 and S_2 is related to that of the connected sum, $S_1 \# S_2$, by the formula*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proof. Assume S_1 and S_2 are triangulated. Form their connected sum by removing from each the interior of a triangle, and then identifying edges and vertices of the boundaries of the removed triangles. The formula then follows by counting vertices, edges, and triangles before and after the formation of the connected sum. More specifically, given equation (3.2.1), by identifying three pairs of vertices and three pairs of edges, we have

$$\chi(S_1 \# S_2) = (v_1 + v_2 - 3) - (e_1 + e_2 - 3) + (f_1 - 1 + f_2 - 1)$$

where v_1, e_1, and f_1 are the numbers of vertices, edges, and faces, respectively, of S_1, and v_2, e_2 and f_2 are the numbers of vertices, edges, and faces, respectively, of S_2. simplifying

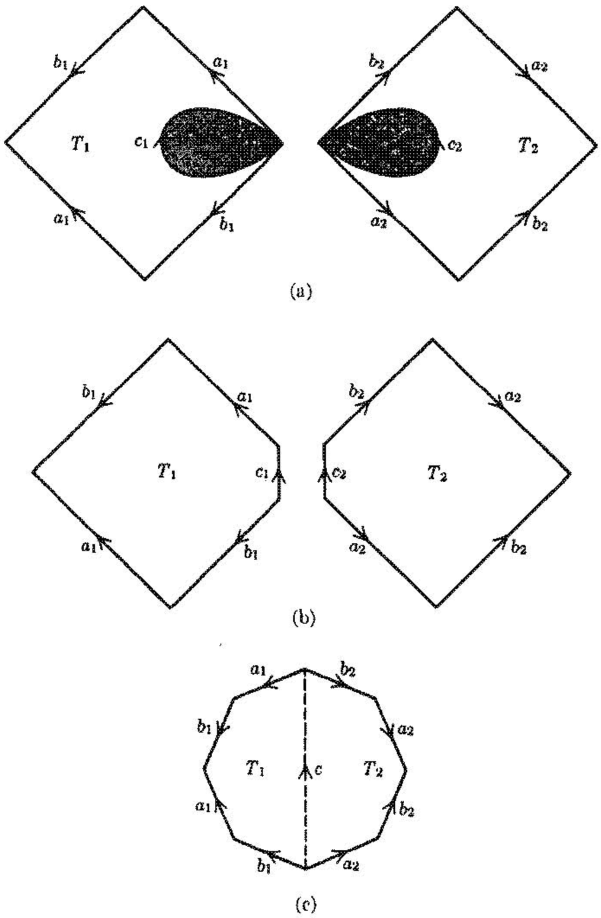


Figure 3.3.1: (a) Two disjoint tori, T_1 and T_2. (b) Disjoint tori with holes cut out. (c) The connected sum $T_1 \# T_2$.

the right hand-side, we have

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

which is the desired result. □

We now return to our main problem, namely finding all compact connected surfaces. Using connected sums there appear to be infinitely many different surfaces. The surfaces obtained in this way are indeed all distinct, as is stated in the following theorem.

Theorem 3.3.3. *(Classification Theorem For Compact Surfaces) Any compact, orientable surface is homeomorphic to a sphere or a connected sum of tori. Any compact, nonori-*

entable surface is homeomorphic to the connected sum of projective planes. And the number of tori respectively projective planes is uniquely determined by the surface.

Then, we have the following important result:

Theorem 3.3.4. *Let S_1 and S_2 be compact surfaces. Then, S_1 and S_2 are homeomorphic if and only if their Euler characteristics are equal and both are orientable or both are nonorientable.*

This is a topological theorem par excellence; it reduces the classification problem for compact surfaces to the determination of the orientability and Euler characteristic. Moreover, Theorem 3.3.4 makes clear what are all possible compact surfaces.

We close this section by giving some standard terminology. A surface that is the connected sum of n tori or n projective planes is said to be of **genus** n, whereas a sphere is of genus 0. By using the Proposition 3.3.2, we can easily prove The relation between the genus g and the Euler characteristic χ of a compact surface:

$$\chi = \begin{cases} 2-2g & \text{in the orientable case,} \\ 2-g & \text{in the nonorientable case.} \end{cases}$$

3.4 The Classification of Compact, Connected Surfaces with Boundary

We define the notion of a manifold with boundary, which is a slight generalization of that of a manifold.

Definition 3.4.1. *A topological n-dimensional manifold with boundary is a Hausdorff space such that each point has an open neighborhood homeomorphic either to the open disk $U^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x| < 1\}$ or to the space $\{x \in U^n : x_1 \geq 0\}$.*
The set of all points that have an open neighborhood homeomorphic to U^n is called the interior of the manifold, and the set of those points p that have an open neighborhood V such that there exists a homeomorphism h of V onto $\{x \in U^n : x_1 \geq 0\}$ with $h(p) = (0, 0, \ldots, 0)$ is called boundary of the manifold.

So we obtain a compact surface with boundary by selecting a finite number of disjoint closed discs in a compact surface and removing their interiors. The number of boundary components is equal to the number of discs chosen.
Conversely, assume that S is a compact surface with boundary and that the boundary has k components $k \geq 1$. Each boundary component is a compact, connected $1-$manifold, i.e., a circle. It is clear that we obtain a compact surface S^* if we take k closed discs and glue the boundary of the ith disc to the ith component of the boundary of S. The topological type of the resulting surface S^* obviously depends only on the topological type of S. What is not so obvious is that a sort of converse statement is true: The topological type of the surface with boundary S depends only on the number of its boundary components and the topological type of the surface S^* obtained by gluing a disc onto each boundary component.
We can state this in another way: If we start with a compact surface S^* and construct a surface with boundary by removing the interiors of k closed discs, which are pairwise

disjoint, then the location of the discs that are to be removed does not matter. The resulting manifold with boundary will be topologically the same no matter how the position of the discs is chosen. We will state this result formally as follows:

Theorem 3.4.2. *Let S_1 and S_2 be compact, connected surfaces with boundary and assume that their boundaries have the same number of components. Then, S_1 and S_2 are homeomorphic if and only if the surfaces S_1^* and S_2^* (obtained by gluing a disc to each boundary component) are homeomorphic.*

The Euler characteristic of a triangulated surface with boundary is defined in exactly the same way as in the case of a surface without boundary. With the use of the Euler characteristic, we can give a complete set of invariants for the classification of compact surfaces with boundary:

Theorem 3.4.3. *Let S_1 and S_2 be compact, connected surfaces with boundary. Then, S_1 and S_2 are homeomorphic if and only if they have the same number of boundary components, they are both orientable or nonorientable, and they have the same Euler characteristic.*

Proof. Let S be a compact, connected surface, with or without boundary. Assume that S is given a definite triangulation, and that we form a new surface S' with boundary by removing the interior of one triangle, which is contained entirely in the interior of S. It is clear that the boundary of S' has one more component than the boundary of S, and that

$$\chi(S') = \chi(S) - 1,$$

i.e., the Euler characteristic is reduced by one.

It follows that, if we start with a triangulated surface S^* (without boundary), and remove the interiors of k pairwise disjoint triangles, we obtain a surface S with boundary such that

$$\chi(S) = \chi(S^*) - k.$$

According to the explanations before Theorem 3.4.2, we obtain in this way every surface with boundary S whose boundary has k components. Thus, we see that the Euler characteristic of S uniquely determines that of S^* and vice versa.

Now, assume that S_1 and S_2 are homeomorphic, then the boundary of S_1 is homeomorphic to the boundary of S_2, it follows that they have the same number of boundary components. By Theorem 3.4.2, S_1^* and S_2^* are homeomorphic and by Theorem 3.3.4 they have the same Euler characteristic and both are either orientable or non-orientable. Because

$$\begin{aligned} \chi(S_1) &= \chi(S_1^*) - k, \\ \chi(S_2) &= \chi(S_2^*) - k, \end{aligned} \tag{3.4.1}$$

S_1 and S_2 have the same Euler characteristic, and both are orientable or non-orientable.

Conversely, let S_1 and S_2 have the same number of boundary components and the same Euler characteristic and assume both are orientable or non-orientable. Then by equations

(3.4.1), S_1^* and S_2^* have the same Euler characteristic and both are orientable or non-orientable. It follows from Theorem 3.3.4 that they are homeomorphic and from Theorem 3.4.2 that S_1 and S_2 homeomorphic. $\square$

Definition 3.4.4. *The genus of a compact surface S with boundary is defined to be the genus of the compact surface S^* obtained by attaching a disc to each boundary component of S.*

3.5 Remarks on Non-Compact Surfaces

We can divide non-compact surfaces into two large classes: those that have a countable basis for their topology, and those that do not. Usually, it is assumed that there is a countable basis of open sets and a theorem of Rado' asserts that a surface can be triangulated if and only if its topology has a countable basis.

Triangulation of a non-compact surface means the same as triangulation of a compact surface, except that the number of triangles is necessarily infinite, and that it is further required that each point has a neighborhood that meets only finitely many triangles.

The existence of triangulations for surfaces having a countable basis is very important and many of the known results in the subject are only proved by using this fact. For the remainder of this chapter we shall only consider such surfaces.

Because there is a classification theorem for compact surfaces, it is natural to inquire wether or not there is a classification theorem for non-compact surfaces. Actually, there is a classification theorem, but it does not seem to be easily applicable to problems that arise in the subject. Although it would take us too far afield to give all the details, we can explain the idea behind this theorem.

Definition 3.5.1. *Let S be a non-compact surface. As usual, by a compactification of S we mean a compact Hausdorff space X, which contains S as an open, dense subspace. Two compactifications, X and Y, are regarded as equivalent if there exists a homeomorphism h of X onto Y such that $h|S$ is the identity map.*

Example 3.5.2. *Let S' be a compact surface with boundary, and let S be its interior. Then S' is a compactification of S.*

To state our next theorem, we need one more definition.

Definition 3.5.3. *Let X be a topological space and let A be a subspace. A is said to be nonseparating on X if, for any open connected subset U of X, $U - A$ is connected.*

Theorem 3.5.4. *Let S be a non-compact surface. There exists a compactification X of S, which has the following three properties:*

1) X is locally connected, i.e., for any point x of X and neighborhood U of x, there is a neighborhood V of x such that V is connected and contained in U.

2) $\beta(S) = X - S$ is totally disconnected, i.e., the connected components in $\beta(S)$ are one-point sets.

3) $\beta(S)$ is nonseparating on X.

Moreover, any two compactifications X_1 and X_2 of S having these three properties are equivalent.

Example 3.5.5. *Let S' be a compact, connected surface, and let A be a closed, totally disconnected subset of S'. For example, A could a finite subset. Let $S = S' - A$, then it is plausible, and can be proved, that S' is a compactification of S having the three properties stated in Theorem 3.5.4. Hence, we may take $X = S'$ and $\beta(S) = A$. In general, however, X will not be a surface.*

The space $\beta(S)$ is called the ideal boundary or set of ends of S, its points called boundary components or ends. Correspondingly, X is sometimes called the end-compactification of S.

3.6 The Topology of the Reference Surface Σ

Now, we go back to our reference surface Σ to give more remarks on the relationship between the total Gauss curvature and the Euler characteristic of the surface. Recall that Σ is assumed to be a complete, orientable, connected, non-compact surface with finite total Gauss curvature.

The famous Gauss-Bonnet Theorem asserts that the total Gauss curvature of a compact orientable two-dimensional manifold without boundary is a constant multiple of its Euler characteristic. If a surface is complete and noncompact but has integrable Gauss curvature, then the total Gauss curvature is no longer a completely topological invariant. In 1957, Huber proved the following:

Theorem 3.6.1. *A complete, non-compact surface Σ with integrable curvature is conformally equivalent to a compact Riemannian surface with finitely many punctures (removing points).*

In particular, Huber also showed that in this case

$$\int_\Sigma K \leq 2\pi\chi(\Sigma), \tag{3.6.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, we will define it later. This theorem is related to Theorem 3.5.4: There, the compact Riemannian surface is X and the finitely many punctures are $\beta(S)$. Moreover, the finitely many punctures correspond to the ends of Σ, i.e., for any puncture p there is a corresponding end $E_p = B \setminus \{p\}$, where B is a geodesic ball around p in X. Since

$$\chi(\Sigma) = \chi(\text{of a compact Riemannian surface with boundary}),$$

by Theorem 3.4.3

$$\chi(\Sigma) = \chi(\text{of a compact Riemannian surface}) - (\text{the number of boundary component}),$$

and it follows that

$$\chi(\Sigma) = 2 - 2g - e, \tag{3.6.2}$$

where e is the number of ends (the number of the boundary components), and g is the genus of Σ.

Let us denote the ends by $\{E_1, \ldots, E_k\}$ and define the corresponding isoperimetric constants λ_i by

$$\lambda_i = \lim_{r \to \infty} \frac{\text{area}(B_r \cap E_i)}{\pi r^2},$$

relative to any fixed point $p \in \Sigma$ with respect to which the geodesic ball B_r of radius r is taken. It is easy to show that this definition is independent of the specific choice of ends E_i.

The contribution of the ends to the deficit in (3.6.1) is given by the following formula.

Theorem 3.6.2 (Hartman)**.** *Let Σ be a complete, non-compact surface with integrable Gauss curvature. Then*

$$\int_\Sigma K = 2\pi\Big(\chi(\Sigma) - \sum_{i=1}^{k} \lambda_i\Big), \tag{3.6.3}$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface.

Remark 3.6.3. *Since the Gauss curvature of Σ is assumed to be integrable, $\chi(\Sigma)$ is finite and it follows that Σ must have finite topological type: it must have finite genus and only finitely many ends.*

Remark 3.6.4. *We defined the Euler characteristic by $\chi(\Sigma) = \sum(-1)^k b_k$, where b_k is the k-th Betti number, and we saw that for a connected, non-compact surface $\chi(\Sigma) = 1 - b_1$, where $b_1 = 2g$, and in the case that Σ also has integrable Gauss curvature we see from Hartman's formula that, in fact,*

$$\int_\Sigma K \leq 2\pi. \tag{3.6.4}$$

Moreover, if we assume that $\int_\Sigma K > 0$, then by (3.6.3) we must have $b_1 = 0$ as well. This means, $\chi(\Sigma) = 1$, which via the uniformization theorem for surfaces "Any simply connected Riemannian surface is conformally equivalent to the unit sphere in Euclidean 3-space, the Euclidean plane or the Euclidean disc" implies that Σ is conformally equivalent to $\mathbb{R}^2$. Therefore, we see that positive total Gauss curvature surfaces are topologically very simple, they are essentially just $\mathbb{R}^2$.

Chapter 4

The Essential Spectrum of the Dirichlet Laplacian on Quantum Layers

It was shown in Chapter 2 that the essential spectrum of the planar layer $\Omega_0 = \mathbb{R}^2 \times (-a, a)$ starts from the lowest eigenvalue $\kappa_1^2 = (\frac{\pi}{2a})^2$. In this section we will compute the infimum of the essential spectrum of a quantum layer Ω around a surface which is asymptotically planar.

We shall localize the essential spectrum of $-\Delta_D^\Omega$ for asymptotically planar layers, so that the curvatures of the underlying surface vanish at infinity :

$$K, M \xrightarrow{\infty} 0 \tag{4.0.1}$$

Here, a function f defined on a non-compact manifold Σ is said to vanish at infinity if

$$\forall \varepsilon > 0 \quad \exists R_\varepsilon > 0, x_\varepsilon \in \Sigma \quad \forall x \in \Sigma \backslash \overline{B(x_\varepsilon, R_\varepsilon)} : \big|f(x)\big| < \varepsilon, \tag{4.0.2}$$

where $B(x_\varepsilon, R_\varepsilon)$ denotes the open ball of center x_ε and radius R_ε. Note that property (4.0.1) is equivalent to the vanishing of the principal curvatures, i.e., $k_1, k_2 \xrightarrow{\infty} 0$.

Remark 4.0.5. *We will always work on $\widetilde{\Omega} := \Sigma \times (-a, a)$ instead of Ω, using the identification $\mathcal{L} : \widetilde{\Omega} \to \Omega$ in Section* 1.

The metric on $\widetilde{\Omega}$ corresponding to the Euclidean metric on Ω is

$$G(X, Y) = g\big((I - uL)X_\Sigma, (I - uL)Y_\Sigma\big) + X'Y'$$

where $X = (X_\Sigma, X')$ and $Y = (Y_\Sigma, Y')$, with volume element

$$d\big(\Sigma \times (-a, a)\big) = (1 - 2Mu + Ku^2)d\Sigma du$$

and the Laplace operator $-\Delta$ associated with G.

Because $\mathcal{L} : (\widetilde{\Omega}, G) \to (\Omega, g_{eucl})$ is an isometry, we have

$$\sigma_{ess}(-\Delta_D^\Omega) = \sigma_{ess}(-\Delta_D^{\widetilde{\Omega}}),$$

and similarly for σ and σ_{disc}.

Theorem 4.0.6. *Let Σ be a complete, connected, non-compact surface embedded in $\mathbb{R}^3$ and suppose its Gauss curvature K satisfies*

$$K \in L^1(\Sigma). \tag{H_1}$$

Let the layer Ω be defined by

$$\Omega := \{z \in \mathbb{R}^3 \mid \operatorname{dist}(z, \Sigma) < a\},$$

i.e., as a tube of radius $a > 0$ about Σ, where

$$a < \rho_m := \Big(max\{\|k_1\|_\infty, \|k_2\|_\infty\}\Big)^{-1}. \tag{H_2}$$

If the curvatures K and M vanish at infinity, then

$$\inf \sigma_{ess}(-\Delta_D^\Omega) = \kappa_1^2.$$

The proof of this theorem will be divided into two steps: Establishing a lower and an upper bound, this is done in Section 4.1 respectively 4.2.

4.1 A Lower Bound on the Essential Spectrum of $-\Delta_D$ on Quantum Layers

To prove a lower bound, we will use the min-max principle in combination with notations from earlier sections.

Theorem 4.1.1. $\inf \sigma_{ess}(-\Delta_D^\Omega) \geq \kappa_1^2$.

Proof. Fix $\varepsilon > 0$. Choose $B(x_\varepsilon, R_\varepsilon)$ as in (4.0.2) for $f = |k_1| + |k_2|$, and consider an open, precompact set $\mathcal{B} \supseteq B(x_\varepsilon, R_\varepsilon)$ with $\mathcal{C}^1$-smooth boundary.

Now, we divide $\widetilde{\Omega}$ into an exterior and an interior part, by putting

$$\widetilde{\Omega}_{int} := \mathcal{B} \times (-a, a) \quad \text{and} \quad \widetilde{\Omega}_{ext} := \widetilde{\Omega} \backslash \overline{\widetilde{\Omega}}_{int}.$$

Then we conclude from (H_2), (1.2.7) and (4.0.1) that

$$\forall (x, u) \in \widetilde{\Omega}_{ext} : (1 - a\varepsilon)^2 \leq 1 - 2M(x)u + K(x)u^2 \leq (1 + a\varepsilon)^2. \tag{4.1.1}$$

Imposing Neumann boundary conditions at the common boundary of $\widetilde{\Omega}_{ext}$ and $\widetilde{\Omega}_{int}$, we arrive at the decoupled operator

$$-\Delta_N := -\Delta_N^{ext} \oplus -\Delta_N^{int}$$

with associated quadratic form $q_N := q_N^{ext} \oplus q_N^{int}$ (see Proposition 1.3.10), where

$$q_N^{ext}(\psi, \phi) := \int_{\widetilde{\Omega}_{ext}} \langle \nabla \psi, \nabla \phi \rangle d\widetilde{\Omega}, \quad D(q_N^{ext}) := \{\psi \in H^1(\widetilde{\Omega}_{ext}) \quad | \quad \psi(\cdot, \pm a) = 0\}$$

and $q_N^{int}(\psi,\phi)$ is given similar formula, by exchanging *ext* for *int*.

By Proposition 2.3.8, we have

$$\sigma_{ess}(-\Delta_N) = \sigma_{ess}(-\Delta_N^{ext}) \cup \sigma_{ess}(-\Delta_N^{int}),$$

but the spectrum of the interior part is purely discrete by Corollary 2.2.3, so the essential components are determined by the exterior part only,

$$\sigma_{ess}(-\Delta_N) = \sigma_{ess}(-\Delta_N^{ext}),$$

and since $-\Delta_N \leq -\Delta_D^{\widetilde{\Omega}}$, the min-max principle gives the following estimate:

$$\inf \sigma(-\Delta_N^{ext}) \leq \inf \sigma_{ess}(-\Delta_N^{ext}) \leq \inf \sigma_{ess}(-\Delta_D^{\widetilde{\Omega}}) \tag{4.1.2}$$

Now, we have

$$q_N^{ext}(\psi,\psi) = \int\limits_{\widetilde{\Omega}_{ext}} \langle \nabla\psi, \nabla\psi \rangle_G d\widetilde{\Omega}$$

$$= \int\limits_{-a}^{a} \int\limits_{\Sigma\setminus\mathcal{B}} \langle \nabla\psi, \nabla\psi \rangle_G d\big(\Sigma \times (-a,a)\big),$$

and by using inequality (4.1.1) and the definition of the metric G, where

$$\Big\langle \nabla\psi, \nabla\psi \Big\rangle_G = \Big\langle \nabla\psi, \nabla\psi \Big\rangle_g + \Big\langle \frac{\partial\psi}{\partial u}, \frac{\partial\psi}{\partial u} \Big\rangle_{du^2},$$

we get

$$q_N^{ext}(\psi,\psi) = \int\limits_{-a}^{a} \int\limits_{\Sigma\setminus\mathcal{B}} \bigg(\Big\langle \nabla\psi, \nabla\psi \Big\rangle_g + \Big(\frac{\partial\psi}{\partial u}\Big)^2 \bigg) \big(1 - 2Mu + Ku^2\big) d\Sigma du$$

$$\geq \int\limits_{-a}^{a} \int\limits_{\Sigma\setminus\mathcal{B}} \bigg(\Big\langle \nabla\psi, \nabla\psi \Big\rangle_g + \Big(\frac{\partial\psi}{\partial u}\Big)^2 \bigg) (1 - a\varepsilon)^2 d\Sigma du. \tag{4.1.3}$$

Now

$$\inf \sigma_{ess}\Big(-\frac{d^2}{du^2}\Big)_D = \inf_{\substack{\psi:\psi\neq 0 \\ \psi(.,\pm a)=0}} \frac{\int_{-a}^{a} \Big(\frac{\partial\psi}{\partial u}\Big)^2 du}{\int_{-a}^{a} \psi^2 du} = \kappa_1^2,$$

and so we obtain

$$\int\limits_{-a}^{a} \Big(\frac{\partial\psi}{\partial u}\Big)^2 du \geq \kappa_1^2 \int\limits_{-a}^{a} \psi^2 du,$$

for any ψ with $\psi(.,\pm a) = 0$. Since clearly $\langle \nabla\psi, \nabla\psi \rangle \geq 0$, (4.1.3) yields:

$$q_N^{ext}(\psi,\psi) \geq (1 - a\varepsilon)^2 \int\limits_{-a}^{a} \int\limits_{\Sigma\setminus\mathcal{B}} \kappa_1^2 \psi^2 d\Sigma du$$

By using inequality (4.1.1) again, we get

$$q_N^{ext}(\psi,\psi) \geq \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2 \int_{-a}^{a}\int_{\Sigma\setminus\mathcal{B}} \psi^2(1+a\varepsilon)^2 d\Sigma du$$

$$\geq \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2 \int_{-a}^{a}\int_{\Sigma\setminus\mathcal{B}} \psi^2(1-2Mu+Ku^2) d\Sigma du$$

$$= \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2 \int_{\widetilde{\Omega}_{ext}} \psi^2 d\widetilde{\Omega}.$$

And, as an immediate consequence

$$\frac{q_N^{ext}(\psi,\psi)}{\int_{\widetilde{\Omega}_{ext}} \psi^2 d\widetilde{\Omega}} \geq \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2,$$

which implies

$$\inf \sigma(-\Delta_N^{ext}) \geq \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2.$$

Therefore, from inequality (4.1.2), we get

$$\inf \sigma_{ess}(-\Delta_N^{\widetilde{\Omega}}) \geq \frac{(1-a\varepsilon)^2}{(1+a\varepsilon)^2}\kappa_1^2.$$

The claim then follows from the fact that ε can be chosen arbitrarily small. $\square$

4.2 An Upper Bound on the Essential Spectrum of $-\Delta_D$ on Quantum Layers

At the beginning of the proof of the upper bound, we will use Theorem 2.3.3 and the following theorem from [LZ11].

Theorem 4.2.1. *Let Σ be a complete, non-compact Riemannian surface. Assume that*

$$\lim_{x\to\infty} K(x) = 0.$$

Then, the essential spectrum of the Laplacian $-\Delta$ of Σ is the interval $[0,\infty)$.

Then, we complete the proof by using Weyl's Theorem, Theorem 2.3.5.

Remark 4.2.2. *We will use the corresponding normalized eigenfunction given explicitly by*

$$\chi_1(u) := \sqrt{\frac{1}{a}}\cos\kappa_1 u.$$

By using the identities $|\nabla u| = 1$ *and the equation* (1.4.2)*, we get*

$$-\Delta u = 2M_u$$

and

$$-\Delta\chi_1(u) = 2M_u\chi_1'(u) + \kappa_1^2\chi_1(u), \tag{4.2.1}$$

where

$$M_u = \frac{M - Ku}{1 - 2Mu + Ku^2}.$$

Theorem 4.2.3. $\inf \sigma_{ess}(-\Delta_D^\Omega) \le \kappa_1^2$.

Proof. It follows from Theorem 4.2.1 that if K vanishes at infinity then the threshold of the essential spectrum of the Laplacian on Σ equals 0. Thus, by Theorem 2.3.3, for any $\varepsilon > 0$ there exists an infinite dimensional subspace $S_g \subseteq H^2(\Sigma)$ such that

$$\forall\varphi \in S_g : \|-\Delta\varphi\|_g \le \varepsilon\|\varphi\|_g, \tag{4.2.2}$$

and by using Green's formula and the Cauchy-Schwartz inequality, we get for $\varphi \in S_g$

$$\begin{aligned}\|\nabla\varphi\|_g^2 = (\nabla\varphi, \nabla\varphi)_g &= (-\Delta\varphi, \varphi)_g \\ &\le \|-\Delta\varphi\|_g\|\varphi\|_g \\ &\le \varepsilon\|\varphi\|_g^2.\end{aligned} \tag{4.2.3}$$

Now, $\forall\varphi \in H^2(\Sigma)$ and by using equation (1.4.2) and Green's formula, we have

$$\begin{aligned}q[\varphi\chi_1] &= \big(\varphi\chi_1, -\Delta(\varphi\chi_1)\big) \\ &= \big(\varphi\chi_1, (-\Delta_\Sigma + 2M_u\partial_u - \partial_u^2)\varphi\chi_1\big) \\ &= \big((\nabla\varphi)\chi_1, (\nabla\varphi)\chi_1\big) + \big(\varphi\chi_1, 2M_u\varphi(\partial_u\chi_1)\big) + \big(\varphi\chi_1, \varphi(-\partial_u^2\chi_1)\big) \\ &= \big\|(\nabla\varphi)\chi_1\big\|^2 + \big(\varphi\chi_1, 2M_u\varphi(\partial_u\chi_1)\big) + \kappa_1^2\big\|\varphi\chi_1\big\|^2.\end{aligned} \tag{4.2.4}$$

Firstly, we will compute the second term on the right-hand side of equation (4.2.4), we get

$$\begin{aligned}\big(\varphi\chi_1, 2M_u\varphi\chi_1'\big) &= \int_\Sigma \varphi^2 \int_{-a}^{a} 2M_u\chi_1\chi_1'(1 - 2Mu + Ku^2)d\Sigma du \\ &= \int_\Sigma \varphi^2 \int_{-a}^{a} 2(M - Ku)\chi_1\chi_1' d\Sigma du,\end{aligned}$$

and by integrating by parts with respect to u, it follows that

$$
\begin{aligned}
(\varphi\chi_1, 2M_u\varphi\chi_1') &= \int\limits_{\Sigma} \varphi^2 \Big[(M - Ku)\chi_1^2 \Big|_{-a}^{a} + \int\limits_{-a}^{a} K\chi_1^2 du \Big] d\Sigma \\
&= \int\limits_{\Sigma} \varphi^2 \int\limits_{-a}^{a} \frac{K}{(1 - 2Mu + Ku^2)} \chi_1^2 (1 - 2Mu + Ku^2) d\Sigma du \\
&= \int\limits_{\widetilde{\Omega}} \varphi^2 K_u \chi_1^2 d\widetilde{\Omega} \\
&= (\varphi\chi_1, K_u\varphi\chi_1), \qquad (4.2.5)
\end{aligned}
$$

where K_u is defined in Lemma 1.4.1.

Secondly, we can compute the first term on the right-hand side of equation (4.2.4) as the following Lemma

Lemma 4.2.4.

$$\left\|(\nabla\varphi)\chi_1\right\|^2 \leq \varepsilon(C_+ \setminus C_-)\|\varphi\chi_1\|^2$$

where $C_\pm = (1 \pm a\rho_m^{-1})^2$.

Proof. By using the estimates (1.2.7) and (4.2.3), we get

$$\begin{aligned}
\left\|(\nabla\varphi)\chi_1\right\|^2 &= \int\limits_{\Sigma}\int\limits_{-a}^{a}\left|(\nabla\varphi)\chi_1\right|^2\sqrt{G}dudx \\
&\leq C_+\int\limits_{-a}^{a}\chi_1^2du\int\limits_{\Sigma}|\nabla\varphi|^2\sqrt{g}dx \\
&\leq \varepsilon(C_+)\int\limits_{-a}^{a}\chi_1^2du\int\limits_{\Sigma}|\varphi|^2\sqrt{g}dx \\
&= \varepsilon(C_+\setminus C_-)\int\limits_{-a}^{a}\chi_1^2du\int\limits_{\Sigma}|\varphi|^2(C_-)\sqrt{g}dx \\
&\leq \varepsilon(C_+\setminus C_-)\int\limits_{\Sigma}\int\limits_{-a}^{a}|\varphi\chi_1|^2\sqrt{G}dxdu \\
&= \varepsilon(C_+\setminus C_-)\|\varphi\chi_1\|^2. \quad \square
\end{aligned}$$

Using equations (4.2.4), (4.2.5) and Lemma 4.2.4, we see that

$$\forall\varphi\in S_g:\left\|\nabla(\varphi\chi_1)\right\|^2\leq\varepsilon(C_+\setminus C_-)\|\varphi\chi_1\|^2+(\varphi\chi_1,K_u\varphi\chi_1)+\kappa_1^2\|\varphi\chi_1\|^2, \tag{4.2.6}$$

hence for any $\varepsilon>0$ there exists $S:=S_g\otimes\chi_1\subset H^2(\widetilde{\Omega})$ such that

$$\forall\psi\in S:\|\nabla\psi\|^2-(\psi,K_u\psi)\leq\left(\kappa_1^2+\varepsilon(C_+\setminus C_-)\right)\|\psi\|^2.$$

This proves that

$$\inf\sigma_{ess}(-\Delta_D^{\widetilde{\Omega}}-K_u)\leq\kappa_1^2.$$

The proof of Theorem 4.2.3 is completed by showing that K_u is relatively compact with respect to $-\Delta_D^{\widetilde{\Omega}}$. Since K_u vanishes at infinity by assumption (4.0.1), the operator $K_u(-\Delta_D^{\widetilde{\Omega}}+1)^{-1}$ is compact in $L^2(\widetilde{\Omega})$, and by the Weyl's Theorem, Theorem 2.3.5

$$\sigma_{ess}(-\Delta_D^{\widetilde{\Omega}})=\sigma_{ess}(-\Delta_D^{\widetilde{\Omega}}-K_u).$$

Thus

$$\inf\sigma_{ess}(-\Delta_D^{\widetilde{\Omega}})\leq\kappa_1^2. \quad \square$$

Remark 4.2.5. *Notice that only $K\xrightarrow{\infty}0$ is needed in order to establish the upper bound.*

Proof (of Theorem 4.0.6): This is a direct consequence of Theorem 4.1.1 and 4.2.3.

Chapter 5

Existence of Discrete Spectrum of the Dirichlet Laplacian on Quantum Layers

The aim of this chapter is the following: establishing certain geometric conditions which guarantee the existence of discrete spectrum for the Dirichlet Laplacian on the quantum layer Ω (which is asymptotically planar) below κ_1^2. Since we have shown that the spectral threshold for layers around asymptotically planar surfaces is just κ_1^2, the spectrum below this value consists of discrete eigenvalues.

The proof is based on the variational idea of finding a test function ψ in the domain of $-\Delta_D^{\widetilde{\Omega}}$ such that

$$q_1[\psi] := q[\psi] - \kappa_1^2 \|\psi\|^2 < 0. \tag{5.0.1}$$

The important technical tool needed to construct ψ is the existence of appropriate mollifiers on Σ. This is ensured by the following lemma

Lemma 5.0.6. *Let Σ be a surface as in Theorem 4.0.6 and assume (H_1). Then, there exists a sequence $\{\varphi_n\}_{n \in N}$ of smooth functions with compact supports in Σ such that*

(1) $\forall n \in \mathbb{N} : 0 \le \varphi_n \le 1$,

(2) $\|\nabla \varphi_n\|_g \xrightarrow[n\to\infty]{} 0$,

(3) $\varphi_n \xrightarrow[n\to\infty]{} 1$ *uniformly on compact subsets of* Σ.

Proof. see [CEK04].

This sequence enables us to cut off a generalized test function which would give formally a negative value of the statement (5.0.1).

Remark 5.0.7. *In the special case when $\Sigma = K' \cup \Sigma'$, where K' is compact and Σ' is isometric to $\mathbb{R}^2 \setminus B_R(0)$ for some $R > 0$, we can construct φ_n as follows:*

Firstly, we define $\widetilde{\varphi}_n \in C_0^\infty(\Sigma)$ *by*

$$\widetilde{\varphi}_n(x) = \begin{cases} 1, & x \in K' \quad or \quad R \le r \le nR \\ \frac{1}{\log nR} \log \frac{n^2R^2}{r}, & nR \le r \le n^2R^2 \\ 0, & r \ge n^2R^2 \end{cases} \tag{5.0.2}$$

where $r = |x|$*. Then we define* $\varphi_n(x)$ *by smoothing* $\widetilde{\varphi}_n(x)$ *on an interval of length* 1 *around* nR *and* n^2R^2*.*
Now, we want to prove that $\varphi_n(x)$ *satisfies the three properties in Lemma 5.0.6:*

(1) It is clear that $0 \le \varphi_n \le 1$*, for all* $n \in \mathbb{N}$*.*

(2) We start by calculating $\nabla\widetilde{\varphi}_n(x)$*:*

$$\nabla\widetilde{\varphi}_n(x) = \begin{cases} 0, & x \in K' \quad or \quad R \le r \le nR \\ -\frac{1}{r}\frac{1}{\log nR}\frac{\partial}{\partial r}, & nR \le r \le n^2R^2 \\ 0, & r \ge n^2R^2 \end{cases}$$

Then,

$$\|\nabla\widetilde{\varphi}_n\|^2 = \int_\Sigma \left|\nabla\widetilde{\varphi}_n(x)\right|^2 d\Sigma = \frac{1}{(\log nR)^2}\int_{nR}^{n^2R^2} \frac{1}{r^2} r dr$$

$$= \frac{1}{(\log nR)^2} \log r\Big|_{nR}^{n^2R^2} = \frac{1}{(\log nR)^2}(\log n^2R^2 - \log nR)$$

$$= \frac{1}{(\log nR)^2}\log nR = \frac{1}{\log nR}.$$

Since, we have

$$a)\left|\nabla\varphi_n(x)\right| = 0 \qquad for \quad R \le r < nR - \frac{1}{2} \quad or \quad r > n^2R^2 + \frac{1}{2}$$

$$b)\left|\nabla\varphi_n(x)\right| \le \frac{C}{(\log nR)nR} \qquad for \quad nR - \frac{1}{2} < r < nR + \frac{1}{2}$$

$$c)\left|\nabla\varphi_n(x)\right| = \left|\nabla\widetilde{\varphi}_n(x)\right| \qquad for \quad nR + \frac{1}{2} < r < n^2R^2 - \frac{1}{2}$$

$$d)\left|\nabla\varphi_n(x)\right| \le \frac{C'}{(\log nR)n^2R^2} \qquad for \quad n^2R^2 - \frac{1}{2} < r < n^2R^2 + \frac{1}{2}$$

we arrive at,

$$\|\nabla\varphi_n\|^2 = \int_{nR-\frac{1}{2}}^{nR+\frac{1}{2}} \left|\nabla\varphi_n(x)\right|^2 rdr + \int_{nR+\frac{1}{2}}^{n^2R^2-\frac{1}{2}} \left|\nabla\widetilde{\varphi}_n(x)\right|^2 rdr + \int_{n^2R^2-\frac{1}{2}}^{n^2R^2+\frac{1}{2}} \left|\nabla\varphi_n(x)\right|^2 rdr$$

$$\leq 1 \max_{nR-\frac{1}{2}<r<nR+\frac{1}{2}} \left(\left|\nabla\varphi_n(x)\right|^2 r\right) + \frac{1}{\log nR} + 1. \max_{n^2R^2-\frac{1}{2}<r<n^2R^2+\frac{1}{2}} \left(\left|\nabla\varphi_n(x)\right|^2 r\right)$$

$$= \frac{C^2}{(\log nR)^2(nR)^2}(nR+\frac{1}{2}) + \frac{1}{\log nR} + \frac{C'^2}{(\log nR)^2(n^2R^2)^2}(n^2R^2+\frac{1}{2})$$

$$\leq \frac{C_1^2}{(\log nR)^2(nR)} + \frac{1}{\log nR} + \frac{C_1'^2}{(\log nR)^2(n^2R^2)}.$$

Thus,

$$\lim_{n\to\infty} \|\nabla\varphi_n\|^2 = 0.$$

(3) For all compact sets $K \subset\subset \Sigma$, there exists n_1, such that $K \cap \Sigma' \subset B_{n_1R-\frac{1}{2}}(0)$, i.e., we have $R \leq r \leq n_1R - \frac{1}{2}$ for all $x \in K \cap \Sigma'$. Therefore, for all $n \geq n_1$, we get that $R \leq r \leq nR - \frac{1}{2}$, for all $x \in K$, and hence $\varphi_n(x) = 1$ on K.

5.1 The First Condition

Theorem 5.1.1. *Let Σ be a surface as in Theorem 4.0.6. Assume (H_1), (H_2), and suppose that Σ is not a plane. If the surface has non-positive total Gauss curvature, i.e., $\mathcal{K} \leq 0$, then*

$$\inf \sigma(-\Delta_D^{\Omega}) < \kappa_1^2.$$

Consequently, if the surface Σ is not a plane but its curvatures vanish at infinity, then the condition $\mathcal{K} \leq 0$ is sufficient to guarantee that $-\Delta_D^{\Omega}$ has at least one eigenvalue of finite multiplicity below the threshold of its essential spectrum, i.e. $\sigma_{disc}(-\Delta_D^{\Omega}) \neq \emptyset$.

Before we are going into the proof we will give the following lemma which told us that in our case, the mean curvature M could not be zero every where:

Lemma 5.1.2. *If $M = 0$ on the whole surface and $\mathcal{K} = 0$ then Σ is a plane.*

Proof. Since $M(x) = 0$ every where then $k_1(x) = -k_2(x)$ for all $x \in \Sigma$, and we have

$$\mathcal{K} = \int_\Sigma K(x)d\Sigma = \int_\Sigma -k_1^2(x)d\Sigma = 0$$

that follows $k_1(x) = k_2(x) = 0$ for all $x \in \Sigma$, then Σ is a plane.

Proof of Theorem 5.1.1. We begin the construction of ψ by considering

$$\psi = \varphi_n\chi_1 \in q(-\Delta_D^{\widetilde{\Omega}}) = \left\{\psi \in H^1(\widetilde{\Omega}), \psi(.,\pm a) = 0\right\},$$

where φ_n is defined as in Lemma 5.0.6. Since $-\Delta_\Omega = -\Delta_\Sigma + 2M_u\partial_u - \partial_u^2$, then we have:

$$\begin{aligned} q[\psi] &= \big(\varphi_n\chi_1, -\Delta(\varphi_n\chi_1)\big) \\ &= \big\|(\nabla\varphi_n)\chi_1\big\|^2 + \big(\varphi_n\chi_1, 2M_u\varphi_n\chi_1'\big) + \kappa_1^2\big\|\varphi_n\chi_1\big\|^2 \end{aligned} \tag{5.1.1}$$

Firstly, we will calculate the second term in the right-hand side of equation (5.1.1). By means of (1.4.3), integration by parts and since $\int_{-a}^{a} \chi_1^2 du = 1$, we get

$$\begin{aligned} \big(\varphi_n\chi_1, 2M_u\varphi_n\chi_1'\big) &= \int_\Sigma \varphi_n^2 \int_{-a}^{a} 2M_u\chi_1\chi_1'(1-2Mu+Ku^2)d\Sigma du \\ &= \int_\Sigma \varphi_n^2 \int_{-a}^{a} 2(M-Ku)\chi_1\chi_1' d\Sigma du \\ &= \int_\Sigma \varphi_n^2 \Big[(M-Ku)\chi_1^2\big|_{-a}^{a} + \int_{-a}^{a} K\chi_1^2 du\Big] d\Sigma \\ &= \int_\Sigma \varphi_n^2 K d\Sigma \int_{-a}^{a} \chi_1^2 du \\ &= (\varphi_n, K\varphi_n)_g. \end{aligned}$$

By means of (1.4.3), integration by parts and since $\int_{-a}^{a} \chi_1^2 du = 1$.

By using inequality (1.2.7), we can estimate the first term at the right-hand side of (5.1.1) as follows

$$\begin{aligned} \big\|(\nabla\varphi_n)\chi_1\big\|^2 &= \int_{-a}^{a}\int_\Sigma (\nabla\varphi_n)^2\chi_1^2(1-2Mu+Ku^2)d\Sigma du \\ &\le (1+a\rho^{-1})^2 \int_\Sigma (\nabla\varphi_n)^2 d\Sigma \int_{-a}^{a} \chi_1^2 du \\ &= (1+a\rho^{-1})^2\|\nabla\varphi_n\|_g^2. \end{aligned}$$

Then, it follows from above that

$$q_1[\varphi_n\chi_1] \le (1+a\rho^{-1})^2\|\nabla\varphi_n\|_g^2 + (\varphi_n, K\varphi_n)_g. \tag{5.1.2}$$

By using the Lemma 5.0.6 and the dominated convergence theorem, we get:

$$(1 + a\rho^{-1})^2 \|\nabla\varphi_n\|_g^2 \xrightarrow[n\to\infty]{} 0$$

$$(\varphi_n, K\varphi_n)_g \xrightarrow[n\to\infty]{} \int\limits_{\Sigma} K d\Sigma = \mathcal{K}.$$

If $\mathcal{K} < 0$, then this shows that we can find $n_0 \in \mathbb{N}$ such that $q_1[\varphi_{n_0}\chi_1] < 0$.

In order to deal with the case $\mathcal{K} = 0$, we construct the test function by adding a small deformation term to ψ. For a real number ε, which will be specified later, we set

$$\psi_\varepsilon = \psi + \varepsilon\theta,$$

where

$$\theta(x, u) := j(x)u\chi_1(u),$$

and j is a smooth positive function on Σ with compact support in a region where the mean curvature M is nonzero and does not change sign.

Since θ is clearly a function in $D(q)$ as well, we may write

$$q_1[\varphi_n\chi_1 + \varepsilon\theta] = q_1[\varphi_n\chi_1] + 2\varepsilon q_1(\theta, \varphi_n\chi_1) + \varepsilon^2 q_1[\theta]. \tag{5.1.3}$$

Since $\mathcal{K} = 0$, the first term in the right-hand side of this identity tends to zero as $n \to \infty$ as we have seen above. We now calculate the quadratic form in the second term. By means of (1.4.2) and by using Green's formula, we have

$$\begin{aligned}
q_1(\theta, \varphi_n\chi_1) &= q(\theta, \varphi_n\chi_1) - \kappa_1^2(\theta, \varphi_n\chi_1) \\
&= \big(\theta, -\Delta(\varphi_n\chi_1)\big) - \kappa_1^2(\theta, \varphi_n\chi_1) \\
&= \big(\theta, (-\Delta\varphi_n)\chi_1\big) + (\theta, 2M_u\varphi_n\partial_u\chi_1) - (\theta, \varphi_n\partial_u^2\chi_1) - \kappa_1^2(\theta, \varphi_n\chi_1) \\
&= \big(\nabla\theta, (\nabla\varphi_n)\chi_1\big) + (\theta, 2M_u\varphi_n\chi_1') + \kappa_1^2(\theta, \varphi_n\chi_1) - \kappa_1^2(\theta, \varphi_n\chi_1) \\
&= \big(\nabla\theta, (\nabla\varphi_n)\chi_1\big) + (\theta, 2M_u\varphi_n\chi_1')
\end{aligned}$$

In the following, we will calculate each of these terms:

a) By means of (1.4.3), we have

$$(\theta, 2M_u\varphi_n\chi_1') = \int_\Sigma \int_{-a}^{a} 2\theta M_u\varphi_n\chi_1'(1-2Mu+Ku^2)d\Sigma du$$

$$= \int_\Sigma \int_{-a}^{a} 2u\chi_1\chi_1' j\varphi_n(M-Ku)d\Sigma du$$

$$= \int_\Sigma j\varphi_n M d\Sigma \int_{-a}^{a} 2u\chi_1\chi_1' du - \int_\Sigma j\varphi_n K d\Sigma \int_{-a}^{a} 2u^2\chi_1\chi_1' du$$

and it is easy to check that

$$\begin{aligned} &\int_{-a}^{a} 2u\chi_1\chi_1' du = -1, \\ &\int_{-a}^{a} 2u^2\chi_1\chi_1' du = 0 \end{aligned} \tag{5.1.4}$$

which shows that

$$(\theta, 2M_u\varphi_n\chi_1') = -(j, M\varphi_n)_g. \tag{5.1.5}$$

b) We calculate

$$\big(\nabla\theta, (\nabla\varphi_n)\chi_1\big) = \int_\Sigma \int_{-a}^{a} \nabla\theta\nabla\varphi_n\chi_1(1-2Mu+Ku^2)d\Sigma du$$

$$= \int_\Sigma \nabla j\nabla\varphi_n \int_{-a}^{a} u\chi_1^2(1-2Mu+Ku^2)d\Sigma du$$

$$+ \int_\Sigma j\nabla\varphi_n \int_{-a}^{a} \chi_1^2(1-2Mu+Ku^2)d\Sigma du$$

$$+ \int_\Sigma j\nabla\varphi_n \int_{-a}^{a} 2u\chi_1'\chi_1(1-2Mu+Ku^2)d\Sigma du,$$

and it is a straightforward exercise to compute:

$$\int_{-a}^{a} u\chi_1^2 du = 0$$
$$\int_{-a}^{a} u^2\chi_1^2 du = \frac{a^2}{3} - \frac{1}{2\kappa_1^2} \tag{5.1.6}$$
$$\int_{-a}^{a} u^3\chi_1^2 du = 0$$
$$\int_{-a}^{a} 2u^3\chi_1\chi_1' du = -3\Big(\frac{a^2}{3} - \frac{1}{2\kappa_1^2}\Big)$$

Then, we get

$$\big(\nabla\theta, (\nabla\varphi_n)\chi_1\big) = -2\Big(\frac{a^2}{3} - \frac{1}{2\kappa_1^2}\Big)\Big[\int_\Sigma \nabla j \nabla\varphi_n M d\Sigma + \int_\Sigma j\nabla\varphi_n K d\Sigma\Big]. \tag{5.1.7}$$

By means of the Schwarz inequality, boundedness of K and M, properties of j and Lemma 5.0.6, it follows that the last equation tend to zero as $n \to \infty$. Moreover, the equation (5.1.5) tend to

$$(\theta, 2M_u\varphi_n\chi_1') \xrightarrow[n\to\infty]{} -(j, M)_g.$$

Now, we go back to equation (5.1.3). Since θ does not depend on n, one gets

$$q_1[\varphi_n\chi_1 + \varepsilon\theta] \xrightarrow[n\to\infty]{} -2\varepsilon(j, M)_g + \varepsilon^2 q_1[\theta],$$

and since $(j, M)_g \neq 0$ which may be made negative by choosing ε sufficiently close to 0 and of an appropriate sign. □

An interesting new spectral result then follows from the observation that making the topology of Σ more complicated than that of a plane, one always achieves the basic condition $\mathcal{K} \leq 0$ is satisfied:

Lemma 5.1.3. *Let Σ be a complete, connected but not simply connected and non-compact surface , then its total Gauss curvature is non-positive, i.e., $\mathcal{K} \leq 0$.*

Proof. In 1957, Huber showed that the total Gauss curvature of this surface Σ is given by

$$\mathcal{K} = \int_\Sigma K d\Sigma \leq 2\pi\chi(\Sigma),$$

and we need only show that $\chi(\Sigma) \leq 0$. The Euler characteristic of non-compact manifold is given by

$$\chi = 2 - 2g - e$$

where g is the genus of Σ and e is the number of ends. Since the surface is not simply connected, $g \geq 1$. Then, for all $e \in \mathbb{N}$, the Euler characteristic of a non-compact, not simply connected surface is non-positive. □

Corollary 5.1.4. *Under the assumptions of Theorem 4.0.6, one has* $\inf \sigma(-\Delta_D^\Omega) < \kappa_1^2$ *whenever* Σ *is not homeomorphic to a plane.*

5.2 The Second and Third Conditions

We establish the next two conditions in the following theorem.

Theorem 5.2.1. *Let* Σ *be a surface as in the Theorem 4.0.6. Assume* (H_1), (H_2), *and suppose that* Σ *is asymptotically a plane. Then, any of the conditions*

(a) The radius of the layer a *is small enough and* $\nabla M \in L^2_{loc}(\Sigma)$,

(b) $\mathcal{M} = \infty$ *and* $\nabla M \in L^2(\Sigma)$

is sufficient to guarantee that

$$\inf \sigma(-\Delta_D^\Omega) < \kappa_1^2.$$

Proof. For the two conditions, we will use the test function

$$\psi_n(x,u) = \big(1 + M(x)u\big)\varphi_n(x)\chi_1(u),$$

where $\varphi_n(x)$ is a function satisfying Lemma 5.0.6. Since

$$\begin{aligned}\nabla\psi_n(.,u) =&(1+Mu)(\nabla\varphi_n)\chi_1(u) + (\nabla M)u\varphi_n\chi_1(u)\\ &+ \big((1+Mu)\varphi_n\chi_1'(u) + M\varphi_n\chi_1(u)\big)\nabla u,\end{aligned}$$

it is easy to see that $\psi_n \in D(q)$ provided $\nabla M \in L^2_{loc}(\Sigma)$.

Now, we want to prove that

$$q_1[\psi_n] = q[\psi_n] - \kappa_1^2\|\psi_n\|^2 < 0.$$

Since $\nabla\varphi_n$ and ∇M are orthogonal to ∇u, then we have

$$\begin{aligned}q[\psi_n] = \|\nabla\psi_n\|^2 =&\big\|(1+Mu)|\nabla\varphi_n|\chi_1 + |\nabla M|u\varphi_n\chi_1\big\|^2\\ &+ \big\|(1+Mu)\varphi_n\chi_1' + M\varphi_n\chi_1\big\|^2\end{aligned} \tag{5.2.1}$$

By using Minkovski's inequality $\big(|f+g|^2 \le 2(|f|^2+|g|^2)\big)$ on the first line at the right-hand side of (5.2.1), and since the curvatures K and M are uniformly bounded due to (H_2), we obtain

$$\big\|(1+Mu)|\nabla\varphi_n|\chi_1 + |\nabla M|u\varphi_n\chi_1\big\|^2 \le 2\Big(\big(1+a\|M\|_\infty\big)^2\big\||\nabla\varphi_n|\chi_1\big\|^2 + a^2\big\||\nabla M|\varphi_n\chi_1\big\|^2\Big).$$

The second term at the right-hand side of (5.2.1) can be rewritten as follows:

$$\begin{aligned}\big\|(1+Mu)\varphi_n\chi_1' + M\varphi_n\chi_1\big\|^2 =&\big\|(1+Mu)\varphi_n\chi_1'\big\|^2 + \big\|M\varphi_n\chi_1\big\|^2\\ &+ 2\big\|M(1+Mu)\varphi_n^2\chi_1\chi_1'\big\|\end{aligned} \tag{5.2.2}$$

We start by calculating the first term at the right-hand side of (5.2.2).

$$\left\|(1+Mu)\varphi_n\chi_1'\right\|^2 = \int_\Sigma \int_{-a}^{a} (1+Mu)^2\varphi_n^2\chi_1'^2(1-2Mu+Ku^2)dud\Sigma$$

and the terms in the right-hand side of the this last equation amount to:

$$a)\int_\Sigma \varphi_n^2\Big[\int_{-a}^{a}\chi_1'^2(1-2Mu+Ku^2)du\Big]d\Sigma = \int_\Sigma \varphi_n^2\Big[\int_{-a}^{a}\chi_1'^2du\Big]d\Sigma - 2\int_\Sigma M\varphi_n^2\Big[\int_{-a}^{a}u\chi_1'^2du\Big]d\Sigma$$

$$+\int_\Sigma K\varphi_n^2\Big[\int_{-a}^{a}u^2\chi_1'^2du\Big]d\Sigma$$

$$=\kappa_1^2\int_\Sigma \varphi_n^2 d\Sigma + 0 + \Big(\frac{\kappa_1^2a^2}{3}+\frac{1}{2}\Big)\int_\Sigma K\varphi_n^2d\Sigma$$

$$b)2\int_\Sigma M\varphi_n^2\Big[\int_{-a}^{a}u\chi_1'^2(1-2Mu+Ku^2)du\Big]d\Sigma = 2\int_\Sigma M\varphi_n^2\Big[\int_{-a}^{a}u\chi_1'^2du\Big]d\Sigma - 4\int_\Sigma M^2\varphi_n^2\Big[\int_{-a}^{a}u^2\chi_1'^2du\Big]d\Sigma$$

$$+2\int_\Sigma MK\varphi_n^2\Big[\int_{-a}^{a}u^3\chi_1'^2du\Big]d\Sigma$$

$$=0-4\Big(\frac{\kappa_1^2a^2}{3}+\frac{1}{2}\Big)\int_\Sigma M^2\varphi_n^2d\Sigma+0$$

$$c)\int_\Sigma M^2\varphi_n^2\Big[\int_{-a}^{a}u^2\chi_1'^2(1-2Mu+Ku^2)du\Big]d\Sigma = \int_\Sigma M^2\varphi_n^2\Big[\int_{-a}^{a}u^2\chi_1'^2du\Big]d\Sigma - 2\int_\Sigma M^3\varphi_n^2\Big[\int_{-a}^{a}u^3\chi_1'^2du\Big]d\Sigma$$

$$+\int_\Sigma M^2K\varphi_n^2\Big[\int_{-a}^{a}u^4\chi_1'^2du\Big]d\Sigma$$

$$=\Big(\frac{\kappa_1^2a^2}{3}+\frac{1}{2}\Big)\int_\Sigma M^2\varphi_n^2d\Sigma-0$$

$$+\Big(\frac{\kappa_1^2a^4}{5}+a^2-\frac{3}{2\kappa_1^2}\Big)\int_\Sigma M^2K\varphi_n^2d\Sigma$$

Thus

$$\big\|(1+Mu)\varphi_n\chi_1'\big\|^2 = \kappa_1^2\int\limits_\Sigma \varphi_n^2 d\Sigma + \Big(\frac{\kappa_1^2a^2}{3}+\frac{1}{2}\Big)\int\limits_\Sigma K\varphi_n^2 d\Sigma - 3\Big(\frac{\kappa_1^2a^2}{3}+\frac{1}{2}\Big)\int\limits_\Sigma M^2\varphi_n^2 d\Sigma$$

$$+\Big(\frac{\kappa_1^2a^4}{5}+a^2-\frac{3}{2\kappa_1^2}\Big)\int\limits_\Sigma M^2K\varphi_n^2 d\Sigma$$

In the same way, by using equations (5.1.4) and (5.1.6) we calculate the second and third terms at the right hand side of (5.2.2), and also the term $-\kappa_1^2\|\psi_n\|^2$. We get

$$\|M\varphi_n\chi_1\|^2 = \int\limits_\Sigma M^2\varphi_n^2 d\Sigma + \Big(\frac{a^2}{3}-\frac{1}{2\kappa_1^2}\Big)\int\limits_\Sigma M^2K\varphi_n^2 d\Sigma$$

$$2\big\|M(1+Mu)\varphi_n^2\chi_1\chi_1'\big\| = \int\limits_\Sigma M^2\varphi_n^2 d\Sigma - 6\Big(\frac{a^2}{3}-\frac{1}{2\kappa_1^2}\Big)\int\limits_\Sigma M^2K\varphi_n^2 d\Sigma$$

$$-\kappa_1^2\|\psi_n\|^2 = -\kappa_1^2\int\limits_\Sigma \varphi_n^2 d\Sigma + 3\Big(\frac{a^2\kappa_1^2}{3}-\frac{1}{2}\Big)\int\limits_\Sigma M^2\varphi_n^2 d\Sigma - \Big(\frac{a^2\kappa_1^2}{3}-\frac{1}{2}\Big)\int\limits_\Sigma K\varphi_n^2 d\Sigma$$

$$-\Big(\frac{a^4\kappa_1^2}{5}-a^2+\frac{3}{2\kappa_1^2}\Big)\int\limits_\Sigma M^2K\varphi_n^2 d\Sigma$$

By adding (5.2.2) to the the term $-\kappa_1^2\|\psi_n\|^2$, we obtain

$$\big\|(1+Mu)\varphi_n\chi_1' + M\varphi_n\chi_1\big\|^2 - \kappa_1^2\big\|\psi_n\big\|^2 = \big(\varphi_n,(K-M^2)\varphi_n\big)_g + \frac{\pi^2-6}{12\kappa_1^2}\big(\varphi_n,KM^2\varphi_n\big)_g.$$

Therefore,

$$\begin{aligned} q_1[\psi_n] \quad &\le 2\Big(\big(1+a\|M\|_\infty\big)^2\big\||\nabla\varphi_n|\chi_1\big\|^2 + a^2\big\||\nabla M|\varphi_n\chi_1\big\|^2\Big) \\ &\quad +\big(\varphi_n,(K-M^2)\varphi_n\big)_g + \tfrac{\pi^2-6}{12\kappa_1^2}\big(\varphi_n,KM^2\varphi_n\big)_g. \end{aligned} \tag{5.2.3}$$

Proof of Theorem 5.2.1 under Condition (a). To begin with the integral containing $K-M^2$ in (5.2.3) is always negative for any nonplanar and noncompact surface, since this term can be rewritten by means of the principle curvatures, i.e.,

$$K-M^2 = -\frac{1}{4}(k_1-k_2)^2.$$

Secondly, by applying Lemma 5.0.6 and (1.2.7) to the first term at the right-hand side of (5.2.3), we have

$$\begin{aligned}\big(1+a\|M\|_\infty\big)^2\big\||\nabla\varphi_n|\chi_1\big\|^2 &= \big(1+a\|M\|_\infty\big)^2\int\limits_\Sigma\int\limits_{-a}^{a}|\nabla\varphi_n|^2\chi_1^2(1-2Mu+Ku^2)dud\Sigma\\ &\le \big(1+a\|M\|_\infty\big)^2 C_+\int\limits_\Sigma|\nabla\varphi_n|^2\int\limits_{-a}^{a}\chi_1^2 dud\Sigma\\ &= \big(1+a\|M\|_\infty\big)^2 C_+\int\limits_\Sigma|\nabla\varphi_n|^2 d\Sigma \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

The remaining terms vanish for n fixed as $a\to 0$. (For the latter,note that κ_1^{-2} is proportional to a^2). Hence we can find a sufficiently large $n_0\in\mathbb{N}$ such that the sum of the first term in the right-hand side of (5.2.3) and the first integral at the second line of (5.2.3) is negative, and then choose the layer half-width a sufficiently small so that $q_1[\psi_n]<0$.

Proof of Theorem 5.2.1 under Condition (b). Since $\nabla M\in L^2(\Sigma)$ and (H_1) holds true, all the terms at the right-hand side of (5.2.3) except the first integral at the second line tend to finite values as $n\to\infty$. In more detail, by using Lemma 5.0.6, (1.2.7), and the dominated convergence theorem, we see that

$$\big(1+a\|M\|_\infty\big)^2\big\||\nabla\varphi_n|\chi_1\big\|^2 \le \big(1+a\|M\|_\infty\big)^2 C_+\int\limits_\Sigma|\nabla\varphi_n|^2 d\Sigma \xrightarrow[n\to\infty]{} 0,$$

$$a^2\big\||\nabla M|\varphi_n\chi_1\big\|^2 \le a^2C_+\int\limits_\Sigma|\nabla M|^2\varphi_n^2 d\Sigma \xrightarrow[n\to\infty]{} a^2C_+\int\limits_\Sigma|\nabla M|^2 d\Sigma<\infty,$$

$$(\varphi_n, KM^2\varphi_n)_g=\int\limits_\Sigma KM^2\varphi_n^2 d\Sigma \xrightarrow[n\to\infty]{} \int\limits_\Sigma KM^2 d\Sigma<\infty.$$

For the remaining term, we arrive at

$$\big(\varphi_n,(K-M^2)\varphi_n\big)_g=\int\limits_\Sigma(K-M^2)\varphi_n^2 d\Sigma \xrightarrow[n\to\infty]{} \int\limits_\Sigma(K-M^2)=-\infty,$$

since $\mathcal{M}=\int\limits_\Sigma M^2 d\Sigma=\infty$. Hence we can find $n_0\in\mathbb{N}$ such that $q_1[\psi_{n_0}]<0$. $\square$

By combining Theorem 5.2.1 with White's Proposition "Let Σ be an embedded surface in $\mathbb{R}^3$. If $K\in L^1(\Sigma)$ and $\int\limits_\Sigma K>0$, then $\int\limits_\Sigma M^2=\infty$", we obtain the following:

Corollary 5.2.2. *Let Σ be an embedded surface in $\mathbb{R}^3$ of integrable Gauss curvature. If $\int\limits_\Sigma K>0$ and $\nabla_g M\in L^2(\Sigma)$, then* $\inf\sigma(-\Delta_D^\Omega)<\kappa_1^2$.

Chapter 6

Estimating the Number of Eigenvalues of the Dirchlet Laplacian on Layers

As we have seen in Chapter 5 if our reference surface Σ has at lest one genus then its total Gauss curvature is non-positive $\mathcal{K} \leq 0$. Therefore, the spectrum of $-\Delta_D^\Omega$ has at least one eigenvalue below $\kappa_1^2 = (\frac{\pi}{2a})^2$.

In this chapter we want to find a lower bound on the number of the eigenvalues below $\kappa_1^2 = (\frac{\pi}{2a})^2$ of the Dirichlet Laplacian on a quantum layer Ω in terms of the genus g of Σ. We now define a geometric handle of a plane as follows

Definition 6.0.3. *A geometric handle H of a plane is a surface with boundary which has genus* 1*, whose boundary is connected, and so that a neighborhood U of ∂H is isometric to the annulus $\overline{B_{R+1}(0)} \setminus B_R(0)$ in $\mathbb{R}^2$ for some $R > 0$, where we call the smallest such R the exterior radius of H.*

6.1 A Lower Bound on the Number of Eigenvalues of the Dirichlet Laplacian on Layers

Firstly, we will display the following theorem which gives us a sufficient condition to estimate the number of eigenvalues of self-adjoint operators bounded from below.

Theorem 6.1.1. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, that is bounded from below. Assume that $W \subset D(A)$, with* $\dim W = k$, *and*

$$\forall \psi \in W, \quad \textit{if} \quad \psi \neq 0, \quad \frac{q[\psi]}{\|\psi\|^2} \leq E.$$

If $E < \inf \sigma_{ess}(A)$, then there are at least k eigenvalues $\leq E$.

Proof. For any given functions $\varphi_1, \dots, \varphi_{k-1} \in \mathcal{H}$, let L be the linear map, defined by

$$\begin{aligned} L : W &\to \mathbb{R}^{k-1} \\ \psi &\mapsto \big((\psi, \varphi_1), \dots, (\psi, \varphi_{k-1})\big). \end{aligned}$$

Since

$$\dim W = \dim \mathrm{Im}(L) + \dim \ker(L)$$

and $\dim W = k$, $\dim \mathrm{Im}(L) \leq k-1$, we get $\dim \ker(L) \geq 1$. Thus, there exists $\psi \in \ker(L)$ where $\psi \neq 0$. Therefore, we can find $\psi \in W \cap [\varphi_1, \dots, \varphi_{k-1}]^{\perp}$, such that

$$\psi \neq 0, \quad \frac{q[\psi]}{\|\psi\|^2} \leq E.$$

Then, we get

$$\inf_{\substack{\psi \in W, \psi \neq 0 \\ \psi \in [\varphi_1, \dots, \varphi_{k-1}]^{\perp}}} \frac{q[\psi]}{\|\psi\|^2} \leq E,$$

and because $W \subset D(A)$, it follows that

$$\sup_{\varphi_1, \dots, \varphi_{k-1}} \inf_{\substack{\psi \in D(A), \psi \neq 0 \\ \psi \in [\varphi_1, \dots, \varphi_{k-1}]^{\perp}}} \frac{q[\psi]}{\|\psi\|^2} \leq E.$$

By min-max principle, Theorem 2.2.1, for

$$\mu_k = \sup_{\varphi_1, \dots, \varphi_{k-1}} \inf_{\substack{\psi \in D(A), \psi \neq 0 \\ \psi \in [\varphi_1, \dots, \varphi_{k-1}]^{\perp}}} \frac{q[\psi]}{\|\psi\|^2}.$$

either

a) $\mu_k < \inf \sigma_{ess}(A)$, then there are at least k eigenvalues below the bottom of the essential spectrum;

   or

b) $\mu_k = \inf \sigma_{ess}(A)$, and in that case there are at most $k-1$ eigenvalues below μ_k.

Since $\mu_k \leq E$ and $E < \inf \sigma_{ess}(A)$, μ_k satisfies the first case, i.e., there are at least k eigenvalues $\leq E$. $\square$

Now, we arrive at the first main result of this thesis, in which we relate the topological structure of the reference surface to the spectrum of quantum layers around these surfaces.

Theorem 6.1.2. (Main Theorem 1) *Let $m \in \mathbb{N}$. There is a constant $\mathcal{C}$, such that: If the reference surface Σ is a Euclidean plane with m-handles $H_1, \dots, H_m$, whose distance to each other is greater than or equal to $\mathcal{C}$, then there are at least m eigenvalues of the Dirichlet Laplacian $-\Delta_D$ on the quantum layer Ω less than κ_1^2.*

Proof. To start with, we will make the following assumptions for all $i = 1, \dots, m$:

We take Σ_i to be a Euclidean plane $\mathbb{R}^2$ with one handle H_i, so that each Σ_i has genus 1, one end and negative total Gauss curvature, where

$$\mathcal{K} = \int_{\Sigma_i} K d\Sigma_i = -4\pi.$$

Let R_i be a smallest exterior radius of H_i. Then, we may write $\Sigma_i = H_i \cup \Sigma_i'$, where Σ_i' is isometric to $\mathbb{R}^2 \setminus B_{R_i}(0)$. Therefore, by Remark 5.0.7 we can construct a mollifier φ_{i_n} as in equation (5.0.2). We have already shown in the proof of Theorem 5.1.1 that

$$q_1[\varphi_{i_n}\chi_1] \leq C_+\|\nabla\varphi_{i_n}\|^2 + (\varphi_{i_n}, K\varphi_{i_n}).$$

Now, we want to find $n_o \in \mathbb{N}$ such that $q_1[\varphi_{i_n}\chi_1] < 0$. We have seen in Remark 5.0.7 that

$$\|\nabla\varphi_{i_n}\|^2 \leq \frac{Q}{\log nR_i},$$

where Q is a positive constant, and since $K = 0$ outside the handle and $\varphi_{i_n} = 1$ on H_i,

$$(\varphi_{i_n}, K\varphi_{i_n}) = \int\limits_{\Sigma_i} K\varphi_{i_n} d\Sigma_i = \int\limits_{H_i} K d\Sigma_i = \int\limits_{\Sigma_i} K d\Sigma_i = \mathcal{K} = -4\pi.$$

Therefore, we have

$$C_+\|\nabla\varphi_{i_n}\|^2 + (\varphi_{i_n}, K\varphi_{i_n}) < 0$$

if

$$\frac{Q}{\log nR_i} < \frac{4\pi}{C_+} \quad \text{equivalently,} \quad n > \frac{e^{\frac{QC_+}{2\pi}}}{R_i} = \frac{C_o}{R_i}.$$

Thus, there exist $n_0 \in \mathbb{N}$, such that $n_0 > \frac{C_0}{R_i}$ for all $i = 1, ..., m$, and for $f_i = \varphi_{i_{n_0}}$ we have

$$q[f_i\chi_1] < \kappa_1^2\|f_i\chi_1\|^2. \tag{6.1.1}$$

Then one may consider the f_i as functions on Σ (the surface with m-handles $H_i, i = 1, \ldots, m$). Also, let $p_i \in H_i$ and $p_j \in H_j$, where $i \neq j$, $i, j = 1, \ldots, m$ and assume that

$$\operatorname{dist}(p_i, p_j) \geq \mathcal{C} = n_0^2 R_i^2 + n_0^2 R_j^2 > \left(\frac{C_0}{R_i}\right)^2 R_i^2 + \left(\frac{C_0}{R_j}\right)^2 R_j^2 = 2C_0^2.$$

Therefore, each two different test functions f_i and f_j have disjoint compact supports, and so

$$q(f_i\chi_1, f_j\chi_1) = 0.$$

Now, apply Theorem 6.1.1 to $W = \operatorname{span}\{f_1, \ldots, f_m\} \subset H^2(\Sigma)$. If $\psi \in W$,

$$\psi = \sum_{i=1}^{m} \alpha_i f_i,$$

we have

$$q[\psi\chi_1] = q\big[(\alpha_1 f_1 + \ldots + \alpha_m f_m)\chi_1\big]$$

$$= \alpha_1^2 q[f_1\chi_1] + \ldots + \alpha_m^2 q[f_m\chi_1],$$

and so

$$\|\psi\chi_1\|^2 = \alpha_1^2\|f_1\chi_1\|^2 + \ldots + \alpha_m^2\|f_m\chi_1\|^2. \tag{6.1.2}$$

by using (6.1.1), we may set

$$E = \max\left\{\frac{q[f_i\chi_1]}{\|f_i\chi_1\|^2}, i = 1, \dots, m\right\} < \kappa_1^2,$$

and from (6.1.2), we get

$$q[\psi\chi_1] \leq E\big(\alpha_1^2\|f_1\chi_1\|^2 + \ldots + \alpha_m^2\|f_m\chi_1\|^2\big)$$

$$= E\|\psi\chi_1\|^2.$$

Now, because $\dim W = m$ there are at least m eigenvalues $\leq E < \kappa_1^2$. $\square$

Chapter 7

Approximation of Eigenvalues

Our goal in this section is to approximate the discrete spectrum and the associated eigenfunctions of the Hamiltonian $H^a = -a^2\Delta_D^\Omega$ by using an effective Schrödinger operator H^a_{eff} on $L^2(\Sigma)$. Here $-\Delta_D^\Omega$ is the Dirichlet Laplacian on the quantum layer $\Omega = \Omega^a$ of width $2a$ around Σ and the factor a^2 has been put in for convenience because otherwise the spectrum of H^a would diverge in the limit $a \to 0$.

We derive an effective Schrödinger equation on the surface Σ and we prove that the eigenvalues of the corresponding effective Hamiltonian H^a_{eff} below a certain value coincide up to errors of order a^3 with those of Hamiltonian H^a.

Approximation of Eigenvalues of Hamiltonian has been done in a more general situation in [WT09] and [WTL10].

7.1 Approximation of Eigenvalues in a General Case

In this section, we discuss the general result of [WTL10], Theorem 3.1, which uses the effective Hamiltonian to approximate certain parts of the discrete spectrum and the associated eigenfunctions of the Hamiltonian H^a. This result shows how to obtain the approximations of eigenvalues of H^a from the eigenvalues of H^a_{eff} and vice versa.

This theorem is stated in greater generality than we need here: Let $(\mathcal{A}, G)$ be a Riemannian manifold of dimension $d + k$ $(d, k \in \mathbb{N})$ with metric G, $C \subset \mathcal{A}$ be a smooth submanifold without boundary of dimension d equipped with the induced metric $g = G_{|C}$. C may be compact or non-compact.
At C there is a natural decomposition $T\mathcal{A}_{|C} = TC \times NC$ of $\mathcal{A}$'s tangent bundle into the tangent and the normal bundle of C. Furthermore, let $\Omega \subset \mathcal{A}$ be the tube of radius $a > 0$ around C. (In [WTL10] more general tubes with varying radii and shapes of cross-sections are considered).

Now, we will state Theorem 3.1 in [WTL10] for $\mathcal{A} = \mathbb{R}^3$, g_{eucl} the Euclidean metric and C an orientable submanifold of dimension 2. In this case we may identify NC with $C \times \mathbb{R}$ by virtue of a unit normal vector field N, identifying the point $(x, tN) \in NC$ with $(x, t) \in C \times \mathbb{R}$.

Moreover, let

$$\widehat{\Omega} := \{(x,t) : x \in C, |t| < 1\} = C \times (-1,1) \subset NC,$$

$$\widetilde{\Omega} := \{(x,u) = (x,at) : (x,t) \in \widehat{\Omega}\} = C \times (-a,a) \subset NC,$$

and

$$\begin{aligned} \Phi : NC &\to \mathbb{R}^3 \\ (x,u) &\mapsto x + uN_x, \end{aligned}$$

be the exponential map. We assume that there exists $a_0 > 0$, so that $\Phi|_{\widetilde{\Omega}}$ is a diffeomorphism onto its image. The quantum layer Ω around C is defined, for $a \leq a_0$, as

$$\Omega := \Phi(\widetilde{\Omega}).$$

If g_{eucl} denoted the Euclidean metric on Ω, we define the metric and the Hamiltonian on $\widetilde{\Omega}$ for each $a > 0$, as

$$\begin{aligned} G &:= (\Phi)^*(g_{eucl}), \\ \widetilde{H}^a &:= a^2 \widetilde{H} = -a^2 \Delta^{\widetilde{\Omega}}, \end{aligned}$$

where $-\Delta^{\widetilde{\Omega}}$ is the Laplace operator on $\widetilde{\Omega}$ for the metric G.

Moreover, we can also scale any subset of $\widehat{\Omega}$ in the normal direction via

$$\begin{aligned} \phi|_{\widehat{\Omega}} : \widehat{\Omega} &\to \widetilde{\Omega} \\ (x,t) &\mapsto (x,at) = (x,u). \end{aligned}$$

Now, we want to define the Hamiltonian $\widehat{H}^a$ on $\widehat{\Omega}$ via an unitary transform as in the following definition:

Definition 7.1.1. *Let $d\widetilde{\Omega}$ be the volume element of $\widetilde{\Omega}$ for the metric G, and $d\overline{\Omega} = dCdu$ be the volume element of the product metric $g + du^2$ on $C \times (-a,a)$.*

i) The unitary transform $\widehat{U}$ is defined by

$$\begin{aligned} \widehat{U} : L^2(\widetilde{\Omega}, d\overline{\Omega}) &\to L^2(\widetilde{\Omega}, d\widetilde{\Omega}) \\ \psi &\mapsto \sigma^{-\frac{1}{2}}\psi, \end{aligned}$$

where

$$\sigma = \frac{d\widetilde{\Omega}}{d\overline{\Omega}}.$$

ii) The dilation operator $\widehat{D}$ is defined by

$$\begin{aligned} \widehat{D} : L^2(\widehat{\Omega}, dCdt) &\to L^2(\widetilde{\Omega}, d\overline{\Omega}) \\ (\widehat{D}\psi)(x,u) &:= a^{-\frac{1}{2}}\psi(x, \frac{u}{a}). \end{aligned}$$

Then, we define the Hamiltonian $\widehat{H}^a$ as

$$\widehat{H}^a = \widehat{D}^* \widehat{U}^* \widetilde{H}^a \widehat{U} \widehat{D} \qquad \text{on} \quad L^2(\widehat{\Omega}, dCdt).$$

In [WT09] has been shown that, the leading order of $\widehat{H}^a$ may be split as

$$\widehat{H}^a = -a^2\Delta_C + V - \partial_t^2 + O(a^3), \tag{7.1.1}$$

Where $-\Delta_C$ is the Laplacian on C and V is a function depending on the coordinates x, only.

For $x \in C$, let

$$\widehat{\Omega}(x) = \widehat{\Omega} \cap N_x C.$$

The restriction of

$$\widehat{G} := (\phi)^*(G)$$

to $N_x C$ is the Euclidean metric, which it is the metric on $\widehat{\Omega}$ and $\widehat{\Omega}(x)$ is its unit ball.
We denote by $\chi(x,.)$ the first L^2-normalized eigenfunction of the Laplacian $-\partial_t^2$ on $\widehat{\Omega}(x)$ and by E the associated eigenvalue. E and χ are independent of x, so that $\chi(x,t) = \chi(t)$ for $t \in (-1,1)$.
The associated space

$$\begin{aligned}\mathcal{P} &= \ker(-\partial_t^2 - EI)\\ &= \{\varphi(x)\chi(t), \varphi(x) \in L^2(C,g)\} \subset L^2(\widehat{\Omega}, \widehat{G}),\end{aligned}$$

may be identified with $L^2(C,g)$ via the unitary operator

$$\begin{aligned} U : \mathcal{P} &\to L^2(C,g)\\ \varphi(x)\chi(t) &\mapsto \varphi(x)\end{aligned} \tag{7.1.2}$$

$\mathcal{P}$ is approximately invariant under $\widehat{H}^a$ in the following sense: the projection P onto $\mathcal{P}$ is the spectral projection of $-\partial_t^2$ and we know that $[\partial_t^2, P] = [E, P] = 0$. Hence

$$[P, \widehat{H}^a] = [P, -a^2\Delta_C + V] + O(a^3) = O(a^3).$$

Remark 7.1.2. *In [WTL10], more general operators are considered where the function V does not depend only on x, the codimension of C is $k \neq 1$ and Hamiltonian $H^a = -a^2\Delta_C + V - \Delta_n + O(a)$ where $-\Delta_n$ is the normal Hamiltonian. Therefore $[P, \widehat{H}^a] = O(a)$. Then, they use adiabatic perturbation theory to get a better approximation of the spectrum:*

Fix $E_{max} < \infty$. Via adiabatic perturbation theory it is possible to construct a projection

$$P^a = P + aP^1 + a^2P^2 \tag{7.1.3}$$

and a unitary

$$U^a : \mathcal{P}^a \to L^2(C,g) \tag{7.1.4}$$

such that

$$[P^a, \widehat{H}^a]_{\chi(-\infty,E_{max}]}(\widehat{H}^a) = O(a^3),$$

where $\chi(-\infty, E_{max}]$ is the characteristic function of $(-\infty, E_{max}]$.

The Theorem 3.1 in [WTL10] in a general case is as follows, where the latin indices $i, j, ...$ running from 1 to d for the normal coordinates on C.

Theorem 7.1.3. *Let E be the first eigenvalue of $-\Delta_n$ and $E_{max} < \infty$. There are $c > 0$ and $a_0 > 0$ such that for $a < a_0$ there exist a Riemannian metric g^a on C, an orthogonal projection P^a and a unitary operator U^a as in (7.1.3) and (7.1.4), respectively. Then the operator H^a_{eff} defined by*

$$H^a_{\mathit{eff}} := U^a P^a \widehat{H}^a P^a U^{a*} \quad \text{with domain} \quad U^a D(\widehat{H}^a)$$

satisfies the following:

For all functions $a \mapsto E^a$ and $a \mapsto \widehat{E}^a$ defined for $a \in (0, a_0)$ with $\limsup_a E^a < E_{max}$ and $\limsup_a \widehat{E}^a < E_{max}$ one has

(i) $H^a_{\mathit{eff}}\varphi^a = E^a\varphi^a \Rightarrow \big\|(\widehat{H}^a - E^a)U^{a*}\varphi^a\big\| \leq ca^3\|U^{a*}\varphi^a\|$,

(ii) $\widehat{H}^a\psi^a = \widehat{E}^a\psi^a \Rightarrow \big\|(H^a_{\mathit{eff}} - \widehat{E}^a)U^a P^a\psi^a\big\| \leq ca^3\|\psi^a\|$.

For $\varphi_1 = \chi_{(-\infty, E_{max}]}(-a^2\Delta_C + E)\varphi_1$ the effective Hamiltonian H^a_{eff} is given by

$$\begin{aligned}(\varphi_2, H^a_{\mathit{eff}}\varphi_1)_C = \int_C & \big[(g^a)^{ij}(p^a)^i\varphi_2(p^a)^j\varphi_1 + \varphi_2 E\varphi_1 - a^2\varphi_2 {U^a_1}^* R_{-\Delta_t}(E)U^a_1\varphi_1 \\ & + a^2\varphi_2(V_{geom} + V_{BH} + V_{amb})\varphi_1\big]dC + O(a^3),\end{aligned} \tag{7.1.5}$$

where

$$(g^a)^{ij} = g^{ij} + aA^{ij} + a^2(\textit{term contain } \overline{R}) + a^2 B_1^{ij},$$

$$(p^a)^i = a\partial_i - a\big(\chi(t), \nabla^h_i\chi(t)\big) - a^2(\textit{term contain } \overline{R}) + a^2 B_2^{ij},$$

$$R_{-\partial_t^2}(E) = (1 - P^a)(-\Delta_n - E)^{-1}(1 - P^a),$$

$$U^a_1 = 2g^{ij}\nabla^h_i\chi(t)a\partial_j + a^2 B_3^{ij},$$

$$V_{geom} = -\frac{1}{4}\eta^2 + \frac{1}{2}R^{ij}_{ij} + (\textit{term contain } \overline{R})$$

$$V_{BH} = g^{ij}\big(\nabla^h_i\chi(t), (1 - p)\nabla^h_j\chi(t)\big),$$

$$V_{amb} = \textit{term contain } \overline{R},$$

with $\frac{1}{2}\eta$ the mean curvature, $R, \overline{R}$ the Riemann tensor of C and $\mathcal{A}$, and terms $A^{ij}, B_1^{ij}, B_2^{ij}, B_3^{ij}$ which can be compute from the curvature data of C and $\mathcal{A}$.

At the beginning, we assumed that $\mathcal{A} = \mathbb{R}^3$ with an Euclidean metric, hence all the terms that contain $\overline{R}$ are vanish because the Riemann tensor with respect to the Euclidean metric equals zero. Moreover, let $v \in T_xC$, and a function $\chi(t)$ on NC, the lift of v to NC has no t-component, and thus $\nabla^h_v\chi(t) = 0$. Then, we can rewrite the above Theorem in the special case such that

Theorem 7.1.4. *Let E be the first Dirichlet eigenvalue of $-\partial_t^2$ on $[-1,1]$, P the orthogonal projection and U the unitary operator associated with E as in* (7.1.2). *Then the effective Hamiltonian H^a_{eff} defined by*

$$H^a_{\mathit{eff}} := UP\widehat{H}^a PU^* \quad \text{with domain} \quad UD(\widehat{H}^a)$$

$$= -a^2\Delta_C + E + V_{geom} + O(a^3).$$

satisfies the following:

For all functions $a \mapsto E^a$ and $a \mapsto \widehat{E}^a$ defined for $a \in (0, a_0)$ with $\limsup_a E^a < E_{max}$ and $\limsup_a \widehat{E}^a < E_{max}$ one has

(i) $H^a_{\mathit{eff}}\varphi = E^a\varphi \Rightarrow \big\|(\widehat{H}^a - E^a)U^*\varphi\big\| \le ca^3\|U^*\varphi\|,$

(ii) $\widehat{H}^a\psi = \widehat{E}^a\psi \Rightarrow \big\|(H^a_{\mathit{eff}} - \widehat{E}^a)UP\psi\big\| \le ca^3\|\psi\|.$

7.2 Approximation of the Eigenvalues of the Dirichlet Laplacian on Thin Layers

In this section we will apply the above statements to our case where $\mathcal{A} = \mathbb{R}^3$, g_{eucl} is the Euclidean metric and $C = \Sigma$ is a complete, connected and non-compact surface embedded in $\mathbb{R}^3$.
Recall that the quantum layer Ω of width $2a$ around Σ is the image of the mapping

$$\begin{aligned}\mathcal{L} : \Sigma \times (-a,a) &\to \mathbb{R}^3\\ (x,u) &\mapsto x + un,\end{aligned}$$

where $\widetilde{\Omega} = \Sigma \times (-a,a)$.

We also considered the Dirichlet Laplacian $-\Delta_D^{\widetilde{\Omega}}$ on $L^2(\widetilde{\Omega})$, which we can write by using the coordinate (x,u) (see Chapter 1, Section: The Hamiltonian) as

$$-\Delta^{\widetilde{\Omega}} = -\frac{1}{\sqrt{\det G}}\partial_\mu\sqrt{\det G}G^{\mu\nu}\partial_\nu + 2\frac{M - Ku}{1 - 2Mu + Ku^2}\partial_u - \partial_u^2,$$

on $L^2\big(\Sigma \times (-a,a), d\widetilde{\Omega}\big)$, where $d\widetilde{\Omega} = (1 - 2Mu + Ku^2)d\Sigma du$.

At the same time, we can define an alternative form of the Hamiltonian by using the following unitary transformation,

$$\begin{aligned}\widehat{U} : L^2\big(\Sigma \times (-a,a), d\Sigma du\big) &\to L^2\big(\Sigma \times (-a,a), d\widetilde{\Omega}\big)\\ \psi(x,u) &\mapsto \sigma^{-\frac{1}{2}}\psi(x,u),\end{aligned}$$

where

$$\sigma = \frac{d\widetilde{\Omega}}{d\Sigma du} = (1 - 2Mu + Ku^2).$$

This leads to the unitary equivalent operator

$$\widehat{H}_1 = (\widehat{U})^{-1}\widetilde{H}\widehat{U}.$$

Simplifying notation, we will denote $\partial_u\sigma = \sigma'$ and $\partial_u^2\sigma = \sigma''$, then we can write $\widehat{H}_1$ as

$$\widehat{H}_1 = \sigma^{\frac{1}{2}}\Big(- \frac{1}{\sqrt{\det G}}\partial_\mu\sqrt{\det G}G^{\mu\nu}\partial_\nu - \frac{\sigma'}{\sigma}\partial_u - \partial_u^2\Big)\sigma^{-\frac{1}{2}} \tag{7.2.1}$$

and by using the equations

$$(\sigma^{-\frac{1}{2}})' = -\frac{1}{2}\sigma^{-\frac{3}{2}}\sigma',$$

$$\begin{aligned}(\sigma^{-\frac{1}{2}})'' &= \frac{3}{4}\sigma^{-\frac{5}{2}}\sigma'^2 - \frac{1}{2}\sigma^{-\frac{3}{2}}\sigma'' \\ &= \frac{1}{4}\sigma^{-\frac{5}{2}}(3\sigma'^2 - 2\sigma\sigma''),\end{aligned}$$

we can calculate the second and the third terms in the right hand side of equation (7.2.1):

$$\sigma^{\frac{1}{2}}\big(- \frac{\sigma'}{\sigma}\partial_u\big)\sigma^{-\frac{1}{2}} = \frac{1}{2}\sigma^{-2}\sigma'^2 - \frac{\sigma'}{\sigma}\partial_u,$$

$$\sigma^{\frac{1}{2}}(-\partial_u^2)\sigma^{-\frac{1}{2}} = -\frac{1}{4}\sigma^{-2}(3\sigma'^2 - 2\sigma\sigma'') + \frac{\sigma'}{\sigma}\partial_u - \partial_u^2.$$

By adding the last two equations, we get

$$\begin{aligned}\sigma^{\frac{1}{2}}(-\partial_u^2)\sigma^{-\frac{1}{2}} + \sigma^{\frac{1}{2}}\big(- \frac{\sigma'}{\sigma}\partial_u\big)\sigma^{-\frac{1}{2}} &= \frac{1}{4}\sigma^{-2}(-\sigma'^2 + 2\sigma\sigma'') - \partial_u^2 \\ &= \frac{1}{4(1-2Mu+Ku^2)^2}\Big[- (-2M + 2Ku)^2 \\ &\qquad + 2(1-2Mu+Ku^2)(2K)\Big] - \partial_u^2 \\ &= \frac{K - M^2}{(1-2Mu+Ku^2)^2} - \partial_u^2.\end{aligned}$$

Also, the first term in the right hand side of equation (7.2.1), can be rewritten as

$$\begin{aligned}\sigma^{\frac{1}{2}}\Big(- \frac{1}{(\sqrt{\det G})}\partial_\mu\sqrt{\det G}G^{\mu\nu}\partial_\nu\Big)\sigma^{-\frac{1}{2}} &= \sigma^{\frac{1}{2}}\Big(- \sigma^{-\frac{1}{2}}\frac{1}{\sqrt{\det g}}\partial_\mu\sigma^{\frac{1}{2}}\sqrt{\det g}G^{\mu\nu}\partial_\nu\Big)\sigma^{-\frac{1}{2}} \\ &= -\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det G}G^{\mu\nu}(\partial_\nu\sigma^{-\frac{1}{2}}).\end{aligned}$$

Then, we have

$$\widehat{H}_1 = -\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det G}G^{\mu\nu}(\partial_\nu\sigma^{-\frac{1}{2}}) + \frac{K - M^2}{(1-2Mu+Ku^2)^2} - \partial_u^2.$$

Now, we define the dilation operator

$$\widehat{D} : L^2\big(\Sigma \times (-1,1), d\Sigma dt\big) \to L^2\big(\Sigma \times (-a,a), d\Sigma du\big)$$

by

$$(\widehat{D}\psi)(x,u) := a^{-\frac{1}{2}}\psi\big(x,\frac{u}{a}\big).$$

For the function

$$\psi(x,t) = \varphi(x)\chi(t) = \varphi(x)\cos\big(\frac{\pi t}{2}\big)$$

we get

$$(\widehat{D}\psi)(x,u) = a^{-\frac{1}{2}}\psi\big(x,\frac{u}{a}\big) = a^{-\frac{1}{2}}\varphi(x)\chi\big(\frac{u}{a}\big) = \sqrt{\frac{1}{a}}\varphi(x)\cos\big(\frac{\pi u}{2a}\big)$$

Moreover, the dilation Hamiltonian, is given by

$$\begin{aligned}\widehat{H}_2 &:= (\widehat{D})^{-1}\widehat{H}_1\widehat{D}\\ &= (\widehat{D})^{-1}\Big[-\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det G}G^{\mu\nu}(\partial_\nu\sigma^{-\frac{1}{2}})\\ &\qquad + \frac{K-M^2}{(1-2Mu+Ku^2)^2} - \partial_u^2\Big]\widehat{D}.\end{aligned}$$

We will calculate each term of the last equation, denoting

$$\begin{aligned}\widehat{\sigma} &= (\widehat{D})^{-1}\sigma\widehat{D} = 1 - 2M(at) + K(at)^2,\\ \det\widehat{G} &= (\widehat{D})^{-1}(\det G)\widehat{D}.\end{aligned}$$

Now,

$$\begin{aligned}&(\widehat{D})^{-1}\Big[-\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det G}G^{\mu\nu}(\partial_\nu\sigma^{-\frac{1}{2}})\Big]\widehat{D}\\ &\qquad = -\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det \widehat{G}}G^{\mu\nu}(\partial_\nu\widehat{\sigma}^{-\frac{1}{2}}),\end{aligned}$$

$$(\widehat{D})^{-1}\Big[\frac{K-M^2}{(1-2Mu+Ku^2)^2}\Big]\widehat{D} = \frac{K-M^2}{\big(1-2M(at)+K(at)^2\big)^2},$$

$$(\widehat{D})^{-1}\big[-\partial_u^2\big]\widehat{D} = -\partial_{at}^2 = -\frac{1}{a^2}\partial_t^2,$$

and thus

$$\widehat{H}_2 = -\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}G^{\mu\nu}\partial_\nu - \frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det \widehat{G}}G^{\mu\nu}(\partial_\nu\widehat{\sigma}^{-\frac{1}{2}}) + \frac{K-M^2}{\big(1-2M(at)+K(at)^2\big)^2} - \frac{1}{a^2}\partial_t^2,$$

where, the first three terms depend on x and t and ∂_t^2 independent of x. For a thin layers the Hamiltonian decouples into a sum of two operators

$$\widehat{H}_{2_x} = -\frac{1}{\sqrt{\det g}}\partial_\mu\sqrt{\det g}g^{\mu\nu}\partial_\nu + (K-M^2),$$

$$\widehat{H}_{2_t} = -\frac{1}{a^2}\partial_t^2$$

Therefore, it follows that

$$\widehat{H}_2 = -\frac{1}{\sqrt{\det g}}\partial_\mu \sqrt{\det g} g^{\mu\nu}\partial_\nu + (K - M^2) - \frac{1}{a^2}\partial_t^2 + O(a).$$

Writing $\widehat{H}^a = a^2 \widehat{H}_2$, so we get

$$\widehat{H}^a = -a^2\Delta_\Sigma + a^2(K - M^2) - \partial_t^2 + O(a^3).$$

Now, we will check that this coincides with Theorem 7.1.4 when applied to our case. To begin with, we will compute the effective Hamiltonian H^a_{eff} of $\widehat{H}^a$. The first eigenvalue of $-\partial_t^2$ is $E = (\frac{\pi}{2})^2$ and the corresponding eigenfunction is $\chi_1(t) = \cos\frac{\pi t}{2}$, thus the associated eigenspace is given by

$$\mathcal{P} = \ker\Big(-\partial_t^2 - \big(\frac{\pi}{2}\big)^2\Big) = \Big\{\varphi(x)\chi_1(t), \varphi(x) \in L^2(\Sigma)\Big\} \tag{7.2.2}$$

and the unitary operator U is:

$$\begin{aligned} U : \mathcal{P} &\to L^2(\Sigma) \\ \varphi(x)\chi_1(t) &\mapsto \varphi(x). \end{aligned} \tag{7.2.3}$$

Furthermore, the effective Hamiltonian H^a_{eff} is given in the following theorem

Lemma 7.2.1. *For $E = (\frac{\pi}{2})^2$, the first Dirichlet eigenvalue of $-\partial_t^2$ on $[-1, 1]$, and for all $\varphi \in L^2(\Sigma)$, the quadratic form corresponding to effective Hamiltonian H^a_{eff} of $\widehat{H}^a$, is given by*

$$(\varphi, H^a_{\mathit{eff}}\varphi)_\Sigma = \int_\Sigma \Big[a^2(\nabla\varphi)^2 + \big(\frac{\pi}{2}\big)^2\varphi^2 + a^2(K - M^2)\varphi^2\Big]d\Sigma + O(a^3).$$

Proof. For $i, j = x_1, x_2$ the coordinates on Σ, we will calculate the components in the equation (7.1.5) as follows:

$$(g^a)^{ij} = g^{ij} + O(a),$$

$$(p^a)^i = a\nabla_i + O(a^2),$$

$$U_1^a = O(a^2),$$

$$V_{\text{BH}} = 0,$$

$$V_{\text{amb}} = 0.$$

Moreover, since R is the Riemann tensor of Σ and $R^{ij}_{ij} = h_{jj}h_{ii} - h_{ij}h_{ji}$, where (h_{ij}) is the matrix associated to the second fundamental form then

$$\Sigma_{ij}R^{ij}_{ij} = 0 + \kappa_1\kappa_2 + \kappa_1\kappa_2 + 0 = 2K,$$

and since $M = \frac{1}{2}\eta$ is the mean curvature, then we have

$$V_{\text{geom}} = K - M^2.$$

By using equation (7.1.5), we obtain

$$H^a_{\text{eff}} = -a^2\Delta_\Sigma + a^2(K - M^2) + \big(\frac{\pi}{2}\big)^2 + O(a^3). \quad \square$$

At the close of this section, we rewrite Theorem 7.1.4 adopted to our case.

Theorem 7.2.2. *Let $E = (\frac{\pi}{2})^2$ be the first eigenvalue of $-\partial_t^2$, P an orthogonal projection and U a unitary operator defined in 7.2.3. Then the effective Hamiltonian*

$$H^a_{\mathit{eff}} := UP\widehat{H}^a PU^* \quad \text{with domain} \quad UD(\widehat{H}^a)$$

$$= -a^2\Delta_\Sigma + a^2(K - M^2) + \big(\frac{\pi}{2}\big)^2 + O(a^3).$$

satisfies the following:

For all functions $a \mapsto E^a$ and $a \mapsto \widehat{E}^a$ defined for $a \in (0, a_0)$ with $\limsup_a E^a < E$ and $\limsup_a \widehat{E}^a < E$ one has

(i) $H^a_{\mathit{eff}}\varphi = E^a\varphi \Rightarrow \big\|(\widehat{H}^a - E^a)U^*\varphi\big\| \leq ca^3\|U^*\varphi\|,$

(ii) $\widehat{H}^a\psi = \widehat{E}^a\psi \Rightarrow \big\|(H^a_{\mathit{eff}} - \widehat{E}^a)UP\psi\big\| \leq ca^3\|\psi\|.$

Remark 7.2.3. *We recall that for any self-adjoint operator H the bound $\big\|(H - \lambda)\psi\big\| < \delta\|\psi\|$ for $\lambda \in \mathbb{R}$, implies that H has spectrum in the interval $[\lambda - \delta, \lambda + \delta]$. So that the statement (i) in Theorem 7.2.2 entails that $\widehat{H}^a$ has an eigenvalue in the interval of length $2ca^3$ around the eigenvalue E^a of H^a_{eff}. Then for a given family of eigenvalues E^a of H^a_{eff} with $\limsup_a E^a < E$ and a family of corresponding eigenfunctions φ, we may write the corresponding eigenvalues of $\widehat{H}^a$ as*

$$\begin{aligned}\lambda &= E^a + O(a^3)\\ &= \text{eigenvalue of } \big(-a^2(\Delta_\Sigma - (M^2 - K))\big) + \big(\frac{\pi}{2}\big)^2 + O(a^3).\end{aligned}$$

Remark 7.2.4. *Since By means of (1.2.6) and $\psi(x,t) = \varphi_n(x)\chi_1(t)$, we have*

$$\|\psi\|^2 = \int_{-1}^{1}\int_{\Sigma} \varphi^2\chi_1^2\sqrt{G}dtdx$$

$$\leq C_+\int_{-1}^{1}\chi_1^2 dt\int_{\Sigma}\varphi^2\sqrt{g}dx$$

$$= C_+ \backslash C_- \int_{\Sigma}\varphi^2 C_-\sqrt{g}dx$$

$$\leq C_+ \backslash C_- \int_{\Sigma}\varphi^2\sqrt{G}dx = C_+\backslash C_-\|\varphi\|^2 = C\|UP\psi\|^2.$$

Then, we can also derive bounds for the eigenvalues of H^a_{eff} from the eigenvalue of $\widehat{H}^a$.

Chapter 8

Estimating the Number of Eigenvalues of the Dirichlet Laplacian on Thin Layers

The goal of this chapter is to give a lower bound on the number of eigenvalues of the Dirichlet Laplacian on a thin quantum layer Ω around the reference surface Σ with bounded Gauss curvature from below, in terms of the genus g of the surface Σ.

We have seen in Theorem 7.2.2 that we may write the eigenvalues of the Dirichlet Laplacian $-\Delta_D$ on the thin quantum layer Ω in the following way:

$$\lambda = \left(\frac{\pi}{2a}\right)^2 + E + O(a) \qquad \text{as} \quad a \to 0,$$

where E is the corresponding eigenvalues of the Schrödinger operator $L = -\Delta_\Sigma - V$, where

$$V = M^2 - K = \left(\frac{k_1 - k_2}{2}\right)^2.$$

In order to obtain a lower bound on the number of eigenvalues of $-\Delta_D^\Omega$ which are less than $\kappa_1^2 = (\frac{\pi}{2a})^2$, we therefore need to find a lower bound on the number of negative eigenvalues of the Schrödinger operator L.

For more details on how to find negative eigenvalues of Schrödinger operators see [GNY04].

8.1 Introduction and Statement

8.1.1 Covering Property

Definition 8.1.1. *Given $l > 1$ and a positive integer N, we say that a metric space (X, d) satisfies the (l, N)-covering property if, for any ball $B(x, r)$ in X, there exists a family of at most N balls of radius r/l, which cover $B(x, r)$.*

Lemma 8.1.2. *If a metric space (X, d) satisfies the (l, N)-covering property then it satisfies the (λ, Q)-covering property for any $\lambda > 1$ and some $Q = Q(\lambda, l, N)$.*

Proof. Indeed, let j be a positive integer such that $l^{j-1} < \lambda < l^j$. Since the (λ, Q)-covering property is monotone in λ, it suffices to assume that $\lambda = l^j$ where $j \in \mathbb{N}$. If $j = 1$ then the claim is trivial. Let us make the inductive step from j to $j+1$. By the inductive hypothesis, any ball $B(x,r)$ can be covered by at most Q_j balls $B(x_i, r/l^j)$ and each ball $B(x_i, r/l^j)$ can be covered by at most N balls of radius r/l^{j+1} each. Hence, $B(x,r)$ can be covered by at most $Q_{j+1} := NQ_j$ balls of radius r/l^{j+1} each, which settles the claim.

For example, if $Q = Q(\lambda, 2, N)$ and $\lambda < 2^j$ then a ball of radius r can be covered by at most $Q = N^j$ balls of radius r/λ.

8.1.2 Finding Disjoint Annuli in a Metric Space

Let (X,d) be a metric space. By an annulus in X we mean any set $A \subset X$ of the form

$$A = \{x \in X : r \le d(x,a) < R\},$$

where $a \in X$ (in particular, if $r = 0$ then A is the ball $B(x,R)$). Also, denote by $2A$ the following annulus:

$$2A = \{x \in X : \frac{1}{2} r \le d(x,a) < 2R\}.$$

The following theorem proves the existence of disjoint annuli and is key to our approach to estimating the number of eigenvalues.

Theorem 8.1.3. *Let (X,d) be a metric space and ν a non-atomic Radon measure on X. Assume that the metric space (X,d) satisfies the following:*

1. All metric balls $B(x,r) = \{y \in X : d(x,y) < r\}$ in X are precompact.

2. (X,d) has the $(2,N)$- covering property.

Then for any positive integer k, there exists a sequence $\{A_i\}_{i=1}^k$ of k annuli in X such that

(i) the annuli $2A_i$ are disjoint,

(ii) $\nu(A_i) \ge c_N \frac{\nu(X)}{k}$, for any $i = 1, 2, \ldots, k$.

Here c_N is a positive constant depending on N. (For example, one can define it by $c_N^{-1} = 2 + 4Q(1600, 2, N)$ where $Q = N^{11}$ by Lemma 8.1.2).

Remark 8.1.4. *A measure ν on X is called a Radon measure if ν is inner regular, defined on all Borel sets on X and is finite on all compact sets.*

8.1.3 Eigenvalues of Schrödinger Operators

Let X be a Riemannian manifold and $-\Delta$ be the Laplace operator on X. Given a non-negative function V, where $V \in L^1_{loc}(X, \mu)$ and μ is the Riemannian measure on X, consider the Schrödinger operator

$$L = -\Delta - V$$

and the associated quadratic form

$$q[f] := \int_X f(Lf) d\mu = \int_X \left(|\nabla f|^2 - V f^2\right) d\mu,$$

defined for all $f \in H^1(X)$.

For any λ, define the counting function $N_\lambda(L)$ as the supremum of the dimensions of all vector spaces $\mathcal{V} \subset H^1(X)$ such that

$$q[f] < \lambda \|f\|^2 \qquad \text{for all} \quad f \in \mathcal{V}, f \neq 0.$$

In particular, if the spectrum of L below λ is discrete, then $N_\lambda(L)$ is just the number of eigenvalues of L which are smaller than λ, counted with multiplicities. In the case $\lambda = 0$, we will also use the notation

$$Neg(L) = N_0(L).$$

8.2 Estimating the Number of Negative Eigenvalues of Schrödinger Operators

We start this section by the following remark

Remark 8.2.1. *If we apply the Corollary 5.6 of [GNY04], which say "If Σ is a Riemann surface which is conformal to $(\Sigma^g \setminus P)$, where Σ^g is a closed orientable Riemann surface of genus g and P is a finite subset of Σ^g, then, for any non-negative function $V \in L^1_{loc}(\Sigma)$, we have*

$$Neg(-\Delta_\Sigma - V) \geq \frac{\int_\Sigma V d\Sigma}{C(g+1)}, \tag{8.2.1}$$

where C is a positive constant.", in our case where Σ is a complete connected non-compact surface with finite total Gauss curvature. Since, By Theorem 3.6.1, our surface Σ is conformally equivalent to a compact Riemann surface with finitely many punctures and $V = M^2 - K$ is a non negative function in $L^1_{loc}(\Sigma)$. Then we have:

$$Neg\big(-\Delta_\Sigma - (M^2 - K)\big) \geq \frac{\int_\Sigma (M^2 - K) d\Sigma}{C(g+1)}, \tag{8.2.2}$$

and

$$\int_\Sigma (M^2 - K) d\Sigma \geq -\int_\Sigma K d\Sigma \geq 2\pi(e + 2g - 2), \tag{8.2.3}$$

and we also have $\chi(\Sigma) < 0$ and hence $e + 2g \geq 3$, which implies

$$e + 2g - 2 \geq \frac{1}{2}(g+1). \tag{8.2.4}$$

Indeed, if (8.2.4) fails then $2e + 3g \leq 4$, which is not compatible with $e + 2g \geq 3$. Substituting (8.2.4) and (8.2.3) into (8.2.2) we obtain

$$Neg(L) \geq \frac{\pi}{C} = C', \tag{8.2.5}$$

where C' is a positive constant.

But this result is not very useful because the bound on $Neg(L)$ is independent of the genus g of the surface Σ and C' could well be less than 1 which showed that $Neg(L) \geq 1$. Then, we have at least one eigenvalue of $-\Delta_D^\Omega$ and this we have already proved in Corollary 5.1.4.

Therefore, we might need more assumptions to get stronger results. First, we will recall some information from [GNY04].

The following result gives a lower bound on the number of negative eigenvalues of the Schrödinger operator in a complete Rimannian manifold:

Theorem 8.2.2. *Let X be a complete Riemannian manifold. Assume that for some constants N and M, the following is true:*

(i) any ball $B(x,r)$ in X can be cover by at most N balls of radii $\frac{r}{2}$,

(ii) for all $x \in X$ and $r > 0$

$$\operatorname{vol}\big(B(x,r)\big) \leq Mr^2. \tag{8.2.6}$$

Then, for any function V is defined as above,

$$Neg(-\Delta - V) \geq \big\lfloor C_{N,M} \int\limits_X V d\mu \big\rfloor, \tag{8.2.7}$$

where the constant $C_{N,M}$ only depends on N and M.

Proof. to begin with, let us describe an approach to the proof. Since

$$Neg(L) = \sup\big\{ \dim \mathcal{V} : \mathcal{V} \subset H^1(X) \quad \text{and} \quad q(f) < 0 \quad \forall f \in \mathcal{V}\backslash\{0\}\big\},$$

it suffices to construct a subspace $\mathcal{V}$ of $H^1(X)$ such that $q[f]$ is negative on $\mathcal{V}$ and

$$\dim(\mathcal{V}) \geq C \int\limits_X V d\mu, \qquad \text{where} \quad C = C_{N,M}.$$

We will construct $\mathcal{V}$ as the span of functions f_i, where $\{f_i\}_{i=1}^k$ is a sequence of functions with disjoint compact supports such that $q[f_i] < 0$. Then $q[f] < 0$ will be true for any non-zero function $f \in \operatorname{span}\{f_i\}$, and $\dim \mathcal{V} = k$. Hence it suffices to construct a sequence $\{f_i\}_{i=1}^k$ of functions with disjoint compact supports such that, for any $i = 1, ..., k$,

$$\int\limits_X |\nabla f_i|^2 d\mu < \int\limits_X V f_i^2 d\mu,$$

and

$$k \geq C \int\limits_X V d\mu.$$

For an annulus $A = \big\{x \in X : r \leq d(x,a) < R\big\}$, consider the following function

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin 2A, \\ \frac{1}{\ln 2} \ln \frac{2|x|}{r} & \frac{r}{2} \leq |x| \leq r, \\ \frac{1}{\ln 2} \ln \frac{2R}{|x|} & R \leq |x| \leq 2R. \end{cases} \tag{8.2.8}$$

where $|x|$ means the distance $d(x,a)$.

Lemma 8.2.3. *Under assumption* (ii) *of Theorem 8.2.2, the function* f *of* (8.2.8) *satisfies*

$$\int_X |\nabla f|^2 d\mu \leq CM,$$

for some constant C.

Proof. From the definition of f, it is clear that $\nabla f = 0$ in A or outside $2A$ and

$$\left|\nabla f(x)\right| \leq \frac{C'}{r} \qquad \text{if} \quad \frac{r}{2} \leq |x| \leq r,$$

$$\left|\nabla f(x)\right| \leq \frac{C'}{R} \qquad \text{if} \quad R \leq |x| \leq 2R,$$

where C' independent of r and R. By using hypothesis (ii) of the Theorem 8.2.2 then, we have

$$\begin{aligned}\int_X |\nabla f|^2 d\mu &= \int_{\frac{r}{2}\leq d(x,a)\leq r} |\nabla f|^2 d\mu + \int_{R\leq d(x,a)\leq 2R} |\nabla f|^2 d\mu \\ &\leq \frac{C'^2}{r^2}\operatorname{vol}\left(\frac{r}{2} \leq d(x,a) \leq r\right) + \frac{C'^2}{R^2}\operatorname{vol}\left(R \leq d(x,a) \leq 2R\right) \\ &\leq \frac{C'^2}{r^2}\operatorname{vol}\left(d(x,a) \leq r\right) + \frac{C'^2}{R^2}\operatorname{vol}\left(d(x,a) \leq 2R\right) \\ &\leq \frac{C'^2}{r^2} M r^2 + \frac{4C'^2}{R^2} M R^2 \\ &\leq CM. \quad \square\end{aligned}$$

Now we need a sequence of annuli $\{A_i\}_{i=1}^k$, but such that the annuli $2A_i$ are mutually disjoint. Then, we define f_i for each pair $(A_i, 2A_i)$ as above.
By applying Theorem 8.1.3 with the measure ν, where $d\nu = V d\mu$, we know that for any positive integer k, there exist k annuli A_i in M such that

$$\int_{A_i} V d\mu \geq c_N \frac{\nu(X)}{k},$$

and the annuli $2A_i$ are disjoint. k will be chosen later. Also note that

$$\int_X V f_i^2 d\mu \geq \int_{A_i} V f_i^2 d\mu = \int_{A_i} V d\mu.$$

Hence by Lemma 8.2.3, the condition

$$\int_X |\nabla f_i|^2 d\mu < \int_X V f_i^2 d\mu$$

will be satisfied if

$$\int\limits_{A_i} V d\mu \geq c_N \frac{\nu(X)}{k} > CM. \tag{8.2.9}$$

For simplicity, let $v = \frac{CM}{c_N}$. We have two cases:

i) If $\nu(X) \leq v$ then such a k cannot be constructed.

ii) If $\nu(X) > v$, then there exists an integer $k \in \left[\frac{1}{2}\frac{\nu(X)}{v}, \frac{\nu(X)}{v}\right)$. This k satisfies (8.2.9), hence there are k annuli satisfying the hypotheses of Theorem 8.1.3, and we can define the test function f_i for each pair $(A_i, 2A_i)$ as above. In this way we obtain a sequence of functions with disjoint supports and with

$$\int\limits_X |\nabla f|^2 d\mu \leq CM.$$

Now we consider the linear space $\mathcal{V}$ spanned by above test functions f_i. $\mathcal{V}$ is a k-dimensional subspace of $H^1(X)$ such that for any $f \in \mathcal{V}\backslash\{0\}$

$$\int\limits_X |\nabla f|^2 d\mu < \int\limits_X V f^2 d\mu,$$

i.e., $q[f] < 0$, whence it follows that

$$Neg(-\Delta - V) \geq k.$$

Thus, we obtain

$$Neg(-\Delta - V) \geq \frac{c_N}{2CM}\nu(X) \geq \left\lfloor C_{N,M} \int\limits_X V d\mu \right\rfloor.$$

Also in the case where $\nu(X) \leq v$ we have

$$\frac{c_N}{2CM} \int\limits_X V d\mu < 1,$$

and

$$Neg(-\Delta - V) \geq 0 = \left\lfloor C_{N,M} \int\limits_X V d\mu \right\rfloor,$$

is trivially true. $\square$

Remark 8.2.4. *For comparison theorem of Cwickel-Lieb-Rozenblum states that, for any non-negative function V in $\mathbb{R}^n$, $n > 2$,*

$$Neg(L) \leq C \int\limits_{\mathbb{R}^n} V^{\frac{n}{2}} d\mu, \tag{8.2.10}$$

where C depends only on the dimension n. It is known that (8.2.10) *is not true in $\mathbb{R}^2$ whereas, by Theorem 8.2.2, the opposite inequality* (8.2.7) *holds in $\mathbb{R}^2$.*

8.3 A Lower Bound on the Number of the Eigenvalues of $-\Delta_D$ on Thin Layers

We want to apply Theorem 8.2.2 to our case where Σ is a complete, connected and non-compact surface embedded in $\mathbb{R}^3$ with finite total Gauss curvature.

First, Σ needs to satisfy the $(2, N)$-covering property which is hypothesis (i) in Theorem 8.2.2. Hence we need more assumptions on Σ one of them being that the Gauss curvature of Σ bounded from below. One of the most useful consequences of a lower bound on the Ricci curvature (which in 2 dimensions is the Gauss curvature) is the Bishop-Gromov Theorem:

Theorem 8.3.1. *Let X be a complete n-dimensional Riemannian manifold, such that $Ric \geq (n-1)\kappa$, where $\kappa \in \mathbb{R}$. Denote by $b(r)$ the volume of a ball of radius r in a simply connected n-dimensional manifold with constant curvature κ. For $0 < r < R$, we have*

$$\frac{\operatorname{vol}\big(B(R)\big)}{\operatorname{vol}\big(B(r)\big)} \leq \frac{b(R)}{b(r)}. \tag{8.3.1}$$

Furthermore,

$$\frac{\operatorname{vol}\big(B(r)\big)}{b(r)} \to 1 \quad \text{as} \quad r \to 0,$$

and hence $\operatorname{vol}\big(B(r)\big) \leq b(r)$.

One of the direct applications of this theorem is Gromov's packing lemma.

Lemma 8.3.2. *Let X be as in Theorem 8.3.1. For each $r > 0$, there exists a number N of balls of radius $\frac{r}{2}$ that are covering a ball of radius r, such that $N \leq C(r, \kappa)$ where $C(r, \kappa)$ is a constant depends only on r and κ.*

Proof. Fix $B(r)$, a ball of radius r in X, and consider a maximal family $\big\{B(x_i, r/4)\big\}_{i=1}^N$ of disjoint balls inside $B(5r/4)$. Then the corresponding family of balls of radius $r/2$, $\big\{B(x_i, r/2)\big\}_{i=1}^N$, cover $B(r)$.
To prove this, we assume that the corresponding family of balls of radius $r/2$, $\big\{B(x_i, r/2)\big\}_{i=1}^N$, dose not cover $B(r)$. Then, there is at least one point $\widetilde{x} \in B(r)$ with

$$\widetilde{x} \notin \cup_{i=1}^N B(x_i, r/2), \quad \text{i.e.,} \quad d(\widetilde{x}, x_i) > r/2 \quad \text{for all} \quad i = 1, \ldots, N,$$

and the ball $B(\widetilde{x}, r/4)$ is disjoint with $B(x_i, r/4)$ for all $i = 1, \ldots, N$ and contained in $B(x, 5r/4)$. But then the family $\big\{B(x_i, r/4)\big\}_{i=1}^N$ is not maximal, which contradicts the assumption.

Moreover, by virtue of the Bishop-Gromve Theorem, we have

$$\operatorname{vol} B(r/4) \geq C(r, \kappa)^{-1} \operatorname{vol} B(r),$$

where $C(r, k)$ is a constant depending only on r and κ. Since the balls $B(x_i, r/4)$ are disjoint, we have

$$\operatorname{vol}\big(B(r)\big) \geq \sum_{i=1}^N \operatorname{vol} B(x_i, r/4) \geq N C(r, k)^{-1} \operatorname{vol} B(r),$$

and therefore $N \leq C(r, k)$. $\square$

The following theorem will give us an insight into which additional assumptions we should put on Σ.

Theorem 8.3.3. (Main Theorem 2) *Let Σ be a complete, connected and non-compact surface embedded in $\mathbb{R}^3$ with integrable Gauss curvature. Assume that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 have a common boundary such that:*

i) $\Sigma_1 = \mathbb{R}^2 \setminus B_R(0)$, *for $R > 0$, with the Euclidean metric.*

ii) Σ_2 *is a compact surface with boundary, it has bounded diameter, volume and bounded Gauss curvature from below,*

$$\begin{aligned} diam(\Sigma_2) &\leq D \\ \mathrm{vol}(\Sigma_2) &\leq \vartheta \\ K &\geq -\kappa^2, \end{aligned}$$

where $D, \vartheta > 0$ and $\kappa \in \mathbb{R}$, (a very simple example of such a surface, see Fig 8.3.1).

Let $r' = 8\max\{D - R, R\}$, then there is a constant $C_{(\kappa,\vartheta,r')} > 0$, only depending on κ, ϑ, r', such that we have:

$$Neg\big(-\Delta_\Sigma - (M^2 - K)\big) \geq \Big\lfloor C_{(\kappa,\vartheta,r')} \int\limits_\Sigma (M^2 - K) d\Sigma \Big\rfloor.$$

In particular,

$$Neg\big(-\Delta_\Sigma - (M^2 - K)\big) \geq \lfloor C_{(\kappa,\vartheta,r')} g \rfloor, \tag{8.3.2}$$

where g is the genus of Σ.

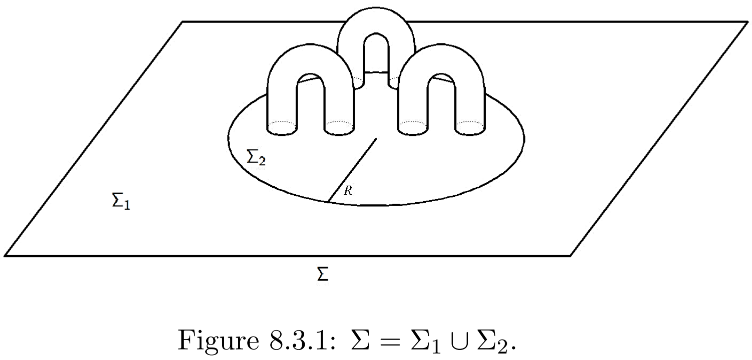


Figure 8.3.1: $\Sigma = \Sigma_1 \cup \Sigma_2$.

Remark 8.3.4. *In the proof of the theorem, we will explain why we take $r' = 8\max\{D - R, R\}$ like that, where this assumption is necessary to prove that the surface Σ satisfies the $(2, N)$-covering property when the radius of the ball $B(x, r)$ in Σ is very large, for all $x \in \Sigma$.*

Proof. We begin by proving that Σ satisfies hypothesis (ii) of Theorem 8.3.3. Therefore, we want to find a constant M, such that for all $x \in \Sigma$ and $r > 0$,

$$\operatorname{vol} B(x,r) \leq Mr^2.$$

We distinguish two cases, because the volume of the ball $B(x,r)$ for r small is dominated by the handles and for $r >> R$ is dominated by the Euclidean volume.

a) $r \leq r'$. Since $K \geq -\kappa^2$, it follows from the Bishop-Gromov Theorem that, for all $x \in \Sigma$,

$$\operatorname{vol} B(x,r) \leq b(x,r),$$

where $b(x,r)$ is the volume of the ball of radius r centered at x in a simply connected 2-dimensional hyperbolic manifold with constant curvature $-\kappa^2$. This quantity equals

$$\frac{2\pi}{\kappa^2}\Big(\cosh(r\kappa) - 1\Big),$$

and using the Taylor series

$$\cosh(r\kappa) = 1 + \frac{(r\kappa)^2}{2!} + \frac{(r\kappa)^4}{4!} + \cdots ,$$

we can write

$$\begin{aligned}
\operatorname{vol} B(x,r) &\leq \frac{2\pi}{\kappa^2}\Big(\frac{(r\kappa)^2}{2!} + \frac{(r\kappa)^4}{4!} + \cdots\Big) \\
&= 2\pi r^2\Big(\frac{1}{2!} + \frac{r^2\kappa^2}{4!} + \frac{r^4\kappa^4}{6!} + \cdots\Big) \\
&\leq 2\pi r^2\Big(\frac{1}{2!} + \frac{r'^2\kappa^2}{4!} + \frac{r'^4\kappa^4}{6!} + \cdots\Big) \\
&\leq \pi r^2\Big(1 + \frac{r'^2\kappa^2}{2!} + \frac{r'^4\kappa^4}{4!} + \cdots\Big) \\
&\leq \pi e^{r'\kappa} r^2 \\
&= M_1(r',\kappa) r^2,
\end{aligned}$$

where $M_1(r',\kappa) = \pi e^{r'\kappa}$ is clearly positive and only depends on r' and κ.

b) $r \geq r'$. For all $x \in \Sigma$, we have

$$B(x,r) = \big(\Sigma_1 \cap B(x,r)\big) \cup \big(\Sigma_2 \cap B(x,r)\big).$$

Since

$$\operatorname{vol}\big(\Sigma_1 \cap B(x,r)\big) \leq \pi r^2,$$

$$\operatorname{vol}\big(\Sigma_2 \cap B(x,r)\big) \leq \operatorname{vol}(\Sigma_2) \leq \vartheta,$$

we arrive at

$$\begin{aligned}\operatorname{vol} B(x,r) &\leq \vartheta + \pi r^2,\\ &= \pi r^2\Big(\frac{\vartheta}{\pi r^2}+1\Big),\\ &\leq \pi r^2\Big(\frac{\vartheta}{\pi r'^2}+1\Big),\\ &= M_2(r',\vartheta)r^2,\end{aligned}$$

where $M_2(r',\vartheta) = \pi(\frac{\vartheta}{\pi r'^2}+1)$ is positive and only depends on r' and ϑ.
In conclusion, using $M = \max\big\{M_1(r',\kappa), M_2(r',\vartheta)\big\}$, we get $\operatorname{vol} B(x,r) \leq Mr^2$.

In this next step, we again use the theorem of Bishop-Gromov and the packing lemma to prove that Σ satisfies the (2,N)-covering property. Here, we also distinguish the two cases above:

a') $r \leq r'$. For all $x \in \Sigma$, we have

$$\frac{\operatorname{vol} B(x,r)}{\operatorname{vol} B(x,r/2)} \leq \frac{b(x,r)}{b(x,r/2)} = \frac{\cosh(r\kappa)-1}{cosh(r\kappa/2)-1},$$

and similar to the above, we obtain

$$\begin{aligned}\cosh(r\kappa)-1 &\leq r^2\kappa^2\Big(\frac{1}{2!}+\frac{r^2\kappa^2}{4!}+\cdots\Big)\\ &\leq r^2\kappa^2\Big(\frac{1}{2!}+\frac{r'^2\kappa^2}{4!}+\cdots\Big)\\ &\leq \kappa^2 e^{r'\kappa}r^2\\ &\leq c_1(r',\kappa)r^2,\end{aligned}$$

where $c_1(r',\kappa)$ is a positive constant and only depends on r' and κ. Also,

$$\cosh(r\kappa/2)-1 \geq \frac{(r\kappa/2)^2}{2} = \frac{r^2\kappa^2}{8},$$

which gives

$$\frac{\operatorname{vol} B(x,r)}{\operatorname{vol} B(x,r/2)} \leq N_1(r',\kappa),$$

for $N_1(r',\kappa) = 8e^{r'\kappa}$.

b') $r \geq r'$. We want to prove that there is a constant $N_2(r',\vartheta)$, such that for all $x \in \Sigma$,

$$\frac{\operatorname{vol} B(x,r)}{\operatorname{vol} B(x,r/2)} \leq N_2(r',\vartheta).$$

As we have seen in b), for all $x \in \Sigma$,

$$\operatorname{vol} B(x,r) \leq M_2(r',\vartheta)r^2.$$

We now prove the existence of a constant $c_2(r', \vartheta)$ such that $\operatorname{vol} B(x,r) \geq r^2 c_2(r', \vartheta)$:

a") If $x \notin \Sigma_2$, we have

$$\frac{1}{2} B(x, r/2) \subset \Sigma_1$$

and then

$$\operatorname{vol} B(x, r/2) \geq \frac{\pi (r/2)^2}{2} = \frac{\pi r^2}{8}.$$

b") If $x \in \Sigma_2$, in this case $r' = 8 \max\{D - R, R\}$ is a sufficient condition to prove that there exists a constant $c_2(r', \vartheta)$, satisfying that $\operatorname{vol} B(x, r/2) \geq c_2(r', \vartheta) r^2$ for r very large, because we can find an annulus

$$A = \left\{ x_0 \in \Sigma_1 : R < |x_0| < (r/2) - (D - R) \right\}$$

is contained in $B(x, r/2)$:

To prove that: Suppose $R < |x_0| < (r/2) - (D - R)$. Then, let P be the point with $|p| = R$ and closest to x_0, i.e.,

$$\begin{aligned} d(x_0, p) &= |x_0| - R \\ d(p, x) &\leq D. \end{aligned}$$

Then it follows that

$$\begin{aligned} d(x_0, x) &\leq d(x_0, p) + d(p, x) \\ &\leq |x_0| - R + D < r/2. \end{aligned}$$

Now, we obtain

$$\begin{aligned} \operatorname{vol} B(x, r/2) &\geq \operatorname{vol}(A) \\ &= \pi \Big[\big(r/2 - (D - R) \big)^2 - R^2 \Big], \end{aligned}$$

since $D - R \leq r'/8 \leq r/8$, and so

$$\begin{aligned} r/2 - (D - R) &\geq r/2 - r/8 = 3r/8 \\ R &\leq r'/8 \leq r/8. \end{aligned}$$

Therefore,

$$\big(r/2 - (D - R) \big)^2 - R^2 \geq \frac{r^2}{8}$$

and we get

$$\operatorname{vol} B(x, r/2) \geq r^2 c_2(r', \vartheta),$$

where $c_2(r', \vartheta) = \frac{\pi}{8}$. From the above, we then have

$$\frac{\operatorname{vol} B(x, r)}{\operatorname{vol} B(x, r/2)} \leq N_2(r', \vartheta).$$

where $N_2(r', \vartheta) = 8(\frac{\vartheta}{\pi r'^2} + 1)$ is a constant depending on r' and ϑ, only.

Now, using the packing lemma and the constant $N = \max\{N_1(r', \kappa), N_2(r', \vartheta)\}$, we know that each ball of radius r can be covered by at most N balls of radius $r/2$.

By Theorem 8.2.2, there is a constant $C_{N,M}$ depending on N and M only such that for $V = M^2 - K$, we have

$$Neg\big(-\Delta_\Sigma - (M^2 - K)\big) \geq \lfloor C_{N,M} \int_\Sigma V d\Sigma \rfloor.$$

Since M and N depend only on κ, ϑ, r', we may write $C_{N,M} = C_{(\kappa,\vartheta,r')}$. Finally, as we have seen at the beginning of this chapter

$$\int_\Sigma (M^2 - K) \geq \pi(g+1),$$

which yields

$$Neg\big(-\Delta_\Sigma - (M^2 - K)\big) \geq \lfloor C_{(\kappa,\vartheta,r')} g \rfloor. \quad \square$$

Corollary 8.3.5. *Let Σ be a surface as in the Theorem 8.3.3, and Ω be the quantum layer of radius a around Σ. Then, there exists $a_0 > 0$ such that for all $a \leq a_0$, the number of eigenvalues of $-\Delta_D^\Omega$ is at least $\lfloor C_{(\kappa,\vartheta,r')} g \rfloor$.*

Chapter 9

Concluding Remarks

The main interest of this thesis was to find a lower bound on the number of eigenvalues of the Dirichlet Laplacian, $-\Delta_D^\Omega$, on a quantum layer Ω in terms of the genus g of a surface Σ.

In the first Main Theorem 6.1.2, we found a lower bound which depends only on the genus of Σ, if Σ is a Euclidean plane with a finite number of handles whose distance to each other is at least a constant.

The statement of the second main result is, if Σ is a Euclidean plane outside a compact set with nontrivial topology, then the number of eigenvalues of $-\Delta_D$ on thin layer is greater than or equal to g multiplied by a constant C. Since C depends on bounds for some geometrical quantities, it would be desirable to study the relation of these bounds to the genus.

However, the question how to evaluate the number Cg stays open. Further investigation may involve the question if the first result is true if Σ is not a Euclidean plane outside the handles.

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Lebenslauf

Persönliche Daten

Name	Diala Yacoub
Geburtstag	01.09.1980
Geburtsort	Latakia, Syrien

Bildungsgang

1985 – 1997	Schulbesuch in Latakia, Syrien, Abschluss: Abitur
1997 – 2001	Studium der Mathematischen Wissenschaften an der Universität Latakia, Abschluss: Bachelor
2001 – 2002	Diplomstudium Mathematik und Informatik an der Universität Latakia, Abschluss: Diplom
2007 – 2008	Vorbereitung zur Promotion an der Mathematischen Fakultät Universität Göttingen
seit 2008	Promotionsstudium Mathematik an der Mathematischen Fakultät Universität Oldenburg.

Erklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und nur die angegeben Hilfmittel benutzt habe.

Oldenburg, den 01. September 2014

Diala Yacoub