VECTOR-VALUED FOURIER HYPERFUNCTIONS

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Zusammenfassend sollte von diesen kümmerlichen Worten eines im Gedächtnis bleiben:

Danke!

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1 Introduction

The aim of the present work is the development of the theory of Fourier hyperfunctions in one variable with values in a non-necessarily metrizable locally convex space E and to find necessary and sufficient conditions such that a reasonable theory of E-valued Fourier hyperfunctions is possible. In particular, it is shown that, if E is an ultrabornological PLS-space, such a theory exists if and only if E satisfies the so-called property (PA). It will turn out that the vector-valued Fourier hyperfunctions can be realized as the sheaf generated by equivalence classes of certain compactly supported E-valued functionals and interpreted as boundary values of slowly increasing holomorphic functions.

Scalar-valued Fourier hyperfunctions $\mathcal{R}$ were introduced by Kawai [28] in 1970. He constructed them as a flabby sheaf on D_n, where D_n means the radial compactification of $\mathbb{R}^n$, using cohomology theory and Hörmander's L^2-estimates [20]. He proved that the global sections are stable under Fourier transformation $\mathscr{F}$, i.e. $\mathscr{F}\colon\mathcal{R}(D_n)\to\mathcal{R}(D_n)$ is an isomorphism. This sheaf is a generalization of the sheaf $\mathcal{B}$ of hyperfunctions on $\mathbb{R}^n$ which was developed by Sato [55] (and [56]); in particular, $\mathcal{R}\big|_{\mathbb{R}^n}=\mathcal{B}$ holds. Hyperfunctions emerged as an useful tool in the theory of partial differential equations (see [33]), in particular, in the solution of the abstract Cauchy problem. Komatsu developed the theory of Laplace hyperfunctions, a theory of operator valued generalized functions with a suitable Laplace transform, more precisely, for operators in Banach spaces, and the abstract Cauchy problem was solved by a condition on the resolvent of the operator which characterized the generators of hyperfunction semigroups (see [34], [35], [36] and [37]). This theory was improved and extended beyond operators in Banach spaces by Domański and Langenbruch (see [14], [15]). Since some partial differential equations can be taken as ordinary vector-valued equations (e.g. [50], [51]), the question arose whether there was a vector-valued counterpart for the theory of (Fourier) hyperfunctions. Whereas Schwartz achieved this in the analogous theory of distributions by tensor products [58], one faces a crucial problem in the development of such a theory of vector-valued, in short, E-valued where E is a locally convex space, Fourier hyperfunctions, namely, the lack of a natural linear topology on the scalar-valued (Fourier) hyperfunctions (with the exception of the space of global sections in the case of Fourier hyperfunctions). Despite of this difficulty, Ion and Kawai [21](1975) developed a theory of hyperfunctions with values in Fréchet spaces, Ito and Nagamachi [24](1975) a theory of Fourier hyperfunctions with values in Hilbert spaces, which was used by Mugibayashi and Nagamachi ([48], [49]) for an axiomatic formulation of quantum field theory in terms of Fourier hyperfunctions, and Junker [26](1979) a theory of Fourier hyperfunctions with values in Fréchet spaces. Although Ito tried to extend the theory of Fourier hyperfunctions to non-Fréchet spaces E ([23], [22]), his effort has some mathematical gaps. He only realizes them in form of a presheaf satisfying the sheaf condition (S1) of Bredon [9, 1.5, p. 5] and then defines the sheaf of general Fourier hyperfunctions as the associated sheaf (see [9, 1.3, p. 5]). Even worse, it is not certain whether the restrictiction maps he defines exist for a general locally convex space E, so it is not even sure if his E-valued Fourier hyperfunctions form a presheaf at all (see Remark 6.3 and the remarks before it). But their existence can be assured by an additional condition for E, viz. E has to be strictly admissible which is explained in the forthcoming. As we will see, a reasonable theory of E-valued Fourier hyperfunctions is indeed impossible, at least in one variable, for ultrabornological PLS-spaces E not satisfying the so-called property (PA).

Domański and Langenbruch [13](2008) not only overcame these obstacles and developed a theory of vector-valued hyperfunctions beyond the class of Fréchet spaces, but also found natural limits of this kind of theory. They characterized in a large natural class of locally convex spaces those spaces for which a reasonable theory of E-valued hyperfunctions exists at all (see [13, Theorem 8.9, p. 1139]). To be more precise: they state that a reasonable theory of E-valued hyperfunctions should generate a flabby sheaf with the property that the set of sections supported by a compact subset $K \subset \mathbb{R}^n$ should coincide with $L(\mathcal{A}(K), E)$, the space of linear continuous operators from $\mathcal{A}(K)$ to E where $\mathcal{A}(K)$ denotes the space of germs of real analytic functions on K. Transferring this condition to the theory of Fourier hyperfunctions, I am convinced that a reasonable theory of E-valued hyperfunctions (in one variable) should produce a flabby sheaf such that the set of sections supported by a compact subset $K \subset \overline{\mathbb{R}}$ should coincide with "the space of E-valued $\mathcal{P}_*$-functionals" $L(\mathcal{P}_*(K), E)$ where $\overline{\mathbb{R}}$ is the radial compactification of $\mathbb{R}$ and $\mathcal{P}_*(K)$ the space of rapidly decreasing holomorphic germs near K (see Definition 3.1). If one restricts such a sheaf to $\mathbb{R}$, the restricted sheaf fulfills the condition of Domański and Langenbruch for a reasonable theory of E-valued hyperfunctions, since $\mathcal{P}_*(K) = \mathcal{A}(K)$ for compact $K \subset \mathbb{R}$, which is desirable in the spirit of the property $\mathcal{R}|_{\mathbb{R}} = \mathcal{B}$ of the scalar-valued case. Furthermore, the global sections of such a sheaf are stable under Fourier transformation (see Theorem 4.6). This implies that for those spaces E, for which a reasonable theory of E-valued hyperfunctions is impossible, a reasonable theory of E-valued Fourier hyperfunctions is impossible as well. A long list of examples of spaces E for which a reasonable theory of E-valued Fourier hyperfunctions is possible resp. impossible can be found in Example 5.26 resp. Example 5.27(a).

In the approach of Domański and Langenbruch the existence of an E-valued sheaf of hyperfunctions is deeply connected with the solvability of the E-valued Laplace equation; namely, if the $(n+1)$-dimensional Laplace operator

$$\Delta_{n+1} : C^{\infty}(\Omega, E) \to C^{\infty}(\Omega, E)$$

is surjective for every open set $\Omega \subset \mathbb{R}^{n+1}$ where $C^{\infty}(\Omega, E)$ is the space of smooth E-valued functions on Ω, then a reasonable theory of E-valued hyperfunctions on $\mathbb{R}^n$ is possible (see [13, Theorem 6.9, p. 1125]). For E-valued Fourier hyperfunctions in one variable the corresponding counterpart is the following. A complete locally convex space E is called admissible if the Cauchy-Riemann operator

$$\overline{\partial} : \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$ where $\overline{\mathbb{C}} := \mathbb{R} + i\overline{\mathbb{R}}$ and $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ is, roughly speaking, the space of smooth E-valued slowly increasing functions outside K (see Definition 3.2). E is called strictly admissible if E is admissible and, in addition,

$$\overline{\partial} : C^{\infty}(\Omega, E) \to C^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$. We will prove that E being strictly admissible yields to the existence of a reasonable theory of E-valued Fourier hyperfunctions in one variable. Hence the whole Section 5 is dedicated to this problem and culminates in the main theorem of this section, Theorem 5.25, where all the classes of strictly admissible spaces known so far are collected, in

particular, ultrabornological PLS-spaces with property (PA). The main tools of this section are the splitting theory for Fréchet spaces of Vogt [63] and new results on the splitting theory for PLS-spaces by Bonet and Domański [8] as well as results on tensor products obtained in Section 3 (see Theorem 3.11).
In correspondence with the scalar-valued case, the E-valued Fourier hyperfunctions are defined in Section 6 from two different points of view for a strictly admissible space E. On the one hand as the sheaf generated by equivalence classes of E-valued $\mathcal{P}_{*}$-functionals, and on the other as the sheaf of boundary values of the elements of $\mathcal{O}^{exp}\left(U\smallsetminus\overline{\mathbb{R}},E\right)$. This is, to put it roughly, the space of holomorphic E-valued slowly increasing functions on U outside an open set $\Omega\subset\overline{\mathbb{R}}$, where U is an open set in $\overline{\mathbb{C}}$ with $U\cap\overline{\mathbb{R}}=\Omega$ (see Definition 6.7). The construction of these sheaves will benefit from some kind of Köthe duality established in Section 4 (see Theorem 4.1) and it will turn out that both sheaves are isomorphic (see Theorem 6.11). At the end of the section, it will turn out, as already mentioned in parts, that, if E is an ultrabornolgical PLS-space, a reasonable theory of E-valued Fourier hyperfunctions in one variable exists if and only if E satisfies the property (PA) (see Theorem 6.14).

2 Notation and preliminaries

By E we will (almost) always denote a complete locally convex space equipped with the system of semi-norms $(p_\alpha)_{\alpha\in A}$, where A is a directed set. The only exceptions are Theorem 5.1, the Lemmas 5.3-5.5, Remark 5.9 and a short remark right after Example 5.12, where E denotes a fundamental solution.

Basic notations for sets, the spaces $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$

We denote by $\overline{\mathbb{R}}$ the radial compactifaction of $\mathbb{R}$ defined as follows. We set $\overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$ and equip this space with the topology given by:
A set $\Omega\subset\overline{\mathbb{R}}$ is open iff

- $\Omega\cap\mathbb{R}$ is open in $(\mathbb{R},|\cdot|)$ and
- there exists $a\in\mathbb{R}$ such that $[a,\infty]\subset\Omega$, if $\infty\in\Omega$, or $[-\infty,a]\subset\Omega$, if $-\infty\in\Omega$.

Remark that $\overline{\mathbb{R}}$, equipped with this topology, is a compact space. Furthermore, we define

$$\inf\varnothing := \infty \quad \text{and} \quad \sup\varnothing := -\infty$$

as well as $[a,b] := \varnothing$ for $a,\ b\in\mathbb{R}$, $a>b$. Moreover, we set $-\infty + a := -\infty$ and $\infty + a := \infty$ for $a\in\mathbb{R}$. In addition, we define $\lceil x\rceil := \min\{k\in\mathbb{Z}\mid x\le k\}$ for $x\in\mathbb{R}$.
Further, we define $\overline{\mathbb{C}} := \overline{\mathbb{R}} + i\mathbb{R}$ and equip it with the product topology. In particular, this means that an open set $U\subset\overline{\mathbb{C}}$ contains ∞ or $-\infty$ iff there exist $a\in\mathbb{R}$ and $\varepsilon>0$ such that $([a,\infty]+i]-\varepsilon,\varepsilon[)\subset U$ resp. $([-\infty,a]+i]-\varepsilon,\varepsilon[)\subset U$.
Let $z\in\mathbb{C}$. Then there are $x,\ y\in\mathbb{R}$ such that $z = x+iy$. We use the usual notation $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$. In short, we often just use for an element $z\in\mathbb{C}$ the notation $z = z_1 + iz_2$ without previously pointing that out. Furthermore, we also use a notation of mixed-type

$$z = z_1 + iz_2 = (z_1, z_2) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

hence consider $\mathbb{C}$ as the vector space $\mathbb{R}^2$ equipped with the usual multiplication. Further, we denote by $\langle z|w\rangle$ the usual scalar product of $z,\ w\in\mathbb{R}^2$. We define the distance of two subsets $M_0,\ M_1\subset\mathbb{C}$ via

$$\mathrm{d}(M_0,M_1) := \begin{cases} \inf\limits_{z\in M_0,\, y\in M_1} |z-y|, & \text{if } M_0,\, M_1 \neq \varnothing, \\ \infty, & \text{if } M_0 = \varnothing \text{ or } M_1 = \varnothing. \end{cases}$$

If M_0 is compact and M_1 closed such that $M_0\cap M_1 = \varnothing$, then there exists $z_0\in M_0$ with $\mathrm{d}(M_0,M_1) = \mathrm{d}(z_0,M_1)$, in particular, $\mathrm{d}(M_0,M_1)>0$ (see for example [31, 1.4.II. Beispiel, p. 31]).
Moreover, we denote by $D_r(z) := \{w\in\mathbb{C}\mid |w-z|<r\}$ the ball around $z\in\mathbb{C}$ with radius $r>0$.
For an universal set (basic set) U we denote the absolute complement of a subset $M\subset U$ by

$M^C := U \setminus M$. Further, we denote by $\#M$ the number of elements of a set M and for a subset M of a topological space X the set of inner points of M by $\mathring{M}$, the closure of M by $\overline{M}$ and the boundary of M by ∂M.

Spaces of continuous linear operators

By $L(E,F)$ we denote the space of continuous linear operators from X to Y where X and Y are locally convex spaces. If $F = \mathbb{C}$, we just write $E' := L(E,\mathbb{C})$ for the dual space. By $L_\sigma(E,F)$, $L_c(E,F)$, $L_{co}(E,F)$, $L_\tau(E,F)$, $L_e(E,F)$ and $L_b(E,F)$ we denote the space $L(E,F)$ equipped with the weak topoplogy, the topology of uniform convergence on precompact subsets of E, the compact open topology, the Mackey topology, the topology of uniform convergence on equicontinuous subsets of E and the strong topology. Sometimes we also use the symbol $\sigma(E',E)$ for the weak topology and the symbol $\lambda(E',E)$ for the topology of uniform convergence on precompact sets on E'. But, if not otherwise stated, $L(E,F)$ resp. E' is always equipped with the strong topology and we only write $L_b(E,F)$, if we want to emphasize this fact.
Moreover, we often use the notation

$$\langle T,x\rangle := T(x)$$

for $T \in L(E,F)$ and $x \in E$.
Later, when we are concerned with tensor products, we will use the so-called ε-product which is defined by $E\varepsilon F := L_e(E'_c,F)$ for complete locally convex spaces E and F. Remark that $E'_c = E'_{co}$ if E is complete. This definition of the ε-product coincides with the original one by Schwartz [58, Définition, p. 18] if E is (quasi-)complete. For the references to the book [25] of Jarchow in this context, whose definition of the ε-product also differs in general from the one given here, we remark that they coincide if E is complete by [25, 9.3.7. Theorem, p. 179], which is the case we are interested in.

Infinetely partial differentiable functions, distributions and holomorphic functions

Let $n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ be open. By $\mathrm{C}^\infty(U)$ and $\mathrm{C}^\infty(U,E)$ we denote the spaces of scalar- and E-valued infinitely differentiable functions on U. For $f \in \mathrm{C}^\infty(U,E)$ we use the usual notation

$$\partial^\alpha f(x) := \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x)$$

with $\alpha \in \mathbb{N}_0^n$ and denote by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ the order of differentiation. Further, we use for α, $\beta \in \mathbb{N}_0^n$ the notation

$$\beta \leq \alpha \;:\Leftrightarrow\; \forall\, 0 \leq i \leq n:\ \beta_i \leq \alpha_i$$

and define

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}$$

if $\beta \leq \alpha$, where the right hand side is defined by ordinary binomial coefficients. We remark that $\beta \leq \alpha$ implies $|\beta| \leq |\alpha|$. This notation is useful when we are concerned with partial derivatives of

products (Leibniz rule).
The space of infinitely differentiable functions with compact support in U is defined by

$$\mathrm{C}_0^\infty(U) := \mathcal{D}(U) := \limind_{K \subset U \text{ compact}} \mathrm{C}_0^\infty(K)$$

where

$$\mathrm{C}_0^\infty(K) := \{ f \in \mathrm{C}^\infty(\mathbb{R}^n) \mid f(x) = 0 \;\forall\, x \notin K \}.$$

Every element f of $\mathcal{D}(U)$ can be regarded as an element of $\mathcal{D}(\mathbb{R}^n)$ just by the trivial setting $f := 0$ on U^C. Moreover, we set for $k \in \mathbb{N}_0$ and $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$

$$|||f|||_k := \sup_{\substack{x \in \mathbb{R}^n \\ \alpha \in \mathbb{N}_0^n,\, |\alpha| \leq k}} |\partial^\alpha f(x)|.$$

$|||\cdot|||_k$ is a norm on $\mathrm{C}_0^\infty(\mathbb{R}^n)$ for any k.
The dual $\mathcal{D}'(U) := \mathcal{D}(U)'$ is called the space of distributions on U. The Dirac distribution δ is defined via $\delta(\varphi) := \varphi(0)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and for a locally integrable function $f \in L^1_{loc}(U)$ we denote by T_f the regular distribution defined by

$$T_f(\varphi) := \int_{\mathbb{R}} f(x)\varphi(x)\,dx, \;\; \varphi \in \mathcal{D}(U).$$

The partial derivatives of a distribution $T \in \mathcal{D}'(U)$ are defined by

$$\partial^\alpha T(\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi), \;\; \varphi \in \mathcal{D}(U).$$

The convolution $T * \varphi$ of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ and a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is defined by

$$(T * \varphi)(x) := T(\varphi(x - \cdot)), \;\; x \in \mathbb{R}^n.$$

In particular, we have $\delta * \varphi = \varphi$ and

$$\left(T_f * \varphi\right)(x) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)\,dy, \;\; x \in \mathbb{R}^n, \tag{2.1}$$

for $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
Furthermore, $\partial^\alpha(T * \varphi) = (\partial^\alpha T) * \varphi = T * (\partial^\alpha \varphi)$ is valid for $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. For more details on the theory of distributions see [18].
Let $U \subset \mathbb{C}$ be open. By $\mathcal{O}(U)$ and $\mathcal{O}(U,E)$ we denote the spaces of scalar- and E-valued holomorphic functions on U.

PLS-spaces, Proj^1 and Ext^1 functor and splitting of exact sequences

Let us recall that a locally convex space X is a PLS-space (PLN-space) if $X = \operatorname{lim\,proj}_{N \in \mathbb{N}} X_N$ where X_N are DFS-spaces, i.e. the strong duals of Fréchet-Schwartz spaces (DFN-spaces, i.e. the strong duals of nuclear Fréchet spaces). X_N being a DFS-space is equivalent by [45, Theorem 25.20, p. 286] to the existence of a sequence of Banach spaces $(X_{N,n})_{n \in \mathbb{N}}$ such that

$X_N = \lim\operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ with compact linking maps. For this reason DFS-spaces are also called LS-spaces. Examples of PLS-spaces are the space of distributions $\mathcal{D}'(\Omega)$ and the space of real analytic functions $\mathcal{A}(\Omega)$. In particular, every Fréchet-Schwartz space is a PLS-space (for a short proof see [13, Proof of Proposition 4.3, p. 1113-1114]). For more examples see Example 5.26 and Example 5.27. Further, we recall that a LFS-space is an inductive limit of a sequence of Fréchet-Schwartz spaces. For more details on PLS-spaces we refer to [11].
We repeat some homological tools, mostly working in the background of the results at the end of Section 6. The Proj^1 functor is defined as follows. Let $X := \lim\operatorname{proj}_{N\in\mathbb{N}} X_N$ where (X_N) is a sequence of locally convex spaces with a sequence of linking maps $i_N^{N+1}\colon X_{N+1}\to X_N$. One defines

$$\operatorname{Proj}^1_{N\in\mathbb{N}}(X_N) := \prod_{N\in\mathbb{N}} X_N/\operatorname{im}\sigma, \quad \sigma\colon \prod_{N\in\mathbb{N}} X_N \to \prod_{N\in\mathbb{N}} X_N,$$
$$\sigma((x_N)) := \left(i_N^{N+1}x_{N+1} - x_N\right)_{N\in\mathbb{N}}.$$

For reduced spectra of DFS-spaces or Banach spaces, meaning that $i_N\colon X\to X_N$ has a dense range for every $N\in\mathbb{N}$, Proj^1 depends only on X and not on the spectrum itself. We remark that a PLS-space X is ultrabornological iff $\operatorname{Proj}^1 X = 0$ by [65, Corollary 3.3.10, p. 46].
We have the following relation between the terminology of homological algebra and the theory of locally convex spaces, which can be found in [53, §4+§9] and [65, Chap. 2+5]. The category $\mathcal{LCS}$ of locally convex spaces resp. the category $\mathcal{F}$ of Fréchet spaces consists of (not necessarily Hausdorff) locally convex spaces resp. Fréchet spaces (over the same scalar field $\mathbb{R}$ or $\mathbb{C}$) as objects and continuous linear maps (operators) as morphisms. So, we use the notation $L(X,Y)$ for $\operatorname{Hom}(X,Y)$, where X and Y are locally convex spaces resp. Fréchet spaces, and the group structure is given by the usual addition. In the following, let $\mathcal{K}$ be $\mathcal{LCS}$ or $\mathcal{F}$. An operator $f\colon X\to Y$, X, $Y\in\mathcal{K}$, is a monomorphism iff it is injective. An operator $f\colon X\to Y$, where X, $Y\in\mathcal{LCS}$ (resp. X, $Y\in\mathcal{F}$), is an epimorphism, iff it is surjective (resp. it has dense range). In $\mathcal{K}$ every operator $f\colon X\to Y$ has a kernel, namely, the subspace $f^{-1}(\{0\})\subset X$ equipped with the induced topology, it has a cokernel, namely, in $\mathcal{LCS}$ the quotient space $Y/f(X)$ and in $\mathcal{F}$ the quotient space $Y/\overline{f(X)}$, equipped with the quotient topology. Accordingly, the subspace $f(X)\subset Y$, equipped with the induced topology, can be interpreted as the image of f and the quotient space $X/f^{-1}(\{0\})$, equipped with the quotient topology, as the coimage of f (for $\mathcal{F}$ keep in mind that $\overline{f^{-1}(\{0\})} = f^{-1}(\{0\})$). Further, the morphism

$$\tilde{f}\colon X/f^{-1}(\{0\}) \to f(X),\ [x]\mapsto f(x),$$

is a bimorphism, i.e. it is a monomorphism as well as an epimorphism. Thus the category $\mathcal{K}$ is semi-abelian. $f\colon X\to Y$ is a homomorphism iff it is open onto its range. A space I in $\mathcal{K}$ is called injective iff for every $f\in L(X,I)$, where X in $\mathcal{K}$, and every monohomomorphism (i.e. a topological embedding) $i\colon X\to Y$, where Y in $\mathcal{K}$, there is an extension $\tilde{f}\in L(Y,I)$ of f, i.e. $\tilde{f}\circ i = f$. The crucial point is that the category $\mathcal{K}$ has many injective objects by [53, Corollary 4.1, p. 23] resp. [65, Theorem 2.2.1 and a subsequent remark, p. 13-14], i.e. for every object X of $\mathcal{K}$ there exist an injective object I and a monohomomorphism $i\colon X\to I$. In particular, this implies that every object X in $\mathcal{K}$ has an injective resolution, i.e. there exists an exact complex

$$0\to X\xrightarrow{i} I_0 \xrightarrow{i_0} I_1 \xrightarrow{i_1} I_2 \to\cdots,$$

where I_k are injective objects in $\mathcal{K}$ for every k. For a fixed locally convex space (resp. Fréchet space) E one considers the functor $L(E,\cdot)$ assigning to a locally convex space (resp. Fréchet space) X the linear space $L(E,X)$ and to an operator $T{:}X\to Y$ the linear map $T^{*}{:}L(E,X)\to L(E,Y)$, $f\mapsto f\circ T$. This covariant functor is injective and additive. Since $\mathcal{K}$ has many injective objects, the construction of the right derived functors of $L(E,\cdot)$ is possible and they are denoted with $\operatorname{Ext}^k(E,\cdot)$. For any injective resolution of X we have

$$\operatorname{Ext}^k(E,X)\cong \ker i_k^{*}/\operatorname{im} i_{k-1}^{*},\; k\geq 1,$$

where $i_k^{*}{:}L(X,I_k)\to L(X,I_{k+1})$, $i^{*}(T):=i_k\circ T$, and $\operatorname{Ext}^0(E,X)=L(E,X)$. Then the following Theorem (see [53, Proposition 2.1, p. 13] and [53, Proposition 9.1, p. 49] or [65, Theorem 2.1.1, p. 11-12] and [65, Theorem 5.1.1, p. 77]) is valid:

Theorem. *1) Let E be a locally convex space (resp. Fréchet space) and*

$$0\to X\xrightarrow{I} Y\xrightarrow{Q} Z\to 0$$

an exact sequence sequence of locally convex spaces (resp. Fréchet spaces). Then there is an exact complex

$$0\to L(E,X)\xrightarrow{I^{*}}L(E,Y)\xrightarrow{Q^{*}}L(E,Z)\xrightarrow{\delta^0}\operatorname{Ext}^1(E,X)\to\operatorname{Ext}^1(E,Y)\to\operatorname{Ext}^1(E,Z)\xrightarrow{\delta^1}\operatorname{Ext}^2(E,X)\to\cdots$$

where $I^{*}{:}L(E,X)\to L(E,Y)$, $I^{*}(T):=I\circ T$, and $Q^{*}{:}L(E,Y)\to L(E,Z)$, $Q^{*}(T):=Q\circ T$.
2) Let X be a locally convex space (resp. Fréchet space) and

$$0\to E\xrightarrow{i} G\xrightarrow{q} F\to 0$$

an exact sequence of locally convex spaces (resp. Fréchet spaces). Then there is an exact complex

$$0\to L(F,X)\xrightarrow{q_{*}}L(G,X)\xrightarrow{i_{*}}L(E,X)\xrightarrow{\delta_0}\operatorname{Ext}^1(F,X)\to\operatorname{Ext}^1(G,X)\to\operatorname{Ext}^1(E,X)\xrightarrow{\delta_1}\operatorname{Ext}^2(F,X)\to\cdots$$

where $q_{*}{:}L(F,X)\to L(G,X)$, $q_{*}(T):=T\circ q$, and $i_{*}{:}L(G,X)\to L(E,X)$, $i_{*}(T):=T\circ i$.

The connection to splitting theory is now as follows. Let E and F be locally convex spaces (resp. Fréchet spaces). Then the following conditions are equivalent by [65, Proposition 5.1.3, p. 79] resp. [63, 1.8. Theorem, p. 11]:

- Every exact sequence
$$0\to E\xrightarrow{i} G\xrightarrow{q} F\to 0 \tag{2.2}$$
splits (i.e. q has a right inverse or, equivalently, i has a left inverse) where G is a locally convex space (resp. Fréchet space).
- $\operatorname{Ext}^1(F,E)=0$

Furthermore, we remark that an exact sequence (2.2) is always topologically exact in $\mathcal{F}$, i.e. the continuous, linear maps i and q are open onto their image due to the open mapping theorem. If E and F are PLS-spaces and every topologically exact sequence (2.2), where G is a PLS-space, splits, then this is correspondingly denoted by $\operatorname{Ext}^1_{PLS}(F,E)=0$. Note that this really is a difference. There is a close relation between the functors Proj^1 and Ext^1 (for Fréchet spaces) resp.

Ext^1_{PLS} or, more precisely, their vanishing (see for example [63, 1.2 Theorem, p. 9] and [8, Theorem 3.4, p. 9]). This is only a short summary. For more details see [52], [53], [65], for splitting theory of Fréchet spaces [63] and for splitting theory of PLS-spaces [8].

For the classical theory of hyperfunctions see [27], [57] or [47]. For the sheaf theory see [9] or [39]. For the theory of locally convex spaces see [45] or [16]. For the theory of tensor products see [25] or [60].
Since the theory of E-valued hyperfunctions developed by Domański and Langenbruch in [13] was the initial point of my work, many theorems proven here were obtained by modifying the proofs of their counterparts in [13]. Thus it is refered to these counterparts in a footnote of the following kind,

counterpart: [13, Theorem X.Y, p. xy].

3 Vector-valued functions with exponential growth conditions

This section is dedicated to some basic topological properties of the spaces already mentioned in the introduction, the space $\mathcal{P}_*(K)$ of rapidly decreasing holomorphic germs near K, the space $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right)$ of smooth E-valued slowly increasing functions outside K and the space $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right)$ of holomorphic E-valued slowly increasing functions outside K where K is a compact subset of $\overline{\mathbb{R}}$. In particular, it will turn out that the space $\mathcal{P}_*(K)$ is a DFS-space and that $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, \mathbb{C}\right)$ as well as $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, \mathbb{C}\right)$ are nuclear Fréchet spaces. Using the nuclearity, we obtain a representation of $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right)$ via tensor products at the end of this section, namely,

$$\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right) \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, \mathbb{C}\right)\varepsilon E \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, \mathbb{C}\right)\hat{\otimes}_\varepsilon E \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, \mathbb{C}\right)\hat{\otimes}_\pi E$$

and the same for $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K, E\right)$.
Then the results of this section will be used as auxiliary tools in the following sections.
We begin with the definitions of the spaces above. For a compact set $K \subset \overline{\mathbb{R}}$ and $n \in \mathbb{R}$, $n \geq 1$, we define the sets

$$U_n(K) := \{z \in \mathbb{C} \mid \mathrm{d}(z, K\cap\mathbb{C}) < 1/n\} \cup \begin{cases} \varnothing, & \text{if } K \subset \mathbb{R}, \\ ]n,\infty[+i]-1/n, 1/n[, & \text{if } \infty \in K,\ -\infty \notin K, \\ ]-\infty,-n[+i]-1/n, 1/n[, & \text{if } \infty \notin K,\ -\infty \in K, \\ (]-\infty,-n[\cup]n,\infty[)+i]-1/n, 1/n[, & \text{if } \infty \in K,\ -\infty \in K, \end{cases}$$

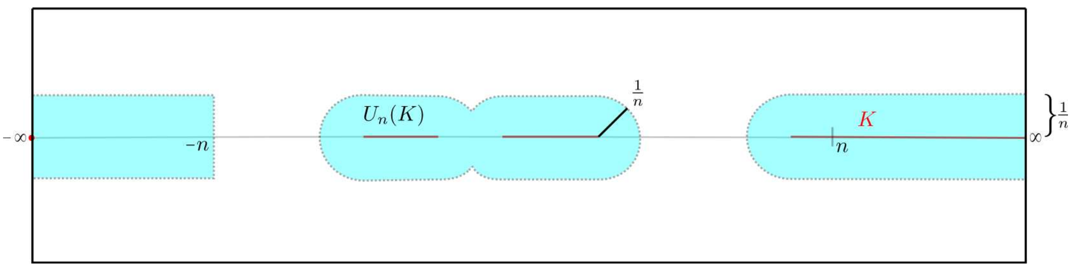

Figure 3.1: $U_n(K)$ for $\pm\infty \in K$

and

$$S_n(K) := \left(\overline{U_n(K)}\right)^{\complement} \cap \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n\}$$

where the closure and the complement are taken in $\mathbb{C}$.

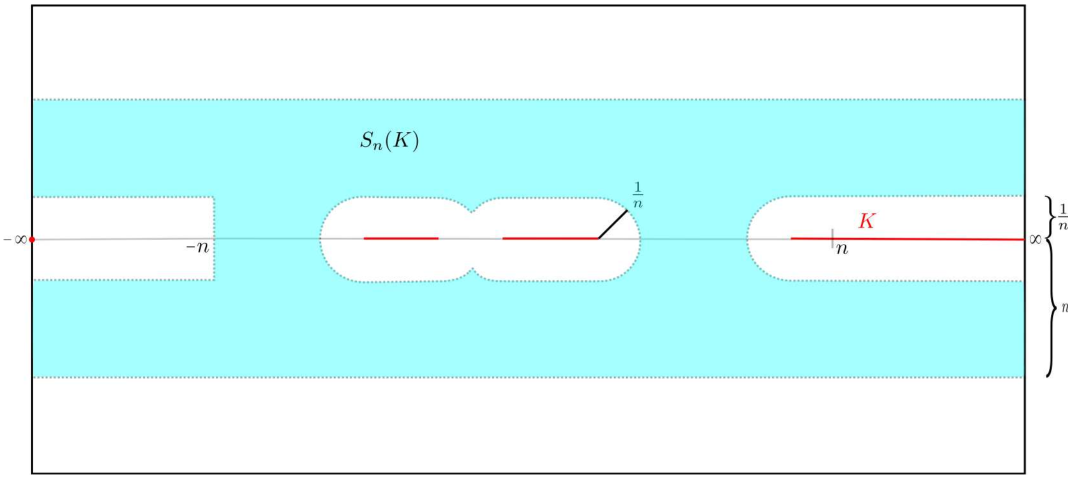

Figure 3.2: $S_n(K)$ for $\pm\infty \in K$

3.1 Definition (rapidly decreasing holomorphic germs)**.** Let $K \subset \overline{\mathbb{R}}$ be a compact set. The space of rapidly decreasing holomorphic germs near K is defined as

$$\mathcal{P}_*(K) := \lim_{\substack{\longrightarrow \\ n\in\mathbb{N}}} \operatorname{ind} \mathcal{O}_n(U_n(K))$$

where

$$\mathcal{O}_n(U_n(K)) := \{f \in \mathcal{O}(U_n(K)) \cap C\left(\overline{U_n(K)}\right) \mid \|f\|_n := \sup_{z\in\overline{U_n(K)}} |f(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} < \infty\}$$

and $\mathcal{O}(\varnothing) := 0$ and the spectral mappings are given by

$$\pi_{n,k}: \mathcal{O}_n(U_n(K)) \to \mathcal{O}_k(U_k(K)),\ \pi_{n,k}(f) := f|_{U_k(K)},\ n \le k.$$

Recall that E is a complete locally convex space equipped with the system of semi-norms $(p_\alpha)_{\alpha\in A}$.

3.2 Definition (vector-valued slowly increasing infinitely continuously differentiable resp. holomorphic functions)**.** Let $K \subset \overline{\mathbb{R}}$ be a compact set.

a) For $n \in \mathbb{R}$, $n > 1$, we define

$$\mathcal{E}_n^{exp}(S_n(K), E) := \{f \in C^\infty(S_n(K), E) \mid \forall \alpha \in A,\ m \in \mathbb{N}_0 : |f|_{K,n,m,\alpha} < \infty\}$$

where

$$|f|_{K,n,m,\alpha} := \sup_{\substack{z\in S_n(K) \\ \beta\in\mathbb{N}_0^2, |\beta|\le m}} p_\alpha\left(\partial^\beta f(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}.$$

The space of vector-valued slowly increasing infinitely continuously differentiable functions outside K is defined as

$$\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) := \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}_{\ge 2}}} \operatorname{proj} \mathcal{E}_n^{exp}(S_n(K), E)$$

where the spectral mappings are given by

$$\pi_{n,k}: \mathcal{E}_k^{exp}(S_k(K), E) \to \mathcal{E}_n^{exp}(S_n(K), E),\ \pi_{n,k}(f) := f|_{S_n(K)},\ n \le k.$$

b) For $n \in \mathbb{R}$, $n > 1$, we define

$$\mathcal{O}_n^{exp}(S_n(K), E) := \{f \in \mathcal{O}(S_n(K), E) \mid \forall \alpha \in A : |f|_{K,n,\alpha} < \infty\}$$

where

$$|f|_{K,n,\alpha} := \sup_{z \in S_n(K)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}.$$

The space of vector-valued slowly increasing holomorphic functions outside K is defined as

$$\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) := \lim_{\substack{\longleftarrow \\ n \in \mathbb{N}_{\geq 2}}} \operatorname{proj} \mathcal{O}_n^{exp}(S_n(K), E)$$

where the spectral mappings are given by

$$\pi_{n,k} \colon \mathcal{O}_k^{exp}(S_k(K), E) \to \mathcal{O}_n^{exp}(S_n(K), E),\ \pi_{n,k}(f) := f|_{S_n(K)},\ n \leq k.$$

If not necessary, the subscript K in the notation of the semi-norms is omitted and in the Banach-valued, particularly, scalar-valued, case the subscript α as well. The notation for the spaces in the scalar-valued case is $\mathcal{E}_n^{exp}(S_n(K)) := \mathcal{E}_n^{exp}(S_n(K), \mathbb{C})$, $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) := \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, \mathbb{C}\right)$ as well as $\mathcal{O}_n^{exp}(S_n(K)) := \mathcal{O}_n^{exp}(S_n(K), \mathbb{C})$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) := \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, \mathbb{C}\right)$.

The mappings $\pi_{n,k}$ in the Definitions 3.1 and 3.2 are obviously linear and continuous.
Now we take a closer look at the sets $U_n(K)$ and $S_n(K)$.

3.3 Remark. Let $K \subset \overline{\mathbb{R}}$ be compact and $n \in \mathbb{R}$, $n \geq 1$.

(1) The set $U_n(K)$ is open and has finitely many components.

(2) Let $K \neq \varnothing$ and Z be a component of $U_n(K)$. We define $a := \min Z \cap K$ and $b := \max Z \cap K$ if existing (in $\mathbb{R}$).

  a) If Z is bounded, there exists $0 < R \leq 1/n$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(z, [a,b]) < r\} \subset Z$

  b) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and a exists, there exists $0 < R \leq 1/n$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(z, [a, \infty[) < r\} \subset Z$

  c) If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and b exists, there exists $0 < R \leq 1/n$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(z,]-\infty, b]) < r\} \subset Z$

  d) If $Z \cap \mathbb{R}$ is unbounded from below and above, there exists $0 < R \leq 1/n$ such that for all $0 < r \leq R$: $\{z \in \mathbb{C} \mid \mathrm{d}(z, \mathbb{R}) < r\} \subset Z$

  e) If $Z \cap \mathbb{R}$ is bounded from below and unbounded from above and a does not exist, then $Z =]n, \infty[+ i]-1/n, 1/n[$. If $Z \cap \mathbb{R}$ is bounded from above and unbounded from below and b does not exist, then $Z =]-\infty, -n[+ i]-1/n, 1/n[$.

(3) Let $q \in \mathbb{R}$, $q > n > 1$, $M \subset \overline{S_n(K)}$ and O a component of M^C such that $O \cap \overline{S_n(K)}^C \neq \varnothing$. Then $O \cap \overline{S_q(K)}^C \neq \varnothing$.

(4) Let $q \in \mathbb{R}$, $q > n \geq 1$. Then

$$\mathrm{d}\left(\partial U_q(K), \partial U_n(K)\right) = \frac{1}{n} - \frac{1}{q},\ q > n \geq 1,\ K \neq \varnothing,$$

and

$$\mathrm{d}\left(\partial S_q(K), \partial S_n(K)\right) = \begin{cases} \frac{1}{n} - \frac{1}{q}, & q > n > 1,\ K \neq \varnothing, \\ q - n, & q > n > 1,\ K = \varnothing. \end{cases}$$

Proof. (1) Consider the case $\infty \in K, -\infty \notin K$. Obviously $U_n(K)$ is an open set. Let $\left(Z_i^n\right)_{i \in I}$ denote the components of $U_n(K)$. Then $U_n(K) = \dot\bigcup_{i \in I} Z_i^n$ and by definition of a component there is $j \in I$ such that Z_j^n is the only component including $]n, \infty[+i]-1/n, 1/n[$. Furthermore, there exists $m \in \mathbb{R}$ with $\bigcup_{i \in I \setminus \{j\}} \left(Z_i^n \cap \mathbb{R}\right) \subset [m, n]$ by assumption. For $i \neq j$ the length $\lambda\left(Z_i^n \cap \mathbb{R}\right)$ of the interval $Z_i \cap \mathbb{R}$, where λ denotes the Lebesgue measure, is estimated from below by $\lambda\left(Z_i^n \cap \mathbb{R}\right) \geq 2/n$ by definition of $U_n(K)$. Since all Z_i^n are pairwise disjoint, this implies that I has to be finite. The others cases can analogously be proven.

(2) a) Since $Z \cap K$ is closed in $\mathbb{R}$ and therefore compact, a and b exist. Hence $[a, b] \subset Z$ by definition of $U_n(K)$ and as Z is connected. $[a, b]$ being a compact subset of the open set Z implies there is $\varepsilon > 0$ such that $([a, b] + i] - \varepsilon, \varepsilon[) \subset Z$ by the tube lemma. The choice of $R := \min\left(\varepsilon, \frac{1}{n}\right)$ completes the proof by definiton of $U_n(K)$ and since $a,\ b \in Z \cap K$.

b) If $Z \cap K \cap]-\infty, n] \neq \varnothing$, then a exists and analogously to a) there exists $R > 0$ such that for all $0 < r \leq R$:

$$\{z \in \mathbb{C} |\ \mathrm{d}(z, [a, n]) < r\} \subset Z$$

By definition of $U_n(K)$ this brings forth $\{z \in \mathbb{C} |\ \mathrm{d}(z, [a, \infty[) < r\} \subset Z$.
If $Z \cap K \cap]-\infty, n] = \varnothing$ and a exists, the desired $R > 0$ exists by definition of $U_n(K)$ since $n \notin Z \cap K$ and $Z \cap K$ is closed in $\mathbb{R}$, thus $\mathrm{d}(n, Z \cap K) > 0$.

c) Analogously to b).

d) By the assumptions $Z \cap K \cap [-n, n] \neq \varnothing$. Analogously to a) there exists $R > 0$ such that for all $0 < r \leq R$:

$$\{z \in \mathbb{C} |\ \mathrm{d}(z, [-n, n]) < r\} \subset Z$$

Like in b) and c) this brings forth $\{z \in \mathbb{C} |\ \mathrm{d}(z, \mathbb{R}) < r\} \subset Z$.

e) This follows directly by the definition of $U_n(K)$ and as Z is a component of $U_n(K)$.

(3) By definition we have $\overline{S_n(K)}^C = U_n(K) \dot\cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > n\}$ and $\overline{S_q(K)}^C = U_q(K) \dot\cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > q\}$. So the components of $\overline{S_n(K)}^C$ are $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > n\}$, $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < -n\}$ and the components of $U_n(K)$. The components of $\overline{S_q(K)}^C$ are $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > q\}$, $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < -q\}$ and the components of $U_q(K)$. If $O \cap U_n(K) \neq \varnothing$, then there is a component Z^n of $U_n(K)$ such that $O \cap Z^n \neq \varnothing$. To be more precise, $Z^n \subset O$ by virtue of the properties of M and O being a component of M^C. If this component contains a point $x \in K$, then there is a component Z^q of $U_q(K)$ which contains this point as well. Hence we have

$$x \in (Z^q \cap Z^n) \subset Z^n \subset O$$

and $O \cap \overline{S_q(K)}^C \neq \varnothing$. If Z^n does not contain a point of K, then it must be an unbounded component. So we have

$$(]q,\infty[+i[-1/q,1/q]) \subset (]n,\infty[+i]-1/n,1/n[) \subset Z^n$$

or

$$(]-\infty,-q[+i[-1/q,1/q]) \subset (]-\infty,-n[+i]-1/n,1/n[) \subset Z^n.$$

Thus there is a component Z^q of $U_q(K)$ with $Z^q \cap Z^n \neq \varnothing$ implying $O \cap \overline{S_q(K)}^C \neq \varnothing$ like above.
If $O \cap \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > n\} \neq \varnothing$, the statement is obvious.

(4) In the following keep in mind that

$$q-n \geq \frac{1}{n}-\frac{1}{q} \tag{3.1}$$

since $qn \geq 1$.
Let $K \neq \varnothing$. By virtue of (1) we have $U_j(K) = \dot{\bigcup}_{i=1}^{k_j} Z_i^j$, $k_j < \infty$, and thus $\partial U_j(K) = \bigcup_{i=1}^{k_j} \partial Z_i^j$ for $j \in \{n,q\}$ with the notations from the proof of (1). By definition every Z_i^q is contained in some $Z_{i_0}^n$.
If $Z_{i_0}^n$ is bounded, we obtain

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \frac{1}{n}-\frac{1}{q}$$

by definition of $U_j(K)$.
If $Z_{i_0}^n$ is unbounded, w.l.o.g. $Z_{i_0}^n \cap \mathbb{R}$ is bounded from below and unbounded from above, we define $a := \min Z_{i_0}^n \cap K$ and $a_0 := \min Z_i^q \cap K$ if existing (in $\mathbb{R}$). If a exists, we have the following cases:
<u>1. case:</u> $n \leq a-\frac{1}{n}$, $q \leq a-\frac{1}{q}$
Then $Z_{i_0}^n =]n,\infty[+i]-1/n,1/n[$, $Z_i^q =]q,\infty[+i]-1/q,1/q[$ and

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \min\left(q-n, \frac{1}{n}-\frac{1}{q}\right) = \frac{1}{n}-\frac{1}{q}.$$

<u>2. case:</u> $n \leq a-\frac{1}{n}$, $a-\frac{1}{q} < q < a$
Then $Z_{i_0}^n =]n,\infty[+i]-1/n,1/n[$, $Z_i^q = D_{1/q}(a) \cup (]q,\infty[+i]-1/q,1/q[)$ and

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \min\left(\frac{1}{n}-\frac{1}{q}, \underbrace{a-n-\frac{1}{q}}_{\geq \frac{1}{n}-\frac{1}{q}}\right) = \frac{1}{n}-\frac{1}{q}.$$

<u>3. case:</u> $n \leq a-\frac{1}{n}$, $a \leq q$
If a_0 exists, we have $a \leq a_0$. If Z_i^q is bounded or unbounded and $a_0-\frac{1}{q} < q < a_0$ or $a_0 \leq q$, then

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \min\left(\frac{1}{n}-\frac{1}{q}, \underbrace{a_0-n-\frac{1}{q}}_{\geq a-n-\frac{1}{q} \geq \frac{1}{n}-\frac{1}{q}}\right) = \frac{1}{n}-\frac{1}{q}.$$

If $q \leq a_0 - \frac{1}{q}$ or a_0 does not exist, then

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \min\left(q-n, \frac{1}{n} - \frac{1}{q}\right) = \frac{1}{n} - \frac{1}{q}.$$

<u>4. case:</u> $a - \frac{1}{n} < n < a$, $q \leq a - \frac{1}{q}$
Then $Z_{i_0}^n = D_{1/n}(a) \cup (]n,\infty[+i]-1/n, 1/n[)$, $Z_i^q =]q,\infty[+i]-1/q, 1/q[$, and

$$\begin{aligned}\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) &= \min\Big(\mathrm{d}\left(\partial Z_i^q \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq q\}, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq q\}\right),\\ &\qquad \underbrace{\mathrm{d}\left(\partial Z_i^q \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq q\}, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq q\}\right)}_{=\frac{1}{n}-\frac{1}{q}}\Big)\\ &= \min\Big(\underbrace{\mathrm{d}\left(q+i]-1/q, 1/q[, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq q\}\right)}_{\geq q-n}, \frac{1}{n} - \frac{1}{q}\Big)\\ &= \frac{1}{n} - \frac{1}{q}.\end{aligned}$$

(This case is not possible if $q, n \in \mathbb{N}$.)
<u>5. case:</u> $a - \frac{1}{n} < n < a$, $a - \frac{1}{q} < q < a$
Then $Z_{i_0}^n = D_{1/n}(a) \cup (]n,\infty[+i]-1/n, 1/n[)$, $Z_i^q = D_{1/q}(a) \cup (]q,\infty[+i]-1/q, 1/q[)$ and so

$$\begin{aligned}\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) &= \min\Big(\mathrm{d}\left(\partial Z_i^q \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq a\}, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq a\}\right),\\ &\qquad \underbrace{\mathrm{d}\left(\partial Z_i^q \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq a\}, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq a\}\right)}_{=\frac{1}{n}-\frac{1}{q}}\Big)\\ &= \min\Big(\underbrace{\mathrm{d}\left(\partial Z_i^q \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq a\}, \partial Z_{i_0}^n \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq a\}\right)}_{\geq q-n}, \frac{1}{n} - \frac{1}{q}\Big)\\ &= \frac{1}{n} - \frac{1}{q}.\end{aligned}$$

<u>6. case:</u> $a - \frac{1}{n} < n < a \leq q$
We get the same equalities/estimates like in the fifth case by replacing a with a_0 or q if a_0 does not exist.
<u>7. case:</u> $a \leq n$
Then $Z_{i_0}^n = \{z \in \mathbb{C} \mid \mathrm{d}(z, [a,\infty[\cap K) < 1/n\} \cup (]n,\infty[+i]-1/n, 1/n[)$ and several different cases are possible for the structure of Z_i^q. But, in all cases, we have

$$\mathrm{d}\left(\partial Z_i^q, \partial Z_{i_0}^n\right) = \frac{1}{n} - \frac{1}{q}$$

due to (3.1).
If a does not exist, we have $Z_{i_0}^n =]n,\infty[+i]-1/n, 1/n[$ and $Z_i^q =]q,\infty[+i]-1/q, 1/q[$, hence the same situation like in the first case. Combining these results, we get the first statement of (4). The second statement of (4) follows by (3.1) and the definitions of the sets involved,

except for $K=\varnothing$. In this case it is obvious.

□

The previous remark will often be useful, amongst others, for the choice of paths of integrals in Section 4. The next lemma describes the relation between partial and complex derivatives of higher order of a holomorphic function.

3.4 Lemma. *Let $U\subset\mathbb{C}$ be an open set and $f\in\mathcal{O}(U)$. For $z\in U$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}_0^2$ one has*

$$\partial^{\alpha}f(z)=i^{\alpha_2}f^{(|\alpha|)}(z) \tag{3.2}$$

where $f^{(|\alpha|)}$ denotes the $|\alpha|$-th complex derivative of f.

Proof. This lemma can be proven by induction over $|\alpha|$. For $|\alpha|=0$ this is obviously true. Now assume that (3.2) holds for $|\alpha|=n$ with $n\in\mathbb{N}_0$. For $|\alpha|=n+1$ one has $\alpha=\beta+\gamma$ with $\beta,\gamma\in\mathbb{N}_0^2$ and $|\beta|=n$ and $|\gamma|=1$. The assumption and the fact that $f\in\mathcal{O}(U)$ implies $f^{(|\beta|)}\in\mathcal{O}(U)$ lead to

$$\begin{aligned}
\partial^{\alpha}f(z)=\partial^{\gamma}\left(\partial^{\beta}f\right)(z)&\underset{\text{assumption}}{=}i^{\beta_2}\partial^{\gamma}\left(f^{(|\beta|)}\right)(z)\\
&=\begin{cases} i^{\alpha_2}\partial_1\left(f^{(|\beta|)}\right)(z), & \text{if } \gamma=(1,0)\\ i^{\beta_2}\partial_2\left(f^{(|\beta|)}\right)(z), & \text{if } \gamma=(0,1)\end{cases}\\
&=\begin{cases} i^{\alpha_2}f^{(|\beta|+1)}(z), & \text{if } \gamma=(1,0)\\ i^{\beta_2}if^{(|\beta|+1)}(z), & \text{if } \gamma=(0,1)\end{cases}\\
&=i^{\alpha_2}f^{(|\alpha|)}(z).
\end{aligned}$$

□

Let $U\subset\mathbb{C}$ be open and $f\in\mathcal{O}(U,E)$. For $z\in U$ and $n\in\mathbb{N}_0$ we denote the point evaluation of complex derivatives with $\delta_z^{(n)}f:=f^{(n)}(z)$ and for $\alpha\in\mathbb{N}_0^2$ the point evaluation of partial derivatives with $\delta_z^{(\alpha)}f:=\partial^{\alpha}f(z)$.
Next, we prove that $\mathcal{P}_*(K)$ is a DFS-space, which is only mentioned by Kawai ([28, p. 469]). Part ii) and the hint to use [6] in part iv) of the proof of statement (1) of the following theorem can be found in the proof of [26, 1.11 Satz, p. 11]. For the sake of completeness and since the ideas of the proof will be used in the following, a full proof of statement (1) is given here.

3.5 Theorem. *Let $K\subset\overline{\mathbb{R}}$, $K\neq\varnothing$, be a compact set.*

(1) $\mathcal{P}_*(K)$ is a DFS-space.

(2) The set of point evaluations of complex derivatives $\left\{\delta_{x_0}^{(n)}\mid x_0\in K\cap\mathbb{R},n\in\mathbb{N}_0\right\}$ is total in $\mathcal{P}_*(K)_b'$, i.e. span $\left\{\delta_{x_0}^{(n)}\mid x_0\in K\cap\mathbb{R},n\in\mathbb{N}_0\right\}$ *is dense in $\mathcal{P}_*(K)_b'$, if $K\subset\mathbb{R}$ or $\infty\in K$, $-\infty\notin K$ and ∞ is not an isolated point of K or $-\infty\in K$, $\infty\notin K$ and $-\infty$ is not an isolated point of K or $\pm\infty\in K$ and ∞ and $-\infty$ are not isolated points of K. The same is true for the set of point evaluations of partial derivatives $\left\{\delta_{x_0}^{(\alpha)}\mid x_0\in K\cap\mathbb{R},\alpha\in\mathbb{N}_0^2\right\}$.*

(3) The set of point evaluations $\{\delta_{x_0}\mid x_0\in K\cap\mathbb{R}\}$ is total in $\mathcal{P}_*(K)_b'$ if K has no isolated points.

Proof. (1) i) $\|\cdot\|_n$ is a norm on $\mathcal{O}_n(U_n(K))$. Next, we show that this space is complete. Now let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{O}_n(U_n(K))$. Let $\varepsilon > 0$ and $M \subset U_n(K)$ compact. Then there exists $N \in \mathbb{N}$ such that for all $k, m \geq N$

$$\begin{aligned}\varepsilon > \|f_k - f_m\|_n &= \sup_{z\in\overline{U_n(K)}} |f_k(z) - f_m(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \\ &\geq \sup_{z\in\overline{U_n(K)}} |f_k(z) - f_m(z)| \\ &\geq \sup_{z\in M} |f_k(z) - f_m(z)|.\end{aligned}$$

Thus (f_k) is also a Cauchy sequence in $\mathrm{CB}\left(\overline{U_n(K)}\right) := \left\{f \in \mathrm{C}\left(\overline{U_n(K)}\right) \mid f \text{ bounded}\right\}$, equipped with the norm

$$\|f\| := \sup_{z\in\overline{U_n(K)}} |f(z)|,$$

as well as in $\mathcal{O}(U_n(K))$, equipped with the topology induced by the semi-norms

$$p_M(f) := \sup_{z\in M} |f(z)|,\ M \subset U_n(K) \text{ compact}, \tag{3.3}$$

and has a limit f resp. F in these spaces since they are complete by [45, 5.16 Beispiele (3), p. 35] resp. [60, Example II, p. 91]. The functions f and F coincide on $U_n(K)$ because for all $z \in U_n(K)$ and all $\varepsilon_0 > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$

$$|f(z) - F(z)| \leq |f(z) - f_k(z)| + |f_k(z) - F(z)| \leq \|f - f_k\| + p_{\{z\}}(f_k - F) < 2\varepsilon_0.$$

Hence f is holomorphic on $U_n(K)$ and continuous on the closure. Since every Cauchy sequence is bounded, there exists $C = C(n) \geq 0$ with $|f_k(z)| e^{1/n|\mathrm{Re}(z)|} \leq C$ for all $z \in \overline{U_n(K)}$ and $k \in \mathbb{N}$ implying $f \in \mathcal{O}_n(U_n(K))$ by pointwise convergence. Using the pointwise convergence again, we get for all $z \in \overline{U_n(K)}$ and for all $k \geq N$

$$|f_k(z) - f(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} = \lim_{m\to\infty} |f_k(z) - f_m(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \leq \lim_{m\to\infty} \|f_k - f_m\|_n < \varepsilon$$

and therefore $\|f_k - f\|_n < \varepsilon$. This means $(f_k)_{k\in\mathbb{N}}$ converges to f in $\mathcal{O}_n(U_n(K))$ as well, connoting this space to be a Banach space.

ii) The mappings $\pi_{n,m}: \mathcal{O}_n(U_n(K)) \to \mathcal{O}_m(U_m(K))$, $n \leq m$, are injective by virtue of the identity theorem and the definition of sets $U_n(K)$, $n \in \mathbb{N}$. Thus the considered spectrum is an embedding spectrum.

iii) Let $M \subset U_n(K)$ compact. For all $f \in B_n := \{g \in \mathcal{O}_n(U_n(K)) \mid \|g\|_n \leq 1\}$ and all $z \in M$

$$|f(z)| \leq \|f\|_n \leq 1,$$

so $\sup_{f\in B_n} p_M(f) \leq 1$. Thus B_n is bounded in $\mathcal{O}(U_n(K))$ with respect to the semi-norms (3.3). As this space is a Fréchet-Montel space by [60, Proposition 34.4, p. 357], B_n is relatively compact and hence relatively sequentially compact by [45, 4.8 Satz, p. 19].

iv) What remains to be shown, is that for all $n \in \mathbb{N}$ there exists $m > n$ such that

$\pi_{n,m}\colon \mathcal{O}_n(U_n(K)) \to \mathcal{O}_m(U_m(K))$ is a compact mapping. Because the considered spaces are Banach spaces by i), it suffices, due to [45, 4.10 Corollar, p. 20], to show the existence of $m > n$ such that $(\pi_{n,m}(f_k))_{k\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{O}_m(U_m(K))$ for every sequence $(f_k)_{k\in\mathbb{N}}$ in B_n. According to [6, Th. (b), p. 67-68] set $m := 2n$. Let $\varepsilon > 0$ and choose $Q := \overline{U_{2n}(K)} \cap \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq \max(0, -2n\ln\varepsilon)\}$. Then $Q \subset U_n$ compact and

$$\sup_{z\in\overline{U_{2n}(K)}\setminus Q} \frac{e^{\frac{1}{2n}|\mathrm{Re}(z)|}}{e^{\frac{1}{n}|\mathrm{Re}(z)|}} = \sup_{z\in\overline{U_{2n}(K)}\setminus Q} e^{-\frac{1}{2n}|\mathrm{Re}(z)|} \underset{\substack{\text{choice}\\ \text{of } Q}}{<} \varepsilon. \tag{3.4}$$

In addition, we observe that

$$C(\varepsilon) := \sup_{z\in Q} e^{\frac{1}{2n}|\mathrm{Re}(z)|} \leq e^{\frac{1}{2n}\max(0,-2n\ln\varepsilon)} = \begin{cases} 1, & \varepsilon \geq 1, \\ \frac{1}{\varepsilon}, & \varepsilon < 1. \end{cases} \tag{3.5}$$

Now let $(f_k)_{k\in\mathbb{N}}$ be a sequence in B_n. By iii) it has a convergent subsequence $\left(f_{k_l}\right)_{l\in\mathbb{N}}$ with respect to the semi-norms (3.3). Then there exists $N \in \mathbb{N}$ such that for $l, j \geq N$

$$p_Q\left(f_{k_l} - f_{k_j}\right) = \sup_{z\in Q} |f_{k_l}(z) - f_{k_j}(z)| < \varepsilon^2 \tag{3.6}$$

and therefore

$$\begin{aligned}
&\left\|\pi_{n,2n}\left(f_{k_l}\right) - \pi_{n,2n}\left(f_{k_j}\right)\right\|_{2n} \\
&\leq \sup_{z\in Q} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{2n}|\mathrm{Re}(z)|} + \sup_{z\in\overline{U_{2n}(K)}\setminus Q} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{2n}|\mathrm{Re}(z)|} \\
&\leq C(\varepsilon) \sup_{z\in Q} |f_{k_l}(z) - f_{k_j}(z)| + \sup_{z\in\overline{U_{2n}(K)}\setminus Q} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \frac{e^{\frac{1}{2n}|\mathrm{Re}(z)|}}{e^{\frac{1}{n}|\mathrm{Re}(z)|}} \\
&\underset{(3.6)}{\leq} C(\varepsilon)\varepsilon^2 + \sup_{z\in\overline{U_{2n}(K)}\setminus Q} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \sup_{z\in\overline{U_{2n}(K)}\setminus Q} e^{-\frac{1}{2n}|\mathrm{Re}(z)|} \\
&\underset{(3.4)}{\leq} C(\varepsilon)\varepsilon^2 + \varepsilon \sup_{z\in\overline{U_{2n}(K)}\setminus Q} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq C(\varepsilon)\varepsilon^2 + \varepsilon \sup_{z\in\overline{U_n(K)}} |f_{k_l}(z) - f_{k_j}(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq C(\varepsilon)\varepsilon^2 + \varepsilon\left(\left\|f_{k_l}\right\|_n + \left\|f_{k_j}\right\|_n\right) \\
&\underset{f_k\in B_n}{\leq} C(\varepsilon)\varepsilon^2 + 2\varepsilon \\
&\underset{(3.5)}{\leq} \begin{cases} \varepsilon^2 + 2\varepsilon, & \varepsilon \geq 1, \\ 3\varepsilon, & \varepsilon < 1. \end{cases}
\end{aligned}$$

Hence the subsequence $\left(\pi_{n,2n}\left(f_{k_l}\right)\right)_{l\in\mathbb{N}}$ of $\left(\pi_{n,2n}(f_k)\right)_{k\in\mathbb{N}}$ converges in $\mathcal{O}_{2n}(U_{2n}(K))$ proving the compactness of $\pi_{n,2n}$.

(2) We set $F := \operatorname{span}\left\{\delta_{x_0}^{(n)} \mid x_0 \in K \cap \mathbb{R}, n \in \mathbb{N}_0\right\}$. $\delta_{x_0}^{(n)}$ is linear and for $k \in \mathbb{N}$ and $f \in \mathcal{O}_k(U_k(K))$

$$\begin{aligned}
\left|< \delta_{x_0}^{(n)}, f >\right| &= \left|f^{(n)}(x_0)\right| \\
&= \left|\frac{n!}{2\pi i}\int_{\partial D_{\frac{1}{2k}}(x_0)} \frac{f(z)}{(z-x_0)^{n+1}} dz\right| \\
&\leq n!\,(2k)^n \max_{z\in\partial D_{\frac{1}{2k}}(x_0)} |f(z)| \\
&\leq n!\,(2k)^n \max_{z\in\partial B_{\frac{1}{2k}}(x_0)} e^{-\frac{1}{k}|\mathrm{Re}(z)|} \sup_{z\in U_k(K)} |f(z)|\, e^{\frac{1}{k}|\mathrm{Re}(z)|} \\
&\leq n!\,(2k)^n \|f\|_k .
\end{aligned}$$

Hence $\delta_{x_0}^{(n)}$ is continuous on $\mathcal{O}_k(U_k(K))$ for arbitrary k and so on $\mathcal{P}_*(K)$ implying $F \subset \mathcal{P}_*(K)'$. As $\mathcal{P}_*(K)$ is a DFS-space, it is reflexive by [45, 25.19 Satz (1), p. 285] which means that the canonical embedding $J\colon \mathcal{P}_*(K) \to \mathcal{P}_*(K)''$, $f \mapsto J(f)$, defined by $J(f)\colon \mathcal{P}_*(K)' \to \mathbb{C}$, $T \mapsto T(f)$, is a topological isomorphism. Hence for the polar set F° of F one has

$$\begin{aligned}
F^\circ &= \left\{y \in \mathcal{P}_*(K)'' \mid \forall T \in F : y(T) = 0\right\} \\
&\cong \{f \in \mathcal{P}_*(K) \mid \forall T \in F : J(f)(T) = 0\} \\
&= \{f \in \mathcal{P}_*(K) \mid \forall T \in F : T(f) = 0\} =: M.
\end{aligned}$$

For $f \in M$ and $T := \delta_{x_0}^{(n)} \in F$

$$0 = T(f) = f^{(n)}(x_0)$$

is valid. Thus f is identical to zero on a neighbourhood of x_0 (by Taylor series expansion) since $n \in \mathbb{N}_0$ is arbitrary and f is holomorphic near $x_0 \in U_n(K)$. Due to the assumptions every component of $U_n(K)$ contains a point $x_0 \in K \cap \mathbb{R}$ so f is identical to zero on $U_n(K)$ by the identity theorem. Therefore $F^\circ = \{0\}$ and thus F is dense in $\mathcal{P}_*(K)'$ by the bipolar theorem. The adjunct is due to (3.2).

(3) The proof is similar to (2). We define $F := \operatorname{span}\{\delta_{x_0} \mid x_0 \in K \cap \mathbb{R}\}$. Then, like above, for $f \in \{g \in \mathcal{P}_*(K) \mid \forall T \in F : T(g) = 0\}$ and $T := \delta_{x_0} \in F$

$$0 = T(f) = f(x_0).$$

Due to the assumptions every component Z of $U_n(K)$ contains a point $x_0 \in K \cap \mathbb{R}$ and every point in $Z \cap K \cap \mathbb{R}$ is an accumulation point of this set. So f is identical to zero on $U_n(K)$ by the identity theorem.

□

The proof of the first part of the next theorem is due to Junker [26, 1.4 Lemma (2), p. 5], but there for a Fréchet space E and on the level of the projective limit (here, the second part) and we need results on the level of the projective spectra. The proof of the fourth part can be found in [26, 1.4 Lemma (1), p. 5], but again for the projective limit. Since we need it as well on the level of the projective spectrum plus the appearing inequality resp. the idea of the proof will be used several times, it is given here.

3.6 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be a compact set.*

(1) Let $n \in \mathbb{R}$, $n > 1$. Then $\mathcal{E}_n^{exp}(S_n(K), E)$ and $\mathcal{O}_n^{exp}(S_n(K), E)$ are complete locally convex spaces. In particular, they are Fréchet spaces if E is a Fréchet space and the latter is a Banach space if E is a Banach space.

(2) $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ are complete locally convex spaces. In particular, they are Fréchet spaces if E is a Fréchet space.

(3) $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ is a Fréchet-Schwartz space.

(4) (a) Let $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$, $\alpha \in A$ and $k \in \mathbb{N}$, $k > n$. Then there exists $C = C(n,k,m) > 0$ such that

$$|f|_{n,m,\alpha} \leq C|f|_{k,\alpha} \tag{3.7}$$

for all $f \in \mathcal{O}_k^{exp}(S_k(K), E)$.
In particular, we have $\mathcal{O}_k^{exp}(S_k(K), E) \subset \mathcal{E}_n^{exp}(S_n(K), E)$.

(b) $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ is a topological subspace of $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$.

Proof. (1) Obviously $|f|_{n,m,\alpha}$, $m \in \mathbb{N}_0$, resp. $|f|_{n,\alpha}$ are semi-norms for $\alpha \in A$ and the spaces $\mathcal{E}_n^{exp}(S_n(K), E)$ resp. $\mathcal{O}_n^{exp}(S_n(K), E)$ equipped with these systems of semi-norms are locally convex. What remains to be shown, is that they are complete.
i) Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}_n^{exp}(S_n(K), E)$. The space $\mathrm{C}^\infty(S_n(K), E)$ equipped with the system of semi-norms

$$p_{M,m,\alpha}(f) := \sup_{\substack{z\in M \\ \beta\in\mathbb{N}_0^2, |\beta|\leq m}} p_\alpha\left(\partial^\beta f(z)\right), \tag{3.8}$$

$M \subset S_n(K)$ compact, $m \in \mathbb{N}_0$ and $\alpha \in A$, is complete by [60, Proposition 44.1, p. 446]. The inclusion $\mathcal{E}_n^{exp}(S_n(K), E) \hookrightarrow \mathrm{C}^\infty(S_n(K), E)$ is continuous since for all $M \subset S_n(K)$ compact, $m \in \mathbb{N}_0$ and $\alpha \in A$

$$p_{M,m,\alpha}(f) \leq \sup_{z\in M} e^{\frac{1}{n}|\mathrm{Re}(z)|}|f|_{n,m,\alpha}$$

for all $f \in \mathcal{E}_n^{exp}(S_n(K), E)$. Thus $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}^\infty(S_n(K), E)$ as well and has a limit f in this space due to the completeness. Let $\varepsilon > 0$, $m \in \mathbb{N}_0$, $\alpha \in A$ and $z \in S_n(K)$. As this convergence implies pointwise convergence, there exists $N(z) \in \mathbb{N}$ such that for all $l \geq N(z)$

$$p_\alpha\left(\partial^\beta f_l(z) - \partial^\beta f(z)\right) < \frac{\varepsilon}{2} \tag{3.9}$$

for all $\beta \in \mathbb{N}_0^2$, $|\beta| \leq m$. Furthermore, there exists $N_0 \in \mathbb{N}$ such that for all $k, l \geq N_0$

$$|f_k - f_l|_{n,m,\alpha} < \frac{\varepsilon}{2} \tag{3.10}$$

by assumption. Hence we get for all $k \geq N_0$ by choosing $l \geq \max(N(z), N_0)$

$$\begin{aligned}
&p_\alpha\left(\partial^\beta f(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} - p_\alpha\left(\partial^\beta f_k(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq p_\alpha\left(\partial^\beta f_k(z) - \partial^\beta f(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq p_\alpha\left(\partial^\beta f_k(z) - \partial^\beta f_l(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + p_\alpha\left(\partial^\beta f_l(z) - \partial^\beta f(z)\right) \underbrace{e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{\leq 1} \\
&\leq \sup_{w \in S_n(K)} p_\alpha\left(\partial^\beta f_k(w) - \partial^\beta f_l(w)\right) e^{-\frac{1}{n}|\mathrm{Re}(w)|} + p_\alpha\left(\partial^\beta f_l(z) - \partial^\beta f(z)\right) \\
&\underset{(3.9)}{<} \sup_{\substack{w \in S_n(K) \\ \gamma \in \mathbb{N}_0^2, |\gamma| \leq m}} p_\alpha\left(\partial^\gamma f_k(w) - \partial^\gamma f_l(w)\right) e^{-\frac{1}{n}|\mathrm{Re}(w)|} + \frac{\varepsilon}{2} \\
&= |f_k - f_l|_{n,m,\alpha} + \frac{\varepsilon}{2} \\
&\underset{(3.10)}{<} \varepsilon
\end{aligned}$$

for all $|\beta| \leq m$ and so $|f_k - f|_{n,m,\alpha} < \varepsilon$ as well as $|f|_{n,m,\alpha} < \varepsilon + |f_k|_{n,m,\alpha}$ for all $k \geq N_0$. This means that $f \in \mathcal{E}_n^{exp}(S_n(K), E)$ and that $(f_k)_{k \in \mathbb{N}}$ converges to f in $\mathcal{E}_n^{exp}(S_n(K), E)$ as k tends to ∞.
ii) The completeness of $\mathcal{O}_n^{exp}(S_n(K), E)$ can be proven in the same way using the completeness of the space $\mathcal{O}(S_n(K), E)$ equipped with the system of semi-norms

$$p_{M,\alpha}(f) := \sup_{z \in M} p_\alpha(f(z)), \tag{3.11}$$

$M \subset S_n(K)$ compact and $\alpha \in A$. The remaining endorsement is evident.

(2) This follows by (1) and [16, 2.4 Korollar, p. 36].

(3) This proof follows the ideas of the proof of Theorem 3.5(1). By [32, Remark 6, p. 380] we have to show that for all $n \in \mathbb{N}_{\geq 2}$ exists $p > n$ such that $\pi_{n,p}: \mathcal{O}_p^{exp}\left(S_p(K)\right) \to \mathcal{O}_n^{exp}(S_n(K))$ is a compact mapping. Because the considered spaces are Banach spaces by (1), it suffices to show the existence of $p > n$ such that $\left(\pi_{n,p}(f_k)\right)_{k \in \mathbb{N}}$ has a convergent subsequence in $\mathcal{O}_p^{exp}\left(S_p(K)\right)$ for every sequence $(f_k)_{k \in \mathbb{N}}$ in B_p where $B_p := \left\{g \in \mathcal{O}_p^{exp}\left(S_p(K)\right) \mid |g|_p \leq 1\right\}$. Choose $p := 2n$. Let $\varepsilon > 0$ and choose $Q := \overline{S_n(K)} \cap \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq \max\{0, -2n \ln \varepsilon\}\}$. Then $Q \subset S_{2n}(K)$ compact and

$$\sup_{z \in S_n(K) \setminus Q} \frac{e^{-\frac{1}{n}|\mathrm{Re}(z)|}}{e^{-\frac{1}{2n}|\mathrm{Re}(z)|}} = \sup_{z \in S_n(K) \setminus Q} e^{-\frac{1}{2n}|\mathrm{Re}(z)|} \underset{\substack{\text{choice} \\ \text{of } Q}}{<} \varepsilon.$$

Let $M \subset S_{2n}(K)$ be compact and p_M like in (3.11). For all $f \in B_{2n}$ we have

$$p_M(f) \leq \underbrace{\sup_{z \in M} e^{\frac{1}{n}|\mathrm{Re}(z)|}}_{=: C_{M,n}} \underbrace{\|f\|_n}_{\leq 1} \leq C_{M,n},$$

so $\sup_{f \in B_n} p_M(f) \leq C_{M,n}$. Thus B_{2n} is bounded in $\mathcal{O}(S_{2n}(K))$ with respect to the semi-norms (3.11). As this space is a Fréchet-Montel space, B_{2n} is relatively compact and hence relatively sequentially compact. The rest of the proof is analogous to part iv) of the proof of Theorem 3.5(1) using in the last step the completeness of $\mathcal{O}_n^{exp}(S_n(K))$ by (1).

(4) (a) Choose $r := \frac{k-n}{2nk}$, $K \neq \varnothing$, resp. $r := \frac{k-n}{2}$, $K = \varnothing$. By the Cauchy inequality we have for every $f \in \mathcal{O}_k^{exp}(S_k(K), E)$

$$
\begin{aligned}
|f|_{n,m,\alpha} &= \sup_{\substack{z\in S_n(K)\\ \beta\in\mathbb{N}_0^2, |\beta|\leq m}} p_\alpha\Big(\underbrace{\partial^\beta f(z)}_{\underset{(3.2)}{=} i^{\beta_2} f^{(|\beta|)}(z)}\Big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq \sup_{\substack{z\in S_n(K)\\ \beta\in\mathbb{N}_0^2, |\beta|\leq m}} \frac{|\beta|!}{r^{|\beta|}}\max_{|\zeta-z|=r} p_\alpha(f(\zeta))e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq e^{\frac{r}{n}} \sup_{\substack{z\in S_n(K)\\ \beta\in\mathbb{N}_0^2, |\beta|\leq m}} \frac{|\beta|!}{r^{|\beta|}}\max_{|\zeta-z|=r} p_\alpha(f(\zeta))e^{-\frac{1}{n}|\mathrm{Re}(\zeta)|}\\
&\leq \underbrace{e^{\frac{r}{n}} \sup_{\beta\in\mathbb{N}_0^2, |\beta|\leq m} \frac{|\beta|!}{r^{|\beta|}}}_{=:C} |f|_{k,\alpha} < \infty,
\end{aligned}
$$

thus $\mathcal{O}_k^{exp}(S_k(K), E) \subset \mathcal{E}_n^{exp}(S_n(K), E)$.

(b) By part (a) $\mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus K, E\big)$ is included in $\mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus K, E\big)$ and the induced topology is not finer than the initial one. On the other hand, for all $\alpha \in A$ and $n \in \mathbb{N}_{\geq 2}$

$$|f|_{n,\alpha} = |f|_{n,0,\alpha}$$

holds for any $f \in \mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus K, E\big)$ which proves the statement.

□

3.7 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact. Then the spaces $\mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus K\big)$ and $\mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus K\big)$ are nuclear.*

Proof. Since $\mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus K\big)$ is a topological subspace of $\mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus K\big)$ by Theorem 3.11(4)(b) and nuclearity is inherited by topological subspaces of nuclear spaces due to [16, 27.2.1. Satz, p. 155], we only need to show that $\mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus K\big)$ is nuclear. The following proof is inspired by a proof of nuclearity for the space of (non-weighted) $\mathcal{C}^\infty$-functions in [45, 28.9 Beispiele (1), p. 330].

1. Let $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_0$. First of all, we construct a partition of unity. Choose $0 < h < \frac{1}{2\sqrt{2}n}$ and set $\varepsilon := \frac{h}{2}$. For $j \in \mathbb{Z}^2$ and $r > 0$ we define

$$Q(j,r) := \varepsilon\left[j + \left(\frac{1}{2}, \frac{1}{2}\right)\right] + Q_{\frac{r}{2}}(0) = \varepsilon j + \frac{1}{2}(\varepsilon - r, \varepsilon - r) + [0,r]^2.$$

Further, we define $J := \{j \in \mathbb{Z}^2 \mid Q(j,\varepsilon) \cap S_n(K) \neq \varnothing\}$. For all $j \in J$ it follows, by the choice of h, that $Q(j, 3\varepsilon) \subset S_{2n}(K)$, since $2\sqrt{2}\varepsilon = \sqrt{2}h < \frac{1}{2n} < n$, and that

$$S_n(K) \subset \bigcup_{j\in J} Q(j,\varepsilon) \subset \bigcup_{j\in J} Q(j,3\varepsilon) \subset S_{2n}(K).$$

Choose $0<\tilde{\varepsilon}<\mathrm{d}(Q(j,\varepsilon),Q(j,3\varepsilon))=\varepsilon$ (in particular this choice is independent of j). The set J is countably infinite, i.e. there exists a bijection $a\colon\mathbb{N}\to J$, where we write $a_k:=a(k)$ for $k\in\mathbb{N}$. The construction of the partition of unity is now done like in [18, Theorem 1.4.1, p. 25].

(i) Let v_k be the characteristic function of $Q(a_k,\varepsilon)+\overline{D_{\tilde{\varepsilon}/2}(0)}$ and $\chi\in C_0^\infty(D_1(0))$ a non-negative function such that $\int\chi dx=1$. Then $\chi_{\tilde{\varepsilon}/4}$, defined by $\chi_{\tilde{\varepsilon}/4}(x):=(\tilde{\varepsilon}/4)^{-2}\chi\left(\frac{x}{\tilde{\varepsilon}/4}\right)$, has its support in the ball $D_{\tilde{\varepsilon}/4}(0)$ and $\int\chi_{\tilde{\varepsilon}/4}dx=1$, so the convolution

$$\psi_k:=v_k*\chi_{\tilde{\varepsilon}/4}\in C_0^\infty\left(Q(a_k,\varepsilon)+\overline{D_{\frac{3}{4}\tilde{\varepsilon}}(0)}\right),$$

$0\leq\psi_k\leq1$ and $\psi_k=1$ on $Q(a_k,\varepsilon)+\overline{D_{\tilde{\varepsilon}/4}(0)}$ since $1-\psi_k=(1-v_k)*\chi_{\tilde{\varepsilon}/4}$ vanishes on $Q(a_k,\varepsilon)+\overline{D_{\tilde{\varepsilon}/4}(0)}$.

(ii) By virtue of this construction we get ([18, (1.4.2), p. 25]): For all $k\in\mathbb{N}$ and all $\alpha\in\mathbb{N}_0^2$ there exists a constant $C_\alpha>0$ only depending on α, especially not on k, such that

$$|\partial^\alpha\psi_k|\leq C_\alpha\left(\frac{\tilde{\varepsilon}}{4}\right)^{-|\alpha|}.$$

(iii) Like in the proofs of [18, Theorem 1.4.4, p. 27, Theorem 1.4.10, p. 30] we define

$$\varphi_k:=\psi_k(1-\psi_1)\ldots(1-\psi_{k-1}).$$

Due to (i) we have $\operatorname{supp}\varphi_k\subset\operatorname{supp}\psi_k$ and $\varphi_k\in C_0^\infty\left(Q(a_k,\varepsilon)+\overline{D_{\frac{3}{4}\tilde{\varepsilon}}(0)}\right)$.

(iv) For $\zeta\in\bigcup_{j\in J}Q(j,\varepsilon)=\bigcup_{k\in\mathbb{N}}Q(a_k,\varepsilon)$ we define $M(\zeta):=\{k\in\mathbb{N}\mid\zeta\in\operatorname{supp}\varphi_k\}$. For all ζ we have $M(\zeta)\subset\{k\in\mathbb{N}\mid\zeta\in\operatorname{supp}\psi_k\}$ and hence $\#M(\zeta)\leq9$ by the construction of ψ_k and the definition of the squares. In other words, all but a finite number of functions φ_k vanish identically on any compact subset of $\bigcup_{j\in J}Q(j,\varepsilon)$.

(v) We have on $\bigcup_{k\in\mathbb{N}}Q(a_k,\varepsilon)$

$$\left(\sum_{k\in\mathbb{N}}\varphi_k\right)-1\underset{\substack{(iii),\\(iv)}}{=}-\prod_{k\in\mathbb{N}}(1-\psi_k)\underset{(i)}{=}0$$

and thus $\sum_{k\in\mathbb{N}}\varphi_k=1$.

2. Let $f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$, $\alpha\in\mathbb{N}_0^2$, $|\alpha|\leq m$, and $z\in Q(a_k,\varepsilon)$. We have $Q(a_k,\varepsilon)\subset Q(a_k,3\varepsilon)$ and

there exist b_i, c_i, $i=1,2$, such that $Q(a_k,3\varepsilon)=[b_1,c_1]\times[b_2,c_2]$. We get (by induction)

$$\begin{aligned}
|\partial^{\alpha}(\varphi_k f)(z)| &= \big|\partial^{\alpha}(\varphi_k f)(z_1,z_2)-\underbrace{\partial^{\alpha}(\varphi_k f)(b_1,z_2)}_{\underset{(iii)}{=}0}\big| \\
&= \left|\int_{b_1}^{z_1}\partial^{(\alpha_1+1,\alpha_2)}(\varphi_k f_1)(\zeta_0,z_2)\,d\zeta_0 + i\int_{b_1}^{z_1}\partial^{(\alpha_1+1,\alpha_2)}(\varphi_k f_2)(\zeta_0,z_2)\,d\zeta_0\right| \\
&= \left|\int_{b_1}^{z_1}\partial^{(\alpha_1+1,\alpha_2)}(\varphi_k f)(\zeta_0,z_2)\,d\zeta_0\right| \\
&= \left|\int_{b_1}^{z_1}\int_{b_1}^{\zeta_0}\cdots\int_{b_1}^{\zeta_{m-\alpha_1-1}}\partial^{(m+1,\alpha_2)}(\varphi_k f)(\zeta_{m-\alpha_1},z_2)\,d\zeta_{m-\alpha_1}\ldots d\zeta_1 d\zeta_0\right| \\
&\leq |z_1-b_1|^{m-\alpha_1}\int_{b_1}^{z_1}\big|\partial^{(m+1,\alpha_2)}(\varphi_k f)(\zeta_1,z_2)\big|\,d\zeta_1 \\
&\leq \underbrace{|z_1-b_1|^{m-\alpha_1}|z_1-b_2|^{m-\alpha_2}}_{\leq(3\varepsilon)^{2m-|\alpha|}}\int_{b_1}^{z_1}\int_{b_2}^{z_2}\big|\partial^{(m+1,m+1)}(\varphi_k f)(\zeta_1,\zeta_2)\big|\,d\zeta_2 d\zeta_1 \\
&\underset{\text{Fubini}}{\leq}(3\varepsilon)^{2m-|\alpha|}\int_{Q(a_k,3\varepsilon)}\big|\partial^{(m+1,m+1)}(\varphi_k f)(\zeta)\big|\,d\zeta.
\end{aligned} \tag{3.12}$$

Furthermore, we get for all $z\in S_n(K)$

$$\begin{aligned}
|\partial^{\alpha}f(z)| &\underset{(v)}{=}\Big|\partial^{\alpha}\Big(\sum_{k\in\mathbb{N}}\varphi_k f\Big)(z)\Big| \underset{(iv)}{=}\Big|\sum_{k\in\mathbb{N}}\partial^{\alpha}\big(\varphi_k f\big)(z)\Big| \\
&\leq \sum_{k\in\mathbb{N}}\big|\partial^{\alpha}\big(\varphi_k f\big)(z)\big| \underset{(iv)}{=}\sum_{k\in M(z)}\big|\partial^{\alpha}\big(\varphi_k f\big)(z)\big| \\
&\underset{(3.12)}{\leq}(3\varepsilon)^{2m-|\alpha|}\sum_{k\in M(z)}\int_{Q(a_k,3\varepsilon)}\big|\partial^{(m+1,m+1)}(\varphi_k f)(\zeta)\big|\,d\zeta.
\end{aligned} \tag{3.13}$$

Now we denote by $t_k:=\varepsilon a_k+\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)$ the center of the squares $Q(a_k,\varepsilon)$ and $Q(a_k,3\varepsilon)$ and consider the mapping

$$\Phi\colon\mathbb{R}^2\to\mathbb{R}^2,\ \Phi(w):=3w-2t_k.$$

Then Φ is a C^1-diffeomorphism, $(D\Phi)(w)=\begin{pmatrix}3&0\\0&3\end{pmatrix}$ as Jacobian matrix and $\Phi(Q(a_k,\varepsilon))=Q(a_k,3\varepsilon)$. Moreover, we obtain via chain rule for all $\beta\in\mathbb{N}_0^2$

$$\partial^{\beta}(\varphi_k f)(\Phi(w))=3^{-|\beta|}\partial^{\beta}(\varphi_k f\circ\Phi)(w). \tag{3.14}$$

Since

$$\operatorname{supp}(\varphi_k f\circ\Phi)\subset\operatorname{supp}(\varphi_k\circ\Phi)\subset\mathring{Q}(a_k,\varepsilon) \tag{3.15}$$

for all $k\in\mathbb{N}$ by (iii) and the definition of Φ, we get for k, $l\in\mathbb{N}$, $k\neq l$,

$$[\operatorname{supp}(\varphi_k\circ\Phi)\cap\operatorname{supp}(\varphi_l\circ\Phi)]\subset\big[\mathring{Q}(a_k,\varepsilon)\cap\mathring{Q}(a_l,\varepsilon)\big]=\varnothing$$

and thus by (3.14)

$$\operatorname{supp}\big(\big[\partial^{\beta}(\varphi_k f)\big]\circ\Phi\big)\cap\operatorname{supp}\big(\big[\partial^{\beta}(\varphi_l f)\big]\circ\Phi\big)=\varnothing. \tag{3.16}$$

In addition, we have for $z \in S_n(K)$ and all $w \in Q(a_k, \varepsilon)$, $k \in M(z)$,

$$|\mathrm{Re}(w)| - |\mathrm{Re}(z)| \le |\mathrm{Re}(w) - \mathrm{Re}(z)| \le 2\varepsilon$$

and so

$$-\frac{1}{n}|\mathrm{Re}(z)| \le -\frac{1}{n}|\mathrm{Re}(w)| + \frac{2\varepsilon}{n}. \tag{3.17}$$

Applying the transformation formula to (3.13), we obtain for all $z \in S_n(K)$

$$\begin{aligned}
&|\partial^\alpha f(z)| e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&= 9(3\varepsilon)^{2m-|\alpha|} \sum_{k \in M(z)} \int_{Q(a_k,\varepsilon)} \left|\partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| dw e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\underset{(3.17)}{\le} 9(3\varepsilon)^{2m-|\alpha|} e^{\frac{2\varepsilon}{n}} \sum_{k \in M(z)} \int_{Q(a_k,\varepsilon)} \left|\partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw\\
&\underset{(3.15)}{=} 9(3\varepsilon)^{2m-|\alpha|} e^{\frac{2\varepsilon}{n}} \sum_{k \in M(z)} \int_{S_{2n}(K)} \left|\partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw\\
&\underset{(iv)}{=} 9(3\varepsilon)^{2m-|\alpha|} e^{\frac{2\varepsilon}{n}} \int_{S_{2n}(K)} \sum_{k \in M(z)} \left|\partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw\\
&\le 9(3\varepsilon)^{2m-|\alpha|} e^{\frac{2\varepsilon}{n}} \int_{S_{2n}(K)} \sum_{k \in \mathbb{N}} \left|\partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw\\
&\underset{(3.16)}{=} 9(3\varepsilon)^{2m-|\alpha|} e^{\frac{2\varepsilon}{n}} \int_{S_{2n}(K)} \left|\sum_{k \in \mathbb{N}} \partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw\\
&\le \underbrace{9\max(3\varepsilon, 1)^{2m} e^{\frac{2\varepsilon}{n}}}_{=:D} \int_{S_{2n}(K)} \left|\sum_{k \in \mathbb{N}} \partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
|f|_{n,m} &= \sup_{\substack{z \in S_n(K),\\ \alpha \in \mathbb{N}_0^2,\, |\alpha| \le m}} |\partial^\alpha f(z)| e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\le D \int_{S_{2n}(K)} \left|\sum_{k \in \mathbb{N}} \partial^{(m+1,m+1)}(\varphi_k f)(3w - 2t_k)\right| e^{-\frac{1}{n}|\mathrm{Re}(w)|} dw.
\end{aligned} \tag{3.18}$$

3. For $\zeta \in S_{2n}(K)$ set

$$\Delta(\zeta)[f] := \sum_{k \in \mathbb{N}} \partial^{(m+1,m+1)}(\varphi_k f)(3\zeta - 2t_k) e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}.$$

By (3.16) there is at most one $k_0 \in \mathbb{N}$ such that $\zeta \in \mathrm{supp}\left(\left[\partial^\beta\left(\varphi_{k_0} f\right)\right] \circ \Phi\right)$. Otherwise we have $\Delta(\zeta)[f] = 0$ and choose an arbitrary $k_0 \in \mathbb{N}$. Due to (ii), (iii) and the Leibniz rule, we also have, with the notation $\tilde{m} := (m+1, m+1)$, that there exists a constant $C > 0$, depending on m and $\tilde{\varepsilon}$ (not on ζ), such that

$$
\begin{aligned}
&\left|\partial^{(m+1,m+1)}\left(\varphi_{k_0}f\right)\left(3\zeta-2t_{k_0}\right)\right| \\
&=\Big|\sum_{\gamma\leq\tilde{m}}\binom{\tilde{m}}{\gamma}\partial^{\tilde{m}-\gamma}\varphi_{k_0}\left(3\zeta-2t_{k_0}\right)\partial^{\gamma}f\left(3\zeta-2t_{k_0}\right)\Big| \\
&\leq\left(\sum_{\gamma\leq\tilde{m}}\binom{\tilde{m}}{\gamma}\left|\partial^{\tilde{m}-\gamma}\varphi_{k_0}\left(3\zeta-2t_{k_0}\right)\right|\right)\sup_{\substack{w\in Q\left(a_{k_0},3\varepsilon\right),\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq|\tilde{m}|}}\left|\partial^{\beta}f(w)\right| \\
&\leq C\sup_{\substack{w\in Q\left(a_{k_0},3\varepsilon\right),\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq 2(m+1)}}\left|\partial^{\beta}f(w)\right|.
\end{aligned}
$$

This implies, keeping $\zeta\in Q\left(a_{k_0},\varepsilon\right)$ by (3.15) in mind,

$$
\begin{aligned}
|\Delta(\zeta)[f]|&=\left|\sum_{k\in\mathbb{N}}\partial^{(m+1,m+1)}(\varphi_k f)(3\zeta-2t_k)\,e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}\right| \\
&\leq\left|\partial^{(m+1,m+1)}\left(\varphi_{k_0}f\right)\left(3\zeta-2t_{k_0}\right)\right|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|} \\
&\leq C\sup_{\substack{w\in Q\left(a_{k_0},3\varepsilon\right),\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq 2(m+1)}}\left|\partial^{\beta}f(w)\right|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|} \\
&\leq\underbrace{Ce^{\frac{\varepsilon}{n}}}_{=:C_1}\sup_{\substack{w\in Q\left(a_{k_0},3\varepsilon\right),\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq 2(m+1)}}\left|\partial^{\beta}f(w)\right|e^{-\frac{1}{2n}|\mathrm{Re}(w)|} \\
&\leq C_1\,|f|_{2n,2(m+1)}\,.
\end{aligned}
\tag{3.19}
$$

If we set $V:=\left\{f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)\mid|f|_{2n,2(m+1)}\leq\frac{1}{C_1}\right\}$, then V is an absolutely convex neighbourhood of zero in $\mathcal{E}^{exp}(K)$. We claim that the mapping

$$\Delta\colon S_{2n}(K)\to\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)'_{\sigma}$$

is continuous. Let $\varepsilon_0>0$ and $\zeta\in S_{2n}(K)$ and w.l.o.g. there is k_0 with $\zeta\in\operatorname{supp}\left(\varphi_{k_0}f\circ\Phi\right)\subset\mathring{Q}\left(a_{k_0},\varepsilon\right)$ (see (3.15)). Then choose $0<\delta_1<\mathrm{d}\left(\zeta,\partial Q\left(a_{k_0},\varepsilon\right)\right)$. For all $w\in S_{2n}(K)$, such that $|w-\zeta|<\delta_1$, the following is valid for all $f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)$:

$$
\begin{aligned}
&|\Delta(w)[f]-\Delta(\zeta)[f]| \\
&=\left|\partial^{(m+1,m+1)}\left(\varphi_{k_0}f\right)\left(3w-2t_{k_0}\right)e^{-\frac{1}{2n}|\mathrm{Re}(w)|}-\partial^{(m+1,m+1)}\left(\varphi_{k_0}f\right)\left(3\zeta-2t_{k_0}\right)e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}\right|
\end{aligned}
$$

Since $\partial^{(m+1,m+1)}\left(\varphi_{k_0}f\right)\left(3\cdot-2t_{k_0}\right)e^{-\frac{1}{2n}|\mathrm{Re}(\cdot)|}$ is continuous on $S_{2n}(K)$, there is $\delta_2=\delta_2(f)>0$ such that

$$|\Delta(w)[f]-\Delta(\zeta)[f]|<\varepsilon_0$$

for all $w\in S_{2n}(K)$ with $|w-\zeta|<\min(\delta_1,\delta_2)=:\delta_0(f)$. For a finite set $M\subset\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)$

define $\delta := \min_{f\in M}\delta_0(f) > 0$. Then we get for all $w \in S_{2n}(K)$, such that $|w-\zeta| < \delta$,

$$\sup_{f\in M}|\Delta(w)[f]-\Delta(\zeta)[f]| < \varepsilon_0$$

which proves the claim. In addition, we have $\Delta(S_{2n}(K)) \subset V^\circ$ by (3.19). Next we set

$$u\colon \mathrm{C}(V^\circ)\to\mathbb{C},\ u(g) := D\int_{S_{2n}(K)} g(\Delta(\zeta))e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta.$$

Due to the Alaoğlu-Bourbaki theorem V° is $\sigma\left(\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)',\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)\right)$–compact and $g\circ\Delta$ is Lebesgue measurable since g and Δ are continuous. Moreover, u is linear and, if we equip $\mathrm{C}(V^\circ)$ with the norm

$$|||g||| := \sup_{y\in V^\circ}\left|g(y)\right|,\ g\in\mathrm{C}(V^\circ),$$

continuous as

$$\begin{aligned}|u(g)| &\leq D\int_{S_{2n}(K)}\left|g(\underbrace{\Delta(\zeta)}_{\in V^\circ})\right|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta \leq D\int_{S_{2n}(K)}\sup_{y\in V^\circ}\left|g(y)\right|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta\\ &= \underbrace{D\int_{S_{2n}(K)}e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta}_{\leq 4nD\int_{-\infty}^{\infty}e^{-\frac{1}{2n}|x|}dx=:D_1<\infty}|||g||| \leq D_1|||g|||.\end{aligned}$$

Hence there exists a (positive) measure μ on $\left(V^\circ,\ \sigma\left(\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)',\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)\right)\right)$ and a Borel measurable function $h_0\colon V^\circ\to\mathbb{C}$ with $|h_0(y)| = 1$ for all $y\in V^\circ$ by [45, 13.10 Satz von Riesz, p. 102], such that

$$u(g) = \int_{V^\circ} gh_0 d\mu. \tag{3.20}$$

4. Altogether, we obtain, keeping in mind that every $f\in\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)$ defines a continuous (linear) functional $J(f)\colon V^\circ\to\mathbb{C},\ y\mapsto y(f)$, that

$$\begin{aligned}|f|_{n,m} &\underset{(3.18)}{\leq} D\int_{S_{2n}(K)}|\Delta(\zeta)[f]|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta = D\int_{S_{2n}(K)}|J(f)(\Delta(\zeta))|e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}d\zeta\\ &= u(|J(f)|) \underset{(3.20)}{=} \int_{V^\circ}|J(f)|h_0 d\mu = \int_{V^\circ}|y(f)|h_0(y)d\mu(y)\\ &= \left|\int_{V^\circ}|y(f)|h_0(y)d\mu(y)\right| \leq \int_{V^\circ}|y(f)|\underbrace{|h_0(y)|}_{=1}d\mu(y)\\ &= \int_{V^\circ}|y(f)|d\mu(y).\end{aligned}$$

Therefore, $\mathcal{E}^{exp}(\overline{\mathbb{C}}\setminus K)$ is nuclear by [45, § 28, Definition, p. 324].[1]

□

[1]Usually it is required that the measure μ is a positive Radon measure, see for example [54, 4.1.5. Satz, p. 64]. But this is not needed due to [45, § 28, Definition, p. 324], [45, 28.4 Satz, p. 327] and [54, 4.1.1. Lemma, p. 62].

3.8 Remark.

1. A direct proof of nuclearity of $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ can be done in a similar, but easier (without partition of unity), way using the mean value property of holomorphic functions.

2. A different proof of nuclearity can be found in [26, 1.6 Folgerung, p. 7].

3. Since every nuclear space is a Schwartz space by [45, 28.5 Corollar, p. 328], we have by Theorem 3.6(2) that $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ are Fréchet-Schwartz spaces, so for the latter one another proof of this property (see Theorem 3.6(3)). Furthermore, they are Montel spaces by [45, 24.24 Bemerkung (b), p. 267] and [45, 23.23 Satz, p. 253].

For a subset M of a vector space X we denote by $\Gamma(M)$ the absolutely convex hull of M.

3.9 Lemma. *Let X be a complete Montel space and $\left(Y,\left(|\!|\!|\cdot|\!|\!|_{\beta}\right)_{\beta\in B}\right)$ a locally convex space. Then $L(X'_{\tau},Y)=L(X'_{c},Y)$ holds algebraically. In particular, this is true if X is a nuclear Fréchet space.*

Proof. Let $y\in L(X'_{\tau},Y)$. For $\beta\in B$ there exists $M\subset X$ absolutely convex and weakly compact and $C>0$ such that

$$|\!|\!|y(x')|\!|\!|_{\beta}\leq C\sup_{x\in M}|x'(x)|\leq C\sup_{x\in\overline{M}}|x'(x)|$$

for all $x'\in X'$. Since M is weakly compact, it is weakly bounded and hence by the Mackey theorem bounded with respect to the initial topology of X. So M is relative compact as X is Montel space. Thus $\overline{M}$ is compact and precompact as well by [25, 3.5.3. Corollary, p. 65].
On the other hand, let $y\in L(X'_{c},Y)$. For $\beta\in B$ there exists $M\subset X$ precompact and $C>0$ such that

$$|\!|\!|y(x')|\!|\!|_{\beta}\leq C\sup_{x\in M}|x'(x)|\leq C\sup_{x\in\overline{\Gamma(M)}}|x'(x)|$$

for all $x'\in X'$. By [25, 6.7.1. Proposition, p. 112] $\overline{\Gamma(M)}$ is precompact since M is precompact. Because X is complete and $\overline{\Gamma(M)}$ is precompact, it follows by [25, 3.5.3. Corollary, p. 65] that $\overline{\Gamma(M)}$ is compact, in particular, weakly compact. Moreover, this set is absolutely convex due to [25, 6.2.1. Proposition, p. 102].
Since every nuclear space is a Schwartz space by [45, 28.5 Corollar, p. 328] and every Fréchet space is barrelled by [45, 23.23 Satz, p. 253], we have that every nuclear Fréchet space is a Montel space by [45, 24.24 Bemerkung (b), p. 267] connoting the apposition. □

The next aim is to prove that the spaces $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K,E\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K,E\right)$ are topologically isomorphic to $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$ resp. $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$. The proof is based on an analysis of the proofs of an analogous statement for continuous functions on a compact space resp. weighted continuous functions by Bierstedt [3, 4.4 Lemma, 4.5 Lemma, 4.6 Folgerung, 4.11 Satz, p. 44-50] and [7, Lemma 5.7, Consequence 5.9, Proposition 5.10, p. 29-32] resp. [4, 4.2 Lemma, 4.3 Folgerung, p. 199-200] and [5, 2.1 Satz, 2.2 Bemerkung, p. 137-138] as well as upon analyzing the proof of the so-called Grothendieck's weak-strong principle [17, Chap. II, § 3, n° 3, Lemme 8, Théorème 13, p. 78-80].

3.10 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be compact.*

(1) *The sets* $\left\{\delta_z^{(\beta)} \mid z \in \mathbb{C} \setminus K,\ \beta \in \mathbb{N}_0^2\right\}$ *and* $\left\{\delta_z^{(\beta)} e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in \mathbb{C} \setminus K,\ \beta \in \mathbb{N}_0^2\right\}$ *are contained in* $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'$.
In addition, $\{\delta_z \mid z \in \mathbb{C} \setminus K\}$ *and* $\left\{\delta_z e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in \mathbb{C} \setminus K\right\}$ *are subsets of* $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'$.

(2) *Let* $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$ *and* $\Delta_{n,m}\left(S_n(K)\right) := \left\{\delta_z^{(\beta)} e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in S_n(K),\ \beta \in \mathbb{N}_0^2,\ |\beta| \leq m\right\}$.
Then $\Gamma\left(\Delta_{n,m}\left(S_n(K)\right)\right)$ *is dense in* $B_{n,m}^{\circ}$ *with respect to* $\sigma\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)', \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)\right)$ *and* $\lambda\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)', \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)\right)$ *where* $B_{n,m} := \left\{f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \mid |f|_{n,m} \leq 1\right\}$.

(3) *Let* $n \in \mathbb{N}_{\geq 2}$ *and define* $\Delta^n\left(S_n(K)\right) := \left\{\delta_z e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in S_n(K)\right\}$.
Then $\Gamma\left(\Delta^n\left(S_n(K)\right)\right)$ *is dense in* B_n° *with respect to* $\sigma\left(\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)', \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)\right)$ *and* $\lambda\left(\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)', \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)\right)$ *where* $B_n := \{f \in \mathcal{O}^{exp}(K) \mid |f|_n \leq 1\}$.

(4) i) *The topology of* $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$ *is given by the system of semi-norms*

$$q_{n,m,\alpha}(u) := \sup_{\substack{z \in S_n(K),\\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(u\left(\delta_z^{(\beta)}\right)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|},\ n \in \mathbb{N}_{\geq 2},\ m \in \mathbb{N}_0,\ \alpha \in A,$$

ii) *and the topology of* $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$ *by*

$$q_{n,\alpha}(u) := \sup_{z \in S_n(K)} p_\alpha\left(u(\delta_z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|},\ n \in \mathbb{N}_{\geq 2},\ \alpha \in A.$$

Proof. (1) Let $\beta \in \mathbb{N}_0^2$, $z \in \mathbb{C} \setminus K$ and $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$. Then there exists $n \in \mathbb{N}_{\geq 2}$ such that $z \in S_n(K)$ and

$$\begin{aligned}\left|\delta_z^{(\beta)}(f)\right| e^{-\frac{1}{n}|\mathrm{Re}(z)|} \leq \left|\delta_z^{(\beta)}(f)\right| &= \left|\partial^\beta f(z)\right| e^{-\frac{1}{n}|\mathrm{Re}(z)|} e^{\frac{1}{n}|\mathrm{Re}(z)|} \\ &\leq \underbrace{e^{\frac{1}{n}|\mathrm{Re}(z)|}}_{=:C(z)} \sup_{\substack{x \in S_n(K),\\ \gamma \in \mathbb{N}_0^2, |\gamma| \leq |\beta|}} \left|\partial^\gamma f(x)\right| e^{-\frac{1}{n}|\mathrm{Re}(x)|} = C(z)\,|f|_{n,|\beta|} < \infty\end{aligned}$$

implying the statement. The other proof is analogous.

(2) We have

$$\begin{aligned}&\Delta_{n,m}\left(S_n(K)\right)^{\circ} \\ &= \left\{\delta_z^{(\beta)} e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in S_n(K),\ |\beta| \leq m\right\}^{\circ} \\ &= \left\{f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \mid \forall z \in S_n(K)\ \forall \beta \in \mathbb{N}_0^2,\ |\beta| \leq m : \left|\delta_z^{(\beta)} e^{-\frac{1}{n}|\mathrm{Re}(z)|}(f)\right| \leq 1\right\} \\ &= \left\{f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \mid \forall z \in S_n(K)\ \forall \beta \in \mathbb{N}_0^2,\ |\beta| \leq m : \left|\partial^\beta f(z) e^{-\frac{1}{n}|\mathrm{Re}(z)|}\right| \leq 1\right\} \\ &= \left\{f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \mid |f|_{n,m} \leq 1\right\} \\ &= B_{n,m}.\end{aligned}$$

The polar $B_{n,m}^{\circ}$ is equicontinuous in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)'$ and thus the topologies $\sigma\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)$ and $\lambda\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)$ coincide on $B_{n,m}^{\circ}$ by [16, 3.3 Satz, p. 53]. Due to the bipolar theorem we get

$$\begin{aligned}\overline{\Gamma\left(\Delta_{n,m}\left(S_n\left(K\right)\right)\right)}^{\lambda\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)} &= \overline{\Gamma\left(\Delta_{n,m}\left(S_n\left(K\right)\right)\right)}^{\sigma\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)}\\ &= \Delta_{n,m}\left(S_n\left(K\right)\right)^{\circ\circ} = B_{n,m}^{\circ}\end{aligned}$$

where the polar sets are taken with respect to the dual system $\left\langle\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right\rangle$.

(3) Analogously to the proof of (2).

(4) i) By [25, 8.4, p. 152, 16.1, p. 344] the system of semi-norms

$$\tilde{q}_{n,m,\alpha}\left(u\right) := \sup_{y\in B_{n,m}^{\circ}} p_{\alpha}\left(u\left(y\right)\right),\ n\in\mathbb{N}_{\geq 2},\ m\in\mathbb{N}_0,\ \alpha\in A,$$

gives the topology on $\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$. As every $u\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$ is continuous on $B_{n,m}^{\circ}$, we may replace $B_{n,m}^{\circ}$ by a $\lambda\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)',\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)$-dense subset. Therefore, we gain by (2)

$$\tilde{q}_{n,m,\alpha}\left(u\right) = \sup_{y\in\Gamma\left(\Delta_{n,m}\left(S_n\left(K\right)\right)\right)} p_{\alpha}\left(u\left(y\right)\right).$$

Let $\lambda_j\in\mathbb{C}$, $\beta_j\in\mathbb{N}_0^2$, $\left|\beta_j\right|\leq m$, $z_j\in S_n\left(K\right)$, $1\leq j\leq k$, and $\sum_{j=1}^{k}\left|\lambda_j\right|\leq 1$, $k\in\mathbb{N}$. Then we have for $u\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$

$$\begin{aligned}&p_{\alpha}\left(u\Big(\sum_{j=1}^{k}\lambda_j\delta_{z_j}^{(\beta_j)}e^{-\frac{1}{n}\left|\mathrm{Re}(z_j)\right|}\Big)\right)\\ &= p_{\alpha}\left(\sum_{j=1}^{k}\lambda_j u\big(\delta_{z_j}^{(\beta_j)}\big)e^{-\frac{1}{n}\left|\mathrm{Re}(z_j)\right|}\right) \leq \sum_{j=1}^{k}\left|\lambda_j\right| p_{\alpha}\big(u\big(\delta_{z_j}^{(\beta_j)}\big)\big)e^{-\frac{1}{n}\left|\mathrm{Re}(z_j)\right|}\\ &\leq \underbrace{\sum_{j=1}^{k}\left|\lambda_j\right|}_{\leq 1} \sup_{\substack{z\in S_n(K),\\ \beta\in\mathbb{N}_0^2,\left|\beta\right|\leq m}} p_{\alpha}\big(u\big(\delta_z^{(\beta)}\big)\big)e^{-\frac{1}{n}\left|\mathrm{Re}(z)\right|} \leq q_{n,m,\alpha}\left(u\right),\end{aligned}$$

thus $\tilde{q}_{n,m,\alpha}\left(u\right)\leq q_{n,m,\alpha}\left(u\right)$. On the other hand we obtain

$$\begin{aligned}\tilde{q}_{n,m,\alpha}\left(u\right) &= \sup_{y\in\Gamma\left(\Delta_{n,m}\left(S_n\left(K\right)\right)\right)} p_{\alpha}\left(u\left(y\right)\right) \geq \sup_{y\in\Delta_{n,m}\left(S_n\left(K\right)\right)} p_{\alpha}\left(u\left(y\right)\right)\\ &= \sup_{\substack{z\in S_n(K),\\ \beta\in\mathbb{N}_0^2,\left|\beta\right|\leq m}} p_{\alpha}\left(u\left(\delta_z^{(\beta)}\right)\right)e^{-\frac{1}{n}\left|\mathrm{Re}(z)\right|} = q_{n,m,\alpha}\left(u\right).\end{aligned}$$

ii) Analogously to the proof of i), using (3) instead of (2).

□

3.11 Theorem. *Let $K \subset \mathbb{R}$ be compact. Then we have the following topological isomorphisms:*

$$\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E \quad \text{and} \quad \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \cong \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$$

Proof. 1. We will prove that the mapping

$$T\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right),\ u \longmapsto u \circ \Delta,$$

where

$$\Delta\colon \mathbb{C} \setminus K \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_c',\ z \longmapsto \delta_z,$$

is defined by $\delta_z(f) := f(z)$, is the desired isomorphism.

a) At first, we will show that the mapping T is well-defined and that $\partial^{\beta} T(u) = u \circ \Delta^{(\beta)}$, where $\Delta^{(\beta)}(z) := \delta_z^{(\beta)}$, is valid for all $\beta \in \mathbb{N}_0^2$.
By Lemma 3.10(1) the term $u \circ \Delta^{(\beta)}$ is defined for all $\beta \in \mathbb{N}_0^2$. Let $u \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$ and $h \in \mathbb{R}$, $h \neq 0$, such that $\overline{D_{|h|}(z)} \subset \mathbb{C} \setminus K$. Then we have

$$\frac{T(u)(z) - T(u)(z + he_k)}{h} = \frac{u(\delta_z) - u\left(\delta_{z+he_k}\right)}{h} = u\left(\frac{\delta_z - \delta_{z+he_k}}{h}\right)$$

where e_k, $k = 1,2$, denote the unit vectors in $\mathbb{R}^2$.
We remark that a subset of $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ is precompact iff it is relatively compact due to [45, 4.10 Corollar, p. 20] since $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ is a Fréchet space by Theorem 3.6(2).
Now let $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$. Then

$$\begin{aligned}
&\left|\frac{\delta_z(f) - \delta_{z+he_k}(f)}{h} - \delta_z^{(e_k)}(f)\right| \\
&= \left|\frac{\delta_z(f) - \delta_{z+he_k}(f)}{h} - \delta_z^{(e_k)}(f)\right| = \left|\frac{f(z) - f(z+he_k)}{h} - \partial^{e_k} f(z)\right| \\
&\to 0,\ h \to 0.
\end{aligned}$$

This means that $\frac{\delta_z - \delta_{z+he_k}}{h}$ converges to $\delta_z^{(e_k)}$ in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_\sigma'$ as h tends to 0 and in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_c'$ as well due to [16, 10.3.4 Satz, p. 53] since the Fréchet space $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ is barrelled and by the remark about precompactness above. So we obtain

$$\begin{aligned}
\partial^{e_k} T(u)(z) &= \lim_{h \to 0} \frac{T(u)(z) - T(u)(z+he_k)}{h} = u\left(\lim_{h \to 0} \frac{\delta_z - \delta_{z+he_k}}{h}\right) = u\left(\delta_z^{(e_k)}\right) \\
&= u \circ \Delta^{(e_k)}(z)
\end{aligned}$$

in E with respect to $(p_\alpha)_{\alpha \in A}$ for all $z \in \mathbb{C} \setminus K$. By induction over $|\beta|$ we get that $T(u) \in \mathrm{C}^\infty(\mathbb{C} \setminus K, E)$ and that $\partial^{\beta} T(u) = u \circ \Delta^{(\beta)}$.

Furthermore, we get by Lemma 3.10(4)i) for every $n\in\mathbb{N}_{\geq 2}$, $m\in\mathbb{N}_0$ and $\alpha\in A$

$$|T(u)|_{n,m,\alpha}=\sup_{\substack{z\in S_n(K),\\ \beta\in\mathbb{N}_0^2,|\beta|\leq m}} p_\alpha\Big(\underbrace{\partial^\beta T(u)(z)}_{=u\circ\Delta^{(\beta)}(z)=u\left(\delta_z^{(\beta)}\right)}\Big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}=q_{n,m,\alpha}(u)<\infty, \tag{3.21}$$

implying $T(u)\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K,E\right)$ for every $u\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$. Hence the map T is defined and continuous.

b) injectivity: Let $u\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\varepsilon E$ and $T(u)=0$, i.e. $u\circ\Delta(z)=u(\delta_z)=0$ for all $z\in\mathbb{C}\smallsetminus K$. By differentiating we get due to the first part of the proof

$$u\circ\Delta^{(\beta)}(z)=u\left(\delta_z^{(\beta)}\right)=0$$

for all $z\in\mathbb{C}\smallsetminus K$ and all $\beta\in\mathbb{N}_0^2$. By virtue of Lemma 3.10(3)i) this implies for every $n\in\mathbb{N}_{\geq 2}$, $m\in\mathbb{N}_0$ and $\alpha\in A$ that $q_{n,m,\alpha}(u)=0$ and hence $u=0$.

c) surjectivity: For $f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K,E\right)$ define the mapping

$$u_f\colon\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)'\to E'^{*},\ y\mapsto u_f(y),$$

where E'^{*} is the algebraic dual of E', plus

$$u_f(y)\colon E'\to\mathbb{C},\ e'\mapsto y(e'\circ f),$$

and $e'\circ f$ is defined by $(e'\circ f)(z):=e'(f(z))$ for all $z\in\mathbb{C}\smallsetminus K$.
Let $B_\alpha:=\{x\in E\mid p_\alpha(x)<1\}$ for $\alpha\in A$. The first step is to prove that the mapping u_f is well-defined and that $u_f\in L\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)'_\tau,\left(E'^{*},\left(p_{B_\alpha^\circ}\right)_{\alpha\in A}\right)\right)$ where E'^{*} is equipped with the system of semi-norms

$$p_{B_\alpha^\circ}(x):=\sup_{e'\in B_\alpha^\circ}|x(e')|,\ \alpha\in A.$$

We clearly have $e'\circ f\in C^\infty(\mathbb{C}\smallsetminus K)$ for $e'\in E'$ and $f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K,E\right)$ and there are $C>0$, $\alpha\in A$, such that

$$|e'\circ f|_{n,m}=\sup_{\substack{z\in S_n(K),\\ \beta\in\mathbb{N}_0^2,|\beta|\leq m}}\underbrace{\left|\partial^\beta e'\circ f(z)\right|}_{\substack{=\left|e'\left(\partial^\beta f(z)\right)\right|\\ \leq Cp_\alpha\left(\partial^\beta f(z)\right)}}e^{-\frac{1}{n}|\mathrm{Re}(z)|}\leq C|f|_{n,m,\alpha}$$

for every $n\in\mathbb{N}_{\geq 2}$ and $m\in\mathbb{N}_0$. Thus $e'\circ f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ and $u_f(y)$ is defined for any $y\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)'$ as well as obviously linear, so the mapping u_f is defined.
Let $\alpha\in A$, $n\in\mathbb{N}_{\geq 2}$ and $m\in\mathbb{N}_0$. For all $z\in S_n(K)$ and $\beta\in\mathbb{N}_0^2$, $|\beta|\leq m$, we have

$$p_\alpha\left(\frac{\partial^\beta f(z)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}{2|f|_{n,m,\alpha}}\right)=\frac{1}{2|f|_{n,m,\alpha}}p_\alpha\left(\partial^\beta f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\leq\frac{|f|_{n,m,\alpha}}{2|f|_{n,m,\alpha}}=\frac{1}{2}<1$$

if $|f|_{n,m,\alpha} \neq 0$. Under this condition we get

$$\sup_{e' \in B_\alpha^\circ} |e' \circ f|_{n,m} = 2|f|_{n,m,\alpha} \sup_{e' \in B_\alpha^\circ} \sup_{\substack{z \in S_n(K), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \underbrace{\left| e'\left(\frac{\partial^\beta f(z) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}{2|f|_{n,m,\alpha}} \right) \right|}_{\leq 1} \leq 2|f|_{n,m,\alpha} < \infty.$$

If $|f|_{n,m,\alpha} = 0$, we have

$$p_\alpha\left(\partial^\beta f(z) e^{-\frac{1}{n}|\mathrm{Re}(z)|}\right) \leq |f|_{n,m,\alpha} = 0$$

for all $z \in S_n(K)$, $\beta \in \mathbb{N}_0^2$, $|\beta| \leq m$, and thus

$$\left\{\partial^\beta f(z) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \mid z \in S_n(K),\, \beta \in \mathbb{N}_0^2,\, |\beta| \leq m\right\} \subset B_\alpha$$

implying

$$\sup_{e' \in B_\alpha^\circ} |e' \circ f|_{n,m} = \sup_{e' \in B_\alpha^\circ} \sup_{\substack{z \in S_n(K), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \left| e'\Big(\underbrace{\partial^\beta f(z) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{\in B_\alpha} \Big) \right| \leq 1.$$

Hence, in both cases, the set $M_\alpha := \{e' \circ f \mid e' \in B_\alpha^\circ\}$ is bounded in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$. Furthermore, the closure of the absolutely convex hull $\overline{\Gamma(M_\alpha)}$ of M_α is bounded by [25, 6.7.1. Proposition, p. 112] and absolutely convex by [25, 6.2.1. Proposition, p. 103]. The set $\overline{\Gamma(M_\alpha)}$ is (weakly) compact since it is bounded and closed plus $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ a Montel space by Remark 3.8. This and

$$p_{B_\alpha^\circ}\left(u_f(y)\right) = \sup_{e' \in B_\alpha^\circ} |y(e' \circ f)| = \sup_{x \in M_\alpha} |y(x)| \leq \sup_{x \in \overline{\Gamma(M_\alpha)}} |y(x)| \tag{3.22}$$

imply that $u_f \in L\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_\tau', \left(E'^*, \left(p_{B_\alpha^\circ}\right)_{\alpha \in A}\right)\right)$.
The locally convex space $\left(J(E), \left(p_{B_\alpha^\circ}\right)_{\alpha \in A}\right)$, where J denotes the canonical embedding $J\colon E \to E'^*$, is complete since E is complete and for all $\alpha \in A$, $x \in E$,

$$p_{B_\alpha^\circ}(J(x)) = \sup_{e' \in B_\alpha^\circ} \Big|\underbrace{J(x)(e')}_{=e'(x)}\Big| = p_\alpha(x) \tag{3.23}$$

by [45, 22.14 Satz, p. 237]. Especially, $J(E)$ is closed in E'^*. The set $\{\delta_z \mid z \in \mathbb{C} \setminus K\}$ is total in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_\sigma'$ by the bipolar theorem and thus in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_\tau'$ as well. Since for all $e' \in E'$

$$u_f(\delta_z)(e') = \delta_z(e' \circ f) = e'(f(z)) = J(f(z))(e')$$

and u_f is linear and continuous plus $J(E)$ closed, we get more precisely that
$u_f \in L\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)_\tau', \left(J(E), \left(p_{B_\alpha^\circ}\right)_{\alpha \in A}\right)\right)$.
Therefore, we obtain by setting $\left(J^{-1} \circ u_f\right)(y) := J^{-1}\left(u_f(y)\right)$, $y \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$, for all

$\alpha \in A$

$$p_\alpha\left(\left(J^{-1} \circ u_f\right)(y)\right) = p_{B_\alpha^\circ}\left(J\left(\left(J^{-1} \circ u_f\right)(y)\right)\right) = p_{B_\alpha^\circ}\left(u_f(y)\right) \le \sup_{x \in \overline{\Gamma(M_\alpha)}} |y(x)|$$

by (3.23) and (3.22). Hence $J^{-1} \circ u_f \in L\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'_\tau, E\right) = L\left(\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'_c, E\right)$ by virtue of Lemma 3.9 and so we have $J^{-1} \circ u_f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$. In addition, we gain for $z \in \mathbb{C} \setminus K$

$$T\left(J^{-1} \circ u_f\right)(z) = \left(\left(J^{-1} \circ u_f\right) \circ \Delta\right)(z) = J^{-1}\left(u_f(\delta_z)\right) = J^{-1}\left(J(f(z))\right) = f(z),$$

thus $T\left(J^{-1} \circ u_f\right) = f$ proving the surjectivity of T.

d) continuity of T^{-1} : Looking at (3.21), we get that the inverse of T is also continuous.

2. The proof for the weighted holomorphic functions is analogous to the one above using Lemma 3.10(4)ii) instead of Lemma 3.10(4)i).

□

3.12 Corollary. *Let $K \subset \overline{\mathbb{R}}$ be compact. Then we have the following topological ismorphisms:*

1.

$$\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\varepsilon E \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$$

2.

$$\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \cong \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\varepsilon E \cong \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$$

Proof. As $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ is nuclear by Theorem 3.7, we have $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\varepsilon E \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$ by [60, Theorem 50.1, p. 511]. Due to the nuclearity $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ has the approximation property by [25, 21.2.2. Corollary, p. 483]. Furthermore, $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ and E are complete locally convex spaces and thus we get $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\varepsilon E \cong \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \varepsilon E$ by [38, §43, 3.(7), p. 243]. The statement follows then by Theorem 3.11. The same arguments are valid for $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$. □

3.13 Remark.

(1) Let E be a Fréchet space and F the nuclear Fréchet space $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ resp. $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$. For $f \in F$ and $x \in E$ we define $f \otimes x : \mathbb{C} \setminus K \to E$, $(f \otimes x)(z) := f(z)x$. Then we can describe the statement of the Corollary above in a more concrete manner. The injection

$$\chi : F \otimes_\pi E \to F \varepsilon E, \ \sum_{i=1}^{k} f_i \otimes e_i \mapsto \left(y \mapsto \sum_{i=1}^{k} \langle y, f_i \rangle \otimes e_i\right),$$

is continuous by [25, 16.1, p. 344, 16.1.3. Proposition, p. 345] and so it has a unique, continuous linear extension $\tilde{\chi} : F \hat{\otimes}_\pi E \to F \varepsilon E$ by [25, 3.4.2. Theorem, p. 61]. This extension is a topological isomorphism due to the nuclearity of F. By virtue of [17, Chap. I, § 2, n° 1, Théorème 1, p. 51] every element $f \in F \hat{\otimes}_\pi E$ is the sum of an absolutely convergent series

$$f = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i$$

where $(\lambda_i) \in \ell_1$ and (f_i) and (e_i) are null sequences in F resp. E.[2] Remark that this representation is not unique. Now we define the mapping

$$\chi_0 : F \hat{\otimes}_\pi E \to F\varepsilon E,\ f = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i \mapsto \left(y \mapsto \sum_{i=1}^{\infty} \lambda_i \langle y, f_i \rangle e_i \right).$$

This mapping is well-defined, i.e. it does not depend on the representation of f and series appearing on the right hand side converge in E since we have for $f \in F \hat{\otimes}_\pi E$ and $\alpha \in A$ (A countable)

$$p_\alpha (\chi_0 (f)(y)) \le \sum_{i=1}^{\infty} |\lambda_i| |\langle y, f_i \rangle| p_\alpha (e_i) \le \sum_{i=1}^{\infty} |\lambda_i| \sup_{n \in \mathbb{N}} |\langle y, f_n \rangle| \sup_{n \in \mathbb{N}} p_\alpha (e_n)$$

for all $y \in F'$. Furthermore, $\sup_{n \in \mathbb{N}} p_\alpha (e_n) < \infty$, as (e_n) is a null sequence, and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, because $(\lambda_i) \in \ell_1$. The set $M := \{f_n \mid n \in \mathbb{N}\}$ is precompact in F by a comment in [60, p. 54], since (f_n) is a Cauchy sequence in F. So we get that there is a constant $C > 0$ such that

$$p_\alpha (\chi_0 (f)(y)) \le \sum_{i=1}^{\infty} p_\alpha (\lambda_i \langle y, f_i \rangle e_i) \le C \sup_{x \in M} |y(x)| < \infty.$$

This implies that $\left(\sum_{i=1}^{k} \lambda_i \langle y, f_i \rangle e_i\right)_k$ is a Cauchy sequence in E and thus convergent by the completeness of E as well as $\chi_0 (f) \in F\varepsilon E$. The independence of the representation results now from the totality of $\left\{\delta_z^{(\beta)} \mid z \in \mathbb{C} \setminus K,\ \beta \in \mathbb{N}_0^2\right\}$ resp. $\{\delta_z \mid z \in \mathbb{C} \setminus K\}$ in F_c'. For $F = \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)$ one denotes with $|\cdot|_{n,m} \hat{\otimes}_\pi p_\alpha$ the continuation of $|\cdot|_{n,m} \otimes_\pi p_\alpha$ and these semi-norms form a fundamental system of semi-norms of $\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K) \hat{\otimes}_\pi E$. Then one gets for $f \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K) \hat{\otimes}_\pi E$

$$\left(|\cdot|_{n,m} \hat{\otimes}_\pi p_\alpha\right)(f) = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| |f_i|_{n,m} p_\alpha (e_i) \mid f = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i \right\}$$

where the infimum runs through all such representations by [61, 6.5 Theorem, p. 65] resp. [7, Corollary 8.4, p. 54].[3] For $f = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i$ we obtain by Lemma 3.10(4)

$$\begin{aligned}
&q_{n,m,\alpha}(\chi_0 (f)) \\
&= \sup_{\substack{z \in S_n(K),\\ \beta \in \mathbb{N}_0^2, |\beta| \le m}} p_\alpha \left(\chi_0 (f) \left(\delta_z^{(\beta)} \right) \right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} = \sup_{\substack{z \in S_n(K),\\ \beta \in \mathbb{N}_0^2, |\beta| \le m}} p_\alpha \left(\sum_{i=1}^{\infty} \lambda_i \partial^\beta f_i (z) e_i \right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\le \sum_{i=1}^{\infty} |\lambda_i| \sup_{\substack{z \in S_n(K),\\ \beta \in \mathbb{N}_0^2, |\beta| \le m}} \partial^\beta f_i (z) e^{-\frac{1}{n}|\mathrm{Re}(z)|} p_\alpha (e_i) = \sum_{i=1}^{\infty} |\lambda_i| |f_i|_{n,m} p_\alpha (e_i)
\end{aligned}$$

and, therefore, $q_{n,m,\alpha}(\chi_0 (f)) \le \left(|\cdot|_{n,m} \hat{\otimes}_\pi p_\alpha\right)(f)$ implying the continuity of χ_0. The analogous result is valid for $F = \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)$. By the remark about the uniqueness in the

[2] A similar series representation for the injective tensor product can be found in [3, p. 37] combined with [4, 4. Bemerkung, 6. Satz, p. 196-197] resp. [7, Proposition 3.23, p. 22-23].
[3] The proof is based on the one of [54, 7.5.1. Theorem, p. 105].

beginning we get $\tilde{\chi} = \chi_0$. Combining this topological isomorphism with the topological isomorphism T of Theorem 3.11, we gain for $f = \sum_{i=1}^{\infty} \lambda_i f_i \otimes e_i$ and $z \in \mathbb{C} \smallsetminus K$

$$(T \circ \tilde{\chi})(f)(z) = T(\chi_0(f))(z) = [\chi_0(f) \circ \Delta](z) = \chi_0(f)(\delta_z) = \sum_{i=1}^{\infty} \lambda_i f_i(z) e_i = f(z),$$

so $(T \circ \tilde{\chi}) = \mathrm{id}$. Hence the topological isomorphism between $F \hat{\otimes}_\pi E$ and $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$ (resp. $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$) is nothing else but the identity if E is a Fréchet space.

(2) Junker [26, 1.7 Satz, p. 8-9] gave a proof of Theorem 3.11 resp. Corollary 3.12 in the case that E is a Fréchet space which is in some parts similar to the proof given here. But his proof of surjectivity is less transparent. Moreover, though he proves that T (there called k) is an algebraic isomorphism and hence, using the nuclearity, that there exists an algebraic isomorphism $j : \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right) \hat{\otimes}_\pi E \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$, he does not explicitly state or prove how j looks like. But he treats j like the identity in the proof of continuity, which is confirmed in the first remark here, and then uses the open mapping theorem to get that j is topological. Apart from these inconsistencies, one can not use the open mapping theorem so easily for the proof of the general statement given here if E is not a Fréchet space, since one would need, that $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right) \varepsilon E$ $\left(\text{resp. } \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right) \varepsilon E\right)$ is a webbed space and $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$ $\left(\text{resp. } \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)\right)$ ultrabornological. The first condition is at least fulfilled if E is webbed by Theorem 3.6(2), Theorem 3.7 and [10, Théorème 4, p. 79], but, aside from the case that E is a Fréchet space and hence $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$ (resp. $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$) as well by Theorem 3.6(2), there are no other cases known (to me), when the latter condition is fulfilled.

4 Vector-valued $\mathcal{P}_*$-functionals and a duality theorem

The aim of this section is to prove that the spaces $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ and $L_b\left(\mathcal{P}_*\left(K\right),E\right)$ are topologically isomorphic for any non-empty compact set $K\subset\overline{\mathbb{R}}$ and any complete locally convex space E, i.e. to find a topological isomorphism

$$H\colon\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\to L_b\left(\mathcal{P}_*\left(K\right),E\right).$$

For special cases like $E=\mathbb{C}$ [28, Theorem 3.2.1, p. 480] and Fréchet spaces E [26, 3.9 Satz, p. 41] it is already known that these spaces are isomorphic. The approach in this section will differ from the aforementioned ones and establish a kind of Köthe duality between the spaces $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ and $L_b\left(\mathcal{P}_*\left(K\right),E\right)$ for arbitrary complete locally convex spaces E. At least in the special case $K=[a,\infty]$, $a\in\mathbb{R}$, and $E=\mathbb{C}$ this duality is already known [46, Theorem 3.3, p. 85-86] and serves as initial point of the considerations that follow. From a later point of view (see Section 6) the isomorphism H just expresses that the E-valued Fourier hyperfunctions defined as boundary values whose support is contained in K coincide with the ones defined via E-valued $\mathcal{P}_*$-functionals with support in K. This section is closed by the definition of the Fourier transformation on $L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right),E\right)$.

For $f:=[F]\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ we define $H(f)\colon\mathcal{P}_*\left(K\right)\to E$ as follows. For $\varphi\in\mathcal{P}_*\left(K\right)$ exists $n\in\mathbb{N}$ such that $\varphi\in\mathcal{O}_n\left(U_n\left(K\right)\right)$. A component Z of $U_n\left(K\right)$ fulfills one of the cases of Remark 3.3 (2) and so for $0<r<R$ (in the cases a)-d)) resp. $0<r<1/n$ (in the case e)) we define

$$V_r(Z):=\begin{cases}\{z\in\mathbb{C}\mid\operatorname{dist}(z,[a,b])<r\}, & \text{if Z fulfills a)},\\ \{z\in\mathbb{C}\mid\operatorname{dist}(z,[a,\infty[)<r\}, & \text{if Z fulfills b)},\\ \{z\in\mathbb{C}\mid\operatorname{dist}(z,]-\infty,b])<r\}, & \text{if Z fulfills c)},\\ \{z\in\mathbb{C}\mid\operatorname{dist}(z,\mathbb{R})<r\}, & \text{if Z fulfills d)},\\ (1/r,\infty)\times]-r,r[, & \text{if } Z=]n,\infty[\times]-1/n,1/n[,\\ ]-\infty,-1/r[\times]-r,r[, & \text{if } Z=]-\infty,-n[\times]-1/n,1/n[,\end{cases}$$

where Z fulfills e) in the last two cases. By Remark 3.3 (1) there is $k\in\mathbb{N}$ with $U_n\left(K\right)=\bigcup_{j=1}^{k}Z_j$ where the Z_js denote the components of $U_n\left(K\right)$. Now let

$$\tilde{H}_K\left(F\right)\left(\varphi\right):=\int_{\gamma_K}F\left(\zeta\right)\varphi\left(\zeta\right)d\zeta \tag{4.1}$$

where $\gamma_K:=\sum_{j=1}^{k}\gamma_j$ and γ_j is the path along the boundary of $V_{r_j}\left(Z_j\right)$ in $\mathbb{C}$ in the positive sense (counterclockwise). If not necessary, the subscript K in the notation of $\tilde{H}_K$ and the path γ_K is omitted.

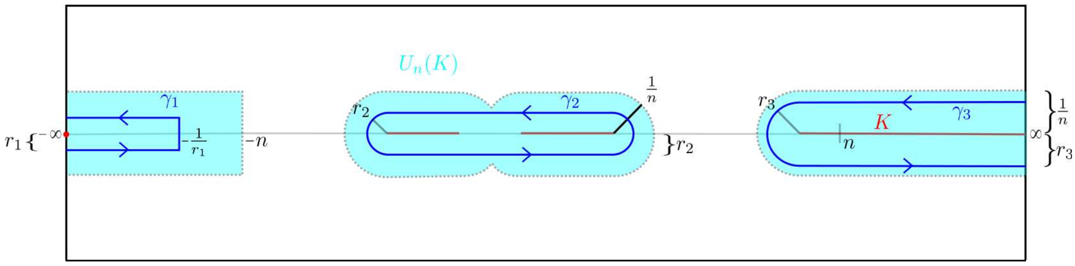

Figure 4.1: Path γ_K for $\pm\infty \in K$

Let $F, G \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ with $F - G \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$. Consider the case $\infty \in K$, $-\infty \notin K$. By the Cauchy integral theorem and the assumptions we have

$$\int_{\gamma} (F-G)(\zeta)\varphi(\zeta)d\zeta = \int_{\gamma_k} (F-G)(\zeta)\varphi(\zeta)d\zeta$$

where Z_k is the unbounded component and $Z_j, 1 \leq j < k$, are the bounded components of $U_n(K)$. For $\gamma_0 \colon [-r_k, r_k] \to \mathbb{C}$, $\gamma_0(t) := x + it$, where $x > 1/r_k$, one has $x > 1/r_k > n$ and thus $\gamma_0([-r_k, r_k]) \subset Z_k$. For $\alpha \in A$ there are $C_1, C_2 > 0$ such that

$$\begin{aligned} p_\alpha\left(\int_{\gamma_0} (F-G)(\zeta)\varphi(\zeta)d\zeta\right) &\leq \int_{-r_k}^{r_k} p_\alpha((F-G)(x+it))|\varphi(x+it)|dt \\ &\leq \int_{-r_k}^{r_k} C_1 e^{(1/2n)x} C_2 e^{-(1/n)x} dt \\ &= 2C_1C_2 r_k e^{-(1/2n)x} \\ &\underset{x\to\infty}{\to} 0 \end{aligned}$$

due to the assumptions. Thus again by the Cauchy integral theorem

$$\int_{\gamma_k} (F-G)(\zeta)\varphi(\zeta)d\zeta = 0$$

holds. The others cases follow analogously. Hence for $f = [F]$ the definition

$$H_K(f)(\varphi) := \tilde{H}_K(F)(\varphi) = \int_{\gamma_K} F(\zeta)\varphi(\zeta)d\zeta \tag{4.2}$$

is independent of the choice of the representative F of f. Again the subscript K in the notation is omitted, if not necessary.
For a component Z of $U_n(K)$ let $0 < r, s < R$ (in the cases a)-d)) with R of Remark 3.3 (2) resp. $0 < r, s < 1/n$ (in the case e)). With the definitions from above consider the case $\infty \in K$, $-\infty \notin K$. By the Cauchy integral theorem

$$\begin{aligned} &\int_{\gamma} F(\zeta)\varphi(\zeta)d\zeta - \int_{\tilde{\gamma}} F(\zeta)\varphi(\zeta)d\zeta \\ &= \int_{\gamma k} F(\zeta)\varphi(\zeta)d\zeta - \int_{\tilde{\gamma} k} F(\zeta)\varphi(\zeta)d\zeta \end{aligned} \tag{4.3}$$

is valid where $\tilde{\gamma} := \sum_{j=1}^{k} \tilde{\gamma}_j$ and $\tilde{\gamma}_j$ is the path along the boundary of $V_{s_j}(Z_j)$ in $\mathbb{C}$ in the positive sense. W.l.o.g. $s_k < r_k$. Now let $m := 2\lceil 1/s_k \rceil$. Then $m \in \mathbb{N}_{\geq 2}$, $m > 2n$ and $1/m < s_k < r_k < m$. For $\gamma_0 \colon [s_k, r_k] \to \mathbb{C}$, $\gamma_0(t) := x + it$, $x > 1/s_k$, and $\alpha \in A$ there are $C_1, C_2 >$ such that

$$\begin{aligned} p_\alpha\left(\int_{\gamma_0} F(\zeta)\varphi(\zeta)d\zeta\right) &\leq \int_{s_k}^{r_k} p_\alpha(F(x+it))|\varphi(x+it)|dt \\ &\underset{\text{def.}\,m}{\leq} \int_{s_k}^{r_k} C_1 e^{(1/m)x} C_2 e^{-(1/n)x} dt \\ &\underset{\text{def.}\,m}{\leq} C_1 C_2 (r_k - s_k) e^{-(1/2n)x} \\ &\underset{x\to\infty}{\to} 0. \end{aligned}$$

Analogously for $\gamma_1 \colon [-r_k, -s_k] \to \mathbb{C}$, $\gamma_1(t) := x + it$, $x > 1/s_k$, one has

$$p_\alpha\left(\int_{\gamma_1} F(\zeta)\varphi(\zeta)d\zeta\right) \underset{x\to\infty}{\to} 0$$

and thus the right hand side of (4.3) is equal to zero. Again the others cases follow analogously. Hence the definition of $H(f)$ (and $\tilde{H}(F)$) is independent of the choice of r corresponding to a component Z of $U_n(K)$ and thus well-defined on $\mathcal{P}_*(K)$.

4.1 Theorem. [1] *For any non-empty compact set $K \subset \overline{\mathbb{R}}$ the mapping*

$$H \colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) / \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right) \to L_b(\mathcal{P}_*(K), E)$$

is a topological isomorphism.

Proof. i) First we have to take a look at the quotient space above. To speak of a topological isomorphism, one is in need of a reasonable locally convex topology on the quotient space. We denote by

$$q \colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) / \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$$

the quotient map and equip the quotient space with usual system of quotient semi-norms $\left(|\cdot|^{\wedge}_{l,\alpha}\right)_{l \in \mathbb{N}_{\geq 2}, \alpha \in A}$ given by

$$|f|^{\wedge}_{l,\alpha} := \inf_{F \in q^{-1}(f)} |F|_{l,\alpha}.$$

This quotient space, equipped with these semi-norms, is locally convex iff $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$ is closed in $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ with respect to the induced topology. This condition is indeed fulfilled since we will prove at the end of part i) that

$$\tilde{H} \colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to L_b(\mathcal{P}_*(K), E)$$

is continuous and at the end of part iii) that $\ker\left(\tilde{H}\right) = \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$.

We only consider the case $\infty \in K, -\infty \notin K$. The proof for the other cases is analogous. Let $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) / \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$, $n \in \mathbb{N}$ and $\varphi \in \mathcal{O}_n(U_n(K))$. Then, with the definitions from before, $U_n(K) = \bigcup_{j=1}^{k} Z_j$. For the length of $\gamma_j, 1 \leq j < k$, one has

$$l\left(\gamma_j\right) = 2\left(b_j - a_j + \pi r_j\right) \tag{4.4}$$

[1] counterpart: [13, Theorem 5.2, p. 1119]

and for the length of γ_-

$$l(\gamma_-) = \begin{cases} 2(x-a_k)+\pi r_k, & \text{if } Z_k \text{ fulfills b)},\\ 2(x-1/r_k+r_k), & \text{if } Z_k \text{ fulfills e)}, \ Z_k =]n,\infty[\times]-1/n, 1/n[, \end{cases} \tag{4.5}$$

where $x > 1/r_k$ is fixed, γ_- denotes the part of γ_k with $\mathrm{Re}(\gamma_k) \subset (-\infty, x)$ and γ_+ the part of γ_k with $\mathrm{Re}(\gamma_k) \subset [x,\infty)$. Set $m := 2\max_{1\le j\le k}\lceil 1/r_j\rceil$ and w.l.o.g. Z_k fulfills b).
Then for any $\alpha \in A$ and any $F \in q^{-1}(f)$

$$\begin{aligned}
p_\alpha(H(f)(\varphi)) &= p_\alpha\big(\tilde{H}(F)(\varphi)\big) = p_\alpha\left(\int_\gamma F(\zeta)\varphi(\zeta)\,d\zeta\right)\\
&\le \sum_{j=1}^{k-1} p_\alpha\left(\int_{\gamma_j} F(\zeta)\varphi(\zeta)\,d\zeta\right) + p_\alpha\left(\int_{\gamma_-+\gamma_+} F(\zeta)\varphi(\zeta)\,d\zeta\right)\\
&\le \sum_{j=1}^{k-1} l(\gamma_j) \sup_{\zeta\in\mathrm{range}(\gamma_j)} p_\alpha(F(\zeta))|\varphi(\zeta)| + l(\gamma_-) \sup_{\zeta\in\mathrm{range}(\gamma_-)} p_\alpha(F(\zeta))|\varphi(\zeta)|\\
&\quad + p_\alpha\left(-\int_x^\infty F(t+ir_k)\varphi(t+ir_k)\,dt\right) + p_\alpha\left(\int_x^\infty F(t-ir_k)\varphi(t-ir_k)\,dt\right)\\
&\overset{(4.4)}{\underset{(4.5)}{=}} \sum_{j=1}^{k-1} 2\big(b_j - a_j + \pi r_j\big) \sup_{\zeta\in\mathrm{range}(\gamma_j)} p_\alpha(F(\zeta))|\varphi(\zeta)|\\
&\quad + (2(x-a_k)+\pi r_k) \sup_{\zeta\in\mathrm{range}(\gamma_-)} p_\alpha(F(\zeta))|\varphi(\zeta)|\\
&\quad + \int_x^\infty p_\alpha(F(t+ir_k))|\varphi(t+ir_k)|\,dt + \int_x^\infty p_\alpha(F(t-ir_k))|\varphi(t-ir_k)|\,dt\\
&\underset{\mathrm{def}.m}{\le} 2\sum_{j=1}^{k-1}\big(b_j-a_j+\pi r_j\big) \sup_{\zeta\in\mathrm{range}(\gamma_j)} |F|_{m,\alpha}\|\varphi\|_n e^{\left(\frac{1}{m}-\frac{1}{n}\right)|\mathrm{Re}(\zeta)|}\\
&\quad + (2(x-a_k)+\pi r_k) \sup_{\zeta\in\mathrm{range}(\gamma_-)} |F|_{m,\alpha}\|\varphi\|_n e^{\left(\frac{1}{m}-\frac{1}{n}\right)|\mathrm{Re}(\zeta)|}\\
&\quad + 2\int_x^\infty |F|_{m,\alpha}\|\varphi\|_n e^{\left(\frac{1}{m}-\frac{1}{n}\right)t}\,dt\\
&\underset{\mathrm{def}.m}{\le} 2|F|_{m,\alpha}\|\varphi\|_n \sum_{j=1}^{k-1}\big(b_j-a_j+\pi r_j\big) \sup_{\zeta\in\mathrm{range}(\gamma_j)} e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}\\
&\quad + |F|_{m,\alpha}\|\varphi\|_n (2(x-a_k)+\pi r_k) \sup_{\zeta\in\mathrm{range}(\gamma_-)} e^{-\frac{1}{2n}|\mathrm{Re}(\zeta)|}\\
&\quad + 2|F|_{m,\alpha}\|\varphi\|_n \underbrace{\int_x^\infty e^{-\frac{1}{2n}t}\,dt}_{=2ne^{-\frac{1}{2n}x}}\\
&\le \left(2\sum_{j=1}^{k-1}\big(b_j-a_j+\pi r_j\big) + (2(x-a_k)+\pi r_k) + 4ne^{-\frac{1}{2n}x}\right)|F|_{m,\alpha}\|\varphi\|_n.
\end{aligned}$$

Hence there exists $C = C(n,K) > 0$ with

$$p_\alpha\left(\tilde{H}(F)(\varphi)\right) = p_\alpha(H(f)(\varphi)) \leq C|F|_{m,\alpha}\|\varphi\|_n, \tag{4.6}$$

thus $H(f) = \tilde{H}(F) \in L(\mathcal{O}_n(U_n(K)),E)$ and therefore $H(f) = \tilde{H}(F) \in L(\mathcal{P}_*(K),E)$ by [16, 3.6 Satz, p. 117] since $n \in \mathbb{N}$ is arbitrary. As $F \in q^{-1}(f)$ is arbitrary, (4.6) also yields to

$$p_\alpha(H(f)(\varphi)) \leq C \inf_{F \in q^{-1}(f)} |F|_{m,\alpha}\|\varphi\|_n = C|f|^{\wedge}_{m,\alpha}\|\varphi\|_n \tag{4.7}$$

for any $n \in \mathbb{N}$ and $\alpha \in A$. Now let $M \subset \mathcal{P}_*(K)$ be a bounded set. Since the sequence $(B_n)_{n\in\mathbb{N}}$ of closed unitballs B_n of $\mathcal{O}_n(U_n(K))$ is a fundamental system of bounded sets in $\mathcal{P}_*(K)$ by [45, 25.19 Satz (2), p. 286], there exist $n \in \mathbb{N}$ and $\lambda > 0$ with $M \subset \lambda B_n$. Hence by (4.6) one gets

$$\begin{aligned}\sup_{\varphi\in M} p_\alpha\left(\tilde{H}(F)(\varphi)\right) &\leq \sup_{\varphi\in\lambda B_n} p_\alpha\left(\tilde{H}(F)(\varphi)\right) = |\lambda| \sup_{\varphi\in B_n} p_\alpha\left(\tilde{H}(F)(\varphi)\right)\\ &\leq |\lambda| C|F|_{m,\alpha}\end{aligned}$$

and by (4.7)

$$\begin{aligned}\sup_{\varphi\in M} p_\alpha(H(f)(\varphi)) &\leq \sup_{\varphi\in\lambda B_n} p_\alpha(H(f)(\varphi)) = |\lambda| \sup_{\varphi\in B_n} p_\alpha(H(f)(\varphi))\\ &\leq |\lambda| C|f|^{\wedge}_{m,\alpha}\end{aligned}$$

proving the continuity of $\tilde{H}$ resp. of H, but in the latter case only if we prove, in addition, that $\ker\left(\tilde{H}\right) = \mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ (see the remark in the beginning).
Moreover, we observe the following: Let $K_1 \subset K \subset \overline{\mathbb{R}}$ be arbitrary compact sets. For every $F \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K_1, E\right)$ and every $\varphi \in \mathcal{P}_*(K)$

$$H_K([F]) = \int_{\gamma_K} F(z)\varphi(z)\,dz = \int_{\gamma_{K_1}} F(z)\varphi(z)\,dz = H_{K_1}([F])$$

holds by the Cauchy integral theorem implying

$$H_K\big|_{\mathcal{O}^{exp}(\overline{\mathbb{C}}\setminus K_1,E)/\mathcal{O}^{exp}(\overline{\mathbb{C}},E)} = H_{K_1}. \tag{4.8}$$

ii) For $z \in \mathbb{C} \setminus K$ and $\zeta \in \mathbb{C} \setminus \{z\}$ define $g(z,\zeta) := \frac{e^{-(z-\zeta)^2}}{z-\zeta}$. Then $g(z,\cdot) \in \mathcal{O}(\mathbb{C} \setminus \{z\})$. Let $z_0 \in \mathbb{C} \setminus K$. Now choose $n \in \mathbb{N}$ such that

$$\mathrm{d}\left(z_0, \overline{U_n(K)}\right) = \mathrm{dist}\left(z_0, \overline{U_n(K)}\right) > 0 \tag{4.9}$$

which is possible by the choice of z_0 and the definition of $U_n(K)$. Since (4.9) means that there exists $\varepsilon > 0$ with $D_\varepsilon(z_0) \cap \overline{U_n(K)} = \varnothing$, one has $\mathrm{d}\left(w, \overline{U_n(K)}\right) > 0$ for all $w \in D_\varepsilon(z_0)$.

With $w := x+iy$ and $\zeta := \mu + i\eta$, where $x, y, \mu, \eta \in \mathbb{R}$, we get

$$\begin{aligned}
\|g(w,\cdot)\|_n &= \sup_{\zeta \in \overline{U_n(K)}} \frac{e^{-\operatorname{Re}\left((w-\zeta)^2\right)}}{|w-\zeta|} e^{\frac{1}{n}|\operatorname{Re}(\zeta)|} \\
&\le \frac{1}{\mathrm{d}\left(w, \overline{U_n(K)}\right)} e^{-x^2+y^2} \sup_{\mu+i\eta \in \overline{U_n(K)}} e^{-\mu^2+\eta^2+2x\mu-2y\eta+\frac{1}{n}|\mu|} \\
&\le \frac{1}{\mathrm{d}\left(w, \overline{U_n(K)}\right)} e^{-x^2+y^2+\frac{2}{n}|y|+\frac{1}{n^2}} \sup_{\mu \in \mathbb{R}} e^{-\mu^2+|\mu|\left(2|x|+\frac{1}{n}\right)} \\
&= \frac{1}{\mathrm{d}\left(w, \overline{U_n(K)}\right)} e^{-x^2+y^2+\frac{2}{n}|y|+\frac{1}{n^2}} e^{-\left(|x|+\frac{1}{2n}\right)^2+\left(|x|+\frac{1}{2n}\right)\left(2|x|+\frac{1}{n}\right)} \\
&= \frac{1}{\mathrm{d}\left(w, \overline{U_n(K)}\right)} e^{\frac{1}{n}|x|+y^2+\frac{2}{n}|y|+\frac{5}{4n^2}} < \infty,
\end{aligned} \tag{4.10}$$

thus $g(w,\cdot) \in \mathcal{P}_*(K)$. Hence the expression $< T, g(w,\cdot) >$ is defined for $T \in L(\mathcal{P}_*(K), E)$ and so the corresponding function

$$< T^{\#}, g >: \mathbb{C} \setminus K \to E,\ z \mapsto < T, g(z,\cdot) > .$$

For the function $\left(\frac{\partial}{\partial z} g\right)(z_0,\cdot): \mathbb{C} \setminus \{z_0\} \to \mathbb{C},\ \zeta \mapsto \left(\frac{\partial}{\partial z} g\right)(z_0,\zeta)$, where $\left(\frac{\partial}{\partial z} g\right)$ denotes the complex derivative of g with respect to z, one gets like in (4.10) (with $w = z_0$)

$$\begin{aligned}
\left\|\left(\frac{\partial}{\partial z} g\right)(z_0,\cdot)\right\|_n &= \sup_{\zeta \in \overline{U_n(K)}} \left| -\left(2 + \frac{1}{(z_0-\zeta)^2}\right) e^{-(z_0-\zeta)^2} \right| e^{\frac{1}{n}|\operatorname{Re}(\zeta)|} \\
&\le \left(2 + \frac{1}{\mathrm{d}\left(z_0, \overline{U_n(K)}\right)^2}\right) e^{\frac{1}{n}|x|+y^2+\frac{2}{n}|y|+\frac{5}{4n^2}} \\
&< \infty,
\end{aligned}$$

so $\left(\frac{\partial}{\partial z} g\right)(z_0,\cdot) \in \mathcal{P}_*(K)$. Hence the limit

$$\begin{aligned}
\lim_{h\to 0} \frac{< T, g(z_0+h,\cdot) > - < T, g(z_0,\cdot) >}{h} &\underset{T \in L(\mathcal{P}_*(K),E)}{=} < T, \lim_{h \to 0} \frac{g(z_0+h,\cdot) - g(z_0,\cdot)}{h} > \\
&= < T, \left(\frac{\partial}{\partial z} g\right)(z_0,\cdot) >
\end{aligned}$$

exists meaning $< T^{\#}, g > \in \mathcal{O}(\mathbb{C} \setminus K, E)$.

Then for $l \in \mathbb{N}_{\ge 2}$ define $M := \left\{ g(z,\cdot) e^{-\frac{1}{l}|\operatorname{Re}(z)|} \mid z \in S_l(K) \right\} \subset \mathcal{O}(U_{2l}(K))$. By (4.10) and Remark 3.3(4)

$$\sup_{\varphi \in M} \|\varphi\|_{2l} = \sup_{z \in S_l(K)} \|g(z,\cdot)\|_{2l} e^{-\frac{1}{l}|\operatorname{Re}(z)|}$$

$$\begin{aligned}
&\leq \sup_{x+iy\in S_l(K)} \frac{1}{\operatorname{dist}\left(x+iy,\overline{U_{2l}(K)}\right)} e^{\frac{1}{2l}|x|+y^2+\frac{1}{l}|y|+\frac{5}{16l^2}-\frac{1}{l}|x|} \\
&\leq \frac{e^{l^2+1+\frac{5}{16l^2}}}{\frac{1}{l}-\frac{1}{2l}} \sup_{x\in\mathbb{R}} e^{-\frac{1}{2l}|x|} \\
&= 2le^{l^2+1+\frac{5}{16l^2}} \\
&< \infty
\end{aligned} \tag{4.11}$$

holds, hence M is bounded in $\mathcal{P}_*(K)$ by [45, 25.19 Satz (2), p. 286] again.
Since $T\in L(\mathcal{P}_*(K),E)$, the following is valid by [16, 23.3.6 Satz, p. 117]:

$$\forall\, n\in\mathbb{N}\ \forall\, \alpha\in A\ \exists\, C>0:\ p_\alpha(T(\varphi))\leq C\|\varphi\|_{n,\alpha} \quad \text{for all } \varphi\in\mathcal{O}_n(U_n(K))$$

So for $l\in\mathbb{N}_{\geq 2}$ and $\alpha\in A$ there is $C>0$ such that

$$\begin{aligned}
\left|\left[<T^{\#},g>\right]\right|^{\wedge}_{l,\alpha} \leq |<T^{\#},g>|_{l,\alpha} &= \sup_{z\in S_l(K)} p_\alpha(<T,g(z,\cdot)>)e^{-\frac{1}{l}|\mathrm{Re}(z)|} \\
&= \sup_{z\in S_l(K)} p_\alpha\left(<T,g(z,\cdot)e^{-\frac{1}{l}|\mathrm{Re}(z)|}>\right) = \sup_{\varphi\in M} p_\alpha(<T,\varphi>) \\
&\leq C\sup_{\varphi\in M}\|\varphi\|_{2l} \underset{(4.11)}{<} \infty
\end{aligned}$$

and therefore the mapping

$$S_K\colon L_b(\mathcal{P}_*(K),E)\to \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right),\ S_K(T):=\left[\frac{1}{2\pi i}<T^{\#},g>\right]$$

is defined and continuous. Further, we observe that $\mathcal{P}_*\left(\overline{\mathbb{R}}\right)$ is dense in $\mathcal{P}_*(K)$ for any non-empty compact set $K\subset\overline{\mathbb{R}}$ by [28, Theorem 2.2.1, p. 474]. Thus the embedding of $\mathcal{P}_*(K)$ into $\mathcal{P}_*(K_1)$ is continuous and dense, hence defines the embedding of $L(\mathcal{P}_*(K_1),E)$ into $L(\mathcal{P}_*(K),E)$ (the density of the first embedding implies the injectivity of the latter one) for arbitrary compact sets $K_1\subset K\subset\overline{\mathbb{R}}$, and we have

$$S_K(T)=S_{K_1}(T) \quad \text{for all } T\in L(\mathcal{P}_*(K_1),E) \tag{4.12}$$

just by the definition of g. Therefore, we will normally omit the subscript K in what follows. The map S is also called (weighted) Cauchy transformation for obvious reasons (see [46]).

iii) The aim of the next two parts is to show that S is the inverse mapping of H. First, the injectivity of H: For this it suffices to show $S\circ H=\mathrm{id}$ on $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$. For $z:=x+iy\in\mathbb{C}\setminus K$, $x,y\in\mathbb{R}$, choose $n\in\mathbb{N}$ like in (4.9) and define $\Gamma:=\Gamma_- - \Gamma_+$ with

$$\Gamma_\pm\colon\mathbb{R}\to\mathbb{C},\ \Gamma_\pm(t):=t\pm ip$$

where $\max_{1\leq j\leq k}\{r_j,|\mathrm{Im}(z)|,2\}<p$.

Let $f:=[F]\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$. Consider the case $\infty\in K$, $-\infty\notin K$. Now set $m:=2\max\{\lceil 1/r_k\rceil,p\}$. For $\gamma_0\colon[r_k,p]\to\mathbb{C}$, $\gamma_0(t):=u+it$, $u>\max\{1/r_k,x\}$, there is $C_1>0$

such that

$$p_\alpha\left(\int_{\gamma_0} F(\zeta)\frac{e^{-(z-\zeta)^2}}{\zeta-z}d\zeta\right)\le\int_{r_k}^{p} C_1 e^{\frac{1}{m}u}\frac{1}{|z-u-it|}e^{-x^2+y^2-u^2+t^2+2xu-2yt}dt$$

$$\le\underbrace{C_1e^{-x^2+y^2}(p-r_k)}_{=:C_2}\frac{1}{u-x}e^{p^2+2|y|p}e^{-u^2+\left(\frac{1}{m}+2|x|\right)|u|}$$

$$=C_2\frac{1}{u-x}e^{-u^2+\left(\frac{1}{m}+2|x|\right)|u|}$$

$$\underset{u\to\infty}{\to}0.$$

Analogously for $\gamma_1\colon[-p,-r_k]\to\mathbb{C}$, $\gamma_1(t):=u+it$, $u>\max\{1/r_k,x\}$, and $\gamma_2\colon[-p,p]\to\mathbb{C}$, $\gamma_2(t):=u-it$, $u<\min_{1\le j\le k}\{a_j-r_j,x\}$, if Z_k fulfills b), resp. $u<\min_{1\le j<k}\{a_j-r_j,1/r_k,x\}$, if Z_k fulfills e) and $Z_k=]n,\infty[\times]-1/n,1/n[$, one gets

$$p_\alpha\left(\int_{\gamma_i}F(\zeta)\frac{e^{-(z-\zeta)^2}}{\zeta-z}d\zeta\right)\underset{u\to\infty}{\to}0,\ i=1,2,$$

and hence by the Cauchy integral formula

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma-\gamma}F(\zeta)\frac{e^{-(z-\zeta)^2}}{\zeta-z}d\zeta=-\frac{1}{2\pi i}\int_{\Gamma-\gamma}F(\zeta)g(z,\zeta)d\zeta.$$

Notice that the right hand side does not depend on the choice of p in the definition of Γ by the Cauchy integral theorem and considerations like above. Then

$$\begin{aligned}G(z):&=<\frac{1}{2\pi i}H([F]),g(z,\cdot)>-F(z)\\&=\frac{1}{2\pi i}<H([F]),g(z,\cdot)>+\frac{1}{2\pi i}\int_{\Gamma-\gamma}F(\zeta)g(z,\zeta)d\zeta\\&=\frac{1}{2\pi i}\int_{\gamma}F(\zeta)g(z,\zeta)d\zeta+\frac{1}{2\pi i}\int_{\Gamma-\gamma}F(\zeta)g(z,\zeta)d\zeta\\&=\frac{1}{2\pi i}\int_{\Gamma}F(\zeta)g(z,\zeta)d\zeta\\&=\frac{1}{2\pi i}\int_{\Gamma}F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}d\zeta\end{aligned}\tag{4.13}$$

is valid. But the right hand side of (4.13), as a function in z, is holomorphic on $S_p(\varnothing)=\{z\in\mathbb{C}\mid|\mathrm{Im}(z)|<p\}$ (differentiation under the integral sign), so G is extended to a function in $\mathcal{O}(\mathbb{C},E)$ by the right hand side which shall also be denoted with G.
For $l\in\mathbb{N}_{\ge2}$ choose $m:=2\max\{p,l\}$. Then for $z\in S_{\frac{p}{2}}(\varnothing)$ there exists $C_1>0$ such that

$$2\pi p_\alpha(G(z))=p_\alpha\left(\int_\Gamma F(\zeta)\frac{e^{-(z-\zeta)^2}}{z-\zeta}d\zeta\right)$$

$$\le\int_{-\infty}^{\infty}p_\alpha(F(t-ip))\left|\frac{e^{-(z-t+ip)^2}}{z-t+ip}\right|dt+\int_{-\infty}^{\infty}p_\alpha(F(t+ip))\left|\frac{e^{-(z-t-ip)^2}}{z-t-ip}\right|dt$$

$$
\begin{aligned}
&\leq C_1 e^{-x^2+y^2+p^2}\left(\frac{1}{|y+p|}e^{2yp}+\frac{1}{|y-p|}e^{-2yp}\right)\int_{-\infty}^{\infty} e^{\frac{1}{m}|t|-t^2+2xt}dt\\
&\leq 2C_1\frac{1}{p-|y|}e^{-x^2+y^2+p^2+2|y|p}\int_{-\infty}^{\infty} e^{-t^2+\left(\frac{1}{m}+2|x|\right)|t|}dt\\
&\leq 4C_1\frac{1}{p-|y|}e^{-x^2+y^2+p^2+2|y|p+\left(\frac{1}{2m}+|x|\right)^2}\int_{0}^{\infty} e^{-\left(t-\left(\frac{1}{2m}+|x|\right)\right)^2}dt\\
&= 4C_1\frac{1}{p-|y|}e^{-x^2+y^2+p^2+2|y|p+\left(\frac{1}{2m}+|x|\right)^2}\left(\int_{0}^{\infty} e^{-t^2}dt+\int_{-\left(\frac{1}{2m}+|x|\right)}^{0} e^{-t^2}dt\right)\\
&\leq 4C_1\sqrt{\pi}\frac{1}{p-|y|}e^{y^2+p^2+2|y|p+\frac{1}{4m^2}+\frac{1}{m}|x|}
\end{aligned}
$$

and so

$$
\begin{aligned}
\sup_{\substack{0\leq|y|\leq\frac{p}{2}\\ x\in\mathbb{R}}} p_\alpha(G(x+iy))e^{-\frac{1}{l}|x|} &\leq \frac{2C_1}{\sqrt{\pi}}\sup_{\substack{0\leq|y|\leq\frac{p}{2}\\ x\in\mathbb{R}}}\frac{1}{p-|y|}e^{y^2+p^2+2|y|p+\frac{1}{4m^2}-\left(\frac{1}{l}-\frac{1}{m}\right)|x|}\\
&\leq \frac{4C_1}{p\sqrt{\pi}}e^{\frac{9}{4}p^2+\frac{1}{4m^2}}\sup_{x\in\mathbb{R}}e^{-\frac{1}{2l}|x|}\\
&= \frac{4C_1}{p\sqrt{\pi}}e^{\frac{9}{4}p^2+\frac{1}{4m^2}}
\end{aligned}
$$

yielding to

$$
|G|_{\varnothing,l,\alpha}=\sup_{z\in S_l(\varnothing)} p_\alpha(G(z))e^{-\frac{1}{l}|\mathrm{Re}(z)|}\underset{\frac{p}{2}\geq\frac{1}{l}}{\leq}\max\left(|G|_{K,l,\alpha},\sup_{\substack{0\leq|y|\leq\frac{p}{2}\\ x\in\mathbb{R}}} p_\alpha(G(x+iy))e^{-\frac{1}{l}|x|}\right)<\infty.
$$

Hence $G\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ and thus

$$
(S\circ H)(f)=\left[<\frac{1}{2\pi i}H(f)^{\#},g>-F\right]+f=[G]+f=f,
$$

i.e. H is injective. In particular, this means that $\ker\left(\tilde{H}\right)=\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$ proving the statement in the beginning of part i) as well.

iv) What remains to be shown, is the surjectivity of H. For this it suffices to show $H\circ S=\mathrm{id}$ on $L(\mathcal{P}_*(K),E)$. Due to the Hahn-Banach theorem (see for example [45, 22.12 Satz (c), p. 236]) this is equivalent to the condition that

$$
e'((H\circ S(T))(\varphi))=e'(T(\varphi))
$$

holds for any $T\in L(\mathcal{P}_*(K),E)$, $\varphi\in\mathcal{P}_*(K)$ and $e'\in E'$.
Since

$$
\begin{aligned}
e'((H\circ S(T))(\varphi)) &= e'\left(H\left(\left[\frac{1}{2\pi i}<T^{\#},g>\right]\right)(\varphi)\right)\\
&= e'\left(\frac{1}{2\pi i}\int_\gamma <T,g(z,\cdot)>\varphi(z)\,dz\right)
\end{aligned}
$$

$$= \frac{1}{2\pi i}\int_\gamma < e'\circ T, g(z,\cdot) > \varphi(z)\,dz$$
$$= (H\circ S(e'\circ T))(\varphi)$$

where one uses Riemann sums and the Cauchy integral theorem for the third equation and $e'\circ T\in\mathcal{P}_*(K)'$, it suffices to show the result for $E=\mathbb{C}$.
First let us consider the case $K=\overline{\mathbb{R}}$. As the set of point evaluations $\{\delta_{x_0}\mid x_0\in\mathbb{R}\}$ is total in $\mathcal{P}_*\left(\overline{\mathbb{R}}\right)'$ by Theorem 3.5(3), one has to show that $(H\circ S(\delta_{x_0}))(\varphi) =< \delta_{x_0},\varphi >$ for $\varphi\in\mathcal{P}_*\left(\overline{\mathbb{R}}\right)$.
Now we have

$$(H\circ S(\delta_{x_0}))(\varphi) = \frac{1}{2\pi i}\int_\gamma < \delta_{x_0}, g(z,\cdot) > \varphi(z)\,dz. \tag{4.14}$$

Let us take a closer look at the integral on the right hand side of (4.14). Let $m\in\mathbb{N}_{\geq 2}$ and $z\in S_m(\{x_0\})$. Then

$$\begin{aligned}|< \delta_{x_0}, g(z,\cdot) >|_{\{x_0\},m} &= \sup_{z\in S_m(\{x_0\})} |< \delta_{x_0}, g(z,\cdot) >| e^{-\frac{1}{m}|\mathrm{Re}(z)|} = \sup_{z\in S_m(\{x_0\})} |g(z,x_0)| e^{-\frac{1}{m}|\mathrm{Re}(z)|}\\ &= \sup_{z\in S_m(\{x_0\})} \frac{1}{|z-\zeta|} e^{-x^2+y^2-x_0^2+2xx_0-\frac{1}{m}|x|} \leq m e^{m^2-x_0^2} \sup_{x\in\mathbb{R}} e^{-x^2+|x|\left(2|x_0|+\frac{1}{m}\right)}\\ &= m e^{m^2-x_0^2} e^{-\left(|x_0|+\frac{1}{2m}\right)^2+\left(|x_0|+\frac{1}{2m}\right)\left(2|x_0|+\frac{1}{m}\right)} < \infty,\end{aligned}$$

thus $< \delta^{\#}_{x_0}, g >\in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\{x_0\}\right)$. This means that the path of the integral on the right hand side of (4.14) can be deformed using the Cauchy integral theorem and one gets for $R>0$ sufficiently small, so that $\overline{D_R(x_0)}\subset U_n(K)$ for $\varphi\in\mathcal{O}_n(U_n(K))$, $n\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma < \delta_{x_0}, g(z,\cdot) > \varphi(z)\,dz &= \frac{1}{2\pi i}\int_{\partial D_R(x_0)} < \delta_{x_0}, g(z,\cdot) > \varphi(z)\,dz\\ &= \frac{1}{2\pi i}\int_{\partial D_R(x_0)} g(z,x_0)\,\varphi(z)\,dz\\ &= \frac{1}{2\pi i}\int_{\partial D_R(x_0)} \frac{e^{-(z-x_0)^2}\varphi(z)}{z-x_0}dz\\ &= \varphi(x_0)\\ &=< \delta_{x_0},\varphi >\end{aligned}$$

by the Cauchy integral formula.
Now let $K\neq\varnothing$ be an arbitrary compact subset of $\overline{\mathbb{R}}$. Since $\mathcal{P}_*(K)'$ is embbeded in $\mathcal{P}_*\left(\overline{\mathbb{R}}\right)'$ and $H_{\overline{\mathbb{R}}}\circ S_{\overline{\mathbb{R}}} = \mathrm{id}_{\mathcal{P}_*\left(\overline{\mathbb{R}}\right)'}$, it suffices to show that $H_K\circ S_K = \left(H_{\overline{\mathbb{R}}}\circ S_{\overline{\mathbb{R}}}\right)\big|_{\mathcal{P}_*(K)'}$. By (4.12) we have $S_{\overline{\mathbb{R}}}\big|_{\mathcal{P}_*(K)'} = S_K$ and by (4.8) $H_{\overline{\mathbb{R}}}\big|_{\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)} = H_K$. So the theorem is finally proven.

□

4.2 Remark. For K like in Theorem 3.5(2) a different proof of part iv) is also possible. As the set of point evaluations of complex derivatives $\left\{\delta^{(n)}_{x_0}\mid x_0\in K\cap\mathbb{R}, n\in\mathbb{N}_0\right\}$ is total in $\mathcal{P}_*(K)'$ for K like in Theorem 3.5 (2), one has to show that $\left(H\circ S\left(\delta^{(n)}_{x_0}\right)\right)(\varphi) =< \delta^{(n)}_{x_0},\varphi >$ for $\varphi\in\mathcal{P}_*(K)$.

Now we have

$$\left(H\circ S\left(\delta_{x_0}^{(n)}\right)\right)(\varphi)=\frac{1}{2\pi i}\int_\gamma<\delta_{x_0}^{(n)},g(z,\cdot)>\varphi(z)\,dz. \tag{4.15}$$

Let us take a closer look at the integral on the right hand side of (4.15). Let $m\in\mathbb{N}_{\geq 2}$ and $z\in S_m(\{x_0\})$. Then $g(z,\cdot)\in\mathcal{O}\left(D_{\frac{1}{m}}(x_0)\right)$. Using the notation $g_z(\zeta):=g(z,\zeta)$ for $\zeta\in D_{\frac{1}{m}}(x_0)$, the Cauchy inequality yields to

$$\begin{aligned}
\left|g_z^{(n)}(x_0)\right| &\leq n!\,(2m)^n\max_{\zeta\in\partial D_{\frac{1}{2m}}(x_0)}|g(z,\zeta)|\\
&= n!\,(2m)^n\max_{\zeta\in\partial D_{\frac{1}{2m}}(x_0)}\frac{1}{|z-\zeta|}e^{-x^2+y^2-\mu^2+\eta^2+2x\mu-2y\eta}\\
&\leq n!\,(2m)^{n+1}\max_{\zeta\in\partial D_{\frac{1}{2m}}(x_0)}e^{-x^2+y^2+\mu^2+\eta^2+2|x||\mu|+2|y||\eta|}\\
&\leq n!\,(2m)^{n+1}\max_{\zeta\in\partial D_{\frac{1}{2m}}(x_0)}e^{-x^2+y^2+|\zeta|^2+2|\zeta|(|x|+|y|)}\\
&\leq n!\,(2m)^{n+1}e^{-x^2+y^2+\left(\frac{1}{2m}+|x_0|\right)^2+2\left(\frac{1}{2m}+|x_0|\right)(|x|+|y|)}\\
&= \underbrace{n!\,(2m)^{n+1}e^{\left(\frac{1}{2m}+|x_0|\right)^2}}_{=:C_0}e^{-x^2+y^2+\left(\frac{1}{m}+2|x_0|\right)(|x|+|y|)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sup_{z\in S_m(\{x_0\})}\left|g_z^{(n)}(x_0)\right|e^{-\frac{1}{m}|\mathrm{Re}(z)|} &\leq C_0\sup_{z\in S_m(\{x_0\})}e^{-x^2+y^2+\left(\frac{1}{m}+2|x_0|\right)(|x|+|y|)-\frac{1}{m}|x|}\\
&\leq \underbrace{C_0e^{m^2+\left(\frac{1}{m}+2|x_0|\right)m}}_{=:C_1}\sup_{x\in\mathbb{R}}e^{-x^2+2|x_0||x|}\\
&= C_1e^{-x_0^2+2x_0^2}=C_1e^{x_0^2}
\end{aligned}$$

and so $\left(z\mapsto<\delta_{x_0}^{(n)},g(z,\cdot)>\right)\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\{x_0\}\right)$. This means that the path of the integral on the right hand side of (4.15) can be deformed using the Cauchy integral theorem and one gets for $R>0$ sufficiently small, so that $\overline{D_R(x_0)}\subset U_l(K)$ for $\varphi\in\mathcal{O}_l(U_l(K))$, $l\in\mathbb{N}$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma<\delta_{x_0}^{(n)},g(z,\cdot)>\varphi(z)\,dz &= \frac{1}{2\pi i}\int_{\partial D_R(x_0)}<\delta_{x_0}^{(n)},g(z,\cdot)>\varphi(z)\,dz\\
&= \frac{1}{2\pi i}\int_{\partial D_R(x_0)}g_z^{(n)}(x_0)\,\varphi(z)\,dz.
\end{aligned}$$

The Laurent series of $g(z,\cdot)$ in ζ is

$$g(z,\zeta)=\frac{1}{z-\zeta}+\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}(z-\zeta)^{2j-1}$$

and so

$$g_z^{(n)}(x_0) = \frac{n!}{(z-x_0)^{n+1}} + h(z,x_0)$$

where $h(\cdot,x_0)$ is an entire function. By the Cauchy integral theorem and the Cauchy integral formula for derivatives we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_R(x_0)} g_z^{(n)}(x_0)\,\varphi(z)\,dz \\
&= \frac{1}{2\pi i}\int_{\partial D_R(x_0)} \left(\frac{n!}{(z-x_0)^{n+1}} + h(z,x_0)\right)\varphi(z)\,dz \\
&= \frac{n!}{2\pi i}\int_{\partial D_R(x_0)} \frac{\varphi(z)}{(z-x_0)^{n+1}}dz = \varphi^{(n)}(x_0) = <\delta_{x_0}^{(n)},\varphi> .
\end{aligned}$$

Observe that this kind of proof is not possible if, for example, $K = \{\infty\}$ since a counterpart of Theorem 3.5(2) is missing.

By [28, Theorem 2.2.1, p. 474] $\mathcal{P}_*(\mathbb{R})$ is dense in $\mathcal{P}_*(K)$ for a compact set $K \subset \overline{\mathbb{R}}$, $K \neq \varnothing$. So for different compact sets $K,J \subset \overline{\mathbb{R}}$ we may identify elements of $L(\mathcal{P}_*(K),E)$ and $L(\mathcal{P}_*(J),E)$ by means of their restrictions to $\mathcal{P}_*(\mathbb{R})$. Then the following result defining the support of a vector-valued $\mathcal{P}_*$-functional is valid:

4.3 Proposition. [2] *Let $K,J \subset \overline{\mathbb{R}}$ be compact sets and $K \cap J \neq \varnothing$.*

(1) $L(\mathcal{P}_*(K),E) \cap L(\mathcal{P}_*(J),E) = L(\mathcal{P}_*(K\cap J),E)$

(2) For any $T \in L(\mathcal{P}_*(K),E)$ there is a minimal compact set $J \subset K$ such that $T \in L(\mathcal{P}_*(J),E)$. The set J is called the support of T.

Proof. (1) Let $T \in L(\mathcal{P}_*(K),E) \cap L(\mathcal{P}_*(J),E)$. Then

$$\begin{aligned}
H^{-1}(T) &\in \left(\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\right) \cap \left(\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus J,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\right) \\
&= \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus (K\cap J),E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)
\end{aligned}$$

and $T \in L(\mathcal{P}_*(K\cap J),E)$ by Theorem 4.1 (and (4.12)). The other inclusion is obvious.

(2) This is clear by Theorem 4.1 since for any $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)$ there is a minimal J such that $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus J,E\right)$.

□

4.4 Remark. Let $K \subset \overline{\mathbb{R}}$ be a non-empty compact set. Then $\mathcal{P}_*(K)$ is nuclear and $\mathcal{P}_*(K)'_b$ a nuclear Fréchet-Schwartz space. Furthermore, we have

$$L_b(\mathcal{P}_*(K),E) \cong \mathcal{P}_*(K)'_b \hat{\otimes}_\pi E \cong \mathcal{P}_*(K)'_b \hat{\otimes}_\varepsilon E \cong \mathcal{P}_*(K)'_b \varepsilon E.$$

Proof. By Theorem 4.1 $\mathcal{P}_*(K)'_b$ is topologically isomorphic to $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$. This quotient space is nuclear by [60, Proposition 50.1 (50.4), p. 514] since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)$ is nuclear by

[2] counterpart: [13, Proposition 5.3, p. 1121]

Theorem 3.7 and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ a closed subspace. Hence $\mathcal{P}_*(K)_b'$ is nuclear as well. It is a Fréchet-Schwartz space because $\mathcal{P}_*(K)$ is a DFS-space by Theorem 3.5(1). So due to [60, Proposition 50.6, p. 523] $\left(\mathcal{P}_*(K)_b'\right)_b'$ is nuclear and, as $\mathcal{P}_*(K)$ is reflexive, $\mathcal{P}_*(K)$, too. Since $\mathcal{P}_*(K)$ is a DFS-space, in particular reflexive, thus barrelled by [45, 23.22 Satz, p. 253], and complete, plus E complete as well as $\mathcal{P}_*(K)_b'$ complete and nuclear, we obtain

$$L_b(\mathcal{P}_*(K),E) \cong \mathcal{P}_*(K)_b' \hat{\otimes}_\pi E$$

by [60, Proposition 50.5, p. 522]. The remaining isomorphisms are due to the nuclearity of $\mathcal{P}_*(K)$. □

For a different proof of this statement see [26, 1.11 Satz, p. 11] and [26, 3.9 Satz, p. 41].

For $K = \overline{\mathbb{R}}$ we look at the duality of Theorem 4.1 once again, but from a different point of view. Let $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}, E\right)$. In the spirit of [40] and [57, Chapitre II, p. 77-97] we assign the boundary value

$$\langle R(f),\varphi\rangle := \lim_{t,t'\searrow 0}\int_{\mathbb{R}} (f(x+it)-f(x+it'))\,\varphi(x)\,dx, \quad \varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right),$$

to this function, if existing. Furthermore, we define the upper boundary value by

$$\langle R^+(f),\varphi\rangle := \lim_{t\searrow 0}\int_{\mathbb{R}} f(x+it)\,\varphi(x)\,dx, \quad \varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right),$$

and the lower boundary value by

$$\langle R^-(f),\varphi\rangle := \lim_{t\searrow 0}\int_{\mathbb{R}} f(x-it)\,\varphi(x)\,dx, \quad \varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right),$$

if existing.

4.5 Theorem. *(1) The boundary values $R(f)$, $R^+(f)$ and $R^-(f)$ exist. They are elements of $L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right),E\right)$ and*

$$R(f) = R^+(f) - R^-(f) = -\tilde{H}(f)$$

for all $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}, E\right)$.

(2) The mapping $[f] \mapsto R(f)$ is a topological isomorphism between $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}, E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$ and $L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right),E\right)$.

Proof. (1) Let $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}, E\right)$ and $n \in \mathbb{N}$. For $t > 0$ we define

$$R_t^{\pm}(f)(\varphi) := \int_{\mathbb{R}} f(x \pm it)\,\varphi(x)\,dx$$

for any $\varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right)$. Let $\alpha \in A$ and $\varphi \in \mathcal{O}_n\left(U_n\left(\overline{\mathbb{R}}\right)\right)$, $n \in \mathbb{N}$. We choose $m \in \mathbb{N}_{\geq 2}$ such that $m > 2\max(n,t)$ and $\frac{1}{m} < t$. Then we obtain

$$\begin{aligned} p_\alpha(R_t^{\pm}(f)(\varphi)) &\leq \int_{\mathbb{R}} p_\alpha(f(x\pm it))\,|\varphi(x)|\,dx \leq |f|_{m,\alpha}\,\|\varphi\|_n \int_{-\infty}^{\infty} e^{\frac{1}{m}|x\pm it|-\frac{1}{n}|x|}dx \\ &\leq 2e^{\frac{1}{2n}t}\,|f|_{m,\alpha}\,\|\varphi\|_n \int_0^{\infty} e^{-\frac{1}{2n}|x|}dx = 4ne^{\frac{1}{2n}t}\,|f|_{m,\alpha}\,\|\varphi\|_n < \infty. \end{aligned}$$

Hence $R_t^{\pm}(f) \in L\big(\mathcal{O}_n\big(U_n\big(\overline{\mathbb{R}}\big)\big), E\big)$ for every $n \in \mathbb{N}$ connoting $R_t^{\pm}(f) \in L\big(\mathcal{P}_*\big(\overline{\mathbb{R}}\big), E\big)$.
Now set $\varphi_t^{\pm}(x) := \varphi(x \pm it)$. Then the functions

$$t \mapsto R_t^{\pm}(f)(\varphi_t^{\pm}) = \int_{\mathbb{R}} f(x \pm it)\,\varphi(x \pm it)\,dx \tag{4.16}$$

are defined for $\varphi \in \mathcal{O}_n\big(U_n\big(\overline{\mathbb{R}}\big)\big)$, $n \in \mathbb{N}$, on $]0, 1/n[$ and constant by the remarks above Theorem 4.1. Thus the limits $\lim_{t\searrow 0} R_t^{\pm}(f)(\varphi_t^{\pm})$ exist in E for every $\varphi \in \mathcal{P}_*\big(\overline{\mathbb{R}}\big)$.
Let $\alpha \in A$, $n \in \mathbb{N}$, and $\varphi \in \mathcal{O}_n\big(U_n\big(\overline{\mathbb{R}}\big)\big)$. For $0 < t < \frac{1}{3n}$ and $z \in \overline{U_{3n}(K)}$ we have

$$\begin{aligned}
|\varphi(z) - \varphi(z \pm it)|e^{\frac{1}{3n}|\mathrm{Re}(z)|} &= \left|\int_{[z\pm it,z]_L} \varphi'(w)\,dw\right| e^{\frac{1}{3n}|\mathrm{Re}(z)|} \le t \sup_{w\in[z\pm it,z]_L} |\varphi'(w)|e^{\frac{1}{3n}|\mathrm{Re}(z)|}\\
&\le t \sup_{w\in[z\pm it,z]_L} 6n \max_{|\zeta-w|=\frac{1}{6n}} |\varphi(\zeta)|e^{\frac{1}{3n}|\mathrm{Re}(z)|}\\
&\le 6ne^{\frac{1}{18n^2}} t \underbrace{\sup_{w\in[z\pm it,z]_L} \max_{|\zeta-w|=\frac{1}{6n}} |\varphi(\zeta)|e^{\frac{1}{3n}|\mathrm{Re}(\zeta)|}}_{\le\|\varphi\|_n} \le 6ne^{\frac{1}{18n^2}}\|\varphi\|_n t
\end{aligned}$$

by the Cauchy integral formula and the Cauchy inequality where we denote by $[z \pm it, z]_L$ the line segment from $z \pm it$ to z. Hence we get

$$\|\varphi - \varphi(\cdot \pm it)\|_{3n} \le 6ne^{\frac{1}{18n^2}}\|\varphi\|_n t. \tag{4.17}$$

Further, we have for $0 < t < \frac{1}{3n}$ and $x \in \mathbb{R}$

$$\left|\mathrm{Im}\left(x \pm i\frac{1}{3n}\right)\right| = \frac{1}{3n} \quad \text{plus} \quad 6n > \frac{1}{n} > \left|\mathrm{Im}\left(x \pm t \pm i\frac{1}{3n}\right)\right| = t + \frac{1}{3n} > \frac{1}{6n}.$$

Due to the Cauchy integral theorem we obtain for all $0 < t < \frac{1}{3n}$

$$\begin{aligned}
&p_\alpha(R_t^{\pm}(f)(\varphi) - R_t^{\pm}(f)(\varphi_t^{\pm}))\\
&= p_\alpha\left(\int_{\mathbb{R}} f(x \pm it)(\varphi(x) - \varphi(x \pm it))\,dx\right)\\
&= p_\alpha\left(\int_{\mathbb{R}} f\left(x \pm it \pm i\frac{1}{3n}\right)\left(\varphi\left(x \pm i\frac{1}{3n}\right) - \varphi\left(x \pm it \pm i\frac{1}{3n}\right)\right)dx\right)\\
&\le |f|_{6n,\alpha}\|\varphi - \varphi(\cdot \pm it)\|_{3n} \int_{-\infty}^{\infty} e^{\frac{1}{6n}|x\pm it| - \frac{1}{3n}|x|}dx\\
&\le 12ne^{\frac{1}{6n}t}|f|_{6n,\alpha}\|\varphi - \varphi(\cdot \pm it)\|_{3n}\\
&\underset{(4.17)}{\le} \left(72n^2e^{\frac{1}{18n^2}}\|\varphi\|_n|f|_{6n,\alpha}\right)e^{\frac{1}{6n}t}t\\
&\underset{t\searrow 0}{\to} 0.
\end{aligned}$$

Since the limits $\lim_{t\searrow 0} R_t^{\pm}(f)(\varphi_t^{\pm})$ exist in E for every $\varphi \in \mathcal{P}_*\big(\overline{\mathbb{R}}\big)$, this implies that the limits $\langle R^{\pm}(f),\varphi\rangle = \lim_{t\searrow 0} R_t^{\pm}(f)(\varphi)$ exist in E, more precisely,

$$\langle R^{\pm}(f),\varphi\rangle = \lim_{t\searrow 0} R_t^{\pm}(f)(\varphi) = \lim_{t\searrow 0} R_t^{\pm}(f)(\varphi_t^{\pm}).$$

The space $\mathcal{P}_*(\overline{\mathbb{R}})$ is a DFS-space by Theorem 3.5(1) and hence a Montel space due to [32, Theorem 6´, p. 375]. Thus it is barrelled by [45, 24.24 Bemerkung (a), p. 267] and by the Banach-Steinhaus theorem we obtain $R^{\pm}(f) \in L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$. Actually we even have that $R_t^{\pm}(f)$ converges to $R^{\pm}(f)$ in $L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$ as $t \searrow 0$ by virtue of [16, 10.3.4 Satz, p. 53] because every bounded set in $\mathcal{P}_*(\overline{\mathbb{R}})$, being a Montel space, is relatively compact. Furthermore, we get

$$\begin{aligned}\langle R(f),\varphi\rangle &= \lim_{t,t'\searrow 0}\left(R_t^{+}(f)(\varphi) - R_{t'}^{-}(f)(\varphi)\right) = \lim_{t\searrow 0} R_t^{+}(f)(\varphi) - \lim_{t\searrow 0} R_t^{-}(f)(\varphi)\\ &= \langle R^{+}(f),\varphi\rangle - \langle R^{-}(f),\varphi\rangle = \lim_{t\searrow 0}(R_t^{+}(f)(\varphi_t^{+}) - R_t^{-}(f)(\varphi_t^{-}))\\ &= \lim_{t\searrow 0}\left(\int_{\mathbb{R}} f(x+it)\,\varphi(x+it)\,dx - \int_{\mathbb{R}} f(x-it)\,\varphi(x-it)\,dx\right) = -\tilde{H}(f)(\varphi) \quad (4.18)\end{aligned}$$

for every $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$ by the definition of $\tilde{H}$ in (4.1) and the remarks above Theorem 4.1. In particular, this means that $R(f) \in L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$ for every $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}},E)$.

(2) By the first part the considered map coincides with $-H$ and the statement follows directly by Theorem 4.1.

□

In particular, this theorem contains, at least in one variable, [28, Theorem 3.2.9, p. 483-484] for $E = \mathbb{C}$ and [26, Satz 3.13, p. 44] for Fréchet spaces E, where it is stated that the map

$$\tilde{R}\colon \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}},E)/\mathcal{O}^{exp}(\overline{\mathbb{C}},E) \to L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E),$$

defined by

$$\tilde{R}([f])(\varphi) := R_t^{+}(f)(\varphi_t^{+}) - R_t^{-}(f)(\varphi_t^{-})$$

for $f \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}},E)$ and $\varphi \in \mathcal{P}_*(\overline{\mathbb{R}})$ and fixed t small enough, is an isomorphism. This result is contained since the functions in (4.16) are constant and due to (4.18).
Finally, we define the Fourier transformation on $L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$. By [28, Proposition 3.2.4, p. 483] the Fourier transformation $\mathscr{F}\colon \mathcal{P}_*(\overline{\mathbb{R}}) \to \mathcal{P}_*(\overline{\mathbb{R}})$ defined by

$$\mathscr{F}(\varphi)(\zeta) := \hat{\varphi}(\zeta) := \int_{\mathbb{R}} \varphi(x)e^{ix\zeta}dx, \quad \varphi \in \mathcal{O}_n(U_n(\overline{\mathbb{R}})), \zeta \in U_n(\overline{\mathbb{R}}),$$

is a topological isomorphism. For an easy proof see the one of [27, Proposition 8.2.2, p. 376] which only needs some slight modifications to be applied here. The Fourier transformation on $L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$ is now defined by transposition and we obtain:

4.6 Theorem. *The Fourier transformation $\mathscr{F}_d\colon L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E) \to L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E)$ defined by*

$$\mathscr{F}_d(T)(\varphi) := \langle T,\mathscr{F}(\varphi)\rangle, \quad T \in L_b(\mathcal{P}_*(\overline{\mathbb{R}}),E),\ \varphi \in \mathcal{P}_*(\overline{\mathbb{R}}),$$

is a topological isomorphism.

Proof. Let $T \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$ and let (φ_n) be a sequence in $\mathcal{P}_*\left(\overline{\mathbb{R}}\right)$ converging to $\varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right)$. Then $\mathscr{F}(\varphi_n)$ converges to $\mathscr{F}(\varphi)$ since $\mathscr{F}$ is continuous. By the continuity of T we get

$$\lim_{n\to\infty} \mathscr{F}_d(T)(\varphi_n) = \lim_{n\to\infty} \langle T, \mathscr{F}(\varphi_n)\rangle = \left\langle T, \lim_{n\to\infty} \mathscr{F}(\varphi_n)\right\rangle = \langle T, \mathscr{F}(\varphi)\rangle = \mathscr{F}_d(T)(\varphi).$$

So, as $\mathscr{F}_d(T)$ is obviously linear, we have $\mathscr{F}_d(T) \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$ and therefore the map $\mathscr{F}_d$ is well-defined and also linear. Next, define

$$\mathscr{F}_d^{-1}: L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right) \to L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$$

by $\mathscr{F}_d^{-1}(T)(\varphi) := \langle T, \mathscr{F}^{-1}(\varphi)\rangle$ for $T \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$ and $\varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right)$. Like above we have $\mathscr{F}_d^{-1}(T) \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$, since $\mathscr{F}^{-1}$ is continuous, and thus the map $\mathscr{F}_d^{-1}$ is well-defined as well. Furthermore, the equations

$$\mathscr{F}_d^{-1}(\mathscr{F}_d(T))(\varphi) = \left\langle \mathscr{F}_d(T), \mathscr{F}^{-1}(\varphi)\right\rangle = \left\langle T, \mathscr{F}\left(\mathscr{F}^{-1}(\varphi)\right)\right\rangle = \langle T, \varphi\rangle$$

and

$$\mathscr{F}_d\left(\mathscr{F}_d^{-1}(T)\right)(\varphi) = \left\langle \mathscr{F}_d^{-1}(T), \mathscr{F}(\varphi)\right\rangle = \left\langle T, \mathscr{F}^{-1}(\mathscr{F}(\varphi))\right\rangle = \langle T, \varphi\rangle$$

hold for every $T \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$ and every $\varphi \in \mathcal{P}_*\left(\overline{\mathbb{R}}\right)$ implying that $\mathscr{F}_d$ is an algebraic isomorphism. It is also topological since one has for arbitrary $\alpha \in A$ and bounded $B \subset \mathcal{P}_*\left(\overline{\mathbb{R}}\right)$ that

$$\sup_{\varphi\in B} p_\alpha(\mathscr{F}_d(T)(\varphi)) = \sup_{\varphi\in B} p_\alpha(\langle T, \mathscr{F}(\varphi)\rangle) = \sup_{\varphi\in\mathscr{F}(B)} p_\alpha(\langle T, \varphi\rangle)$$

plus

$$\sup_{\varphi\in B} p_\alpha\left(\mathscr{F}_d^{-1}(T)(\varphi)\right) = \sup_{\varphi\in B} p_\alpha\left(\left\langle T, \mathscr{F}^{-1}(\varphi)\right\rangle\right) = \sup_{\varphi\in\mathscr{F}^{-1}(B)} p_\alpha(\langle T, \varphi\rangle)$$

are valid for all $T \in L_b\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right), E\right)$ where the sets $\mathscr{F}(B)$ and $\mathscr{F}^{-1}(B)$ are bounded due to the linearity and continuity of $\mathscr{F}$ resp. $\mathscr{F}^{-1}$ and the boundedness of B. $\square$

5 Strictly admissible spaces

We recall from the introduction that a complete locally convex space E is called *admissible*, if the Cauchy-Riemann operator

$$\overline{\partial}: \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$. E is called *strictly admissible* if E is admissible and if, in addition,

$$\overline{\partial}: \mathrm{C}^{\infty}(\Omega, E) \to \mathrm{C}^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$.
As a first step it is shown that $\mathbb{C}$ is admissible. Let $n \in \mathbb{N}_{\geq 2}$ and denote by $\mathcal{E}^{exp}_{n,\overline{\partial}}(S_n(K), E)$ the topological subspace $\left\{f \in \mathcal{E}^{exp}_n(S_n(K)) \mid \overline{\partial} f = 0\right\}$ of $\mathcal{E}^{exp}_n(S_n(K), E)$. Observe that

$$\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) = \lim_{\substack{\longleftarrow \\ n \in \mathbb{N}_{\geq 2}}} \mathrm{proj}\, \mathcal{E}^{exp}_{n,\overline{\partial}}(S_n(K), E)$$

as topological spaces by Theorem 3.6(4).

We will prove that the spaces of the projective spectrum on the right hand side have some kind of density property and that for every $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ and every $n \in \mathbb{N}_{\geq 2}$ there is a $u \in \mathcal{E}^{exp}_n(S_n(K))$ such that $\overline{\partial} u = f$ on $S_n(K)$. The combination of these results then yields the admissibility of $\mathbb{C}$ via the Mittag-Leffler procedure. By classical theory of tensor products of Fréchet spaces as well as splitting theory of Fréchet spaces resp. PLS-spaces further admissible spaces are obtained and at the end of this section it is proven that the admissible spaces found so far are already strictly admissible. In addition, a list of concrete examples of locally convex spaces that are strictly admissible or that are not strictly admissible, from the view of Theorem 6.14, is provided.

We begin with the proof of the already announced density theorem. The underlying idea of the proof was to analyze a proof of Hörmander, [18, Theorem 4.4.5, p. 112], in a comparable situation for C^{∞}−functions. The proof is split into several parts to enhance comprehensibility and clarity.

5.1 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be a compact set and $k, p, n \in \mathbb{R}$ with $k > p > n > 1$. Then $\pi_{n,k}\left(\mathcal{E}^{exp}_{k,\overline{\partial}}(S_k(K))\right)$ is dense in $\pi_{n,p}\left(\mathcal{E}^{exp}_{p,\overline{\partial}}(S_p(K))\right)$ with respect to $\left(|\cdot|_{n,m}\right)_{m \in \mathbb{N}_0}$.*

In order to gain access to the theory of distributions in this approach, we prove another density statement first.

5.2 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be a compact set and $p, j \in \mathbb{R}$, $p > j > 1$. Then $\pi_{j,p}\left(\mathrm{C}^{\infty}_0(S_p(K))\right)$ is dense in $\pi_{j,p}\left(\mathcal{E}^{exp}_p(S_p(K))\right)$ with respect to $\left(|f|_{j,m}\right)_{m \in \mathbb{N}_0}$.*

Proof. Let $f \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$ and $\varepsilon > 0$. Choose $s, t \in \mathbb{R}$ with $p > s > t > j$ and set

$$Q_0 := \overline{S_t(K)} \cap \left\{ z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq \max\left(0, \frac{jp \ln \varepsilon}{j-p}\right) \right\},$$
$$Q_1 := S_s(K) \cap \left\{ z \in \mathbb{C} \mid |\mathrm{Re}(z)| < \max\left(0, \frac{jp \ln \varepsilon}{j-p}\right) + 1 \right\}.$$

Then Q_0 is compact, $Q_0 \subset Q_1 \subset S_p(K)$ and

$$\sup_{z \in S_j(K) \setminus Q_0} \frac{e^{-\frac{1}{j}|\mathrm{Re}(z)|}}{e^{-\frac{1}{p}|\mathrm{Re}(z)|}} = \sup_{z \in S_j(K) \setminus Q_0} e^{\frac{j-p}{jp}|\mathrm{Re}(z)|} \underset{\substack{\text{choice} \\ \text{of } Q_0}}{<} \varepsilon. \tag{5.1}$$

Like in the proof of [18, Theorem 1.4.1, p. 25] one can find, by using Remark 3.3(4), $\varphi \in \mathrm{C}_0^\infty\left(S_p(K)\right)$, $0 \leq \varphi \leq 1$, such that $\varphi \equiv 1$ near Q_0, $\varphi \equiv 0$ near Q_1^C and

$$|\partial^\alpha \varphi| \leq C_\alpha \left(\frac{\varepsilon_0}{4}\right)^{-|\alpha|} \tag{5.2}$$

for all $\alpha \in \mathbb{N}_0^2$ where

$$\varepsilon_0 := \begin{cases} \min\left(\frac{1}{t} - \frac{1}{s}, 1\right), & K \neq \varnothing, \\ \min(s-t, 1), & K = \varnothing, \end{cases} = \begin{cases} \frac{1}{t} - \frac{1}{s}, & K \neq \varnothing, \\ \min(s-t, 1), & K = \varnothing, \end{cases}$$

and $C_\alpha > 0$ is a constant only depending on α.
Then $\varphi f \in \mathrm{C}_0^\infty\left(S_p(K)\right)$ and for $m \in \mathbb{N}_0$

$$\begin{aligned}
|\varphi f - f|_{j,m} &\leq \sup_{\substack{z \in Q_1 \setminus Q_0 \\ |\alpha| \leq m}} |\partial^\alpha(\varphi f)(z) - \partial^\alpha f(z)| e^{-\frac{1}{j}|\mathrm{Re}(z)|} + \sup_{\substack{z \in S_j(K) \setminus Q_1 \\ |\alpha| \leq m}} |\partial^\alpha f(z)| e^{-\frac{1}{j}|\mathrm{Re}(z)|} \\
&\leq \sup_{\substack{z \in Q_1 \setminus Q_0 \\ |\alpha| \leq m}} \left| \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \partial^{\alpha-\gamma}\varphi(z) \partial^\gamma f(z) \right| e^{-\frac{1}{j}|\mathrm{Re}(z)|} + 2 \sup_{\substack{z \in S_p(K) \setminus Q_0 \\ |\alpha| \leq m}} |\partial^\alpha f(z)| e^{-\frac{1}{p}|\mathrm{Re}(z)|} \frac{e^{-\frac{1}{j}|\mathrm{Re}(z)|}}{e^{-\frac{1}{p}|\mathrm{Re}(z)|}} \\
&\underset{(5.1)}{\leq} \sup_{|\alpha| \leq m} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \sup_{z \in Q_1 \setminus Q_0} |\partial^{\alpha-\gamma}\varphi(z)| \left(\sup_{\substack{z \in S_p(K) \setminus Q_0 \\ |\beta| \leq m}} \left| \partial^\beta f(z) \right| e^{-\frac{1}{j}|\mathrm{Re}(z)|} \right) + 2\varepsilon |f|_{p,m} \\
&\underset{(5.2)}{\leq} \underbrace{\sup_{|\alpha| \leq m} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{\alpha-\gamma} \left(\frac{\varepsilon_0}{4}\right)^{-|\alpha-\gamma|}}_{:= C(m, \varepsilon_0)} \varepsilon |f|_{p,m} + 2\varepsilon |f|_{p,m} \\
&= (C(m, \varepsilon_0) + 2) |f|_{p,m} \varepsilon
\end{aligned}$$

holds where $C(m, \varepsilon_0)$ is independent of ε proving the density. □

The next lemma is devoted to a special fundamental solution of the $\overline{\partial}$-operator and its properties (used already in the proof of Theorem 4.1).

5.3 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be a compact set, $q \in \mathbb{R}$, $q > 1$, and $E\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $E(z) := \frac{e^{-z^2}}{\pi z}$.*

a) $\overline{\partial} T_E = \delta$.

b) Let $\varepsilon > 0$, $x \notin \overline{S_{q+\varepsilon}(K)}$ and $\alpha \in \mathbb{N}_0^2$. Then $\partial_x^{\alpha}[E(\cdot - x)] \in \mathcal{E}_{q,\overline{\partial}}^{exp}\left(S_q(K)\right)$.

c) Let $N \subset \mathbb{C}$ be a compact set and $m \in \mathbb{N}_0$. Then there exists a constant $B_1 = B_1(q,N)$ such that

$$|T_E * \psi|_{q,m} \leq B_1 |||\psi|||_m \tag{5.3}$$

for all $\psi \in C_0^{\infty}(N)$ with the convolution from (2.1).
Especially, one gets $T_E * \psi \in \mathcal{E}_q^{exp}\left(S_q(K)\right)$.

d) Let $p, j, n \in \mathbb{R}$ with $p > j > n > 1$.

i) There exists $\varphi \in C^{\infty}(\mathbb{C})$, $0 \leq \varphi \leq 1$, such that $\varphi \equiv 1$ near $\overline{S_n(K)}$ and $\varphi \equiv 0$ near $S_j(K)^C$ plus

$$|\partial^{\alpha}\varphi| \leq \tilde{C}_{\alpha}\left(\frac{\tilde{\varepsilon}}{4}\right)^{-|\alpha|} \tag{5.4}$$

for all $\alpha \in \mathbb{N}_0^2$ where

$$\tilde{\varepsilon} := \begin{cases} \frac{1}{n} - \frac{1}{j}, & K \neq \varnothing, \\ j - n, & K = \varnothing, \end{cases}$$

and $\tilde{C}_{\alpha} > 0$ is a constant only depending on α.

ii) Choose $\varphi \in C^{\infty}(\mathbb{C})$ like in i). Let $s, t \in \mathbb{R}$ with $p \geq s \geq t > 0$ and $m \in \mathbb{N}_0$. Then there exists a constant $C_1 = C_1(j,n,s,t,m)$ such that

$$|T_E * (\varphi f)|_{t,m} \leq C_1 |f|_{s,j,m}. \tag{5.5}$$

for all $f \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$ where

$$|f|_{s,j,m} := \sup_{\substack{z \in S_j(K) \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \left|f^{(\beta)}(z)\right| e^{-\frac{1}{s}|\mathrm{Re}(z)|}$$

and the convolution is defined by the right hand side of (2.1) and we set $\varphi f := 0$ outside $S_p(K)$.
Especially, one gets $T_E * (\varphi f) \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$.

Proof. a) Let $\varphi \in C_0^{\infty}(\mathbb{C})$ and set $E_0(z) := \frac{1}{\pi z}$ plus $g(z) := e^{-z^2}$. Using $g \in \mathcal{O}(\mathbb{C})$ and the fact that T_{E_0} is a fundamental solution of the $\overline{\partial}$-operator by [18, (3.1.12), p. 63], one gets

$$\begin{aligned}\left\langle \overline{\partial} T_E, \varphi \right\rangle &= -\left\langle T_E, \overline{\partial}\varphi \right\rangle = -\left\langle T_{E_0}, g\overline{\partial}\varphi \right\rangle = -\left\langle T_{E_0}, \overline{\partial}(g\varphi) \right\rangle = \left\langle \overline{\partial} T_{E_0}, g\varphi \right\rangle \\ &= \langle \delta, g\varphi \rangle = g(0)\varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle .\end{aligned}$$

b) Since $x \notin \overline{S_{q+\varepsilon}(K)}$, it follows $\partial_x^\alpha [E(\cdot - x)] \in \mathcal{O}\left(S_q(K)\right)$. Let $z \in S_q(K)$ and $\beta \in \mathbb{N}_0^2$. With $r := \frac{1}{2} \mathrm{d}\left(\partial S_{q+\varepsilon}(K), \partial S_q(K)\right) > 0$ one has by the Cauchy inequality

$$
\begin{aligned}
\left|\partial_z^\beta \partial_x^\alpha [E(z-x)]\right| &\underset{(3.2)}{=} \left|i^{\alpha_2+\beta_2} (-1)^{|\alpha|} E^{(|\alpha+\beta|)}(z-x)\right| \le \frac{|\alpha+\beta|!}{r^{|\alpha+\beta|}} \max_{|\zeta-(z-x)|=r} |E(\zeta)| \\
&\le \frac{1}{\pi} \frac{|\alpha+\beta|!}{r^{|\alpha+\beta|+1}} \max_{|\zeta-(z-x)|=r} e^{-\mathrm{Re}\left(\zeta^2\right)} = \frac{1}{\pi} \frac{|\alpha+\beta|!}{r^{|\alpha+\beta|+1}} \max_{|\zeta-(z-x)|=r} e^{-\zeta_1^2+\zeta_2^2} \\
&\le \underbrace{\frac{1}{\pi} \frac{|\alpha+\beta|!}{r^{|\alpha+\beta|+1}} e^{(r+k+|x_2|)^2}}_{=:A_0} \max_{|\zeta-(z-x)|=r} e^{-\zeta_1^2} \\
&= A_0 \max_{t\in[0,1]} e^{-(2rt+z_1-x_1-r)^2} \\
&= A_0 \max_{t\in[0,1]} e^{-4r^2t^2-4rtz_1+4rtx_1+4r^2t-z_1^2+2x_1z_1+2rz_1-x_1^2-2rx_1-r^2} \\
&\le A_0 \underbrace{e^{-x_1^2+6r|x_1|+3r^2}}_{\le e^{12r^2}} e^{-z_1^2+2(3r+|x_1|)|z_1|} \\
&\le \underbrace{A_0 e^{12r^2}}_{=:A_1(\beta)} e^{-z_1^2+2(3r+|x_1|)|z_1|}
\end{aligned}
$$

and hence

$$
\begin{aligned}
|\partial_x^\alpha [E(\cdot - x)]|_{q,m} &\le \sup_{|\beta|\le m} A_1(\beta) \sup_{z\in S_q(K)} e^{-z_1^2+2(3r+|x_1|)|z_1|} e^{-\frac{1}{q}|z_1|} \\
&\le \sup_{|\beta|\le m} A_1(\beta) e^{\left(3r+|x_1|-\frac{1}{2q}\right)^2} < \infty. \qquad (5.6)
\end{aligned}
$$

c) Due to the compactness of N, there exists $B_0 > 0$ such that $|z| \le B_0$ for all $z \in N$. By definition of distributional convolution $T_E * \psi \in \mathrm{C}^\infty(\mathbb{C})$ and for $x \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^2$ and $\varepsilon > 0$ the following inequalities hold

$$
\begin{aligned}
&|\partial^\alpha (T_E * \psi)(x)| \\
&= \left|\int_{\mathbb{C}} E(y) (\partial^\alpha \psi)(x-y)\, dy\right| \le |||\psi|||_{|\alpha|} \int_{x-N} |E(y)|\, dy \\
&= \frac{1}{\pi} |||\psi|||_{|\alpha|} \int_N \left|\frac{e^{-(x-y)^2}}{x-y}\right| dy \\
&\le \frac{1}{\pi} |||\psi|||_{|\alpha|} \left(\int_{D_\varepsilon(x)} \left|\frac{e^{-(x-y)^2}}{x-y}\right| dy + \int_{N\setminus D_\varepsilon(x)} \left|\frac{e^{-(x-y)^2}}{x-y}\right| dy\right) \\
&\le \frac{1}{\pi} |||\psi|||_{|\alpha|} \left(\int_0^{2\pi} \int_0^\varepsilon \frac{e^{-r^2\cos(2\nu)}}{r} r\, dr\, d\nu + \frac{1}{\varepsilon} \int_{N\setminus D_\varepsilon(x)} \left|e^{-(x-y)^2}\right| dy\right) \\
&\le \frac{1}{\pi} |||\psi|||_{|\alpha|} \left(2\pi\varepsilon e^{\varepsilon^2} + \frac{1}{\varepsilon} \int_{N\setminus D_\varepsilon(x)} e^{-\left(x_1^2-x_2^2-2(x_1y_1-x_2y_2)+y_1^2-y_2^2\right)} dy\right)
\end{aligned}
$$

$$
\begin{aligned}
&\underset{\text{Fubini}}{\leq} \frac{1}{\pi} |||\psi|||_{|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2} + \frac{1}{\varepsilon}e^{x_2^2}\int_{\mathbb{R}} e^{-y_1^2+2x_1y_1-x_1^2}dy_1 \int_{[-B_0,B_0]} e^{y_2^2-2x_2y_2}dy_2\right)\\
&\leq \frac{1}{\pi} |||\psi|||_{|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2} + \frac{2B_0}{\varepsilon}e^{x_2^2+2|x_2|B_0+B_0^2}\int_{\mathbb{R}} e^{-(y_1-x_1)^2}dy_1\right)\\
&= \frac{1}{\pi} |||\psi|||_{|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2} + \frac{2B_0\sqrt{\pi}}{\varepsilon}e^{x_2^2+2|x_2|B_0+B_0^2}\right)
\end{aligned}
$$

and hence for fixed $\varepsilon > 0$

$$
\begin{aligned}
|T_E * \psi|_{q,m} &\leq \underbrace{\frac{1}{\pi}\left(2\pi\varepsilon e^{\varepsilon^2} + \frac{2B_0\sqrt{\pi}}{\varepsilon}e^{q^2+2qB_0+B_0^2}\right)}_{=:B_1} \underbrace{\sup_{x\in S_q(K)} e^{-\frac{1}{q}|\mathrm{Re}(x)|}}_{\leq 1} |||\psi|||_m\\
&\leq B_1 |||\psi|||_m . \qquad (5.7)
\end{aligned}
$$

d) i) φ exists by the proof of [18, Theorem 1.4.1, p. 25] and Remark 3.3(4).
ii) The proof is similar to the proof of (5.3). Let $x \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^2$ and $\varepsilon > 0$. Then we have

$$
\begin{aligned}
&\left|\int_{\mathbb{C}} E(y)\,\partial_x^{\alpha}[(f\varphi)(x-y)]\,dy\right|\\
&\leq \int_{\mathbb{C}} |E(y)\,\partial^{\alpha}(f\varphi)(x-y)|e^{-\frac{1}{s}|x_1-y_1|}e^{\frac{1}{s}|x_1-y_1|}dy\\
&\leq \sup_{z\in S_j(K)} |\partial^{\alpha}(f\varphi)(z)|e^{-\frac{1}{s}|\mathrm{Re}(z)|}\int_{x-S_j(K)} |E(y)|e^{\frac{1}{s}|x_1-y_1|}dy\\
&\leq \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\sup_{z\in S_j(K)}|\partial^{\alpha-\gamma}\varphi(z)|\Big(\sup_{\substack{z\in S_j(K)\\ |\beta|\leq|\alpha|}}\left|\partial^{\beta}f(z)\right|e^{-\frac{1}{s}|\mathrm{Re}(z)|}\Big)\int_{x-S_j(K)}|E(y)|e^{\frac{1}{s}|x_1-y_1|}dy\\
&\underset{(5.4)}{\leq} \underbrace{\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\tilde{C}_{\alpha-\gamma}\left(\frac{\tilde{\varepsilon}}{4}\right)^{-|\alpha-\gamma|}}_{=:C_0}|f|_{s,j,|\alpha|}\int_{x-S_j(K)}|E(y)|e^{\frac{1}{s}|x_1-y_1|}dy\\
&\leq \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(\int_{D_\varepsilon(x)}\left|\frac{e^{-(x-y)^2}}{x-y}\right|e^{\frac{1}{s}|y_1|}dy + \int_{S_j(K)\smallsetminus D_\varepsilon(x)}\left|\frac{e^{-(x-y)^2}}{x-y}\right|e^{\frac{1}{s}|y_1|}dy\right)\\
&\leq \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(\int_0^{2\pi}\int_0^{\varepsilon}\frac{e^{-r^2}\cos(2v)}{r}e^{\frac{1}{s}|x_1+r\cos(v)|}r\,dr\,dv + \frac{1}{\varepsilon}\int_{S_j(K)\smallsetminus D_\varepsilon(x)}\left|e^{-(x-y)^2}\right|e^{\frac{1}{s}|y_1|}dy\right)\\
&\leq \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2+\frac{1}{s}(\varepsilon+|x_1|)} + \frac{1}{\varepsilon}\int_{S_j(K)\smallsetminus D_\varepsilon(x)} e^{-\left(x_1^2-x_2^2-2(x_1y_1-x_2y_2)+y_1^2-y_2^2\right)+\frac{1}{s}|y_1|}dy\right)\\
&\underset{\text{Fubini}}{\leq} \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2+\frac{1}{s}(\varepsilon+|x_1|)} + \frac{1}{\varepsilon}e^{x_2^2}\int_{\mathbb{R}} e^{-y_1^2+2x_1y_1-x_1^2+\frac{1}{s}|y_1|}dy_1\int_{[-j,j]} e^{y_2^2-2x_2y_2}dy_2\right)\\
&\leq \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2+\frac{1}{s}(\varepsilon+|x_1|)} + \frac{2j}{\varepsilon}e^{x_2^2+2j|x_2|+j^2}e^{\frac{1}{s}|x_1|+\frac{1}{4s^2}}\underbrace{\int_{\mathbb{R}} e^{-\left(|y_1|-(|x_1|+\frac{1}{2s})\right)^2}dy_1}_{\leq 2\sqrt{\pi}}\right)\\
&\leq \frac{1}{\pi}C_0|f|_{s,j,|\alpha|}\left(2\pi\varepsilon e^{\varepsilon^2+\frac{1}{s}(\varepsilon+|x_1|)} + \frac{4j\sqrt{\pi}}{\varepsilon}e^{x_2^2+2j|x_2|+j^2+\frac{1}{4s^2}+\frac{1}{s}|x_1|}\right). \qquad (5.8)
\end{aligned}
$$

Thus $T_E * (\varphi f) \in \mathrm{C}^\infty(\mathbb{C})$ and $\partial^\alpha (T_E * (\varphi f))(x) = \int_{\mathbb{C}} E(y)\,\partial_x^\alpha [(f\varphi)(x-y)]\,dy$ (differentiation under the integral sign) as well as for fixed $\varepsilon > 0$

$$
\begin{aligned}
&|T_E * (\varphi f)|_{t,m} \\
&\leq \underbrace{\left(2\varepsilon e^{\varepsilon^2+\frac{1}{s}\varepsilon} + \frac{4j}{\varepsilon\sqrt{\pi}} e^{t^2+2jt+j^2+\frac{1}{4s^2}}\right)\left(\sup_{|\alpha|\leq m} C_0(\alpha)\right)}_{=:C_1} \underbrace{\left(\sup_{x\in S_t(K)} e^{\left(\frac{1}{s}-\frac{1}{t}\right)|x_1|}\right)}_{\leq 1,\, t\leq s} |f|_{s,j,m} \\
&= C_1 |f|_{s,j,m}.
\end{aligned}
$$

Especially, one gets $T_E * (\varphi f) \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$ for $t = s = p$.

□

The next step is to define different kinds of convolutions and study their relations and properties which shall be exploited in the proof of the density theorem.

5.4 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be a compact set and E like in Lemma 5.3. Let $k,p,j,n \in \mathbb{R}$ with $k > p > j > n > 1$ and $w \in \left(\pi_{n,p}\left(\mathcal{E}_p^{exp}\left(S_p(K)\right)\right), \left(|\cdot|_{n,m}\right)_{m\in\mathbb{N}_0}\right)'$.*

a) For $\psi \in \mathrm{C}_0^\infty(\mathbb{C})$ we define

$$\left\langle w *_1 T_{\check{E}}, \psi\right\rangle := \left\langle w, (T_E * \psi)|_{S_n(K)}\right\rangle.$$

Then $w *_1 T_{\check{E}} \in \mathcal{D}'(\mathbb{C})$.

b) Let $\varepsilon > 0$. For $x \notin \overline{S_{k+\varepsilon}(K)}$ we define

$$\left(w *_2 \check{E}\right)(x) := \left\langle w, E(\cdot - x)|_{S_n(K)}\right\rangle.$$

Then $w *_2 \check{E} \in \mathrm{C}^\infty\left(\overline{S_{k+\varepsilon}(K)}^C\right)$ and for $\alpha \in \mathbb{N}_0^2$

$$\partial_x^\alpha \left(w *_2 \check{E}\right)(x) = \left\langle w, \partial_x^\alpha [E(\cdot - x)]|_{S_n(K)}\right\rangle. \tag{5.9}$$

c) Let $\varepsilon > 0$. For $\psi \in \mathrm{C}_0^\infty(\mathbb{C})$ with $\operatorname{supp}\psi \subset \overline{S_{k+\varepsilon}(K)}^C$ the definitions of convolution above are consistent, i.e.

$$\left\langle w *_1 T_{\check{E}}, \psi\right\rangle = \left\langle T_{w *_2 \check{E}}, \psi\right\rangle.$$

d) Choose φ like in Lemma 5.3d), let $m \in \mathbb{N}_0$ and for $f \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$ we define

$$\left\langle w *_\varphi T_{\check{E}}, f\right\rangle := \left\langle w, [T_E * (\varphi f)]|_{S_n(K)}\right\rangle.$$

Then there exists a constant $C_2 = C_2(j,n,m) > 0$ such that

$$\left|\left\langle w *_\varphi T_{\check{E}}, f\right\rangle\right| \leq C_2 |f|_{j,m}. \tag{5.10}$$

Proof. a) $w *_1 T_{\check{E}}$ is defined by Lemma 5.3c) $(q = p)$. Let $N \subset \mathbb{C}$ be compact. Since w is continuous, there exist $B_2 > 0$ and $m \in \mathbb{N}_0$ such that

$$\left|\left\langle w *_1 T_{\check{E}}, \psi\right\rangle\right| = \left|\left\langle w, (T_E * \psi)\big|_{S_n(K)}\right\rangle\right| \leq B_2 \|T_E * \psi\|_{n,m} \underset{(5.3),\, q=n}{\leq} B_1 B_2 |||\psi|||_m$$

for all $\psi \in C_0^\infty(N)$, thus $w *_1 T_{\check{E}} \in \mathcal{D}'(\mathbb{C})$.

b) $w *_2 \check{E}$ and the right hand side of (5.9) are defined by Lemma 5.3b) $(q = k,\, k > p)$. For $h \in \mathbb{R}$, $h \neq 0$, and $x \notin \overline{S_{k+\varepsilon}(K)}$ we define

$$\psi_h(x) : S_p(K) \to \mathbb{C},\ \psi_h(x)(y) := \frac{E(y - (x + he_l)) - E(y - x)}{h}$$

where $e_l := \begin{cases} (1,0), & l = 1, \\ (0,1), & l = 2. \end{cases}$ For $0 < |h| < \mathrm{d}(\partial S_{k+\varepsilon}(K), \partial S_k(K)) =: \varepsilon_0$ one has $x + he_l \notin \overline{S_k(K)}$ and so $E(\cdot - (x + he_l)) \in \mathcal{E}_p^{exp}(S_p(K))$ by Lemma 5.3b) $(q = p,\, k > p)$. Hence one gets $\psi_h(x) \in \mathcal{E}_p^{exp}(S_p(K))$.
The underlying idea is

$$\begin{aligned}
\frac{\left(w *_2 \check{E}\right)(x + he_l) - \left(w *_2 \check{E}\right)(x)}{h} &= \left\langle w, \frac{E(\cdot - (x + he_l)) - E(\cdot - x)}{h}\bigg|_{S_n(K)}\right\rangle \\
&= \left\langle w, \psi_h(x)\big|_{S_n(K)}\right\rangle.
\end{aligned}$$

So, if we show, that $\psi_h(x)$ converges to $\partial_{x_l}[E(\cdot - x)]$ in $\mathcal{E}_p^{exp}(S_p(K))$ as h tends to 0, we get, keeping $|\cdot|_{n,m} \leq |\cdot|_{p,m}$ in mind,

$$\partial_l\left(w *_2 \check{E}\right)(x) = \left\langle w, \partial_{x_l}[E(\cdot - x)]\big|_{S_n(K)}\right\rangle.$$

Then the general statement follows by induction over $|\alpha|$.
Let $y \in S_p(K)$ and $\beta \in \mathbb{N}_0^2$. Since

$$|y - x| > \mathrm{d}\left(\partial S_{k+\varepsilon}(K), \partial S_p(K)\right) =: \varepsilon_1,$$

we get $0 \notin D_{\varepsilon_1}(y - x)$. Moreover, $\varepsilon_0 < \varepsilon_1$ by Remark 3.3(4) and so

$$|y - (x + he_l) - (y - x)| = |h| < \varepsilon_0 < \varepsilon_1.$$

Thus $y - (x + he_l) \in \overline{D_{|h|}(y - x)} \subset D_{\varepsilon_0}(y - x)$ and $0 \notin \overline{D_{|h|}(y - x)}$. By the mean value theorem there exist $\zeta_i \in [y - (x + he_l), y - x]_L \subset \overline{D_{|h|}(y - x)}$, $i = 1, 2$, where $[y - (x + he_l), y - x]_L$ denotes the line segment from $y - (x + he_l)$ to $y - x$, such that

$$\begin{aligned}
\partial_y^\beta \psi_h(x)(y) &= \frac{\left(\partial^\beta E\right)(y - (x + he_l)) - \left(\partial^\beta E\right)(y - x)}{h} \\
&= \frac{1}{h}\begin{pmatrix} \langle \operatorname{grad}\left(\partial^\beta E_1\right)(\zeta_1) | -he_l \rangle \\ \langle \operatorname{grad}\left(\partial^\beta E_2\right)(\zeta_2) | -he_l \rangle \end{pmatrix} = -\begin{pmatrix} \partial_l \partial^\beta E_1(\zeta_1) \\ \partial_l \partial^\beta E_2(\zeta_2) \end{pmatrix}
\end{aligned}$$

as well as $\zeta_{ii} \in [\zeta_i, y-x]_L \subset \overline{D_{|h|}(y-x)}$, $i = 1,2$, such that

$$\begin{aligned}\partial_y^\beta \psi_h(x)(y) - \partial_y^\beta \partial_{x_l}[E(y-x)] &= -\begin{pmatrix}\partial_l \partial^\beta E_1(\zeta_1)\\ \partial_l \partial^\beta E_2(\zeta_2)\end{pmatrix} - \partial^\beta(-\partial_l E)(y-x)\\ &= \begin{pmatrix}\langle \operatorname{grad}(\partial_l \partial^\beta E_1)(\zeta_{11})|y-x-\zeta_1\rangle\\ \langle \operatorname{grad}(\partial_l \partial^\beta E_2)(\zeta_{22})|y-x-\zeta_2\rangle\end{pmatrix}.\end{aligned} \tag{5.11}$$

Then

$$\begin{aligned}&\left|\begin{pmatrix}\langle \operatorname{grad}(\partial_l \partial^\beta E_1)(\zeta_{11})|y-x-\zeta_1\rangle\\ \langle \operatorname{grad}(\partial_l \partial^\beta E_2)(\zeta_{22})|y-x-\zeta_2\rangle\end{pmatrix}\right|\\ &\le \left|\langle \operatorname{grad}(\partial_l \partial^\beta E_1)(\zeta_{11})|y-x-\zeta_1\rangle\right| + \left|\langle \operatorname{grad}(\partial_l \partial^\beta E_2)(\zeta_{22})|y-x-\zeta_2\rangle\right|\\ &\le \left|\operatorname{grad}(\partial_l \partial^\beta E_1)(\zeta_{11})\right|\underbrace{|y-x-\zeta_1|}_{\le |h|} + \left|\operatorname{grad}(\partial_l \partial^\beta E_2)(\zeta_{22})\right|\underbrace{|y-x-\zeta_2|}_{\le |h|}\\ &\le \left(\left|\partial_1\partial_l\partial^\beta E_1(\zeta_{11})\right| + \left|\partial_2\partial_l\partial^\beta E_1(\zeta_{11})\right| + \left|\partial_1\partial_l\partial^\beta E_2(\zeta_{22})\right| + \left|\partial_2\partial_l\partial^\beta E_2(\zeta_{22})\right|\right)|h|\\ &\le \left(\left|\partial_1\partial_l\partial^\beta E(\zeta_{11})\right| + \left|\partial_2\partial_l\partial^\beta E(\zeta_{11})\right| + \left|\partial_1\partial_l\partial^\beta E(\zeta_{22})\right| + \left|\partial_2\partial_l\partial^\beta E(\zeta_{22})\right|\right)|h|\\ &\underset{(3.2)}{=} 2\left(\left|E^{(|\beta|+2)}(\zeta_{11})\right| + \left|E^{(|\beta|+2)}(\zeta_{22})\right|\right)|h|\end{aligned} \tag{5.12}$$

is valid. By choosing $\varepsilon_0 < r < \varepsilon_1$, one gets due to Cauchy's integral formula

$$\begin{aligned}\left|E^{(|\beta|+2)}(\zeta_{ii})\right| &= \frac{(|\beta|+2)!}{2\pi}\left|\int_{\partial D_r(y-x)} \frac{E(z)}{(z-\zeta_{ii})^{|\beta|+3}}dz\right| \le \frac{r(|\beta|+2)!}{(r-\varepsilon_0)^{|\beta|+3}} \max_{|z-(y-x)|=r}\left|\frac{e^{-z^2}}{\pi z}\right|\\ &\le \frac{r(|\beta|+2)!}{\pi(r-\varepsilon_0)^{|\beta|+3}(\varepsilon_1-r)} \max_{|z-(y-x)|=r} e^{-z_1^2+z_2^2} \le \underbrace{\frac{r(|\beta|+2)!e^{(r+p+|x_2|)^2}}{\pi(r-\varepsilon_0)^{|\beta|+3}(\varepsilon_1-r)}}_{=:A_{0,0}} \max_{|z-(y-x)|=r} e^{-z_1^2}\\ &= A_{0,0}\max_{t\in[0,1]} e^{-(2rt+y_1-x_1-r)^2}\\ &= A_{0,0}\max_{t\in[0,1]} e^{-4r^2t^2-4rty_1+4rtx_1+4r^2t-y_1^2+2x_1y_1+2ry_1-x_1^2-2rx_1-r^2}\\ &\le \underbrace{A_{0,0}e^{-x_1^2+6r|x_1|+3r^2}}_{\le A_{0,0}e^{12r^2}=:A_{0,1}(\beta)} e^{-y_1^2+2(3r+|x_1|)|y_1|}.\end{aligned} \tag{5.13}$$

Hence by combining (5.11), (5.12) and (5.13), we have for $m \in \mathbb{N}_0$

$$\begin{aligned}\left|\psi_h(x) - \partial_{x_l}[E(\cdot - x)]\right|_{p,m} &\le 4\sup_{|\beta|\le m} A_{0,1}(\beta) \sup_{y\in S_p(K)} e^{-y_1^2+2(3r+|x_1|)|y_1|}e^{-\frac{1}{p}|y_1|}|h|\\ &\le 4\sup_{|\beta|\le m} A_{0,1}(\beta)e^{\left(3r+|x_1|-\frac{1}{2p}\right)^2}|h|\\ &\underset{h\to 0}{\to} 0.\end{aligned} \tag{5.14}$$

This means that $\psi_h(x)$ converges to $\partial_{x_l}[E(\cdot - x)]$ in $\mathcal{E}_p^{exp}(S_p(K))$ and so with respect to

$\left(|\cdot|_{n,m}\right)_{m\in\mathbb{N}_0}$ as well since $|\cdot|_{n,m} \leq |\cdot|_{p,m}$.

c) i) For $h>0$ small enough we define

$$S_h(\psi)\colon S_p(K) \to \mathbb{C}, \quad S_h(\psi)(y) := \sum_{m\in\mathbb{Z}^2} E(y-mh)\,\psi(mh)\,h^2,$$

where $E(0)\,\psi(mh) = E(0)\,0 := 0$. The first part of the proof is to show that $S_h(\psi)$ converges to $T_E * \psi$ in $\mathcal{E}_p^{exp}\left(S_p(K)\right)$ as h tends to 0.

Set $Q_m := mh + [0,h]^2$ and let $N \subset \overline{S_{k+\varepsilon}(K)}^C$ be compact. By the compactness there exists $\tilde{A}_0 > 0$ such that $|z| \leq \tilde{A}_0$ for all $z \in N$. Now we define $M_N := \{m \in \mathbb{Z}^2 \mid Q_m \cap N \neq \varnothing\}$. Due to this definition we have

$$Q_m \cap \overline{S_{k+\varepsilon}(K)}^C \neq \varnothing \tag{5.15}$$

for $m \in M_N$,

$$\left\{m \in \mathbb{Z}^2 \mid mh \in N\right\} \subset M_N \tag{5.16}$$

and

$$|x_1|,\,|x_2| \leq \left\lceil \frac{\tilde{A}_0}{h} \right\rceil h \leq \left(\frac{\tilde{A}_0}{h} + 1\right) h = \tilde{A}_0 + h \tag{5.17}$$

for $x \in Q_m$ plus

$$\#M_N \leq \left(2\left\lceil \frac{\tilde{A}_0}{h} \right\rceil\right)^2 \leq 4\left(\frac{\tilde{A}_0}{h} + 1\right)^2 \tag{5.18}$$

where $\#M_N$ denotes the number of elements of M_N.

We define $\mathrm{d}_k\colon [0,\varepsilon] \to \mathbb{R}_{\geq 0}$, $\mathrm{d}_k(t) := \mathrm{d}\left(\partial S_{k+\varepsilon}(K), \partial S_{k+t}(K)\right)$. By Remark 3.3(4) we have

$$\mathrm{d}_k(t) = \begin{cases} \frac{1}{k+t} - \frac{1}{k+\varepsilon}, & K \neq \varnothing, \\ \varepsilon - t, & K = \varnothing, \end{cases}$$

so d_k is continuous on $[0,\varepsilon]$ as well as strictly monotonically decreasing.

Let $0 < h < \frac{1}{\sqrt{2}} \mathrm{d}\left(\partial S_{k+\varepsilon}(K), \partial S_k(K)\right)$. Then

$$\mathrm{d}_0(0) = \mathrm{d}\left(\partial S_{k+\varepsilon}(K), \partial S_k(K)\right) > \sqrt{2}h \quad \text{and} \quad \mathrm{d}_k(\varepsilon) = 0$$

and thus there exist $\varepsilon_0, \varepsilon_1 \in]0,\varepsilon[$, $\varepsilon_0 < \varepsilon_1$, such that

$$\sqrt{2}h < \mathrm{d}_k(\varepsilon_1) < \mathrm{d}_k(\varepsilon_0) < \mathrm{d}_k(0)$$

by the intermediate value theorem. Hence for Q_m with $Q_m \cap \overline{S_{k+\varepsilon}(K)}^C \neq \varnothing$ the following is valid

$$Q_m \subset S_{k+\varepsilon_1}(K)^C \subset \overline{S_{k+\varepsilon_0}(K)}^C \subset S_k(K)^C. \tag{5.19}$$

Therefore, we obtain with $\varepsilon_2 := \mathrm{d}\left(\partial S_k(K), \partial S_p(K)\right)$

$$|y - x| > \varepsilon_2 \quad \text{for all } y \in S_p(K),\, x \in Q_m, \tag{5.20}$$

and for $y \in S_p(K)$, $x \in Q_m$, $m \in M_N$, $\beta \in \mathbb{N}_0^2$ and $r := \frac{1}{2}\varepsilon_2$ we get analogously to the proof of

Lemma 5.3b)

$$\left|\partial_y^\beta\left[E(y-x)\right]\right| \underset{(5.15),(5.20)}{\leq} \frac{1}{\pi}\frac{|\beta|!}{r^{|\beta|+1}}e^{(r+p+|x_2|)^2}\max_{|\zeta-(y-x)|=r} e^{-\zeta_1^2}$$
$$\underset{(5.17)}{\leq} \underbrace{\frac{1}{\pi}\frac{|\beta|!}{r^{|\beta|+1}}e^{\left(r+p+\tilde{A}_0+\varepsilon_0\right)^2}}_{=:\tilde{A}_1}\max_{t\in[0,1]} e^{-(2rt+y_1-x_1-r)^2}$$
$$\leq \underbrace{\tilde{A}_1 e^{12r^2}}_{=:\tilde{A}_2(\beta)} e^{-y_1^2+2(3r+|x_1|)|y_1|} \underset{(5.17)}{\leq} \tilde{A}_2(\beta)\, e^{-y_1^2+2\left(3r+\tilde{A}_0+\varepsilon\right)|y_1|}, \tag{5.21}$$

while here $\tilde{A}_2$ does not depend on $|x_2|$.
Let $\psi\in C_0^\infty(N)$ and $m_0\in\mathbb{N}_0$. Then we have

$$\left|\partial_y^\beta S_h(\psi)(y)\right| \underset{(5.16),(5.18)}{=} \sum_{m\in M_N}\partial_y^\beta\left[E(y-mh)\right]\psi(mh)h^2$$
$$\underset{(5.21),\, mh\in Q_m}{\leq} h^2\tilde{A}_2(\beta)\, e^{-y_1^2+2\left(3r+\tilde{A}_0+\varepsilon\right)|y_1|}\sum_{m\in M_N}\underbrace{|\psi(mh)|}_{\leq |||\psi|||_0}$$
$$\underset{(5.18)}{\leq} 4h^2\left(\frac{\tilde{A}_0}{h}+1\right)^2\tilde{A}_2(\beta)\, e^{-y_1^2+2\left(3r+\tilde{A}_0+\varepsilon\right)|y_1|}\,|||\psi|||_0$$
$$= 4\left(\tilde{A}_0+h\right)^2\tilde{A}_2(\beta)\, e^{-y_1^2+2\left(3r+\tilde{A}_0+\varepsilon\right)|y_1|}\,|||\psi|||_0$$

and therefore

$$|S_h(\psi)|_{p,m_0} \leq \underbrace{4\left(\tilde{A}_0+h\right)^2\sup_{|\beta|\leq m_0}\tilde{A}_2(\beta)}_{=:\tilde{A}_3}\,|||\psi|||_0\sup_{y\in S_p(K)} e^{-y_1^2+2\left(3r+\tilde{A}_0+\varepsilon\right)|y_1|}e^{-\frac{1}{p}|y_1|}$$
$$\leq \tilde{A}_3 e^{\left(3r+\tilde{A}_0+\varepsilon-\frac{1}{2p}\right)^2}\,|||\psi|||_0 \tag{5.22}$$

bringing forth $S_h(\psi)\in\mathcal{E}_p^{exp}\left(S_p(K)\right)$. Further, the following equations hold:

$$\left|\partial^\beta\left(S_h(\psi)-E*\psi\right)(y)\right|$$
$$=\Big|\sum_{m\in M_N}\underbrace{\partial_y^\beta\left[E(y-mh)\right]\psi(mh)h^2}_{=\int_{Q_m}\partial_y^\beta[E(y-mh)]\psi(mh)dx} - \underbrace{\int_{\mathbb{C}}\partial_y^\beta\left[E(y-x)\right]\psi(x)\,dx}_{=\sum_{m\in M_N}\int_{Q_m}\partial_y^\beta[E(y-x)]\psi(x)dx}\Big|$$
$$=\left|\sum_{m\in M_N}\int_{Q_m}\left(\partial^\beta E\right)(y-mh)\,\psi(mh)-\left(\partial^\beta E\right)(y-x)\,\psi(x)\,dx\right|$$
$$=\Big|\sum_{m\in M_N}\int_{Q_m}\left[\left(\partial^\beta E\right)(y-mh)-\left(\partial^\beta E\right)(y-x)\right]\psi(mh)$$
$$+\left[\psi(mh)-\psi(x)\right]\left(\partial^\beta E\right)(y-x)\,dx\Big| \tag{5.23}$$

The next steps are similar to the proof of b). By the mean value theorem there exist $x_{0,i},\, x_{1,i} \in [x,mh]_L \subset Q_m$, $i = 1,2$, such that

$$|\psi(mh) - \psi(x)| = \left|\begin{pmatrix} \langle \operatorname{grad}(\psi_1)(x_{0,1}) | mh - x\rangle \\ \langle \operatorname{grad}(\psi_2)(x_{0,2}) | mh - x\rangle \end{pmatrix}\right| \leq 4 |||\psi|||_1 |mh - x| \leq 4\sqrt{2}h |||\psi|||_1 \tag{5.24}$$

and

$$\begin{aligned} \left|(\partial^\beta E)(y - mh) - (\partial^\beta E)(y - x)\right| &= \left|-\begin{pmatrix} \langle \operatorname{grad}(\partial^\beta E_1)(y - x_{1,1}) | mh - x\rangle \\ \langle \operatorname{grad}(\partial^\beta E_2)(y - x_{1,2}) | mh - x\rangle \end{pmatrix}\right| \\ &\leq 2\left(E^{(|\beta|+1)}(y - x_{1,1}) + E^{(|\beta|+1)}(y - x_{1,2})\right) |mh - x| \\ &\underset{(5.21)}{\leq} 4\sqrt{2}h\tilde{A}_2(\beta) e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} \end{aligned} \tag{5.25}$$

analogously to (5.12). Thus by combining (5.23), (5.24) and (5.25), one obtains

$$\begin{aligned} &\left|\partial^\beta (S_h(\psi) - E * \psi)(y)\right| \\ &\leq \sum_{m \in M_N} \int_{Q_m} 4\sqrt{2}h\left(\tilde{A}_2(\beta) e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} |\psi(mh)| + |||\psi|||_1 \left|(\partial^\beta E)(y - x)\right|\right) dx \\ &\underset{(5.21)}{\leq} \sum_{m \in M_N} 4\sqrt{2}h\tilde{A}_2(\beta) e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} (|||\psi|||_0 + |||\psi|||_1) \underbrace{\lambda(Q_m)}_{=h^2} \\ &\underset{(5.18)}{\leq} 16\sqrt{2}\, \underbrace{h^3 \left(\frac{\tilde{A}_0}{h} + 1\right)^2}_{=(\tilde{A}_0^2 + 2\tilde{A}_0 h + h^2)h} \tilde{A}_2(\beta) e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} \underbrace{(|||\psi|||_0 + |||\psi|||_1)}_{\leq 2|||\psi|||_1} \\ &\leq \underbrace{32\sqrt{2}\left(\tilde{A}_0^2 + 2\tilde{A}_0\varepsilon + \varepsilon^2\right)}_{=:\tilde{A}_4} h\tilde{A}_2(\beta) |||\psi|||_1 e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} \end{aligned}$$

and so for $m_0 \in \mathbb{N}_0$

$$\begin{aligned} |S_h(\psi) - E * \psi|_{p,m_0} &\leq \tilde{A}_4 \underbrace{\sup_{|\beta| \leq m_0} \tilde{A}_2(\beta)}_{=:\tilde{A}_5} |||\psi|||_1 \sup_{y \in S_p(K)} e^{-y_1^2 + 2(3r + \tilde{A}_0 + \varepsilon)|y_1|} e^{-\frac{1}{p}|y_1|} h \\ &\leq \tilde{A}_5 |||\psi|||_1 e^{\left(3r + \tilde{A}_0 + \varepsilon - \frac{1}{2p}\right)^2} h \\ &\underset{h \to 0}{\to} 0 \end{aligned} \tag{5.26}$$

proving the convergence of $S_h(\psi)$ to $T_E * \psi$ in $\mathcal{E}_p^{exp}(S_p(K))$ and hence with respect to $\left(|\cdot|_{n,m_0}\right)_{m_0 \in \mathbb{N}_0}$ as well.
ii) The next part of the proof is to show that

$$\lim_{h \to 0} \sum_{m \in M_N} \left(w *_2 \check{E}\right)(mh)\, \psi(mh)\, h^2 = \int_{\mathbb{C}} \left(w *_2 \check{E}\right)(x)\, \psi(x)\, dx.$$

We begin with

$$
\begin{aligned}
&\left|\sum_{m\in M_N}\left(w *_2 \check{E}\right)(mh)\,\psi(mh)\,h^2-\int_{\mathbb{C}}\left(w *_2 \check{E}\right)(x)\,\psi(x)\,dx\right|\\
&=\left|\sum_{m\in M_N}\int_{Q_m}\left(w *_2 \check{E}\right)(mh)\,\psi(mh)-\left(w *_2 \check{E}\right)(x)\,\psi(x)\,dx\right|\\
&=\left|\sum_{m\in M_N}\int_{Q_m}\left[\left(w *_2 \check{E}\right)(mh)-\left(w *_2 \check{E}\right)(x)\right]\psi(mh)\right.\\
&\qquad\left.+\left[\psi(mh)-\psi(x)\right]\left(w *_2 \check{E}\right)(x)\,dx\right|.
\end{aligned}
\tag{5.27}
$$

Again, by the mean value theorem there exist $x_{0,i},\ x_{1,i}\in[x,mh]_L\subset Q_m,\ i=1,2$, such that

$$
|\psi(mh)-\psi(x)|=\left|\begin{pmatrix}\langle \operatorname{grad}(\psi_1)(x_{0,1})|mh-x\rangle\\ \langle \operatorname{grad}(\psi_2)(x_{0,2})|mh-x\rangle\end{pmatrix}\right|\le 4\sqrt{2}h\,|||\psi|||_1 \tag{5.28}
$$

and, taking account of (5.19) and b) $(q=k,\varepsilon=\varepsilon_0)$,

$$
\begin{aligned}
&\left|\left(w *_2 \check{E}\right)(mh)-\left(w *_2 \check{E}\right)(x)\right|\\
&=\left|\begin{pmatrix}\langle \operatorname{grad}\left(\left(w *_2 \check{E}\right)_1\right)(x_{1,1})|mh-x\rangle\\ \langle \operatorname{grad}\left(\left(w *_2 \check{E}\right)_2\right)(x_{1,2})|mh-x\rangle\end{pmatrix}\right|\\
&\le\left(\left|\operatorname{grad}\left(\left(w *_2 \check{E}\right)_1\right)(x_{1,1})\right|+\left|\operatorname{grad}\left(\left(w *_2 \check{E}\right)_2\right)(x_{1,2})\right|\right)\sqrt{2}h\\
&\le\underbrace{\left(p_{N+\overline{D_{d_k(\varepsilon_1)}(0)},0}\left(\operatorname{grad}\left(\left(w *_2 \check{E}\right)_1\right)\right)+p_{N+\overline{D_{d_k(\varepsilon_1)}(0)},0}\left(\operatorname{grad}\left(\left(w *_2 \check{E}\right)_2\right)\right)\right)}_{=:\tilde{A}_6}\sqrt{2}h
\end{aligned}
\tag{5.29}
$$

with the usual semi-norms like in (3.8) where we used $d_k(\varepsilon_1)>\sqrt{2}h$ and $x_{1,i}\in Q_m,\ m\in M_N$, in the last inequality. Due to (5.27), (5.28) and (5.29) we gain

$$
\begin{aligned}
&\left|\sum_{m\in M_N}\left(w *_2 \check{E}\right)(mh)\,\psi(mh)\,h^2-\int_{\mathbb{C}}\left(w *_2 \check{E}\right)(x)\,\psi(x)\,dx\right|\\
&\le\sum_{m\in M_N}\left(\tilde{A}_6\sqrt{2}h\,|||\psi|||_0+4\sqrt{2}h\,|||\psi|||_1\,p_{N+\overline{D_{d_k(\varepsilon_1)}(0)},0}\left(w *_2 \check{E}\right)\right)h^2\\
&\underset{(5.18)}{\le}\left(\tilde{A}_0^2+2\tilde{A}_0\varepsilon+\varepsilon^2\right)\left(\tilde{A}_6\sqrt{2}\,|||\psi|||_0+4\sqrt{2}\,|||\psi|||_1\,p_{N+\overline{D_{d_k(\varepsilon_1)}(0)},0}\left(w *_2 \check{E}\right)\right)h\\
&\underset{h\to 0}{\to}0.
\end{aligned}
$$

iii) Merging i) and ii), we get for $\psi\in C_0^\infty(N)$

$$
\begin{aligned}
\left\langle w *_1 T_{\check{E}},\psi\right\rangle&=\left\langle w,(T_E*\psi)\big|_{S_n(K)}\right\rangle\underset{i)}{=}\lim_{h\to 0}\left\langle w,S_h(\psi)\big|_{S_n(K)}\right\rangle\\
&=\lim_{h\to 0}\left\langle w,\sum_{m\in M_N}E(\cdot-mh)\big|_{S_n(K)}\psi(mh)\,h^2\right\rangle
\end{aligned}
$$

$$\underset{(5.18)}{=} \lim_{h\to 0} \sum_{m\in M_N} \underbrace{\left\langle w, E(\cdot - mh)|_{S_n(K)} \right\rangle}_{=\left(w*_2\check{E}\right)(mh)} \psi(mh)h^2 \underset{ii)}{=} \int_{\mathbb{C}} \left(w *_2 \check{E}\right)(x)\psi(x)\,dx$$
$$= \left\langle T_{w*_2\check{E}}, \psi \right\rangle.$$

d) $w *_\varphi T_{\check{E}}$ is defined by Lemma 5.3d). Because w is continuous, there exist $C_2 > 0$ and $m \in \mathbb{N}_0$ such that

$$\left|\left\langle w *_\varphi T_{\check{E}}, f\right\rangle\right| = \left|\left\langle w, [T_E * (\varphi f)]|_{S_n(K)} \right\rangle\right| \le C_2 |T_E * (\varphi f)|_{n,m}$$
$$\underset{(5.5),t=n,s=j}{\le} C_1 C_2 |f|_{j,m}.$$

□

5.5 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be a compact set and E like in Lemma 5.3. Let $k,p,n \in \mathbb{R}$ with $k > p > n > 1$ and $w \in \left(\pi_{n,p}\left(\mathcal{E}_p^{exp}\left(S_p(K)\right)\right), \left(|\cdot|_{n,m}\right)_{m\in\mathbb{N}_0}\right)'$.*
If $w|_{\pi_{n,k}\left(\mathcal{E}_{k,\overline{\partial}}^{exp}(S_k(K))\right)} = 0$, then $\operatorname{supp}\left(w *_1 T_{\check{E}}\right) \subset \overline{S_n(K)}$, where the support is meant in the distributional sense.

Proof. (i) For all $\psi \in C_0^\infty(\mathbb{C})$ and $m \in \mathbb{N}_0$ we have

$$|\psi|_{p,m} = \sup_{\substack{z\in S_p(K)\\ |\beta|\le m}} \left|\partial^\beta \psi(z)\right| e^{-\frac{1}{p}|\mathrm{Re}(z)|} \le \sup_{\substack{z\in\mathbb{C}\\ |\beta|\le m}} \left|\partial^\beta \psi(z)\right| = \|\!|\psi|\!\|_m < \infty,$$

hence $\psi|_{S_p(K)} \in \mathcal{E}_p^{exp}\left(S_p(K)\right)$. Now we define

$$w_0 \colon C_0^\infty(\mathbb{C}) \to C_0^\infty(\mathbb{C}),\ w_0(\psi) := w\left(\psi|_{S_n(K)}\right).$$

Then we obtain by the assumptions on w that there exist $m \in \mathbb{N}_0$ and $C > 0$ such that

$$|w_0(\psi)| = \left|w\left(\psi|_{S_n(K)}\right)\right| \le C|\psi|_{n,m} \le C|\psi|_{p,m} \le C\|\!|\psi|\!\|_m,$$

and therefore $w_0 \in \mathcal{D}'(\mathbb{C})$ as well as $\operatorname{supp} w_0 \subset \overline{S_n(K)}$.
(ii) Let $\psi \in C_0^\infty(\mathbb{C})$. Then we get

$$\left\langle \overline{\partial}\left(w *_1 T_{\check{E}}\right), \psi \right\rangle \underset{5.4a)}{=} \left\langle w *_1 T_{\check{E}}, -\overline{\partial}\psi \right\rangle = -\left\langle w, \left(T_E * \overline{\partial}\psi\right)|_{S_n(K)} \right\rangle = -\left\langle w, \left(\overline{\partial}T_E * \psi\right)|_{S_n(K)} \right\rangle$$
$$\underset{5.3a)}{=} -\left\langle w, (\delta * \psi)|_{S_n(K)} \right\rangle = -\left\langle w, \psi|_{S_n(K)} \right\rangle \underset{(i)}{=} -\langle w_0, \psi \rangle,$$

thus $\overline{\partial}\left(w *_1 T_{\check{E}}\right) = -w_0$ and so $\overline{\partial}\left(w *_1 T_{\check{E}}\right) = 0$ on $C_0^\infty(\mathbb{C} \setminus \operatorname{supp} w_0)$ due to [18, Theorem 2.2.1, p. 41]. Hence, by virtue of the ellipticity of the $\overline{\partial}$-operator, it exists $u \in \mathcal{O}(\mathbb{C} \setminus \operatorname{supp} w_0)$ such that $T_u = w *_1 T_{\check{E}}$ ([19, Theorem 11.1.1, p. 61]).
Let $\varepsilon > 0$. Then (i) yields to $\operatorname{supp} w_0 \subset \overline{S_n(K)}$ and therefore we get $\overline{S_{k+\varepsilon}(K)}^C \subset (\operatorname{supp} w_0)^C$ and

hence $\mathrm{C}_0^\infty\left(\overline{S_{k+\varepsilon}(K)}^C\right) \subset \mathrm{C}_0^\infty\left((\operatorname{supp} w_0)^C\right)$. It follows by Lemma 5.4c) that

$$T_u = w *_1 T_{\check{E}} = T_{w*_2\check{E}}$$

on $\mathrm{C}_0^\infty\left(\overline{S_{k+\varepsilon}(K)}^C\right)$ implying $u = w *_2 \check{E}$ on $\overline{S_{k+\varepsilon}(K)}^C$ by Lemma 5.4b). This means that we have for $x \in \overline{S_{k+\varepsilon}(K)}^C$ and $\alpha \in \mathbb{N}_0^2$

$$\begin{aligned} u^{(|\alpha|)}(x) &= \left(w *_2 \check{E}\right)^{(|\alpha|)}(x) \underset{(3.2)}{=} i^{-\alpha_2}\partial^\alpha\left(w *_2 \check{E}\right)(x) \underset{(5.9)}{=} i^{-\alpha_2}\left\langle w, \partial_x^\alpha [E(\cdot - x)]\big|_{S_n(K)}\right\rangle \\ &\underset{5.3b)}{=} 0 \end{aligned}$$

by the assumptions on w. Hence $u = 0$ in every component O of $(\operatorname{supp} w_0)^C$ with $O \cap \overline{S_{k+\varepsilon}(K)}^C \neq \varnothing$ by the identity theorem. Denote by O_i, $i \in I$, the components of $(\operatorname{supp} w_0)^C$ and let $I_0 := \left\{i \in I \mid O_i \cap \overline{S_n(K)}^C \neq \varnothing\right\}$. Due to Remark 3.3(3) we get $u = 0$ on

$$\bigcup_{i\in I_0} O_i \supset \left(\bigcup_{i\in I_0} O_i\right) \cap \overline{S_n(K)}^C = \underbrace{\left(\bigcup_{i\in I} O_i\right)}_{=(\operatorname{supp} w_0)^C} \cap \overline{S_n(K)}^C = \overline{S_n(K)}^C.$$

Since $T_u = w *_1 T_{\check{E}}$ on $\mathrm{C}_0^\infty\left((\operatorname{supp} w_0)^C\right)$, this implies $\operatorname{supp}\left(w *_1 T_{\check{E}}\right) \subset \overline{S_n(K)}$. □

Now we are finally able to prove the density theorem.

Proof of Theorem 5.1. Let $\tilde{w} \in \pi_{n,k}\left(\mathcal{E}^{exp}_{k,\overline{\partial}}(S_k(K))\right)^\circ$, the polar set of $\pi_{n,k}\left(\mathcal{E}^{exp}_{k,\overline{\partial}}(S_k(K))\right) \subset \pi_{n,p}\left(\mathcal{E}^{exp}_{p,\overline{\partial}}(S_p(K))\right)$. By the Hahn-Banach theorem there exists $w \in \pi_{n,p}\left(\mathcal{E}^{exp}_p(S_p(K))\right)'$ such that $w|_{\pi_{n,p}\left(\mathcal{E}^{exp}_{p,\overline{\partial}}(S_p(K))\right)} = \tilde{w}$.

Let $f \in \mathcal{E}^{exp}_{p,\overline{\partial}}(S_p(K))$, choose $j \in \mathbb{R}$, $n < j < p$, and φ like in Lemma 5.3d) ($t = n$, $s = j$). By Lemma 5.2 there exists a sequence $(\psi_l)_{l\in\mathbb{N}}$, $\psi_l \in \mathrm{C}_0^\infty(S_p(K))$, such that $(\psi_l)_{l\in\mathbb{N}}$ converges to f with respect to $(|\cdot|_{j,m})_{m\in\mathbb{N}_0}$ and so $\left(\overline{\partial}\psi_l\right)_{l\in\mathbb{N}}$ to $\overline{\partial} f$ as well since

$$\overline{\partial}\colon \mathcal{E}^{exp}_j(S_j(K)) \to \mathcal{E}^{exp}_j(S_j(K))$$

is continuous. Thus we obtain

$$\begin{aligned} \left\langle \tilde{w}, f|_{S_n(K)}\right\rangle &= \left\langle w, f|_{S_n(K)}\right\rangle \underset{n<j}{=} \lim_{l\to\infty}\left\langle w, \psi_l|_{S_n(K)}\right\rangle = \lim_{l\to\infty}\left\langle w, (\delta * \psi_l)|_{S_n(K)}\right\rangle \\ &\underset{5.3a)}{=} \lim_{l\to\infty}\left\langle w, \left(T_E * \overline{\partial}\psi_l\right)\big|_{S_n(K)}\right\rangle = \lim_{l\to\infty}\left\langle w *_1 T_{\check{E}}, \overline{\partial}\psi_l\right\rangle \underset{5.5}{=} \lim_{l\to\infty}\left\langle w *_1 T_{\check{E}}, \varphi\overline{\partial}\psi_l\right\rangle \\ &= \lim_{l\to\infty}\left\langle w, \left(T_E * \varphi\overline{\partial}\psi_l\right)\big|_{S_n(K)}\right\rangle = \lim_{l\to\infty}\left\langle w *_\varphi T_{\check{E}}, \overline{\partial}\psi_l\right\rangle \underset{(5.10)}{=} \left\langle w *_\varphi T_{\check{E}}, \overline{\partial} f\right\rangle \\ &= 0, \end{aligned}$$

so $\tilde{w}=0$ connoting the statement due to the bipolar theorem. □

This theorem implies, amongst others, that the initial spectrum of $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ is in a weak sense reduced (a projective spectrum is called reduced if its projective limit is dense in all spaces of the spectrum [16, 26.1.4, p. 143]).

5.6 Corollary. *Let $K\subset\overline{\mathbb{R}}$ be compact and $n\in\mathbb{N}_{\geq 2}$. The space $\pi_n\left(\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\right)$ is dense in $\pi_{n,2n}\left(\mathcal{O}_{2n}^{exp}\left(S_{2n}(K)\right)\right)$ with respect to $|\cdot|_n$ where*

$$\pi_n\colon\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)\to\mathcal{O}_n^{exp}\left(S_n(K)\right),\ \pi_n(f):=f|_{S_n(K)}.$$

Proof. The restriction mappings are omitted during the proof. Due to Theorem 3.11(4) the space $\mathcal{O}_{2n}^{exp}(S_{2n}(K))$ is included in $\mathcal{E}_{n+1,\overline{\partial}}^{exp}(S_{n+1}(K))$. Let $\varepsilon>0$ and $f_0\in\mathcal{O}_{2n}^{exp}(S_{2n}(K))$. For all $j\in\mathbb{N}$ there exist $f_j\in\mathcal{E}_{n+1+j,\overline{\partial}}^{exp}\left(S_{n+1+j}(K)\right)\subset\mathcal{O}_{n+1+j}^{exp}\left(S_{n+1+j}(K)\right)$ such that

$$\left|f_j-f_{j-1}\right|_{n+j-1}=\left|f_j-f_{j-1}\right|_{n+j-1,0}<\frac{\varepsilon}{2^{j+1}} \tag{5.30}$$

by Theorem 5.1. Therefore, we obtain for every $k\in\mathbb{N}$

$$\begin{aligned}|f_k-f_0|_n&=\left|\sum_{j=1}^{k}f_j-f_{j-1}\right|_n\leq\sum_{j=1}^{k}\left|f_j-f_{j-1}\right|_n\leq\sum_{j=1}^{k}\left|f_j-f_{j-1}\right|_{n+j-1}\\&\underset{(5.30)}{\leq}\sum_{j=1}^{k}\frac{\varepsilon}{2^{j+1}}=\frac{\varepsilon}{2}\left(1-\frac{1}{2^k}\right)<\frac{\varepsilon}{2}.\end{aligned} \tag{5.31}$$

Now let $\varepsilon_0>0$ and $l\in\mathbb{N}_{\geq 2}$. Choose $l_0\in\mathbb{N}$, $l_0\geq l$, such that $\frac{\varepsilon}{2^{l_0+1}}<\varepsilon_0$. Similarly we get for all $p\geq k\geq l_0$

$$\begin{aligned}\left|f_p-f_k\right|_l&\leq\left|f_p-f_k\right|_{l_0}=\left|\sum_{j=k+1}^{p}f_j-f_{j-1}\right|_{l_0}\leq\sum_{j=k+1}^{p}\left|f_j-f_{j-1}\right|_{l_0}\\&\underset{\substack{l_0\leq k\leq j-1\\<n+j-1}}{\leq}\sum_{j=k+1}^{p}\left|f_j-f_{j-1}\right|_{n+j-1}\underset{(5.30)}{\leq}\sum_{j=k+1}^{p}\frac{\varepsilon}{2^{j+1}}=\frac{\varepsilon}{2}\left(\frac{1}{2^k}-\frac{1}{2^p}\right)\\&<\frac{\varepsilon}{2^{k+1}}\leq\frac{\varepsilon}{2^{l_0+1}}<\varepsilon_0.\end{aligned}$$

Hence $(f_k)_{k\geq n_0}$ is a Cauchy sequence in $\mathcal{O}_{n+1+n_0}^{exp}\left(S_{n+1+n_0}(K)\right)$ for all $n_0\in\mathbb{N}_0$ and, since these spaces are complete by Theorem 3.6(1), it has a limit $F_{n_0}\in\mathcal{O}_{n+1+n_0}^{exp}\left(S_{n+1+n_0}(K)\right)$. These limits coincide on their common domain because for every $n_1,n_2\in\mathbb{N}_0$, $n_1<n_2$, and $\varepsilon_1>0$ there exists $N\in\mathbb{N}$ such that for all $k\geq N$

$$\begin{aligned}|F_{n_1}-F_{n_2}|_{n+1+n_1}&\leq|F_{n_1}-f_k|_{n+1+n_1}+|f_k-F_{n_2}|_{n+1+n_1}\leq|F_{n_1}-f_k|_{n+1+n_1}+|f_k-F_{n_2}|_{n+1+n_2}\\&<\frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{2}=\varepsilon_1.\end{aligned}$$

So the limit function f, defined by $f:=F_{n_0}$ on $S_{n+1+n_0}(K)$ for all $n_0\in\mathbb{N}_0$, is well defined and we have $f\in\operatorname{lim\,proj}_{n_0\in\mathbb{N}_0}\mathcal{O}_{n+1+n_0}^{exp}\left(S_{n+1+n_0}(K)\right)=\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus K\right)$ by [16, 2.5. Satz, p. 37]. By definition of f there exists $N \in \mathbb{N}$ such that for every $k \geq N$

$$|f - f_0|_n \leq |f - f_k|_n + |f_k - f_0|_n \underset{n<n+1}{<} \frac{\varepsilon}{2} + |f_k - f_0|_n \underset{(5.31)}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

proving the statement. □

Since analogons of Theorem 5.1 and Corollary 5.6 for other growth conditions are of some interest as well, we will make a short digression. The following definition of a weight function is given by Langenbruch in [41, Definition 2.1, p. 225].

5.7 Definition. A continuous function $\nu\colon \mathbb{C} \to [0,\infty[$ is called weight function if it fulfills the following conditions:

a) For all $z \in \mathbb{C}$ one has $\nu(z) = \nu(|\mathrm{Re}(z)|)$.

b) $\nu\colon [0,\infty[\to [0,\infty[$ is strictly monotonic increasing.

c) One has

$$\ln(1+|x|) = o(\nu(x)).$$

d) There are constants $\Gamma > 1$ and $C > 0$ such that for all $x \geq 0$

$$\nu(x+1) \leq \Gamma\nu(x) + C.$$

5.8 Definition. Let $K \subset \overline{\mathbb{R}}$ be compact and $\tau\colon \mathbb{R}_{>1} \to \mathbb{R}_{>0}$ or $\tau\colon \mathbb{R}_{>1} \to \mathbb{R}_{<0}$ strictly monotonic increasing.

a) For $n \in \mathbb{R}$, $n > 1$, we define the space

$$\mathcal{E}_{\nu,\tau(n)}(S_n(K)) := \{f \in C^\infty(S_n(K)) \mid \forall m \in \mathbb{N}_0 : |f|_{\nu,\tau,n,m} < \infty\},$$

where

$$|f|_{\nu,\tau,n,m} := \sup_{\substack{z \in S_n(K)\\ \alpha \in \mathbb{N}_0^2, |\alpha| \leq m}} |\partial^\alpha f(z)| e^{\tau(n)\nu(z)},$$

and the space

$$\mathcal{E}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right) := \lim_{\substack{\longleftarrow\\ n \in \mathbb{N}_{\geq 2}}} \mathcal{E}_{\nu,\tau(n)}(S_n(K)).$$

b) For $n \in \mathbb{R}$, $n > 1$, we define the space

$$\mathcal{O}_{\nu,\tau(n)}(S_n(K)) := \{f \in \mathcal{O}(S_n(K)) \mid |f|_{\nu,\tau,n} < \infty\},$$

where

$$|f|_{\nu,\tau,n} := \sup_{z \in S_n(K)} |f(z)| e^{\tau(n)\nu(z)},$$

and the space

$$\mathcal{O}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right) := \lim_{\substack{\longleftarrow\\ n \in \mathbb{N}_{\geq 2}}} \mathcal{O}_{\nu,\tau(n)}(S_n(K)).$$

In both cases the spectral mappings $\pi_{n,k}, n \leq k$, are again the restrictions.
Further, we define $\mathcal{E}_{\nu,\tau(n),\overline{\partial}}(S_n(K)) := \left\{ f \in \mathcal{E}_{\nu,\tau(n)}(S_n(K)) \mid \overline{\partial} f = 0 \right\}$.

In particular, $\nu\colon \mathbb{C} \to [0,\infty[$, $\nu(z) := |\mathrm{Re}(z)|$, satisfies the conditions of Definition 5.7 and $\tau\colon \mathbb{R}_{>1} \to \mathbb{R}, \tau(n) := -1/n$, is strictly monotonic increasing, so we have

$$\mathcal{E}_{\nu,\tau(n)}(S_n(K)) = \mathcal{E}_n^{exp}(S_n(K)) \quad \text{and} \quad \mathcal{E}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right) = \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right),$$
$$\mathcal{O}_{\nu,\tau(n)}(S_n(K)) = \mathcal{O}_n^{exp}(S_n(K)) \quad \text{and} \quad \mathcal{O}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right) = \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right).$$

Replace in Theorem 3.6(1), the Lemmas 5.2-5.5, Theorem 5.1 and Corollary 5.6 the spaces $\mathcal{E}_n^{exp}$ by $\mathcal{E}_{\nu,\tau(n)}$, $\mathcal{E}_{n,\overline{\partial}}^{exp}$ by $\mathcal{E}_{\nu,\tau(n),\overline{\partial}}$, $\mathcal{O}_n^{exp}$ by $\mathcal{O}_{\nu,\tau(n)}$ and $\mathcal{O}^{exp}$ by $\mathcal{O}_{\nu,\tau}$. Then we have the following observations.

5.9 Remark. Let ν be a weight function.

1. Theorem 3.6(1) is valid for $E = \mathbb{C}$ (the other parts of the theorem as well). Go on like in the proof of Theorem 3.6(1) and for $\tau > 0$ replace (3.9) by

$$\left|\partial^\beta f_l(z) - \partial^\beta f(z)\right| e^{\tau(n)\nu(z)} < \frac{\varepsilon}{2}.$$

2. Lemma 5.2 is valid due to Definition 5.7(a), (b) and (c), implying that $\nu\colon [0,\infty[\to [0,\infty[$ is bijective and strictly monotonic increasing, with

$$Q_0 := \overline{S_t(K)} \cap \left\{ z \in \mathbb{C} \mid \nu(|\mathrm{Re}(z)|) \leq \max\left(0, \frac{\ln \varepsilon}{\tau(j) - \tau(p)}\right)\right\},$$
$$Q_1 := S_s(K) \cap \left\{ z \in \mathbb{C} \mid \nu(|\mathrm{Re}(z)|) < \max\left(0, \frac{\ln \varepsilon}{\tau(j) - \tau(p)}\right) + 1 \right\}.$$

3. Looking at (5.6), we get that Lemma 5.3b) is valid for $\tau < 0$ since

$$\begin{aligned} |\partial_x^\alpha [E(\cdot - x)]|_{\nu,\tau,q,m} &\leq \sup_{|\beta| \leq m} A_1(\beta) \sup_{z \in S_q(K)} e^{-z_1^2 + 2(3r + |x_1|)|z_1|} \underbrace{e^{\tau(q)\nu(|z_1|)}}_{\leq 1} \\ &\leq \sup_{|\beta| \leq m} A_1(\beta) e^{(3r+|x_1|)^2} < \infty. \end{aligned}$$

For $\tau > 0$ it is valid if there exist $D_1, D_2 > 0$ and $0 < a < 2$ such that $\nu(x) \leq D_1 x^a + D_2$ for all $x \geq 0$. Then we have

$$\begin{aligned} |\partial_x^\alpha [E(\cdot - x)]|_{\nu,\tau,q,m} &\leq \sup_{|\beta| \leq m} A_1(\beta) \sup_{z \in S_q(K)} e^{-z_1^2 + 2(3r + |x_1|)|z_1|} \underbrace{e^{\tau(q)\nu(|z_1|)}}_{\leq e^{\tau(q)\left(D_1|z_1|^a + D_2\right)}} \\ &\leq \underbrace{\sup_{|\beta| \leq m} A_1(\beta) e^{\tau(q)D_2}}_{=:A_2} \sup_{z_1 \in \mathbb{R}} e^{-z_1^2 + 2(3r+|x_1|)|z_1| + \tau(q)D_1|z_1|^a}. \end{aligned}$$

As

$$\frac{z_1^2}{2} > 2\max\left(\tau(q)D_1, 2(3r + |x_1|)\right)|z_1|^{\max(1,a)}$$

for $|z_1| > [4\max(\tau(q)D_1, 2(3r+|x_1|))]^{\frac{1}{2-\max(1,a)}} =: A_3$, we get

$$
\begin{aligned}
&|\partial_x^\alpha [E(\cdot - x)]|_{\nu,\tau,q,m} \\
&\le A_2\Big(\underbrace{\sup_{|z_1|\le \max(A_3,1)} e^{-z_1^2+2(3r+|x_1|)|z_1|+\tau(q)D_1|z_1|^a}}_{=:A_4} + \sup_{|z_1|>\max(A_3,1)} \underbrace{e^{-z_1^2+2(3r+|x_1|)|z_1|+\tau(q)D_1|z_1|^a}}_{\le e^{-z_1^2+z_1^2/2}=e^{-z_1^2/2}}\Big) \\
&\le A_2\big(A_4+1\big) < \infty.
\end{aligned}
$$

4. Looking at (5.7), we get that Lemma 5.3c) is valid for $\tau < 0$ since

$$
|T_E * \psi|_{\nu,\tau,q,m} \le B_1 \sup_{z\in S_q(K)} \underbrace{e^{\tau(q)\nu(|z_1|)}}_{\le 1} |||\psi|||_m \le B_1 |||\psi|||_m .
$$

For $\tau > 0$ modify the inequalities above (5.7) in the following manner

$$
|\partial^\alpha (T_E * \psi)(x)| \le \sup_{z\in N} e^{\lambda\tau(q)\nu(z)} |||\psi|||_{|\alpha|} \int_{x-N} |E(y)| e^{-\lambda\tau(q)\nu(x-y)} dy
$$

where $\lambda > 0$ has to be chosen. Then one sees that the following integrals need to be estimated:

$$
\int_0^{2\pi}\int_0^{\varepsilon} e^{-r^2\cos(2\mu)} e^{-\lambda\tau(q)\nu(|x_1+r\cos(\mu)|)} dr d\mu \quad \text{and} \quad \int_{\mathbb{R}} e^{-y_1^2+2|x_1y_1|-x_1^2-\lambda\tau(q)\nu(|y_1|)} dy_1
$$

Taking a look at (5.8), these integrals must also be estimated for checking the validity of Lemma 5.3d)ii), there for $q = s$ and $\lambda = 1$. So let $\lambda = 1$. If there exist $D_1, D_2 > 0$ and $0 < a < 2$ such that $\nu(x) \le D_1 x^a + D_2$ for all $x \ge 0$, then

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} e^{-y_1^2+2|x_1y_1|-x_1^2-\tau(q)\nu(|y_1|)} e^{\tau(q)\nu(|x_1|)} dy_1 \\
&= \int_0^{\frac{|x_1|}{2}} e^{-(y_1-|x_1|)^2-\tau(q)\nu(|y_1|)} e^{\tau(q)\nu(|x_1|)} dy_1 + \int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2-\tau(q)\nu(|y_1|)} e^{\tau(q)\nu(|x_1|)} dy_1
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{\frac{|x_1|}{2}} e^{-(y_1-|x_1|)^2-\tau(q)\nu(|y_1|)} e^{\tau(q)\nu(|x_1|)} dy_1 &\le \begin{cases} \int_0^{\frac{|x_1|}{2}} e^{-\frac{|x_1|^2}{4}-\tau(q)\nu\left(\frac{|x_1|}{2}\right)} dy_1, & \tau(q) < 0, \\ \int_0^{\frac{|x_1|}{2}} e^{-\frac{|x_1|^2}{4}+\tau(q)\nu(|x_1|)} dy_1, & \tau(q) > 0, \end{cases} \\
&\le \begin{cases} \frac{|x_1|}{2} e^{-\frac{|x_1|^2}{4}-\tau(q)\left(D_1\frac{|x_1|^a}{2^a}+D_2\right)}, & \tau(q) < 0, \\ \frac{|x_1|}{2} e^{-\frac{|x_1|^2}{4}+\tau(q)(D_1|x_1|^a+D_2)}, & \tau(q) > 0. \end{cases}
\end{aligned}
$$

As

$$
\frac{|x_1|^2}{8} \ge \begin{cases} -\tau(q)D_1\frac{|x_1|^a}{2^a}, & \tau(q) < 0, \\ \tau(q)D_1|x_1|^a, & \tau(q) > 0, \end{cases}
$$

for

$$|x_1| \geq D_3 := \begin{cases} \left(-8\tau(q)D_1\frac{1}{2^a}\right)^{\frac{1}{2-a}}, & \tau(q)<0, \\ (8\tau(q)D_1)^{\frac{1}{2-a}}, & \tau(q)>0, \end{cases}$$

we obtain

$$\begin{aligned}
&\int_0^{\frac{|x_1|}{2}} e^{-(y_1-|x_1|)^2-\tau(q)\nu(|y_1|)}e^{\tau(q)\nu(|x_1|)}dy_1 \\
&\leq \begin{cases} e^{-\tau(q)D_2}\left(\sup_{|x_1|\leq D_3}\frac{|x_1|}{2}e^{-\frac{|x_1|^2}{4}-\tau(q)D_1\frac{|x_1|^a}{2^a}}+\sup_{|x_1|\geq D_3}\frac{|x_1|}{2}e^{-\frac{|x_1|^2}{8}}\right), & \tau(q)<0, \\ e^{\tau(q)D_2}\left(\sup_{|x_1|\leq D_3}\frac{|x_1|}{2}e^{-\frac{|x_1|^2}{4}+\tau(q)D_1|x_1|^a}+\sup_{|x_1|\geq D_3}\frac{|x_1|}{2}e^{-\frac{|x_1|^2}{8}}\right), & \tau(q)>0, \end{cases} \\
&\leq \begin{cases} e^{-\tau(q)D_2}\left(\frac{D_3}{2}e^{-\tau(q)D_1\frac{D_3^a}{2^a}}+\frac{4}{D_3}\right), & \tau(q)<0, \\ e^{\tau(q)D_2}\left(\frac{D_3}{2}e^{\tau(q)D_1D_3^a}+\frac{4}{D_3}\right), & \tau(q)>0, \end{cases} \\
&=: D_4 = D_4(\operatorname{sign}(\tau)) < \infty, \qquad (5.32)
\end{aligned}$$

where D_4 does not depend on x_1. Furthermore, if there exist $C_1, C_2 > 0$ and $0 < b < 2$ such that $|\nu(y)-\nu(x)| \leq C_1|y-x|^b + C_2$ for all $x,\, y \geq 0$, we get

$$\begin{aligned}
&\int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2-\tau(q)\nu(|y_1|)}e^{\tau(q)\nu(|x_1|)}dy_1 \\
&= \begin{cases} \int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2+|\tau(q)|(\nu(|y_1|)-\nu(|x_1|))}dy_1, & \tau(q)<0, \\ \int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2+|\tau(q)|(\nu(|x_1|)-\nu(|y_1|))}dy_1, & \tau(q)>0, \end{cases} \\
&\leq \int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2+|\tau(q)||\nu(|y_1|)-\nu(|x_1|)|}dy_1 \\
&\leq \int_{\frac{|x_1|}{2}}^{\infty} e^{-(y_1-|x_1|)^2+|\tau(q)|\left(C_1\left|\overbrace{|y_1|}^{=y_1}-|x_1|\right|^b+C_2\right)}dy_1 \\
&= e^{|\tau(q)|C_2}\int_{-\frac{|x_1|}{2}}^{\infty} e^{-\zeta^2+|\tau(q)|C_1|\zeta|^b}d\zeta \\
&\leq \underbrace{2e^{|\tau(q)|C_2}}_{:=C_3}\int_0^{\infty} e^{-\zeta^2+|\tau(q)|C_1\zeta^b}d\zeta \\
&\leq C_3\left(\left(\max_{0\leq\zeta\leq\left(2|\tau(q)|C_1\right)^{\frac{1}{2-b}}} e^{-\zeta^2+|\tau(q)|C_1\zeta^b}\right)+\int_{\left(2|\tau(q)|C_1\right)^{\frac{1}{2-b}}}^{\infty} e^{-\frac{\zeta^2}{2}}d\zeta\right) \\
&\leq C_3\left(e^{|\tau(q)|C_1\left(2|\tau(q)|C_1\right)^{\frac{b}{2-b}}}+\underbrace{\int_0^{\infty} e^{-\frac{\zeta^2}{2}}d\zeta}_{=\sqrt{\pi/2}}\right) =: C_4 < \infty, \qquad (5.33)
\end{aligned}$$

where C_4 does not depend on x_1. Under the conditions made on ν, (5.32) and (5.33) yield to

$$\int_{\mathbb{R}} e^{-y_1^2+2|x_1y_1|-x_1^2-\tau(q)\nu(|y_1|)}dy_1 \leq 2(D_4+C_4)e^{-\tau(q)\nu(|x_1|)}. \qquad (5.34)$$

Now let us turn to the still pending integral and choose $\varepsilon = 1$. If ν satisfies condition d) of Definition 5.7 with $\Gamma = 1$, we gain for $\tau(q) < 0$

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 e^{-r^2\cos(2\mu)}e^{-\tau(q)\nu(|x_1+r\cos(\mu)|)}drd\mu \\
&\leq 2\pi e\int_0^1 e^{-\tau(q)\nu(|x_1|+r)}dr \\
&\leq 2\pi e e^{-\tau(q)\nu(|x_1|+1)} \\
&\leq \underbrace{2\pi e^{-\tau(q)C+1}}_{=:C_5} e^{-\tau(q)\nu(|x_1|)}
\end{aligned} \tag{5.35}$$

and for $\tau(q) > 0$

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 e^{-r^2\cos(2\mu)}e^{-\tau(q)\nu(|x_1+r\cos(\mu)|)}drd\mu \\
&\leq 2\pi e\int_0^1 e^{-\tau(q)\nu(||x_1|-r|)}dr \\
&\leq \begin{cases} 2\pi e e^{-\tau(q)\nu(|x_1|-1)}, & |x_1| \geq 2, \\ 2\pi e, & |x_1| < 2, \end{cases} \\
&\leq \begin{cases} \underbrace{2\pi e^{\tau(q)C+1}}_{=:C_6} e^{-\tau(q)\nu(|x_1|)}, & |x_1| \geq 2, \\ 2\pi e, & |x_1| < 2. \end{cases}
\end{aligned} \tag{5.36}$$

Thus, under the conditions made on ν, looking at the inequalities above (5.7), we get that Lemma 5.3c) is valid for $\tau > 0$ by (5.34) and (5.36) because

$$\begin{aligned}
&|T_E * \psi|_{\nu,\tau,q,m} \\
&\leq \frac{1}{\pi}\underbrace{\sup_{z\in N} e^{\tau(q)\nu(z)}}_{\leq e^{\tau(q)\nu(B_0)}}\left(2\pi e \underbrace{\sup_{|x_1|\leq 2} e^{\tau(q)\nu(|x_1|)}}_{\leq e^{\tau(q)\nu(2)}} + C_6 + 4B_0\left(e^{q^2+2qB_0+B_0^2}\right)(C_4+D_4)\right)|||\psi|||_m
\end{aligned}$$

and analogously, looking at (5.8), we get that Lemma 5.3d)ii) is valid for $\tau > 0$ by (5.34) and (5.36) since

$$\begin{aligned}
|T_E * (\varphi f)|_{\nu,\tau,t,m} \leq \frac{1}{\pi}C_0\Bigg(& 2\pi e\sup_{|x_1|\leq 2} e^{\tau(t)\nu(|x_1|)} + C_6 \underbrace{\sup_{|x_1|>2} e^{(-\tau(s)+\tau(t))\nu(|x_1|)}}_{\leq 1,\, t\leq s} \\
&+ 4j\left(e^{t^2+2jt+j^2}\right)(C_4+D_4)\underbrace{\sup_{x_1\in\mathbb{R}} e^{(-\tau(s)+\tau(t))\nu(|x_1|)}}_{\leq 1,\, t\leq s}\Bigg)|f|_{\nu,\tau,s,j,m}
\end{aligned}$$

as well as for $\tau < 0$ by (5.35) because

$$|T_E * (\varphi f)|_{\nu,\tau,t,m} \leq \frac{1}{\pi} C_0 \Bigg(C_5 \underbrace{\sup_{x_1 \in \mathbb{R}} e^{(-\tau(s)+\tau(t))\nu(|x_1|)}}_{\leq 1,\, t \leq s} + 4j \left(e^{t^2+2jt+j^2} \right) (C_4 + D_4) \underbrace{\sup_{x_1 \in \mathbb{R}} e^{(-\tau(s)+\tau(t))\nu(|x_1|)}}_{\leq 1,\, t \leq s} \Bigg) |f|_{\nu,\tau,s,j,m}$$

where $|f|_{\nu,\tau,s,j,m}$ is defined analogously to the definition given in Lemma 5.3d)ii).

5. Taking a look at (5.14), (5.22) and (5.26), we get that Lemma 5.4 is valid, under the conditions on ν stated in 4), with the same arguments like in 3), and therefore Lemma 5.5, at which we have in part (i) for $\tau > 0$ the inequality

$$|\psi|_{p,m} \leq \sup_{z \in N} e^{\tau(p)\nu(z)} |||\psi|||_m$$

for $\psi \in C_0^\infty(N)$, $N \subset \mathbb{C}$ compact, too.

So by the remark above we get the more general versions of Theorem 5.1 and Corollary 5.6:

5.10 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact, $k, p, n \in \mathbb{R}$ with $k > p > n > 1$ and ν a weight function which satisfies*

(1) condition d) of Definition 5.7 with $\Gamma = 1$.

(2) There exist $C_1, C_2 > 0$ and $0 < a, b < 2$ such that

$$\nu(x) \leq C_1 x^a + C_2 \quad \textit{and} \quad |\nu(y) - \nu(x)| \leq C_1 |y - x|^b + C_2$$

for all $x, y \geq 0$.

Then $\pi_{n,k}\left(\mathcal{E}_{\nu,\tau(k),\overline{\partial}}(S_k(K))\right)$ is dense in $\pi_{n,p}\left(\mathcal{E}_{\nu,\tau(p),\overline{\partial}}(S_p(K))\right)$ with respect to $\left(|\cdot|_{\nu,\tau,n,m}\right)_{m \in \mathbb{N}_0}$.

5.11 Corollary. *Let $K \subset \overline{\mathbb{R}}$ be compact and ν like in Theorem 5.10. The space $\pi_n\left(\mathcal{O}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right)\right)$ is dense in $\pi_{n,2n}\left(\mathcal{O}_{\nu,\tau(2n)}(S_{2n}(K))\right)$ with respect to $|\cdot|_{\nu,\tau,n}$ for every $n \in \mathbb{N}_{\geq 2}$ where*

$$\pi_n \colon \mathcal{O}_{\nu,\tau}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{O}_{\nu,\tau(n)}(S_n(K)),\ \pi_n(f) := f|_{S_n(K)}.$$

5.12 Example. For all $0 < \gamma \leq 1$ the function ν, defined by $\nu(x) := x^\gamma$, $x \geq 0$, is a weight function which satisfies the conditions of Theorem 5.10.

Proof. It is a weight function by [41, Example 5.2, p. 238]. It satisfies the conditions of Theorem 5.10 by the mean value theorem with $C := 2$ in (1) and $C_1 := 1$, $a := b := 1$ and $C_2 := 2$ in (2). □

An idea to extend the results to other weight functions would be to use other fundamental solutions like $E(z) := \frac{e^{-z^k}}{\pi z}$, k even, or $E(z) := \frac{e^{-e^z+1}}{\pi z}$, $z \in \mathbb{C} \setminus \{0\}$.

Now we return to the initial problem, namely, to prove that $\mathbb{C}$ is admissible. For applying Hörmander's solution of the weighted $\overline{\partial}$-problem (see [20, Chap. 4]), it is appropriate to consider L^2-(semi-)norms.
Let P be a polynomial in d real variables with complex coefficients, i.e. there are $n \in \mathbb{N}_0$, $c_\alpha \in \mathbb{C}$, $|\alpha| \le n$, such that

$$P(\zeta) = \sum_{\substack{\alpha \in \mathbb{N}_0^d, \\ |\alpha| \le n}} c_\alpha \zeta^\alpha$$

for all $\zeta \in \mathbb{R}^d$, where $\zeta^\alpha := \zeta_1^{\alpha_1} \cdots \zeta_d^{\alpha_d}$. Further, we set $(-i\partial)^\alpha := (-i)^{|\alpha|} \partial^\alpha$ and $P(D) := P(-i\partial)$.

5.13 Lemma. *Let $V \subset \mathbb{R}^d$ be open and $P(D)$ be a hypoelliptic operator. Let $\{K_n \mid n \in \mathbb{N}\}$ be a compact exhaustion of V. Then*

$$\mathrm{id}\colon \mathrm{C}^\infty(V) \to F(V) := \left\{ f \in L^2_{loc}(V) \mid \forall \alpha \in \mathbb{N}_0^d : \partial^\alpha P(D) f \in L^2_{loc}(V) \right\}$$

is a topological isomorphism where the first space is equipped with the system of semi-norms $\{p_{K_n,m} \mid n \in \mathbb{N}, m \in \mathbb{N}_0\}$ defined by

$$p_{M,m}(f) := \sup_{\substack{x \in M \\ \alpha \in \mathbb{N}_0^d, |\alpha| \le m}} |\partial^\alpha f(x)|, \ M \subset V \text{ compact}, \ m \in \mathbb{N}_0, \tag{5.37}$$

and the latter with the system

$$\left\{ \|\cdot\|_{L^2(K_n)} + s_{K_n,m} \mid n \in \mathbb{N}, \ m \in \mathbb{N}_0 \right\} \tag{5.38}$$

defined for $f = [F]$ by

$$\|f\|_{L^2(M)} := \|F\|_{\mathcal{L}^2(M)} := \left(\int_M |F|^2 \, d\lambda \right)^{\frac{1}{2}} \quad \text{and} \quad s_{M,m}(f) := \sup_{\alpha \in \mathbb{N}_0^d, |\alpha| \le m} \|\partial^\alpha P(D) f\|_{L^2(M)}$$

for $M \subset V$ compact and $m \in \mathbb{N}_0$.

Proof. (i) First let us remark the following: $\mathrm{id}\colon \mathrm{C}^\infty(V) \to F(V)$ means the mapping $f \mapsto [f]$. The derivatives in the definition of $F(V)$ are considered in the distributional sense and $\partial^\alpha P(D) f \in L^2_{loc}(V)$ means that there exists $g \in L^2_{loc}(V)$ such that $\partial^\alpha P(D) T_f = T_g$. The definition of the semi-norm $\|\cdot\|_{L^2(M)}$ does not depend on the chosen representative. As usual there will be made no strict difference between an element of $L^2_{loc}(V)$ and its representatives resp. the corresponding regular distribution, if not necessary.

(ii) $\mathrm{C}^\infty(V)$, equipped with the system of semi-norms (5.37), is a Fréchet space by [25, 2.10 G, p. 51, 3.6.10 Proposition, p.73]. The space $F(V)$, equipped with the system of semi-norms (5.38), is locally convex. Let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in $F(V)$. By definition of $F(V)$ we get that for all $\beta \in \mathbb{N}_0^d$ there exists a sequence $(g_{k,\beta})_{k \in \mathbb{N}}$, $g_{k,\beta} \in L^2_{loc}(V)$, such that $\partial^\beta P(D) T_{f_k} = T_{g_{k,\beta}}$. Therefore (see (5.38)), $(f_k)_{k \in \mathbb{N}}$ and $(g_{k,\beta})_{k \in \mathbb{N}}$, $\beta \in \mathbb{N}_0^d$, are especially Cauchy sequences in $\left(L^2_{loc}(V), \left(\|\cdot\|_{L^2(K_n)} \right)_{n \in \mathbb{N}} \right)$, which is a Fréchet space (use for example [16, 5.17 Lemma, p. 36]), so they have a limit f resp. g_β in this space. Since $(f_k)_{k \in \mathbb{N}}$

converges to $f \in L^2_{loc}(V)$, it follows that $\left(T_{f_k}\right)_{k\in\mathbb{N}}$ converges to T_f in $\mathcal{D}'_\sigma(V)$. Hence we get

$$\partial^\beta P(D)\, T_f \underset{k\to\infty}{\leftarrow} \partial^\beta P(D)\, T_{f_k} = T_{g_{k,\beta}} \underset{k\to\infty}{\rightarrow} T_{g_\beta}$$

in $\mathcal{D}'_\sigma(V)$ implying $f \in F(V)$ and the convergence of $(f_k)_{k\in\mathbb{N}}$ to f in $F(V)$ with respect to the semi-norms (5.38) as well. Thus this space is complete and so a Fréchet space.

(iii) id is obviously linear and injective. It is continuous since for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we have

$$\|f\|^2_{L^2(K_n)} \leq \lambda(K_n)\, p_{n,0}(f)^2$$

and there exists $C > 0$, depending on the coefficients and the number of summands of $P(D)$, such that

$$s_{n,m}(f)^2 \leq C\lambda(K_n)\, p_{n,\deg P+m}(f)^2$$

for all $f \in \mathrm{C}^\infty(V)$ where λ denotes the Lebesgue measure.

(iv) The next step is to prove that id is surjective. Let $f \in F(V)$. Then we have $P(D)f \in \mathrm{W}^\infty_{loc}(V)$ where

$$\mathrm{W}^\infty_{loc}(V) := \left\{ f \in L^2_{loc}(V) \mid \forall\, \alpha \in \mathbb{N}_0^d : \partial^\alpha f \in L^2_{loc}(V) \right\}$$

and so $P(D)f \in \mathrm{C}^\infty(V)$ by the Sobolev embedding theorem [18, Theorem 4.5.13, p. 123]. To be precise, this means that the regular distribution $P(D)f$ has a representative in $\mathrm{C}^\infty(V)$. Due to the hypoellipticity of $P(D)$ we obtain $f \in \mathrm{C}^\infty(V)$, more precisely, that f has a representative in $\mathrm{C}^\infty(V)$, so id is surjective.

(v) The statement is proven by (ii)-(iv) and the open mapping theorem.

□

5.14 Corollary. *Let $0 < r_0 < r_1 < r_2$ and $P(D)$ be a hypoelliptic differential operator. Then we have:*

$\forall\, m \in \mathbb{N}_0\ \exists\, p \in \mathbb{N}_0,\ C > 0\ \forall\, \alpha \in \mathbb{N}_0^d,\ |\alpha| \leq m :$

$$p_{Q_{r_0}(0),0}(\partial^\alpha f) \leq C\left(\|f\|_{\mathcal{L}^2(Q_{r_1}(0))} + \sup_{\beta\in\mathbb{N}_0^d, |\beta|\leq p} \left\|\partial^\beta P(D) f\right\|_{\mathcal{L}^2(Q_{r_1}(0))} \right)$$

for all $f \in \mathrm{C}^\infty\left(\mathring{Q}_{r_2}(0)\right)$ where $Q_r(0) := [-r,r]^d$, $r > 0$.

Proof. Let $V := \mathring{Q}_{r_1}(0)$. Then the sets $K_n := Q_{r_1 - \frac{1}{n+1/r_1}}(0)$, $n \in \mathbb{N}$, form a compact exhaustion of V and there exists $n_0 = n_0(r_0, r_1) \in \mathbb{N}$ such that $Q_{r_0}(0) \subset K_{n_0}$. Since $\mathrm{id}^{-1}: F(V) \to \mathrm{C}^\infty(V)$ is continuous by Lemma 5.13, there are $N \in \mathbb{N}$, $p \in \mathbb{N}_0$ and $C > 0$ such that

$$\begin{aligned} p_{Q_{r_0}(0),0}(\partial^\alpha f) \le p_{K_{n_0},m}(f) = p_{K_{n_0},m}\left(\mathrm{id}^{-1}([f])\right) &\le C\left(\|[f]\|_{L^2(K_N)} + s_{K_N,p}([f])\right) \\ &= C\left(\|f\|_{\mathcal{L}^2(K_N)} + \sup_{\beta\in\mathbb{N}_0^d,|\beta|\le p} \left\|\partial^\beta P(D) f\right\|_{\mathcal{L}^2(K_N)}\right) \\ &\le C\left(\|f\|_{\mathcal{L}^2(Q_{r_1}(0))} + \sup_{\beta\in\mathbb{N}_0^d,|\beta|\le p} \left\|\partial^\beta P(D) f\right\|_{\mathcal{L}^2(Q_{r_1}(0))}\right) \end{aligned}$$

for all $f \in \mathrm{C}^\infty\left(\mathring{Q}_{r_2}(0)\right)$. $\square$

Due to this corollary we can switch to types of L^2-semi-norms which induce the same topology on $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ as the sup-semi-norms. Further, we get an useful inequality.

5.15 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be compact.*

(1) For $n \in \mathbb{N}_{\ge 2}$ we define the locally convex space

$$\mathcal{C}_n^{exp}(S_n(K)) := \{f \in \mathrm{C}^\infty(S_n(K)) \mid \forall\, m \in \mathbb{N}_0 : r_{n,m}(f) < \infty\}$$

where

$$r_{n,m}(f) := \sup_{\alpha\in\mathbb{N}_0^2,|\alpha|\le m} \left(\int_{S_n(K)} |\partial^\alpha f(z)|^2 e^{-\frac{1}{n}|\mathrm{Re}(z)|} dz\right)^{\frac{1}{2}}$$

for $f \in \mathrm{C}^\infty(S_n(K))$ and $m \in \mathbb{N}_0$.
Let $n \in \mathbb{N}_{\ge 2}$, $P(D)$ be hypoelliptic and $f \in \mathrm{C}^\infty(S_{2n}(K))$ such that $r_{2n,0}(f) < \infty$ and $P(D) f \in \mathcal{C}_{2n}^{exp}(S_{2n}(K))$. Then we have $f \in \mathcal{E}_n^{exp}(S_n(K))$. More precisely:

$$\forall\, m \in \mathbb{N}_0\ \exists\, p \in \mathbb{N}_0,\ C_0 > 0 : |f|_{n,m} \le C_0\left[r_{2n,0}(f) + r_{2n,p}(P(D) f)\right]$$

(2) We define the space

$$\mathcal{C}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) := \lim_{\substack{\longleftarrow \\ n\in\mathbb{N}_{\ge 2}}} \mathrm{proj}\, \mathcal{C}_n^{exp}(S_n(K))$$

where the spectral mappings are given by

$$\pi_{n,k} : \mathcal{C}_k^{exp}(S_k(K)) \to \mathcal{C}_n^{exp}(S_n(K)),\ \pi_{n,k}(f) := f\big|_{S_n(K)},\ n \le k.$$

Then $\mathcal{C}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) = \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ as topological vector spaces.

Proof. (1) Let $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^2$, $|\alpha| \le m$. Choose $0 < h < \frac{1}{2\sqrt{2n}}$ and set $\varepsilon := \frac{h}{3}$. For $j \in \mathbb{Z}^2$ and $r > 0$ we define

$$Q(j,r) := \varepsilon\left[j + \left(\frac{1}{2}, \frac{1}{2}\right)\right] + Q_{\frac{r}{2}}(0) = \varepsilon j + \frac{1}{2}(\varepsilon - r, \varepsilon - r) + [0,r]^2$$

and $J := \{j \in \mathbb{Z}^2 \mid Q(j,\varepsilon) \cap S_n(K) \neq \varnothing\}$. For all $j \in J$ it follows, by the choice of h, that $Q(j,3\varepsilon) \subset \mathring{Q}(j,5\varepsilon) \subset S_{2n}(K)$ since $3\sqrt{2}\varepsilon = \sqrt{2}h < \frac{1}{2n} < n$. Thus $f \in \mathrm{C}^\infty\left(\mathring{Q}(j,5\varepsilon)\right)$ resp.

$f\left(\varepsilon\left[j+\left(\frac{1}{2},\frac{1}{2}\right)\right]+\cdot\right)\in\mathcal{C}^{\infty}\left(\mathring{Q}_{\frac{5\varepsilon}{2}}(0)\right)$.
Let $j\in J$. By Corollary 5.14 there exist $p\in\mathbb{N}_0$ and $C>0$, C independent of j, such that

$$\begin{aligned}
&p_{Q(j,\varepsilon),0}\left((\partial^{\alpha}f)e^{-\frac{1}{n}|\mathrm{Re}(\cdot)|}\right)\\
&\leq e^{-\frac{\varepsilon}{n}|\mathrm{Re}(j)|+\frac{\varepsilon}{n}}p_{Q(j,\varepsilon),0}(\partial^{\alpha}f)\\
&=e^{-\frac{\varepsilon}{n}|\mathrm{Re}(j)|+\frac{\varepsilon}{n}}p_{Q_{\frac{\varepsilon}{2}}(0),0}\left(\partial^{\alpha}f\left(\varepsilon\left[j+\left(\frac{1}{2},\frac{1}{2}\right)\right]+\cdot\right)\right)\\
&\underset{5.14}{\leq}Ce^{-\frac{\varepsilon}{n}|\mathrm{Re}(j)|+\frac{\varepsilon}{n}}\Bigg[\left\|f\left(\varepsilon\left[j+\left(\frac{1}{2},\frac{1}{2}\right)\right]+\cdot\right)\right\|_{\mathcal{L}^2\left(Q_{\frac{3\varepsilon}{2}}(0)\right)}\\
&\quad+\sup_{\beta\in\mathbb{N}_0^2,|\beta|\leq p}\left\|\partial^{\beta}P(D)f\left(\varepsilon\left[j+\left(\frac{1}{2},\frac{1}{2}\right)\right]+\cdot\right)\right\|_{\mathcal{L}^2\left(Q_{\frac{3\varepsilon}{2}}(0)\right)}\Bigg]\\
&=Ce^{-\frac{\varepsilon}{n}|\mathrm{Re}(j)|+\frac{\varepsilon}{n}}\left[\|f\|_{\mathcal{L}^2(Q(j,3\varepsilon))}+\sup_{\beta\in\mathbb{N}_0^2,|\beta|\leq p}\left\|\partial^{\beta}P(D)f\right\|_{\mathcal{L}^2(Q(j,3\varepsilon))}\right]\\
&=Ce^{\frac{\varepsilon}{n}}\Bigg[\Big(\int_{Q(j,3\varepsilon)}|f(z)|^2\underbrace{e^{-\frac{2\varepsilon}{n}|\mathrm{Re}(j)|}}_{\leq e^{-\frac{\varepsilon}{n}|\mathrm{Re}(j)|}\leq e^{-\frac{1}{n}|\mathrm{Re}(z)|+\frac{4\varepsilon}{n}}}dz\Big)^{\frac{1}{2}}+\sup_{\substack{\beta\in\mathbb{N}_0^2,\\|\beta|\leq p}}\Big(\int_{Q(j,3\varepsilon)}\left|\partial^{\beta}P(D)f(z)\right|^2\underbrace{e^{-\frac{2\varepsilon}{n}|\mathrm{Re}(j)|}}_{\leq e^{-\frac{1}{n}|\mathrm{Re}(z)|+\frac{4\varepsilon}{n}}}dz\Big)^{\frac{1}{2}}\Bigg]\\
&\leq Ce^{\frac{3\varepsilon}{n}}\Bigg[\Big(\int_{Q(j,3\varepsilon)}|f(z)|^2\underbrace{e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{\leq e^{-\frac{1}{2n}|\mathrm{Re}(z)|}}dz\Big)^{\frac{1}{2}}+\sup_{\substack{\beta\in\mathbb{N}_0^2,\\|\beta|\leq p}}\Big(\int_{Q(j,3\varepsilon)}\left|\partial^{\beta}P(D)f(z)\right|^2e^{-\frac{1}{n}|\mathrm{Re}(z)|}dz\Big)^{\frac{1}{2}}\Bigg]\\
&\leq Ce^{\frac{3\varepsilon}{n}}\left[r_{2n,0}(f)+r_{2n,p}(P(D)f)\right]
\end{aligned}\tag{5.39}$$

and so we get

$$\begin{aligned}
|f|_{n,m}&\leq\sup_{\substack{z\in\bigcup_{j\in J}Q(j,\varepsilon),\\\alpha\in\mathbb{N}_0^2,\,|\alpha|\leq m}}|\partial^{\alpha}f(z)|e^{-\frac{1}{n}|\mathrm{Re}(z)|}=\sup_{\substack{j\in J,\\\alpha\in\mathbb{N}_0^2,\,|\alpha|\leq m}}p_{Q(j,\varepsilon),0}\left((\partial^{\alpha}f)e^{-\frac{1}{n}|\mathrm{Re}(\cdot)|}\right)\\
&\underset{(5.39)}{\leq}\underbrace{Ce^{\frac{3\varepsilon}{n}}}_{=:C_0}\left[r_{2n,0}(f)+r_{2n,p}(P(D)f)\right].
\end{aligned}$$

(2) Let $f\in\mathcal{C}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)$ and $P(D):=\overline{\partial}$. Then f satisfies the conditions of (1) for all $n\in\mathbb{N}_{\geq2}$. So for all $n\in\mathbb{N}_{\geq2}$ and $m\in\mathbb{N}_0$ there exist $p\in\mathbb{N}_0$ and $C_0>0$ such that

$$|f|_{n,m}\leq C_0\left[r_{2n,0}(f)+r_{2n,p}\left(\overline{\partial}f\right)\right]\leq C_0\left[r_{2n,0}(f)+r_{2n,p+1}(f)\right]\leq 2C_0r_{2n,p+1}(f).$$

On the other hand let $f\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus K\right)$. For every $n\in\mathbb{N}_{\geq2}$ and $m\in\mathbb{N}_0$ we have

$$r_{n,m}(f)\leq\sup_{\substack{z\in S_n(K),\\\alpha\in\mathbb{N}_0^2,\,|\alpha|\leq m}}|\partial^{\alpha}f(z)|e^{-\frac{1}{4n}|\mathrm{Re}(z)|}\underbrace{\left(\int_{S_n(K)}e^{\left(\frac{1}{2n}-\frac{1}{n}\right)|\mathrm{Re}(z)|}dz\right)^{\frac{1}{2}}}_{=:C_1<\infty}\leq C_1|f|_{4n,m}.$$

□

By now all ingredients that are required to prove the admissibility of $\mathbb{C}$ are provided.

5.16 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact. Then*

$$\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$$

is surjective.

Proof. (i) Let $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$, $n \in \mathbb{N}_{\geq 2}$ and set

$$\varphi_n\colon \mathbb{C} \to \mathbb{C},\ \varphi_n(z) := \frac{1}{4n}|z|.$$

Then φ_n is subharmonic on $\mathbb{C}$ by [20, Corollary 1.6.6, Theorem 1.6.7, p. 18], particularly, plurisubharmonic. The set $S_{4n}(K)$ is open and pseudoconvex since every open set in $\mathbb{C}$ is a domain of holomorphy by [20, Corollary 1.5.3, p. 15] and hence pseudoconvex by [20, Theorem 4.2.8, p. 88]. For the differential form $g := f\,d\bar{z}$ we have $\overline{\partial} g = 0$ in the sense of differential forms and by Lemma 5.15(2)

$$\int_{S_{4n}(K)} |f(z)|^2 e^{-\varphi_n(z)} dz \leq r_{4n,0}(f)^2 < \infty.$$

Thus by [20, Theorem 4.4.2, p. 94] there is a solution $u_n \in L^2_{loc}(S_{4n}(K))$ of $\overline{\partial} u_n = f|_{S_{4n}(K)}$ in the distributional sense such that

$$\int_{S_{4n}(K)} |u_n(z)|^2 e^{-\varphi_n(z)} \left(1+|z|^2\right)^{-2} dz \leq \int_{S_{4n}(K)} |f(z)|^2 e^{-\varphi_n(z)} dz.$$

Since $\overline{\partial}$ is hypoelliptic, it follows that $u_n \in \mathrm{C}^\infty(S_{4n}(K))$, resp. u_n has a representative which is C^∞. For all $C_0 > 0$ exists $C_1 > 0$ such that

$$2\ln\left(1+|z|^2\right) \leq C_0|z| + C_1$$

for all $z \in \mathbb{C}$. So, for $C_0 := \frac{1}{4n}$ exists $C_1 > 0$ such that

$$\varphi_n(z) + 2\ln\left(1+|z|^2\right) \leq \varphi_n(z) + \frac{1}{4n}|z| + C_1 = \frac{1}{2n}|z| + C_1 \tag{5.40}$$

for all $z \in \mathbb{C}$. Therefore, we gain

$$\begin{aligned} r_{2n,0}(u_n)^2 &= \int_{S_{2n}(K)} |u_n(z)|^2 e^{-\frac{1}{2n}|\mathrm{Re}(z)|} dz \leq e^{1+C_1} \int_{S_{2n}(K)} |u_n(z)|^2 e^{-\frac{1}{2n}|z| - C_1} dz \\ &\underset{(5.40)}{\leq} e^{1+C_1} \int_{S_{2n}(K)} |u_n(z)|^2 e^{-\varphi_n(z) - 2\ln\left(1+|z|^2\right)} dz \\ &\leq e^{1+C_1} \int_{S_{4n}(K)} |u_n(z)|^2 e^{-\varphi_n(z)} \left(1+|z|^2\right)^{-2} dz < \infty. \end{aligned}$$

So the conditions of Lemma 5.15(1) are fulfilled for all $n \in \mathbb{N}_{\geq 2}$ implying $u_n \in \mathcal{E}^{exp}_n(S_n(K))$.

(ii) The next step is to prove the surjectivity of $\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ via the Mittag-Leffler procedure. Due to (i) we have for every $q \in \mathbb{N}_{\geq 2}$ a function $u_q \in \mathcal{E}^{exp}_q\left(S_q(K)\right)$ such

that $\overline{\partial} u_q = f|_{S_q(K)}$.
Now we inductively construct $g_n \in \mathcal{E}^{exp}_{n+2}(S_{n+2}(K))$, $n \in \mathbb{N}$, such that

(1) $\overline{\partial} g_n = f|_{S_{n+2}(K)}$, $n \geq 1$,

(2) $|g_n - g_{n-1}|_{n,n} \leq \frac{1}{2^n}$, $n \geq 2$.

For $n = 1$ set $g_1 := u_3$. Then we have $g_1 \in \mathcal{E}^{exp}_3(S_3(K))$ and $\overline{\partial} g_1 = f|_{S_3(K)}$ by part (i).
Let g_n with (1) for $n \geq 1$ be given. Since

$$\overline{\partial}(u_{n+3} - g_n)|_{S_{n+2}(K)} = \overline{\partial} u_{n+3}|_{S_{n+2}(K)} - \overline{\partial} g_n|_{S_{n+2}(K)} \underset{(i),(1)}{=} f|_{S_{n+3}(K)}\Big|_{S_{n+2}(K)} - f|_{S_{n+2}(K)} = 0,$$

it follows $u_{n+3} - g_n \in \mathcal{E}^{exp}_{n+2,\overline{\partial}}(S_{n+2}(K))$ and by Theorem 5.1 there is $h_{n+1} \in \mathcal{E}^{exp}_{n+3,\overline{\partial}}(S_{n+3}(K))$ such that

$$|u_{n+3} - g_n - h_{n+1}|_{n+1,n+1} \leq \frac{1}{2^{n+1}}.$$

Set $g_{n+1} := u_{n+3} - h_{n+1} \in \mathcal{E}^{exp}_{n+3}(S_{n+3}(K))$. Condition (2) is satisfied by above and condition (1) as well because

$$\overline{\partial} g_{n+1} = \overline{\partial} u_{n+3} - \underbrace{\overline{\partial} h_{n+1}}_{=0} = \overline{\partial} u_{n+3} \underset{(i)}{=} f|_{S_{n+3}(K)}.$$

Now let $\varepsilon > 0$, $l \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_0$. Choose $l_0 \in \mathbb{N}$, $l_0 \geq \max(l, m)$, such that $\frac{1}{2^{l_0}} < \varepsilon$. For all $p \geq k \geq l_0$ we get

$$\begin{aligned} |g_p - g_k|_{l,m} &\leq |g_p - g_k|_{l_0,l_0} = \left|\sum_{j=k+1}^{p} g_j - g_{j-1}\right|_{l_0,l_0} \leq \sum_{j=k+1}^{p} |g_j - g_{j-1}|_{l_0,l_0} \\ &\underset{l_0 \leq k \leq j}{\leq} \sum_{j=k+1}^{p} |g_j - g_{j-1}|_{j,j} \underset{(2)}{\leq} \sum_{j=k+1}^{p} \frac{1}{2^j} < \frac{1}{2^k} \\ &\leq \frac{1}{2^{l_0}} < \varepsilon. \end{aligned}$$

Hence $(g_n)_{n \geq \max(l-2,1)}$ is a Cauchy sequence in $\mathcal{E}^{exp}_l(S_l(K))$ for all $l \in \mathbb{N}_{\geq 2}$ and, since these spaces are complete by Theorem 3.6(1), it has a limit function $g \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)$ by the same arguments like in the proof of Corollary 5.6.
Thus we have for all $l \in \mathbb{N}_{\geq 2}$

$$f|_{S_l(K)} \underset{\substack{(1),\\ n \geq \max(l-2,1)}}{=} \overline{\partial} g_n|_{S_l(K)} \underset{n \to \infty}{\to} \overline{\partial} g|_{S_l(K)}$$

and hence the existence of $g \in \mathcal{E}^{exp}(K)$ with $\overline{\partial} g = f$ on $\mathbb{C} \setminus K$ is proven.

□

Moreover, we are already able to show that Fréchet spaces are admissible just by using classical theory of tensor products of Fréchet spaces.

5.17 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact and E a Fréchet space. Then*

$$\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

is surjective.

Proof. Let $g \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$. Then $g \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$ due to Remark 3.13(1). The mappings $\mathrm{id}_E\colon E \to E$ and $\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{E}^{exp}(K)$ are linear, continuous and surjective, the latter one by Theorem 5.16. Moreover, E and $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ are Fréchet spaces, so $\overline{\partial} \hat{\otimes}_\pi \mathrm{id}_E\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$ is surjective by [61, 6.6 Theorem, p. 65], i.e. there is $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \hat{\otimes}_\pi E$ such that $\left(\overline{\partial} \hat{\otimes}_\pi \mathrm{id}_E\right)(f) = g$. Again we have $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ by Remark 3.13(1) and $\overline{\partial}(f) = \left(\overline{\partial} \hat{\otimes}_\pi \mathrm{id}_E\right)(f) = g$. □

In order to obtain further classes of admissible spaces by the splitting theory for Fréchet spaces of Vogt (see [63]) resp. PLS-spaces by Bonet and Domański (see [8]), we have to prove that the space $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ satisfies (Ω) for every compact set $K \subset \overline{\mathbb{R}}$. Let us recall that a Fréchet space F with an increasing fundamental system of semi-norms $(|||\cdot|||_k)_{k \in \mathbb{N}}$ satisfies (Ω) if

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \forall\, k \in \mathbb{N}\ \exists\, n \in \mathbb{N}, C > 0\ \forall\, r > 0 :\ U_q \subset C r^n U_k + \frac{1}{r} U_p \tag{5.41}$$

where $U_k := \{x \in F \mid |||x|||_k \leq 1\}$.
By [45, 29.13 Lemma, p. 349] this is equivalent to

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \forall\, k \in \mathbb{N}\ \exists\, 0 < \theta < 1\ C > 0 :\ \|y\|_q^* \leq C \|y\|_p^{*\,1-\theta} \|y\|_k^{*\,\theta}, \quad \forall y \in F', \tag{5.42}$$

where

$$\|y\|_k^* := \sup\{|y(x)| \mid |||x|||_k \leq 1\} \in \mathbb{R} \cup \{\infty\}$$

is the dual norm. In this context we remark:

5.18 Remark. Let F be as above and $y \in F'$. Then the following assertions are equivalent:

(1) $\|y\|_k^* < \infty$

(2) $\exists\, C > 0 :\ |y(x)| \leq C |||x|||_k \quad \forall\, x \in F$

Proof. (1)⇒(2): Set $C := \|y\|_k^* + 1$. Then $C \in \mathbb{R}_{>0}$ by assumption. Let $x \in F$, $x \neq 0$. Then

$$|y(x)| = \left| y\left(\frac{x}{|||x|||_k}\right)\right| |||x|||_k \leq \|y\|_k^* \, |||x|||_k < C \, |||x|||_k .$$

For $x = 0$ this is obvious.
(2)⇒(1): Let $x \in F$, $|||x|||_k \leq 1$. Then we get by assumption

$$|y(x)| \leq C |||x|||_k \leq C$$

and thus $\|y\|_k^* \leq C < \infty$. □

At first, the (Ω)-property will be proven for $K=\varnothing$ and then for arbitrary compact sets $K\subset\overline{\mathbb{R}}$. For $K=\varnothing$ a decomposition theorem of Langenbruch, [41, Theorem 2.2, p. 225], will be used which is stated below for the purpose of more clarity and comprehensibility. With the definition, analogous to Definition 5.8b),

$$H_{\tau}(V_t):=\left\{f\in\mathcal{O}(V_t)\mid \|f\|_{\tau,t}:=\sup_{z\in V_t}|f(z)|e^{\tau\nu(z)}<\infty\right\}$$

for $t>0$ and $\tau\in\mathbb{R}$, where $V_t:=\{z\in\mathbb{C}\mid |\mathrm{Im}(z)|<t\}$ and ν is a weight function in the sense of Definition 5.7, the following is valid:

5.19 Theorem. [1]*[41, Theorem 2.2, p. 225] There are $\tilde{t}$, K_1, $K_2>0$ such that for any $\tau_0<\tau<\tau_2$ there is $C_0=C_0(\mathrm{sign}(\tau))$ such that for any $0<2t_0<t<t_2<\tilde{t}$ with*

$$t_0\leq\min\left[K_1,K_2\sqrt{\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0}}\right]$$

there is $C_1\geq 1$ such that for any $r\geq 0$ and any $f\in H_{\tau}(V_t)$ with $\|f\|_{\tau,t}\leq 1$ the following holds: there are $f_2\in\mathcal{O}(V_{t_2})$ and $f_0\in\mathcal{O}(V_{t_0})$ such that $f=f_0+f_2$ on V_{t_0} and

$$\|f_0\|_{C_0\tau_0,t_0}\leq C_1e^{-Gr}\quad\text{and}\quad\|f_2\|_{\tau_2,t_2}\leq e^{r}$$

where

$$G:=K_1\min\left[1,\frac{t-t_0}{2\tilde{t}},\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0}\right].$$

With this notation we have $\mathcal{O}^{exp}(\overline{\mathbb{C}})=\lim\mathrm{proj}_{n\in\mathbb{N}}H_{-\frac{1}{n}}(V_n)$ and $|\cdot|_n=\|\cdot\|_{-\frac{1}{n},n}$, $n>1$, for $\nu:=|\mathrm{Re}(\cdot)|$. To apply this theorem, one has to know the constants involved. In the following, the notations of [41] are used and it is referred to the corresponding positions resp. conditions for these constants. We have

$$\tilde{t}:=\frac{1}{4\ln(\Gamma)}$$

by [41, Lemma 2.4, (2.15), p. 228] with $\Gamma>1$ from Definition 5.7. For $\nu:=|\mathrm{Re}(\cdot)|$ every $\Gamma>1$ is possible. By [41, Corollary 2.6, p. 230-231] one has

$$C_0:=\begin{cases}4\Gamma B_3=\frac{64\cosh(1)}{\cos(1/2)}\Gamma^2>1, & \tau<0,\\ \frac{1}{4\Gamma B_3}=\frac{\cos(1/2)}{64\cosh(1)\Gamma^2}<1, & \tau\geq 0,\end{cases}$$

where $B_3:=\frac{16\cosh(1)}{\cos(1/2)}\Gamma$ by [41, Lemma 2.4, p. 228-229].[2] To get the constants K_1 and K_2, one has to analyze the conditions for t_0 in the proof of [41, Theorem 2.2, p. 225]. By the assumptions on τ_0, τ and τ_2 and the choice of C_0 we obtain

$$\tau_2-C_0\tau_0>\tau_2-C_0\tau\geq\tau_2-\tau>0 \tag{5.43}$$

and

$$\tau-C_0\tau_0>\tau-C_0\tau=\tau(1-C_0)>0. \tag{5.44}$$

[1]A superfluous constant depending on $\mathrm{sign}(\tau_0)$ is omitted.
[2]An error in part b) of the Lemma, p. 229, is corrected here such that the term $\cos(1/2)$ appears.

By choosing $D>0$ in the proof of [41, Theorem 2.2, (2.22) pp., p. 232-233] as $D:=\frac{\tau-C_0\tau_0}{(\tau_2-C_0\tau_0)2\Gamma_0}$,

$$D=\frac{\tau-C_0\tau_0}{(\tau_2-C_0\tau_0)\,2\Gamma_0}=\min\left(\frac{1}{2\tilde{\Gamma}},\frac{1}{2\hat{\Gamma}}\right)\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0}\underset{(5.43),\,(5.44)}{\leq}\min\left(\frac{1}{2\tilde{\Gamma}},\frac{1}{2\hat{\Gamma}}\right)\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau}$$

holds where $\Gamma_0:=\max\left(\tilde{\Gamma},\hat{\Gamma}\right)$ with $\tilde{\Gamma}$, $\hat{\Gamma}>1$ from the proof. For $v:=|\mathrm{Re}(\cdot)|$ all $\tilde{\Gamma}$, $\hat{\Gamma}>1$ are possible. With $\theta\geq\frac{t-t_0}{2\tilde{t}}$ (p. 232) one gets on p. 233, below (2.24), due to the condition $t_0\leq T_0:=\min\left(\frac{t}{2},\frac{1}{4a^2B_1\tilde{t}}\right)$,

$$\begin{aligned}\min\left(\frac{\theta}{2},D,1\right)&\geq\underbrace{\min\left(\frac{1}{2},\frac{1}{2\Gamma_0}\right)}_{=\frac{1}{2\Gamma_0}}\min\left(\theta,\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0},1\right)\geq\frac{1}{2\Gamma_0}\min\left(\frac{t-t_0}{2\tilde{t}},\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0},1\right)\\&\underset{\text{def. }T_0}{\geq}\min\left(\frac{1}{2\Gamma_0},\frac{1}{4a^2B_1\tilde{t}}\right)\min\left(\frac{t-t_0}{2\tilde{t}},\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0},1\right)\\&=\underbrace{\min\left(\frac{1}{2\Gamma_0},\frac{1}{2\cosh(1)\ln(\Gamma)}\right)}_{=:K_1}\min\left(\frac{t-t_0}{2\tilde{t}},\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0},1\right)=:G\end{aligned}$$

where $a:=\ln(\Gamma)$ (at the top of p. 231) and $B_1:=2\cosh(1)$ by the proof of [41, Lemma 2.3, p. 226-227].[3] Looking at the condition $t_0\leq T_1:=\sqrt{\frac{D}{a^2B_1}}$ (p. 232), one gets

$$T_1=\frac{1}{\sqrt{2\Gamma_0a^2B_1}}\sqrt{\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0}}=\frac{1}{\underbrace{2\sqrt{\cosh(1)\Gamma_0}\ln(\Gamma)}_{=:K_2}}\sqrt{\frac{\tau-C_0\tau_0}{\tau_2-C_0\tau_0}}.$$

5.20 Theorem. *$\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ satisfies (Ω).*

Proof. a) Let $p\in\mathbb{N}$. Choose $q\in\mathbb{N}$, $q>\frac{128\cosh(1)}{\cos(1/2)}\Gamma^2p$. To use the theorem above, one needs a linear transformation between strips to get the decomposition on the desired strip, desired in the spirit of Corollary 5.6. Choose $T>0$ such that

$$T<\min\left(\frac{1}{4\Gamma_0p},\frac{1}{2\sqrt{B_1\Gamma_0pq}\ln(\Gamma)},\frac{1}{8\max(q,2k)\ln(\Gamma)}\right).$$

Let $\tau:=-\frac{1}{qT}$, so $\tau<0$, and set

$$\tau_0:=-\frac{1}{2C_0pT}\quad\text{and}\quad\tau_2:=\frac{1}{2}\max\left(\tau,-\frac{1}{2kT}\right),$$
$$t_0:=2pT\quad\text{and}\quad t:=qT\quad\text{and}\quad t_2:=2\max(q,2k)T.$$

[3] The term $\frac{t}{2}$ in the definition of T_0 appears in the theorem as the condition $2t_0<t$.

By the choice of q and since $\tau < 0$ and $T > 0$, we have

$$\tau_0 = -\frac{1}{\frac{128\cosh(1)}{\cos(1/2)}\Gamma^2 pT} < -\frac{1}{qT} = \tau < \frac{1}{2}\max\left(\tau, -\frac{1}{2kT}\right) = \tau_2.$$

By the choice of q and the last term in the choice of T we get

$$0 < 2t_0 = 4pT < 128pT < \frac{128\cosh(1)}{\cos(1/2)}\Gamma^2 pT < qT = t < 2\max(q, 2k)T < \frac{1}{4\ln(\Gamma)} = \tilde{t}.$$

Now the last condition for t_0 has to be checked. First of all we obtain

$$\begin{aligned}
\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0} &\underset{\substack{(5.43),\\(5.44)}}{\geq} \frac{\tau - C_0\tau}{\tau_2 - C_0\tau_0} \underset{\tau_2,\tau_0<0}{\geq} \frac{\tau - C_0\tau}{-C_0\tau_0} = \frac{\tau}{\tau_0}\left(1 - \frac{1}{C_0}\right) = \frac{2C_0 p}{q}\left(1 - \frac{1}{C_0}\right) = \frac{2p}{q}(C_0 - 1) \\
&= \frac{2p}{q}\underbrace{\left(\frac{64\cosh(1)}{\cos(1/2)} - 1\right)}_{\geq 63} > \frac{2p}{q}
\end{aligned} \tag{5.45}$$

and hence

$$\begin{aligned}
&\min\left[K_1, K_2\sqrt{\frac{\tau - C_0\tau_0}{\tau_2 - C_0\tau_0}}\right] \\
&\underset{(5.45)}{\geq} \min\left[K_1, K_2\sqrt{\frac{2p}{q}}\right] = \min\left[\frac{1}{2\Gamma_0}, \frac{1}{2\cosh(1)\ln(\Gamma)}, \frac{1}{2\sqrt{\cosh(1)\Gamma_0}\ln(\Gamma)}\sqrt{\frac{2p}{q}}\right] \\
&= \min\left[\frac{1}{2\Gamma_0}, \frac{1}{2\ln(\Gamma)}\min\left(\frac{1}{\cosh(1)}, \underbrace{\frac{1}{\sqrt{\cosh(1)\Gamma_0}}\sqrt{\frac{2p}{q}}}_{\underset{\substack{\text{choice}\\ \text{of } q}}{\leq} \frac{1}{\sqrt{\cosh(1)\Gamma_0}} \frac{1}{\sqrt{\cosh(1)}} \leq \frac{1}{\cosh(1)}}\right)\right] \\
&= \min\left[\frac{1}{2\Gamma_0}, \frac{1}{2\ln(\Gamma)}\frac{1}{\sqrt{\cosh(1)\Gamma_0}}\sqrt{\frac{2p}{q}}\right] = \min\left[\frac{1}{2\Gamma_0}, \frac{\sqrt{2p}}{\sqrt{2B_1\Gamma_0 q}\ln(\Gamma)}\right] \\
&\underset{\substack{\text{choice}\\ \text{of } T}}{>} 2pT = t_0.
\end{aligned}$$

b) Let $r \geq 0$ and $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ such that $\|f\|_{-\frac{1}{q},q} \leq 1$. Then it follows that

$$1 \geq \|f\|_{-\frac{1}{q},q} = \sup_{z\in V_q}|f(z)|e^{-\frac{1}{q}|\mathrm{Re}(z)|} = \sup_{z\in V_{qT}}|\underbrace{f\left(\frac{z}{T}\right)}_{=:\tilde{f}(z)}|e^{-\frac{1}{qT}|\mathrm{Re}(z)|} = \|\tilde{f}\|_{\tau,t},$$

where $\tilde{f} \in \mathcal{O}(V_t)$, and thus by Theorem 5.19 there are $\tilde{f}_j \in \mathcal{O}\left(V_{t_j}\right)$, $j = 0, 2$, such that

$$\tilde{f} = \tilde{f}_0 + \tilde{f}_2 \text{ on } V_{t_0} \tag{5.46}$$

and

$$C_1 e^{-Gr} \geq \left\| \tilde{f}_0 \right\|_{C_0\tau_0, t_0} = \sup_{z\in V_{t_0}} |\tilde{f}_0(z)| e^{C_0\tau_0|\mathrm{Re}(z)|} = \sup_{z\in V_{t_0/T}} |\tilde{f}_0(Tz)| e^{C_0\tau_0 T|\mathrm{Re}(z)|}$$
$$\underset{\substack{\text{def.}\\ \text{of } t_0,\, \tau_0}}{=} \sup_{z\in V_{2p}} |\underbrace{\tilde{f}_0(Tz)}_{=:f_0(z)}| e^{-\frac{1}{2p}|\mathrm{Re}(z)|} = \| f_0 \|_{-\frac{1}{2p}, 2p}, \tag{5.47}$$

where $f_0 \in \mathcal{O}\left(V_{2p}\right)$, as well as

$$e^r \geq \left\| \tilde{f}_2 \right\|_{\tau_2, t_2} = \sup_{z\in V_{t_2}} |\tilde{f}_2(z)| e^{\tau_2|\mathrm{Re}(z)|} = \sup_{z\in V_{t_2/T}} |\tilde{f}_2(Tz)| e^{\tau_2 T|\mathrm{Re}(z)|}$$
$$\underset{\substack{\text{def.}\\ \text{of } t_2,\, \tau_2}}{\geq} \sup_{z\in V_{2k}} |\underbrace{\tilde{f}_2(Tz)}_{=:f_2(z)}| e^{-\frac{1}{2k}|\mathrm{Re}(z)|} = \| f_2 \|_{-\frac{1}{2k}, 2k}, \tag{5.48}$$

where $f_2 \in \mathcal{O}(V_{2k})$. Furthermore, for $z \in V_{t_0/T} = V_{2p}$ the equation

$$f(z) = \tilde{f}(Tz) \underset{(5.46)}{=} \tilde{f}_0(Tz) + \tilde{f}_2(Tz) = f_0(z) + f_2(z)$$

holds, thus $f = f_0 + f_2$ on V_{2p}. By virtue of Corollary 5.6 the following is valid:

$$\forall\, \varepsilon > 0\ \exists\, \hat{f}_0,\, \hat{f}_2 \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right) : i)\ \ \left\| \hat{f}_0 - f_0 \right\|_{-\frac{1}{p}, p} < \varepsilon$$
$$ii)\ \left\| \hat{f}_2 - f_2 \right\|_{-\frac{1}{k}, k} < \varepsilon \tag{5.49}$$

Now we have to consider two cases. Let $\varepsilon := C_1 e^{-Gr}$. For $k \leq p$ we obtain via (5.49)$i)$

$$f = \hat{f}_0 + \left(f_2 + f_0 - \hat{f}_0\right) \text{ on } V_{2p},$$

so

$$f_2 + f_0 - \hat{f}_0 = f - \hat{f}_0 =: \bar{f}_2 \text{ on } V_{2p} \tag{5.50}$$

where $\bar{f}_2 \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ and thus an holomorphic extension of the left hand side on $\mathbb{C}$. Hence we clearly have $f = \hat{f}_0 + \bar{f}_2$ and

$$\left\| \hat{f}_0 \right\|_{-\frac{1}{p}, p} \leq \left\| \hat{f}_0 - f_0 \right\|_{-\frac{1}{p}, p} + \| f_0 \|_{-\frac{1}{p}, p} \underset{(5.49)i)}{\leq} \varepsilon + \| f_0 \|_{-\frac{1}{p}, p} \leq \varepsilon + \| f_0 \|_{-\frac{1}{2p}, 2p}$$
$$\underset{(5.47)}{\leq} \underbrace{2C_1}_{=:C_2} e^{-Gr} \tag{5.51}$$

as well as

$$\left\| \bar{f}_2 \right\|_{-\frac{1}{k}, k} \leq \| \underbrace{\bar{f}_2 - f_2}_{\underset{(5.50),\, k\leq p}{=} f_0 - \hat{f}_0} \|_{-\frac{1}{k}, k} + \| f_2 \|_{-\frac{1}{k}, k} \leq \left\| f_0 - \hat{f}_0 \right\|_{-\frac{1}{p}, p} + \| f_2 \|_{-\frac{1}{2k}, 2k} \underset{(5.49)i)}{\leq} \varepsilon + \| f_2 \|_{-\frac{1}{2k}, k}$$
$$\underset{(5.48)}{\leq} C_1 \underbrace{e^{-Gr}}_{\leq 1} + e^r \leq \underbrace{(C_1 + 1)}_{=:C_3} e^r. \tag{5.52}$$

Analogously for $k > p$ we obtain via (5.49)*ii*)

$$f = \hat{f}_2 + \left(f_0 + f_2 - \hat{f}_2\right) \text{ on } V_{2p},$$

so

$$f_0 + f_2 - \hat{f}_2 = f - \hat{f}_2 =: \bar{f}_0 \text{ on } V_{2p} \tag{5.53}$$

where $\bar{f}_0 \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ and thus an holomorphic extension of the left hand side on $\mathbb{C}$. Hence we clearly have $f = \bar{f}_0 + \hat{f}_2$ and

$$\begin{aligned}\|\bar{f}_0\|_{-\frac{1}{p},p} &\underset{(5.53)}{=} \|\underbrace{f - \hat{f}_2}_{f_0+f_2-\hat{f}_2}\|_{-\frac{1}{p},p} = \|f_0 + f_2 - \hat{f}_2\|_{-\frac{1}{p},p} \leq \|f_2 - \hat{f}_2\|_{-\frac{1}{p},p} + \|f_0\|_{-\frac{1}{p},p} \\ &\underset{k>p}{\leq} \|f_2 - \hat{f}_2\|_{-\frac{1}{k},k} + \|f_0\|_{-\frac{1}{2p},2p} \underset{(5.49)ii)}{\leq} \varepsilon + \|f_0\|_{-\frac{1}{2p},p} \underset{(5.47)}{\leq} 2C_1 e^{-Gr} = C_2 e^{-Gr}\end{aligned} \tag{5.54}$$

as well as

$$\|\hat{f}_2\|_{-\frac{1}{k},k} \leq \|\hat{f}_2 - f_2\|_{-\frac{1}{k},k} + \|f_2\|_{-\frac{1}{k},k} \underset{(5.49)ii)}{\leq} \varepsilon + \|f_2\|_{-\frac{1}{2k},2k} \underset{(5.48)}{\leq} C_1 e^{-Gr} + e^r \leq C_3 e^r. \tag{5.55}$$

c) Now set $n := \lceil 1/G \rceil$ and $C := C_3 e^{\frac{1}{G}\ln(C_2)}$. Let $\tilde{r} > 0$. For $\tilde{r} \geq 1$ there is $r \geq 0$ such that

$$\tilde{r} = e^{Gr - \ln(C_2)} = C_2^{-1} e^{Gr}$$

and we have by (5.51) and (5.52) for $k \leq p$

$$\|\hat{f}_0\|_{-\frac{1}{p},p} \leq C_2 e^{-Gr} = \frac{1}{\tilde{r}},$$

$$\|\hat{f}_2\|_{-\frac{1}{k},k} \leq C_3 e^r = \underbrace{C_3 e^{\frac{1}{G}\ln(C_2)}}_{=C} e^{\frac{1}{G}(Gr - \ln(C_2))} = C\tilde{r}^{\frac{1}{G}} \underset{\tilde{r}\geq 1}{\leq} C\tilde{r}^n,$$

as well as by (5.54) and (5.55) for $k > p$

$$\|\bar{f}_0\|_{-\frac{1}{p},p} \leq \frac{1}{\tilde{r}}, \quad \|\hat{f}_2\|_{-\frac{1}{k},k} \leq C\tilde{r}^n.$$

For $0 < \tilde{r} < 1$ we have, since $q \geq p$,

$$\|f\|_{-\frac{1}{p},p} \leq \|f\|_{-\frac{1}{q},q} \leq 1 < \frac{1}{\tilde{r}}.$$

Thus the theorem is proven.

□

The analogous result can be proven for the spaces $\mathcal{O}_{v,\tau}\left(\overline{\mathbb{C}}\right)$ of Definition 5.8 using Corollary 5.11 instead of Corollary 5.6 where v is a weight function in the sense of Definition 5.7, at least, if v satisfies the conditions of Theorem 5.10 (see [59, Satz 2.2.1, Definition 2.2.2, p. 43, Satz 2.2.3, p. 44]).
The next lemma, in particular the main ideas, is due to Langenbruch (oral communication).

5.21 Lemma. *Let $K \subset \overline{\mathbb{R}}$ be non-empty and compact.*

(1) $\forall\, p \in \mathbb{N}\ \forall\, q > p\ \forall\, k > q\ \exists\, 0 < \theta < 1,\, C > 0 : \ \|f\|_q \le C \|f\|_p^{1-\theta} \|f\|_k^{\theta} \quad \forall\, f \in \mathcal{O}_p\left(U_p(K)\right)$

(2) $\mathcal{P}_*(K)_b'$ *satisfies* (Ω).

Proof. (1) Let $p < q < k$ and $f \in \mathcal{O}_p\left(U_p(K)\right)$. Considering the components of $U_p(K)$ we have to distinguish three different cases.

(a) Let Z_p be a bounded component of $U_p(K)$. By Remark 3.3(1) there are only finitely many components Z_q of $U_q(K)$ with $Z_q \subset Z_p$. For every such component Z_q choose $\zeta_0 \in Z_q \cap K$, which exists since Z_q is bounded. Let Z_k be the (unique) component of $U_k(K)$ which contains ζ_0. Z_p is a proper simply connected subset of $\mathbb{C}$. Thus there exists a biholomorphic mapping $\psi_0 : Z_p \to D_1(0)$ with $\psi_0(\zeta_0) = 0$ due to the Riemann mapping theorem (and Möbius transformation). In addition, Z_p and $D_1(0)$ are Jordan domains (for the definition see [2, 2.8.5. Lemma, p. 193, 1.8.5. Jordan Curve Theorem, p. 68]) and so there exists a homeomorphism $\psi : \overline{Z_p} \to \overline{D_1(0)}$ such that $\psi|_{Z_p} = \psi_0$ by [2, 2.8.8. Theorem (Caratheodory), p. 195]. Since $\psi\left(\overline{Z_q}\right) \subset \psi\left(Z_p\right) = D_1(0)$ and $\psi\left(\overline{Z_q}\right)$ is compact, as $\overline{Z_q}$ is compact and ψ continuous, there is $0 < r_q < 1$ such that $\psi\left(\overline{Z_q}\right) \subset \overline{D_{r_q}(0)}$. Moreover, there exists $0 < r_k < r_q$ such that $\overline{D_{r_k}(0)} \subset \psi(Z_k)$ since $0 \in \psi(Z_k)$, $\psi(Z_k)$ is open by the open mapping theorem and $\psi(Z_k) \subset \psi\left(Z_q\right)$. The function $u := f \circ (\psi^{-1})$ is holomorphic on $D_1(0)$ and continuous on $\overline{D_1(0)}$, in particular, $|u|$ is subharmonic on $D_1(0)$ and continuous on $\overline{D_1(0)}$. Setting

$$M(r) := \sup_{|z|=r} |u(z)|,\ 0 < r \le 1,$$

we obtain by virtue of [2, 4.4.32. Proposition (Hadamard's Three Circles Theorem), p. 338]

$$\ln M\left(r_q\right) \le \frac{\ln\left(1/r_q\right)}{\ln\left(1/r_k\right)} \ln M(r_k) + \frac{\ln\left(r_q/r_k\right)}{\ln\left(1/r_k\right)} \ln M(1)$$

and hence

$$M\left(r_q\right) \le M(r_k)^{\theta} M(1)^{1-\theta}$$

where $\theta := \frac{\ln(1/r_q)}{\ln(1/r_k)}$. Because $0 < r_k < r_q < 1$, we get $0 < \theta < 1$. By the maximum principle we have

$$\begin{aligned} M\left(r_q\right) &= \sup_{|z|\le r_q} |u(z)| \ge \inf_{|z|\le r_q} e^{-\frac{1}{q}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|} \sup_{|z|\le r_q} \left|f\left(\psi^{-1}(z)\right)\right| e^{\frac{1}{q}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|} \\ &\underset{\psi\left(\overline{Z_q}\right)\subset\overline{D_{r_q}(0)}}{\ge} \underbrace{\inf_{|z|\le r_q} e^{-\frac{1}{q}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}}_{=:C_0>0} \sup_{z\in\overline{Z_q}} |f(z)| e^{\frac{1}{q}|\mathrm{Re}(z)|} \end{aligned}$$

as well as

$$\begin{aligned}
&M(r_k)^{\theta}M(1)^{1-\theta}\\
&=\sup_{|z|\le r_k}|u(z)|^{\theta}\sup_{|z|\le 1}|u(z)|^{1-\theta}\le\left(\sup_{|z|\le r_k}e^{-\frac{1}{k}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{\theta}\left(\sup_{|z|\le r_k}\left|f\left(\psi^{-1}(z)\right)\right|e^{\frac{1}{k}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{\theta}\\
&\quad\left(\sup_{|z|\le 1}e^{-\frac{1}{p}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{1-\theta}\left(\sup_{|z|\le 1}\left|f\left(\psi^{-1}(z)\right)\right|e^{\frac{1}{p}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{1-\theta}\\
&\underset{\overline{D_{r_k}(0)\subset\psi(\overline{Z_k})}}{\le}\underbrace{\left(\sup_{|z|\le r_k}e^{-\frac{1}{q}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{\theta}\left(\sup_{|z|\le 1}e^{-\frac{1}{p}\left|\mathrm{Re}\left(\psi^{-1}(z)\right)\right|}\right)^{1-\theta}}_{=:C_1}\\
&\quad\left(\sup_{z\in\overline{Z_k}}|f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}\left(\sup_{z\in\overline{Z_p}}|f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{z\in\overline{Z_q}}|f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}&\le\frac{C_1}{C_0}\left(\sup_{z\in\overline{Z_k}}|f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}\left(\sup_{z\in\overline{Z_p}}|f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}\\
&\le\frac{C_1}{C_0}\|f\|_k^{\theta}\,\|f\|_p^{1-\theta}. \qquad (5.56)
\end{aligned}$$

(b) Let Z_p be an unbounded component of $U_p(K)$, w.l.o.g. the real part of Z_p is bounded from below and unbounded from above. Let $\zeta_0\in\mathbb{R}$ such that $|\zeta_0|\ge k+\frac{1}{k}$. Then we have $\overline{D_{\frac{1}{j}}(\zeta_0)}\subset\left([j,\infty[+i[-\frac{1}{j},\frac{1}{j}]\right)$ for $j\in\{p,q,k\}$. Applying Hadamard's Three Circles Theorem on $u:=|f|$ again, we get $M\left(\frac{1}{q}\right)\le M\left(\frac{1}{k}\right)^{\theta}M\left(\frac{1}{p}\right)^{1-\theta}$ where

$$\theta:=\frac{\ln\left(\frac{1}{p}/\frac{1}{q}\right)}{\ln\left(\frac{1}{p}/\frac{1}{k}\right)}=\frac{\ln(q/p)}{\ln(k/p)}$$

and $0<\theta<1$. Furthermore, the following inequalities are valid: For $z\in D_r(\zeta_0)$, $r>0$,

$$r\ge|z-\zeta_0|\underset{\zeta_0\in\mathbb{R}}{\ge}|\mathrm{Re}(z)-\zeta_0|\ge\begin{cases}|\mathrm{Re}(z)|-|\zeta_0|\\|\zeta_0|-|\mathrm{Re}(z)|\end{cases}$$

implying

$$-r-|\zeta_0|\le-|\mathrm{Re}(z)|\le r-|\zeta_0|. \qquad (5.57)$$

Like in the first part of the proof we obtain

$$M\left(\frac{1}{q}\right)\ge\inf_{|z-\zeta_0|\le\frac{1}{q}}e^{-\frac{1}{q}|\mathrm{Re}(z)|}\sup_{|z-\zeta_0|\le\frac{1}{q}}|f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}\underset{(5.57)}{\ge}e^{-\frac{1}{q}\left(\frac{1}{q}+|\zeta_0|\right)}\sup_{|z-\zeta_0|\le\frac{1}{q}}|f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}$$

and

$$M\left(\frac{1}{k}\right)^{\theta} M\left(\frac{1}{p}\right)^{1-\theta} \leq \underbrace{\left(\sup_{|z-\zeta_0|\leq\frac{1}{k}} e^{-\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}}_{\underset{(5.57)}{\leq}\ \left(e^{\frac{1}{k}\left(\frac{1}{k}-|\zeta_0|\right)}\right)^{\theta}} \underbrace{\left(\sup_{|z-\zeta_0|\leq\frac{1}{p}} e^{-\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}}_{\underset{(5.57)}{\leq}\ \left(e^{\frac{1}{p}\left(\frac{1}{p}-|\zeta_0|\right)}\right)^{1-\theta}}$$
$$\left(\sup_{|z-\zeta_0|\leq\frac{1}{k}} |f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}\left(\sup_{|z-\zeta_0|\leq\frac{1}{p}} |f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}.$$

Combining these inequalities, we get

$$\sup_{|z-\zeta_0|\leq\frac{1}{q}} |f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}$$
$$\leq e^{\frac{1}{q^2}+\frac{\theta}{k^2}+\frac{1-\theta}{p^2}+|\zeta_0|\left(\frac{1}{q}-\left(\frac{\theta}{k}+\frac{1-\theta}{p}\right)\right)}\left(\sup_{|z-\zeta_0|\leq\frac{1}{k}} |f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}\left(\sup_{|z-\zeta_0|\leq\frac{1}{p}} |f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}. \tag{5.58}$$

The next step is to prove that

$$\frac{1}{q}-\left(\frac{\theta}{k}+\frac{1-\theta}{p}\right)\leq 0. \tag{5.59}$$

Since $p<q$, we have $q=cp$, where $c:=\frac{q}{p}>1$, and by definition of θ

$$\frac{1}{q}-\left(\frac{\theta}{k}+\frac{1-\theta}{p}\right)=\frac{k-p}{kp}\theta+\frac{1-c}{cp}=\frac{k-p}{kp}\frac{\ln(c)}{\ln(k/p)}+\frac{1-c}{cp}.$$

Hence (5.59) is equivalent to $1-\frac{p}{k}+\frac{c-1}{c\ln(c)}\ln\left(\frac{p}{k}\right)\leq 0$. Now we take a closer look at the function $g\colon\mathbb{R}_{>0}\to\mathbb{R}$, $g(x):=1-x+\frac{c-1}{c\ln(c)}\ln x$. Then $g'(x)=-1+\frac{c-1}{c\ln(c)x}$ and $g'(x)\geq 0$ is equivalent to $x\leq\frac{c-1}{c\ln(c)}$. The only zeros of g are $x=\frac{1}{c}<1$ and $x=1$. As $\frac{1}{c}\leq\frac{c-1}{c\ln(c)}$, we have $g(x)\leq 0$ for $0<x\leq\frac{1}{c}$. Keeping in mind that $cp=q<k$, we gain $\frac{p}{k}<\frac{1}{c}$ and so

$$1-\frac{p}{k}+\frac{c-1}{c\ln(c)}\ln\left(\frac{p}{k}\right)=g\left(\frac{p}{k}\right)\leq 0$$

which proves (5.59). Merging (5.58) and (5.59), it follows

$$\sup_{|z-\zeta_0|\leq\frac{1}{q}} |f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}$$
$$\leq e^{\frac{1}{q^2}+\frac{\theta}{k^2}+\frac{1-\theta}{p^2}}\left(\sup_{|z-\zeta_0|\leq\frac{1}{k}} |f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\right)^{\theta}\left(\sup_{|z-\zeta_0|\leq\frac{1}{p}} |f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\right)^{1-\theta}$$

and thus

$$\begin{aligned}
&\sup_{\substack{z\in\mathbb{C},\\ \mathrm{d}(z,[k+1/k,\infty[)\leq 1/q}} |f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|} = \sup_{\substack{\zeta_0\in\mathbb{R},\\ \zeta_0\geq k+1/k}} \sup_{|z-\zeta_0|\leq\frac{1}{q}} |f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}\\
&\leq e^{\frac{1}{q^2}+\frac{\theta}{k^2}+\frac{1-\theta}{p^2}}\Bigg(\sup_{\substack{z\in\mathbb{C},\\ \mathrm{d}(z,[k+1/k,\infty[)\leq 1/k}} |f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}\Bigg)^{\theta}\Bigg(\sup_{\substack{z\in\mathbb{C},\\ \mathrm{d}(z,[k+1/k,\infty[)\leq 1/p}} |f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}\Bigg)^{1-\theta}\\
&\leq e^{\frac{1}{q^2}+\frac{\theta}{k^2}+\frac{1-\theta}{p^2}}\,\|f\|_k^{\theta}\,\|f\|_p^{1-\theta}. \qquad (5.60)
\end{aligned}$$

(c) Let Z_p be w.l.o.g. like in part (b) and define $\tilde{Z}_p := Z_p\cap(]-\infty,k+1/k[+i\mathbb{R})$. By Remark 3.3(1) there are only finitely many components $\tilde{Z}_q$ of $U_q(K)\cap(]-\infty,k+1/k[+i\mathbb{R})$ with $\tilde{Z}_q\subset\tilde{Z}_p$. For every such component $\tilde{Z}_q$ choose $\zeta_0\in\tilde{Z}_q\cap\mathbb{R}$. Let $\tilde{Z}_k$ be the (unique) component of $U_k(K)\cap(]-\infty,k+1/k[+i\mathbb{R})$ which contains ζ_0. The rest is analogous to part (a) and thus there are $\tilde{C}_0$, $\tilde{C}_1>0$ and $0<\theta<1$ such that

$$\sup_{z\in\overline{\tilde{Z}_q}}|f(z)|e^{\frac{1}{q}|\mathrm{Re}(z)|}\leq\frac{\tilde{C}_1}{\tilde{C}_0}\|f\|_k^{\theta}\,\|f\|_p^{1-\theta}. \qquad (5.61)$$

(d) Let us first remark the following: Let B be a set, $B_0\subset B$, $0<\theta_0<\theta_1<1$, $h\colon B_0\to\mathbb{R}_{\geq 0}$, $g\colon B\to\mathbb{R}_{\geq 0}$ and $h\leq g$ on B_0. Then

$$\Big(\sup_{z\in B_0}h(z)\Big)^{\theta_1}\Big(\sup_{z\in B}g(z)\Big)^{1-\theta_1}\leq\Big(\sup_{z\in B_0}h(z)\Big)^{\theta_0}\Big(\sup_{z\in B}g(z)\Big)^{1-\theta_0}.$$

Now take the minimum of all the θs which appear in part (a)-(c). There are finitely many of them and denote this minimum again with θ. Take the maximum of the constants $\frac{C_1}{C_0}$, $e^{\frac{1}{q^2}+\frac{\theta}{k^2}+\frac{1-\theta}{p^2}}$ and $\frac{\tilde{C}_1}{\tilde{C}_0}$ which appear in part (a)-(c). There are again finitely many of them and denote this maximum with C. We apply the remark above on $B_0:=\overline{U_k(K)}$, $B:=\overline{U_p(K)}$, $h(z):=|f(z)|e^{\frac{1}{k}|\mathrm{Re}(z)|}$ and $g(z):=|f(z)|e^{\frac{1}{p}|\mathrm{Re}(z)|}$. Then we get due to (5.56), (5.60) and (5.61)

$$\|f\|_q\leq C\|f\|_k^{\theta}\,\|f\|_p^{1-\theta},$$

so the statement is proven.

(2) Let $p\in\mathbb{N}$ and choose $q>p$. Let $k\in\mathbb{N}$. If $k\leq p$, then we get for an arbitrary $0<\theta<1$ and all $y\in\big(\mathcal{P}(K)_b'\big)'$ by definition

$$\|y\|_q^*\underset{p<q}{\leq}\|y\|_p^*=\|y\|_p^{*\,1-\theta}\|y\|_p^{*\,\theta}\underset{k\leq p}{\leq}\|y\|_p^{*\,1-\theta}\|y\|_k^{*\,\theta}.$$

Let $k>p$. If $k\leq q$, we have for an arbitrary $0<\theta<1$ and all $y\in\big(\mathcal{P}(K)_b'\big)'$ by definition

$$\|y\|_q^*\underset{k\leq q}{\leq}\|y\|_k^*=\|y\|_k^{*\,1-\theta}\|y\|_k^{*\,\theta}\underset{p<k}{\leq}\|y\|_p^{*\,1-\theta}\|y\|_k^{*\,\theta}.$$

Let $k > q$ and $y \in \left(\mathcal{P}(K)_b'\right)'$. If $\|y\|_p^* = \infty$, then (5.42) is obviously fulfilled. Let $\|y\|_p^* < \infty$. As $\mathcal{P}_*(K)$ is a DFS-space by Theorem 3.5(1), the sets $B_n := \{f \in \mathcal{O}_n(U_n(K)) \mid \|f\|_n \leq 1\}$, $n \in \mathbb{N}$, are a fundamental system of bounded sets of $\mathcal{P}_*(K)$ and hence the semi-norms

$$|\!|\!|x|\!|\!|_n := \sup_{f \in B_n} |x(f)|, \, x \in \mathcal{P}_*(K)',$$

form a fundamental system of semi-norms of $\mathcal{P}_*(K)_b'$. Furthermore, $\mathcal{P}_*(K)$ is reflexive and thus there is an unique $f \in \mathcal{P}_*(K)$ such that $J(f) = y$ where $J\colon \mathcal{P}_*(K) \to \mathcal{P}_*(K)''$ denotes the canonical embedding. Then we obtain by [45, 22.14 Satz, p. 237] for all $n \geq p$

$$\begin{aligned}\infty > \|y\|_p^* \underset{p \leq n}{\geq} \|y\|_n^* &= \sup\Big\{ \underbrace{|y(x)|}_{=|J(f)(x)|=|x(f)|} \mid |\!|\!|x|\!|\!|_n \leq 1\Big\} = \sup\left\{|x(f)| \mid x \in B_n^\circ\right\} = \|f\|_{B_n}\\ &= \inf\{t > 0 \mid f \in tB_n\}\end{aligned}$$

where $\|f\|_{B_n}$ denotes the Minkowski functional of B_n. In particular, this means that $\{t > 0 \mid f \in tB_n\} \neq \varnothing$ and thus we have $f \in \mathcal{O}_n(U_n(K))$ as well as

$$\|y\|_n^* = \inf\{t > 0 \mid f \in tB_n\} = \|f\|_n$$

for all $n \geq p$. So by part (1), there are $C > 0$ and $0 < \theta < 1$, only depending on p, q and k, such that

$$\|y\|_q^* = \|f\|_q \leq C \|f\|_p^{1-\theta} \|f\|_k^\theta = C \|y\|_p^{*\,1-\theta} \|y\|_k^{*\,\theta}.$$

□

5.22 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact. Then $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)$ satisfies (Ω).*

Proof. By Theorem 4.1 $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ is topologically isomorphic to $\mathcal{P}_*(K)_b'$. Since (Ω) is a linear-topological invariant by [45, 29.11 Lemma (1), p. 347], $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ satisfies (Ω) due to Lemma 5.21(2). The sequence

$$0 \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right) \xrightarrow{i} \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right) \xrightarrow{q} \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right) \to 0$$

is an exact sequence of Fréchet spaces where i means the inclusion and q the quotient mapping. $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ satisfies (Ω) by Theorem 5.20 and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ as well, thus $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K\right)$ by [64, 1.7. Lemma, p. 230], too. □

The next lemma provides some useful relations between spaces of operators.

5.23 Lemma.

(1) Let X be a complete, reflexive locally convex space and $\left(Y, (|\!|\!|\cdot|\!|\!|_n)_{n\in\mathbb{N}}\right)$ a Fréchet space. Then $L_b\left(X_b', Y_b'\right) \cong L_b\left(Y, \left(X_b'\right)_b'\right) (\cong L_b(Y, X))$ via taking adjoints.

(2) Let X be a complete Montel space. Then $L_b\left(X_b', E\right) \cong X \varepsilon E$, where the topological isomorphism is the identity mapping.

Proof. (1) a) Consider the mapping

$$^{t}(\cdot)\colon L_b\left(X_b',Y_b'\right)\to L_b\left(Y,\left(X_b'\right)_b'\right),\ u\mapsto {}^{t}u,$$

defined by ${}^{t}u(y)(x'):=u(x')(y)$ for $y\in Y$ and $x'\in X'$.
Let $y\in Y$. Since $u\in L\left(X_b',Y_b'\right)$ and $\{y\}$ is bounded in Y, there is a bounded set $B\subset X$ and a constant $C>0$ such that

$$|{}^{t}u(y)(x')|=|u(x')(y)|\leq C\sup_{x\in B}|x'(x)|$$

for all $x'\in X'$. Thus ${}^{t}u(y)\in\left(X_b'\right)'$.
The canonical embedding $J\colon Y\to\left(Y_b'\right)_b'$ is a topological isomorphism between Y and $J(Y)$ by [45, 25.10 Corollar, p. 280] because Y is a Fréchet space. For a bounded set $M\subset X_b'$ we have

$$\sup_{x'\in M}|{}^{t}u(y)(x')|=\sup_{x'\in M}|u(x')(y)|=\sup_{x'\in M}|\langle J(y),u(x')\rangle|.$$

The next step is to prove that $u(M)$ is bounded in Y_b'. Let $N\subset Y$ bounded. Since $u\in L\left(X_b',Y_b'\right)$, there is again a bounded set $B\subset X$ and a constant $C>0$ such that

$$\sup_{x'\in M}\sup_{y\in N}|u(x')(y)|\leq C\sup_{x'\in M}\sup_{x\in B}|x'(x)|<\infty,$$

where the last estimate is due to the boundedness of $M\subset X_b'$. By the remark about the canonical embedding there are $n\in\mathbb{N}$ and $C_0>0$ such that

$$\sup_{x'\in M}|{}^{t}u(y)(x')|=\sup_{y'\in u(M)}|\langle J(y),y'\rangle|\leq C_0\,|||y|||_n,$$

so ${}^{t}u\in L\left(Y,\left(X_b'\right)_b'\right)$ and the mapping ${}^{t}(\cdot)$ is well-defined.

b) injectivity: Let $u,\ v\in L\left(X_b',Y_b'\right)$ with ${}^{t}u={}^{t}v$. This is equivalent to

$$u(x')(y)={}^{t}u(y)(x')={}^{t}v(y)(x')=v(x')(y)$$

for all $y\in Y$ and all $x'\in X'$. This implies $u(x')=v(x')$ for all $x'\in X'$, hence $u=v$.

c) surjectivity: Consider the mapping

$$^{t}(\cdot)\colon L_b\left(Y,\left(X_b'\right)_b'\right)\to L_b\left(X_b',Y_b'\right),\ u\mapsto {}^{t}u,$$

defined by ${}^{t}u(x')(y):=u(y)(x')$ for $x'\in X'$ and $y\in Y$.
Let $x'\in X'$. Since $u\in L_b\left(Y,\left(X_b'\right)_b'\right)$ and $\{x'\}$ is bounded in X', there are $n\in\mathbb{N}$ and a constant $C>0$ such that

$$|{}^{t}u(x')(y)|=|u(y)(x')|\leq C\,|||y|||_n$$

for all $y\in Y$. Thus ${}^{t}u(x')\in Y'$.

Let $B \subset Y$ bounded. The reflexivity of X implies that for every $u(y)$, $y \in B$, there is a unique $x_y \in X$ such that $J_0(x_y) = u(y)$ where $J_0: X \to (X_b')_b'$ denotes the canonical embedding. Then we get

$$\sup_{y\in B} |{}^t u(x')(y)| = \sup_{y\in B} |u(y)(x')| = \sup_{y\in B} |\langle J_0(x_y), x'\rangle| = \sup_{y\in B} |x'(x_y)|.$$

We claim that $D := \{x_y \mid y \in B\}$ is a bounded set in X. Let $N \subset X'$ be finite. Then the set $M := \{{}^t u(x') \mid x' \in N\} \subset Y'$ is finite. The set B is weakly bounded since it is bounded. We have

$$\sup_{y\in B} \sup_{x'\in N} |x'(x_y)| = \sup_{y\in B} \sup_{x'\in N} |{}^t u(x')(y)| = \sup_{y\in B} \sup_{y'\in M} |y'(y)| < \infty,$$

where the last estimate follows from the fact that B is weakly bounded. Thus D is weakly bounded and by the Mackey theorem bounded in X. Therefore, we obtain

$$\sup_{y\in B} |{}^t u(x')(y)| = \sup_{y\in B} |x'(x_y)| = \sup_{x\in D} |x'(x)|$$

for all $x' \in X'$ connoting ${}^t u \in L(X_b', Y_b')$.
Let $u \in L\left(Y, (X_b')_b'\right)$. Then we have ${}^t u \in L_b(X_b', Y_b')$. In addition, for all $y \in Y$ and all $x' \in X'$

$${}^t({}^t u)(y)(x') = {}^t u(x')(y) = u(y)(x')$$

is valid and so ${}^t({}^t u)(y) = u(y)$ for all $y \in Y$ proving the surjectivity.

d) continuity: Let $M \subset Y$ and $B \subset X_b'$ be bounded sets. Then

$$\begin{aligned}\sup_{y\in M} \sup_{x'\in B} |{}^t u(y)(x')| &= \sup_{y\in M} \sup_{x'\in B} |u(x')(y)| = \sup_{x'\in B} \sup_{y\in M} |u(x')(y)| \\ &= \sup_{x'\in B} \sup_{y\in M} |{}^t({}^t u)(x')(y)|\end{aligned}$$

holds for all $u \in L(X_b', Y_b')$. Therefore, ${}^t(\cdot)$ and its inverse are continuous.

The adjunct in brackets follows by the reflexivity of X, since the mapping

$$S: L_b\left(Y, (X_b')_b'\right) \to L_b(Y, X),$$

defined by $S(u)(y) := J_0^{-1}(u(y))$ for $u \in L_b\left(Y, (X_b')_b'\right)$ and $y \in Y$, is a topological isomorphism.

(2) Let $T \in L(X_b', E)$. For $\alpha \in A$ there are a bounded set $B \subset X$ and $C > 0$ such that

$$p_\alpha(T(x')) \le C \sup_{x\in B} |x'(x)| \le C \sup_{x\in \overline{B}} |x'(x)|$$

for every $x' \in X'$. The set $\overline{B}$ is compact, since B is bounded and X a Montel space, and thus precompact by [25, 3.5.3. Corollary, p. 65]. Hence we gain $T \in L(X_c', E)$.
Let $M \subset X'$ be equicontinuous. Due to [25, 8.5.1. Theorem (a), p. 156] M is bounded in X_b'. Therefore,

$$\mathrm{id}: L_b(X_b', E) \to L_e(X_c', E) = X \varepsilon E$$

is continuous.
Let $T \in L(X_c', E)$. For $\alpha \in A$ there are a precompact set $B \subset X$ and $C > 0$ such that

$$p_\alpha(T(x')) \leq C \sup_{x \in B} |x'(x)|$$

for every $x' \in X'$. By [60, Chap. 14, Corollary 1, p. 137] B is bounded in X since it is precompact and so we get $T \in L\left(X_b', E\right)$.
Let M be a bounded set in X_b'. Then M is equicontinuous by virtue of [60, Theorem 33.2, p. 349], as X, being a Montel space, is barreled by [45, 24.24 Bemerkung (a), p. 267]. Thus

$$\mathrm{id}\colon L_e(X_c', E) \to L_b\left(X_b', E\right)$$

is continuous.

□

Now we use the results obtained so far and splitting theory to enlarge our collection of admissible spaces. We recall that a Fréchet space $\left(F, (|||\cdot|||_k)_{k\in\mathbb{N}}\right)$ has (DN) by [45, Chap. 29, Definition, p. 338] if:

$$\exists\, p \in \mathbb{N}\ \forall k \in \mathbb{N}\ \exists\, n \in \mathbb{N}, C > 0\ \forall\, x \in F:\ \ |||x|||_k^2 \leq C\, |||x|||_p\, |||x|||_n$$

A PLS-space $X = \lim\mathrm{proj}_{N\in\mathbb{N}} X_N$, where $X_N = \lim\mathrm{ind}_{n\in\mathbb{N}} \left(X_{N,n}, |||\cdot|||_{N,n}\right)$ are DFS-spaces, has (PA), if:

$$\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \forall\, \eta > 0\ \exists\, k, C, r_0 > 0\ \forall\, r > r_0\ \forall\, x' \in X_N':$$
$$|||x' \circ i_N^M|||_{M,m}^* \leq C\left(r^\eta\, |||x' \circ i_N^K|||_{K,k}^* + \frac{1}{r}\, |||x'|||_{N,n}^*\right)$$

where $|||\cdot|||^*$ denotes the dual norm of $|||\cdot|||$ [8, Section 4, (24), p. 577].

5.24 Theorem. *Let $K \subset \overline{\mathbb{R}}$ be compact. If $E := F_b'$ where F is a Fréchet space satisfying (DN) or E is an ultrabornological PLS-space satisfying (PA), then*

$$\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

is surjective.

Proof. The sequence

$$0 \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \overset{i}{\to} \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \overset{\overline{\partial}}{\to} \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to 0 \tag{5.62}$$

where i means the inclusion, is an exact sequence of Fréchet spaces by Theorem 5.16 and hence topologically exact as well. Denote by $J_0\colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''$ and $J_1\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''$ the canonical embeddings which are topological isomorphisms since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ and $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ are reflexive. Then the exactness of (5.62) implies that

$$0 \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'' \overset{i_0}{\to} \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'' \overset{\overline{\partial}_1}{\to} \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)'' \to 0, \tag{5.63}$$

where $i_0 := J_0 \circ i \circ J_0^{-1}$ and $\overline{\partial}_1 := J_1 \circ \overline{\partial} \circ J_1^{-1}$, is an exact topological sequence. Topological as the bidual of a Fréchet space is again a Fréchet space by [45, 25.10 Corollar, p. 280].

- Let $E := F_b'$ where F is a Fréchet space with (DN). Then $\operatorname{Ext}^1\left(F, \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) = 0$ by [63, 5.1. Theorem, p. 186] since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ satisfies (Ω) due to Theorem 5.22 and therefore $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''$ as well. Combined with the exactness of (5.63) this implies that the sequence

$$0 \to L\left(F, \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \overset{i_0^*}{\to} L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \overset{\overline{\partial}_1^*}{\to} L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \to 0$$

is exact by [53, Proposition 2.1, p. 13-14] where $i_0^*(B) := i_0 \circ B$ and $\overline{\partial}_1^*(D) := \overline{\partial}_1 \circ D$ for $B \in L\left(F, \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right)$ and $D \in L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right)$. In particular, we obtain that

$$\overline{\partial}_1^*: L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \to L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \tag{5.64}$$

is surjective. Via Theorem 3.11 and Lemma 5.23 ($X = \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ and $Y = F$) we have, with the notation used there, the isomorphism

$$\psi := T \circ {}^t(\cdot): L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right),\ \psi(u) = [T \circ {}^t(\cdot)](u) = {}^t u \circ \Delta,$$

and the inverse

$$\psi^{-1}(f) = (T \circ {}^t(\cdot))^{-1}(f) = \left[{}^t(\cdot) \circ T^{-1}\right](f) = {}^t\left(J^{-1} \circ u_f\right), \quad f \in \mathcal{E}^{exp}(K, E),$$

where $J: E \to E''$ is the canonical embedding.
Let $g \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$. Then $\psi^{-1}(g) \in L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right)$ and by the surjectivity of (5.64) there is $u \in L\left(F, \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)''\right)$ such that $\overline{\partial}_1^* u = \psi^{-1}(g)$. So we get $\psi(u) \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$. Now we want to show that $\overline{\partial}\psi(u) = g$ is valid. Consider the equation

$$\left[\frac{{}^t u(\delta_z) - {}^t u\left(\delta_{z+he_k}\right)}{h}\right](x) = {}^t u\left(\frac{\delta_z - \delta_{z+he_k}}{h}\right)(x) = u(x)\left(\frac{\delta_z - \delta_{z+he_k}}{h}\right)$$

for $x \in F$, $z \in \mathbb{C} \setminus K$, $h \neq 0$ and e_k denoting the unit vectors in $\mathbb{R}^2$. Since every bounded set in $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$, being a Montel space by Remark 3.8, is relatively compact, we get like in the first part of the proof of Theorem 3.11 $\partial^\beta(\psi(u))(z) = {}^t u\left(\delta_z^{(\beta)}\right)$ for $\beta \in \mathbb{N}_0^2$ by virtue of [16, 10.3.4 Satz, p. 53] and hence $\overline{\partial}(\psi(u))(z) = {}^t u\left(\overline{\partial}\delta_z\right)$ where $\overline{\partial}\delta_z(f) := \delta_z\left(\overline{\partial} f\right) = \overline{\partial} f(z)$ for $f \in \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$. So for all $x \in F$ and $z \in \mathbb{C} \setminus K$ we have

$$\begin{aligned}
\overline{\partial}(\psi(u))(z)(x) &= {}^t u\left(\overline{\partial}\delta_z\right)(x) = u(x)\left(\overline{\partial}\delta_z\right) = \left\langle \overline{\partial}\delta_z, J_1^{-1}(u(x))\right\rangle = \left\langle \delta_z, \overline{\partial} J_1^{-1}(u(x))\right\rangle \\
&= \left\langle \left[J_1 \circ \overline{\partial} \circ J_1^{-1}\right](u(x)), \delta_z\right\rangle = \left\langle \left(\overline{\partial}_1 \circ u\right)(x), \delta_z\right\rangle = \left\langle \left(\overline{\partial}_1^* u\right)(x), \delta_z\right\rangle \\
&= \psi^{-1}(g)(x)(\delta_z) = {}^t\left(J^{-1} \circ u_g\right)(x)(\delta_z) = \left(J^{-1} \circ u_g\right)(\delta_z)(x) \\
&= g(z)(x).
\end{aligned}$$

For the last equation see the end of the proof of part 1)c) of Theorem 3.11.
Thus $\overline{\partial}(\psi(u))(z) = g(z)$ for every $z \in \mathbb{C} \setminus K$ which proves the surjectivity.

- Let E be an ultrabornological PLS-space satisfying (PA). Since $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)$ is a Fréchet-Schwartz space, its strong dual $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)'$ is a DFS-space (also called LS-space). By [8, Theorem 4.1, p. 577] we obtain $\mathrm{Ext}^1_{PLS}\left(\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)', E\right) = 0$ as the bidual $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)''$ satisfies (Ω) and E is a PLS-space satisfying (PA). Moreover, we have $\mathrm{Proj}^1 E = 0$ due to [65, Corollary 3.3.10, p. 46] because E is an ultrabornological PLS-space. Then the exactness of the sequence (5.63), [8, Theorem 3.4, p. 567] and [8, Lemma 3.3, p. 567] (in the lemma condition (c) is fulfilled, since $\mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)'$ is the strong dual of a nuclear Fréchet space, and one chooses $H = \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)''$ and $F = G = \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''$), imply that the sequence

$$0 \to L\left(E', \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \xrightarrow{i_0^*} L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \xrightarrow{\overline{\partial}_1^*} L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \to 0$$

is exact. The mappings i_0^* and $\overline{\partial}_1^*$ are defined like in the first part. Especially, we get that

$$\overline{\partial}_1^* \colon L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \to L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \tag{5.65}$$

is surjective.
By [13, Remark 4.4, p. 1114] we have $L_b\left(\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)', E''\right) \cong L_b\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right)$ via taking adjoints, since $\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)$, being a Fréchet-Schwartz space, is a PLS-space and hence its strong dual a LFS-space, which is regular by [65, Corollary 6.7, p. 114], and E is an ultrabornological PLS-space. In addition, the mapping

$$S \colon L_b\left(\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)', E''\right) \to L_b\left(\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)', E\right),$$

defined by $S(u)(y) := J^{-1}(u(y))$ for $u \in L_b\left(\mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)', E''\right)$ and $y \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)'$, is a topological isomorphism because E is reflexive by [11, Theorem 3.2, p. 58]. Due to Theorem 3.11 and Lemma 5.23(2) we obtain the isomorphism

$$\psi := T \circ J^{-1} \circ {}^t(\cdot) \colon L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right) \to \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E),$$
$$\psi(u) = \left[T \circ J^{-1} \circ {}^t(\cdot)\right](u) = J^{-1} \circ {}^t u \circ \Delta,$$

with the inverse

$$\psi^{-1}(f) = (T \circ J \circ {}^t(\cdot))^{-1}(f) = \left[{}^t(\cdot) \circ J \circ T^{-1}\right](f) = {}^t\left(J \circ J^{-1} \circ u_f\right) = {}^t u_f$$

for $f \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E)$.
Let $g \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E)$. Then $\psi^{-1}(g) \in L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right)$ and by the surjectivity of (5.65) there exists $u \in L\left(E', \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K)''\right)$ such that $\overline{\partial}_1^* u = \psi^{-1}(g)$. So we have $\psi(u) \in \mathcal{E}^{exp}(\overline{\mathbb{C}} \setminus K, E)$. The last step is to show that $\overline{\partial}\psi(u) = g$ is valid. Like in the first part we

gain for $z \in \mathbb{C} \setminus K$
$$\overline{\partial}\left(\psi(u)\right)(z) = J^{-1}\left[{}^{t}u\left(\overline{\partial}\delta_z\right)\right]$$
and for $x \in E'$
$$\begin{aligned}{}^{t}u\left(\overline{\partial}\delta_z\right)(x) &= u(x)\left(\overline{\partial}\delta_z\right) = \psi^{-1}(g)(x)(\delta_z) = {}^{t}u_g(x)(\delta_z) = \delta_z(x \circ g) = x(g(z))\\ &= J(g(z))(x).\end{aligned}$$
Thus we have ${}^{t}u\left(\overline{\partial}\delta_z\right) = J(g(z))$ and therefore $\overline{\partial}\left(\psi(u)\right)(z) = g(z)$ for all $z \in \mathbb{C} \setminus K$.

□

Now let us consider the non-weighted case, i.e. the question for which locally convex spaces E is
$$\overline{\partial}\colon \mathrm{C}^{\infty}(U,E) \to \mathrm{C}^{\infty}(U,E) \tag{5.66}$$
surjective for every open set $U \subset \mathbb{C}$. By [20, Theorem 1.4.4, p. 12] this is fulfilled for $E = \mathbb{C}$. Furthermore, $\mathcal{O}(U)$ and $\mathrm{C}^{\infty}(U)$, both equipped with the topology of uniform convergence on compact subsets (of all partial derivatives for the latter one), are nuclear Fréchet spaces by [45, 5.18 Beipiele (3)+(4), p. 38, 28.9 Beispiele (1)+(4), p. 330-331] and one has
$$\mathcal{O}(U,E) \cong \mathcal{O}(U)\varepsilon E \cong \mathcal{O}(U)\hat{\otimes}_{\varepsilon}E \cong \mathcal{O}(U)\hat{\otimes}_{\pi}E$$
plus
$$\mathrm{C}^{\infty}(U,E) \cong \mathrm{C}^{\infty}(U)\varepsilon E \cong \mathrm{C}^{\infty}(U)\hat{\otimes}_{\varepsilon}E \cong \mathrm{C}^{\infty}(U)\hat{\otimes}_{\pi}E$$
by [25, 16.7.5 Corollary, p. 366] resp. [60, Theorem 44.1, p. 449] for any complete locally convex space E. Like in Theorem 5.17 this implies that the $\overline{\partial}$-operator in (5.66) is surjective if E is a Fréchet space. If $E := F_b'$ where F is a Fréchet space satisfying (DN) or E is an ultrabornological PLS-space satisfying (PA), this holds due to [62, 2.6 Theorem and remarks in the beginning] resp. [13, Corollary 3.9, p.1112] as well.
Summarizing this remark, Theorem 5.17 and Theorem 5.24, we obtain:

5.25 Theorem. *The following spaces E are strictly admissible:*

- *Fréchet spaces*
- *$E := F_b'$ where F is a Fréchet space satisfying (DN).*
- *Ultrabornological PLS-spaces satisfying (PA)*

We will now provide examples of ultrabornological PLS-spaces satisfying (PA) and examples of such spaces which do not have (PA) as well as examples of LFS-space not being strictly admissible. These examples are directly taken from [13, Corollary 4.8, p. 1116] and [13, Corollary 4.9, p. 1117].

5.26 Example. The following spaces are ultrabornological PLS-spaces with property (PA), in particular, strictly admissible:

- an arbitrary Fréchet-Schwartz space;
- the strong dual of a power series space of inifinite type $\Lambda_{\infty}'(\alpha)$;

- a PLS-type power series space $\Lambda_{r,s}(\alpha,\beta)$ whenever $s=\infty$ or $\Lambda_{r,s}(\alpha,\beta)$ is a Fréchet space;
- the strong dual of any space of holomorphic functions $\mathcal{O}(U)'$, where U is a Stein manifold with the strong Liouville property (for instance, for $U=\mathbb{C}^n$);
- the space of germs of holomorphic functions $\mathcal{O}(K)$ where K is a completely pluripolar compact subset of a Stein manifold (for instance K consists of one point);
- the space of tempered distributions $\mathcal{S}'$ and the space of Fourier ultra-hyperfunctions $\mathcal{P}'_{**}$;
- the space of distributions $\mathcal{D}'(U)$ and ultradistributions of Beurling type $\mathcal{D}'_{(\omega)}(U)$ for any open set $U\subset\mathbb{R}^n$;
- the weighted distribution spaces $(K\{pM\})'$ of Gelfand and Shilov if the weight M satisfies
$$\sup_{|y|\leq 1} M(x+y)\leq C\inf_{|y|\leq 1} M(x+y) \text{ if } x\in\mathbb{R}^n.$$
- the kernel of any linear partial differential operator with constant coefficients in $\mathcal{D}'(U)$ or in $\mathcal{D}'_{(\omega)}(U)$ when $U\subset\mathbb{R}^n$ is open and convex;
- the space $L_b(X,Y)$ where X has (DN), Y has (Ω) and both are nuclear Fréchet spaces. In particular, $L_b(\Lambda_\infty(\alpha),\Lambda_\infty(\beta))$ if both spaces are nuclear.

5.27 Example.

(a) The following ultrabornological PLS-spaces do not have (PA):
  - the strong dual of power series space of finite type $\Lambda_0'(\alpha)$;
  - the space of ultradifferentiable functions of Roumieu type $\mathcal{E}_{\{\omega\}}(U)$, where ω is a non-quasianalytic weight and $U\subset\mathbb{R}^n$ is an arbitrary open set;
  - the strong dual of any space of holomorphic functions $\mathcal{O}(U)'$ where U is a Stein manifold which does not have the strong Liouville property (for instance, $U=\mathbb{D}^n$ the polydisc, $U=\mathbb{B}_n$ the unit ball etc.);
  - the space of germs of holomorphic functions $\mathcal{O}(K)$ where K is compact and not completely pluripolar (for instance, $K=\overline{\mathbb{D}}^n$ or $K=\overline{\mathbb{B}}_n$);
  - the space of distributions (or ultradistributions) with compact support $\mathcal{E}'(U)$ $\left(\text{or } \mathcal{E}'_{(\omega)}(U)\right)$ for $U\subset\mathbb{R}^n$ open;
  - the space of real analytic functions $\mathcal{A}(U)$ for any open set $U\subset\mathbb{R}^n$.

(b) For the following LFS-spaces E the map (5.66) is not surjective and thus E not strictly admissible:
  - the space of test functions $\mathcal{D}(U)$;
  - the spaces of test functions for ultradistributions $\mathcal{D}_{(\omega)}(U)$, the space of ultradistributions of Roumieu type with comp. support $\mathcal{E}'_{\{\omega\}}(U)$, where ω is a non-quasianalytic weight, $U\subset\mathbb{R}^n$ is an arbitrary open set;
  - the strong dual $\mathcal{A}(U)'_b$ for an arbitrary open set $U\subset\mathbb{R}^n$.

In Theorem 6.11 we will see that a reasonable theory of $E-$valued Fourier hyperfunctions is possible if E is strictly admissible. This raises the question if the condition of E being strictly admissible is also necessary for a reasonable theory of $E-$valued Fourier hyperfunctions. At least for ultrabornological PLS-spaces E the answer will be given by Theorem 6.14, namely, a reasonable theory of $E-$valued Fourier hyperfunctions is possible, if and only if E is strictly admissible, and E is strictly admissible, if and only if E has (PA). In particular, for all spaces in Example 5.26 a theory of that kind is possible whereas for the spaces in Example 5.27(a) no construction of a reasonable sheaf of $E-$valued Fourier hyperfunctions exists. For the spaces in Example 5.27(b) this question is still open.

6 The duality method

In this section we construct E-valued Fourier hyperfunctions in one variable as the sheaf generated by equivalence classes of compactly supported $\mathcal{P}_*$-functionals. This construction relies on the Köthe duality proven in Section 4 (see Theorem 4.1) and the method is sometimes called duality method (see [22] and [13]). Furthermore, a description of E-valued Fourier hyperfunctions as boundary values of slowly increasing holomorphic functions is provided and finally the necessity of the conditions that were used for the construction of vector-valued Fourier hyperfunctions will be examined.

6.1 Definition. For an open set $\Omega \subset \overline{\mathbb{R}}$, $\Omega \neq \varnothing$, and a locally convex space E we define the space of E-valued Fourier hyperfunctions on Ω by

$$\mathcal{R}(\Omega,E) := L\big(\mathcal{P}_*\big(\overline{\Omega}\big),E\big)/L\left(\mathcal{P}_*(\partial\Omega),E\right)$$

plus $\mathcal{R}(\varnothing,E) := 0$.

For $T \in L\big(\mathcal{P}_*\big(\overline{\Omega}\big),E\big)$ we denote by $[T]$ the corresponding element of $\mathcal{R}(\Omega,E)$. If the set Ω is equipped with an index, then we sometimes do the same with the corresponding equivalence class, to distinguish between different classes. Further, we use the notation $\mathcal{R}(\Omega) := \mathcal{R}(\Omega,\mathbb{C})$.
We observe that

$$L\left(\mathcal{P}_*(\varnothing),E\right) = L(0,E) = 0$$

by Definition 3.2 and hence $\mathcal{R}\big(\overline{\mathbb{R}},E\big) = L\big(\mathcal{P}_*\big(\overline{\mathbb{R}}\big),E\big)$ (more precisely, we identify $L\big(\mathcal{P}_*\big(\overline{\mathbb{R}}\big),E\big)$ and $\big\{\{T\} \mid T \in L\big(\mathcal{P}_*\big(\overline{\mathbb{R}}\big),E\big)\big\}\big)$.
For $\Omega \neq \overline{\mathbb{R}}$ there is no reasonable locally convex topology on $\mathcal{R}(\Omega,E)$ by [26, 3.10 Bemerkung, p. 41-42] (Using the bipolar theorem, the reflexivity, the Hahn-Banach theorem and the identity theorem, one sees that $\mathcal{P}_*(\partial\Omega)'$ is dense in $\mathcal{P}_*\big(\overline{\Omega}\big)_b'$. Then Remark 4.4 and [25, 16.2.5 Proposition (a), p. 349] imply the statement.).

Let us first take a look at the scalar case. Let $\Omega_1 \subset \overline{\mathbb{R}}$ be open. We claim that the addition

$$+ : \mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big)' \times \mathcal{P}_*\big(\overline{\Omega}\big)' \to \mathcal{P}_*\big(\overline{\Omega}_1\big)',\ (T_1,T_2) \mapsto T_1 + T_2,$$

is surjective for any open $\Omega \subset \Omega_1$. Remark that the spaces appearing above are Fréchet spaces by Theorem 3.5(1). The first step is to determine the dual map

$$+' : \mathcal{P}_*\big(\overline{\Omega}_1\big)'' \to \Big(\mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big)' \times \mathcal{P}_*\big(\overline{\Omega}\big)'\Big)'$$

of $+$. We denote by $J_0 : \mathcal{P}_*\big(\overline{\Omega}_1\big) \to \mathcal{P}_*\big(\overline{\Omega}_1\big)''$, $J_1 : \mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big) \to \mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big)''$ and $J_2 : \mathcal{P}_*\big(\overline{\Omega}\big) \to \mathcal{P}_*\big(\overline{\Omega}\big)''$ the canonical embeddings which are topological isomorphisms due to reflexivity by Theorem 3.5(1). Further, it is easily seen that the linear map

$$\Phi : \mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big)'' \times \mathcal{P}_*\big(\overline{\Omega}\big)'' \to \Big(\mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega\big)' \times \mathcal{P}_*\big(\overline{\Omega}\big)'\Big)',$$

defined by

$$\Phi(y_1,y_2)(T_1,T_2) := y_1(T_1)+y_2(T_2),\ (y_1,y_2)\in \mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)''\times\mathcal{P}_*\left(\overline{\Omega}\right)'',$$
$$(T_1,T_2)\in\left(\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)'\times\mathcal{P}_*\left(\overline{\Omega}\right)'\right)',$$

is an isomorphism. Let $y\in\mathcal{P}_*\left(\overline{\Omega}_1\right)''$. Then there is an unique element $f_0\in\mathcal{P}_*\left(\overline{\Omega}_1\right)$ such that $y=J_0(f_0)$. For $+'(y)$ there are $f_1\in\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)$ and $f_2\in\mathcal{P}_*\left(\overline{\Omega}\right)$ such that

$$+'(y)=\Phi(J_1(f_1),J_2(f_2)).$$

So for arbitrary $(T_1,T_2)\in\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)'\times\mathcal{P}_*\left(\overline{\Omega}\right)'$ we have on the one hand

$$+'(y)(T_1,T_2)=J_0(f_0)(T_1+T_2)=T_1(f_0)+T_2(f_0)$$

and on the other

$$\Phi(J_1(f_1),J_2(f_2))(T_1,T_2)=J_1(f_1)(T_1)+J_2(f_2)(T_2)=T_1(f_1)+T_2(f_2),$$

thus

$$T_1(f_0-f_1)+T_2(f_0-f_2)=0$$

implying $f_1=f_0|_{\overline{\Omega}_1\setminus\Omega}$ and $f_2=f_0|_{\overline{\Omega}}$ by the Hahn-Banach theorem since T_1 and T_2 are arbitrary. Here we used the notation $f_0|_{\overline{\Omega}_1\setminus\Omega}$ and $f_0|_{\overline{\Omega}}$ for f_0 regarded as an element of $\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)$ resp. $\mathcal{P}_*\left(\overline{\Omega}\right)$ via embedding. Hence we can interpret the dual map of $+$ as

$$+':\mathcal{P}_*\left(\overline{\Omega}_1\right)\to\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)\times\mathcal{P}_*\left(\overline{\Omega}\right),\ f\mapsto\left(f|_{\overline{\Omega}_1\setminus\Omega},f|_{\overline{\Omega}}\right).$$

This map is injective by the identity theorem. Let $(+'(f_n))_{n\in\mathbb{N}}$ be a sequence converging in $\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)\times\mathcal{P}_*\left(\overline{\Omega}\right)$ with respect to the product topology. Then it follows that $f_n|_{\overline{\Omega}_1\setminus\Omega}$ converges in $\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)$ as well as $f_n|_{\overline{\Omega}}$ in $\mathcal{P}_*\left(\overline{\Omega}\right)$ and their limits coincide on there common domain. Hence there is a well-defined function $f\in\mathcal{P}_*\left(\overline{\Omega}_1\right)$ such that

$$\lim_{n\to\infty}+'(f_n)=\left(\lim_{n\to\infty}f_n|_{\overline{\Omega}_1\setminus\Omega},\lim_{n\to\infty}f_n|_{\overline{\Omega}}\right)=\left(f|_{\overline{\Omega}_1\setminus\Omega},f|_{\overline{\Omega}}\right)=+'(f)$$

connoting that $+'$ has closed range. By [45, 26.3 Satz vom abgeschlossenen Wertebereich, p. 290] this means that $+$ has closed range and $R(+)=\ker(+')^\circ$ where we used the notation $R(+)$ for the range of $+$. Therefore, we obtain

$$R(+)=\overline{R(+)}=R(+)^{\circ\circ}=\left(\ker(+')^\circ\right)^{\circ\circ}=\{0\}^\circ=\mathcal{P}_*\left(\overline{\Omega}_1\right)'$$

by the bipolar theorem and as $+'$ is injective proving the surjectivity of $+$.
The surjectivtiy of $+$ is equivalent to the surjectivity of the canonical map

$$I:\mathcal{P}_*(\overline{\Omega})'/\mathcal{P}_*(\partial\Omega)'\to\mathcal{P}_*(\overline{\Omega}_1)'/\mathcal{P}_*\left(\overline{\Omega}_1\setminus\Omega\right)'.$$

I is injective by Proposition 4.3(1) (more detailed in the following Lemma) and hence an algebraic isomorphism due to the surjectivity of $+$. So restrictions and a sheaf structure may be defined on $\mathcal{R}_{\Omega_1} := \{\mathcal{R}_{\Omega} \mid \Omega \subset \Omega_1 \text{ open}\}$ like in Definition 6.4. It is not known whether the corresponding mapping I in the vector-valued case is always an isomorphism (see Remark 6.3 as well as the remarks before it). But this holds if we additionally assume that

$$\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \smallsetminus K, E\right)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$, i.e. that E is admissible. Let us turn to the already indicated statement:

6.2 Lemma. [1] *Let E be admissible, $\Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$ be open and $\Omega_2 \neq \varnothing$. Then the canonical mapping*

$$I\colon L\left(\mathcal{P}_*(\overline{\Omega}_2), E\right)/L\left(\mathcal{P}_*(\partial\Omega_2), E\right) \to L\left(\mathcal{P}_*(\overline{\Omega}_1), E\right)/L\left(\mathcal{P}_*\left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right),$$
$$[T]_2 \mapsto [T]$$

is an algebraic isomorphism.

Proof. This mapping is well-defined, in particular, independent of the choice of the representative since $\mathcal{P}_*(\overline{\Omega}_1)$ is continuously and densely embedded in $\mathcal{P}_*(\overline{\Omega}_2)$ (see the remark right above Proposition 4.3) and thus the embedding of $L\left(\mathcal{P}_*\left(\overline{\Omega}_2\right), E\right)$ into $L\left(\mathcal{P}_*\left(\overline{\Omega}_1\right), E\right)$ is defined as well as the mapping of $L\left(\mathcal{P}_*(\partial\Omega_2), E\right)$ into $L\left(\mathcal{P}_*\left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right)$ in this manner.
If $\mathbb{R} \subset \Omega_2$, then $\overline{\Omega}_2 = \overline{\Omega}_1 = \overline{\mathbb{R}}$ and therefore $\overline{\Omega}_1 \smallsetminus \Omega_2 = \overline{\Omega}_2 \smallsetminus \Omega_2 = \partial\Omega_2$. Hence the statement is obviously true. Now let $\mathbb{R} \not\subset \Omega_2$.
Let $T \in L\left(\mathcal{P}_*(\overline{\Omega}_2), E\right)$ with $[T] = 0$. Then we get by Proposition 4.3(1)

$$T \in L\left(\mathcal{P}_*(\overline{\Omega}_2), E\right) \cap L\left(\mathcal{P}_*\left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right) = L\left(\mathcal{P}_*\left(\overline{\Omega}_2 \cap \left(\overline{\Omega}_1 \smallsetminus \Omega_2\right)\right), E\right) = L\left(\mathcal{P}_*(\partial\Omega_2), E\right)$$

and thus $[T]_2 = 0$ implying the injectivity of I.
The surjectivity of I is equivalent to the surjectivity of the mapping

$$I_0\colon L\left(\mathcal{P}_*\left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right) \times L\left(\mathcal{P}_*(\overline{\Omega}_2), E\right) \to L\left(\mathcal{P}_*(\overline{\Omega}_1), E\right),\ (T_1, T_2) \mapsto T_1 + T_2.$$

By Theorem 4.1 the surjectivity of I_0 is equivalent to the surjectivity of

$$I_1\colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right) \times \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_2, E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$$
$$\to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_1, E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right),$$
$$(f_1, f_2) \mapsto f_1 + f_2,$$

and thus to the surjectivity of

$$I_2\colon \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \left(\overline{\Omega}_1 \smallsetminus \Omega_2\right), E\right) \times \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_2, E\right) \to \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \smallsetminus \overline{\Omega}_1, E\right),\ (f_1, f_2) \mapsto f_1 + f_2.$$

The proof is now done in several steps beginning with the construction of a cut-off function.

(i) If $\infty \notin \overline{\Omega}_2$ or $-\infty \notin \overline{\Omega}_2$, then there is $x_i \in \mathbb{R}$ such that $[x_0, \infty] \subset \overline{\Omega}_2^{C}$ resp. $[-\infty, x_1] \subset \overline{\Omega}_2^{C}$ since $\overline{\Omega}_2^{C} \subset \overline{\mathbb{R}}$ is open. If $\infty \in \Omega_2$ or $-\infty \in \Omega_2$, then there is $\tilde{x}_i \in \mathbb{R}$ such that $[\tilde{x}_0, \infty] \subset \Omega_2$

[1] counterpart: [13, Lemma 6.2, p. 1122]

resp. $[-\infty,\tilde{x}_1]\subset\Omega_2$, since Ω_2 is open, and thus $[\tilde{x}_0,\infty]\subset\left(\overline{\Omega}_1\setminus\Omega_2\right)^C$ resp. $[-\infty,\tilde{x}_1]\subset\left(\overline{\Omega}_1\setminus\Omega_2\right)^C$. We define the sets

$$F_0 := \begin{cases} (\mathbb{R}\setminus\Omega_2)\cup([x_0+2,\infty[\times[-1,1]), & \infty\in\overline{\Omega_1}\wedge\infty\notin\overline{\Omega_2}\wedge\left(-\infty\notin\overline{\Omega_1}\vee-\infty\in\overline{\Omega_2}\right),\\ (\mathbb{R}\setminus\Omega_2)\cup(]-\infty,x_1-2]\times[-1,1]), & -\infty\in\overline{\Omega_1}\wedge-\infty\notin\overline{\Omega_2}\wedge\left(\infty\notin\overline{\Omega_1}\vee\infty\in\overline{\Omega_2}\right),\\ (\mathbb{R}\setminus\Omega_2)\cup(]-\infty,x_1-2]\cup[x_0+2,\infty[)\times[-1,1], & \pm\infty\in\overline{\Omega_1}\wedge\pm\infty\notin\overline{\Omega_2},\\ \mathbb{R}\setminus\Omega_2, & \text{else},\end{cases}$$

and

$$F_1 := \begin{cases} \left(\mathbb{R}\cap\overline{\Omega}_2\right)\cup([\tilde{x}_0+2,\infty[\times[-1,1]), & \infty\in\Omega_2\wedge\left(-\infty\notin\overline{\Omega_1}\vee-\infty\notin\Omega_2\right),\\ \left(\mathbb{R}\cap\overline{\Omega}_2\right)\cup(]-\infty,\tilde{x}_1-2]\times[-1,1]), & -\infty\in\Omega_2\wedge\left(\infty\notin\overline{\Omega_1}\vee\infty\notin\Omega_2\right),\\ \left(\mathbb{R}\cap\overline{\Omega}_2\right)\cup(]-\infty,\tilde{x}_1-2]\cup[\tilde{x}_0+2,\infty[)\times[-1,1], & \pm\infty\in\Omega_2,\\ \mathbb{R}\cap\overline{\Omega}_2, & \text{else}.\end{cases}$$

If we number the appearing cases in the definition of F_0 from above to below by 1A, ... , 4A and in the definition of F_1 by 1B, ... , 4B, then we have as possible combinations:

Table 6.1: Combinations

	1A	2A	3A	4A
1B	×	✓	×	✓
2B	✓	×	×	✓
3B	×	×	×	✓
4B	✓	✓	✓	✓

legend:
✓ – possible
× – impossible

The sets F_0 and F_1 are non-empty and closed in $\mathbb{R}^2$, $F_0\cap\mathbb{R}=\mathbb{R}\setminus\Omega_2$, $F_1\cap\mathbb{R}=\overline{\Omega}_2\cap\mathbb{R}$ and $F_0\cap F_1=\partial\Omega_2\cap\mathbb{R}$. By [18, Theorem 1.4.10, p. 30, Corollary 1.4.11, p. 31] there exists $\varphi\in C^\infty\left((F_0\cap F_1)^C\right)=C^\infty(\partial\Omega_2\cap\mathbb{R})$, $0\le\varphi\le1$, such that $\varphi=0$ on V_0 and $\varphi=1$ on V_1 where V_0, $V_1\subset\mathbb{R}^2$ are open and

$$V_0\supset F_0\setminus(F_0\cap F_1)=F_0\setminus\partial\Omega_2\supset\left(\mathbb{R}\setminus\overline{\Omega}_2\right)\text{ and }V_1\supset F_1\setminus(F_0\cap F_1)=F_1\setminus\partial\Omega_2\supset(\mathbb{R}\cap\Omega_2).$$

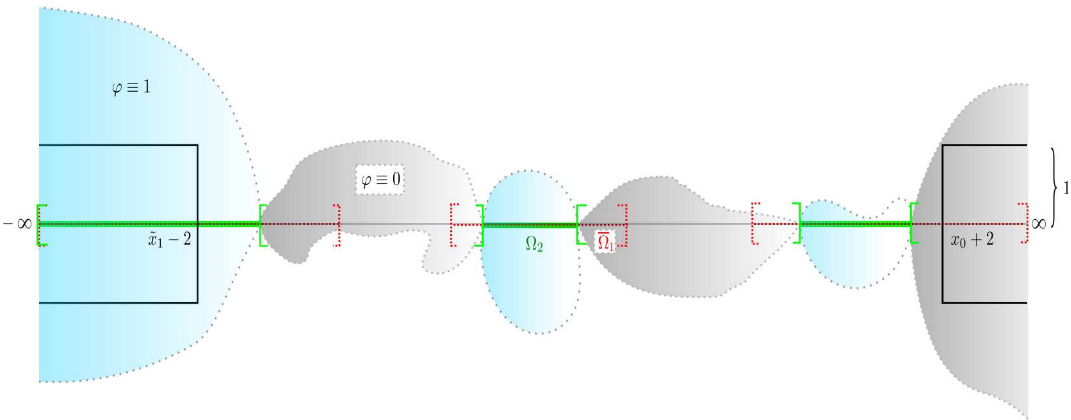

Figure 6.1: case: $\pm\infty\in\overline{\Omega}_1$, $-\infty\in\Omega_2$, $\infty\notin\Omega_2$

Furthermore,

$$\left|\varphi^{(k)}(z)\left(y^{1},\cdots,y^{k}\right)\right| \leq C^{k}\left|y^{1}\right|\cdots\left|y^{k}\right|\frac{\mathrm{d}(z)^{-k}}{d_{1}\cdots d_{k}} \tag{6.1}$$

for all $z\in\mathbb{R}^2\smallsetminus\partial\Omega_2$ and all $y^i\in\mathbb{R}^2$, $1\leq i\leq k$, $k\in\mathbb{N}_0$, where $\varphi^{(k)}$ denotes the differential of order k of φ, $C>0$ is a constant independent of z, y^i, and k,

$$\mathrm{d}(z):=\max\left\{\mathrm{d}(z,F_0),\mathrm{d}(z,F_1)\right\}=\max\left(\min_{x\in F_0}|z-x|,\min_{x\in F_1}|z-x|\right)$$

and $(d_n)_{n\in\mathbb{N}}$ is an arbitrary decreasing sequence with $\sum_{n=1}^{\infty}d_n=1$, e.g. $d_n:=(1/2)^n$. We observe that

$$\varphi^{(k)}(z)\left(y^{1},\cdots,y^{k}\right)=\sum_{i_1=1}^{2}\cdots\sum_{i_k=1}^{2}\partial_{i_1}\cdots\partial_{i_k}\varphi(z)\,y^{1}_{i_1}\cdots y^{k}_{i_k}=\sum_{\substack{|\alpha|=k,\\ \alpha\in\mathbb{N}_0^2}}\partial^{\alpha}\varphi(z)\sum_{\substack{i_1,\cdots,i_k,\\ \#\{i_j\,|\,i_j=1\}=\alpha_1,\\ \#\{i_j\,|\,i_j=2\}=\alpha_2}}y^{1}_{i_1}\cdots y^{k}_{i_k}$$

with the notation $y^i=\left(y^i_1,y^i_2\right)$. In particular, we have for $\beta=(\beta_1,\beta_2)\in\mathbb{N}_0^2$

$$\partial^{\beta}\varphi(z)=\varphi^{(|\beta|)}(z)\left(\underbrace{\begin{pmatrix}1\\0\end{pmatrix},\cdots,\begin{pmatrix}1\\0\end{pmatrix}}_{\text{no.}=\beta_1},\underbrace{\begin{pmatrix}0\\1\end{pmatrix},\cdots,\begin{pmatrix}0\\1\end{pmatrix}}_{\text{no.}=\beta_2}\right)$$

and so as a special case of (6.1)

$$\left|\partial^{\beta}\varphi(z)\right|\leq C^{|\beta|}\frac{\mathrm{d}(z)^{-|\beta|}}{d_1\cdots d_{|\beta|}} \tag{6.2}$$

for all $z\in\mathbb{R}^2\smallsetminus\partial\Omega_2$.

Let us take a closer look at the right hand side of this inequality. For $z\in\mathbb{R}^2\smallsetminus\partial\Omega_2$ there is $z_i\in F_i$ such that $\mathrm{d}(z,F_i)=|z-z_i|$, $i=1,2$, since F_0 and F_1 are closed. Let $n\in\mathbb{N}_{\geq 2}$. We claim that

$$\mathrm{d}(z)\geq\frac{1}{n}\quad\text{for all } z\in S_n(\partial\Omega_2).$$

Let $z\in S_n(\partial\Omega_2)$.

<u>1. case:</u> z_0, $z_1\in\mathbb{R}$

Let us assume that $\mathrm{d}(z)<\frac{1}{n}$. The definition of the set $S_n(\partial\Omega_2)$ implies $z_i\notin\partial\Omega_2\cap\mathbb{R}$. Thus we get by definition of the sets F_i that $z_0\in(\mathbb{R}\smallsetminus\mathring{\Omega}_2)$ and $z_1\in\left(\mathbb{R}\cap\mathring{\overline{\Omega}}_2\right)$, in particular, $z_0\neq z_1$. W.l.o.g. $z_0<z_1$. Then $O_0:=]z_0,z_1[\cap(\mathbb{R}\smallsetminus\mathring{\Omega}_2)$ and $O_1:=]z_0,z_1[\cap\left(\mathbb{R}\cap\mathring{\overline{\Omega}}_2\right)$ are disjoint, open sets in $\mathbb{R}$. Assume that there is no $\tilde{z}\in\partial\Omega_2\cap\mathbb{R}$ with $z_0<\tilde{z}<z_1$. Due to this assumption we obtain

$$O_0\cup O_1=]z_0,z_1[\cap\underbrace{\left[(\mathbb{R}\smallsetminus\mathring{\Omega}_2)\cup\cap\left(\mathbb{R}\cap\mathring{\overline{\Omega}}_2\right)\right]}_{=\mathbb{R}\smallsetminus\partial\Omega_2}=]z_0,z_1[$$

and hence, as $]z_0,z_1[$ is connected, $]z_0,z_1[\subset O_0$ or $]z_0,z_1[\subset O_1$. If $]z_0,z_1[\subset O_0$, we get $z_1\notin\left(\mathbb{R}\cap\mathring{\overline{\Omega}}_2\right)$, and if $]z_0,z_1[\subset O_1$, we have $z_0\notin(\mathbb{R}\smallsetminus\mathring{\Omega}_2)$, which is a contradiction. So there must be a $\tilde{z} \in \partial\Omega_2 \cap \mathbb{R}$ with $z_0 < \tilde{z} < z_1$. The convexity of $D_{\mathrm{d}(z)}(z)$ implies $\tilde{z} \in]z_0, z_1[\subset D_{\mathrm{d}(z)}(z)$, but then the following is valid

$$\frac{1}{n} < |z - \tilde{z}| \leq \max\{|z - z_0|, |z - z_1|\} = \mathrm{d}(z) < \frac{1}{n}$$

which is again a contradiction.
<u>2. case:</u> $(z_0 \notin \mathbb{R},\ z_1 \in \mathbb{R})$ or $(z_0 \in \mathbb{R},\ z_1 \notin \mathbb{R})$
We only consider the first case above, the latter one is analogous. If so, these assumptions can not occur in the cases (4A,YB), Y=1,⋯,4.
In the cases (1A,2B) and (1A,4B) we have $z_1 < x_0$ and $\mathrm{Re}(z_0) \geq x_0 + 2$. Therefore, we get

$$|z_1 - \mathrm{Re}(z_0)| \geq |x_0 - (x_0 + 2)| = 2.$$

In the cases (2A,1B) and (2A,4B) we have $z_1 > x_1$ and $\mathrm{Re}(z_0) \leq x_1 - 2$. Therefore, we get

$$|z_1 - \mathrm{Re}(z_0)| \geq |x_1 - (x_1 - 2)| = 2.$$

In the case (3A,4B) we have $x_1 < z_1 < x_0$ and $\mathrm{Re}(z_0) \leq x_1 - 2$ or $\mathrm{Re}(z_0) \geq x_0 + 2$. We gain like above $|z_1 - \mathrm{Re}(z_0)| \geq 2$.
If $|z - z_0| < \frac{1}{n}$, we obtain by the estimates above

$$\begin{aligned} \mathrm{d}(z) &\geq |z - z_1| \geq |\mathrm{Re}(z - z_1)| = |\mathrm{Re}(z) - z_1| \geq |z_1 - \mathrm{Re}(z_0)| - \underbrace{|\mathrm{Re}(z_0) - \mathrm{Re}(z)|}_{\leq |z - z_0| < \frac{1}{n}} \\ &> |z_1 - \mathrm{Re}(z_0)| - \frac{1}{n} \geq 2 - \frac{1}{n} \geq \frac{1}{n}. \end{aligned}$$

<u>3. case:</u> $z_i \notin \mathbb{R},\ i = 1, 2$
These assumptions can only occur in the cases (1A,2B) and (1B,2A). If $|z - z_0| < \frac{1}{n}$, we have in the case (1A,2B)

$$\mathrm{Re}(z_1) \leq \tilde{x}_1 - 2 < \tilde{x}_1 < x_0 < x_0 + 2 - \frac{1}{n} \leq \mathrm{Re}(z)$$

and thus get

$$\mathrm{d}(z) \geq |z - z_1| \geq |\mathrm{Re}(z) - \mathrm{Re}(z_1)| \geq 2 > \frac{1}{n}$$

and in the case (1B,2A)

$$\mathrm{Re}(z_1) \geq \tilde{x}_0 + 2 > \tilde{x}_0 > x_1 > x_1 - 2 + \frac{1}{n} \geq \mathrm{Re}(z)$$

plus $\mathrm{d}(z) \geq 2 > \frac{1}{n}$ as well.
Hence the claim is proven and via (6.2) we obtain

$$\left|\partial^\beta \varphi(z)\right| \leq C^{|\beta|} \frac{n^{|\beta|}}{d_1 \cdots d_{|\beta|}} \tag{6.3}$$

for all $z \in S_n(\partial\Omega_2)$.

(ii) Let $f \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\Omega}_1, E\right)$. Due to the choice of φ the function $\overline{\partial}(\varphi f)$ may be regarded as an element of $\mathcal{C}^\infty(\mathbb{R}^2 \setminus \partial\Omega_2, E)$ by $\mathcal{C}^\infty$-continuation via $\overline{\partial}(\varphi f) := 0$ on $\mathbb{R} \setminus \partial\Omega_2$. Furthermore,

$$\overline{\partial}(\varphi f)(z) = \begin{cases} 0, & z \in V_0 \cup V_1, \\ \left(\overline{\partial}\varphi\right)(z) f(z), & \text{else}, \end{cases}$$

is valid.

Let $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$ and $\alpha \in A$. First we observe the following. For $\gamma, \beta \in \mathbb{N}_0^2$ with $\gamma \leq \beta$ and $|\beta| \leq m$ we have $|\gamma| \leq |\beta| \leq m$ and $\binom{\beta}{\gamma} = \binom{\beta_1}{\gamma_1}\binom{\beta_2}{\gamma_2} \leq \beta_1!\beta_2! \leq (m!)^2$ as well as

$$\max(|\beta - \gamma + (1,0)|, |\beta - \gamma + (0,1)|) \leq |(\beta_1 + 1, \beta_2 + 1)| = |\beta| + 2 \leq m + 2.$$

Define the set $S(n) := S_n(\partial\Omega_2) \setminus (V_0 \cup V_1)$. By applying the Leibniz rule, we obtain

$$\begin{aligned}
&\left|\overline{\partial}(\varphi f)\right|_{\partial\Omega_2, n, m, \alpha} \\
&= \sup_{\substack{z \in S_n(\partial\Omega_2), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(\partial^\beta \overline{\partial}(\varphi f)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq \sup_{\substack{z \in S_n(\partial\Omega_2) \setminus (V_0 \cup V_1), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta} \overbrace{\binom{\beta}{\gamma}}^{\leq (m!)^2} \underbrace{\left|\partial^{\beta - \gamma}\overline{\partial}\varphi(z)\right|}_{\substack{\leq \frac{1}{2}\left|\partial^{\beta-\gamma+(1,0)}\varphi(z)\right| \\ + \frac{1}{2}\left|\partial^{\beta-\gamma+(0,1)}\varphi(z)\right|}} p_\alpha\Big(\overbrace{\partial^\gamma f(z)}^{\overset{(3.2)}{=} i^{\gamma_2} f^{(|\gamma|)}(z)}\Big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq (m!)^2 \sum_{|\gamma| \leq m+2} \sup_{z \in S(n)} |\partial^\gamma \varphi(z)| \underbrace{\sup_{\substack{z \in S(n), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=: C(f)} \\
&\underset{(6.3)}{\leq} (m!)^2 C(f) \sum_{|\gamma| \leq m+2} C^{|\gamma|} \frac{n^{|\gamma|}}{d_1 \cdots d_{|\gamma|}} \\
&\leq (m!)^2 \frac{[\max(C,1)]^{m+2}}{d_1 \cdots d_{m+2}} C(f) \sum_{|\gamma| \leq m+2} n^{|\gamma|}
\end{aligned} \tag{6.4}$$

where we used the properties of $(d_n)_n$, which imply $0 < d_n < 1$, in the last estimate.

Now we have to take a closer look at $C(f)$. We decompose the set $S(n)$ in the following manner:

$$S(n) = \underbrace{\left[S(n) \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{\subset S_{2n}(\overline{\Omega}_1)} \cup \underbrace{\left[S(n) \setminus \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{=: M}$$

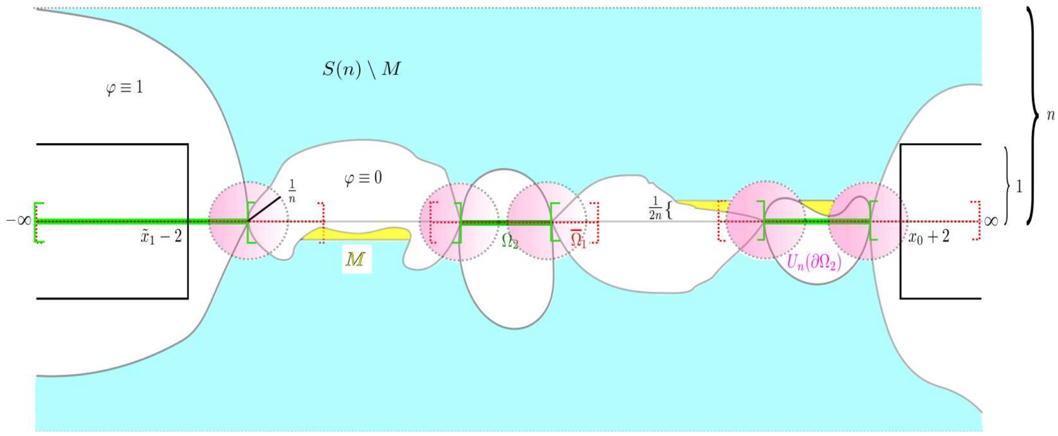

Figure 6.2: case: $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$, $\infty \notin \Omega_2$

Due to the Cauchy inequality we get like in the proof of (3.7) of Theorem 3.11(4) for $r := \frac{1}{2}\left(\frac{1}{2n} - \frac{1}{3n}\right)$

$$\begin{aligned} C(f) &\leq \sup_{\substack{z \in S_{2n}(\overline{\Omega}_1), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + \sup_{\substack{z \in M, \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\ &\leq \underbrace{e^{\frac{r}{n}} \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{r^{|\beta|}}}_{=:C_0} |f|_{\overline{\Omega}_1, 3n, \alpha} + \sup_{\substack{z \in M, \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}. \end{aligned} \tag{6.5}$$

Let us turn our attention to the set M. First we observe that

$$\mathbb{R} \subset \Big[\underbrace{V_0 \cup V_1}_{\supset \mathbb{R} \setminus \partial\Omega_2} \cup \bigcup_{x \in \partial\Omega_2 \cap \mathbb{R}} D_{1/n}(x)\Big] =: V.$$

$V \subset \mathbb{R}^2$ is open as the union of open sets and so we get by definition of the set M

$$\overline{M} \subset \overline{V^C} = V^C \subset \left(\mathbb{R}^2 \setminus \mathbb{R}\right). \tag{6.6}$$

If $\infty \notin \overline{\Omega}_1$ or $-\infty \notin \overline{\Omega}_1$, then there is $a \in \mathbb{R}$ or $b \in \mathbb{R}$ such that $[a, \infty] \subset \overline{\Omega}_1^C \subset (\partial\Omega_2)^C$ resp. $[-\infty, b] \subset \overline{\Omega}_1^C \subset (\partial\Omega_2)^C$ since $\overline{\Omega}_1^C \subset \overline{\mathbb{R}}$ is open. If so, we have

$$\left]a + \frac{1}{n}, \infty\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\subset S_n\left(\overline{\Omega}_1\right) \subset S_n(\partial\Omega_2) \tag{6.7}$$

or

$$\left]-\infty, b - \frac{1}{n}\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\subset S_n\left(\overline{\Omega}_1\right) \subset S_n(\partial\Omega_2) \tag{6.8}$$

by definition of the sets $S_n(\cdot)$ and thus

$$M = \underbrace{\left[M \cap \left(\left]a + \frac{1}{n}, \infty\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\right)\right]}_{\subset S_n(\overline{\Omega}_1)} \cup \underbrace{\left[M \setminus \left(\left]a + \frac{1}{n}, \infty\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\right)\right]}_{=: M_0(a)},$$

$$M = \underbrace{\left[M \cap \left(\left]-\infty, b-\frac{1}{n}\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\right)\right]}_{\subset S_n(\overline{\Omega}_1)} \cup \underbrace{\left[M \setminus \left(\left]-\infty, b-\frac{1}{n}\right[\times \left]-\frac{1}{n}, \frac{1}{n}\right[\right)\right]}_{=:M_0(b)}$$

or

$$\begin{aligned} M = &\underbrace{M \cap \left[\left(\left]-\infty, b-\frac{1}{n}\right[\cup \left]a+\frac{1}{n}, \infty\right[\right) \times \left]-\frac{1}{n}, \frac{1}{n}\right[\right]}_{\subset S_n(\overline{\Omega}_1)} \\ &\cup \underbrace{M \setminus \left[\left(\left]-\infty, b-\frac{1}{n}\right[\cup \left]a+\frac{1}{n}, \infty\right[\right) \times \left]-\frac{1}{n}, \frac{1}{n}\right[\right]}_{=:M_0(a,b)}. \end{aligned}$$

By virtue of the definition of the set $S(n)$ and (6.7) we have $\mathrm{Re}(z) \leq a + \frac{1}{n}$ for all $z \in M_0(a)$ resp. by (6.8) $\mathrm{Re}(z) \geq b - \frac{1}{n}$ for all $z \in M_0(b)$ resp. by (6.7) and (6.8) $b - \frac{1}{n} \leq \mathrm{Re}(z) \leq a + \frac{1}{n}$ for all $z \in M_0(a,b)$.
We claim that M is bounded or $M \subset \left(S_n\left(\overline{\Omega}_1\right) \cup M_0\right)$ where M_0, defined as above, is bounded. As $|\mathrm{Im}(z)| \leq 1/2n$ for every $z \in M$ resp. $z \in M_0$, it suffices to prove that there is $C_1 > 0$ such that $|\mathrm{Re}(z)| \leq C_1$ for every $z \in M$ resp. $z \in M_0$. The assumption $\Omega_2 \subset \Omega_1$ and the choice of the sets F_0 and F_1 ensure the existence of C_1 which can be read off the following chart.

Table 6.2: Bounds for the real part of M resp. M_0

base case	1. subcase	2. subcase	$C_1 = \max(\cdot)$
$\pm\infty \in \partial\Omega_2$			$\lvert -n\rvert$, n
$\infty \in \partial\Omega_2$, $-\infty \notin \partial\Omega_2$	$-\infty \in \Omega_2$		$\lvert\tilde{x}_1 - 2\rvert$, n
	$-\infty \notin \overline{\Omega}_2$	$-\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert$, n
		$-\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert$, n
$\infty \notin \partial\Omega_2$, $-\infty \in \partial\Omega_2$	$\infty \in \Omega_2$		$\lvert -n\rvert$, $\lvert\tilde{x}_0 + 2\rvert$
	$\infty \notin \overline{\Omega}_2$	$\infty \in \overline{\Omega}_1$	$\lvert -n\rvert$, $\lvert x_0 + 2\rvert$
		$\infty \notin \overline{\Omega}_1$	$\lvert -n\rvert$, $\lvert a + \frac{1}{n}\rvert$
$\pm\infty \notin \partial\Omega_2$	$\pm\infty \in \Omega_2$		$\lvert\tilde{x}_1 - 2\rvert$, $\lvert\tilde{x}_0 + 2\rvert$
	$\infty \in \Omega_2$, $-\infty \notin \overline{\Omega}_2$	$-\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert$, $\lvert\tilde{x}_0 + 2\rvert$
		$-\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert$, $\lvert\tilde{x}_0 + 2\rvert$
	$\infty \notin \overline{\Omega}_2$, $-\infty \in \Omega_2$	$\infty \in \overline{\Omega}_1$	$\lvert\tilde{x}_1 - 2\rvert$, $\lvert x_0 + 2\rvert$
		$\infty \notin \overline{\Omega}_1$	$\lvert\tilde{x}_1 - 2\rvert$, $\lvert a + \frac{1}{n}\rvert$
	$\pm\infty \notin \overline{\Omega}_2$	$\pm\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert$, $\lvert x_0 + 2\rvert$
		$\infty \in \overline{\Omega}_1$, $-\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert$, $\lvert x_0 + 2\rvert$
		$\infty \notin \overline{\Omega}_1$, $-\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert$, $\lvert a + \frac{1}{n}\rvert$
		$\pm\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert$, $\lvert a + \frac{1}{n}\rvert$

Hence $\overline{M}$ or $\overline{M}_0$ is compact and we have by (6.6), since $f \in \mathcal{O}^{exp}\left(\mathbb{C} \setminus \overline{\Omega}_1, E\right)$ implies the continuity of $f^{(|\beta|)}$ on $\mathbb{R}^2 \setminus \mathbb{R}$ for all $\beta \in \mathbb{N}_0^2$,

$$\sup_{\substack{z \in M, \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \leq \sup_{\substack{z \in \overline{M}, \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

or, since $\overline{M}_0 \subset \overline{M}$,

$$
\begin{aligned}
&\sup_{\substack{z\in M,\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\big(f^{(|\beta|)}(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq \underbrace{\sup_{\substack{z\in S_n(\overline{\Omega}_1),\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\big(f^{(|\beta|)}(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|f|_{\overline{\Omega}_1,n,m,\alpha}} + \sup_{\substack{z\in \overline{M}_0,\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\big(f^{(|\beta|)}(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq C_2|f|_{\overline{\Omega}_1,2n,\alpha} + \sup_{\substack{z\in \overline{M}_0,\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\big(f^{(|\beta|)}(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty
\end{aligned}
$$

where $C_2 > 0$ is a constant existing by the proof of Theorem 3.6(4). Thus $C(f) < \infty$ by (6.5) and therefore $\left|\overline{\partial}(\varphi f)\right|_{\partial\Omega_2,n,m,\alpha} < \infty$ for all $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$ and $\alpha \in A$ by (6.4) connoting $\overline{\partial}(\varphi f) \in \mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus \partial\Omega_2, E\big)$. As E is admissible, there is $g \in \mathcal{E}^{exp}\big(\overline{\mathbb{C}} \setminus \partial\Omega_2, E\big)$ such that

$$
\overline{\partial} g = \overline{\partial}(\varphi f). \tag{6.9}
$$

(iii) We set $f_1 := (1-\varphi)f + g$ and $f_2 := \varphi f - g$. It remains to be proven that $f_1 \in \mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus \big(\overline{\Omega}_1 \setminus \Omega_2\big), E\big)$ and $f_2 \in \mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus \overline{\Omega}_2, E\big)$. The proof is quite similar to part (ii). f_1 is defined on $\mathbb{C} \setminus \big(\overline{\Omega}_1 \setminus \Omega_2\big)$ (by setting $(1-\varphi)f := 0$ on Ω_2) and can be regarded as an element of $\mathcal{O}\big(\mathbb{C} \setminus \big(\overline{\Omega}_1 \setminus \Omega_2\big), E\big)$ due to (6.9).
Let $n \in \mathbb{N}_{\geq 2}$ and set $S(n) := S_n\big(\overline{\Omega}_1 \setminus \Omega_2\big) \setminus V_1$. Remark that $S_n\big(\overline{\Omega}_1 \setminus \Omega_2\big) \subset S_n(\partial\Omega_2)$ and

$$
S(n) = \underbrace{\left[S(n) \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{\subset S_{2n}(\overline{\Omega}_1)} \cup \underbrace{\left[S(n) \setminus \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{=:M}.
$$

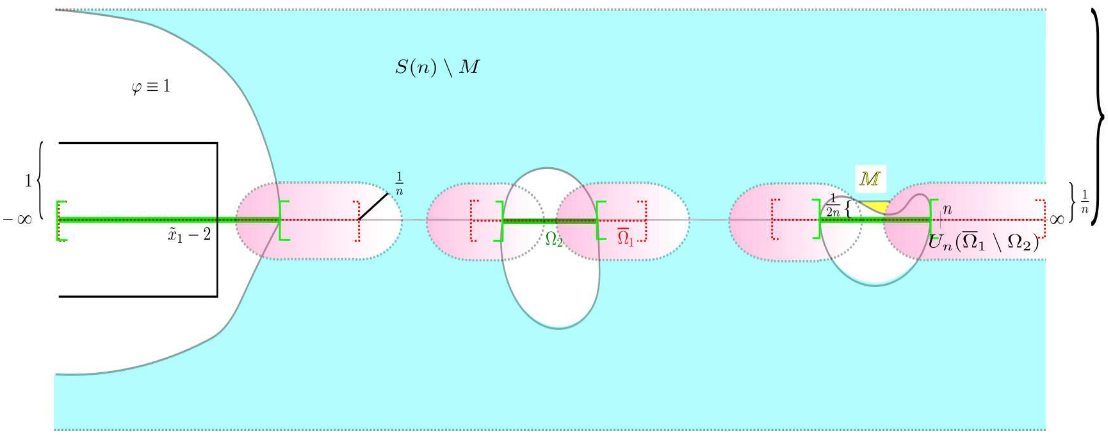

Figure 6.3: case: $\pm\infty \in \overline{\Omega}_1$, $-\infty \in \Omega_2$, $\infty \notin \Omega_2$

For $\alpha \in A$ we have by the choice of φ

$$
\begin{aligned}
|f_1|_{\overline{\Omega}_1\setminus\Omega_2,n,\alpha} &= \sup_{z\in S_n\left(\overline{\Omega}_1\setminus\Omega_2\right)} p_\alpha\big(f_1(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq \underbrace{\sup_{z\in S_n(\partial\Omega_2)} p_\alpha\big(g(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\partial\Omega_2,n,0,\alpha}} + \sup_{z\in S_n\left(\overline{\Omega}_1\setminus\Omega_2\right)} p_\alpha\big((1-\varphi)f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&= |g|_{\partial\Omega_2,n,0,\alpha} + \sup_{z\in S(n)} \underbrace{|1-\varphi(z)|}_{\leq 1} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq |g|_{\partial\Omega_2,n,0,\alpha} + \underbrace{\sup_{z\in S_{2n}\left(\overline{\Omega}_1\right)} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|f|_{\overline{\Omega}_1,2n,\alpha}} + \sup_{z\in M} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&= |g|_{\partial\Omega_2,n,0,\alpha} + |f|_{\overline{\Omega}_1,2n,\alpha} + \sup_{z\in M} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}. \qquad (6.10)
\end{aligned}
$$

Again we have to take a closer look at the set M. First we observe that

$$\mathbb{R}\cap\overline{\Omega}_1 \subset \Big[\underbrace{V_1}_{\supset\mathbb{R}\cap\Omega_2} \cup \bigcup_{x\in\left(\overline{\Omega}_1\setminus\Omega_2\right)\cap\mathbb{R}} D_{1/n}(x)\Big] =: V.$$

$V\subset\mathbb{R}^2$ is open and so we get by definition of the set M

$$\overline{M}\subset\overline{V^C} = V^C \subset \left(\mathbb{R}^2\setminus\overline{\Omega}_1\right).$$

Once again we define the sets M_0 analogously to part (ii) and replace in (6.7) and (6.8) the inclusion $S_n\left(\overline{\Omega}_1\right)\subset S_n(\partial\Omega_2)$ by $S_n\left(\overline{\Omega}_1\right)\subset S_n\left(\overline{\Omega}_1\setminus\Omega_2\right)$. The rest of the proof is analogous to part (ii) where we have as corresponding chart:

Table 6.3: Bounds for the real part of M resp. M_0

base case	subcase	$C_1 = \max(\cdot)$
$\pm\infty\in\overline{\Omega}_1\setminus\Omega_2$		$\lvert -n\rvert$, n
$\infty\in\overline{\Omega}_1\setminus\Omega_2$, $-\infty\notin\overline{\Omega}_1\setminus\Omega_2$	$-\infty\in\Omega_2$ $-\infty\notin\overline{\Omega}_1$	$\lvert\tilde{x}_1-2\rvert$, n $\lvert b-\frac{1}{n}\rvert$, n
$\infty\notin\overline{\Omega}_1\setminus\Omega_2$, $-\infty\in\overline{\Omega}_1\setminus\Omega_2$	$\infty\in\Omega_2$ $\infty\notin\overline{\Omega}_1$	$\lvert -n\rvert$, $\lvert\tilde{x}_0+2\rvert$ $\lvert -n\rvert$, $\lvert a+\frac{1}{n}\rvert$
$\pm\infty\notin\overline{\Omega}_1\setminus\Omega_2$	$\pm\infty\in\Omega_2$ $-\infty\notin\overline{\Omega}_1$, $\infty\in\Omega_2$ $\infty\notin\overline{\Omega}_1$, $-\infty\in\Omega_2$ $\pm\infty\notin\overline{\Omega}_1$	$\lvert\tilde{x}_1-2\rvert$, $\lvert\tilde{x}_0+2\rvert$ $\lvert b-\frac{1}{n}\rvert$, $\lvert\tilde{x}_0+2\rvert$ $\lvert\tilde{x}_1-2\rvert$, $\lvert a+\frac{1}{n}\rvert$ $\lvert b-\frac{1}{n}\rvert$, $\lvert a+\frac{1}{n}\rvert$

Since $f\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega}_1,E\right)$ and so f is continuous on $\mathbb{R}^2\setminus\overline{\Omega}_1$, we obtain again

$$\sup_{z\in M} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty.$$

Thus we get $|f_1|_{\overline{\Omega}_1\setminus\Omega_2,n,\alpha} < \infty$ for every $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ by (6.10) implying $f_1 \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\left(\overline{\Omega}_1\setminus\Omega_2\right),E\right)$.
f_2 is defined on $\mathbb{C}\setminus\overline{\Omega}_2$ (by setting $\varphi f := 0$ on $\overline{\Omega}_1\setminus\overline{\Omega}_2$) and can be regarded as an element of $\mathcal{O}\left(\mathbb{C}\setminus\overline{\Omega}_2,E\right)$ due to (6.9). Let $n\in\mathbb{N}_{\geq 2}$. We set $S(n) := S_n\left(\overline{\Omega}_2\right)\setminus V_0$ and remark that $S_n\left(\overline{\Omega}_2\right)\subset S_n\left(\partial\Omega_2\right)$ as well as

$$S(n) = \underbrace{\left[S(n)\cap\left\{z\in\mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{\subset S_{2n}\left(\overline{\Omega}_1\right)} \cup \underbrace{\left[S(n)\setminus\left\{z\in\mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{2n}\right\}\right]}_{=:M}.$$

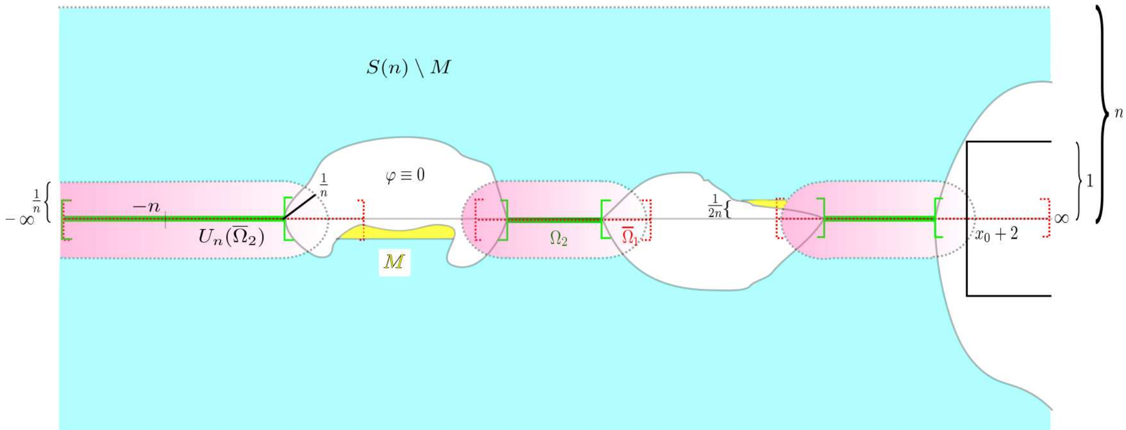

Figure 6.4: case: $\pm\infty\in\overline{\Omega}_1$, $-\infty\in\Omega_2$, $\infty\notin\Omega_2$

For $\alpha\in A$ we have by the choice of φ

$$\begin{aligned}
|f_2|_{\overline{\Omega}_2,n,\alpha} &= \sup_{z\in S_n\left(\overline{\Omega}_2\right)} p_\alpha\left(f_2(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq \underbrace{\sup_{z\in S_n\left(\partial\Omega_2\right)} p_\alpha\left(g(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\partial\Omega_2,n,0,\alpha}} + \sup_{z\in S_n\left(\overline{\Omega}_2\right)} p_\alpha\left(\varphi f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&= |g|_{\partial\Omega_2,n,0,\alpha} + \sup_{z\in S(n)} \underbrace{|\varphi(z)|}_{\leq 1} p_\alpha\left(f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq |g|_{\partial\Omega_2,n,0,\alpha} + \underbrace{\sup_{z\in S_{2n}\left(\overline{\Omega}_1\right)} p_\alpha\left(f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|f|_{\overline{\Omega}_1,2n,\alpha}} + \sup_{z\in M} p_\alpha\left(f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&= |g|_{\partial\Omega_2,n,0,\alpha} + |f|_{\overline{\Omega}_1,2n,\alpha} + \sup_{z\in M} p_\alpha\left(f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}.
\end{aligned}\tag{6.11}$$

Again we have to take a closer look at the set M and observe that

$$\mathbb{R}\subset\Big[\underbrace{V_0}_{\supset\mathbb{R}\setminus\overline{\Omega}_2} \cup \bigcup_{x\in\overline{\Omega}_2\cap\mathbb{R}} D_{1/n}(x)\Big] =: V.$$

$V \subset \mathbb{R}^2$ is open and so we get by definition of the set M

$$\overline{M} \subset \overline{V^C} = V^C \subset \left(\mathbb{R}^2 \setminus \mathbb{R}\right).$$

Define the sets M_0 analogously to part (ii) and replace in (6.7) and (6.8) the inclusion $S_n\left(\overline{\Omega}_1\right) \subset S_n\left(\partial\Omega_2\right)$ by $S_n\left(\overline{\Omega}_1\right) \subset S_n\left(\overline{\Omega}_2\right)$. The rest of the proof is analogous to part (ii) where we have as corresponding chart:

Table 6.4: Bounds for the real part of M resp. M_0

base case	subcase	$C_1 = \max(\cdot)$
$\pm\infty \in \overline{\Omega}_2$		$\lvert -n\rvert,\ n$
$\infty \in \overline{\Omega}_2,\ -\infty \notin \overline{\Omega}_2$	$-\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert,\ n$
	$-\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert,\ n$
$\infty \notin \overline{\Omega}_2,\ -\infty \in \overline{\Omega}_2$	$\infty \in \overline{\Omega}_1$	$\lvert -n\rvert,\ \lvert x_0 + 2\rvert$
	$\infty \notin \overline{\Omega}_1$	$\lvert -n\rvert,\ \lvert a + \frac{1}{n}\rvert$
$\pm\infty \notin \overline{\Omega}_2$	$\pm\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert,\ \lvert x_0 + 2\rvert$
	$\infty \in \overline{\Omega}_1,\ -\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert,\ \lvert x_0 + 2\rvert$
	$\infty \notin \overline{\Omega}_1,\ -\infty \in \overline{\Omega}_1$	$\lvert x_1 - 2\rvert,\ \lvert a + \frac{1}{n}\rvert$
	$\pm\infty \notin \overline{\Omega}_1$	$\lvert b - \frac{1}{n}\rvert,\ \lvert a + \frac{1}{n}\rvert$

Again we gain

$$\sup_{z \in M} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

and thus get $|f_2|_{\overline{\Omega}_2,n,\alpha} < \infty$ for every $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ by (6.11) implying $f_2 \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\Omega}_2, E\right)$. Obviously $f_1 + f_2 = f$ which completes the proof due to part (i).

□

Ito (see [23, remarks above Proposition 3.2, p. 15]) states that Lemma 6.2 is always valid if E is complete, but he does not prove that I is surjective. Nevertheless, he states as an open problem (see [23, Problem A, p. 17]) if for two compact sets K_1, $K_2 \subset \overline{\mathbb{R}}$ the mapping

$$L : L(\mathcal{P}_*(K_1), E) \times L(\mathcal{P}_*(K_2), E) \to L(\mathcal{P}_*(K_1 \cup K_2), E),$$

given by $L(T_1, T_2) := T_1 - T_2$, is surjective.

6.3 Remark. [2] Let $\Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$ be open. Then the following assertions are equivalent:

1. The canonical mapping

$$I : L\big(\mathcal{P}_*(\overline{\Omega}_2), E\big) / L\big(\mathcal{P}_*(\partial\Omega_2), E\big) \to L\big(\mathcal{P}_*(\overline{\Omega}_1), E\big) / L\big(\mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega_2\big), E\big)$$

   is an algebraic isomorphism.

2. The mapping

$$L : L\big(\mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega_2\big), E\big) \times L\big(\mathcal{P}_*\big(\overline{\Omega}_2\big), E\big) \to L\big(\mathcal{P}_*\big(\overline{\Omega}_1\big), E\big)$$

[2] counterpart: [13, Remark 6.3, p. 1123]

is surjective.

Proof. I is obviously surjective if and only if L is surjective. Moreover, I is always linear and injective by Proposition 4.3(1). □

Using Lemma 6.2, we can define restrictions on $\mathcal{R}(\Omega,E)$, if E is admissible, as follows:

6.4 Definition. Let E be admissible. For open sets $\Omega_2 \subset \Omega_1 \subset \overline{\mathbb{R}}$, $\Omega_2 \neq \varnothing$, we denote by

$$q: L\big(\mathcal{P}_*(\overline{\Omega}_1),E\big)/L(\mathcal{P}_*(\partial\Omega_1),E) \to L\big(\mathcal{P}_*(\overline{\Omega}_1),E\big)/L\big(\mathcal{P}_*\big(\overline{\Omega}_1 \setminus \Omega_2\big),E\big)$$

the canonical quotient map.
We define the restriction mappings via Lemma 6.2 by

$$R_{\Omega_1,\Omega_2}: \mathcal{R}(\Omega_1,E) \to \mathcal{R}(\Omega_2,E),\; R_{\Omega_1,\Omega_2}([T]) := [T]\big|_{\Omega_2} := I^{-1}(q([T])),$$

and for an open set $\Omega_1 \subset \overline{\mathbb{R}}$

$$R_{\Omega_1,\varnothing}: \mathcal{R}(\Omega_1,E) \to \mathcal{R}(\varnothing,E),\; R_{\Omega_1,\varnothing}([T]) := [T]\big|_{\varnothing} := 0.$$

6.5 Lemma. [3] *Let E be admissible and $\Omega \subset \overline{\mathbb{R}}$ be open.*
The spaces $\mathcal{R}_\Omega(E) := \{\mathcal{R}(\omega,E) \mid \omega \subset \Omega \text{ open}\}$, equipped with the restrictions of Definition 6.4, form a presheaf on Ω satisfying the condition $(S1)$ (see [9, 1.5, p. 5]):
For every family of open sets $\{\omega_j \subset \Omega \mid j \in J\}$ with $\omega := \bigcup_{j\in J}\omega_j$ holds: If $[T] \in \mathcal{R}(\omega,E)$ such that $R_{\omega,\omega_j}([T]) = 0$ for all $j \in J$, then $[T] = 0$.

Proof. (i) We clearly have $R_{\omega,\omega} = \mathrm{id}_{\mathcal{R}(\omega,E)}$. Let $\omega_3 \subset \omega_2 \subset \omega_1 \subset \omega$ be open. We have to show that $R_{\omega_2,\omega_3} \circ R_{\omega_1,\omega_2} = R_{\omega_1,\omega_3}$ is valid. This is obvious if one of the sets is empty, so let them all be non-empty. Let $T \in L(\mathcal{P}_*(\overline{\omega}_1),E)$. Let $T_0 \in L(\mathcal{P}_*(\overline{\omega}_3),E)$ be a representative of $R_{\omega_1,\omega_3}([T]_1)$, let $T_1 \in L(\mathcal{P}_*(\overline{\omega}_2),E)$ be a representative of $R_{\omega_1,\omega_2}([T]_1)$ and $T_2 \in L(\mathcal{P}_*(\overline{\omega}_3),E)$ a representative of $R_{\omega_2,\omega_3} \circ R_{\omega_1,\omega_2}([T]_1) = R_{\omega_2,\omega_3}([T_1]_2)$. By definition of the restrictions the following is true:

(a) $T_0 - T \in L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3),E)$

(b) $T_1 - T \in L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_2),E)$

(c) $T_2 - T_1 \in L(\mathcal{P}_*(\overline{\omega}_2 \setminus \omega_3),E)$

We first observe that

$$T_0 - T_2 \in L(\mathcal{P}_*(\overline{\omega}_3),E). \tag{6.12}$$

It remains to be shown that $T_0 - T_2 \in L(\mathcal{P}_*(\partial\omega_3),E)$. The equality

$$T_0 - T_2 = (T_0 - T) + (T - T_1) + (T_1 - T_2)$$

holds on $\mathcal{P}_*\big(\overline{\mathbb{R}}\big)$ and the right hand side is an element of

$$L\big(\underbrace{\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3) \cap \mathcal{P}_*(\overline{\omega}_1 \setminus \omega_2) \cap \mathcal{P}_*(\overline{\omega}_2 \setminus \omega_3)}_{=\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3)},E\big) = L(\mathcal{P}_*(\overline{\omega}_1 \setminus \omega_3),E)$$

[3]counterpart: [13, Lemma 6.5, p. 1124]

by $(a)-(c)$ and as $\omega_3 \subset \omega_2 \subset \omega_1$. So due to the remark above Proposition 4.3 $T_0 - T_2$ can be regarded as an element of $L(\mathcal{P}_*(\overline{\omega}_1 \smallsetminus \omega_3), E)$ as well and thus we get by Proposition 4.3(1) and (6.12)

$$\begin{aligned} T_0 - T_2 \in L(\mathcal{P}_*(\overline{\omega}_3), E) \cap L(\mathcal{P}_*(\overline{\omega}_1 \smallsetminus \omega_3), E) &= L(\mathcal{P}_*(\overline{\omega}_3 \cap (\overline{\omega}_1 \smallsetminus \omega_3)), E) \\ &= L(\mathcal{P}_*(\partial \omega_3), E). \end{aligned}$$

(ii) Let T be like in $(S1)$ and $j \in J$. Then for a representative T_j of $R_{\omega,\omega_j}([T])$ it holds $T_j \in L\big(\mathcal{P}_*(\partial \omega_j), E\big)$, since $R_{\omega,\omega_j}([T]) = 0$, and $T - T_j \in L\big(\mathcal{P}_*(\overline{\omega} \smallsetminus \omega_j), E\big)$ by definition of the restriction. Again the equality

$$T = \big(T - T_j\big) + T_j$$

holds on $\mathcal{P}_*\big(\overline{\mathbb{R}}\big)$ and the right hand side is an element of

$$L\big(\underbrace{\mathcal{P}_*(\overline{\omega} \smallsetminus \omega_j) \cap \mathcal{P}_*(\partial \omega_j)}_{=\mathcal{P}_*(\overline{\omega} \smallsetminus \omega_j)}, E\big) = L\big(\mathcal{P}_*(\overline{\omega} \smallsetminus \omega_j), E\big)$$

By the same argument as in part (i) we can regard T as an element of $L\big(\mathcal{P}_*(\overline{\omega} \smallsetminus \omega_j), E\big)$ and get

$$\operatorname{supp} T \subset \overline{\omega} \smallsetminus \omega_j$$

where the support is meant in the sense of Proposition 4.3(2). Since this is valid for all $j \in J$, we obtain

$$\operatorname{supp} T \subset \bigcap_{j \in J} \overline{\omega} \smallsetminus \omega_j = \overline{\omega} \smallsetminus \bigcup_{j \in J} \omega_j = \overline{\omega} \smallsetminus \omega = \partial \omega$$

and thus $T \in L(\mathcal{P}_*(\partial \omega), E)$, i.e. $[T] = 0$.

□

For the special case $\Omega = \overline{\mathbb{R}}$ we use the notation $\mathcal{R}(E) := \mathcal{R}_{\overline{\mathbb{R}}}(E)$. We will see that the presheaf $\mathcal{R}_\Omega(E)$, which satisfies $(S1)$, is already a sheaf, so satisfies, in addition, the sheaf condition $(S2)$ (see [9, 1.5, p. 6]) if we assume that E is not only admissible, but strictly admissible. The next statement will turn out to be an useful tool in this context.

6.6 Proposition. *Let X be a topological space, $\big(\mathcal{G}, R^{\mathcal{G}}\big)$ a presheaf and $\big(\mathcal{F}, R^{\mathcal{F}}\big)$ a sheaf on X. Let $h{:}\,\mathcal{G} \to \mathcal{F}$ be a homomorphism of presheaves such that $h_\Omega{:}\,\mathcal{G}(\Omega) \to \mathcal{F}(\Omega)$ is an isomorphism for every open set $\Omega \subset X$. Then $\big(\mathcal{G}, R^{\mathcal{G}}\big)$ is a sheaf (and h an isomorphism of sheaves).*

Proof. First we remark the following. $h{:}\,\mathcal{G} \to \mathcal{F}$ is a homomorphism of presheaves, i.e. the diagram

$$\begin{array}{ccc} \mathcal{G}(\Omega) & \xrightarrow{h_\Omega} & \mathcal{F}(\Omega) \\ {\scriptstyle R^{\mathcal{G}}_{\Omega,\Omega_1}}\downarrow & & \downarrow{\scriptstyle R^{\mathcal{F}}_{\Omega,\Omega_1}} \\ \mathcal{G}(\Omega_1) & \xrightarrow[h_{\Omega_1}]{} & \mathcal{F}(\Omega_1) \end{array}$$

commutes for open sets $\Omega_1 \subset \Omega \subset X$. Let $f \in \mathcal{F}(\Omega)$. Since h_Ω and h_{Ω_1} are isomorphisms by assumption, one has

$$h_{\Omega_1}^{-1}\left(f|_{\Omega_1}\right) = h_{\Omega_1}^{-1}\left(h_\Omega\left(h_\Omega^{-1}(f)\right)|_{\Omega_1}\right) = h_{\Omega_1}^{-1}\left(h_{\Omega_1}\left(h_\Omega^{-1}(f)|_{\Omega_1}\right)\right) = h_\Omega^{-1}(f)|_{\Omega_1}$$

since h is a homomorphism of presheaves which means that the diagram

$$\begin{tikzcd} \mathcal{G}(\Omega) \arrow[d, "R^{\mathcal{G}}_{\Omega,\Omega_1}"'] & \mathcal{F}(\Omega) \arrow[l, "h_\Omega^{-1}"'] \arrow[d, "R^{\mathcal{F}}_{\Omega,\Omega_1}"] \\ \mathcal{G}(\Omega_1) & \mathcal{F}(\Omega_1) \arrow[l, "h_{\Omega_1}^{-1}"] \end{tikzcd}$$

commutes as well, so h^{-1} is homomorphism of presheaves.
$(S1)$: Let $\{\Omega_j \mid j \in J\}$ be a familiy of open subsets of X and $\Omega := \bigcup_{j\in J}\Omega_j$. Let $f \in \mathcal{G}(\Omega)$ such that $f|_{\Omega_j} = 0$ for all $j \in J$. Then $h_\Omega(f) \in \mathcal{F}(\Omega)$ and

$$h_\Omega(f)|_{\Omega_j} = h_\Omega\left(f|_{\Omega_j}\right) = h_\Omega(0) = 0$$

for all $j \in J$ due to the assumption and since h is a homomorphism of presheaves. As $\mathcal{F}$ is a sheaf, hence satisifies $(S1)$, we obtain $h_\Omega(f) = 0$. Due to the injectivity of h_Ω we get $f = 0$.
$(S2)$: Let $(\Omega_j)_{j\in J}$ and Ω be like above. Let $f_j \in \mathcal{G}(\Omega_j)$ such that $f_j|_{\Omega_j\cap\Omega_k} = f_k|_{\Omega_j\cap\Omega_k}$ for all $j,k \in J$. Then $h_{\Omega_j}(f_j) \in \mathcal{F}(\Omega_j)$ and

$$h_{\Omega_j}(f_j)|_{\Omega_j\cap\Omega_k} - h_{\Omega_k}(f_k)|_{\Omega_j\cap\Omega_k} = h_{\Omega_j\cap\Omega_k}\left(f_j|_{\Omega_j\cap\Omega_k}\right) - h_{\Omega_j\cap\Omega_k}\left(f_k|_{\Omega_j\cap\Omega_k}\right) = 0$$

for all $j,k \in J$ by the assumption and since h is a homomorphism of presheaves. As $\mathcal{F}$ is a sheaf, hence satisifies $(S2)$, there exists $G \in \mathcal{G}(\Omega)$ such that $G|_{\Omega_j} = h_{\Omega_j}(f_j)$ for every $j \in J$. Now we define $F := h_\Omega^{-1}(G) \in \mathcal{F}(\Omega)$. By virtue of the remark in the beginning we gain

$$F|_{\Omega_j} = h_\Omega^{-1}(G)|_{\Omega_j} = h_{\Omega_j}^{-1}\left(G|_{\Omega_j}\right) = h_{\Omega_j}^{-1}\left(h_{\Omega_j}(f_j)\right) = f_j$$

for all $j \in J$.
Therefore, $\mathcal{G}$ is a sheaf and thus h an isomorphism of sheaves. □

We will use this proposition to show that $\mathcal{R}(E)$ satisfies the condition $(S2)$ and is furthermore a flabby sheaf if E is strictly admissible. For this purpose we introduce a boundary value representation of $\mathcal{R}(E)$ in the following way: Let $\Omega \subset \overline{\mathbb{R}}$, $\Omega \neq \varnothing$, be an open set and we define

$$\mathcal{U}(\Omega) := \left\{U \mid U \subset \overline{\mathbb{C}} \text{ open}, U \cap \overline{\mathbb{R}} = \Omega\right\}.$$

Now we define, similar to Definition 3.2, spaces of vector-valued slowly increasing holomorphic functions on $U \setminus \overline{\mathbb{R}}$ resp. U for $U \in \mathcal{U}(\Omega)$.

- If $-\infty \in \Omega$ or $\infty \in \Omega$, we define

$$\mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right) := \lim_{\substack{\longleftarrow \\ n\to\infty}} \operatorname{proj} \mathcal{O}^n\left(S_n(U), E\right)$$

where

$$\mathcal{O}^n(S_n(U),E) := \{f \in \mathcal{O}(S_n(U),E) \mid \forall \alpha \in A: |||f|||_{U^*,n,\alpha} < \infty\}, \quad n \in \mathbb{N}_{\geq 2},$$

with

$$|||f|||_{U^*,n,\alpha} := \sup_{z \in S_n(U)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

and

$$S_n(U) := \begin{cases} U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n,\ \mathrm{Re}(z) > -n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & -\infty \notin \Omega, \infty \in \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n,\ \mathrm{Re}(z) < n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & -\infty \in \Omega, \infty \notin \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & \pm\infty \in \Omega. \end{cases}$$

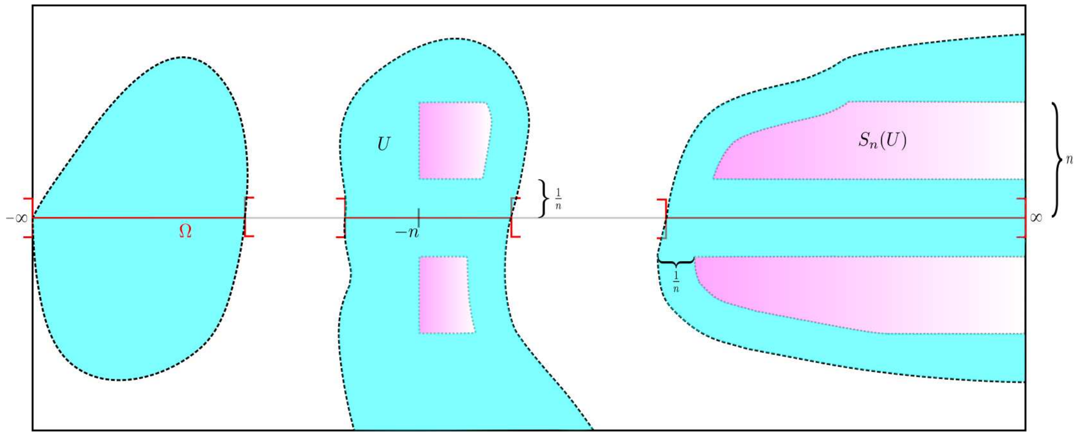

Figure 6.5: $S_n(U)$ for $\infty \in \Omega$, $-\infty \notin \Omega$

- If $\pm\infty \notin \Omega$, we define

$$\mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right) := \mathcal{O}\left(\left(U \setminus \overline{\mathbb{R}}\right) \cap \mathbb{C}, E\right).$$

- If $-\infty \in \Omega$ or $\infty \in \Omega$, we define

$$\mathcal{O}^{exp}(U,E) := \lim_{\substack{\longleftarrow \\ n \to \infty}} \mathrm{proj}\, \mathcal{O}^n(T_n(U),E)$$

where

$$\mathcal{O}^n(T_n(U),E) := \{f \in \mathcal{O}(T_n(U),E) \mid \forall\ \alpha \in A: |||f|||_{U,n,\alpha} < \infty\}, \quad n \in \mathbb{N}_{\geq 2},$$

with

$$|||f|||_{U,n,\alpha} := \sup_{z \in T_n(U)} p_\alpha(f(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}$$

and

$$T_n(U) := \begin{cases} U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n,\ \mathrm{Re}(z) > -n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & -\infty \notin \Omega, \infty \in \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n,\ \mathrm{Re}(z) < n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & -\infty \in \Omega, \infty \notin \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n,\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n}\right\}, & \pm\infty \in \Omega. \end{cases}$$

- If $\pm\infty \notin \Omega$, we define
$$\mathcal{O}^{exp}(U,E) := \mathcal{O}(U \cap \mathbb{C}, E).$$

We remark that $\mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right)$ and $\mathcal{O}^{exp}(U,E)$ are complete locally convex spaces by an analogous proof to the one of Theorem 3.6, if $-\infty \in \Omega$ or $\infty \in \Omega$. If $\pm\infty \notin \Omega$, then this is obviously valid for the corresponding spaces as well if equipped with the topology of uniform convergence on compact subsets. Moreover, if $U = \overline{\mathbb{C}}$, so $\Omega = \overline{\mathbb{R}}$, then the definition of $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}, E\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$ in the just introduced sense coincides with the one in the sense of Definition 3.2 (and therefore the spaces have the same symbol).

6.7 Definition. For an open set $\Omega \subset \overline{\mathbb{R}}$, $\Omega \neq \varnothing$, and $U \in \mathcal{U}(\Omega)$ we define the space of boundary values by
$$bv(\Omega,E) := \mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right) / \mathcal{O}^{exp}(U,E)$$
plus $bv(\varnothing,E) := 0$.

6.8 Lemma. [4] *The definition of $bv(\Omega,E)$ is independent of the choice of $U \in \mathcal{U}(\Omega)$, if E is admissible.*

Proof. Let U, $U_1 \in \mathcal{U}(\Omega)$, w.l.o.g. $U_1 := \left(\overline{\mathbb{C}} \setminus \overline{\mathbb{R}}\right) \cup \Omega$. Then $U \subset U_1$. The canonical mapping
$$J : \mathcal{O}^{exp}\left(U_1 \setminus \overline{\mathbb{R}}, E\right) / \mathcal{O}^{exp}(U_1,E) \to \mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right) / \mathcal{O}^{exp}(U,E),\ [f] \mapsto \left[f\big|_{(U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}} \right],$$
is well-defined since $\mathcal{O}^{exp}(U_1,E) \subset \mathcal{O}^{exp}(U,E)$.
Let $f \in \mathcal{O}^{exp}\left(U_1 \setminus \overline{\mathbb{R}}, E\right)$ with $\left[f\big|_{(U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}} \right] = 0$, i.e. $f\big|_{(U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}} \in \mathcal{O}^{exp}(U,E)$. Then
$$f \in \mathcal{O}^{exp}\left(\left(U_1 \setminus \overline{\mathbb{R}}\right) \cup U, E\right) = \mathcal{O}^{exp}(U_1,E)$$
and therefore $[f] = 0$ connoting the injectivity of J.
The proof of surjectivity resembles the one of Lemma 6.2, but it is sometimes necessary to use two cut-off functions.

(i) If $\infty \notin \overline{\Omega}$ or $-\infty \notin \overline{\Omega}$, then there is $x_i \in \mathbb{R}$ such that $[x_0,\infty] \subset \overline{\Omega}^C$ resp. $[-\infty,x_1] \subset \overline{\Omega}^C$ since $\overline{\Omega}^C \subset \overline{\mathbb{R}}$ is open. If $\infty \in \Omega$ or $-\infty \in \Omega$, then there are $\tilde{x}_i \in \mathbb{R}$ and $\varepsilon_i > 0$ such that $[\tilde{x}_0,\infty] \subset \Omega$ resp. $[-\infty,\tilde{x}_1] \subset \Omega$ and $[\tilde{x}_0,\infty] \times [-\varepsilon_0,\varepsilon_0] \subset U$ resp. $[-\infty,\tilde{x}_1] \times [-\varepsilon_1,\varepsilon_1] \subset U$ since Ω is open

[4] counterpart: [13, Lemma 6.7, p. 1124]

and $U \in \mathcal{U}(\Omega)$. We define the sets

$$F_0 := \begin{cases} (U^C \cap \mathbb{R}^2) \cup (]-\infty, x_1 - 2] \times \mathbb{R}), & -\infty \notin \overline{\Omega},\ \infty \in \partial\Omega, \\ (U^C \cap \mathbb{R}^2) \cup (]-\infty, x_1 - 2] \times \mathbb{R}) \cup \left[\mathbb{R} \times \left(\mathbb{R} \setminus]-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}[\right)\right], & -\infty \notin \overline{\Omega},\ \infty \in \Omega, \\ (U^C \cap \mathbb{R}^2) \cup ([x_0 + 2, \infty[\times \mathbb{R}), & \infty \notin \overline{\Omega},\ -\infty \in \partial\Omega, \\ (U^C \cap \mathbb{R}^2) \cup ([x_0 + 2, \infty[\times \mathbb{R}) \cup \left[\mathbb{R} \times \left(\mathbb{R} \setminus]-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}[\right)\right], & \infty \notin \overline{\Omega},\ -\infty \in \Omega, \\ (U^C \cap \mathbb{R}^2) \cup \left[\mathbb{R} \times \left(\mathbb{R} \setminus]-\frac{\min(\varepsilon_0,\varepsilon_1)}{2}, \frac{\min(\varepsilon_0,\varepsilon_1)}{2}[\right)\right], & \pm\infty \in \Omega, \\ U^C \cap \mathbb{R}^2, & \pm\infty \in \partial\Omega, \\ (U^C \cap \mathbb{R}^2) \cup \left[\mathbb{R} \times \left(\mathbb{R} \setminus]-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}[\right)\right], & \infty \in \Omega,\ -\infty \in \partial\Omega, \\ (U^C \cap \mathbb{R}^2) \cup \left[\mathbb{R} \times \left(\mathbb{R} \setminus]-\frac{\varepsilon_1}{2}, \frac{\varepsilon_1}{2}[\right)\right], & -\infty \in \Omega,\ \infty \in \partial\Omega, \\ (U^C \cap \mathbb{R}^2) \cup (]-\infty, x_1 - 2] \cup [x_0 + 2, \infty[) \times \mathbb{R}, & \pm\infty \notin \overline{\Omega}, \end{cases}$$

and

$$F_1 := \begin{cases} \left(\mathbb{R} \cap \overline{\Omega}\right) \cup \left(]-\infty, \tilde{x}_1 - 2] \times \left[-\frac{\varepsilon_1}{4}, \frac{\varepsilon_1}{4}\right]\right), & -\infty \in \Omega,\ \infty \notin \Omega, \\ \left(\mathbb{R} \cap \overline{\Omega}\right) \cup \left([\tilde{x}_0 + 2, \infty[\times \left[-\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{4}\right]\right), & \infty \in \Omega,\ -\infty \notin \Omega, \\ \left(\mathbb{R} \cap \overline{\Omega}\right) \cup (]-\infty, \tilde{x}_1 - 2] \cup [\tilde{x}_0 + 2, \infty[) \times \left[-\frac{\min(\varepsilon_0,\varepsilon_1)}{4}, \frac{\min(\varepsilon_0,\varepsilon_1)}{4}\right], & \pm\infty \in \Omega, \\ \mathbb{R} \cap \overline{\Omega}, & \pm\infty \notin \Omega. \end{cases}$$

If we number the appearing cases in the definition of F_0 from above to below by 1A, ... , 9A and in the definition of F_1 by 1B, ... , 4B, then we have the following possible combinations:

Table 6.5: Combinations

	1A	2A	3A	4A	5A	6A	7A	8A	9A
1B	×	×	×	✓	×	×	×	✓	×
2B	×	✓	×	×	×	×	✓	×	×
3B	×	×	×	×	✓	×	×	×	×
4B	✓	×	✓	×	×	✓	×	×	✓

legend:
✓ – possible
× – impossible

The sets F_0 and F_1 are closed in $\mathbb{R}^2$, $F_1 \neq \varnothing$ and $F_0 \cap F_1 = \partial\Omega \cap \mathbb{R}$. F_0 is empty iff $U^C \cap \mathbb{R}^2 = \varnothing$ and $\pm\infty \in \partial\Omega$. This implies $U^C \subset \{\pm\infty\} \times \mathbb{R}$, thus $\mathbb{R}^2 \subset U$ and so $\Omega = \mathbb{R}$. If $F_0 \neq \varnothing$, then by [18, Theorem 1.4.10, p. 30, Corollary 1.4.11, p. 31] there exists $\varphi_0 \in \mathcal{C}^\infty\left((F_0 \cap F_1)^C\right) = \mathcal{C}^\infty(\mathbb{R}^2 \setminus \partial\Omega)$, $0 \leq \varphi_0 \leq 1$, such that $\varphi_0 = 0$ on V_0 and $\varphi_0 = 1$ on V_1 where V_0, $V_1 \subset \mathbb{R}^2$ are open and

$$V_0 \supset F_0 \setminus (F_0 \cap F_1) = F_0 \setminus \partial\Omega \supset \left(\mathbb{R} \setminus \overline{\Omega}\right)$$

and

$$V_1 \supset F_1 \setminus (F_0 \cap F_1) = F_1 \setminus \partial\Omega \supset (\mathbb{R} \cap \Omega)$$

as well as

$$\left|\partial^\beta \varphi_0(z)\right| \leq C^{|\beta|} \frac{\mathrm{d}(z)^{-|\beta|}}{d_1 \cdots d_{|\beta|}} \tag{6.13}$$

for all $z \in \mathbb{R}^2 \setminus \partial\Omega$ and all $\beta \in \mathbb{N}_0^2$ where C, d and $(d_n)_{n\in\mathbb{N}}$ are like in part (i) of the proof of Lemma 6.2. If $F_0 = \varnothing$, we set $V_0 := \varnothing$, $V_1 := \mathbb{R}^2$ and $\varphi_0 := 1$ on V_1.

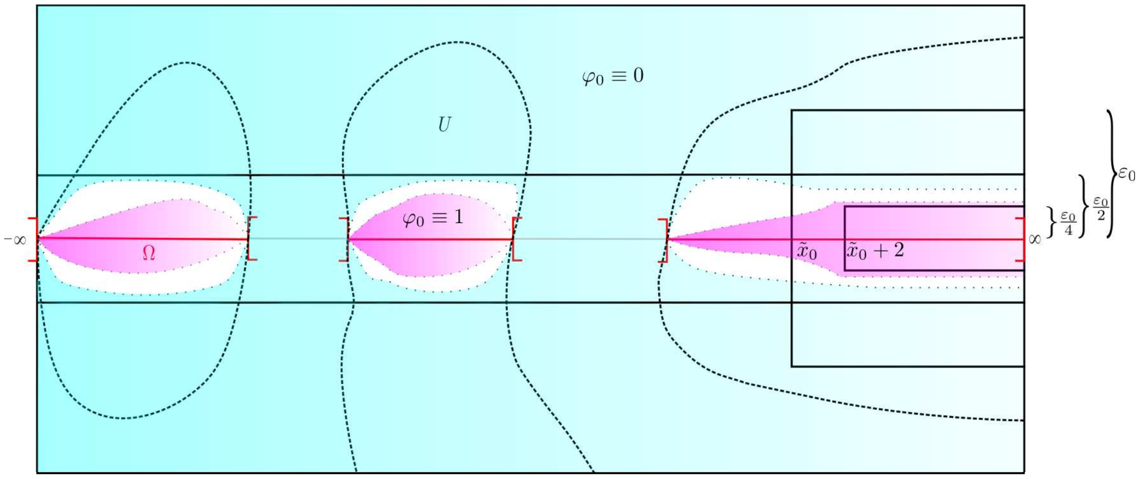

Figure 6.6: case (7A,2B): $\infty \in \Omega$, $-\infty \in \partial\Omega$

Furthermore, we define the sets $K_0 := \left\{(x,y) \in \mathbb{R}^2 \mid y \le -2e^{-|x|} \vee y \ge 2e^{-|x|}\right\}$ and $K_1 := \left\{(x,y) \in \mathbb{R}^2 \mid -e^{-|x|} \le y \le e^{-|x|}\right\}$ as well as

$$\tilde{F}_0 := \begin{cases} K_0 \cup [\mathbb{R}_{\ge 0} \times (\mathbb{R} \setminus]-2,2[)], & -\infty \in \partial\Omega,\ \infty \notin \partial\Omega, \\ K_0 \cup [\mathbb{R}_{\le 0} \times (\mathbb{R} \setminus]-2,2[)], & -\infty \notin \partial\Omega,\ \infty \in \partial\Omega, \\ K_0, & \pm\infty \in \partial\Omega, \end{cases}$$

plus

$$\tilde{F}_1 := \begin{cases} K_1 \cup (\mathbb{R}_{\ge 0} \times [-1,1]), & -\infty \in \partial\Omega,\ \infty \notin \partial\Omega, \\ K_1 \cup (\mathbb{R}_{\le 0} \times [-1,1]), & -\infty \notin \partial\Omega,\ \infty \in \partial\Omega, \\ K_1, & \pm\infty \in \partial\Omega. \end{cases}$$

The sets $\tilde{F}_0$ and $\tilde{F}_1$ are non-empty and closed in $\mathbb{R}^2$ and $\tilde{F}_0 \cap \tilde{F}_1 = \varnothing$. Like above there is $\varphi_1 \in \mathrm{C}^\infty\left(\left(\tilde{F}_0 \cap \tilde{F}_1\right)^C\right) = \mathrm{C}^\infty(\mathbb{R}^2)$, $0 \le \varphi_1 \le 1$, such that $\varphi_1 = 0$ on W_0 and $\varphi_1 = 1$ on W_1 where $W_0, W_1 \subset \mathbb{R}^2$ are open and

$$W_0 \supset \tilde{F}_0 \setminus \left(\tilde{F}_0 \cap \tilde{F}_1\right) = \tilde{F}_0$$

and

$$W_1 \supset \tilde{F}_1 \setminus \left(\tilde{F}_0 \cap \tilde{F}_1\right) = \tilde{F}_1$$

as well as

$$\left|\partial^\beta \varphi_1(z)\right| \le \tilde{C}^{|\beta|} \frac{\tilde{\mathrm{d}}(z)^{-|\beta|}}{d_1 \cdots d_{|\beta|}} \tag{6.14}$$

for all $z \in \mathbb{R}^2$ and all $\beta \in \mathbb{N}_0^2$, where $\tilde{C}$, $\tilde{\mathrm{d}}$ and $(d_n)_{n\in\mathbb{N}}$ are like above. If $\pm\infty \notin \partial\Omega$, we set $W_0 := \varnothing$, $W_1 := \mathbb{R}^2$ and $\varphi_1 := 1$ on W_1.

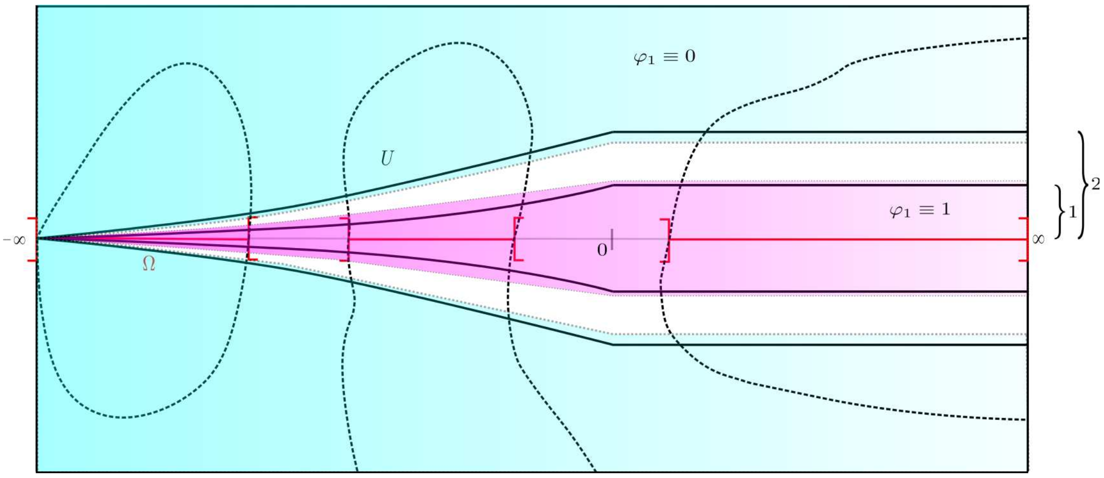

Figure 6.7: case (7A,2B): $\infty \in \Omega$, $-\infty \in \partial\Omega$

Again we take a closer look at the right hand side of (6.13) resp. (6.14) and claim

(a)
$$B := \inf_{z \in S_n(\partial\Omega)} \mathrm{d}(z) > 0, \tag{6.15}$$

(b)
$$D := \inf_{z \in S_n(\partial\Omega)} \tilde{\mathrm{d}}(z) > 0. \tag{6.16}$$

(a) <u>case:</u> $\infty \in \Omega$, $-\infty \notin \overline{\Omega}$, i.e. (2A,2B)

(1) For $z \in S_n(\partial\Omega)$ with $\mathrm{Re}(z) \leq x_1 - 2$ we have

$$\mathrm{d}(z) = \max(\mathrm{d}(z, F_0), \mathrm{d}(z, F_1)) = \mathrm{d}(z, F_1) \geq 2 - \frac{1}{n} \geq 1$$

and with $\mathrm{Re}(z) \geq \tilde{x}_0 + 2$

$$\begin{aligned} \mathrm{d}(z) &\geq \begin{cases} \frac{\varepsilon_0}{4}, & z \in F_0, \\ \min\left(\frac{1}{2}\frac{\varepsilon_0}{4}, 2\right), & z \notin F_0,\ z \notin F_1, \\ \min\left(\frac{\varepsilon_0}{4}, 2\right), & z \in F_1, \end{cases} \\ &= \min\left(\frac{\varepsilon_0}{8}, 2\right). \end{aligned}$$

(2) For $z \in S_n(\partial\Omega)$ with $\mathrm{Re}(z) \leq \tilde{x}_0$ and $|\mathrm{Im}(z)| \geq \frac{1}{n}$ we get

$$\mathrm{d}(z) \geq \mathrm{d}(z, F_1) \geq \min\left(2, \frac{1}{n}\right) = \frac{1}{n}.$$

(3) By Remark 3.3(1) the set $U_n(\partial\Omega)$ has finitely many components Z_j, so there exists $k \in \mathbb{N}$ with $U_n(\partial\Omega) = \bigcup_{j=1}^{k} Z_j$. Since $\pm\infty \notin \partial\Omega$, all Z_j are bounded. Let $a_j := \min Z_j \cap \partial\Omega$ and $b_j := \max Z_j \cap \partial\Omega$. W.l.o.g. $a_j < a_{j+1}$ for $1 \leq j \leq k$ (otherwise renumber). We observe that $b_k = \max_{1 \leq j \leq k} b_j < \tilde{x}_0$. Due to Remark 3.3(2) there is $0 < r_j < {}^1\!/_n$ such that

$\left\{z \in \mathbb{C} \mid \mathrm{d}\left(z, \left[a_j, b_j\right]\right) \leq r_j\right\} \subset Z_j$ for all $1 \leq j \leq k$.
Let $z \in S_n(\partial\Omega)$ such that $x_1 - 2 < \mathrm{Re}(z) < b_k$ and $|\mathrm{Im}(z)| < \frac{1}{n}$.

- If $a_j < b_j$, we therefore obtain for z with $a_j < \mathrm{Re}(z) < b_j$

$$\mathrm{d}(z) \geq \mathrm{d}(z, F_1) \geq r_j.$$

If $k \geq 2$, consider z with $b_j < \mathrm{Re}(z) < a_{j+1}$ for $1 \leq j \leq k-1$. If $\mathrm{d}(z) \leq \frac{1}{2n}$, we have with $N_0 := \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| > \frac{3}{2n}\right\}$ and $N_1 := \left\{w \in \mathbb{C} \mid \mathrm{d}(\partial\Omega \cap \mathbb{R}) < \frac{1}{3n}\right\}$

$$\mathrm{d}(z, F_1) = \mathrm{d}\left(z, F_1 \setminus N_1\right) = \mathrm{d}\left(z, \underbrace{\left(\left[b_j, a_{j+1}\right] \cap \overline{\Omega}\right) \setminus N_1}_{=:K_{1,j}}\right)$$

and

$$\begin{aligned}\mathrm{d}(z, F_0) &= \mathrm{d}\left(z, F_0 \setminus (N_0 \cup N_1)\right) \\ &= \mathrm{d}\left(z, \underbrace{\left(F_0 \cap \left\{w \in \mathbb{C} \mid b_j - \frac{1}{2n} \leq \mathrm{Re}(w) \leq a_{j+1} + \frac{1}{2n}\right\}\right) \setminus (N_0 \cup N_1)}_{=:K_{0,j}}\right).\end{aligned}$$

$K_{0,j}$ and $K_{1,j}$ are bounded and closed sets in $\mathbb{R}^2$, thus compact, and disjoint. Hence $c_j := \mathrm{d}\left(K_{0,j}, K_{1,j}\right) > 0$ yielding to

$$\mathrm{d}(z) = \max_{i \in \{0,1\}} \mathrm{d}\left(z, K_{i,j}\right) \geq \frac{c_j}{2} > 0$$

for all $1 \leq j \leq k-1$. Combining these results, we obtain

$$\mathrm{d}(z) \geq \min\left(\min_{1 \leq j \leq k} r_j, \min_{1 \leq j \leq k-1} \frac{c_j}{2}, \frac{1}{2n}\right) > 0$$

for $z \in S_n(\partial\Omega)$ with $|\mathrm{Im}(z)| \leq 1/n$ and $a_1 < \mathrm{Re}(z) < b_k$.

- Consider z with $\mathrm{Re}(z) < a_1$. Then

$$\mathrm{d}(z) \geq \mathrm{d}(z, F_1) = |z - a_1| > \frac{1}{n}$$

is valid.

Let $z \in S_n(\partial\Omega)$ such that $b_k \leq \mathrm{Re}(z) < \tilde{x}_0 + 2$. If $\mathrm{d}(z) \leq \frac{1}{2n}$, we get with
$N_2 := \left\{w \in \mathbb{C} \mid \mathrm{Re}(w) > \tilde{x}_0 + 2 + \frac{1}{2n}\right\}$ and
$N_3 := \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| > n + \frac{1}{2n} \vee \mathrm{Re}(w) < b_k - \frac{1}{2n}\right\}$

$$\begin{aligned}\mathrm{d}(z, F_1) &= \mathrm{d}\left(z, F_1 \setminus \left(D_{\frac{1}{3n}}(b_k) \cup N_2\right)\right) \\ &= \mathrm{d}\left(z, \underbrace{\{w \in F_1 \mid \mathrm{Re}(w) \geq b_k\} \setminus \left(D_{\frac{1}{3n}}(b_k) \cup N_2\right)}_{=:\tilde{K}_1}\right)\end{aligned}$$

as well as

$$\mathrm{d}(z,F_0)=\mathrm{d}\Bigg(z,\underbrace{F_0\setminus\Big(D_{\frac{1}{3n}}(b_k)\cup N_2\cup N_3\Big)}_{=:\tilde{K}_0}\Bigg).$$

$\tilde{K}_0$ and $\tilde{K}_1$ are compact and disjoint. Thus we have $c_0:=\mathrm{d}\big(\tilde{K}_0,\tilde{K}_1\big)>0$ implying

$$\mathrm{d}(z)=\max_{i\in\{0,1\}}\mathrm{d}\big(z,\tilde{K}_i\big)\geq\frac{c_0}{2}>0.$$

(4) Merging (1)-(3), we gain

$$\inf_{z\in S_n(\partial\Omega)}\mathrm{d}(z)\geq\min\left(1,\min\left(\frac{\varepsilon_0}{8},2\right),\frac{1}{n},\min\left(\min_{1\leq j\leq k}r_j,\min_{1\leq j\leq k-1}\frac{c_j}{2},\frac{1}{2n}\right),\frac{c_0}{2}\right)>0.$$

The proof of the other eight cases can be done quite analogously keeping the definition of $U_n(\partial\Omega)$ in mind and that, if $-\infty\in\partial\Omega$ or $\infty\in\partial\Omega$, we have for $z\in S_n(\partial\Omega)$ with $\mathrm{Re}(z)\leq -n$ resp. $\mathrm{Re}(z)\geq n$

$$\mathrm{d}(z)\geq\mathrm{d}(z,F_1)\geq\frac{1}{n}.$$

(b) We only consider the case $-\infty\in\partial\Omega$ and $\infty\notin\Omega$. The proof for the other two cases is similar.

(1) For $z\in S_n(\partial\Omega)$ with $\mathrm{Re}(z)\geq 1$ we have

$$\begin{aligned}\tilde{\mathrm{d}}(z)=\max\Big(\mathrm{d}\big(z,\tilde{F}_0\big),\mathrm{d}\big(z,\tilde{F}_1\big)\Big)&\geq\begin{cases}1, & z\in\tilde{F}_0,\\ \min\left(\frac{1}{2},1\right), & z\notin F_0,\ z\notin F_1,\\ \min(1,1), & z\in F_1,\end{cases}\\ &=\frac{1}{2}.\end{aligned}$$

(2) Let $z\in S_n(\partial\Omega)$ such that $0\leq\mathrm{Re}(z)<1$. If $\tilde{\mathrm{d}}(z)\leq\frac{1}{2n}$, then

$$\mathrm{d}\big(z,\tilde{F}_0\big)=\mathrm{d}\big(z,\tilde{F}_0\setminus(N_0\cup N_1)\big)\quad\text{and}\quad\mathrm{d}\big(z,\tilde{F}_1\big)=\mathrm{d}\big(z,\tilde{F}_1\setminus N_1\big)$$

where $N_0:=\left\{w\in\mathbb{C}\,\middle|\,|\mathrm{Im}(w)|>n+\frac{1}{2n}\right\}$ and $N_1:=\left\{w\in\mathbb{C}\,\middle|\,\mathrm{Re}(w)<-\frac{1}{n}\vee\mathrm{Re}(w)>1+\frac{1}{n}\right\}$. The sets $\tilde{F}_0\setminus(N_0\cup N_1)$ and $\tilde{F}_1\setminus N_1$ are compact and disjoint, thus we gain $c_0:=\mathrm{d}\big(\tilde{F}_0\setminus(N_0\cup N_1),\tilde{F}_1\setminus N_1\big)>0$ and therefore

$$\tilde{\mathrm{d}}(z)\geq\frac{c_0}{2}>0.$$

(3) Let $z\in S_n(\partial\Omega)$ with $\mathrm{Re}(z)<0$. If $\tilde{\mathrm{d}}(z)\leq\frac{1}{2n}$, then

$$\mathrm{d}\big(z,\tilde{F}_0\big)=\mathrm{d}\big(z,\tilde{F}_0\setminus(N_0\cup N_2)\big)\quad\text{and}\quad\mathrm{d}\big(z,\tilde{F}_1\big)=\mathrm{d}\big(z,\tilde{F}_1\setminus N_2\big)$$

with N_0 from part (2) and
$N_2:=\left\{w\in\mathbb{C}\mid\left(|\mathrm{Im}(w)|<\frac{1}{3n}\wedge\mathrm{Re}(w)<-n-\frac{1}{2n}\right)\vee\mathrm{Re}(w)>\frac{1}{n}\right\}$.

The sets $\tilde{F}_0 \setminus (N_0 \cup N_2)$ and $\tilde{F}_1 \setminus N_2$ are compact and disjoint, so we obtain $c_1 := \mathrm{d}\left(\tilde{F}_0 \setminus (N_0 \cup N_2), \tilde{F}_1 \setminus N_2\right) > 0$ and hence $\tilde{\mathrm{d}}(z) \geq \frac{c_1}{2} > 0$.

(4) By combining these results, we have

$$\inf_{z \in S_n(\partial\Omega)} \tilde{\mathrm{d}}(z) \geq \min\left(\frac{1}{2}, \frac{1}{2n}, \frac{c_0}{2}, \frac{c_1}{2}\right) > 0.$$

(ii) Let $f \in \mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}}, E\right)$. By the choice of φ_0 and φ_1 the function $\overline{\partial}(\varphi_1 \varphi_0 f)$ may be regarded as an element of $\mathrm{C}^\infty(\mathbb{R}^2 \setminus \partial\Omega, E)$ by C^∞-continuation via $\overline{\partial}(\varphi_1\varphi_0 f) := 0$ on $[(U^C \cap \mathbb{R}^2) \cup \mathbb{R}] \setminus \partial\Omega$. Moreover, with the definition

$$V := (V_0 \cup W_0) \cup (V_1 \cap W_1),$$

the equation

$$\overline{\partial}(\varphi_1\varphi_0 f)(z) = \begin{cases} 0, & z \in V, \\ \left[\left(\overline{\partial}\varphi_1\right)\varphi_0 f + \left(\overline{\partial}\varphi_0\right)\varphi_1 f\right](z), & \text{else}, \end{cases}$$

is valid.
The next step is similar to (6.4). Let $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$ and $\alpha \in A$. We define the set $S(n) := S_n(\partial\Omega) \setminus V$ and $C_m := \#\left\{\gamma \in \mathbb{N}_0^2 \mid |\gamma| \leq m\right\}$.
If $\varphi_i \not\equiv 1$, $i = 1, 2$, on $\mathbb{R}^2$, we obtain by applying the Leibniz rule twice

$$\begin{aligned}
&\left|\overline{\partial}(\varphi_1\varphi_0 f)\right|_{\partial\Omega,n,m,\alpha} \\
&= \sup_{\substack{z \in S_n(\partial\Omega),\\ \beta \in \mathbb{N}_0^2,\, |\beta| \leq m}} p_\alpha\left(\partial^\beta \overline{\partial}(\varphi_1\varphi_0 f)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq (m!)^2 \sup_{\substack{z \in S(n),\\ \beta \in \mathbb{N}_0^2,\, |\beta| \leq m}} \sum_{\gamma \leq \beta} \left|\partial^{\beta-\gamma}\left[\left(\overline{\partial}\varphi_1\right)\varphi_0 + \left(\overline{\partial}\varphi_0\right)\varphi_1\right](z)\right| \underbrace{\sup_{\substack{z \in S(n),\\ \beta \in \mathbb{N}_0^2,\, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=:C(f)} \\
&\leq (m!)^4 C(f) \sup_{\substack{z \in S(n),\\ \beta \in \mathbb{N}_0^2,\, |\beta| \leq m}} \sum_{\gamma \leq \beta} \sum_{\tau \leq \beta-\gamma} \left|\partial^\tau\left(\overline{\partial}\varphi_1\right)(z)\,\partial^{\beta-\gamma-\tau}\varphi_0(z) + \partial^\tau\left(\overline{\partial}\varphi_0\right)(z)\,\partial^{\beta-\gamma-\tau}\varphi_1(z)\right| \\
&\leq (m!)^4 C(f) \sum_{|\gamma| \leq m} \sum_{|\tau| \leq m+2} \sup_{z \in S(n)} |\partial^\tau \varphi_1(z)| \sup_{\substack{z \in S(n),\\ \upsilon \in \mathbb{N}_0^2,\, |\upsilon| \leq m}} |\partial^\upsilon \varphi_0(z)| + \sup_{z \in S(n)} |\partial^\tau \varphi_0(z)| \sup_{\substack{z \in S(n),\\ \upsilon \in \mathbb{N}_0^2,\, |\upsilon| \leq m}} |\partial^\upsilon \varphi_1(z)| \\
&\underset{(6.13),\,(6.14)}{\leq} (m!)^4 C_m C(f) \sum_{|\tau| \leq m+2} \tilde{C}^{|\tau|} \sup_{z \in S(n)} \frac{\tilde{\mathrm{d}}(z)^{-|\tau|}}{d_1 \cdots d_{|\tau|}} \sup_{\substack{z \in S(n),\\ \upsilon \in \mathbb{N}_0^2,\, |\upsilon| \leq m}} C^{|\upsilon|} \frac{\mathrm{d}(z)^{-|\upsilon|}}{d_1 \cdots d_{|\upsilon|}} \\
&\quad + C^{|\tau|} \sup_{z \in S(n)} \frac{\mathrm{d}(z)^{-|\tau|}}{d_1 \cdots d_{|\tau|}} \sup_{\substack{z \in S(n),\\ \upsilon \in \mathbb{N}_0^2,\, |\upsilon| \leq m}} \tilde{C}^{|\upsilon|} \frac{\tilde{\mathrm{d}}(z)^{-|\upsilon|}}{d_1 \cdots d_{|\upsilon|}}
\end{aligned}$$

$$\underset{\substack{(6.15),\\(6.16)}}{\leq}(m!)^4 C_m \frac{\left[\max\left(C,\tilde{C},1\right)\right]^{m+2}}{(d_1\cdots d_{m+2})^2} C(f)\sum_{|\tau|\leq m+2} D^{-|\tau|}\sup_{\substack{\upsilon\in\mathbb{N}_0^2,\\|\upsilon|\leq m}} B^{-|\upsilon|}+B^{-|\tau|}\sup_{\substack{\upsilon\in\mathbb{N}_0^2,\\|\upsilon|\leq m}} D^{-|\upsilon|}. \tag{6.17}$$

If $\varphi_0 \not\equiv 1$ and $\varphi_1 \equiv 1$ on $\mathbb{R}^2$, then

$$\left|\overline{\partial}(\varphi_1\varphi_0 f)\right|_{\partial\Omega,n,m,\alpha} \underset{\substack{(6.13),\\(6.15)}}{\leq}(m!)^2 \frac{[\max(C,1)]^{m+2}}{d_1\cdots d_{m+2}} C(f)\sum_{|\gamma|\leq m+2} B^{-|\gamma|},$$

and if $\varphi_0 \equiv 1$ and $\varphi_1 \not\equiv 1$ on $\mathbb{R}^2$, then

$$\left|\overline{\partial}(\varphi_1\varphi_0 f)\right|_{\partial\Omega,n,m,\alpha} \underset{\substack{(6.14),\\(6.16)}}{\leq}(m!)^2 \frac{\left[\max\left(\tilde{C},1\right)\right]^{m+2}}{d_1\cdots d_{m+2}} C(f)\sum_{|\gamma|\leq m+2} D^{-|\gamma|}.$$

Now we have to take a closer look at $C(f)$. First of all we remark that

$$\begin{aligned}\left[\left(U^C\cup\overline{\mathbb{R}}\right)\cap\mathbb{R}^2\right] &= \left(\left[\left(U^C\cap\mathbb{R}^2\right)\cup\left(\mathbb{R}\cap\overline{\Omega}\right)\right]\setminus\partial\Omega\right)\cup(\partial\Omega\cap\mathbb{R})\\ &\subset\left[(V_0\cup W_0)\cup(V_1\cap W_1)\cup\bigcup_{x\in\partial\Omega\cap\mathbb{R}} D_{\frac{1}{n}}(x)\right]\\ &= V\cup\bigcup_{x\in\partial\Omega\cap\mathbb{R}} D_{\frac{1}{n}}(x) =: W.\end{aligned}$$

W is an open set in $\mathbb{R}^2$ as the union of open sets and we get

$$\overline{S(n)} = \overline{[S_n(\partial\Omega)\setminus V]} \subset \overline{W^C} = W^C \subset \left[\left(U\setminus\overline{\mathbb{R}}\right)\cap\mathbb{R}^2\right]. \tag{6.18}$$

In the following we will prove that either $S(n)$ is already bounded or that there are $k\in\mathbb{N}_{\geq 2}$ and $M_0\subset S(n)$ bounded plus $M_1\subset S_k(U)$ such that

$$S(n)\subset(M_0\cup M_1).$$

As $|\mathrm{Im}(z)|\leq 1/n$ for every $z\in S(n)$, it suffices to prove that there is $C_1>0$ such that $|\mathrm{Re}(z)|\leq C_1$ for every $z\in S(n)$ resp. $z\in M_0$.

<u>1. case:</u> $\pm\infty\notin\partial\Omega$

<u>1.1. case:</u> $\pm\infty\notin\overline{\Omega}$, i.e. (9A,4B).

The set $S(n)$ is bounded, since for all $z\in S(n)$:

$$|\mathrm{Re}(z)|\leq\max(|x_1-2|,|x_0+2|)$$

<u>1.2. case:</u> $(\infty\in\Omega,\ -\infty\notin\Omega)$ or $(\infty\notin\Omega,\ -\infty\in\Omega)$ or $\pm\infty\in\Omega$, i.e. (2A,2B) or (4A,1B) or (5A,3B).

We define

$$M := \begin{cases}\{z\in\mathbb{C}\mid \mathrm{Re}(z)>\tilde{x}_0+2\}, & \infty\in\Omega,\ -\infty\notin\Omega,\\ \{z\in\mathbb{C}\mid \mathrm{Re}(z)<\tilde{x}_1-2\}, & \infty\notin\Omega,\ -\infty\in\Omega,\\ \{z\in\mathbb{C}\mid \mathrm{Re}(z)>\tilde{x}_0+2\ \vee\ \mathrm{Re}(z)<\tilde{x}_1-2\}, & \pm\infty\in\Omega,\end{cases}$$

and decompose the set $S(n)$ as follows

$$S(n) = \underbrace{[S(n) \setminus M]}_{=:M_0} \cup \underbrace{[S(n) \cap M]}_{=:M_1}.$$

M_0 is bounded because

$$|\mathrm{Re}(z)| \leq \begin{cases} \max(|x_1 - 2|, |\tilde{x}_0 + 2|), & \infty \in \Omega,\ -\infty \notin \Omega, \\ \max(|\tilde{x}_1 - 2|, |x_0 + 2|), & \infty \notin \Omega,\ -\infty \in \Omega, \\ \max(|\tilde{x}_1 - 2|, |\tilde{x}_0 + 2|), & \pm\infty \in \Omega, \end{cases}$$

for every $z \in M_0$. Let $\varepsilon_2 := \min(\varepsilon_0, \varepsilon_1)$ and

$$r := \frac{1}{2} \min\left(2, \frac{\varepsilon_i}{2}, \frac{\varepsilon_i}{4}\right) = \min\left(1, \frac{\varepsilon_i}{8}\right),\ i = 0, 1, 2,$$

and choose $k \in \mathbb{N}$ with $k > \max(n, \varepsilon_i)$ and $\frac{1}{k} < \frac{\varepsilon_i}{8}$, $i = 0, 1, 2$, in the corresponding cases plus, in addition,

$$-k < \tilde{x}_0, \text{ if } \infty \in \Omega,\ -\infty \notin \Omega, \quad \text{resp.} \quad k > \tilde{x}_1, \text{ if } \infty \notin \Omega,\ -\infty \in \Omega.$$

Then

$$\frac{1}{k} < \min\left(\frac{1}{n}, \frac{\varepsilon_i}{8}\right) \leq \min\left(1, \frac{\varepsilon_i}{4} - r\right) \leq \min\left(1, \frac{3\varepsilon_i}{8}\right)$$

is valid and thus for all $z \in M_1$

$$\begin{aligned}
&\overline{D_r(z)} \\
&\subset \{w \in \mathbb{C} \mid \mathrm{d}(w, M_1) \leq r\} \\
&\subset \begin{cases} \left([\tilde{x}_0 + 2 - r, \infty[\times \left[-\frac{\varepsilon_0}{2} - r, \frac{\varepsilon_0}{2} + r\right]\right) \setminus \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| < \frac{\varepsilon_0}{4} - r\right\}, \ \infty \in \Omega,\ -\infty \notin \Omega, \\ \left(]-\infty, \tilde{x}_1 - 2 + r] \times \left[-\frac{\varepsilon_1}{2} - r, \frac{\varepsilon_1}{2} + r\right]\right) \setminus \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| < \frac{\varepsilon_1}{4} - r\right\}, \ \infty \notin \Omega,\ -\infty \in \Omega, \\ \left((]-\infty, \tilde{x}_1 - 2 + r] \cup [\tilde{x}_0 + 2 - r, \infty[) \times \left[-\frac{\varepsilon_2}{2} - r, \frac{\varepsilon_2}{2} + r\right]\right) \setminus \left\{w \mid |\mathrm{Im}(w)| < \frac{\varepsilon_2}{4} - r\right\}, \ \pm\infty \in \Omega, \end{cases} \\
&\subset \begin{cases} \left([\tilde{x}_0 + 1, \infty[\times \left[-\frac{5\varepsilon_0}{8}, \frac{5\varepsilon_0}{8}\right]\right) \setminus \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| < \frac{\varepsilon_0}{8}\right\}, & \infty \in \Omega,\ -\infty \notin \Omega, \\ \left(]-\infty, \tilde{x}_1 - 1] \times \left[-\frac{5\varepsilon_1}{8}, \frac{5\varepsilon_1}{8}\right]\right) \setminus \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| < \frac{\varepsilon_1}{8}\right\}, & \infty \notin \Omega,\ -\infty \in \Omega, \\ \left((]-\infty, \tilde{x}_1 - 1] \cup [\tilde{x}_0 + 1, \infty[) \times \left[-\frac{5\varepsilon_2}{8}, \frac{5\varepsilon_2}{8}\right]\right) \setminus \left\{w \in \mathbb{C} \mid |\mathrm{Im}(w)| < \frac{\varepsilon_2}{8}\right\}, & \pm\infty \in \Omega, \end{cases} \\
&\subset S_k(U) \subset \left(U \setminus \overline{\mathbb{R}}\right) \cap \mathbb{R}^2.
\end{aligned} \tag{6.19}$$

Due to the Cauchy inequality we get like in Theorem 3.6(4)

$$\begin{aligned}
&\sup_{\substack{z \in M_1, \\ \beta \in \mathbb{N}_0^2,\, |\beta| \leq m}} p_\alpha\left(f^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq \underbrace{e^{\frac{r}{n}} \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{r^{|\beta|}}}_{=:A_0} \sup_{z \in M_1} \max_{|\zeta - z| = r} p_\alpha\left(f(\zeta)\right) e^{-\frac{1}{n}|\mathrm{Re}(\zeta)|}
\end{aligned}$$

$$\le A_0 \sup_{\zeta\in S_k(U)} p_\alpha\big(f(\zeta)\big)e^{-\frac{1}{k}|\mathrm{Re}(\zeta)|}$$
$$= A_0 \|\!|f\|\!|_{U^*,k,\alpha}.$$

<u>2. case:</u> $(\infty\in\partial\Omega,\ -\infty\notin\partial\Omega)$ or $(\infty\notin\partial\Omega,\ -\infty\in\partial\Omega)$
<u>2.1. case:</u> $(\infty\in\partial\Omega,\ -\infty\in\Omega)$ or $(\infty\in\Omega,\ -\infty\in\partial\Omega)$, i.e. (8A,1B) or (7A,2B).
We define the set

$$M := \begin{cases} \{z\in\mathbb{C} \mid \mathrm{Re}(z) < \tilde{x}_1 - 2\}, & \infty\in\partial\Omega,\ -\infty\in\Omega, \\ \{z\in\mathbb{C} \mid \mathrm{Re}(z) > \tilde{x}_0 + 2\}, & \infty\in\Omega,\ -\infty\in\partial\Omega, \end{cases}$$

and decompose $S(n)$ in the same way like before, i.e.

$$S(n) = \underbrace{[S(n)\smallsetminus M]}_{=:M_0} \cup \underbrace{[S(n)\cap M]}_{=:M_1}.$$

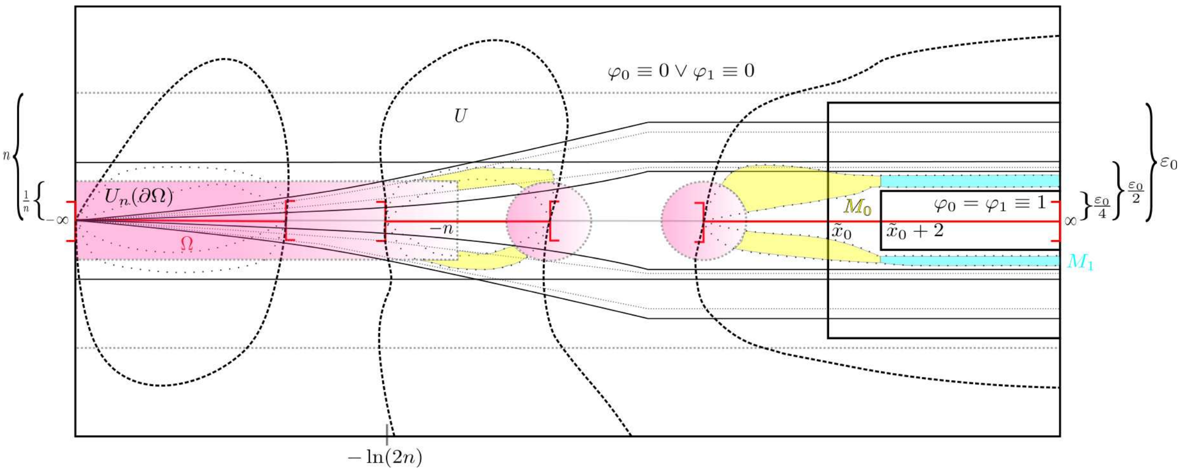

Figure 6.8: case (7A,2B): $\infty\in\Omega$, $-\infty\in\partial\Omega$

We observe that the inequality $1/n \ge 2e^{-|x|}$ is equivalent to $\ln(2n) \le |x|$ for all $x\in\mathbb{R}$. Hence M_0 is bounded, since

$$|\mathrm{Re}(z)| \le \begin{cases} \max(|\tilde{x}_1 - 2|, n, \ln(2n)), & \infty\in\partial\Omega,\ -\infty\in\Omega, \\ \max(|-n|, |-\ln(2n)|, |\tilde{x}_0 + 2|), & \infty\in\Omega,\ -\infty\in\partial\Omega, \end{cases}$$

for all $z\in M_0$. Using the same r and k like in case 1.2 (only for $i = 0,1$), we get again by the Cauchy inequality

$$\sup_{\substack{z\in M_1,\\ \beta\in\mathbb{N}_0^2,\ |\beta|\le m}} p_\alpha\big(f^{(|\beta|)}(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} \le A_0 \|\!|f\|\!|_{U^*,k,\alpha}$$

in the corresponding cases where A_0 is defined like before.

2.2. case: $(\infty\in\partial\Omega,\ -\infty\notin\overline{\Omega})$ or $(\infty\notin\overline{\Omega},\ -\infty\in\partial\Omega)$, i.e. (1A,4B) or (3A,4B). The observation of the previous case yields to the boundedness of $S(n)$ since

$$|\mathrm{Re}(z)|\leq\begin{cases}\max\left(|x_1-2|,n,\ln(2n)\right), & \infty\in\partial\Omega,\ -\infty\notin\overline{\Omega},\\ \max\left(|-n|,|-\ln(2n)|,|x_0+2|\right), & \infty\notin\overline{\Omega},\ -\infty\in\partial\Omega,\end{cases}$$

for every $z\in S(n)$.
3. case: $\pm\infty\in\partial\Omega$, i.e. (6A,4B) or $\varphi_0\equiv 1$.
In both cases the set $S(n)$ is bounded because for all $z\in S(n)$

$$|\mathrm{Re}(z)|\leq\max\left(|-n|,|-\ln(2n)|,\ln(2n),n\right)=\max\left(n,\ln(2n)\right).$$

So in all cases it follows either that $\overline{S(n)}\subset\left[\left(U\setminus\overline{\mathbb{R}}\right)\right]\cap\mathbb{R}^2$ is compact and

$$C(f)\leq\sup_{\substack{z\in\overline{S(n)},\\ \beta\in\mathbb{N}_0^2,\ |\beta|\leq m}}p_\alpha\left(f^{(|\beta|)}(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty$$

or that there are $\overline{M}_0\subset\left[\left(U\setminus\overline{\mathbb{R}}\right)\right]\cap\mathbb{R}^2$ compact and $k\in\mathbb{N}_{\geq 2}$ such that

$$C(f)\leq\sup_{\substack{z\in\overline{M}_0,\\ \beta\in\mathbb{N}_0^2,\ |\beta|\leq m}}p_\alpha\left(f^{(|\beta|)}(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}+A_0\,|||f|||_{U^*,k,\alpha}<\infty$$

by (6.18) and since $f\in\mathcal{O}^{exp}\left(U\setminus\overline{\mathbb{R}},E\right)$, in particular, that all (complex) derivatives are continuous on $\left(U\setminus\overline{\mathbb{R}}\right)\cap\mathbb{R}^2$. Due to (6.17) (and the two subsequent inequalities) this implies that $\left|\overline{\partial}(\varphi_1\varphi_0 f)\right|_{\partial\Omega,n,m,\alpha}<\infty$ for all $n\in\mathbb{N}_{\geq 2}$, $m\in\mathbb{N}_0$ and $\alpha\in A$ and thus $\overline{\partial}(\varphi_1\varphi_0 f)\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$. As E is admissible, there exists $g\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$ such that

$$\overline{\partial}g=\overline{\partial}(\varphi_1\varphi_0 f).\tag{6.20}$$

(iii) We set $F:=\varphi_1\varphi_0 f-g$. The next step is to show that $F\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)$, in particular, $F\in\mathcal{O}^{exp}\left(U_1\setminus\overline{\mathbb{R}},E\right)$ since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)\subset\mathcal{O}^{exp}\left(U_1\setminus\overline{\mathbb{R}},E\right)$, and that $f-F\in\mathcal{O}^{exp}(U,E)$. F is defined on $\mathbb{C}\setminus\overline{\Omega}$ (by setting $\varphi_1\varphi_0 f:=0$ on $\left[\left(U^C\cup\overline{\Omega}\right)\setminus\partial\Omega\right]\cap\mathbb{C}$) and can be regarded as an element of $\mathcal{O}\left(\mathbb{C}\setminus\overline{\Omega},E\right)$ due to (6.20).
Let $n\in\mathbb{N}_{\geq 2}$. We set $V:=V_0\cup W_0$, $S(n):=S_n\left(\overline{\Omega}\right)\setminus V$ and remark that $S_n\left(\overline{\Omega}\right)\subset S_n(\partial\Omega)$. For $\alpha\in A$ we have by the choice of φ_i, $i=1,2$,

$$\begin{aligned}|F|_{\overline{\Omega},n,\alpha}&=\sup_{z\in S_n\left(\overline{\Omega}\right)}p_\alpha\left(F(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\&\leq\underbrace{\sup_{z\in S_n(\partial\Omega)}p_\alpha\left(g(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\partial\Omega,n,0,\alpha}}+\sup_{z\in S_n\left(\overline{\Omega}\right)}p_\alpha\left(\varphi_1\varphi_0 f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\&=|g|_{\partial\Omega,n,0,\alpha}+\sup_{z\in S(n)}\underbrace{|(\varphi_1\varphi_0)(z)|}_{\leq 1}p_\alpha\left(f(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\end{aligned}$$

$$\le |g|_{\partial\Omega,n,0,\alpha} + \sup_{z\in S(n)} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}. \tag{6.21}$$

First we observe that

$$\big(U^C \cup \overline{\mathbb{R}}\big)\cap\mathbb{C} \subset \Big[V_0 \cup \bigcup_{x\in\overline{\Omega}\cap\mathbb{R}} D_{1/n}(x)\Big] =: W.$$

$W\subset\mathbb{C}$ is open and so we get by definition of the set $S(n)$

$$\overline{S(n)} \subset \overline{W^C} = W^C \subset \big(U\setminus\overline{\mathbb{R}}\big)\cap\mathbb{C}.$$

Again we claim that the set $S(n)$ is bounded or that there are $M_0\subset S(n)$ bounded and $k\in\mathbb{N}_{\ge 2}$ and $M_1\subset S_k(U)$ such that $S(n)=M_0\cup M_1$. For the boundedness we just have to prove that there is $C_1>0$ such that $|\mathrm{Re}(z)|\le C_1$ for every $z\in S(n)$ resp. $z\in M_0$. At first we consider the cases where $S(n)$ is bounded. This occurs if $\pm\infty\notin\overline{\Omega}$ or $\pm\infty\in\partial\Omega$ or $\infty\in\partial\Omega$, $-\infty\notin\overline{\Omega}$ or $-\infty\in\partial\Omega$, $\infty\notin\overline{\Omega}$. We get by definition of V_0 resp. V_1

$$\mathrm{Re}(z)\in\begin{cases}[x_1-2,x_0+2], & \pm\infty\notin\overline{\Omega},\\ [\min(-n,-\ln(2n)),\max(n,\ln(2n))], & \pm\infty\in\partial\Omega,\\ [x_1-2,\max(n,\ln(2n))], & \infty\in\partial\Omega,\ -\infty\notin\overline{\Omega},\\ [\min(-n,-\ln(2n)),x_0+2], & -\infty\in\partial\Omega,\ \infty\notin\overline{\Omega},\end{cases}$$

for all $z\in S(n)$ implying the boundedness. Therefore, $\overline{S(n)}\subset\big(U\setminus\overline{\mathbb{R}}\big)\cap\mathbb{C}$ is compact.
If $\infty\in\Omega$ or $-\infty\in\Omega$, we choose $k\in\mathbb{N}$ such that $k\ge n$ and

$$1/k < \varepsilon_0/2 < k,\ \text{if } \infty\in\Omega, \quad \text{resp.} \quad 1/k<\varepsilon_1/2<k,\ \text{if } -\infty\in\Omega,$$

plus, in addition, $-k<\tilde{x}_0+2$, if $\infty\in\Omega$, $-\infty\notin\Omega$, resp. $k>\tilde{x}_1-2$, if $-\infty\in\Omega$, $\infty\notin\Omega$. Then we decompose the set $S(n)$ as follows

$$S(n) = \underbrace{[S(n)\setminus S_k(U)]}_{=:M_0}\cup\underbrace{[S(n)\cap S_k(U)]}_{=:M_1}.$$

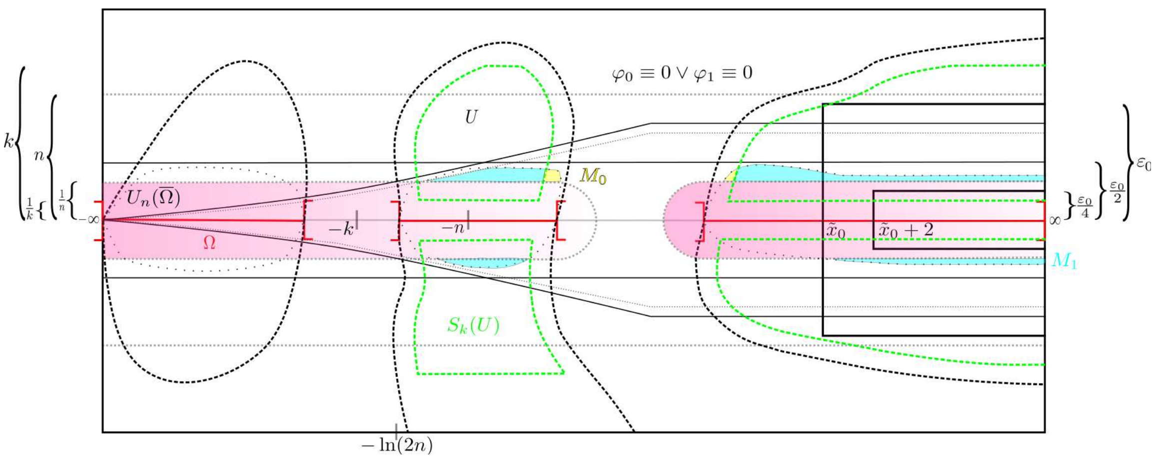

Figure 6.9: case (7A,2B): $\infty\in\Omega$, $-\infty\in\partial\Omega$

Obviously $M_1 \subset S_k(U)$ and $\overline{M}_0 \subset \overline{S(n)} \subset \left(U \setminus \mathbb{R}\right) \cap \mathbb{C}$. Consider the case $\infty \in \Omega$ and $-\infty \in \partial\Omega$. By the choice of V_0 we have

$$M_0 = [S(n) \setminus S_k(U)] \subset \left(S_n\left(\overline{\Omega}\right) \setminus V_0\right) \subset \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \frac{\varepsilon_0}{2}\right\} \tag{6.22}$$

and by the choice of W_0 and since $-\infty \in \partial\Omega$

$$M_0 \subset \left(S_n\left(\overline{\Omega}\right) \setminus W_0\right) \subset \{z \in \mathbb{C} \mid \mathrm{Re}(z) > \min(-n, -\ln(2n))\}. \tag{6.23}$$

Let $z \in S(n)$ with $|\mathrm{Im}(z)| < \frac{\varepsilon_0}{2}$ and $\mathrm{Re}(z) \geq \tilde{x}_0 + 2$. Then

$$z \in \left([\tilde{x}_0 + 2, \infty[\times \left[-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}\right]\right) \subset ([\tilde{x}_0, \infty[\times [-\varepsilon_0, \varepsilon_0]) \subset U$$

and therefore

$$\mathrm{d}(z, \partial U \cap \mathbb{C}) \geq \min\left(2, \frac{\varepsilon_0}{2}\right) > \frac{1}{k}$$

by the choice of k. Furthermore,

$$k > n > |\mathrm{Im}(z)| > \frac{1}{n} > \frac{1}{k}$$

as $[\tilde{x}_0, \infty] \subset \Omega$ and due to the choice of k. In addition, $\mathrm{Re}(z) \geq \tilde{x}_0 + 2 > -k$ and $z \in U$ by the choice of k and since $z \in S(n) \subset U$. Hence we obtain $z \in S_k(U)$. So it follows by (6.22)

$$M_0 = [S(n) \setminus S_k(U)] \subset \{z \in \mathbb{C} \mid \mathrm{Re}(z) < \tilde{x}_0 + 2\}$$

and due to (6.23) we gain the claim with $C_1 := \max(n, \ln(2n), |\tilde{x}_0 + 2|)$. For the still pending cases the proof can be done analogously and we only note the constant

$$C_1 := \begin{cases} \max(n, \ln(2n), |\tilde{x}_1 - 2|), & \infty \in \partial\Omega,\ -\infty \in \Omega, \\ \max(|x_1 - 2|, |\tilde{x}_0 + 2|), & \infty \in \Omega,\ -\infty \notin \overline{\Omega}, \\ \max(|\tilde{x}_1 - 2|, |x_0 + 2|), & \infty \notin \overline{\Omega},\ -\infty \in \Omega, \\ \max(|\tilde{x}_1 - 2|, |\tilde{x}_0 + 2|), & \pm\infty \in \Omega. \end{cases}$$

By the same arguments as in part (ii) we get $\sup_{z \in S(n)} p_\alpha\left(f(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$ and by (6.21) that $F \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\Omega}, E\right)$.

(iv) $f - F$ is defined on $U \cap \mathbb{C}$ (by the setting in the beginning of part (iii)) and can be regarded as an element of $\mathcal{O}(U \cap \mathbb{C}, E)$ due to (6.20). If $\pm\infty \notin \Omega$, then we already have $f - F \in \mathcal{O}^{exp}(U, E)$ just by definition. So let $\infty \in \Omega$ or $-\infty \in \Omega$. Let $n \in \mathbb{N}_{\geq 2}$. We set $V := V_1 \cap W_1$ and $T(n) := T_n(U) \setminus V$. With

$$R := \begin{cases} \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq n\}, & \infty \in \partial\Omega, \\ \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq -n\}, & -\infty \in \partial\Omega, \\ \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \geq n\}, & \pm\infty \in \partial\Omega, \\ \varnothing, & \pm\infty \notin \partial\Omega, \end{cases}$$

we have

$$\left[\overline{U_n(\partial\Omega)} \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\}\right] \subset \left[R \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\} \cup \bigcup_{x \in \partial U \cap \mathbb{C}} \overline{D_{\frac{1}{n}}(x)}\right] =: \tilde{R}$$

and thus

$$T_n(U) \subset \left[(U \cap \mathbb{C}) \setminus \tilde{R}\right] \subset \left(\mathbb{C} \setminus \left[\overline{U_n(\partial\Omega)} \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq n\}\right]\right) = S_n(\partial\Omega). \tag{6.24}$$

For $\alpha \in A$ we have by the choice of φ_i, $i = 1,2$,

$$\begin{aligned}
|||f - F|||_{U,n,\alpha} &= \sup_{z \in T_n(U)} p_\alpha\big([(1-\varphi_1\varphi_0)f + g](z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq \underbrace{\sup_{z \in S_n(\partial\Omega)} p_\alpha\big(g(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\partial\Omega,n,0,\alpha}} + \sup_{z \in T_n(U)} p_\alpha\big((1-\varphi_1\varphi_0)f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&= |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in T(n)} \underbrace{|1-(\varphi_1\varphi_0)(z)|}_{\leq 1} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq |g|_{\partial\Omega,n,0,\alpha} + \sup_{z \in T(n)} p_\alpha\big(f(z)\big)e^{-\frac{1}{n}|\mathrm{Re}(z)|}.
\end{aligned} \tag{6.25}$$

Let $\varepsilon_2 := \min(\varepsilon_0, \varepsilon_1)$. We choose $k \in \mathbb{N}$ such that

$$\frac{1}{k} < \begin{cases} \min\left(\frac{1}{n}, \frac{\varepsilon_2}{4}\right), & \pm\infty \in \Omega, \\ \min\left(\frac{1}{n}, \frac{\varepsilon_0}{4}\right), & \infty \in \Omega,\ -\infty \notin \Omega, \\ \min\left(\frac{1}{n}, \frac{\varepsilon_1}{4}\right), & \infty \notin \Omega,\ -\infty \in \Omega. \end{cases}$$

First we observe that

$$\left(U^C \cup \overline{\mathbb{R}}\right) \cap \mathbb{C} \subset \left[V \cup \bigcup_{x \in U^C \cap \mathbb{C}} D_{1/n}(x)\right] =: W.$$

The set $W \subset \mathbb{C}$ is open and thus we get by definition of the set $T(n)$

$$T(n) = T_n(U) \setminus V = \underbrace{\left[T_n(U) \setminus \left(\bigcup_{x \in U^C \cap \mathbb{C}} D_{1/n}(x)\right)\right]}_{=T_n(U)} \setminus V \subset W^C$$

and so

$$\overline{T(n) \setminus S_k(U)} \subset \overline{T(n)} \subset \overline{W^C} = W^C \subset \left(U \setminus \overline{\mathbb{R}}\right) \cap \mathbb{C}. \tag{6.26}$$

Then we can decompose the set $T(n)$ in the following manner

$$T(n) = \underbrace{[T(n) \setminus S_k(U)]}_{=:M_0} \cup \underbrace{[T(n) \cap S_k(U)]}_{=:M_1}.$$

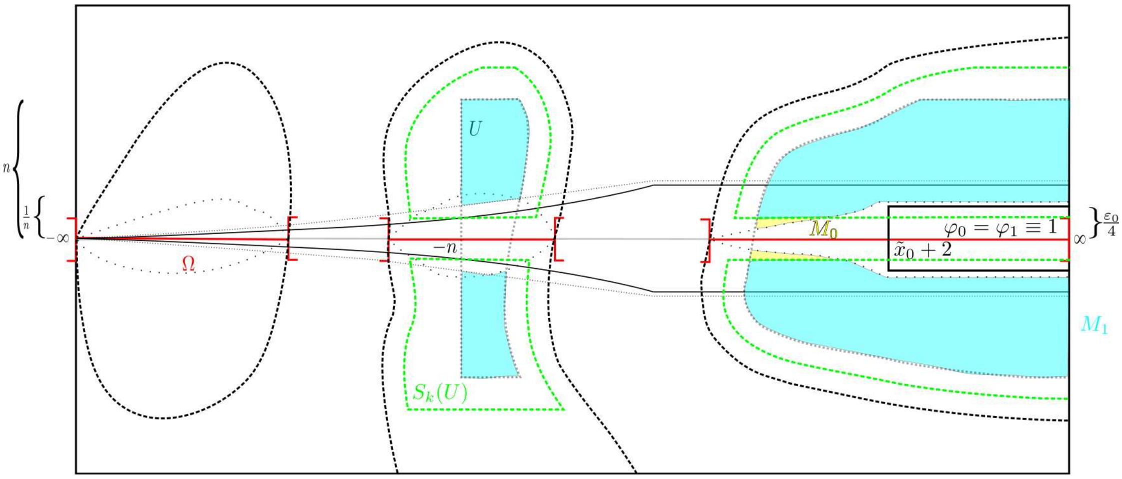

Figure 6.10: case (7A,2B): $\infty \in \Omega$, $-\infty \in \partial\Omega$

We claim that the set M_0 is bounded. Again we just have to prove that there is $C_1 > 0$ such that $|\mathrm{Re}(z)| \leq C_1$ for every $z \in M_0$. By the choice of k and the definition of V_1 and W_1 (for the cases that $-\infty \in \partial\Omega$ or $\infty \in \partial\Omega$ keep in mind that $1/k < 1$) we have

$$\mathrm{Re}(z) \in \begin{cases} [\tilde{x}_1 - 2, \tilde{x}_0 + 2], & \pm\infty \in \Omega, \\ [-n, \max(0, \tilde{x}_0 + 2)], & \infty \in \Omega,\ -\infty \in \partial\Omega, \\ [-n, \tilde{x}_0 + 2], & \infty \in \Omega,\ -\infty \notin \overline{\Omega}, \\ [\min(0, \tilde{x}_1 - 2), n], & -\infty \in \Omega,\ \infty \in \partial\Omega, \\ [\tilde{x}_1 - 2, n], & -\infty \in \Omega,\ -\infty \notin \overline{\Omega}, \end{cases}$$

for every $z \in M_0$ proving the claim. Therefore, $\overline{M}_0$ is compact and by (6.26) we get $\overline{M}_0 \subset (U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}$. Then

$$\begin{aligned} \sup_{z \in T(n)} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} &\leq \sup_{z \in \overline{M}_0} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + \sup_{z \in M_1} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\ &\leq \sup_{z \in \overline{M}_0} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + |f|_{U^*,k,\alpha} < \infty \end{aligned}$$

for all $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ since $f \in \mathcal{O}^{exp}\big(U \setminus \overline{\mathbb{R}}, E\big)$. Hence we obtain by (6.25) that $f - F \in \mathcal{O}^{exp}(U, E)$.

So we have found $F \in \mathcal{O}^{exp}\big(\overline{\mathbb{C}} \setminus \overline{\Omega}, E\big) \subset \mathcal{O}^{exp}\big(U_1 \setminus \overline{\mathbb{R}}, E\big)$ such that $\Big[F|_{(U \setminus \overline{\mathbb{R}}) \cap \mathbb{C}}\Big] = [f]$ proving the surjectivity of J. For arbitrary U, $U_0 \in \mathcal{U}(\Omega)$ we have, with U_1 from the proof,

$$\begin{aligned} \mathcal{O}^{exp}\big(U \setminus \overline{\mathbb{R}}, E\big) / \mathcal{O}^{exp}(U, E) &\cong \mathcal{O}^{exp}\big(U_1 \setminus \overline{\mathbb{R}}, E\big) / \mathcal{O}^{exp}(U_1, E) \\ &\cong \mathcal{O}^{exp}\big(U_0 \setminus \overline{\mathbb{R}}, E\big) / \mathcal{O}^{exp}(U_0, E) \end{aligned}$$

algebraically, connoting the general statement.

□

6.9 Remark. The same result is still valid if we modify the spaces involved in the following way: Let $U \subset \overline{\mathbb{C}}$ be open and $-\infty \in U$ or $\infty \in U$. We say that U is even-tempered if it satisfies one of the following conditions:

(a)
$$\forall\, n \in \mathbb{N}\; \exists\, N \in \mathbb{N}\; \forall\, z \in \left(U^C \cap \mathbb{C}\right) \setminus \bigcup_{x \in (\partial U \cap \mathbb{R})} D_{\frac{1}{n}}(x) : \; |\mathrm{Im}(z)| > \frac{1}{N} \tag{6.27}$$

(b)
$$\forall\, n \in \mathbb{N} : \; \left(U^C \cap \mathbb{C}\right) \setminus \bigcup_{x \in (\partial U \cap \mathbb{R})} D_{\frac{1}{n}}(x) = \varnothing \tag{6.28}$$

We replace in the definition of the spaces the set $\mathcal{U}(\Omega)$ by

$$\mathrm{U}(\Omega) := \begin{cases} \left\{U \mid U \subset \overline{\mathbb{C}} \text{ open}, \; U \cap \overline{\mathbb{R}} = \Omega\right\}, & \pm\infty \notin \Omega, \\ \left\{U \mid U \subset \overline{\mathbb{C}} \text{ open and even-tempered}, \; U \cap \overline{\mathbb{R}} = \Omega\right\}, & -\infty \in \Omega \,\vee\, \infty \in \Omega, \end{cases}$$

and in the definition of $|||f|||_{U^*,n,\alpha}$ the set $S_n(U)$ by

$$s_n(U) := \begin{cases} U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n, \; \mathrm{Re}(z) > -n\right\}, & -\infty \notin \Omega, \infty \in \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n, \; \mathrm{Re}(z) < n\right\}, & -\infty \in \Omega, \infty \notin \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid \frac{1}{n} < |\mathrm{Im}(z)| < n\right\}, & \pm\infty \in \Omega, \end{cases}$$

plus in the definition of $|||f|||_{U,n,\alpha}$ the set $T_n(U)$ by

$$t_n(U) := \begin{cases} U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n, \; \mathrm{Re}(z) > -n, \; \mathrm{d}(z, \partial U \cap \mathbb{R}) > \frac{1}{n}\right\}, & -\infty \notin \Omega, \infty \in \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n, \; \mathrm{Re}(z) < n, \; \mathrm{d}(z, \partial U \cap \mathbb{R}) > \frac{1}{n}\right\}, & -\infty \in \Omega, \infty \notin \Omega, \\ U \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n, \; \mathrm{d}(z, \partial U \cap \mathbb{R}) > \frac{1}{n}\right\}, & \pm\infty \in \Omega. \end{cases}$$

Now we take a look at the proof and the positions which are in need of a modification.

- *part (i):* If $-\infty \in \Omega$ or $\infty \in \Omega$, then the set U_1 from the beginning of the proof is even-tempered since it fulfills (6.28).
- *part (ii):* In (6.19) we have $\overline{D_r(z)} \subset s_k(U) \subset \left(U \setminus \overline{\mathbb{R}}\right) \cap \mathbb{R}^2$ as well.
- *part (iii):* We claim that the set $S(n)$ is bounded or there are $M_0 \subset S(n)$ bounded and $M_1 \subset s_n(U)$ such that $S(n) = M_0 \cup M_1$. We define the set

$$M := \begin{cases} \left\{z \in \mathbb{C} \mid \mathrm{Re}(z) > -n, \; |\mathrm{Im}(z)| > \frac{1}{n}\right\}, & (\infty \in \Omega \wedge -\infty \in \partial\Omega) \vee \left(\infty \in \Omega \wedge -\infty \notin \overline{\Omega}\right), \\ \left\{z \in \mathbb{C} \mid \mathrm{Re}(z) < n, \; |\mathrm{Im}(z)| > \frac{1}{n}\right\}, & (\infty \in \partial\Omega \wedge -\infty \in \Omega) \vee \left(\infty \in \Omega \wedge \infty \notin \overline{\Omega}\right), \\ \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{n}\right\}, & \pm\infty \in \Omega, \end{cases}$$

and decompose the set $S(n)$ in these cases as follows

$$S(n) = \underbrace{[S(n) \setminus M]}_{=:M_0} \cup \underbrace{[S(n) \cap M]}_{=:M_1}.$$

Then

$$M_1 \subset [U \cap (M \cup \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < n\})] = s_n(U)$$

and $\overline{M}_0 \subset \overline{S(n)} \subset \left[\left(U \setminus \mathbb{R}\right) \cap \mathbb{C}\right]$. So we just have to prove that there is $C_1 > 0$ such that $|\mathrm{Re}(z)| \leq C_1$ for every $z \in S(n)$ resp. $z \in M_0$. This constant is noted in the following chart:

Table 6.6: Bounds for the real part of $S(n)$ resp. M_0

base case	subcase	$C_1 = \max(\cdot)$
$\pm\infty \in \overline{\Omega}$	$\infty \in \Omega,\ -\infty \in \partial\Omega$	$\lvert -n \rvert,\ \lvert -\ln(2n) \rvert,\ n$
	$\infty \in \partial\Omega\ -\infty \in \Omega$	$\lvert -n \rvert,\ \ln(2n),\ n$
	$\pm\infty \in \Omega$	$\lvert -n \rvert,\ \lvert -\ln(2n) \rvert,\ \ln(2n),\ n$
	$\pm\infty \in \Omega$	$\lvert -n \rvert, n$
$\infty \in \overline{\Omega},\ -\infty \notin \overline{\Omega}$	$\infty \in \Omega$	$\lvert x_1 - 2 \rvert,\ n$
	$\infty \in \partial\Omega$	$\lvert x_1 - 2 \rvert,\ \ln(2n),\ n$
$\infty \notin \overline{\Omega},\ -\infty \in \overline{\Omega}$	$-\infty \in \Omega$	$\lvert -n \rvert,\ \lvert x_0 + 2 \rvert$
	$-\infty \in \partial\Omega$	$\lvert -n \rvert,\ \lvert -\ln(2n) \rvert,\ \lvert x_0 + 2 \rvert$
$\pm\infty \notin \overline{\Omega}$		$\lvert x_1 - 2 \rvert,\ \lvert x_0 + 2 \rvert$

- *part (iv):* Define $t(n) := t_n(U) \setminus V$ and replace in the definition of $\tilde{R}$ the set $\bigcup_{x \in \partial U \cap \mathbb{C}} \overline{D_{\frac{1}{n}}(x)}$ by $\bigcup_{x \in \partial U \cap \mathbb{R}} \overline{D_{\frac{1}{n}}(x)}$. Then we have in (6.24)

$$t_n(U) = \left[(U \cap \mathbb{C}) \setminus \tilde{R}\right].$$

Furthermore, we choose $k \in \mathbb{N}$ such that

$$\frac{1}{k} \leq \begin{cases} \min\left(\frac{1}{n}, \frac{\varepsilon_2}{4}\right), & \pm\infty \in \Omega, \\ \min\left(\frac{1}{n}, \frac{\varepsilon_0}{4}\right), & \infty \in \Omega,\ -\infty \notin \Omega, \\ \min\left(\frac{1}{n}, \frac{\varepsilon_1}{4}\right), & \infty \notin \Omega,\ -\infty \in \Omega, \end{cases}$$

and, in addition,

$$-k < \tilde{x}_0 + 2, \text{ if } \infty \in \Omega,\ -\infty \notin \Omega, \quad \text{resp.} \quad k > \tilde{x}_1 - 2, \text{ if } \infty \notin \Omega,\ -\infty \in \Omega,$$

plus $k > N$, if U satifies condition (6.27). Then we remark that

$$\left(U^C \cup \mathbb{R}\right) \cap \mathbb{C} \subset \left[V \cup \left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > \frac{1}{k}\right\} \cup \bigcup_{x \in \partial U \cap \mathbb{R}} D_{1/n}(x)\right] =: W$$

by the choice of k and since U is even-tempered. The set $W \subset \mathbb{C}$ is open and thus we get by the definition of the set $t(n)$

$$\overline{t(n) \cap \left\{z \in \mathbb{C} \mid |\mathrm{Im}| \leq \frac{1}{k}\right\}} \subset \overline{W^C} = W^C \subset \left(U \setminus \mathbb{R}\right) \cap \mathbb{C}. \tag{6.29}$$

We define the sets

$$N_0 := \begin{cases} \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \tilde{x}_0 + 2,\ \operatorname{Re}(z) < \tilde{x}_1 - 2\}, & \pm\infty \in \Omega, \\ \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \max(0, \tilde{x}_0 + 2)\}, & \infty \in \Omega,\ -\infty \in \partial\Omega, \\ \{z \in \mathbb{C} \mid \operatorname{Re}(z) > \tilde{x}_0 + 2\}, & \infty \in \Omega,\ -\infty \notin \overline{\Omega}, \\ \{z \in \mathbb{C} \mid \operatorname{Re}(z) < \min(0, \tilde{x}_1 - 2)\}, & \infty \in \partial\Omega,\ -\infty \in \Omega, \\ \{z \in \mathbb{C} \mid \operatorname{Re}(z) < \tilde{x}_1 - 2\}, & \infty \notin \overline{\Omega},\ -\infty \in \overline{\Omega}, \end{cases}$$

and

$$N_1 := \begin{cases} \left\{z \in \mathbb{C} \mid \tilde{x}_1 - 2 \leq \operatorname{Re}(z) \leq \tilde{x}_0 + 2,\ |\operatorname{Im}(z)| > \frac{1}{k}\right\}, & \pm\infty \in \Omega, \\ \left\{z \in \mathbb{C} \mid -k < \operatorname{Re}(z) \leq \max(0, \tilde{x}_0 + 2),\ |\operatorname{Im}(z)| > \frac{1}{k}\right\}, & \infty \in \Omega,\ -\infty \in \partial\Omega, \\ \left\{z \in \mathbb{C} \mid -k < \operatorname{Re}(z) \leq \tilde{x}_0 + 2,\ |\operatorname{Im}(z)| > \frac{1}{k}\right\}, & \infty \in \Omega,\ -\infty \notin \overline{\Omega}, \\ \left\{z \in \mathbb{C} \mid \min(0, \tilde{x}_1 - 2) \leq \operatorname{Re}(z) < k,\ |\operatorname{Im}(z)| > \frac{1}{k}\right\}, & \infty \in \partial\Omega,\ -\infty \in \Omega, \\ \left\{z \in \mathbb{C} \mid \tilde{x}_1 - 2 \leq \operatorname{Re}(z) < k,\ |\operatorname{Im}(z)| > \frac{1}{k}\right\}, & \infty \notin \overline{\Omega},\ -\infty \in \overline{\Omega}, \end{cases}$$

and

$$N_2 := \begin{cases} \left\{z \in \mathbb{C} \mid \tilde{x}_1 - 2 \leq \operatorname{Re}(z) \leq \tilde{x}_0 + 2,\ |\operatorname{Im}(z)| \leq \frac{1}{k}\right\}, & \pm\infty \in \Omega, \\ \left\{z \in \mathbb{C} \mid -k < \operatorname{Re}(z) \leq \max(0, \tilde{x}_0 + 2),\ |\operatorname{Im}(z)| \leq \frac{1}{k}\right\}, & \infty \in \Omega,\ -\infty \in \partial\Omega, \\ \left\{z \in \mathbb{C} \mid -k < \operatorname{Re}(z) \leq \tilde{x}_0 + 2,\ |\operatorname{Im}(z)| \leq \frac{1}{k}\right\}, & \infty \in \Omega,\ -\infty \notin \overline{\Omega}, \\ \left\{z \in \mathbb{C} \mid \min(0, \tilde{x}_1 - 2) \leq \operatorname{Re}(z) < k,\ |\operatorname{Im}(z)| \leq \frac{1}{k}\right\}, & \infty \in \partial\Omega,\ -\infty \in \Omega, \\ \left\{z \in \mathbb{C} \mid \tilde{x}_1 - 2 \leq \operatorname{Re}(z) < k,\ |\operatorname{Im}(z)| \leq \frac{1}{k}\right\}, & \infty \notin \overline{\Omega},\ -\infty \in \overline{\Omega}. \end{cases}$$

Then we can decompose the set $t(n)$ in the following manner

$$t(n) = \bigcup_{i=0}^{2} \underbrace{(t(n) \cap N_i)}_{=:M_i}.$$

By the choice of k we have

$$M_i \subset \left\{z \in U \cap \mathbb{C} \mid \frac{1}{k} < |\operatorname{Im}(z)| < k\right\} \subset s_k(U)$$

for $i = 0, 1$. The set M_2 is obviously bounded in $\mathbb{C}$, therefore, $\overline{M}_2$ compact, and by (6.29) we get $\overline{M}_2 \subset \big(U \setminus \mathbb{R}\big) \cap \mathbb{C}$. Then

$$\begin{aligned} &\sup_{z \in t(n)} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \\ \leq &\sup_{z \in M_0 \cup M_1} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\operatorname{Re}(z)|} + \sup_{z \in \overline{M}_2} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\operatorname{Re}(z)|} \\ \leq &|f|_{U^*,k,\alpha} + \sup_{z \in \overline{M}_2} p_\alpha\big(f(z)\big) e^{-\frac{1}{n}|\operatorname{Re}(z)|} < \infty \end{aligned}$$

for all $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$.

By virtue of Lemma 6.8 we may define restrictions in $bv(\Omega,E)$ in the following manner:

6.10 Definition. Let E be admissible and Ω, $\Omega_1\subset\overline{\mathbb{R}}$, $\Omega_1\subset\Omega$, be open. For $\Omega_1\neq\varnothing$ let $[f]\in bv(\Omega,E)=\mathcal{O}^{exp}\left(U\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U,E)$ where $U\in\mathcal{U}(\Omega)$. Setting $U_1:=U\cap(\Omega_1\times\mathbb{R})$, we may define the restriction map by

$$R_{\Omega,\Omega_1}([f]):=[f]\big|_{\Omega_1}:=\left[f\big|_{(U_1\setminus\overline{\mathbb{R}})\cap\mathbb{C}}\right]\in\mathcal{O}^{exp}\left(U_1\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U_1,E)=bv(\Omega_1,E).$$

In addition, we define for an open set $\Omega\subset\overline{\mathbb{R}}$

$$R_{\Omega,\varnothing}\colon bv(\Omega,E)\to bv(\varnothing,E),\ R_{\Omega,\varnothing}([f]):=[f]\big|_{\varnothing}:=0.$$

We denote the space $\left\{bv(\Omega,E)\mid\Omega\subset\overline{\mathbb{R}}\text{ open}\right\}$ by $bv(E)$.

6.11 Theorem. [5] *Let E be strictly admissible.*

a) $bv(E)$, equipped with the restrictions from Definition 6.10, is a sheaf on $\overline{\mathbb{R}}$.

b) $bv(E)$ is flabby.

c) $bv(E)$ is isomorphic to $\mathcal{R}(E)$; in particular, $\mathcal{R}(E)$ is a sheaf.

Proof. a) (i) For $\Omega\subset\overline{\mathbb{R}}$ open the mapping $R_{\Omega,\Omega}$ can be regarded as $\mathrm{id}_{bv(\Omega,E)}$ by Lemma 6.8. Let $\Omega_3\subset\Omega_2\subset\Omega_1\subset\overline{\mathbb{R}}$ be open. We have to prove that $R_{\Omega_2,\Omega_3}\circ R_{\Omega_1,\Omega_2}=R_{\Omega_1,\Omega_3}$ holds. This is obviously true if one of the sets is empty, so let them all be non-empty. Let $[f]\in bv(\Omega_1,E)=\mathcal{O}^{exp}\left(U_1\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U_1,E)$ where $U_1\in\mathcal{U}(\Omega_1)$. With $U_2:=U_1\cap(\Omega_2\times\mathbb{R})$ and

$$U_3:=U_2\cap(\Omega_3\times\mathbb{R})=[U_1\cap(\Omega_2\times\mathbb{R})]\cap(\Omega_3\times\mathbb{R})\underset{\Omega_3\subset\Omega_2}{=}U_1\cap(\Omega_3\times\mathbb{R})\tag{6.30}$$

we get

$$R_{\Omega_2,\Omega_3}\circ R_{\Omega_1,\Omega_2}([f])=R_{\Omega_2,\Omega_3}\left(\left[f\big|_{(U_2\setminus\overline{\mathbb{R}}\cap\mathbb{C})}\right]\right)\underset{U_3\subset U_2}{=}\left[f\big|_{(U_3\setminus\overline{\mathbb{R}})\cap\mathbb{C}}\right]\underset{(6.30)}{=}R_{\Omega_1,\Omega_3}([f]).$$

(ii) $(S1)$: Let $\left\{\Omega_j\subset\overline{\mathbb{R}}\mid j\in J\right\}$ be a family of open sets and $\Omega:=\bigcup_{j\in J}\Omega_j$.
Let $[f]\in bv(\Omega,E)=\mathcal{O}^{exp}\left(U\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U,E)$, where $U\in\mathcal{U}(\Omega)$, such that $R_{\Omega,\Omega_j}([f])=0$ for all $j\in J$. The assumption $R_{\Omega,\Omega_j}([f])=0$ is equivalent to $f\in\mathcal{O}^{exp}\left(U_j,E\right)$ for every $j\in J$ where $U_j:=U\cap\left(\Omega_j\times\mathbb{R}\right)$. Thus we obtain

$$f\in\mathcal{O}^{exp}\left(\left[U\setminus\overline{\mathbb{R}}\right]\cup\bigcup_{j\in J}\Omega_j,E\right)=\mathcal{O}^{exp}\left(\left[U\setminus\overline{\mathbb{R}}\right]\cup\Omega,E\right)\underset{U\in\mathcal{U}(\Omega)}{=}\mathcal{O}^{exp}(U,E)$$

and hence $[f]=0$.
(iii) $(S2)$: Let $\left(\Omega_j\right)_{j\in J}$ and Ω be like in part (ii).
Let $\left[f_j\right]\in bv\left(\Omega_j,E\right)=\mathcal{O}^{exp}\left(U_j\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}\left(U_j,E\right)$, where $U_j\in\mathcal{U}\left(\Omega_j\right)$, such that

[5] counterpart: [13, Theorem 6.9, p. 1125]

$[f_j]|_{\Omega_j\cap\Omega_k} = [f_k]|_{\Omega_j\cap\Omega_k}$. Hence we have

$$g_{jk} := f_j|_{[(U_j\cap U_k)\smallsetminus\overline{\mathbb{R}}]\cap\mathbb{C}} - f_k|_{[(U_j\cap U_k)\smallsetminus\overline{\mathbb{R}}]\cap\mathbb{C}} \in \mathcal{O}^{exp}\left(U_j\cap U_k, E\right)$$

plus $g_{jk} = -g_{kj}$ as well as $g_{jk}+g_{kl}+g_{lj} = 0$ on $U_j\cap U_k\cap U_l$ by easy calculation.
If $\pm\infty\notin\Omega$ and thus $\pm\infty\notin\Omega_j$, then exactly like in [20, Theorem 1.4.5, p. 13], where one uses that E is *strictly* admissible instead of [20, Theorem 1.4.4, p. 12], there are $g_j\in\mathcal{O}\left(U_j\cap\mathbb{C}, E\right)$ such that $g_{jk} = g_k - g_j$ on $U_j\cap U_k\cap\mathbb{C}$ (here the adjunct *strictly* is needed). The setting $F_j := f_j + g_j$ defines a function $F\in\mathcal{O}\left(\left(U\smallsetminus\overline{\mathbb{R}}\right)\cap\mathbb{C}, E\right) = \mathcal{O}^{exp}\left(U\smallsetminus\overline{\mathbb{R}}, E\right)$ since

$$F_j - F_k = f_j + g_j - f_k - g_k = \underbrace{f_j - f_k}_{=g_{jk}} + \underbrace{g_j - g_k}_{=-g_{jk}} = 0$$

on $U_j\cap U_k\cap\mathbb{C}$ such that

$$[F]|_{\Omega_j} = \left[F|_{(U_j\smallsetminus\overline{\mathbb{R}})\cap\mathbb{C}}\right] = \left[f_j|_{(U_j\smallsetminus\overline{\mathbb{R}})\cap\mathbb{C}}\right] + \left[g_j|_{(U_j\smallsetminus\overline{\mathbb{R}})\cap\mathbb{C}}\right] = [f_j] \text{ for any } j\in J.$$

Now let $-\infty\in\Omega$ or $\infty\in\Omega$, i.e. there exists $j\in J$ such that $-\infty\in\Omega_j$ or $\infty\in\Omega_j$. We only consider the case that there are $j_0,\ j_1\in J$ such that $-\infty\in\Omega_{j_0}$ and $\infty\in\Omega_{j_1}$. For the other two cases the proof is analogous. Then there are $x_0,\ x_1\in\mathbb{R}$ and $\varepsilon_0,\ \varepsilon_1 > 0$ such that $[-\infty, x_0]\times[-\varepsilon_0,\varepsilon_0]\subset U_{j_0}$ and $[x_1,\infty]\times[-\varepsilon_1,\varepsilon_1]\subset U_{j_1}$. Now let $x := \max(|x_0|,|x_1|)$ and $\varepsilon := \min(\varepsilon_0,\varepsilon_1)$. We define the sets

$$G_0 := \left(]-\infty,-x-1[\times]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[\right)^C, \quad H_0 :=]-\infty,-x-2]\times\left[-\frac{\varepsilon}{4},\frac{\varepsilon}{4}\right]$$

as well as

$$G_1 := \left(]x+1,\infty[\times]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[\right)^C, \quad H_1 := [x+2,\infty[\times\left[-\frac{\varepsilon}{4},\frac{\varepsilon}{4}\right].$$

By the proof of [18, Theorem 1.4.1, p. 25] there are $\varphi_i\in\mathrm{C}^\infty(\mathbb{R}^2)$, $i = 0,1$, such that $0\leq\varphi_i\leq 1$ and $\varphi_i = 0$ near G_i plus $\varphi_i = 1$ near H_i as well as $\left|\partial^\beta\varphi_i\right|\leq C_{i,\beta}\tilde{\varepsilon}^{-|\beta|}$ for all $\beta\in\mathbb{N}_0^2$ where $\tilde{\varepsilon} := \frac{1}{4}\min\left(\frac{\varepsilon}{4},1\right)$ and $C_{i,\beta} > 0$.

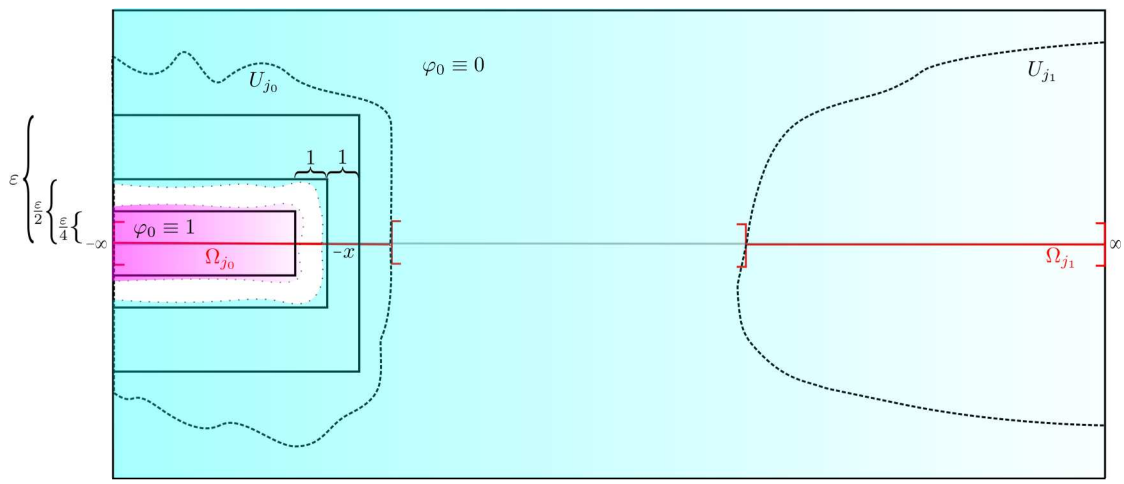

Figure 6.11: case: $-\infty\in\Omega_{j_0}$, $\infty\in\Omega_{j_1}$

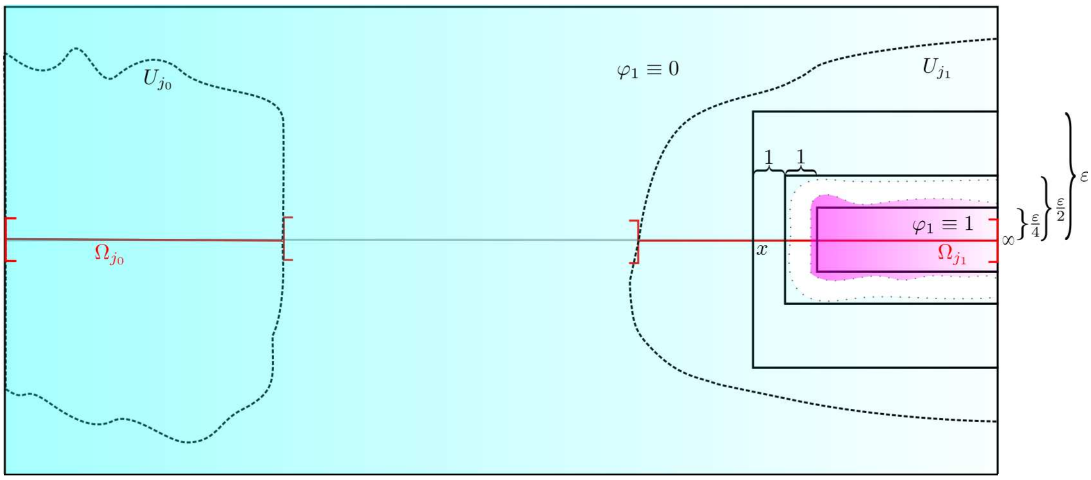

Figure 6.12: case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

Due to the first case there is $F \in \mathcal{O}\left(\left(U \setminus \overline{\mathbb{R}}\right) \cap \mathbb{C}, E\right)$ such that $[F]|_{\Omega_j \cap \mathbb{R}} = [f_j]|_{\Omega_j \cap \mathbb{R}}$ for every $j \in J$. By the proof of Lemma 6.8 there exists $\tilde{F} \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\Omega}, E\right)$ with $F - \tilde{F} \in \mathcal{O}(U \cap \mathbb{C}, E)$. Thus we obtain

$$f_j - \tilde{F} = \underbrace{(f_j - F)}_{\in \mathcal{O}(U_j \cap \mathbb{C}, E)} + \underbrace{(F - \tilde{F})}_{\in \mathcal{O}(U \cap \mathbb{C}, E)} \in \mathcal{O}(U_j \cap \mathbb{C}, E) \tag{6.31}$$

for all $j \in J$. So by the choice of φ_i we can regard $\overline{\partial}\left(\varphi_0\left(f_{j_0} - \tilde{F}\right) + \varphi_1\left(f_{j_1} - \tilde{F}\right)\right)$ as an element of $C^\infty(\mathbb{R}^2, E)$ (set $\varphi_i\left(f_{j_i} - \tilde{F}\right) := 0$ outside U_{j_i}). Let $n \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}_0$ and $\alpha \in A$. Then we obtain by applying the Leibniz rule and the choice of φ_i like in (6.4) resp. (6.17)

$$\begin{aligned}
&\left|\overline{\partial}\left(\varphi_0\left(f_{j_0} - \tilde{F}\right) + \varphi_1\left(f_{j_1} - \tilde{F}\right)\right)\right|_{\varnothing, n, m, \alpha} \\
&= \sup_{\substack{z \in S_n(\varnothing), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(\partial^\beta \overline{\partial}\left(\varphi_0\left(f_{j_0} - \tilde{F}\right) + \varphi_1\left(f_{j_1} - \tilde{F}\right)\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq (m!)^2 \sum_{i=0}^{1} \sup_{\substack{z \in S_n(\varnothing) \setminus (G_i \cup H_i), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} \sum_{\gamma \leq \beta} \left|\partial^{\beta - \gamma}\left(\overline{\partial}\varphi_i\right)(z)\right| p_\alpha\left(\partial^\gamma\left(f_{j_i} - \tilde{F}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq (m!)^2 \sum_{i=0}^{1} \sum_{|\gamma| \leq m+2} \underbrace{\sup_{z \in S_n(\varnothing) \setminus (G_i \cup H_i)} |\partial^\gamma \varphi_i(z)|}_{\leq C_{i,\gamma} \tilde{\varepsilon}^{-|\gamma|}} \underbrace{\sup_{\substack{z \in S_n(\varnothing) \setminus (G_i \cup H_i), \\ \beta \in \mathbb{N}_0^2, |\beta| \leq m}} p_\alpha\left(\left(f_{j_i} - \tilde{F}\right)^{(|\beta|)}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=: C_i^*} \\
&\leq (m!)^2 \left(C_0^* + C_1^*\right) \sum_{|\gamma| \leq m+2} \left(C_{0,\gamma} + C_{1,\gamma}\right) \tilde{\varepsilon}^{-|\gamma|}.
\end{aligned} \tag{6.32}$$

Now we have to take a closer look at C_i^*. By the choice of the sets G_i and H_i

$$\begin{aligned}
&S_n(\varnothing) \setminus (G_0 \cup H_0) \\
&\subset \underbrace{\left\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \leq \varepsilon/2,\ -x-2 \leq \mathrm{Re}(z) < -x-1\right\}}_{=: N_0} \cup \underbrace{\left\{z \in \mathbb{C} \mid \varepsilon/4 < |\mathrm{Im}(z)| < \varepsilon/2,\ \mathrm{Re}(z) < -x-2\right\}}_{=: M_0}
\end{aligned}$$

and

$$\begin{aligned}&S_n(\varnothing)\setminus(G_1\cup H_1)\\ \subset\,&\underbrace{\{z\in\mathbb{C}\mid |\mathrm{Im}(z)|\leq \varepsilon/2,\ x+1<\mathrm{Re}(z)\leq x+2\}}_{=:N_1}\cup\underbrace{\{z\in\mathbb{C}\mid \varepsilon/4<|\mathrm{Im}(z)|<\varepsilon/2,\ \mathrm{Re}(z)>x+2\}}_{=:M_1}\end{aligned}$$

is valid.

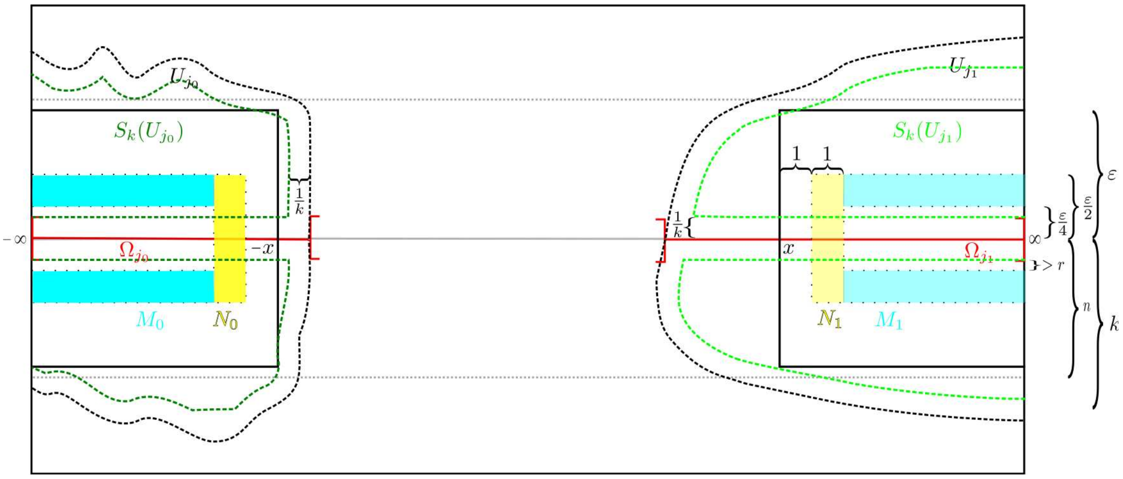

Figure 6.13: case: $-\infty\in\Omega_{j_0}$, $\infty\in\Omega_{j_1}$

The sets N_i are clearly bounded and $\overline{N_0}\subset U_{j_0}$ as well as $\overline{N_1}\subset U_{j_1}$. This implies

$$\sup_{\substack{z\in N_i,\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\left(\left(f_{j_i}-\tilde{F}\right)^{(|\beta|)}(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty,\ i=0,1, \tag{6.33}$$

by (6.31). If we set

$$r:=\frac{1}{2}\min\left(2,\frac{\varepsilon}{2},\frac{\varepsilon}{4}\right)=\min\left(1,\frac{\varepsilon}{8}\right)$$

and choose $k\in\mathbb{N}$ with $k>\max(n,\varepsilon)$ and $\frac{1}{k}<\frac{\varepsilon}{8}$ plus $-k<x$, if $\infty\notin\Omega_{j_0}$ resp. $-\infty\notin\Omega_{j_1}$, then

$$\overline{D_r(z)}\subset S_k\left(U_{j_i}\right)\subset S_k\left(\overline{\Omega}\right),\ i=0,1,$$

holds for all $z\in M_i$ like in (6.19). Due to the Cauchy inequality we get like in Theorem 3.6(4) for $i=0,1$

$$\sup_{\substack{z\in M_i,\\ \beta\in\mathbb{N}_0^2,\,|\beta|\leq m}} p_\alpha\left(\left(f_{j_i}-\tilde{F}\right)^{(|\beta|)}(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\leq A_0\left(|||f_{j_i}|||_{U_{j_i}^*,k,\alpha}+\left|\tilde{F}\right|_{\overline{\Omega},k,\alpha}\right)<\infty \tag{6.34}$$

where $A_0:=e^{\frac{r}{n}}\sup_{\beta\in\mathbb{N}_0^2,\,|\beta|\leq m}\frac{|\beta|!}{r^{|\beta|}}$. So we get $C_i^*<\infty$, $i=1,2$, by (6.33) and (6.34) which implies $\overline{\partial}\left(\varphi_0\left(f_{j_0}-\tilde{F}\right)+\varphi_1\left(f_{j_1}-\tilde{F}\right)\right)\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}},E\right)$ by virtue of (6.32). Since E is admissible, there is $g\in\mathcal{E}^{exp}\left(\overline{\mathbb{C}},E\right)$ such that

$$\overline{\partial}g=\overline{\partial}\left(\varphi_0\left(f_{j_0}-\tilde{F}\right)+\varphi_1\left(f_{j_1}-\tilde{F}\right)\right). \tag{6.35}$$

Set $G := \varphi_0\left(f_{j_0} - \tilde{F}\right) + \varphi_1\left(f_{j_1} - \tilde{F}\right) - g$. Then $G \in \mathcal{O}(\mathbb{C}, E)$ by (6.35) and for all $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ we have

$$
\begin{aligned}
|G|_{\{\pm\infty\},n,\alpha} &\leq \sum_{i=0}^{1} \sup_{z \in S_n(\{\pm\infty\}) \setminus G_i} p_\alpha\left(\varphi_i\left(f_{j_i} - \tilde{F}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + \underbrace{\sup_{z \in S_n(\varnothing)} p_\alpha(g(z)) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\varnothing,n,\alpha}} \\
&\leq \sum_{i=0}^{1} \sup_{z \in S_n(\{\pm\infty\}) \setminus G_i} p_\alpha\left(\left(f_{j_i} - \tilde{F}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + |g|_{\varnothing,n,\alpha}. \qquad (6.36)
\end{aligned}
$$

Furthermore, if we choose $k \in \mathbb{N}$ such that $k > n$ and $\frac{1}{k} < \min\left(1, \frac{\varepsilon}{2}\right)$ plus $-k < x+1$, if $\infty \notin \Omega_{j_0}$ resp. $-\infty \notin \Omega_{j_1}$, then $[S_n(\{\pm\infty\}) \setminus G_i] \subset \left[M_i \cup S_k\left(U_{j_i}\right)\right]$, $i = 0, 1$, where

$$
M_i := \begin{cases} \varnothing, & n \leq x+1, \\ \left\{z \in \mathbb{C} \mid -n \leq \mathrm{Re}(z) \leq -x-1,\ |\mathrm{Im}(z)| \leq \frac{1}{n}\right\}, & n > x+1,\ i = 0, \\ \left\{z \in \mathbb{C} \mid x+1 \leq \mathrm{Re}(z) \leq n,\ |\mathrm{Im}(z)| \leq \frac{1}{n}\right\}, & n > x+1,\ i = 1, \end{cases}
$$

is a compact subset of $U_{j_i} \cap \mathbb{C}$.

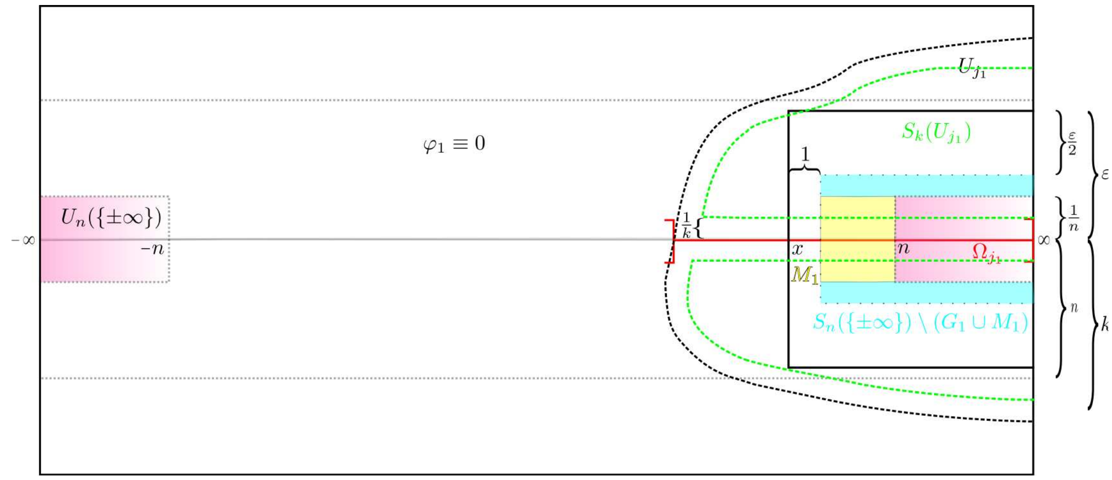

Figure 6.14: case: $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$, $n > x+1$, $i = 1$

In addition, $S_k\left(U_{j_i}\right) \subset S_k\left(\overline{\Omega}\right)$ and hence, keeping (6.31) in mind,

$$
\begin{aligned}
&\sup_{z \in S_n(\{\pm\infty\}) \setminus G_i} p_\alpha\left(\left(f_{j_i} - \tilde{F}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} \\
&\leq \sum_{i=0}^{1} \sup_{z \in M_i} p_\alpha\left(\left(f_{j_i} - \tilde{F}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|} + \underbrace{\sup_{z \in S_k\left(U_{j_i}\right)} p_\alpha\left(\left(f_{j_i}\right)(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|f_{j_i}|_{U_{j_i}^*,k,\alpha}} \\
&\quad + 2 \underbrace{\sup_{z \in S_k\left(\overline{\Omega}\right)} p_\alpha\left(\tilde{F}(z)\right) e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|\tilde{F}|_{\overline{\Omega},n,\alpha}}
\end{aligned}
$$

$$\leq 2\left|\tilde{F}\right|_{\overline{\Omega},n,\alpha}+\sum_{i=0}^{1}\left|f_{j_i}\right|_{U^{*}_{j_i},k,\alpha}+\sum_{i=0}^{1}\sup_{z\in M_i}p_{\alpha}\left(\left(f_{j_i}-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty.$$

So we gain $G\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus\{\pm\infty\},E\right)$ by (6.36).
Now we define the function $F^{*}:=\tilde{F}+G$. Then we have

$$F^{*}=\tilde{F}+G\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus\overline{\Omega},E\right)\subset\mathcal{O}^{exp}\left(U\smallsetminus\overline{\mathbb{R}},E\right).$$

The last step is to prove that F^{*} has the desired property, i.e. $[F^{*}]|_{\Omega_j}=\left[f_j\right]$ for any $j\in J$. If $j\in J$ with $\pm\infty\notin\Omega_j$, then

$$f_j-F^{*}=\left(f_j-\tilde{F}\right)-G\in\mathcal{O}\left(U_j\cap\mathbb{C},E\right)$$

by (6.31) and since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\smallsetminus\{\pm\infty\},E\right)\subset\mathcal{O}(\mathbb{C},E)$. Thus we have $[F^{*}]|_{\Omega_j}=\left[f_j\right]$.
Let $j\in J$ such that $-\infty\in\Omega_j$ or $\infty\in\Omega_j$. Then we have for $n\in\mathbb{N}_{\geq 2}$ and $\alpha\in A$

$$\begin{aligned}
&\left|f_j-F^{*}\right|_{U_j,n,\alpha}\\
&=\sup_{z\in T_n(U_j)}p_{\alpha}\left(\left(f_j-\tilde{F}-\varphi_0\left(f_{j_0}-\tilde{F}\right)-\varphi_1\left(f_{j_1}-\tilde{F}\right)+g\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq\sum_{i=0}^{1}\sup_{z\in T_n(U_j)\cap H_i}p_{\alpha}\left(\left(f_j-f_{j_i}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}+\underbrace{\sup_{z\in S_n(\varnothing)}p_{\alpha}(g(z))e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|g|_{\varnothing,n,\alpha}}\\
&\quad+\sup_{z\in T_n(U_j)\smallsetminus(H_0\cup H_1)}p_{\alpha}\left(\left(f_j-\tilde{F}-\varphi_0\left(f_{j_0}-\tilde{F}\right)-\varphi_1\left(f_{j_1}-\tilde{F}\right)\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}
\end{aligned}\tag{6.37}$$

where we used $T_n\left(U_j\right)\subset S_n(\varnothing)$ plus

$$H_0\subset G_1\quad\text{and}\quad H_1\subset G_0.\tag{6.38}$$

Moreover, the following estimate holds

$$\begin{aligned}
&\sup_{z\in T_n(U_j)\smallsetminus(H_0\cup H_1)}p_{\alpha}\left(\left(f_j-\tilde{F}-\varphi_0\left(f_{j_0}-\tilde{F}\right)-\varphi_1\left(f_{j_1}-\tilde{F}\right)\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\leq\sup_{z\in T_n(U_j)\smallsetminus(H_0\cup H_1)}p_{\alpha}\left(\left(f_j-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\
&\quad+\sum_{i=0}^{1}\sup_{z\in T_n(U_j)\smallsetminus(H_i\cup G_i)}p_{\alpha}\left(\left(f_{j_i}-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}
\end{aligned}\tag{6.39}$$

by the triangular inequality, (6.38) and the properties of φ_i. Choose $k\in\mathbb{N}$ such that $k>\max\left(n,\frac{\varepsilon}{2}\right)$ and $\frac{1}{k}<\frac{\varepsilon}{4}$ and, in addition, $-k<x+1$, if $\infty\notin\Omega_{j_0}$ resp. $-\infty\notin\Omega_{j_1}$. Remark that

$$T_n\left(U_j\right)\smallsetminus(H_i\cup G_i)\subset\begin{cases}\left(]-\infty,-x-1]\times\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\right)\smallsetminus\left(]-\infty,-x-2]\times\left[-\frac{\varepsilon}{4},\frac{\varepsilon}{4}\right]\right), & i=0,\\ \left([x+1,\infty[\times\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\right)\smallsetminus\left([x+2,\infty[\times\left[-\frac{\varepsilon}{4},\frac{\varepsilon}{4}\right]\right), & i=1,\end{cases}$$

$$\subset S_k\left(U_{j_i}\right) \cup M_i,\ i = 0, 1,$$

with

$$M_i := \begin{cases} \left\{ z \in \mathbb{C} \mid -x-2 < \mathrm{Re}(z) < -x-1,\ |\mathrm{Im}(z)| \leq \frac{1}{k} \right\}, & i = 0, \\ \left\{ z \in \mathbb{C} \mid x+1 < \mathrm{Re}(z) < x+2,\ |\mathrm{Im}(z)| \leq \frac{1}{k} \right\}, & i = 1, \end{cases}$$

by the choice of k.

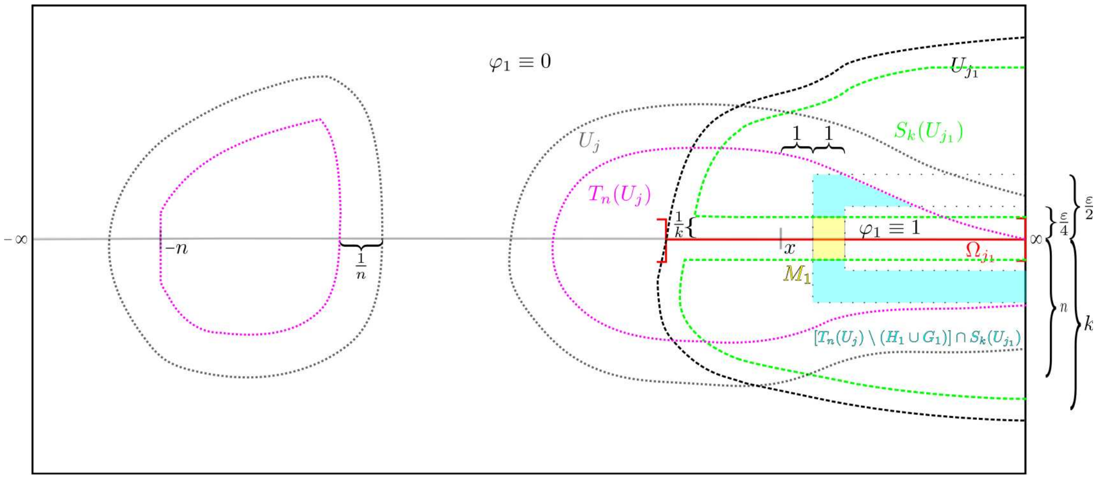

Figure 6.15: case $i = 1$: $\infty \in \Omega_j$, $-\infty \notin \Omega_j$, $\infty \in \Omega_{j_1}$, $-\infty \notin \Omega_{j_1}$

The sets M_i, $i = 0, 1$, are obviously bounded and $\overline{M}_i \subset \left(U_{j_i} \cap \mathbb{C}\right)$. Further, we define the set

$$M_2 := \left[T_n\left(U_j\right) \setminus \left(H_0 \cup H_1\right)\right] \setminus S_k\left(U_j\right)$$

which is bounded, since $M_2 \subset \{z \in \mathbb{C} \mid -x-2 < \mathrm{Re}(z) < x+2,\ |\mathrm{Im}(z)| \leq 1/k\}$ due to the choice of k, and one has $\overline{M}_2 \subset \overline{T_n\left(U_j\right)} \subset \left(U_j \cap \mathbb{C}\right)$.

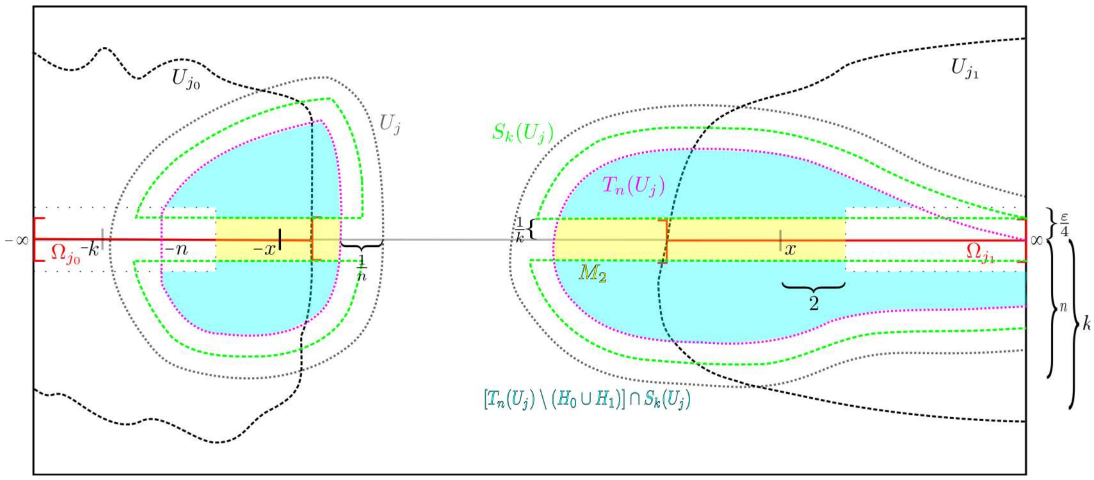

Figure 6.16: case: $\infty \in \Omega_j$, $-\infty \notin \Omega_j$, $-\infty \in \Omega_{j_0}$, $\infty \in \Omega_{j_1}$

These results yield to

$$\begin{aligned}&\sup_{z\in T_n(U_j)\smallsetminus(H_i\cup G_i)} p_\alpha\left(\left(f_{j_i}-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\&\leq \left|f_{j_i}\right|_{U_{j_i}^*,k,\alpha}+\left|\tilde{F}\right|_{\overline{\Omega},k,\alpha}+\sup_{z\in\overline{M}_i} p_\alpha\left(\left(f_{j_i}-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty\end{aligned}$$

for $i=0,1$ and

$$\begin{aligned}&\sup_{z\in T_n(U_j)\smallsetminus(H_0\cup H_1)} p_\alpha\left(\left(f_j-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\&\leq \left|f_j\right|_{U_j^*,k,\alpha}+\left|\tilde{F}\right|_{\overline{\Omega},k,\alpha}+\sup_{z\in\overline{M}_2} p_\alpha\left(\left(f_j-\tilde{F}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty\end{aligned}$$

by triangular inequality and (6.31). Thus the right hand side of (6.39) is bounded from above.
Let us turn to the still pending estimates in (6.37), so we have to take a look at the sets $T_n\left(U_j\right)\cap H_i$, $i=0,1$.

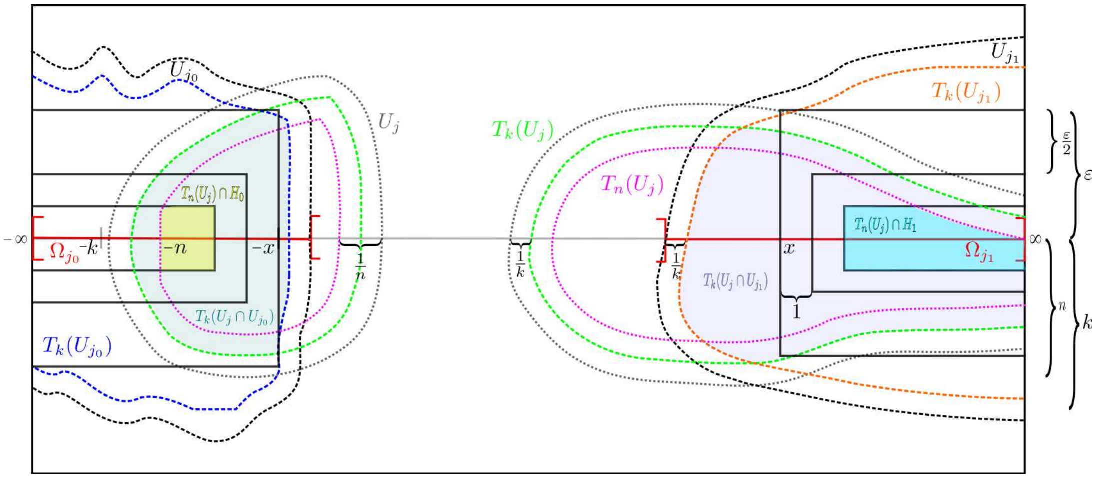

Figure 6.17: case: $\infty\in\Omega_j$, $-\infty\notin\Omega_j$, $\infty\notin\Omega_{j_0}$, $-\infty\in\Omega_{j_0}$, $\infty\in\Omega_{j_1}$, $-\infty\notin\Omega_{j_1}$

Choose $k\in\mathbb{N}$ such that $k>n$ and $\frac{1}{k}<\min\left(1,\frac{\varepsilon}{2}\right)$ and, in addition, $-k<x+1$, if $\infty\notin\Omega_{j_0}$ resp. $-\infty\notin\Omega_{j_1}$. Let $z\in H_i$, $i=0,1$, with $|\mathrm{Im}(z)|<k$. Then $z\in U_{j_i}$ and

$$\mathrm{Re}(z)\leq -x-1<k,\ \text{ if } i=0,\ \infty\notin\Omega_{j_0},\quad \text{resp.}\quad \mathrm{Re}(z)\geq x+1>-k,\ \text{ if } i=1,\ -\infty\notin\Omega_{j_1},$$

by the choice of k as well as

$$\mathrm{d}\left(z,\partial U_{j_i}\cap\mathbb{C}\right)\geq\min\left(1,\frac{\varepsilon}{2}\right)>\frac{1}{k}$$

implying $z\in T_k\left(U_{j_i}\right)$. Since $k>n$, we have $T_n\left(U_j\right)\subset T_k\left(U_j\right)$ and thus $\left(T_n\left(U_j\right)\cap H_i\right)\subset\left[T_k\left(U_j\right)\cap T_k\left(U_{j_i}\right)\right]$. Now let $z\in T_k\left(U_j\right)\cap T_k\left(U_{j_i}\right)$. Then $z\in U_j\cap U_{j_i}$ and $|\mathrm{Im}(z)|<k$. Since

$\partial\left(U_j\cap U_{j_i}\right)\cap\mathbb{C}$ is closed, there is $z_0\in\partial\left(U_j\cap U_{j_i}\right)\cap\mathbb{C}$ with

$$\mathrm{d}\left(z,\partial\left(U_j\cap U_{j_i}\right)\cap\mathbb{C}\right)=|z-z_0|.$$

Moreover,

$$\left[\partial\left(U_j\cap U_{j_i}\right)\cap\mathbb{C}\right]\subset\left(\partial U_j\cap\mathbb{C}\right)\cup\left(\partial U_{j_i}\cap\mathbb{C}\right)$$

and thus we obtain

$$\mathrm{d}\left(z,\partial\left(U_j\cap U_{j_i}\right)\cap\mathbb{C}\right)=|z-z_0|\geq\left\{\begin{array}{ll}\mathrm{d}\left(z,\partial U_j\cap\mathbb{C}\right), & z_0\in\partial U_j\cap\mathbb{C},\\ \mathrm{d}\left(z,\partial U_{j_i}\cap\mathbb{C}\right), & z_0\in\partial U_{j_i}\cap\mathbb{C},\end{array}\right\}>\frac{1}{k}.$$

If $\pm\infty\notin\Omega_j\cap\Omega_{j_i}$, we have in addition $-k<\mathrm{Re}(z)<k$. Therefore, $T_k\left(U_j\right)\cap T_k\left(U_{j_i}\right)$ is bounded and the closure a subset of $U_j\cap U_{j_i}\cap\mathbb{C}$, if $\pm\infty\notin\Omega_j\cap\Omega_{j_i}$, and $\left[T_k\left(U_j\right)\cap T_k\left(U_{j_i}\right)\right]\subset T_k\left(U_j\cap U_{j_i}\right)$, if $-\infty\in\Omega_j\cap\Omega_{j_i}$ or $\infty\in\Omega_j\cap\Omega_{j_i}$. This yields to

$$\begin{aligned}&\sup_{z\in T_n\left(U_j\right)\cap H_i} p_\alpha\left(\left(f_j-f_{j_i}\right)(z)\right)e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\ &\leq\begin{cases}\sup_{z\in\overline{T_k\left(U_j\right)\cap T_k\left(U_{j_i}\right)}} p_\alpha\left(\left(f_j-f_{j_i}\right)(z)\right)e^{-\frac{1}{k}|\mathrm{Re}(z)|}, & \pm\infty\notin\Omega_j\cap\Omega_{j_i},\\ \left|f_j-f_{j_i}\right|_{U_j\cap U_{j_i},k,\alpha}, & \text{else},\end{cases}\\ &<\infty\end{aligned}$$

since $f_j-f_{j_i}\in\mathcal{O}^{exp}\left(U_j\cap U_{j_i},E\right)$, if $\Omega_j\cap\Omega_{j_i}\neq\varnothing$, by assumption (or $U_j\cap U_{j_i}=\varnothing$). Combining the results obtained, we have $\left|f_j-F^*\right|_{U_j,n,\alpha}<\infty$ for all $n\in\mathbb{N}_{\geq2}$ and $\alpha\in A$ by (6.37) and thus $f_j-F^*\in\mathcal{O}^{exp}\left(U_j,E\right)$, i.e. $[F^*]|_{\Omega_j}=\left[f_j\right]$.

b) Let $[f]\in bv(\Omega,E)=\mathcal{O}^{exp}\left(U\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U,E)$ where $U\in\mathcal{U}(\Omega)$ and $\Omega\subset\overline{\mathbb{R}}$ open. By virtue of the proof of Lemma 6.8 there exists a function $F\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)\subset\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\mathbb{R}},E\right)$ such that $f-F\in\mathcal{O}^{exp}(U,E)$. Hence $[F]\in bv\left(\overline{\mathbb{R}},E\right)$ is an extension of $[f]$ to $\overline{\mathbb{R}}$.

c) For an open set $\Omega\subset\overline{\mathbb{R}}$, $\Omega\neq\varnothing$, one has the following (algebraic) isomorphisms

$$\begin{aligned}\mathcal{R}(\Omega,E)=L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)/L\left(\mathcal{P}_*(\partial\Omega),E\right)&\cong\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)\\ &\cong\mathcal{O}^{exp}\left((\Omega\times\mathbb{R})\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(\Omega\times\mathbb{R},E)=bv(\Omega,E).\end{aligned}$$

The first isomorphism is due to Theorem 4.1 and given by the map

$$\begin{gathered}G_\Omega\colon L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)/L\left(\mathcal{P}_*(\partial\Omega),E\right)\to\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right),\\ [T]\mapsto\left[\tilde{T}\right]_\sim,\quad\left[\tilde{T}\right]_{\overline{\Omega}}=H_{\overline{\Omega}}^{-1}(T),\end{gathered}$$

where H is the isomorphism from Theorem 4.1 and we denote by $[\cdot]$ the equivalence classes in $L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)/L\left(\mathcal{P}_*(\partial\Omega),E\right)$, with $[\cdot]_\sim$ the ones in $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$ and with $[\cdot]_{\overline{\Omega}}$ the ones in $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)$.
well-defined: Let $T_0,T_1\in L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)$ with $[T_0]=[T_1]$, i.e. $T_0-T_1\in L\left(\mathcal{P}_*(\partial\Omega),E\right)$.

Then

$$H_{\overline{\Omega}}^{-1}(T_0-T_1)=H_{\partial\Omega}^{-1}(T_0-T_1)$$

by (4.8) and

$$\begin{aligned}\left[\tilde{T}_0-\tilde{T}_1\right]_{\overline{\Omega}}&=\left[\tilde{T}_0\right]_{\overline{\Omega}}-\left[\tilde{T}_1\right]_{\overline{\Omega}}=H_{\overline{\Omega}}^{-1}(T_0)-H_{\overline{\Omega}}^{-1}(T_1)=H_{\overline{\Omega}}^{-1}(T_0-T_1)\\&=H_{\partial\Omega}^{-1}(T_0-T_1)\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\end{aligned}$$

holds. Thus $\tilde{T}_0-\tilde{T}_1\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$, i.e. $\left[\tilde{T}_0-\tilde{T}_1\right]_{\sim}=0$.
On the other hand, let $T\in L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)$ and $\tilde{T}_0$, $\tilde{T}_1\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)$ such that $\left[\tilde{T}_0\right]_{\overline{\Omega}}=\left[\tilde{T}_1\right]_{\overline{\Omega}}=H_{\overline{\Omega}}^{-1}(T)$. Then $\tilde{T}_0-\tilde{T}_1\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\subset\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$ and hence $\left[\tilde{T}_0-\tilde{T}_1\right]_{\sim}=0$.
injectivity: Let $T\in L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)$ with $G_{\Omega}(T)=\left[\tilde{T}\right]_{\sim}=0$. Then $\tilde{T}\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$ and thus

$$H_{\overline{\Omega}}^{-1}(T)=\left[\tilde{T}\right]_{\overline{\Omega}}\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right).$$

Therefore, we get

$$T=H_{\overline{\Omega}}\left(H_{\overline{\Omega}}^{-1}(T)\right)=H_{\partial\Omega}\left(H_{\overline{\Omega}}^{-1}(T)\right)\in L\left(\mathcal{P}_*(\partial\Omega),E\right)$$

by (4.8) and so $[T]=0$.
surjectivity: Let $T_0\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)$. Then we have $H_{\overline{\Omega}}\left([T_0]_{\overline{\Omega}}\right)\in L\left(\mathcal{P}_*\left(\overline{\Omega}\right),E\right)$ by Theorem 4.1. Then we define $T:=H_{\overline{\Omega}}\left([T_0]_{\overline{\Omega}}\right)$ and get

$$H_{\overline{\Omega}}^{-1}(T)=H_{\overline{\Omega}}^{-1}\left(H_{\overline{\Omega}}\left([T_0]_{\overline{\Omega}}\right)\right)=[T_0]_{\overline{\Omega}}$$

by Theorem 4.1 again. This means that $G_{\Omega}([T])=[T_0]_{\sim}$.
The second isomorphism is defined by the map

$$\begin{gathered}J_{\Omega}\colon\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)\to\mathcal{O}^{exp}\left((\Omega\times\mathbb{R})\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(\Omega\times\mathbb{R},E),\\ [f]_{\sim}\mapsto\left[f|_{((\Omega\times\mathbb{R})\setminus\overline{\mathbb{R}})\cap\mathbb{C}}\right]_{\Omega}.\end{gathered}$$

This map is well-defined since $\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)\subset\mathcal{O}^{exp}(\Omega\times\mathbb{R},E)$.
injectivity: Let $f\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\overline{\Omega},E\right)$ with $J_{\Omega}([f]_{\sim})=0$, i.e. $f\in\mathcal{O}^{exp}(\Omega\times\mathbb{R},E)$. Then it follows that $f\in\mathcal{O}(\mathbb{C}\setminus\partial\Omega,E)$. Further, the estimate

$$|f|_{\partial\Omega,n,\alpha}\leq\underbrace{\sup_{z\in S_n(\overline{\Omega})}p_{\alpha}(f(z))e^{-\frac{1}{n}|\mathrm{Re}(z)|}}_{=|f|_{\overline{\Omega},n,\alpha}}+\sup_{z\in S_n(\partial\Omega)\setminus S_n(\overline{\Omega})}p_{\alpha}(f(z))e^{-\frac{1}{n}|\mathrm{Re}(z)|}\tag{6.40}$$

holds for all $n\in\mathbb{N}_{\geq 2}$ and $\alpha\in A$.

Let us examine the set $S_n(\partial\Omega)\setminus S_n(\overline{\Omega})$. We have for $z\in S_n(\partial\Omega)\setminus S_n(\overline{\Omega})$

$$\mathrm{Re}(z)\in\begin{cases}[\min\partial\Omega\cap\mathbb{R},\max\partial\Omega\cap\mathbb{R}], & \pm\infty\notin\overline{\Omega},\\ [-n,n], & \pm\infty\in\partial\Omega,\\ ]-\infty,n], & -\infty\in\Omega,\ \infty\in\partial\Omega,\\ [-n,\infty[, & -\infty\in\partial\Omega,\ \infty\in\Omega,\\ \mathbb{R}, & \pm\infty\in\Omega,\\ [-n,\max\partial\Omega\cap\mathbb{R}], & -\infty\in\partial\Omega,\ \infty\notin\overline{\Omega},\\ ]-\infty,\max\partial\Omega\cap\mathbb{R}], & -\infty\in\Omega,\ \infty\notin\overline{\Omega},\\ [\min\partial\Omega\cap\mathbb{R},n], & -\infty\notin\overline{\Omega},\ \infty\in\partial\Omega,\\ [\min\partial\Omega\cap\mathbb{R},\infty[, & -\infty\notin\overline{\Omega},\ \infty\in\Omega,\end{cases}$$

and $|\mathrm{Im}(z)|\leq\frac{1}{n}$. Furthermore, we observe that $W:=\bigcup_{x\in\partial\Omega\cap\mathbb{R}}D_{\frac{1}{n}}(x)$ is open and

$$\overline{S_n(\partial\Omega)\setminus S_n(\overline{\Omega})}=\overline{\left(\left[S_n(\partial\Omega)\setminus S_n(\overline{\Omega})\right]\setminus W\right)}\subset\overline{W^C}=W^C\subset\mathbb{C}\setminus\partial\Omega. \tag{6.41}$$

So, if $\pm\infty\notin\Omega$, then $\overline{S_n(\partial\Omega)\setminus S_n(\overline{\Omega})}$ is a compact subset of $\mathbb{C}\setminus\partial\Omega$. Due to (6.40) and since $f\in\mathcal{O}(\mathbb{C}\setminus\partial\Omega,E)$, we get $|f|_{\partial\Omega,n,\alpha}<\infty$ in this case.
Let $-\infty\in\Omega$ or $\infty\in\Omega$. Then there are $x_i\in\mathbb{R}$, $i=0,1$, such that $[-\infty,x_0]\subset\Omega$ resp. $[x_1,\infty]\subset\Omega$. Choose $k\in\mathbb{N}$ such that $k>n$ and, in addition,

$$k>x_0,\ \text{if}\ -\infty\in\Omega,\ \infty\notin\Omega,\quad\text{resp.}\quad -k<x_1,\ \text{if}\ -\infty\notin\Omega,\ \infty\in\Omega.$$

Then we obtain for $z\in\left[S_n(\partial\Omega)\setminus S_n(\overline{\Omega})\right]\setminus T_k(\Omega\times\mathbb{R})=:M$

$$|\mathrm{Re}(z)|\leq\begin{cases}\max(|x_0|,n), & -\infty\in\Omega,\ \infty\in\partial\Omega,\\ \max(|x_1|,n), & -\infty\in\partial\Omega,\ \infty\in\Omega,\\ \max(|x_0|,|x_1|), & \pm\infty\in\Omega,\\ \max(|x_0|,|\max\partial\Omega\cap\mathbb{R}|) & -\infty\in\Omega,\ \infty\notin\overline{\Omega},\\ \max(|\min\partial\Omega\cap\mathbb{R}|,|x_1|), & -\infty\notin\overline{\Omega},\ \infty\in\Omega,\end{cases}$$

by the choice of k and as $\partial\Omega\subset\Omega^C$. Hence M is bounded, thus $\overline{M}$ compact, and $\overline{M}\subset(\mathbb{C}\setminus\partial\Omega)$ by (6.41). Therefore, we gain

$$\begin{aligned}&\sup_{z\in S_n(\partial\Omega)\setminus S_n(\overline{\Omega})}p_\alpha(f(z))e^{-\frac{1}{n}|\mathrm{Re}(z)|}\\ &\leq\underbrace{\sup_{z\in T_k(\Omega\times\mathbb{R})}p_\alpha(f(z))e^{-\frac{1}{k}|\mathrm{Re}(z)|}}_{=|f|_{\Omega\times\mathbb{R},k,\alpha}}+\sup_{z\in\overline{M}}p_\alpha(f(z))e^{-\frac{1}{n}|\mathrm{Re}(z)|}<\infty\end{aligned}$$

since $f\in\mathcal{O}^{exp}(\Omega\times\mathbb{R},E)$ and $f\in\mathcal{O}(\mathbb{C}\setminus\partial\Omega,E)$. By (6.40) we have $|f|_{\partial\Omega,n,\alpha}<\infty$ in this case as well and thus $f\in\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus\partial\Omega,E\right)$ proving the injectivity of J_Ω.
surjectivity: Let $[f]_\Omega\in\mathcal{O}^{exp}\left((\Omega\times\mathbb{R})\setminus\overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(\Omega\times\mathbb{R},E)$. By the proof of Lemma 6.8

there is $F \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \overline{\Omega}, E\right)$ such that $f - F \in \mathcal{O}^{exp}(\Omega \times \mathbb{R}, E)$, i.e. $J_{\Omega}([F]_{\sim}) = [f]_{\Omega}$.
The last step is to prove that these isomorphisms, which we denote by $h_{\Omega} := J_{\Omega} \circ G_{\Omega}$, are compatible with the respective restrictions, i.e. for open sets $\Omega_1 \subset \Omega \subset \overline{\mathbb{R}}$ the diagram

$$\begin{tikzcd}
\mathcal{R}(\Omega, E) \arrow[r, "h_{\Omega}"] \arrow[d, "R^{\mathcal{R}}_{\Omega,\Omega_1}"'] & bv(\Omega, E) \arrow[d, "R^{bv}_{\Omega,\Omega_1}"] \\
\mathcal{R}(\Omega_1, E) \arrow[r, "h_{\Omega_1}"'] & bv(\Omega_1, E)
\end{tikzcd}$$

commutes. Let $T \in L\left(\mathcal{P}_*\left(\overline{\Omega}\right), E\right)$. Choose a representative T_0 of $R^{\mathcal{R}}_{\Omega,\Omega_1}([T])$. By definition of the restriction

$$T_0 - T \in L\left(\mathcal{P}_*\left(\overline{\Omega} \setminus \Omega_1\right), E\right) \tag{6.42}$$

is valid. Let $\tilde{T}_0$ be a representative of $H^{-1}_{\overline{\Omega}}(T_0)$. Then we have

$$\left(h_{\Omega_1} \circ R^{\mathcal{R}}_{\Omega,\Omega_1}\right)([T]) = h_{\Omega_1}([T_0]_1) = J_{\Omega_1} \circ G_{\Omega_1}([T_0]_1) = \left[\tilde{T}_0|_{((\Omega_1 \times \mathbb{R}) \setminus \overline{\mathbb{R}}) \cap \mathbb{C}}\right]_{\Omega_1}.$$

On the other hand, let $\tilde{T}$ be a representative of $H^{-1}_{\overline{\Omega}}(T)$. Then we get

$$\left(R^{bv}_{\Omega,\Omega_1} \circ h_{\Omega}\right)([T]) = R^{bv}_{\Omega,\Omega_1}\left(\left[\tilde{T}|_{((\Omega \times \mathbb{R}) \setminus \overline{\mathbb{R}}) \cap \mathbb{C}}\right]_{\Omega}\right) = \left[\tilde{T}|_{((\Omega_1 \times \mathbb{R}) \setminus \overline{\mathbb{R}}) \cap \mathbb{C}}\right]_{\Omega_1}.$$

Further,

$$\left[\tilde{T}_0 - \tilde{T}\right]_{\overline{\Omega}} = H^{-1}_{\overline{\Omega}}(T_0 - T) = H^{-1}_{\overline{\Omega} \setminus \Omega_1}(T_0 - T) \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \left(\overline{\Omega} \setminus \Omega_1\right), E\right) / \mathcal{O}^{exp}\left(\overline{\mathbb{C}}, E\right)$$

by (6.42) and (4.8). Therefore, $\tilde{T}_0 - \tilde{T} \in \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus \left(\overline{\Omega} \setminus \Omega_1\right), E\right) \subset \mathcal{O}^{exp}(\Omega_1 \times \mathbb{R}, E)$ which implies $\left(h_{\Omega_1} \circ R^{\mathcal{R}}_{\Omega,\Omega_1}\right)([T]) = \left(R^{bv}_{\Omega,\Omega_1} \circ h_{\Omega}\right)([T])$. By virtue of Proposition 6.6 it follows that $\mathcal{R}(E)$ is a sheaf which is isomorphic to $bv(E)$.

□

Immediately we get the following corollary.

6.12 Corollary. [6] *Let E be strictly admissible, $\Omega \subset \overline{\mathbb{R}}$ open. The spaces $\{\mathcal{R}(\omega, E) \mid \omega \subset \Omega \text{ open}\}$, equipped with the restrictions of Definition 6.4, form a flabby sheaf.*

Corollary 6.12 provides an answer to a problem stated by Ito, at least for E-valued Fourier hyperfunctions in one variable (see [23, Problem B, p. 18]).

Now we want to describe the sections with support in a given compact set $K \subset \overline{\mathbb{R}}$. We recall the definition of the support of a section of a sheaf. Let X be a topological space, $\left(\mathcal{F}, R^{\mathcal{F}}\right)$ a sheaf on X and $f \in \mathcal{F}(X)$ a section of a sheaf. Then the support of f, denoted by $\operatorname{supp}_{\mathcal{F}} f$ or shortly $\operatorname{supp} f$, is the complement of the largest open subset of X on which $f = 0$, i.e.

$$\operatorname{supp} f = \Big(\bigcup_{V \in Z_f} V\Big)^C$$

[6] counterpart: [13, Corollary 6.10, p. 1126]

where $Z_f := \{V \mid V \subset X \text{ open}, f|_V = 0\}$ (condition $(S1)$ is used in this definition). This directly yields to the following description of the support of an element of $bv(\Omega,E)$ for an open set $\Omega \subset \overline{\mathbb{R}}$ and a strictly admissible space E:
Let $f = [F] \in \mathcal{O}^{exp}(U \setminus \overline{\mathbb{R}}, E)/\mathcal{O}^{exp}(U,E)$, where $U \in \mathcal{U}(\Omega)$, and $\Omega_1 \subset \Omega$ be open. If $-\infty \in \Omega$ or $\infty \in \Omega$, we define the set

$$S_n(U,\Omega_1) := \left\{z \in U \cap \mathbb{C} \mid \mathrm{d}\left(z, \left(\overline{\Omega} \cap \mathbb{R}\right) \setminus \Omega_1\right) > \frac{1}{n},\ \mathrm{d}(z, \partial U \cap \mathbb{C}) > \frac{1}{n},\ |\mathrm{Im}(z)| < n\right\}$$
$$\cap \begin{cases} \mathbb{C}, & \pm\infty \in \Omega, \\ \{z \in \mathbb{C} \mid \mathrm{Re}(z) > -n\}, & \infty \in \Omega,\ -\infty \notin \Omega, \\ \{z \in \mathbb{C} \mid \mathrm{Re}(z) < n\}, & \infty \notin \Omega,\ -\infty \in \Omega, \end{cases}$$
$$\setminus \begin{cases} (]-\infty,-n] \cup [n,\infty[) + i\left[-\frac{1}{n},\frac{1}{n}\right], & \pm\infty \notin \Omega_1, \\ ]-\infty,-n] + i\left[-\frac{1}{n},\frac{1}{n}\right], & -\infty \notin \Omega_1,\ \infty \in \Omega_1, \\ [n,\infty[+ i\left[-\frac{1}{n},\frac{1}{n}\right], & \infty \notin \Omega_1,\ -\infty \in \Omega_1, \\ \varnothing, & \pm\infty \in \Omega_1, \end{cases}$$

for $n \in \mathbb{N}_{\geq 2}$.
If $-\infty \in \Omega$ or $\infty \in \Omega$, then $f|_{\Omega_1} = 0$ is equivalent to

(a) F can be extended to a holomorphic function on $\left[\left(U \setminus \overline{\mathbb{R}}\right) \cup \Omega_1\right] \cap \mathbb{C}$ if $\pm\infty \notin \Omega_1$.

(b) F can be extended to a holomorphic function on $\left[\left(U \setminus \overline{\mathbb{R}}\right) \cup \Omega_1\right] \cap \mathbb{C}$ and

$$|F|_{U,\Omega_1,n,\alpha} := \sup_{z \in S_n(U,\Omega_1)} p_\alpha(F(z))\, e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty \tag{6.43}$$

for every $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ if $-\infty \in \Omega_1$ or $\infty \in \Omega_1$.

We remark that (6.43) is valid in (a) as well.
If $\pm\infty \notin \Omega$, then $f|_{\Omega_1} = 0$ is equivalent to statement (a).
Observing that

$$\left[\left(U \setminus \overline{\mathbb{R}}\right) \cup Z_f\right] \cap \mathbb{C} = \left[\left(U \setminus \overline{\mathbb{R}}\right) \cup (\Omega \setminus \operatorname{supp} f)\right] \cap \mathbb{C} = (U \setminus \operatorname{supp} f) \cap \mathbb{C},$$

since $U \in \mathcal{U}(\Omega)$, and

$$\left(\overline{\Omega} \cap \mathbb{R}\right) \setminus Z_f = \overline{\Omega} \cap \operatorname{supp} f \cap \mathbb{R} = (\partial\Omega \cup \operatorname{supp} f) \cap \mathbb{R},$$

we get $F \in \mathcal{O}((U \setminus \operatorname{supp} f) \cap \mathbb{C}, E)$ and, if $-\infty \in \Omega_1$ or $\infty \in \Omega_1$, in addition,

$$|F|_{U,Z_f,n,\alpha} = \sup_{z \in S_n(U,Z_f)} p_\alpha(F(z))\, e^{-\frac{1}{n}|\mathrm{Re}(z)|} < \infty$$

for every $n \in \mathbb{N}_{\geq 2}$ and $\alpha \in A$ where we have

$$\mathrm{d}\left(z, \left(\overline{\Omega} \cap \mathbb{R}\right) \setminus \Omega_1\right) = \mathrm{d}(z, (\partial\Omega \cup \operatorname{supp} f) \cap \mathbb{R})$$

in the definition of $S_n(U, Z_f)$.

Now let $K \subset \Omega$ be compact and set

$$bv_K(\Omega,E) := \{f \in bv(\Omega,E) \mid \operatorname{supp} f \subset K\}$$

plus for $U \in \mathcal{U}(\Omega)$

$$\mathcal{O}^{exp}(U \setminus K,E) := \left\{f \in \mathcal{O}((U \setminus K) \cap \mathbb{C},E) \mid \forall\, n \in \mathbb{N}_{\geq 2}\ \forall\, \alpha \in A : |F|_{U,\Omega\setminus K,n,\alpha} < \infty\right\},$$

if $-\infty \in \Omega$ or $\infty \in \Omega$, resp.

$$\mathcal{O}^{exp}(U \setminus K,E) := \mathcal{O}((U \setminus K) \cap \mathbb{C},E),$$

if $\pm\infty \notin \Omega$.
Due to the considerations above and Lemma 6.8 we gain the following description of $bv_K(\Omega,E)$.

6.13 Lemma. *Let E be strictly admissible, $\Omega \subset \overline{\mathbb{R}}$ be open and $K \subset \Omega$ compact. For any $U \in \mathcal{U}(\Omega)$ we have the (algebraic) isomorphism:*

$$bv_K(\Omega,E) \cong \mathcal{O}^{exp}(U \setminus K,E)/\mathcal{O}^{exp}(U,E)$$

In particular, we have

$$bv_K\left(\overline{\mathbb{R}},E\right) \cong \mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right) \cong L(\mathcal{P}_*(K),E).$$

Proof. Using Lemma 6.8, we represent $bv(\Omega,E)$ by $\mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U,E)$. Then the identity-mapping

$$\begin{aligned}\mathrm{id}\colon \left\{f = [F] \in \mathcal{O}^{exp}\left(U \setminus \overline{\mathbb{R}},E\right)/\mathcal{O}^{exp}(U,E) \mid \operatorname{supp} f \subset K\right\} &\to \mathcal{O}^{exp}(U \setminus K,E)/\mathcal{O}^{exp}(U,E),\\ [F] &\to [F],\end{aligned}$$

is (well-)defined and surjective by the considerations above and obviously injective.
Now let $\Omega := \overline{\mathbb{R}}$, set $\Omega_1 := \overline{\mathbb{R}} \setminus K$ and choose $U := \overline{\mathbb{C}}$. We claim that the definition of the space $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K,E\right)$ in the sense above and in the sense of Definition 3.2 coincide (and therefore the spaces have the same symbol). Let $n \in \mathbb{N}_{\geq 2}$. Then

$$\mathrm{d}\left(z,\left(\overline{\Omega} \cap \mathbb{R}\right) \setminus \Omega_1\right) = \mathrm{d}\left(z,\left(\partial\overline{\mathbb{R}} \cup K\right) \cap \mathbb{R}\right) = \mathrm{d}(z,K \cap \mathbb{R})$$

and

$$\mathrm{d}(z,\partial U \cap \mathbb{C}) = \mathrm{d}(z,\varnothing) = \infty > \frac{1}{n}$$

holds for $z \in \mathbb{C}$. Further,

$$\left.\begin{aligned}\pm\infty &\notin \overline{\mathbb{R}} \setminus K\\ -\infty &\notin \overline{\mathbb{R}} \setminus K\\ \infty &\notin \overline{\mathbb{R}} \setminus K\\ \pm\infty &\in \overline{\mathbb{R}} \setminus K\end{aligned}\right\} \quad \text{is equivalent to} \quad \left\{\begin{aligned}\pm\infty &\in K\\ -\infty &\in K\\ \infty &\in K\\ \pm\infty &\notin K\end{aligned}\right.$$

and hence we obtain $S_n\left(\overline{\mathbb{C}},\overline{\mathbb{R}}\setminus K\right)=S_n(K)$. Thus the claim is proven. Therefore,

$$bv_K\left(\overline{\mathbb{R}},E\right)\cong\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\setminus K,E\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}},E\right)\cong L(\mathcal{P}_*(K),E)$$

holds by Theorem 4.1 which proves the endorsement. □

Remark that this isomorphism induces a reasonable locally convex topology on $bv_K\left(\overline{\mathbb{R}},E\right)$ since $L(\mathcal{P}_*(K),E)$ has such a topology.

As already mentioned, I am convinced that a reasonable theory of E-valued Fourier hyperfunctions (in one variable) should produce a flabby sheaf $\mathcal{F}$ on $\overline{\mathbb{R}}$ such that the set of sections supported by a compact subset $K\subset\overline{\mathbb{R}}$ should coincide, in the sense of being isomorphic, with $L(\mathcal{P}_*(K),E)$ since the restricted sheaf $\mathcal{F}|_{\mathbb{R}}$ then satisfies the conditions of Domański and Langenbruch for a reasonable theory of E-valued hyperfunctions. In addition, the map $\mathscr{F}:\mathcal{F}\left(\overline{\mathbb{R}}\right)\to\mathcal{F}\left(\overline{\mathbb{R}}\right)$, defined by $\mathscr{F}:=J^{-1}\circ\mathscr{F}_d\circ J$, where $J:\mathcal{F}\left(\overline{\mathbb{R}}\right)\to L\left(\mathcal{P}_*\left(\overline{\mathbb{R}}\right),E\right)$ is an isomorphism existing by assumption and $\mathscr{F}_d$ the Fourier transformation of Theorem 4.6, can be regarded as Fourier transformation on the space of global sections and is an isomorphism.
If E is strictly admissible, the sheaves $bv(E)$ and $\mathcal{R}(E)$ satisfy this condition for a reasonable theory of E-valued Fourier hyperfunctions by Theorem 6.11 and Lemma 6.13 (For $\mathcal{R}(E)$ remark that sheaf isomorphisms preserve supports, so the definition of a support in Proposition 4.3(2) was well-chosen.). The next theorem confirms that the sufficient condition of E being strictly admissible is also necessary for a reasonable theory of E-valued Fourier hyperfunctions in one variable if E is an ultrabornological PLS-space and describes further equivalent sufficient and necessary conditions.

6.14 Theorem. [7] *Let E be an ultrabornological PLS-space. Then the following assertions are equivalent:*

(a) There is a flabby sheaf $\mathcal{F}$ on some open set $\varnothing\neq\Omega\subset\overline{\mathbb{R}}$ such that

$$\begin{aligned}\mathcal{F}_K(\Omega):&=\{T\in\mathcal{F}(\Omega)\mid\operatorname{supp}_{\mathcal{F}}(T)\subset K\}\\&\cong L(\mathcal{P}_*(K),E)\quad\textit{for any compact } K\subset\Omega.\end{aligned}$$

(b) There is a flabby sheaf $\mathcal{F}$ on $\overline{\mathbb{R}}$ such that

$$\begin{aligned}\mathcal{F}_K\left(\overline{\mathbb{R}}\right):&=\left\{T\in\mathcal{F}\left(\overline{\mathbb{R}}\right)\mid\operatorname{supp}_{\mathcal{F}}(T)\subset K\right\}\\&\cong L(\mathcal{P}_*(K),E)\quad\textit{for any compact } K\subset\overline{\mathbb{R}}.\end{aligned}$$

(c) E is strictly admissible.

(d) $P(D):\mathrm{C}^\infty(U,E)\to\mathrm{C}^\infty(U,E)$ is surjective for some (any) elliptic operator $P(D)$ and some (any) open set $U\subset\mathbb{R}^n$ and some (any) $n\in\mathbb{N}_{\geq 2}$.

(e) E has (PA).

Proof. $(e)\Leftrightarrow(d)$: [13, Corollary 4.1, p. 1113] resp. [13, Corollary 3.9, p. 1112]
$(e)\Rightarrow(c)$: Theorem 5.25
$(c)\Rightarrow(b)$: Theorem 6.11 and Lemma 6.13

[7] counterpart: [13, Theorem 8.9, p. 1139]

$(b) \Rightarrow (a)$: Obvious with $\Omega := \overline{\mathbb{R}}$.
$(a) \Rightarrow (e)$: Let there be a flabby sheaf $\mathcal{F}$ on an open set $\varnothing \neq \Omega \subset \overline{\mathbb{R}}$ such that

$$\begin{aligned}\mathcal{F}_K(\Omega) &= \{T \in \mathcal{F}(\Omega) \mid \operatorname{supp}_{\mathcal{F}}(T) \subset K\}\\ &\cong L(\mathcal{P}_*(K), E) \quad \text{for any compact } K \subset \Omega.\end{aligned}$$

Then the restriction $\mathcal{F}|_{\Omega\cap\mathbb{R}}$ of $\mathcal{F}$ to $\Omega \cap \mathbb{R}$ is a flabby sheaf as well such that

$$\begin{aligned}\left(\mathcal{F}|_{\Omega\cap\mathbb{R}}\right)_K(\Omega\cap\mathbb{R}) &= \left\{T \in \mathcal{F}|_{\Omega\cap\mathbb{R}}(\Omega\cap\mathbb{R}) \mid \operatorname{supp}_{\mathcal{F}|_{\Omega\cap\mathbb{R}}}(T) \subset K\right\}\\ &\cong L(\mathcal{A}(K), E) \quad \text{for any compact } K \subset (\Omega\cap\mathbb{R})\end{aligned}$$

since $\mathcal{P}_*(K) = \mathcal{A}(K)$ for every compact set $K \subset \mathbb{R}$. By virtue of [13, Theorem 8.9, p. 1139] this implies that E has (PA). $\square$

7 Summary and outlook

We have seen that a reasonable theory of E-valued Fourier hyperfunctions in one variable exists for a complete locally convex space E if E is strictly admissible, i.e. if the Cauchy-Riemann operator

$$\overline{\partial}: \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

is surjective for any compact set $K \subset \overline{\mathbb{R}}$ and, in addition,

$$\overline{\partial}: \mathrm{C}^{\infty}(\Omega, E) \to \mathrm{C}^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$. At first this problem was solved for $E = \mathbb{C}$ by combining Hörmander's solution of the weighted $\overline{\partial}$-problem and the Mittag-Leffler procedure (Theorem 5.16). By virtue of representations of $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ and $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ as tensor products (Theorem 3.11) the corresponding result is also valid for Fréchet spaces E (Theorem 5.17 resp. Junker, [26]). In order to extend this result beyond the class of Fréchet spaces by the splitting theory of Vogt and the one of Bonet and Domański, it was necessary to prove that the space $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ satisfies the condition (Ω) for any compact set $K \subset \overline{\mathbb{R}}$. For $K = \varnothing$ this was done by using a decomposition result of Langenbruch (Theorem 5.20) and in combination with a duality established between the spaces $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right) / \mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ and $\mathcal{P}_{*}(K)_{b}'$ (Theorem 4.1) the general result was obtained (Theorem 5.22). Due to the condition of E being strictly admissible, the theory of vector-valued Fourier hyperfunctions (in one variable) could be extended far beyond the class of Fréchet spaces and they are realized on the one hand as the sheaf generated by equivalence classes of compactly supported E-valued $\mathcal{P}_{*}$-functionals and on the other as boundary values of E-valued slowly increasing holomorphic functions (Theorem 5.24 and Theorem 6.11). Furthermore, natural limits of this kind of theory were found in the class of ultrabornological PLS-spaces, namely, if E is an ultrabornological PLS-space, a reasonable theory of E-valued Fourier hyperfunctions in one variable is possible if and only if E satisfies (PA) (Theorem 6.14). For many classical spaces in analysis it is well-known whether they have (PA) or not, in particular every Fréchet-Schwartz space has (PA) (Example 5.26 and Example 5.27).

Obviously the question arises if such a theory is also possible in several variables. By the results of Junker (see [26, Section 3, p. 32-46]) we know that a reasonable theory of Fréchet-valued Fourier hyperfunctions in several variables exists. Domański and Langenbruch could construct an E-valued sheaf of hyperfunctions in d variables under the assumption that the $(d+1)$-dimensional Laplace operator

$$\Delta_{d+1}: \mathrm{C}^{\infty}(\Omega, E) \to \mathrm{C}^{\infty}(\Omega, E)$$

is surjective for every open set $\Omega \subset \mathbb{R}^{d+1}$. Perhaps it is possible to create a reasonable theory of E-valued Fourier hyperfunctions in d variables if we additionally assume that

$$\Delta_{d+1}: \mathcal{E}^{exp}\left((D_d \times \mathbb{R}) \setminus K, E\right) \to \mathcal{E}^{exp}\left((D_d \times \mathbb{R}) \setminus K, E\right)$$

is surjective for any compact set $K \subset D_d$ where D_d is the radial compactification of $\mathbb{R}^d$ and, if we write points $\zeta \in \mathbb{R}^{d+1}$ as $\zeta = (x,y) \in \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned}&\mathcal{E}^{exp}((D_d \times \mathbb{R}) \setminus K, E)\\ &:= \left\{f \in \mathrm{C}^\infty\left([(D_d \times \mathbb{R}) \setminus K] \cap \mathbb{R}^{d+1}, E\right) \mid \forall\, \alpha \in A,\ n \in \mathbb{N},\ m \in \mathbb{N}_0 :\ r_{n,m,\alpha}(f) < \infty\right\}\end{aligned}$$

where $(p_\alpha)_{\alpha \in A}$ is a fundamental system of semi-norms on E and

$$r_{n,m,\alpha}(f) := \sup_{\substack{(x,y) \in R_n(K) \cap \mathbb{R}^{d+1},\\ \beta \in \mathbb{N}_0^{d+1},\, |\beta| \leq m}} p_\alpha\left(\partial^\beta f(x,y)\right) e^{-\frac{1}{n}|x|}$$

plus

$$R_n(K) := \left\{(x,y) \in (D_d \times \mathbb{R}) \setminus K \mid |y| < n \text{ and } \inf_{w \in K} \rho((x,y), w) > \frac{1}{n}\right\}$$

where ρ denotes the canonical metric on $D_d \times \mathbb{R}$. Following [13], let

$$\tilde{\mathcal{E}}^{exp}_{\Delta}((D_d \times \mathbb{R}) \setminus K, E) := \{f \in \mathcal{E}^{exp}((D_d \times \mathbb{R}) \setminus K, E) \mid \Delta f = 0,\ f(x,y) = f(x,-y)\}$$

denote the E-valued slowly increasing harmonic functions outside K which are even with respect to the last variable and

$$\mathcal{P}_*(K) := \lim_{\substack{\longrightarrow\\ n \in \mathbb{N}}} \mathcal{O}(U_n(K))$$

denote the rapidly decreasing holomorphic germs near K where

$$\mathcal{O}(U_n(K)) := \{f \in \mathcal{O}\left(U_n(K) \cap \mathbb{C}^d\right) \cap \mathrm{C}\left(\overline{U_n(K)} \cap \mathbb{C}^d\right) \mid \|f\|_n := \sup_{z \in \overline{U_n(K)} \cap \mathbb{C}^d} |f(z)| e^{\frac{1}{n}|\mathrm{Re}(z)|} < \infty\}$$

with

$$U_n(K) := \left\{z \in D_d \times \mathbb{R}^d \mid \inf_{w \in K} \rho_d(z,w) < \frac{1}{n}\right\}$$

where ρ_d denotes the canonical metric on $D_d \times \mathbb{R}^d$. Further, let

$$\tilde{\mathcal{P}}_\Delta(K) := \lim_{\substack{\longrightarrow\\ n \in \mathbb{N}}} \tilde{\mathcal{P}}_\Delta(V_n(K)),$$

where

$$\tilde{\mathcal{P}}_\Delta(V_n(K)) := \{f \in \mathrm{C}^\infty\left(V_n(K) \cap \mathbb{R}^{d+1}\right) \mid \Delta f = 0,\ f(x,y) = f(x,-y),\ |||f|||_n < \infty\}$$

with

$$|||f|||_n := \sup_{(x,y) \in \overline{V_n(K)} \cap \mathbb{R}^{d+1}} |f(x,y)| e^{\frac{1}{n}|x|}$$

and

$$V_n(K) := \left\{(x,y) \in D_d \times \mathbb{R} \mid \inf_{w \in K} \rho((x,y), w) < \frac{1}{n}\right\},$$

denote the rapidly decreasing harmonic germs near K which are even with respect to the last variable.
In order to gain counterparts of [13, Lemma 5.1, p. 1118] (see also [12, Proposition 2.3, p. 44]),

[13, Theorem 5.3, p. 1119] resp. Theorem 4.1 (see also [1, Satz 2, p. 376]), one faces the open problems:

7.1 Problem. Let $K \subset D^d$ be compact and E a complete locally convex space.

(a) Are $\mathcal{P}_*(K)$ and $\tilde{\mathcal{P}}_\Delta(K)$ topologically isomorphic?

(b) Are $\tilde{\mathcal{E}}^{exp}_\Delta((D_d \times \mathbb{R}) \setminus K, E)/\tilde{\mathcal{E}}^{exp}_\Delta(D_d \times \mathbb{R}, E)$ and $L_b(\tilde{\mathcal{P}}_\Delta(K), E)$ topologically isomorphic?

One difficulty in answering the second question is that it still lacks in a fundamental solution of Δ_{d+1} with the right growth conditions.
A different idea would be to represent Fourier hyperfunctions as boundary values of solutions of the heat equation (see [29], [42], [43], [44] and [30]).

Zusammenfassung und Ausblick

Wie gesehen, ist eine vernünftige Theorie E-wertiger Fourier Hyperfunktionen möglich, wenn E streng zulässig (strictly admissible) ist, d.h. wenn der Cauchy-Riemann Operator

$$\overline{\partial}\colon \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right) \to \mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$$

für jede kompakte Menge $K \subset \overline{\mathbb{R}}$ surjektiv ist und außerdem

$$\overline{\partial}\colon \mathrm{C}^{\infty}(\Omega, E) \to \mathrm{C}^{\infty}(\Omega, E)$$

für jede offene Menge $\Omega \subset \mathbb{C}$ surjektiv ist. Im ersten Schritt wurde dieses Problem für den skalarwertigen Fall gelöst, indem Hörmanders Lösung des gewichteten $\overline{\partial}$-Problems mit dem Mittag-Leffler Verfahren kombiniert wurde (Theorem 5.16). Dank der Darstellung der Räume $\mathcal{E}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ und $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K, E\right)$ mittels Tensorprodukten (Theorem 3.11) gilt das entsprechende Resultat auch für Fréchet-Räume E (Theorem 5.17 bzw. Junker, [26]). Um es durch die Splitting-Theorie von Vogt bzw. die von Bonet und Domański über die Klasse der Fréchet-Räume hinaus auszuweiten, war es notwendig zu zeigen, dass der Raum $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)$ die Eigenschaft (Ω) für jede kompakte Menge $K \subset \overline{\mathbb{R}}$ besitzt. Im Fall $K = \varnothing$ wurde dazu ein Zerlegungsresultat von Langenbruch verwendet (Theorem 5.20) und durch die Kombination mit einer zwischen den Räumen $\mathcal{O}^{exp}\left(\overline{\mathbb{C}} \setminus K\right)/\mathcal{O}^{exp}\left(\overline{\mathbb{C}}\right)$ und $\mathcal{P}_*(K)'_b$ bewiesenen Dualität (Theorem 4.1) erhielt man die allgemeine Aussage (Theorem 5.22). Vermöge der Bedingung der strengen Zulässigkeit von E konnte somit die Theorie der vektorwertigen Fourier Hyperfunktionen (in einer Variablen) weit über die Klasse der Fréchet-Räume hinaus erweitert werden und sie werden einerseits als von Äquivalenzklassen E-wertiger $\mathcal{P}_*$-Funktionale mit kompaktem Träger erzeugte Garbe dargestellt und andererseits als Randwerte E-wertiger langsam wachsender holomorpher Funktionen (Theorem 5.24 and Theorem 6.11). Desweiteren wurden natürliche Grenzen dieser Art von Theorie in der Klasse der ultrabornolgischen PLS-Räume gefunden, nämlich, wenn E ein ultrabornologischer PLS-Raum ist, dann ist eine vernünftige Theorie E-wertiger Fourier Hyperfunktionen in einer Variablen genau dann möglich, wenn E die Eigenschaft (PA) hat (Theorem 6.14). Für viele Standardräume der Analysis ist bekannt, ob sie (PA) haben oder nicht, insbesondere hat jeder Fréchet-Schwartz Raum (PA) (Example 5.26 and Example 5.27).

Offensichtlich stellt sich die Frage, ob eine solche Theorie auch in mehreren Variablen möglich ist. Aufgrund der Ergebnisse von Junker (siehe [26, Kapitel 3, S. 32-46]) wissen wir, dass eine vernünftige Theorie Fréchet-wertiger Fourier Hyperfunktionen auch in mehreren Variablen möglich ist. Domański und Langenbruch gelang es, eine E-wertige Garbe von Hyperfunktionen in d Variablen unter der Annahme, dass der $(d+1)$-dimensionale Laplace Operator

$$\Delta_{d+1}\colon \mathrm{C}^{\infty}(\Omega, E) \to \mathrm{C}^{\infty}(\Omega, E)$$

für jede offene Menge $\Omega \subset \mathbb{R}^{d+1}$ surjektiv ist, zu konstruieren. Vielleicht ist es möglich, eine sinnvolle Theorie E-wertiger Fourier Hyperfunktionen in d Variablen zu erschaffen, wenn wir zusätzlich annehmen, dass

$$\Delta_{d+1}\colon \mathcal{E}^{exp}((D_d\times\mathbb{R})\setminus K,E)\to\mathcal{E}^{exp}((D_d\times\mathbb{R})\setminus K,E)$$

für jede kompakte Menge $K\subset D_d$ surjektiv ist, wobei D_d die radiale Kompaktifizierung von $\mathbb{R}^d$ bezeichne und, wenn wir Punkte $\zeta\in\mathbb{R}^{d+1}$ schreiben als $\zeta=(x,y)\in\mathbb{R}^d\times\mathbb{R}$,

$$\begin{aligned}&\mathcal{E}^{exp}((D_d\times\mathbb{R})\setminus K,E)\\ &:=\left\{f\in\mathrm{C}^\infty\left([(D_d\times\mathbb{R})\setminus K]\cap\mathbb{R}^{d+1},E\right)\mid\forall\,\alpha\in A,\,n\in\mathbb{N},\,m\in\mathbb{N}_0:\,r_{n,m,\alpha}(f)<\infty\right\},\end{aligned}$$

wobei $(p_\alpha)_{\alpha\in A}$ ein Fundamentalsystem von Halbnormen auf E sei, und

$$r_{n,m,\alpha}(f):=\sup_{\substack{(x,y)\in R_n(K)\cap\mathbb{R}^{d+1},\\ \beta\in\mathbb{N}_0^{d+1},\,|\beta|\le m}}p_\alpha\left(\partial^\beta f(x,y)\right)e^{-\frac{1}{n}|x|}$$

sowie

$$R_n(K):=\left\{(x,y)\in(D_d\times\mathbb{R})\setminus K\mid |y|<n\text{ and }\inf_{w\in K}\rho((x,y),w)>\frac{1}{n}\right\},$$

wobei ρ die kanonische Metrik auf $D_d\times\mathbb{R}$ bezeichne. Dem Vorgehen in [13] folgend, bezeichne

$$\tilde{\mathcal{E}}^{exp}_{\Delta}((D_d\times\mathbb{R})\setminus K,E):=\{f\in\mathcal{E}^{exp}((D_d\times\mathbb{R})\setminus K,E)\mid\Delta f=0,\ f(x,y)=f(x,-y)\}$$

die E-wertigen langsam wachsenden harmonischen Funktionen außerhalb von K, welche gerade bezüglich der letzten Variablen sind, und

$$\mathcal{P}_*(K):=\lim_{\substack{\longrightarrow\\ n\in\mathbb{N}}}\operatorname{ind}\mathcal{O}(U_n(K))$$

seien die schnell fallenden holomorphen Keime nahe K, wobei

$$\mathcal{O}(U_n(K)):=\{f\in\mathcal{O}\left(U_n(K)\cap\mathbb{C}^d\right)\cap\mathrm{C}\left(\overline{U_n(K)}\cap\mathbb{C}^d\right)\mid\|f\|_n:=\sup_{z\in\overline{U_n(K)}\cap\mathbb{C}^d}|f(z)|e^{\frac{1}{n}|\mathrm{Re}(z)|}<\infty\}$$

mit

$$U_n(K):=\left\{z\in D_d\times\mathbb{R}^d\mid\inf_{w\in K}\rho_d(z,w)<\frac{1}{n}\right\}$$

sei und ρ_d die kanonische Metrik auf $D_d\times\mathbb{R}^d$ bezeichne. Weiter bezeichne

$$\tilde{\mathcal{P}}_\Delta(K):=\lim_{\substack{\longrightarrow\\ n\in\mathbb{N}}}\operatorname{ind}\tilde{\mathcal{P}}_\Delta(V_n(K)),$$

wobei

$$\tilde{\mathcal{P}}_\Delta(V_n(K)):=\{f\in\mathrm{C}^\infty\left(V_n(K)\cap\mathbb{R}^{d+1}\right)\mid\Delta f=0,\ f(x,y)=f(x,-y),\ |\!|\!|f|\!|\!|_n<\infty\}$$

mit

$$|\!|\!|f|\!|\!|_n:=\sup_{(x,y)\in\overline{V_n(K)}\cap\mathbb{R}^{d+1}}|f(x,y)|e^{\frac{1}{n}|x|}$$

und

$$V_n(K) := \left\{(x,y) \in D_d \times \mathbb{R} \mid \inf_{w \in K} \rho((x,y),w) < \frac{1}{n}\right\},$$

den Raum der schnell fallenden harmonischen Keime nahe K, welche gerade bezüglich der letzen Variablen sind.
Beim Versuch Gegenstücke zu [13, Lemma 5.1, S. 1118] (siehe auch [12, Proposition 2.3, S. 44]), [13, Theorem 5.3, S. 1119] bzw. Theorem 4.1 (siehe auch [1, Satz 2, S. 376]) zu gewinnen, wird man mit folgendem offenen Problem konfrontiert:

7.2 Problem. Sei $K \subset D^d$ kompakt und E ein vollständiger lokal konvexer Raum.

(a) Sind $\mathcal{P}_*(K)$ und $\tilde{\mathcal{P}}_\Delta(K)$ topologisch isomorph?

(b) Sind $\tilde{\mathcal{E}}^{exp}_\Delta((D_d \times \mathbb{R}) \setminus K, E) / \tilde{\mathcal{E}}^{exp}_\Delta(D_d \times \mathbb{R}, E)$ und $L_b(\tilde{\mathcal{P}}_\Delta(K), E)$ topologisch isomorph?

Eine Schwierigkeit in der Beantwortung der zweiten Frage liegt darin, dass es momentan noch an einer Fundamentallösung von Δ_{d+1} mit den richtigen Wachstumseigenschaften mangelt.
Eine andere Idee wäre es, Fourier Hyperfunktionen als Randwerte von Lösungen der Wärmeleitungsgleichung darzustellen (siehe [29], [42], [43], [44] und [30]).

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