Vector Valued Bohnenblust-Hille Inequalities

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angenomme Dissertation

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Abstract

Bohnenblust and Hille stated in their famous $\frac{2M}{M+1}$-inequality that for each $M, N \in \mathbb{N}$ and every M-linear mapping $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{C}$ we have

$$\left(\sum_{i_1,\ldots,i_M=1}^{N} |A(e_{i_1}, \ldots, e_{i_M})|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq \sqrt{2}^{M-1} \|A\|$$

and that this exponent $\frac{2M}{M+1}$ is optimal. Since every M-homogeneous polynomial $P : \ell_\infty^N \to \mathbb{C}$ can be uniquely defined by a symmetric M-linear mapping $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{C}$ through $P(z) = A(z, \ldots, z)$, an analogues inequality for the coefficients of M-homogeneous polynomials $P : \ell_\infty^N \to \mathbb{C}$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$ follows directly via ploarization from the upper multilinear inequality. Bohnenblust and Hille originally designed this polynomial inequality to solve Bohr's absolute convergence problem, asking for the maximal width S of the strip on which a Dirichlet series converges uniformly but not absolutely. Bohr, Bohnenblust and Hille showed that S equals $\frac{1}{2}$.

In this work we study the multilinear as well as the polynomial Bohnenblust-Hille inequality in vector valued settings and we will see that this inequality has deep applications in various fields of analysis. For example, for several sets $\mathcal{A}$ of M-linear mappings $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to X$ we give an answer to the question of the existence of an exponent $1 \leq r < \infty$ and a constant $C > 0$ (both depending on M and $\mathcal{A}$ but not on N) such that for every $A \in \mathcal{A}$ we have

$$\left(\sum_{i_1,\ldots,i_M=1}^{N} \|A(e_{i_1}, \ldots, e_{i_M})\|^r \right)^{\frac{1}{r}} \leq C\|A\|.$$

Bohr himself provided the upper estimate $S \leq \frac{1}{2}$ for the width of his strip of uniform but not absolute convergence. His essential idea was to translate his problem concerning Dirichlet series into a problem about power series in infinitely many variables. But he was not able to give a lower estimate for S, since for this he needed a better understanding of power series in infinitely many variables. So he started to study power series in one variable and got as a byproduct of his effort what we now call Bohr's power series theorem. For every bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$ we have

$$\sum_{n=1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \leq \sup_{z \in \mathbb{D}} |f(z)|$$

and the valued $\frac{1}{3}$ is optimal. In this work we analyze Bohr's power series theorem for holomorphic functions of the N-dimensional unit disc $\mathbb{D}^N$ with values in a Banach space X. Here our vector valued Bohnenblust-Hille inequalities play a fundamental role.

Another phenomenon which is closely related to Bohr's strip is an observation of Maurizi and Queffélec which says that the maximal width S of Bohr's strip

equals the infimum taken over all $\sigma \geq 1$ for which there exists a constant $C > 0$ such that for each choice of $a_1, \ldots, a_N \in \mathbb{C}$ we have

$$\sum_{n=1}^{N} |a_n| \leq N^{\sigma} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

In this work we study such inequalities for finite Dirichlet polynomials in Banach spaces and again our vector valued Bohnenblust-Hille inequalities will be one of our main tools.

Zusammenfassung

Die bedeutende $\frac{2M}{M+1}$-Ungleichung von Bohnenblust und Hille besagt, dass für alle $M, N \in \mathbb{N}$ und alle M-linearen Abbildungen $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{C}$ gilt

$$\left(\sum_{i_1,\ldots,i_M=1}^{N} |A(e_{i_1}, \ldots, e_{i_M})|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq \sqrt{2}^{M-1} \|A\|,$$

wobei der Exponent $\frac{2M}{M+1}$ optimal ist. Da jedes M-homogene Polynom $P : \ell_\infty^N \to \mathbb{C}$ eindeutig über eine symmetrische M-lineare Abbildung $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{C}$ durch $P(z) = A(z, \ldots, z)$ definiert ist, lässt sich mittels Polarisation sofort eine analoge Ungleichung für die Monomialkoeffiezienten M-homogener Polynome $P : \ell_\infty^N \to \mathbb{C}$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$, herleiten. Diese polynomielle Ungleichung wurde ursprünglich von Bohnenblust und Hille dazu benutzt Bohrs absolutes Konvergenzproblem zu lösen, das nach der maximalen Breite S des Streifens der unbedingten aber nicht absoluten Konvergenz einer Dirichletreihe fragt. Bohr, Bohnenblust und Hille zeigten, dass $S = \frac{1}{2}$ ist.

In dieser Arbeit untersuchen wir die Bohnenblust-Hille-Ungleichung sowohl für multilineare Abbildungen als auch für M-homogene Polynome mit Werten in Banachräumen und wir werden sehen, dass diese Ungleichung in unterschiedlichen Gebieten der Analysis zur Anwendung kommt. Zum Beispiel untersuchen wir gegebene Mengen $\mathcal{A}$ von M-linearen Abbildungen $A : \ell_\infty^N \times \ldots \times \ell_\infty^N \to X$ auf die Existenz eines Exponenten $1 \leq r < \infty$ und einer Konstanten $C > 0$ (beide in Abhängigkeit von $\mathcal{A}$ und M aber unabhängig von N), sodass für alle $A \in \mathcal{A}$ gilt

$$\left(\sum_{i_1,\ldots,i_M=1}^{N} \|A(e_{i_1}, \ldots, e_{i_M})\|^r \right)^{\frac{1}{r}} \leq C\|A\|.$$

Bohr selbst lieferte bereits die obere Abschätzung $S \leq \frac{1}{2}$ der Breite des Streifens von unbedingter aber nicht absoluter Konvergenz. Hierzu formulierte er das Problem über Dirichletreihen in eines über Potenzreihen unendlich vieler Variablen um. Jedoch war er nicht in der Lage eine untere Abschätzung für S zu geben, da er dazu ein tieferes Verständnis von Potenzreihen unendlich vieler Variablen benötigte als es seinerzeit möglich war. So begann er zunächst mit dem Studium von Potenzreihen einer Variablen und bekam als Nebenprodukt seiner Bemühungen was heute als Bohrs Potenzreihensatz bekannt ist. Für jede beschränkte holomorphe Funktion $f : \mathbb{D} \to \mathbb{C}$ gilt

$$\sum_{n=1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \leq \sup_{z \in \mathbb{D}} |f(z)|$$

und der Wert $\frac{1}{3}$ ist optimal. In dieser Arbeit werden wir Bohrs Potenzreihensatz für holomorphe Funktionen auf der N-dimensionalen Einheitsschreibe $\mathbb{D}^N$ mit Werten in einem Banachraum X untersuchen. Hier spielen unsere vektorwertigen Bohnenblust-Hille-Ungleichungen eine fundamentale Rolle.

Ein weiteres Phänomen, das im engen Zusammenhang zu Bohrs Streifen steht, ist eine Beobachtung von Maurizi und Queffélec. Diese besagt, dass die maximale Breite S von Bohrs Streifen dem Infimum aller $\sigma \geq 1$ entspricht, für die es eine Konstante $C > 0$ gibt, sodass für alle $a_1, \ldots, a_N \in \mathbb{C}$ gilt

$$\sum_{n=1}^{N} |a_n| \leq N^{\sigma} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

Ein weiteres Ziel dieser Arbeit ist es, solche Ungleichungen für endliche Dirichletpolynome in Banachräumen zu untersuchen. Hierbei sind die vektorwertigen Bohnenblust-Hille-Ungleichungen wieder von zentraler Bedeutung.

An dieser Stelle möchte ich die Gelegenheit nutzen, den Menschen zu danken, die zu dem Gelingen dieser Dissertation beigetragen haben. An erster Stelle gebührt mein Dank Andreas Defant für die herausragende Betreuung meiner Promotion, bei der er mir zu jeder Zeit mit seinem breiten Wissen und zahlreichen Anregungen zur Verfügung stand. Außerdem danke ich Pablo Sevilla Peris für die vielen Ideen und Verbesserungsvorschläge während seiner Zeit hier in Oldenburg wie auch aus der Ferne. Es war mir eine große Freude, in einer stets angenehmen Atmosphäre an unserem gemeinsamen Forschungsprojekt mitwirken zu können.

I like to thank Manuel Maestre and Dumitru Popa who worked with me on the papers which form parts of my thesis.

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Contents

Notations

- $\mathcal{L}_M(X_1, \ldots, X_M; Y)$ is the Banach space of all continuous, M-linear mappings from $\prod_{m=1}^M X_m$ to Y provided with the norm $\|A\| := \sup_{\|x_m\| \le 1} \|A(x_1, \ldots, x_M)\|$. If $X_1 = \ldots = X_M = X$ we write $\mathcal{L}_M(X; Y)$ and $\mathcal{L}_M^s(X; Y)$ for the subspace of all symmetric mappings.
- $\mathcal{P}_M(X; Y)$ is the Banach space of all continuous M-homogeneous polynomials $P : X \to Y$ between normed spaces X, Y provided with the norm $\|P\| = \sup_{\|x\| \le 1} \|P(x)\|$ .... p. 53
- $H(U, Y)$ is the vector space of all holomorphic functions $f : U \to Y$, $H(U) := H(U, \mathbb{C})$ .... Def. 4.1
- $H_\infty(U, Y)$ is the Banach space of all $f \in H(U, Y)$ which are bounded on U provided with the norm $\|f\|_\infty := \sup_{x \in U} \|f(x)\|$ .... Def. 4.1
- $\mathbb{D}^N = \big\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_n| \le 1 \big\}$.
- $\mathbb{T}^N = \big\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_n| = 1 \big\}$, μ^N is the normalized Lebesgue measure on $\mathbb{T}^N$ .... p. 60
- $\mathcal{D}(X)$ is the set of all Dirichlet series in a Banach space X, $\mathcal{D} := \mathcal{D}(\mathbb{C})$. $\mathcal{D}_M(X)$ is the set of all M-homogeneous Dirichlet series in X, $\mathcal{D}_M := \mathcal{D}_M(\mathbb{C})$. p. 69
- $[\mathrm{Re} > \sigma] := \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \sigma \}$
- $\mathcal{H}^\infty(X)$ the Banach space of all Dirichlet series $f(s) = \sum a_n n^{-s}$ in X that are analytic and bounded on $[\mathrm{Re}\, s > 0]$ equipped with the norm $\|f\|_\infty = \sup_{[\mathrm{Re}\, s > 0]} \|f(s)\|$. .... p. 76
- $p = (p_n)$ is the sequence of all prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5 \ldots$
- $\Omega(n)$ is the number of prime factors of n according to their multiplicity. . p. 69
- $\pi(n)$ is the number of prime factors less or equal to n. .... p. 71
- $\mathscr{M}(C, N) = \{ \mathbf{i} = (i_k)_{k \in C} \mid 1 \le i_k \le N \text{ for each } k \in C \}$, $\mathscr{M}(M, N) = \mathscr{M}(\{1, \ldots, M\}, N)$. .... p. 29
- $\complement C := \{1, \ldots, M\} \setminus C$ denotes the complement of a set C in $\{1, \ldots, M\}$.
- $\mathbb{N}_0^{(\mathbb{N})}$ is the set of all multi-indices of finite length,
  $\mathbb{N}_0^N$ is the set of all multi-indices of length N,
  $\mathscr{J}(M, N) = \{ \mathbf{i} \in \mathscr{M}(M, N) \mid i_1 \le \ldots \le i_M \}$,
  $\Lambda(M, N) = \big\{ \alpha \in \mathbb{N}_0^N \mid |\alpha| = M \big\}$, where $|\alpha| = \alpha_1 + \ldots + \alpha_N$. .... p. 53

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- $\cot(X)$ is the optimal cotype of a Banach space X,
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- $K_{p,q}$ is the optimal constant in the Kahane inequality .... (2.18)
- $\mathfrak{K}_p$ is the optimal constant in the Khintchine inequality .... Lem. 2.8
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- $M_p(X)$, $M^p(X)$ are the p-convexity and p-concavity constant of X ... Def. 3.8
- $a_N \prec b_N :\Leftrightarrow \exists c \forall N \in \mathbb{N} : a_N \le c\, b_N$,
$a_N \asymp b_N :\Leftrightarrow a_N \prec b_N$ and $b_N \prec a_N$.
- $o(f)$, $\mathcal{O}(f)$ are the Landau symbols:
$f \in o(g) \Leftrightarrow \forall c > 0 \exists x_o \forall x > x_0 : |f(x)| \le c|g(x)|$.
$f \in \mathcal{O}(f) \Leftrightarrow \exists c > 0 \exists x_o \forall x > x_0 : |f(x)| \le c|g(x)|$.
- $p^* = \frac{p}{p-1}$ is the conjugate of $1 < p < \infty$.

Introduction

A (scalar valued) Dirichlet series is a series of the form

$$\sum_{n\geq 1} a_n \frac{1}{n^s},$$

where the coefficients a_n are in $\mathbb{C}$ and s is a complex variable. There are three different types of convergence concerning Dirichlet series, convergence, absolute convergence, and uniform convergence. Maximal domains where a Dirichlet series $D(s) = \sum a_n n^{-s}$ converges, converges absolutely, or converges uniformly are half-planes $[\mathrm{Re} > \sigma]$, where $\sigma = \sigma_c(D)$, $\sigma_a(D)$, or $\sigma_u(D)$ is defined to be the infimum of all $r \in \mathbb{R}$ such that on $[\mathrm{Re} > r]$ we have convergence of requested type.

Harald Bohr asked 1913 in [13] for the largest possible width of the strip in the complex plane on which a scalar valued Dirichlet series converges uniformly but not absolutely. More precisely he asked for the exact value of the number

$$S = \sup_{D\in\mathcal{D}} \left(\sigma_a(D) - \sigma_u(D)\right),$$

where $\mathcal{D}$ denotes the set of all scalar valued Dirichlet series. Bohr himself was able to show that

$$S \leq \frac{1}{2},$$

but the problem was then to find the exact value of S. He did not even have an example of a Dirichlet series for which the abscissas σ_u and σ_a do not coincide. This problem was open for a long time and finally solved by Bohnenblust and Hille in 1931, who showed that $\frac{1}{2}$ is the exact value of S. So we have what we now call the Bohr-Bohnenblust-Hille Theorem.

Theorem (Bohr-Bohnenblust-Hille)**.** *The maximal width of the strip of uniform but not absolute convergence for scalar valued Dirichlet series is*

$$S = \frac{1}{2}.$$

For the proof Bohnenblust and Hille established in [12, Theorem I] their famous $\frac{2M}{M+1}$-inequality, which is of independent high interest.

Theorem (Bohnenblust-Hille, 1931)**.** *For every $M, N \in \mathbb{N}$ and every continuous M-linear mapping $A \in \mathcal{L}_M(\ell_\infty^N; \mathbb{C})$ we have*

$$\left(\sum_{i\in\mathscr{M}(M,N)} |A(e_{i_1},\ldots,e_{i_M})|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \leq \sqrt{2}^{M-1} M^{\frac{M-1}{2M}} \|A\|$$

and moreover the exponent $\frac{2M}{M+1}$ *is optimal.*

The case $M = 2$ is Littlewood's famous $\frac{4}{3}$-inequality in [57, Theorem 1], and both inequalities have deep application in various fields of analysis (see e.g. [8, 9, 26, 29, 31, 33, 36, 38, 39, 49, 53, 71]). In fact, Bohnenblust and Hille needed a polynomial version of their inequality, which they deduced via polarization from the multilinear Bohnenblust-Hille inequality.

Theorem. *For every* $M \in \mathbb{N}$ *there is a constant* K_M *such that for every* $N \in \mathbb{N}$ *and every* M*-homogeneous polynomial* $P \in \mathcal{P}_M(\ell_\infty^N; \mathbb{C})$, $P(z) = \sum c_\alpha z^\alpha$, *we have*

$$\left(\sum_{\alpha \in \Lambda(M,N)} |c_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq K_M \|P\|$$

and the exponent $\frac{2M}{M+1}$ *is optimal as well.*

Quite recently, Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip gave in [29, Theorem 1] a substantial improvement of the best constant in the polynomial Bohnenblust-Hille inequality, by showing that the inequality is hypercontractive. By this we mean that, if $\mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{pol}}$ denotes the optimal constant in the upper inequality, there is a universal constant C such that $\mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{pol}} \leq C^M$.

This work is divided into two parts. In part I we study vector valued variants of both the multilinear as well as the polynomial Bohnenblust-Hille inequality. In Chapter 2 we study the following question. Let $\mathcal{A}$ be a set of M-linear mappings $A \in \mathcal{L}_M(\ell_\infty^N; X)$ with values in a Banach space X, is there an exponent $1 \leq r < \infty$ and a constant $C > 0$ (both depending only on M and $\mathcal{A}$ and not on N) such that for every $A \in \mathcal{A}$ we have

$$\left(\sum_{\mathbf{i} \in \mathscr{M}(M,N)} \|A(e_{i_1}, \dots, e_{i_M})\|^r \right)^{\frac{1}{r}} \leq C\|A\| \, ? \tag{1}$$

And of course this immediately results in the questions: For a given class $\mathcal{A}$, what are the optimal exponent r and constant $C > 0$ such that (1) holds.

These problems are analyzed in Chapter 2 within the theory of (multiple) summing operators in Banach spaces. We introduce the new notion of *coordinatewise* multiple summing operators and use it to give various vector valued extensions of the multilinear Bohnenblust-Hille inequality. This approach has already appeared in our publication

> [35] *Coordinatewise multiple summing operators in Banach spaces.* A. Defant, D. Popa, and U. Schwarting. J. Funct. Anal., 259(1): 220-242, 2010.

But Chapter 2 also contains some important refinements of the main results given in [35]. Besides these vector valued generalizations of the Bohnenblust-Hille inequality the methods introduced in Chapter 2 will also lead to a fundamental improvement of the optimal constant in the classical Bohnenblust-Hille inequality. This causes a new development in the field of quantum information theory.

In Chapter 3 we study vector valued variants of the polynomial Bohnenblust-Hille inequality. Here we study a similar question to that in the multilinear case. Given a set $\mathcal{B}$ of M-homogeneous polynomials $P \in \mathcal{P}_M(\ell_\infty^N; X)$ with values in a Banach space X, what are the best exponents $1 \le r < \infty$ and constants $C > 0$ (both depending only on M and $\mathcal{B}$ and not on N) such that for every M-homogeneous polynomial $P \in \mathcal{B}$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$ we have

$$\Big(\sum_{\alpha \in \Lambda(M,N)} \|c_\alpha\|^r \Big)^{\frac{1}{r}} \le C\|P\| \,?$$

Among others we show how any result in the multilinear case causes an analogue result for polynomials. And using a different approach Chapter 3 also contains a far reaching vector valued extension of the Defant-Frerick-Ortega-Cerdà-Ounaïes-Seip Theorem mentioned above. The latter is part of our publication

[34] *Bohr radii of vector valued holomorphic functions.* A. Defant, M. Maestre, and U. Schwarting. Adv. Math., 231(5): 2837-2857, 2012.

In part II we discuss two topics which arose from Bohr's absolute convergence problem in vector valued settings. The first one, which is also part of the publication [34], is motivated by Bohr's famous power series theorem.

Theorem (Bohr's power series theorem)**.** *For each bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$ we have that*

$$\sup_{z \in \frac{1}{3}\mathbb{D}} \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} z^n \right| \le \sup_{z \in \mathbb{D}} |f(z)|$$

and the value $\frac{1}{3}$ is optimal.

This motivates the following definition of the N-dimensional Bohr radius, which was first given by Boas and Khavinson in [11]. For $N \in \mathbb{N}$ the Nth Bohr radius K_N is the supremum taken over all $0 \le r \le 1$ such that for each holomorphic function $f \in H(\mathbb{D}^N)$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \mathbb{N}_0^N} |c_\alpha(f) z^\alpha| \le \|f\|_\infty \,,$$

where $c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!}$ are the coefficients of the monomial series expansion of f. The search for the optimal asymptotic of the N-dimensional Bohr radius had a long development started by Dineen and Timoney in [42], continued by Boas an Khavinson [11], by Boas [10] and by Defant and Frerick [27] and finally ended by Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip in [29, Theorem 2]. Using the hypercontractive Bohnenblust-Hille inequality mentioned above they showed that

$$K_N \asymp \sqrt{\frac{\log N}{N}}.$$

In Chapter 5 we study the N-dimensional Bohr radius of vector valued holomorphic mappings. More precisely, given an operator $v : X \to Y$ between Banach spaces and a $\lambda \geq \|v\|$ we ask for the asymptotic behaviour of the numbers $K_N(v, \lambda)$ defined to be the supremum over all $r \geq 0$ such that for all holomorphic functions $f \in H(\mathbb{D}^N, X)$, $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha$, we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \mathbb{N}_0^N} \|v c_\alpha(f) z^\alpha\|_Y \leq \lambda \sup_{z \in \mathbb{D}^N} \Big\| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha \Big\|_X.$$

In Chapter 5 we prove upper and lover estimates for the numbers $K_N(v, \lambda)$ for certain operators $v : x \to Y$. These are for example

- v the identity id_X on any Banach space X.
- v any of the embeddings $\ell_p \hookrightarrow \ell_q$ with $1 \leq p \leq q < \infty$.
- v an arbitrary operator $\ell_1 \to \ell_q$ with $1 \leq q < \infty$.

We show that for finite dimensional Banach spaces X the asymptotic behaviour of $K_N(\mathrm{id}_X, \lambda)$ is exactly as in the scalar case. But for an infinite dimensional Banach space X the logarithmic term disappears and the asymptotic decay of $K_N(\mathrm{id}_X, \lambda)$ depends on the geometry of X. If X is of cotype q then we have with constants only depending on X and λ

$$\frac{1}{N^{1-\frac{1}{q}}} \prec K_N(\mathrm{id}_X, \lambda) \prec \frac{1}{N^{1-\frac{1}{\cot(X)}}}.$$

In particular, if X has no finite cotype then $K_N(\mathrm{id}_X, \lambda) \asymp \frac{1}{N}$. This immediately gives the optimal asymptotic of the N-dimensional Bohr radius of the ℓ_p spaces. With constants only depending on p and λ we have that

$$K_N(\mathrm{id}_{\ell_p}, \lambda) \asymp \frac{1}{N^{1-\frac{1}{\max\{2,p\}}}}.$$

If we now look at the N-dimensional Bohr radius for certain operators $v : X \to Y$ between Banach spaces we see in Chapter 5 that the logarithmic term in some situations turns up again. For example we show that if $1 \leq p < q < \infty$ we have with constants depending only on λ and p, q

$$K_N(\ell_p \hookrightarrow \ell_q, \lambda) \asymp \begin{cases} \sqrt{\frac{\log N}{N}} & \text{if } p < 2 \\ \left(\frac{1}{N}\right)^{1-\frac{1}{p}} & \text{if } p \geq 2. \end{cases}$$

We study the N-dimensional Bohr radius $K_N(v, \lambda)$ in a more general setting within the theory of $(r, 1)$-summing operators $v : X \to Y$ between Banach spaces. Here the vector valued extension of the hypercontractive Bohnenblust-Hille inequality of Part I will play a fundamental role.

The second topic of Part II is motivated by an observation of Maurizi and Queffélec on Bohr's strip S of uniform but not absolute convergence for Dirichlet series. They showed in [60, Theorem 2.4] that the maximal width S of Bohr's strip equals the infimum of all $\sigma \geq 0$ for which there exists a constant $C > 0$ such that for all N and all $a_1, \ldots, a_N \in \mathbb{C}$ we have

$$\sum_{n=1}^{N} |a_n| \leq C N^{\sigma} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

This motivates the following definition. Given a natural number N, we define the Nth Queffélec number Q_N to be the best constant $C_N \geq 1$ such that for each choice of $a_1, \ldots, a_N$ in $\mathbb{C}$

$$\sum_{n=1}^{N} |a_n| \leq C_N \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

By using the hypercontractive Bohnenblust-Hille theorem mentioned above Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip gave in [29, Theorem 3] the following optimal lower and upper bound for the Nth Queffélec number Q_N. The result completed a long process started by Queffélec [71] in the mid nineties, continued by Konyagin and Queffélec [53] in 2002 and by de la Bretèche [26] in 2008.

Theorem (Defant-Frerick-Ortega-Ounaïes-Seip)**.**

$$Q_N = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}} + o(1))\sqrt{\log N \log\log N}}}. \tag{2}$$

Chapter 6 discusses the Queffélec numbers of Dirichlet series in Banach spaces and of operators between Banach spaces and is part of the upcoming publication

> [37] *Estimates for finite Dirichlet polynomials in Banach spaces.* A. Defant, U. Schwarting, and P. Sevilla-Peris, preprint, 2013.

More precisely, given a non-zero operator $v : X \to Y$ between Banach spaces, we are interested in the asymptotic behaviour of the numbers $Q_N(v)$ defined to be the best constant $C_N \geq 1$ such that for each choice of $a_1, \ldots, a_N \in X$ we have

$$\sum_{n=1}^{N} \|v a_n\|_Y \leq C_N \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_X.$$

In Chapter 6 we give upper and lower estimates for the numbers $Q_N(v)$ for certain operators $v : X \to Y$ between Banach spaces, which are again for example

- v the identity id_X on a Banach space X.
- v any of the embeddings $\ell_p \hookrightarrow \ell_q$ with $1 \leq p \leq q < \infty$
- v an arbitrary operator $\ell_1 \to \ell_q$ with $1 \leq q < \infty$.

The phenomena which appear here are quite similar to the results on the N-dimensional Bohr radii in the preceding Chapter 5. To be a little less vague, we show that in the finite dimensional case the asymptotic behaviour of $Q_N(\mathrm{id}_X)$ is exactly as in the scalar case whereas in the infinite dimensional case the exponential term disappears. If X is an infinite dimensional Banach space and $\varepsilon > 0$ we have with constants only depending on X

$$N^{1-\frac{1}{\cot(X)}} \prec Q_N(\mathrm{id}_X) \prec N^{1-\frac{1}{\cot(X)+\varepsilon}}.$$

And again we will see that for certain operators between Banach spaces $v : X \to Y$ the exponential term shows up again. Just to name one example, we show in Chapter 6 that for any operator $v : \ell_1 \to \ell_p$, $1 \leq p \leq 2$, we have

$$Q_N(v) \leq \frac{\sqrt{N}}{e^{\left(\sqrt{1-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}$$

In Chapter 6 we study the numbers $Q_N(v)$ within the theory of $(r,1)$-summing operators and again the vector valued extension of the hypercontractive Bohnenblust-Hille inequaltiy of Part I will be of fundamental importance.

Part I

Vector Valued Bohnenblust-Hille Inequalities

1. Introduction and Motivation

Bohnenblust and Hille (1931) proved in their ingenious paper [12, Theorem I] their following famous inequality.

Theorem 1.1 (Bohnenblust-Hille, 1931)**.** *For every $M \in \mathbb{N}$ there is a constant $C_M \geq 1$ such that for every $N \in \mathbb{N}$ and every $A \in \mathcal{L}_M(\ell_\infty^N; \mathbb{C})$ we have*

$$\left(\sum_{\mathbf{i} \in \mathscr{M}(M,N)} |A(e_{i_1}, \dots, e_{i_M})|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq C_M \|A\|,$$

and moreover the exponent $\frac{2M}{M+1}$ is optimal.

This inequality was originally designed to solve Bohr's absolute convergence problem within the theory of Dirichlet series, but it turned out to be of independent high interest with deep applications in various fields of analysis (see e.g. [8, 9, 26, 29, 31, 33, 36, 38, 39, 49, 53, 71]). Actually, the Bohnenblust-Hille inequality was long forgotten and rediscovered more than forty years later by Davie [25, Section 2] and Kaijser [51, Lemma (1.1)]. The proof in [25] and [51] is slightly different from the original one and delivers a better constant than the one given by Bohnenblust and Hille in [12] (see below).

More recently, Defant and Sevilla-Peris began in [38] to study and generalize the Bohnenblust-Hille inequality within the setting of multiple summing M-linear mappings between Banach spaces. This work was continued and improved in the paper

> [35] *Coordinatewise multiple summing operators in Banach spaces.* A. Defant, D. Popa, and U. Schwarting. J. Funct. Anal., 259(1): 220-242, 2010.

which will be presented here. In [35] we introduce the new notion of coordinatewise multiple summing operators between Banach spaces and give a far reaching vector valued extension of the Bohnenblust-Hille Theorem. This new approach refines even the proof of the scalar case and aroused interest in the (optimal) constant of the classical Bohnenblust-Hille inequality. Here and from now on we denote the optimal constants in the Bohnenblust-Hille inequality by $\mathrm{B}^{\mathrm{mult}}_{\mathbb{R},\mathrm{M}}$, $\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}}$ and $\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}}$ respectively (note that Theorem 1.1 also holds for every $A \in \mathcal{L}_M(\ell_\infty^N; \mathbb{R})$). Until recently the best known estimates for the Bohnenblust-Hille constant depended exponentially on M:

$$\begin{aligned} \mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} &\leq M^{\frac{M+1}{2M}} 2^{\frac{M-1}{2}} \quad \text{(Bohnenblust-Hille [12], 1931)} \\ \mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} &\leq 2^{\frac{M-1}{2}} \quad \text{(Davie [25] and Kaijser [51], 1970s)} \\ \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} &\leq \left(2/\sqrt{\pi}\right)^{M-1} \quad \text{(Queffélec [71], 1995)} \end{aligned} \tag{1.1}$$

Inspired by the ideas of [35] the authors in [66], [43] and [63] were able to proof a polynomial dependence of $\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}}$ on M, in particular Nuñez-Alarcón, Pellegrino, Seoane-Sepúlveda and Serrano-Rodríguez showed in [63, Corollary 8.5 and Section 9] that

$$\begin{aligned} \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},\mathrm{M}} &\leq 1.65(M-1)^{0.516322} + 0.13 \quad \text{(NPSS [63], 2013)} \\ \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} &\leq 1.41(M-1)^{0.304975} - 0.04 \quad \text{(NPSS [63], 2013).} \end{aligned} \tag{1.2}$$

In [61, Theorem 5] Montanaro shows that this new estimate has an application in the field of quantum information theory (see Section 2.2.5).

In Chapter 2 we present the ideas of our publication [35] and motivated by these ground-breaking results in the scalar case we are now able to improve the main results of [35] with refined constants. In order to explain the ideas in Chapter 2 first of all we recall the classical definition of $(r,1)$-summing (linear) operators. We denote the weak ℓ_1-norm of N vectors $x(1),\ldots,x(N)$ in a Banach space X by

$$\mathsf{w}_1(x) := \mathsf{w}_1\left((x(n))_{n=1}^N\right) = \sup_{x'\in B_{X'}} \sum_{n=1}^N |x'(x(n))| .$$

Definition 1.2 ($(r,1)$-summing)**.** Let X and Y be Banach spaces and $1 \leq r < \infty$. An operator $u \in \mathcal{L}(X;Y)$ is said to be $(r,1)$-summing if there exists a constant $C > 0$ such that for any choice of finitely many vectors $x_1,\ldots,x_N \in X$ we have

$$\left(\sum_{n=1}^N \|u(x_n)\|^r\right)^{\frac{1}{r}} \leq C\mathsf{w}_1\left((x_n)_{n=1}^N\right).$$

By $\Pi_{r,1}(X;Y)$ we denote the set of all $(r,1)$-summing operators in $\mathcal{L}(X;Y)$, which becomes a Banach space under the norm $\pi_{r,1}(u) := \inf C$, the infimum taken over all possible constants in the upper inequality.

The theory of absolutely summing operators plays a fundamental role in Banach space theory (see e.g. [40] for a detailed view on this topic). The $(1,1)$-summing operators are called absolutely summing and were first introduced by Alexandre Grothendieck. In this language his famous inequality [46, Théorème 1, §4.2] says that every bounded linear operator $u : \ell_1 \to \ell_2$ is absolutely summing. Littlewood's $\frac{4}{3}$-inequality implies that each canonical embedding $\ell_1 \hookrightarrow \ell_{4/3}$ is $(\frac{4}{3},1)$-summing, therefore, it can be seen as a forerunner of Grothendieck's inequality. There are many extensions of these results to the general ℓ_p case. The most important ones are due to Kwapień [54, §1] and Bennett-Carl (independently proved in [6] and [22, §2]).

Theorem 1.3 (Kwapień, 1968)**.** *For $1 \leq p \leq \infty$, every bounded linear operator $u : \ell_1 \to \ell_p$ is $(r,1)$-summing, where the optimal r is defined by*

$$\frac{1}{r} = 1 - \left|\frac{1}{p} - \frac{1}{2}\right| .$$

Theorem 1.4 (Bennett-Carl, 1973/74)**.** *For $1 \leq p \leq q \leq \infty$, the inclusion $\ell_p \hookrightarrow \ell_q$ is $(r,1)$-summing, where the optimal r is given by*

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \max\left\{\frac{1}{2}, \frac{1}{q}\right\}.$$

The following definition of multiple summing multilinear operators was first independently given by Bombal, Pérez-García and Villanueva in [16, Definition 2.1] and Matos in [58, Definition 2.2]. The case $M = 1$ is again the classical definition of $(r,1)$-summing operators.

Definition 1.5 (multiple $(r,1)$-summing)**.** Let $X_1, \ldots, X_M$ and Y be Banach spaces and $1 \leq r < \infty$. $A \in \mathcal{L}_M(X_1, \ldots, X_M; Y)$ is said to be multiple $(r,1)$-summing if there is a constant C such that for any choice of finitely many vectors $(x_m(i_m))_{i_m=1}^{N_m}$ in X_m, $1 \leq m \leq M$, we have

$$\left(\sum_{i_1,\ldots,i_M=1}^{N_1,\ldots,N_M} \|A\left(x_1(i_1), \ldots, x_M(i_M)\right)\|_Y^r\right)^{\frac{1}{r}} \leq C \prod_{m=1}^{M} \mathsf{w}_1(x_m).$$

In that case we define the multiple $(r,1)$-summing norm $\pi_{r,1}^{mult}(A)$ of A to be the infimum of all such constants in the upper inequality. It is easily seen that the class $\Pi_{r,1}^{mult}(X_1, \ldots, X_M; Y)$ of all multiple $(r,1)$-summing operators with the norm $\pi_{r,1}^{mult}$ is a Banach space.

Let us explain the connection between Bohnenblust-Hille type inequalities and multiple summing operators. If we define for N vectors $x(1), \ldots, x(N) \in X$ the linear operator $T : \ell_\infty^N \to X$ by $T(e_n) = x(n)$ we have that $\|T\| = \mathsf{w}_1\left((x(n))_{n=1}^N\right)$. This allows us the following reformulation of the Bohnenblust-Hille inequality. Given Banach spaces $X_1, \ldots, X_M$ and $A \in \mathcal{L}_M(X_1, \ldots, X_M; \mathbb{C})$, for every choice of finitely many vectors $(x_m(i_m))_{i_m=1}^{N_m}$ in X_m, $1 \leq m \leq M$, we have

$$\left(\sum_{i_1,\ldots,i_M=1}^{N_1,\ldots,N_M} |A\left(x_1(i_1), \ldots, x_M(i_M)\right)|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \leq \mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{mult}} \|A\| \prod_{m=1}^{M} \mathsf{w}_1(x_m); \tag{1.3}$$

in short, every M-linear form $A \in \mathcal{L}_M(X_1, \ldots, X_M; \mathbb{C})$ is multiple $\left(\frac{2M}{M+1}, 1\right)$-summing.

For finite dimensional Banach spaces Y every $A \in \mathcal{L}_M(X_1, \ldots, X_M; Y)$ turns out to be still multiple $\left(\frac{2M}{M+1}, 1\right)$-summing. In the infinite dimensional case this exponent is connected to the geometry of the space Y. In this case we need some definitions, that we recall now. We denote by $(r_n)_{n\in\mathbb{N}}$ the sequence of Rademacher functions

$$r_n : [0,1] \to \mathbb{R},$$

defined by

$$r_n(t) := \operatorname{sgn}\left(\sin 2^n \pi t\right). \tag{1.4}$$

Definition 1.6 (cotype)**.** Given $2 \le q < \infty$, a Banach space X is said to be of cotype q if there exists a constant $C > 0$ such that for any choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\Big(\sum_{n=1}^{N} \|x_n\|^q \Big)^{\frac{1}{q}} \le C \Big(\int_0^1 \Big\| \sum_{n=1}^{N} x_n r_n(t) \Big\|^2 dt \Big)^{\frac{1}{2}}.$$

With $C_q(X)$ we denote the optimal constant in the preceding inequality and with $\cot(X)$ the infimum of all q such that X has cotype q. Note that every Banach space X has cotype ∞ and whenever $\cot(X) = \infty$ we denote $\frac{1}{\cot(X)} = 0$.

Example. *It is well-known that*

$$\cot(\ell_p) = \begin{cases} 2 & \text{if } p \le 2, \\ p & \text{if } p \ge 2. \end{cases} \tag{1.5}$$

The following vector valued variant of the Bohnenblust-Hille result was shown by Bombal, Pérez-García and Villanueva in [16, Theorem 3.2].

Theorem 1.7 (Bombal-Pérez-Villanueva, 2004)**.** *Let $X_1, \dots, X_M$ be Banach spaces and Y a cotype q space. Then every $A \in \mathcal{L}_M(X_1, \dots, X_M; Y)$ is multiple $(q,1)$-summing and $\pi_{q,1}^{mult}(A) \le C_q(Y)^M \|A\|$.*

Later, Defant and Sevilla-Peris in [38, Theorem 1] gave a multilinear extension of the Bennett-Carl Theorem 1.4.

Theorem 1.8 (Defant-Sevilla, 2009)**.** *For $1 \le p \le q \le \infty$ the optimal r such that the composition $I \circ A$ of the canonical embedding $I : \ell_p \hookrightarrow \ell_q$ with any multilinear $A \in \mathcal{L}_M(X_1, \dots, X_M; \ell_p)$ is multiple $(r,1)$-summing is given by*

$$r = \begin{cases} \frac{2M}{M + 2(\frac{1}{p} - \max\{\frac{1}{q}, \frac{1}{2}\})} & \text{if } p \le 2 \\ p & \text{if } p \ge 2. \end{cases} \tag{1.6}$$

Our main result of Chapter 2 unifies and extends all above scalar valued and vector valued results in terms of *coordinatewise* multiple summing operators. In order to give a first impression we mention a direct consequence of this main result.

Corollary 1.9. *For any Banach space Y of cotype q and $1 \le r \le q < \infty$ every multilinear mapping $A \in \mathcal{L}_M(X_1, \dots, X_M; Y)$ is multiple $\Big(\frac{qrM}{q+(M-1)r}, 1\Big)$-summing whenever A is separately $(r,1)$-summing, i. e. for every $1 \le m \le M$ the restriction of A to X_m is $(r,1)$-summing (fixing all the coordinates outside m).*

This result obviously contains the Bohnenblust-Hille Theorem 1.1 but also the vector valued variant of Bombal, Pérez-García and Villanueva, since by a well known fact due to Maurey (see e. g. [40, 11.17]) every bounded linear operator with values in a cotype q

space is $(q,1)$-summing. Combining the upper theorem with the Bennett-Carl Theorem 1.4 we immediately get the multilinear Bennett-Carl Theorem of Defant and Sevilla-Peris in Theorem 1.8. But Chapter 2 contains a lot more. For example we give a multilinear extension of the Kwapień Theorem 1.3. Namely, we show that for $1 \le p \le \infty$ the composition $T(A_1, \dots, A_K)$ of a K-linear mapping $T \in \mathcal{L}_K(\ell_1; \ell_p)$ with K many M-linear mappings $A_k \in \mathcal{L}_M(\ell_\infty; \ell_1)$ is multiple $(r,1)$-summing with

$$r = \begin{cases} \frac{2M}{M+2-\frac{2}{p}} & \text{if } 1 \le p \le 2, \\ \frac{2M}{\frac{2M}{p}+1} & \text{if } 2 \le p \le \frac{2M}{M-1}, \\ 2 & \text{if } \frac{2M}{M-1} \le p \le \infty. \end{cases} \tag{1.7}$$

Chapter 3 deals with polynomial versions of vector valued Bohnenblust-Hille type inequalities. In order to solve Bohr's absolute convergence problem Bohnenblust and Hille in fact needed and proved the following version of their inequality.

Theorem 1.10 (Bohnenblust-Hille, 1931). *For every M there is a constant C_M such that for every $N \in \mathbb{N}$ and every $P \in \mathcal{P}_M(\ell_\infty^N; \mathbb{C})$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$, we have*

$$\Big(\sum_{\alpha \in \Lambda(M,N)} |c_\alpha|^{\frac{2M}{M+1}} \Big)^{\frac{M+1}{2M}} \le C_M \, \|P\| \, , \tag{1.8}$$

and moreover the exponent $\frac{2M}{M+1}$ is optimal.

Let us denote by $\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},\mathrm{M}}$ the optimal constant in (1.8). Then it is easy to deduce from the multilinear Bohnenblust-Hille inequality (Theorem 1.1) and an estimate of Harris for the polarization constant (see [48, Theorem 1] or (3.2)) that

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},\mathrm{M}} \le (\sqrt{2})^{M-1} \frac{M^{\frac{M}{2}} (M+1)^{\frac{M+1}{2}}}{2^M (M!)^{\frac{M+1}{2M}}}.$$

Using Sawa's Khintchine-type inequality for Steinhaus variables ([72, Theorem B] or Lemma 2.9), Queffélec [71, Theorem III-1] obtained the slightly better estimate

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},\mathrm{M}} \le \left(\frac{2}{\sqrt{\pi}} \right)^{M-1} \frac{M^{\frac{M}{2}} (M+1)^{\frac{M+1}{2}}}{2^M (M!)^{\frac{M+1}{2M}}}. \tag{1.9}$$

Finally Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip proved in [29, Theorem 1] a substantial improvement for the constant $\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},\mathrm{M}}$. Using a different approach they managed to show that the polynomial Bohnenblust-Hille inequality is hypercontractive, this means the following:

Theorem 1.11 (Defant-Frerick-Ortega-Ounaïes-Seip, 2011). *There is a constant $C \ge 1$ such that for every M we have*

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},\mathrm{M}} \le C^M.$$

This improvement founds deep application in the theory of the N-dimensional Bohr radius and in estimates for finite Dirichlet polynomials, and it marks an endpoint of a long development there. This will be discussed in Part II in detail.

In Chapter 3 we take a close look at vector valued variants of the polynomial Bohnenblust-Hille inequality by means of the concept of summing operators. Note that Definition 1.2 can be rephrased as follows. An operator $v : X \to Y$ is $(r, 1)$-summing, $1 \leq r < \infty$, if and only if there is a constant $C > 0$ such that for any N and any choice of N many vectors $x_\alpha \in X$, $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = 1$, we have

$$\Big(\sum_{|\alpha|=1} \|vx_\alpha\|^r \Big)^{\frac{1}{r}} \leq C \sup_{x \in \mathbb{D}^N} \Big\| \sum_{|\alpha|=1} x_\alpha z^\alpha \Big\|.$$

The following main result of Chapter 3 is a far reaching extension of the Defant-Frerick-Ortega-Ounaïes-Seip Theorem and it shows under which additional assumptions on the underling space Y the upper inequality can be extended to sets of indices of order M instead of order 1.

Theorem 1.12. *Let Y be a q-concave Banach lattice with $2 \leq q < \infty$ and $v : X \to Y$ an $(r, 1)$-summing operator with $1 \leq r \leq q$. Then there is a constant $C > 0$ such that for any choice of $x_\alpha \in X$, $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = M$ we have*

$$\Big(\sum_{|\alpha|=M} \|vx_\alpha\|^{\frac{qrM}{q+(M-1)r}} \Big)^{\frac{q+(M-1)r}{qrM}} \leq C^M \sup_{x \in \mathbb{D}^N} \Big\| \sum_{|\alpha|=M} x_\alpha z^\alpha \Big\|.$$

As an immediate consequence of Theorem 1.12 combined with the Bennett-Carl Theorem 1.4 we have the following polynomial version of the Bennett-Carl Theorem, which was originally proved by Defant and Sevilla-Peris in [38, Theorem 4] but now we can show that the received inequality is even hypercontractive.

Theorem 1.13. *Given $1 \leq p \leq q \leq \infty$, there exists a constant $C > 0$ such that for every M, N and every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; \ell_p)$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$ we have*

$$\Big(\sum_{\alpha \in \Lambda(M,N)} \|c_\alpha\|_q^r \Big)^{\frac{1}{r}} \leq C^M \|P\|,$$

where r defined as in 1.6.

This Theorem is contained in Chapter 3 as well as a polynomial version of Kwapień's Theorem 1.3.

Theorem 1.14. *Given $1 \leq p < \infty$, there exists a constant $C > 0$ such that for each composition $QP : \ell_\infty \to \ell_1$, $PQ(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$, of an M-homogeneous polynomial $P : \ell_\infty \to \ell_1$ with an K-homogeneous polynomial $Q : \ell_1 \to \ell_p$ we have*

$$\Big(\sum_{|\alpha|=M} \|c_\alpha\|_p^r \Big)^{\frac{1}{r}} \leq C \|P\| \|Q\|,$$

where r is defined as in 1.7.

2. Coordinatewise Multiple Summing Operators

2.1 Definitions and Notation

For natural numbers M and N and a finite subset $C \subset \mathbb{N}$ we define the index set

$$\mathscr{M}(C,N) = \{\mathbf{i} = (i_k)_{k\in C} \mid 1 \leq i_k \leq N \text{ for each } k \in C\}$$

and for $C = \{1,\dots,M\}$ we abbreviate

$$\mathscr{M}(M,N) = \mathscr{M}(\{1,\dots,M\},N).$$

For two disjoint subsets $C_1, C_2 \subset C$ with $C = C_1 \cup C_2$ and indices $\mathbf{i}^1 \in \mathscr{M}(C_1,N)$, $\mathbf{i}^2 \in \mathscr{M}(C_2,N)$ we define the index $\mathbf{i} = (\mathbf{i}^1,\mathbf{i}^2) \in \mathscr{M}(C,N)$ in the obvious way:

$$i_k = \begin{cases} i_k^1 & \text{if } k \in C_1 \\ i_k^2 & \text{if } k \in C_2. \end{cases}$$

Let $X_1,\dots,X_M$ be Banach spaces and $C \subset \{1,\dots,M\}$ a non-void and proper subset of coordinates, then we abbreviate the cartesian product $\prod_{k\in C} X_k$ by X^C (which endowed with the sup norm is a Banach space). Also, for any $x \in X^C$ we define $\tilde{x} \in X^{\{1,\dots,M\}}$ through

$$\tilde{x}(k) = \begin{cases} x(k) & \text{if } k \in C \\ 0 & \text{if } k \in \complement C, \end{cases}$$

where $\complement C$ denotes the complement of C in $\{1,\dots,M\}$. It is obvious that for any $A \in \mathcal{L}_M(X_1,\dots,X_M;Y)$ the mapping

$$\begin{aligned} A^C : X^{\complement C} &\to \mathcal{L}_{|C|}(X^C;Y) \\ x &\mapsto [y \mapsto A(\tilde{y}+\tilde{x})] \end{aligned}$$

is well defined and multilinear. One should realize that $A^C x$ for each $x \in X^{\complement C}$ is nothing else than the restriction of A to the coordinates of C fixing the coordinates in $\complement C$ through x.

Definition 2.1 (multiple $(r,1)$-summing in the coordinates of C)**.** Given $1 \leq r < \infty$ and a non-void and proper subset of coordinates $C \subset \{1,\dots,M\}$, we say that $A \in \mathcal{L}_M(X_1,\dots,X_M;Y)$ is multiple $(r,1)$-summing in the coordinates of C whenever A^C has its range in $\Pi^{mult}_{r,1}(X^C;Y)$. Moreover, we call A separately $(r,1)$-summing if A is $(r,1)$-summing in all one point subsets of $\{1,\dots,M\}$.

It will be important to note that A is multiple $(r,1)$-summing in the coordinates of C if and only if

$$A^C : X^{\complement C} \to \Pi_{r,1}^{mult}(X^C;Y)$$

is a well defined multilinear operator (which then by a standard close graph argument is also bounded).

We continue with some more notations. Define for $q \geq 2$ the functions

$$\omega : [1,q)^2 \to \mathbb{R}_{\geq 0}, \qquad \omega(x,y) := \frac{q^2(x+y) - 2qxy}{q^2 - xy}$$

and

$$f : [1,q)^2 \to \mathbb{R}_{\geq 0}, \qquad f(x,y) := \frac{q^2 x - qxy}{q^2(x+y) - 2qxy}. \tag{2.1}$$

For $K \in \mathbb{N}$ we define inductively the function

$$\omega_K : [1,q)^K \to \mathbb{R}$$

through $\omega_1(r_1) := r_1$, $\omega_2(r_1,r_2) := \omega(r_1,r_2)$ and for $K \geq 3$

$$\omega_K(r_1,\ldots,r_K) := \omega_2(r_K, \omega_{K-1}(r_1,\ldots,r_{K-1})). \tag{2.2}$$

By abuse of notation we often abbreviate $\omega_K(r_1,\ldots,r_K)$ by ω_K.

Also, we define for each $K \in \mathbb{N}_{\geq 2}$ functions

$$f_K := (f_K^1,\ldots,f_K^K) : [1,q)^K \to \mathbb{R}_{\geq 0}^K;$$

first

$$f_2(r_1,r_2) := (f(r_1,r_2), f(r_2,r_1)),$$

and second inductively the function f_K (a function in the K variables $r_1,\ldots,r_K$) through the function f_{K-1} (a function in the $K-1$ variables $r_1,\ldots,r_{K-1}$) by

$$f_K^k := f_{K-1}^k \cdot f(\omega_{K-1}, r_K), \ 1 \leq k \leq K-1 \tag{2.3}$$

and

$$f_K^K := f(r_K, \omega_{K-1})$$

Properties of ω_K and f_K

Note that for $r_1 = \ldots = r_K = r$ we have

$$\omega_K(r, \ldots, r) = \frac{qrK}{q + (K-1)r} \tag{2.4}$$

and

$$f_K^1 = \ldots = f_K^K = \frac{1}{K}\,. \tag{2.5}$$

For any choice of $(r_1, \ldots, r_K) \in [1, q)^K$ it can be checked easily by induction that

$$f_K^1 + \ldots + f_K^K = 1\,.$$

Furthermore it will be important to note that for $x, y \in [1, q)$

$$\max(x, y) \leq \omega(x, y) < q$$

and hence for any $(r_1, \ldots, r_K) \in [1, q)^K$

$$\max_{1\leq k\leq K} r_k \leq \omega_K(r_1, \ldots, r_K) < q. \tag{2.6}$$

An easy calculation shows that for $x, y, z \in [1, q)$

$$\omega(z, \omega(x, y)) = \omega(\omega(y, z), x) \tag{2.7}$$

and by induction we get that for $1 \leq L < K$ and $(r_1, \ldots, r_K) \in [1, q)^K$

$$\omega_K(r_1, \ldots, r_K) = \omega\left(\omega_L(r_1, \ldots, r_L), \omega_{K-L}(r_{L+1}, \ldots, r_K)\right). \tag{2.8}$$

By (2.8) and the symmetry of ω we have that ω_K is symmetric, i.e. $\omega_K(r_1, \ldots, r_K) = \omega_K(r_{\sigma(1)}, \ldots, r_{\sigma(K)})$for any permutation σ of $\{1, \ldots, K\}$.

One the other hand one can show that for $x, y, z \in [1, q)$

$$f\left(\omega(x, y), z\right) \cdot f\left(y, x\right) = f\left(y, \omega(x, z)\right). \tag{2.9}$$

This gives that for $1 \leq L < K$

$$f_K^k(r_1, \ldots, r_K) = \begin{cases} f_L^k(r_1, \ldots, r_L) \cdot f\left(\omega_L(r_1, \ldots, r_L), \omega_{K-L}(r_{L+1}, \ldots, r_K)\right) & \text{if } k \leq L, \\ f_{K-L}^{k-L}(r_{L+1}, \ldots, r_K) \cdot f\left(\omega_{K-L}(r_{L+1}, \ldots, r_K)\omega_L(r_1, \ldots, r_L)\right) & \text{if } k > L. \end{cases} \tag{2.10}$$

Proof. For $1 \leq i < j \leq K$ we abbreviate $\omega_{j-i+1}(r_i, \ldots, r_j)$ by $\Omega_{i,j}$. Now if $k \leq L$ we have

$$\begin{aligned} f_K^k(r_1, \ldots, r_K) &= f_{K-1}^k(r_1, \ldots, r_{K-1}) f(\Omega_{1,K-1}, r_K) \\ &= f_{K-2}^k(r_1, \ldots, r_{K-2}) f(\Omega_{1,K-2}, r_{K-1}) f(\Omega_{1,K-1}, r_K) \\ &= f_{K-2}^k(r_1, \ldots, r_{K-2}) f(\Omega_{1,K-2}, r_{K-1}) f(\omega(\Omega_{1,K-2}, r_{K-1}), r_K). \end{aligned}$$

By (2.9) we thus have

$$f_K^k(r_1, \dots, r_K) = f_{K-2}^k(r_1, \dots, r_{K-2}) f(\Omega_{1,K-2}, \Omega_{K-1,K}).$$

Inductively we get

$$f_K^k(r_1, \dots, r_K) = f_L^k(r_1, \dots, r_L) f(\Omega_{1,L}, \Omega_{L+1,K}).$$

If $L < k < K$ we have with the same arguments as in the upper case that

$$\begin{aligned} f_K^k(r_1, \dots, r_K) &= f_k^k(r_1, \dots, r_k) f(\Omega_{1,k}, \Omega_{k+1,K}) \\ &= f(r_k, \Omega_{1,k-1}) f(\omega(\Omega_{1,k-1}, r_k), \Omega_{k+1,K}). \end{aligned}$$

By (2.9) this is

$$\begin{aligned} &= f(r_k, \omega_{K-1}(r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_K)) \\ &= f(r_k, \omega(\omega_{K-L-1}(r_{L+1}, \dots, r_{k-1}, r_{k+1}, \dots, r_K), \Omega_{1,L})) \end{aligned}$$

which again by (2.9) equals

$$\begin{aligned} &= f(r_k, \omega_{K-L-1}(r_{L+1}, \dots, r_{k-1}, r_{k+1}, \dots, r_K)) f(\Omega_{L+1,K}, \Omega_{1,L}) \\ &= f(r_k, \Omega_{L+1,k-1}) f(\Omega_{L+1,k}, \Omega_{k+1,K}) f(\Omega_{L+1,K}, \Omega_{1,L}) \\ &= f_{k-L}^{k-L}(r_{L+1}, \dots, r_k) f(\Omega_{L+1,k}, \Omega_{k+1,K}) f(\Omega_{L+1,K}, \Omega_{1,L}) \\ &= f_{K-L} k - L(r_{L+1}, \dots, r_K) f(\Omega_{L+1,K}, \Omega_{1,L}). \end{aligned}$$

And finally

$$\begin{aligned} f_K^K(r_1, \dots, r_K) &= f(r_K, \Omega_{1,K-1}) \\ &= f(r_K, \omega(\Omega_{L+1,K-1}, \Omega_{1,L})). \end{aligned}$$

Using (2.9) we complete the argument

$$\begin{aligned} &= f(r_K, \Omega_{L+1,K-1}) f(\Omega_{L+1,K}, \Omega_{1,L}) \\ &= f_K^K(r_{L+1}, \dots, r_K) f(\Omega_{L+1,K}, \Omega_{1,L}). \end{aligned}$$

□

2.2 A Vector Valued Bohnenblust-Hille Type Theorem

The main theorem of this chapter is the following universal vector valued Bohnenblust-Hille type theorem.

Theorem 2.2. *Let $\{1,\dots,M\}$ be a disjoint union of K non-void subsets C_k, Y a cotype q Banach space and $1 \le r_1,\dots,r_K < q$. Assume that $U \in \mathcal{L}_M(X_1,\dots,X_M;Y)$ is multiple $(r_k,1)$-summing in each set of coordinates C_k, $1 \le k \le K$. Then U is multiple $(\omega_K,1)$-summing and*

$$\pi^{mult}_{\omega_K,1}(U) \le \sigma_K(r_1,\dots,r_K) \prod_{k=1}^{K} \left\| U^{C_k} : X^{\complement C_k} \to \Pi^{mult}_{r_k,1}\left(X^{C_k};Y\right) \right\|^{f^k_K},$$

where $\sigma_K(r_1,\dots,r_K)$ only depends on $K, |C_1|,\dots,|C_K|, r_1,\dots,r_K, q$ and $C_q(Y)$. More precisely, for $1 \le s < q$ and $N \in \mathbb{N}$ let $A^N_{q,s}(Y) = C_q(Y)^N K^N_{s,2}$ be the constant obtained in Lemma 2.4. Then $\sigma_1(r) = 1$ and for K even with $s_1 := \omega_{\frac{K}{2}}(r_1,\dots,r_{\frac{K}{2}})$, $s_2 := \omega_{\frac{K}{2}}(r_{\frac{K}{2}+1},\dots,r_K)$

$$\sigma_K(r_1,\dots,r_K) = \left(A^{|\bigcup_{k=\frac{K}{2}+1}^{K} C_k|}_{q,s_1}(Y) \cdot \sigma_{\frac{K}{2}}(r_1,\dots,r_{\frac{K}{2}}) \right)^{f(s_1,s_2)} \cdot \left(A^{|\bigcup_{k=1}^{\frac{K}{2}} C_k|}_{q,s_2}(Y) \cdot \sigma_{\frac{K}{2}}(r_{\frac{K}{2}+1},\dots,r_K) \right)^{f(s_2,s_1)}, \tag{2.11}$$

for K odd with $s_1 := \omega_{\frac{K-1}{2}}(r_1,\dots,r_{\frac{K-1}{2}})$, $s_2 := \omega_{\frac{K+1}{2}}(r_{\frac{K+1}{2}},\dots,r_K)$

$$\sigma_K(r_1,\dots,r_K) = \left(A^{|\bigcup_{k=\frac{K+1}{2}}^{K} C_k|}_{q,s_1}(Y) \cdot \sigma_{\frac{K-1}{2}}(r_1,\dots,r_{\frac{K-1}{2}}) \right)^{f(s_1,s_2)} \cdot \left(A^{|\bigcup_{k=1}^{\frac{K-1}{2}} C_k|}_{q,s_2}(Y) \cdot \sigma_{\frac{K+1}{2}}(r_{\frac{K+1}{2}},\dots,r_K) \right)^{f(s_2,s_1)}. \tag{2.12}$$

We take the time to make a short historical trip on the proof of the Bohnenblust-Hille inequality. Within the last eighty years there have been two fundamental improvements of the estimate of the Bohnenblust-Hille constant:

$\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} \le M^{\frac{M+1}{2}} 2^{\frac{M-1}{2}}$ (Bohnenblust-Hille [12], 1931)
$\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} \le 2^{\frac{M-1}{2}}$ (Davie [25] and Kaijser [51], 1970s)
$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \le 1.41(M-1)^{0.304975} - 0.04$ (Nuñez-Pellegrino-Seoane-Serrano [63], 2013)
$\mathrm{B}^{\mathrm{mult}}_{\mathbb{R},\mathrm{M}} \le 1.65(M-1)^{0.526322} + 0.13$ ([63], 2013).

In the following we show two different proofs of Theorem 2.2. In the first step we illustrate the ideas with which Kaijser reproved the Bohnenblust-Hille result [51, Lemma (1.1)], which is originally formulated in terms of tensor products. His ideas were for example picked by Defant and Sevilla in [38, Section 3] to give their vector valued variants of the Bohnenblust-Hille inequality. Here we translate the proofs into the language of inequalities and generalize them on a very abstract level. So the first proof of Theorem 2.2, a combination of Lemma 2.3 and Lemma 2.5, can be seen as a far reaching extension of Kaijser's reproof of the original Bohnenblust-Hille inequality. The proof leads to the constants

$$\sigma_K = \prod_{k=1}^{K} A_{q,r_k}^{|\complement C_k|}(Y)^{f_K^k}. \tag{2.13}$$

In section 2.2.4 we take a closer look at the constant $A_{q,r_k}(Y)$ and we will see that in the setting of the classical Bohnenblust-Hille inequality (i.e. $Y = \mathbb{K}$, $K = M$ $r_1 = \ldots = r_M = 1$, $q = 2$) Lemma 2.5 just delivers the old results of Davie and Kaijser $\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} \leq \sqrt{2}^{M-1}$ and of Queffélec [71] $\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \leq (2/\sqrt{\pi})^{M-1}$ mentioned in the introduction.

In the second step we give the proof of Theorem 2.2 which delivers the recursive formula for the constants σ_K given in (2.11) and (2.12). Although we are not able to proof that this recursive formula produces constants that are generally better than the constants (2.13) in Lemma 2.5, we will see that at least in some special cases Theorem 2.2 delivers fundamental better constants (see Theorem 2.10 for the complex case and also Corollary 2.14). The proof of Theorem 2.2 has its origin in our publication [35, Theorem 5.1]. There we formulate the problem in terms of coordinatewise multiple summing operators for the first time, which opened up a new point of view on the Bohnenblust-Hille type inequalities. The fact that even in the scalar case the results of [35] are of worth was first seen by Pellegrino and Seoane-Sepúlveda in [66]. With the methods introduced in [35] Pellegrino and Seoane-Sepúlveda proved in [66] and together with Nuñez-Alarcón in [64] the following recursive formula for the optimal Bohnenblust-Hille constants. In fact, even if not explicitly calculated there, this formula is already a direct consequence of [35, Theorem 4.1] (we show this in Section 2.2.4).

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \leq \begin{cases} (2/\sqrt{\pi})^{M-1} & \text{for } M = 2,3 \\ \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}}{2}} \mathfrak{S}_{\frac{2M}{M+2}}^{\frac{M}{2}} & \text{for } M \text{ even} \\ \left(\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}-1}{2}} \mathfrak{S}_{\frac{2M-2}{M+1}}^{\frac{M+1}{2}}\right)^{\frac{M-1}{2M}} \left(\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}+1}{2}} \mathfrak{S}_{\frac{2M+2}{M+3}}^{\frac{M-1}{2}}\right)^{\frac{M+1}{2M}} & \text{for } M \text{ odd} \end{cases} \tag{2.14}$$

Here $\mathfrak{S}_s$ denotes the optimal constant in the Khintchine inequality for Steinhaus random variables (Lemma 2.9) and using the optimal constant in the classical Khintchine inequality (Lemma 2.8) instead, one gets an analog result for the real valued Bohnenblust-Hille constants $\mathrm{B}^{\mathrm{mult}}_{\mathbb{R},\mathrm{M}}$. This two formulas deliver the new estimates of Nuñez-Alarcón, Pellegrino, Seoane-Sepúlveda and Serrano-Rodríguez in [63] for the optimal Bohnenblust-Hille constant mentioned above (see also [43, 62, 65, 74] for previous versions and [61] for a shorter but weaker proof). In Theorem 2.2 we pick the ideas of Pellegrino and Seoane-Sepúlveda and improve the constants given in [35]. Note that in the case $Y = \mathbb{K}$, $K = M$, $r_1 = \ldots r_K = 1$, $q = 2$ we still receive the recursion formulas of Pellegrino and Seoane-Sepúlveda (see Section 2.2.4).

2.2.1 Two Lemmata

The proof of Theorem 2.2 needs two independently interesting lemmata. The first one is a strong modification of an inequality due to Blei [8, Lemma 5.3]. In the case $r_1 = \ldots = r_K = 1$ and $q = 2$ Blei in fact ascribes this inequality to an advice of Kaijser (in the comments to [8, Lemma 5.3]) and taking a close look to Kaijser's reproof of the Bohnenblust-Hille inequality [51, Lemma 1.1] one can see that this inequality is a key observation in Kaijser's proof. That is why we from now on call it the Blei-Kaijser inequality. Note, that some weaker versions of Lemma 2.3 are already shown in our publications [35, Lemma 3.1] and [34, Lemma 5.1].

Lemma 2.3 (Blei-Kaijser inequality)**.** *Let $C_1, \ldots, C_K$ be K disjoint finite index sets, $C := \bigcup_{k=1}^{K} C_k$ and $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathscr{M}(C,N)}$ a scalar matrix with positive entries. Then we have for every $r_1, \ldots, r_K \geq 1$ and $q > \max\{r_k \mid 1 \leq k \leq K\}$ that*

$$\Bigg(\sum_{\boldsymbol{i}\in\mathscr{M}(C,N)} a_{\boldsymbol{i}}^{\omega_K}\Bigg)^{\frac{1}{\omega_K}} \leq \prod_{k=1}^{K}\Bigg(\sum_{\boldsymbol{i}\in\mathscr{M}(C_k,N)}\Bigg(\sum_{\boldsymbol{j}\in\mathscr{M}(\complement C_k,N)} a_{(\boldsymbol{i},\boldsymbol{j})}^{q}\Bigg)^{\frac{r_k}{q}}\Bigg)^{\frac{f_K^k}{r_k}}, \tag{2.15}$$

where ω_K and f_K^k are defined in (2.2) *and* (2.3)*.*[1]

Proof. We split the exponent ω_K in two summands

$$\begin{aligned}\omega_K = \omega_2(r_K, \omega_{K-1}) &= \frac{q^2(r_K + \omega_{K-1}) - 2q\, r_K\, \omega_{K-1}}{q^2 - r_K\, \omega_{K-1}}\\ &= \frac{q^2 r_K - q\, r_K\, \omega_{K-1}}{q^2 - r_K\, \omega_{K-1}} + \frac{q^2\, \omega_{K-1} - q\, r_K\, \omega_{K-1}}{q^2 - r_K\, \omega_{K-1}}\\ &= f(r_K, \omega_{K-1})\, \omega_K + f(\omega_{K-1}, r_K)\, \omega_K,\end{aligned}$$

(where $\omega_{K-1} = r_1$ if $K = 2$) and write

$$\sum_{\mathbf{i}\in\mathscr{M}(C,N)} a_{\mathbf{i}}^{\omega_K} = \sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)}\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{f(r_K,\omega_{K-1})\,\omega_K}\, a_{(\mathbf{i},\mathbf{j})}^{f(\omega_{K-1},r_K)\,\omega_K}.$$

Applying Hölder's inequality, first to the inner sum with $p = \frac{r_K}{f(r_K,\omega_{K-1})\omega_K} = \frac{q^2 - r_K\omega_{K-1}}{q^2 - q\omega_{K-1}}$ and $p^* = \frac{q}{f(\omega_{K-1},r_K)\omega_K} = \frac{q^2 - r_K\omega_{K-1}}{q\omega_{K-1} - r_K\omega_{K-1}}$, the upper term is majorized by

$$\sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)}\Bigg(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{r_K}\Bigg)^{\frac{f(r_K,\omega_{K-1})\,\omega_K}{r_K}}\Bigg(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{q}\Bigg)^{\frac{f(\omega_{K-1},r_K)\,\omega_K}{q}},$$

[1]**Note added in proof.** The result of Lemma 2.3 was independently shown by D. Popa and G. Sinnamon in [68, Corollary 2.2]. Additionally, [68, Corollary 2.2] provides the following formulas for the exponents ω_K and f_K^k

$$\omega_K = \frac{qR}{1+R} \quad \text{and} \quad f_K^k = \frac{r_k}{R(q - r_k)}, \tag{2.16}$$

where $R = \sum_{k=1}^{K} \frac{r_k}{q - r_k}$.

and second to the outer sum with $p = \frac{q}{f(r_K,\omega_{K-1})\omega_K} = \frac{q^2 - r_K\omega_{K-1}}{qr_K - r_K\omega_{K-1}}$ and $p^* = \frac{\omega_{K-1}}{f(\omega_{K-1},r_K)\omega_K} = \frac{q^2-r_K\omega_{K-1}}{q^2-qr_K}$, then the upper term is majorized by

$$\Bigg(\sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{r_K}\Big)^{\frac{q}{r_K}}\Bigg)^{\frac{f(r_K,\omega_{K-1})\,\omega_K}{q}} \cdot \Bigg(\sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{q}\Big)^{\frac{\omega_{K-1}}{q}}\Bigg)^{\frac{f(\omega_{K-1},r_K)\,\omega_K}{\omega_{K-1}}}.$$

Applying Minkowski's inequality on the first factor we finally get that

$$\sum_{\mathbf{i}\in\mathscr{M}(C,N)} a_{\mathbf{i}}^{\omega_K} \le \Bigg(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)}\Big(\sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{q}\Big)^{\frac{r_K}{q}}\Bigg)^{\frac{f(r_K,\omega_{K-1})\,\omega_K}{r_K}} \cdot \Bigg(\sum_{\mathbf{i}\in\mathscr{M}(\complement C_K,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(C_K,N)} a_{(\mathbf{i},\mathbf{j})}^{q}\Big)^{\frac{\omega_{K-1}}{q}}\Bigg)^{\frac{f(\omega_{K-1},r_K)\,\omega_K}{\omega_{K-1}}}. \tag{2.17}$$

In the case $K = 2$ this is the conclusion is. For $K > 2$ we proceed by induction over K. Assuming that (2.15) holds for any $K-1$, $K \ge 3$, the right factor of (2.17) is then majorized by

$$\left[\prod_{k=1}^{K-1}\Bigg(\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)} a_{(\mathbf{i},\mathbf{j})}^{q}\Big)^{\frac{r_k}{q}}\Bigg)^{\frac{f_{K-1}^k}{r_k}}\right]^{f(\omega_{K-1},r_K)\,\omega_K}$$

which proofs our assertion. □

For the second lemma recall the definitions of the Rademacher functions in (1.4) and note that that by Kahane's inequality (see [50] or [40, 11.1]), if $0 < p, q < \infty$, there is a constant $C > 0$ such that regardless of the choice of a Banach space X and for finitely many vectors $x_1, \ldots, x_N$ in X we have

$$\Big(\int_0^1 \Big\|\sum_{n=1}^N a_n r_n(t)\Big\|^q\Big)^{\frac{1}{q}} \le C\Big(\int_0^1 \Big\|\sum_{n=1}^N a_n r_n(t)\Big\|^p\Big)^{\frac{1}{p}}; \tag{2.18}$$

with $K_{p,q}$ we denote the optimal constant in the upper inequality.

The following lemma is originally shown in [67, Lemma 3] and with a slightly better constant in [35, Lemma 2.2]. Note that an immediate consequence of this lemma is that every bounded $A \in \mathcal{L}_M(X_1, \ldots, X_M; Y)$ with values in a cotype q space Y is multiple $(q,1)$-summing – the Bombal-Pérez-Villanueva Theorem 1.7.

Lemma 2.4. *Let Y be a Banach space of cotype q, $1 \leq r \leq q$ and $(a_i)_{i_1,\dots,i_M=1}^{N_1,\dots,N_M}$ a matrix in Y. Then*

$$\Big(\sum_{i_1,\dots,i_M=1}^{N_1,\dots,N_M} \|a_i\|^q \Big)^{\frac{1}{q}} \leq A_{q,r}^M(Y) \Big(\int_0^1 \dots \int_0^1 \Big\| \sum_{i_1,\dots,i_M=1}^{N_1,\dots,N_M} a_i r_{i_1}(t_1) \cdots r_{i_M}(t_M) \Big\|^r dt_1 \dots dt_M \Big)^{\frac{1}{r}}$$

where $A_{q,r}^M(Y) := C_q(Y)^M K_{r,2}^M$.

Proof. The proof follows by induction in M. The case $M = 1$ is simply the fact that Y is of cotype q and Kahane's inequality (2.18). Let us now assume that the inequality holds for $M-1$. Then the conclusion for M follows by the following chain of inequalities: By induction hypothesis we get

$$\Big(\sum_{i_1,\dots,i_M=1}^{N_1,\dots,N_M} \|a_{\mathbf{i}}\|^q \Big)^{\frac{1}{q}} = \Bigg[\sum_{i_1=1}^{N_1} \Bigg(\Big(\sum_{i_2,\dots i_M=1}^{N_2,\dots N_M} \|a_{\mathbf{i}}\|^q \Big)^{\frac{1}{q}} \Bigg)^q \Bigg]^{\frac{1}{q}}$$

$$\leq A_{q,r}^M(Y) \Bigg(\sum_{i_1=1}^{N_1} \Big(\int_0^1 \dots \int_0^1 \Big\| \sum_{i_2,\dots,i_M=1}^{N_2,\dots,N_M} a_{\mathbf{i}} r_{i_2}(t_2) \cdots r_{i_M}(t_M) \Big\|^r dt_2 \dots dt_M \Big)^{\frac{q}{r}} \Bigg)^{\frac{1}{q}}$$

and by the continuous Minkowski inequality we have that

$$\Bigg(\sum_{i_1=1}^{N_1} \Big(\int_0^1 \dots \int_0^1 \Big\| \sum_{i_2,\dots,i_M=1}^{N_2,\dots,N_M} a_{\mathbf{i}} r_{i_2}(t_2) \cdots r_{i_M}(t_M) \Big\|^r dt_2 \dots dt_M \Big)^{\frac{q}{r}} \Bigg)^{\frac{1}{q}}$$

$$\leq \Bigg(\int_0^1 \dots \int_0^1 \sum_{i_1=1}^{N_1} \Big(\Big\| \sum_{i_2,\dots,i_M=1}^{N_2,\dots,N_M} a_{\mathbf{i}} r_{i_2}(t_2) \cdots r_{i_M}(t_M) \Big\|^q \Big)^{\frac{r}{q}} dt_2 \dots dt_M \Bigg)^{\frac{1}{r}}.$$

But since Y is of cotype q

$$\Big(\sum_{i_1=1}^{N_1} \Big\| \sum_{i_2,\dots,i_M=1}^{N_1,\dots,N_M} a_{\mathbf{i}} r_{i_2}(t_2) \cdots r_{i_M}(t_M) \Big\|^q \Big)^{\frac{1}{q}}$$

$$\leq C_q(Y) \Big(\int_0^1 \Big\| \sum_{i_1=1}^{N_1} r_{i_1}(t_1) \sum_{i_2,\dots,i_M=1}^{N_1,\dots,N_M} a_{\mathbf{i}} r_{i_2}(t_2) \cdots r_{i_M}(t_M) \Big\|^2 dt_1 \Big)^{\frac{1}{2}}$$

and so Kahane's inequality (2.18) and Fubini's theorem finally give the conclusion. □

2.2.2 The Fundamental Lemma

Lemma 2.5. *Let $\{1, \dots, M\}$ be a disjoint union of K non-void subsets C_k, Y a cotype q Banach space and $1 \leq r_1, \dots, r_K < q$. Assume that $U \in \mathcal{L}_M(X_1, \dots, X_M; Y)$ is multiple $(r_k, 1)$-summing in each set of coordinates C_k, $1 \leq k \leq K$. Then U is multiple $(\omega_K, 1)$-summing and*

$$\pi_{\omega_K,1}^{mult}(U) \leq \prod_{k=1}^K \Big(A_{q,r_k}^{|\complement C_k|}(Y) \Big\| U^{C_k} : X^{\complement C_k} \to \Pi_{r_k,1}^{mult} \big(X^{C_k}; Y \big) \Big\| \Big)^{f_K^k},$$

where $A_{q,r_k}^N(Y) = C_q(Y)^N K_{r_k,2}^{N-1}$ *is the constant obtained in Lemma 2.4.*[2]

Proof. For each $1 \le m \le M$ take $(x_m(i_m))_{i_m=1}^N \subset X_m$ with

$$\mathsf{w}_1\left((x_m(i_m))_{i_m=1}^N\right) \le 1.$$

For $\mathbf{i} \in \mathscr{M}(C_k, N)$ and $\mathbf{j} \in \mathscr{M}(\complement C_k, N)$ define $(x(\mathbf{i}), x(\mathbf{j})) = ((x(\mathbf{i}), x(\mathbf{j}))(m))_{m=1}^M \in \prod_{m=1}^M X_m$ by

$$(x(\mathbf{i}), x(\mathbf{j}))(m) := \begin{cases} x_m(i_m) & \text{if } m \in C_k \\ x_m(j_m) & \text{if } m \in \complement C_k. \end{cases}$$

Since $\max(r_1, \dots, r_K) < q$ we obtain from Lemma 2.3 that

$$\left(\sum_{\mathbf{i}\in\mathscr{M}(M,N)} \|U(x_1(i_1),\dots,x_M(i_M))\|_Y^{\omega_K}\right)^{\frac{1}{\omega_K}}$$
$$\le \prod_{k=1}^K \left(\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)} \left(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)} \|U(x(\mathbf{i}), x(\mathbf{j}))\|_Y^q\right)^{\frac{r_k}{q}}\right)^{\frac{f_K^k}{r_k}}.$$

For a fixed $k \in \{1, \dots, K\}$ and fixed $\mathbf{i} \in \mathscr{M}(C_k, N)$ we deduce from Lemma 2.4 that

$$\left(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)} \|U(x(\mathbf{i}), x(\mathbf{j}))\|_Y^q\right)^{\frac{1}{q}}$$
$$\le A_{q,r_k}^{|\complement C_k|}(Y)\left(\int_{[0,1]^{|\complement C_k|}} \Big\|\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)} U(x(\mathbf{i}), x(\mathbf{j})) \prod_{m\in\complement C_k} r_{j_m}(t_m)\Big\|_Y^{r_k} dt\right)^{\frac{1}{r_k}}$$

For each $1 \le m \le M$ we define the random variable

$$R_m : [0,1] \to X_m, \quad R_m(s) = \sum_{i_m=1}^N r_{i_m}(s) x_m(i_m)$$

and for $C \subset \{1, \dots, M\}$ the random vector

$$R_C : [0,1]^{|C|} \to \prod_{m\in C} X_m, \quad R_C(t) = (R_m(t_m))_{m\in C}.$$

As above we define for $\mathbf{i} \in \mathscr{M}(C_k, N)$ the vectors $\left(x(\mathbf{i}), R_{\complement C_k}(t)\right)$ by

$$\left(x(\mathbf{i}), R_{\complement C_k}(t)\right)(m) := \begin{cases} x_m(i_m) & \text{if } m \in C_k \\ R_m(t_m) & \text{if } m \in \complement C_k. \end{cases}$$

[2]**Note added in proof.** The result of Lemma 2.5 was independently show by D. Popa and G. Sinnamon in [68, Theorem 3.2], providing the explicit formulas in (2.16) for the exponents ω_K and f_K^k.

Since U is multilinear we get (remember that k and $\mathbf{i}$ are fixed)

$$\Big(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)}\|U(x(\mathbf{i}),x(\mathbf{j}))\|_Y^q\Big)^{\frac{1}{q}}\le A_{q,r_k}^{|\complement C_k|}(Y)\Big(\int_{[0,1]^{|\complement C_k|}}\big\|U\big(x(\mathbf{i}),R_{\complement C_k}(t)\big)\big\|_Y^{r_k}\,dt\Big)^{\frac{1}{r_k}},$$

and summing over all $\mathbf{i}\in\mathscr{M}(C_k,N)$ we get

$$\begin{aligned}&\Bigg(\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)}\|U(x(\mathbf{i}),x(\mathbf{j}))\|_Y^q\Big)^{\frac{r_k}{q}}\Bigg)^{\frac{1}{r_k}}\\&\le A_{q,r_k}^{|\complement C_k|}(Y)\Big(\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)}\int_{[0,1]^{|\complement C_k|}}\big\|U\big(x(\mathbf{i}),R_{\complement C_k}(t)\big)\big\|_Y^{r_k}\,dt\Big)^{\frac{1}{r_k}}\\&=A_{q,r_k}^{|\complement C_k|}(Y)\Big(\int_{[0,1]^{|\complement C_k|}}\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)}\big\|U\big(x(\mathbf{i}),R_{\complement C_k}(t)\big)\big\|_Y^{r_k}\,dt\Big)^{\frac{1}{r_k}}\end{aligned}\tag{2.19}$$

Now, since by assumption U is multiple $(r_k,1)$-summing in the coordinates of C_k we know that

$$U^{C_k}:X^{\complement C_k}\to\Pi_{r_k,1}^{mult}\big(X^{C_k};Y\big)$$

is well defined and bounded, hence (2.19) is

$$\begin{aligned}&\le A_{q,r_k}^{|\complement C_k|}(Y)\Big(\int_{[0,1]^{|\complement C_k|}}\pi_{r_k,1}^{mult}\big(U(\cdot,R_{\complement C_k}(t))\big)^{r_k}\,dt\Big)^{\frac{1}{r_k}}\\&\le A_{q,r_k}^{|\complement C_k|}(Y)\,\Big\|U^{C_k}:X^{\complement C_k}\to\Pi_{r_k,1}^{mult}(X^{C_k};Y)\Big\|\Big(\int_{[0,1]^{|\complement C_k|}}\big\|R_{\complement C_k}(t)\big\|_{X^{\complement C_k}}^{r_k}\,dt\Big)^{\frac{1}{r_k}}.\end{aligned}$$

But since $\big\|R_{\complement C_k}(t)\big\|_{X^{\complement C_k}}\le\prod_{m\in\complement C_k}\mathsf{w}_1(x_m)\le 1$ we finally get

$$\begin{aligned}&\Bigg(\sum_{\mathbf{i}\in\mathscr{M}(C_k,N)}\Big(\sum_{\mathbf{j}\in\mathscr{M}(\complement C_k,N)}\|U(x(\mathbf{i}),x(\mathbf{j}))\|_Y^q\Big)^{\frac{r_k}{q}}\Bigg)^{\frac{1}{r_k}}\\&\le A_{q,r_k}^{|\complement C_k|}(Y)\,\Big\|U^{C_k}:X^{\complement C_k}\to\Pi_{r_k,1}^{mult}(X^{C_k};Y)\Big\|\end{aligned}$$

All in all we have shown that

$$\begin{aligned}&\Big(\sum_{\mathbf{i}\in\mathscr{M}(M,N)}\|U(x_1(i_1),\dots,x_m(i_m))\|_Y^{\omega_K}\Big)^{\frac{1}{\omega_K}}\\&\le\prod_{k=1}^{K}\Big(A_{q,r_k}^{|\complement C_k|}(Y)\,\Big\|U^{C_k}:X^{\complement C_k}\to\Pi_{r_k,1}^{mult}(X^{C_k};Y)\Big\|\Big)^{f_K^k}.\end{aligned}$$

□

It turns out that the following special case of Lemma 2.5, the result from [35, Theorem 4.1], is the most important one.

Corollary 2.6. *Let Y be a cotype q space and $1 \leq s_1, s_2 \leq q$. Assume that $U \in \mathcal{L}_M(X_1, \ldots, X_M; Y)$ is multiple $(s_1, 1)$-summing in the coordinates of S_1 and multiple $(s_2, 1)$-summing in the coordinates of S_2 where S_1 and S_2 form a partition of non-void subsets of $\{1, \ldots, M\}$. Then U is multiple $(\omega(s_1, s_2), 1)$-summing and*

$$\pi^{mult}_{\omega(s_1,s_2),1}(U) \leq \sigma_2(s_1, s_2) \left\| U^{S_2} : X^{S_1} \to \Pi^{mult}_{s_2,1}\left(X^{S_2}, Y\right) \right\|^{f(s_2,s_1)} \cdot \left\| U^{S_1} : X^{S_2} \to \Pi^{mult}_{s_1,1}\left(X^{S_1}, Y\right) \right\|^{f(s_1,s_2)},$$

where

$$\sigma_2(s_1, s_2) = \left(A^{|S_2|}_{q,s_1}(Y)\right)^{f(s_1,s_2)} \left(A^{|S_1|}_{q,s_2}(Y)\right)^{f(s_2,s_1)}.$$

2.2.3 The Proof of Theorem 2.2

Now we are in the position to give the proof of Theorem 2.2

Proof. The case $K = 2$ is Corollary 2.6. Now we proceed by induction in K. But in contrast to the proof of [35, Theorem 5.1] we deduce the case K from the case $\frac{K}{2}$ if K is even and from the cases $\frac{K+1}{2}$, $\frac{K-1}{2}$ if K is odd.

We start with the case K even. Let us assume that the result is true for any M and any disjoint unit of $K/2$ subsets of coordinates. Given $U \in \mathcal{L}_M(X_1, \ldots, X_M; Y)$ and a disjoint composition $\{1, \ldots, M\} = \bigcup_{k=1}^{K} C_k$ such that U is multiple $(r_k, 1)$-summing in each of the coordinates C_k. We define the sets of coordinates

$$S_1 := \bigcup_{k=1}^{\frac{K}{2}} C_k, \quad S_2 := \bigcup_{k=\frac{K}{2}+1}^{K} C_k$$

and the numbers

$$s_1 := \omega_{\frac{K}{2}}(r_1, \ldots, r_{\frac{K}{2}}), \quad s_2 := \omega_{\frac{K}{2}}(r_{\frac{K}{2}+1}, \ldots, r_K).$$

Note that by (2.8) we have that

$$\omega_K(r_1, \ldots, r_K) = \omega_2(s_1, s_2). \tag{2.20}$$

We first intend to show that the mappings

$$U^{S_1} : X^{S_2} \longrightarrow \Pi^{mult}_{s_1,1}(X^{S_1}, Y)$$
$$U^{S_2} : X^{S_1} \longrightarrow \Pi^{mult}_{s_2,1}(X^{S_2}, Y)$$

are defined and moreover

$$\|U^{S_1}\| \le \sigma_{\frac{K}{2}}(r_1, \dots, r_{\frac{K}{2}}) \prod_{k\in S_1} \|U^{C_k}\|^{f^k_{\frac{K}{2}}(r_1,\dots,r_{\frac{K}{2}})} \tag{2.21}$$

$$\|U^{S_2}\| \le \sigma_{\frac{K}{2}}(r_{\frac{K}{2}+1}, \dots, r_K) \prod_{k\in S_2} \|U^{C_k}\|^{f^k_{\frac{K}{2}}(r_{\frac{K}{2}+1},\dots,r_K)}. \tag{2.22}$$

Take $(x_k)_{k\in S_2} \in X^{S_2}$. Then from the assumption that U is multiple $(r_k, 1)$-summing in each set of coordinates C_k it is straightforward that the mapping

$$T_1 := U^{S_1}\left((x_k)_{k\in S_2}\right) : X^{S_1} \to Y$$

for each $1 \le j \le K/2$ is multiple $(r_j, 1)$-summing in the coordinates of C_j and

$$\|T_1^{C_j} : X^{S_1\setminus C_j} \to \Pi^{mult}_{r_j,1}(X^{C_j}, Y)\| \le \prod_{k\in S_2} \|x_k\| \|U^{C_j} : X^{\bigcup_{k=1}^K C_k\setminus C_j} \to \Pi^{mult}_{r_j,1}(X^{C_j}, Y)\|$$

By induction we have that the mapping T_1 is hence multiple $(s_1, 1)$-summing and

$$\pi^{mult}_{s_1,1}(T_1) \le \sigma_{\frac{K}{2}}(r_1 \dots, r_{\frac{K}{2}}) \prod_{j\in S_1} \|T_1^{C_j}\|^{f^j_{\frac{K}{2}}(r_1,\dots,r_{\frac{K}{2}})}$$

Together with the preceding inequality we obtain as desired (2.21)

$$\begin{aligned}\pi^{mult}_{s_1,1}\left(U^{S_1}\left((x_k)_{k\in S_2}\right)\right) &\le \sigma_{\frac{K}{2}}(r_1 \dots, r_{\frac{K}{2}}) \prod_{j\in S_1} \|T_1^{C_j}\|^{f^j_{\frac{K}{2}}(r_1,\dots,r_{\frac{K}{2}})} \\ &\le \sigma_{\frac{K}{2}}(r_1 \dots, r_{\frac{K}{2}}) \prod_{j\in S_1} \|U^{C_j}\| \prod_{k\in S_2} \|x_k\|.\end{aligned}$$

With exactly the same argument we get (2.22). Hence U is multiple $(s_1, 1)$-summing in the coordinates of S_1 and multiple $(s_2, 1)$-summing in the coordinates of S_2. Therefore from Corollary 2.6 and (2.20) we deduce that U is multiple $(\omega_K, 1)$-summing and

$$\pi^{mult}_{\omega_K,1}(U) \le A^{|S_2|}_{q,s_1}(Y)^{f(s_1,s_2)} A^{|S_1|}_{q,s_2}(Y)^{f(s_2,s_1)} \|U^{S_1}\|^{f(s_1,s_2)} \|U^{S_2}\|^{f(s_2,s_1)}$$

By induction hypothesis and the formula (2.10) we get that

$$\begin{aligned}\pi^{mult}_{\omega_K,1}(U) \le &\left(A^{|S_2|}_{q,s_1}(Y)\sigma_{\frac{K}{2}}(r_1, \dots, r_{\frac{K}{2}})\right)^{f(s_1,s_2)} \left(A^{|S_1|}_{q,s_2}(Y)\sigma_{\frac{K}{2}}(r_{\frac{K}{2}+1}, \dots, r_K)\right)^{f(s_2,s_1)} \\ &\cdot \prod_{k=1}^K \|U^{C_k}\|^{f^k_K}.\end{aligned}$$

If K is odd we set

$$S_1 := \bigcup_{k=1}^{\frac{K-1}{2}} C_k, \quad S_2 := \bigcup_{k=\frac{K+1}{2}}^{K} C_k$$

and

$$s_1 := \omega_{\frac{K}{2}}(r_1, \dots, r_{\frac{K-1}{2}}), \quad s_2 := \omega_{\frac{K}{2}}(r_{\frac{K+1}{2}}, \dots, r_K).$$

With exactly the same arguments as in the upper case, we get the desired result. □

2.2.4 On the scalar Bohnenblust-Hille constants

We consider Theorem 2.2 in the case $X = \mathbb{K}$. We have that $\mathbb{K}$ is of cotype 2 with $C_2(\mathbb{K}) = 1$ and since each continuous functional on any Banach space X_m, $1 \leq m \leq M$ is $(1,1)$-summing the case $r_1 = \ldots = r_K = 1$, $q = 2$ and $K = M$ delivers the classical Bohnenblust-Hille Theorem. In order to understand the constant obtained in (2.11) and (2.12) first of all note that the numbers s_1, s_2, $f(s_1, s_2)$ and $f(s_2, s_1)$ can be given explicitly. For each $M \in \mathbb{N}$ we have by (2.4) that $\omega_M = \frac{2M}{M+1}$ and thus

$$\omega_{\frac{M}{2}} = \frac{2M}{M+2}, \quad \omega_{\frac{M-1}{2}} = \frac{2M-2}{M+1}, \quad \omega_{\frac{M+1}{2}} = \frac{2M+2}{M+3}.$$

From (2.5) we know that

$$f(\omega_{\frac{M}{2}}, \omega_{\frac{M}{2}}) = \frac{1}{2}$$

and by the definition of f (2.1) we have

$$f(\omega_{\frac{M-1}{2}}, \omega_{\frac{M+1}{2}}) = \frac{M-1}{2M}, \quad f(\omega_{\frac{M+1}{2}}, \omega_{\frac{M-1}{2}}) = \frac{M+1}{2M}.$$

Thus Theorem 2.2 delivers the following recursive formula for the Bohnenblust-Hille constants

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} \leq \begin{cases} A^{\frac{M}{2}}_{2,\frac{2M}{M+2}}(\mathbb{K}) \mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\frac{\mathrm{M}}{2}} & \text{for } M \text{ even} \\ \left(A^{\frac{M+1}{2}}_{2,\frac{2M-2}{M+1}}(\mathbb{K}) \mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\frac{\mathrm{M}-1}{2}} \right)^{\frac{M-1}{2M}} \left(A^{\frac{M-1}{2}}_{2,\frac{2M+2}{M+3}}(\mathbb{K}) \mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\frac{\mathrm{M}+1}{2}} \right)^{\frac{M+1}{2M}} & \text{for } M \text{ odd,} \end{cases} \tag{2.23}$$

with $\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}} = 1$.

Remark 2.7. Note that this recursive formula is already a consequence of the result in our publication [35, Theorem 4.1] (which is here Corollary 2.6). For even M we choose S_1 and S_2 in such a way that $|S_1| = |S_2| = \frac{M}{2}$. With $s_1 = s_2 = \frac{2M}{M+2}$ we get the desired result. For odd M we choose S_1 and S_2 in such a way that $|S_1| = \frac{M-1}{2}$ and $|S_2| = \frac{M+1}{2}$. Finally, $s_1 = \frac{2M-2}{M+1}$ and $s_2 = \frac{2M+2}{M+3}$ give the conclusion.

In the scalar case the constants $A_{2,s}(\mathbb{K})$ are well-known for every $1 \leq s \leq 2$. Note that in the cases $X = \mathbb{K}$ we clearly have that $\mathbb{K}$ is of cotype 2 with cotype constant $C_2(\mathbb{K}) = 1$ and the Kahane inequality in Lemma 2.4 can be replaced by the Khintchine inequality.

Lemma 2.8 (Khintchine inequality)**.** *For $1 \leq p < \infty$ there is a constant $\mathfrak{K}_p \geq 1$ such that for every $N \in \mathbb{N}$ and every $a_1, \ldots, a_N \in \mathbb{C}$ we have*

$$\left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \leq \mathfrak{K}_p \left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}}.$$

The optimal constants in the Khintchine inequality are due to Haagerup [47]. They are exactly the same if only real a_n are considered:

$$\mathfrak{K}_p = \begin{cases} 2^{\frac{1}{p}-\frac{1}{2}} \le \mathfrak{K}_1 = \sqrt{2} & \text{if } 1 \le p \le p_0 \\ 2^{-\frac{1}{2}}\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{p+1}{2}\right)}\right)^{\frac{1}{p}} \le \mathfrak{K}_1 & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \le p < \infty \text{ (obvious).} \end{cases}$$

The number $p_0 \approx 1,84742$ is the solution of $\Gamma((p+1)/2) = \sqrt{\pi}$ in $]1,2[$.

This gives that for $1 \le s \le 2$

$$A_{2,s}(\mathbb{K}) = \mathfrak{K}_s \quad \text{and in particular} \quad A_{2,1}(\mathbb{K}) = \sqrt{2}. \tag{2.24}$$

Let now $(S_n)_n$ be an i.i.d. sequence of Steinhaus random variables $S_n : (\Omega, P) \to \mathbb{T}$ on a probability space (Ω, P) which are equidistributed on the circle $\mathbb{T}$. Replacing the Rademacher variables in the Khintchine inequality by Steinhaus random variables the following inequality holds.

Lemma 2.9 (Khintchine inequality for complex Steinhaus variables)**.** *For $1 \le p < \infty$ there is a constant $\mathfrak{S}_p \ge 1$ such that for every $N \in \mathbb{N}$ and every $a_1, \ldots, a_N \in \mathbb{C}$ we have*

$$\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} \le \mathfrak{S}_p \left(\int_\Omega \Big|\sum_{n=1}^{N} a_n S_n\Big|^p dP\right)^{\frac{1}{p}}.$$

The optimal constants in the upper inequality are due to Sawa [72, Theorem B] for p=1 and König [52] for $1 \le p < \infty$ (see also [1, Section 2])

$$\mathfrak{S}_p = \begin{cases} \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{-\frac{1}{p}} & \text{if } 1 \le p < 2 \\ 1 & \text{if } 2 \le p < \infty \text{ (obvious).} \end{cases}$$

With essentially the same proof as Lemma 2.4 we have that for every, $1 \le s \le 2$ and every matrix $(a_{\mathbf{i}})_{i_1,\ldots,i_M=1}^{N_1,\ldots,N_M}$ in $\mathbb{C}$

$$\left(\sum_{\mathbf{i} \in \mathscr{M}(M,N)} |a_{\mathbf{i}}|^2\right)^{\frac{1}{2}} \le A'_{2,s}(\mathbb{C})^{M-1} \left(\int_{\mathbb{T}^N} \Big| \sum_{\mathbf{i} \in \mathscr{M}(M,N)} a_{\mathbf{i}} z_{i_1} \cdots z_{i_M}\Big|^s\right)^{\frac{1}{s}},$$

where

$$A'_{2,s}(\mathbb{C}) = \mathfrak{S}_s \quad \text{and in particular} \quad A'_{2,1}(\mathbb{C}) = \frac{2}{\sqrt{\pi}}. \tag{2.25}$$

Thus, Lemma 2.5 delivers the following estimates for the scalar Bohnenblust-Hille constants (the case $Y = \mathbb{K}$, $K = M$, $r_1 = \ldots = r_K = 1$, $q = 2$)

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{R},\mathrm{M}} \le A_{2,1}(\mathbb{R})^{M-1} = \sqrt{2}^{M-1}$$
$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \le A'_{2,1}(\mathbb{C})^{M-1} = \left(\frac{2}{\sqrt{\pi}}\right)^{M-1},$$

which are the estimates of Davie and Kaijser and Queffélec for the scalar Bohnenblust-Hille constants mentioned in (1.1). The formula (2.23) delivers with (2.24) and (2.25) the formula of Nuñez-Alarcón, Pellegrino and Seoane-Sepúlveda mentioned in (2.14).

Since the gamma function is difficult to handle the following weaker estimate is very useful. For every $0 < p < q < \infty$ and every $a_1, \ldots, a_N \in \mathbb{C}$ we have that

$$\Big(\int_{\mathbb{T}^N} \Big| \sum_{n=1}^{N} a_n z_n \Big|^q d\mu^N(z) \Big)^{\frac{1}{q}} \leq \sqrt{\frac{q}{p}} \Big(\int_{\mathbb{T}^N} \Big| \sum_{n=1}^{N} a_n z_n \Big|^p d\mu^N(z) \Big)^{\frac{1}{p}}, \tag{2.26}$$

which gives us that for every $0 < s < 2$ we have

$$A'_{2,s}(\mathbb{C}) \leq \sqrt{\frac{2}{s}}. \tag{2.27}$$

This inequality can be shown on a direct way but it is already a consequence of a Theorem in the next Chapter. In the case $M = 1$ of Theorem 3.9 we have that the upper inequality holds for every $0 < p < q < \infty$ and every (1-homogeneous polynomial) $\sum_{n=1}^{N} a_n z_n \in \mathcal{P}_1(\ell_\infty^N; \mathbb{C})$.

The following result of Nuñez-Alarcón, Pellegrino, Seoane-Sepúlveda and Serrano-Rodríguez in [63, Corollary 8.5 and Section 9] (see also (1.2)) says that the optimal constants in the multilinear Bohnenblust-Hille inequality $\mathrm{B}^{\mathrm{mult}}_{\mathbb{K},\mathrm{M}}$ are of sub-polynomial instead of sub-exponential growth. We give a short proof based entirely on Corollary 2.6. For simplicity we only handle the complex case (see e.g. [61, Theorem 17] for the real case).

Theorem 2.10. *There is a universal constant $C > 0$ such that*

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \leq M^C.$$

With the estimate (2.27) the following corollary is a direct consequence of (2.23).

Corollary 2.11. *For every M we have*

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \leq \begin{cases} \sqrt{\frac{M+2}{M}}^{\frac{M}{2}} \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}}{2}} & \text{if } M \text{ even,} \\ \Big(\sqrt{\frac{M+1}{M-1}}^{\frac{M+1}{2}} \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}-1}{2}} \Big)^{\frac{M-1}{2M}} \Big(\sqrt{\frac{M+3}{M+1}}^{\frac{M-1}{2}} \mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}+1}{2}} \Big)^{\frac{M+1}{2M}} & \text{if } M \text{ odd.} \end{cases}$$

Now we are in the position to give a proof of Theorem 2.10

Proof of Theorem 2.10. Let us first assume that M is a power of 2, i.e. $M = 2^l$. Clearly, $\big(\frac{M+2}{M}\big)^{\frac{M}{4}}$ converges to $\sqrt{e}$ and hence Corollary 2.11 gives

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\mathrm{M}} \leq \sqrt{e}\,\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},\frac{\mathrm{M}}{2}} \leq \sqrt{e}^{\,l} = \sqrt{e}^{\,\log_2 M}.$$

This gives as desired

$$\mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{mult}} \leq M^{\log_2 \sqrt{e}}.$$

Let now M be arbitrary. Then $\mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{mult}} \leq \mathrm{B}_{\mathbb{C},2^{[\log_2 \mathrm{M}]}}^{\mathrm{mult}}$ where $[\log_2 M]$ is smallest integer $\geq \log_2 M$. Then a straight forward argument shows that $\mathrm{B}_{\mathbb{C},\mathrm{M}}^{\mathrm{mult}} \leq M^{\log_2 e}$. □

2.2.5 An application in Quantum Information Theory

In this section we report on a result of Montanaro [61, Theorem 5] that provides an application of the new estimate of Nuñez-Alarcón, Pellegrino, Seoane-Sepúlveda and Serrano-Rodríguez (1.2) for the optimal Bohnenblust-Hille constants in the field of quantum information theory. Thus this section contains no new conclusions but it points out an interesting aspect of the Bohnenblust-Hille inequality. While most applications of this inequality aim for estimating the sum of coefficients of a multilinear mapping, in this application the Bohnenblust-Hille inequality is used the other way round. In the quantum information theory appears a problem which asks for upper and lower estimates of the norm $\|T\|_\infty$ for a given an M-linear mapping $T \in \mathcal{L}_M(\ell_\infty^N; \mathbb{R})$ defined by a matrix $A = (a_{i_1 \dots i_M})$ of entries $a_{i_1 \dots i_M} \in \{-1, 1\}$ and a probability distribution $\pi : \{1, \dots, N\}^M \to [0, 1]$ through the coefficients

$$T(e_{i_1}, \dots, e_{i_M}) := a_{i_1 \dots i_M} \pi(i_1, \dots, i_M).$$

Here the Bohnenblust-Hille inequality is going to be a lot of use. But let us take the time to understand the background of this problem which is settled in the field of the so-called nonlocal games. The nonlocal games are designed to explore the differences between classical and quantum mechanics and were first introduced by Cleve, Høyer, Toner and Watrous in [24]. In his PhD thesis [21, Chapter 1] Briët gives a very clear and understandable introduction to the whole topic which we recommend to the reader for more detailed information and on which this presentation is based.

In the following we only pick out special type of the nonlocal games, the XOR games, which are of interest for our purpose. A two player XOR game (XOR stands for the exclusive OR) consists of a matrix $A = (a_{ij}) \in \{-1, 1\}^{N \times N}$, two finite sets $\mathcal{I}, \mathcal{J}$ both of cardinality N and a joint probability distribution $\pi : \mathcal{I} \times \mathcal{J} \to [0, 1]$. The game involves three parties: two players usually called Alice and Bob and an external person, the referee. At the beginning of the game the referee picks a pair of questions $(i, j) \in \mathcal{I} \times \mathcal{J}$ according to the probability distribution π and sends the question i to Alice and j to Bob. The players reply upon them independently with the answers $x, y \in \{-1, 1\}$. Alice and Bob (together) win the game if the product of their answers equals the corresponding entry in the matrix, more precisely if for the posed questions $(i, j) \in \mathcal{I} \times \mathcal{J}$ the product of their answers is

$$xy = a_{ij}.$$

During the game the players are not allowed to communicate with each other but the probability distribution π and the matrix A are known to all participants in advance so that Alice and Bob may decide a common strategy before the beginning of the game. One can now distinguish between two different classes of strategies, the classical strategies belonging to the field of classical mechanics and the entangled strategies in the framework of quantum mechanics.

In a classical strategy the players simply determine the answer to each possible question before the beginning of the game, i. e. they define their answers through the deterministic maps $x : \mathcal{I} \to \{-1,1\}$ and $y : \mathcal{J} \to \{-1,1\}$. Then the probability of winning the XOR game is given by

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} a_{ij}\pi(i,j)x(i)y(j).$$

One can show that the players can not increase their maximal chance of winning by using shared or private randomness, for example flipping coins some of which both players can see and the others are private, since such a process always produces a probability distribution over a deterministic classical strategy (see [24]).

In contrast to that the players in the world of quantum mechanics can produce answers that are correlated in a way which is impossible in the world of classical mechanics. The players may base their answers on the outcome of a local measurement on a shared entangled state, which in some situations enables the players to increase their maximal chance of winning the game (see for example [23]). But since the entangled strategies will not be the focus of our attention we refer the interested reader to [21] for more details.

Analogously, an M-player XOR game is specified in the following way. An M-player XOR game $\mathcal{G} = (\pi, A)$ is given by a matrix $A = (a_{\mathbf{i}})_{\mathbf{i}\in\mathscr{M}(M,N)}$ for which each entry $a_{\mathbf{i}} \in \{-1,1\}$ and a probability distribution $\pi : \mathscr{M}(M,N) \to [0,1]$. During the game the referee picks an M-tuple $(i_1,\dots,i_M) \in \mathscr{M}(M,N)$ according to the probability distribution π and sends each question i_m to the player m, $1 \le m \le M$. Using a classical strategy each player m answers upon his question with an (deterministic) answer map $y_m : \{1,\dots,N\} \to \{-1,1\}$ The players win the game if and only if the product of their answers equals the corresponding entry in the matrix A, that is if

$$y_1(i_1)\cdots y_M(i_M) = a_{i_1\dots i_M}.$$

In order to describe the complexity of a given XOR game $\mathcal{G} = (\pi, A)$ one defines the bias $\beta(\mathcal{G})$ to be the greatest difference between the chance of winning and the chance of losing the game for the optimal classical strategy. Thus the classical bias of an M-player XOR game is given by

$$\beta(\mathcal{G}) = \max_{y_1,\dots,y_M\in\{-1,1\}^N} \Big| \sum_{\mathbf{i}\in\mathscr{M}(M,N)} \pi(\mathbf{i})a_{\mathbf{i}}y_1(i_1)\cdots y_M(i_M)\Big|.$$

Or in other words, if we define the M-linear mapping $T \in \mathcal{L}_M(\ell_\infty^N;\mathbb{R})$ by the coefficients $T(e_{i_1},\dots,e_{i_M}) := a_{i_1\dots i_M}\pi(i_1,\dots,i_M)$ then the bias is given by

$$\beta(\mathcal{G}) = \|T\|_\infty.$$

The question is now to find the game for which the classical bias is minimised. It is known there exists an M-player XOR game for which

$$\beta(\mathcal{G}) \leq \frac{1}{N^{\frac{M-1}{2}}}$$

(see [45]). It is now very easy to show that the Bohnenblust-Hille inequality delivers a lower estimate for the classical bias of an M-player XOR game (see Montanaro [61, Theorem 5]). For sake of completeness we show the proof here.

Theorem 2.12. *For every M-player XOR game $\mathcal{G} = (\pi, A)$ we have that*

$$\beta(\mathcal{G}) \geq \frac{1}{\mathrm{B}_{\mathbb{R},\mathrm{M}}^{\mathrm{mult}} N^{\frac{M-1}{2}}},$$

in particular, there exists a constant $C > 0$ such that

$$\beta(\mathcal{G}) \geq \frac{1}{M^C N^{\frac{M-1}{2}}}.$$

Proof. For any M-player XOR game $\mathcal{G} = (\pi, A)$ define the M-linear mapping $T \in \mathcal{L}_M(\ell_\infty^N; \mathbb{R})$ by the coefficients $T(e_{i_1}, \ldots, e_{i_M}) := a_{i_1 \ldots i_M} \pi(i_1, \ldots, i_M)$. Then we have that

$$\sum_{\mathbf{i} \in \mathscr{M}(M,N)} |T(e_{i_1}, \ldots, e_{i_M})| = \sum_{\mathbf{i} \in \mathscr{M}(M,N)} \pi(i_1, \ldots, i_M) = 1$$

With the Hölder inequality in the first step and the Bohnenblust-Hille inequality (Theorem 1.1) in the second step we have that

$$\begin{aligned} \sum_{\mathbf{i} \in \mathscr{M}(M,N)} |T(e_{i_1}, \ldots, e_{i_M})| &\leq \Big(\sum_{\mathbf{i} \in \mathscr{M}(M,N)} |T(e_{i_1}, \ldots, e_{i_M})|^{\frac{2M}{M+1}} \Big)^{\frac{M+1}{2M}} \Big(\sum_{\mathbf{i} \in \mathscr{M}(M,N)} 1 \Big)^{\frac{M-1}{2M}} \\ &\leq \mathrm{B}_{\mathbb{R},\mathrm{M}}^{\mathrm{mult}} N^{\frac{M-1}{2}} \|T\|. \end{aligned}$$

This gives the desired result. In particular, we get with (1.2) or Theorem 2.10 that there is a constant $C > 0$ such that $\beta(\mathcal{G}) \geq M^{-C} N^{-(M-1)/2}$. □

2.3 Some Consequences

Recall that by the Bombal-Pérez-Villanueva Theorem 1.7 every bounded M-linear operator with values in a cotype q space is multiple $(q, 1)$-summing. As an immediate consequence of Theorem 2.2 we now have the following

Corollary 2.13. *Let Y be a Banach space of cotype q, and $1 \leq r < q$. For*

$$\rho_M := \omega_M(r, \ldots, r) = \frac{qrM}{q + (M-1)r},$$

each separately $(r,1)$*-summing* $U \in \mathcal{L}_M(X_1,\dots,X_M;Y)$ *is multiple* $(\rho_M,1)$*-summing and*

$$\pi^{mult}_{\rho_M,1}(U) \le \sigma_M \prod_{m=1}^{M} \left\| U^{\{m\}} : X^{\complement\{m\}} \to \Pi_{r,1}\left(X^{\{m\}};Y\right)\right\|^{\frac{1}{M}}$$

with $\sigma_1 = 1$ *and for* $M \ge 2$

$$\sigma_M = \begin{cases} A_{q,\rho_{\frac{M}{2}}}(Y)^{\frac{M}{2}}\sigma_{\frac{M}{2}} & \textit{if } M \textit{ is even} \\ \left(A_{q,\rho_{\frac{M-1}{2}}}(Y)^{\frac{M+1}{2}}\sigma_{\frac{M-1}{2}}\right)^{\frac{M-1}{2M}}\left(A_{q,\rho_{\frac{M+1}{2}}}(Y)^{\frac{M-1}{2}}\sigma_{\frac{M+1}{2}}\right)^{\frac{M+1}{2M}} & \textit{if } M \textit{ is odd.} \end{cases}$$

Corollary 2.13 includes the scalar valued and vector valued results on multiple summing operators mentioned in the introduction: Since every continuous functional on X_m is $(1,1)$-summing, the case $Y = \mathbb{C}$ delivers the classical Bohnenblust-Hille result Theorem 1.1 (see also (1.3). And with the Bennett-Carl Theorem 1.4 we easily deduce the multilinear Bennett-Carl version of Defant and Sevilla-Peris [38, Theorem 1] (see also Theorem 1.8). Given $M \in \mathbb{N}$ and $1 \le p \le q \le \infty$, define

$$\rho_M = \begin{cases} \frac{2M}{M+2(\frac{1}{p}-\max\{\frac{1}{q},\frac{1}{2}\})} & \text{if } p \le 2 \\ p & \text{if } p \ge 2. \end{cases}$$

Then there is a constant $C_M > 0$ such that for every $A \in \mathcal{L}_M(\ell_\infty;\ell_p)$ we have

$$\left(\sum_{i_1,\dots,i_M=1}^{\infty} \|A(e_{i_1},\dots,e_{i_M})\|_q^{\rho_M}\right)^{\frac{1}{\rho_M}} \le C_M\|A\|. \tag{2.28}$$

If C_M denotes the optimal constant in (2.28) the proof of Defant and Sevilla-Peris gives that $C_M \le C^M$ for some universal constant $C > 0$. But in the case $1 \le p \le 2$, $q = 2$ Corollary 2.13 gives that the constant C_M is of sub-polynomial instead of sub-exponential growth.

Corollary 2.14. *Given* $M \in \mathbb{N}$ *and* $1 \le p \le 2$*, define*

$$\rho_M = \frac{2M}{M+2(\frac{1}{p}-\frac{1}{2})}.$$

Then there exists a constant $C > 0$ *such that for every* $A \in \mathcal{L}_M(\ell_\infty;\ell_p)$ *we have*

$$\left(\sum_{i_1,\dots,i_M=1}^{\infty} \|A(e_{i_1},\dots,e_{i_M})\|_2^{\rho_M}\right)^{\frac{1}{\rho_M}} \le M^C\|A\|.$$

Proof. First of all recall that by the Bennett-Carl Theorem 1.4 the composition $I \circ A$ of A with the canonical embedding $I : \ell_p \hookrightarrow \ell_2$ is separately $(p,1)$-summing. On the other hand by the orthonormality of the z_n in $\mathscr{L}_2(\mu^N)$, the inequality (2.26) and the Minkowski inequality we get for all $1 \leq s \leq 2$

$$\begin{aligned}\Big(\sum_{n=1}^{N} \|a_n\|_2^2\Big)^{\frac{1}{2}} &= \Big\| \Big(\sum_{n=1}^{N} |a_n|^2\Big)^{\frac{1}{2}} \Big\|_2 \\ &= \Big\| \Big(\int_{\mathbb{T}^N} \Big|\sum_{n=1}^{N} a_n z_n\Big|^2 d\mu^N(z)\Big)^{\frac{1}{2}} \Big\|_2 \\ &\leq \sqrt{\frac{2}{s}} \Big\| \Big(\int_{\mathbb{T}^N} \Big|\sum_{n=1}^{N} a_n z_n\Big|^s d\mu^N(z)\Big)^{\frac{1}{s}} \Big\|_2 \\ &\leq \sqrt{\frac{2}{s}} \Big(\int_{\mathbb{T}^N} \Big\|\sum_{n=1}^{N} a_n z_n\Big\|_2^s d\mu^N(z)\Big)^{\frac{1}{s}}.\end{aligned}$$

Thus the constant in Lemma 2.4 can be estimated by

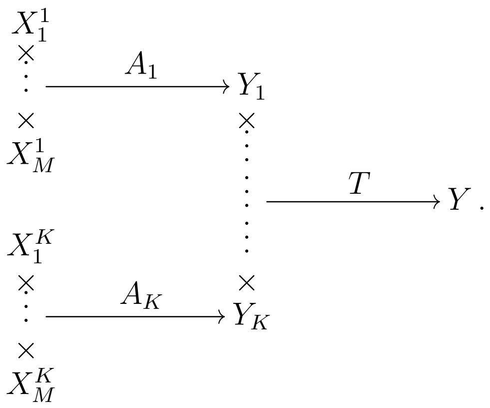
$$A_{2,s}(\ell_2) \leq \sqrt{\frac{2}{s}} \quad \text{for all } 1 \leq s \leq 2. \tag{2.29}$$

Let now M be a power of 2. Then (2.29) and Corollary 2.13 give that

$$C_M \leq \sqrt{\frac{M + 4(\frac{1}{p} - \frac{1}{2})}{M}}^{\frac{M}{2}} C_{\frac{M}{2}}.$$

With exactly the same arguments as in the proof of Theorem 2.10 we get the conclusion. □

But we can do a lot more. For example, recall that by Kwapień's Theorem 1.3 every bounded linear operator $v : \ell_1 \to \ell_p$ is $(r,1)$-summing with $\frac{1}{r} = 1 - |\frac{1}{p} - \frac{1}{2}|$. The following theorem, also as a consequence of Theorem 2.2, is a multilinear extension of Kwapień's theorem. For an M-linear mapping $T \in \mathcal{L}_K(Y_1, \ldots, Y_K; Y)$ and K many M-linear mappings $A_k \in \mathcal{L}_M(X_1^k, \ldots, X_M^k; Y_k)$ we write $T(A_1, \ldots, A_K)$ for the composition defined in the following way



Theorem 2.15. *Take $T \in \mathcal{L}_K(\ell_1; \ell_p)$ (with $1 \le p \le \infty$) and K many M-linear $A_k \in \mathcal{L}_M(\ell_\infty; \ell_1)$, for $1 \le k \le K$. Then the composition $T(A_1, ..., A_K)$ is multiple $(r_M, 1)$-summing for*

$$r_M = \begin{cases} \frac{2M}{M+2-\frac{2}{p}} & \text{if } 1 \le p \le 2, \\ \frac{2M}{\frac{2M}{p}+1} & \text{if } 2 \le p \le \frac{2M}{M-1}, \\ 2 & \text{if } \frac{2M}{M-1} \le p \le \infty. \end{cases}$$

Moreover,

$$\pi^{mult}_{r_M,1}(T(A_1, ..., A_K)) \le \sigma \|T\| \prod_{k=1}^{K} \|A_k\|,$$

where $\sigma > 1$ is a constant only depending on p, M and K.

The proof of Theorem 2.15 is a consequence of this independently interesting composition theorem.

Corollary 2.16. *Let Y be a Banach space of cotype q and $1 \le r \le q$. Assume that $T \in \mathcal{L}_K(Y_1, \dots, Y_K; Y)$ is multiple $(r,1)$-summing and take K many M-linear mappings $A_k \in \mathcal{L}_M(X_1^k, \dots, X_M^k; Y_k)$, $1 \le k \le K$. Define*

$$\rho_M := \frac{qrM}{q + (M-1)r}.$$

Then the composition $T(A_1, \dots, A_K)$ is multiple $(\rho_M, 1)$-summing and

$$\pi^{mult}_{\rho_M,1}(T(A_1, \dots, A_K)) \le \sigma_M \pi^{mult}_{r,1}(T) \prod_{k=1}^{K} \|A_k\|,$$

where σ_M is the constant from Theorem 2.2 (depending on M, K, r, q and $C_q(Y)$).

Proof. Without loss of generality we may assume that $r < q$. For the case $q = r$ see the remark after Lemma 2.4. Since T is $(r,1)$-summing the composition $T(A_1, \dots, A_K)$ is multiple $(r,1)$-summing in each of the following M sets of coordinates

$$\begin{aligned} &\{1, M+1, \dots, (K-1)M+1\}, \\ &\{2, M+2, \dots, (K-1)M+2\}, \\ &\qquad \vdots \\ &\{M, 2M, \dots, KM\}. \end{aligned}$$

Then the claim is an immediate consequence of Theorem 2.2. □

Proof of Theorem 2.15. Note first that by Theorem 1.7 each M-linear operator with values in a cotype q space is multiple $(q,1)$-summing. Hence all the A_k, $1 \le k \le K$

are multiple $(2,1)$-summing, which gives that $T(A_1,\ldots,A_K)$ is at least multiple $(2,1)$-summing. Hence it suffices to show that $T(A_1,\ldots,A_K)$ is multiple $(r,1)$-summing with

$$r = \begin{cases} \frac{2M}{M+2-\frac{2}{p}} & \text{if } 1 \le p \le 2, \\ \frac{2M}{\frac{2M}{p}+1} & \text{if } 2 \le p < \infty; \end{cases} \tag{2.30}$$

note that for $\frac{2M}{M-1} \le p \le \infty$ we have that $2 \le 2M/(\frac{2M}{p}+1)$. From [20, Corollary 4.3] and Kwapień's theorem we know that the claim of Theorem 2.15 holds for $M = 1$ and arbitrary K, in other words T itself is multiple $(r,1)$-summing, $\frac{1}{r} = 1 - |\frac{1}{p} - \frac{1}{2}|$. But now we easily deduce from Corollary 2.16 that $T(A_1,\ldots,A_K)$ is multiple $(r,1)$-summing with r defined as in (2.30). $\square$

3. Polynomial Versions

A function $P : E \to F$ between two normed spaces E and F is said to be an M-homogeneous polynomial if there exists an M-linear mapping $A : \prod_{m=1}^{M} E \to F$ such that $P(x) = A(x, \ldots, x)$ for every $x \in E$. The space of all continuous M-homogeneous polynomials from E to F is denoted by $\mathcal{P}_M(E; F)$, which together with the norm $\|P\| = \sup_{\|x\|\leq 1} \|P(x)\|$ forms a Banach space. For every M homogeneous polynomial $P \in \mathcal{P}_M(E; F)$ there exists a unique symmetric M-linear mapping A^s such that $A^s(x, \ldots, x) = P(x)$. Each M-linear mapping $A \in \mathcal{L}_M(E; F)$ can be symmetrized by $A^s(x_1, \ldots, x_M) = \frac{1}{M!} \sum_{\sigma \in S_M} A(x_{\sigma(1)}, \ldots, x_{\sigma(M)})$, where S_M denotes the set of all permutations of $\{1, \ldots, M\}$.

It will be important to relate the norm of an M-homogeneous polynomial $P \in \mathcal{P}_M(E; F)$ with the norm of its associated M-linear mapping $A \in \mathcal{L}_M(E; F)$. It is well known (see e.g. [41, Proposition 1.8]) that the norms of P and A satisfy

$$\|P\| \leq \|A\| \leq \frac{M^M}{M!} \|P\| . \tag{3.1}$$

On the other hand, it is shown by Harris in [48, Theorem 1] that for an M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty; \mathbb{C})$ and its associated $A \in \mathcal{L}_M(\ell_\infty; \mathbb{C})$ we have for every $m_1, \ldots, m_l \in \mathbb{N}$ with $m_1 + \ldots + m_l = M$ and $z_1, \ldots, z_l \in B_{\ell_\infty}$ that

$$\left| A(z_1, \overset{m_1}{\ldots}, z_1, \ldots, z_l, \overset{m_l}{\ldots}, z_l) \right| \leq \frac{m_1! \cdots m_l!}{m_1^{m_1} \cdots m_l^{m_l}} \frac{M^M}{M!} \|P\|. \tag{3.2}$$

3.1 The Basic Notations

For $M, N \in \mathbb{N}$ and a finite subset $C \subset \mathbb{N}$, recall the definition of the index sets

$$\begin{aligned} \mathscr{M}(C, N) &= \{\mathbf{i} = (i_k)_{k \in C} \mid 1 \leq i_k \leq N \text{ for each } k \in C\} \\ \mathscr{M}(M, N) &= \mathscr{M}(\{1, \ldots, M\}, N) \end{aligned}$$

(see page 29). By $\mathbb{N}_0^{(\mathbb{N})}$ we denote the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of finite length, by $\mathbb{N}_0^N$ the set of all multi-indices of length N. Furthermore we define the following index sets

$$\begin{aligned} \Lambda(M, N) &= \left\{\alpha \in \mathbb{N}_0^N \mid |\alpha| = M\right\} \\ \mathscr{J}(M, N) &= \{i \in \mathscr{M}(M, N) \mid i_1 \leq \ldots \leq i_M\} . \end{aligned}$$

There is a one-to-one correspondence between $\Lambda(M, N)$ and $\mathscr{J}(M, N)$. For each $\alpha \in \Lambda(M, N)$ is $\mathbf{j}_\alpha = (1, \overset{\alpha_1\text{-times}}{\ldots}, 1, 2, \overset{\alpha_2\text{-times}}{\ldots}, 2, \ldots, N, \overset{\alpha_N\text{-times}}{\ldots}, N)$ the associated index in the set $\mathscr{J}(M,N)$ and on the other hand for $\mathbf{j} \in \mathscr{J}(M,N)$ the associated multi-index ${}^{\mathbf{j}}\alpha \in \Lambda(M,N)$ is given by ${}^{\mathbf{j}}\alpha_r = |\{k \mid j_k = r\}|$. We define the following equivalence relation in $\mathscr{M}(M,N)$: $\mathbf{i} \sim \mathbf{j}$ if there is a permutation $\sigma \in S_M$ such that $i_{\sigma(k)} = j_k$ for all $1 \leq k \leq M$. For a given $\mathbf{i} \in \mathscr{M}(M,N)$ we denote by $[\mathbf{i}]$ the equivalence class. Note that for each $\mathbf{i} \in \mathscr{M}(M,N)$ there is a unique $\mathbf{j} \in \mathscr{J}(M,N)$ such that $[\mathbf{i}] = [\mathbf{j}]$, thus $\mathscr{M}(M,N) = \dot{\bigcup}_{\mathbf{j} \in \mathscr{J}(M,N)} [\mathbf{j}]$. We denote by $|\mathbf{i}|$ the cardinality of $[\mathbf{i}]$ that is the number of different indices belonging to $[\mathbf{i}]$. Note that $|\mathbf{j}_\alpha| = \frac{M!}{\alpha!}$ for every $\alpha \in \Lambda(M,N)$.

Every polynomial $P \in \mathcal{P}_M(E,Y)$ on a Banach sequence space E with values in some Banach space Y has a monomial series expansion $P(z) = \sum_{\alpha \in \Lambda(M,N)} c_\alpha^{(N)} z^\alpha$ whenever it is restricted to any finite dimensional section E_N (the span of the first N basis vectors $e_1 \ldots, e_N$). Indeed, given a polynomial $P \in \mathcal{P}_M(E;Y)$ and its associated symmetric M-linear mapping $A \in \mathcal{L}_M^s(E;Y)$ we have that for every $z \in E_N$

$$\begin{aligned}
P(z) &= A(z,\ldots,z) \\
&= \sum_{\mathbf{i} \in \mathscr{M}(M,N)} A(e_{i_1},\ldots,e_{i_M}) z_{i_1} \cdots z_{i_M} \\
&= \sum_{\mathbf{j} \in \mathscr{J}(M,N)} \Big(\sum_{\mathbf{i} \in [\mathbf{j}]} A(e_{i_1},\ldots,e_{i_M}) \Big) z_{j_1} \cdots z_{j_M} \\
&= \sum_{\alpha \in \Lambda(M,N)} |\mathbf{j}_\alpha| \, A(e_{j_1},\ldots,e_{j_M}) z_1^{\alpha_1} \cdots z_N^{\alpha_N} \\
&=: \sum_{\alpha \in \Lambda(M,N)} c_\alpha^{(N)}(P) z^\alpha
\end{aligned}$$

Clearly we have that $c_\alpha^{(N)}(P) = c_\alpha^{(N+1)}(P)$ for every $\alpha \in \mathbb{N}_0^N \subset \mathbb{N}_0^{N+1}$ and thus there is a unique family $(c_\alpha(P))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ in Y such that for all N and all $z \in E_N$

$$P(z) = \sum_{\alpha \in \Lambda(M,N)} c_\alpha(P) z^\alpha.$$

The power series $\sum c_\alpha(P) z^\alpha$ is called the monomial series expansion of P and $c_\alpha = c_\alpha(P)$ are its monomial coefficients.

If it is not stated differently we will always denote the monomial coefficients of an M-homogeneous polynomial $P \in \mathcal{P}_M(E;F)$ with c_α or with $b_{j_1 \ldots j_M} := c_\alpha$ where $(j_1,\ldots,j_M) = \mathbf{j}_\alpha$, and $a_{i_1,\ldots,i_M} := A(e_{i_1},\ldots,e_{i_M})$ will be the coefficients defining the symmetric M-linear mapping A associated with P. We will switch between these different notations as it fits in our situation: for each $z \in E_N$ we have

$$\begin{aligned} P(z) &= \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha = \sum_{\mathbf{j} \in \mathscr{J}(M,N)} b_{\mathbf{j}} z j_1 \cdots z_{j_M} \\ &= \sum_{\mathbf{j} \in \mathscr{J}(M,N)} |\mathbf{j}|\, a_{\mathbf{j}} z_{j_1} \cdots z_{j_M} = \sum_{\mathbf{i} \in \mathscr{M}(M,N)} a_{\mathbf{i}} z_{i_1} \cdots i_{i_M}. \end{aligned}$$

Note that in particular we have

$$c_\alpha = |\mathbf{j}_\alpha| A(e_{i_1}, \ldots, e_{i_M}) = |\mathbf{j}_\alpha| a_{j_\alpha}.$$

3.2 Deduction from the Multilinear Case

The following theorem shows on an abstract level how to deduce results on the summability of the monomial coefficients of vector valued polynomials on ℓ_∞ from corresponding results on multiple summing operators on ℓ_∞ via polarization. For $T \in \mathcal{L}_K(Y_1, \ldots, Y_K; Y)$ and K many M-homogeneous polynomials $P_k \in \mathcal{P}_M(\ell_\infty; Y_k)$ we define an MK-homogeneous polynomial $T(P_1, \ldots, P_K) : \ell_\infty \longrightarrow Y$ by

$$T(P_1, \ldots, P_K)(z) := T(P_1(z), \ldots, P_K(z)).$$

Theorem 3.1. *Let $K, M \in \mathbb{N}$ and $T \in \mathcal{L}_K(Y_1, \ldots, Y_K; Y)$ be given. Assume that there is an exponent $1 \leq \omega < \infty$ and a constant $C > 0$ such that*

$$\left(\sum_{i_1, \ldots, i_{MK}=1}^{\infty} \| T(A_1, \ldots, A_K)(e_{i_1}, \ldots, e_{i_{MK}}) \|^\omega \right)^{\frac{1}{\omega}} \leq C \prod_{k=1}^{K} \|A_k\|$$

for all M-linear mappings $A_k \in \mathcal{L}_M(\ell_\infty; Y_k)$, $1 \leq k \leq K$. Then there exists a constant $D > 0$ such that

$$\left(\sum_{|\alpha| = MK} \| c_\alpha (T(P_1, \ldots, P_K)) \|^\omega \right)^{\frac{1}{\omega}} \leq D \prod_{k=1}^{K} \|P_k\| \tag{3.3}$$

for all M-homogeneous polynomials $P_k \in \mathcal{P}_M(\ell_\infty; Y_k)$, $1 \leq k \leq K$. The proof shows that the best constant D in the upper inequality can be estimated by

$$D \leq C (MK)!^{\frac{\omega-1}{\omega}} \left(\frac{M^M}{M!} \right)^K.$$

Proof. Note first that by definition the monomial coefficients $c_\alpha(\varphi)$ of the polynomial $\varphi := T(P_1 \ldots, P_K) \in \mathcal{P}_{MK}(\ell_\infty; Y)$ are given through the monomial coefficients of the restriction of φ to any ℓ_∞^N. It suffices to estimate the ω-norm on the left side of (3.3) only for finitely many multi-indices α of order MK. Combining these two facts we see that it is enough to prove (3.3) for each choice of M-homogeneous polynomials $P_k \in \mathcal{P}_M(\ell_\infty^N; Y_k)$, $1 \leq k \leq K$. Fix K many such polynomials P_k and denote their associated symmetric M-linear mappings by $A_k \in \mathcal{L}_M^s(\ell_\infty^N; Y_k)$, $1 \leq k \leq K$. Define

$$\Phi : \prod_{i=1}^{MK} \ell_\infty^N \longrightarrow Y, \quad \Phi = T(A_1, \dots, A_K)$$

and its symmetrization

$$\Phi^s(x_1, \dots, x_{MK}) := \frac{1}{(MK)!} \sum_{\sigma \in S_{MK}} \Phi(x_{\sigma(1)} \dots, x_{\sigma(MK)}).$$

Note that Φ^s is the (unique) symmetric MK-linear mapping associated with the polynomial φ, and hence $c_\alpha(\varphi) = |\mathbf{j}_\alpha| \Phi^s(e_{(j_\alpha)_1}, \dots, e_{(j_\alpha)_{MK}})$. Since $|\mathbf{j}_\alpha|^\omega = \left(\frac{(MK)!}{\alpha!}\right)^\omega \leq |\mathbf{j}_\alpha| \, (MK)!^{\omega-1}$ we get

$$\begin{aligned}\left(\sum_{\alpha \in \Lambda(MK,N)} \|c_\alpha(\varphi)\|^\omega\right)^{\frac{1}{\omega}} &= \left(\sum_{\alpha \in \Lambda(MK,N)} |\mathbf{j}_\alpha|^\omega \left\|\Phi^s(e_{(j_\alpha)_1}, \dots, e_{(j_\alpha)_{MK}})\right\|^\omega\right)^{\frac{1}{\omega}} \\ &\leq (MK!)^{\frac{\omega-1}{\omega}} \left(\sum_{\alpha \in \Lambda(MK,N)} |\mathbf{j}_\alpha| \left\|\Phi^s(e_{(j_\alpha)_1}, \dots, e_{(j_\alpha)_{MK}})\right\|^\omega\right)^{\frac{1}{\omega}},\end{aligned}$$

and hence by the definition of the symmetric mapping Φ^s and Minkowski's inequality

$$\begin{aligned}&\left(\sum_{\alpha \in \Lambda(MK,N)} \|c_\alpha(\varphi)\|^\omega\right)^{\frac{1}{\omega}} \\ \leq\ & (MK)!^{\frac{\omega-1}{\omega}} \left(\sum_{\mathbf{i} \in \mathscr{M}(MK,N)} \|\Phi^s(e_{i_1}, \dots, e_{i_{MK}})\|^\omega\right)^{\frac{1}{\omega}} \\ =\ & (MK)!^{\frac{\omega-1}{\omega}} \left(\sum_{\mathbf{i} \in \mathscr{M}(MK,N)} \left\|\frac{1}{(MK)!} \sum_{\sigma \in S_{MK}} \Phi(e_{\sigma(i_1)}, \dots, e_{\sigma(i_{MK})})\right\|^\omega\right)^{\frac{1}{\omega}} \\ \leq\ & (MK)!^{\frac{\omega-1}{\omega}} \frac{1}{(MK)!} \sum_{\sigma \in S_{MK}} \left(\sum_{\mathbf{i} \in \mathscr{M}(MK,N)} \left\|\Phi(e_{\sigma(i_1)}, \dots, e_{\sigma(i_{MK})})\right\|^\omega\right)^{\frac{1}{\omega}} \\ =\ & (MK)!^{\frac{\omega-1}{\omega}} \left(\sum_{\mathbf{i} \in \mathscr{M}(MK,N)} \|\Phi(e_{i_1}, \dots, e_{i_{MK}})\|^\omega\right)^{\frac{1}{\omega}}.\end{aligned}$$

Finally, we see by assumption and (3.1) that

$$\begin{aligned}\left(\sum_{\alpha \in \Lambda(MK,N)} \|c_\alpha(\varphi)\|^\omega\right)^{\frac{1}{\omega}} &\leq C(MK)!^{\frac{\omega-1}{\omega}} \prod_{k=1}^K \|A_k\| \\ &\leq C(MK)!^{\frac{\omega-1}{\omega}} \left(\frac{M^M}{M!}\right)^K \prod_{k=1}^K \|P_k\|.\end{aligned}$$

□

3.2.1 Some Consequences

An immediate consequence of Theorem 3.1 combined with Corollary 2.16 is the following important special case.

Corollary 3.2. *Let $T \in \mathcal{L}_K(Y_1, \dots, Y_K; Y)$ be multiple $(r,1)$-summing and Y be a Banach space of cotype q with $1 \le r \le q$. Define*

$$\rho_M := \frac{qrM}{q + (M-1)r}$$

Then there is a constant $D \ge 1$ such that for each choice of polynomials $P_k \in \mathcal{P}_M(\ell_\infty; Y_k)$, $1 \le k \le K$ we have

$$\Bigg(\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = MK}} \| c_\alpha \left(T(P_1, \dots, P_K) \right) \|^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} \le D \prod_{k=1}^{K} \|P_k\| \,.$$

The proof shows that the best constant D in the upper inequality can be estimated by

$$D \le \sigma_M \pi_{r,1}^{mult}(T) (MK)!^{\frac{(q-1)M - q(\frac{1}{r} - \frac{1}{q})}{qM}} \left(\frac{M^M}{M!} \right)^K ,$$

where σ_M is the constant from Theorem 2.2 (depending only on M, K, r, q and $C_q(Y)$).

As a direct consequence of Corollary 3.1 we get the polynomial version of the Bennett-Carl Theorem given by Defant and Sevilla-Peris in [38, Theorem 4]. But since we will be able to give a better constant than in [38] we will come to this later (see Theorem 3.14).

Theorem 3.1 also implies a polynomial version of Kwapień's Theorem 1.3. If $S : \ell_\infty \to \ell_1$ and $R : \ell_1 \to \ell_p$ are operators, then he monomial series expansion of the 1-homogeneous polynomial $RS : \ell_\infty \to \ell_p$ is obviously given by $\sum_k RS(e_k) z_k$. By Kwapień's Theorem 1.3 we have that $\sum_k \|RS(e_k)\|_p^r < \infty$, where the (optimal) r is given by $\frac{1}{r} = 1 - |\frac{1}{p} - \frac{1}{2}|$. Combining Theorem 3.1 with the multilinear version of Kwapień's Theorem 2.15 we get the following extension.

Theorem 3.3. *Given $1 \le p < \infty$, there exists a constant $C > 0$ such that for each composition $QP : \ell_\infty \to \ell_p$ of an M-homogeneous polynomial $P : \ell_\infty \to \ell_1$ with an K-homogeneous polynomial $Q : \ell_1 \to \ell_p$ we have*

$$\Bigg(\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = MK}} \| c_\alpha(QP) \|_p^{r_M} \Bigg)^{\frac{1}{r_M}} \le C \|Q\| \|P\|,$$

where r_M is defined as in Theorem 2.15 by

$$r_M = \begin{cases} \frac{2M}{M+2-\frac{2}{p}} & \text{if } 1 \le p \le 2, \\ \frac{2M}{\frac{2M}{p}+1} & \text{if } 2 \le p \le \frac{2M}{M-1}, \\ 2 & \text{if } \frac{2M}{M-1} \le p \le \infty. \end{cases}$$

3.3 A Hypercontractive Bohnenblust-Hille Type Inequality

In this section we present a vector valued extension of the result of Defant, Frerick, Ortega, Ounaïes and Seip in Theorem 1.11, which states that there is a constant $C > 0$ such that for every $P \in \mathcal{P}_M(\ell_\infty^N; \mathbb{C})$ we have

$$\left(\sum_{|\alpha|=M} |c_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq C^M \|P\|.$$

In particular, we show that if Y is a q-concave Banach lattice then the inequality in Corollary 3.2 is in the special case $K = 1$ hypercontractive. By this we mean the following Theorem.

Theorem. *Let Y be a q-concave Banach lattice, with $2 \leq q < \infty$, and $v : X \to Y$ an $(r, 1)$-summing operator with $1 \leq r \leq q$. Then there is a constant $C \geq 1$ such that for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty; X)$ we have*

$$\left(\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha|=M}} \|vc_\alpha\|_Y^{\frac{qrM}{q+(M-1)r}} \right)^{\frac{q+(M-1)r}{qrM}} \leq C^M \|P\|.$$

But first of all we give a short survey of the basics of Banach lattices. For more details check [56] and [73].

3.3.1 An Excursus on Banach Lattices

Definition 3.4 (vector lattice, Banach lattice)**.** A partially ordered vector space X is called *vector lattice* if the following axioms are satisfied

(B1) For all $x, y, z \in X$: $x \leq y \Rightarrow x + z \leq y + z$.

(B2) For every $x \geq 0$ and every $a \in \mathbb{R}_+$: $ax \geq 0$.

(B3) For all $x, y \in X$ exist a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

A vector lattice X is called *Banach lattice* if X is a Banach space and

(B4) For all $x, y \in X$: $|x| \leq |y| \Rightarrow ||x|| \leq ||y||$, where the absolute value $|x|$ of $x \in X$ is defined by $|x| = x \vee (-x)$.

Definition 3.5 (lattice homomorphism)**.** Let X, Y be vector lattices and $T : X \longrightarrow Y$ a linear map. T is called a *lattice homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ and $T(X \wedge y) = T(x) \wedge T(y)$.

Remark 3.6. If $T : X \longrightarrow Y$ is a lattice homomorphism $x \leq y$ implies $Tx \leq Ty$. Indeed, $x \leq y$ implies $x - y = (x - y) \vee 0$ and hence $T(x) - T(y) = T((x - y) \vee 0) = T(x - y) \vee T(0) \geq 0$.

Krivine's functional calculus

The idea of this section is to define elements

$$\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$$

in a Banach lattice X, where $1 \leq p < \infty$ and $x_1, \dots, x_N \in X$. Therefore we shortly introduce a method known as Krivine's functional calculus, which is described [56, p. 40 ff] or [40, p. 327 ff.] in more detail.

Let $C(S_{\ell_\infty^N})$ be the set of all continuous functions on the unit sphere of ℓ_∞^N and $C_1(\mathbb{R}^N)$ the set of all continuous, 1-homogeneous functions $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ (i.e. for each $\lambda \geq 0$ is $g(\lambda t_1, \dots, \lambda t_N) = \lambda g(t_1, \dots, t_N)$), both of which are clearly Banach lattices. For each function $f \in C(S_{\ell_\infty^N})$ we can define the continuous 1-homogeneous function $\tilde{f} : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\tilde{f}(t) := \begin{cases} \|t\|_\infty f\left(\frac{t}{\|t\|_\infty}\right) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Thereby we get a lattice isomorphism

$$\Psi_N : C(S_{\ell_\infty^N}) \longrightarrow C_1(\mathbb{R}^N)$$
$$f \longmapsto \tilde{f}.$$

Theorem 3.7 (Krivine's functional calculus)**.** *Let $x_1, \dots, x_N$ be elements of a Banach lattice X. Then there exists a unique lattice homomorphism*

$$\Phi_X^{x_1,\dots,x_N} : C(S_{\ell_\infty^N}) = C_1(\mathbb{R}^N) \longrightarrow X$$

such that

- $\Phi_X^{x_1,\dots,x_N}(\varphi_i) = x_i$ *for* $1 \leq i \leq N$*, where* $\varphi_i(t_1, \dots, t_N) = t_i$.
- $\|\Phi_X^{x_1,\dots,x_N}(\varphi_i)(f)\| \leq \||x_1| \vee \dots \vee |x_N|\| \, \|f\|_{C_1(\mathbb{R}^N)}$ *for every* $f \in C_1(\mathbb{R}^N)$.

It is natural and convenient to denote the element $\Phi_X^{x_1,\dots,x_N}(f)$ by $f(x_1, \dots, x_N)$.

The following concept of q-concavity plays a fundamental role in the theory of Banach lattices and is closely related to the notion cotype.

Definition 3.8 (q-concave)**.** For $1 \leq q < \infty$ a Banach lattice X is called *q-concave* if there exists a constant $C > 0$ such that

$$\left(\sum_{n=1}^{N} \|x_n\|_X^q\right)^{\frac{1}{q}} \leq C \left\| \left(\sum_{n=1}^{N} |x_n|^q\right)^{\frac{1}{q}} \right\|_X$$

for every choice of vectors $x_1, \dots, x_N \in X$. The smallest constant in the upper inequality is denoted by $M_q(X)$.

Note that we have the following relations in Banach lattices.

1. A q-concave Banach lattice X with $q \geq 2$ is of cotype q (see e.g. [56, Proposition 1.f.3.] for the proof).
2. A Banach lattice X is of cotype 2 if and only if X is 2 concave (see e.g. [56, Proposition 1.f.16] for the proof).

3.3.2 An Inequality Due to Bayart

One crucial point in the proof of the Defant-Frerick-Ounaïes-Ortega-Seip Theorem 1.11 is the following inequality which is essentially due to Bayart [4, Theorem 9] (see also [5, Section 3.2]). We denote by μ^N the normalized Lebesgue measure on the N-dimensional Torus $\mathbb{T}^N := \{(z_1, \dots, z_N) \in \mathbb{C}^N \mid |z_n| = 1\}$.

Theorem 3.9. *Let $0 < p < q < \infty$. For every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; \mathbb{C})$ we have*

$$\left(\int_{\mathbb{T}^N} |P(z)|^q \, d\mu^N(z) \right)^{\frac{1}{q}} \leq \sqrt{\frac{q}{p}}^M \left(\int_{\mathbb{T}^N} |P(z)|^p \, d\mu^N(z) \right)^{\frac{1}{p}}.$$

Even if this result itself does not appear in [4] all arguments can be found there: A result of Weissler [76, Corollary 2.1] (and Bonami [19, Chapitre III Théorème 7] for the special case $p = 1$ and $q = 2$) states that for every $0 < p < q < \infty$ and for every polynomial $\sum_{n=1}^N a_n z^n$ in one complex variable we have

$$\left(\int_{\mathbb{T}} \Big| \sum_{n=1}^N a_n \left(\sqrt{\frac{p}{q}} z \right)^n \Big|^q d\mu(z) \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{T}} \Big| \sum_{n=1}^N a_n z^n \Big|^p d\mu(z) \right)^{\frac{1}{p}}. \tag{3.4}$$

Following precisely the proof of [4, Theorem 9] one gets the desired result (see also [28, Lemma 3.6]). To be more specific, if $P \in \mathcal{P}_M(\ell_\infty^N; \mathbb{C})$ then (3.4) and the Minkowski inequality give that

$$\begin{aligned}
&\sqrt{\frac{p}{q}}^M \left(\int_{\mathbb{T}^N} |P(z)|^q \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{T}^{N-1}} \int_{\mathbb{T}} \left| P\left(z_1 \sqrt{\frac{p}{q}}, \dots, z_N \sqrt{\frac{p}{q}} \right) \right|^q d\mu(z_N) d\mu^{N-1}(z_1, \dots, z_{N-1}) \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{T}^{N-1}} \left(\int_{\mathbb{T}} \left| P\left(z_1 \sqrt{\frac{p}{q}}, \dots, z_{N-1} \sqrt{\frac{p}{q}}, z_N \right) \right|^p d\mu(z_N) \right)^{\frac{q}{p}} d\mu^{N-1}(z_1, \dots, z_{N-1}) \right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}^{N-1}} \left| P\left(z_1 \sqrt{\frac{p}{q}}, \dots, z_{N-1} \sqrt{\frac{p}{q}}, z_N \right) \right|^q d\mu^{N-1}(z_1, \dots, z_{N-1} \right)^{\frac{p}{q}} d\mu(z_N)) \right)^{\frac{1}{p}}.
\end{aligned}$$

Repeating the same argument for the other coordinates $z_1, \dots, z_{N-1}$ we get the desired inequality in Theorem 3.9.

Bayart's Theorem in Banach lattices

With the Krivine calculus we can give a vector valued variant of Bayart's inequality (Theorem 3.9) in Banach lattices. To do this just observe that the functions

$$(c_\alpha)_{\alpha\in\Lambda(M,N)} \mapsto \Big(\int_{T^N} |P(z)|^p \, d\mu^N(z)\Big)^{\frac{1}{p}},$$

are 1-homogeneous and apply Theorem 3.7 to see that for $0 < p < q < \infty$ and every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$ with values in a Banach lattice X we have

$$\Big(\int_{\mathbb{T}^N} |P(z)|^q \, d\mu^N(z)\Big)^{\frac{1}{q}} \le \sqrt{\frac{q}{p}}^M \Big(\int_{\mathbb{T}^N} |P(z)|^p \, d\mu^N(z)\Big)^{\frac{1}{p}}. \tag{3.5}$$

The following vector valued variant of Bayart's inequality will be used frequently.

Lemma 3.10. *Let X be a q-concave Banach lattice, $2 \le q < \infty$. Then we have for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$ that*

$$\Big(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|_X^q\Big)^{\frac{1}{q}} \le M_q(X)\sqrt{2}^M \int_{\mathbb{T}^N} \|P(z)\|_X d\mu^N(z).$$

Proof. Since Y is q-concave with $q \ge 2$ and by the orthonormality of the monomials z^α in $\mathscr{L}_2(\mu^N)$ we have

$$\begin{aligned}\Big(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|_X^q\Big)^{\frac{1}{q}} &\le M_q(X)\Big\|\Big(\sum_{\alpha\in\Lambda(M,N)} |c_\alpha|^q\Big)^{\frac{1}{q}}\Big\|_X \\ &\le M_q(X)\Big\|\Big(\sum_{\alpha\in\Lambda(M,N)} |c_\alpha|^2\Big)^{\frac{1}{2}}\Big\|_X \\ &= M_q(X)\Big\|\Big(\int_{\mathbb{T}^N} |P(z)|^2 \, d\mu^N(z)\Big)^{\frac{1}{2}}\Big\|_X\end{aligned}$$

Then we have by (3.5)

$$\begin{aligned}\Big(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|_X^q\Big)^{\frac{1}{q}} &\le M_q(X)\sqrt{2}^M \Big\|\int_{\mathbb{T}^N} |P(z)| \, d\mu^N(z)\Big\|_X \\ &\le M_q(X)\sqrt{2}^M \int_{\mathbb{T}^N} \|P(z)\|_X \, d\mu^N(z).\end{aligned}$$ □

3.3.3 Mixed-Norm Inequalities

In this section we give some mixed-norm inequalities which will be of interest later on.

Lemma 3.11. *Let Y be a q-concave Banach lattice with $2 \leq q < \infty$, $M \in \mathbb{N}$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Then there is a constant C such that for every $N \in \mathbb{N}$ and every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$*

$$\left(\sum_{i_M=1}^{N} \Big(\sum_{\mathbf{j} \in \mathscr{J}(M-1,N)} \| \, |\mathbf{j}| \, v a_{\mathbf{j},i_M} \|_Y^q \Big)^{\frac{r}{q}} \right)^{\frac{1}{r}} \leq C \, \|P\| \, ,$$

where $a_{i_1,\dots,i_M} = A(e_{i_1}, \dots e_{i_M})$ denote the coefficients of the symmetric M-linear mapping A associated to P. The proof shows that for the best constant C in the upper inequality we have

$$C \leq M_q(Y) \sqrt{2}^{M-1} \pi_{r,1}(v) \left(1 + \frac{1}{M-1} \right)^{M-1} .$$

Proof. Since Y is a q-concave Banach lattice, with $2 \leq q < \infty$, and

$$\sum_{\mathbf{j} \in \mathscr{J}(M-1,N)} |\mathbf{j}| \, v a_{\mathbf{j},i_M} z_{\mathbf{j}} \in \mathcal{P}_{M-1}(\ell_\infty^N; Y)$$

Lemma 3.10 gives

$$\begin{aligned} & \left(\sum_{i_M=1}^{N} \Big(\sum_{\mathbf{j} \in \mathscr{J}(M-1,N)} \| |\mathbf{j}| \, v a_{\mathbf{j},i_M} \|_Y^q \Big)^{\frac{r}{q}} \right)^{\frac{1}{r}} \\ \leq \;& M_q(Y) \sqrt{2}^{M-1} \left(\sum_{i_M=1}^{N} \Big(\int_{\mathbb{T}^N} \Big\| \sum_{\mathbf{j} \in \mathscr{J}(M-1,N)} |\mathbf{j}| \, v a_{\mathbf{j},i_M} z_{\mathbf{j}} \Big\|_Y d\mu^N(z) \Big)^r \right)^{\frac{1}{r}} \end{aligned}$$

which by Minkowski's inequality and by the symmetry of the $a_{i_1 \dots i_M}$ is

$$\begin{aligned} \leq \;& M_q(Y) \sqrt{2}^{M-1} \int_{\mathbb{T}^N} \Big(\sum_{i_M=1}^{N} \Big\| \sum_{\mathbf{j} \in \mathscr{J}(M-1,N)} |\mathbf{j}| \, v a_{\mathbf{j},i_M} z_{\mathbf{j}} \Big\|_Y^r \Big)^{\frac{1}{r}} d\mu^N(z) \\ = \;& M_q(Y) \sqrt{2}^{M-1} \int_{\mathbb{T}^N} \Big(\sum_{i_M=1}^{N} \Big\| v \Big(\sum_{\mathbf{i} \in \mathscr{M}(M-1,N)} a_{\mathbf{i},i_M} z_{\mathbf{i}} \Big) \Big\|_Y^r \Big)^{\frac{1}{r}} d\mu^N(z). \end{aligned}$$

The operator v is $(r,1)$-summing. Hence for each $z \in \mathbb{T}^N$

$$\begin{aligned} \Big(\sum_{i_M=1}^{N} \Big\| v \Big(\sum_{\mathbf{i} \in \mathscr{M}(M-1,N)} a_{\mathbf{i},i_M} z_{\mathbf{i}} \Big) \Big\|_Y^r \Big)^{\frac{1}{r}} \;&\leq\; \pi_{r,1}(v) \sup_{x' \in B_{X'}} \sum_{i_M=1}^{N} \Big| x' \Big(\sum_{\mathbf{i} \in \mathscr{M}(M-1,N)} a_{\mathbf{i},i_M} z_{\mathbf{i}} \Big) \Big| \\ &=\; \pi_{r,1}(v) \sup_{x' \in B_{X'}} \sup_{y \in B_{\ell_\infty^N}} |x'(A(z, \dots, z, y))| \, . \end{aligned}$$

By the estimate of Harris (3.2) we get that this is

$$\begin{aligned} &\leq \pi_{r,1}(v)\left(1+\frac{1}{M-1}\right)^{M-1} \sup_{x'\in B_{X'}} |x'\circ P| \\ &= \pi_{r,1}(v)\left(1+\frac{1}{M-1}\right)^{M-1} \|P\|\,, \end{aligned}$$

what we wanted to show. □

Theorem 3.12. *Let Y be a q-concave Banach lattice with $2 \leq q < \infty$, $M \in \mathbb{N}$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Then there is a constant C such that for every $N \in \mathbb{N}$ and every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$, $P(z) = \sum b_{j_1\dots j_M} z_{j_1}\cdots z_{j_M}$, we have*

$$\left(\sum_{j_M=1}^{N}\left(\sum_{\substack{j_1,\dots,j_{M-1}:\\ j_1\leq\dots\leq j_M}} \|vb_{j_1\dots j_M}\|_Y^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \leq C\,\|P\|\,.$$

The proof shows that the best constant XC in the upper inequality can be estimated by

$$C \leq M M_q(Y)\sqrt{2}^{M-1}\pi_{r,1}(v)\left(1+\frac{1}{M-1}\right)^{M-1}.$$

Proof. We have

$$\begin{aligned} \left(\sum_{j_M=1}^{N}\left(\sum_{\substack{j_1,\dots,j_{M-1}:\\ j_1\leq\dots\leq j_M}} \|vb_{j_1\dots j_M}\|_Y^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} &= \left(\sum_{j_M=1}^{N}\left(\sum_{\substack{j_1,\dots,j_{M-1}:\\ j_1\leq\dots\leq j_M}} \|\,|j_1,\dots,j_M|\,va_{j_1,\dots,j_M}\|_Y^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{i_M=1}^{N}\left(\sum_{\mathbf{j}\in\mathscr{J}(M-1,N)} \|\,|\mathbf{j},i_M|\,va_{\mathbf{j},i_M}\|_Y^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}. \end{aligned}$$

For each $\mathbf{j} \in \mathscr{J}(M-1,N)$ and $1 \leq i_M \leq N$ we have that $|\mathbf{j}, i_M| \leq M|\mathbf{j}|$ since

$$\frac{|\mathbf{j},i_M|}{|\mathbf{j}|} = \frac{M!}{(M-1)!}\cdot\frac{|\{k \mid \mathbf{j}_k = 1\}|!}{|\{k \mid (\mathbf{j},i_M)_k = 1\}|!}\cdots\frac{|\{k \mid \mathbf{j}_k = N\}|!}{|\{k \mid (\mathbf{j},i_M)_k = N\}|!} \leq M. \tag{3.6}$$

Hence this is

$$\leq M\left(\sum_{i_M=1}^{N}\left(\sum_{\mathbf{j}\in\mathscr{J}(M-1,N)} \|\,|\mathbf{j}|\,va_{\mathbf{j},i_M}\|_Y^q\right)^{\frac{r}{q}}\right)^{\frac{1}{r}}.$$

Finally, Lemma 3.11 gives the conclusion. □

3.3.4 The Proof of the Hypercontractive Bohnenblust-Hille Type Inequality

Now we are in the position to prove the hypercontractive vector valued Bohnenblust-Hille theorem which we already mentioned above. Note that this theorem already appears in our publication [34, Theorem 5.3], but the constant we receive here is slightly better than the one given in [34].

Theorem 3.13. *Let Y be a q-concave Banach lattice, with $2 \leq q < \infty$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Define*

$$\rho_M := \frac{qrM}{q + (M-1)r}.$$

Then for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty; X)$ we have

$$\Bigg(\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = M}} \|vc_\alpha\|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} \leq M^{1-\frac{1}{\rho_M}} M_q(Y) \sqrt{2}^{M-1} \pi_{r,1}(v) \left(1 + \frac{1}{M-1}\right)^{M-1} \|P\|. \tag{3.7}$$

In particular Theorem 3.13 shows that there is a constant $C \geq 1$ such that for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty; X)$ the following holds

$$\Bigg(\sum_{\substack{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \\ |\alpha| = M}} \|vc_\alpha\|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} \leq C^M \|P\|.$$

Proof. Note that as in the proof of Theorem 3.1 it suffices to show that (3.7) holds for every N and every polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$. The case $r = q$ is a direct consequence of Lemma 3.10. Let now be $r < q$. We have

$$\begin{aligned}
\Bigg(\sum_{\alpha \in \Lambda(M,N)} \|vc_\alpha\|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} &= \Bigg(\sum_{\mathbf{j} \in \mathcal{J}(M,N)} \| |[\mathbf{j}]| \, va_{\mathbf{j}} \|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} \\
&= \Bigg(\sum_{\mathbf{i} \in \mathcal{M}(M,N)} \frac{1}{|[\mathbf{i}]|} \| |[\mathbf{i}]| \, va_{\mathbf{i}} \|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}} \\
&= \Bigg(\sum_{\mathbf{i} \in \mathcal{M}(M,N)} \Big\| |[\mathbf{i}]|^{1-\frac{1}{\rho_M}} \, va_{\mathbf{i}} \Big\|_Y^{\rho_M} \Bigg)^{\frac{1}{\rho_M}}
\end{aligned}$$

and with Lemma 2.3 and the symmetry of A this is

$$\begin{aligned}
&\leq \left[\prod_{m=1}^{M} \Bigg(\sum_{\mathbf{i} \in \mathcal{M}(\{m\},N)} \Bigg(\sum_{\mathbf{j} \in \mathcal{M}(\complement\{m\},N)} \Big\| |[(\mathbf{i},\mathbf{j})]|^{1-\frac{1}{\rho_M}} \, va_{(\mathbf{i},\mathbf{j})} \Big\|_Y^q \Bigg)^{\frac{r}{q}} \Bigg)^{\frac{1}{r}} \right]^{\frac{1}{M}}. \\
&= \Bigg(\sum_{i_M=1}^{N} \Bigg(\sum_{\mathbf{j} \in \mathcal{M}(M-1,N)} \Big\| |[\mathbf{j}, i_M]|^{1-\frac{1}{\rho_M}} \, va_{\mathbf{j},i_M} \Big\|_Y^q \Bigg)^{\frac{r}{q}} \Bigg)^{\frac{1}{r}}
\end{aligned}$$

Note that for any multi-index $\mathbf{j} \in \mathscr{M}(M-1,N)$ and $1 \leq i_M \leq N$ we have $|\mathbf{j}, i_M| \leq M\,|\mathbf{j}|$ (see (3.6)) and that $1 - \frac{1}{\rho_M} \leq 1 - \frac{1}{q}$. Hence,

$$\begin{aligned}\Big(\sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha\|_Y^{\rho_M}\Big)^{\frac{1}{\rho_M}} &\leq M^{1-\frac{1}{\rho_M}} \Bigg(\sum_{i_M=1}^{N}\Big(\sum_{\mathbf{j}\in\mathscr{M}(M-1,N)} \Big\||\mathbf{j}|^{1-\frac{1}{q}}\, va_{\mathbf{j},i_M}\Big\|_Y^{q}\Big)^{\frac{r}{q}}\Bigg)^{\frac{1}{r}}. \\ &= M^{1-\frac{1}{\rho_M}} \Bigg(\sum_{i_M=1}^{N}\Big(\sum_{\mathbf{j}\in\mathscr{J}(M-1,N)} \||\mathbf{j}|\, va_{\mathbf{j},i_M}\|_Y^{q}\Big)^{\frac{r}{q}}\Bigg)^{\frac{1}{r}}.\end{aligned}$$

Then Lemma 3.11 gives the conclusion. □

Recall that by the Bennett-Carl Theorem 1.4, for $1 \leq p \leq q \leq \infty$, the inclusion $\ell_p \hookrightarrow \ell_q$ is $(r,1)$-summing, where the optimal r is given by $1/r = 1/2 + 1/p - \max\{1/2, 1/q\}$. Thus, Theorem 3.13 implies the polynomial version of the Bennett-Carl Theorem given by Defant and Sevilla-Peris in [38, Theorem 4], but now we are able to show that this theorem is even hypercontractive.

Theorem 3.14. *Given $M \in \mathbb{N}$ and $1 \leq p \leq q \leq \infty$, define*

$$r_M = \begin{cases} \frac{2M}{M+2(\frac{1}{p}-\max\{\frac{1}{q},\frac{1}{2}\})} & \text{if } p \leq 2 \\ p & \text{if } p \geq 2. \end{cases}$$

Then there exists a constant $C > 0$ such that for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty;\ell_p)$ we have

$$\Big(\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=M}} \|c_\alpha\|_q^{r_M}\Big)^{\frac{1}{r_M}} \leq C^M \|P\|.$$

Proof. We consider three different cases.
The case $\mathbf{1 \leq p \leq q \leq 2}$. By the Bennett-Carl Theorem 1.4 the inclusion $\ell_p \hookrightarrow \ell_q$ is $(r,1)$-summing where $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$ and ℓ_q is known to be 2-concave. Then we can apply Theorem 3.13 to get the desired result.
The case $\mathbf{1 \leq p < 2 \leq q}$. By the Bennett-Carl Theorem $\ell_p \hookrightarrow \ell_2$ is $(p,1)$-summing. We can now use Theorem 3.13 together with $\|\cdot\|_q \leq \|\cdot\|_2$.
The case $\mathbf{2 \leq p}$. By Lemma 3.10 we have for every $P \in \mathcal{P}_M(\ell_\infty;\ell_p)$ that

$$\Big(\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=M}} \|c_\alpha\|_q^{p}\Big)^{\frac{1}{p}} \leq \Big(\sum_{\substack{\alpha\in\mathbb{N}_0^{(\mathbb{N})}\\|\alpha|=M}} \|c_\alpha\|_p^{p}\Big)^{\frac{1}{p}} \leq C^M \|P\|.$$

□

Part II

Vector Valued Dirichlet Series

4. Introduction and Motivation

A vector valued Dirichlet series is a series of the form

$$D(s) = \sum_{n \geq 1} a_n \frac{1}{n^s},$$

where the coefficients a_n are in a Banach space X and s is a complex variable. With $\mathcal{D}(X)$ we will denote the set of all Dirichlet series in X, and we abbreviate the set of all Dirichlet series in $\mathbb{C}$ with $\mathcal{D}$. A Dirichlet series is called M-homogeneous whenever the coefficients $a_n = 0$ for all indices n which do not have exactly M prime divisors according to their multiplicity. More precisely, every natural n has a unique prime factorization $n = p_1^{\alpha_1(n)} \cdots p_r^{\alpha_r(n)}$, where $p = (p_n)_n$ denotes the sequence $p_1 < p_2 < ..$ of prime numbers. If we write

$$\Omega(n) := \alpha_1(n) + \ldots + \alpha_r(n)$$

then an M-homogeneous Dirichlet series is of the form

$$\sum_{n:\, \Omega(n)=M} a_n \frac{1}{n^s}.$$

By $\mathcal{D}_M(X)$ we denote the set of all M-homogenous Dirichlet series in X and if $X = \mathbb{C}$ we write $\mathcal{D}_M$.

It turns out that if a Dirichlet series $D(s) = \sum a_n n^{-s}$ converges for some $s_0 \in \mathbb{C}$ it converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \operatorname{Re} s_0$, thus the maximal sets of convergence are half-planes in $\mathbb{C}$. This gives us three abscissas concerning the convergence of a Dirichlet series $D(s) = \sum a_n \frac{1}{n^s}$, the abscissa of convergence, the abscissa of absolute convergence and the abscissa of uniform convergence defined by

$$\begin{aligned}
\sigma_c(D) &:= \inf \{\sigma \in \mathbb{R} \mid D(\sigma) \text{ converges}\} \\
\sigma_a(D) &:= \inf \{\sigma \in \mathbb{R} \mid D(\sigma) \text{ converges absolutely}\} \\
\sigma_u(D) &:= \inf \{\sigma \in \mathbb{R} \mid D \text{ converges uniformly on } [\operatorname{Re} > \sigma]\}.
\end{aligned}$$

Clearly we have that

$$-\infty \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq \infty \quad \text{and} \quad \sigma_a - \sigma_c \leq 1.$$

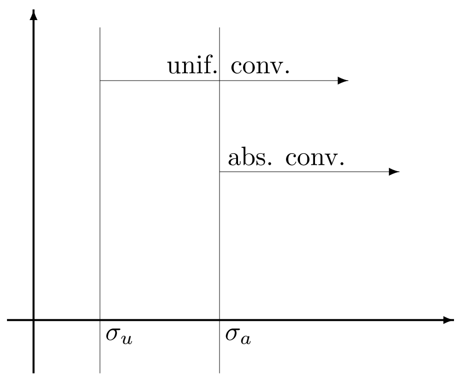
As in the scalar case (see e.g. [71, (1.4)]) these three abscissas are given by the following Hadamard type formulas. If $D(s) = \sum a_n \frac{1}{n^s}$ diverges at zero, we have that

$$\begin{aligned}
\sigma_c(D) &= \limsup_{N\to\infty} \frac{\log\left(\left\|\sum_{n=1}^N a_n\right\|_X\right)}{\log N} \\
\sigma_a(D) &= \limsup_{N\to\infty} \frac{\log\left(\sum_{n=1}^N \|a_n\|_X\right)}{\log N} \\
\sigma_u(D) &= \limsup_{N\to\infty} \frac{\log\left(\sup_{t\in\mathbb{R}}\left\|\sum_{n=1}^N a_n n^{-it}\right\|_X\right)}{\log N}.
\end{aligned}$$

There exists another characterization of the uniform convergence abscissa. To get it, we define the abscissa of boundedness $\sigma_b(D)$ to be the infimum of all $\sigma \geq \sigma_c$ such that the holomorphic function defined by D on the half-plane $[\mathrm{Re} > \sigma_c]$ is bounded on the smaller half-plane $[\mathrm{Re} > \sigma]$. It turns out that as is in the scalar case (proved by Bohr in [14, Satz 1]) the abscissa of boundedness and the abscissa of uniform convergence coincide (see e.g. [30]); for each Dirichlet series D we have that

$$\sigma_u(D) = \sigma_b(D). \tag{4.1}$$

Harald Bohr's so called *absolute convergence problem* from [13] asked for the largest possible width of the strip in $\mathbb{C}$ on which a (scalar valued) Dirichlet series converges uniformly but not absolutely:



In other terms, Bohr asked for the precise value of the number

$$S := \sup_{D\in\mathcal{D}} \left(\sigma_a(D) - \sigma_u(D)\right)$$

and managed in [13, Satz X] to show that

$$S \leq \frac{1}{2}.$$

Let us have a look at Bohr's methods adapted to the vector valued setting. Every $n \in \mathbb{N}$ defines a unique multi-index $\alpha(n) \in \mathbb{N}_0^{(\mathbb{N})}$ through its prime factorization $n = p_1^{\alpha_1(n)} \cdots p_r^{\alpha_r(n)}$. Thus there is a one-to-one correspondence between Dirichlet series and (formal) power series in infinitely many variables

$$\sum_{n \geq 1} a_n \frac{1}{n^s} \leftrightsquigarrow \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha, \quad \text{where } c_\alpha = a_{p^\alpha}$$

Note that for a Dirichlet series $\sum a_n n^{-s}$ we also sometimes write the corresponding power series as

$$\sum_{n \geq 1} a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$$

and switch between these two notations as it fits in the situation. Bohr's great idea was now to show that, if $D(s) = \sum_{n=1}^N a_n n^{-s}$ is a (finite) Dirichlet polynomial in X and if $P(z) = \sum_{n=1}^N a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)}$ is the corresponding polynomial, we have what in the literature is often called the *local version of Bohr's trick* (the vector valued case is a simple consequence of the Hahn-Banach Theorem)

$$\sup_{t \in \mathbb{R}} \|D(it)\| = \sup_{z \in B_{\ell_\infty^r}} \|P(z)\| \tag{4.2}$$

(see e.g. [30, Lemma 5]); note that the number r of variables of P is given by $r = \pi(N)$, where $\pi(N)$ as usual denotes the number of prime numbers less or equal to N.

Let us now describe the meaning of Bohr's absolute convergence problem in terms of power series in infinitely many variables, translated by modern terminology into infinite dimensional holomorphy.

Definition 4.1 (holomorphic function in Banach spaces)**.** For Banach spaces X and Y and an open subset U of X a function $f : U \to Y$ is called holomorphic if its Fréchet derivative exists. More precisely, if for all $\xi \in U$ there is a continuous linear mapping $A : X \to Y$ such that

$$\lim_{x \to \xi} \frac{\|f(x) - f(\xi) - A(x - \xi)\|}{\|x - \xi\|} = 0.$$

With $H(U, Y)$ we denote the vector space of all holomorphic functions $f : U \to Y$. We will denote $H(U) = H(U, \mathbb{C})$. Moreover, $H_\infty(U, Y)$ denotes the Banach spaces of all holomorphic functions $f \in H(U, Y)$ which are bounded on U endowed with norm $\|f\|_\infty := \sup_{x \in U} \|f(x)\|$.

In finite dimensions it is well-known that every holomorphic function $f : \mathbb{D}^N \to \mathbb{C}$ has a monomial series expansion, i.e. there is a unique family $(c_\alpha(f))_{\alpha \in \mathbb{N}_0^N}$ of coefficients such that for every $z \in \mathbb{D}^N$ we have

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha. \tag{4.3}$$

These coefficients can be calculated by the Cauchy integral formula

$$c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!} = \frac{1}{(2\pi i)^N} \int_{|z_1|=r} \cdots \int_{|z_N|=r} \frac{f(z)}{z^{\alpha+1}} d\mu^N(z),$$

where $0 < r < 1$ is arbitrary. For holomorphic functions $f : B_{\ell_\infty} \to \mathbb{C}$ in infinitely many variables, if restricted to $\mathbb{D}^N$, we can thus find an unique family $(c_\alpha^{(N)}(f))_{\alpha \in \mathbb{N}_0^N}$ such that (4.3) holds for every $z \in \mathbb{D}^N$. And by the Cauchy integral formula it is easily seen that $c_\alpha^{(N)}(f) = c_\alpha^{(N+1)}(f)$ for all $\alpha \in \mathbb{N}_0^N \subset \mathbb{N}_0^{N+1}$. Hence there is a unique family of coefficients $(c_\alpha(f))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ and f defines a formal power series $\sum c_{\alpha(f)} z^\alpha$. This clearly converges to $f(z)$ for finite sequences $z \in B_{\ell_\infty}$ and a natural question is: for which other sequences does this monomial series of f converge? We call the set $\operatorname{dom}(f) = \{z \in B_{\ell_\infty} \mid \sum |c_\alpha(f) z^\alpha| < \infty\}$ the domain of convergence of f and define

$$\operatorname{dom} H(B_{\ell_\infty}) = \bigcap_{f \in H(B_{\ell_\infty})} \operatorname{dom}(f).$$

An intensive study of domains of convergence for holomorphic functions defined on arbitrary Banach sequence spaces can be found in [70] or [33]. Back to Bohr's absolute convergence problem, he defined the number

$$K := \sup\{1 \le p \le \infty \mid \ell_p \cap B_{\ell_\infty} \subset \operatorname{dom} H(B_{\ell_\infty})\}$$

and used the prime number theorem to show that (see [13, Satz IX])

$$S = \frac{1}{K}.$$

Bohr was able to establish that $K \ge 2$ (and hence $S \le \frac{1}{2}$), but his problem was to find the exact value of K. He didn't even know if $K < \infty$, or in other words, he had no example of a Dirichlet series for which the abscissas σ_u and σ_a do not coincide. In [13, p. 446] he says

> "Um dies Problem zu erledigen ist ein tieferes Eindringen in die Theorie der Potenzreihen unendlich vieler Variablen nötig, als es mir in §3 gelungen ist."

Since Bohr did not find any reasonable way to determine the precise value of K, a problem in infinitely many variables, he returned to one dimension and got as a by-product of his effort what is nowadays called Bohr's power series theorem ([15, Section 3]).

Theorem 4.2 (Bohr's power series theorem). *For each bounded holomorphic function $f : \mathbb{D} \to \mathbb{C}$ we have*

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \le \sup_{z \in \mathbb{D}} |f(z)|$$

and the value $\frac{1}{3}$ is optimal.

Note first that the left side of the upper inequality can be rephrased as

$$\sup_{z\in\frac{1}{3}\mathbb{D}} \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} z^n \right| = \sum_{n=1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n}.$$

This motivates the following definition of the Bohr radius K_N for holomorphic functions on the N-dimensional polydisc $\mathbb{D}^N$ (also called the N-dimensional Bohr radius) which is due to Boas and Khavinson [11]. For $N \in \mathbb{N}$ the Nth Bohr radius K_N is the supremum taken over all $0 \leq r \leq 1$ such that for each holomorphic function $f \in H(\mathbb{D}^N)$ we have

$$\sup_{z\in r\mathbb{D}^N} \sum_{\alpha\in\mathbb{N}_0^N} |c_\alpha(f) z^\alpha| \leq \|f\|_\infty ,$$

where $c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!}$ are the coefficients of the monomial series expansion of f (see (4.3)). With this notation Bohr's power series theorem obviously reads

$$K_1 = \frac{1}{3}.$$

Boas and Khavinson established in [11, Theorem 2] that for $N > 1$

$$\frac{1}{3}\frac{1}{\sqrt{N}} \leq K_N \leq 2\sqrt{\frac{\log N}{N}}$$

(see [42, Theorem 3.2] of Dineen and Timoney for an earlier weaker version initiating the previous one). In [10, p. 239] Boas conjectured that *"... presumably this logarithmic factor, an artifact of the proof, should not really be present"*. This conjecture was disproved by Defant and Frerick in [27, Theorem 1.1]:

$$\sqrt{\frac{\log N}{N \log\log N}} \prec K_N.$$

The final result was recently given by Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip in [29, Theorem 2] using the hypercontractivity of the Bohnenblust-Hille inequality (Theorem 1.11):

$$K_N \asymp \sqrt{\frac{\log N}{N}}. \tag{4.4}$$

In Chapter 5 we define and give upper and lower estimates for the N-dimensional Bohr radius $K_N(v, \lambda)$ of certain operators $v : X \to Y$ between Banach spaces which is for $\lambda > \|v\|$ defined to be the supremum of all $r \geq 0$ such that such that for all holomorphic functions $f \in H(\mathbb{D}^N, X)$, $f(z) = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f) z^\alpha$, we have

$$\sup_{z\in r\mathbb{D}^N} \sum_{\alpha\in\mathbb{N}_0^N} \|v c_\alpha(f) z^\alpha\|_Y \leq \lambda \sup_{z\in\mathbb{D}^N} \Big\| \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f) z^\alpha \Big\|_X.$$

If v is the identity on X we use the notation $K_N(X,\lambda)$. The first step of Chapter 5 is to study the N-dimensional Bohr radius of Banach spaces. We show that in the finite dimensional case the asymptotic behaviour of $K_N(X,\lambda)$ is exactly the same as in the scalar case. In contrast to that, for infinite dimensional Banach spaces X the logarithmic term disappears. To be more specific, we show the following theorem.

Theorem 4.3. *Let X be a complex Banach space and $\lambda > 1$. With constants depending only on X and λ we have:*

(1) If X is finite dimensional then

$$K_N(X,\lambda) \asymp \sqrt{\frac{\log N}{N}}.$$

(2) If X is infinite dimensional of cotype q then

$$\frac{1}{N^{1-\frac{1}{q}}} \prec K_N(X,\lambda) \prec \frac{1}{N^{1-\frac{1}{\cot(X)}}}.$$

(3) In particular, if X has no finite cotype then

$$K_N(X,\lambda) \asymp \frac{1}{N}.$$

In the second part of Chapter 5 we are interested in the N-dimensional Bohr radius of certain operators $v : X \to Y$ between Banach spaces. Here we see that the logarithmic term which disappears in case of infinite dimensional Banach spaces sometimes turns up again. Using the vector valued extension of the hypercontractive Bohnenblust-Hille inequality from Part I Theorem 3.13 we study the numbers $K_N(v,\lambda)$ within the theory of $(r,1)$-summing operators and get to the following lower estimates.

Theorem 4.4. *Let $v : X \to Y$ be a bounded operator between Banach spaces and $\lambda > \|v\|$. With constants only depending on v, λ ,X, and Y we have*

(1) If X or Y are of cotype q, then

$$K_N(v,\lambda) \succ \left(\frac{1}{N}\right)^{1-\frac{1}{q}}.$$

(2) If Y is a q-concave Banach lattice with $2 \le q < \infty$ and there is an $1 \le r < q$ such that v is $(r,1)$-summing, then

$$K_N(v,\lambda) \succ \left(\frac{\log N}{N}\right)^{1-\frac{1}{q}}.$$

The second topic of this part also arises from Bohr's absolute convergence problem which asks for the maximal width of the strip of uniform but not absolute convergence

$$S = \sup_{D \in \mathcal{D}} (\sigma_u(D) - \sigma_a(D)).$$

Bohnenblust and Hille solved in 1931 Bohr's absolute convergence theorem by giving an example of a Dirichlet series for which the difference $\sigma_a - \sigma_u$ equals $\frac{1}{2}$. More precisely, in [12, Theorem VII] they used their famous inequality (1.8) to create for any given $0 \leq \sigma \leq \frac{1}{2}$ a Dirichlet series such that $\sigma_a - \sigma_u = \sigma$. This gives us what we now call the Bohr-Bohnenblust-Hille theorem.

Theorem 4.5 (Bohr-Bohnenblust-Hille)**.** *The maximal width of the strip of uniform but not absolute convergence for Dirichlet series is*

$$S = \frac{1}{2}.$$

The ingenious idea of Bohnenblust and Hille was to reduce the problem to M-homogeneous Dirichlet series. Implicitly, they defined in their proof the width of Bohr's strip restricted to M-homogeneous Dirichlet series

$$S_M = \sup_{D \in \mathcal{D}_M} \left(\sigma_a(D) - \sigma_u(D)\right),$$

and showed that

$$S_M = \frac{M-1}{2M}.$$

Letting M tend to infinity this gives, since $S_M \leq S$, that $S \geq \frac{1}{2}$.

In [30] Defant, García, Maestre, and Pérez-García started to study the width of Bohr's strip for vector valued Dirichlet series. That is, given a non-zero operator $v : X \to Y$ between two Banach spaces, the number

$$S(v) := \sup_{D \in \mathcal{D}(X)} \left(\sigma_a(vD) - \sigma_u(D)\right),$$

where, given a Dirichlet series $D(s) = \sum a_n n^{-s}$ in X, vD denotes the Dirichlet series $\sum v a_n n^{-s}$ in Y. And similarly we define for $M \in \mathbb{N}$ the number

$$S_M(v) := \sup_{D \in \mathcal{D}_M(X)} \left(\sigma_a(vD) - \sigma_u(D)\right).$$

If v is the identity on X we write $S(X)$ and $S_M(X)$. It turns out that for any finite dimensional X we still have that $S(X) = \frac{1}{2}$ and $S_M(X) = \frac{M-1}{2M}$. But if X is infinite dimensional these two numbers coincide and only depend on the optimal cotype of X. More precisely, the main result from [30, Theorem 1.1] says that

$$S(X) = S_M(X) = 1 - \frac{1}{\cot(X)}. \tag{4.5}$$

For many Banach spaces this optimal cotype is well-known. As an example we have

$$S(\ell_p) = S_M(\ell_p) = \begin{cases} \frac{1}{2} & \text{if } 1 \leq p \leq 2 \\ 1 - \frac{1}{p} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

This means that for Bohr's strips of infinite dimensional Banach spaces there is no difference between arbitrary Dirichlet series and M-homogeneous Dirichlet polynomials. In [39] Defant and Sevilla-Peris made a careful study of the ℓ_p case and showed in [39, Theorem 1.1] how to "find the polynomials back"

$$S_M(\ell_p \hookrightarrow \ell_q) = \begin{cases} \frac{M-2(\frac{1}{p}-\frac{1}{q})}{2M} & \text{if } 1 \leq p \leq q \leq 2 \\ \frac{M-2(\frac{1}{p}-\frac{1}{2})}{2M} & \text{if } 1 \leq p \leq 2 \leq q \leq \infty \\ 1 - \frac{1}{p} & \text{if } 2 \leq p \leq q \leq \infty. \end{cases}$$

Maurizi and Queffélec observed in [60, Theorem 2.4] that the maximal width S of Bohr's strip equals the infimum of all $\sigma \geq 0$ for which there exists a constant $C > 0$ such that for all N and all $a_1, \ldots, a_N \in \mathbb{C}$ we have

$$\sum_{n=1}^{N} |a_n| \leq C N^{\sigma} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

This motivates the following definition. Given a natural number N, let Q_N be the best constant $C_N \geq 1$ such that for each choice of $a_1, \ldots, a_N$ in $\mathbb{C}$

$$\sum_{n=1}^{N} |a_n| \leq C_N \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

By using the hypercontractive Bohnenblust-Hille theorem mentioned above Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip gave in [29, Theorem 3] the following optimal lower and upper bound for the Nth Queffélec number Q_N. The result completed a long process started by Queffélec [71] in the mid nineties, continued by Konyagin and Queffélec [53, Theorem 4.3] in 2002 and by de la Bretèche [26, Théorèm 1.1] in 2008.

Theorem 4.6 (Defant-Frerick-Ortega-Ounaïes-Seip)**.**

$$Q_N = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}} + o(1))\sqrt{\log N \log\log N}}}.$$

The above theorem has an important consequence on Bohr's strip itself. Recall that by (4.1) for each Dirichlet series $D(s) = \sum a_n n^{-s}$ the abscissa $\sigma_u(D)$ of uniform convergence equals the abscissa $\sigma_b(D)$ of boundedness. When discussing the Bohr-Bohnenblust-Hille theorem it is therefore quite natural to introduce the space $\mathcal{H}^\infty$, which consists of those bounded and analytic functions f on $[\mathrm{Re} > 0]$ such that f can be represented by a Dirichlet series on some half-plane (and then as a consequence even on

[Re > 0]). The Bohr-Bohnenblust-Hille Theorem states that the Dirichlet series defining a function $f \in \mathcal{H}^\infty$ converges absolutely on [Re $= \frac{1}{2} + \varepsilon$] and the value $\frac{1}{2}$ is optimal. By a deep result of Balasubramanian, Calado, and Queffélec [2, Theorem 1.2] each such Dirichlet series even converges absolutely on the vertical line [Re $= \frac{1}{2}$]. The following result is an improved version of the Balasubramanian-Calado-Queffélec Theorem and allows us to look at Bohr's strip in microscopic detail.

Corollary 4.7 (Defant-Frerick-Ortega-Ounaïes-Seip)**.** *The supremum of the set of real numbers numbers C such that for every $\sum a_n n^{-s} \in \mathcal{H}^\infty$ we have*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\frac{1}{2}}} e^{C\sqrt{\log n \log\log n}} < \infty$$

equals $\frac{1}{\sqrt{2}}$.

In Chapter 6 we define and study the Queffélec numbers $Q_N(v)$ for operators $v : X \to Y$ between Banach spaces, which are defined to be the best constant $C_N \geq 1$ such that for every choice of $a_1, \ldots, a_N \in X$ we have

$$\sum_{n=1}^{N} \|va_n\|_Y \leq C_N \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_X .$$

If v is the identity on X we use the notation $Q_N(X)$. We will show that the numbers $Q_N(v)$ still characterize the width $S(v)$ of Bohr's strip. For example we show that the following formula holds

$$S(v) = \limsup \frac{\log Q_N(v)}{\log N}.$$

In the first part of Chapter 6 we study the Queffélec numbers of Banach spaces. We show that for every finite dimensional Banach space X the asymptotic behaviour of $Q_N(X)$ is exactly the same as in the scalar case. In contrast to that in the infinite dimensional case the exponential term disappears and the asymptotic growth on $Q_N(X)$ depends on the geometry of X. More precisely, we show the following

Theorem 4.8. *Let X be a Banach space. Then with constants depending only on X we have:*

(1) For finite dimensional X

$$Q_N(X) = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}} + o(1))\sqrt{\log N \log\log N}}} .$$

(2) For infinite dimensional X and any $\varepsilon > 0$

$$N^{1 - \frac{1}{\cot(X)}} \prec Q_N(X) \prec N^{1 - \frac{1}{\cot(X) + \varepsilon}} .$$

In the second part of Chapter 6 we study the numbers $Q_N(v)$ for certain operators $v : X \to Y$ between Banach spaces, within the theory of $(r, 1)$-summing operators. Here the vector valued extension of the hypercontractive Bohnenblust-Hille inequality will play a fundamental role. We will see that in some situations the exponential term for infinite dimensional Banach spaces turns up again. Among others we show the following theorem.

Theorem 4.9. *Let Y be a q-concave Banach lattice and $v : X \to Y$ an $(r, 1)$-summing operator with $1 \leq r < q$. Then*

$$Q_N(v) \leq \frac{N^{1-\frac{1}{q}}}{e^{\left(2\frac{q-1}{q}\sqrt{\frac{1}{r}-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}.$$

5. Bohr Radii of Vector Valued Holomorphic Functions

The aim of this chapter is to give upper and lower estimates for the N-dimensional Bohr radius of vector valued holomorphic functions.

Definition 5.1. Let $0 \neq v : X \to Y$ be a bounded (linear) operator between complex Banach spaces, $N \in \mathbb{N}$, and $\lambda \geq \|v\|$. The λ-Bohr radius of v, denoted by $K_N(v, \lambda)$, is the supremum over all $r \geq 0$ such that for all holomorphic functions $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha$ on $\mathbb{D}^N$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \mathbb{N}_0^N} \|v c_\alpha(f) z^\alpha\|_Y \leq \lambda \sup_{z \in \mathbb{D}^N} \Big\| \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha(f) z^\alpha \Big\|_X .$$

We write $K_N(v)$ whenever $\lambda = 1$. If v is the identity on X we use the notation $K_N(X, \lambda)$ and $K_N(X)$ and if $X = \mathbb{C}$ we write $K_N(\lambda)$ and K_N. The above inequality trivially holds for every non bounded holomorphic function on $\mathbb{D}^N$. Thus, in the sequel, we will assume the holomorphic function f to be bounded on $\mathbb{D}^N$.

We start with some comments on the case $N = 1$ and $X = \mathbb{C}$. Recall that Bohr's power series theorem states that for each bounded holomorphic function $f \in H(\mathbb{D})$ we have

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \leq \|f\|_\infty$$

and that the value $\frac{1}{3}$ is optimal. So, if $\frac{1}{3} \leq r < 1$, it is natural to ask for the existence (and the optimality) of a constant $C(r) \geq 1$ such that for each $f \in H(\mathbb{D})$ we have

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| r^n \leq C(r) \|f\|_\infty .$$

The Cauchy-Schwarz inequality immediately gives that $C(r) \leq (1-r^2)^{-1/2}$ for $\frac{1}{3} \leq r < 1$, since

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| r^n &\leq \Big(\sum_{n=0}^{\infty} r^{2n} \Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \Big)^{\frac{1}{2}} \\ &\leq (1-r^2)^{-\frac{1}{2}} \|f\|_2 \leq (1-r^2)^{-\frac{1}{2}} \|f\|_\infty ; \end{aligned}$$

here we assume without loss of generality that f is defined on $\bar{\mathbb{D}}$ and $\|f\|_2$ denotes the L_2-norm of f with respect to the normalized Lebesgue measure on the Torus $\mathbb{T}$. By a result of Bombieri in [17] the exact value of this constant in the range $1/3 \leq r \leq 1/\sqrt{2}$ is given by the formula

$$C(r) = \frac{1}{r}\left(3 - \sqrt{8(1-r^2)}\right),$$

and later Bombieri and Bourgain proved in [18, Theorem 1.1, 1.2] that

$$C(r) < (1-r^2)^{-1/2} \quad \text{for } r > 1/\sqrt{2},$$

and

$$C(r) \asymp (1-r^2)^{-1/2} \quad \text{as } r \to 1.$$

Note that the strictly increasing function $K_1(\mathbb{C}, \cdot) : [1, \infty[\to [1/3, 1[$ has as its inverse the function $C(\cdot) : [1/3, 1[\to [1, \infty[$. Hence Bombieri's result implies that for all $1 \leq \lambda \leq \sqrt{2}$

$$K_1(\mathbb{C}, \lambda) = \frac{1}{3\lambda - 2\sqrt{2(\lambda^2 - 1)}},$$

and for λ close to ∞

$$K_1(\mathbb{C}, \lambda) \asymp \frac{\sqrt{\lambda^2 - 1}}{\lambda}.$$

On the other hand, Blasco [7, Theorem 1.2] showed that for $X = \ell_p^2$ we have that

$$K_1(X, 1) = 0 \quad \text{for every } 1 \leq p \leq \infty.$$

This explains why we implemented the constant λ in the definition of the vector valued Bohr radii. We will see later, in Theorem 5.5, that for $\lambda > \|v\|$ we always have that $K_N(v, \lambda) > 0$.

5.1 Basic Properties of Bohr Radii

In order to obtain non trivial estimates for Bohr radii it is a fruitful strategy to study M-homogeneous polynomials first. Recall the basic notations on M-homogeneous polynomials given in Section 3.1, especially that c_α will always denote the monomial coefficients of the M-homogeneous polynomial $P \in \mathcal{P}_M(\mathbb{C}^N; X)$, $P(z) = \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha$.

The following definition is the M-homogeneous counterpart of Definition 5.1.

Definition 5.2 ($K_N^M(v)$)**.** Let $0 \neq v : X \to Y$ be a bounded (linear) operator between Banach spaces, $N, M \in \mathbb{N}$, and $\lambda \geq \|v\|$. We define $K_N^M(v, \lambda)$ to be the supremum of all $r \geq 0$ such that for every M-homogeneous polynomial $P \in \mathcal{P}_M(\mathbb{C}^N; X)$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \Lambda(M,N)} \|v c_\alpha z^\alpha\|_Y \leq \lambda \sup_{z \in \mathbb{D}^N} \Big\| \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha \Big\|_X.$$

The following basic facts hold for $K_N(v,\lambda)$

$$K_N(X,\lambda) \leq K_N(\mathbb{C},\lambda), \tag{5.1}$$

$$\max\left\{K_N\left(X,\lambda/\|v\|\right), K_N\left(Y,\lambda/\|v\|\right)\right\} \leq K_N(v,\lambda). \tag{5.2}$$

An it is straightforward to show that $K_N^M(v,\lambda)$ satisfies

$$K_N^M(v,\lambda) = \sup\left\{r \geq 0 \mid \forall P \in \mathcal{P}_M(\mathbb{C}^N;X) : \sup_{z\in\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha z^\alpha\| \leq \frac{\lambda}{r^M}\|P\|\right\}, \tag{5.3}$$

$$K_N^M(v,\lambda) = \sqrt[M]{\lambda} K_N^M(v,1). \tag{5.4}$$

Lemma 5.3. *For all $N, M \in \mathbb{N}$ and all $\lambda \geq 1$ we have*

$$K_N^{M+1}(v,\lambda) \leq \left(K_N^M(v,\lambda)\right)^{\frac{M}{M+1}}.$$

Proof. If $0 < r < K_N^{M+1}(v,\lambda)$ and $P : \mathbb{C}^N \longrightarrow X$ is an M-homogeneous polynomial, then $Q(z) = z_1 P(z)$ is an $(M+1)$-homogeneous polynomial. Thus

$$\sup_{z\in r\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha z^{\alpha+(1,0,\ldots,0)}\|_Y \leq \lambda \sup_{z\in\mathbb{D}^N} \Big\| \sum_{\alpha\in\Lambda(M,N)} c_\alpha z^{\alpha+(1,0,\ldots,0)} \Big\|_X.$$

In particular, by the maximum modulus theorem,

$$\sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha\|_Y r^{M+1} \leq \lambda \sup_{z\in\mathbb{T}^N} |z_1|\|P(z)\|_X = \lambda \sup_{z\in\mathbb{D}^N} \|P(z)\|_X.$$

We have obtained that

$$\sup_{z\in r^{1+\frac{1}{M}}\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha z^\alpha\|_Y = r \sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha\|_Y r^M \leq \lambda \sup_{z\in\mathbb{D}^N} \|P(z)\|_X,$$

i.e. $r^{1+\frac{1}{M}} \leq K_N^M(v,\lambda)$. Hence $K_N^{M+1}(v,\lambda)^{\frac{M+1}{M}} \leq K_N^M(v,\lambda)$. □

The following lemma links the N-dimensional Bohr radius $K_N(v,\lambda)$ with its M-homogeneous counterparts.

Lemma 5.4. *For $\lambda > \|v\|$ we have*

$$\frac{\lambda - \|v\|}{2\lambda - \|v\|} \inf_{M\in\mathbb{N}} \left\{K_N^M(v,\lambda)\right\} \leq K_N(v,\lambda) \leq \inf_{M\in\mathbb{N}} \left\{K_N^M(v,\lambda)\right\} \tag{5.5}$$

$$\frac{\lambda - \|v\|}{\lambda - \|v\| + 1} \inf_{M\in\mathbb{N}} \left\{K_N^M(v)\right\} \leq K_N(v,\lambda) \leq \lambda \inf_{M\in\mathbb{N}} \left\{K_N^M(v)\right\} \tag{5.6}$$

Proof. The right inequality of (5.5) is clear. We concentrate on the proof of the left one of (5.5). Take $f \in H_\infty(\mathbb{D}^N, X)$, $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha$, and $z_0 \in \mathbb{C}^N$ such that $\|z_0\|_\infty \leq \inf_{M \in \mathbb{N}} \left\{ K_N^M(v, \lambda) \right\}$. Clearly $0 < \frac{\lambda - \|v\|}{2\lambda - \|v\|} < 1$. Since $z_0 \in K_N^M(v,\lambda)\mathbb{D}^N$ for all M and by the Cauchy inequalities (see e.g. [41, p. 148]) we have

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^N} \left\| v c_\alpha \left(\tfrac{\lambda - \|v\|}{2\lambda - \|v\|} z_0 \right)^\alpha \right\|_Y &= \|v(c_0)\| + \sum_{M=1}^\infty \left(\tfrac{\lambda - \|v\|}{2\lambda - \|v\|} \right)^M \sum_{\alpha \in \Lambda(M,N)} \|v c_\alpha\|_Y |z_0^\alpha| \\ &\leq \|v\| \|c_0\| + \sum_{M=1}^\infty \left(\tfrac{\lambda - \|v\|}{2\lambda - \|v\|} \right)^M \lambda \Big\| \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha \Big\|_{\mathbb{D}^N} \\ &\leq \left(\|v\| + \lambda \sum_{M=1}^\infty \left(\tfrac{\lambda - \|v\|}{2\lambda - \|v\|} \right)^M \right) \|f\|_{\mathbb{D}^N} = \lambda \|f\|_{\mathbb{D}^N}. \end{aligned}$$

Thus

$$\frac{\lambda - \|v\|}{2\lambda - \|v\|} \inf_{M \in \mathbb{N}} \left\{ K_N^M(v, \lambda) \right\} \leq K_N(v, \lambda).$$

The proof for (2) is similar. Since $K_N^M(v,\lambda) = \sqrt[M]{\lambda} K_N^M(v)$ we actually get

$$K_N(v, \lambda) \leq \inf_{M \in \mathbb{N}} \left\{ \sqrt[M]{\lambda} K_N^M(v) \right\}.$$

For the other inequality we proceed as above, taking $\|z_0\| \leq \inf_{M \in \mathbb{N}} \left\{ K_N^M(v) \right\}$ and using that $z_0 \in K_N^M(v)\mathbb{D}^N$ for all M. Then we get that

$$\sum_{\alpha \in \mathbb{N}_0^N} \left\| v c_\alpha \left(\tfrac{\lambda - \|v\|}{\lambda - \|v\| + 1} z_0 \right)^\alpha \right\| \leq \lambda \|f\|_{\mathbb{D}^N}. \qquad \square$$

Now we show that for every bounded operator v and every $\lambda > \|v\|$ the λ-Bohr radius of v is positive.

Theorem 5.5. *Let $v : X \to Y$ be a non-null bounded operator between complex Banach spaces and $\lambda > \|v\|$, then there exists $C > 0$ such that for all $N \in \mathbb{N}$*

$$K_N(v, \lambda) \geq C \frac{1}{N},$$

where

$$C = \begin{cases} \max \left\{ \frac{\lambda - \|v\|}{2\lambda - \|v\|}, \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1)\|v\|} \right\}, & \text{if } \|v\| \geq 1. \\ \\ \max \left\{ \frac{\lambda - \|v\|}{2\lambda - \|v\|}, \frac{\lambda - \|v\|}{\lambda - \|v\| + 1} \right\}, & \text{if } 0 < \|v\| < 1. \end{cases}$$

Proof. For every M-homogeneous polynomial $P \in \mathcal{P}_M(\mathbb{C}^N; X)$ we have that

$$\begin{aligned}\sup_{z \in \frac{1}{N}\mathbb{D}^N} \sum_{\alpha \in \Lambda(M,N)} \|v(c_\alpha) z^\alpha\| &\leq \max_{\alpha \in \Lambda(M,N)} \|v(c_\alpha)\| \sup_{z \in \frac{1}{N}\mathbb{D}^N} \sum_{\alpha \in \Lambda(M,N)} |z^\alpha| \\ &\leq \|v\| \max_{\alpha \in \Lambda(M,N)} \|c_\alpha\| \sup_{z \in \frac{1}{N}\mathbb{D}^N} \big(|z_1| + \ldots + |z_N|\big)^M \\ &\leq \lambda \max_{\alpha \in \Lambda(M,N)} \|c_\alpha\|.\end{aligned}$$

But

$$\|c_\alpha\| = \Big\| \int_{|z_1|=1} \cdots \int_{|z_N|=1} \frac{P(z)}{z^{\alpha+(1,\ldots,1)}} dz_1 \ldots dz_N \Big\| \leq \|P\|_{\mathbb{D}^N},$$

which implies that

$$K_N^M(v, \lambda) \geq \frac{1}{N}$$

for all M. Hence, by (5.5) we have

$$K_N(v, \lambda) \geq \frac{\lambda - \|v\|}{(2\lambda - \|v\|)} \frac{1}{N}.$$

On the other hand arguing as above, we get for every M-homogeneous polynomial $P \in \mathcal{P}_M(\ell_\infty^N; X)$ that

$$\sup_{z \in \frac{1}{\sqrt[M]{\|v\|}N}\mathbb{D}^N} \sum_{\alpha \in \Lambda(M,N)} \|v(c_\alpha) z^\alpha\| = \frac{1}{\|v\|} \sup_{z \in \frac{1}{N}\mathbb{D}^N} \sum_{\alpha \in \Lambda(M,N)} \|v(c_\alpha) z^\alpha\| \leq \|P\|_{\mathbb{D}^N}.$$

Thus we have for every M

$$K_N^M(v) \geq \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1) \sqrt[M]{\|v\|}} \frac{1}{N}.$$

Now by using (5.6) we obtain

$$K_N(v, \lambda) \geq \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1)\|v\|} \frac{1}{N}, \quad \text{if } \|v\| \geq 1$$

and

$$K_N(v, \lambda) \geq \frac{\lambda - \|v\|}{\lambda - \|v\| + 1} \frac{1}{N}, \quad \text{if } 0 < \|v\| < 1. \qquad \square$$

Let us observe that

$$\max\Big\{\frac{\lambda - \|v\|}{2\lambda - \|v\|}, \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1)\|v\|}\Big\} = \begin{cases} \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1)\|v\|} & \text{if } 1 \leq \|v\| \leq 2 \\ \frac{\lambda - \|v\|}{2\lambda - \|v\|} & \text{if } 2 < \|v\| \end{cases}$$

and that

$$\max\left\{\frac{\lambda - \|v\|}{2\lambda - \|v\|}, \frac{\lambda - \|v\|}{(\lambda - \|v\| + 1)}\right\} = \begin{cases} \frac{\lambda - \|v\|}{2\lambda - \|v\|} & \text{if } 0 < \|v\| < \lambda \leq 1 \\ \frac{\lambda - \|v\|}{\lambda - \|v\| + 1} & \text{if } 0 < \|v\| < 1 < \lambda. \end{cases}$$

In particular, we obtain the following lower estimate for the Bohr radius of a complex Banach space.

Corollary 5.6. *Let X be a complex Banach space and $\lambda > 1$, then*

$$K_N(X, \lambda) \geq \frac{\lambda - 1}{\lambda} \frac{1}{N},$$

for all $N \in \mathbb{N}$.

This corollary gives us the following general estimates for the Bohr radius $K_N(v, \lambda)$. For every $\lambda > \|v\|$

$$\frac{1}{N} \prec K_N(v, \lambda) \prec \sqrt{\frac{\log N}{N}}, \tag{5.7}$$

where the upper bound is an almost immediate consequence of the upper bound for the scalar case given in (4.4).

5.2 Bohr Radii of Banach Spaces

In this section we focus on the case $v = \mathrm{id}_X$. We show that for finite dimensional Banach spaces X the asymptotic behaviour of $K_N(X, \lambda)$ is exactly like in the scalar case (4.4). But for infinite dimensional Banach spaces the logarithmic term disappears.

Theorem 5.7. *Let X be a complex Banach space and $\lambda > 1$. With constants depending only on X and λ we have:*

(1) If X is finite dimensional then

$$K_N(X, \lambda) \asymp \sqrt{\frac{\log N}{N}}.$$

(2) If X is infinite dimensional of cotype q then

$$\frac{1}{N^{1-\frac{1}{q}}} \prec K_N(X, \lambda) \prec \frac{1}{N^{1-\frac{1}{\cot(X)}}}.$$

(3) In particular, if X has no finite cotype then

$$K_N(X, \lambda) \asymp \frac{1}{N}.$$

Proof of (1). For the lower bound we show that there is a constant $C(X) > 0$, only depending on X, such that for every N

$$C(X)\frac{\lambda-1}{2\lambda-1}\sqrt{\frac{\log N}{N}} \leq K_N(X,\lambda).$$

By (5.5) it is enough to show that for every N

$$\inf_M \left\{K_N^M(X,\lambda)\right\} \geq C(X)\sqrt{\frac{\log N}{N}}.$$

Take an $M \in \mathbb{N}$ and an M-homogeneous polynomial $P \in \mathcal{P}_M(\mathbb{C}^N; X)$. Then we have that

$$\sup_{z\in\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|c_\alpha z^\alpha\| = \sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\| \leq \left(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \left(\sum_{\alpha\in\Lambda(M,N)} 1\right)^{\frac{M-1}{2M}} \tag{5.8}$$

We consider the left factor of the upper product. Since X is finite dimensional the identity on X is absolutely summing and hence (see [40, Theorem 10.4]) s-summing for every $s \geq 1$ with $\pi_s(\mathrm{id}_X) \leq \pi_1(\mathrm{id}_X)$. Hence

$$\left(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \leq \pi_1(\mathrm{id}_X) \sup_{x'\in B_{X'}} \left(\sum_{\alpha\in\Lambda(M,N)} |x'(c_\alpha)|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}}.$$

The hypercontractivity of the Bohnenblust-Hille inequality (Theorem 1.11) gives us that there is a constant $C_1 > 0$ such that for all M and all $P \in \mathcal{P}_M(\mathbb{C}^N; X)$ we have

$$\begin{aligned}\left(\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha\|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} &\leq \pi_1(\mathrm{id}_X) C_1^M \sup_{x'\in B_{X'}} \sup_{z\in\mathbb{D}^N} \left| x'\Big(\sum_{\alpha\in\Lambda(M,N)} c_\alpha z^\alpha\Big)\right| \\ &= \pi_1(\mathrm{id}_X) C_1^M \|P\|_{\mathbb{D}^N}.\end{aligned}$$

For the right factor of (5.8) note that

$$\sharp\Lambda(M,N) = \binom{N+M-1}{M} \leq \frac{M^M}{M!}\cdot\frac{(N+M)^M}{M^M} \leq e^M\left(1+\frac{N}{M}\right)^M \tag{5.9}$$

This gives us that

$$\sup_{z\in\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|c_\alpha z^\alpha\| \leq \pi_1(\mathrm{id}_X) C_1^M e^{\frac{M-1}{2}} \left(1+\frac{N}{M}\right)^{\frac{M-1}{2}} \|P\|_{\mathbb{D}^N}.$$

and by (5.3) we obtain that

$$K_N^M(X,\lambda) \geq \sqrt[M]{\lambda}\left(1+\frac{N}{M}\right)^{-\frac{M-1}{2M}} \frac{1}{\sqrt[M]{\pi_1(\mathrm{id}_X)} C_1 e^{\frac{M-1}{2M}}}.$$

In short, there exists a constant $C_2(X) > 0$, only depending on X, such that for all M

$$K_N^M(X,\lambda) \geq C_2(X) \sqrt[M]{\lambda} \left(1 + \tfrac{N}{M}\right)^{-\frac{M-1}{2M}} .$$

Minimizing the right side of this inequality in M gives the conclusion: Obviously, we have for all M that

$$\begin{aligned}\left(1 + \tfrac{N}{M}\right)^{-\frac{M-1}{2M}} &\geq 2^{-\frac{M-1}{2M}} \min\left\{1, \left(\tfrac{N}{M}\right)^{-\frac{M-1}{2M}}\right\} \\ &\geq \frac{1}{\sqrt{2}} \min\left\{1, \left(\frac{MN^{1/M}}{N}\right)^{\frac{1}{2}}\right\}\end{aligned}$$

The function $f : (0, +\infty) \to \mathbb{R}$, $f(x) = xN^{\frac{1}{x}}$ attains a strict minimum $e \log N$ at $x = \log N$. Hence there is a constant $C(X)$ such that for all M and N

$$K_N^M(X,\lambda) \geq C(X) \sqrt[M]{\lambda} \sqrt{\frac{\log N}{N}},$$

and we obtain as desired

$$\inf_M \left\{K_N^M(X,\lambda)\right\} \geq C(X) \sqrt{\frac{\log N}{N}}.$$

For the upper bound we show that there is a constant $B > 0$ such that for all N

$$K_N(X,\lambda) \leq B\lambda^2 \sqrt{\frac{\log N}{N}}.$$

We know from the proof of [32, Lemma 4.3] that for the Arithmetic Bohr radius $A(\mathbb{D}^N, \lambda)$, which is defined in [33] as

$$A(\mathbb{D}^N, \lambda) = \sup\left\{\frac{1}{N}\sum_{i=1}^N r_i \mid r \in \mathbb{R}^N_{\geq 0}, \forall f \in H^\infty(\mathbb{D}^N) : \sum_{\alpha \in \mathbb{N}_0^N} |c_\alpha(f)| r^\alpha \leq \lambda \|f\|_{\mathbb{D}^N}\right\},$$

the inequality

$$K_N(\mathbb{C}, \lambda) \leq A(\mathbb{D}^N, \lambda)$$

holds for all $\lambda \geq 1$ and all N. But from [32, Theorem 4.7] there exists a uniform constant $B > 0$ such that

$$A(\mathbb{D}^N, \lambda) \leq B\lambda^{\frac{2}{\log N}} \sqrt{\frac{\log N}{N}}. \tag{5.10}$$

Thus (5.1) gives the desired result. □

The proof of (2) and (3) calls for the following definition.

Definition 5.8 (Finite factorization of $\ell_q \hookrightarrow \ell_\infty$)**.** For $2 \leq q < \infty$ we say that a Banach space X *finitely factors* $\ell_q \hookrightarrow \ell_\infty$ *for* $0 < \varepsilon < 1$ if for every $N \in \mathbb{N}$ we can find vectors $x_1, \ldots, x_N$ such that for every $z \in \mathbb{C}^N$

$$\frac{1}{1+\varepsilon} \left\| z \right\|_\infty \leq \Big\| \sum_{n=1}^{N} z_n x_n \Big\| \leq \left\| z \right\|_q .$$

In fact, if a Banach space X finitely factors $\ell_q \hookrightarrow \ell_\infty$ for some $0 < \varepsilon < 1$, then it does so for all of them (see e.g. [40, Proposition 14.4]).

Proof of (2) and (3). For the upper bound of both claims we use the following deep result of Maurey and Pisier in [59, Théorèm 1.1] (see also [40, Theorem 14.5]):

$$\cot(X) = \max \left\{ 2 \leq q \leq \infty \mid X \text{ finitely factors } \ell_q \hookrightarrow \ell_\infty \right\}.$$

This means in particular that for every ε and N there exist $x_1, \ldots, x_N$ such that on the one hand for every $1 \leq n \leq N$

$$\frac{1}{1+\varepsilon} = \frac{1}{1+\varepsilon} \left\| (0, \ldots, 1, 0, \ldots) \right\|_\infty \leq \| x_n \|$$

and on the other hand for every $z \in \mathbb{D}^N$

$$\Big\| \sum_{n=1}^{N} z_n x_n \Big\| \leq \| z \|_{\cot(X)}.$$

Hence,

$$\frac{N}{1+\varepsilon} \leq \sum_{n=1}^{N} \| x_n \| \leq \frac{1}{K_N^1(X)} \sup_{z \in \mathbb{D}^N} \Big\| \sum_{n=1}^{N} x_n z_n \Big\| \leq \frac{1}{K_N^1(X)} \sup_{z \in \mathbb{D}^N} \| z \|_{\cot(X)} = \frac{N^{\frac{1}{\cot(X)}}}{K_N^1(X)}$$

(meaning $N^{\frac{1}{\infty}} = 1$). This proves that for every $1 > \varepsilon > 0$

$$K_N^1(X) \leq \frac{1+\varepsilon}{N^{1 - \frac{1}{\cot(X)}}},$$

and then

$$K_N(X, \lambda) \leq K_N^1(X, \lambda) = \lambda K_N^1(X) \leq \frac{\lambda(1+\varepsilon)}{N^{1-\frac{1}{\cot(X)}}}. \tag{5.11}$$

For Banach spaces X without finite cotype the lower bound for $K_N(X, \lambda)$ was already stated in Corollary 5.6. We finally assume that X has cotype q. Given an M-homogeneous polynomial $P \in \mathcal{P}_M(\mathbb{C}^N; X)$, $P(z) = \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha$, we denote by $A \in \mathcal{L}_M(\mathbb{C}^N; X)$ its associated symmetric M-linear mapping. By a result of Bombal, Pérez-García, and Villanueva in [16, Theorem 3.2] we have

$$\Big(\sum_{\mathbf{i} \in \mathcal{M}(M,N)} \| A(e_{i_1}, \ldots, e_{i_M}) \|^q \Big)^{\frac{1}{q}} \leq C_q(X)^M \| A \|.$$

Consider $q^* > 1$ such that $\frac{1}{q^*} + \frac{1}{q} = 1$. Then, applying Hölder's inequality, we have for all $z \in \mathbb{C}^N$ that

$$\begin{aligned}\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha z^\alpha\| &= \sum_{\mathbf{j}\in\mathcal{J}(M,N)} |[\mathbf{j}]|\, \|A(e_{j_1},\ldots,e_{j_M})\| |z_{j_1}\ldots z_{j_M}| \\ &= \sum_{\mathbf{i}\in\mathcal{M}(M,N)} \|A(e_{i_1},\ldots,e_{i_M})\| |z_{i_1}\ldots z_{i_M}| \\ &\le \Big(\sum_{\mathbf{i}\in\mathcal{M}(M,N)} \|A(e_{i_1},\ldots,e_{i_M})\|^q\Big)^{\frac{1}{q}} \Big(\sum_{\mathbf{i}\in\mathcal{M}(M,N)} |z_{i_1}\ldots z_{i_M}|^{q^*}\Big)^{\frac{1}{q^*}} \\ &\le C_q(X)^M \|A\| \Big(\sum_{\mathbf{i}\in\mathcal{M}(M,N)} |z_{i_1}\ldots z_{i_M}|^{q^*}\Big)^{\frac{1}{q^*}} \\ &= C_q(X)^M \|A\| \big(|z_1|^{q^*} + \ldots + |z_N)|^{q^*}\big)^{\frac{M}{q^*}}.\end{aligned}$$

Hence, as a consequence of the polarization formula (see e.g. [41, Proposition 1.8]), we obtain for all $z \in \frac{1}{C_q(X)} B_{\ell_{q^*}^N}$

$$\sum_{\alpha\in\Lambda(M,N)} \|c_\alpha z^\alpha\| \le \|A\| \le \frac{M^M}{M!} \|P\| \le e^M \|P\|.$$

But $\frac{1}{C_q(X)N^{1/q^*}}\mathbb{D}^N$ is contained in $\frac{1}{C_q(X)} B_{\ell_{q^*}^N}$. Thus for every M-homogeneous polynomial P we get

$$\sup\Big\{ \sum_{\alpha\in\Lambda(M,N)} \|c_\alpha z^\alpha\| \mid z \in \frac{1}{eC_q(X)N^{1/q^*}}\mathbb{D}^N \Big\} \le \|P\|,$$

which implies that $K_N^M(X) \ge 1/eC_q(X)N^{1/q^*}$ for every M. Now by applying (5.6) we obtain that for every $N > 1$ and every $\lambda > 1$

$$\frac{\lambda-1}{\lambda} \frac{1}{eC_q(X)N^{1-\frac{1}{q}}} \le K_N(X,\lambda),$$

the conclusion. □

Note in particular that if $\cot(X)$ is attained we have that with constants only depending on λ and $\cot(X)$

$$K_N(X,\lambda) \asymp \frac{1}{N^{1-\frac{1}{\cot(X)}}}.$$

As an immediate consequence of Theorem 5.7 we obtain the asymptotically correct order of the Nth Bohr radius of ℓ_p-spaces.

Corollary 5.9. *With constants only depending on λ and p we have*

$$K_N(\ell_p,\lambda) \asymp \frac{1}{N^{1-\frac{1}{\max\{2,p\}}}}.$$

5.3 Bohr Radii of Operators

In this section we discuss the Bohr Radius of operators $v : X \to Y$ between Banach spaces within the theory of $(r, 1)$-summing operators (recall the Definition 1.2). Here our vector valued hypercontractive Bohnenblust-Hille Theorem 3.13 from part I will play a central role. We mainly focus our interest on the following two kind of operators

- v any of the embeddings $\ell_p \hookrightarrow \ell_q$ with $1 \leq p \leq q < \infty$
- v an arbitrary operator $\ell_1 \to \ell_q$ with $1 \leq q < \infty$.

These operators are well-understood and central in the theory of summing operators. Recall that the Bennett-Carl Theorem 1.4 states that the embedding $\ell_p \hookrightarrow \ell_q$, $1 \leq p \leq q < \infty$, is $(r, 1)$-summing with $1/r = 1/2 + 1/p - \max\{1/2, 1/q\}$ and by the Kwapień Theorem 1.3 every operator $v : \ell_1 \to \ell_p$, $1 \leq p \leq \infty$, is $(r, 1)$-summing with $1/2 = 1 - |1/p - 1/2|$.

Our results are the following.

Theorem 5.10. *Let $1 \leq p < q < \infty$. Then with constants depending only on λ and p, q*

$$K_N(\ell_p \hookrightarrow \ell_q, \lambda) \asymp \begin{cases} \sqrt{\frac{\log N}{N}} & \text{if } p < 2 \\ \left(\frac{1}{N}\right)^{1-\frac{1}{p}} & \text{if } p \geq 2 \end{cases}$$

Note that for $p < 2$ again a logarithmic term appears which is in contrast to the case $p = q$ and the case $p \geq 2$ where no such term appears.

Theorem 5.11. *Let $1 \leq q < \infty$ and $v : \ell_1 \to \ell_q$ be any bounded operator. Then with constants only depending on λ and v we have*

$$\left(\frac{\log N}{N}\right)^{1-\frac{1}{\max\{q,2\}}} \prec K_N(v, \lambda) \prec \sqrt{\frac{\log N}{N}}.$$

In particular, for $1 < q \leq 2$ we have

$$K_N(v, \lambda) \asymp \sqrt{\frac{\log N}{N}}.$$

Due to the theorems of Bennett-Carl and Kwapień it seems reasonable to handle the the upper theorems in the context of $(r, 1)$-summing operators. The lower bounds are each consequences of the following more abstract theorem.

Theorem 5.12. *Let $v : X \to Y$ be a bounded operator between Banach spaces and $\lambda > \|v\|$. With constants only depending on v, λ ,X, and Y we have*

(1) If X or Y are of cotype q, then

$$K_N(v, \lambda) \succ \left(\frac{1}{N}\right)^{1-\frac{1}{q}}.$$

(2) If Y is a q-concave Banach lattice with $2 \leq q < \infty$ and there is an $1 \leq r < q$ such that v is $(r,1)$-summing, then

$$K_N(v,\lambda) \succ \left(\frac{\log N}{N}\right)^{1-\frac{1}{q}}.$$

Proof. Note that (1) is already proved: In (5.2) and Theorem 5.7 we showed that

$$\begin{aligned} K_N(v,\lambda) &\geq \max\{K_N(X,\lambda/\|v\|), K_N(Y,\lambda/\|v\|)\} \\ &\geq \frac{\lambda-\|v\|}{\lambda}\frac{1}{e\min\{C_q(X),C_q(Y)\}}\left(\frac{1}{N}\right)^{1-\frac{1}{q}}. \end{aligned}$$

For the proof of (2) we follow the proof of Theorem 5.7 (1). By Hölder's inequality and our vector valued hypercontractive Bohnenblust-Hille type Theorem 3.13 and the we have that there is a constant $C > 0$, only depending on Y and v, such that for every $P \in \mathcal{P}_M(\ell_\infty^N; X)$

$$\begin{aligned} \sup_{z\in\mathbb{D}^N} \sum_{\alpha\in\Lambda(M,N)} \|vc_\alpha z^\alpha\|_Y &\leq \Big(\sum_{\alpha\in\Lambda(N,M)} 1\Big)^{\frac{(q-1)M-\frac{q}{r}+1}{qM}} \Big(\sum_{\alpha\in\Lambda(N,M)} \|c_\alpha\|_Y^{\frac{qrM}{q+(M-1)r}}\Big)^{\frac{q+(M-1)r}{qrM}} \\ &\leq \Big(\sum_{\alpha\in\Lambda(N,M)} 1\Big)^{\frac{(q-1)M-\frac{q}{r}+1}{qM}} C^M \|P\|_{D^N}. \end{aligned}$$

By (5.3), (5.4) and (5.9) there exists a constant $E > 0$ such that every M, N and $\lambda > \|v\|$

$$K_M^N(v,\lambda) \geq E\sqrt[M]{\lambda}\left(1+\frac{N}{M}\right)^{-\frac{(q-1)M-\frac{q}{r}+1}{qM}}.$$

Minimizing the right side of this inequality, as it is done in the proof of Theorem 5.7 (1), we see that there is a constant C such that for every N and $\lambda > \|v\|$

$$\inf_M \left\{K_M^N(v,\lambda)\right\} \geq C\left(\frac{\log N}{N}\right)^{1-\frac{1}{q}}$$

and finally Lemma 5.4 (1) gives the conclusion. □

We are now in the position to give the proofs of Theorem 5.10 and Theorem 5.11.

Proof of Theorem 5.10. We start with the upper bounds: Recall from (5.7) that for any operator $v : X \to Y$ and any $\lambda > \|v\|$we have

$$K_N(v,\lambda) \prec \sqrt{\frac{\log N}{N}}.$$

For the case $2 \leq p < \infty$ note first that

$$K_N(\ell_p \hookrightarrow \ell_q, \lambda) \leq K_N^1(\ell_p \hookrightarrow \ell_q, \lambda).$$

For each N have that

$$\begin{aligned} N = \sum_{n=1}^{N} \|e_n\|_q &\leq \frac{\lambda}{K_N^1(\ell_p \hookrightarrow \ell_q, \lambda)} \sup_{z \in \mathbb{D}^N} \Big\| \sum_{n=1}^{N} e_n z_n \Big\|_p \\ &= \frac{\lambda}{K_N^1(\ell_p \hookrightarrow \ell_q, \lambda)} \sup_{z \in \mathbb{D}^N} \|z\|_p = \frac{N^{\frac{1}{p}}}{K_N^1(\ell_p \hookrightarrow \ell_q, \lambda)}, \end{aligned}$$

which gives the remaining upper estimate. For the proof of the lower bounds we consider three different cases.

The case $\mathbf{1 \leq p < q \leq 2}$. By the Bennett-Carl Theorem 1.4 the inclusion $\ell_p \hookrightarrow \ell_q$ is $(r, 1)$-summing where $\frac{1}{r} == \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$. Since ℓ_q is known to be 2-concave, the lower estimate is a consequence of Theorem 5.12 (2).

The case $\mathbf{1 \leq p < 2 \leq q}$. Clearly, $K_N(\ell_p \hookrightarrow \ell_q, \lambda) \geq K_N(\ell_p \hookrightarrow \ell_2, \lambda)$, hence this case follows from the preceding one.

The case $\mathbf{2 \leq p}$. Note that $K_N(\ell_p \hookrightarrow \ell_q, \lambda) \geq K_N(\ell_p \hookrightarrow \ell_p, \lambda)$. Then we conclude the desired estimate from Corollary 5.9. □

Proof of Theorem 5.11. As in the preceding proof the upper estimate follows from (5.7). For the lower estimate we only have to combine Kwapień's Theorem 1.3, Theorem 5.12, and the well known fact that ℓ_q is $\max\{2, q\}$-concave. □

6. Estimates for Dirichlet polynomials in Banach spaces

Maurizi and Queffélec observed in [60, Theorem 2.4] that the maximal width S of Bohr's strip equals the infimum of all $\sigma \geq 0$ for which there exists a constant $C > 0$ such that for all N and all $a_1, \dots, a_N \in X$ we have

$$\sum_{n=1}^{N} |a_n| \leq C N^{\sigma} \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

The same holds for S_M, except that $a_n \neq 0$ implies that $\Omega(n) = M$. This motivates the following definition.

Definition 6.1 (Queffélec numbers $Q_N(v)$ and $Q_N^M(v)$)**.** Given natural N, M and a non-zero operator $v : X \to Y$ between Banach spaces, we define the Nth (M-homogeneous) Queffélec number $Q_N(v)$ (resp. $Q_N^M(v)$) of v to be the best constant $C \geq 1$ such that for each choice of $a_1, \dots, a_N \in X$ (resp. $a_1, \dots, a_N \in X$ such that $a_n = 0$ whenever $\Omega(n) \neq M$) we have

$$\sum_{n=1}^{N} \|v a_n\|_Y \leq C \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_X.$$

If v is the identity on X, the notation will be $Q_N(X)$ and $Q_N^M(X)$. Note that $Q_N(\mathbb{C}) = Q_N$ and $Q_N^M(\mathbb{C}) = Q_N^M$.

Following the ideas of Maurizi and Queffélec we get the following straight forward extension of their result, which shows that the Queffélec numbers $Q_N(v)$ and $Q_N^M(v)$ in a sense graduate the width of Bohr's strips $S(v)$ and $S_M(v)$ (defined in Chapter 4).

Proposition 6.2. *If $v : X \to Y$ is a non-zero operator between Banach spaces, then*

$$S(v) = \inf \{ \sigma \geq 0 \mid \exists C_\sigma \forall N : Q_N(v) \leq C_\sigma N^\sigma \}$$

and

$$S_M(v) = \inf \left\{ \sigma \geq 0 \mid \exists C_\sigma \forall N : Q_N^M(v) \leq C_\sigma N^\sigma \right\}.$$

The preceding proposition affords an interesting formula of the width $S(v)$ of Bohr's strip, similar to the Hadamard type formulas mentioned in Chapter 4.

Proposition 6.3. *If $v : X \to Y$ is a non-zero operator between Banach spaces, then*

$$S(v) = \limsup_{N \to \infty} \frac{\log Q_N(v)}{\log N}.$$

Proof. Take an $r > \limsup \frac{\log Q_N(v)}{\log N}$. Clearly, there is an N_0 such that $\log Q_N(v) \leq \log N^r$ for every $N \geq N_0$. So there is a constant C such that $Q_N(v) \leq CN^r$ for every N, which implies that $r \in \{\sigma \geq 0 \mid \exists C_\sigma \forall N : Q_N(v) \leq C_\sigma N^\sigma\}$. Hence

$$\limsup_{N\to\infty} \frac{\log Q_N(v)}{\log N} \geq \inf \{\sigma \geq 0 \mid \exists C_\sigma \forall N : Q_N(v) \leq C_\sigma N^\sigma\} = S(v).$$

On the other hand, we know from Proposition 6.2 that for every $\sigma > S(v)$ there is a constant C_σ such that $Q_N(v) \leq C_\sigma N^\sigma$ for every N, which implies that

$$\limsup_{N\to\infty} \frac{\log Q_N(v)}{\log N} \leq \limsup_{N\to\infty} \left(\frac{\log C_\sigma}{\log N} + \frac{\sigma \log N}{\log N} \right) = \sigma.$$

Hence

$$\limsup_{N\to\infty} \frac{\log Q_N(v)}{\log N} \leq S(v).$$ □

The aim of this chapter is to study the asymptotic behaviour of the Queffélec numbers $Q_N(v)$ and $Q_N^M(v)$ for certain operators $v : X \to Y$ between Banach spaces. In section 6.1 we deal with the case

- v the identity id_X on a Banach space X.

In the sections 6.2 and 6.3 we analyze the numbers $Q_N(v)$ and $Q_N^M(v)$ within the theory of $(r,1)$-summing operators (recall the Definition 1.2) by using the results of part I. Here we focus our interest on the important and well understood operators (Recall the Theorems 1.3 and 1.4 of Kwapień and Bennett-Carl)

- v any of the embeddings $\ell_p \hookrightarrow \ell_q$ with $1 \leq p \leq q < \infty$
- v an arbitrary operator $\ell_1 \to \ell_q$ with $1 \leq q < \infty$.

Note that the following upper and lower estimates hold for every non-zero operator $v : X \to Y$

$$\frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{\log N \log\log N}}} \leq Q_N(v) \prec N. \tag{6.1}$$

Proof. The lower estimate can be easily deduced from the scalar case (2). For the upper estimate first of all note that an easy calculation shows that for every $b_1, \ldots, b_N \in \mathbb{C}$ we have

$$\left(\sum_{n=1}^N |b_n|^2 \right)^{\frac{1}{2}} = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T \Big| \sum_{n=1}^N b_n n^{-it} \Big|^2 dt \right)^{\frac{1}{2}}; \tag{6.2}$$

just observe that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^T e^{it\log\frac{m}{n}} dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

For $a_1, \dots, a_N \in X$ the following chain of inequalities gives the conclusion

$$\begin{aligned}
\sum_{n=1}^{N} \|va_n\|_Y &\le N\|v\| \max_{1\le n\le N} \|a_n\|_X = N\|v\| \max_{1\le n\le N} \sup_{x'\in B_{X'}} |x'(a_n)| \\
&\le N\|v\| \sup_{x'\in B_{X'}} \Big(\sum_{n=1}^{N} |x'(a_n)|^2 \Big)^{\frac{1}{2}} = N\|v\| \sup_{x'\in B_{X'}} \Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Big| \sum_{n=1}^{N} x'(a_n) n^{-it} \Big|^2 dt \Big)^{\frac{1}{2}} \\
&\le N\|v\| \sup_{x'\in B_{X'}} \sup_{t\in\mathbb{R}} \Big| x'\Big(\sum_{n=1}^{N} a_n n^{-it} \Big) \Big| = N\|v\| \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_X . \qquad \square
\end{aligned}$$

6.1 Queffélec Numbers of Banach Spaces

Theorem 6.4. *Let X be a Banach space. Then with constants depending only on X we have:*

(1) For finite dimensional X

$$Q_N(X) = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{\log N \log\log N}}} .$$

(2) For infinite dimensional X and any $\varepsilon > 0$

$$N^{1-\frac{1}{\cot(X)}} \prec Q_N(X) \prec N^{1-\frac{1}{\cot(X)+\varepsilon}} .$$

Proof of (1). The lower estimate is always true (see (6.1)). For the upper estimate note that by the Dvoretzky-Rogers Theorem [44, Theorem 1] a Banach space X is finite dimensional if and only if the identity on X is absolutely summing, i.e. $(1,1)$-summing (see also [40, Theorem 1.2]). Hence for every choice of $a_1, \dots, a_N \in X$ we have

$$\begin{aligned}
\sum_{n=1}^{N} \|a_n\| &\le \pi_{1,1}(\mathrm{id}_X) \sup_{x'\in B_{X'}} \sum_{n=1}^{N} |x'(a_n)| \\
&\le \pi_{1,1}(\mathrm{id}_X) \sup_{x'\in B_{X'}} Q_N \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} x'(a_n) n^{-it} \Big| \\
&= \pi_{1,1}(\mathrm{id}_X) Q_N \sup_{t\in\mathbb{R}} \sup_{x'\in B_{X'}} \Big| x'\Big(\sum_{n=1}^{N} a_n n^{-it} \Big) \Big| \\
&= \pi_{1,1}(\mathrm{id}_X) Q_N \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\| .
\end{aligned}$$

Proof of (2). For the upper estimate we know from (4.5) and Proposition 6.2 that

$$1 - \frac{1}{\cot(X)} = \inf\{\sigma \ge 0 \mid \exists C_\sigma \forall N : Q_N(X) \le C_\sigma N^\sigma\} .$$

Hence for every $p > \cot(X)$ there is a constant C_p such that for all N

$$Q_N(X) \leq C_p N^{1-\frac{1}{p}}.$$

For the lower estimate recall the definition of the vector valued Bohr radius of M-homogeneous polynomials given in Definition 5.2: For $M, N \in \mathbb{N}$ and $\lambda \geq 0$ the number $K_N^M(X, \lambda)$ is defined to be the supremum of all $r > 0$ such that for all M-homogeneous polynomials $P \in \mathcal{P}({}^M\ell_\infty^N; X)$

$$\sup_{z \in B_{\ell_\infty^N}} \sum_{\alpha \in \Lambda(M,N)} \|c_\alpha z^\alpha\| \leq \frac{\lambda}{r^M} \sup_{z \in B_{\ell_\infty^N}} \Big\| \sum_{\alpha \in \Lambda(M,N)} c_\alpha z^\alpha \Big\|.$$

It is easily seen that

$$\sup_{z \in B_{\ell_\infty^N}} \sum_{\alpha \in \Lambda(M,N)} \|c_\alpha z^\alpha\| = \sum_{\alpha \in \Lambda(M,N)} \|c_\alpha\|$$

and hence

$$K_N^1(X, \lambda) = \sup\{r > 0 | \forall a_1, \ldots, a_N \in X : \sum_{n=1}^N \|a_n\| \leq \frac{\lambda}{r} \sup_{z \in B_{\ell_\infty^N}} \Big\| \sum_{n=1}^N a_n z_n \Big\| \}.$$

We know from (5.11) that $K_N^1(X, \lambda) \leq \frac{\lambda(1+\varepsilon)}{N^{1-\frac{1}{\cot(X)}}}$ for every $1 > \varepsilon > 0$ and by definition of $Q_N(X)$ we have that

$$\sum_{n=1}^N \|a_n\| \leq Q_N(X) \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^N a_n n^{-it} \Big\| \leq Q_N(X) \sup_{z \in B_{\ell_\infty^N}} \Big\| \sum_{n=1}^N a_n z_n \Big\|.$$

Hence hence for every $1 > \varepsilon < 0$

$$Q_N(X) \geq \frac{\lambda}{K_N^1(X, \lambda)} \geq \frac{N^{1-\frac{1}{\cot(X)}}}{1+\varepsilon}. \qquad \square$$

For ℓ_p spaces we know the optimal cotype 1.5. Then Theorem 6.4 gives quite precise estimates for the growth of $Q_N(\ell_p)$. In the case $p \geq 2$ we can even give the precise asymptotic behaviour.

Corollary 6.5. *With constants depending only on p we have that*

(1) *If* $1 \leq p < 2$

$$N^{\frac{1}{2}} \prec Q_N(\ell_p) \prec N^{1-\frac{1}{2+\varepsilon}},$$

(2) *If* $2 \leq p \leq \infty$

$$Q_N(\ell_p) \asymp N^{1-\frac{1}{p}}.$$

Proof. We show that in the case $p \geq 2$ there is no ε needed. The case $p = \infty$ is shown in (6.1). Using (6.2) we have for $2 \leq p < \infty$

$$\left(\sum_{n=1}^{N} \|a_n\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{l=1}^{\infty}\sum_{n=1}^{N} |a_n(l)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{l=1}^{\infty}\left(\sum_{n=1}^{N} |a_n(l)|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$
$$= \left(\sum_{l=1}^{\infty} \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{N} a_n(l)n^{-it}\Big|^2 dt\right)^{\frac{p}{2}}\right)^{\frac{1}{p}},$$

but for all L

$$\left(\sum_{l=1}^{L} \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{N} a_n(l)n^{-it}\Big|^2 dt\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \lim_{T\to\infty}\left(\sum_{l=1}^{L}\left(\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{n=1}^{N} a_n(l)n^{-it}\Big|^2 dt\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

which is by Minkowski's inequality

$$\leq \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}\left(\sum_{l=1}^{L}\Big|\sum_{n=1}^{N} a_n(l)n^{-it}\Big|^p\right)^{\frac{2}{p}} dt\right)^{\frac{1}{2}}$$
$$\leq \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big\|\sum_{n=1}^{N} a_n n^{-it}\Big\|_p^2 dt\right)^{\frac{1}{2}}$$
$$\leq \sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N} a_n n^{-it}\Big\|_p .$$

Hence

$$\sum_{n=1}^{N} \|a_n\|_p \leq N^{1-\frac{1}{p}}\left(\sum_{n=1}^{N} \|a_n\|^p\right)^{\frac{1}{p}} \leq N^{1-\frac{1}{p}} \sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N} a_n n^{-it}\Big\|_p .$$ □

6.2 Queffélec Numbers of Operators

In this section we give upper estimates for the Queffélec numbers

- $Q_N(\ell_p \hookrightarrow \ell_q)$ with $1 \leq p < q \leq \infty$,
- $Q_N(\ell_1 \to \ell_q)$ with $1 < q \leq 2$.

Again we recall the definition of an $(r,1)$-summing operator (Definition 1.2) and the results of Bennett-Carl (Theorem 1.4) and Kwapień (Theorem 1.3). Due to this it seems reasonable to attack our problem in the more abstract setting of $(r,1)$-summing operators.

Theorem 6.6. *Let Y be a q-concave Banach lattice with $2 \leq q < \infty$ and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r < q$. Then*

$$Q_N(v) \leq \frac{N^{\frac{q-1}{q}}}{e^{\left(2\frac{q-1}{q}\sqrt{\frac{1}{r}-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}$$

The proof follows carefully the proof of de la Bretèche in [26] in the scalar case together with its improvement of Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip in [29]. The crucial point there is the hypercontractivity of the Bohnenblust-Hille inequality (Theorem 1.11). Here we will use our vector valued hypercontractive Bohnenblust-Hille inequality from Theorem 3.13 in the following variant for Dirichlet series.

Lemma 6.7. *Let Y be a q-concave Banach lattice, with $2 \leq q < \infty$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Define*

$$\rho_M := \frac{qrM}{q+(M-1)r}.$$

Then there is a constant $C > 0$ such that for every $a_1, \ldots, a_N \in X$ we have

$$\Big(\sum_{\substack{n=1\\ \Omega(n)=M}}^{N} \|va_n\|^{\rho_M}\Big)^{\frac{1}{\rho_M}} \leq C^M \sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N} a_n n^{-it}\Big\|. \tag{6.3}$$

Proof. As a straightforward consequence of Theorem 3.13 and (4.2) we get that there is a constant C such that for every choice of $a_1, \ldots, a_N \in X$ we have

$$\Big(\sum_{\substack{n=1\\ \Omega(n)=M}}^{N} \|va_n\|^{\rho_M}\Big)^{\frac{1}{\rho_M}} \leq C^M \sup_{t\in\mathbb{R}}\Big\|\sum_{\substack{n=1\\ \Omega(n)=M}}^{N} a_n n^{-it}\Big\|.$$

We use the methods of Queffélec in [71, Theorem III-1] to show that

$$\sup_{t\in\mathbb{R}}\Big\|\sum_{\substack{n=1\\ \Omega(n)=M}}^{N} a_n n^{-it}\Big\| \leq \sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N} a_n n^{-it}\Big\|:$$

Consider for given $a_1, \ldots, a_N \in X$ and $r = \pi(N)$ the two polynomials

$$P(z) = \sum_{n=1}^{N} a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)}, \quad P_M(z) = \sum_{\substack{n=1\\ \Omega(n)=M}}^{N} a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)}.$$

Then we have for every $z = (z_1, \ldots, z_r) \in B_{\ell_\infty^r}$ that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P(z_1e^{it}, z_2e^{it}, \ldots, z_re^{it})\, e^{-Mit}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big(a_1 + \sum_{j=1}^{r} a_{p_j} z_j e^{it} + \sum_{j_1,j_2=1}^{r} a_{p_{j_1}p_{j_2}} z_{j_1} z_{j_2} e^{2it} + \ldots\Big)\, e^{-Mit}dt \\
&= P_M(z)
\end{aligned}$$

and hence

$$\sup_{t\in\mathbb{R}} \Big\| \sum_{\substack{n=1\\ \Omega(n)=M}}^{N} a_n n^{-it} \Big\| = \|P_M\| \leq \|P\| = \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\| \qquad \square$$

Remark 6.8. For $1 \leq r < q < \infty$, $N \in \mathbb{N}$, $0 < \lambda < 1$, and $y := \frac{(\log N)^\lambda}{\log\log N}$ we define the sequence in m

$$\begin{aligned} h^{q,r}_{N,y}(m) &:= m \log C - (\tfrac{1}{r} - \tfrac{1}{q}) \tfrac{1}{m} \log N - \tfrac{q-1}{q} m \log y \\ &\quad + (\tfrac{1}{r} - \tfrac{1}{q}) \log y + \tfrac{q-1}{q}(y-1) \log\log N + \tfrac{q-1}{q} D y, \end{aligned}$$

where D is a universal constant and $C = C(q,r)$ is a constant depending on q and r. For each N large enough the sequence $h^{q,r}_{N,y}(m)$ is increasing for $m \leq M_0 := \sqrt{\frac{q}{\lambda(q-1)}(\frac{1}{r} - \frac{1}{q}) \frac{\log N}{\log\log N}}$ decreasing afterwards.

Proof. As a function on $\mathbb{R}_{>0}$

$$\frac{\mathrm{d}}{\mathrm{d}m} h^{r,q}_{N,y}(m) = \log C + (\tfrac{1}{r} - \tfrac{1}{q}) \tfrac{1}{m^2} \log N - \tfrac{q-1}{q} \log y.$$

Hence $h^{r,q}_{N,y}$ has its maximum at

$$m = \sqrt{\frac{q}{\lambda(q-1)}(\tfrac{1}{r} - \tfrac{1}{q}) \frac{\log N}{\log\log N} \left(1 - \frac{\frac{1}{\lambda}\log\log\log N}{\log\log N} - \frac{\frac{q}{\lambda(q-1)}\log C}{\log\log N}\right)^{-1}}. \qquad \square$$

Lemma 6.9. *For* $y = \frac{\sqrt{\log N}}{\log\log N}$

$$h^{r,q}_{N,y}(M_0) \leq \left(-\sqrt{2\tfrac{q-1}{q}(\tfrac{1}{r} - \tfrac{1}{q})} + o(1)\right) \sqrt{\log N \log\log N}$$

and for any other $y = \frac{(\log N)^\lambda}{\log\log N}$ *the estimate is worse.*

Proof. For $\lambda \leq \frac{1}{2}$ we get

$$\begin{aligned} &\tfrac{1}{\sqrt{\log N \log\log N}}\Big(M_0 \log C - (\tfrac{1}{r} - \tfrac{1}{q}) \tfrac{1}{M_0} \log N - \tfrac{q-1}{q} M_0 \log y \\ &+ (\tfrac{1}{r} - \tfrac{1}{q}) \log y + \tfrac{q-1}{q}(y-1) \log\log N + \tfrac{q-1}{q} D y\Big) \\ &= -(\tfrac{1}{r} - \tfrac{1}{q}) \tfrac{1}{M_0} \sqrt{\tfrac{\log N}{\log\log N}} - \tfrac{q-1}{q} M_0 \tfrac{\log y}{\sqrt{\log N \log\log N}} + o(1) \\ &= -2\sqrt{\lambda \tfrac{q-1}{q}(\tfrac{1}{r} - \tfrac{1}{q})} + o(1) \end{aligned}$$

Note that for $\lambda > \frac{1}{2}$ the summands $\frac{(q-1) y \log\log N}{q\sqrt{\log N \log\log N}}$ and $\frac{(q-1) D y}{q\sqrt{\log N \log\log N}}$ do not converge. $\square$

Remark 6.10. For each natural n we clearly have that $n \geq 2^{\Omega(n)}$ and hence $\Omega(n) \leq \frac{\log n}{\log 2}$.

Now we are in the position to give the Proof of Theorem 6.6. We will follow the ideas of de la Bretèche's proof of the scalar case given in [26].

Proof of Theorem 6.6. For any natural n let $P^+(n)$ be the greatest prime factor of n and $P^-(n)$ the smallest prime factor of n, with the convention $P^+(1) = P^-(1) = 1$. For $N, m \in \mathbb{N}$ and any $y \leq N$ we define the following sets

$$\begin{aligned} S(N,y) &:= \big\{n \leq N \mid P^+(n) \leq y\big\} \\ T(N,y) &:= \big\{n \leq N \mid P^-(n) > y\big\} \\ T_m(N,y) &:= \{n \in T(N,y) \mid \Omega(n) = m\} \end{aligned}$$

Moreover, if we define

$$N_m(N,y) := |\{n \in T(N,y) \mid \Omega(n) \geq m\}| = \sum_{j \geq m} |T_j(N,y)|$$

we have by a far more general result due to Balazard [3, Corollaire 1] that there is an absolute constant $D > 0$ such that for any N, M, y

$$|T_m(N,y)| \leq N_m(N,y) \prec \frac{N}{y^m} (\log N)^{y-1} e^{Dy}. \tag{6.4}$$

Note that for any $y \leq N$ each $n \in \{1, \ldots, N\}$ can be uniquely decomposed in

$$n = kl, \quad \text{where } k \in S(N,y) \text{ and } l \in T(\tfrac{N}{k}, y), \tag{6.5}$$

more precisely, if $n = p_1^{\alpha_1} \cdots p_\mu^{\alpha_\mu}$ and $\nu = \pi(y)$ then n can be uniquely decomposed in $n = kl$ where $k = p_1^{\alpha_1} \cdots p_\nu^{\alpha_\nu}$ and $l = p_{\nu+1}^{\alpha_{\nu+1}} \cdots p_\mu^{\alpha_\mu}$. We take $N \in \mathbb{N}$ and define $y := \sqrt{\log N}/\log\log N$. Given a Dirichlet polynomial $D(s) = \sum_{n=1}^N a_n n^{-s}$ in X, let P be the associated polynomial $P(z) = \sum_{n=1}^N a_n z_1^{\alpha_1(n)} \cdots z_\mu^{\alpha_\mu(n)}$, where $\mu = \pi(N)$. With the decomposition in (6.5) we have that

$$P(z) = \sum_{n=1}^N a_n z_1^{\alpha_1(n)} \cdots z_\mu^{\alpha_\mu(n)} = \sum_{k \in S(N,y)} z_1^{\alpha_1(k)} \cdots z_\nu^{\alpha_\nu(k)} \sum_{l \in T(\frac{N}{k}, y)} a_{kl} z_{\nu+1}^{\alpha_{\nu+1}(l)} z_\mu^{\alpha_\mu(l)},$$

where $\nu = \pi(y)$. For $k \in S(N,y)$ we define $P_k(z_{\nu+1}, \ldots, z_\mu) = \sum_{l \in T(\frac{N}{k}, y)} a_{kl} z_{\nu+1}^{\alpha_{\nu+1}(l)} z_\mu^{\alpha_\mu(l)}$ and the associated $D_k(s) = \sum_{l \in T(\frac{N}{k}, y)} a_{kl} l^{-s}$. By (4.2) we clearly have that $\|D\|_\infty = \|P\|_\infty$ and $\|D_k\|_\infty = \|P_k\|_\infty$. We follow now the methods of [53, Proof of Theorem 4.3] to get

$$\|D_k\|_\infty = \|P_k\| \leq \|P\| = \|D\|_\infty : \tag{6.6}$$

An easy calculation shows that

$$P_k(z) = \frac{1}{(2\pi)^\nu} \int_0^{2\pi} \cdots \int_0^{2\pi} P(e^{it_1}, \ldots, e^{it_\nu}, z) e^{-i(\alpha_1(k)t_1 + \ldots + \alpha_\nu(k)t_\nu)} dt_1 \ldots dt_\nu,$$

which gives the desired inequality (6.6). With the decomposition (6.5) we have that

$$\begin{aligned}\sum_{n=1}^{N}\|va_n\|_Y &= \sum_{k\in S(N,y)}\sum_{m\geq 1}\sum_{l\in T_m(\frac{N}{k},y)}\|va_{kl}\| \\ &\leq |S(N,k)| \sup_{k\in S(N,y)}\sum_{m\geq 1}\sum_{l\in T_m(\frac{N}{k},y)}\|va_{kl}\|_Y .\end{aligned}$$

Using the Hölder inequality with $\rho_m = \frac{qrm}{q+(m-1)r}$ and $\rho_m^* = \frac{qm}{(q-1)m-\frac{q}{r}+1}$ we get that for each $m \in \mathbb{N}$

$$\sum_{l\in T_m(\frac{N}{k},y)}\|va_{kl}\|_Y \leq \Big(\sum_{l\in T_m(\frac{N}{k},y)}\|va_{kl}\|_Y^{\rho_m}\Big)^{\frac{1}{\rho_m}} |T_m(N,y)|^{\frac{1}{\rho_{m*}}} .$$

(6.3), (6.6), and (6.4) give

$$\begin{aligned}&\prec \sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N}a_n n^{-it}\Big\| C^m\left(\frac{N}{y^m}(\log N)^{y-1}e^{Dy}\right)^{\frac{q-1}{q}-\frac{1}{m}(\frac{1}{r}-\frac{1}{q})} \\ &\leq N^{\frac{q-1}{q}}\sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N}a_n n^{-it}\Big\| e^{h_{N,y}^{q,r}(m)},\end{aligned}$$

where

$$\begin{aligned}h_{N,y}^{q,r}(m) =\ &\log C^m + \tfrac{q-1}{q}\left(-\log y^m + (y-1)\log\log N + Dy\right) \\ &- (\tfrac{1}{r}-\tfrac{1}{q})\left(\tfrac{1}{m}\log N - \log y\right).\end{aligned}$$

Remark 6.8 and Lemma 6.9 give

$$\sum_{l\in T_m(\frac{N}{k},y)}\|va_{kl}\|_Y \leq N^{\frac{q-1}{q}}\sup_{t\in\mathbb{R}}\Big\|\sum_{n=1}^{N}a_n n^{-it}\Big\| e^{\left(-\sqrt{2\frac{q-1}{q}(\frac{1}{r}-\frac{1}{q})}+o(1)\right)\sqrt{\log N\log\log N}} .$$

On the other hand we have that for any $y' \leq N$

$$|S(N,y')| \leq \left(1+\frac{\log N}{\log 2}\right)^{\pi(y')} ,$$

since each element $n = p_1^{\alpha_1(n)}\cdots p_{\pi(y')}^{\alpha_{\pi(y')}(n)} \in S(N,y')$ is generated by the first $\pi(y')$ prime numbers and the multiplicity of each prime factor is at least 0 and at most $\log N/\log 2$ (see Remark 6.10). For $y = \sqrt{\log N}/\log\log N$ the prime number theorem gives that there is a constant $d > 0$ such that

$$|S(N,y)| \leq e^{d\frac{\sqrt{\log N}}{\log\log N}} .$$

Moreover Remark 6.10 implies that $T_m(\frac{N}{k}, y) = \emptyset$ for each $m > \frac{\log N}{\log 2}$. Hence

$$\begin{aligned}\sum_{n=1}^{N} \|va_n\|_Y &\leq |S(N,y)| \sup_{k\in S(N,y)} \sum_{m\geq 1} \sum_{l\in T_m(\frac{N}{k},y)} \|va_{kl}\|_Y . \\ &\leq N^{\frac{q-1}{q}} \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\| e^{d\frac{\sqrt{\log N}}{\log\log N} + \log\frac{\log N}{\log 2} + \left(-\sqrt{2\frac{q-1}{q}(\frac{1}{r}-\frac{1}{q})}+o(1)\right)\sqrt{\log N \log\log N}} \\ &= N^{\frac{q-1}{q}} \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\| e^{\left(-\sqrt{2\frac{q-1}{q}(\frac{1}{r}-\frac{1}{q})}+o(1)\right)\sqrt{\log N \log\log N}}\end{aligned}$$

This proves our assertion. □

The next result needs the following Lemma which follows directly from [2, Theorem 1.1] and the Hahn-Banach Theorem.

Lemma 6.11. *There is a constant $C > 0$ such that for every $\sum_n a_n n^{-s} \in \mathcal{H}^\infty(X)$ and $N \geq 2$ we have*

$$\Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_\infty \leq C \log N \Big\| \sum_{n=1}^{\infty} a_n n^{-s} \Big\|_\infty .$$

As a consequence of Theorem 6.6 we get the following vector valued version of the Defant-Frerick-Ortega-Ounaïes-Seip Theorem mentioned in Corollary 4.7.

Corollary 6.12. *Let Y be a q-concave Banach lattice, with $2 \leq q < \infty$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r < q$. Define $\tau := 2\frac{q-1}{q}\sqrt{\frac{1}{r} - \frac{1}{q}}$. If $D(s) = \sum a_n n^{-s}$ belongs to $\mathcal{H}^\infty(X)$ then we have for every $\varepsilon > 0$ that*

$$\sum_{n=1}^{\infty} \frac{\|va_n\|_Y}{n^{\frac{q-1}{q}}} e^{(\tau-\varepsilon)\sqrt{\log n \log\log n}} < \infty.$$

Proof. Since the sequence $e^{(\tau-\varepsilon)\sqrt{\log n \log\log n}} n^{\frac{1}{q}-1}$ is decreasing from an $n_0 \in \mathbb{N}$ we have that

$$\sum_{n=1}^{\infty} \frac{\|va_n\|_Y}{n^{\frac{q-1}{q}}} e^{(\tau-\varepsilon)\sqrt{\log n \log\log n}} \prec \sum_{k=0}^{\infty} \frac{e^{(\tau-\varepsilon)\sqrt{\log 2^k \log\log 2^k}}}{2^{k\frac{q-1}{q}}} \sum_{n=1}^{2^{k+1}} \|va_n\|$$

Then we have with Theorem 6.6 and Lemma 6.11 that this is

$$\prec \sum_{k=0}^{\infty} \frac{e^{(\tau-\varepsilon)\sqrt{\log 2^k \log\log 2^k}}}{e^{(\tau+o(1))\sqrt{\log 2^{k+1} \log\log 2^{k+1}}}} \log 2^{k+1} \prec \sum_{k=0}^{\infty} \frac{k+1}{e^{\frac{\varepsilon}{2}\sqrt{k \log k}}} < \infty$$

□

Now we are in the position to give upper estimates for the Queffélec numbers $Q_N(\ell_p \hookrightarrow \ell_q)$ and $Q_N(v : \ell_1 \to \ell_q)$.

Theorem 6.13. *Let $1 \le p < q \le \infty$. Then with constants depending only on p, q we have:*

$$Q_N(\ell_p \hookrightarrow \ell_q) \le \begin{cases} \dfrac{\sqrt{N}}{e^{\left(\sqrt{\frac{1}{p} - \max\left\{\frac{1}{2}, \frac{1}{q}\right\}} + o(1)\right)\sqrt{\log N \log\log N}}} & \text{if } 1 \le p < 2 \\ N^{1-\frac{1}{p}} & \text{if } p \ge 2. \end{cases}$$

Proof. We consider three different cases.
The case $\mathbf{1 \le p < q \le 2}$. By the Bennett-Carl Theorem 1.4 the inclusion $\ell_p \hookrightarrow \ell_q$ is $(r,1)$-summing where $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$. Since ℓ_q is known to be 2-concave, the estimate is a consequence of Theorem 6.6.
The case $\mathbf{1 \le p < 2 \le q}$. This case follows from the preceding once, since we clearly have that $Q_N(\ell_p \hookrightarrow \ell_q) \le Q_N(\ell_p \hookrightarrow \ell_2)$.
The case $\mathbf{2 \le p}$. We have $Q_N(\ell_p \hookrightarrow \ell_q) \le Q_N(\ell_p \hookrightarrow \ell_p)$. Then we conclude the desired estimate from Corollary 6.5. □

Theorem 6.14. *Let $v : \ell_1 \to \ell_q$ be a non zero operator and $1 < q \le 2$. Then*

$$Q_N(v) \le \frac{\sqrt{N}}{e^{\left(\sqrt{1-\frac{1}{q}} + o(1)\right)\sqrt{\log N \log\log N}}}$$

Proof. Combine Kwapień's Theorem 1.3, Theorem 6.6 and the fact that ℓ_q is 2-concave for $q \le 2$. □

6.3 M-homogeneous Queffélec Numbers

In this section we give an upper estimate for the Queffélec number $Q_N^M(\ell_p \hookrightarrow \ell_q)$, $1 \le p < q < \infty$, $1 \le p \le 2$. Again we start with a general result on $(r,1)$-summing operators.

Theorem 6.15. *Let Y be a q-concave Banach lattice with $2 \le q < \infty$, $v : X \to Y$ an $(r,1)$-summing operator with $1 \le r < q$ and take $0 < \lambda < \frac{q-1}{q}(M-1)$. Define*

$$\omega_M := \frac{(q-1)M - q\left(\frac{1}{r} - \frac{1}{q}\right)}{qM}.$$

Then there is a constant C such that for every $N \in \mathbb{N}$ and every M-homogeneous Dirichlet series $\sum a_n \frac{1}{n^s}$

$$\sum_{n=1}^{N} \frac{\|va_n\|_Y}{n^{\omega_M}} (\log n)^{\lambda} \le C \sup_{t\in\mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_X.$$

Proof. Recall that $p = (p_n)_n$ denotes the sequence $p_1 < p_2 < \ldots$ of all prime numbers. Given $N \in \mathbb{N}$ we define the M-homogeneous polynomial

$$P : \ell_\infty^N \to X, \quad P(z) = \sum_{\mathbf{j} \in \mathscr{J}(M,N)} b_{\mathbf{j}} z_{j_1} \cdots z_{j_M}$$

where

$$b_{\mathbf{j}} = b_{j_1 \ldots j_M} := \begin{cases} a_{p_{j_1} \cdots p_{j_M}} & \text{if } p_{j_1} \cdots p_{j_M} \leq N, \\ 0 & \text{else.} \end{cases}$$

Then we have that

$$\begin{aligned} &\sum_{n=1}^{N} \frac{\|v a_n\|_Y}{n^{\omega_M}} (\log n)^\lambda \\ &= \sum_{\mathbf{j} \in \mathscr{J}(M,N)} \frac{\|v b_{\mathbf{j}}\|_Y}{(p_{j_1} \cdots p_{j_M})^{\omega_M}} (\log p_{j_1} \cdots p_{j_M})^\lambda \\ &\leq \sum_{j_M=1}^{N} \frac{(M \log p_{j_M})^\lambda}{p_{j_M}{}^{\omega_M}} \sum_{\substack{j_1, \ldots, j_{M-1} \\ j_1 \leq \ldots \leq j_M}} \frac{\|v b_{\mathbf{j}}\|_Y}{(p_{j_1} \cdots p_{j_{M-1}})^{\omega_M}} \end{aligned}$$

and by Hölder's inequality this is

$$\leq M^\lambda \sum_{j_M=1}^{N} \frac{(\log p_{j_M})^\lambda}{p_{j_M}{}^{\omega_M}} \Bigg(\sum_{\substack{j_1, \ldots, j_{M-1}: \\ j_1 \leq \ldots \leq j_M}} \|v b_{\mathbf{j}}\|_Y^q \Bigg)^{\frac{1}{w} q} \Bigg(\sum_{\substack{j_1, \ldots, j_{M-1}: \\ j_1 \leq \ldots \leq j_M}} \big(p_{j_1} \cdots p_{j_{M-1}}\big)^{-q^* \omega_M} \Bigg)^{\frac{1}{q^*}} \tag{6.7}$$

Using the fact that for $0 < \alpha < 1$

$$\sum_{p \leq x} p^{-\alpha} \underset{x \to \infty}{\asymp} \frac{x^{1-\alpha}}{\log x},$$

see e.g. [69, I. Satz 4.2], we get that

$$\begin{aligned} \Bigg(\sum_{\substack{j_1, \ldots, j_{M-1}: \\ j_1 \leq \ldots \leq j_M}} \big(p_{j_1} \cdots p_{j_{M-1}}\big)^{-q^* \omega_M} \Bigg)^{\frac{1}{q^*}} &\leq \Bigg(\sum_{j \leq j_M} p_j{}^{-q^* \omega_M} \Bigg)^{\frac{M-1}{q^*}} \\ &\prec \left(\frac{p_{j_M}{}^{1-q^* \omega_M}}{\log p_{j_M}} \right)^{\frac{M-1}{q^*}} \end{aligned}$$

Hence (6.7) is

$$\prec M^\lambda \sum_{j_M=1}^{N} \frac{(\log p_{j_M})^{\lambda - \frac{M-1}{q^*}}}{p_{j_M}{}^{\frac{r-1}{r}}} \Bigg(\sum_{\substack{j_1, \ldots, j_{M-1} \\ j_1 \leq \ldots \leq j_M}} \|v b_{\mathbf{j}}\|_Y^q \Bigg)^{\frac{1}{q}}$$

and applying again Hölder's inequality we get that this is

$$\prec M^\lambda \left(\sum_{j_M=1}^N \frac{(\log p_{j_M})^{r^*\left(\lambda - \frac{M-1}{q^*}\right)}}{p_{j_M}} \right)^{\frac{1}{r^*}} \left(\sum_{j_M=1}^N \left(\sum_{\substack{j_1,\dots,j_{M-1} \\ j_1 \le \dots \le j_M}} \|vb_{\mathbf{j}}\|_Y^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}.$$

The left factor in the upper product converges for all $\lambda < \frac{M-1}{q^*} = \frac{q-1}{q}(M-1)$ since, by the prime number theorem, we have that

$$\left(\sum_{j_M=1}^N \frac{(\log p_{j_M})^{r^*\left(\lambda - \frac{M-1}{q^*}\right)}}{p_{j_M}} \right)^{\frac{1}{r^*}} \prec \left(\sum_{n=1}^N \frac{(\log(n \log n))^{r^*\left(\lambda - \frac{M-1}{q^*}\right)}}{n \log n} \right)^{\frac{1}{r^*}}$$

For the right factor Theorem 3.12 and Bohr's trick (4.2) finally give that

$$\begin{aligned} \left(\sum_{j_M=1}^N \left(\sum_{\substack{j_1,\dots,j_{M-1} \\ j_1 \le \dots \le j_M}} \|vb_{\mathbf{j}}\|_Y^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} &\prec \sup_{z \in B_{\ell_\infty^N}} \Big\| \sum_{\mathbf{j} \in \mathscr{J}(M,N)} b_{\mathbf{j}} z_{j_1} \cdots z_{j_M} \Big\|_X \\ &= \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^N a_n n^{-it} \Big\|_X, \end{aligned}$$

the conclusion. □

Theorem 6.16. *For $1 \le p \le 2$, $p < q \le \infty$, $M \in \mathbb{N}$, and $\lambda < \frac{M-1}{2}$ we define*

$$\omega_M = \frac{M - 2\left(\frac{1}{p} - \max\{\frac{1}{q}, \frac{1}{2}\}\right)}{2M}.$$

Then we have that

$$\frac{N^{\omega_M}}{(\log N)^{M\omega_M}} \prec Q_N^M(\ell_p \hookrightarrow \ell_q) \prec \frac{N^{\omega_M}}{(\log N)^\lambda}.$$

Proof of the upper bound. For $1 \le p < q \le 2$ the space ℓ_q is 2-concave and, by the Bennett-Carl Theorem 1.4, the embedding $\ell_p \hookrightarrow \ell_q$ is $(r,1)$-summing for $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$. Then Theorem 6.15 implies that for every M-homogenous Dirichlet series $\sum a_n n^{-s}$ and $\lambda < \frac{M-1}{2}$ we have that

$$\begin{aligned} \sum_{n=1}^N \|a_n\|_q &\le \frac{N^{\omega_M}}{(\log N)^\lambda} \sum_{n=1}^N \|a_n\|_q \frac{(\log n)^\lambda}{n^{\omega_M}} \\ &\prec \frac{N^{\omega_M}}{(\log N)^\lambda} \sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^N a_n n^{-it} \Big\|_p. \end{aligned}$$

If $q \ge 2$ we clearly have that $Q_N^M(\ell_p \hookrightarrow \ell_q) \le Q_N^M(\ell_p \hookrightarrow \ell_2)$. This proofs our assertion. □

The proof of the lower bound is based on the ideas of Maurizi and Queffélec in the scalar case (see [60, Theorem 3.1]). We start with two lemmata.

Lemma 6.17. *Given $M \in \mathbb{N}$ there exists a constant $C > 0$ such that for every $N \in \mathbb{N}$ and every choice of scalars $a_1, \ldots, a_N \in \mathbb{C}$ such that $a_n \neq 0$ implies that $\Omega(n) = M$ we have*

$$\int \Big\| \sum_{n=1}^{N} a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big\|_{\mathcal{P}_M(\ell_\infty^r;\mathbb{C})} d\omega \leq C \left[\max_{n \leq N} |a_n| \sqrt{1 + \log N} + \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{\frac{1}{2}} \sqrt{r \log\log N} \right],$$

where $g_1, \ldots, g_N$ are independent gaussian random variables.

Proof. We choose independent gaussian random variables $g_1, \ldots, g_N$ and consider the M-homogeneous polynomial

$$P_\omega(z) = \sum_{n=1}^{N} a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)}.$$

By rewriting $\|P_\omega\|_\infty$ as follows

$$\begin{aligned} \|P_\omega\|_\infty &= \sup_{z \in B_{\ell_\infty^r}} \Big| \sum_{n=1}^{N} a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big| = \sup_{z \in \mathbb{T}^r} \Big| \sum_{n=1}^{N} a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big| \\ &= \sup_{t_j \in [0,2\pi]} \Big| \sum_{n=1}^{N} a_n g_n(\omega) e^{i(\alpha_1(n)t_1 + \ldots + \alpha_r(n)t_r)} \Big| \end{aligned}$$

and since

$$\sup_{t_j \in [0,2\pi]} \left| a_n e^{i(\alpha_1(n)t_1 + \ldots + \alpha_r(n)t_r)} \right| = |a_n|$$

a theorem of Kahane in [50, Chapter 6, Theorem 3] implies that

$$\mathfrak{P} \left(\|P_\omega\|_\infty \geq C \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{\frac{1}{2}} \sqrt{r \log M} \right) \leq \frac{1}{M^2 e^r}. \tag{6.8}$$

But by a theorem of Ledoux and Talagrand [55, Proposition 6.8] we have that

$$\int \|P_\omega\|_\infty d\omega \leq 6 \int \max_{n \leq N} \left\| a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \right\|_\infty d\omega + 6 t_0,$$

where

$$t_0 = \inf \left\{ t > 0 \mid \mathfrak{P} \left(\|P_\omega\|_\infty > t \right) \leq \frac{1}{8,3} \right\}.$$

On the one hand, (6.8) and the fact that for $M, r \geq 2$ we have $M^2 e^r \geq 8,3$ give that $t_0 \leq C'(\sum_{n=1}^N |a_n|^2)^{\frac{1}{2}} \sqrt{r \log M}$. Furthermore it is clear that the degree M of P_ω is at most $\frac{\log N}{\log 2}$ (see Remark 6.10). On the other hand

$$\begin{aligned}\int \max_{n\leq N} \left\| a_n g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \right\|_\infty &= \int \max_{n\leq N} |a_n g_n(\omega)| \sup_{z\in B_{\ell_\infty^r}} |z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)}| d\omega \\ &\leq \max_{n\leq N} |a_n| \int \max_{n\leq N} |g_n(\omega)| d\omega \\ &= \max_{n\leq N} |a_n| \int \Big\| \sum_{n=1}^N g_n(\omega) e_n \Big\|_{\ell_\infty^N} d\omega.\end{aligned}$$

and by Tomczak-Jaegermann [75, Proposition 45.1]

$$\int \Big\| \sum_{n=1}^N g_n(\omega) e_n \Big\|_{\ell_\infty^N} d\omega \leq C'' \left(1 + \log N\right)^{\frac{1}{2}}.$$

This gives the conclusion. □

Lemma 6.18. *Given $M \in \mathbb{N}$ there exists a constant $C > 0$ such that for every $N, K \in \mathbb{N}$, $1 \leq p \leq 2$ and every choice of $a_1, \ldots, a_N \in \mathbb{C}$ such that $a_n \neq 0$ implies that $\Omega(n) = M$ we have*

$$\begin{aligned}&\int \Big\| \sum_{n=1}^N a_n \Big(\sum_{k=1}^K \varepsilon_{kn}(\omega) e_k \Big) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big\|_{\mathcal{P}_M(\ell_\infty^r; \ell_p^K)} d\omega \\ &\leq C \Bigg[\Big(\max_{n\leq N} |a_n| \sqrt{1 + \log N} + \Big(\sum_{n=1}^N |a_n|^2 \Big)^{\frac{1}{2}} \sqrt{r \log \log N} \Big) K^{\frac{1}{p} - \frac{1}{2}} \\ &+ \sup_{n\leq N} \left(|a_n| \sqrt{\frac{\alpha!}{M!}} \right) r^{\frac{M}{2}} K^{\frac{1}{p}} \Bigg],\end{aligned}$$

where $(\varepsilon_{nk})_{n,k}$ is a family of Rademacher random variables.

Proof. We choose independent Gaussian random variables $(g_{nk})_{n,k=1}^{N,K}$. It is a well known fact that the Rademacher averages are dominated by the Gaussian averages (see [40, Proposition 12.11]):

$$\begin{aligned}&\int \Big\| \sum_{n=1}^N a_n \Big(\sum_{k=1}^K \varepsilon_{kn}(\omega) e_k \Big) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big\|_{\mathcal{P}_M(\ell_\infty^r; \ell_p^K)} d\omega \\ &\leq C' \int \Big\| \sum_{n=1}^N a_n \Big(\sum_{k=1}^K g_{kn}(\omega) e_k \Big) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big\|_{\mathcal{P}_M(\ell_\infty^r; \ell_p^K)} d\omega \\ &= C' \int \Big\| \sum_{n,k=1}^{NK} g_{kn}(\omega) \left(a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \otimes e_k \right) \Big\|_{\mathcal{P}_M(\ell_\infty^r) \otimes_\varepsilon \ell_p^K} d\omega.\end{aligned}$$

By Chevét's inequality [75, Corollary 3.2] we have that

$$\int \Big\| \sum_{n,k=1}^{NK} g_{kn}(\omega)(a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \otimes e_k) \Big\|_{\mathcal{P}_M(\ell_\infty^r)\otimes_\varepsilon \ell_p^K} d\omega$$

$$\leq C' \Big[\sup_{\substack{y' \in \mathcal{P}_M(\ell_\infty^r)' \\ \|y'\| \leq 1}} \Big(\sum_{n=1}^{N} |y'(a_n z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)})|^2 \Big)^{\frac{1}{2}} \int \Big\| \sum_{k=1}^{K} g_k(\omega) e_k \Big\|_{\ell_p^K} d\omega$$

$$+ \sup_{y' \in B_{\ell_p^K}{}'} \Big(\sum_{k=1}^{K} |y'(e_k)|^2 \Big)^{\frac{1}{2}} \int \Big\| \sum_{n=1}^{N} g_n(\omega) z_1^{\alpha_1(n)} \cdots z_r^{\alpha_r(n)} \Big\|_{\mathcal{P}_M(\ell_\infty^r)} d\omega \Big]$$

Note that for $1 \leq p \leq 2$

$$\sup_{y' \in B_{\ell_p^K}{}'} \Big(\sum_{k=1}^{K} |y'(e_k)|^2 \Big)^{\frac{1}{2}} = \sup_{x \in B_{\ell_{p'}^K}} \Big(\sum_{k=1}^{K} |x_k|^2 \Big)^{\frac{1}{2}} = \|\operatorname{id} : \ell_{p'}^K \to \ell_2^K \| = K^{\frac{1}{p} - \frac{1}{2}}.$$

and from the proof of [39, Lemma 4.2] we know that

$$\sup_{\substack{y' \in \mathcal{P}_M(\ell_\infty^r)' \\ \|y'\| \leq 1}} \Big(\sum_{n=1}^{N} |y'(a_n z^{\alpha(n)})|^2 \Big)^{\frac{1}{2}} \leq \sup_{n \leq N} \Big(|a_n| \sqrt{\frac{\alpha(n)!}{M!}} \Big) \|\operatorname{id} : \ell_\infty^r \to \ell_2^r \|^M$$

$$= \sup_{n \leq N} \Big(|a_n| \sqrt{\frac{\alpha(n)!}{M!}} \Big) r^{\frac{M}{2}}.$$

Moreover, [75, Proposition 45.1] gives that

$$\int \Big\| \sum_{k=1}^{K} g_k(\omega) e_k \Big\|_{\ell_p^K} d\omega \leq C'' K^{\frac{1}{p}}.$$

Finally, Lemma 6.17 gives the conclusion. □

Proof the lower bound of Theorem 6.16. **The case $\mathbf{1 \leq p < q \leq 2}$.** We set

$$r = \pi(N^{\frac{1}{M}})$$

and define the set

$$A = \{ n \in \mathbb{N} \mid n = p_{i_1} \cdots p_{i_M},\, 1 \leq i_1 < i_2 \ldots < i_m \leq r \}.$$

Note that by the definition of r each $n \in A$ holds $n \leq p_r^M \leq N$. Given $K \in \mathbb{N}$ and Rademacher random variables $(\varepsilon_{nk})_{n,k=1}^{NK}$, we define the M-homogenous Dirichlet polynomial f_ω in ℓ_p^K by

$$f_\omega(s) = \sum_{n=1}^{N} c_n n^{-s} := \sum_{n=1}^{N} a_n \Big(\sum_{k=1}^{K} \varepsilon_{nk}(\omega) e_k \Big) n^{-s},$$

where

$$a_n = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

Then we have that

$$\|c_n\|_q = \Big(\sum_{k=1}^{K} |a_n \varepsilon_{nk} e_k|^q\Big)^{\frac{1}{q}} = |a_n| K^{\frac{1}{q}}$$

and hence

$$\sum_{n=1}^{N} \|c_n\|_q = |A| K^{\frac{1}{q}}.$$

Moreover we denote with P_ω the M-homogenous polynomial associated to f_ω. By Lemma 6.18 and Bohr's trick (4.2) we have

$$\begin{aligned} |A|K^{\frac{1}{q}} &\prec Q_N^M(\ell_p \hookrightarrow \ell_q) \int \|f_\omega\|_\infty d\omega = Q_N^M(\ell_p \hookrightarrow \ell_q) \int \|P_\omega\|_\infty d\omega \\ &\prec Q_N^M(\ell_p \hookrightarrow \ell_q) \Big[\Big(\max_{n \le N} |a_n| \sqrt{1 + \log N} + \Big(\sum_{n=1}^{N} |a_n|^2\Big)^{\frac{1}{2}} \sqrt{r \log\log N} \Big) K^{\frac{1}{p} - \frac{1}{2}} \\ &\quad + \sup_{n \le N} \Big(|a_n| \sqrt{\frac{\alpha(n)!}{M!}} \Big) r^{\frac{M}{2}} K^{\frac{1}{p}} \Big] \end{aligned}$$

We choose now $K = r$. The prime number theorem implies that $K = r = \pi(N^{\frac{1}{M}}) \asymp \frac{N^{\frac{1}{M}}}{\log N}$ and hence $|A| = \binom{r}{M} \asymp \frac{r^M}{M!} \asymp \frac{N}{(\log N)^M}$. Note also that $\sup_{\alpha \in \Lambda(M,r)} \sqrt{\frac{\alpha!}{M!}} \le 1$. Then we get

$$\begin{aligned} \frac{N}{(\log N)^M} \Big(\frac{N^{\frac{1}{M}}}{\log N} \Big)^{\frac{1}{q}} \prec Q_N^M(\ell_p \hookrightarrow \ell_q) \Big[\Big(\sqrt{1 + \log N} + \frac{N^{\frac{M+1}{2M}} \sqrt{\log\log N}}{(\log N)^{\frac{M+1}{2}}} \Big) \Big(\frac{N^{\frac{1}{M}}}{\log N} \Big)^{\frac{1}{p} - \frac{1}{2}} \\ + \frac{N^{\frac{1}{2}}}{(\log N)^{\frac{M}{2}}} \Big(\frac{N^{\frac{1}{M}}}{\log N} \Big)^{\frac{1}{p}} \Big] \end{aligned}$$

It is easy to see that the last factor increases faster. Then

$$\frac{N^{1 + \frac{1}{Mq}}}{(\log N)^{M + \frac{1}{q}}} \prec Q_N^M(\ell_p \hookrightarrow \ell_q) \frac{N^{\frac{1}{2} + \frac{1}{Mp}}}{(\log N)^{\frac{M}{2} + \frac{1}{p}}}$$

and this gives the conclusion.

The case $\mathbf{1 \le p \le 2 < q}$. First of all we show that for every $K \in \mathbb{N}$ there exists an M-homogeneous polynomial $P : \ell_\infty^{MK} \to \ell_p^K$, $P(z) = \sum_{|\alpha|=M} c_\alpha z^\alpha$, such that

$$\sum_{\alpha \in \Lambda(M, MK)} \|c_\alpha\|_2 = K^M \tag{6.9}$$

and

$$\sup_{z\in B_{\ell_\infty^{MK}}} \Big\| \sum_{\alpha\in\Lambda(M,MK)} c_\alpha z^\alpha \Big\|_p \leq K^{\frac{M}{2}+\frac{1}{p}-\frac{1}{2}} . \tag{6.10}$$

Let $(a_{ij})_{i,j}$ be a complex $K\times K$-matrix satisfying

$$\sum_{k=1}^{K} a_{kl}\bar{a}_{km} = K\delta_{lm} \quad \text{and} \quad |a_{lm}| = 1 \quad \text{for all } l,m\in\mathbb{N}$$

(for example take $a_{ml} = e^{\frac{2\pi iml}{K}}$). With this we define the M-homogeneous polynomial $P:\ell_\infty^{MK}\to\ell_p^K$ by

$$P(z) = \sum_{\mathbf{i}\in\mathscr{M}(M,MK)} a_{1i_M}a_{i_Mi_{M-1}}\cdots a_{i_2i_1}z_{i_1}z_{K+i_2}\cdots z_{(M-1)K+i_M}e_{i_M}$$

Let us show that P satisfies (6.10). Indeed, if $z\in B_{\ell_\infty^{MK}}$, we have by the conditions of the matrix $(a_{ij})_{i,j}$ that

$$\begin{aligned}
\|P(z)\|_2^2 &= \sum_{i_M=1}^{K}\Big|\sum_{\mathbf{i}\in\mathscr{M}(M-1,K)} a_{1i_M}a_{i_Mi_{M-1}}\cdots a_{i_2i_1}z_{i_1}\cdots z_{(M-1)K+i_M}\Big|^2\\
&\leq \sum_{i_M=1}^{K}\Big|\sum_{\mathbf{i}\in\mathscr{M}(M-1,K)} a_{i_Mi_{M-1}}\cdots a_{i_2i_1}z_{i_1}\cdots z_{(M-2)K+i_{M-1}}\Big|^2\\
&= \sum_{i_M=1}^{K}\sum_{\mathbf{i},\mathbf{j}\in\mathscr{M}(M-1,K)} a_{i_Mi_{M-1}}\bar{a}_{i_Mj_{M-1}}\cdots z_{(M-2)K+i_{M-1}}\bar{z}_{(M-2)K+j_{M-1}}\\
&=\Big(\sum_{\mathbf{i},\mathbf{j}\in\mathscr{M}(M-1,K)} a_{i_{M-1}i_{M-2}}\bar{a}_{j_{M-1}j_{M-2}}\cdots z_{(M-2)K+i_{M-1}}\bar{z}_{(M-2)K+j_{M-1}}\\
&\quad\cdot\sum_{i_M=1}^{K} a_{i_Mi_{M-1}}\bar{a}_{i_Mj_{M-1}}\Big)\\
&=K\sum_{i_{M-1}=1}^{K}|z_{(M-2)K+i_{M-1}}|^2\Big|\sum_{\mathbf{i},\mathbf{j}\in\mathscr{M}(M-2,K)} a_{i_{M-1}i_{M-2}}\cdots z_{(M-3)K+i_{M-2}}\Big|^2\\
&\leq K\sum_{i_{M-1}=1}^{K}\Big|\sum_{\mathbf{i},\mathbf{j}\in\mathscr{M}(M-2,K)} a_{i_{M-1}i_{M-2}}\cdots a_{i_2i_1}z_{i_1}\cdots z_{(M-3)K+i_{M-2}}\Big|^2
\end{aligned}$$

Repeating this argument we finally end up in

$$\begin{aligned}\|P(z)\|_2^2 &\leq K^{M-2} \sum_{i_2=1}^{K} \Big| \sum_{i_1=1}^{K} a_{i_2 i_1} z_{i_1} \Big|^2 \\ &= K^{M-2} \sum_{i_1, j_1 = 1}^{K} \Big(\sum_{i_2=1}^{K} a_{i_2 i_1} \bar{a}_{i_2 j_1} \Big) z_{i_1} \bar{z}_{j_1} \\ &= K^{M-1} \sum_{i_1=1}^{K} |z_{i_1}|^2 \leq K^M.\end{aligned}$$

Thus,

$$\sup_{z \in B_{\ell_\infty^{KM}}} \|P(z)\|_p \leq \| \operatorname{id} : \ell_p^K \hookrightarrow \ell_2^K \| \sup_{z \in B_{\ell_\infty^{KM}}} \|P(z)\|_2 \leq K^{\frac{1}{p} - \frac{1}{2}} K^{\frac{M}{2}}.$$

On the other hand P satisfies (6.9) since

$$\begin{aligned}\sum_{\alpha \in \Lambda(M, MK)} \|c_\alpha\|_2 &= \sum_{\mathbf{i} \in \mathscr{M}(M,K)} \|a_{1 i_M} a_{i_M i_{M-1}} \cdots a_{i_2 i_1} e_{i_M}\|_2 \\ &= \sum_{\mathbf{i} \in \mathscr{M}(M,K)} \Big(\sum_{k=1}^{K} |a_{1 i_M} a_{i_M i_{M-1}} \cdots a_{i_2 i_1} e_{i_M}(k)|^2 \Big)^{\frac{1}{2}} \\ &= \sum_{\mathbf{i} \in \mathscr{M}(M,K)} 1 = K^M.\end{aligned}$$

If we now choose $K = \big[\frac{\pi(N^{1/M})}{M} \big]$ (where $[\,]$ denotes the integer part) we have that for every $\alpha \in \Lambda(M, MK)$ that $p^\alpha \leq p_{MK}^M \leq N$. We define the M-homogenous Dirichlet-polynomial $\sum_{n=1}^{N} a_n n^{-s}$ by

$$a_{p^\alpha} = \begin{cases} c_\alpha & \text{if } \alpha \in \Lambda(M, MK) \\ 0 & \text{else .} \end{cases}$$

Clearly,

$$\sum_{n=1}^{N} \|a_n\|_2 = \sum_{\alpha \in \Lambda(M, MK)} \|c_\alpha\|_2 = K^M,$$

and by Bohr's trick (4.2)

$$\sup_{t \in \mathbb{R}} \Big\| \sum_{n=1}^{N} a_n n^{-it} \Big\|_p = \sup_{z \in B_{\ell_\infty^{MK}}} \Big\| \sum_{\alpha \in \Lambda(M, MK)} c_\alpha z^\alpha \Big\|_p \leq K^{\frac{M}{2} + \frac{1}{p} - \frac{1}{2}}.$$

Hence

$$Q_N^M(\ell_p \hookrightarrow \ell_2) \geq K^{\frac{M}{2} - (\frac{1}{p} - \frac{1}{2})}$$

By the definition of K and the prime number theorem there is a constant C such that $K \geq C \frac{N^{\frac{1}{M}}}{M \log\left(N^{\frac{1}{M}}\right)} \geq C \frac{N^{\frac{1}{M}}}{\log N}$, i.e.

$$Q_N^M(\ell_p \hookrightarrow \ell_2) \geq C' \left(\frac{N^{\frac{1}{M}}}{\log N} \right)^{\frac{M-2(\frac{1}{p}-\frac{1}{2})}{2}} . \qquad \square$$

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Hiermit erkläre ich, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

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