# Pupils' competencies in proof and argumentation

## – Differences between Korea and Germany at the lower secondary level

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# ABSTRACT

TIMSS (Third International Mathematics and Science Study) and PISA (Programme for International Student Assessment) studies which research international comparisons of mathematics achievement show that Asian pupils such as Korean and Japanese outperform their Western counterparts. After the international comparative studies some western countries looked at Asian mathematics classrooms and tried to find the reasons for this inquiry. International studies of student achievement are extraordinarily complex researches that are difficult to organise and analyse. However, the importance of those studies is increased, because they might explain the factors which have influence on achievement or give new ideas to improve the curricular.

Proof plays an important role in mathematics and also in mathematics class. Moreover, it is an important topic in the mathematics curriculum and an essential aspect of mathematical competence. In school mathematics, proof is taught to develop deductive reasoning and to promote the understanding of mathematics. However, proof is one of the difficult issues for pupils to learn. Recent studies have revealed wide gaps in pupils' understanding of proofs (Senk, 1985; Martin & Harel, 1989; Harel & Sowder, 1998; Healy & Hoyles, 1998; Reiss & Thomas 2001). Moreover, researches indicate that students both at high school and university level have difficulty, not only in producing proofs, but also even in recognizing what proof is (Galbraith, 1981; Fishbein and Kedem, 1982; Vinner, 1983; Chazan, 1993; Moore 1994).

My thesis aims at finding and investigating more reasons for the difference in pupil performance on mathematics in international study. In detail, my thesis aims at examining and comparing Korean and German pupils' competencies in proof and argumentation about geometry, especially at the lower secondary level. In addition, pupils' belief about mathematics will be discussed. To help explain my data, I compared the Korean educational system and mathematics curriculum, in particular the geometry component, with those of Germany. Moreover, I reviewed the results of international comparative research and the related literature.

The data of 659 German 7$^{th}$ grade pupils (8$^{th}$ grade: 528) in 27 classes and 189 Korean 7$^{th}$ grade pupils (8$^{th}$ grade: 182) in 5 classes were collected. In addition, 22 German teachers and 58 Korean teachers answered the same questionnaire on beliefs about mathematics. The German data is from the BIQUA Project on reasoning and proof in the geometry classroom which was headed by Prof. Reiss and funded by DFG (cf. Reiss, Hellmich & Thomas, 2001).

Participation in the interview research worked on a relatively voluntary basis; however, the different levels of achievement of the pupils were taken into account. So the 15 Korean 8$^{th}$ grade pupils (7 girls and 8 boys) who participated between January and February 2003, five pupils were assigned to the upper, five to the middle, and five to the lower achievement group. However, 18 German pupils (9 girls and 9 boys) did not participate in the quantitative test, and so some pupils participated on a voluntary basis and some were selected according to their previous marks for mathematics.

The result indicates that in the test for both graders, the German pupils (M= 7.39 for the 7$^{th}$ graders, M= 6.44 for the 8$^{th}$ graders) performed significantly better on those problems relating to basic competence than the Korean pupils (M=5.65 for the 7$^{th}$ graders, M=4.96 for the 8$^{th}$ graders). On the other hand, the Korean pupils (M=7. 45 for the 7$^{th}$ graders, M=5.54 for the 8$^{th}$ graders) performed significantly better on those problems concerning competence in proof and argumentation than the German pupils (M=5.23 for the 7$^{th}$ graders, M=4.19 for the 8$^{th}$ graders). Our findings indicate that most pupils are highly competent at appreciating correct proofs to be correct and accepting their generality. However, pupils have greater difficulty in recognising incorrect arguments to be incorrect than recognising correct proofs to be correct. This is consistent with the findings of the study on grade 13 pupils (Reiss, Klieme & Heinze, 2001) and with the findings of Healy and Holyes (1998).

From a factoranalysis relevant to beliefs about mathematics, the three categories application, formalism, and process were taken to represent these beliefs. To sum up, one might conclude that the Korean pupils and the German pupils shared roughly similar beliefs. Although many research papers have assumed that pupils' beliefs are one of the most important factors which have influence on their achievement, one might conclude from this study that it is not necessarily true. Moreover, our data suggest that teachers' beliefs do not have necessarily influence on those of their pupils. There is such a tendency in the case of the Korean teachers and pupils, but there are considerable differences between the views of the German teachers and those of the German pupils. It could be explained that the German teachers are aware of the importance of the application of mathematics in the real society or aware that the application of mathematics is the social wishes, however, in the classroom, it might be not well embedded. Therefore, the German pupils have little appreciate application view than other factors.

There was indeed a significant correlation between process-oriented belief and achievement for both the Korean and the German pupils. However its coefficient was low, therefore it could be said that there is no meaningful relationship between the two. In addition, a significant correlation between application-oriented belief and methodological competence was apparent in the case of the German pupils, even though the correlation coefficient was not high. However, for the German pupils, there was no significant correlation between formalism-oriented belief and any of the three cognitive variables defined. As regards the Korean pupils, there were significant correlations between process-oriented belief and two of the cognitive variables, namely basic competence and competence in proof and argumentations, as well as between formalism-oriented belief and basic competence. However, there was no significant correlation between application-oriented belief and any of the cognitive variables for the Korean pupils.

Pupils have certain preferences as regards the approach they use in arguments. In this study, visual arguments, enactive arguments, arguments by calculation and geometrical arguments are examined. The pupils' processes of argumentation do not necessarily begin with the typical deductive methods the experts might expect. Some pupils depend more on their intuition, some depend, for example in the case of geometry, on the visual appearance of an object. This may discourage pupils from trying to think logically; however, pupils might do learn how logical thinking and previous knowledge can be used together with their intuition and pre-assumption. The pupils who made visual arguments might benefit from more time to attempt to give algebraic or geometrical arguments. In this way, they might gain and then develop a feel for strategies they might use in formal proofs, and improve their recall of definitions and results.

According to the results of the quantitative test and the interview study in this thesis, the Korean and the German pupils struggle with the deductive formal proof. Moreover, this study was prompted in part by a widespread view that the Korean and the German pupils' performance on proofs are poor. Besides, the pupils' answers in the achievement test showed that they had a limited range of strategies for proofs and a poor understanding of concepts. In addition, pupils' proof-writing skills do not seem to be sufficiently well-developed, and they do not always seem to be able to apply the correct properties appropriately.

Therefore, pupils should also be given more time to explore arguments on their own and should be encouraged to write down the processes in their arguments, for example by stating the properties which are used and by explaining strategies orally. What those pupils who are not used to writing such detailed proofs might need at first is time to familiarise themselves with this, and then they need their understanding of how to write proofs.

# ACKNOWLEDGEMENTS

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# Preface

This thesis is part of the doctoral programme sponsored by the Ministry of Science and Culture of Lower Saxony, in which our research paradigm "Educational Reconstruction model" is worked out. This model was developed by Kattmann, et al (1997) which places the structure of the learner above the structure of the subject matter. On this basis, the teaching conditions which best encourage learning are investigated. In this program, there are different subject areas, including mathematics education, biology education, pedagogy and other subjects.

## 0.1. The model of educational reconstruction

This model consists of three interrelated components: comprehending the pupils' perspective, the scientific clarification, and educational structuring. (See, the Figure 1-1)

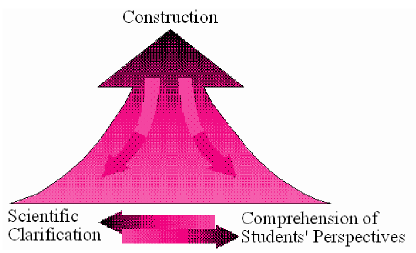


*Figure 1-1 The model of educational reconstruction*

- **Comprehending the pupils' perspective**

In this part, pupils' own views on their subjects as well as their learning process and strategies for contents are empirically investigated. This investigation therefore focuses on cognitive, affective, and psychomotor components, as well as the chronological development of the pupils' perspectives.

- **Scientific clarification**

In contrast to comprehending the pupils' perspective, in this research area the experts' theories, methods, and terminology will be observed or reconstructed. Material discussed in this investigation will comprise research including the most recent studies - both theoretical and practical work taken from publications, textbooks, instructions for practical studies, and working processes are discussed in this investigation.

- **Educational structuring**

In this part, learning methods for the classroom will be reconstructed or produced from a didactical perspective by modifying the structures of the subject matter in accordance with pupils' perspectives. The objective is an empirical investigation of the effectiveness of differently structured learning environments, which can be characterised as having teaching plans, guidelines, principles of teaching, detailed teaching units, or curriculum elements that are either theme-oriented or pupil-oriented.

This model systematically relates the way learners tackle a subject with subject structures that are clarified in a technical way and uses these structures for the construction of a programme of study for the pupils. In the case of my work, the first part is more important. Accordingly, I will mainly concentrate on the comprehension of the pupils' perspectives.

# Chapter 1 Introduction

## 1.1. The purpose of this study

In the TIMSS (Third International Mathematics and Science Study) and PISA (Programme for International Student Assessment) studies which make international comparisons of mathematics achievement, Korean and Japanese pupils have outperformed their Western counterparts. In the wake of these international comparative studies, some western countries took a greater interest in how mathematics is taught in Asian classrooms and tried to find the reasons for this discrepancy. One of the comparative studies was the TIMSS video study that compared mathematics instruction in Japan, Germany, and the United States. This study revealed significant differences between the teaching styles of Japan on the one hand and Germany and the United States on the other. Different teaching styles stem from various teacher-related and pupil-related cognitive and non-cognitive variables. How mathematical beliefs contribute to different instructional patterns is a topic of intense discussion (c.f. Pehkonen, 1996, Ernest, 1988).

International studies of student achievement are extraordinarily complex researches that are difficult to organise and analyse. However, these studies are becoming more widely valued. They arguably explain the factors which have influence on achievement and provide a source for new ideas designed to improve curricula.

However, it is difficult to compare the education systems of the two countries because education is influenced by social traditions in many ways. Leung (2001) asserted that it is even more difficult to compare education between East Asia and the West[1], because of the different cultures and educational philosophies. Moreover, education is not simply a matter of import and export. These cultural differences provide a unique opportunity to gain a deeper understanding of students' learning and performances. Therefore, we should try to investigate them.

Geometry is one of the most important parts of Korean and German school mathematics, because deductive reasoning is used to greater effect in geometry than in other subjects. Moreover geometrical concepts have close connections to other subjects. In addition, there are often a variety of solutions to geometrical problems, and so geometry develops creativity and the ability to investigate.

Proof and argumentation are central to geometry. Proof plays an important role in mathematics and also in mathematics class. Moreover, it is an important topic in mathematics curriculum and an essential aspect of mathematical competence. In school mathematics, the notion of proof is taught to develop deductive reasoning and to encourage a better understanding of mathematics generally. However, proof is one of the more difficult concepts for pupils to understand. Recent studies have revealed shortcomings in pupils' understanding of proofs (Senk, 1985; Martin & Harel, 1989; Harel & Sowder, 1998; Healy & Hoyles, 1998; Reiss & Thomas 2001). Moreover, research indicates that students both at high school and university level have difficulty not only in producing proofs, but also even in recognizing what a proof is (Galbraith, 1981; Fishbein and Kedem, 1982; Vinner, 1983; Chazan, 1993; Moore 1994).

My thesis aims at examining and comparing Korean and German pupils' competencies in proofs and argumentation in geometry, especially at the lower secondary level. Moreover, I will compare the different educational systems and the respective curricula of Korea and Germany as supporting data.

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[1] Leung (2001) defined East Asia and the West according to cultural demarcations rather than geographic divisions, roughly identified as the Confucian tradition on one hand, and the Greek/Latin/Christian tradition on the other.

## 1.2. Research questions

- Recent studies suggest that East Asian pupils are on average better than their Western counterparts not only at basic skills of computation but also routine problem solving. But, do Western pupils perform better than their East Asian counterparts on those problems concerning proofs and argumentation about geometry? This question has yet to be satisfactorily addressed. In this study, the competencies in proof and arguments of the Korean and the German $7^{th}$ and $8^{th}$ grade pupils will be compared.

  (1-a) Which basic competence do the $7^{th}$ and the $8^{th}$ graders have?
  (1-b) Which competence in proof and argumentation do the $7^{th}$ and the $8^{th}$ graders have?
  (1-b-1) Are the pupils able to recall names of several concepts of geometry?
  (1-b-2) Are the pupils capable to formulate proving?
  (1-c) Are there differences in pupils' abilities in proving and arguments between the two countries?

- Beliefs about mathematics are currently regarded as an important factor in mathematics achievement. However, the relationship between achievement and beliefs is unclear, as is the relationship between beliefs and methodological competence. The results of Korean and German pupils in these categories will be compared in this study.

  (2-a) Is there the relationship between achievement and beliefs? If then, which factor of beliefs could have influence on achievement?
  (2-b) Is there the relationship between methodological competence and beliefs? If then, which factor of beliefs could have influence on methodological competence?

- There have been few research papers which examine the educational systems, curricula and textbooks of Korea and Germany, even though these are important factors with regards to a pupil's achievement. In this study, differences and similarities between the educational systems of the two countries will be analysed. In particular, the Korean national curriculum and a selected German curriculum (that of Bavaria) will be compared, as will their respective textbooks, especially the sections on geometry.

  (3-a) Are there differences and similarities in the educational systems of the two countries?
  (3-b) Are there differences and similarities in the curricula of the two countries?
  (3-c) What is the difference in content of geometry between mathematics textbooks of the two countries?

- From the results I will gather in order to answer the first research question above, the way in which pupils go about proving results will be investigated in more depth. The kind of approach the pupils prefer, for example, as well as the difficulties they encounter in the process of proving and their understanding of concepts will be focused on.

  (4-a) What are the pupils' understandings of concepts?
  (4-b) What kind of arguments do pupils prefer?
  (4-c) What kind of difficulties in proving do pupils have?

## 1.3. Outline of the report

This study is divided into nine chapters. This chapter provides an overview of the study, including the purpose of the study and the research questions.

Chapter II details the educational systems of the two countries. Similarities and differences are pointed out.

Chapter III is an analysis of the mathematics curricula of Korea and Bavaria. It compares the geometry components of the curricula and textbooks.

Chapter IV reviews tests conducted to enable an international comparison of achievement, in particular TIMSS and PISA and discusses their results.

Chapter V is a review of literature about proof, about some of the schemas defining different levels of proof and about understanding proofs. Beliefs (*Mathematische Weltbilder*) are discussed and Korean and German empirical investigations into pupils' understanding of proofs are examined.

Chapter VI describes the methodological procedures used in this study. This includes research questions, general administration, and the structure of the research.

Chapter VII contains the quantitative results and analyses the relationship between achievement and beliefs as well as the relationship between methodological competence and beliefs.

Chapter VIII contains the interview results, an analysis of the processes pupils use to form proofs and difficulties in proving details.

Finally, chapter IX presents a summary of the study, conclusions, limitations and implications for the further research.

# Chapter 2 Educational Tradition

"Pupils should not even stand in the shadow of their teachers"
- An old Korean saying

In any social environment, education is influenced in many ways by the traditions of the given environment. Therefore, it is quite difficult to compare the education systems of two countries. Leung (2001) maintained that it is even more difficult to compare the education systems of East Asia with those of the West[2], such as England and USA because of the different cultures and educational philosophies. Moreover, education is not simply a matter of import and export. These cultural differences provide us with a unique opportunity to gain a deeper understanding of pupils' learning and performances. Therefore, we should try to invest time in researching these differences.

## 2.1. Short history of educational philosophy

The above Korean proverb shows how much respect the Korean pupils should have for their teachers. The idea that pupils should have so much respect for their teachers that they even show the teachers' shadows the same respect goes back to Confucianism. Confucianism is regarded as having been and being one of the main cultural traditions in East Asia, including Korea (Yang, 1999). In Korea in particular, Confucianism has served as a glue to ensure the cohesiveness of the family and of socio-economic hierarchies. Many of the Confucian values, such as respect for authority and sustained discipline are shared in the classroom as well as in education in Korea (Kim, 1992). Although such values are slowly disappearing because of the globalisation of the world, the Confucian kinship networks still work as a central pillar of society.

Apart from Confucianism, there is another educational ideology in Korea called "Hongik Ingan", which means "a man's life should benefit all of humanity". This was the idea of the national foundation started by Tan-gun, the god-king, about 5000 years ago. All educational policies are planned and coordinated under the educational ideology of "Hongik Ingan" (MOE, 2001).

The German educational philosophies are characterised by the development of two approaches in the 17$^{th}$ and in the 19$^{th}$ century. First, there is the realistic-oriented approach which emphasises a utility principle and preparation for future life. This approach was used in *Volkschulen* and *Realschulen*. Second, there is the humanistic-oriented approach of Humboldt. Humboldt's humanistic concept of "*Bildung*" searches for rational understanding of the order of the natural world and promotes the unity of academic knowledge and moral education. It aims to provide a general education, but also to develop an elite (cf. Pepin 1999a).

### 2.1.1. Mathematics in early

There are evident differences in cultural traditions in mathematics between East Asia and the West. The western culture, which derives from ancient Greece, considered mathematics to be an important part of education for a long time. For example, at the *Gymnasium* in Germany in the 19$^{th}$ century, mathematics was taught for "general human education". An important point is that pure mathematics and its formalism were taught more than applications.

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[2] Leung (2001) defined East Asia and the West according to cultural demarcations rather than geographic divisions, roughly identified as the Confucian tradition on one hand, and the Greek/Latin/Christian tradition on the other.

In Korea, on the other hand, less value was attached to mathematics. The social (or human) sciences and the philosophy of Confucianism were considered to be more important, and state examinations only existed in these subjects. Those students who passed state examinations could attain the high position of an official civil servant. Mathematics known as “a subject for calculation” was studied by “business people” who were in the second class, but not by the upper class people such as philosophers. Pure mathematics was not developed, but mathematics was applied in making calendars, in architecture and in taking measurements. After the Second World War, mathematics took on an important position in the school system.

## 2.2. Early educational system

### 2.2.1. Korea

A Korean school system originated in the 4th century, but was modelled on ancient Chinese schools. From the very beginning to the end of the Joseon Dynasty (1392-1910), before the reformation to a system similar to the western one, there were public and private institutions where children from the upper class and working class respectively prepared themselves for entrance to the bureaucracy, the most favoured form of employment. The both institutions taught Chinese classics to children and in so doing fulfilled conditions for the qualifications required for civil service. A kind of elitism dominated the system. In these institutions, mathematics was not taught at all.

The traditional education was based on the words form of classical humanism, whereas schools from after the reformation, steeped in an ethos of modernization and industrialization characteristic of Western civil culture, were typically imbued with a form of rationalistic scientism (Lee, 2001). Since 1883, the modern schools similar to western ones have been established by the government as well as by the citizens. The educational system and the associated opportunities have expanded with a remarkable speed since then.

### 2.2.2. Germany

According to Weidig (1992), German education has a long tradition. Historically, as in the case of Korea, only a small proportion of the population could enjoy the highest levels of education. The majority of Germans received a basic education followed by a specific training in a certain trade.

During the early middle Ages, in the schools which were part of the clerical system, the teaching of mathematics for the trade and guild system was limited to geometry. In the 12th century, municipal schools were founded so that education would be more centred on practical abilities. At the beginning of the 19th century, teacher-training institutes were founded in order to support general education. It was believed that only well-trained teachers could improve the school system. Further types of schools were developed after the establishment of *Realschulen*, the humanistically-oriented *Gymnasium* and the regular school, known as the *Volkschule*. In the first half of the century, the middle school (*Mittelschule*) developed and was introduced to provide a better education than the *Volkschule*. At the beginning of the 20th century, there were three main types of schools for general education, corresponding to the three main classes of society and their educational ideals:

- Schools preparing pupils for university (today called *Gymnasium*): schools with humanistic education as their goal for the children of the upper class.
- the middle school (today called *Realschule*): for the children from the middle classes preparing for the requirements of a trade in industry
- *Volkschule*: for the children of workers and farmers, emphasizing practical abilities.

(cf. Weidig, 1992)

Attending the first class of the chosen school dictated a child's career. Therefore, advancement to a higher type of school was nearly impossible to most pupils. In order to reduce the limitations of this system, the comprehensive elementary school (*Grundschule*) was introduced in 1920. Since then, all pupils have had to attend an elementary school for the first four years. They have then gone on to a *Volkschule* (today *Hauptschule*), *Realschule*, or *Gymnasium* (Weidig, 1992).

At present, this traditional model applies to lower secondary education; however, there are still differences in social classes. According to PISA, over 50% of pupils from the upper professional class (or service class) attend Gymnasium. By contrast, pupils from the semi-skilled and unskilled worker family go to *Haupt-* and *Berufschule* (see figure 2-1). Moreover, German pupils whose parents have the best jobs score on average about the best-performing country.

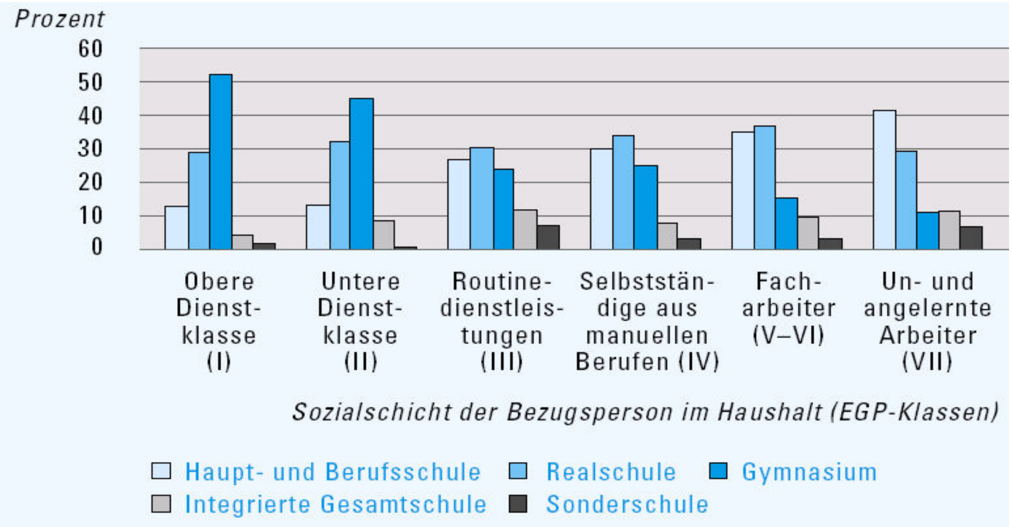



*Figure 2-1 15 years-old pupils after social layer affiliation (EGP classes) and types of schools (PISA 2000)*

A social movement in the late 1960s brought about a rise in an alternative system to the traditional one. The comprehensive schools (called "*Gesamtschule*") promoted the idea of more egalitarian access to education and were a form consisting of the integration of these three traditional schools. However, these schools are located mostly in middle and northern Germany. In Bavaria, only two or three *Gesamtschule,* which were established in the 1970s, are still active, and the Bavarian government does not even intend to keep these schools.

## 2.3. The current educational system

### 2.3.1. Korea

Korea has a 6-3-3-4 school ladder system which is a unified structure connecting the different school levels. The main track of the system includes six years of elementary school, three years of middle school, three years of high school, and four years of university education (two- or three year junior colleges).

School System

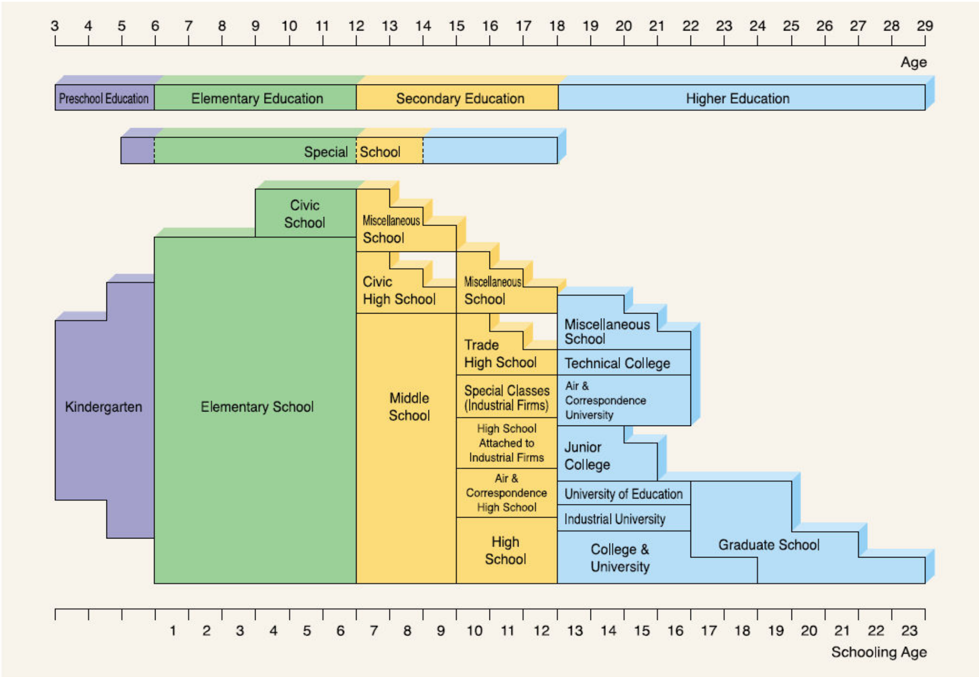



*Figure 2-2 Korean school system (MOE 2001)*

- *Elementary schools (grades 1-6)*

Elementary education in Korea is free, compulsory, and provides the general rudimentary education necessity in life. As of the ages of five or six, depending on the parents' decision, children receive a notification of admission from a school located in their residential area. Upon the entrance to elementary school, children automatically advance to the next grade each year. Recently, an accelerated grade advancement system was introduced to allow gifted children to skip a grade and an advance to the next grade. However, accelerated advancement is allowed only twice during individual's elementary and middle school years (MOE, 2001).

- *Middle schools (grades 7-9)*

The purpose of middle school is to provide standard secondary education using the basis of elementary education. Since 1969, there has been no limitation on entrance to middle school and all elementary school graduates who want to enter middle school are assigned to the school nearest to their place of residence. Therefore, entrance to middle school from elementary school has now reached over 99 percent. In 1985, free and compulsory middle school education was started in farming and fishing areas and has now expanded to all middle school pupils nationwide (MOE, 2001).

- *High schools (grades 10-12)*

Middle school graduates or those who have an equivalent academic background may enter high schools. The period of study is three years and the pupils bear the expenses of the education. Admission into the high school used to depend on marks in an entrance examination, but now works slightly differently, as clarified in Section 2.4.

In Korea, pupils as well as parents consider the three years of high school to be an important period because high grades are needed on the entrance examination for university.

High school education is mainly aimed at improving pupils' ability to establish their own career path according to the aptitudes and talents developed during their education, and secondly at developing the qualities required to become a global citizen. Those goals are more precisely stated as follows (MOE, 2001):

- To form a balanced character resulting from a sound mind and body and to have a mature sense of oneself.
- To acquire the ability to think and a logical, critical and creative attitude in order to be prepared for the world of learning and living.
- To be trained in the knowledge and skills of various fields and to improve the ability to establish one's career path according to individual aptitudes and talents.
- To seek the development of the Korean tradition and culture on a world level.
- To work for the development of the national community and to have the awareness and attitude of a global citizen.

High schools are divided into several categories according to their aims or functions: there are general, vocational, science, and special high schools. The curriculum at the general high school is mainly designed for pupils who want to continue their studies at university level.

- Vocational high schools provide an advanced general education as well as vocational training in agriculture, technology, commerce, fishery and oceanography, industry, and home economics. The curriculum at the vocational schools consists of about 40-60% general courses, with the remainder being vocational courses.
- Science high schools were established in a response to a suddenly increased desire to improve the education of youngsters with scientific talents and in this way to respond to the nation's great interest in making scientific and technological advances. Those who have completed two years in a science high school can be admitted to the bachelor's program at the Korea Advanced Institute for Science and Technology.
- Special High schools are the foreign language high schools, arts high schools, and athletics high school by so-called special purposes for gifted pupils. The pupils who attend these schools major in foreign language, music, arts, sports, and dance.

### 2.3.2. Germany

The German educational system cannot be so easily explained because of the "*Kulturhoheit der Länder*" which means that each state (*Land*) has the right and the obligation to determine it's own objectives in education. So a deeper understanding of the German educational system can only truly be obtained by focusing on the systems in each of the 16 states. The Basic Law of 1949 stated that every German person had the right to self-fulfilment, that is, the right to select the type of education they prefer. The states are required to provide equal educational opportunities and quality education for all in various educational institutions (Führ 1989). At present it is even more difficult to explain definitely about an educational system of even one state because discussions on reform are ongoing. These discussions on reform are carried out at both *Länder* and Federal level, by virtue of the federal structure of the German state.

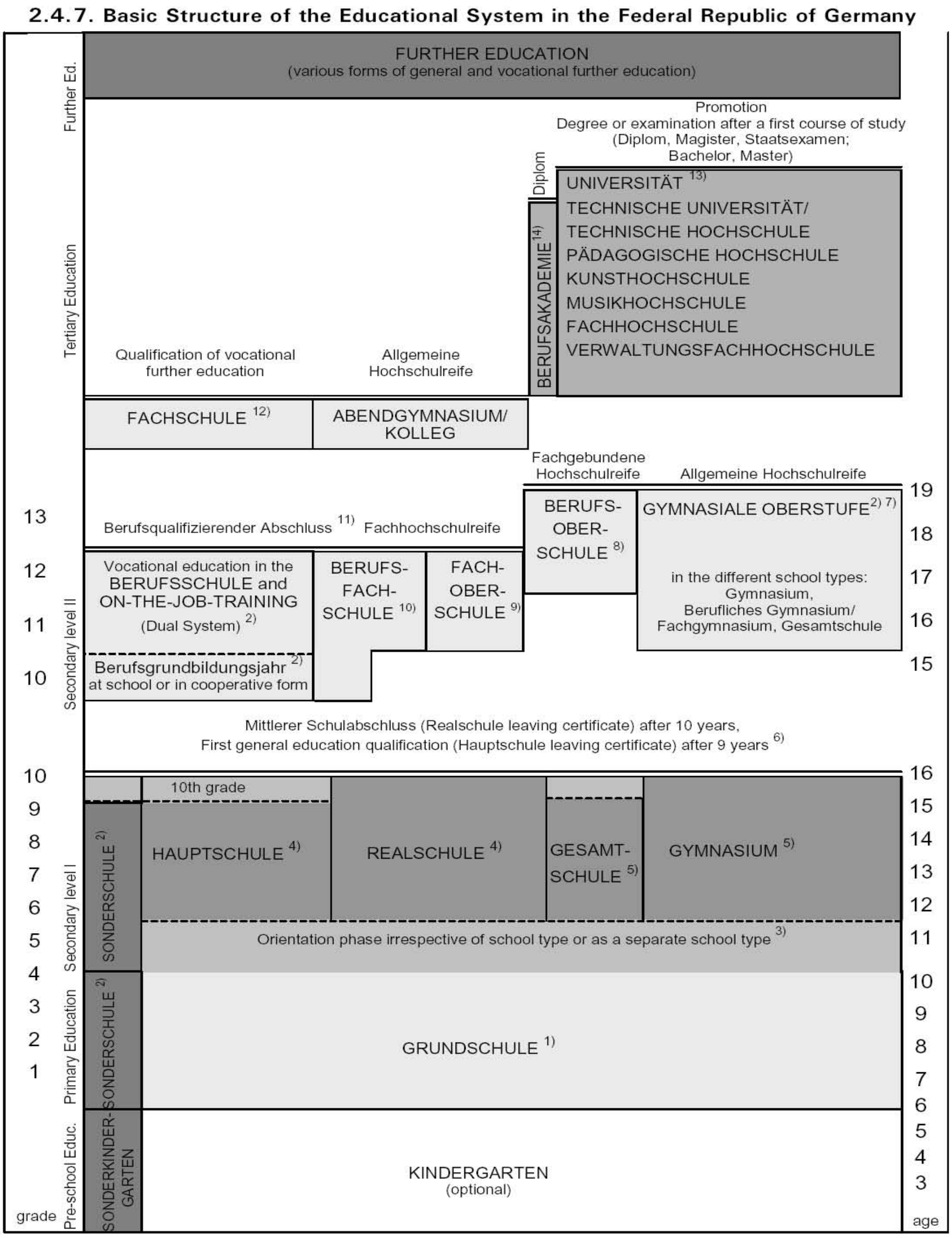



*Figure 2-3 German school system (see "Annotations" in Appendix A) (KMK, 2003)*

- *Grundschule (grades first-fourth, in most states)*

At the age of six, parents must enrol their child at the elementary school of their residential area. They will study there for four years in most states or six years in Berlin and Brandenburg. In order to foster the equality of educational opportunity, there is no tracking at the elementary level. Instruction aims to foster pupils' individual talents, to build the basis for independent learning and community living, and to impart basic knowledge and skills. Emphasis is placed on linking school material and extracurricular experiences (KMK, 2003).

Teachers recommend a particular school to each pupil based on criteria such as academic achievement and potential, personal characteristics, and the ability to work independently and with self-confidence.

However, in most states, a child's parents make the final decision on which of the three possible types of school the child will attend after the fourth grade. Some parents will choose to go against the teacher's recommendation because they believe a higher academic level offer their children more opportunities. In Bavaria in particular, academic achievement is the most important factor determining whether or not the *Realschule* and *Gymnasium* accept a given pupil.

The orientation stage called *Orientierungsstufe* or *Förderstufe* is organised as a separate organisational unit independent of the standard school types in some *Länder*, mostly in *Niedersachsen.* Irrespective of school type, grades 5 and 6 constitute a phase of particular support, supervision and orientation with regard to the pupil's future educational path and its particular focuses (KMK, 2003). However, it abolishes because it did not carry out what it should be since June 2003 in *Niedersachsen*. Therefore in Germany the *Orientierungsstufe* independent of type of school could not be seen any more.

***Secondary education***

In Germany, secondary education separates into the lower secondary level (*Sekundarstufe I*), which comprises education from grades five to nine (or seven to nine) in school and the upper secondary level (*Sekundarstufe II*), which comprises all the courses of education that build on the foundations laid in the lower secondary level (KMK, 2003). There are several types of secondary level education: *Hauptschule*, *Realschule*, *Gymnasium*, and *Gesamtschule*. All the courses of education at lower secondary level have the primary function to prepare pupils for courses at upper secondary level which are required for vocational school or university entrance qualification.

- *Hauptschule (grades five-nine)*

The *Hauptschule* emphasises a practical, skill-based, basic general education for those children whose achievement is lowest in elementary school. They receive a slower-paced and more basic instruction in the same primary academic subjects such as German, Mathematics, a Foreign Language (usually English), Physics/Chemistry, and many other subjects which are also taught at the *Realschule* and *Gymnasium*. An additional possible route at the *Hauptschule* is a vocational orientation.

In most states, pupils enrol in the *Hauptschule* at the beginning of the fifth grade and continue their education at the *Hauptschule* through the ninth grade.

- *Realschule (grades five-ten, in most states)*

The *Realschule,* which consists of grades five through ten, provides pupils with a pragmatic character of education. The emphasis is placed on more extensive general education. Its name is translated as "school of the reality" and shows its nature as a real-life school[3]. In *Länder* with six years of elementary school or with an *Orientierungsstufe* regardless of the type of school, it starts with grade seven and pupils can get the certification in grade ten.

The *Realschule* is divided up into the *Unterstufe* (lower level), which incorporates the 5$^{th}$, 6$^{th}$, and 7$^{th}$ grades and the *Oberstufe* (upper level), which consists of the 8$^{th}$, 9$^{th}$, and 10$^{th}$ grades. The lower level has a more pedagogical emphasis, while the upper level is more closely oriented to particular disciplines. There is also a three- or four-year form of the *Realschule* for *Hauptschule* pupils who may transfer to the *Realschule* after grade six or seven (KMK, 2003).

- *Gymnasium (grades five-twelve, in most states)*

Study at a Gymnasium in Germany normally lasts nine years (grades five to thirteen), but the standard period of study will soon be changed to eight years. It is becoming increasingly common for the primary and secondary stages of school education in Germany to last 12 years.

[3] See more http://www.unav.es/grice/HP/texto/pdfpaises/germanyeducation.pdf

The *Gymnasium* provides pupils with an intensified general education and traditionally leads to study at university. The final two years of *Gymnasium* (grades eleven-twelve in most states) are called the *Oberstufe* (upper level). The most common educational tracks offered by standard *Gymnasien* are Languages, Literature, Social Sciences, Mathematics and Natural Science.

Courses in the upper level of the *Gymnasium* in Germany are taught at basic (*Grundkurse*) and advanced level (*Leistungskurse*). The advanced courses approach the material in a more comprehensive manner and the material covered is significantly harder than at the basic level. For example, at the advanced level pupils may be required to write and present a report or conduct an experiment they have planned themselves. Basic courses, on the other hand, cover the same subject matter using simpler examples to illustrate key concepts and techniques.

There is also some scope for pupils to transfer from the *Hauptschule* to the *Realschule*, or from the *Realschule* to the *Gymnasium*. Mathematics and Foreign languages are therefore frequently taught in sets according to the pupils' aptitude.

- *Gesamtschule*

*Gesamtschule* are known as comprehensive schools. There are many *Gesamtschulen* in most *Länder*, but in others there are only a few. These schools offer an alternative to the traditional system. The central idea is to make education accessible to everyone; these schools consequently have a curriculum which is designed to provide foundation for all possible future careers. These schools may be either cooperative or integrated schools. The cooperative *Gesamtschule* retains the traditional hierarchical structure by providing different study routes within a single school. In other words, it combines the *Hauptschule*, *Realschule*, and *Gymnasium* in one unit. This structure caters for a range of abilities while allowing greater mobility between different routes. The integrated *Gesamtschule* forms one organisational and educational unit, which has no system of different study routes, and in which pupils with different levels of academic achievement are taught in the same classes. Pupils in these schools all attend the same classes in the fifth and sixth years, and thereafter are allocated to honours courses in subjects such as mathematics, German, a foreign language (the first they have been taught), and science (namely physics or chemistry) depending upon their performance. The allocations to the various courses are consequently very important to pupils (KMK, 2003).

***Upper secondary level***

A variation of the traditional *Gymnasium* is the *Berufliches Gymnasium*, which offers specialised study routes in areas such as economics and the technological sciences in addition to core academic courses. Pupils who successfully complete the course of study at a *Gymnasium* (or *Berufliches Gymnasium*) and pass the wide-ranging examinations receive the *Abitur* for study of univerisity.

***Full-time vocational schools***

Full-time vocational schools include the *Berufsfachschule*, the *Fachoberschule*, the *Berufliches Gymnasium* or *Fachgymnasium*, the *Berufsoberschule*, the *Fachschule* and other types of schools which exist only in certain *Länder*. At upper secondary level the vocational school offers a wide range of study routes and courses of varying duration. It prepares the *Realschule* or *Gymnasial* pupils for a specific occupation at different levels of qualification.

***Vocational training in the dual system***

According to KMK (2003), two-thirds of young people in Germany undergo vocational training in the dual system *(duales System)* which typically lasts three years, depending on the chosen occupation.

The *dual system* is so called because training is carried out in two places of learning: at the workplace and in a vocational school *(Berufsschule)*. The aim of training in the dual system is to provide a broadly-based vocational training and to enable pupils to attain the qualifications and skills needed to practise a skilled occupation within a structured course of training.

There are no other prerequisites for admission to the dual system; it is generally open to everyone. Learning in the dual system is based on a training contract under private law between the training company and the trainee. The trainees spend three or four days a week at the company and up to two days at the *Berufsschule*. The training companies assume the costs of the on-the-job training and pay pupils an allowance in accordance with the collective bargaining agreement in the given sector. The allowance increases each year and is, on average, about a third of the starting salary for a specialist trained in the given occupation (KMK, 2003).

The skills and knowledge that are to be acquired during the course of training at the workplace are set out in the *Ausbildungsordnung* (training regulations). *Berufsschule* classes cover the material for each recognised occupation for which formal training is required, as set out in the training regulations of the *Rahmenlehrplan* (framework curriculum).

- *Berufsschule*

This is a vocational school at upper secondary level generally providing part-time education and vocational subjects to pupils receiving vocational education and training through the dual system.

Sixty percent of Korean upper secondary pupils enrol in general academic high schools designed mainly to prepare pupils for tertiary education and the remaining forty percent of pupils go to vocational high schools which prepare them for the labour market (MOE, 2003). By contrast, more than half of German upper secondary pupils attend vocational high schools.

## 2.4. Entrance Examinations, Assessment & Repeating a Year

### 2.4.1. Middle school

As mentioned above, at the present time both middle schools in Korea and secondary education (*Gymnasien, Realschule, and Hauptschule)* in Germany have no obligatory public examinations.

- Germany

In Germany however, in order to attend the *Gymnasium*, pupils finishing elementary school must either have sufficient grades or the recommendation of a teacher. However, the pupil's achievement is more important than the teacher's recommendation. An improved performance can lead to a transfer to a more demanding school. A pupil can just as easily be transferred to a less demanding school if they cannot satisfy the required performance standards.

- Korea

By comparison, no such transfer is possible in Korea. Consequently, even if pupils have performed very poorly one year, they still go on to the next grade. It means Korean pupils who graduated elementary school may have a more opportunity than German pupils to further chance to learn more and far up to grade nine.

### 2.4.2. High school

- Korea

Before 1995, Korean pupils were admitted to high school according to the grades they had achieved in the public examination. Pupils who did not attain a sufficiently high grade were not selected by the general high school. Instead, they would attend another type of school, for example a vocational high school. The general high schools selected the pupils who achieve a certain grade through a multiple application lottery system in each school district. However, as of the revision of the education law in 1995, there are now various methods of selection, such as recognition of the school activities records. The special high schools, such as foreign language schools and independent private schools which have a clear and special philosophy and for which pupils are required to pay fees, have the right to select pupils. Pupils who want to be admitted to these schools are therefore required to write an entrance examination which is made by those schools. The goal of most middle school graduates is to attend a general high school, which can be used as a stepping-stone to a university. Over 80 percent of graduates from general high schools attain either a university or college place, 4 percent enter employment as soon as they have finished general high school, and around 14 percent become unemployed, go to government training institutions, or start forms of employment or training unknown to those who collected these figures (MOE, 2001). Vocational high schools often take those pupils who have been rejected by general high schools or whose parents cannot afford general high school fees.

- Germany

The pupils who attend any type of school for secondary education could still continue to attend to 12$^{th}$ grade, if they do not drop the schools (see 2.4.4.).

### 2.4.3. University

- Korea

For university and other academic study routes in Korea, public examinations are used as a method of selecting pupils. Korea has only one central examination (*Su-neung*) for certification and selection to university education, which is held during the final year of high school (grade twelve). Korean, Mathematics, English, and Natural Sciences are mandatory subjects for this central examination. In addition, pupils must select a social or cultural science and a second foreign language. This means that pupils must take an examination in most of the subjects he/she studies. An oral test is also administered. However, this is not considered to be of great importance.

Furthermore, a university could select the pupil based on their achievement of school activities record. As a result, some pupils, with a high level of academic achievement might not be selected by that university.

- Germany

In Germany, there is an exam called the "*Abitur*" which is the school-leaving exam that follows attendance of a Gymnasium. However, the *Abitur* is not the same in all the *Länder*. Two of the southern states, Bavaria and Baden-Württemberg, have their own centralised examination (called "central *Abitur"*), which is developed by some selected or recommended teachers. This central exam is the primary mechanism for monitoring and controlling standards of education at the Gymnasium level. The other states, including Lower Saxony, do not have a central examination at present. Yet each school creates its own exam. This allows each school to adapt its own *Lehrpläne* to the abilities of its pupils. The central *Abitur* therefore might result higher standards - pupils must adapt to certain standards rather than having the *Abitur* tailored to them. The *Nidersachesische* Ministry of education would like to change this *Abitur* to central *Abitur*.

In Bavaria, where the central *Abitur* is used, pupils must select three subjects for written examinations. Two of these subjects should be selected from the advanced courses (*Leistungskurse*) the pupil is taking. There is also an oral test (*mündliche Prüfung*). Bavarian pupils therefore prepare only four subjects in which to take the *central Abitur.* These subjects can be selected from such subject areas as Foreign Languages, Social/Cultural Science, Mathematics, and Natural Sciences (Physics, Biology, etc.).

In the TIMSS and PISA studies, the German *Länder*, for example, *Bayern* and *Baden-Wüttemburg* which use a *central Abitur* performed better on average than the other *Länder* such as *Nidersachsen, Sachen-Anhalt,* and *Bremen*.

#### 2.4.4. Assessment & Repeating a Year

- Korea

Korean middle school pupils have two exams per semester in all subjects and up to four exams for the primary subjects, namely Korean, Mathematics, English and Science. Reviews of homework, quizzes and minor tests can also be graded, but these only account for 10 to 20 percent of the final grade. Oral exams are seldom used in Korean classes. This has been suggested as a reason why pupils struggle to express their opinions orally. Choe (2003) also asserted that Korean pupils had difficulties in explaining their thoughts logically when addressing open realistic (or practical) problems. They also had trouble justifying their answers. On this basis, oral exams arguably have a big impact on pupils' learning.

- Germany

*Gymnasium* pupils in Germany have two to four in-class exams per semester for the primary subjects English, German, and Mathematics; two per semester for Biology; and at most one in secondary subjects such as Music, Geography, and History. Teachers periodically ask pupils to review their homework orally and then grade them on their presentation. Moreover, oral exams have become an increasingly important tool for assessment in the 12th grade, when pupils start preparing for the *Abitur* exam.

In Germany pupils can be made to repeat a school year even in elementary school. German pupils who have to repeat a year three times are required to attend another school which better corresponds to their ability. Pupils who are given a 5 (on the standard mark scale of 1 to 6) in two major subjects[4] must repeat the school year. However, they then have the opportunity to avoid this eventuality by passing examinations in the same subjects at the beginning of the following school year. This examination (*Nachprüfung*) allows pupils to catch up over the summer vacation. If they are successful, they go on to the next school year. By contrast, in Korea, school years are not repeated.

## 2.5. Discussion

In the wake of the TIMSS and PISA results, many researchers attempted to explain why Asian pupils performed better than their western counterparts in mathematics. Possible reasons and pupils' school life schedule will be outlined, as these might explain the difference between the two cultures.

#### 2.5.1. Studying hard and enjoyable learning?

Studying is considered to be a serious endeavour in Korea. Pupils are expected to work hard and persevere in their studies. Stevenson et al. (1987) commented that "Asian parents teach their children early that the route to success lies in hard work". Indeed, Park and Yim (2002) stated that in East Asia,

[4] It depends on schools. The major subjects could be mathematics, German, English and so on.

learning is viewed as "the process of training in hardship". It is common knowledge for pupils that many successful and famous people have recovered from hard times by studying. To some extent this encourages pupils to study hard in spite of a lack of interest.

By contrast, learning is considered to be an "activity which is enjoyed" in western countries (Park & Yim, 2002). Enjoyable learning has been a slogan in a number of Western countries. Leung (2001) explained this phenomenon by using a philosophy of education which focuses on the child. Western educators generally believe that children learn effectively when they enjoy learning. In general, it is considered to be important that learning be a pleasurable experience.

### 2.5.2. Attitudes

One potential problem regarded as a virtue in East Asia is that young people listen carefully to and remember the teaching of sages and old masters (Park & Yim, 2002). This attitude can also be widely observed in school. A disadvantage of this is that pupils can then become accustomed to learn somewhat passively. There may be little interaction between teachers and pupils, although teachers that ask lots of questions can counter this problem.

### 2.5.3. Motivation to learn

One of best-known conclusions of TIMSS and PISA is that Korean pupils do not have as great an interest in learning mathematics as their western counterparts. Despite this, they study hard. They have more motivations extrinsic to the classroom, for example they try to perform well on exams to improve their chances of admission to university and to satisfy their parents' expectations. These are important sources of motivation for East Asian pupils - academic achievement has always been considered to be a means of bringing honour to parents and family.

Educators in the West treasure intrinsic motivations in learning mathematics (Leung, 2001) Educators consider interest to be the best way of motivating pupils to learn and to study. Extrinsic motivation derived from examination pressure causes anxiety and it is thought that this form of motivation can have an adverse effect on pupils' learning (Leung, 2001).

### 2.5.4. Day school vs. half-day school

In Korea, pupils have school from Monday to Friday and with a half-day on Saturday. The Korean government is at present trying to encourage schools to work to a five-day week. So far, this has only been introduced at a few trial schools. A typical school day is from 8:30 a.m. to 4:00 p.m. in middle school and from 9:00 a.m. to 4:00 p.m. in high school. There are also some special lessons for the primary subjects before and after these hours. Consequently, high school may start at 8:00 a.m. and last until 9:00 p.m. - 10:00 p.m. depending on schools. Pupils may also receive private tuition at home or go to a private institute to learn more about primary subjects.

General high school is often criticised by parents, as they argue that the Korean educational system only prepares pupils for the university admission exams. Here is a typical day of a Korean pupil in 10th grade as given on the Internet by a parent criticising the educational system.

- 8:00-8:50 a special lesson so-called 0-lesson
- 9:00-13:00 1-4 lessons
- 13:00-14:00 lunch time
- 14:00-18:00 5-8 lessons
- 18:00-19:00 dinner time
- 19:00-19:50 studying and listening median lesson

- 20:00-21:00 learning in the classroom
- 21:30-23:30 private institute

Many parents and pupils as well as some teachers are unhappy with the Korean school system. However, most schools keep it because it is thought to avoid the disadvantages of private institutes as described below.

By contrast, schooling in Germany is five days a week. For the most part, school lasts for just half a day. A typical school day starts at 8:00 a.m. and finishes at 1:00 p.m. At upper secondary level, schooling consists of 28 to 35 lessons per week, depending on the type of school. However, after TIMSS and PISA, German politicians discussed the introduction of day schools which would finish at 3:00 or 4:00 p.m.

As shown in PISA, Korean pupils do homework after school lessons. 27% of Korean pupils do homework or study mathematics over 3 hours per week. It is almost same percentage of pupils do homework 1 to 3 hours (27%) or less one hour (27.10%) per week (PISA 2000).

German pupils do homework or study mathematics 2.74 hours per week on average (PISA 2000).

### 2.5.5. Private institute

Germany has private institutes and private teachers who help pupils who fail to attain high marks. In Korea, most pupils go to private institutes or are taught by private teachers after school whether their grades at school are good or bad. The subjects taught are mostly Mathematics, English, and Sciences. This means that pupils may cover the same topics twice; in school and at the private institute. This may help pupils to achieve higher marks. However, some pupils depend on private institutes more than on school. They believe they can learn better in private institutes. According to Lee (1999), Korean families spent 25 billion U.S. dollars in 1996 (fully 150% of the government's educational budget) on private institute (including private tuition at home) outside the school system. A national survey in 1997 showed that the average middle school pupil (grades seven to nine) took about 10.3 hours of lessons per week outside the regular school system (Kim, Yang, Kim & Lee, 2001, p.50). The government is trying to avoid expanding these private institutes and attempting to consolidate public education. As a result, in 1997, the Ministry of Education announced the 7$^{th}$ Curriculum in order to ease the burden on private institutes. In addition, as mentioned above, pupils watch and study the median lesson in school instead of going to private institutes.

A significant problem with Korean school education is the importance placed on university admittance. Teachers concentrate on strategies to increase the number of pupils who score highly on examinations such as the examination for admission to university (Choe, 2003). Choe (2003) argued that Korean teachers and private tutors misuse their opportunities in that they require pupils to memorise formulae in order to attain high scores in various examinations. So 'rote-learning' and a lack of creativity are common in Korean schools.

A private institute is not an alternative to compulsory public education. It is an extra institute outside the school system. A thorough analysis of private institutes is therefore not relevant to the topic of this thesis. Private institutes play an important role in Korea, but in my opinion a relatively harmful one.

## 2.6. Summary

Traditionally, Korea has maintained a uniform educational system which focuses on teachers and on marks. Korea's policy makers have tried to shift the educational system towards a more pupil-centred set-up, but do not want to make radical changes. They want to keep the appearance of the system as it is and to change little of the system's true nature. Similarly, Germany's educational system is at present classroom-centred; however, it is slowly changed to pupil-centred and has good connections with vocational training.

One great difference in the educational philosophies of the two countries is apparent in the subject of elite education. The prevailing educational philosophy in Korea is to train everyone equally. By contrast, Germany's philosophy is to train the best academic achievers according to Humboldt's humanistic concept. Germany therefore has a tri-partite school system with certain ideals. They are set out by a pyramid of academic achievement.

Park and Yim (2002) used similar arguments. In East Asia, there have traditionally been standards which all members of society are supposed to reach. Therefore, everyone should try to reach this standard. East Asians generally believe that all people can become holy or wise. This may well be influenced by Confucian philosophy. Westerners, on the other hand, in general believe that the ability to learn consists of very different levels. The Bible states that everyone has different talents and it is thought that the social and educational systems should reflect this. This totally contradicts the philosophy of East Asia.

It is important to note that there are regional differences in the educational system in Germany. Each state's school structure has been influenced to some extent by different historical and political events.

# Chapter 3 Curriculum and textbooks

"… The treatment of school mathematics curricula in international comparative investigations is a story of increased efforts to take aspects of curriculum complexity into account. It is also, however, a story of persistent failure to probe sufficiently below the surface of, and to challenge assumptions about, what is to be understood as curriculum." (Keitel & Kilpatrick, 1999, p.242)

Why do we investigate mathematics curricula? According to Schmidt et al (1997), because the interaction between teachers and pupils in the classroom does not occur randomly. Instead, based on visions of what education should be, on ideas of how to bring about formative educational experiences, and on opportunities provided to make these experiences possible, they are deliberately shaped. Curriculum is the plan that expresses educational aims and intentions and serves as the broad course that runs throughout formal schooling.

Among the fundamental questions about mathematics curricula, I am interested in how different curricula affect the learning of mathematics in different countries. Educational researchers have for a long time accepted that pupils' performance on tests is closely related to what they experience in classrooms. I have therefore analysed curriculum guides, which are official documents that most clearly reflect the intentions, visions, and aims of countries, and textbooks which give significant insights into intentions and aims, despite the fact they are not official in the way curriculum guides are.

Of course there has been an in-depth investigation into this. The Third International Mathematics and Science Study (TIMSS) was a very widely conducted comparative study. One aim of this study was to ascertain some of the curricular and classroom factors that have influence on pupils' learning in mathematics and science. This analysis gave a broad spectrum of answers from suggesting steps which could be taken to facilitating.

Over the past twenty years, prior to TIMSS, there have been numerous international studies and comparisons of attainment in mathematics education. The First International Mathematics Study (FIMS) was carried out in 1964, with the participation of twelve countries (Husén, 1967). Freudenthal (1975) pointed out that cross-national comparisons are not valid without considering curricular aspects. He argued that a country's success in such a comparison depended to a large extent on the degree to which the test instrument (what is tested and how it is tested) was aligned with the mathematics curriculum of the particular country. Taking into consideration such criticisms, the Second International Mathematics Study (SIMS), which was carried out between 1980 and 1982 in twenty countries, differentiated between the intended, the implemented, and the attained curriculum (see Travers & Westbury, 1989).

In 1995 TIMSS was carried out, in over 40 countries (see Schmidt, et al, 1997). The framework for the TIMSS differentiates among the intended curriculum, the implemented curriculum, and the attained curriculum.

- The intended curriculum, at the level of educational system; that is seen in national policies and official documents which reflect of societal visions, educational planning, and official or political sanctioning for educational objectives, i.e. it refers to an educational system's goals and means.
- The implemented curriculum, at the level of the class; in many cases, teachers need to interpret and modify the intended curriculum according to their perceptions of the needs and abilities of their classes. It refers to practices, activities and institutional arrangements within the educational context of schools and classrooms.
- The attained curriculum, at the level of pupils; this refers to the outcomes of schooling, the result of what takes place in classrooms. Academic achievement and pupils' belief are two measures of

what has been attained. At the individual level, it is about the amount of homework a pupil does, the effort the pupil expends, the pupil's classroom behavioural patterns, and so on.

Research has shown that the implemented curriculum, even in highly regulated educational systems, is not identical to the intended curriculum.

Schmidt et al (1997) investigated mathematics curricula of participating in TIMSS and reported on differences in curricular organization. Some countries, such as Korea, are highly centralised, with the ministry of education (or the highest authority in the education system) being exclusively responsible for the major decisions governing the direction in which education is moving. In Germany, on the other hand, such decisions are made by "*Länder*" (the individual states). It is therefore possible to have different curricula even within a single country. It is also possible to make curricula cater to communities perceived as having different needs.

## 3.1. The aims in Mathematics Curriculum

The whole of the educational system in Korea has been closely controlled by central government and developed through a series of five-year national plans. Korea has a single curriculum for all schools, which was revised in 1997 for the purpose of attaining four goals; firstly, to enrich elementary and basic education; secondly, to increase self-directed ability; thirdly, to encourage learner-centred education; and fourthly, to increase autonomy at the local and school level. Even though Korea stresses local implementation, the curriculum gives a detailed guide to grades to be awarded for individual topics, which are themselves subdivided.

While Germany is a relatively homogeneous nation, its constitution guarantees the cultural sovereignty of each *Land*. Each of its 16 states has its own ministry of education and a distinctive set of political, religious, and cultural traditions. The responsibility for primary and secondary schooling in Germany rests with state and local authorities in the ministries of culture and education (*Ministerien für Kultur und Bildung*). This chapter will discuss the curricula of two German states: namely that of Lower Saxony (*Niedersachsen*), which is located in Northern Germany, and that of Bavaria (*Bayern*), which is located in Southern Germany.

### 3.1.1. A short history of Korean mathematics curriculum (focusing 6$^{th}$ and 7$^{th}$ national mathematics curriculum)

Since 1946, Korea has had a strong national system of education and had a common curriculum across the country for each type of school. Since then, however, the curriculum has been revised every 7 to 8 years to adapt to social changes and new educational needs.

| Amendment | Period | Main Focus |
|---|---|---|
| 0 | 1946-1954 | Progressivism |
| 1 | 1955-1962 | Real-life-centred |
| 2 | 1963-1972 | Systematic-learning |
| 3 | 1973-1981 | "New Math" |
| 4 | 1982-1988 | Back to basic |
| 5 | 1989-1994 | More Back to basic |
| 6 | 1995-1999 | Problem solving |
| 7 | 2000-now | Differentiated curriculum |

*Table 3-1 Periods of National Curricula of Mathematics in Korea, Choe (2003)*

I will outline two recently revised curricula in Korea. The 6$^{th}$ curriculum was developed in 1992 and went into effect in the spring semester of 1995 in middle schools and in 1996 in high schools. The main focus of this teacher-centred curriculum was “problem-solving”. As such, the planning of mathematical content, instructional methods, and evaluation comments were clearly stated in this curriculum. The basic characteristics of the 6th Mathematics curriculum were (1) the development of mathematical thinking and problem-solving, (2) the emphasis on realism in mathematics, (3) the use of calculation and computers as mathematical tools, (4) the consideration of pupils’ abilities and career goals, and (5) the variety of didactical learning and evaluation methods.

The 7th curriculum was revised in 1997 and was introduced in 2001. The feature of the 7th Curriculum is its emphasis on the learner-centred education. The major characteristics of the curriculum are (1) From the first to the tenth grade, the common course requirements are based on national standards (2) the introduction of the concept of “differentiated curriculum” in major subjects such as Korean, English, Mathematics, Science, and Social Studies. This means that from grade one to ten, the curriculum is differentiated on the basis of the pupils’ academic capabilities, and from grades eleven to twelve, the curriculum is differentiated on the basis of the pupils’ interests and career goals. Further characteristics of the curriculum include: (3) the concept of “elective-course program” introduced for grades eleven and twelve, which means pupils can select courses according to their interests and career goals, (4) the expansion of “school discretionary time” to encourage self-motivated learning, independent studies and other creative activities, (5) the reduction of the mathematics content, and (6) the establishment of the quality control system based on curriculum evaluation.

The main difference between the 6th and the 7th curriculum is that the 7th curriculum focuses on individual differences. Many Korean educators had earlier pointed out that teaching and learning in the classroom was carried out without considering individual pupils’ abilities and interests. This concern was finally taken into account during the preparation of the new curriculum. The current curriculum suggests that pupils should actively participate in classroom activities rather than being subjected to passive activities lead by teachers.

In addition, the mathematics content has been reduced by 30% compared to the former curriculum, because many administrators in the government thought that the level of difficulty of school mathematics in Korea was much higher than in Western countries. Consequently, the number of Mathematics lessons in the 9th grade was also reduced.

| | Grade 7 | Grade 8 | Grade 9 | |
|---|---|---|---|---|
| 6th | 136<br>(4 hours per week) | 136<br>(4 hours per week) | 136<br>(4 hours per week) | 34weeks per year |
| 7th | 136<br>(4 hours per week) | 136<br>(4 hours per week) | 102<br>(3 hours per week) | One hour means<br>45 minute |

*Table 3-2 The numbers of mathematics lesson in Korea*

- **Korea – Curriculum guidelines of middle school (MOE, 1997)**

The 7th Mathematics Curriculum implies that pupils should develop “Mathematical Power” not only through problem solving, but also through communication, reasoning, and connections. “Mathematical Power” which is taken in the 7th Curriculum is a more general concept than “problem solving” which is the focus of the 6th curriculum. It is more than a collection of concepts and skills. The three main aims of the 7th curriculum are:

- Understanding knowledge: Korean pupils are required to understand basic concepts, procedures and principles, as well as to improve their understanding of how these ideas are connected by using their experience of various physical objects encountered in daily life.
- Application: Korean pupils should observe, analyse, and structure the daily problems mathematically and get the solution of the problems of everyday life by applying mathematical knowledge and functions in a rational manner.
- Attitude: Pupils should show interest and pay attention in mathematics.

### 3.1.2. A short history of German mathematics curriculum

As mentioned, each *Land* in Germany has its individual curriculum. Even though the framework guidelines differ considerably from *Land* to *Land*, in some respects, similar developmental tendencies can be observed in the long-term (Klieme et al, 2003).

In the late 1960s and the 1970s, mathematics curricula in western German *Länder* and the GDR focused on conceptual aspects and analytical considerations. With varying exactness depending on the individual Land, the detailed guidelines outlined goals, methods and corresponding assessments that were deemed to be suitable by the authors (Klieme et al, 2003).

In the 1990s, the detail-oriented approach of curricula was changed. This approach focused on content and left the actual elaboration of the content largely in the hands of teachers. However, as a result, the pedagogical and didactical theory of teaching and learning failed to emerge (Klieme et al, 2003). Plans for a new curriculum for the *Gymnasien* in Bavaria are currently being discussed.

In the proposed curriculum, more attention would be paid to pupil's activities so that pupils might "discover" mathematical content. This suggests the influence – however, limited trends that conceive of teaching and learning as a constructive process and of a more process- and less product-oriented view of mathematics. Similar tendencies can be observed in many of the German states (Klieme et al, 2003).

During the last few years the curricula have undergone many changes. However, the trends apparent in these changes had more to do with approach than content. Thus, the new orientation for mathematics teaching is designed to

- Present mathematics both as a theoretical study and as a tool for solving problems in the Natural and Social Sciences.
- Provide experience of the fundamental mathematical idea of generalisation, of the need for proofs, of structural aspects, algorithms, the idea of infinity, and of both deterministic and stochastic thinking.
- Use inductive and deductive reasoning, various methods for constructing proofs, axioms, normalisations, generalisations, specifications, and heuristic work.
- Provide variation in argumentation and representation in all fields and aspects of mathematics teaching.
- Teach historical aspects of mathematics.

These goals, taken as common learning objectives for the teaching of mathematics, can be regarded as constituting a consensus among mathematics educators. Moreover, among others, these goals are explained in mathematics teaching in the upper secondary level (Borneleit et al., 2001). The main difference between basic and advanced courses is that objectives formulated for advanced courses are oriented more towards mathematics as a science, whereas those for basic courses stress algorithms and mathematical formulae. Both types of courses are based on the same three pillars of mathematics education: Calculus, Linear Algebra/Analytical Geometry, and Probability and Statistics. Of these three, Calculus is seen as being the most important and takes up the largest portion of teaching time. Both Calculus and Linear Algebra/Analytical Geometry were a part of curricula as early as the beginning of the 20th century, but probability and statistics have been added only in the last 10 to15 years. These three main components of the teaching of mathematics are paramount in the standards set out for the German baccalaureate (*Abitur*) in all states of the German Republic. Further specifications and differentiations of goals vary very much from state to state.

- **Lower Saxony – Framework guidelines (*Rahmenrichtlinien*, in *Niedersächsisches Kultusministerium*, 1989)**

The curriculum of Lower Saxony for mathematics, called the *Rahmenrichtlinien* (Framework guidelines), consists of three goals which should be achieved in mathematics class at the lower secondary level:

(1) The understanding of basic knowledge and the application of elementary skills

This means that pupils should master the appropriate mathematical language to enable them to know, describe, and form a precise idea of mathematical concepts and procedures. Most importantly, pupils should understand the concepts covered in each grade. Pupils should be able to transform mathematical facts into suitable representational forms and extract relevant information from the different representational forms (*Niedersächsisches Kultusministerium*, 1989).

(2) Guidance in problem solving, abstractive and constructive thinking.

Abstractive and constructive thinking are of great importance for problem solving, arguing and proving mathematical problems in the classes of lower secondary. Moreover, all of these aspects concern the conversion to mathematical problem of certain problems occurring in nature. Pupils should learn, for example, to divide a complex problem into simpler sub-problems in a meaningful way in order to develop different solutions gradually. These solutions must then be compared with each other and applied correctly to any new problems. Another aspect involved in this goal is guidance in the formulation of hypotheses and mathematical theorems which should be proved if necessary. Furthermore, pupils should be trained to recognise logical errors and to look for exceptions to mathematical rules as well as to examine the results on compatibility with highly ranked criteria (*Niedersächsisches Kultusministerium*, 1989).

(3) Encouragement of attitude in non-cognitive range

Further to the expression of constructive criticism and the acknowledgment and acquisition of correct solutions, the use of precise formulations and a logical and sequential representation is of particular importance. Such an attitude is particularly helpful whilst arguing and justifying as well as whilst outlining the steps in a proof (*Niedersächsisches Kultusministerium*, 1989).

- **Bavaria – Syllabi (*Lehrpläne*)**

In Bavarian Syllabi, mathematics is understood as “a subject which has been developed over thousands of years as a cultural achievement common to all humans. It takes aspects of reality and compiles concepts, theories, structures, and models. … Mathematics is traditionally a characteristic part of the language of the natural sciences and of technology. In addition, in economics and politics as well as in the social sciences, well-known statements frequently form the basis for decisions of great importance with mathematical methods.”

The goals of mathematics education in Bavaria are to improve self-esteem, to further pupils’ ability to apply what they have learnt, and to foster enjoyment in learning. The curriculum recommends that high-school pupils become familiar with mathematical objects and concepts as expressed in mathematical language, formulae, and diagrams, as they should be acquainted with and understand the possibilities of using deduction in real life”. They should acquire the ability to answer questions from different fields and judge results correctly. They should become aware that many problems of our time can be approached using a rational perspective, and that mathematical thinking and procedures are applied in most sciences, in most vocational fields, and last but not least in day-to-day life. A further goal is to make young pupils understand mathematics as much as possible, for them to take enjoyment in tackling mathematical topics, to stimulate discussions about mathematics and to arouse their curiosity.

## 3.2. The Aims and Expectations of Geometry class in the curriculum

Geometry is one of the most important parts in Korean and German school mathematics since deductive reasoning is used greater effect in this subject than in others. Geometry provides a rich context for the development of investigative, creative, and reasoning skills, including making conjectures and validating them. Moreover, geometrical concepts have close connections to other subjects.

Geometry is increasingly used to solve problems outside the mathematics classroom. The possible solutions to geometrical problems are therefore varied.

The geometry content in Korean and German curricula extends well beyond merely identifying geometric shapes or using procedures to apply spatial visualization skills in order to understand relationships. Spatial sense is a fundamental component in assessing geometry. If pupils understand spatial relationships, they can use the dynamic nature of geometry to connect mathematics to the world around them.

### 3.2.1. Korea

Pupils should intuitively understand the concepts of "plane" and "space" by looking at natural phenomena and should practice drawing and measuring. Pupils later gain experience in problem-solving using deductive reasoning. It is important to understand some basic facts about the geometrical figures in the plane and in space and to see that deductive reasoning is more useful in geometry than in other subjects. Moreover, geometrical concepts have close connections to other subjects. Korean teachers focus on the development of pupils' ability to visualise figures in space and in the plane. It is important to understand some basic facts about geometrical figures in plane and space.

Pupils should be able to;

- Intuitively understand the concepts of the plane and space
- Understand basic properties of plane and space
- Understand the relationships between figures in plane geometry and those in solid geometry, and properties common to both.
- Prove propositions relating to figures by deductive reasoning
- Verify propositions relating to figures by checking they hold for given examples by measuring
- Apply these propositions to other problems

Mathematics teachers in Korea ask pupils to be familiar with and understand a large range of concepts, axioms, theorems and related formulae, and to memorise and master their relationships.

Korean pupils begin to study logical reasoning and to learn how to prove geometrical propositions in the $8^{th}$ grade.

In grade 7, Korean pupils learn the simple properties (axioms) of basic figures in terms of points, lines, sides, and angles. They also study the congruence of triangles and learn about polygons and polyhedrons. Pupils might learn some basic properties by means of intuitive experimental activities.

In grade 8, Korean pupils learn the properties of quadrilaterals using the congruence of triangles and also learn the properties determining the similarity of two triangles; however, for Korean pupils, proofs showing in given examples that these properties hold are not emphasized. After showing that these properties imply similarity, pupils are encouraged to check that this is true by looking at examples and by applying them to solve certain other problems.

### 3.2.2. Germany

It is clear from the Bavarian curriculum and from German curricula in general that geometry class makes an important contribution to general education. It does so by helping to develop pupils' intuition and their ability to apply and extend their logical reasoning, as well as their ability to generalise the connections they have already made. It also links what they have learnt about shapes with numerical and algebraic methods or with metric and coordination. Plane and solid figures are covered and an interrelation between concrete experience and mental activity is initiated. This is achieved for example by measuring, by construction, with the use of compasses and set squares, by constructing polyhedrons and through the use of geometry software. Geometry classes stimulate thinking and the communication processes which lead to such heuristic activities as assuming, justifying, disproving and generalizing.

In grade 7 in Germany, mathematics is taught more systematically and the analytic thinking of pupils is therefore more significant. A greater emphasis is placed on forming sequences of reasoning. Following up on their previous knowledge pupils start to make connections in the geometry of polygons and polyhedrons. It is at this point that they should begin to enjoy geometry and develop an appreciation for it. In particular, this additional knowledge makes them more careful and accurate in their own constructions.

In grade 8, German pupils systematize and generalize that which has already been learnt. To justify mathematical statements, pupils practice the logical argumentation they have come to recognise in grade 7 as the most important characteristic of mathematical working. In geometry pupils deepen their understanding of the measurement of surfaces and of space by examining circles and cylinders.

## 3.3. Proofs in curriculum

From a general educational rather than mathematical perspective, Sowder and Harel (1998) argued against limiting pupils' experiences with proofs to geometry.

"It seems clear that to delay the exposure to reason-giving until the secondary-school geometry course and to expect at that point an instant appreciation for the more sophisticated mathematical justification is an unreasonable expectation" (p. 674).

Schoenfeld (1994) explained the importance of proof as follow;

"Proof is not a thing separable from mathematics as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricular, at all levels" (p.76).

In both countries, grade seven introduces pupils in geometry class to proofs. The high-attaining pupils in the German *Gymnasium* (grades 7–12) start grade seven with proofs of simple geometrical statements. For pupils in the German *Realschule* (grades 7-10), proofs in geometry are usually introduced in the 8$^{th}$ grade and in general, the pupils in the German *Hauptschule* (grades 7–9) will not advance beyond argumentation (Heinze & Kwak 2002).

Under the instruction of the Korean Ministry of Education (1997), teachers should carefully make clear to pupils that proofs are a means of explaining reasoning logically rather than developing rigorous deductive inference. It is important to develop the ability to explain precisely and concisely the process of inference, but this is not something that can be achieved over a short period. Therefore, the first time pupils are required to write their own proof, it is a case of simple inference or a symbol is introduced by the teacher to simplify the problem.

The German curriculum recommends that rigorous or formal proofs be avoided. Instead, it is recommended that "heuristic activities" such as those given above should be encouraged through the use of dynamic geometry software in class. In activities on the dynamic geometry software, it is advised that pupils not only exhaustively try ideas out, observe, and guess, but also that they include a reasonable justification. The revised curriculum in Lower Saxony recommends and emphasises non-rigorous proofs as follows:

"Assumptions must also be justified, proven or disproved with selected examples. Justifying takes place in the area of conflict between visual and general understanding on the one hand and locally arguing on the other. Therefore, visible situations may be selected, and attention should be paid to ensuring:

- that the visual situations not be too obvious to avoid proving the propositions.
- that no exaggerated formalism (in ways of writing and language) obscure the course of the proof
- that the reasoning used remains altogether visible (modularity)."

(*Niedersächsisches Kultusministerium*, 2003)

In both curricula, teachers are supposed to explain proofs in geometry classes. Rigorous and formal proofs are avoided, whereas convincing proofs are recommended. Even so, in Korea, what happens in the classroom is not the same as what is suggested in the curriculum. Pupils do end up seeing and having to write rigorous and formal proofs, even in their textbook, although for writing such a proof, the steps required for a proof are suggested.

## 3.4. Textbooks

Freundental (1981) has addressed that the factors most influential on mathematics education are textbooks and teacher training. According to Freundental, textbooks, as the conservative operator, are the most powerful tools for providing education. Research has demonstrated the strong influence textbooks have on the mathematics content that is taught and learned (Robitalle & Travers, 1992; Schmidt, McKnight, & Raizen, 1997; Schmidt et al., 2001). Moreover, as shown in TIMSS (Schmidt et al., 1997; Valverde et al., 2002), textbooks are regarded as being close to the implemented curriculum and as a link between intentions and practice which gives a good idea of the intended curriculum.

In an investigation into textbooks, particularly for mathematics at the lower secondary level, Gilbert (1989) gives a brief review of 'traditional approaches' and criticises the reliance of research on text analysis which are removed from their context of use (Pepin, 2001). He argues that 'the analysis of text can point to potential, even likely, outcomes in classroom use of texts, but it can never conclude with confidence that the ideological import of a text as interpreted by the researcher will be similarly realised in the discourse of the classroom' (p.68). Textbooks are an important factor for influencing the questions, issues, and topics covered and discussed in classrooms. Pepin (2001) addressed how textbooks should be analysed as follows: Thus, textbooks should be analysed in terms of both their content and their structure, as well as in terms of the process component; that is, their use in classrooms by pupils and teachers.

On the basis of TIMSS the Korean textbooks are very closely related both to the intended and the implemented curriculum.

Two Korean books and two German books for grades 7 and 8 will now be analysed. The Korean books are called "Su-Hak[5], 7-Na" and "Su-Hak, 8-Na" (mathematics for the second semester in each grade), and the German books are "Anschauliche Geometrie 7" and "Anschauliche Geometrie 8".

| | Korea | Germany (Bavaria) |
|---|---|---|
| Textbook | Su-Hak 7-Na<br>Su-Hak 8-Na | Anschauliche Geometrie 7<br>Anschauliche Geometrie 8 |
| Publisher | Doosan | Oldenbourg Verlag GmbH, München |
| Publishing year | 2001 | 1997 |

*Table 3-3 Investigated textbooks*

### 3.4.1. Nature of textbooks

Korean textbooks are intended to be organised based on the mathematics curriculum. The structure of mathematics textbooks in Korea is quite different from that in Germany. Firstly, Korean mathematics textbooks are organised into three domains, namely numbers and algebra, statistics and probability, and

[5] Su-Hak means mathematics.

geometry. There is one textbook for each semester. For example, in the first semester of the 7th grade, throughout the country, numbers and algebra are taught using a textbook called “Su-Hak, 7-Ga”, whereas in the second semester of the 7th grade, statistics and probability and geometry are taught using a textbook called “Su-Hak, 7-Na”.

In each chapter, sections are roughly divided into three parts:

a) Activities
b) Main contents
c) Exercises

The activities are small investigations and practical or cognitive activities (sometimes bordering on exercises) which are intended to introduce pupils to a certain notion. The main contents correspond to the core material that must be taught and understood, and applied using various kinds of problems. The third part incorporates exercises which enable pupils to evaluate their progress using problems of increasing difficulty.

| | Korea | Germany |
|---|---|---|
| | Global Unit:<br>Introduction (abstract description of unit designed to prepare pupils for learning, including some problems). | |
| | Chapter:<br>Short clarification of the aim of the chapter | Chapter |
| | Section | Section |
| Activities | Experiment activities | Sometimes |
| Main contents | Definitions, theorems, problems | Definition, theorems |
| Exercises | Problems for pupils to evaluate their own progress, supplemental exercises, questions to help pupils improve (at most 5 problems in total) | Extensive problems<br>(Between 3 to 32 in total) |
| | “Let us think harder”,<br>Developing mathematical ability using activities | |
| | Exercise | |
| | Mathematics essay | |
| | Summary | |
| | Summary Exercise | |

*Table 3-4 The layout of the Korean and of the German textbook*

The global unit of the Korean textbook starts with a simple introduction, mentioning various phenomena from the outside world, and some history of mathematical matter which is relevant to the given unit. It tries to evoke the pupils’ interest and then gives its aim as being to increase pupils’ motivation. Finally, in what is called “preparing for study”, there are some simple questions which are designed to check the pupils’ ability.

Each section, which itself is composed of various smaller themes, starts with “activities”, which serves as a natural approach to the theme and gives concrete examples, simple questions, and hints (or explanations). These help to explain definitions in the main text. The section finishes with questions which could be represented in each section. These questions are laid out in order of difficulty from basic to advanced, a scheme which potentially evokes interest and improves the pupils’ ability to learn.

The end of the section consists of various parts containing questions for pupils to evaluate their progress before pupils are able to confirm the learning topics using supplemental and advanced questions. It finishes with more problems in parts called “Lets think harder” and “Developing mathematical ability using activities” and intended to encourage various kinds of mathematical thinking. There is also an exercise part to help pupils to review topics covered.

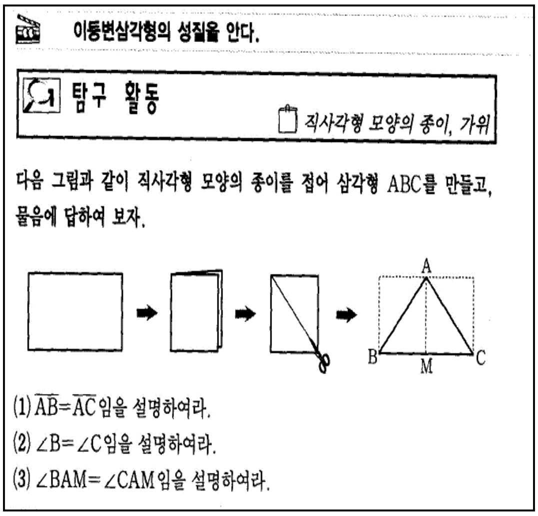
The end of the unit also has several simple problems to check whether pupils know the basic concepts, principles, and theorems. These problems are divided into supplemental and advanced questions.

In Germany, there are three types of textbooks[6] which are geared towards the achievement levels of pupils in the three different types of schools previously discussed. The textbook "Anschauliche Geometrie" which is used in *Gymnasium* in Bavaria is analysed here.

Because this book only deals with geometry, there is no global unit like in a Korean textbook. As in the Korean case, "Anschauliche Geometrie" consists of chapters which each contain several sections.

This textbook is divided into two parts, of which the second is an extensive range of exercises. The introduction in this book does not correspond to those containing small investigations in the Korean book. The introduction in this selected German textbook is contained in the first of the two parts, the main contents, in which definitions and theorems are stated. The majority of the units in each part consist of exercises, the number of which ranges from 3 to over 30 in different units. This book has neither a summary nor a final exercise unit.

One characteristic that distinguishes Korean textbooks from this selected German textbook is the part in the Korean book containing small investigations. Korean textbooks start with activities to enable pupils to understand the theme, whereas this selected German textbook begins with an introduction or activities that are sometimes presented with respect to the topics (See figure 3-1).


이등변삼각형의 성질을 안다.

탐구 활동

직사각형 모양의 종이, 가위

다음 그림과 같이 직사각형 모양의 종이를 접어 삼각형 ABC를 만들고, 물음에 답하여 보자.

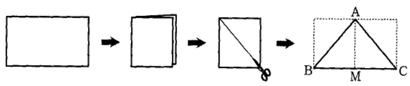


(1) $\overline{AB}=\overline{AC}$임을 설명하여라.
(2) $\angle B=\angle C$임을 설명하여라.
(3) $\angle BAM=\angle CAM$임을 설명하여라.

**Activities**,

Paper formed rectangles, scissors

Let us make triangle ABC by folding the paper formed rectangle as following.
Give the answers the following questions.




(1) Explain why $\overline{AB} = \overline{AC}$
(2) Explain why $\angle B = \angle C$
(3) Explain why $\angle BAM = \angle CAM$

*Figure 3-1 An example of activities in a Korean textbook (p.44, Su-Hak 8-Na (수학 8- 나))*

### 3.4.2. Use of textbooks

It is difficult to establish exactly how textbooks are used in different classrooms. However, based on findings of park and Yim (2002), the following is found for the use of Korean textbooks. For German case, it is summarised from the results of comparative study of textbooks conducted by pepin (2001) who examined textbooks and the use of textbooks, at the lower secondary, in England, French and German classroom and case study conducted by the Center for Human Growth and Development at the University of Michigan.

- Textbooks hold an important position in Korean classrooms and education. Textbooks have official status and the clarity with which they reflect official curriculum is carefully monitored

[6] German textbooks are so various. According to Sträßer (2001), depending on each single class, on every type of lower secondary school, and on a single Land, several textbooks are used.

(Park & Yim, 2002). All the pupils registered in the same year at a school use the same textbook. When using textbooks, teachers rely mainly on the main contents and exercises. Textbooks are the standard instructional resource common to most teachers and pupils in Korea.

- On the other hand, in German classrooms textbooks are available, but they are not necessarily used in class. When teachers use textbooks, it is mainly for exercises in school and for homework. Teachers are expected to provide introductions and explanations of mathematical notions (Pepin, 2001). Indeed, in German *Gymnasium* mathematics textbooks are used mainly for reference and reviewing, not as an integral part of learning (Riley et al, 1999). The book serves as a supplement to which teachers can refer their pupils when they are reviewing material covered on the board.

## 3.5. Geometry contents in curricula and textbooks

To begin with I will classify the geometry content of the Korean curriculum for each grade into different groups and will compare whether German curriculum and textbooks have similar contents. School geometry in Korea and Germany deals with the study of shapes and objects in two and three dimensions, their relationships and properties (see Table 3-5).

| Korea | | | Germany |
|---|---|---|---|
| Grade | Range | Contents | |
| 7 | Basic figures | Simple properties of points, line segments, sides, and angles | o[1] |
| | | Relation of location (situation) between points, lines, and plane | o |
| | | Properties of parallel lines – angles | o |
| | Construction using straightedge and compass and congruence | Simple construction of two dimensional shapes | o |
| | | Construction of triangles and its condition | o |
| | | Properties of congruent figures and Congruence for triangles | o |
| | Plane figures | Properties of polygons | |
| | | Circle and sector; Basic concepts of a circle; the centre, the angle subtended at the centre, sector, segments, Properties of circles | 8 |
| | Solid figures | Polyhedron, regular polyhedron | 8 |
| | | Surface of revolution | 8 |
| | Measure of plane figures | How to measure interior- and exterior angles in polygons<br>How to measure the length of an arc and area of sector | 8 |
| | Measure of solid figures | Surface area and volume of cylinder | 8 |
| | | Surface area and volume of pyramid | 8 |
| | | Surface area and volume of spheres | 8 |
| 8 | Properties of triangle | Propositions and theorems | o |
| | | Properties of an isosceles triangle | o |
| | | Congruence of right-angled triangles | o |
| | | Centre of the circumcircle, centroid of a triangle | o |
| | Properties of quadrilateral | Parallelogram | o |
| | | Various other quadrilaterals | o |
| | Similarity of figures | Similar figures, definition and properties of similarity | 9 |
| | | Conditions of similarity of triangle | 9 |
| | | Drawing of similar figures | 9 |
| | Application of similarity | Triangles and parallel lines | 9 |
| | | Ratios in intersected parallel lines | 9 |

| | | | |
|---|---|---|---|
| | | The segment joining the midpoints of two sides of a triangle and segments divided proportionally | 9 |
| | | Triangle centroid | 9 |
| | | Measuring volume and area of a figure using ratios from similarity | 9 |
| | Vectors | 11 | 8 |
| | Symmetry figure | 5 | 7 |
| | Thales' Theorem | 9 | 8 |

*Table 3-5 Contents of Geometry in Korea and Germany ([1] "o" means the topic begins in same grade in Germany as in Korea)*

As can be seen from Table 3-5, the topics are similar at the lower secondary schools. There are slight differences between the Korean and the German textbook. For example, some topics are introduced earlier in one country than in the other, and certain topics are emphasised more in one country than in the other. In particular, solid geometry (basic concepts and measuring area, volume and surface area etc.) and similarity are introduced earlier in Korea, and Thales' theorem and vectors are introduced earlier in Germany. Emphasis is placed on basic figures in the plane, congruence, and proofs of the properties of a quadrilateral, for which congruence conditions are used in grade 8). Many topics are introduced earlier in Korea than in Germany.

### 3.5.1. Topics introduced earlier in Korea

- Solid geometry

In both countries the main ideas of Euclidean geometry are incorporated in the geometry textbooks.

Korean pupils learn plane and solid geometry in parallel, in the same grade. Their German counterparts, on the other hand, first deal with plane geometry in the 7$^{th}$ grade and then solid geometry in the 8$^{th}$ grade.

In Korea, throughout two sections, polyhedron and surface of revolution, solid geometry is taught. In the section on polyhedrons, five regular polyhedrons, their nets, and the Euler's formula are covered. In addition, spheres and pyramids are introduced in surface of revolution. Pupils learn their definitions and propositions relating to them. Applying this knowledge, for example by measuring, is stressed.

In Germany, this only comes at the end of grade 8, although pupils then go into greater depth, for example by studying a certain projection called a *Schrägbild.*

- Similarity

Korean pupils learn the concept "similarity" and its applications during the 8$^{th}$ grade. They are required to be able to differentiate between the various conditions for congruence. German pupils, on the other hand, learn this in the 9$^{th}$ grade. The new curriculum in Bavaria states that pupils are to learn this concept earlier, so similarity will in future be introduced in the 8$^{th}$ grade. This curriculum also states that pupils should see how geometry can be applied using algebraic methods, for example by considering the consequences of theorems. Pupils should be able to recognise when figures are similar and be able to draw relevant conclusions.

- Symmetrical figures

A comparison of the geometry lessons of the two countries shows that Korean pupils do not study symmetry as much as German pupils. For example, in the 5$^{th}$ grade they briefly learn some properties of axis and point symmetry as part of their studies on plane geometry, but do not develop this in the upper grades.

German 7th grade pupils learn about the construction of reflections and rotations in greater depth. In the Bavarian curriculum, it is stated that "… the analysis of rotation and translation deepens their understanding and leads to concepts relating to the congruence of figures." Congruence is seen as being closely related to symmetry. Moreover, by constructing symmetrical figures, pupils learn about mathematical concepts as well as the culturally and historically meaningful principle of construction. They gradually learn about abstract geometrical ideas. Symmetry is also an important tool for learning about the properties of quadrilaterals as well as for those methods of proof which use congruence. It helps pupils to analyse, to argue consistently and to justify their results. At the same time, since the way pupils think becomes strongly shaped by axioms, pupils gradually move away from the visual and intuitive knowledge they have acquired so far and become more dependent on deductive knowledge.

### 3.5.2. Topics introduced earlier in Germany

- Thales' theorem

Thales' theorem, which states that an inscribed angle in a semicircle is a right angle, was proposed by one of the greatest mathematicians of Ancient Greece, Thales. This theorem is learnt in the 9th grade in Korea. However, in Germany, both the theorem and its converse are taught in the 7th grade with the properties of angles, when German pupils learn about right-angled triangles.

- Vector

In Korea, vectors are introduced in the 11th grade and taught intensively. However, in the southern states of Germany, this concept is briefly introduced in the 8th grade along with that of coordinate axes. At this stage it is confined simply to the definition and its properties, such as existence, commutative and associate laws, the unit vector (the identity element), and its converse. In the 11th grade, the notion of a vector used is a more detailed one and is learned in greater depth. The inner product is also covered. Moreover, the linear equation and plane equation are required to be presented using vectors.

### 3.5.3. Topics taught at the same stage in both countries

- Basic figures

In both countries the definitions of basic concepts such as a point, line, segment and perpendicular are taught in the 7th grade, and there is also a unit on triangles. However, German schools go into greater depth with concepts such as convex, concave, and coordinate system, an idea which Korean pupils learn about in algebra.

In addition, great importance is attached in both countries to the sections on quadrilaterals in the 8th grade, since these, together with the units on triangles form a solid basis for rich geometrical investigations. There are many opportunities for pupils to consolidate and deepen their knowledge of triangle geometry; in Korea, the appropriate properties are introduced and proved using the properties of isosceles and right-angled triangles. The Bavarian curriculum, on the other hand, suggests that this section should be a means of developing proof techniques. Moreover, it suggests that pupils should be able to differentiate clearly between assumption and conclusion, to accomplish simple proofs by themselves and to formulate the converse statements.

- Constructions using straightedge & compass

A geometrical construction consists of a procedure by which a drawing is produced through the use of specific tools and according to specific rules.

Both curricula highlighted the importance of their construction section. It is stated in the Korean curriculum that the development of construction ability caused by learning about figures helps pupils to

understand more precisely the concepts and properties of figures. This concept also develops interest in drawing figures and brings pupils to pay greater attention to the process when doing so.
However, in the Korean section on construction, pupils make only simple constructions, such as drawing in the perpendicular bisector, bisecting an angle, and constructing a triangle. This section is actually meant as a supplement for other sections such as one on the properties of and relationships between figures. Therefore, Korean pupils seldom make constructions, because they neither have to cover it nor apply this skill.

German educators in general believe that pupils develop their imagination and mental agility when they construct triangles, quadrilaterals and other figures. Therefore, German pupils spend a lot of time on their construction section. Moreover, before learning about the properties of angles and figures such as triangles, pupils are asked to construct them.

- Congruence

In the section on congruence for each country, the definition of congruence and the conditions for congruence are taught. Korean pupils are required to learn three congruence conditions, namely SSS, SAS, and ASA. However, German pupils are supposed to learn one more congruence condition, namely SSA. As mentioned earlier, construction plays an important role in Germany, and it is difficult to recognise without a construction that the condition SSA is sufficient for congruence. Since German pupils learn about transformations which preserve congruence such as translation, rotation, and their properties, they are well placed to deal with this extra condition.

- How do Korean and German pupils proof learn?

In Korea, before learning what a proof is and writing proof, pupils are first supposed to learn what propositions and theorems are. Once that is done, the if-then form and the idea of a converse are taught. Similarly, the selected German textbook deals with propositions, the if-then form and its converse before pupils can start to learn about proof.

The following is the definition of proof given in a Korean textbook:

“Experiments and practical work are used to show that a proposition is true. However proof is the process of showing that a proposition is true by using propositions or definitions which are already known, without using an experiment or practical work.”

In the selected German textbook, proof is explained as following:

*”Die Mathematiker haben Methoden entwickelt, die einem die Richtigkeit von Sätzen des Es-Gibt-Typs und des Wenn-Dann-Typs klar machen. Eine solche Methode heißt* ***Beweis****. Es gibt vier wichtige Beweisarten:“*

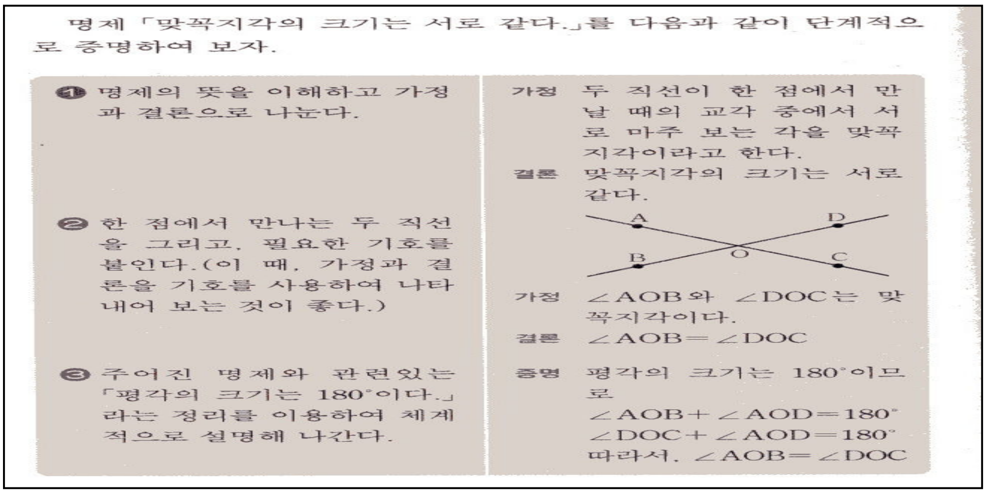
- *Beweis durch Nachrechnen*
- *Widerspruchsbeweis*
- *Symmetriebeweis*
- *Kongruenzbeweis*

“Mathematicians have developed methods which make clear the correctness of theorems of the ‘there-exists’ form and ‘if-then’ forms. Such a method is called a proof. There are four important types of proof”

- Proof by calculation
- Proof by contradiction
- Proof using symmetry
- Proof using congruence

From the two definitions here, the differences between the two countries in the way proof is viewed are apparent. The Korean textbook implies that experiment and practices are not sufficient for a proof, even though they are used to show that a proposition is true. In other word, the Korean textbook might still teach the traditional view of proof to pupils. On the other hand, the selected German textbook does not exclude the empirical arguments. Moreover, the Korean textbook suggests the way of proof generally, but the selected German one suggests the methods, what kind of proof could be used.

As the next step in the Korean textbook, an example of a step of proof is given for the proposition "opposite angles are equal".


명제 「맞꼭지각의 크기는 서로 같다.」를 다음과 같이 단계적으로 증명하여 보자.

| | |
|---|---|
| ❶ 명제의 뜻을 이해하고 가정과 결론으로 나눈다. | 가정 두 직선이 한 점에서 만날 때의 교각 중에서 서로 마주 보는 각을 맞꼭지각이라고 한다.<br>결론 맞꼭지각의 크기는 서로 같다. |
| ❷ 한 점에서 만나는 두 직선을 그리고, 필요한 기호를 붙인다.(이 때, 가정과 결론을 기호를 사용하여 나타내어 보는 것이 좋다.) | A D B O C<br>가정 ∠AOB와 ∠DOC는 맞꼭지각이다.<br>결론 ∠AOB=∠DOC |
| ❸ 주어진 명제와 관련있는 「평각의 크기는 180°이다.」라는 정리를 이용하여 체계적으로 설명해 나간다. | 증명 평각의 크기는 180°이므로<br>∠AOB+∠AOD=180°<br>∠DOC+∠AOD=180°<br>따라서, ∠AOB=∠DOC |

*Figure 3-2 An example of a step of proof in a Korean textbook (p.42, Su-Hak 8-Na (수학 8-나))*

| Let us prove the proposition "the opposite angles are equal", as follows: | |
|---|---|
| 1. Understand the proposition: separate the assumption from the conclusion. | Assumption: If two lines meet at a point, then of the angles formed, the angles which face each other are called opposite angles.<br>Conclusion: the opposite angles are equal. |
| 2. Draw two lines which meet at a point, and label this point and others.<br><br>(It is better to express the assumption and conclusion in symbols). | Assumption: ∠AOB and ∠DOC are opposite angles.<br>Conclusion: ∠AOB=∠DOC |
| 3. Using a theorem "The angle on a straight line add up to 180°" in relation to the given proposition, explain the argumentation of this proposition systematically. | PROOF: Since the angle on a straight line add up to 180°,<br>∠AOB+∠AOD=180°<br>∠DOC+∠AOC=180°<br>therefore, ∠AOB=∠DOC. |

On the left-hand side, the proof process for the given example is explained and on the right-hand side, the details are given. The following is then given as a general step:

- How to prove proposition:

1) Understand the meaning of problem and separate the assumption from the assertion
2) If the problem involves figures, draw these adequately and add any necessary symbols.
3) Consider definitions, theorems and properties related to this problem, to make plans and explain systematically.

The Korean textbook then gives advice as to how a proposition can be proved by means of deduction.

After explaining the definition of proof and some methods of proof, the selected German textbook also gives how the proof is consisted of as follows:

Bei jedem Beweis unterscheidet man drei Teile:

1. Genaue Formulierung aller **Voraussetzungen** (Vor.)
2. Genaue Formulierung der **Behauptung** (Beh.)
3. Begründung der Behauptung; man verwendet dabei die Voraussetzungen und schon bekannte Sätze und Definitionen. (Bew.)

Das Beweisschema sieht dann ungefähr so aus:

Vor.: ..... (V1)
..... (V2)
..... (V3) usw.
Beh.: .....
Bew.: ..... (V1, Satz, ...)
..... (Definition, V2, ...)
..... q.e.d.

Die Schlussformel q.e.d. ist die Abkürzung für den lateinischen Satz »quod erat demonstrandum«, sie heißt auf deutsch »was zu beweisen war«. Seit Euklid stellt man mit ihr erleichtert fest, dass man den Beweis geschafft hat.

*Figure 3-3 An example of a step of proof in a selected German textbook (p.43, Anschauliche Geometrie 8)*

The selected German textbook spends lots of pages to give examples of various methods of proofs as explained above, such as calculation, contradiction and symmetry, as well as congruence, whereas only congruence is mentioned in the Korean textbook. Here are the examples from the selected German textbook. The first is a proposition which can be proved using symmetry.

**Der Symmetriebeweis**

Entdeckt man in einer Figur ein Symmetriezentrum oder eine Symmetrieachse, so wird man beim Beweisen die Symmetrie-Eigenschaften ausnützen.

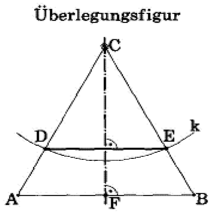
*Figure 3-4 an example of a proof using symmetry in the selected German textbook (p.45, Anschauliche Geometrie 8)*

***Beispiel zur Achsensymmetrie***

**Satz:**

**Schneidet ein Kreis um die Spitze eines gleichschenkligen Dreiecks die Schenkel in zwei Punkten, dann ist ihre Verbindungsstrecke parallel zur Basis.**

Überlegungsfigur




Vor.: $\overline{AC} = \overline{BC}$ (V1)

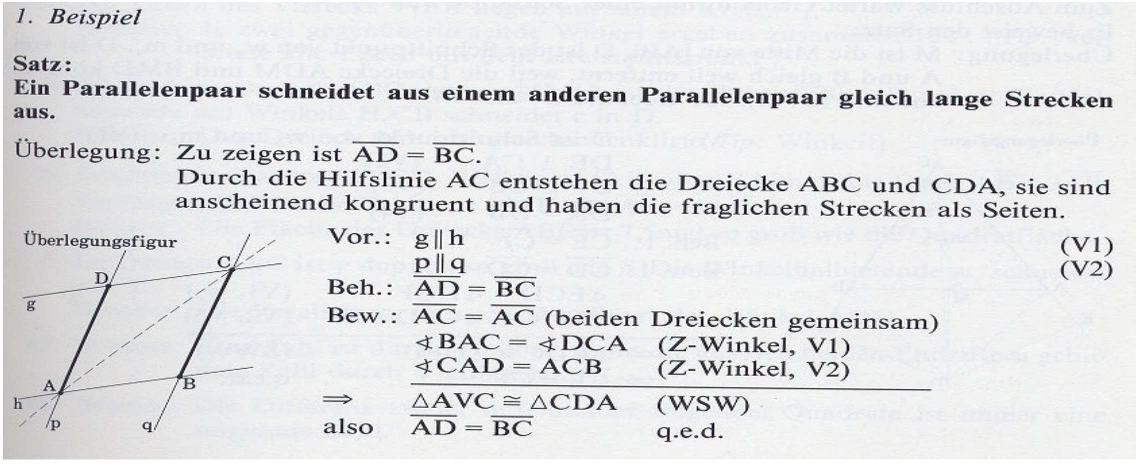
$\overline{DC} = \overline{EC}$ (V2)

Beh.: $DE \parallel AB$

Bew.: CF ist Symmetrieachse des Dreiecks ABC und des Kreises k, folglich liegen die Schnittpunkte D und E symmetrisch zur Achse CF.
Weil die Verbindungslinie zueinander symmetrischer Punkte senkrecht auf der Achse steht, ist CF gemeinsames Lot von DE und AB.
Also sind DE und AB parallel.

*Figure 3-5 Another example of a proof using symmetry in the selected German textbook (p.46, Anschauliche Geometrie 8)*

The next one is an example which is proved using congruence.



*1. Beispiel*

**Satz:**
**Ein Parallelenpaar schneidet aus einem anderen Parallelenpaar gleich lange Strecken aus.**

Überlegung: Zu zeigen ist $\overline{AD} = \overline{BC}$.
Durch die Hilfslinie AC entstehen die Dreiecke ABC und CDA, sie sind anscheinend kongruent und haben die fraglichen Strecken als Seiten.



| | | |
|---|---|---|
| Vor.: | $g \parallel h$ | (V1) |
| | $p \parallel q$ | (V2) |
| Beh.: | $\overline{AD} = \overline{BC}$ | |
| Bew.: | $\overline{AC} = \overline{AC}$ (beiden Dreiecken gemeinsam) | |
| | $\sphericalangle BAC = \sphericalangle DCA$ | (Z-Winkel, V1) |
| | $\sphericalangle CAD = ACB$ | (Z-Winkel, V2) |
| $\Rightarrow$ | $\triangle AVC \cong \triangle CDA$ | (WSW) |
| also | $\overline{AD} = \overline{BC}$ | q.e.d. |

*Figure 3-6 An example of a proof using congruence in the selected German textbook (p.47, Anschauliche Geometrie 8)*

A slight difference could be observed with the Korean textbook, which suggests proof method only using congruence. By contrast, the selected German textbook suggests the various methods with examples as above. Moreover, after one or two examples are given, many problems requiring each method introduced are given in the exercise part in the selected German textbook.

## 3.6. Summary

In comparing curricula which deal with geometry, both countries' curricula can be seen to be connected to the year group and they have similar aims and intentions as regards the learning of mathematics. The goals and expectations are quite explicitly stated in both curricula. Moreover, the guidelines on proving in the two countries are quite explicit and similar. A result of this is that intuitive activities are emphasised in the 7th grade whereas more formal and deductive proofs, but not rigorous proofs, are focused on and applied in the 8th grade.

The layouts of the two textbooks are quite different. The Korean textbook gives simple problems as an introduction. This introduction consists of a few activities and many kinds of problem at varying levels of difficulty. This might be a useful way of bringing pupils to take an interest in mathematics and of evoking a desire to learn mathematics. On the other hand, the selected German textbook does not consist always of introduction part of new theme. However, it includes many exercise problems. This extensive range of exercise might be typical for German textbooks. According to Sträßer (2001), this kind of exercise, sometimes called complemented homework tasks which sometimes fill more than 50% of the pages of a textbook.

In a Korea classroom, textbooks are usually used (Park & Yim, 2002), yet in Germany textbooks do not play such an integral role in the classroom (Riley et al, 1999).

The geometry topics are mostly the same in both books. For example, plane geometry as used in both countries is seen as the appropriate material to help pupils to understand the meaning and methods of proof. However, Korean pupils learn plane and solid geometry in the same grade, in parallel, and these topics are repeated in the following grades. In other words, the Korean textbooks integrate topics more, and significantly, concepts and skills are more closely intertwined.

In both countries, proof using the congruence of triangles is regarded as a powerful method for investigating the properties of figures.

The Korean textbook presents the proof approach only with the deductive way of basis figure and axiomatic system, even though the Korean curriculum suggests that the intuitive way and experiments are implicated instead only deductive way in classroom. This might be a result of Korean pupils losing their interest in mathematics. By contrast, the selected German textbook presents various methods of proof and gives pupils time to consider them.

However, it is interesting to note that both Korean and German authorities plan to improve their curricula in order to create better learning opportunities in mathematics.

As an example, the $7^{th}$ Korean curriculum focuses on pupils' interest in mathematics and individual differences as well as on pupils' active participation rather than passive learning.

Many educators and policy makers recently discussed the state of the German educational system and its prospects for development (Klieme et al, 2003). One principle of interest raised is that of "Binding for all": minimum requirements that are expected of all learners. These minimum standards would apply to all pupils, regardless of the type of school they attend (Klieme et al, 2003). The report in which this principle was put forward (*Nationale Bildungsstandards*) strongly recommended the establishment of a binding minimum level of proficiency in national standards for Germany. German educators emphasise this principle as being of the utmost importance, because of the notable deficiencies at the lower levels of performance in the German educational system at present. The aim of this principle is to ensure that weaker pupils are not left behind. German educators believe that this could make a valuable contribution to reducing disparities in the educational system (Klieme et al, 2003).

# Chapter 4 International Studies of Student Achievement

In this chapter attention shifts to the international comparisons of the performance of eighth graders on mathematics. The studies of the International Association for the Evaluation of Educational Achievement (IEA) have been developed over the course of the past 30 years. However, comparisons of the relative achievement of pupils are not easy. In comparative studies such as TIMSS and PISA, East Asian countries have consistently outperformed western countries from North America, Europe, and Australia (Robitaille and Garden, 1989; Stevenson et al., 1990, 1993; Lapointe et al., 1992; Beaton et al., 1996; Mullis et al, 1997).

## 4.1. First International Mathematics Study & Second International Mathematics Study

The First International Mathematics Study (FIMS) was initiated in 1960 as the first study of the IEA. FIMS was essentially a comparative investigation of the outcomes of schooling with a focus on mathematics achievement as the dependent variable. The twelve countries that participated in the data collection phase in 1964 were almost all located in Europe and were predominantly highly industrialised; for example, Germany and France were among the twelve (Husén, 1967). Korea did not participate. FIMS examined national probability samples from two populations: 13-year-old pupils (population 1) and pupils in their last year of secondary school (population 2). Almost all of the problems developed for FIMS were multiple choices questions and were constructed through a collaborative international effort. FIMS produced numerous findings of interest to mathematics educators and contributed, in a substantial manner, to the development of a better understanding of the immense variability which exists across countries with respect to a number of variables. This variability of course has important implications for the teaching and learning of mathematics (Husén, 1967). This contribution laid the major groundwork for more intense study of the connections between what teachers do and what pupils learn. Nonetheless, details about the sampling procedures used are sparse and response rates are unknown.

The Second International Mathematics Study (SIMS) was designed to provide an international portrait of mathematics education and allowed, at every stage, for significant input and guidance from a wide range of members of the mathematics community. Twenty countries participated in SIMS in one of two ways during the 1981-82 school years. However, neither Germany nor Korea participated. The full study was designed to provide longitudinal data comparing pre-test data collected at the beginning of the school year with post-test data collected at the end of the school year. However, countries could opt to participate only in the post-test phase of the study. Sudents in the study were selected from two populations: population A consisted of pupils of whom the majority were between 13 years and 13 years 11 months old by the middle of the school year. Population B pupils were in their last year of secondary school and were taking mathematics as a substantial part of their academic program. SIMS provided valuable information not only on the extent of growth in pupils' learning but also on a variety of relationships between teaching practices, curricula, and so on.

## 4.2. Third International Mathematics Study

The Third International Mathematics and Science study (TIMSS) which was conducted in 1994-1995 was the largest and most comprehensive comparative international study of education ever undertaken. More than 40 countries including Germany (3464 pupils in the 7th grade and 3419 pupils in the 8th grade) and Korea (2907 pupils in the 7th grade and 2920 pupils in the 8th grade) participated in TIMSS. The mathematics and science performance of national samples of pupils in grades 3 and 4 (population 1), grades 7 and 8 (population 2) and grade 12 (population 3) was assessed and related information was collected from those same pupils, their teachers, and their schools. Germany was represented in populations 2 and 3 (see Baumert, Lehrmann & Lehrke, 1997) and Korea was represented in populations 1 and 2. Using questionnaires, videotapes, and analyses of curriculum materials, TIMSS also investigated the contexts in which mathematics and science were learnt in the participating countries.

The original design of TIMSS, like SIMS, was one that required a pre- and post-test to measure this growth. Unfortunately, most of the participating nations were unable to participate in both a pre- and a post-test, so the study reverted to a simple cross-sectional and single-testing design. The resulting analyses can consequently offer no more than circumstantial evidence on what is important in the learning of mathematics and science.

A Video study on a total of 231 grade 8 mathematics classes in Japan, Germany, and the USA (Kawanaka, Stigler et al 1999) was also part of this study. The Videotape Classroom Observation Study was designed to provide a rich source of information about practices used for mathematics instruction in the classroom in each of the three countries and also to provide contextual background information on the statistical indicators available from the main TIMSS study.

A conceptual framework in TIMSS was developed in which pupils' outcomes in mathematics are placed in context. TIMSS framework has three dimensions.

- Content: this aspect represents the content of school mathematics, such as fractions and arithmetic, algebra, geometry, measurement, data representation analysis and probability, and proportionality.
- Performance expectations: this aspect describes the kind of performance that pupils will be expected to demonstrate while engaged with content. There are five main categories: knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating.
- Perspectives: this aspect has particular relevance for the analysis of documents such as textbooks. It is intended to illustrate curricular goals that focus on the development of pupils' attitudes, interest, and motivations in mathematics teaching.

In FIMS and SIMS, countries were ranked in order of the percentage of correctly answered problems, whereas in the case of TIMSS, scale scores were used.

## 4.3. Third International Mathematics Study Repeat – TIMSS-R (TIMSS 1999)

TIMSS 1999, also known as TIMSS-Repeat or TIMSS-R, is a replication of the TIMSS component from the lower-secondary or middle school level – the eighth grade in most countries. As in the original 1995 study, TIMSS 1999 included a full range of context questionnaires and the TIMSS-R Videotape Classroom Study, which examined instructional practices for mathematics and science in seven nations. 26 of the countries which participated in TIMSS also participated in TIMSS 1999. Germany was not among these countries. For countries which participated in both studies, a comparison between the achievement of 13-year-old pupils in 1995 and in 1999 was possible, as was a review of the progress made by pupils who were in grade 4 in 1995 and in grade 8 in 1999.

Average mathematics achievement across the group of 26 countries increased slightly from a scaled score of 519 in 1995 to 521 in 1999. However, this increase was not statistically significant. The results for some individual countries were similar to those in TIMSS: in particular, Korean pupils maintained their high position among the 38 participating countries. Analysis of the TIMSS-R Korean data shows that the 8th grade Korean pupils attained even higher levels of achievement in mathematics than their predecessors managed in TIMSS. The excellent performance of the Korean pupils is all the more remarkable if their unfavourable learning environment is brought into consideration.

In 1995, TIMSS assessed both fourth- and eighth-grade pupils. This allowed participating countries to compare the performance of their fourth and eighth grade pupils, and gave a cross-sectional perspective on how different grades[7] compare (TIMSS 1999). The fourth grade in 1995 and the eighth grade in 1999 of each of Singapore, Korea, Japan, and Hong Kong all performed significantly above the international average for the respective categories.

## 4.4. PISA (Programme for International Student Assessment)

PISA is a new survey, which takes place every 3 years, of the knowledge and skills of 15-year-olds in the principal industrialised countries. In 2000, 28 OECD (Organization for Economic Cooperation and Development) Member countries and four other countries took part in the first PISA survey. PISA 2000 surveyed reading literacy, mathematical literacy and scientific literacy with a primary focus on reading. The survey is repeated every three years with the primary focus shifting to mathematics in 2003, science in 2006, and back to reading in 2009.

Mathematical literacy is defined in PISA as *an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 1999)*

This definition of mathematical literacy includes the ability to put mathematical knowledge and skills to functional use rather than just to master them within a school curriculum. It also implies that the ability to pose and solve mathematical problems in a variety of situations, as well as the inclination to do so, often relies on personal traits such as self-confidence and curiosity. Several aspects of this definition have a specific meaning in the context of the PISA study. On reading this definition carefully, it is clear that it revolves around wider uses in people's lives rather than simply carrying out mechanical operations.

Literacy in each of reading, mathematics, and science is defined in terms of three dimensions: content, process, and situation. In the case of mathematical literacy, the three dimensions are specified as follows:

- Processes: the focus is on students' abilities to analyse, reason, and communicate ideas effectively by posing, formulating, and solving mathematical problems. Processes are divided into three classes: reproduction, definitions, and computations; connections and integration for problem solving; and mathematisation, mathematical thinking, and generalisation.
- Content: PISA emphasises broad mathematical concepts such as change and growth, space and shape, chance, quantitative reasoning, and uncertainty and dependency relationships.
- Context: an important aspect of mathematical literacy is using mathematics in a variety of situations, including personal life, school life, work and sports, and local community and society.

As a result, PISA mathematics literacy problems seek to measure how well pupils are able to apply a variety of mathematical processes to an assortment of problems – from the kind of problems they solve for a school assignment to the kind they solve to help make home improvements.

[7] The achievement scale for mathematics in the fourth grade in TIMSS is not directly comparable with that for the eighth grade in TIMSS-R by several factors. Consequently, the results for the fourth grade and those for the eighth grade may be compared only in relative terms, for example with reference to the international average for countries that participated in 1995 at both the fourth and eighth grades.

## 4.5. Mathematical competencies

Germany was one of the few countries that took part in all the comparative studies undertaken by the International Association for the Evaluation of Educational Achievement (IEA). The comparative studies in which both Korea and Germany participated are TIMSS and PISA.

With regard to the results of TIMSS and PISA, the competence model is of particular interest, a non-hierarchical list of general mathematical competencies which are relevant and pertinent to all levels of education (OECD, 2001). I will briefly describe the mathematical competencies used in PISA itself and those used in the German analysis of PISA. Klieme et al developed their own competence classes which partially agreed with the international competence class; however, it was classified from the results of German data. Five levels of mathematical competency for $8^{th}$ graders were defined for the German analysis according to the pupils' ability and the difficulty of the problems used (Baumert et al, 2002).

### 4.5.1. TIMSS

Klieme and other co-workers have developed competence levels (sometimes known as ability levels) for pupils in the final year of secondary school. The empirically-based grading in terms of these competence levels complies with the level of literacy which is called in the theoretical discussion. Therefore, mathematical literacy (*Mathematische Grundbildung*) is defined to be somewhere between thinking using experience on the one hand and critical use of mathematical models on the other hand.

- *Level 1 (Score 400): Everyday life-referred consequences*

  The problem can be solved with the sufficient security. It does not require any explicit mathematical operations; only intuitive considerations are needed.

- *Level 2 (Score 500): Application of simple routines*

  Problems which are solved by using ratios or by calculating percentages come under this level. Such material is standard for mathematical instruction at a medium level of achievement.

- *Level 3 (Score 600): Modelling and sequences of operations*

  For Level 3, pupils must be able to carry out a sequence of several mathematical operations such as calculating volume and ratios. At this level, mathematical modelling is possible. Usually, the mathematical approach or idea needed is not contained itself in the formulation of the problem - it must be found and developed by the pupils.

- *Level 4 (Score 700): Mathematical arguments*

  Those who attain this level, the highest defined for mathematical literacy (*Mathematische Grundbilung*) are able to provide diagrams and interpret critically.

Klieme et al apply these 4 levels to $8^{th}$ grade as follows (TIMSS, 1997):

1. Elementary calculation knowledge (below 400): It is assumed that pupils can somewhat master basic operation of arithmetic, as long as the tasks require an operation exclusively with natural numbers and not large numbers. Simple tables and quantitative relationships representing pictograms are understood. The basic tasks of this type are provided regularly with entrance examinations for training circumstances. Between 80% and 90% of German pupils are over the elementary level. However, 15% of $7^{th}$ German graders and 11% of $8^{th}$ German graders are below this elementary level.

2. Intermediate Level I (400-450): mathematical repertoire will be only little expected. An elementary understanding of usual fraction and decimal fraction will be achieved, so that the large differentiates will be recognised. Fraction with a common denominator could be added and subtracted, as long as negative numbers do not appear. The simplest algebraic terms will be understood. About 75% of 8th German graders master the basic repertoire which means 25% German pupils did not still achieve this basic repertoire.

3. Mastering of routine process (450-500): understanding of prescribed material from curriculum for 6th grade to 8th grade: simple linear equation with an unknown could be somewhat surely solved. Terms with an unknown will be understood; simple tasks of proportion can be solved; simple tasks of two and three sentence will be somewhat surely solved. An elementary understanding of properties of a square and a rectangle will be achieved. About 50% of 8th German graders achieve the third level.

4. Intermediate Level II (500-550): pupils master difficult material of 6th grade with adequate certainly. The extension and cancelling of fraction will be now managed. The division with usual fraction and decimal fraction is, however, still serious a problem for pupils. Equivalent algebraic expression will be recognised, and linear equations can be more complex. Tasks of simple similarity and simple mapping can be solved. An elementary understanding of angle relationships in geometrical figure can be encountered. About third of 8th German graders achieve the fourth level.

5. Understanding of mathematical concepts and process (over 600): pupils can successfully deal with arithmetical tasks which require a complex sequel of steps of calculation. Mastering of calculation of percentage can be only expected for the higher achievement groups. Pupils in this group can interpret also demotic (popular) formulated task into algebraic equation. The central terms of mapping geometry and plane geometry (angle, triangles, quadrangle) will be somewhat understood. Only 17% of German pupils achieved to the highest level.

- Some remarks in TIMSS

Working within the mathematics curriculum framework, mathematics test specifications were developed for grades 7 and 8 which included problems from a wide range of mathematics topics and requiring pupils to use a range of skills (Schmidt, et al, 1997, pp.7-8).

This table shows the average percentages for each topic for Korea and Germany.

| | Korea | Germany | International mean |
|---|---|---|---|
| Fractions & number sense | 74 | 58 | 58 |
| Geometry | 75 | 51 | 56 |
| Algebra | 69 | 48 | 52 |
| Data representation analysis & probability | 78 | 64 | 62 |
| Measurement | 66 | 51 | 51 |
| Proportionality | 62 | 42 | 45 |

*Table 4-1 Average percentages by country*

Even though German pupils' achievements in geometry are below the international average, by comparison with the other topics, it is placed on third by national average. Baumert et al (1997) argued the questions in geometry in TIMSS are still a problem for general German 8th grade pupils, even the main focus of geometry lesson is the middle level 8th grade internationally.

TIMSS uses five benchmarks: the top 5 percent, the top 10 percent, the top 25 percent, the upper 75 percent, and the upper 95 percent. The next table shows the cut-off scores for each benchmark for both

Korea and Germany. For example, the top 5 percent of Korean pupils scored 786 or higher in mathematics and the top 5 percent of German pupils scored 661 or higher in mathematics.

| | Average | $5^{th}$ | $25^{th}$ | $75^{th}$ | $90^{th}$ | $95^{th}$ |
|---|---|---|---|---|---|---|
| Korea | 607 (2.4) | 418 (4.0) | 540 (5.0) | 609 (3.9) | 682 (2.7) | 786 (7.1) |
| Germany | 509 (4.5) | 368 (8.2) | 448 (9.4) | 506 (6.3) | 572 (7.5) | 661 (3.4) |

*Table 4-2 Percentiles for mathematics achievement in the respective $8^{th}$ grades, with standard errors in benchmarks (SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1994-95)*

Korean mathematics educators have more interest in affective range than in cognitive range in TIMSS, because most Korean pupils had seldom interest or motivation in mathematics. In TIMSS pupils were asked their perception about doing well in mathematics. 62% of Korean pupils disagreed or strongly disagreed about doing well in mathematics. On the other hand, 31% of German pupils disagreed or strongly disagreed about doing well in mathematics. Only 6% of Korean pupils strongly agreed about doing well in mathematics, by contrast, 36% German pupils strongly agreed about doing well in mathematics (see table 4-3).

| | Strongly disagree | | Disagree | | Agree | | Strongly agree | |
|---|---|---|---|---|---|---|---|---|
| | Percent of pupils | M.A | Percent of pupils | M.A | Percent of pupils | M.A | Percent of pupils | M.A |
| Korea | 9(0.5) | 535(5.7) | 53(1.0) | 572(3.0) | 32(0.9) | 669(3.0) | 6(0.6) | 702(5.7) |
| Ger. | 7(0.5) | 474(7.1) | 24(1.0) | 491(5.2) | 33(1.1) | 511(5.1) | 36(1.1) | 529(5.3) |

*Table 4-3 Percentages for pupil' self-perceptions about usually doing well in mathematics and mean achievement in the respective $8^{th}$ graders with standard errors (SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1994-95) M.A: mean achievement*

In addition, pupils were asked about the necessity of various attributes or activities to do well in mathematics (see table 4-4). 86% Korean pupils agreed that natural talent or ability was important to do well in mathematics, while 59% German pupils did. Even half of Korean pupils agreed that good luck was important to do well, while 25% German pupils did. 73% Korean pupils agreed with necessity of memorising the textbook or notes, while 47% German pupils agreed that memorisation was important to do well in mathematics.

| | Natural talent / ability | Good luck | Lots of hard work studying at home | Memorise the textbook or notes |
|---|---|---|---|---|
| Korea | 86 (0.7) | 63 (1.0) | 98 (0.2) | 73 (0.7) |
| Germany | 59 (1.5) | 25 (1.1) | 76 (1.1) | 47 (1.5) |

*Table 4-4 Percentages for pupil' reports on things necessary to do well in mathematics in the respective $8^{th}$ graders with standard errors (SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1994-95) M.A: mean achievement*

Pupils also were asked about why they need to do well in mathematics. Pupils could agree with any or all of the three areas of possible motivation presented in table 4-5, including getting a desire a job, to please their parents, and to get into their desired secondary school or university (see table 4-5).

Only 13% Korean pupils strongly agreed that they needed to do well in mathematics to get their desired job, while 39% German pupils strongly did. On the other hand, over 50% Korean pupils disagreed or strongly disagreed that this was a motivating factor for doing well in mathematics, while 30% German pupils did. In addition, 86% Korean pupils agreed or strongly agreed that they needed to do well in mathematics to get their desired secondary school or university, while 65% German pupils did.

| | Get desired job | | | Please parents | | | Get into desired secondary school or university | | |
|---|---|---|---|---|---|---|---|---|---|
| | Strongly agree | Agree | D/ S.D | Strongl y agree | Agree | D/ S.D | Strongl y agree | Agree | D/ S.D |
| Korea | 13(0.8) | 34(0.8) | 53(1.1) | 11(0.7) | 44(1.2) | 44(1.3) | 35(1.2) | 51(1.0) | 14(0.8) |
| Ger. | 39(1.3) | 31(1.1) | 30(1.0) | 25(1.2) | 32(0.9) | 43(1.2) | 32(1.1) | 33(1.1) | 35(1.2) |

*Table 4-5 Percentages for pupil' reports on why they need to do well in mathematics in the respective 8th graders with standard errors (SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1994-95) D: disagree; S.D: strongly disagree*

The following table provides pupils' responses to the question about how much they like or dislike mathematics in relation to their average mathematics achievement. In both countries, Korea and Germany, more than 40% pupils answered they disliked mathematics (see table 4-5).

| | Dislike a lot | | Dislike | | Like | | Like a lot | |
|---|---|---|---|---|---|---|---|---|
| | Percent of pupils | M.A | Percent of pupils | M.A | Percent of pupils | M.A | Percent of pupils | M.A |
| Korea | 6(0.3) | 536(8.0) | 36(1.2) | 569(3.6) | 44(1.2) | 628(3.3) | 14(0.8) | 676(5.0) |
| Ger. | 23(1.2) | 481(4.8) | 22(1.1) | 508(6.8) | 31(1.1) | 525(5.0) | 24(1.1) | 522(5.7) |

*Table 4-6 Percentages for pupil' reports on how much they like mathematics and mean achievement in the respective 8th graders with standard errors (SOURCE: IEA Third International Mathematics and Science Study (TIMSS), 1994-95) M.A: mean achievement*

TIMSS also asked pupils' attitudes toward mathematics. 50% Korean pupils expressed negative or strongly negative attitudes about mathematics, while 43% German pupils did. In addition, only 5% Korean pupils had strongly positive attitudes towards mathematics, while 13% German pupils did.

The international as well as the national results of Germany show long-term deficiencies in mathematics education in Germany (Baumert, J., Lehmann, R & Lehrke, 1997):

- As one might expect, there is a large difference in basic cognitive abilities between pupils from different types of school. However, the wide overlap in the distributions for different schools is of particular interest. 30 percents of the *Realschule* pupils and 25 percents of the *Gesamtschule* pupils lie above the average level of the *Gymnasiasten* for basic cognitive abilities.
- The improvement in achievement from one type of school to another varies, yet these differences correspond to what one might expect. The extent to which achievement levels are similar in some different types of school suggests that the different materials have similar sections. The knowledge acquisition runs a little cumulatively.
- The mathematics achievement of the 7th and 8th grade *Gymnasium* pupils is on average considerably better than at each of the *Realschule*, *Hauptschule* and *Gesamtschule*. Nevertheless, the overlaps in the achievement distributions are large. A good 40 percent of the *Realschule* pupils and 25 percent of the *Gesamtschule* pupils achieve the middle level of *Gymnasium* mathematics.
- The average achievement level of students of *Gymnasium* in mathematics lies somewhat below of ability level, which is expected a sufficient understanding of mathematical concepts and procedures.

The relatively poor performance of German pupils for mathematics in TIMSS tends to be perceived as a consequence of deficiencies in the curriculum, as could be seen from the TIMSS video study (TIMSS, 1997). It means there might be large gap between the German implemented curriculum and the intended curriculum.

### 4.5.2. PISA (2000) – international test

The first major aspect of the OECD/PISA mathematical literacy framework is mathematical competencies. They define three competence classes including the following elements: (1) Mathematical thinking skill, (2) Mathematical argumentation skill, (3) Problem posing and solving skill, (4) Representation skill, (5) Symbolic, formal, and technical skill, (6) Communication skill, and (7) Aids and tools skill (OECD 2000).

These skills are given in non-hierarchical lists; however, they are relevant to all levels of education. The following three competence classes are also not intended to form a strict hierarchy – competencies[8] will vary according to the individual student.

**Class 1 competencies: reproduction, definitions, computations**

The class includes knowledge of facts, representation, recognising when objects or expressions are equivalent, recalling the forms of mathematical objects and their properties, performing routine procedures, applying standard algorithms, and developing technical skills. Manipulating expressions containing symbols and formulae and calculations are also competencies which belong in this class.

**Class 2 competencies: connections and integration for problem solving**

One of the important competences in this class is the connections between the different strands and domains in mathematics, and the information that must be integrated to solve simple problems. Students will therefore, have to choose strategies and mathematical tools to use. They are also expected to handle different methods of representation, according to the situation and purpose. The connections component also requires students to be able to distinguish and to relate different statements such as definitions, claims, examples, conditioned assertions and proofs. This class relates to the several of the mathematical skills mentioned above, "mathematical argumentation skills" for examples require some reasoning, "modelling skills" and "problem-posing and –solving skills" and "representation skills". Problems in this class are often placed within a context and engage students in mathematical decision making.

One of the most important competencies in this class is the ability to combine information from different mathematical topics in order to solve simple problems. Pupils are therefore required, as part of this class of competencies, to choose which strategies and mathematical tools they will use. They are also expected to deal with different methods of representation according to the situation. The component of combining information as described above also requires pupils to be able to distinguish and to relate different statements such as definitions, claims, examples, assertions with conditions, and proofs. This class relates to the several of the mathematical skills listed above: "mathematical argumentation skills", which require some reasoning, and "modelling skills", "skills in posing and solving problems" and "representation skills". Problems in this class are often placed within a context and engage pupils in making mathematical decisions.

**Class 3 competencies: mathematical thinking, generalisation, and insight**

In this class, pupils are asked to "mathematise" situations, that is, to recognise and extract the mathematics embedded in the situation and to use mathematics to solve the problem, to analyse, to interpret, to develop their own models and strategies, and to present mathematical arguments including proofs and generalisations.

### 4.5.3. PISA national test in Germany

Germany has an additional national-test called PISA-E which is comparable between individual *Länder*. In this national test, $9^{th}$ grade pupils participate as well as 15-year-old pupils (Baumert, Klieme &

[8] These competencies which are classified in international PISA text are different from the competence level used in PISA-E.

Neubrand 2001). Five levels of mathematical competency are defined by pupils' ability and by the difficulty value assigned to problems (PISA 2000 – *Die Länder der Bundesrepublik Deutschland im Vergleich*, 2002).

| | Competence level | Range of scales of ability |
|---|---|---|
| I | Computation at primary school level | 329 - 420 |
| II | Modelling at primary school level | 421 - 511 |
| III | Modelling and forming conceptual relationships at lower secondary level | 512 - 603 |
| IV | Extensive Modelling of the basis on demanding concepts | 604 - 695 |
| V | Complex Modelling and inner-mathematical argument (argumentation confined to mathematics) | above 696 |

*Table 4-7 Competence model as defined in PISA-E*

Individuals at the lowest competency level are able to quickly retrieve and to apply their arithmetical knowledge. Those at the highest competency level, by contrast, are capable of complex modelling and mathematical argumentation. The PISA competency level model (Klieme, Neubrand & Lüdtke 2001, p. 160) defines the following five levels:

- *Competence level I* (from 329 to 420 points): Computation at primary school level

  Pupils at this level have only an elementary arithmetical and geometrical knowledge. They can recall and directly apply this knowledge if it is clear from the outset how the problem should be "standard-mathematised". They are not capable of conceptual modelling.

- *Competence level II* (from 421 to 511 points): Modelling at primary school level

  This level deals with simple conceptual modelling which is embedded into an outside mathematical context. Pupils at this competence level can find the suitable solution under several possible answers, if by graphics, tables and drawing, etc., a structure is given. At this level only elementary school mathematics are covered.

- *Competence level III* (from 512 to 603 points): Modelling and forming conceptual relationships at lower secondary level

  From level II to level III a qualitative jump in several respects takes place. Pupils at this level have covered some lower secondary school material, including standard material common to the curricula of all school forms. They can connect concepts from different mathematical topics and use the solution of problem posing, if visual representations support the process of solutions. German mathematics experts regard this level as the standard level of mathematical literacy.

- *Competence level IV* (from 604 to 695 points): Extensive Modelling of the basis on demanding concepts

  If pupils at this competence level master more extensive processes of operation in technical areas, then they can also construct a solution over several intermediate results. Moreover, pupils can manage open modelling problems in which they must find a solution by selecting a correct solution method from several different possibilities. Strengthened inner-mathematical conceptual relationships can be modelled.

- *Competence level V* (above 696 points): Complex Modelling and inner-mathematical argumentation

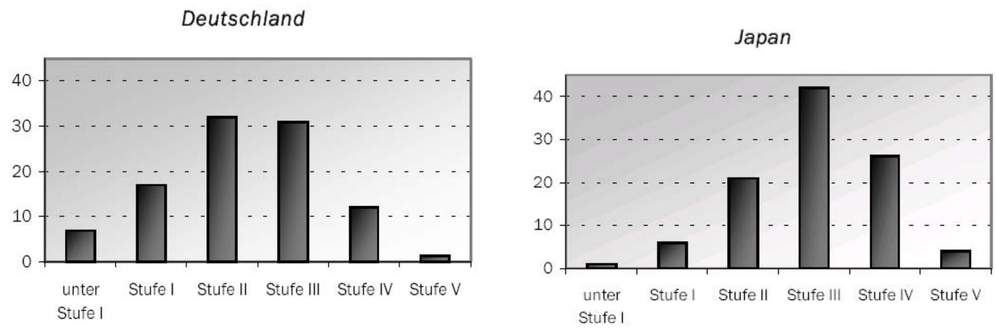
  Pupils who belong to this competence level can also manage open formulated problems, in which the pupil must decide which model to use and construct it herself. Conceptual modelling at this level often involves reasoning and proving as well as considering the process of modelling itself.

German PISA-Experts believe that competence level III reflects the average standard of mathematical literacy in PISA, and that this level should be attained by 15 year-old pupils.

- Some remarks in PISA

German educators analysed the results of the PISA study in order to measure the competencies of the pupils of participating countries. They compared the German results with those of Austria and Switzerland, countries that neighbour Germany, and with countries from western and northern Europe such as France, England, and Sweden as well as with the USA and Japan (PISA 2000 pp.167-172).

Even though Korea participated in PISA, however, unfortunately there are no results comparing Korea and Germany with respect to competence level directly. Since Korea and Japan have similar mean performances (TIMSS), a comparison of the competence levels of Germany and Japan is potentially useful as a means of comparing Korea and Germany indirectly (see Graph 4-1).




*Graph 4-1 Competence levels of German and Japanese pupils*

From this graph we can see that over 60% of the German pupils belong to either competence level II or level III. Moreover, competence level II is the most common level for German pupils. The percentage of German pupils who could be said to have sufficient literacy is less than 50%. Moreover, the percentage of German pupils who either belong to competence level or are below this standard is 24%. This means that the German educational system does not appear to be particularly successful as regards teaching weaker pupils effectively.

The next table breaks down these results according to school types (PISA 2000).

| | Hauptschule | Integrierte Gesamtschule | Realschule | Gymnasium |
|---|---|---|---|---|
| Stufe V | 0,0 | 0,6 | 0,5 | 4,2 |
| Stufe IV | 0,4 | 4,1 | 6,5 | 31,9 |
| Stufe III | 6,5 | 24,2 | 36,1 | 48,0 |
| Stufe II | 37,1 | 40,7 | 42,4 | 14,8 |
| Stufe I | 38,6 | 24,6 | 12,7 | 1,1 |
| unter Stufe I | 17,4 | 6,2 | 2,0 | 0,0 |

*Table 4-8 Competence levels by type of German school*

Over 15% of *Hauptschule* pupils are at a level lower than competence level I, while there are no *Gymnasium* pupils in this category. Moreover, about 30% of *Gymnasium* pupils are at competence level 4, while only 6% of *Realschule* pupils (0.4% of *Hauptschule* pupils) belong to the same level.

If *Gymnasium* pupils alone had been compared with Japanese pupils, the results would have been quite different (PISA 2000). For example, the percentage of Gymnasium pupils (31.9%) at competence level 4 is higher than that of Japanese pupils (below 30%).

The next table shows the results of the PISA assessment. In the PISA assessment, the top five percent of all OECD pupils achieved 655 points or more. The top five percent of Korean pupils scored 676 or higher and the top five percent of the German pupils scored at least 649.

| | Average | $5^{th}$ | $10^{th}$ | $25^{th}$ | $75^{th}$ | $90^{th}$ | $95^{th}$ |
|---|---|---|---|---|---|---|---|
| Korea | 547 (2.8) | 400 (6.1) | 438 (5.0) | 493 (4.2) | 606 (3.4) | 650 (4.3) | 676 (5.3) |
| Germany | 490 (2.5) | 311 (7.9) | 349 (6.9) | 423 (3.9) | 563 (2.7) | 619 (3.6) | 649 (3.9) |
| OECD average | 500 (0.7) | 326 (1.5) | 367 (1.0) | 435 (1.0) | 571 (0.8) | 625 (0.8) | 655 (1.2) |

*Table 4-9 Average mathematics literacy scores by percentiles with standard errors.*

- Korea is among the top countries for mathematics literacy. Germany is one of a large group of 16 countries that rank close to the overall average and have an achievement level between 470 and 520 points.
- Germany has a relatively large gap between the $75^{th}$ and $25^{th}$ percentiles: 140 points. Korea, on the other hand, shows comparatively small disparities, with 113 points separating its $75^{th}$ from its $25^{th}$ percentiles.
- The gap about the $95^{th}$ percentiles between the two countries is only 27 points, while the gap about $5^{th}$ percentiles between the two is 89 points.

## 4.6. Discussion

By comparing the percentage for Korea and Germany of test problems from the TIMSS which closely match the respective intended mathematics curriculum, it is clear that Germany's percentage is higher than that of Korea.

| | % Problems addressing national curriculum (n=157) |
|---|---|
| Korea | 92 |
| Germany | 96 |

*Table 4-10 Percentage of test problems from the TIMSS matching the intended mathematics curriculum for each country*

As previously mentioned, the problems of TIMSS are curricular-based ones. Nevertheless, there is no great difference in the two percentages above. The difference in achievement between the two countries, however, is quite large. As mentioned in chapter 3, the Korean textbook, shaped by the curriculum, is a standard instruction material used by teachers and pupils and problems similar to those in TIMSS are explicitly taught in the classroom. One might therefore assume that the implemented curriculum is very similar to the intended curriculum in Korea. However, it might be said that the German intended curriculum is far removed from what happens in the classroom.

There are wide differences within Germany itself. As explained above, in Germany there are three different types of school for different academic levels. The achievements of *Gymnasien* pupils were better than those of pupils in the other school forms, the *Realschule* and *Hauptschule* (Baumert, Bos et al. 1998). As a result, the performance of pupils within Germany varies more widely than in any other PISA country (Klieme et al, 2003). Moreover, the additional study PISA-E demonstrates that there are significant differences between the individual states (*Länder*) in Germany. According to the German national PISA test, Bavarian pupils' achievements were above the international average (516 points), while the achievements of pupils in *Niedersachsen* were below the international average (478 points).

Despite the fact that Korean pupils came second for mathematics in the PISA study, the Korean Ministry of education worried about the higher-attaining pupils in Korea, because Korea's best pupils lag behind the best pupils in New Zealand, Japan, Switzerland, and Australia. There are few pupils at the extreme ends of the performance scale.

# Chapter 5 Literature Review

Proof has been considered a fundamental part of the practice of mathematics since ancient times. It is an important topic in any mathematics curriculum and an essential aspect of mathematical competence. However, proof is one of the most difficult ideas for pupils to learn about. Moreover, recent studies have revealed wide gaps in students' understanding of proofs (Williams, 1979; Senk, 1985; Martin & Harel, 1989; Harel & Sowder, 1998).

In this chapter I will briefly review the relevant literature. The aim is to examine mathematical proof from an educational point of view, as well as different proof schemes.

## 5.1. What is a proof?

> "For mathematicians, a proof is a coherent chain of argumentation in which one or more conclusions are deduced, in accord with certain well specified rules of deduction, from two sets of givens: (a) a set of hypotheses and (b) a set of accepted facts, consisting of either axioms or results that are known to have been proven true"
>
> - Schoenfeld (1988, p.157)

Schoenfeld (1988) describes a mathematician's definition of proof above. This mathematician's definition is a typical traditional view which agrees with that of Hilbert: "a mathematical proof is a formal and logical line of reasoning which begins with a set of axioms and moves through logical steps to a conclusion" (1923).

Rav (1999) also gives a description of proof as follows:

"Proofs are the mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution" (p.13).

However, he suggests that we think of proofs as "a network of roads in a public transportation system and regard statements of theorems as bus stops". His metaphors indicate that a proof is important precisely because it provides a way forward (Hanna, 2000).

Here we should note the idea of a plausible proof as suggested by Polya (1969) and Lakatos (1976). The word proof is not always necessarily used as a synonym for formal proof. Polya (1969) explained that didactical proof should be plausible before it is axiomatic. Moreover, many mathematics educators attached great importance to this didactical proof. For example, Hanna (1995, p.47) noted that the best proof is one that also helps understand the meaning of the theorem being proved: to see not only that it is true, but also why it is true.

According to Martin and Harel (1989), in everyday life people consider proof to be "what convinces me". Harel and Sowder (1998) also gave a similar definition:

"Proving, or justifying, a result involves ascertaining – that is, convincing oneself – and persuading, that is, convincing others. An individuals' proof scheme consists of whatever constitutes ascertaining and persuading for that person. … The word *proof* in *proof scheme* is used in the broader, psychological sense of *justification* rather than in the narrower sense of *mathematical proof*".

What pupils think of proof should now be considered, since it may well prove to be very different to the mathematicians' definition of proof. I will first deal with the philosophy of mathematics about proof.

The philosophical views many mathematicians have about proof provide one of the most important aspects to discuss when analysing how proof is taught in school.

Of all the sciences, mathematics has a unique relation to philosophy (Ross[9]). In Ross' opinion, since antiquity philosophers have envied mathematics as the model of logical perfection because of the clarity of its concepts and the certainty of its conclusions, and have devoted much effort to explaining the nature of mathematics.

### 5.1.1. Proof in philosophy of mathematics

The philosophy of mathematics has been studied for many centuries and has been concerned with the issue of the nature of mathematics. In the last century, the nature of mathematics became a central issue for educationalists. An individual's philosophy of education was thought to determine how he lived his life. A personal philosophy of mathematics education is now thought to ascertain the way we learn and teach mathematics within the classroom and the school environment (Southwell, 1999).

Thus the question of interest becomes how the philosophy of mathematics has developed. In addition, how has the way in which proofs are viewed changed in the philosophy of mathematics?

As concepts and views of mathematics have changed, the view of mathematical proof has changed as well. According to Sekiguchi (1991), mathematical proof has come to be viewed as a social practice and has been analysed for its intra-and interpersonal meanings in mathematics education.

I will now introduce four philosophies of absolute view of mathematics. The first one is Platonism that prevailed widely until the 18$^{th}$ century; the second one is logicism which originated in England; the third one is formalism which started in Germany; and the fourth one is the intuitionism of Netherlands in the 19$^{th}$ century.

According to Monk (1970), the mathematical world is populated with 65% Platonists, 30% formalists, and 5% constructivists. In the book of David and Hersh (1981), they suggested that the typical working mathematician is a Platonist on weekdays and a formalist on Sundays, i.e. a secret Platonist with a formalist mask which he puts on when the occasion calls for it. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all (p.321, David and Hersh, 1981). It means that even Platonism and formalism are two distinct and basically irreconcilable views for philosophy of mathematics, however, in the working mathematician's views, they are often simultaneously present same degree (Kutrovátz, 2001).

- Platonism

The term Platonism is used because such a view is seen to parallel Plato's belief in a "heaven of ideas", an unchanging ultimate reality which the everyday world can only imperfectly approximate. Mathematical objects are abstract entities in the Platonic universe. In Greece where Plato lived, mathematics meant geometry; therefore, the philosophy of mathematics according to Plato is actually philosophy of geometry.

The following six points represent the core of Platonism (Brown, 1999).

(1) Mathematical objects are perfectly real and exist independently of us.
(2) Mathematical objects exist outside of space and time.

[9] Foundations Study Guide: Philosophy of Mathematics, http://ios.org/articles/foundations_phil-of-mathematics.asp

(3) Mathematical entities are abstract in one sense but not in another.
(4) We can intuit mathematical objects and grasp mathematical truths.
(5) Mathematics is a priori, not empirical.
(6) Even though mathematics is a priori, it is not necessarily certain.
(7) Platonism, more than any other view of mathematics, is open to the possibility of an endless variety of investigative techniques.

The proof procedure in Platonism consists of the derivation of a theorem by following valid inferences from trivial truths (axioms, or postulates), from perfectly clear descriptions of essences (definitions), and from previously derived theorems (Lee, 2002).

- Logicism

During the development of areas of mathematics such as set theory and function theory, several paradoxes arose which could not be explained using the Platonic axioms. Logicism was therefore established to explain this phenomenon.

Logicism is basically a form of Platonist realism in which mathematics is seen as a set of abstract realms that exist externally to human creation. Logicism claims that logic is the proper foundation of mathematics and that all mathematical statements are logical truths. According to logicists, all mathematical concepts can be reduced to abstract properties which can be derived through logical principles, i.e. by “reducing mathematics to logic”.

The founder of logicism was Gottlob Frege. He built up arithmetic from a system of logic using Basic Law V[10], a principle which he took to be acceptable as part of logic. However, his construction was defective. Russell discovered that Basic Law V is inconsistent and the statement of this is known as Russell’s Paradox.

Logicists considered symbols to be an aid to achieve strictly accurate demonstrative reasoning. As a result, proof becomes a process of reasoning being represented as a valid derivation in symbolic logic (Lee, 2002). In logicism, the role of proof is to show that certain propositions are precisely of the same type. Logicians aim to show that their axiomatic system is powerful enough to deduce these propositions. The main concern of logicians is to deduce from an axiomatic system theorems in the usual view of mathematics which have already been proved. Therefore, the function of the logicians’ proof is also to demonstrate the strength of their axiomatic system (Lee, 2002).

Goodman (1986) explained why logicism couldn’t be an adequate philosophy of mathematics as follows. External mathematical structures are real in practice; however, the essence of logicism denies the objective reality of any such structure.

- Brouwer’s intuitionism

Subsequent to logicism, intuitionism was conceived of by Dutch topologist L. E. J. Brouwer around 1908. Brouwer held the idealist view that mathematical concepts are admissible only if they are adequately grounded in intuition and that mathematical theories are significant only if they concern entities which are constructed out of something which is immediately clear intuitively (Bell, 1999). Thus Brouwer held that mathematical theorems are synthetic a priori truths.

However, intuitionists are not always clear about the meaning and philosophical foundations of their positions; they attend to mathematical details at the expense of philosophical exposition (Ross).

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[10] For concepts F and G, the extension of F equals the extension of G if and only if for all objects a, Fa if and only if Ga

In the tradition of intuitionism, mathematics is perceived as an intellectual activity in which mathematical concepts are seen as mental constructions regulated by natural laws. These constructions are regarded as abstract objects which do not necessarily depend on proofs.

Brouwer rejected the classical stance of categorising proofs as either true or false and instead argued that other possibilities for claiming mathematical truth should be allowed as academically acceptable. For him, mathematical induction comes before and it is independent of logic. Likewise, intuition and imagination are developed at an early stage and are necessary psychological stages in the process of invention (Goodman, 1986).

In the intuitionist's view, a mathematical proposition is demonstrated true only when a mental construction by one's fundamental intuition shows the proposition to be true. Logic and language are considered as having a secondary position in proof. They are used only to describe mathematical thought when remembering or communicating. In addition, description is always considered to be incomplete. Therefore, mathematical reasoning requires mathematical insight (or intuition) and mental construction.

Goodman (1986) formulated his argument against intuitionism by saying that the essence of intuitionism denies the objective reality of mathematical truth.

- Formalism

Formalists share logicists' view that logic is necessary. However, they argue that mathematical knowledge is brought about through the manipulation of symbols which operate by prescribed rules and formulae and whose understanding should be accepted *a priori*.

A famous proponent of formalism was David Hilbert, whose goal was a complete and consistent axiomatisation of all of mathematics. Formalists seek to express mathematics in terms of strictly formal logical systems and to study them without attempting to understand their meaning. This distinguishes them from logicists, who seek to establish the meaning of mathematical notions by defining them in terms of concepts of logic (Ross).

The formalists hoped to express the mathematics of infinite sets in such a system and to establish the consistency of that system by finite methods. For a formalist, the motivation for formalising proof is to establish absolute rigour in the proof procedure. The formalists believed that an objective standard of proof could be achieved, reducing the proof procedure to symbolic manipulation (Lee 2002).

The mathematical ideas of formalism are arranged only using symbols and mathematical systems such as definitions, properties, and rules. Therefore, formalism has been criticised because of the scarce space left for creative thinking.

**Remark:** The classical philosophies of mathematics that are logicism, intuitionism and formalism cannot be held to provide complete accounts of the nature of mathematics. However, they each give an expression of an important truth about its nature: *Logicism* is a concept in which mathematical truth and logical demonstration go hand in hand, *intuitionism* holds that mathematical activity proceeds by the performance of mental constructions, and *formalism* claims that the consequence of these constructions is presented through the medium of formal symbols (Bell, 1999).

- Quasi-empiricism

In his thesis "proof and refutation", Lakatos (1976) argued that dogmatic philosophies of mathematics (such as logicism or formalism) are unacceptable. He suggested that mathematics should be practical, fallible, situated, and constructed by individuals and by society against the traditional views of mathematics as an abstract, absolute, universal, and infallible system. He made a strong attack on formalism and logicism. He suggested that mathematics is a human creation born and nurtured from practical experience, always growing and changing, open to revision and challenge, and whose claims of truth depend on “guessing by speculation and criticism, by the logic of proofs and refutations…" (p. 5). This is called “quasi-empiricism”.

Lakatos developed his cycle of proofs and refutation to represent accurately what proving is to mathematicians. He did so in opposition to what he called the Euclidean model, which portrays mathematical research as a process of beginning with a set of assumptions, and then proving theorems from them with absolute certainty. This view might have been derived from the assertions of formalist philosophers of mathematics. The Euclidean model bears a superficial resemblance to a cycle that begins with making a proof.

The quasi-empiricist position can be described as a sceptical and fallibilist philosophy of mathematics. It asserts that we have no way of knowing what is true, or even that there is truth in mathematics; that we can only put forward tentative guesses or conjectures which are then to be tested. The process of testing either proves or refutes our initial conjectures.

- Social constructivism or Social realism

Social constructivism arises as an alternative perspective to radical constructivism in the work of some researchers into the psychology of mathematics education. The roots of social psychology can be found in the ideas of the radical constructivist (Piagetian) and in Vygotskian theory (Ernest, 1999).

Radical constructivism can be described as being based on the concept of an evolving and adapting organism, but one which is isolated in the environment (Ernest, 1993)

Ernest (1998) has recently developed an epistemological view of social constructivism. According to him, the social constructivist thesis states that mathematics is a social construction and a cultural product which is fallible like any other branch of knowledge. It involves two claims. First, that the origins of mathematics are social and cultural. And second, that a justification of mathematical knowledge depends on its quasi-empirical basis (Ernest, 1998).

According to the views of social constructivism, mathematical knowledge is influenced by human activities and grows out of a community composed of individual mathematicians (Ernest, 1998). Ernest strongly opposes the absolutist view and suggests that mathematical knowledge and logic are fallible, because mathematical proofs are the result of social discourse within the mathematical community. He emphasises the importance of language which has certain features such as rules of participation, behavioural patterns, and linguistic usage from which meaning is established according to social patterns.

**Summary:** Platonism focuses on the absolute existence and absolute truth of mathematics. Also the other absolute philosophies of mathematics such as logicism, intuitivism, and formalism assert that mathematics is a body of absolute and certain knowledge.

For followers of these three absolute philosophies, the absolute basic or truth of mathematics should be fixed as inducing the mathematical knowledge through the proof which satisfies their own standards, namely the axioms of logic for logicism, the constructive activities based on the human intuition for intuitivism, and the operation of symbols excluding meaning in formalism.

However, the problem for absolute philosophy of mathematics is that it only stresses the deductive reasoning, i.e., the process from the hypothesis to conclusion with postulates, axioms and definitions. Moreover, if the theorem is proved, then the process of proof is finished and is not continued any further. However, absolute certainty cannot be gained in this way.

In contrast to the absolute philosophies, quasi-empirism holds that mathematical knowledge is a set of conjectures which are provisionally accepted as being true. Therefore, proving a theorem does not contribute to mathematical knowledge. Moreover, after proving the theorem, it is necessary to analyse the proof and the constant process which improves the conjecture and proof. According to Lakatos, the proof is not a means to justify some theorems, but a means of improving the conjecture by analysing the proof or of establishing a new concept by refuting the proof itself.

However, many mathematicians have criticised Lakatos' philosophy, because the typical working mathematician are Platonist or formalist as mentioned earlier. On the other hand, researchers of the didactics of mathematics supported Lakatos' philosophy. Moreover, they assumed that the methods of Lakatos could be applied broadly in the classroom.

Social constructivism points out the limitation of proof being seen as formal and having logic as its foundations, as is asserted by absolute philosophies. In social constructivism, proof is considered as a means to conviction and understanding. It is suggested that teachers help pupils to understand the use of proof for giving insight into why a theorem is true. Therefore, the function of proof in the classroom is not as the process which occurs superficially to attain a certain level of rigor, but as an explanation to improve pupils' ability in proving and to help pupils understand theorems and proofs.

### 5.1.2. The role of proof

The role of proofs has been investigated by a number of researchers since two decades. One might expect the role of proofs in the classroom to figure in these investigations in some way. But these functions described as below are not all relevant to learning mathematics to the same degree, so naturally they should not all be taught with the same time and effort (de Villiers, 1990, Hersh, 1993).

Bell (1976, 1979) described proof not only as verification but also as an illumination and systematization. De Villiers (1990) explicitly described the separate roles of proofs as follows:

- Verification (concerned with the truth of a statement)
- Explanation (providing insight into why it is true)
- Systematization (the organization of various results into a deductive system of axioms, major concepts, and theorems)
- Discovery (the discovery or invention of new results)
- Communication (the passing-on of mathematical knowledge and meaning)

De Villiers recommends that pupils use proof for explanation and discovery before they use it for verification and systemization.

(a) Verification

As Bell (1976) states, verification is the most obvious role carried out by mathematical proof. Verifying or providing conviction as to the truth of mathematical conjectures was traditionally seen as the primary reason for proving. However, de Villiers (1990) suggests that conviction is probably a prerequisite for the finding of a proof. Indeed, this role is the most familiar one to pupils. However, many studies conclude that only a few pupils see verifying as a use for proving (Bell, 1976; Braconne & Dionne, 1987; Fischbein, 1982; de Villiers, 1992; Senk, 1985).

De Villiers, Hanna and other researchers have asserted that the role of proof cannot be restricted to verification alone.

(b) Explanation

Bell (1976) also pointed out that a proof is effective if it provides a good explanation when he stated that proof is expected "to convey an insight into why the proposition is true."

Hanna (1989, 1995) and de Villiers (1991) both stressed the importance of proving as a way of *explaining* in educational contexts. Hanna asserted that proofs used by teachers in lessons should be chosen on the basis of both their explanatory and verificatory qualities: "the best proof is one which also helps mathematicians understand the meaning of the theorem being proved: to see not only that it is true, but also why it is true" (Hanna 1995, p.47).

Moreover, mathematicians expect the role of proof to extend beyond the verification of results to providing explanation.

"Mathematicians are interested in more than *whether* a conjecture is correct, mathematicians want to know *why* it is correct" (Hersh, 1993, p. 390).

This role is important from a didactical perspective.

(c) Systematisation

Systematisation, "the most characteristically mathematical role of proof", was defined by Bell (1976). According to him, systematization consists of "the organisation of results into a deductive system of axioms, major concepts and theorems, and minor results derived from these" (p. 24). It seems to be difficult for pupils to recognise axiomatic structures, or systematizations, even if they have covered certain axiomatic systems such as Euclidean geometry in secondary school geometry lessons. De Villiers (1990) explained that proof is an indispensable tool for systematizing various known results into a deductive system of axioms, definitions, and theorems. Although he mentioned the importance of systematisation for mathematicians, he did not provide evidence to support his hypothesis that pupils prove in order to satisfy a need to systematise.

(d) Discovery or creation

The ideas for solving problems or for proving theorems have been sometimes discovered by intuition, quasi-empirical methods, or by accident, as in the case of Aristotle's "Eureka"-realisation. However, there are numerous other examples in the history of mathematics, where proof, together with formal deductive processes such as axiomatisation and definition, has led to new results, for example in the field of non-Euclidean geometries as de Villiers (1999) noted. Therefore, de Villiers (1999) suggested that to a mathematician, proof should be a means of exploring, analyzing, and discovering or inventing new results, not merely as a means of verifying results he is already aware of. What about in the case of pupils? Chazan & Yerushalmy (1998) addressed that the role of proof in introducing mathematics pupils have not yet covered is beginning to play a larger part in many secondary school geometry classrooms, particularly those classrooms in which pupils can utilise dynamic geometry software.

(e) Communication

De Villiers (1990) also suggests that communication is an important role of proof. Moreover, several researchers have stressed the importance of the role of proof in communication.

Hanna (1989) noted that "the acceptance of a theorem by practising mathematicians is a social process" (p. 21). As such, proof as a social construct and product of mathematical discourse has been to some extent accepted in the mathematics community (Hanna, 1983; Hersh, 1993). Moreover, Davis (1976) has also mentioned that one of the real values of proof is that it creates a forum for critical debate.

From a social perspective of this kind, the structure of a mathematical proof is a means to the epistemological end of providing a persuasive justification and of justification for the truth of a mathematical proposition (Ernest, 2000). To fulfil this function, a mathematical proof must satisfy the appropriate community, namely mathematicians, that it follows the currently accepted criteria for a mathematical proof. Moreover, in the classroom, a mathematical proof can be a form of discourse and a means of communication between teachers and pupils and among pupils themselves.

## 5.2. Proof schemes

This section is concerned with examining proof schemes within the literature of mathematics education. In the last decade, a number of research papers related to proof schemes have been published. For example, some researchers have classified or arranged proofs (Balacheff, 1988; Braconne and Dionne, 1987), while some researches have focused on proofs which are meaningful to pupils, regardless of the quality of the reasons given (Wittman and Müller, 1988; Blum and Kirsh 1991).

### 5.2.1. Lakatos' quasi-empirical proof

Lakatos published his masterwork "Proofs and Refutations" in 1976. In it, he criticised the way in which the nature of proof had come to be dominated by logicism and formalism. In his introduction, he claimed that the history of mathematics and the logic of mathematical discovery couldn't be developed without criticism and the ultimate rejection of formalism.

He wrote his dissertation in the form of a dialogue between a teacher and several pupils, who studied and discussed the famous Euler-Descartes formula V-E+F=2 for a polyhedron.

The teacher explained the traditional proof, to which was the pupils immediately gave counterexamples. In response to these counterexamples, the statement of the theorem was modified and proved. New counterexamples arose, definitions were proposed and revised, and new adjustments were made.

Lakatos gave a simple pattern of the growth of informal mathematical theories, consisting of the following stages:

1. Primitive conjecture.
2. Proof: a rough experiment or argument decomposing the primitive conjecture into sub-conjectures or lemmas.
3. "Global" counterexamples emerge, that is, counterexamples to the primitive conjecture.
4. Proof re-examined: the 'guilty lemma' to which the global counterexample is a 'local' counterexample is spotted. This guilty lemma may have previously remained 'hidden' or may have been misidentified. Now it is made explicit and built into the primitive conjecture as a condition. The theorem – the improved conjecture – supersedes the primitive conjecture with the new proof-generated concept as its paramount new feature (Lakatos, 1976, p. 127).

These four stages constitute the essential kernel of proof analysis. But there are some further standard stages which frequently occur:

5. Proofs of other theorems are examined to see if the newly found lemma or the new proof-generated concept occurs in them: this concept may be found lying at cross-roads of different proofs, and thus emerge as of basic importance.
6. The hitherto accepted consequences of the original and now refuted conjecture are checked.
7. Counterexamples are turned into new examples – new fields of inquiry open up.

Lakatos uses proof as a "thought-experiment" or a "quasi-experiment" which suggests a decomposition of the original conjecture into sub-conjectures or lemmas, thus embedding it is a possibly quite distant body of knowledge (1976, p.9). He asserts that the thought-experiment is the most ancient form of a mathematical proof.

In the Euclidean tradition, the activities of guessing and proving are rigidly separated, since this tradition holds that 'non-rigorous proofs' cannot be classed as proofs at all. Lakatos criticised this Euclidean tradition and introduced a proof scheme which relied on the degrees of formality of the proof, namely informal, quasi-empirical, and formal, depending on the format of the proof and the level of axiomatisation within the deductive system in which it is situated. He suggested the term "quasi-formal proof" or "formal proof with gaps" to describe proofs that are in principle formalisable, and reserved the term "informal proof" to describe a proof stemming from informal theory (Lakatos, 1986, pp.159-160).

According to Lakatos, informal proof is nothing other than a proof in an axiomatised mathematical theory which has already taken the shape of a hypothetico-deductive system[11], but which leaves its underlying logic unspecified (Lakatos, 1978, p. 62-63). Informal proof could be formalised without too much efforts and logic.

In the proof schemes of Lakatos, quasi-empirical proof could be understood as a different way of viewing an axiomatic system. Within the perspective of quasi-empiricism, new mathematical concepts are developed by improving and speculating proofs and conjectures by the use of counterexamples gleaned from re-examining the proofs.

To Lakatos, "proof" in this context of informal mathematics does not mean a mechanical procedure which carries truth in an unbreakable chain from assumptions to conclusions. Instead, it means explanations, justifications and elaborations which make the conjecture more plausible and convincing, while it is being made more detailed and accurate under the influence of counterexamples (David and Hersh, 1981). Lakatos applied his epistemological analysis not to formal mathematics but to informal mathematics: mathematics in a process of growth and discovery which is of course mathematics, as it is known to mathematicians and pupils of mathematics.

### 5.2.2. Balacheff's pragmatic and conceptual proofs

The different levels of pupils' ability to reason or give proofs have been studied from many different perspectives. Balacheff identified three categories: "pragmatic proofs", "conceptual proofs" and "demonstration" (1987, 1988).

> "Pragmatic proofs are those having recourse to actual actions or showings, and by contrast, conceptual proofs are those which do not involve action and rest on formulations of the properties in question and relations between them" (1988, p.217).

> "Demonstration requires a specific status of knowledge which must be organised in a theory and recognised as such by a community: the validity of definitions, theorems, and deductive rules is socially shared" (1987, p. 30).

He determined four approaches. Coe and Ruthven (1994, p.44) summarised them as follows:

- Naive empiricism in which the truth of a result is asserted after verifying several cases;
- The crucial experiment in which a proposition is verified for a particular case recognised to be typical but non-trivial;
- The generic example in which the reasons for the truth of an assertion are made explicit in a prototypical case; and
- The thought experiment in which the operations and foundational relations of the proof are indicated in some other way than by the result of their use.

Here, the "generic example" is of particular note, because Balacheff classed this case as a conceptual proof. However, it is necessary to find the "germination" in pragmatic proofs, and to give it a theoretical

[11] Philosopher Karl Popper suggested that it is impossible to prove a scientific theory true by means of induction, because no amount of evidence assures us that contrary evidence will not be found. Instead, Karl Popper proposed that proper science is accomplished by deduction. Deduction involves the process of falsification. Falsification is a particular specialized aspect of hypothesis testing. It involves stating some output from theory in specific and then finding contrary cases using experiments or observations. The methodology proposed by Popper is commonly known as the *hypothetico-deductive method*.

position as a level of proof. Therefore, a generic example has an important role in the shift from pragmatic proofs to conceptual proofs.

Balacheff's idea is that it is possible to establish some intermediate levels between proving and demonstrating through which pupils can and should shift from a fundamental level to a more advanced level, and finally reach a level of demonstration.

### 5.2.3. Blum and Kirsch's preformal proofs

Wittmann distinguished three levels of proving, in accordance with Branford (1908):

- Experimental "proofs": An experimental proof consists of the verification of a finite number of examples which are not sure to be general.
- "*Inhaltlich-anschaulich*" or "intuitional" proofs: These are based on constructions and operations, which are intuitive and recognizable which apply examples and get definite conclusions.
- Formal ("scientific") proofs

Wittmann called for an emphasis on "*inhaltlich-anschaulich*" proofs in school and in teacher training.

Blum and Kirsch (1991) kept to these levels. However, they also defined 'preformal proof' which consists of "inhaltlich-anschaulich" and what are known as "action proof".

- Experimental "proofs"
- Preformal proof
- Formal ("scientific") proofs:

Their "experimental proofs" are similar to Balacheff's "naive empiricism", namely the verification of a finite number of cases.

In accordance with Z. Semadeni's concept of "action proofs"[12], they defined a 'preformal proof' as "*a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises*". For them, the important criteria for a proof to constitute a preformal proof are correctly constructed arguments and valid premises based on reality or intuition.

Particular examples of such premises include concrete things such as the given real objects, geometric-intuitive facts and fundamental ideas which are based on reality, or more intuitively, "commonly intelligible" and "psychologically obvious" statements (p.187). According to how the premises of the conclusions were represented, they gave three different kinds of preformal proofs:

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[12] Semadeni (1984) introduces a form of generic proof called an action proof, the purpose of which is to provide "a general scheme for devising primary-school proofs" (p.32). Describing an action proof as "an idealized, simplified version of a recommended way in which children can convince themselves of the validity of a statement" (p.32), and thus identifying proof as that which convinces, Semadeni then presents the following description of what an action proof actually consists of:
An action proof of a statement S should consist of the following steps:
1: Choose a special case of S. The case should be generic (that is without special features), not too complicated, and not too simple (trivial examples may later be particularly hard to generalize). Choose an enactive and/or iconic representation of this case or a paradigmatic example (in the sense of Freudenthal [1980]) Perform certain concrete, physical actions (manipulation objects, drawing pictures, moving the body, etc.) so as to verify the statement in the given case.
2: Choose other examples, keeping the general schema permanent but varying the constants involved. In each case verify the statement, trying to use the same method as in 1.
3: When you no longer need physical actions, continue performing them mentally until you are convinced that you know how to do the same for many other examples.
4: Try to determine the class of cases for which this works. (p.32)

- Action proof: consists of certain concrete actions (actually carried out or only imagined) with a concretely given paradigmatic, generic example, where the actions correspond to correct mathematical arguments.
- Geometric-intuitive proof: refers to basic geometric conceptions and to intuitively evident facts such as areas and their properties.
- Reality-oriented proof: basic ideas which have meaning in reality and are easily accessible for learners are used, such as the derivative as a local rate of change.

Blum and Kirsch explained the standards which constitute real objects and how a statement has to be judged as "psychologically obvious" or "commonly intelligible".

> "What are "intuitive" or "obvious" bases for argumentation has to be decided in each individual case by the persons involved on the basis of their knowledge. Such bases can, of course, be changed in course of time, in particular by learning or experience" (p.187)

In their paper, two questions came up. The first one is: "How can learners judge for themselves the validity or non-validity of a preformal proof they are given or they construct themselves?" Their answer to this question is to consult someone who has a "higher order" of knowledge and abilities at his disposal.

> "It requires a competent mathematician to judge whether a given preformal proof is acceptable" (p.189)

The second question is: "How can learners find a preformal proof for themselves?" Their didactical answer stated that it is certainly not sufficient to simply demonstrate several such proofs for learners and to hope that they will pick up this idea themselves. Therefore, Blum and Kirsch suggested some necessary prerequisites for pupils to understand, judge, and in some cases independently find preformal proof:

- To place value on manifold kinds of representations of mathematical content, especially to stress reality-oriented basic ideas and to impart geometric intuitive basic conceptions
- To frequently provide preformal proofs in the classroom
- To provide examples of formal proofs
- To formalise non-formal arguments and to ikonicise or enacitivise formal arguments, or to interpret them in the real world
- To discuss proofs and proving with pupils, especially about different levels thereby, i.e. to help pupils reflect upon what is going on in the classroom.

### 5.2.4. Harel and Sowder' three categories of proof schemes

Harel and Sowder (1998) defined a proof scheme to be the arguments that a person uses to convince her and others of the truth or falseness of a mathematical statement. They investigated the proof schemes of college students using classroom observations, tests, and interviews. They characterised seven major types of proof schemes which each come under one of three categories of proof schemes, namely, external conviction, empirical proof schemes, and analytical proof schemes:

- External conviction: pupils convince themselves or others using something external to them.

- Ritual proof scheme: the convincing is due to the form of the proof, not its content. For instance, many pre-service teachers believe a geometry argument must be presented in a two-column format in order for it to be an acceptable proof (Martin and Harel, 1989)
- Authoritarian proof scheme: a scheme that is presented or approved by an established authority, such as a teacher, the books and so on.
- Symbolic proof scheme: this consists of a symbolic manipulation behind which there may or may not be a meaning.

- Empirical proof schemes: these take either inductive or perceptual

- Inductive proof scheme: pupils consider one or more examples to be convincing evidence of the truth of the general case.
- Perceptual proof scheme: pupils make inferences that are based on rudimentary mental images and are not fully supported by deduction. They can visually demonstrate that a certain property holds by an appropriate diagram. Harel and Sowder (1998) noted: "The important characteristic of rudimentary mental images is that they ignore transformations on objects or are incapable of anticipating results of transformations completely or accurately" (p. 255).

- Analytical proof schemes: these take either a transformational or axiomatic form

- Transformational proof scheme: Deductive processes in which pupils consider general aspects, apply goal-oriented and anticipatory mental operations, and transform images.
- Axiomatic proof scheme: this proof goes beyond a transformational one, in that pupils also recognise that mathematical systems rest on statements which are accepted without proof.

Harel and Sowder (1998) argued that instructional activities which educate pupils on reasons about situations in terms of *transformational proof schemes* are crucial to pupils' mathematical development. They also suggested that it is important to begin at an early age.

## 5.3. Process of proof

Mathematical proving is a complex activity which cannot be simply reduced to the correct use of deductive argumentations. It combines, for example, processes of concise logical argumentation as well with the more heuristic processes of producing a conjecture and looking for plausible arguments to support the conjecture.

### 5.3.1. Process of proof by Hanna

Hanna (1983) described five ways in which she believed mathematicians verify propositions. Most mathematicians accept a new theorem when some combinations of the following factors are present:

1. They understand the theorem, the concepts embodied in it, its logical antecedents, and its implications. There is nothing to suggest that it is not true;
2. The theorem is significant enough to have implications in one or more branches of mathematics (and thus important and useful enough to warrant detailed study and analysis);
3. The theorem is consistent with the body of accepted mathematical results;
4. The author has an unimpeachable reputation as an expert in the subject matter of the theorem;
5. There is a convincing argument for it (rigorous or otherwise), of a type which they have encountered before.

If there is a rank order of criteria for admissibility, then these five criteria all rank higher than a rigorous proof. (p. 70)

Hanna (1989) has written about the acceptance of theorems, however, the acceptance of theorems and conjectures often occur by different means. Reid (1995) addressed that most of these are based in the human and social nature of mathematics, not on the use of proving to produce certainty.

### 5.3.2. Process of proof by Boero

Boero (1999) described different phases of mathematicians' activities in producing conjectures and constructing mathematical proofs. The model consists of six phases which define the proving process of an expert, but it is not meant to be a linear model. Conjecturing, exploring, testing results and writing a formal proof has taken to be activities which are performed frequently during the process.

1. The production of a conjecture: this includes the examination of the situation outlined in the problem as well as the identification of arguments which would support the evidence. Boero refered to this phase as "the private side of mathematicians' work", which is usually not shared with the mathematical community.
2. The formulation of the statement according to shared textual conventions: this phase aims at providing a precisely formulated conjecture, which will then be the basis for all further activities. It may be revised in the forthcoming processes but this revision would have consequences for most activities performed by the mathematician.
3. The exploration of the (precisely stated) conjecture and the identification of appropriate arguments for its validation: this is also part of the "private work", since exploration might, for example, lead to errors or at least to complicated formulations in the proof. Only the next three phases are subject to public communication.
4. The selection and combination of coherent arguments in a deductive chain: Boero suggests that this phase is the start of that part of the mathematicians' work which is presented or explained in an informal way to colleagues.
5. The organization of these arguments according to mathematical standards: this phase leads to the production of a text for publication.
6. The proposal of a formal proof

This phase model shows that the proving process of an expert is not step by step process, but a sequence of intertwined activities.

This model illustrates that proving is a complex cognitive activity. It is not only characterised by logical argumentation but also by an exchange between explorative, inductive, and deductive processes. The mathematician has to identify a suitable choice of the elements involved in this process and arrange them in a logically consistent scheme.

The first and second phases seldom occur in normal secondary school mathematics classes. Instead, pupils are presented with theorems which have already been proved, they learn to follow the same routine in their proving process and they apply the proofs or theorems they have been shown to other problems or theorems (Heinze, 2004; Heinze & Reiss, 2004).

### 5.3.3. Cognitive development of representation and proofs by Tall (1995)

Expanding on the ideas of Bruner (1966), Tall (1995) developed the idea of relating the level of a pupil to the type of proof the pupil uses. Tall (1995) describes these arguments as follows:

- Enactive arguments: According to Tall, this argument is part of the most elementary level. It involves carrying out a physical action to demonstrate the truth of a given statement. He explains that the essential motivation for using this argument is the need for physical movement as well as visual and verbal support to show the required relationships.
- Visual (Enactive) Proof of Geometric Statements: Visual proof often involves enactive elements and usually has verbal support.
- Graphic Proof of Numeric and Algebraic Statements: The idea underlying certain arithmetical statements can be "seen" to be true by using visual configurations as prototypes in a generic way.
- Proof in arithmetic by specific and generic calculation: This involves computational activity and the checking of calculations but normally no long deductive logical chains.

- Algebraic proof by algebraic manipulation: Algebra has the capacity to express arithmetical ideas in a general notation and so has more scope for proof than generic arithmetic.
- Euclidean Proof as a verbal translation of generic visual proof: Euclidean proof is often seen as being a good starting point to develop the rigour required for logical proof. As proposed in the books of Euclid it seems to have the form of a major systematic theory. However, as encountered in school, individual proofs are almost always verbal translations of what is seen in a visual picture involving a certain geometric configuration.

Tall argues that the cognitive development of a notion of proof results from the different forms of representation, at various levels of sophistication, of which the learner is capable. Even at the formal level, the use of the single word "proof" disguises the fact that there are many different views of proof, which stem from different historical and cultural contexts (Tall, 1995).

## 5.4. Beliefs about mathematics

"Students' beliefs shape their behaviour in ways that have extraordinarily powerful (and often negative) consequences"
- Schoenfeld (1992, p. 359)

Since the 1970s, research on mathematical beliefs has become an important branch of research in mathematics education (e.g. Thompson, 1992; Törner, 1998). This research has shown that teachers' beliefs have influence on their teaching practices (Lerman, 1983). Similarly, teachers' different teaching philosophies have been shown to lead to different teaching practices in the mathematics classroom (Ernest, 1991). Beliefs about mathematics arguably cause pupils difficulties in solving mathematical tasks. Pehkonen (1995) explained that problem-solving competency depends not only on the pupils' mathematical knowledge and abilities, but also on their beliefs about mathematics. Beliefs may have a powerful impact on how children learn and use mathematics. Therefore, they may also form an obstacle to the effective learning of mathematics. Pupils who have rigid negative beliefs about mathematics and its learning easily become passive learners who emphasise remembering as opposed to understanding in their learning. Accordingly, beliefs about mathematics should be regarded as an important factor which has influence on mathematical problem solving.

### 5.4.1. Belief system

Beliefs have been investigated intensively in the last few years, but the theoretical concept of mathematical beliefs is still widely discussed. Green (1971) was one of the first to classify belief structures and belief systems.

The notion of belief systems can be used as a metaphor for examining and describing the make-up of an individual's beliefs (Green 1971). A belief system is a dynamic system which develops when individuals evaluate their beliefs against their own experiences.

Green (1971) identified three dimensions of belief systems, namely "quasi-logicalness", "psychological centrality", and "cluster structure", which relate not to the content of the beliefs themselves, but to the way in which these dimensions are interrelated within the system. These dimensions reflect firstly the notion that beliefs are not held in total independence of all other beliefs, secondly the degree of conviction with which beliefs are held, and lastly the notion that beliefs are held in "clusters" (Thompson, 1992).

- Quasi-logical structure, consisting of some *primary* beliefs and some *derivative* beliefs.

- Primary beliefs are those which would be held by a teacher who believes that it is important to present mathematics "clearly" to pupils.
- Derivative beliefs are those which would be held by a teacher who believes that it is important both to prepare lessons thoroughly, ensuring a clear, sequential presentation, and also to be prepared to answer readily any question posed by pupils.

- Psychological centrality which relates to the degree of conviction. This can range from central to peripheral:

- Central: convictions representing the most strongly held beliefs.
- Peripheral: convictions which are most susceptible to change or examination

Green noted that logical primacy and psychological centrality are independent dimensions, arguing that they are two different features or properties of a belief. "A belief may be logically derivative and yet be psychologically peripheral". The derivative belief in the importance of being prepared to answer pupil questions may be more important or psychologically central to the teacher for reasons of maintaining authority and credibility ("Teachers are supposed to know their stuff") than for clarifying the subject to pupils (Thompson, 1992).

- Cluster structure

The third of Green's dimensions concerns the claim that "beliefs are held in clusters, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs". This clustering prevents cross-fertilisation among clusters of beliefs as well as confrontations between them, and makes it possible to hold conflicting sets of beliefs. This clustering property may help to explain some of the inconsistencies among the beliefs professed by teachers and documented in several studies (Brown, 1985; Cooney, 1985; Thompson, 1982, 1984).

Green (1971) pointed out that beliefs always come in sets or groups, never incomplete independence of one another. The individual compares his beliefs with new experiences and with the beliefs of other individuals and his beliefs are subjected to continuous evaluation and undergo change. When an individual adopts a new belief, it will automatically form a part of the larger structure of his personal knowledge and of his belief system, since beliefs never fully develop independently (Green, 1971).

As mentioned earlier, there is no exact and common definition of mathematical beliefs. As a consequence of the vague definition of the concept, researchers have often argued on the basis of their own definitions, which were sometimes contradicted those of others. For example, Schoenfeld (1985) stated that "belief systems are one's mathematical world view". He later modified his definition, interpreting beliefs as an individual's understandings and feelings that shape the way he/she conceptualises and engages in mathematical behaviour (Schoenfeld, 1992). Hart (1989) used the word belief "to reflect certain types of judgments about a set of object". Törner and Grigutsch (1994) used the expression belief in the sense of a "mathematical world view (*Mathematische Weltbilder*)", agreeing with the definition of Schoenfeld (see Pehkonen, 1996). This paper will keep to the interpretation of Törner and Grigutsch, which will itself be presented.

### 5.4.2. Mathematical World Views (*Mathematische Weltbilder*)

The definition of beliefs is still disputed by researchers. The most commonly held view maintains first that it is the effort to separate beliefs from cognition. Schoenfeld (1985) pointed out that the purely cognitive components of his framework for the analysis of mathematical behaviour failed to predict the problem-solving processes of pupils accurately. Second is that there is what Green called the "quasi-logical structure of belief," namely the way in which the affective dimension of beliefs has influence on the role and meaning of each belief in the individual's belief system. Third problem is with the term "belief" which can also differ from scientific knowledge that can be expressed with logical sentences (Furinghetti & Pehkonen, 2002).

A person's *view of mathematics* is formed based on his experiences and observations that make him conclude statements on different phenomena and their nature (cf. Malinen 2000 cited in Pehkonen & Pietilä, 2003). Schoenfeld (1985) used the concept "mathematical world view". According to this, an individual's view of mathematics is a mixture of knowledge, beliefs, perceptions, attitudes, and feelings. It is the filter that regulates his thinking and actions in mathematics-related situations.

Törner and Grigutsch have characterised beliefs as follows:

"Attitude is a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (action) aspect." (Törner & Grigutsch, 1994, p.213).

This wide spectrum of beliefs around mathematics consists of at least four main components which are also relevant for mathematics teaching:

(1) Beliefs about mathematics;
(2) Beliefs about teaching mathematics;
(3) Beliefs about learning mathematics.
(4) Beliefs about ourselves as practitioners of mathematics (one's concept of oneself as a mathematics practitioner: a self-evaluation of one's abilities and the causes of one's individual success and failure)

These main groups of beliefs can in turn be divided into smaller units. It is evident that these "dimensions of beliefs" are interrelated (cf. Törner, 2002).

## 5.5. Empirical investigations on pupils' performance

This section will examine the pupils' performance on proving. As chronically, the researches will be addressed. In particular, some results of the researches in Korea and Germany will be introduced. However, those empirical investigations showed that the pupils' performance in mathematical reasoning and proof are relatively poor.

- Williams (1979)

The research of Williams made clear the reality that even highly achieving pupils were not aware of the distinctions between empirical and deductive proof. The subjects of Williams' investigation were eleventh-grade Canadian pupils whose achievements were amongst the best in their schools and who had studied using a modern high school curriculum designed to prepare them for the study of mathematics in college or university.

For one problem, 68% of the group believed that an empirical argument was a sufficient proof; only 6.4 % saw the need for a deductive proof. For another problem which was more familiar to the pupils, the results were similar. 54% of the sample accepted an empirical argument; only 14% of the sample accepted the deductive proof.

On a problem that was designed to assess pupils' understanding of the generality of the given proofs, Williams found that 20% of the pupils did not realise that the given deductive proof proved a relationship for all triangles. 31% seemed to accept the generality of the given proofs.

- Senk (1985, 1989)

Senk (1985) carried out a multi-school U.S. survey of high-school geometry pupils that showed most American pupils had difficulties in writing proofs down. The study results indicated that most pupils (about 70%) could prove simple geometry results requiring only a single additional step, indicating that they had mastered at least the basic concept and the syntax of proofs. Successful performance on longer proofs was much rare. Even at the end of a year of teaching, barely 20% of the pupils could construct proofs with great complexity. Senk also found that proof-writing achievement was strongly dependent on the school itself and that 30% were proficient in proof writing for those problems which were similar to the ones in the textbook.

Also Senk (1989) has conducted an investigation into students' ability to write geometry proofs based on the model of van Hiele levels. Pupils took tests both before and after a course in geometry. Senk gave tests at the beginning of a high school geometry course to determine students' van Hiele level and background knowledge. Then, at the end of the year, the students were tested again, this time on van Hiele level, geometry knowledge, and their ability to write proofs. Senk also found that many students enter high school geometry with a low van Hiele level of understanding, and suggests that this may be why geometry gives high school students so much difficulty.

- Vinner (1983)

Vinner (1983) focused on the following question: what makes a given sequence of correct mathematical arguments a mathematical proof in the eyes of high school pupil? He asked pupils to give their preference for proving a particular case of a previously proved statement. He found they preferred using a particular case of the deductive proof to using the general result. The general proof was viewed as a method to examine and to verify a particular case. Vinner further observed that pupils judged a mathematical proof according to how formal its layout was.

- Chazan (1993)

Chazan (1993) carried out in-depth clinical interviews, in which empirical evidence was employed, with seventeen high-school pupils participating in geometry classes. He found out that pupils tended to see empirical evidence as sufficient for a proof and mathematical proof simply as evidence.

Five of the seventeen pupils thought evidence could amount to a proof and the other pupils argued that it could not constitute a proof, giving the following reasons: there may be counterexamples, every example is a special case and measurement is not exact.

Some pupils believed that their views of proof are just evidence since they thought that counterexamples to deductive proofs could be found.

Chazan suggested that in geometry classes it might be valuable to begin with explanatory proofs as Hanna (1989) pointed out.

- Healy and Holyes (1998)

Healy and Hoyles (1998) made a significant contribution to the field with their recent systematic investigation of pupils' understanding of proofs, ability to construct proofs and views on the role of proof. Their investigation with 2459 high-attaining tenth grade pupils in England showed that even these pupils had great difficulties in generating proofs. The empirical investigations showed that the pupils' abilities in mathematical reasoning and proof were rather poor. Pupils were not proficient in constructing mathematical proofs, however, more pupils were able to identify a correct proof than were able to write one by themselves. Similarly to the findings of Williams, pupils were more likely to rely on empirical verification. However, most of them were well aware that once a statement had been proved, it held for all cases within its domain of validity.

In multiple-choice questions concerning various given arguments (empirical, exhaustive, visual, narrative, and formal), pupils were asked to identify both their own favourites and the choices which they thought would get the best mark. From these questions it was clear that the pupils believed that a formal presentation of a proof would receive the best mark. However, they were significantly more likely to select empirical arguments for their own approach.

Healy and Holyes indicated that even though many pupils came to value general and explanatory arguments through these investigative activities, the explanatory arguments did not help pupils to mentally construct a mathematical proof and successfully set out mathematical arguments in a logical manner.

### 5.5.1. Research in Germany

Reiss, Klieme & Heinze (2001) investigated the performance of 13th grade pupils on proof. 81 pupils took part in the study, of which 59 attended a regular mathematics course and 22 an advanced course. They were asked to solve TIMSS geometry problems and took additional tests in areas such as meta-cognitive assessment, declarative knowledge, understanding of proof, and spatial reasoning.

Methodological knowledge was assessed using problems from Healy and Hoyles' proof questionnaire (Healy and Hoyles, 1998). The problem dealt with the question of whether a given triangle could be proved to be isosceles. Pupils were presented with a correct formal proof, a correct narrative proof, and two incorrect arguments. They were then asked to assess the correctness and generality of each of the four arguments.

As a result, 53% of German pupils were able to construct correct proofs using Euclidean geometry; however, they found that judging given proofs is much easier than constructing their own proofs. 57% of the pupils recognised the correct formal proof to be correct, and 57% of pupils correctly appreciated its generality.

- Heinze & Reiss (2003) – Interview study

Reiss conducted the next study of interest in the form of a series of interviews. The sample consisted of 26 pupils at the end of their upper secondary education, in the case of the German pupils, in grade 13. Pupils solved the problems in individual interviews. They were asked to verbalise their problem-solving steps and were further encouraged if they were quiet for a short time, but the interviewer did not interrupt their problem-solving processes. Only after they told the interviewer that they were finished, he asked specific questions concerning errors he had observed during their work on the problem. All sessions were videotaped and transcriptions were made of each session.

The data gathered reveal that pupils are barely aware that mathematical reasoning is not guided by plausibility arguments. Moreover, they are not able to examine the given situation and discuss arguments that might possibly lead to a solution. The findings suggest that pupils' difficulties in mathematics may stem from the inadequate form of mathematical argumentation they use. Heinze and Reiss (2003) was able to show evidence for the proposition that pupils' difficulties in mathematical problems cannot be attributed to a lack of basic and declarative knowledge but rather to a lack of methodological knowledge. The constraints of mathematical argumentation and in particular mathematical proof are unfamiliar to pupils and they obviously feel uncomfortable during mathematical problem-solving activities which ask for argumentation and proof.

Heinze and Reiss (2003) also found that pupils at the end of upper secondary level have insufficient methodological knowledge about proof which then results in problems with judging proofs. They describe three aspects of methodological knowledge about proof: the proof scheme, the proof structure, and the logical chain, which they consider to be important components of competence in constructing proof. Their empirical data support their claim that all three aspects of methodological knowledge are important when pupils judge the validity of other proofs.

### 5.5.2. Research in Korea

Na (1996) has investigated potential ways of improving proof teaching by analysing the nature of proof and the practice of proof teaching. Her results can be summed up as follows:

- First of all, proof in Korean mathematics class is taught in a synthetic manner. Na argued that this makes Korean pupils accept proof as something to memorise, not as something to explore.
- Second, Korean mathematics teachers usually use proof as a means of justification. She argued that the use of proof only as a means of justification has advantages and limitations for pupils trying to understand the nature of proofs. It makes more difficult for pupils to learn proofs.
- Third, she argued that the manner in which pupils are taught to construct proofs in Korea is too hurried, which means that having started to learn about proof, it is usually a long time before pupils begin to learn the basic concepts required to prove statements.

She gives suggestions on what teachers should do to improve their proof teaching. First the teachers should teach proof as a dynamic reasoning process unifying analytic thought and synthetic thought, and avoiding the current mistake of seeing proof as a unilaterally synthetic method. Moreover, they should make pupils consider and fulfil the needs of proof in a natural way, for example by providing them with the contexts of both justification and rediscovery (conjecture) simultaneously. Therefore, various aspects of proof such as systematisation and conviction should be introduced in mathematics class.

Seo (1998) analysed and examined the constituents of proof which are used in $8^{th}$ and $9^{th}$ grades and tried to find reasons why pupils have trouble learning to prove mathematical properties. He conducted examinations, an analysis of the textbooks used, and interviews. His results can be summarised as follows:

- Most pupils do not distinguish a definition from a property very well, so they have difficulties in using symbols, in distinguishing an assumption from a conclusion and in using the appropriate diagrams. The basic symbols and principles are taught in the first year of middle school and pupils use them to prove results after about one year. This could be a reason why pupils often do not know the exact names of principles and cannot apply them correctly.
- Textbooks do not deal with counterexamples clearly enough, but they contain some problems that are solved only by using counterexamples. Some pupils realise that a single counterexample is sufficient to disprove a false proposition, but they prefer not to use one.
- Most constituents relating to the meaning of proof are only mentioned briefly in textbooks if at all, and many pupils therefore do not have an exact understanding of them. In particular, the pupils accept the results of experiments or measurements as proof and prefer these to the logical proofs given in textbooks.

Park (1999) carried out a study on 126 $9^{th}$ grade pupils with the following two aims. The first was to investigate a process of mathematical proof. The second was to investigate some types of mistakes that middle school pupils make in the processes by which they form a mathematical proof. Three classes of middle school pupils took part in a test and were interviewed. The results of the test and the interviews can be summed up as follows.

- Most pupils had difficulty in understanding the meaning of mathematical proofs. They did not distinguish between empirical proof and proof using a counterexample. Moreover, they did not realise that once a mathematical statement or proposition was proved, it was a truth without exception. This caused difficulties when the pupils tried to apply theorems to solve problems and for particular applications.
- They could not utilise useful mathematical concepts in the process of problem-solving, even though they knew some of them.
- Their skills in mathematical symbolisation were limited. Some of them could explain certain mathematical concepts in a normal sentence, that is, in a narrative way, and yet using the appropriate symbols.

- Their strategies for mathematical proofs were very limited; the pupils were familiar with few strategies and used these strategies repeatedly. When asked to write down their thought processes for some mathematical proofs, they successfully recognised prerequisites for the statements which were to be proved and drew the appropriate drawings; however, pupils lacked the skills or strategies to keep to the structured process of mathematical proof. They failed to arrive at an algebraic formula or to apply a certain concept.

Park concluded by recommending the following two things. First, it was necessary to teach pupils to apply mathematical concepts proficiently and represent them as mathematical symbols. Second, the process of a mathematical proof was a kind of problem solving; hence it was essential to teach pupils to diverse strategies. Otherwise, pupils will be only able to solve a limited range of problems.

## 5.6. Empirical investigation on Mathematical world view (beliefs)

This section will examine the researches on pupils' beliefs about mathematics. Most researches concerned teachers' beliefs about mathematics and their teaching practices. Moreover, it is shown that teachers' different teaching philosophies lead to different teaching practices in classrooms (Lerman, 1983; Ernest, 1991; Thompson, 1992). Numerous studies were conducted which gave an overview about beliefs or conception about mathematics (cf. Pehknon & Törner, 1999). Hence it will be focused on only the German research on 'mathematical world view'. In addition, it will be introduced some researches on beliefs and attitude in Korea.

The Third International Mathematics and Science Study (TIMSS) (Beaton, et al., 1996), revealed that most teachers believe mathematics is essentially a vehicle to model the real world, that ability in mathematics is innate, and that more than one representation should be used in explaining a mathematical concept. With respect to the emphasis on drill and repetitive practice, teachers around the world did not show a consistent response (in Handal, 2003).

### 5.6.1. Research of mathematical world view (beliefs) in Germany

As mentioned earlier, Törner and Grigutsch have contributed significantly to empirical research on mathematical beliefs. They started a large survey about beliefs with more than 300 German secondary school teachers in 1994. They concentrated on the various attitudes towards mathematics, focusing on four aspects of a 'mathematical world view'. These aspects may be classified as schema, formalism, process, and application. Ten problems, each requiring a methodological and statistical approach, were assigned to each of these four aspects. In this study, they attempted to investigate belief system structures using factoral analysis and they gave a structure of teachers' belief systems defined by the significant partial correlation (Grigutsch, Raatz & Törner, 1995). They also asked same questionnaire to 119 university mathematics teachers. They found five aspects. Four to five aspects were defined by means of factor analysis and subsequently verified as relevant aspects of the view of mathematics. Four aspects are same as above, the formalism aspect, the schema aspect, the process aspect, the application aspect, and the last one can be called the Platonism aspect of mathematics (Grigutsch & Törner, 1998). Attitudes towards these aspects differ on average, so that the 'average' view of mathematics of university teachers is clearly accentuated in these five aspects. The process aspect acclaims the highest agreement, whereby the aspects formalism and application claim an average to above average assessment. In contrast, the Platonic aspect receives only weak to very weak agreement, and the schema aspect is on the whole rejected (Grigutsch & Törner, 1998).

### 5.6.2. Research on beliefs and attitudes in Korea

Nam (1999) investigated the relationship between teachers and pupils with regards to their beliefs and attitudes on mathematics. 129 teachers and their 1290 pupils from 40 middle and high schools participated in his research. The questionnaires were analysed by calculating the variance of the teachers and the pupils and applying the T-test for the characteristics of the teachers and the pupils. In addition, correlation analysis was used to investigate the relationship between the views of the teachers and those of their pupils.

One of the more significant results regarded those teachers who believed that mathematics is a unified body of knowledge and a static immutable product. Their pupils tended more towards the belief that mathematics is mostly facts and procedures that have to be memorised. Moreover, Nam (1999) found that the higher the teachers' scores for beliefs and attitudes as regards the teaching of mathematics, the higher their pupils' scores for beliefs and attitudes as regards the learning of mathematics. One might conclude that teachers' beliefs and attitudes influenced pupils to have similar beliefs and attitudes.

## 5.7. Discussion

In the classroom, the role of proving in discovery, communication and explanation should be emphasised to a greater extent than the role of proving in verification and systematization.

As the concepts and views of mathematics have changed, so have the views of mathematical proof, from a means of verification to justify the truth of mathematical proposition in absolutism, through the discovery involved in quasi-empiricism, to a means of conviction and understanding in social constructivism. Nowadays, social constructivism is dominant in the philosophy of mathematics; therefore, mathematical proof has come to be viewed as a social practice and to be analysed for its intra- and interpersonal significance in mathematics education.

The proof schemes were summarised by reviewing the literature setting out the aspects of formality in classifying proofs. Lakatos, Balacheff, Blum, and Kirsch present a classification scheme for proofs. The ideas 'quasi-empirical proof' by Lakatos, the 'generic example' by Balacheff or 'preformal proof' by Blum and Kirsch should be emphasised for pupils to learn and understand proving.

From empirical researches, including German and Korean investigations, it is clear that most pupils struggle to construct proofs. Heinze and Reiss (2003) showed evidence for the proposition that pupils' difficulties in mathematical problems cannot be attributed to a lack of basic and declarative knowledge but to a lack of methodological knowledge. Heinz and Reiss (2003) described three aspects of methodological knowledge about proof: the proof scheme, the proof structure, and a logical chain, which they considered to be important components of proof competence. By analysing empirical data they claimed that all three aspects of methodological knowledge are important when pupils judge the validity of proofs. As Na (1996) showed, Korean pupils accept a proof as something to memorise, not as something to explore. Moreover, most of the Korean pupils had difficulties in understanding the meaning of mathematical proof (Park, 1999).

# Chapter 6 Research Aims and Methodology

In the previous four chapters I have discussed the educational system and curriculum of Korea and Germany, the performance of these two countries' pupils in the international achievement tests and a number of philosophical and educational issues concerning the nature of mathematical proof from Platonism to the social theory of proof. This theoretical framework will underpin a comparison of Korean and German pupils' performances on proofs and into how pupils go about proving statements.

This chapter describes my primary research aims and the specifics of this research, including details on the sample and the methods of data collection.

This research consists of a quantitative research study and an interview study. A quantitative research design allows flexibility in the treatment of data, in terms of comparative analyses, statistical analyses, and repeatability of data collection in order to verify reliability. However, it has also the disadvantage of failing to provide any further explanation or analysis beyond description. Interview research is therefore conducted in order to provide deeper explanations using the think-aloud method.

## 6.1. Research aims

Korean pupils officially learn a great deal in their school life, yet their knowledge of topics they have covered is limited and even pupils who achieve well in mathematics do not know why they should spend time studying it. Many Korean educators are worrying this. Nevertheless, although mathematics is viewed as one of the most difficult subjects taught in Korean schools, it is also thought to be one of the most important.

In school mathematics in Korea, proof is one of the main methods used to develop deductive reasoning ability and assist the understanding of mathematics. Proof therefore plays an important role in geometry classes in lower secondary school, and is taught at a steady pace over the pupils' time at this school. As TIMSS and other researches (e.g., Senk, 1985; Healy & Hoyles, 1998; Reiss & Thomas 2001) have shown, pupils have difficulties in proving the problems even if they have the necessary specific knowledge.

In the international comparisons of mathematics achievement such as the TIMSS and PISA studies, Korean and Japanese pupils outperformed their Western counterparts. However, the exact reasons why Asian pupils performed better are unclear. Of course, there had been some investigations, for example the TIMSS video study, in which the mathematics instruction in Japan, Germany and in the United States were compared. This study revealed important differences in teaching styles between Japan on the one hand and Germany and the United States on the other. Different teaching styles depend on a number of teacher-related and pupil-related cognitive and non-cognitive variables. The influence of mathematical beliefs on different instructional patterns, for example, is intensively discussed. This paper aims at finding and investigating more reasons for the difference in performance mentioned above.

It is hoped my research will help to explain the pupils' competencies in proof and argumentation, differences and similarities in the proof performances of the pupils between the two countries and give a look at the two educational systems. Moreover, it is hoped my research contributes to developments in the teaching of proofs and argumentation as well as in the two educational systems.

This paper aims to describe the individual cognitive and non-cognitive variables of competence in proof and argumentation and then to go into greater detail with these descriptions and interpret them more deeply in the interview study.

### 6.1.1. Quantitative research

This quantitative research concerns the competencies that pupils have with respect to proving, and how they perform proofs. The aims are to analyse the factors (e.g. basic knowledge, methodological knowledge, etc.) which could have influence on geometrical competence and to identify aspects of geometrical competence of lower secondary pupils in Korea and Germany who are starting to learn to reason and prove mathematically. So the focus is not on high-attaining pupils but on pupils in grade 7 and 8. To be precise, the aims of the research are:

- To analyse how pupils construct proofs
- To investigate pupils' judgments of proofs
- To compare Korean and German pupils on these two points

### 6.1.2. Interview research

Interview research aims to find the reasons for the results of the quantitative research and to obtain further results. The aims are as follows:

- To analyse the reasons behind pupils' construction of proofs
- To investigate the reasons behind pupils' judgments of proofs
- To analyse the appreciation of proofs in mathematics learning
- To compare Korean and German pupils on these three points

## 6.2. Research design

Here it is explicitly explained the samples and structures of each research.

### 6.2.1. Quantitative research

The 7$^{th}$ grade German pupils were tested at first in June 2001. The same pupils then took a different test between December 2001 and February 2002, but by this time the pupils had moved up to the 8$^{th}$ grade. The German schools were located in Oldenburg, Niedersachsen (Northern Germany).

By contrast, due to yearly changes in the Korean classes, the Korean 7$^{th}$ grade sample was not the same as the 8$^{th}$ grade sample. So in December 2001, the 7$^{th}$ grade took an achievement test and answered a questionnaire on beliefs, whereas the 8$^{th}$ grade took an achievement test. The Korean schools were located in Chonju, in the Southwestern of South Korea.

- Sample and administration

The data of 659 German 7$^{th}$ grade pupils (8$^{th}$ grade: 528) in 27 classes and 189 Korean 7$^{th}$ grade pupils (8$^{th}$ grade: 182) in 5 classes were collected. These data partly related to competency in proof and argumentation on geometry questions and partly were taken from the completed questionnaires on beliefs about mathematics. In addition, 22 German teachers and 58 Korean teachers answered the same questionnaire on beliefs about mathematics. The German data is from the BIQUA Project on reasoning and proof in the geometry classroom which was headed by Prof. Reiss and funded by DFG (cf. Reiss, Hellmich & Thomas, 2001).

- Structure

The written test consisted of ten questions on angles, triangles, and quadrilaterals and was designed to be completed in forty-five minutes. The structure of the proof achievement test for the 7th grade was as follows:

- Four open questions in which pupils had to construct direct proofs with familiar mathematical content
- Three open questions, in which pupils had to calculate a certain angle
- Two open questions and one multiple-choice question on symmetry
- One question, in which pupils had to evaluate given proofs. Two proofs were correct and two were incorrect. A variety of forms were used for these proofs– empirical, circular, narrative and formal.

In addition to the pupils' proof achievement test, a belief questionnaire was designed to obtain information about mathematical views. The factors were application, formalism, and process:

- Application: Mathematics has a practical use and mathematical knowledge is important for later life.
- Formalism: Mathematics can be characterised by strictly logical and precise thinking in an exactly defined subject.
- Process: Mathematics is characterised as a process, i.e. an activity whereby problems arose and understood. This is a process of discovery concerning the creation, invention, and reinvention of mathematics.

Similarly to that of the 7th grade test, the structure of the proof achievement test for the 8th grade was as follows:

- Five open questions, in which pupils had to construct direct proofs on familiar mathematical content
- Two open questions, in which pupils had to calculate a certain angle
- One open question and one multiple-choice question on symmetry

### 6.2.2. Interview study

The Korean 8th grade pupils were interviewed between January and February 2003; as for the German 8th grade pupils, their interviews took place in Bavaria in May 2003.

Pupils were interviewed individually in their school for about an hour to 90 minutes. Each interview was videotaped.

- Sample and administration

Participation in this research worked on a relatively voluntary basis; however, the different levels of achievement of the pupils were taken into account. So of the 15 Korean 8th pupils (7 girls and 8 boys) who participated between January and February 2003, five pupils were assigned to the upper, five to the middle, and five to the lower achievement group.

However, 18 of the German pupils (9 girls and 9 boys) did not participate in the quantitative test, therefore, all pupils participated on a voluntary basis and some of them were selected according to their previous marks for mathematics by their teacher.

- Structure

The interview consisted of three parts: proving problems, evaluating given proofs, and a discussion on the pupil's beliefs about proof.

In the first part, pupils were asked to prove 6 geometry problems and while doing so, to comment on all their activities (known as the "think-aloud method"). For each problem, they were asked afterwards whether they thought their proof was correct and where they had encountered difficulties. Pupils were then asked which problem they found to be the easiest and which they found to be the most difficult.

In the second part of the interview, pupils were asked to evaluate four different given 'solutions' to a proving question: one empirical solution, one incorrect version with a circular argument, and two correct proofs, one in a formal and the other in a narrative style. They had to give their own assessment of each proof and say whether it was a correct proof or not. If it wasn't, they were to point out where the fault lay; if it was, they had to say whether the proof was universally valid or not. Finally, they were asked to say which one was the clearest for an explanation of the proof to other pupil, which one would get the best mark from a teacher, and which one was most like the approach they would have used.

In the last part of the interview, they were asked about their understanding of proof; for example, what a proof is for, to what extent they were interested in proof, its importance in school, and so on.

## 6.3. Data analysis

With SPSS program, the descriptive statistics and the frequency distributions of the scores from the achievement test are reported in chapter 7. In chapter 8, the strategies and examples that pupils used in their proofs are summarised and analysed.

## 6.4. Think-aloud method

The think-aloud method has its origins in psychological research (cf. Teger & Biedenkamp 1983). It developed from the earlier introspection method. The introspection method is based on the idea that one can more or less observe events that take place in consciousness by observing events in the outside world (van Someren, Barnard &Sandberg, 1994). The introspection method has theoretical and methodological problems, but some successful research has been carried out using this method.

The quality of the "Think-aloud method" as an instrument for investigation was examined experimentally by Deffner (1984). His investigation suggested that the "Think-aloud method" is particularly successful for exercises which are given orally and which require systematic working in several sequential steps. This method has its faults, but it can provide a rich source of knowledge for making judgements about the knowledge subjects using at a given time. Currently the think-aloud method is accepted as a useful one by many leading psychologists and it also has an important place in the repertoire of many knowledge engineers.

## 6.5. Limitations

Most of the results are quantitative and relied on results obtained from an achievement test and questionnaires on beliefs. I myself translated the questionnaires from German into Korean, and tried to do this as accurately as possible. The major limitation of this study was the sample size. The Korean sample consisted only of five classes from 4 different schools located in a small city in Southwestern Korea. In the German case, the samples were larger, however, those schools are located also a small city and its surroundings. There might therefore be a small but not a negligible difference between the socio-economic statuses of the respective areas and the respective educational facilities.

The German data from the BIQUA project headed by Reiss is actually pre-test and post-test for the same sample. I, therefore, wanted my research to have the same format. However, the times at which the questionnaires should be answered were a problem. In Germany, the school year starts at the beginning of September and usually ends the following July, whereas the school year in Korea starts in March and ends the following February. Also, the 7th graders and the 8th graders in Korea learn geometry at the end of the school year (normally November to February). Due to yearly changes in the Korean classes, it was impossible to make pre-test and post-test with the same sample. However, these different research designs of the two countries do not have disadvantage in this thesis, because it is compared the achievement and beliefs of only same graders between Korea and Germany. It is not a longitudinal study. It was not concerned with process that pupils change their achievement while they change the graders. It would be better to have the same design; however, it was not necessary.

# Chapter 7 Results of Achievement Test

This chapter is devoted to describing the findings of the present study. I will introduce the results of pupils' proof performance. In section 7.1., the descriptive results for the achievement test are presented. In section 7.2., pupils' methodological competence is presented. In section 7.3., teachers' beliefs and pupils' beliefs about mathematics are firstly introduced and then the relationship between achievement test and beliefs and the relationship between methodological competence and beliefs are presented.

As mentioned in chapter 6, 189 Korean and 659 German 7$^{th}$ grade pupils and 182 Korean and 528 German 8$^{th}$ grade pupils took part in the study.

| | Korea | Germany |
|---|---|---|
| 7$^{th}$ grade pupils | 189 | 659 |
| 8$^{th}$ grade pupils | 182 | 528 |
| Teachers | 58 | 22 |

*Table 7-1 Numbers of samples*

This German data is from the BIQUA Project, headed by Prof. Reiss and funded by DFG, on reasoning and proof in the geometry classroom (cf. Reiss, Hellmich & Thomas, 2001).

The competence levels, one of the more important results will be firstly given. Before that, an overview of the various scales will be given, the number of problems in each, and the most important distributional parameters which are all shown in the table 7-2.

| Scale | Number of problems | Mean | | SD | |
|---|---|---|---|---|---|
| | | Korea | Germany | Korea | Germany |
| 7$^{th}$ grade Test | 13 | .50 | .49 | .22 | .20 |
| Basic competence | 6 | .47 | .62 | .19 | .21 |
| Competence in proofs | 7 | .53 | .38 | .30 | .27 |
| 8$^{th}$ grade Test | 13 | .40 | .41 | .27 | .17 |
| Basic competence | 6 | .41 | .54 | .27 | .23 |
| Competence of proving | 7 | .40 | .30 | .30 | .18 |
| Methodological competence | 8 | .61 | .51 | .22 | .21 |

*Table 7-2 Distributional parameters and statistical frequency data for the Korean and the German pupils (normed score)*

Table 7-2 shows that in the achievement test of the 7$^{th}$ grade, the performance of the Korean pupils was a little bit better than that of the German pupils. In contrast to the results of the 7$^{th}$ grade, in the achievement test of the 8$^{th}$ grade, the German pupils performed better than the Korean pupils even if the scores were almost the same.

As the next step, the independent t-test was applied, in order to determine whether there was a significant difference between the Korean and the German pupils with respect to the achievement test.

| | | M | SD | p |
|---|---|---|---|---|
| 7$^{th}$ grade achievement test | Korea | .50 | .22 | .275 |
| | Germany | .49 | .20 | |
| 8$^{th}$ grade achievement test | Korea | .40 | .27 | .821 |
| | Germany | .41 | .17 | |

*Table 7-3 T-test of the 7$^{th}$ and the 8$^{th}$ grade total achievement test (normed score)*

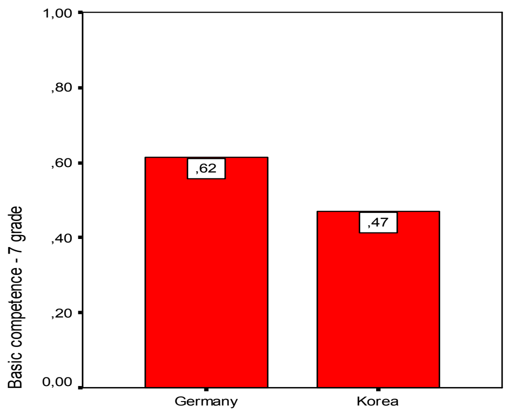
This T-test shows that there was no significant difference between the Korean group and the German group averages with respect to the 7th grade test (p=.275). Moreover, there was no significant difference between the Korean group and the German group averages with respect to the 8th grade test (p=.821).

In the following, we turn to basic competence and competence in proof and argumentation of pupils. The problems in this study are classified as "basic competence" and "competence in proof and argumentation" in our competence model.

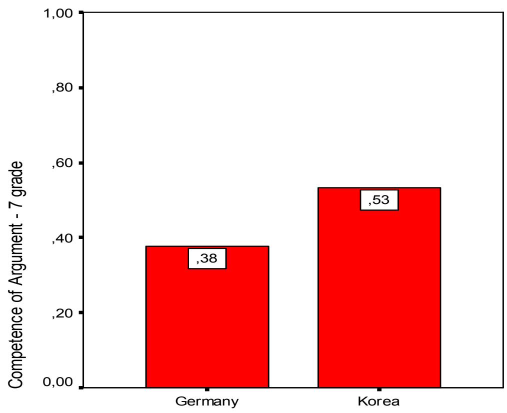
Basic competence can be understood as the general understanding of elementary definitions and concepts and the aptitude for elementary calculation that pupils should develop in class. The problems concern simple calculation of angles and identification of symmetrical figures.

Competence in proof and argumentation can be described as identifying and applying the basic concepts and justifying the statement with creative ideas and logical argumentation. The problems concern the concept of angles and its application in proofs and reasoning.

From now on, the normalised scores are used. First a comparison of the two groups in basic competence and competence in proof and argumentation are presented. The following are the results of the 7th grade pupils.




*Graph 7-1 Normed score for basic competence for the 7th grade pupils*




*Graph 7-2 Normed score for competence in proof for the 7th grade pupils*

As we can see from the graphs, the German pupils performed quite well in terms of basic competence, while the Korean pupils performed better in terms of competence in proof and argumentation than the German pupils.

The independent t-test was applied, in order to determine whether there was a significant difference between the Korean and the German 7th graders with respect to the basic competence and competence in proof and argumentation.

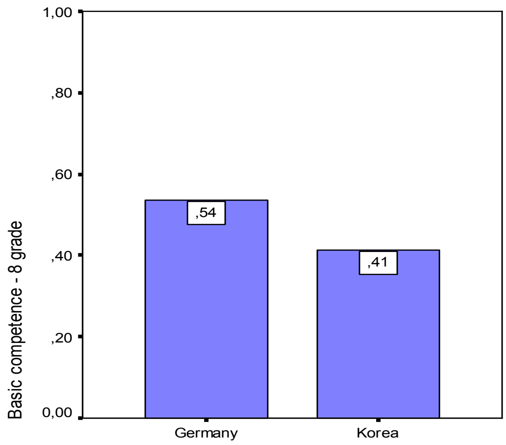
The next table shows the significant difference between the Korean and the German 7th graders with respect to basic competence and competence in proof and argumentation.

| | | M | SD | p |
|---|---|---|---|---|
| Basic competence | Korea | .47 | .19 | .000 |
| | Germany | .62 | .21 | |
| Competence in proof and argumentation | Korea | .53 | .30 | .000 |
| | Germany | .38 | .27 | |

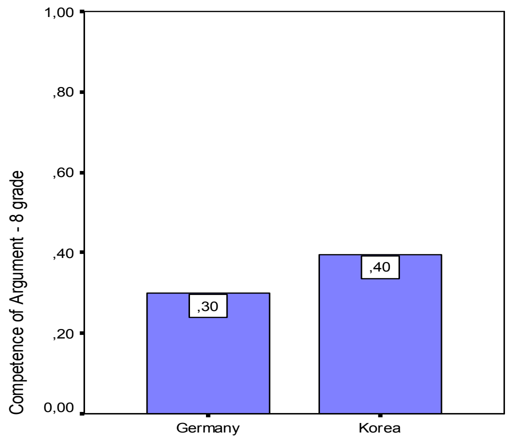
*Table 7-4 T-test for basic competence and competence in proof and argumentation for the 7th grade (normed score)*

While no statistically significant differences emerged for the total achievement test for the 7th grade as seen from table 7-3, however, as regards basic competence and competence in proof and argumentation, respectively, significant differences between the Korean and the German 7th pupils are here observed (p<.001).

The results for the 8th grade pupils are presented in the graphs below:




*Graph 7-3 Normed score for basic competence for the 8th grade pupils*




*Graph 7-4 Normed score for competence in proof for the 8th grade pupils*

The same phenomenon is revealed as for the 7th graders i.e., the German pupils showed better performance on the problems in basic competence, while the Korean pupils performed better on the problems as regards competence in proof and argumentation. However, the score for the 8th grade pupils is lower than that of the 7th grade pupils in both countries.

The next table shows the significant difference between the Korean and the German 8th graders with respect to basic competence and competence in proof and argumentation.

| | | M | SD | p |
|---|---|---|---|---|
| Basic competence | Korea | .41 | .27 | .000 |
| | Germany | .54 | .23 | |
| Competence in proof and argumentation | Korea | .40 | .30 | .000 |
| | Germany | .30 | .18 | |

*Table 7-5 T-test for basic competence and competence in proof and argumentation for the 8th grade (normed score)*

The significant differences between the two groups with respect to basic competence and competence in proof and argumentation are observed, as for the 7th grade test.

- ***Competence model***

In constructing the test Klieme's (2000) model of mathematical competencies was applied. With respect to an achievement test on basic competence and competence in proof and argumentation about geometry, three competence levels were identified. The three competence levels are defined as follows:

- *Competence Level I: Simply use of rules and elementary reason*

In the first competence level, the application of concepts and rules and elementary reasoning (i.e., procedural knowledge) are foremost. Pupils should be able to master simple arithmetic operations and use elementary basic knowledge. Formal mathematical arguing is not required in this level.

- *Competence Level II: Proving and justifying (one-step)*

Certain aspects of declarative knowledge are required for the second competence level. Pupils should reason correctly to solve the given geometrical problems and write down their reasoning appropriately. A flexible application of the concepts and factual knowledge associated with geometry of an intermediate level of difficulty may be used by some pupils.

- *Competence Level III: Proving and justifying (several steps)*

The third competence level is characterised by original and partially creative problem solving, arguing and justifying. On this level, there is a need for the ability to adequately notate the arguments and reasons for a step in a proof as well as the ability to form a meaningful chain of several arguments.

For the achievement test, we classified pupils as belonging to a lower, a middle, or an upper group as regards their achievement and compared their performance with respect to the competence levels.

The following table shows the relationship between competence levels and the 7th grade pupils' performance. The data show that the results for each competence level conform to the grouping of the pupils into an upper, a middle and a lower achievement group.

| | Competence level I Elementary problems | | Competence level II Argument and proof (one step) | | Competence level III Argument & Proof (more steps) | |
|---|---|---|---|---|---|---|
| | Korea | Germany | Korea | Germany | Korea | Germany |
| Mean | 0.47 | 0.69 | 0.71 | 0.56 | 0.40 | 0.24 |
| Lower group | 0.30 | 0. 51 | 0.34 | 0.22 | 0.06 | 0.00 |
| Middle group | 0.47 | 0. 72 | 0.83 | 0.61 | 0.37 | 0.18 |
| Upper group | 0.64 | 0. 85 | 0.93 | 0.89 | 0.76 | 0.50 |

*Table 7-6 Percentage of correct solution for the 7th grade*

In this table, the numbers stand for the percentage of correct responses to all problems in the corresponding level. This table shows that for competence level III, the lower group of the German pupils did not have any correct solutions, while few of the Korean pupils in the corresponding group had correct solutions of tasks for the same competence level. As we move across and down the table, gradual increases from this extreme are evident. This can be regarded as an internal validation of the test (Reiss, Hellmich & Reiss, 2002).

An interesting result is that the middle and upper groups of the Korean 7th grade pupils performed better in competence level II than in competence level I. As for the German pupils, only the upper group performed better in competence level II than in competence level I.

The next table shows the 8th grade pupils' performance with respect to the competence model.

| | Competence level I Elementary problems | | Competence level II Argument and proof (one step) | | Competence level III Argument & Proof (more steps) | |
|---|---|---|---|---|---|---|
| | Korea | Germany | Korea | Germany | Korea | Germany |
| mean | 0.41 | 0.54 | 0.46 | 0.41 | 0.31 | 0.15 |
| Lower group | 0.12 | 0.30 | 0.14 | 0.27 | 0.02 | 0.05 |
| Middle group | 0.40 | 0.57 | 0.46 | 0.37 | 0.22 | 0.10 |
| Upper group | 0.71 | 0.74 | 0.77 | 0.59 | 0.69 | 0.30 |

*Table 7-7 Percentage of correct solution for the 8th grade*

As was the case for the 7th grade pupils, the middle and upper group of the Korean 8th grade pupils performed better in competence level II than in competence level I. For competence level III, the lower group of the German performed slightly better than that of the Korean group in all three competence levels which was not revealed in competence model of the 7th grade.

The middle and upper groups of the Korean 8th grade pupils like their 7th grade counterparts performed better in competence level II than in competence level I. However, for the higher German group this was no longer the case.

- **Discussion**

To sum up, the German pupils in both grades performed significantly better on problems requiring basic competence than the Korean pupils. On the other hand, the Korean 7th and 8th grade pupils performed significantly better on argumentation and proofs problems. However, the results show that pupils in both countries have difficulties in proving and justifying.

As mentioned above, one of the most interesting results is that the 7th grade pupils who belong to the higher group in both countries have better performance in competence level II than competence level I. Moreover, the Korean 7th and 8th grade pupils who belong to the middle group also have better performance in competence level II than competence level I. However, for the German 8th grade pupils, this phenomenon is not apparent. Similarly, the Korean 8th grade pupils who belong to the higher group also performed better in competence level II than competence level I. In both countries, the performance of the 7th grader is better than that of the 8th grader.

In the next section, we will look at pupils' performance in more detail.

## 7.1. Descriptive results of achievement test for the 7th and the 8th grade pupils

This section lays out selected descriptive results and gives more detailed examples of pupils' answers. The remaining results and examples can be found in the Appendix B. Some symbols are used differently in Korean and German test. As the test was translated into Korean, for better understanding, it has to be changed Greek symbols to English symbols such as $\alpha$ to *a*, $\beta$ to *b*, and $\delta$ to *d*. These Greek symbols are not used frequently in Korean middle schools.

### 7.1.1. Achievement Test for the 7th graders

- **Basic competence**

The problems in "basis competence" in the 7th grade have two characteristics. Firstly, there were three problems which are related to symmetric concepts. They are taught in the 5th grade in Korea, but in contrast to Germany, they are repeated in the 7th or the 8th grade. Secondly, there were also the geometry problems which were simply calculated, however, they were half scored if pupils did not give any argumentation. In actual fact, in the test of the 7th grade, pupils are only asked to give an answer to calculation problems. However, the results revealed that pupils who gave the reasons for calculation problems had preformed better on problems concerning proofs and argumentation. It was therefore necessary to take these phenomena into consideration and only give half-marks.

The table below shows the pupils' average for all problems as regards basic competence and the *p*-value for determining the significant difference between the Korean and the German 7th graders for all problems.

| | | | M | SD | $p$-value |
|---|---|---|---|---|---|
| Symmetry | P1 | Korea | .33 | .38 | .000 |
| | | Germany | .73 | .44 | |
| | P2 | Korea | .18 | .28 | .002 |
| | | Germany | .28 | .40 | |
| | P8 | Korea | .56 | .36 | .714 |
| | | Germany | .61 | .35 | |
| Calculation | P3 | Korea | .42 | .27 | .000 |
| | | Germany | .68 | .33 | |
| | P4 | Korea | .47 | .25 | .000 |
| | | Germany | .72 | .37 | |
| | P7 | Korea | .87 | .33 | .000 |
| | | Germany | .68 | .44 | |

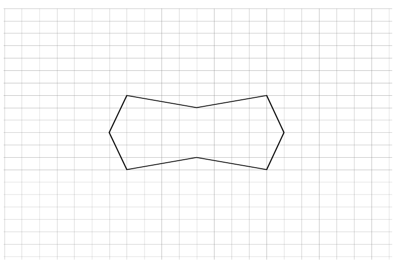
*Table 7-8 The average of correctness in each problem in the 7th grade test (normed score) and t-test*

As can be seen from the table 7-8, except for problem 8, the significant difference between the Korean and the German 7th graders for all problems is observed.

- Symmetry

In this problem, pupils were asked to draw all the lines of symmetry of a given polygon. Pupils needed to understand the concept of symmetry. However, the mathematical concept was not necessarily needed, because the word "symmetric" is often used in everyday life.

Draw the lines of symmetry



Problem 1.

| | Korea | Germany |
|---|---|---|
| No answer | 15.3 | 3.6 |
| Wrong answer | 18.0 | 2.1 |
| More than 2 lines of symmetry | 17.5 | 19.9 |
| Only one line of symmetry | 32.3 | 3.8 |
| Right answer | 16.9 | 70.6 |
| Total | 100.0 | 100.0 |

*Table 7-9 Answers given to problem 1 by percentage*

However, the results of the Korean pupils for this problem were much worse than those of the German pupils. During the test, one of the Korean pupils even asked me what symmetry is. As can be seen from the table 7-9, over 15% of the Korean pupils did not draw the lines, which shows they do not even know the definition, while only 3.6% of the German pupils failed to draw any lines. Moreover, over 70% of the German pupils gave the right answer, while about 17% of the Korean pupils did.

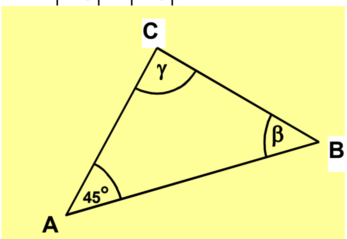
From these results, we might conclude that either the Korean pupils did not learn the exact definition of "symmetry" or that the problems related to daily life was not still applied in the mathematics class. Therefore, symmetry should be contained in the 7th grade curriculum for better understanding congruence as an activity.

- Simple calculation

Another question of basic competence is a simple calculation question. Only two questions were here presented. Pupils were asked only to calculate unknown angles in both problems.

The first question, problem 3, asked pupils to determine the unknown angles in an isosceles triangle. Pupils needed to understand the concept “isosceles triangle”, know properties of the angles in a triangle (sum of the angles is 180°), and correctly calculate the answer.

This is an isosceles triangle with |AC| = |BC|.




Calculate angle β and angle γ.

Problem 3

| | Korea | Germany |
|---|---|---|
| No answer | 4.8 | 3.2 |
| Incorrect working | 6.3 | 1.5 |
| Wrong answers | 13.2 | 5.0 |
| Right answer (without calculation) | 67.2 | 38.4 |
| Right answer (with calculation) | 2.6 | 22.6 |
| Right answer (with incorrect reasoning) | 0.5 | 5.3 |
| Right answer (with sum of angles in a triangle) | 0.5 | 9.0 |
| Right answer (with base angles) | 1.1 | 6.4 |
| Right answer (with both reasons) | 3.7 | 8.6 |
| Total | 100.0 | 100.0 |

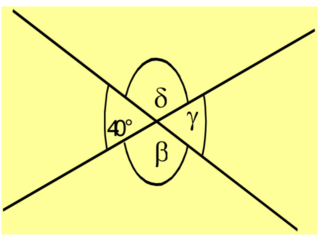
*Table 7-10 Answers given to problem 3 by percentage*

Table 7-10 indicates that over 65% of the Korean pupils and nearly 40% of the German pupils gave the right answer without any calculation. Only 8% of the Korean pupils gave the right answer with calculation or reasons, while about 45% of the German pupils gave the right answer with calculation or argumentation. This result shows that the Korean pupils tend to only give answers without explaining their reasoning. We should take note of the proportion of pupils who did not give any answer or gave wrong answers including incorrect works; of the Korean pupils, 25% were in this group – more than double the proportion of the German pupils in this group as a percentage of the German pupils as a whole.

The second question in this category, problem 4, asked pupils to determine the unknown angles in an intersection of two straight lines. Pupils needed to understand concepts relating to angles such as “opposite angle are equal”, “adjacent angles add up to 180°”, know properties of angles on a straight line (sum of angles is 180°), and correctly calculate the answer.

The results show the same phenomenon as the first question. Over 75% of the Korean and nearly 30% of the German pupils gave only a right answer without any calculation and reasons. 15% of the Korean and 14% of the German pupils either did not give an answer or gave incorrect answers.

Calculate the δ, β, & γ angles.



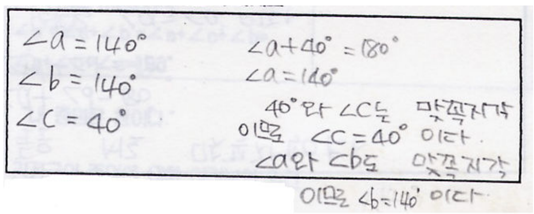

Problem 4.

| | Korea | Germany |
|---|---|---|
| No answer | 3.7 | 0.8 |
| Incorrect working | 1.6 | 1.4 |
| Incorrect answers | 10.6 | 12.4 |
| Right answer (without calculation) | 75.1 | 27.8 |
| Right answer (with calculation) | 3.7 | 25.8 |
| Right answer (with opposite angles) | 1.1 | 7.9 |
| Right answer (with adjacent angles) | 0.5 | 2.3 |
| Right answer (with both reasons) | 3.7 | 21.7 |
| Total | 100.0 | 100.0 |

*Table 7-11 Answers given to problem 4 by percentage*

As can be seen from the table 7-11, 3.7% of the Korean pupils gave an answer with calculation, while 25.8% of the German pupils did. Moreover, over 30% of the German pupils gave an argumentation, whereas only 5% of the Korean pupils did.

Here is an example of a Korean pupil's answer. This answer came up frequently in one class in Korea. In this class, about 8% of the pupils gave such an answer with reasoning. In actual fact, this class showed the best performance of the five Korean classes.




∠ $a$ (δ) =140°, ∠ $a$ +40°=180°
∠ $b$ (β) =140° ∠ $a$ =140°
∠ $c$ (γ) =40° since 40° and ∠ $c$ are opposite angles, ∠ $c$ =40°
since ∠ $a$ and ∠ $b$ are also opposite angles, ∠ $b$ =140°

*Figure 7-1 A Korean pupil's answer to problem 4*

Here are two examples of German pupils' answers. Figure 7-2 shows the answer which was solved using the statement "opposite angles are equal", while figure 7-3 shows the answer which was solved using the rule "adjacent angles add up to 180°".

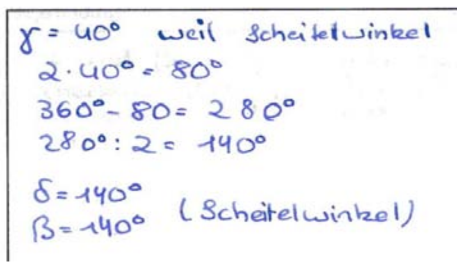



*Figure 7-2 A German pupil's answer to problem 4*

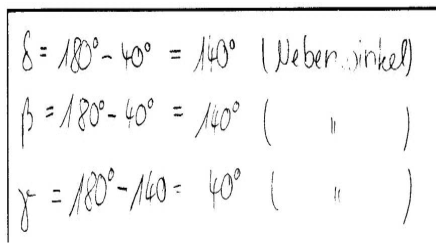



*Figure 7-3 A German pupil's answer to problem 4*

The answers given by the German pupils showed similarities with those given by the Korean pupils. According to Reiss (2001), the frequency of answers like the ones above (figure 7-2 and figure 7-3) are was significantly greater in one class than other classes, and again, this class scored higher than the other classes. She concluded that giving mathematical reasoning and arguments was an important part of the lessons of this class, and that being capable of reasoning and formulating meaningful arguments has a positive influence on achievement.

As we can see from the two tables above (Table 7-10 & Table 7-11), the Korean pupils did not usually write down their argumentation for calculation problems. Hwang (2000) gave one of the reasons: "Korean pupils are used to getting the exact answer, however, they have difficulties in explaining or writing the argumentation or the results." Her argument relates more to algebraic problems, yet we might conclude that the same reason is valid for the two problems discussed here.

- **Competence in proof and argumentation**

This part addresses those problems concerning concepts, i.e. those for which pupils must know the concepts and identify and apply them to other problems. In addition, it consists of more complicated problems which have more steps for proofs and argumentations.

As mentioned above shortly, the Korean pupils performed better than the German pupils on questions concerning competence in proof and argumentation.

The below table shows the pupils' average for each problem relating to competence in proof and argumentation and the corresponding significant difference between the Korean and the German 7th graders for averages of all problems.

| | | | M | SD | $p$-value |
|---|---|---|---|---|---|
| One step | P5 | Korea | .65 | .47 | .058 |
| | | Germany | .58 | .47 | |
| | P6a | Korea | .76 | .43 | .000 |
| | | Germany | .53 | .49 | |
| | P10a | Korea | .71 | .43 | .000 |
| | | Germany | .57 | .47 | |
| Several steps | P6b | Korea | .38 | .47 | .000 |
| | | Germany | .23 | .38 | |
| | P9 | Korea | .46 | .39 | .000 |
| | | Germany | .28 | .34 | |
| | P10b | Korea | .41 | .44 | .000 |
| | | Germany | .23 | .39 | |
| | P10c | Korea | .35 | .38 | .000 |
| | | Germany | .20 | .38 | |

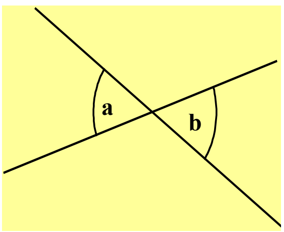
*Table 7-12 The average of correctness in each problem in the 7th grade test (normed score) and t-test*

As can be seen from table 7-12, for Problem 5 no significanct difference between the Korean and the German 7th graders is observed. The strong significant differences between the two groups in the averages for each of the remaining problems are observed.

- Proof with one Step

In this problem pupils were asked to justify why $\angle a = \angle b$ (or $\angle \alpha = \angle \beta$). Pupils needed to understand the concept of "opposite angle" and know properties of angles on a straight line (sum of angles is 180°). It was interesting to see whether pupils gave alternative arguments and explained the concept of "opposite angle".

"Angles facing each other are equal"




Give the reason or justify why $\angle a = \angle b$

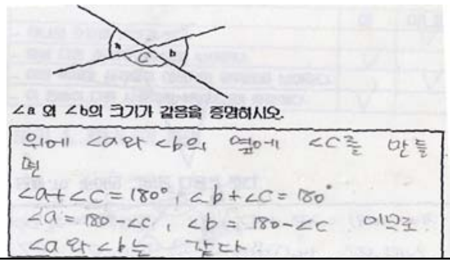
Problem 5

| | Korea | Germany |
|---|---|---|
| No answer | 6.9 | 9.4 |
| Incorrect working | 7.4 | 8.3 |
| Incorrect arguments (e.g. tautological) | 15.9 | 18.8 |
| Measured (without calculation) | 2.6 | 0.9 |
| Basically right, but only described | 4.2 | 9.6 |
| Right (concept of opposite angle mentioned but not explained) | 40.2 | 29.9 |
| Right (concept given and explained) | 16.9 | 21.5 |
| Right answer (corresponding reasons) | 5.8 | 1.5 |
| Total | 100.0 | 100.0 |

*Table 7-13 Answers given to problem 5 by percentage*

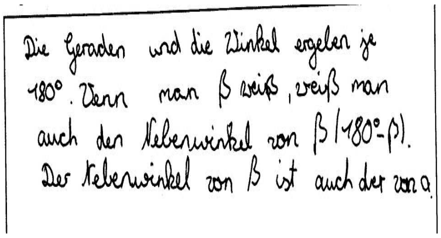
Over 15% of the Korean and 18% of the German pupils gave incorrect arguments, for example they repeated the sentence given in the question: "the angles which are faced each other are same". Moreover, 7.4 % of the Korean and 8.3% of the German pupils referred to concepts such as alternative angles and corresponding angles which were not needed to answer the question. Over 40% of the Korean pupils gave only a concept as a correct answer, as compared with 30% of the German pupils.

Below is the answer of one Korean pupil and of one German pupil. In figure 7-4, the Korean pupil proved the statement using the fact that angles on a straight line add up to 180°, even though this property is not explicitly stated.




If ∠ *c* is constructed between ∠ *a* and ∠ *b*, then ∠ *a* +∠ *c*=180°, ∠ *b* +∠ *c*=180°. Since ∠ *a* = 180 - ∠ *c*, ∠ *b*=180- ∠ *c*, it follows that ∠ *a* and ∠ *b* are equal.

*Figure 7-4 A Korean pupil's answer to problem 5*




The line and the angle are each 180°. If one knows β, one knows also the adjacent angle of β (180° - β).

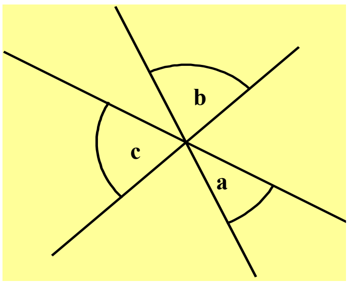
The adjacent angle of β is also that of α.

*Figure 7-5 A German pupil's answer to problem 5*

In figure 7-5, the German pupil proved the statement using the concept of an adjacent angle. He concluded his proof, indirectly, by noting that the adjacent angle of β is the same as that of α. The Korean pupil wrote her proof down as a mixture of a normal explanation and symbols, while the German pupil used only a normal explanation.

- Proof with several steps

In this problem pupils were asked to justify that $\angle a+\angle b+\angle c=180°$ (or $\angle\alpha+\angle\beta+\angle\gamma=180°$). Pupils needed to understand the concept of an "opposite angle" and know that the angles on a straight line add up to 180°.




Prove that ∠ *a*+∠ *b*+∠ *c*=180°

Problem 9

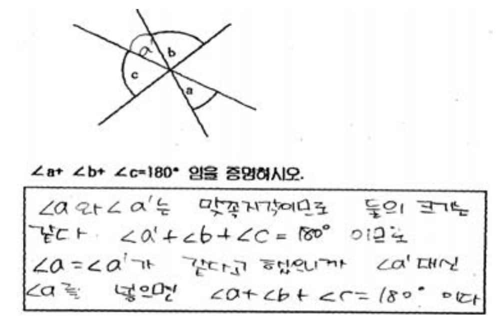
| | Korea | Germany |
|---|---|---|
| No answer | 16.9 | 19.6 |
| Incorrect working | 14.3 | 14.7 |
| Incorrect arguments used | 3.2 | 20.8 |
| Wrong concepts used | 2.1 | 15.5 |
| Only giving the "opposite angle" argument | 27.0 | 14.6 |
| Only giving the "half-circle" argument | 9.5 | 4.2 |
| Correct (using both reasons) | 27.0 | 10.6 |
| Total | 100.0 | 100.0 |

*Table 7-14 Answers given to problem 9 by percentage*

2% of the Korean and 15% of the German pupils used the wrong concepts, such as alternate angles or corresponding angles. In addition, 20% of the German pupils used incorrect arguments, for example by saying that there are 360° in a full turn around the point in the centre, so there must be 180° in a half-turn, and yet failing to give a reason. This could be a visual problem, in that pupils were unable to recognise that the angle between ∠ *b* and ∠ *c* was the angle opposite angle ∠ *a*. 36% of the Korean and 18% of the German pupils gave an argument which it itself was not sufficient for the answer to be correct. Nearly 30% of the Korean and 10% of the German pupils gave the right arguments.

Almost 17% of the Korean and 20% of the German pupils did not give any answer. Unlike for problems of basic competence, the number of the German pupils who did not give any answer was larger than that for Korean pupils; nevertheless, the difference was no larger in this case.

A Korean pupil's answer and a German pupil's answer are shown below. The Korean pupil's answer (Figure 7-6) gave all the steps necessary to prove the given statement. She used the concept of opposite angles and angle on a straight line for her proof.

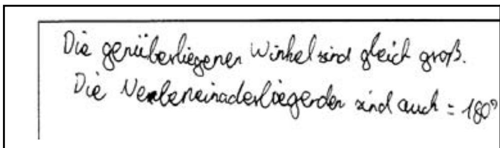



Since $\angle a$ and $\angle a'$ are opposite angles, therefore they are equal.

Since $\angle a' + \angle b + \angle c = 180°$, and
$\angle a = \angle a'$,
If $\angle a'$ is substituted for $\angle a$,
then $\angle a' + \angle b + \angle c = 180°$.

*Figure 7-6 A Korean pupil's answer to problem 9*

In Figure 7-7 all the necessary steps were given. However, the German pupil did not give the name "opposite angle"; instead he wrote "the angles facing each other". In addition, in contrast to Figure 7-6 the German pupil did not use any symbols.




The angles which face each other are equal.
The adjacent angles are also = 180°

*Figure 7-7 A German pupil's answer to problem 9*

**Summary:** In the test for the 7th graders, the German pupils performed significantly better on those problems concerning basic competence than the Korean pupils. On the other hand, the Korean pupils performed significantly better on those problems concerning competence in proof and argumentation.

In particular, the German pupils performed significantly better on the symmetry problem than the Korean pupils. The Korean pupils had difficulties with the symmetry problems. This finding is apparent in TIMSS too. The German pupils performed relatively well in the multiple-choice problem (M02 in TIMSS), in which the pupils' knowledge of the symmetrical properties of a geometrical figure were to be evaluated. 64% of German 8th graders got the right answer; by contrast, only half of Korean pupils answered correctly. Korean pupils performed poorly on this problem as compared with other geometry problems in TIMSS.

For calculation problems, the Korean pupils gave only the answer, while 30-40% of the German pupils wrote the answer along with the process of calculation or relevant reasons. We might conclude that the Korean pupils were only concerned with getting the right answer (as a product), while the German pupils were more concerned with the process of working towards the answer.

Even though many pupils could recall the names of several concepts, there were large proportions (15% of the Koreans and 20% of the Germans) of the pupils who could not recall the concepts themselves. This strongly suggests they have a lack of declarative knowledge (e.g. understanding of geometrical concepts, such as opposite angles and angle on a straight line). According to Reiss, Klieme and Heinze (2001), even pupils at the end of secondary level have considerable failings in declarative knowledge. They often have a vague intuitive understanding of concepts such as "congruence", but this understanding is restricted to examples, and they have no exact mathematical knowledge of the respective definitions and theorems.

### 7.1.2. Achievement Test for the 8th graders

The Test of the 8th graders has also a similar structure as that of the 7th graders.

- **Basic competence**

The questions concerning basis competence in the 8th grade came from two areas. First, there were 2 questions related to symmetric concepts. Second, there were four questions on simple concepts which are explicitly taught in the 7th grade and which came under competence level II in the 7th grade test, for example the concepts concerning angles. However, in the 8th grade test, those problems came under competence level I, because they were no longer new and pupils could be expected to recognise them and apply them to other problems.

The table below gives the pupils' average for each problem relating to basic competence and the *p*-value for determining the significant difference between the Korean and the German 8th graders for all problems.

| | | | M | SD | *p*-value |
|---|---|---|---|---|---|
| Symmetry | P2 | Korea | .38 | .42 | .041 |
| | | Germany | .45 | .38 | |
| | P8 | Korea | .32 | .28 | .000 |
| | | Germany | .56 | .26 | |
| Simple concept | P5a | Korea | .48 | .47 | .794 |
| | | Germany | .49 | .47 | |
| | P6a | Korea | .49 | .48 | .130 |
| | | Germany | .55 | .48 | |
| | P9a | Korea | .55 | .50 | .000 |
| | | Germany | .71 | .44 | |
| | P9b | Korea | .26 | .36 | .000 |
| | | Germany | .45 | .42 | |

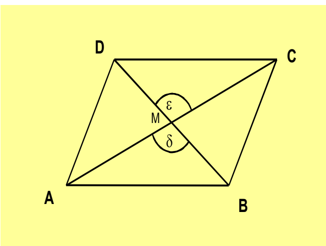
*Table 7-15 The average of correctness in each problem in the 8th grade test (normed score) and t-test*

As can be seen from the table 7-15, in accordance with problem 5a and problem 6a, there is no significant difference between the Korean and the German 8th graders. However, the weak significant difference between the two groups for problem 2 is observed. In addition, there are strong significant differences between the Korean and the German 8th graders in the respective average of the other problems.

- Simple concept

In the next problem pupils were asked to explain why $\angle\, \delta = \angle\, \varepsilon$. Pupils needed to understand the concept of an "opposite angle" and know properties of angles on a straight line ("sum of angles is 180°"). It was interesting to see whether pupils gave alternative arguments using the properties of parallelogram.

ABCD is a rectangle with |MA|=|MC| and |MB| =|MD|




Give a reason: (5a) δ = ε

Problem 5a

| | Korea | Germany |
|---|---|---|
| No answer | 19.2 | 8.5 |
| Incorrect working | 26.4 | 36.3 |
| Measuring | 1.1 | 0.2 |
| Right answer (insufficient reason) | 7.1 | 12.1 |
| Right (concept of opposite angle mentioned but not explained) | 20.3 | 29.7 |
| Right (concept given and explained) | 20.3 | 12.5 |
| Right answer (sufficient reasons) | 5.4 | 0.6 |
| Total | 100.0 | 100.0 |

*Table 7-16 Answers given to problem 5a by percentage*

Table 7-16 shows that 20% of the Korean and 30% of the German pupils gave the name of the concept and presumably took this to be a complete answer. Moreover, 25% of the Korean and 13% of the German pupils gave additional reasons. Some of these pupils explained their reasons giving the properties of parallelogram by using the conditions given in the question.

From an analysis of the incorrect working, we find that there are two types of errors. One is the misuse of the concepts such corresponding angle and alternative angle which were irrelevant to this question. The other is that pupils wrote only the definition "parallelogram" without any reasons, because the given shape looks like parallelogram rather than a rectangle. However, this concept is not relevant to the answer to (5a).

The next table 7-17 compares the answers for Problem 5 (see p.97) in the test for the 7th graders and Problem 5a in the test for the 8th graders. In actual fact, these two problems concern declarative knowledge.

| | Korea | | Germany | |
|---|---|---|---|---|
| | 7th grade | 8th grade | 7th grade | 8th grade |
| No answer | ***6.9*** | ***19.2*** | ***9.4*** | ***8.5*** |
| Incorrect working | ***7.4*** | ***26.4*** | ***8.3*** | ***36.3*** |
| Measures, incorrect argument & Right answers (insufficient reasons) | 22.7 | 8.2 | 29.3 | 12.3 |
| Only concept (opposite angle) | 40.2 | 20.3 | 29.9 | 29.7 |
| Correct answer (concept of opposite angle used and explained) | 16.9 | 20.3 | 21.5 | 12.5 |
| Correct answers (concept of congruent triangles and of angles on a straight line used) | 5.8 | 5.4 | 1.5 | 0.6 |

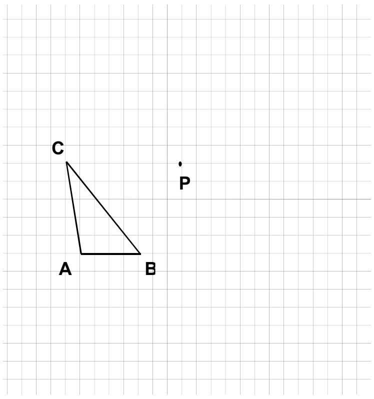
*Table 7-17 A comparison of the 7th and the 8th grade answers by percentage of all pupils in the given group*

More than twice as many Korean 8th graders as 7th graders gave no answer at all, while the percentage of the German pupils failing to give an answer becomes slightly lower for 8th graders than for 7th graders. Moreover, in both countries the proportion of the 8th graders who gave answers with incorrect working was much greater than the proportion for the 7th graders. This may have been due to the visual difficulty mentioned above. Many pupils wrote "parallelogram" without any reasoning or sometimes with the argument: "it looks like a parallelogram" even though this was an irrelevant answer.

- Symmetry and construction

In this problem pupils were asked to rotate a triangle 180° about a point. Pupils needed to understand the term "rotation". This problem is quite advanced as compared with the problems of the 7th grade test. In contrast to the 7th grade test, the pupils were asked to give all the steps in their working.

Rotate triangle ABC 180° about the point P.
Label the rotated triangle A'B'C'.




How did you construct the triangle A'B'C'?
Give all the steps in your working!!!

Problem 2

| | Korea | Germany |
|---|---|---|
| No answer | 19.2 | 11.9 |
| Incorrect working | 9.9 | 8.1 |
| Translation | 0.5 | 1.5 |
| Reflection | 2.7 | 12.1 |
| Wrong centre of rotation | 8.2 | 0.6 |
| Wrong angle of rotation | 9.9 | 0.8 |
| Right construction, points A',B',C' not labelled | 12.6 | 3.6 |
| Right construction, incorrect labelling of points | 1.6 | 4.4 |
| Right construction, steps of working not given | 6.6 | 6.4 |
| Right construction, only one step of working given | 1.6 | 25.2 |
| Right construction, all steps of work given | 26.9 | 25.4 |
| Total | 100.0 | 100.0 |

*Table 7-18 Answers given to problem 2 by percentage*

The German pupils performed better on this problem than the Korean pupils. This is consistent with the results for the symmetry problem in the test for the 7$^{th}$ graders.

Table 7-19 compares the answers by percentage of the whole group for the 7$^{th}$ graders and for the 8$^{th}$ graders on Problem 2 in their respective tests (see Appendix B-1 for Problem 2 for the 7$^{th}$ graders). Both questions concern symmetry.

| | Korea | | Germany | |
|---|---|---|---|---|
| | 7$^{th}$ grade | 8$^{th}$ grade | 7$^{th}$ grade | 8$^{th}$ grade |
| No answer | 23.8 | 19.2 | 24.1 | 11.9 |
| Incorrect working | *36.0* | *9.9* | *12.0* | *8.1* |
| Incorrect answers | *7.9* | *21.4* | *27.9* | *15.0* |
| Right construction, no labelling of points | 22.8 | 12.6 | 1.1 | 3.6 |
| Right construction, incorrect or no labelling | 5.8 | 8.2 | 15.5 | 10.8 |
| Right construction, only one step of working given | 0.0 | 1.6 | 7.6 | 25.2 |
| Right construction, all steps of working given | 3.7 | 26.9 | 11.8 | 25.4 |

*Table 7-19 Comparison of answers given by percentage of the whole group for the 7$^{th}$ and the 8$^{th}$ grade*

In contrast to the problem concerning declarative knowledge, the percentages for 'no answer' and 'incorrect working' are lower for the 8$^{th}$ graders. The 8$^{th}$ grade pupils performed better than the 7th grade pupils in both countries.

- Competence in proof and argumentation

As for the 7$^{th}$ grade test, questions relating to competence in proof and argumentation consisted of four problems which could be solved in one step and three problems which had to be proved with several steps. In addition, the 8$^{th}$ grade pupils were asked to give reasons with their answer.

The table below shows the pupils' average for each problem relating to competence in proof and argumentation and the *p*-value for determining the significant differences between the Korean and the German 8th graders for all problems.

| | | | M | SD | *p*-value |
|---|---|---|---|---|---|
| One step | P1 | Korea | .40 | .40 | .000 |
| | | Germany | .25 | .36 | |
| | P3 | Korea | .46 | .43 | .000 |
| | | Germany | .30 | .36 | |
| | P4 | Korea | .54 | .36 | .005 |
| | | Germany | .63 | .36 | |
| | P7 | Korea | .44 | .47 | .853 |
| | | Germany | .45 | .45 | |
| Several steps | P5b | Korea | .33 | .37 | .000 |
| | | Germany | .13 | .30 | |
| | P6b | Korea | .29 | .44 | .442 |
| | | Germany | .45 | .42 | |
| | P9c | Korea | .33 | .47 | .000 |
| | | Germany | .07 | .23 | |

*Table 7-20 The average of correctness in each problem in the 8th grade test (normed score) and t-test*

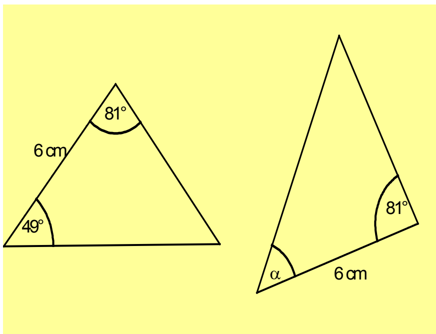
As can be seen from table 7-20, there are no significant differences between the Korean and the German 8th graders for Problem 7 and Problem 6b. On the other hand, strong significant differences between the two groups in the respective averages of the remaining problems can be observed.

- Proof with one Step

In this problem, pupils were asked to determine the unknown angle in a triangle congruent to another given triangle. Pupils therefore needed to understand the concept of congruence, be able to visualise the flipped and rotated figure, know the properties of angles in triangles (sum of angles is 180°), and perform the correct computation.

Problem 1.

The triangles are congruent.




Give the angle α.
Give a reason for your answer!

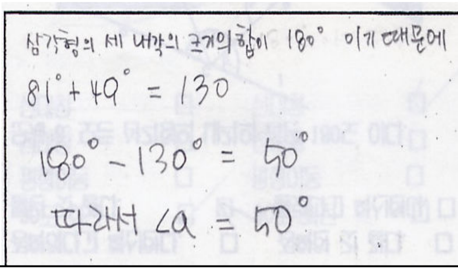
| | Korea | Germany |
|---|---|---|
| No answer | 16.4 | 1.5 |
| Incorrect working | 26.9 | 61.6 |
| Wrong answer (with the right reason) | 0.5 | 5.9 |
| Right answer (with an invalid reason) | 3.3 | 2.3 |
| Right answer (with an insufficient reason) | 29.1 | 14.4 |
| Right answer (with the right reason) | 23.6 | 14.4 |
| Total | 100.0 | 100.0 |

*Table 7-21 Answers given to problem 1 by percentage*

This problem is found in TIMSS too. In TIMSS, Korean pupils (66%) did relatively well on this problem, while German pupils (29%) performed poorly.

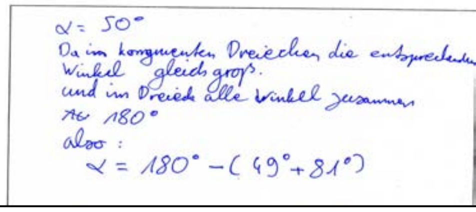
As table 7-21 shows, incorrect working should be taken note. In the German case it happened more often; over 60% of the German pupils gave false answers, double the percentage of the Korean pupils.

The incorrect working of one Korean and of one German pupil is presented below.


삼각형의 세 내각의 크기의 합이 180° 이기 때문에
81°+49° = 130
180° − 130° = 50°
따라서 ∠a = 50°

Since the sum of angles in triangles is 180°,
81°+ 49° = 130°
180° - 130° = 50°
Therefore, ∠ *a* = 50°

*Figure 7-8 A Korean pupil's answer to problem 1*


α = 50°
Da im kongruenten Dreiecken die entsprechenden Winkel gleich groß.
und im Dreieck alle Winkel zusammen 180°
also:
α = 180° − ( 49°+81°)

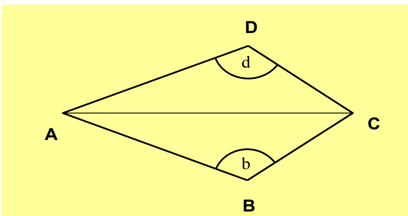
α=50°
Since in congruent triangles, the corresponding angle are equal.
And in a triangle, all angles together are 180°.
So: α= 180° - (49° + 81°)
(α= α+β+γ- (β+γ

*Figure 7-9 A German pupil's answer to problem 1*

From their arguments, it is clear that both pupils knew that this problem could be solved using the sum of the angles in a triangle or the condition of congruence. However, they passed over the fact that the figure is only rotated.

The next problem asked pupils to prove that ∠ *b* = ∠ *d*. This problem is related to declarative knowledge and its application. Pupils first needed to understand the concept of congruence and isosceles triangle and then to recall and apply the properties of congruence as well as isosceles triangles.

In the square ABCD,
|AB|=|AD| and |BC|=|DC|.




Prove that ∠b = ∠d

Problem 3

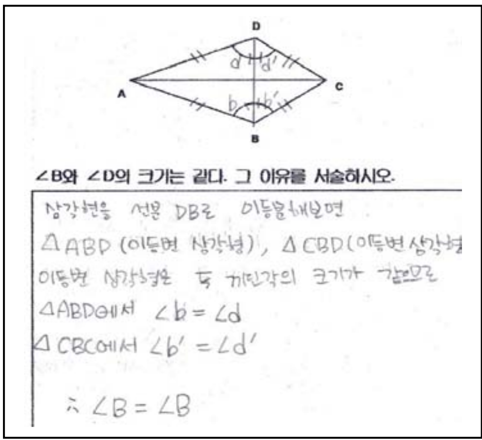
| | Korea | Germany |
|---|---|---|
| No answer | 10.4 | 5.9 |
| Incorrect working | 28.0 | 33.3 |
| Wrong reasons (SSA, SAS) | 3.3 | 13.8 |
| Right answer (with a slight error) | 24.7 | 32.8 |
| Right answer | 33.5 | 14.2 |
| Total | 100.0 | 100.0 |

*Table 7-22 Answers given to problem 3 by percentage*

Table 7-22 shows that nearly 60% of the Korean and 50% of the German pupil proved this problem. Over 10% of the Korean pupils did not give any answer, as compared with 5.9% of the German pupils.

An example of a Korean pupil's answer and one of a German pupil's answer is given below.

The first answer (Figure 7-10) is that of the Korean pupil. She drew an additional line on the diagram to form two triangles. She also added some marks to the diagram to make clear to herself what the assumptions given in the question were. She used the property that the base angles of an isosceles triangle are equal, but she did not recall the correct name for these angles and called them "putting angles" instead.



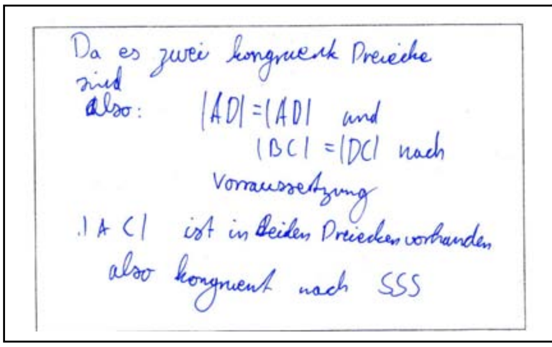

The triangle is divided by line DB,
△ ABD (isosceles), △ CBD (isosceles).

Since isosceles triangle has same angles the two "putting angles" (actually, they are base angles) have to be same,

In △ ABD, ∠b=∠d,
In △ CBC, ∠b'=∠d',
∴ ∠B=∠B

*Figure 7-10 A Korean pupil's answer to problem 3*

The next answer is that of a German pupil. Unlike the Korean pupil whose answer is given here, he argued using the properties of congruence. He used the assumptions given in the question.




Since they are two congruent triangles, because:
|AD|=|AB| and |BC|=|DC| by assumption

|AC| is in both triangles.
Therefore, congruent by SSS

*Figure 7-11 A German pupil's answer to problem 3*

**Summary:** In the test for the 8th graders, the German pupils again performed better on the problems concerning basic competence than Korean pupils. On the other hand, Korean pupils again performed significantly better on those problems concerning competence in proof and argumentation.

In particular, the German 8th graders performed significantly better on the symmetry problems than the Korean 8th graders. On the transformation problem (Problem 2), both the 8th grade Korean and the 8th grade German pupils performed better than the 7th graders. In addition, the proportion of pupils who did not give any answer was smaller for the 8th graders in both Korea and Germany.

Moreover, concerning the problems relating to the simple concepts, the German pupils performed better than the Korean pupils, although significant mean difference between the two groups is not observed.

### 7.1.3. Discussion

Here the results of the Korean and the German 7th and 8th graders' basic competence and competence in proof and argumentation are interpreted in more depth. Our results show that the German pupils performed well on problems for which they could solve using simple arithmetic and procedural knowledge, while the Korean pupils attained high marks on the problems for which they could argue by declarative knowledge and non-routine tasks.

The Korean pupils had great difficulties in solving those problems concerning symmetry and transformation, even if the Korean 8th graders performed slightly better on those problems than the 7th graders. As mentioned earlier in this section, in Korea the concept "Symmetry" is taught in grade five in art class, not mathematics class. One might conclude that Korean pupils do not perform well on problems which are not extensively covered in the curriculum.

By contrast, in Germany it is taught in elementary school, but in secondary school the concept is supposed to be repeated as an aid to understanding congruence.

For the problems that required calculation most Korean pupils gave only the answers, while the German pupils gave the answers and also explained their reasons or their calculation. From my own personal experience, a plausible reason for the Korean pupils only giving the answers might be the structure of the examination. In Korea, there is only a written test which is composed of 20 multiple choice questions and two or three open questions that pupils try to solve in 45 minutes. Korean pupils should get the answers as fast as they can, because they do not have enough time to write down the whole process of calculation. Therefore, without giving their reasoning they write only the answers down. This familiarity with solving problems very quickly may have compelled the Korean pupils to avoid explaining or writing arguments in our test too.

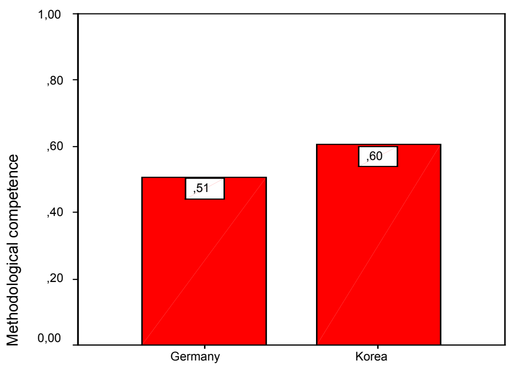
By contrast, in Germany there are two kinds of test, the oral test and written test. Maybe this oral test helps pupils better to explain their arguments or write down the reasons. One German trainee teacher has explained that German pupils are encouraged to write down all their arguments, even if the question is concerned only with simple calculation in the mathematics test at school.

As Reiss and Heinze (2002) confirmed, in the German classes which participated in the test, the pupils who gave working or an explanation for even simple calculation problems performed better on those problems requiring competence in proof and argumentation.

So in Korean mathematics classes too, pupils should be encouraged to include working even when solving simple calculation problems.

## 7.2. Pupils' methodological competence

Methodological knowledge, which is the form of knowledge relating to the validity of mathematical arguments, was also assessed. This problem dealt with the property that the sum of the angles in a given triangle is 180°. Pupils were asked to judge the validity of four solutions: one correct formal proof, one correct narrative proof, one solution with an empirical argument and one circular solution. They were then asked to assess the correctness and generality of each of the four arguments.




*Graph 7-5 The normed average score for methodological competence*

This graph shows the methodological competence of pupils in both countries. From this graph we can observe that the performance of the Korean pupils was slightly better than that of the German pupils.

To find the significant differences between the Korean and the German pupils with respect to four arguments in methodological competence, the independent T-test is applied here.

- T-Test

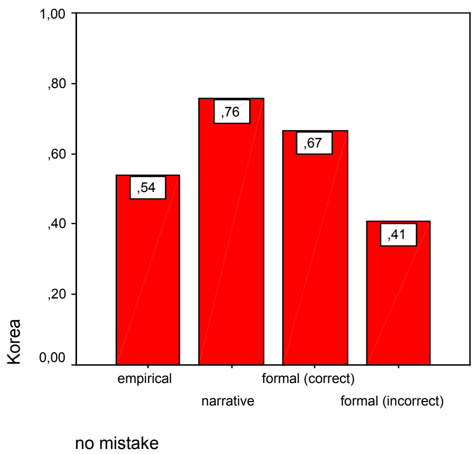
| | | M | SD | p |
|---|---|---|---|---|
| Methodological competence (total) | Korea | .60 | .22 | .000 |
| | Germany | .51 | .21 | |
| Empirical argument | Korea | .54 | .35 | .000 |
| | Germany | .43 | .34 | |
| Narrative proof | Korea | .77 | .38 | .030 |
| | Germany | .70 | .39 | |
| Formal proof | Korea | .72 | .39 | .015 |
| | Germany | .64 | .42 | |
| Formal incorrect argument | Korea | .41 | .36 | .000 |
| | Germany | .29 | .33 | |

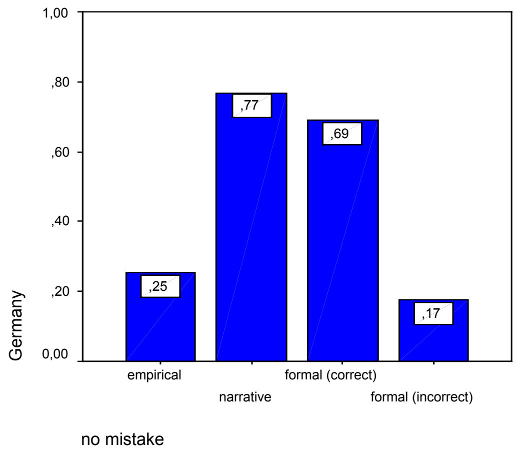
*Table 7-23 T-Test for results for methodological competence (normed score)*

As can be seen from table 7-23, there is a significant difference between the Korean and the German pupils with respect to the methodological competence. In particular, significant differences can be observed with respect to each solution.

In the following part, the results for correctness and generality on each of the four arguments for Korea and for Germany are presented separately. First, here is the result for the correctness of the pupils' judgment on each of the four solutions.

- Correctness




*Graph 7-6 The normed correctness of the Korean and the German pupils' judgements*

As shown in graph 7-6 most pupils in both countries recognised the correct formal proofs to be correct; in particular, 67% of the Koreans and 69% of the Germans, for the formal proof, and 76% of the Koreans and 77% of the Germans for the narrative proof. It is worth noting that pupils seem to be more familiar with narrative proof than formal proof.

On the other hand, pupils had difficulties in recognising that certain arguments were incorrect. Only 25% of the German pupils and 54% of the Korean realised that the empirical argument was incorrect. Moreover, pupils had greater difficulties in recognising the last argument to be incorrect since it seems to be a correct proof because of the formal layout which is used: only 17% of the German and 41% of the Korean pupils correctly said that this argument was incorrect. In other words, the rest of the pupils believed the proof was correct.

Normed scores were given to the Korean and the German pupils for the accuracy of their judgements on the answers they were shown.

To find the significant differences between the two groups with respect to each argument, the independent T-test is applied here.

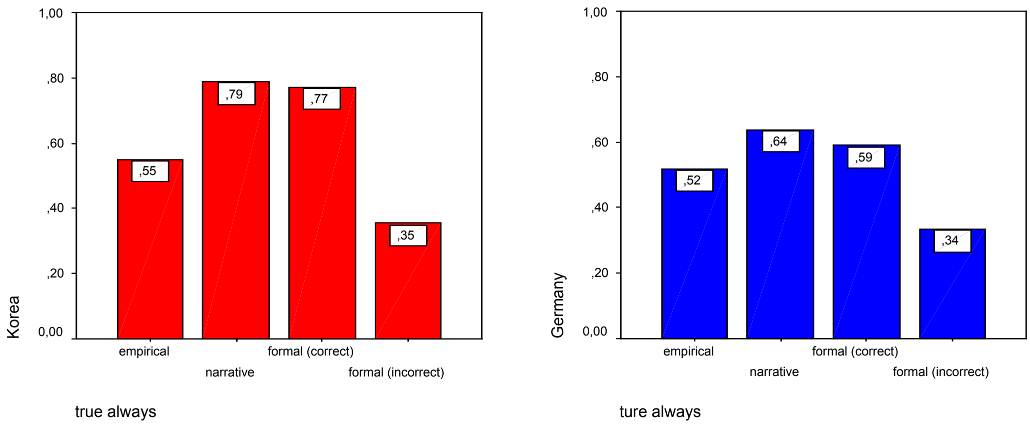
| | | M | SD | p |
|---|---|---|---|---|
| Empirical argument | Korea | .54 | .50 | .000 |
| | Germany | .25 | .43 | |
| Narrative proof | Korea | .76 | .43 | .749 |
| | Germany | .77 | .42 | |
| Formal proof | Korea | .67 | .47 | .562 |
| | Germany | .69 | .46 | |
| Formal incorrect argument | Korea | .41 | .49 | .000 |
| | Germany | .17 | .38 | |

*Table 7-24 T-Test for normed scores given to pupils for their judgements of the given solutions between Korea and Germany*

According to the T-test, there were strongly significant differences between the two groups in their mean scores for the empirical argument ($p<.001$) and for the formal incorrect argument ($p<.001$). However, there is no significant mean difference between the two groups with respect to the formal proof and narrative proof.

The result indicates that the Korean pupils have significantly higher mean scores than the German pupils for their judgements of correctness on the empirical argument and on the formal incorrect argument.

- Generality




*Graph 7-7 The normed score for generality of the Korean and the German pupils' judgements*

As shown in graph 7-7, 79% of the Korean and 64% of the German pupils were rightly satisfied with the generality of the narrative proof. Moreover, 77% of the Korean and 59% of the German pupils accepted the generality of the formal proof. This means that pupils were aware that once a statement had been proved, it was not necessary to verify the statement for particular cases.

However, for empirical arguments, a similar proportion of both groups (52% of the German pupils, 55% of the Koreans) recognised that the given statement could not be generalised. 34% of the German pupils and 35% of the Korean pupils recognised that the incorrect proof with the formal layout could not be generalised.

To determine whether there are significant differences between the Korean and the German pupils with respect to generality of each of the four problems, the independent T-test is applied.

From the T-test it can be observed that there are a strongly significant differences between the means of the two groups for both the formal proof ($p<.001$) and the narrative proof ($p<.001$). However, for the empirical argument and the formal incorrect argument, there are no significant differences between the means of the two groups.

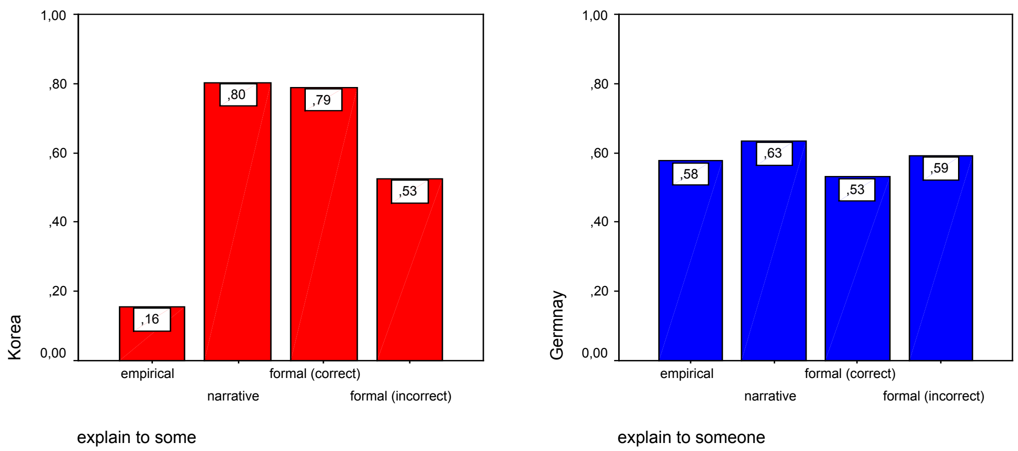
| | | M | SD | p |
|---|---|---|---|---|
| Empirical argument | Korea | .55 | .50 | .448 |
| | Germany | .52 | .50 | |
| Narrative proof | Korea | .79 | .47 | .000 |
| | Germany | .64 | .48 | |
| Formal proof | Korea | .77 | .48 | .000 |
| | Germany | .59 | .49 | |
| Formal incorrect argument | Korea | .35 | .47 | .625 |
| | Germany | .34 | .47 | |

*Table 7-25 T-Test for generality of each problem between Korea and Germany (normed score)*

The T-test shows strongly significant mean differences between the two groups with respect to the formal and the narrative proofs. The data indicate that the Korean pupils have significantly higher mean scores for generality than the German pupils for the formal proof and the narrative proof.

- Explaining to other classmates

In addition to evaluating the correctness and generality of the proofs, pupils were asked to judge whether each of the four arguments was an appropriate means of explaining the particular geometrical content to one of their classmates.




*Graph 7-8 The preferences of the Korean and the German pupils for particular arguments*

The German pupils found that the four arguments were similarly appropriate as explanations for a classmate. On the other hand, the Korean pupils preferred the two correct proofs (narrative and formal proof) as explanations for their classmates. Moreover, pupils in both countries thought that the narrative proof was a more appropriate means of explaining the geometrical content of the proof to a classmate than the formal proof, although for the Korean pupils there is no great difference between narrative and formal proofs.

Only 16% of the Korean pupils chose the empirical argument to explain the content to other pupils, while 58% of the German pupils did. It means most Korean pupils believed that the empirical argument was unsuitable as a way explaining the relevant content to other pupils.

It is interesting to note that the incorrect formal form proof was seen by most pupils in both countries (59% of the German and 53% of the Korean) as a suitable way of explaining the geometrical content to a classmate.

### 7.2.1. Methodological competence and achievement test

The correlation coefficients and associated *p*-values of the Korean pupils for methodological competence from the achievement test are presented in table 7-26.

The results show a significant correlation between methodological competence and each of the cognitive variables for the Korean pupils (basic competence, p=.015; competence in proof and argumentation, p<.001; test, p<.001).

| | Basic competence | Competence in proof and argumentation | Test |
|---|---|---|---|
| Methodological competence | .128*<br>.015 | 198**<br>.000 | .188**<br>.000 |

** p<.05. **p<.01*

*Table 7-26 Correlation coefficients and p-values for the Korean pupils*

The results in table 7-27 show a significant correlation between methodological competence and each of cognitive variables for the German pupils (basic competence, p<.001; competence in proof and argumentation, p<.001; test, p<.001).

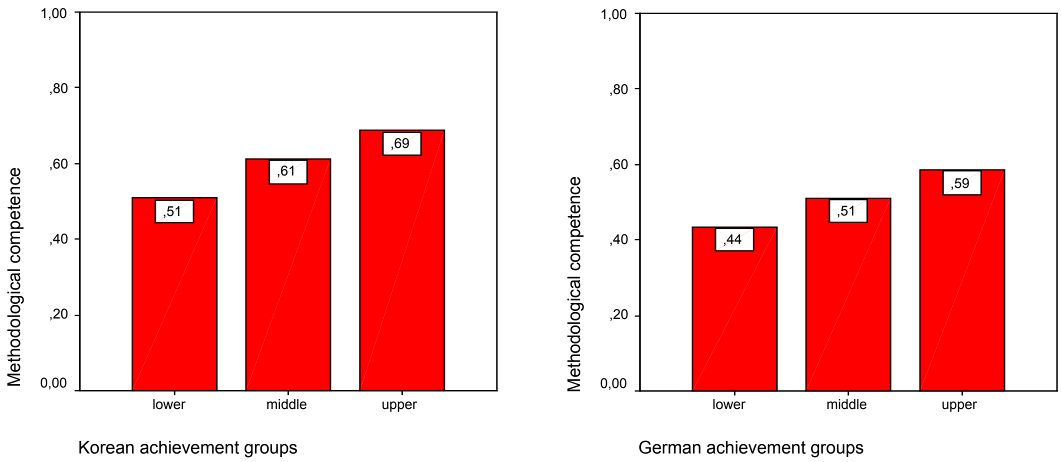
| | Basic competence | Competence in proof and argumentation | Test |
|---|---|---|---|
| Methodological competence | .144**<br>.000 | .263**<br>.000 | .249**<br>.000 |

** p<.05. **p<.01*

*Table 7-27 Correlation coefficients and p-values for the German pupils*

Those two data indicate a significant and positive correlation between methodological competence and each of the cognitive variables. There is only one weak significant difference apparent, namely between the basic competence and methodological competence of the Korean pupils, nevetheless the correlation coefficient is not high.

With respect to the achievement test we can form three achievement groups: a lower, a middle, and an upper group. I will now look for the relationship between these groups with respect to methodological competence.




*Graph 7-9 The relationship between methodological competence and achievement groups*

From the graph 7-9, we might conclude that in both countries, pupils who achieve more highly have a better methodological competence. In order to determine whether the differences in methodological competence between the achievement groups are statistically significant or not, the T-test is applied.

- T-Test for the Korean groups

The one-way analysis of variance (ANOVA) is used here to determine whether or not there were significant differences between the three groups. The results are as follows:

| | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|
| Between groups | .976 | 2 | .488 | 10.805 | .000** |
| Within groups | 8.404 | 186 | .045 | | |
| Total | 9.381 | 188 | | | |

*Table 7-28 Oneway anova for methodological competence*

There are significant differences among the three groups for methodological competence. To find out which pairs of groups show significant mean differences, the Scheffe Process is used. The results are as follows:

| | | Middle group | High group |
|---|---|---|---|
| Methodological competence | Lower group | -.1049* | -.1790* |
| | Middle group | | -.0741 |

*Table 7-29 Scheffe process for comparisons of groups (* p<.05.)*

There is a significant mean difference between the lower group and the middle group and between the lower group and the upper group with respect to the methodological competence.

It can therefore be concluded that Korean pupils from the upper group had a better methodological competence than that in the lower group, and that middle group pupils had a better methodological competence than that in the lower group. However, we cannot make such a statement about the relationship between the middle group and the upper group.

- T-test for the German groups

The data on methodological competence for the three German achievement groups can be analysed using the one-way ANOVA. The corresponding results are as follows:

| | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|
| Between groups<br>Within groups<br>Total | 2.514<br>25.394<br>27.908 | 2<br>656<br>658 | 1.257<br>.039 | 32.475 | .000** |

*Table 7-30 Oneway anova for methodological competence (** p<.01.)*

There are significant differences among the three groups for methodological competence. To find out which pairs of groups show significant mean differences, the Scheffe Process is used. The results are as follows:

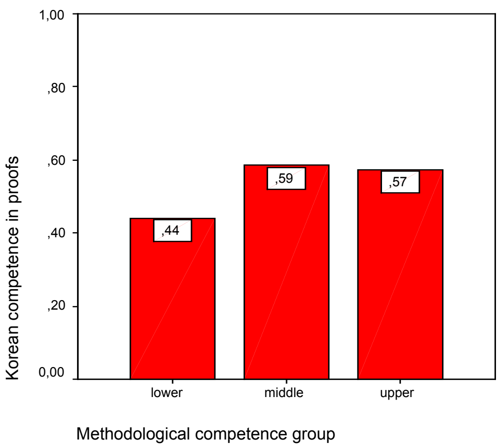
| | | Middle group | High group |
|---|---|---|---|
| Methodological competence | Lower group | -.0764* | -.1507* |
| | Middle group | | -.0743* |

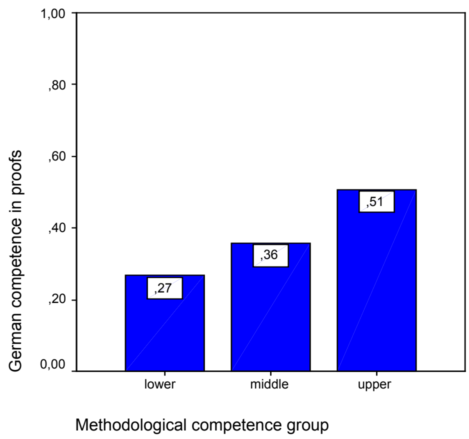
*Table 7-31 Scheffe-process of comparisons of groups (* p<.05.)*

There is a significant mean difference between each of the three possible pairings of groups with respect to the methodological competence.

It can therefore be concluded that the methodological competence of the pupils in the upper group was significantly better than that of those in the lower group, and also that the methodolotical competence of the middle-group pupils was better than that of the lower-group pupils. In contrast to the Korean result, German upper-group pupils showed significantly higher methodological competence than the German middle-group pupils.

I will now look for the relationship between the three groups of methodological competence with respect to competence in proof and argumentation. With respect to the methodological competence we can form three groups: a lower, a middle, and an upper group. However, these three groups need not necessarily coincide with the achievement groups.




*Graph 7-10 The relationship between methodological competence groups and competence in proof*

From the graph 7-10, we might conclude that in Germany, pupils who achieve more high methodological competence have a better competence in proof and argumentation. In contrast, the Korean pupils who belong to middle group of methodological competence have highst competence in proof and argumentation. In order to determine whether the differences between the methodological competence groups with respect to competence in proof and argumentation are statistically significant or not, the T-test is applied.

- T-Test for the Korean groups

The one-way analysis of variance (ANOVA) is used here to determine whether or not there were significant differences between the three groups. The results are as follows:

| | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|
| Between groups<br>Within groups<br>Total | .813<br>15.895<br>16.708 | 2<br>186<br>188 | .407<br>.085 | 4.758 | .010 |

*Table 7-32 Oneway anova for competence in proof and argumentation*

There are significant mean differences among the three groups for competence in proof and argumentation. To find out which pairs of groups show significant mean differences, the Scheffe Process is used. The results are as follows:

| | | Middle group | High group |
|---|---|---|---|
| Competence in proof and argumentation | Lower group | -.1435* | -.1320* |
| | Middle group | | -.0115 |

*Table 7-33 Scheffe process for comparisons of groups (* p<.05.)*

There is a significant mean difference between the lower group and the middle group and between the lower group and the upper group with respect to the competence in proof and argumentation.

It can therefore be concluded that Korean pupils from the upper group had better competence in proof than those in the lower group, and that middle group pupils had better methodological competence than those in the lower group. However, we cannot make such a statement about the relationship between the middle group and the upper group. Therefore the Korean pupils who belong to middle group of methodological competence have highst competence in proof and argumentation is a kind of tendency.

- T-test for the German groups

The data on competence in proof and argumentation for the three German groups of methodological competence can be analysed using the one-way ANOVA. The corresponding results are as follows:

| | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|
| Between groups<br>Within groups<br>Total | 6.166<br>39.958<br>46.124 | 2<br>656<br>658 | 3.083<br>.061 | 50.616 | .000** |

*Table 7-34 Oneway anova for competence in proof and argumentation (** p<.01.)*

There are significant differences among the three groups for competence in proof and argumentation. To find out which pairs of groups show significant mean differences, the Scheffe Process is used. The results are as follows:

| | | Middle group | High group |
|---|---|---|---|
| Competence in proof and argumentation | Lower group | -.0914* | -.2416* |
| | Middle group | | -.1502* |

*Table 7-35 Scheffe-process of comparisons of groups (* p<.05.)*

There is a significant mean difference between each of the three possible pairings of groups with respect to the competence in proof and argumentation.

It can therefore be concluded that the competence in proof and argumentation of the pupils in the upper group was significantly better than that of those in the lower group, and also that the competence in proof and argumentation of the middle-group pupils was better than that of the lower-group pupils. In contrast to the Korean result, the German upper-group pupils showed significantly higher competence in proof and argumentation than the German middle-group pupils.

### 7.2.2. Discussion

Our findings indicate that most pupils are more competent at appreciating correct proofs to be correct and accepting their generality.

Pupils have greater difficulty in recognising incorrect arguments to be incorrect than in recognising correct proofs to be correct. This is consistent with the findings of the study on grade 13 pupils (Reiss, Klieme & Heinze, 2001) and with the findings of Healy and Holyes (1998).

The pupils' judgements on the formal incorrect argument should be noticed. Only 17% of the German pupils and 41% of the Korean pupils recognised it as being incorrect. Also, only a third of the respondents in both countries realised the argument was not generaliseable. So pupils seem to find it just as difficult to accept correct proofs with a non-formal presentation as to reject incorrect but formally-presented proofs. It means that the form of the proof is important when pupils are asked to judge whether it is suitably correct and general or for it to be used to explain a concept to a classmate.

The pupils in both countries preferred to use narrative proof to explain the geometrical content to a classmate. However, we might conclude that for the German pupils, the empirical argument also has a meaningful role to play as an explanation of the geometrical content of the proof to other classmates. The Korean pupils in general believed that the empirical argument is not suitable to use for explaining the concept to other pupils. As for the incorrect formal form argument, this was seen by pupils in both countries (59% German and 53% Korean) as useful for explaining the geometrical content of the proof to a classmate. As compared with the study on German grade 13 pupils, the proportion of pupils selecting the empirical arguments is quite high, perhaps due to the age difference between that sample and the sample in this study.

Lin (2000) has also conducted a study in which the problems used by Healy and Holyes were used to find the Taiwanese pupils' methodological competence and to compare it with that of the English pupils. The result is that none of the Taiwanese pupils who took part in his study chose to use an empirical argument as their own approach in a proof. He suggested that the difference in the styles of teaching and learning in Taiwan and England might be a reason for this. In Taiwan, pupils learn how to prove statements about geometry by developing the incomplete formal argument they were already capable of to a formal proof. However, in England, there are two ways of learning to construct a proof in geometry; the empirical argument and the development of an incomplete argument to a formal proof. This might make British pupils believe that an empirical argument is to be a sufficient proof. This is thought to happen in Germany too.

Heinze and Reiss (2003) confirmed that German pupils have difficulties in bridging the gap between empirical argumentation and formal argumentation, since a preference for inductive arguments stems from the common use of inductive argumentation in elementary school.

Moreover, Heinze and Reiss (2003) found that a correlation exists between the methodological competence of the German pupils and their achievement measures both for basic competence and for competence in proof and argumentation.

## 7.3. Beliefs about mathematics (*Mathematische Weltbilder*)

We used a questionnaire designed by Törner and Grigutsch (1994) and revised by Klieme (2001). This questionnaire consists of 24 items, and was scaled between 1 (= totally agree) and -1 (= totally disagree). From an analysis of factors relevant to beliefs about mathematics, the three categories application, formalism, and process were taken to represent these beliefs. As mentioned earlier, definition of beliefs is taken here to be one's subjective knowledge (which also includes affective ranges) of a certain object or concern by Törner and Grigutsch (1994).

I will now briefly comment on the sample used and on the research questions. The sample for the questionnaire consisted of 189 Korean and 659 German 7th grade pupils and also 58 Korean and 22 German teachers.

- **Research questions:**

(1) Are there any differences between the mathematical beliefs of the Korean and the German teachers?
(2) Are there any differences between the mathematical beliefs of the Korean and the German pupils?
(3) Is there a relationship between beliefs and pupils' achievement?
(4) Is there a relationship between beliefs and pupils' methodological competence?

The differences and relationships were analysed by applying the variance and independent t-test for groups and examining the results.

### 7.3.1. Factor analysis

From the factor analysis of the German pupils' results, three factors were found to be important. As for the orthogonal solution, each factor was determined by problems whose loadings exceeded .39. At first, we were interested in finding out whether the factors were similar in content and whether they could be meaningfully interpreted. From an analysis of factors relevant to beliefs about mathematics, the three categories application, formalism, and process were taken to represent these beliefs (See, Kwak and Reiss, 2002).

- **Application**

| Nr. | Statements | loading |
|---|---|---|
| A3 | Only a few things learned from mathematics can be used later in life. | -.794 |
| A5 | In mathematics classes one does not really learn things which are useful in reality. | -.694 |
| D4 | One needs mathematics only in very few situations. | -.650 |
| A1 | Knowledge of mathematics is very important for pupils later in life. | .592 |
| A2 | Mathematics helps you to solve daily tasks and problems. | .435 |
| A4 | Many parts of mathematics are either of practical use or are directly relevant to application. | .405 |

*Table 7-36 The statements of the German pupils with respect to application analysed using factor analysis*

These pupils consider mathematical knowledge to be important for their future lives; they also believe mathematics helps them to solve everyday tasks and has practical uses – that mathematics is very important in real life. This factor is called the "Application aspect"**.**

- **Formalism**

| Nr. | Problem | loading |
|---|---|---|
| F2 | Mathematics is characterised by very exact terms. | .734 |

| | | |
|---|---|---|
| F1 | In the mathematics classroom, it is very important for one to always think logically and exactly. | .606 |
| F4 | In the mathematics classroom, one must express everything very exactly. | .590 |
| S5 | Mathematics consists of remembering and applying rules, formulae, facts, and processes of calculation. | .531 |
| S2 | Mathematics is a collection of calculation processes and arithmetic rules which indicate exactly how problems are to be solved. | .487 |
| F5 | Mathematical thinking is thinking in formulae and numbers. | .409 |

*Table 7-37 The statements of the German pupils with respect to formalism analysed using factor analysis*

According to these pupils, mathematics can be characterised by strictly logical and precise thinking in an exactly defined subject. This factor is called the "Formalism aspect".

- **Process**

| Nr. | Problem | Loading |
|---|---|---|
| P1 | In mathematics, one can find out and try out many things for him or herself. | .724 |
| P2 | Central aspects of mathematics are contents, ideas, and cognitive processes. | .676 |
| P4 | Mathematics requires inventive thinking. | .620 |
| D1 | Mathematics lessons train you to think in abstract situations. | .455 |

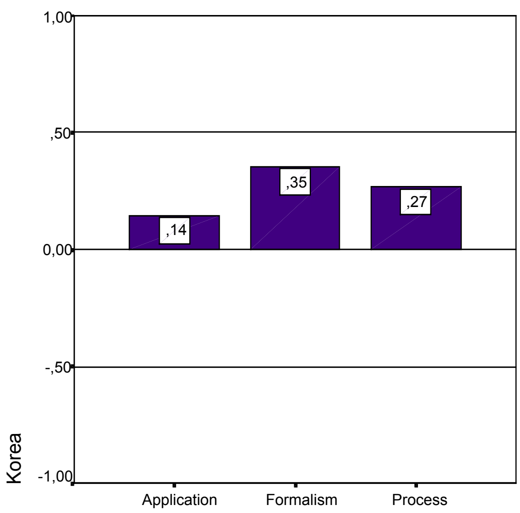
*Table7-38 The statements of the German pupils with respect to process analysed using factor analysis*

For the German pupils, mathematics is characterised as a process, i.e. an activity whereby problems arise and are understood. This is a process of discovery concerning the creation, invention, and reinvention of mathematics. This factor is called the "Process aspect" and represents the view that mathematics is dynamic.

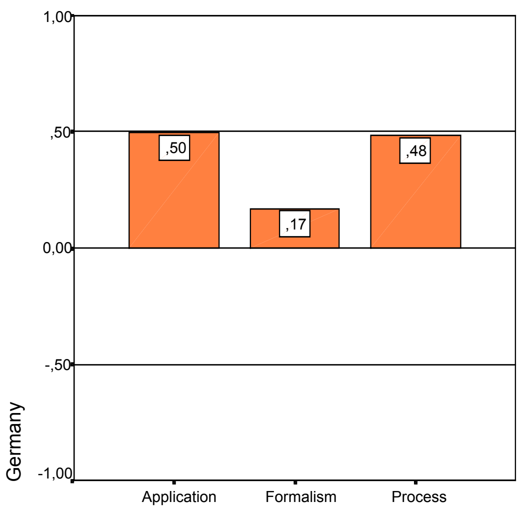
7.3.2. Teachers' Beliefs

- Average of scores

The following graph shows to what extent the Korean and the German teachers agree that mathematics is application-oriented, formalism-oriented, and process-oriented.




*Graph 7-11 Average of the Korean teachers*




*Graph 7-12 Average of the German teachers*

All orientations have positive values but the prevailing views of the Korean and the German teachers differ. Formalism is the dominating factor for the Korean teachers, while the application aspect has the lowest score. On the other hand, for the German teachers, the aspect of application is the dominating factor, whereas the formalism aspect shows the lowest score. We can only speculate on the reasons for these differences. The German teachers are probably more influenced by current discussions on the role of mathematics in the classroom or on realistic mathematics. Also, the Korean teachers seem to be more cautious in expressing strong opinions.

To find the significant differences between the Korean and the German teachers with respect to each of the three factors, the independent T-test is applied here.

- T-test of teachers

| | | M | SD | p |
|---|---|---|---|---|
| Application | Korea | .14 | .50 | .000 |
| | Germany | .50 | .50 | |
| Formalism | Korea | .37 | .47 | .553 |
| | Germany | .17 | .48 | |
| Process | Korea | .27 | .48 | .020 |
| | Germany | .48 | .49 | |

*Table 7-39 T-Test between the Korean and the German teachers on each of the three factors*

This table shows that as regards process and application, there is a significant difference between the Korean and the German teachers.

We might therefore conclude that the German teachers appreciated the application-oriented and process-oriented characteristics of mathematics more than the Korean teachers. Although the Korean teachers rate the aspect of formalism of mathematics more highly than the German teachers do, we cannot observe a significant difference in the attitudes towards this factor from the T-test.

- Correlation

| | Process | Formalism |
|---|---|---|
| Application | .403* | -.157 |
| Process | | .005 |

* Correlation is on the niveau of 0.05 significant

*Table 7-40 Correlation three factors of the German teachers*

| | Process | Formalism |
|---|---|---|
| Application | .032 | -.043 |
| Process | | .217* |

* Correlation is on the niveau of 0.05 significant

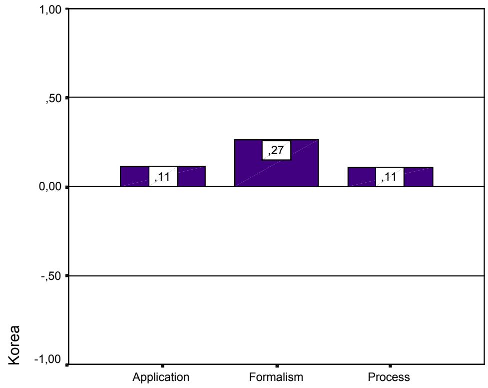
*Table 7-41 Correlation three factors of the Koreans*

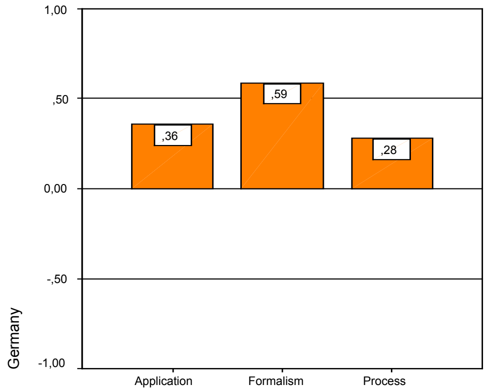
For the German teachers, the significant correlation between application and process is observed. On the other hand, for the Korean teachers, significant correlation between process and formalism is observed, although the correlation coefficient is not high.

### 7.3.3. Pupils' Beliefs

- Average of scores

The following graph shows to what extent the Korean and the German pupils agree that mathematics is application-oriented, formalism-oriented, and process-oriented.




*Graph 7-13 Average of the Korean pupils*

*Graph 7-14 Average of the German pupils*

Interestingly, the Korean and the German pupils share similar model of views about mathematics. However, the ratings given by the German pupils are higher than those given by the Korean pupils. The German pupils agreed more strongly with the given statements than the Korean pupils. The aspect of formalism is the dominating factor for both groups of pupils. The aspect of process is the lowest factor for both groups of pupils, although for the Korean pupils, the aspect of application is also lowest.

To find the significant differences between the Korean and the German pupils with respect to each of the three factors, the independent T-test is applied here.

- Independent samples T-test

| | | M | SD | p |
|---|---|---|---|---|
| Application | Korea | .11 | .50 | .000 |
| | Germany | .36 | .50 | |
| Formalism | Korea | .27 | .47 | .000 |
| | Germany | .59 | .48 | |
| Process | Korea | .11 | .48 | .000 |
| | Germany | .28 | .49 | |

*Table 7-42 T-Test between the Korean and the German pupils on each of the three factors*

As in the case of the teachers, this table also shows that there is a significant difference between the two groups with respect to application and process. But in this case a significant difference between the Korean and the German pupils as regards formalism can also be observed.

Thus we might deduce that the German pupils appreciate the application-oriented, the formalism-oriented and the process-oriented characteristics of mathematics to a greater extent than the Korean pupils do.

- **Correlation and associated *p*-values**

- Korea

The various correlation coefficients and associated *p*-values for the three belief variables and the cognitive values of the Korean pupils are presented in table 7-43.

The results show that application is significantly correlated only with process ($p=.018$). They also show a significant correlation between formalism and process ($p<.001$) and between formalism and basic competence ($p=.011$). In addition, they show that process is significantly correlated with three cognitive variables (basic competence $p=.016$; competence in proof and argumentation, $p=.003$; test, $p<.001$). However, all correlation coefficients are not high.

They show no significant correlation between application and each of the three cognitive variables given here (basic competence, p=.479; competence in proof and argumentation, p=.052; test, p=.081), nor do they show a significant correlation between application and formalism (p=.231). Moreover, they show no significant correlation between formalism and each of the three cognitive variables (competence in proof and argumentation, p=.177; test, p=.052; methodological competence, p=.389). In addition, they show no significant correlation between process and methodological competence (p=.459).

| | Formalism | Process | Basic competence | Competence in proofs | Methodological competence | Test |
|---|---|---|---|---|---|---|
| Application | .064<br>.231 | .128*<br>.018 | .077<br>.143 | .037<br>.487 | .041<br>.036 | .059<br>.252 |
| Formalism | | .263**<br>.000 | .135*<br>.011 | .072<br>.177 | .046<br>.389 | .101<br>.052 |
| Process | | | .129*<br>.016 | .157**<br>.003 | .040<br>.459 | .161**<br>.000 |
| Basic competence | | | | .416**<br>.000 | .128*<br>.015 | .610**<br>.000 |
| Competence in proofs | | | | | .198**<br>.000 | .841**<br>.000 |
| Methodological competence | | | | | | .188**<br>.000 |

*Table 7-43 Correlation coefficients and p-values for the Korean pupils (* p<.05. **p<.01)*

Even though there are significant positive correlations between formalism and basic competence, between process and basic competence and between process and competence in proofs, the coefficients are not high, therefore it might be concluded that there is no great meaningful relationship between each factors of beliefs and cognitive variables.

- Germany

The various correlation coefficients and associated *p*-values for the three belief variables and the cognitive values of the German pupils are presented in table 7-44.

In contrast to the case of the Korean pupils, the results show that application is significantly correlated with formalism (p=.002), with process (p<.001), and with methodological competence (p=.036). They also show that formalism is significantly correlated with process (p<.001) and that process is significantly correlated with test (p=.041).

However, they show no significant correlation between application and each of the three cognitive variables (basic competence, p=.479; competence in proof and argumentation, p=.052; test, p=.081), nor do they show a significant correlation between formalism and each of the three cognitive variables (basic competence, p=.841; competence in proof and argumentation, p=.561; test, p=.730; methodological competence, p=.171). Similarly, they show no significant correlation between process and each of the three cognitive variables (basic competence, p=.479; competence in proof and argumentation, p=.052; test, p=.081).

| | Formalism | Process | Basic competence | Competence in proofs | Methodological competence | Test |
|---|---|---|---|---|---|---|
| Application | .089**<br>.002 | .193**<br>.000 | .020<br>.479 | .055<br>.052 | .059*<br>.036 | .048<br>.081 |
| Formalism | | .250**<br>.000 | .006<br>.841 | -.016<br>.561 | -.039<br>.171 | -.009<br>.730 |
| Process | | | .053<br>.056 | .046<br>.102 | .052<br>.067 | .056*<br>.041 |
| Basic | | | | .296** | .144** | .577** |

| | | | | | | |
|---|---|---|---|---|---|---|
| competence | | | | .000 | .000 | .000 |
| Competence in proofs | | | | | .263**<br>.000 | .761**<br>.000 |
| Methodological competence | | | | | | .249**<br>.000 |

*Table 7-44 Correlation coefficients and p-values for the German pupils (* p<.05. **p<.01) (cf. Reiss, Hellmich & Reiss, 2002)*

Even though there are significant positive correlations between application and methodological competence, between process and achievement test, the coefficients are not high, therefore it might be concluded that there is no great meaningful relationship between each factors of beliefs and cognitive variables.

### 7.3.4. The relationship between the achievement test and beliefs about mathematics

From the results from the achievement test, three groups can be formed according to attainment in the test: a lower, a middle and an upper achievement group. I will examine the relationship between those groups with respect to their beliefs about mathematics.

The one-way analysis of variance (ANOVA) can be used to determine whether or not there were significant differences among the three groups. The results are as follows:

| | | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|---|
| Application | Between groups<br>Within groups<br>Total | .245<br>19.028<br>19.273 | 2<br>183<br>185 | .123<br>.104 | 1.179 | .310 |
| Formalism | Between groups<br>Within groups<br>Total | .615<br>14.066<br>14.681 | 2<br>181<br>183 | .307<br>.078 | 3.956 | .021* |
| Process | Between groups<br>Within groups<br>Total | 1.456<br>18.281<br>19.737 | 2<br>183<br>185 | .728<br>.100 | 7.287 | .001** |

*Table 7-45 The relationships between the Korean achievement groups with respect to the three belief factors*

The results indicate the significant effect of process and formalism with F=7,287 (p=0.001) and F=3,956 (p=0.021) respectively. This means there is a significant difference among the three groups for formalism and process. To find out in which pairings of groups significant differences can be observed, the Scheffe Process can be applied. The result is as follows:

| | | Middle group | High group |
|---|---|---|---|
| Formalism | Lower group | -.0314 | -.1360* |
| | Middle group | | -.1046 |
| Process | Lower group | -.0094 | -.1926* |
| | Middle group | | -.1832* |

*Table 7-46 Scheffe Process for comparisons between groups (* p<.05.)*

A significant difference between the effect of formalism (p<0.05) on the lower group and on the upper group can be observed here. Moreover, a significant difference between the effect of process on the lower and on the upper group (p<0.05) and between the effect of process on the middle group and on the upper group (p<0.05) can also be observed.

It can therefore be concluded that the Korean upper group believed more strongly than the lower group that mathematics is formalism-oriented. We might also deduce that the Korean upper group believed more strongly than the middle group, and the middle group more strongly than the lower group, that mathematics is process-oriented.

However, as shown in table 7-47, none of the corresponding significant differences between the German groups are observed.

| | | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|---|
| Application | Between groups<br>Within groups<br>Total | .451<br>126.061<br>126.512 | 2<br>640<br>642 | .226<br>.197 | 1.146 | .319 |
| Formalism | Between groups<br>Within groups<br>Total | .077<br>138.735<br>138.812 | 2<br>656<br>658 | .038<br>.211 | .182 | .834 |
| Process | Between groups<br>Within groups<br>Total | 1.744<br>244.650<br>246.394 | 2<br>656<br>658 | .872<br>.373 | 2.338 | .097 |

*Table 7-47 The relationships between the German achievement groups with respect to the three belief factors*

### 7.3.5. Beliefs and methodological competence

The pupils can be separated into three groups according to: a lower, a middle and an upper methodological competence group. These three groups need not necessarily coincide with the achievement groups. I will examine the relationships between those groups with respect to beliefs about mathematics.

The data was also analysed using the one-way ANOVA of the three beliefs factors for the three Korean methodological competence groups to determine whether or not there were significant differences among the three groups. The results are shown in the table below:

| | | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|---|
| Application | Between groups<br>Within groups<br>Total | .257<br>19.016<br>19.273 | 2<br>183<br>185 | .129<br>.104 | 1.237 | .293 |
| Formalism | Between groups<br>Within groups<br>Total | .096<br>14.584<br>14.681 | 2<br>181<br>183 | .048<br>.081 | .596 | .552 |
| Process | Between groups<br>Within groups<br>Total | .113<br>19.624<br>19.737 | 2<br>183<br>185 | .057<br>.107 | .528 | .591 |

*Table 7-48 The relationship between the Korean methodological competence groups with respect to the three belief factors*

In contrast to the relationship between the Korean achievement groups and beliefs, this table showed that there are no significant differences between the three Korean methodological competence groups with respect to each of the beliefs factors. Therefore it could be concluded that there is no relationship between beliefs and methodological competence for the Korean pupils.

The next table shows the relationships between the different German methodological competence groups with respect to belief factors.

| | | Sum of squares | df | Mean square | F | *p*-value |
|---|---|---|---|---|---|---|
| Application | Between groups<br>Within groups<br>Total | .497<br>126.016<br>126.512 | 2<br>640<br>642 | .248<br>.197 | 1.261 | .284 |

| | | | | | | |
|---|---|---|---|---|---|---|
| Formalism | Between groups<br>Within groups<br>Total | .667<br>138.145<br>138.812 | 2<br>656<br>658 | .334<br>.211 | 1.585 | .206 |
| Process | Between groups<br>Within groups<br>Total | 4.120<br>242.274<br>246.394 | 2<br>656<br>658 | 2.060<br>.369 | 5.578 | .004* |

*Table 7-49 The relationship between the German methodological competence groups with respect to the three belief factors*

For the German sample, there is a significant difference between the groups for only the process category as seen in table 7-49. To find out in which pairings of groups significant differences can be observed, the Scheffe Process can be applied. The results are as follows:

| | | Middle group | High group |
|---|---|---|---|
| Process | Lower group | -.0196 | -.1814* |
| | Middle group | | -.1618* |

*Table 7-50 Scheffe Process for comparisons between groups (* p<.05)*

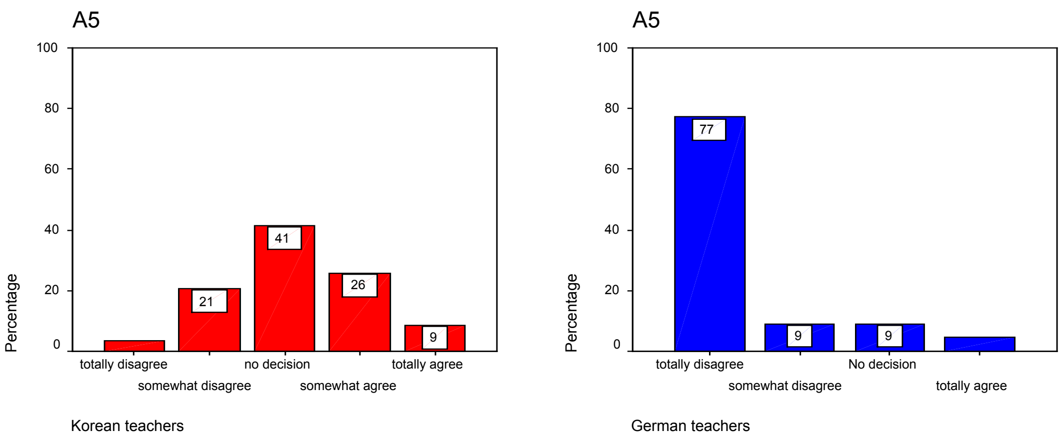
A significant difference between the effect of process on the lower group and on the upper group and between its effect on the higher group and on the middle group can be observed here.

This indicates that those German pupils who belonging to the upper methodological competence group believed more strongly that mathematics is process-oriented than the lower group and the middle group did.

### 7.3.6. Summary

Our data suggest that teachers' beliefs do not necessarily have influence on those of their pupils. There is such a tendency in the case of the Korean teachers and pupils, but there are important differences between the views of the German teachers and those of the German pupils. It could be explained that the German teachers aware the importance of the application of mathematics in the real society or aware that the application of mathematics is the social wishes, however, in the classroom, it might be not well embedded. Therefore, the German pupils have little appreciate this factor than other factors.

Moreover, there is a systematic difference between the Korean teachers and the pupils and the German teachers and the pupils with respect to the extent of their agreement with the statements. I will give one example in which this phenomenon is particularly evident.




*Graph 7-15 Distributions of responses for the statement A5*

This is probably due to cultural differences between the two countries. I suspect that Korean people are reluctant in expressing an absolute agreement or disagreement to a specific statement.

By analysing using the independent T-Test, one could observe that the German teachers tended to attach greater importance to the application-oriented and process-oriented characteristics of mathematics than the Korean teachers did. In addition, the German pupils were more likely to appreciate all three factors to a greater extent than the Korean pupils did.

By analysing the correlations between the two categories, we can see that the Korean pupils who believed that mathematics is formalism-oriented were likely to attain higher scores for basic competence. In addition, those Korean pupils who thought that mathematics is process-oriented were likely to achieve higher scores on the test.

Moreover, the German pupils who believed that mathematics is application-oriented were likely to achieve higher scores for methodological competence. In addition, those German pupils who thought that mathematics is process-oriented were likely to achieve higher scores on the test.

So in both countries, pupils who believed that mathematics is process-oriented were likely to achieve higher scores on the test.

The relationship between beliefs and achievement groups can be summarised by saying that the Korean upper achievement group believed more strongly that mathematics is formalism-oriented than the Korean lower group did. Also, the Korean upper achievement group attached greater importance to the process-oriented quality of mathematics than both the lower group and the middle group. However, there was no significant relationship between the beliefs and the achievement groups of the German pupils.

The relationship between beliefs and methodological competence groups means that the German pupils who belonged to the upper methodological competence group believed more strongly that mathematics is process-oriented than both the lower group and the middle group did. However, there was no significant relationship between the beliefs and the methodological competence groups of the Korean pupils.

# Chapter 8 Interview study

In this study, I will attempt to identify and describe the way pupils' constructed arguments and understanding of geometry concepts, using evidence from introspective think-aloud protocols of pupils as they attempted test problems. 15 Korean and 18 German 8th grade pupils took part in this interview study. I examine differences and similarities in proof strategies and sources of knowledge used to proof problems relating to the three competence levels.

Most of the examples in this chapter are transcribed from videotapes of the interview.

In section 8.1, some problems will be introduced and analysed. In section 8.2, I will focus on pupils' understanding of concepts. In the previous chapter, I explained that most Korean and German pupils are able to recall the concepts, yet a small proportion of the pupils are able to prove them. In section, 8.3, I will look at the different methods of arguments which pupils used. In section 8.4, I will analyse pupils' answers where they have had difficulties with their proof.

Before going further, I will shortly explain the theoretical framework and present the research questions again:

- Theoretical framework

As already explained in chapter 5, proof plays an important role in mathematics and also in mathematics class. The understanding of proofs is an important component of mathematical competence. However, proof is one of the difficult issues for pupils to learn. Recent studies have revealed wide gaps in pupils' understanding of proofs (Senk, 1985; Martin & Harel, 1989; Harel & Sowder, 1998; Healy & Hoyles, 1998; Reiss & Thomas 2001). Moreover, research indicates that students both at high school and university level have difficulty, not only in producing proofs, but also even in recognizing what a proof is (Galbraith, 1981; Fishbein and Kedem, 1982; Vinner, 1983; Chazan, 1993; Moore 1994).

Healy and Holyes (1997) showed that high attaining 10th grade pupils had deficits in their understanding of proofs and their ability to construct proofs (see more detail 5.5). A similar research in Germany with 13th grade pupils showed similar results as Healy and Holyes (Reiss, Klieme & Heinze, 2001) (see 5.5.1). Harel and Sowder (1998) identified 17 different proof schemes, which can be split up into three categories: (1) the external conviction (e.g. reference to a higher authority), (2) the empirical proof scheme (e.g. inductive or perceptual arguments), and (3) the analytical proof scheme (e.g. the deductive-axiomatic proof scheme) (more detail, see 5.2.4).

Tall (1995) described cognitive development of representation and proof. He argues that the cognitive development of a notion of proof should take into account the different forms of representation of which the learner is capable at various levels of sophistication. Even at the formal level, the use of the single word "proof" disguises the fact that there are many different views of proof, which stem from different historical and cultural contexts (Tall, 1995) (see. 5.3.4).

Keith Weber summarised clearly the difficulties even college students have with proving in his article (2003), principally discussing students' conceptions of proof, their notational difficulties and poor conceptual understanding and ineffective proof strategies. For example, one of students' most ubiquitous difficulties with the concept of proof is that students often believe that non-deductive arguments constitute a proof (Weber, 2003). Other common students' beliefs are as follows, those given in Harel and sowder (1998).

- *Ritual.* An argument is a proof if and only if it is in accordance with a specific mathematical convention. For instance, many pre-service teachers believe a geometry argument must be in a two-column format to be a proof (Martin and Harel, 1989).
- *Authoritative.* An argument is a proof if it is presented by or approved by an established authority, such as a teacher or a famous mathematician.
- *Inductive.* Checking that a general statement holds for one example, or perhaps several examples, is sufficient to demonstrate its veracity.
- *Perceptual.* By way of an appropriate diagram, one can visually demonstrate that a certain property holds (Harel and sowder, 1998). For instance, to prove that the sequence $1/n$ converges, some students will draw a diagram illustrating that as $n$ grows large, the terms of $1/n$ will become arbitrarily close to zero.

Another difficulty students have with proving is notational difficulties. Weber (2003) gave an example which is finding of Selden and Selden (1995). 61 students were asked to translate informal mathematical statement into the language of predicate calculus in introductory proof courses. Selden and Selden found that students were successful at this task less than 10% of the time. For instance, not one of the 20 students asked could express the statement "A function $f$ is increasing on an interval $I$ means that for any numbers $x_1$ and $x_2$ in I, if $x_1 < x_2$, then $f(x_1) < f(x_2)$." as a logical sentence (in Weber 2003).

Weber (2003) explained students' poor conceptual understanding using result of Moore (1994). Moore found that students could sometimes state a concept's definition while having little understanding of the concept. These students could not, for instance, describe the concept in their own words or generate a single example of this concept. When these students were asked to write proofs about this concept, they did not know how to begin (in Weber 2003).

- Research questions

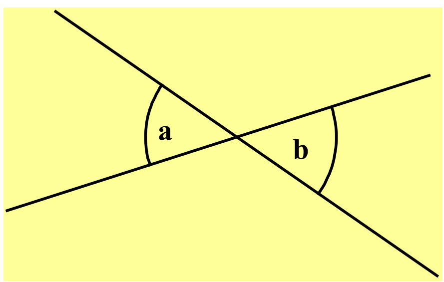
Based on above theoretical framework and the results from the quantitative test described in chapter 7, the interview study conducted with the following questions:

- What are the pupils' understandings of concepts? (see 8.2)
- What kind of arguments do pupils prefer? (see 8.3)
- What kind of difficulties in proving do pupils have? (see 8.4)

## 8.1. Analysis of test problems

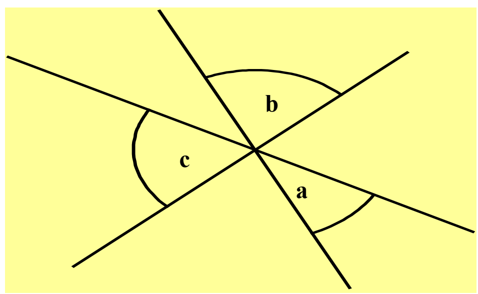
As seen below, the problems used in the interview were open-ended problems, where the pupils wrote down the answer, but could also freely explain orally the reasons for their answer.

Problem1.




Prove that $\angle a = \angle b$

Problem2.




Prove that $\angle a + \angle b + \angle c = 180°$

To solve the two questions above, concepts relating to angles and their properties were needed. The definitions of angles and their properties are introduced in the $7^{th}$ grade in both countries.

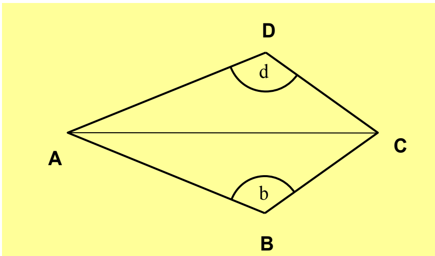
For the first question our focus was on whether or not the pupils knew the concept "opposite angle" and whether they could give alternative arguments as well as use this concept. The alternative arguments could be algebraically solved using the property that angles on a straight line add up to 180° or properties of an adjacent angle.

This question was not only the easiest but also the most difficult one for the Korean pupils. The reason for its easiness is that mentioning the concept alone was enough as an answer. In contrast, the reason for its difficulty is that the Korean pupils thought they should find another complex way to prove it than only giving the concept.

The second question was designed to allow an analysis of whether the pupils were familiar with the concepts of an opposite angle and of "the angles on a straight line" and could apply their properties.

Problem 4

The quadrilateral ABCD is with |AB|=|AD| and |BC|=|DC|.




Prove that $\angle b = \angle d$

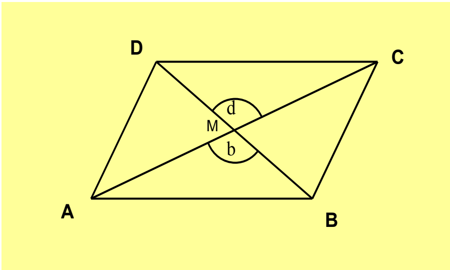
For this question, the focus was on whether or not the pupils understood first the concept of congruence and its conditions and on whether they could describe alternative arguments for example using properties of an isosceles triangle as well as the concept of congruence. In Korea, the concept of congruence and its conditions are taught in the $7^{th}$ grade. However its applications are dealt with frequently in the $8^{th}$ grade for proving some properties of triangles or rectangles.

The Side-Side-Side condition of congruence (SSS) could be used as one method for proving the result. It is given that |AB|=|AD| and |BC|=|DC| and since the line AC is common to the two triangles ADC and ABC, the condition SSS is satisfied and so the two triangles are congruent. Therefore $\angle b = \angle d$.

Another method, this time using the line BD, required pupils to show that triangle ABD is an isosceles triangle, as is triangle CBD. It then follows from the property that the base angles are the same in an isosceles triangle that ∠ADB=∠ABD in triangle ABD and ∠CDB=∠CBD in triangle CDB. Therefore $\angle b = \angle d$.

Problem5.

In the quadrilateral ABCD is with |MA|=|MC| and |MB|=|MD|.




Prove that (5-1) $\angle b = \angle d$ (5-2) |AB|=|CD|

This question can be solved using the geometry knowledge acquired in the 7th grade (using the properties of angles) and in the 8th grade (congruence and properties of a parallelogram). As a first step it was necessary to recall the term 'opposite angle' and its proof. As a second step the equality of the angle *b* and the angle *d*, as well as the given assumption, were used to solve the second question by showing that triangle DMC and BMA are congruent by condition SAS (Side-Angle-Side condition).

Problems 4 and 5 created some difficulties for pupils because of the figure of the quadrilateral ABCD, apparently a kite in Problem 4 and a parallelogram in Problem 5, respectively. In other words, in both cases, there was the chance that pupils might assume that ABCD was a kite or parallelogram without giving no further reasons or give incorrect arguments as a result of this assumption (from visual apprearance).

## 8.2. Understanding concept

The aim of this section is to analyse the reasons behind how pupils constructed their proofs, especially their understanding of concepts and the application of those concepts. In detail, when pupils understand concept, what kind of approach they use, such as visual approach or using properties which they already learnt. In addition, it is also quite interesting to see whether the pupils accept a concept as a possible proof or not.

Table 8-1 shows the frequency of the various answers given to the first problem. For the problems concerning concepts relating to angles, most pupils in Korea (9) and Germany (13) started off their answer with the words 'opposite angle'.

One Korean pupil and two German pupils gave a different and invalid concept for this question such as 'alternative angle' as an answer. They both realised later actually their answer was incorrect, and the correct answer is "opposite angle". This suggests they were familiar with the names of concepts such as 'opposite angle', ''alternate angle', 'corresponding angle' and so on. But they did not know exact name of concepts, i.e., they did not recognise which one was appropriate for this question.

| | | Correct Argument | | | Incorrect Argument | | | No response | Total |
|---|---|---|---|---|---|---|---|---|---|
| | | Concept given | Concept & Proof | Proof | Wrong Concept | Insufficient answer | Proof given, but wrong concept used | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Kor. | Not sure | 7 | 1 | | 1 | | 1 | 1 | 11 |
| | Sure | 2 | | 1 | | 1 | | | 4 |
| Ger. | Not sure | 2 | | | 1 | 1 | | 2 | 6 |
| | Sure | 11 | | | 1 | | | | 12 |

*Table 8-1 Answers given to first problem by frequency*

There is an interesting trend here. In the table we can see that most of the Korean pupils who merely gave a concept as an answer did not think their answers were correct. On the other hand, most German pupils thought that their answers were correct.

Korean pupils explained that there might be another way of solving the problem and that only one word was not enough to constitute a proof. One Korean pupil, Su-Yeon, explained that she learnt what she wrote in elementary school, but she was not sure, because what she had written did not seem to be enough to make for a right answer.

| | |
|---|---|
| Su-Yeon: 초등학교때 배웠는데, 맞꼭지각으로 같으면요, 각이 같다고 배웠거든요? 그러니까, 각 *a* 와 각 *b* 는 맞꼭지각으로 같아요. | Su-Yeon: I learnt in elementary school that if two angles are opposite angles then the angles are the same. Therefore angle *a* and angle *b* are the same because they are opposite angles |
| 그래요, 그러면 문제를 정확하게 풀었다고 생각하나요? | Interviewer: What do you think, did you solve it correctly? |
| Su-Yeon: 뭔가 하나 부족한 것 같은데… | Su-Yeon: No, something seemed to be missing. |
| 자, 그러면, 이 문제가 어려웠어요? | Interviewer: Was it difficult for you? |
| Su-Yeon: 아니요, 좀 당황했어요. | Su-Yeon: No, but it was a little bit confusing. |

*Table 8-2 Statement of Su-Yeon*

Andrea, one German pupil, explained why she was not sure that her answer was correct as follows:

> Andrea: *"ich habe es nicht direkt bewiesen, ich habe halt das, was wir vor längere Zeit schon mal bewiesen haben, einfach jetzt hingeschrieben."*
> Andrea: "*I did not prove it directly, - I only wrote down what we had proved a long time ago.*"

### 8.2.1. Understanding concept using visual approach (for sake of a figure)

In the problems figures were given, because they were used to enable the pupils to better understand the questions. However, some pupils did not use the deductive proof method. For example, some German pupils gave an answer using a visual approach. One of them, Andrea, explained her answer to the fourth problem as follows:

| German | Translate in English |
|---|---|
| Andrea: Das ist ja ein Drachenviereck, und deswegen, beim Drachenviereck sind gegenüberliegenden Winkel gleich groß und die hier werden halbiert, deswegen, ist der Winkel $\delta$ gleich $\beta$. | Andrea: This is a kite, and, therefore in the kite, the angles facing each other are same and here the angle is bisected (by the line), that's why the angle $\delta$ is the same as $\beta$. |
| Interviewer: Und was denkst du, hast du die Aufgabe richtig gelöst? | Interviewer: And what do you think, did you solve the task correctly? |
| Andrea: Ja, das glaub ich jetzt schon | Andrea: Yes, I do think so. |
| Interviewer: Ja, welcher Teil war da besonders schwierig? | Interviewer: Tell me, which part was especially difficult? |

| | |
|---|---|
| Andrea: Bei dem fand ich jetzt nicht schwer, weil das eine symmetrische Figur ist, also ein Drachenviereck. Und man hat schon mal alle Eigenschaft von dem gelernt, daher. | Andrea: I didn't find this one difficult, because of the symmetrical form of a kite. And we have already learnt all its properties that's why. |

*Table 8-3 Transcript of part of Andrea's interview*

Later she was asked to explain how she knew the figure was a kite. At first she could not explain how, but she soon realised that it was a kite because of the given assumptions.

### 8.2.2. Understanding concept using properties (or assumptions)

Another German pupil, Bettina, tried to prove the 4th problem by using the properties of a quadrilateral. At first, she explained that this quadrilateral was a kite, but she was not sure. Therefore she tried to find another way to prove it using an arc, a "*Fasskreisbogen*". Soon she knew that this would not lead her to the answer, either. Finally she wrote down "Because it is a kite" as an answer (or maybe as an argument). However, she was still not sure whether her answer was correct or not. She explained that the fact that "this quadrilateral is a kite" could be easily shown and that in fact one can see this. In addition, she explained "it follows from the condition given in the question that it is a kite".

Nam-Kyung, Korean pupil, recognised in Question 5 that the assumptions implied that the shape was a parallelogram. She began explaining "*the two diagonals bisect each other*", and she knew that the quadrilateral was the parallelogram from this property. She also added that if she had known the conditions for the quadrilateral to be parallelogram, she would have been able to solve the problem easily.

| Korean | Translate in English |
|---|---|
| Nam Kyung: 각 *a* 와 각 *b* 는 맞꼭지각이 되므로 같다고 할 수 있고,<br><br>AB 와 CD 는 이 사각형에서 대각선의 길이가, 대각선이 서로 다른 것을 이등분 하므로, 이 사각형을 평행사변형, 평행사변형이라고 할 수 있는데, AB 의 길이가, 평행사변형은 대변의 길이가 같으므로, AB 와 CD 의 길이가 같다고 할 수 있습니다. (풀이를 서술한다.) | NK (00:10:48) It can be proved that angle *a* and angle *b* are same as a pair of opposite angles.<br><br>AB and CD in this quadrilateral, the length of diagonal… the two diagonals bisect each other. Therefore this quadrilateral could be a parallelogram, and then the lengths of the opposite sides are same, and so the lengths AB and CD are same. [She then started to write the answer.] |

*Table 8-4 Transcript of part of Nam Kyung's interview.*

### 8.2.3. Just a concept or a valid proof?

As explained, most pupils only mentioned the concept and did not give any further arguments. Only one Korean pupil solved the problem correctly. In addition, one Korean and one German pupil tried to prove it with further arguments, but their arguments were insufficient. I will now give three examples of answers given. The first example (Figure 8-1), the answer of a Korean pupil, is a simple arithmetic calculation using the fact that the angles on a straight line add up to 180°. The second one (Figure 8-2 and Figure 8-3) is that of a German pupil, Bettina. Bettina at first gave only a concept as an answer, but on being asked to give further arguments, she explained her reasoning using arithmetic calculation. The last example (Figure 8-4) is that of a Korean pupil. She gave opposite angle as the required concept, but then tried to prove it with another way.

We will now consider the first example. Min-Sung has proved it simply using an angle he labelled '*c*' and the concept of "angles on a straight line". As seen in Figure 8-1, it is a simple but precise answer. However his answer is typical of the answer given in his Korean textbook for proving that opposite angles are equal.

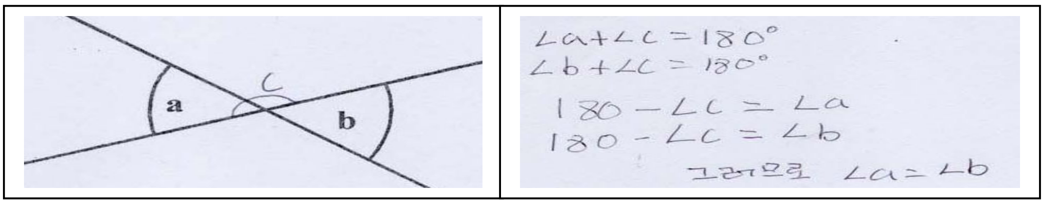



*Figure 8-1 Min-Sungs' answer*

The following examples are those of Bettina. At first Bettina gave only the sentence below as an argument: "Because they are opposite angles," which is a typical answer that most pupils have given (see Figure 8-2).

α = β da sie Scheitelwinkel sind

*Figure 8-2 Bettinas' first answer*

She said that she thought her argument formed a correct answer. In addition, she thought that it was easy to give an argument, because she was always required to give an argument in mathematics class at school.

She was then encouraged to continue by giving additional arguments. She drew a line to bisect the angles adjacent to angle $\alpha$ which she called "*Spiegelung*" (line of reflection). Using this line, and an angle adjacent to this line, she continued her proof as follows:

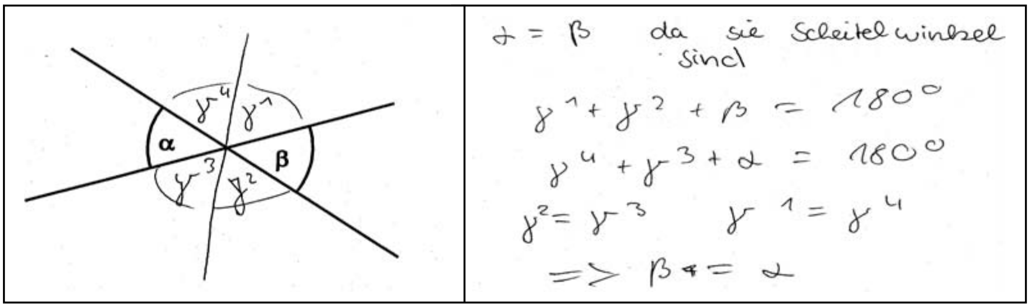
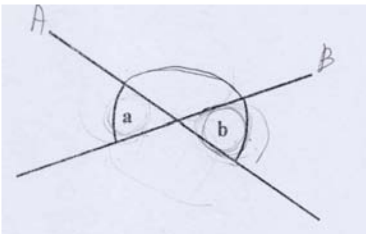
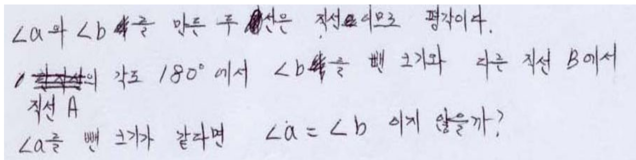



*Figure 8-3 Bettinas' second answer*

Since the "*Spiegelung*" (line of reflection or mirror) bisects both of the two angles adjacent to angle $\alpha$, it follows that $\gamma^2 = \gamma^3$ and $\gamma^1 = \gamma^4$.

The next example shows how a pupil went on to develop an adequate argument, although she had at first thought that the given statement was true simply because her teacher had told her.




The two lines which give rise to $\angle a$ and $\angle b$ also give rise to two more angles. So if the angles on line *A* add up to 180° and the angles on line *B* do too, then the angle above the intersection is 180°– $\angle$ *b* and is also 180° – $\angle$ *a*, then maybe $\angle a = \angle b$?

*Figure 8-4 A-Ras' answer*

A-Ra, one of the Korean pupils, struggled to solve the first question. She explained the concept she used but was not sure whether or not she had proved the statement correctly. Her processes of thought for the proof ranged from an empirical argument to an adequate argument.

> AR (00:00:30): *Prove that $\angle a = \angle b$… This is a line; if it is 180°… may I not use a protractor?*
> I (00:01:03): *We don't have one.*
> AR (00:01:29): *If 180° - $\angle a$ is the same as 180° - $\angle b$, then this angle [the angle above the intersection of the two lines in the diagram] is the same as that angle [the angle below the intersection of the two lines in the diagram], so $\angle a$ and $\angle b$ are equal. What can I do?*
> I (00:01:48): *Think slowly.*
> AR (00:01:52): *Has someone really proved this statement?*
> I (00:02:03): *Please tell me what you are thinking. It's important to tell me everything you think of.*
> AR (00:02:13): *From what my teacher has said, opposite angles are always the same.*
> I (00:02:37): *Ok. Then write down what you are thinking. [She writes part of her answer down.] Please read what you have written down.*
> AR (00:05:38): *The two lines which give rise to $\angle a$ and $\angle b$ also give rise to two more angles. So if the angles on line A add up to 180° and the angles on line B do too, then the angle above the intersection is 180°– $\angle b$ and is also 180°– $\angle a$, then maybe $\angle a = \angle b$?*
> I (00:05:57): *Do you think you have proved the statement correctly?*
> AR (00:06:09): *No, because I do not know the degrees in each angle for sure, therefore I cannot check how many degrees angle b is, although if it could be subtracted…*
> I (00:06:24): *So if degrees of angle were 40° or 50°?*
> AR (00:06:31): *Then I could prove it, but that isn't the case, and I can't use a protractor anyway. So I cannot exactly prove or disprove whether the angles are equal or not.*
> I (00:06:44): *Did you find the statement difficult to prove?*
> AR (00:06:48): *Yes, I always thought opposite angles are the same, it's a fixed idea I have. I never thought why it's the case. So it was difficult to prove.*

In this transcript three kinds of thought processes used to try to obtain a proof are evident. Firstly there is an empirical argument. She asked a protractor to measure the angles before she tried to use arguments for a proof. Secondly there is the assertion that the statement is true, in that her teacher has already told her so. Lastly we see a complete argument, even if the pupil was not sure of her answer. The last sentence makes clear what happens in a Korean classroom. In the classroom, the pupils might just memorise concepts for lack of time to think through or prove these concepts themselves.

### 8.2.4. Discussion

This section details the pupils' understanding of mathematical concepts. The results indicated that:

- Most pupils in Korea and Germany can recall the mathematical concepts; however, they could not argue why the concepts are true. This may be, as some pupils from both Korea and Germany explained, because they learn only the definition in class. Moreover, most Korean pupils did not accept mathematical concepts as a correct proof. They believed that there would be other, more complex methods for proving the result than simply giving a concept, while German pupils tended to convince themselves that they could accept mathematical concepts as a proof.

- Harel and Sowder (1998) identified 17 different proof schemes, which can be split up into three categories: (1) the external conviction (e.g. reference to a higher authority), (2) the empirical proof scheme (e.g. inductive or perceptual arguments), and (3) the analytical proof scheme (e.g. the deductive-axiomatic proof scheme). Even though Problem 1 is a very simple question, A-Ra, a Korean pupil showed how her proof scheme included an empirical argument, the assertion that the statement was true since her teacher had said so, and a deductive proof. It is quite interesting these three categories can be found within a single pupil's reasoning.

- Sometimes pupils merely accepted the fact that a teacher had said a statement was true rather than creating new arguments, as can be seen from the transcripts of A-Ra, Su-Yeon and Andrea. Pupils for whom this was the case might arguably be more likely to write an answer without carefully considering the problem or trying to create other arguments. Moreover, they might tend to view proofs as knowledge that must be memorised.

## 8.3. Process of possible arguments

To decide which procedure should be used to prove a problem, a pupil might draw on his/her previous knowledge and experience with related problems. She might give reasons only according to what she sees. In this section, the strategies or process used by the 8$^{th}$ graders in an attempt to construct a proof will be analysed. In addition, the way pupils constructed arguments will be described.

According to Tall (1995), Bruner (1966) formulated three kinds of representation such as enactive, iconic and symbolic. Following the ideas of Bruner (1966), Tall (1995) went further with the idea of associating the type of proof with the level of the students (See, 5.3.4).

In our interview, these classifications are applied to geometrical concepts and shapes such as opposite angles and parallelograms.

- Here, the **visual argument** is defined as an argument in which a figure is drawn but no logical deductive argument is given. The visual argument is identified with the level 0 which van Hiele called visualisation at the primitive level. This approach could lead a quick conclusion. However, one disadvantage is that it is then unclear whether the pupils have really understood the concepts and whether they can give other arguments.

- The **enactive argument** is used the way as Tall classified. It generally involves carrying out or visualising a physical action to demonstrate the truth of a given statement. The essential factor for this argument is the need for physical movement, real or visualised, as well as visual and verbal support to show the required relationships. These have disadvantages similar to those of visual arguments, since it is not clear whether the pupils have really understood the concepts, and whether they can construct their own proofs.

- **Argument with arithmetic using calculation** is defined by computational activity, usually with little use of geometrical properties. However, this does not mean that geometrical knowledge and properties are excluded from this category. This approach does provide an opportunity to avoid using unsatisfactory arguments such as visual arguments, but a small error may lead to the wrong answer.

- Last, a **geometrical argument** is classified as an argument using only geometrical concepts and properties. It usually includes some kind of geometrical sketch rather than an algebraic expression. Unlike enactive arguments or visual arguments, this approach does not lead to a quick conclusion.

### 8.3.1. Visual arguments

Pupils recognise figures from their appearance alone, but the properties of a figure are not perceived. For problem 4, for example, some pupils explained that the quadrilateral was a kite simply due to its appearance, but they did not really know why it was a kite, even though the condition given in the problem was supposed to be used to show this. Moreover, this visual argument sometimes lead to incorrect arguments since pupils made erroneous assumptions (See, 8.4.1).

Here are two statements made by the same pupil, Elena. The first one is part of her answer to the first problem, in which the concept of "opposite angles" was to be explained. Elena argued as follows:

> Elena: "*Winkel α und Winkel β sind gleich groß, weil die beiden Geraden sich (schreibt Begründung) an einem Punkt treffen, wo die beiden Winkel der Geraden gleich groß sind, deshalb sind sie auch gleich groß*"
> Elena: "*The angle α and angle β are equal, because both lines (She is writing her arguments down) meet at a point, where both of the angles of the lines are the same - that's why they are equal*."

As can be seen from her statement, her approach relied heavily on what she saw. She herself was not convinced by this approach, but she was not able to develop her argument.

The next statement is Elena's incorrect answer to Problem 2. At first she had no idea as to why the three angles add up to 180°. She then said:

> Elena: "*Ja, ich weiß, dass die Innenwinkelsumme im Dreieck z.B. 180° ist. Hier kann man sich so ein Dreieck denken, aber die Winkel würden dann woanders liegen, und keine Ahnung*"
> Elena: "*Well I know that the sum of the interior angles of a triangle is, for example, 180°. Here you can imagine a triangle like that, but the angles would then be somewhere else and I don't know.*"

Her answer does not seem to be relevant to the problem; she seemed to confuse a semi-circle with a triangle in her answer.

### 8.3.2. Enactive arguments

One of the Korean pupils did not actually make a physical movement, however, his argument was still an enactive argument. He explained his answer to the fourth problem as follows:

> Ji-Sang: "*여기 삼각형이요, ADC 하고 ABC 로 나눠서요, ABC 하고 ADC 하고 겹치면요, B 하고 D 하고 같이 포개지면, 같은 각이 나올 것 같아요*"
> Ji-Sang: "*For the triangles here, ADC and ABC, if ABC is folded onto ADC, then point B lies on point D, so the angles might be the same*"

Later I asked him what the term was used for the situation he had explained, but he didn't answer, even though he had heard the term 'congruence' and learnt about it too. He could recall terms. However, his explanation was a little unclear, since he used terms such as "AA congruence and SAS congruence".

However, he did not actually know which condition needed to be used in the given problem. It means he could recall some concepts; however, he could not recognise which one was appropriate for this question.

The misconception he had is clear from looking at his answer. In fact, there is no 'AA condition for congruence', but rather the "AA condition for similarity[13]". This confusion may well have stemmed from the fact that congruence and similarity are taught in the same grade in Korea.

Annika, one of the German pupils, also gave an enactive argument. She explained her answer by constructing a triangle.

> Annika: "*Also es ist ja so, $\overline{AC}$ ist die gemeinsame Strecke dieser 2 Dreiecke, dann haben wir, wenn ich jetzt das konstruieren würde, dann hab' ich jeweils den gleichen Schnittpunkt, weil $\overline{AB} = \overline{AD}$ ist, also angenommen, ich habe jetzt hier eingestochen, dann habe ich hier z.B. den Kreis und der zieht sich dann hier weiter und von C aus ist es auch der gleiche Radius (das ist halt meine Skizze) und dann ergibt sich jeweils D und B. Also hab' ich die 2 Punkte und der Winkel muss gleich sein, weil es ja im Endeffekt nur eine Spiegelung ist an der Achse $\overline{AC}$. Aber Begründen ist jetzt, ...*"
>
> Annika: "*Well, it's like this, $\overline{AC}$ is the common line of these two triangles, then we have, if I construct it now, then I will have the same intersection in each case, because $\overline{AB} = \overline{AD}$, let's suppose I make a point here [the point A], then I have got a circle here, and it is extended to here, and the radius from C is the same - that's on my sketch - then in each case we get D and B. So I have two points and the angle must be the same, because it's basically just a reflection at the axis $\overline{AC}$. But to find reasons for it ...*"

The interviewer then asked Annika what she would write down, to which Annika argued as follows:

> Annika: "*Also B ist Spiegelpunkt von D an der Achse AC, an der Spiegelachse und das ist wegen… weil die Strecken sich halt ähneln und das ergibt dann gleichen Schnittpunkt. Den gleichen Abstand der Achse AC oder zu C und A. So gibt's für mich Sinn, aber ob das jetzt als Begründung unbedingt so angenehm wäre für die Lehrerin, weiß ich nicht.*"
>
> Annika: "*Well B is the reflection of the point D at the axis $\overline{AC}$, in the axis of reflection, because the lines are similar length, therefore, this makes the same intersection. The same distance of the axis $\overline{AC}$ or to C and A. That's how it makes sense to me, but I don't know, whether that would necessarily be an acceptable reason to the teacher.*"

Her answer is actually similar to the answer of the Korean pupil Ji-Sang given above. Although she explained using the line of reflection, her answer was not clear. Her argument is not sufficient for a correct answer. She did not consider the properties she had learnt. It is worth noting that she mentioned the teacher's opinion, even though she was not asked what teacher' opinion was for her answer. Although she was not sure whether her argument would be suitable to justify the statement to be proved to her teacher, she intuitively believed her argument was correct. She also explained that it was difficult for her to write down her argument, although she had it in her head.

### 8.3.3. Arguments by calculation

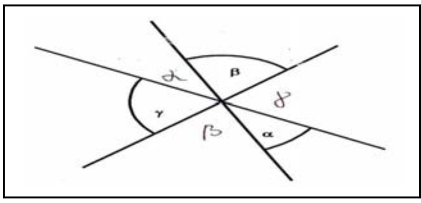
The answer of a German pupil, Bettina, to the second problem, is given here. In this problem, pupils were asked to prove that $\alpha+\beta+\gamma = 180°$. Her argument used the idea of the angle of a full rotation about a point and the concept of an opposite angle.

> Bettina: "*Also die Winkelsumme, das sind ja 360°- eine Drehung, und der Winkel da drüben (zeigt) $=\alpha$ wegen den Scheitelwinkeln*"

[13] AA similarity is when two corresponding vertex angles are the same in two triangles, then two triangles called to be similar in Korean mathematics class.

Bettina: "*So the sum of the angles, that's 360° - a rotation, and the angle up here [she points to it] is* $\alpha$*, because of opposite angles*."

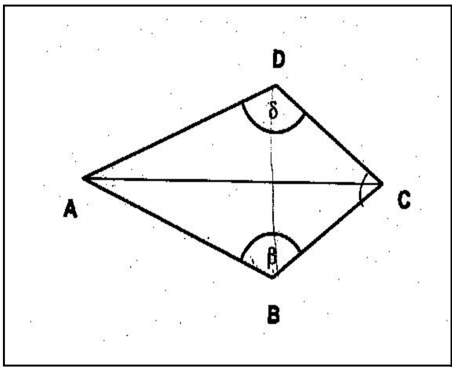
Although she orally mentioned the geometrical concept 'opposite angle', her answer consisted only of calculation as can be seen in Figure 8-5.




$$360° = 2\beta + 2\alpha + 2\gamma$$
$$360° = 2(\beta + \alpha + \gamma) \quad |:2$$
$$180 = \beta + \alpha + \gamma$$

*Figure 8-5 Bettinas' answer to the second problem*

The next answer is again of a German pupil. Katharina realised that the statement in the fourth question could be proved using the fact that the base angles of an isosceles triangles are equal. She began her argument by adding the line DB to the figure. Although her argument contained some mistakes, she tried to prove the statement by calculation too. She seemed to prefer to solve a question of this kind by calculation rather than by using the geometrical properties.




$\Delta$ ABC = gleichseitiges Dreieck

$\Delta$ BCD = "

~~$\measuredangle$ BAD = 180° − $(\frac{1}{2}\delta + \frac{1}{2}\beta)$~~

$\frac{1}{2}\delta + \frac{1}{2}\beta = 180° - \measuredangle BAD$

$\frac{1}{2}\delta + \frac{1}{2}\beta = 180° - \measuredangle DCB$

$\frac{1}{2}\delta + \frac{1}{2}\delta = \frac{1}{2}\beta + \frac{1}{2}\beta$

$\delta = \beta$

*Figure 8-6 Katharina's answer to the fourth problem*

Katharina: "*Ja, ok. Wenn man hier, z.B: wenn man einen Strich hindurch zieht, dann haben wir hier ein gleichschenkliges Dreieck. Ne, ein gleichseitiges Dreieck, das da und das da. Dann ist – ja ok… Also, wenn man diesen Winkel weiß, kann man die Hälfte von diesen und die Hälfte von* $\beta$ *ausrechnen, weil das ist dann Basiswinkel - nee das ist der Basiswinkel, und was heißen die, auf jeden Falls diese Winkeln hier, diese müssen gleich sein, weil, es verteilt ist und hier das genau selbe, wenn man diese zwei zusammen zählt, und diese zwei dann kommt dasselbe raus. Soll ich es aufschreiben?*"

Katharina: "*Yes, ok. If you, for example here, if you draw a line through there, then we have an isosceles triangles here. No, an equilateral triangle, one there and one there [she points to the two triangles]. Then it is – yes, ok… if you know this angle [she points to angle* $\delta$*], you can work out half of this angle and half of* $\beta$*. Because this is a base angle [she points to an angle forming part of* $\beta$*], that is the base angle (she means the angle forming part of* $\delta$*), and what are they called…? In any case, these angles here have to be the same, because it's shared, and exactly the same here, if you count these two together, and these two, then you get the same answer… shall I write that down?*"

Some errors are evident in her written proof. These mistakes will be dealt with more detail in section 8.4. Her answer incorporates an incorrect "definition" and some correct algebric calculation along with some incorrect statements. It seems to be consistent with the idea either that she has not tried to understand it, just made an assumption due to an incorrect definition of an equilateral triangle or that she has become confident that the statement was true just from looking at the figure.

### 8.3.4. Geometrical arguments

The following example, an answer to the second question, is that of a Korean pupil, Nam-Kyung. She only used geometrical knowledge such as the concept of opposite angles and the fact that the angles on a straight line add up to 180°.

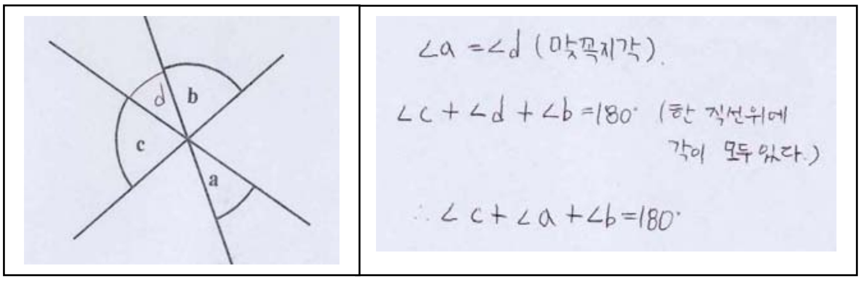
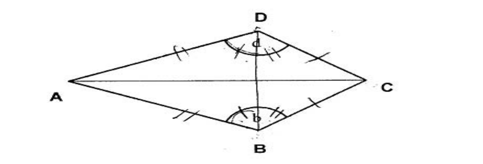
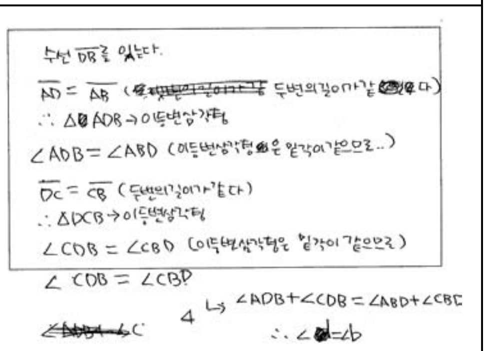



*Figure 8-7 Nam-Kyung's answer to the second problem*

> Nam Kyung: "*교차하는 대각선에서, 맞꼭지각의 크기는 서로 같으므로, a 와 여기, 이 각의 크기와 이 각의 크기는 같고, 직선위에, 직선위에, 직선위에서 각의 크기가, 직선은, 각의 크기가, 대각선에서 각의 크기가 합이 180 도니까, a,b,c 의 합은 180 도이다.*"
> Nam Kyung: "*Since the diagonal lines meet at one point, the opposite angles are same, a and this angle here are the same. For the straight line, the angles on the straight line add up to 180°, therefore the sum of a, b and c is 180°.*"

The next example is the answer of a Korean pupil, Su-Yeon. Like Katharina, Su-Yeon began by drawing the line DB. However, she found that the two triangles (triangle ADB and triangle CDB) are isosceles triangles, using the condition stated in the formulation of the problem. She therefore continued her argument using the properties of isosceles triangle as seen in Figure 8-8.



Draw a line connecting the points D and B.

AD=AB (the lengths of the two sides are the same)
∴ ΔADB → isosceles triangle
∠ ADB = ∠ ABD (since in an isosceles triangle, the base angles are equal)

DC=CB (the lengths of the two sides are the same)
∴ ΔDCB → isosceles triangle
∠ CDB = ∠ CBD (since in the isosceles triangle, base angles are same)
∠ CDB = ∠ CBD →
∠ ADB + ∠ CDB = ∠ ABD + ∠ CBD

∴ *∠d = ∠b*

*Figure 8-8 Su-Yeon's answer to the second question*

Su-Yeon showed a good and exact recall of concepts and strategies, performing confidently in this problem. Her answer is perfectly correct. However, she was not sure whether she had given a correct answer: "*Personally, from the knowledge I have, I don't know whether I've proved it.*" It is quite interesting to note that even though her answer was perfectly correct, she was not convinced of this. She had difficulty in explaining what kinds of properties isosceles triangles have. Moreover, she went on to explain that she was confused by the properties of isosceles triangles.

### 8.3.5. Discussion

As pointed out by Poincaré in the statement below, which is cited in Tall (1995), pupils have certain preferences in the methods of proof with which they tackle questions.

"... Among our pupils we notice the same differences; some prefer to treat their problems 'by analysis,' others 'by geometry.' The first are incapable of 'seeing in space', the others are quickly tired of long calculations and become perplexed" [Poincaré, 1913, page 212.] (Cited in Tall, 1995)

To summarise this section, four different types of argument are explained below:

- Visual arguments: when pupils explain the concepts using figures, it can enable them to feel that they understand, because it comes to seem obvious to them. However, one disadvantage is that it is then unclear whether the pupils have really understood the concepts and whether they can give other arguments (See, transcript by Elena, 8.2.1). Moreover, those pupils who have used a visual argument may not be able to use notation effectively nor recognise particular stages in the proof. In addition, they will not use geometrical properties they have already learnt in class. If the visual arguments are strongly linked to the formal argument, it might then be better for the pupils to improve their proof-writing skills. This brings pupils more confidence than only visual arguments alone.

- Enactive arguments: These have disadvantages similar to those of visual arguments, since it is not clear whether the pupils have really understood the concepts, and whether they can construct their own proofs. Enactive arguments also fail to address weaknesses in logical and adequate arguments.

- Arguments by calculation: This approach does provide an opportunity to avoid using unsatisfactory arguments such as visual arguments, but a small error may lead to the wrong answer (see, Figure 8-6). The pupils are required to refer to the definitions of geometrical concepts, but to attempt to work with calculation.

- Geometrical arguments: Unlike enactive arguments or visual arguments, this approach does not lead to a quick conclusion. Pupils who are able to use the concepts and properties are likely to produce accurate working. Even though pupils who use geometrical arguments have difficulties in writing their arguments down, they are frequently able to produce valid arguments. However, the arguments given may well simply have been learned by rote from textbooks.

The revised Korean and German curricula gave more recognition for pupils' intuition for solving geometrical problems. In particular, the intuitive understanding of concepts and the use of visual arguments were made acceptable as alternative methods to prove statements. However, it maybe made pupils be more likely to give insufficient arguments. Therefore, the intuitive understanding of concepts should be carefully taught. Moreover, pupils who use visual arguments should be given more time to attempt to do other arguments, for example, algebraic arguments or geometrical arguments. In this way, they might develop a better sense of possible strategies for formal proofs and improve their ability to recall definitions and results.

Even though our sample is small, interesting relationships are apparent. Here is the distribution of pupils by arguments used for Problem 2 and Problem 4.

| | Visual arguments | | Enactive arguments | | Argument by calculation | | Geometrical arguments | | No answer | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Korea | Germany | Korea | Germany | Korea | Germany | Korea | Germany | Korea | Germ any |
| P2 | | 1 | 1 | 1 | 2 | 4 | 12 | 8 | 1 | 4 |
| P4 | 1 | 4 | | | 0 | 4 | 12 | 4 | 1 | 6 |

*Table 8-5 Distribution of the second problem and the fourth problem*

As can be seen from table 8-5, visual or enactive arguments were hardly used by the Korean pupils whereas the German pupils used visual arguments more commonly. However, the most interesting feature of the table is that the Korean pupils tend to prefer to use geometrical knowledge to prove the given statements, while the German pupils tend to prefer to use arithmetic calculation to express their arguments. This raises certain questions. What does this difference stem from? Does the type of cognitive development proof depend mainly on the individual? Or is it caused by different curricula?

As explained in chapter 3, the German textbooks hold various methods to be acceptable, such as proof by calculation, by contradiction, or using symmetry or congruence. Therefore it might be concluded that the German pupils have various different possible ways to express their answers.

Perhaps the Korean pupils experience a proof consisting of a certain sequences of steps. Certainly, examinations in Korea are always written, while in Germany there are both written and oral examinations. Therefore, Korean pupils are only required to use formal arguments, which have the disadvantage that some Korean pupils learn proofs by rote. This should be avoided since the way pupils think about geometrical definitions and properties is as important as what they are learning to produce in the way of written work.

In a Korean classroom, the teacher often dominates in the class and determines what the pupils have to learn (Na, 1996). This experience often gives the pupils a feeling of security, but it also makes pupils learn too passively. On the other hand, in a German classroom, pupils play more often an active role and teachers used a more conversational style where they tried to involve the whole class in a discussion (Pepin, 1999b). She explained the tradition of Germany encouraged teachers to teach the class as a whole. Therefore there is a greater interaction between the teacher and the pupils. As a result, by means of communication with the teacher, pupils are able to explain their arguments and select the kind of argument they should use from various related arguments.

However, pupils are not given enough time to prove statements by themselves and to develop their skills in the strategies they use in proofs in a Korean class (Na, 1996). Moreover, pupils have little experience of using various methods in proofs they write independently. This can mean that some pupils only memorise the statements and even the steps of the proof without having any understanding of what they are writing. Sometimes pupils perform poorly simply because they have forgotten whatever they learnt. Therefore, pupils should be encouraged to work on their own more frequently.

The generally different ways in which Korean and German pupils argue can be seen as being a consequence of these different learning environments.

## 8.4. Difficulties in constructing proofs

Here I will analyse some difficulties the pupils in this study had in constructing proofs. There have been many research papers on pupils' difficulties in proving (e.g., Galbraith, 1981). Keith Weber summarised clearly the difficulties pupils have with proving in his article (2003), principally discussing pupils' conceptions of proof and their notational difficulties. The difficulties Weber considered also seem to arise in the answers of pupils in this study.

8.4.1. Intuitive understanding of concepts from visual appearance

One of the main problems in the learning of geometry is that sometimes pupils depend too much on their intuition, which means they seem unable to recall geometrical properties; instead, they appear to try to discern them from a figure.

As briefly mentioned in section 8.2 above, a few pupils understood the concepts from just a visual representation and argued their answers using a sketch, as Andrea did in her answer to the fourth problem (see. 8.2.1).

Elena's incorrect answer to problem 3 is another such example. This problem is broken down into three parts which each concerns concepts relating to angles. Her answer to the first part is given here.

> Elena: "*Das Ganze ist so, weil diese beiden Dreiecke sind ja kongruent und deshalb sind auch die Winkel gleich*."
> Elena: "*The whole thing is the following, because both these triangles are congruent and therefore the angles are the same as well*."

Her explanation of her answer to part (2), using the concept of congruence and the property that the sum of the angles in a triangle is 180°, is given in Figure 8-9 below. However, it is unclear whether the two triangles are really congruent or not, since the lengths of the sides are not given in the question. Elena did not check this, merely assuming that the triangles were congruent from the appearance of the shapes. Her answer clearly showed that she relied on what she saw.

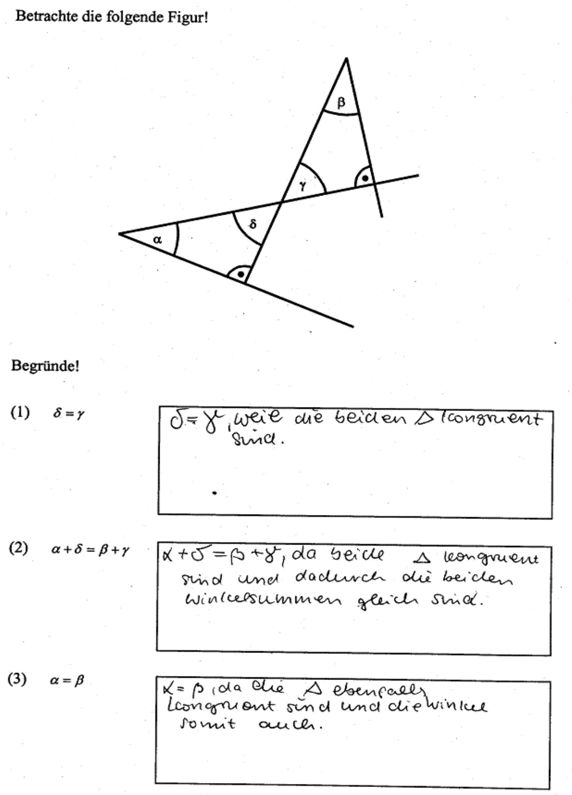



*Figure 8-9 Elena' answer to the third problem*

Another German pupil, Elvira, made the same mistake on parts (2) and (3) as Elena did above. She explained her argument with WSW, that is, ASA (Angle-Side-Angle condition of congruence), one of the conditions for congruence. But, she went on to explain the answer using the angles (90°, $\delta$ and $\gamma$), although she did not explicitly mention the property that the sum of the angles in a triangle is 180°.

> Elvira: "*Und die ($\alpha$) und die ($\beta$) auch gleich. Das sind Winkel-Seite-Winkel Satz, wenn man das hier drüber tun würde. Ok, $\alpha$ ist genau so gross wie $\beta$, weil die 90 Grad sind und dieser Winkel ($\delta$) und ($\gamma$) Scheitelwinkel sind.* "
> Elvira: "*and that ($\alpha$) and that ($\beta$) are also equal. That is the Angle-Side-Angle condition, if you would put this over there. Ok, $\alpha$ is exact same as$\beta$, because they are 90°, and this angle ($\delta$) and ($\gamma$) are opposite angles.*"

When the interviewer then asked her to explain what she was thinking, she gave the following answer:

> Elvira: *" Ja, also ich denke, diese Dreieck (ABC) ist kongruent zu dem (CDE), wenn das so ist…, denn, $\alpha$ muss genau so groß wie $\beta$ sein, und die Länge BC genau so lang, wie die CD, und AC ist genau so lang, wie CE, und AB ist genau so lang wie DE. BC, dieser Winkel ist 90 Grad, deswegen alle... soll ich jetzt aufschreiben?…*"
> Elvira: "*Yes, well I think this triangle (ABC) is congruent to that one (CDE); if that's right…, $\alpha$ should exactly be the same as $\beta$, and BC has exactly the same length as DC, AC has the same length as CE, and AB has the same length as DE. BC…, this angle is 90°, that is why all… Should I start writing now?*"

Her answer is satisfactory, but she did not seem to have made the idea of congruence clear to herself and in my opinion she gave relatively little explanation.

The pupils thought that when the three angles in one triangle are the same as the three angles in another triangle, then the two triangles are congruent. This is not necessarily the case. The pupils failed to notice that they had only satisfied the condition for similarity.

### 8.4.2. An error due to procedural difficulties

This example is Myung-Sun's answer to question 1. As can be seen from Figure 8-10, he had the right idea: he tried to prove the given statement using the fact that angles on a straight line add up to 180°. To do this he introduced angles *c* and *d*.

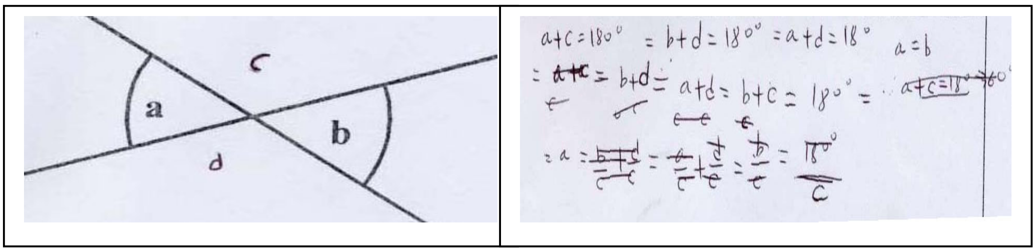



*Figure 8-10 Myung-Suns' answer to the first question*

There are two errors in his answer. The first one is that he used both angle c and angle d in his attempted proof. He should have used only one of them; otherwise he comes to a point at which he can make no further progress. As for the second one, Myung-Sun knew that angle c was common to the two expressions he derived and that it should be eliminated. However, he divided by c rather than subtracting c. Although he was unable to give a precise answer in this question, this was clearly what he was trying to do, and shows that his difficulties were procedural. He did not give any further arguments.

8.4.3. Difficulties in applying known concepts

Some pupils knew and could recall properties, yet they could not apply them appropriately. Ji-Sang, for example, gave an enactive argument for the fourth problem. After finishing all the problems, in the discussion of the proofs he had given, I asked him whether he knew the concept "congruence" and its properties.

> I: *Here in problem 4, you said that if one triangle can be folded onto another, then the angles are also the same, didn't you? What is it called, if one triangle can be exactly folded onto another?*
> Ji-Sang: *I don't know.*
> I: *Have you heard of congruence? Triangles are congruent. What have we learnt about congruence in class?*
> Ji-Sang: *AA congruence, SAS congruence*
> I: *SAS congruence for which two of the sides on one triangle are the same as two of the sides on the other, and the angles between the sides are equal. We learnt about SSS congruence, when the three sides are each the same length as the corresponding side on the other triangle, didn't we?*
> Ji-Sang: *Yes.*
> I: *Then here, which property of congruence must be used?*
> Ji-Sang: *SAS ...*

When he first tried to solve the problem, he made no mention of concept of '*congruence*'. He tried to argue his answer by explaining about one triangle being folded onto another, which is an enactive argument (see 8.3.2). Once I had given him a hint, as shown in the dialogue, it was clear that he was familiar with the conditions for the congruence of two triangles. However, he did not know which of the conditions should be used in this case.

Moreove, as mentioned in Section 8.3.2, there is no such thing as 'AA congruence', but rather an *'AA condition of similarity'*. Perhaps he confused the two.

One more interesting point was that some German pupils wrote or said "*F-Winkel*" (F-angle) to represent a "corresponding angle" and "*Z-Winkel*" (Z-angle) for an "alternative angle". They explained that they had learnt these names in class so that they could more easily remember the concepts. However, they could not recall the conventional names for these concepts; they remembered only the shorter names.

Jonas was later asked what an "*F-Winkel*" was, to which he said "*Außenwinkel*" (exterior angle) at first. Another interviewer corrected him, saying that it was a "*Stufenwinkel*" (corresponding angle). He agreed and explained in addition:

> Jonas: „*Wir haben im Mathe, nur F-Winkel gehört*"
> Jonas: "*We only heard about F-angle in math class.*"

8.4.4. False concept and wrong pre-assumption

Let us return to Katharina's answer to the fourth problem, which as mentioned in Section 8.3.3, contained some mistakes. Two of these are given below:

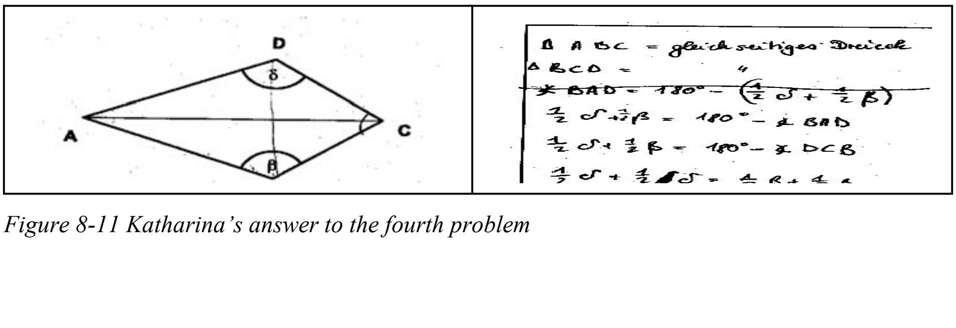



*Figure 8-11 Katharina's answer to the fourth problem*

At first, she wrote down that triangles ABC (by which we might assume she meant triangle ABD) and BDC are equilateral triangles. In fact from the condition given in the question, it only follows that they are isosceles triangles. She may have confused the definition of an equilateral triangle with that of an isosceles triangle (see the transcript below figure 8-6), because she could recall the term base angles. However, she did not exact know which angles the base angles were. So she could recall some concepts, but could not precisely match the terms with their meanings.

She drew an additional line and split both angle $\alpha$ and $\beta$ into "halves" for the purpose of her argument. However, it is not known whether the line DB bisects both angle $\alpha$ and angle $\beta$. Such a property would have needed to be checked, and if correct, proved. Nevertheless, she used it and gave arguments further to this. This was her second mistake.

She did not explain how she arrived at the second equation from last. However, we might speculate that she used a property of an equilateral triangle.

When asked whether she thought that she had proved the given statement correctly, she said that she had not, because:

> Katharina: "*Ja, mit dem Beweisen hier, dass $\beta$ und $\alpha$ überhaupt halt - keine Ahnung. Nicht so ganz geklappt*"
> Katharina: "*Yes, in proving that $\beta$ and $\alpha$ ... I don't know, I did not succeed*."
>
> Katharina: "*oh Gott, alles – ja, keine Ahnung, ja - hier mit dem Winkel. Z.B. Also ich konnte halt davor sagen, ich weiß es nicht, ob es davor richtig war, wie ich's gesagt habe, aber das aufschreiben war also schwieriger. Wenn man die genauere Schreibweise schreiben muss, so…*"
> Katharina: "*Yes, I don't have a clue, here with the angle, for example, well I could say it before, I do not know whether it was right then, the way I said it, but writing it down was even more difficult. If you have to write it in a precise way, then…*"

She felt that writing down her argument was more difficult than merely explaining it orally. Whether pupils generally agree with such a statement may depend on how much time they spend practicing writing proofs.

### 8.4.5. Discussion

Research indicates that pupils both at high school and university level have difficulty not only in producing proofs, but even in recognizing what a proof is (Galbraith, 1981; Fishbein and Kedem, 1982; Vinner, 1983; Chazan, 1993; Moore 1994).

It should be noted that it is sometimes difficult to give errors in proofs a name as I have attempted to above, because these errors are closely related.

The pupils in our sample did not give the impression that they had had much experience of providing mathematical arguments in geometry. For example, for problem 4, even though many pupils, both Korean and German, managed to argue using the properties of a kite, but some others only considered the shape of the figure. Many of those pupils who employed arguments using calculation made a simple mistake in the course of their calculation. Another major problem arose when pupils made an assumption based on an object's visual appearance and consequently gave insufficient reasoning in their answer. However, such a pre-assumption or the use of intuition can sometimes be a useful tool in geometry for finding the correct process to use in a proof. Therefore, pupils should support their logical and previous knowledge with their own intuition and with pre-assumptions they make.

Pupils who used visual arguments, for example Elena and Elvira, seemed to understand the concept of "congruence" well, yet were not able to show that two triangles were congruent in a particular case. Tall (1995) reasoned that such difficulties occur because the enactive or visual form of the proof does not suggest the sequence of deductions which should be used in a formal proof.

Even when pupils are able to understand and recall concepts accurately, this does not necessarily mean that they can construct proofs correctly, as shown by Ji-Sang's answer. It means knowing the definition of a concept does not ensure that pupils can really prove it. Even though he knew what some of the condition for congruence were, he could not actually explain what congruence was and he did not know which of the conditions should be used in Problem 4.

When pupils could not describe the concepts in their own words, they usually did not know how to begin their proof. This has been shown to be the case with college students, too (Moore, 1994). On this basis it is important for pupils to first understand concepts and properties sufficiently before they attempt to prove statements. It is also very important that they be able to recognise errors they make in their own proofs and check their answers thoroughly.

## 8.5. Summary

As noted in chapter 7, Korean and German pupils' answers in the achievement test show a lack of skills in proving and a poor understanding of geometry concepts. In this chapter, I have analysed in greater depth pupils' understanding of geometry concepts, the processes used in their proofs, and the difficulties they encounter in writing their own proofs.

Most of the pupils in both Korea and Germany could recall mathematical concepts such as "opposite angles are equal" and "congruence", yet they could not reason convincingly to show they are true. Moreover, most Korean pupils did not accept the use of mathematical concepts as representing a correct proof. They believed that there would be other, more complex methods for proving the result than simply giving a concept, while German pupils tended to convince themselves that they could accept mathematical concepts as a proof. However, the uncertainties the Korean pupils had may well have made them more capable of adapting their understanding of certain arguments.

In addition, sometimes pupils thought that they did not need to prove concepts, because they had only learned them in the classroom. However, this unnecessarily might let pupils only memorise the concepts. When the pupils memorise concepts without understanding of them, then they could forget them easily. Those who had forgotten what they were taught might then perform poorly for this reason.

However, some pupils prove questions with strategies based on rote learning rather than creative thinking. So the environment in which they were taught was perhaps more geared towards the rote learning of procedures.

Pupils have certain preferences as regards the approach they use in arguments (Tall, 1995). In this study, visual arguments, enactive arguments, arguments by calculation and geometrical arguments are examined. The pupils' processes of argumentation need not necessarily begin with the typical deductive methods the experts might expect.

Some pupils depend more on their intuition, some depend, for example in the case of geometry, on the visual appearance of an object. This may discourage pupils from trying to think logically; however, pupils might do well to learn how logical thinking and previous knowledge can be used together with their intuition and pre-assumption. The pupils who made visual arguments might benefit from more time to attempt to give algebraic or geometrical arguments. In this way, they might gain and then develop a feel for strategies they might use in formal proofs, and improve their recall of definitions and results.

When pupils could not describe the concepts in their own words, they tended not to know how to start proving the statements. This situation is apparent even in college students (Moore, 1994). This seems to suggest that it is important for pupils to fully understand concepts and properties at first, before going on to prove statements. It is also very important for them to recognise errors in the proofs they write and to check through their answers once they have written them.

Pupils seem to find writing proofs particularly difficult. Most of the Korean pupils interviewed for this study explained that this was because they have had little practice in writing proofs. Some Korean research papers (e.g. Na 1996) have showed that in an ordinary Korean classroom, pupils may not have enough time to practice proofs on their own.

The pupils in our sample are unlikely to have had much experience of giving mathematical arguments relating to geometry given their lack of understanding of the basic concepts. Even when pupils knew and could recall certain concepts, they could not apply them correctly. In addition, one Korean pupil (Myung-Sun) incorrectly manipulated an algebraic expression in a way one might expect from an elementary school pupil. Another common mistake came when pupils argued using calculation and made a simple computational error at some point. Pre-assumptions caused by visual appearance were another major problem which often led pupils to think no further argumentation was needed when in fact their answer was insufficient. However, in geometry such pre-assumptions and the related intuition are sometimes crucial to finding the correct process for the proof. Therefore, pupils should complement their ability to think logically and their previous knowledge with their intuition and with pre-assumptions they make.

As described by Polya (1957), an understanding of basic concepts, as well as the ability to apply properties adequately, is needed if a pupil is to be able to make a well-structured plan. According to Beoro (1999), logical argumentation should be part of the pupils' cognitive activity, as should the concurrent use of explorative, inductive and deductive processes. Therefore pupils need as much experience as possible of interaction with other pupils and of an interchange of arguments to prepare them to develop their own arguments independently.

# Chapter 9 Conclusion

This chapter will conclude the study by giving an overview of the results presented in Chapters 7 and 8 along with conclusions, implications and future researches.

The purpose of this thesis was to compare Korean and German pupils' competences in proofs and argumentation at the lower secondary level. Yet, it will be focused not only on showing how well Korean pupils or German pupils do in mathematics and examining on which level they are, but also on analysing the factors that contribute to pupil achievement. Therefore it is hoped to explain the different competence or achievement between Korea and Germany that is already shown in the international comparative study. To help explain my data, I compared the Korean educational system and mathematics curriculum, in particular the geometry component, with those of Germany. Moreover, I reviewed the results of international comparative research and the related literature.

## 9.1. Overview of Results

### 9.1.1. The achievement test (See 7.1)

**Question 1.** Recent studies imply that East Asian pupils on average outperform their Western counterparts at basic skills of computation and routine problem solving. Do Western pupils perform better than their East Asian counterparts on those problems concerning proofs and argumentation about geometry? This question remains to be sufficiently addressed. In this study, the competencies in proof and argumentation of Korean and German 7$^{th}$ and 8$^{th}$ grade pupils will be compared.

To briefly sum up the results, in this study for both the 7$^{th}$ and 8$^{th}$ graders, the German pupils performed significantly better on those problems relating to basic competence than the Korean pupils (cf. section 7.1, Table 7-4 and Table 7-5). On the other hand, the Korean pupils performed significantly better on those problems concerning competence in proof and argumentation (See, 7.1, Table 7-4 and Table 7-5). However, the results show that pupils in both countries have difficulties in proving. Our results show that the German pupils performed well on problems they could solve using simple arithmetic and procedural knowledge, while the Korean pupils attained higher marks for the problems on which they could argue on declarative knowledge and tasks that did not involve calculation alone.

In more detail,

- Basic competence

  The German 7$^{th}$ and 8$^{th}$ graders performed significantly better on the symmetry problem than the Korean pupils (cf. section 7.1). The Korean pupils had difficulties with the symmetry problems. This appeared to be the case in TIMSS, too. For example, German pupils performed relatively well in the multiple-choice problem (M02 in TIMSS), in which the pupils' knowledge of the symmetrical properties of a geometrical figure was tested. 64% of German 8$^{th}$ graders got the right answer; by contrast, only half of Korean pupils answered correctly. Korean pupils performed poorly on this problem as compared with other geometry problems in TIMSS.

As mentioned in Section 7.1, in Korea the concept "symmetry" is taught in grade five in art class, not mathematics class. By contrast, in Germany it is taught in mathematics classes in elementary school, but in secondary school it is recommended that the concept be repeated as an aid to understanding congruence. One might conclude that Korean pupils do not perform well on problems which are not extensively covered in their curriculum.

> For the problems that required calculation, most Korean pupils gave only the answers, neglecting to write down the arguments they used, while the German pupils gave the answers and also explained their reasons or their calculations. We might conclude that the Korean pupils were only concerned with obtaining the end product, namely the right answer, while the German pupils were more concerned with the process of working towards the answer.

Hwang (2000) gave one of the likely reasons: “Korean pupils are used to simply getting the right answer; however, they had difficulty in explaining or writing the argumentation or the results.” This statement was made with algebraic problems in mind, yet we might conclude that the same reason is valid for the two problems (i.e., problem 3 & 4 for the 7$^{th}$ grade test) discussed in Chapter 7. Another plausible reason for the Korean pupils only to give the answers might be the structure of the examination taken at least twice each semester in secondary school. In Korea, there is only a written test lasting 45 minutes, which is composed of about 20 multiple-choice questions and two or three open questions. Korean pupils should try to get the answers as fast as they can, because the questions are too many, so they do not have enough time to write down the whole process of calculation. Therefore, they merely write only the answer down without giving their reasoning. This familiarity with solving problems very quickly may have compelled the Korean pupils to avoid explaining or writing arguments in our test too. By contrast, two kinds of test, an oral test and a written test, are used throughout German secondary education. The oral test arguably helps pupils to explain their arguments and to write down their reasons. One German trainee teacher explained that in the mathematics test at school, German pupils are encouraged to write down all their arguments, even if the question is concerned only with calculation.

- Competence in proof and argumentation

> Even though most pupils in the two countries could recall the names of several concepts, 15% of the 7$^{th}$ grade Koreans and 20% of the 7$^{th}$ grade Germans still had difficulties, which means they have a lack of declarative knowledge (e.g. understanding of geometrical concepts, such as “opposite angles” and “angles on a straight line”). According to Reiss, Klieme and Heinze (2001), even pupils at the end of secondary level have considerable failings in declarative knowledge. They often have a vague intuitive understanding of concepts such as “congruence”, but this understanding is restricted to examples, and they do not have an exact understanding of the related definitions and theorems.

As Reiss and Heinze (2002) confirmed, in the German classes which participated in the test, the pupils who gave working or an explanation for even simple calculation problems performed better on those problems requiring competence in proof and argumentation. Shortly explained in section 7.1, similar situation is found in the Korean class too. 8% of the Korean pupils in the best-performed class gave answer with reasoning. 8% is the small portion, however, it is double as the average percentage for the code “answer with reasoning”. (See, table 7-11, in section 7.1). The obvious conclusion to be drawn is that pupils in Korean mathematics classes should be encouraged to include working even when solving simple calculation problems too. Reiss (2001) concluded that giving mathematical reasoning and arguments was an important part of the lessons of the class used in her study, and that being capable of reasoning and formulating meaningful arguments has a strong positive influence on achievement.

### 9.1.2. Methodological competence (See 7.2)

Our findings indicate that most pupils are highly competent to appreciate correct proofs to be correct and to accept their generality. However, pupils have also greater difficulty in recognising incorrect arguments to be incorrect than recognising correct proofs to be correct. This is consistent with the findings of the study on grade 13 pupils (Reiss, Klieme & Heinze, 2001) and with the findings of Healy and Holyes (1998).

- Empirical argument

Pupils had difficulties in recognising that the empirical argument was incorrect. Only 25% of the German and 54% of the Korean pupils realised that the empirical argument was incorrect. As for generality, a similar proportion of both groups (52% of the German and 55% of the Korean pupils) recognised that the given statement could not be generalised. 58% of the German pupils said that the empirical argument had a meaningful role to play as a potential explanation of the geometrical content of the proof to other classmates. On the other hand, only 16% of the Korean pupils believed that the empirical argument is suitable to use for explaining the concept to other pupils.

- Formal incorrect argument

The judgements on the formal incorrect argument should be noticed. Pupils had greater difficulties in recognising this argument to be incorrect: Only 17% of the German pupils and 41% of the Korean pupils recognised it as being incorrect. This means that pupils find it difficult to reject formally presented but incorrect arguments. In other words, the formal layout makes pupil convince that it seem to be correct. Besides, in both countries only about a third of the respondents realised the argument could not be generalised. However, 53% of the Korean and 59% of the German pupils agree in using this argument to explain the concept to other pupils. This shows that the form of the proof is important for the pupils to judge whether it is suitably accurate and general.

- Narrative proof

Most pupils were able to appreciate the narrative proof as being correct: 76% of the Korean pupils and 77% of the German pupils. Most pupils correctly accepted the generality of the narrative proof: 79% of the Korean pupils and 64% of the German pupils. In this case, a significant difference between the two countries with respect to the generality can be observed. The pupils in both countries (80% of the Koreans and 63% of the Germans) saw the narrative proof as an effective way of explaining the geometrical content to a classmate.

- Formal proof

As for the formal proof, most pupils in both countries recognised the correct formal proof as correct: 67% of the Korean pupils and 69% of the German pupils. Moreover, 77% of the Koreans and 59% of the Germans accepted the generality of this formal proof. In this case, there is a significant difference between the two groups with respect to the generality. 79% of the Koreans and 53% of the Germans saw the formal proof as a potential means of explaining the geometrical content of the proof to their classmates.

The pupils of both countries preferred to use narrative proof to explain the geometrical content to a classmate. The teaching style called the '*fragend-entwickelnder Unterricht*' in German mathematics classroom might be a reason. Some Korean pupils explained that narrative proof is softer or easier than formal proof. That is why they preferred narrative proof.

We might conclude that for the German pupils, the empirical argument also plays a meaningful role to explain the geometrical content of the proof to other classmates. Only 16% of the Korean pupils chose the empirical argument to explain the content to other pupils, while 58% of the German pupils did. The Korean pupils in general believed that the empirical argument is not suitable to use for explaining the content to other pupils. As mentioned in chapter 3, in the Korean textbook, it is clearly written that proof is the process of showing that a proposition is true without using an experiment or pratical work, even though experiments and practical work are used to show that a proposition is true. Because the textbook hold an important position in Korean mathematics classroom, the Korean pupils remembered well the above definition of proof. That is why only small portion of the Korean pupils chose the empirical argument as an explaining factor.

Lin (2000) has also conducted a study to find the Taiwanese pupils' methodological competence and compare it with that of the English pupils by using the problems used by Healy and Holyes (1998). None of the Taiwanese pupils who took part in his study chose to use an empirical argument as their own approach in a proof. He suggested that the difference in the styles of teaching and learning between Taiwan and England might be a reason for this. In Taiwan, pupils learn how to prove statements about geometry by developing from the incomplete formal argument they were already capable of to a formal proof. However, in England, there are two ways of learning to construct a proof in geometry: the empirical argument and the development from an incomplete argument to a formal proof. This might make British pupils believe that an empirical argument is to be sufficient as a proof. This is thought to happen in Germany too. It is also same reason for difference of pupils' view in an empirical argument between Korea and Germany.

### 9.1.3. Beliefs (See 7.3)

From a factoranalysis relevant to beliefs about mathematics, the three categories application, formalism, and process were taken to represent these beliefs. To sum up, one might conclude that the Korean pupils and the German pupils shared roughly similar beliefs.

Our data suggest that teachers' beliefs do not have necessarily influence on those of their pupils. There is such a tendency in the case of the Korean teachers and pupils, but there are considerable differences between the views of the German teachers and those of the German pupils. It could be explained that the German teachers aware the importance of the application of mathematics in the real society or aware that the application of mathematics is the social wishes, however, in the classroom, it might not be well embedded. Therefore, the German pupils have little appreciation application factor than other factors.

From an analysis of the results using the independent T-Test, it is clear that the German teachers tended to attach greater importance to the application-oriented and process-oriented characteristics of mathematics than Korean teachers did. In addition, German pupils were more likely to appreciate all three factors to a greater extent than the Korean pupils did.

**Question 2.** Beliefs about mathematics are currently regarded as an important factor in mathematics achievement. However, the relationship between achievement and beliefs is unclear, as is the relationship between beliefs and methodological competence. The corresponding results of Korean and German pupils will be compared in this study.
(2-a) Is there the relationship between achievement and beliefs? If then, which factor of beliefs could be influenced on achievement?
(2-b) Is there the relationship between methodological competence and beliefs? If then, which factor of beliefs could be influenced on methodological competence?

For the Korean pupils, there was a positive significant correlation between formalism and process ($p<.001$) and between application and process ($p=.018$), but no correlation between formalism and application. On the other hand, in the case of the German pupils, all three beliefs factors were significantly correlated with each other: $p<.001$ for formalism and process, $p=.002$ for formalism and application, and $p<.001$ for application and process.

Those phenomena above could be interpreted as follows:

- The correlation between formalism and process by pupils

The reason for correlation between formalism and process is clear. In school, when pupils solve problems (it means when pupils do process mathematics), they need quite many formulae, rules, verification of statements and logical structure. Therefore it could be explained why there is correlation between the two.

- The correlation between process and application by pupils

Process of mathematics is an activity such as discovery and quasi-empirical argument by Lakatos which is quite related to reality. The correlation between the two for the German pupils and for the German teachers is also observed. Moreover, real mathematics tasks might be more frequently dealt with in German class. Therefore, the German pupils appreciated that they need mathematics for the future, while the Korean pupils did not do well.

- The correlation between formalism and application by pupils

However, in this case, difference in the relation between Korea and Germany is observed. For the Korean pupils, there is no correlation, yet there is for the German pupils. Even though Korean curriculum suggests that real mathematics tasks should be dealt with in class, but it does not well work out. Real mathematics tasks are seldom dealt with in classrooms. Therefore, the Korean pupils might not have a view on application. That is why that many Korean pupils might do not agree that mathematics helps to solve daily tasks and problems. So, formalism including mathematical rigor might not be related with application for the Korean pupils.

For the German pupils, the reason of this correlation will be explained like a similar reason for the correlation between application and process. When the German pupils have tasks which are related to reality, they formulate them using mathematics formulae and so on. Therefore the German pupils might aware well that mathematics is needed in the future.

Analyses were conducted on various relationships between beliefs, methodological competence and mathematics achievement.

- By analysing the individual correlations between the three factors of belief and the cognitive variables, we can see that the Korean pupils who believed that mathematics is formalism-oriented were likely to attain higher scores for basic competence. In addition, those Korean pupils who thought that mathematics is process-oriented were likely to achieve higher scores for basic competence, for competence in proof and arguments and for the test.

- German pupils who believed that mathematics is application-oriented were likely to achieve higher scores for methodological competence, whereas those German pupils who strongly agreed that mathematics is process-oriented were likely to achieve higher scores on the test.

- So in both countries, pupils who believed that mathematics is process-oriented were likely to achieve higher scores on the test.

- **The relationship between beliefs and achievement**

There is no meaningful relationship between beliefs and achievement, however, the significant difference among achievement groups with respect to beliefs can be summarised by saying that the Korean upper achievement group believed more strongly that mathematics is formalism-oriented than the Korean lower group did. Also, the Korean upper achievement group attached greater importance to the process-oriented quality of mathematics than both the lower group and the middle group. However, there was no significant difference among achievement groups of the German pupils with respect to beliefs.

- **The relationship between beliefs and methodological competence**

There is no meaningful relationship between beliefs and the methodological competence, however, the significant difference among methodological competence groups with respect to beliefs can be summarised by saying that the German pupils who belonged to the upper methodological competence group believed more strongly that mathematics is process-oriented than both the lower group and the middle group did. However, there was no significant difference among methodological competence groups of the Korean pupils with respect to beliefs.

### 9.1.4. Curricula and textbooks (See chapter 3)

**Question 3.** There have been few research papers concerning the educational systems, curriculum and textbooks of Korea and Germany, even though these are important variables when considering a pupil's achievement. In this study, differences and similarities in the educational systems of the two countries are analysed. In particular, the Korean national curriculum and a selected German curriculum (that of Bavaria) and the respective textbooks, in particular the corresponding sections on geometry are compared.

(3-a) Are there differences and similarities in the educational systems of the two countries?

(3-b) Are there differences and similarities in the curricula and textbooks of the two countries?

Korea has a 6-3-3-4 school ladder system which is a set structure connecting the different school levels. The main track of the system comprises six years of elementary school, three years of middle school, three years of high school, and four years of university education or alternatively two- or three- year junior colleges. By contrast, the German school system has different ideals; therefore it is split into three-tier at the secondary stage. All children attend elementary school, for four years in most *Länder* but six years in two *Länder*. Children then have a choice between nine years at the *Gymnasium* (although eight years is becoming the norm), six years of *Realschule* or five years of *Hauptschule*. A noteworthy disparity between the two systems is that school years are not repeated in Korea, while in Germany, school years can be repeated even in elementary school. It is recommended that German pupils who repeat a grade more than twice leave their school and attend another school better suited to their academic achievement.

A comparison of the two countries' curricula shows that both curricula are specified precisely and exactly. Topics must be covered in a given school years and both have similar aims and priorities as regards the learning of mathematics. Goals and expectations are quite explicitly described in both curricula. Moreover, the guidelines for proving are precise and similar in the two countries.

The sections relating to geometry in the two textbooks are mostly identical. For example, the section on plane geometry in both textbooks is the proper material to help pupils understand the meaning of proof and the methods which can be employed. But the order in which topics are introduced is very different. The difference between the two systems arises in that Korean pupils learn plane and solid geometry in the same grade, while in Germany, plain geometry is taught in 7$^{th}$ grade and solid geometry is taught in 8$^{th}$ grade. Sometimes this fact may make Korean pupils confuse the congruence and similarity.

The Korean textbook starts by introducing simple problems that enable pupils to find out their knowledge, strengths and weaknesses by themselves. It consists of a few activities and problems of varying with the relative difficulty and on a range of topics. Such a starting to the textbook may be an effective means of encouraging pupils to take an interest in learning mathematics and be keen to continue learning. On the contrary, the German textbook does not always provide a gentle introduction to new topics. However, it does include many exercises for the pupils. Besides, textbooks do not form such an integral part of learning in the German classroom (Riley et al, 1999).

In both Korea and Germany, plane geometry is the main topic used to help 7$^{th}$ and 8$^{th}$ grade pupils understand the meaning of proofs. Pupils learn various methods of proof for statements that are valid in plane geometry. Proofs using the congruence of triangles are therefore regarded in both countries as a valuable method of investigating the properties of geometrical figures. The Korean textbook mainly is used to approach to a deductive proof, even though the Korean curriculum suggests that intuitive thinking and experiments should be used to develop a deductive approach. This might be due to Korean pupils losing their interest in mathematics. On the other hand, the German textbook presents various methods of proof and gives pupils time to consider them.

However, it is interesting to note that both Korean and German authorities plan to improve their curricula. For example, the Korean curriculum will focus more on pupils' interest in mathematics and their individual needs than their achievement, and more on active learning by pupils than on the passive learning characterised by listening to the teachers. In the wake of the TIMSS and PISA results, many German educators and policy makers discussed the state of German education system and its prospects for development. They intend to maintain the three-tier system for secondary education and suggest many principles should have involved the future curriculum. One of these principles is that of "Binding for all" which German educators emphasise as being of the utmost importance. The aim of this principle is to ensure that weaker pupils are not left behind. The German educators believe that this could make a valuable contribution to reducing disparities in the education system (Klieme et al, 2003).

### 9.1.5. Process of pupils' proof (See chapter 8)

**Question 4.** From the results I will gather in order to answer the research questions above, the way in which pupils go about proving results, for example the kind of approach they prefer, as well as the difficulties encountered in the process of proving and the pupils' understanding of concepts, will be investigated in more depth.
(4-a) What are the pupils' understandings of concepts? (see 8.2)
(4-b) What kind of arguments do pupils prefer? (see 8.3)
(4-c) What kind of difficulties in proving do pupils have? (see 8.4)

As explained in chapter 7, the Korean and the German pupils' answers in the achievement test show a lack of proving skills and a poor understanding of geometrical concepts.

Most of the pupils in both countries could recall mathematical concepts such as "opposite angles are equal" and "congruence"; however, they could not explain clearly why they are true. Moreover, most Korean pupils did not accept mathematical concepts as a correct proof. They believed that there would be another way, i.e., more complex methods for proving the result than simply giving a concept, while German pupils tended to convince themselves that they could accept mathematical concepts as a proof. However, these uncertainties of the Korean pupils have made them be more capable of approaching the formal proof and adapting their understanding of certain arguments.

In addition, sometimes pupils thought that they did not need to prove concepts, because they had only learned them in the classroom. However, this unnecessarily might let pupils only memorise the concepts. When the pupils memorise concepts without understanding of them, then they could forget them easily. Those who had forgotten what they were taught might then perform poorly for this reason.

Pupils have certain preferences as regards the approach they use in arguments. In this study, visual arguments, enactive arguments, arguments by calculation and geometrical arguments are examined. The pupils' processes of argumentation do not necessarily begin with the typical deductive methods the experts might expect.

Some pupils depend more on their intuition, some depend, for example in the case of geometry, on the visual appearance of an object. This may discourage pupils from trying to think logically; however, pupils should also learn how logical thinking and previous knowledge could be used together with their intuition and pre-assumption. The pupils who made visual arguments might benefit from more time to attempt to give algebraic or geometrical arguments. In this way, they might gain and then develop ideas for strategies they might use in formal proofs, and improve their recall of definitions and results.

There is a relationship between the understandings of geometrical and those of algebraic concepts, for example, the similarity of rectangles can be expressed in terms of proportions. As it can be observed from the interview study, some pupils' approach might consist only of using geometrical properties or algebraic calculation.

The pupils in our sample are unlikely to have had much experience of giving mathematical arguments relating to geometry given their lack of understanding of the basic concepts. Even when pupils knew and could recall certain concepts, they could not apply them correctly. In addition, one Korean pupil (Myung-Sun) incorrectly manipulated an algebraic expression in a way one might expect from an elementary school pupil. Another common mistake came when pupils argued using calculation, which means they made a simple computational error at some point. Pre-assumptions caused by visual appearance were another major problem which often led pupils to think no further argumentation. In fact it sometimes made pupils' answers insufficient. However, in geometry such pre-assumptions and the related intuition are sometimes crucial to finding the correct process for the proof. Therefore, pupils should complement their ability to think logically and their previous knowledge with their intuition and with pre-assumptions they make.

When pupils could not describe the concepts and proof in their own words, they tended not to know how to start proving the statements. This situation is apparent even in college students (Moore, 1994). This seems to suggest that it be important for pupils to fully understand concepts and properties at first, before going on to prove statements. It is also very important for them to recognise errors in their proofs and to check through their answers once they have written them.

## 9.2. Conclusions

Geometry is seen by teachers and mathematics educators as an important topic to learn and as being relevant to reality. Perhaps geometry is also seen as being one of those topics in mathematics which provide pupils with skills and knowledge which are directly related to their future life. Proof is also considered by mathematics teachers and educators to be an important part of school mathematics and as a useful method that should be understood and developed to help pupils think more creatively in later life.

However, according to the results of both the quantitative test and the interview study in this thesis, pupils struggle with the deductive formal proof. Moreover, the pupils' answers in the achievement test showed that they had a limited range of strategies for proofs and a poor understanding of concepts. In addition, pupils' proof-writing skills do not seem to be sufficiently well-developed, and they do not always seem to be able to apply the correct properties appropriately.

It might be sensible for pupils whose approach consists of visual arguments to write their thought process down as practice, even if it is not asked for in the question or it is not the method which they would normally be encouraged to use. Some pupils tend to rely on what their teachers have told them to be true when trying to determine the correct answer.

Pupils seem to find proof writing relatively difficult. Moore (1994) found that even college students found writing proofs difficult because they had little intuitive understanding of the concepts. Most of the Korean pupils interviewed in this study explained that their difficulties were due to their little practice in writing proofs. Some Korean researches, for example that of Na in 1996, have showed that in an ordinary Korean classroom, pupils may not have enough time to practice proofs on their own. Therefore, Na (1996) recommended that an interest in proof on the part of the pupils should be encouraged.

From the Korean pupils' answers, an impressive ability to memorise proofs is apparent. Perhaps the important role of memorisation, particularly in learning proofs stems from the traditional style of teaching and the examinations in Korea. In addition, some Korean pupils merely accepted the fact that a teacher had said a statement was true rather than they created new arguments by themselves (see 8.2.4). In a Korean classroom, the teacher often dominates in the class and determines what the pupils have to learn (Na, 1996). This experience often gives the pupils a feeling of security, however it also makes pupils learn too passively. The other problem is that Korean pupils are not given enough time to prove statements by themselves and to develop their skills or strategies they used (Na, 1996). Moreover, pupils have little experience of using various methods in proofs they write independently. This can mean that some pupils only memorise the statements and even the steps in the proof without any understanding of what they are writing. Sometimes pupils perform poorly just because they have forgotten whatever they learnt. Therefore, pupils should be encouraged to work on their own more frequently.

On the other hand, in a German classroom, pupils participated in class more actively and teachers used a more conversational style where they tried to involve the whole class in a discussion (Pepin, 1999b). She explained the tradition that encouraged teachers to teach the class as a whole. Therefore there is a greater interaction between the teacher and the pupils.

Moreover, Heinze and Reiss (2004) analysed proof process in the German mathematics classrooms. They identified that the typical proof process in the German mathematics classroom is planned and controlled by the teacher. The teacher leads the pupils through the "maze" of the proof situation. And the role of the pupils is more or less to guess the direction their teacher has in his/her mind. This teaching style is based on a sequence of short questions by the teacher and short answers by the pupils (called '*fragend-entwickelnder Unterricht*') (see, Heinze & Reiss, 2004, p.103). This teaching style is the most popular form of instruction in German secondary schools (cf. TIMSS video study Klieme, Schümer & Knoll, 2001). Heinze and Reiss (2004) also claimed this kind of teaching has the problem which there is no place for in-depth stages which are necessary in the proof process, e.g. for the exploration of the problem situation or the collection of additional information (p.103).

It was unable to identify above characteristics in our study; however, above characteristics (*fragend-entwickelnder Unterricht*) could be one of the reasons that the German pupils' performance is poor. Even though there are greater interaction between teacher and pupils, pupils probably did not have the feedback from the teacher rightly and immediately. That makes probably pupils confuse to reach the real answers.

The generally different ways in which Korean and German pupils argue can be seen as being a consequence of these different learning environments. This suggests that the traditional "teacher-centred approach" which still prevails in Korean classes should be replaced by a "pupil-centred approach". To be more precise, pupils should be encouraged to a greater extent to find things out for themselves and to independently enhance their skills in strategies they use for proofs. It is also important that pupils be encouraged to reflect on proofs they have seen and to find their own ways of thinking up and working through proofs.

Although many research papers have assumed that pupils' beliefs are one of the most important factors which have influence on their achievement, one might conclude from this study that this is not necessarily true. There was indeed a significant correlation between process-oriented belief and achievement for both the Korean and the German pupils, however its coefficient was low, it could be said that there is no meaningful relationship. In addition, a significant correlation between application-oriented belief and methodological competence was apparent in the case of the German pupils, even though the correlation coefficient is not high. However, for the German pupils there were no significant correlations between formalism-oriented belief and any of the three cognitive variables defined. As regards the Korean pupils, there were significant correlations between process-oriented belief and two of the cognitive variables, namely basic competence and competence in proof and argumentations, as well as between formalism-oriented belief and basic competence. However, there were no significant correlations between application-oriented belief and any of the cognitive variables for the Korean pupils.

This section is concluded with implications made by the research.

## 9.3. Implication

This study was prompted in part by a widespread view that the Korean and the German pupils' performance on proofs are poor.

From the results of both the quantitative test and the interview study, it seems that a small, but not a negligible proportion of the pupils in the sample struggled to understand some geometrical concepts.

Moreover, some of them could not recall the exact names of certain concepts. Of course, knowing the name of a concept does not require a full understanding of the concept, but it is the first step which must be taken in order to acquire this understanding. Teachers' encouragement to promote understanding for pupils by giving clear definitions should therefore be taken into consideration. This can also help pupils to generate strategies for proofs in a structured way.

Pupils could be expected to benefit a great deal by understanding why the properties they refer to work and how to apply strategies for proving.

The proof processes of the German pupils are more various than those of the Korean pupils as a tendency. Therefore in the Korean class, the various proof processes should be more suggested in general. Moreover, as our quantitative data shows, many pupils have no idea of this process, for example, Korean 16.9%; German 19.6%, for the problem 9 in the 7$^{th}$ grade test. In the 8$^{th}$ grade test, it is worse. In addition, some Korean pupils felt that writing their argument down is difficult in the interview study. The reasons were they were not often used to doing in the classroom. Therefore, pupils should also be given more time to explore arguments on their own and should be encouraged to write down the processes in their arguments, for example by stating the properties which are used and by explaining strategies orally. What those pupils who are not used to writing such detailed proofs might need at first is time to familiarise themselves with this, and then they need a gradual change in the way they write their answers, which would be an important step forward in their understanding of how to write proofs.

## 9.4. Suggestions for further studies

Even though it might be concluded that there are not meaningful the relationships between beliefs and achievement and between beliefs and methodological competence from this study, the relationship between process and achievement test is observed in both countries. It showed however, positive chance that beliefs could be the influenced factor on the achievement. Therefore, the questionnaire on beliefs used in this study should be more finely revised to a better ways to measure in the further studies.

A more detailed investigation of the use of mathematics textbooks in class might be necessary in order to find the various reasons for pupils' different approaches in proofs.

Improvements in the way in which pupils' learning about proof in secondary school should be focused on more strongly. Environments in which pupils are able to improve the skills they use in strategies for proofs could potentially then be found to have a strong positive effect on pupils' learning generally.

Even though there have been many international comparative studies, there has so far been no research conducted to directly compare Korea and Germany. It is hoped that this thesis contributes one step further in comparing the pupils' competence in proof between Korea and Germany and in analysing the principles behind the practices. Also, I hope that thes thesis contributes something to the exchange between the two countries of the different views which pupils have and their preferences in the arguments they use. It is my hope that the results in this thesis might also contribute to an increased interest in comparative research between Korea and Germany. The educational traditions of the two countries should be taken into consideration so that the advantages of each system are preserved.

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# Appendix A. Annotations (from chapter 2)

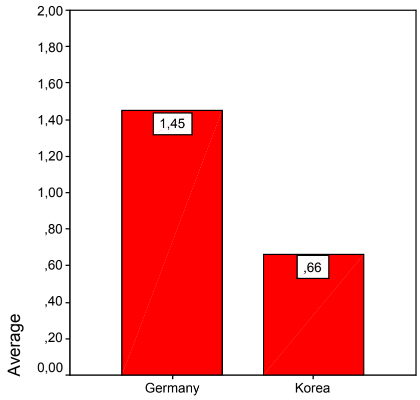
1. In some *Länder* special types of transition from pre-school to primary education *(Vorklassen, Schulkindergärten)* exist. In *Berlin* and *Brandenburg* the primary school comprises 6 grades.
2. The disabled attend special forms of general-education and vocational school types (in some cases integrated with non-handicapped pupils) depending on the type of disability in question. Designation of schools varies according to the law of each *Land*.
3. Irrespective of school type, *grades* five and 6 constitute a phase of particular support, supervision and orientation with regard to the pupil's future educational path and its particular focuses. In some *Länder*, the orientation stage *(Orientierungsstufe* or *Förderstufe)* is organised as a separate organisational unit independent of the standard school types.
4. The *Hauptschule* and *Realschule* courses of education are also offered at schools with several courses of education, for which the names differ from one *Land* to another. The *Mittelschule* (*Sachsen*), *Regelschule* (*Thüringen*), *Sekundarschule* (*Sachsen-Anhalt*), *Erweiterte Realschule* (*Saarland*), *Integrierte Haupt- und Realschule* (*Hamburg*), *Verbundene Haupt- und Realschule* (*Hessen, Mecklenburg-Vorpommern*) and *Regionale Schule* (*Rheinland-Pfalz, Mecklenburg-Vorpommern*), as well as comprehensive schools (*Gesamtschulen*) fall under this category.
5. The *Gymnasium* course of education is also offered at comprehensive schools (*Gesamtschule*). In the cooperative comprehensive schools, the three courses of education (*Hauptschule, Realschule and Gymnasium*) are brought under one educational and organisational umbrella; these form an educational and organisational whole at the *integrated Gesamtschule*. The provision of comprehensive schools (*Gesamtschulen*) varies in accordance with the respective educational laws of the *Länder*.
6. The general education qualifications that may be obtained after grades 9 and 10 carry particular designations in some *Länder*. These certificates can also be obtained in evening classes.
7. Admission to the *Gymnasiale Oberstufe* requires a formal entrance qualification which can generally be obtained after grade 10. At present, in most *Länder* the *Allgemeine Hochschulreife* can still be obtained after the successful completion of 13 consecutive school years. In some *Länder*, the *Allgemeine Hochschulreife* can either be acquired after 12 years of schooling, or the conversion in successive stages to 12 years of school education is currently under way. In other *Länder*, a shorter school education of 12 years up to the *Allgemeine Hochschulreife* is an optional offer.
8. The *Berufsoberschule* has so far only existed in a few *Länder* and offers school-leavers with the *Mittlerer Schulabschluss* who have completed vocational training or five years' working experience which is the opportunity to obtain the *Fachgebundene Hochschulreife*. Pupils can obtain the *Allgemeine Hochschulreife* by proving their proficiency in a second foreign language.
9. The *Fachoberschule* is a school type lasting two years ($11^{th}$ and $12^{th}$ grades) which takes pupils who have completed the *Mittlerer Schulabschluss* and which qualifies them for higher education *Fachhochschulreife*. Pupils who have successfully completed *the Mittlerer Schulabschluss* and who have been through initial vocational training can also enter the *Fachoberschule* directly in the $12^{th}$ grade.
10. *Berufsfachschulen* are full-time vocational schools differing in terms of entrance requirements, duration, and leaving certificates. There is a special form of the two-year *Berufsfachschule* that requires a *Mittlerer Schulabschluss* for admission leading to a state-recognised examination as an assistant. One or two-year courses at *Berufsfachschulen* offer basic vocational training. Under certain conditions the *Fachhochschulreife* can be acquired on completion of a course lasting a minimum of two years.
11. Extension courses are offered to enable pupils to acquire qualifications equivalent to the *Hauptschule* and *Realschule* leaving certificates.
12. *Fachschulen* cater for vocational continuing education (1-3 year duration) and as a rule require the completion of relevant vocational training in a recognised occupation and subsequent employment. In addition, the *Fachhochschulreife* can be acquired under certain conditions.

13. Including institutions of higher education offering courses in particular disciplines at university level (e.g. theology, philosophy, medicine, administration science, sport).
14. The *Berufsakademie* is a tertiary sector institution in eight *Länder* offering academic training at a *Studienakademie* (study institution) combined with practical in-company professional training in keeping with the principle of the dual system.

# Appendix B Descriptive results

## B-1. Test for the 7th graders

Problem 1.

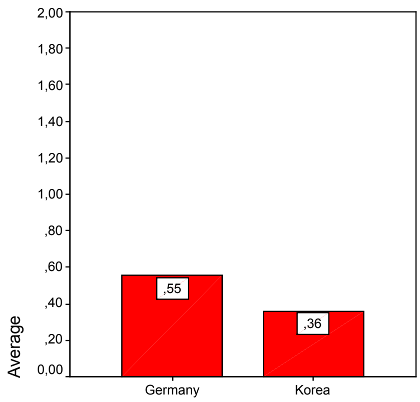



Problem 1

| | Korea | Germany |
|---|---|---|
| No answer | 15.3 | 3.6 |
| Wrong answer | 18.0 | 2.1 |
| More than 2 lines of symmetry | 17.5 | 19.9 |
| Only one line of symmetry | 32.3 | 3.8 |
| Right answer | 16.9 | 70.6 |
| Total | 100.0 | 100.0 |

Table B-1-1 Answers given by percentage to problem 1

Problem 2.

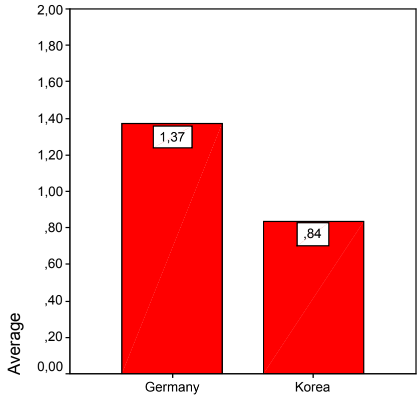



Problem 2

| | Korea | Germany |
|---|---|---|
| No answer | 23.8 | 24.1 |
| Wrong answer | 36.0 | 12.0 |
| Translation | 1.1 | 2.3 |
| Reflection | 2.6 | 12.0 |
| Wrong centre of rotation | 4.2 | 11.2 |
| Wrong angle of rotation | .0 | 2.4 |
| Right constructed, not noted | 22.8 | 1.1 |
| Right construction, wrong noted | 5.8 | 15.5 |
| Right construction (with small error) | | 7.6 |
| Completely right working | 3.7 | 11.8 |
| Total | 100.0 | 100.0 |

Table B-1-2 Answers given by percentage to problem 2

Problem 3.

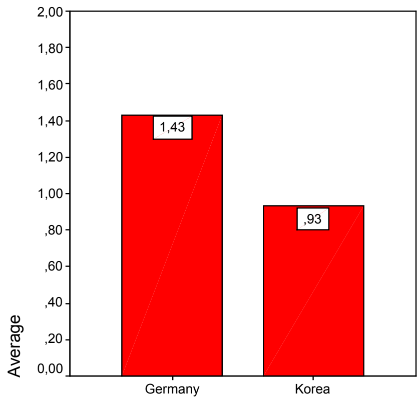



Problem 3

| | Korea | Germany |
|---|---|---|
| No answer | 4.8 | 3.2 |
| Incorrect works | 6.3 | 1.5 |
| Wrong answers | 13.2 | 5.0 |
| Right (without calculation) | 67.2 | 38.4 |
| Right (with calculation) | 2.6 | 22.6 |
| Right (with incorrect reasoning) | 0.5 | 5.3 |
| Right (with sum of angles in a triangle) | 0.5 | 9.0 |
| Right (with base angles) | 1.1 | 6.4 |
| Right answer (with both reasons) | 3.7 | 8.6 |
| Total | 100.0 | 100.0 |

Table B-1-3 Answers given by percentage to problem 3

Problem 4.

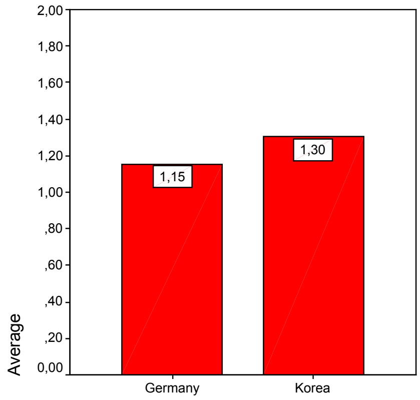



Problem4

| | Korea | Germany |
|---|---|---|
| No answer | 3.7 | 0.8 |
| Incorrect working | 1.6 | 1.4 |
| Incorrect answers | 10.6 | 12.4 |
| Right answer (without calculation) | 75.1 | 27.8 |
| Right answer (with calculation) | 3.7 | 25.8 |
| Right answer (with opposite angles) | 1.1 | 7.9 |
| Right answer (with adjacent angles) | 0.5 | 2.3 |
| Right answer (with both reasons) | 3.7 | 21.7 |
| Total | 100.0 | 100.0 |

Table B-1-4 Answers given by percentage to problem 4

Problem 5.

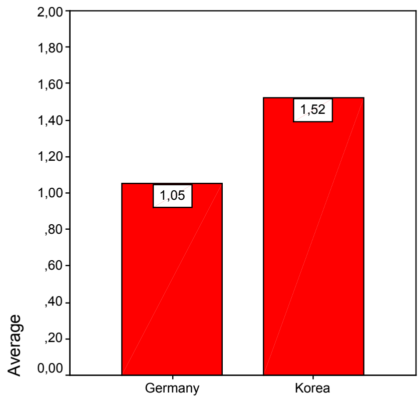


Problem 5

| | Korea | Germany |
|---|---|---|
| No answer | 6.9 | 9.4 |
| Incorrect working | 7.4 | 8.3 |
| Incorrect arguments (e.g. tautological) | 15.9 | 18.8 |
| Measured (without calculation) | 2.6 | 0.9 |
| Basically right, but only described | 4.2 | 9.6 |
| Right (concept of opposite angle mentioned but not explained) | 40.2 | 29.9 |
| Right (concept given and explained) | 16.9 | 21.5 |
| Right answer (corresponding reasons) | 5.8 | 1.5 |
| Total | 100.0 | 100.0 |

Table B-1-5 Answers given by percentage to problem 5

Problem 6a.

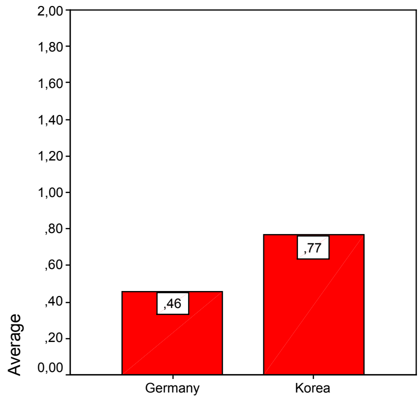


Problem 6a

| | Korea | Germany |
|---|---|---|
| No answer | 15.3 | 26.7 |
| Incorrect working | 6.9 | 6.8 |
| Incorrect arguments (e.g. tautological) | 1.1 | 4.2 |
| Wrong concept (Corresponding angle, opposite angle) | 0.5 | 8.2 |
| Meaningful sentence, not completely | 0.0 | 2.6 |
| Right (concept of alternative angle mentioned but not explained) | 41.3 | 39.9 |
| Right (concept given and explained) | 34.9 | 11.5 |
| Total | 100.0 | 100.0 |

Table B-1-6a Answers given by percentage to problem 6a

Problem 6b.

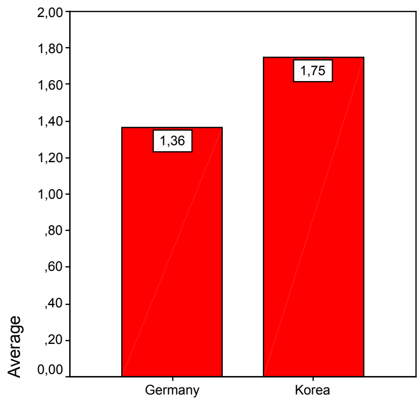


Problem 6b

| | Korea | Germany |
|---|---|---|
| No answer | 29.6 | 43.2 |
| Incorrect working | 29.1 | 26.7 |
| Meaningful sentence, not completely | 5.8 | 14.1 |
| Right reasons with small error | 18.0 | 7.1 |
| Right with formula | 5.3 | 1.8 |
| Right (with triangles or square) | 1.6 | 2.4 |
| Right (with circle or half circle) | 10.6 | 4.6 |
| Total | 100.0 | 100.0 |

Table B-1-6b Answers given by percentage to problem 6b

Problem 7.

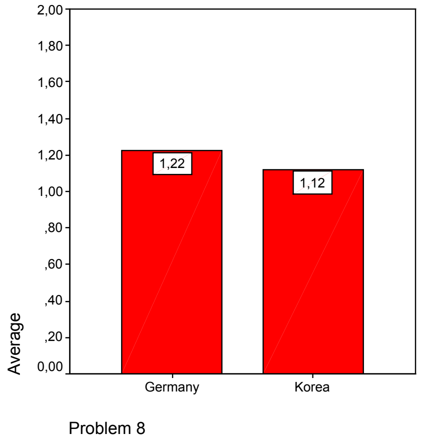


Problem 7

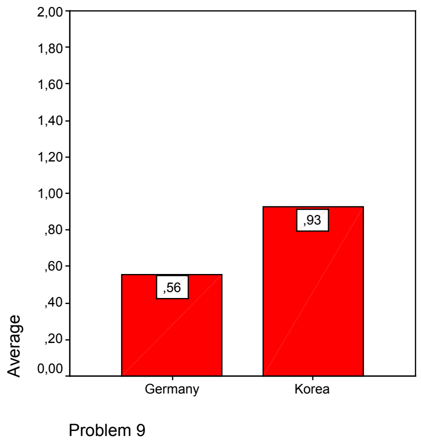
| | Korea | Germany |
|---|---|---|
| No answer | 4.8 | 3.3 |
| Incorrect working | 1.1 | 11.4 |
| Measured | 0.0 | 0.8 |
| Wrong calculation | 6.9 | 12.4 |
| Right answer (wrong reasons) | 0.0 | 7.4 |
| Incorrect answer (right reasons) | 0.0 | 1.2 |
| Right answer (without reasons) | 73.0 | 38.1 |
| Right answer (with wrong concept) | 0.5 | 2.4 |
| Right answer (only corresponding angle or adjacent angle) | 9.5 | 13.2 |
| Right answer (corresponding angle and adjacent angle) | 4.2 | 9.7 |
| Total | 100.0 | 100.0 |

Table B-1-7 Answers given by percentage to problem 7

Problem 8.



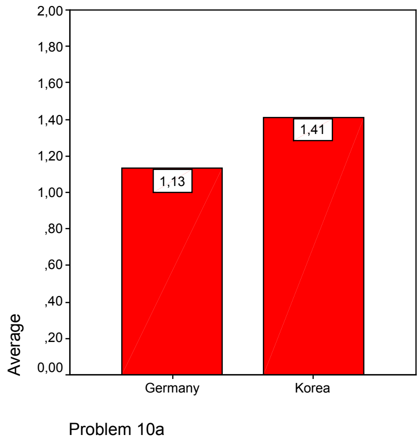

Problem 9.




| | Korea | Germany |
|---|---|---|
| No answer | 16.9 | 19.6 |
| Incorrect working | 14.3 | 14.7 |
| Incorrect arguments used | 3.2 | 20.8 |
| Wrong concepts used | 2.1 | 15.5 |
| Only giving the “opposite angle” argument | 27.0 | 14.6 |
| Only giving the “half-circle” argument | 9.5 | 4.2 |
| Correct (using both reasons) | 27.0 | 10.6 |
| Total | 100.0 | 100.0 |

Table B-1-9 Answers given by percentage to problem 9

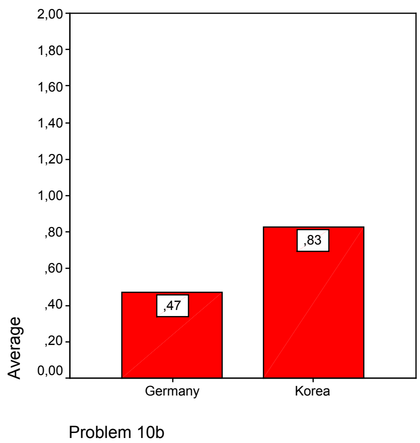
Problem 10a.




| | Korea | Germany |
|---|---|---|
| No answer | 9.0 | 11.5 |
| Incorrect working | 6.3 | 17.3 |
| Incorrect concept (corresponding angle, adjacent angle) | 9.5 | 10.0 |
| Right reasons with small error | 9.0 | 9.1 |
| Right | 66.1 | 52.0 |
| Total | 100.0 | 100.0 |

Table B-1-10a Answers given by percentage to problem 10a

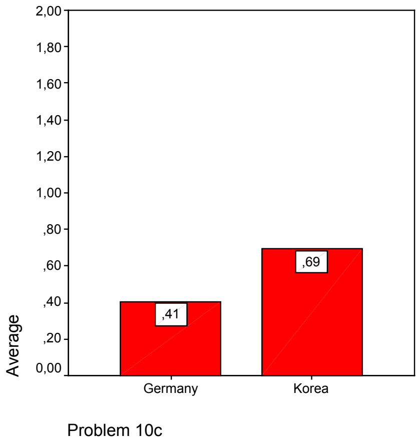
Problem 10b.




| | Korea | Germany |
|---|---|---|
| No answer | 17.5 | 21.5 |
| Incorrect working | 25.4 | 46.0 |
| Incorrect concept (corresponding angle, alternative angle) | 5.8 | 3.6 |
| Right reasons with small error | 20.1 | 10.9 |
| Right (with sum of angle and 10a) | 31.2 | 17.9 |
| Total | 100.0 | 100.0 |

Table B-1-10b Answers given by percentage to problem 10b

Problem 10c.

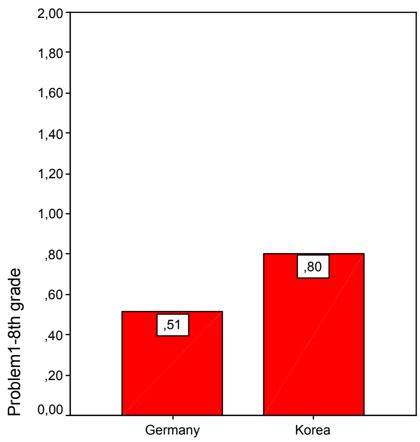



| | Korea | Germany |
|---|---|---|
| No answer | 23.8 | 28.7 |
| Incorrect working | 21.2 | 44.3 |
| Incorrect concept (corresponding angle, alternative angle) | 4.2 | 3.5 |
| Right reasons with small error | 32.3 | 6.5 |
| Right (with sum of angle and 10a) | 18.5 | 17.0 |
| Total | 100.0 | 100.0 |

Table B-1-10c Answers given by percentage to problem 10c

## B-2. Test for the 8th graders

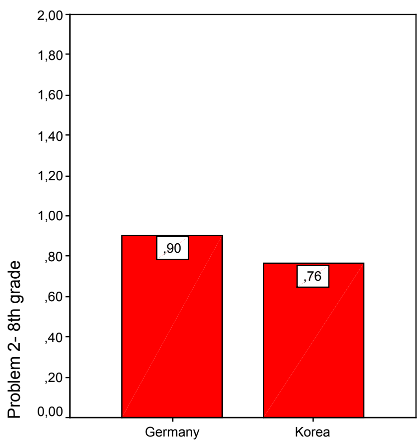
Problem 1.




| | Korea | Germany |
|---|---|---|
| No answer | 16.5 | 1.5 |
| Incorrect working | 26.9 | 61.6 |
| Wrong answer (with the right reason) | 0.5 | 5.9 |
| Right answer (with an invalid reason) | 3.3 | 2.3 |
| Right answer (with an insufficient reason) | 29.1 | 14.4 |
| Right answer (with the right reason) | 23.6 | 14.4 |
| Total | 100.0 | 100.0 |

Table B-2-1 Answers given by percentage to problem 1

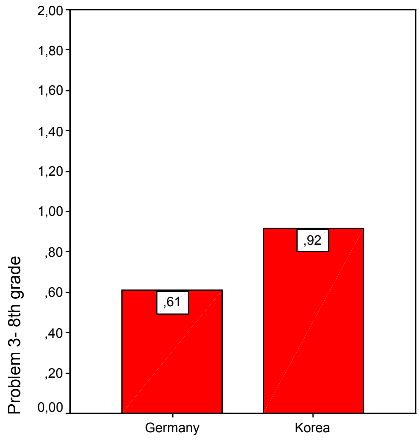
Problem 2.




| | Korea | Germany |
|---|---|---|
| No answer | 19.2 | 11.9 |
| Incorrect working | 9.9 | 8.1 |
| Translation | 0.5 | 1.5 |
| Reflection | 2.7 | 12.1 |
| Wrong centre of rotation | 8.2 | 0.6 |
| Wrong angle of rotation | 9.9 | 0.8 |
| Right construction, points A', B', C' not labelled | 12.6 | 3.6 |
| Right construction, incorrect labelling of points | 1.6 | 4.4 |
| Right construction, steps of working not given | 6.6 | 6.4 |
| Right construction, only one step of working given | 1.6 | 25.2 |
| Right construction, all steps of work given | 26.8 | 25.4 |
| Total | 100.0 | 100.0 |

Table B-2-2 Answers given by percentage to problem 2

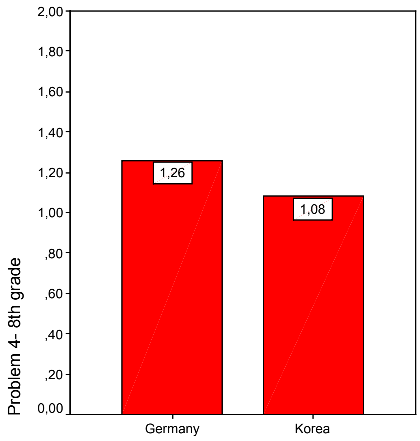
Problem 3.




| | Korea | Germany |
|---|---|---|
| No answer | 10.4 | 5.9 |
| Incorrect working | 28.0 | 33.3 |
| Wrong reasons (SSA. SAS) | 3.3 | 13.8 |
| Right answer (with a slight error) | 24.7 | 32.8 |
| Right answer | 33.5 | 14.2 |
| Total | 100.0 | 100.0 |

Table B-2-3 Answers given by percentage to problem 3

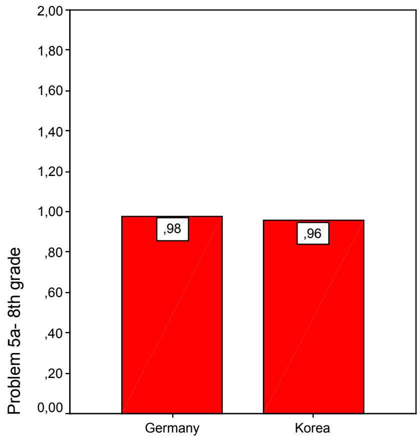
Problem 4.




| | Korea | Germany |
|---|---|---|
| No answer | 12.6 | 9.8 |
| Incorrect working | 9.9 | 4.0 |
| Measured | | 2.5 |
| Wrong answer (with the right reason) | 1.1 | 1.3 |
| Right answer (with an invalid reason or with no reason) | 5.5 | 8.3 |
| Right answer (with an insufficient reason) | 7.1 | 13.1 |
| Right answer (with the calculation reason) | 33.0 | 18.8 |
| Right answer (with the right reason) | 30.8 | 42.2 |
| Total | 100.0 | 100.0 |

Table B-2-4 Answers given by percentage to problem 4

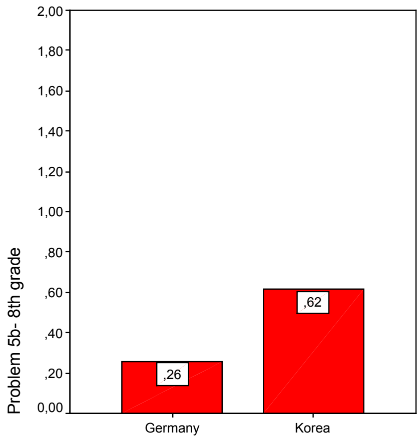
Problem 5a.




| | Korea | Germany |
|---|---|---|
| No answer | 19.2 | 8.5 |
| Incorrect working | 26.4 | 36.3 |
| Measuring | 1.1 | 0.2 |
| Right answer (insufficient reason) | 7.1 | 12.1 |
| Right (concept of opposite angle mentioned but not explained) | 20.3 | 29.7 |
| Right (concept given and explained) | 20.3 | 12.5 |
| Right answer (sufficient reasons) | 5.4 | 0.6 |
| Total | 100.0 | 100.0 |

Table B-2-5a Answers given by percentage to problem 5a

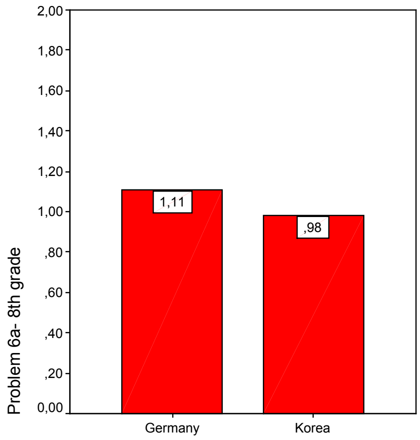
Problem 5b.




| | Korea | Germany |
|---|---|---|
| No answer | 26.2 | 15.2 |
| Incorrect working | 28.0 | 20.1 |
| Measured | | 0.9 |
| Incorrect reason (SSW, WSW) | 0.0 | 5.1 |
| Right with small error | 8.8 | 10.0 |
| Right (SWS) | 15.9 | 7.8 |
| Right answer (corresponding reasons) | 20.9 | 40.9 |
| Total | 100.0 | 100.0 |

Table B-2-5b Answers given by percentage to problem 5b

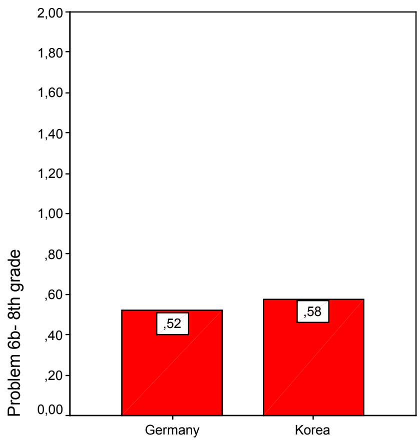
Problem 6a.




| | Korea | Germany |
|---|---|---|
| No answer | 34.1 | 16.3 |
| Incorrect working | 10.4 | 8.5 |
| Incorrect arguments | | 6.1 |
| Wrong concept (Corresponding angle. opposite angle) | 2.7 | 11.7 |
| Meaningful sentence, not completely | 7.1 | 3.8 |
| Right (concept of Wechselwinkel mentioned but not explained) | 19.8 | 43.8 |
| Right (concept given and explained) | 25.8 | 9.8 |
| Total | 100.0 | 100.0 |

Table B-2-6a Answers given by percentage to problem 6a

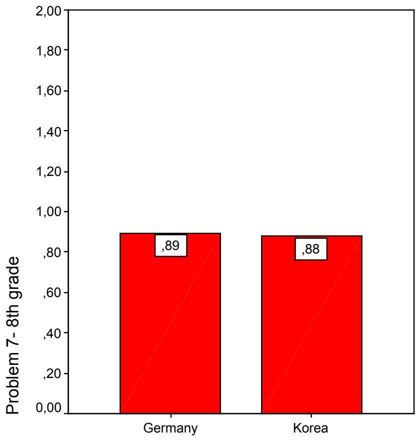
Problem 6b.




| | Korea | Germany |
|---|---|---|
| No answer | 47.8 | 43.4 |
| Incorrect working | 22.0 | 26.1 |
| Meaningful sentence, not completely | 2.7 | 9.1 |
| Right reasons with small error | 2.2 | 4.5 |
| Right with Formula | | 0.6 |
| Right (with triangles or square) | 11.0 | 7.4 |
| Right (with circle or half circle) | 14.3 | 8.9 |
| Total | 100.0 | 100.0 |

Table B-2-6b Answers given by percentage to problem 6b

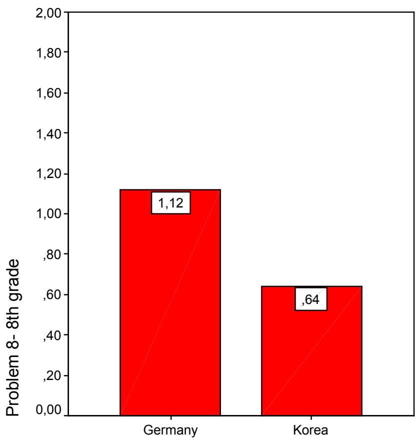
Problem 7.



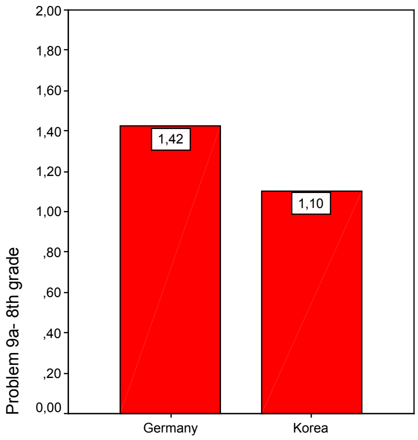

| | Korea | Germany |
|---|---|---|
| No answer | 29.1 | 11.6 |
| Incorrect working | 21.4 | 26.9 |
| Measure | 0.0 | 0.6 |
| Incorrect reason (SSA, SAS) | 0.0 | 7.2 |
| Right with small error | 11.0 | 18.2 |
| Right working (ASA or isosceles triangle) | 38.5 | 35.6 |
| Total | 100.0 | 100.0 |

Table B-2-7 Answers given by percentage to problem 7

Problem 8.



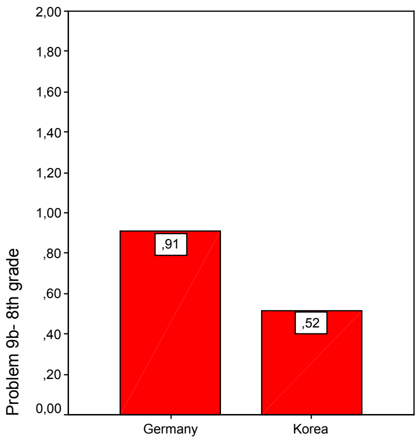
Problem 9a.




| | Korea | Germany |
|---|---|---|
| No answer | 31.3 | 8.1 |
| Incorrect working | 12.1 | 10.2 |
| Incorrect concept (Wechselwinkel, opposite angle) | 1.1 | 7.8 |
| Meaningful writing | 0.5 | 5.5 |
| Right (concept of corresponding angle mentioned but not explained) | 24.2 | 57.0 |
| Right (concept given and explained) | 30.8 | 11.4 |
| Total | 100.0 | 100.0 |

Table B-2-9a Answers given by percentage to problem 9a

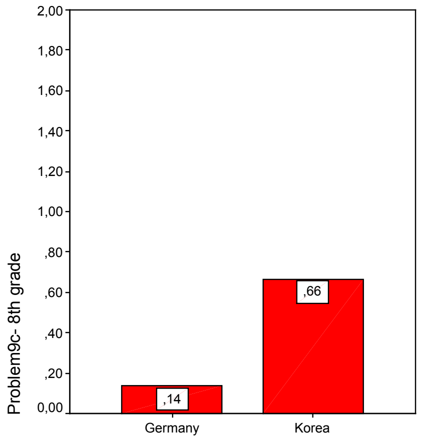
Problem 9b.




| | Korea | Germany |
|---|---|---|
| No answer | 36.8 | 16.1 |
| Incorrect working | 25.3 | 15.9 |
| Incorrect concept (Wechselwinkel, opposite angle) | 24.2 | 9.5 |
| Meaningful writing | 0.0 | 26.3 |
| Right (concept of angle in a straight line mentioned but not explained) | 1.6 | 13.1 |
| Right (concept given and explained) | 12.1 | 19.1 |
| Total | 100.0 | 100.0 |

Table B-2-9b Answers given by percentage to problem 9b

Problem 9c.




| | Korea | Germany |
|---|---|---|
| No answer | 56,0 | 72,0 |
| Incorrect working | 9,9 | 18,6 |
| 2 steps omitted | 0,0 | 2,7 |
| 1 steps omitted | 1,6 | 2,7 |
| Right working | 32,4 | 4,2 |
| Total | 100,0 | 100,0 |

Table B-2-9c Answers given by percentage to problem 9c

# Appendix C Transcript

## C-1. Transcript of the Korean pupils

- ### Su-Yeon

  - **Problem 1**

**00:01:07 초등학교때 배웠는데, 맞꼭지각으로 같으면요, 각이 같다고 배웠거든요? 그러니까, 각 *a* 와 각 *b* 는 맞꼭지각으로 같아요.**
00:01:31 그래요, 그러면 문제를 정확하게 풀었다고 생각하나요?
**00:01:36 아니요.**
00:01:37 아니에요?
**00:01:38 뭔가 하나 부족한 것 같은데...**
00:01:42 그러면 그것을 한번 적어볼래요?
(풀이를 적는다.)
00:02:33 자, 그러면, 이 문제가 어려웠어요?
**00:02:36 아니요**
00:02:37 안 어려웠어요?
**00:02:39 좀 당황했어요.**

**00:01:07** I **learnt in elementary school that if two angles are opposite angles then the angles are the same. Therefore angle *a* and angle *b* are the same because they are opposite angles**
00:01:31 What do you think, did you solve it correctly?
**00:01:36 No**
00:01:37 No?
**00:01:38 Something seemed to be missing...**
00:01:42 Would you write it down?
(She writes the answer down.)
00:02:33 Was it difficult for you?
**00:02:36 No,**
00:02:37 Not difficult?
**00:02:39 No, but it was a little bit confusing.**

- ### Nam-Kyung

  - **Problem 2**

00:01:31 먼저 생각나는 것 이야기 해주고 그 다음에 적어보세요.
**00:01:36 교차하는 대각선에서, 맞꼭지각의 크기는 서로 같으므로, *a* 와 여기, 이 각의 크기와 이 각의 크기는 같고, 직선위에, 직선위에, 직선위에서 각의 크기가, 직선은, 각의 크기가, 대각선에서 각의 크기가 합이 180 도니까, *a, b, c* 의 합은 180 도이다. (풀이를 적는다.)**
00:02:59 생각나는 것 다 적었어요?
**00:03:01 네**
00:03:02 자, 그래요, 그러면 문제를 정확하게 풀었다고 생각하나요?
**00:03:06 이것 정확하게 풀긴 풀었어요. 정확하게.**

00:03:10 정확하게 풀긴 풀었어요? 이 증명문제에 어려운 부분이 있었나요?
**00:03:15 어려운 부분은 없는데.**

00:01:31 Please explain what you think of and then write it down.
**00:01:36 Since the diagonal lines meet at one point, the opposite angles are same, *a* and this angle here are the same. For the straight line, the angles on the straight line add up to 180°, therefore the sum of *a, b* and *c* is 180°. (She writes her answer down.)**
00:02:59 Did you write all what you thought of?
**00:03:01 Yes**
00:03:02 What do you think, did you solve it correctly?
**00:03:06 Yes, I did solve it correctly. Correctly**
00:03:10 Did you? Is there a difficult part in this task?
**00:03:15 No**

- **Problem 5**

00:10:20 그래요, 그럼 다음 문제를 풀어보죠.
**00:10:48 각 *a* 와 각 *b* 는 맞꼭지각이 되므로 같다고 할 수 있고, *AB* 와 *CD* 는 이 사각형에서 대각선의 길이가, 대각선이 서로 다른 것을 이등분 하므로, 이 사각형을 평행사변형, 평행사변형이라고 할 수 있는데, *AB* 의 길이가, 평행사변형은 대변의 길이가 같으므로, *AB* 와 *CD* 의 길이가 같다고 할 수 있습니다. (풀이를 서술한다. )**
00:13:41 자, 그러면, 문제를 정확하게 풀었다고 생각하나요?
**00:13:44 네**
00:13:48 이 증명문제에 어려운 부분이 있었나요?
**00:13:54 그냥 평행사변형의 조건만 알고 쉽게 풀 수 있는 문제인 것 같아요.**

00:10:20 Ok, Let's go the next task.
**00:10:48 Angle *a* and angle *b* are equal because of the opposite angle pair, *AB* and *CD* in this quadrilateral, the length of diagonal… the two diagonals bisect each other. Therefore this quadrilateral could be a parallelogram, and then the lengths of the opposite sides are same, and so the lengths *AB* and *CD* are same.** [She then started to write the answer.]
00:13:41 What do you think, did you solve it correctly?
**00:13:44 Yes**
00:13:48 Is there a difficult part in this task?
**00:13:54 It seems be an easy task to solve, if we know the condition (properties) of parallelogram.**

- **A-Ra**

  - **Problem 1**

**00:00:41 각 *a* 와 각 *b* 가 크기가 같음을 증명하시요. 음, 우선 이게 직선이니까요, 여기 180 도 이면, 이거 각도기로 그런 걸로 하면 안되죠?**
00:01:03 각도기가 없는데요?
**00:01:29 180 도에서 각 *b* 와 각 *a* 를 뺀 크기가 같다면, 음, 이 크기가 같으므로, 이 크기가 같은거고, 그래서 이것도 같은거고, 어… 어떻게 해야지?**
00:01:48 천천히 생각하세요?
**00:01:52 어, 이런 것도 진짜 증명한 사람이 있어요?**
00:02:03 자, 그러면, 그것을 보고 생각나는 것 다 이야기 하는것이 중요하거든요, 생각나는 것 있어요?
**00:02:13 이제 수학 선생님 방법대로 하면, 맞꼭지각은 항상 크기가 같다. 이런 것 뭐…**
00:02:37 자, 그러면 생각나는 것 적어보세요.
(풀이를 적는다. 직선을 A, B 로 표시한다. )

00:05:34 쓴 것을 다시 한번 읽어 볼래요?

**00:05:38 각 *a* 와 각 *b* 를 만든, 두선은 직선이므로 평각이다. 직선선 *a* 의 각도 180 도에서 각 *b* 를 뺀 크기와, 다른 직선에서 각 *a* 를 뺀 크기는 같다면, 각 *a* 와 각 *b* 는 같지 않을까?**

00:05:57 그래요, 그러면 문제를 정확하게 풀었다고 생각하나요?

**00:06:03 아니요.**

00:06:04 아니에요? 왜 아니에요?

**00:06:09 음, 이 각 *b* 가 아닌 각의 크기를 제대로 모르니깐요, 뺀다고 해도 각 *b* 의 크기가 나오지 않을 것 같아요.**

00:06:24 예를 들면, 그 각이 40 도 다 50 도다 라고 정확히 주어졌으면,

**00:06:31 그럴 수도, 증명할 수도 있겠는데요. 이거는 그렇게 안 나왔으니까, 지금 각도기 사용도 못하고 그러니까, 크기를 정확하게 같다고 증명할 수가 없을 것 같아요.**

00:06:44 자, 그러면, 이 증명문제에 어려운 부분이 있었나요?

**00:06:48 음, 너무 고정관념으로 맞꼭지각은 항상 같다고만 생각을 했으니까, 왜 그런지 생각도 안하고, 평소에 생각하던 것이 아니어서, 제가 증명을 하기가 어려웠던 것 같아요.**

**00:00:41 Prove that ∠ *a* = ∠ *b*… This is a line; if it is 180° … may I not use a protractor?**

00:01:03 We don't have one?

**00:01:29 If 180° - ∠ *a* is the same as 180° - ∠ *b*, then this angle [the angle above the intersection of the two lines in the diagram] is the same as that angle [the angle below the intersection of the two lines in the diagram], so ∠ *a* and ∠ *b* are equal. What can I do?**

00:01:48 Think slowly?

**00:01:52 Has someone really proved this statement?**

00:02:03 Please tell me what you are thinking. It's important to tell me everything you think of?

**00:02:13 From what my teacher has said, opposite angles are always the same …**

00:02:37 Ok. Then write down what you are thinking.

[She writes part of her answer down.]

00:05:34 Please read what you have written down

**00:05:**38 **The two lines which give rise to ∠ *a* and ∠ *b* also give rise to two more angles. So if the angles on line A add up to 180° and the angles on line B do too, then the angle above the intersection is 180°– ∠ *b* and is also 180° – ∠ a, then maybe ∠ *a*= ∠ *b*?**

00:05:57 Do you think you have proved the statement correctly?

**00:06:03 No**

00:06:04 No? Why not?

**00:06:09 No, because I do not know the degrees in each angle for sure, therefore I cannot check how many degrees angle b is, although if it could be subtracted…**

00:06:24 So if degrees of angle were 40° or 50°?

**00:06:31 Then I could prove it, but that isn't the case, and I can't use a protractor anyway. So I cannot exactly prove or disprove whether the angles are equal or not.**

00:06:44 Did you find the statement difficult to prove?

**00:06:48 Yes, I always thought opposite angles are the same, it's a fixed idea I have. I never thought why it's the case. So it was difficult to prove.**

- **Ji-Sang**

  - **Problem 4**

**00:08:44 여기 삼각형이요, ADC 하고 ABC 로 나눠서요, ABC 하고 ADC 하고 겹치면요, B 하고 D 하고 같이 포개지면, 같은 각이 나올 것 같아요.**

00:09:50 그래요, 그럼 설명한 것을 한번 적어볼래요?

(풀이를 서술한다.)

00:10:27 그래요, 그러면 이 문제를 정확하게 풀었다고 생각해요?

**00:10:31 네**

00:10:32 이 문제에서 어려운 부분이 있었어요?
**00:10:36 없었어요.**

**00:08:44 For the triangles here, ADC and ABC, if ABC is folded onto ADC, then point B lies on point D, so the angles might be the same.**
00:09:50 ok. Would you please write it down what you explained?
(He writes the answer down.)
00:10:27 Do you think you have proved the statement correctly?
**00:10:31Yes**
00:10:32 Is there a difficult part in this task?
**00:10:36 No**

00:49:52 여기 4 번째에서, 두 삼각형을 겹쳐보면, 포개어 보면, 각이 같다 그랬죠. 구 삼각형을 포개어보면, 두 삼각형이 완전히 겹쳐지는 것을, 뭐라고 하죠?
**00:50:09 잘 모르겠어요.**
00:50:10 닮음 말고, 또 그전에 두 삼각형이 완전히 똑같다 라는 것을 뭐라고 하죠?
00:50:32 합동이라는 것을 들어봤어요? 삼각형이 합동이다. 그러면, 우리 문제에서 삼각형이 합동이면, 수업시간에 뭐 배웠어요?
**00:50:42 AA 합동, SAS 합동**
00:50:45 좀 더 크게 이야기 해 줄래?
**00:50:48 AA 합동, 그런 것**
00:50:50 AA 합동, 또?
**00:50:55 SAS 합동**
00:50:59 SAS 합동, 그러니까 두 변의 길이가 같고, 그 사이에 끼인 각이 같고, 아니면, SSS, 세변의 길이가 다 같고, 그런 것 배웠죠?
**00:51:10 네**
00:51:10 여기에서는 무슨 삼각형 합동 조건일까요? 삼각형의, 무슨 합동 조건일까요?
**00:51:22 SAS**

00:49:52 Here in problem 4, you said that if one triangle can be folded onto another, then the angles are also the same, didn't you? What is it called, if one triangle can be exactly folded onto another??
**00:50:09 I don't know.**
00:50:10 Not similarity, what is called that two triangles are exactly same?
00:50:32 Have you heard of congruence? Triangles are congruent. What have we learnt about congruence in class?
**00:50:42 AA congruence, SAS congruence**
00:50:45 Would you please speak of it aloud?
**00:50:48 AA congruence, something like that…**
00:50:50 AA congruence and then?
**00:50:55 SAS congruence**
00:50:59 SAS congruence…, SAS congruence for which two of the sides on one triangle are the same as two of the sides on the other, and the angles between the sides are equal. We learnt about SSS congruence, when the three sides are each the same length as the corresponding side on the other triangle, didn't we??
**00:51:10 Yes**
00:51:10 *Then here, which property of congruence must be used?*
**00:51:22 SAS congruence**

## C-2. Transcript of the German pupils

- ### Bettina

  - **Problem 1**

00:00:06 Hier sind Beweisaufgaben. Die du bearbeiten sollst. Lies dir die Aufgabe erst einmal durch und sag mir genau, was du dabei denkst. Auch wenn du diese Lösungen im Kopf durchgehst und wenn du diese Lösungen danach schreibst.
**00:00:19 Ok. Ich probiere es mal.**
00:00:23 Das wäre die erste Aufgabe.
**00:00:25 Ja, das stimmt, weil das ein Scheitelwinkel ist. Ja, das wär's. Das einzige, was mir einfällt, jetzt auf die Schnelle. Soll ich das jetzt hinschreiben? (*Die Schülerin schreibt Begründung*)**
00:00:54 Was denkst du, hast du diese Aufgabe richtig gelöst?
**00:00:58 Ja**
00:01:00 Welcher Teil der Beweisaufgabe war besonders schwierig?
**00:01:02 Nein, eigentlich nicht, weil wir das in der Schule schon so gemacht haben und da das eigentlich nur ein kleiner Teil ist, der immer wieder in Schulaufgaben oder so Beweisen drankommt, warum Winkel gleich groß sind, da kommt es öfter vor…**
00:01:17 Und wenn du diese Aufgabe einschätzen musst, wie schwierig war diese Aufgabe insgesamt?
**00:01:21 Nicht sehr schwierig.**

  - **Problem 2**

00:01:26 Kommen wir zur nächsten Aufgabe.
**00:01:33 Also die Winkelsumme, das sind ja 360°- eine Drehung, und der Winkel da drüben (*zeigt*) =$\alpha$ wegen den Scheitelwinkeln.**
00:01:47 Darfst du ruhig was reinschreiben aufs Blatt.
**00:01:52 (*Die Schülerin zeichnet sich auf*). Der Winkel (*zeigt*)= $\beta$ auch wegen den Scheitelwinkeln und der Winkel =$\gamma$ (*zeigt*) weil auch Scheitelwinkel und das stimmt, weil...einfach...kann man da eine Gleichung aufschreiben, z.B. so: (*Die Schülerin schreibt*). 360° =2$\beta$+2$\alpha$+2$\gamma$. Das ist jetzt hier die Winkelsumme. Das habe ich ja jetzt aufgeschrieben. Und dann kann ich die 2 ausklammern und kommt halt raus 180°=$\alpha$+$\beta$+$\gamma$. weil, wenn man das hier durch 2 teilt und da durch 2 dann kommt da 180 raus.**
00:02:57 Was denkst du, hast du diese Aufgabe richtig gelöst?
**00:03:00 Ja**
00:03:02 War da ein Teil dabei, der schwierig war für dich?
**00:03:08 Also nicht wirklich, aber die war schon schwieriger als die davor, weil da musste man das kombinieren, aber sehr schwierig fand ich sie nicht.**
00:03:16 Wie schwierig war diese Aufgabe insgesamt?
**00:03:24 Ja…auch nicht so schwer.**

- ### Katharina

  - **Problem 4**

00:10:13 Gut, kommen wir zu nächsten Aufgabe.
**00:10:27 Ja, ok, also, wenn man hier, z.B.: wenn man strich hindurch zieht, dann haben wir hier ein gleichschenkliges Dreieck. Ne, ein gleichseitiges Dreieck, das da und das da. Dann ist – ja ok… Also, Wenn man diesen Winkel weiß, kann man die Hälfte von diesen und die Hälfte von $\beta$ ausrechnen, weil das ist Basiswinkel - nee, das ist dann Basiswinkel, und was heißen die, auf jeden fall diese Winkeln hier, diese müssen gleich sein, weil, es verteilt ist**

**und hier das genau selbe, wenn man diese zwei zusammen zählt, und diese zwei dann kommt dasselbe raus. Soll ich es aufschreiben?**

00:11:16 Ja.

**00:11:47 Ok. Kann ich den Winkel auch rein tun?**

00:11:49 Ja, natürlich

**00:11:50 Ja, ich kann es ja, sagen**

00:11:56 Du darfst auf den Blättern schreiben, was du möchtest.

**00:11:58 Ok, tja**

00:15:07 Was meinst du, hast du die Aufgabe richtig gelöst?

**00:15:10 Nein, eigentlich nicht.**

00:15:11 Und warum glaubst du, dass du es nicht richtig gelöst hast?

**00:15:14 Ja, mit dem Beweisen hier, dass β und δ überhaupt halt - keine Ahnung. Nicht so ganz geklappt.**

00:15:24 Was meinst du, was war besonders schwierig für dich?

**00:15:30 Oh Gott, alles – ja, keine Ahnung, ja - hier mit dem Winkel. z.B. Also ich konnte halt davor sagen, ich weiß es nicht, ob es davor richtig war, wie ich's gesagt habe, aber das aufschreiben war also schwieriger. Wenn man die genaue Schreibweise schreiben muss, so…**

00:15:56 Und wie schwierig würdest du jetzt diese Aufgabe einschätzen?

**00:16:00 Ja, schwerer als die anderen eigentlich schon.**

- ## Elena

  - ### Problem 1

00:00:00 Lies dir es durch, sag mir laut, was dir dabei durch den Kopf geht und auch, wenn du sie bearbeitest, sprich dazu, was du machst, damit ich deine Gedankengänge erkennen kann.

**00:00:29 Winkel α und Winkel β sind gleich groß, weil die beiden Geraden (*schreibt Begründung)* sich an einem Punkt treffen, wo die beiden Winkel der Geraden gleich groß sind, deshalb sind sie auch gleich groß.**

00:01:04 Wenn du das jetzt so hinschreibst, beiden Geraden, wie würdest du es noch weiter begründen, du hast da in der Hälfte abgebrochen,

**00:01:14 Also, weil die beiden Geraden sich.... (*Die Schülerin schreibt*) So hätte ich das begründet.**

00:01:51 Was denkst du? Hast du die Aufgabe richtig gelöst?

**00:01:55 Nein**

00:01:56 Warum nicht?

**00:02:00 mmh, ja weil das eh nichts… Also, die Begründung kann richtig sein, aber so, wie ich es ausgedrückt, ist es ganz sicher falsch.**

00:02:05 Warum glaubst du das?

**00:02:11 So ein Gefühl halt.**

00:02:14 So ein Gefühl… Was war denn schwierig bei dieser Aufgabe für dich?

**00:02:18 Überhaupt herauszufinden, wieso sie gleich sind.**

00:02:23 Und wie schwierig würdest du jetzt die Aufgabe jetzt insgesamt bewerten? leicht, mittel, ...

**00:02:30 Die ist schon leicht, aber für mich ist sie schwer.**

  - ### Problem 2

00:02:35 Gut, dann schauen wir uns die nächste Aufgabe an.

**00:02:53 Winkel α, Winkel β und Winkel $\gamma$ sind 180° (*schreibt Begründung)*, denn wenn man das hier zu verbinden würde, dann würde das eigentlich…Nein. Keine Ahnung. Ich kenn mich jetzt überhaupt nicht aus, wie die drei Winkel 180° zusammen ergeben sollen.**

00:03:43 Was macht dir denn so große Schwierigkeiten?

**00:03:56 Ja, ich weiß, dass die Innenwinkelsumme im Dreieck z.B. 180° ist. Hier kann man sich so ein Dreieck denken, aber die Winkel würden dann woanders liegen, und, keine Ahnung**.

00:04:10 Und wie schwierig würdest du denn diese Aufgabe einschätzen?
**00:04:12 Die ist schon schwieriger als die letzte. Auf jeden Fall.**
00:04:15 Und von deiner Einschätzung her, eher leicht....
**00:04:18 Mittel**

- **Problem 3**

00:04:20 Mittel. Ok. Gehen wir zur nächsten Aufgabe.
**00:05:00 Das Ganze ist so, weil diese beiden Dreiecke sind ja kongruent und deshalb sich auch die Winkel gleich.**
00:05:12 Kannst du das hinschreiben? In die einzelnen Felder?
*(schreibt Begründung)*
00:05:49 Ja, und wie würdest du 2 begründen? Einfach dass δ gleich β plus γ ist?
**00:05:58 Auch deshalb, weil… Also auch des mit den beiden Dreiecken, weil die kongruent sind, weil wenn man diese Winkel zusammenrechnet und diese Winkel, dann müsste es das selbe ergeben, wie jetzt nur ein Winkel – gleich sind, also… Wenn die beiden Dreiecke gleich sind, dann müssen auch die ganzen Winkel zusammen das gleiche wie im anderen Dreieck.**
00:06:25 Kannst es da auch hinschreiben.
(*schreibt Begründung 2)*
00:07:02 Und bei 3. unten.
(*schreibt Begründung 3)*
**00:07:30 Fertig**
00:07:32 Fertig. Was denkst du? Hast du denn die Aufgabe richtig gelöst?
**00:07:43 Nein.**
00:07:45 Warum nicht?
**00:07:50 Das Denken, also wie ich auf die Lösung gekommen bin, könnte schon richtig sein, aber ich denke mal, ich hab's halt auf jeden Fall falsch hingeschrieben, also die ganzen Begriffe und so…Ich habe einen viel komplizierten Weg gemacht.**
00:08:10 Welcher Teil dieser Aufgabe war denn besonders schwierig für dich?
**00:08:20: Der da *(zeigt auf die Begründung 2)***
00:08:23 Und wenn du jetzt diese Aufgabe einschätzen musst? leicht, mittel, schwer...
**00:08:29 Auch mittelschwer.**

- ## Andrea

- **Problem 1**

00:00:01 Das musst du selber bearbeiten. Du liest jede Aufgabe mal durch, und sagen, was du die dabei denkst. Auch wenn du sie dann bearbeitest, erzähl mir was dabei durch den Kopf geht. Die erste Aufgabe.
**00:00:23 Man lernt, dass Scheitelwinkel sind, das ist ein festgestellter Satz. Den wir bewiesen haben, aber**
00:00:35 Gut, wenn du sie jetzt schriftlich begründest, wie würdest du dann machen?
**00:00:42 Ja, ich würde es schreiben, der Winkel α ist gleich dem Winkel β, wegen dem Scheitelwinkel.**
00:01:04 Ok, was denkst du, hast du jetzt die Aufgabe richtig gelöst?
**00:01:09 Ich habe es nicht direkt bewiesen, ich habe halt das, was wir vor längere Zeit schon mal bewiesen haben, einfach jetzt hingeschrieben.**
00:01:17 Gab es Teil für dich jetzt schwierig war?
**00:01:23 Nein, eigentlich nicht**
00:01:27 Wie würdest du jetzt die Aufgabe insgesamt einschätzen, von der Schwierigkeit?
**00:01:31 Nicht schwer. Weil wenn man jetzt z.B. beweisen musst, dass Dreieck kongruent, und Winkel und so, das haben wir ja schon alles gelernt, z.B., Scheitelwinkel oder Z-Winkel oder so.**

- **Problem 4**

00:08:14 Gut, kommen wir zu nächsten
**00:08:28 Das ist ja ein Drachenviereck, und deswegen, beim Drachenviereck sind die gegenüberliegenden Winkel gleich groß und die hier werden halbiert und deswegen, ist der Winkel δ gleich β.**
00:09:10 Und was denkst du, hast du die Aufgabe richtig gelöst?
**00:09:13 Ja, das glaub ich jetzt schon.**
00:09:16 Ja, welcher Teil war da besonders schwierig?
**00:09:19 Bei dem fand ich es jetzt nicht schwer, weil das eine symmetrische Figur ist, also ein Drachenviereck. Und da hat man schon mal alle Eigenschaft von dem gelernt, daher.**
00:09:29 Und wie schwierig war die Aufgabe jetzt insgesamt?
**00:09:35 So wie die zweite, z.B.. Oder wie die erste. Also eigentlich nicht schwer.**

- ## Annika

- **Problem 3**

00:03:45 Also, bei dem ersten ist das klar, dass es wieder der Scheitelwinkel, weil… *(Die Schülerin schreibt)*
00:04:16 Und beim zweiten ist es ja so, dass ich jeweils einen rechten Winkel habe und die Winkelsumme im Dreieck ist ja 180°, also muss ich jeweils bei jedem Dreieck, es ist ja klar, 180 Grad-90 Grad rechnen muss und dann muss sich jeweils das gleiche ergeben, weil…ja…
00:04:40 Würdest das dann aufschreiben?
00:05:39 Gehen wir zu 3.
**00:05:44 Das ist ja auch wieder klar, weil ich hab ja hier, dass… durch den Scheitelwinkel eigentlich bewiesen, dass die 2 Winkel gleich groß und dann sind die 2 Winkel auch gleich groß, also hab' ich 180 Grad-(90 Grad + $\delta$ (entweder der Winkel oder der Winkel)) ergibt sich α und hier muss ich das genauso rechnen, also 180 Grad-(90 Grad+ $\gamma$ (den Winkel weil die sind ja Scheitelwinkel)), ergibt sich β Winkel wegen der Winkelsumme im Dreieck.**
00:06:39 Können wir das so stehen lassen, oder?
**00:06:40 Also, ich würde Winkelsumme im Dreieck schreiben.**
00:07:15 Glaubst du, wir haben die Aufgaben alle richtig gelöst?
**00:07:18 Ja.**
00:07:19 Gab's einen Teil, der schwieriger war?
**00:07:25 Ich denke, man muss halt ein bisschen logisch nachdenken, weil das ist mit dem rechten Winkel und man kommt man eigentlich auch ganz leicht drauf, sonst wär's nicht so schwer.**

- **Problem 4**

00:07:40 Gucken wir uns mal die nächste an.
**00:08:03 Also es ist ja so, $\overline{AC}$ ist die gemeinsame Strecke dieser 2 Dreiecke, dann haben wir, wenn ich jetzt das konstruieren würde, dann hab' ich jeweils den gleichen Schnittpunkt, weil $\overline{AB} = \overline{AD}$ ist, also angenommen, ich habe jetzt hier eingestochen, dann habe ich hier z.B. den Kreis und der zieht sich dann hier weiter und von C aus ist es auch der gleiche Radius (das ist halt meine Skizze) und dann ergibt sich jeweils D und B. Also hab' ich die 2 Punkte und der Winkel muss gleich sein, weil es ja im Endeffekt nur eine Spiegelung ist an der Achse $\overline{AC}$. Aber begründen ist jetzt, ...**
00:09:05 Wie bringen wir das jetzt zu Blatt?

**00:09:07 Ja genau Also B ist Spiegelpunkt von D an der Achse AC, an der Spiegelachse und das ist wegen… weil die Strecken sich halt ähneln und das ergibt dann den gleichen Schnittpunkt. Den gleichen Abstand der Achse AC oder zu C und A. So gibt's für mich Sinn, aber ob das jetzt als Begründung unbedingt so angenehm wäre für die Lehrerin, weiß ich nicht.**

00:11:30 Glaubst du, wir sind richtig mit deiner Begründung?

**00:11:33 Ja, es macht auf jeden Fall für mich Sinn und ob das jetzt, was da steht, unbedingt für die Lehrerin Sinn macht, weiß ich nicht, aber prinzipiell schon.**

00:11:46 Gab's eine besondere Schwierigkeit?

**00:11:50 Ja, das mit dem Begründen ist schwierig, das man das auf's Blatt kriegt, man hat's eigentlich im Kopf.**

- ## Elvira

  - **Problem 3**

00:06:11 Ja, gut dann zu der nächsten.

00:06:30 Was denkst du, jetzt?

**00:06:37 Also, soweit… dass δ gleich γ ist, die sind gleich, weil ersten sie Scheitelwinkel sind, und das liegt daran, dass die beiden im Dreieck 90° enthalten. Und wegen des 90 Grad hier, sind auch diese Längen gleich, dann ist ja auch α gleich β. Die sind Scheitelwinkel.**

**00:07:42 Können wir denken so, dieser Winkel und dem Winkel auch gleich. Also wegen Winkelseitewinkel Satz, wenn man dieses Dreieck hier drüber tut… würde es treffen.**

**00:08:33 Ok, α ist genauso groß wie β, es liegt daran weil die 90 Grad sind, und dieser Winkel (δ) und (γ) Scheitelwinkel sind.**

00:09:30 Kannst du sagen, was du da denkst?

**00:09:32 Ja, also, ich denke, diese Seite ist genauso, kongruent zu der, wenn das so ist, α genau so groß wie β sein, und weil die Länge BC ist genau so lang wie DC, und AC ist genau so lang wie CE, und AB ist genau so lang wie DE. BC, dieser Winkel auf BC ist 90 Grad, deswegen alle… soll ich jetzt aufschreiben?**

00:10:40 Denkst du, hast du diese Aufgabe richtig gelöst?

**00:10:43 Ja, ich glaube, da fehlt noch die Begründung wieso hier Scheitelwinkel sind, wahrscheinlich, und α gleich β ist schon richtig, und die zwei, α, δ… ich denke auch richtig gemacht, weil das WWS-Satz ist.**

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ERKLÄRUNG

Hiermit erkläre ich, dass ich diese eingereichte Dissertationsschrift selbstständig und ohne unerlaubte Hilfsmittel verfasst und keine anderen als die angegebenen Quellen benutzt habe.

Jee Yi Kwak