Zur Homepage der Dissertation

Generalized Koszul Complexes

Von der Fakultät für Mathematik und Naturwissenschaften
der Carl von Ossietzky Universität Oldenburg
zur Erlangung des Grades und Titels eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
angenommene Dissertation von

Bogdan Ichim
geboren am 23.10.1976
in Galati, Rumänien

Gutachter:	Prof. Dr. Udo Vetter
Zweitgutachter:	Prof. Dr. Winfried Bruns
Tag der Disputation:	17-XII-2004

Abstract

The thesis deals with the following question: given a linear map $\chi : \mathcal{F} \to \mathcal{G}$ of finite free modules over a noetherian ring R and another finite free R-module $\mathcal{H}$, when is there a linear map $\lambda : \mathcal{G} \to \mathcal{H}$ such that $\operatorname{grade} I_\chi \leq \operatorname{grade} I_\lambda$ and $\lambda\chi = 0$? (By I_χ we denote the ideal of maximal minors of χ.) If, for example, $\operatorname{rank} \mathcal{F} = 1$, $\operatorname{rank} \mathcal{G} = n$, and χ is given by a regular sequence $x_1, \ldots, x_n$ in R, then it was proved by Bruns and Vetter in [BV4] that the question has a positive answer if and only if $\operatorname{rank} \mathcal{H} = 1$ and n is even.

The general version of this question should be very hard to be answered. Some approach has been done here.

Assume that $m = \operatorname{rank} \mathcal{F} \leq \operatorname{rank} \mathcal{G}$. Then it turns out that the existence of a satisfying λ is closely connected with the homology of the generalized Koszul complex associated with the induced map $\bar{\lambda} : M = \operatorname{Coker} \chi \to \mathcal{H}$. But what is the generalized Koszul complex?

For technical reasons it is better to start from the dual map $\chi^* : \mathcal{G}^* \to \mathcal{F}^*$. To avoid notational complication we replace $\mathcal{G}^*$, $\mathcal{F}^*$ and χ^*, by G, F and ψ. If G and F are free, with the linear map $\psi : G \to F$ one associates the Eagon-Northcott family of complexes $\mathcal{C}^t(\psi)$. The homology of $\mathcal{C}^t(\psi)$ is well-understood. In particular it is grade sensitive with respect to the ideal I_ψ.

More generally we consider linear maps $\psi : G \to F$, where only F has to be free (weaker assumptions are possible). We construct a family of complexes $\mathcal{C}_\psi(t)$ associated with ψ which generalizes both the Eagon-Northcott family of complexes, and the classical Koszul complex. There is a similar construction of a family of complexes $\mathcal{D}_\varphi(t)$ for a map $\varphi : H \to G$, where H is free. The complexes just mentioned are the *generalized Koszul complexes*. If $H \xrightarrow{\varphi} G \xrightarrow{\psi} F$ is a complex, we can compose $\mathcal{C}_\psi(t)$ and $\mathcal{D}_\varphi(t)$ to our main tool, the bicomplex $\mathcal{C}_{.,.}(t)$.

Further we investigate the homology of $\mathcal{C}_{\bar{\lambda}}(t)$. The most satisfactory result (see Theorem 3.6) is obtained if $\operatorname{grade} I_\chi$ has the greatest possible value $n - m + 1$. The theorem covers a result of Migliore, Nagel and Peterson (see Proposition 5.1 in [MPN]) who proved it partially for Gorenstein rings R, using local cohomology. It also generalizes Theorem 5 in [BV4]. If one further requires that $\operatorname{grade} I_\lambda$ should be big enough, a full answer (necessary and sufficient conditions) to our initial question may be found in Theorem 3.2, a generalization of Corollary 3 in [BV1].

What can be deduced if $\operatorname{grade} I_\chi$ has not the greatest possible value (but is not too small)? In this case theorem 3.11 provides some necessary conditions for the existence of a non-trivial map λ. As a consequence we derive Corollary 3.14, a purely numerical criterion for the non-vanishing of product of matrices.

In the last part we study the homology of $\mathcal{C}_{\bar{\lambda}}(t)$ in the particular case $\operatorname{grade} I_\chi = n - m = \dim R$. We obtain information about the length of the homology in Theorem 3.6, which generalizes unpublished results of Vetter. Finally, in Theorem 3.20, we

give a proof of a Theorem of Naruki (see [Na], Theorem 2.1.1) by purely algebraic methods. A partial (algebraic) proof of this Theorem may be found in [BV1] while a complete proof has already been given by Herzog and Martsinkovsky in [HM].

The thesis is based on results of Bruns and Vetter (see [BV4]). They study the homology of the Koszul complex associated with a linear form on a module of projective dimension 1, using a Koszul bicomplex construction obtained from a Koszul complex and certain Eagon-Northcott complexes. The idea to build and link Koszul bicomplexes appears also in the the paper [HM] of Herzog and Martsinkovsky (see in particular the gluing construction for the residue field of a complete intersection).

Zusammenfassung

Die Arbeit beschäftigt sich mit folgender Frage: Gegeben sei eine lineare Abbildung $\chi : \mathcal{F} \to \mathcal{G}$ endlich erzeugter freier R-Moduln über einem noetherschen Ring R und ein weiterer endlich erzeugter freier R-Modul $\mathcal{H}$; wann gibt es eine lineare Abbildung $\lambda : \mathcal{G} \to \mathcal{H}$, so daß $\operatorname{grade} I_\chi \le \operatorname{grade} I_\lambda$ und $\lambda\chi = 0$ gilt? (Dabei bezeichne I_χ das Ideal der maximalen Minoren von χ.) Ist z.B. $\operatorname{rank} \mathcal{F} = 1$, $\operatorname{rank} \mathcal{G} = n$, und χ die durch eine reguläre Folge $x_1, \dots, x_n$ in R gegebene lineare Abbildung $R \to \mathcal{G}$, dann hat die Ausgangsfrage nach einem Satz von Bruns und Vetter in [BV4] genau dann eine positive Antwort, wenn $\operatorname{rank} \mathcal{H} = 1$ und n gerade ist.

Es dürfte sehr schwierig sein, die Frage in voller Allgemeinheit zu beantworten. Wir geben in unserer Arbeit einige Näherungslösungen an.

Angenommen $m = \operatorname{rank} \mathcal{F} \le \operatorname{rank} \mathcal{G}$. Dann steht die Existenz eines geeigneten λ in engem Zusammenhang mit der Homologie des verallgemeinerten Koszul-Komplexes zur induzierten Abbildung $\bar{\lambda} : M = \operatorname{Coker} \chi \to \mathcal{H}$. Was ist dabei ein verallgemeinerter Koszul-Komplex?

Aus technischen Gründen ist es besser, von der dualen Abbildung $\chi^* : \mathcal{G}^* \to \mathcal{F}^*$ auszugehen. Dabei ersetzen wir zur Vereinfachung $\mathcal{G}^*$, $\mathcal{F}^*$ und χ^*, durch G, F und ψ. Sind G und F frei, dann kann man der linearen Abbildung $\psi : G \to F$ eine Familie von Eagon-Northcott-Komplexen $\mathcal{C}^t(\psi)$ zuordnen. Die Homologie von $\mathcal{C}^t(\psi)$ hängt in wohlbekannter Weise ab vom Grad des Ideals I_ψ.

Wir betrachter allgemeiner lineare Abbildungen $\psi : G \to F$, wobei G nicht notwendig frei sein muß, und konstruieren zu ψ eine Familie $\mathcal{C}_\psi(t)$ von Komplexen, die sowohl eine Verallgemeinerung der Eagon-Northcott-Komplexe als auch der Koszul-Komplexe darstellen. In ähnlicher Weise läßt sich zu einer Abbildung $\varphi : H \to G$ mit einem freien R-Modul H eine Familie $\mathcal{D}_\varphi(t)$ angeben. Die beiden Familien sind unsere *verallgemeinerten Koszul-Komplexe*. Ist $H \xrightarrow{\varphi} G \xrightarrow{\psi} F$ ein Komplex, dann können wir $\mathcal{C}_\psi(t)$ und $\mathcal{D}_\varphi(t)$ zu unserem wichtigsten Werkzeug zusammenfügen, dem Bikomplex $\mathcal{C}_{.,.}(t)$.

Wir untersuchen im weiteren die Homologie von $\mathcal{C}_{\bar{\lambda}}(t)$. Das beste Resultat erhalten wir, wenn der Grad von I_χ den größtmöglichen Wert $n-m+1$ hat (Theorem

3.6). Das Theorem verallgemeinert ein Ergebnis von Migliore, Nagel und Peterson (vgl. Proposition 5.1 in [MPN]), die es teilweise und mittels lokaler Kohomologie für Gorenstein-Ringe R beweisen. Außerdem ist es eine Verallgemeinerung von Theorem 5 in [BV4]. Stellt man an den Grad von I_λ gewisse Minimalitätsanforderungen, dann lassen sich notwendige und hinreichende Bedingungen für eine positive Antwort auf die Ausgangsfrage angeben (Theorem 3.2); hier handelt es sich um eine Verallgemeinerung von Corollary 3 in [BV1].

Was läßt sich sagen, wenn $\operatorname{grade} I_\chi$ nicht maximal (aber nicht zu klein) ist? Theorem 3.11 enthält für diesen Fall einige notwendige Bedingungen für die Existenz eines nicht-trivialen λ. Als Folgerung ergibt sich 3.14, ein numerisches Kriterium für das Nicht-Verschwinden von Matrizen-Produkten.

Im letzten Teil untersuchen wir die Homologie von $\mathcal{C}_{\bar{\lambda}}(t)$ für den Fall $\operatorname{grade} I_\chi = n - m = \dim R$. Theorem 3.6, eine Verallgemeinerung von nicht veröffentlichen Resultaten von Vetter, enthält Informationen über deren Länge. Wir beweisen damit (s. Theorem 3.20) ein Theorem von Naruki (vgl. [NA], Theorem 2.1.1) mittels rein algebraischer Methoden. Ein partieller (algebraischer) Beweis findet sich bereits in [BV1] und ein vollständiger Beweis in [HM].

Die Arbeit basiert auf Ergebnissen von Bruns und Vetter (vgl. [BV4]), die die Homologie des Koszulkomplexes einer Linearform auf einem Modul der projektiven Dimension 1 untersuchen. Sie benutzen dabei eine Koszul-Bikomplex-Konstruktion, die ähnlich der unseren aus einem Koszul-Komplex und gewissen Eagon-Northcott-Komplexen gewonnen wird. Die Idee der Konstruktion und Zusammenfügung von Koszul-Bikomplexen gibt es auch schon in der Arbeit [HM] von Herzog und Martsinkovsky (s. insbesondere die Verklebungs-Konstruktion für den Restklassenkörper eines vollständigen Durchschnitts).

Contents

Introduction

A natural question: When does the product of two matrices not vanish?

In this thesis we answer to this question in a very special setup. Let R be a commutative noetherian ring. If we identify an R-morphism with a representing matrix, we may equivalently ask when the sequence

$$R^l \xrightarrow{\varphi} R^n \xrightarrow{\psi} R^m$$

is not a complex, i.e. when $\psi\varphi \neq 0$. To illustrate the difficulty of the problem, we propose the reader the following exercise.

EXERCISE A. Let $n \in \mathbb{N}$ and let $x, y, z \in \mathbb{Z}$ be non zero elements. Show that if $n > 2$, then the product

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x^{n-1} \\ y^{n-1} \\ -z^{n-1} \end{pmatrix}$$

does not vanish.

The difficulty of the above Exercise is well known. However, the following exercise admits an easy solution.

EXERCISE B. Let $n \in \mathbb{N}$ and let x, y, z be a regular sequence in R. Show that the product

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x^{n-1} \\ y^{n-1} \\ -z^{n-1} \end{pmatrix}$$

does not vanish.

Proof. Remember that $x \in R$ is called a regular element (non-zero divisor) if $xz = 0$ for $z \in R$ implies $z = 0$, and that a sequence $x = x_1, \ldots, x_k$ of elements of R is called a regular sequence if x_i is a regular element of $R/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, k$. Since x, y, z is a regular sequence, if $ax + by = cz$, the definition implies that $c \in (x, y)$. In particular, if $x^n + y^n = z^n$, then $z^{n-1} \in (x, y)$. So $ax + by = z^{n-1}$, and descending induction on the exponent of z provides a contradiction. $\square$

The situation of the second Exercise admits a generalization which we present in this paper. An important tool for studying regular sequences is the Koszul complex. If $x = x_1, \ldots, x_n$ is a sequence in R, then $x_1e_1 + \ldots + x_ne_n \in R^n$, and we may consider the complex

$$\mathcal{K}(x_1, \ldots, x_n) : 0 \to R \to R^n \to \bigwedge^2 R^n \to \cdots \to \bigwedge^n R^n \to 0,$$

where the the differential sends an element a to the element $a \wedge (x_1e_1 + \ldots + x_ne_n)$. Remember that a complex

$$\cdots \longrightarrow M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \longrightarrow \cdots$$

is called exact if $\operatorname{Im} d_{i-1} = \operatorname{Ker} d_i$. The homology modules $H^i := \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ are measuring how far is a complex from being exact. If I is an ideal and $(x_1, \ldots, x_n)$ is a system of generators for I, then we set $g = \operatorname{grade} I := \inf\{i : H^i(\mathcal{K}(x_1, \ldots, x_n)) \neq 0\}$. The number g depends only on the ideal, and not on the chosen system of generators. Every maximal regular sequence in I has length g. The homology of the Koszul complex is *grade sensitive*.

More general, if M is an R-module, and $\psi : M \to R$ is an R-morphism, then we consider the complex

$$\mathcal{K}(\psi) : \ldots \to \bigwedge^k M \to \cdots \to \bigwedge^2 M \to M \to R \to 0,$$

where the the differential sends an element $m_1 \wedge \ldots \wedge m_k$ to the element $\sum_{i=1}^{k} (-1)^{i+1} \psi(m_i) m_1 \wedge \ldots \widehat{m_i} \ldots \wedge m_k$. If M is free, the two complexes are isomorphic, and their homology is well understood. If M has a presentation

$$0 \longrightarrow \mathcal{F} \xrightarrow{\chi} \mathcal{G} \longrightarrow M \longrightarrow 0$$

where $\mathcal{F}$, $\mathcal{G}$ are free modules, then the homology of $\mathcal{K}(\psi)$ is studied in [BV4], and found to be grade sensitive.

Eagon and Northcott have generalized the Koszul complex. If $\psi : G \to F$ is a map of free R modules of ranks n and m, we set $r = n - m$ and consider the complexes

$$\mathcal{C}^t(\psi) : 0 \to (\bigwedge^0 G \otimes S_{r-t}(F))^* \xrightarrow{\partial_\psi^*} \cdots \xrightarrow{\partial_\psi^*} (\bigwedge^{r-t} G \otimes S_0(F))^* \xrightarrow{\nu_\psi} \bigwedge^t G \otimes S_0(F) \xrightarrow{\partial_\psi} \cdots$$
$$\xrightarrow{\partial_\psi} \bigwedge^0 G \otimes S_t(F) \to 0,$$

where $\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes z) = \sum_{i=1}^{p} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes \psi(y_i) z$. If we fix bases $g_1^*, \ldots, g_n^*$ $(f_1^*, \ldots, f_m^*)$ on G^* (F^*), and define $\delta : \bigwedge^n G^* \to R$ by $g_1^* \wedge \cdots \wedge g_n^* \to 1$, then

$$\nu_\psi(x^*)(y^*) = \delta(x^* \wedge y^* \wedge \bigwedge^m \psi^*(f_1^* \wedge \cdots \wedge f_m^*)), \quad x^* \in \bigwedge^{r-t} G^*, \quad y^* \in \bigwedge^t G^*.$$

The complexes $\mathcal{C}^t(\psi)$ are grade sensitive, namely if $I = I_\psi$ is the ideal generated by the maximal minors of a matrix representing ψ, and $g = \operatorname{grade} I$, then $H^i(\mathcal{C}^t(\psi)) = 0$ for $i < g$.

Remark that the Koszul $\mathcal{K}(\psi)$ may be defined for all R-modules M, while the complexes $\mathcal{C}^t(\psi)$ are defined only for maps of free R-modules. In the first chapter we consider an R-homomorphism $\psi : G \to F$, where only F has to be free of rank m. We introduce the complexes

$$\mathcal{C}_\psi(t) : \cdots \to \bigwedge^{t+m+p} G \otimes D_p(F^*) \xrightarrow{\partial_\psi} \cdots \xrightarrow{\partial_\psi} \bigwedge^{t+m} G \otimes D_0(F^*) \xrightarrow{\nu_\psi^{x^*}} \bigwedge^t G \otimes S_0(F) \xrightarrow{\partial_\psi}$$
$$\cdots \xrightarrow{\partial_\psi} \bigwedge^0 G \otimes S_t(F) \to 0,$$

generalizing the Koszul complexes $\mathcal{K}(\psi)$.

There is a similar construction for a map $\varphi : H \to G$, where now H has to be free of rank l. We consider the complexes

$$\mathcal{D}_\varphi(t) : 0 \to D_t(H) \otimes \bigwedge^0 G \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} D_0(H) \otimes \bigwedge^t G \xrightarrow{\nu_x^\varphi} S_0(H^*) \otimes \bigwedge^{t+l} G \xrightarrow{d_\varphi} \cdots$$
$$\xrightarrow{d_\varphi} S_p(H^*) \otimes \bigwedge^{t+l+p} G \to \cdots ,$$

a generalization of the Koszul complex $\mathcal{K}(x_1, \ldots, x_n)$.

Next we consider a complex $H \xrightarrow{\varphi} G \xrightarrow{\psi} F$, H and F being free R-modules. We assemble the complexes $\mathcal{C}_\psi(t)$ and $\mathcal{D}_\varphi(t)$ to our main tool, the bicomplex $\mathcal{C}_{.,.}(t)$.

In the second chapter we analyze $\mathcal{C}_{.,.}(t)$. We suppose G to be also free. ψ^* induces a map $\bar{\varphi}^* : \operatorname{Coker} \varphi^* \to H^*$. We draw technical results concerning the homology of $(\mathcal{C}_{\bar{\varphi}^*}(t))^*$ and we relate the homology of $\mathcal{C}_{\bar{\varphi}^*}(t)$ to the homology of $(\mathcal{C}_{\bar{\varphi}^*}(t))^*$.

The third chapter deals with the dual situation. It is mainly concerned with the study of the homology of $\mathcal{C}_{\bar{\lambda}}(t)$, where $\bar{\lambda}$ is a map from an R-module M with a presentation as above, into a free R-module $\mathcal{H}$. If $\operatorname{grade} I_\chi$ has the greatest possible value $\operatorname{rank} M + 1$, the homology is found to be grade sensitive and the results of [BV4] are generalized.

Further we investigate when a sequence $H \xrightarrow{\varphi} G \xrightarrow{\psi} F$ of free R-modules is a complex provided that $\operatorname{grade} I_\chi$ is not necessarily maximal. Some necessary conditions are found. As a consequence we draw a criterion for the vanishing of a product of matrices, an answer to a very special case of the question formulated at the beginning of the introduction.

Finally we study the homology of $\mathcal{C}_{\bar{\lambda}}(t)$ in a special setup, namely we assume $\operatorname{grade} I_\chi = \operatorname{rank} M = \dim R$. In this case some of the homology modules have finite length, and we are able to deduce information about their length.

The author wishes to thank - in alphabetical order - Winfried Bruns, Violeta Dinescu, Bernolph von Gemmingen-Guttenberg, Nicolae Manolache, Gigel Militaru, Dorin Popescu, Udo Vetter for the direct and indirect support in the writing of this thesis.

1 Koszul Complexes and Koszul Bicomplexes

In the following sections we assume R to be a commutative ring. By $\bigwedge$ we denote the exterior power, by S the symmetric power, and by D the divided power. If not specified, $\otimes$ ($\wedge$, Hom) denotes tensor product (exterior product, homomorphisms) over R. * will always mean R-dual (with the usual exception in the graded case, see section 1.1).

The purpose of this chapter is to introduce a new way of working with the Koszul complexes and the Koszul bicomplexes.

Usually, if G is an R-module and $\psi : G \to R$ is an R-homomorphism, then the Kozsul complex associated to ψ is defined to be the complex

$$\ldots \to \bigwedge^p G \to \cdots \to \bigwedge^2 G \to G \to R \to 0$$

where the differential sends an element $y_1 \wedge \ldots \wedge y_p,\ y_i \in G$ to the element

$$\sum_{i=1}^{p} (-1)^{i+1} \psi(y_i) y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p.$$

In the first section we notice that the differential is exactly the right multiplication (the right inner product) by $\psi \in G^*$ with respect to the standard $\bigwedge G^*$-right module structure on $\bigwedge G$. The interaction between this structure and the $\bigwedge G$-left module structure on $\bigwedge G$ proves to be helpful to the study of the Koszul complex.

Eagon and Northcott have generalized the Koszul complex. They study the case of a homomorphism $\psi : G \to F$ of free modules. Their construction is difficult to generalize to the case in which either G or F is not free. We suggest a generalization for this case in the second section.

In the final section Koszul complexes are linked to Koszul bicomplexes associated to complexes $H \to G \to F$ of R-modules.

1.1 General Definitions and Properties

Let G be an R-module and let $\psi : G \to R$ be an R-homomorphism. Then there is a unique R-antiderivation ∂_ψ of $\bigwedge G$ which extends $\psi : \bigwedge^1 G \to \bigwedge^0 G$. It has degree (-1) with respect to the grading of $\bigwedge G$, and if $y_i \in G$ for $i = 1, \ldots, p$, then

$$\partial_\psi(y_1 \wedge \ldots \wedge y_p) = \sum_{i=1}^{p} (-1)^{i+1} \psi(y_i) y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p.$$

DEFINITION 1.1. If M is an R-module, then we denote by $K_\cdot^R(\psi, M)$ the (chain) complex $(\bigwedge G \otimes M, \, \partial_\psi \otimes 1_M)$. For simplicity the differential of $K_\cdot^R(\psi, M)$ is denoted by ∂_ψ or ∂_ψ^R. As it is a common practice, we sometimes write

$$\cdots \to \bigwedge\nolimits^p G \otimes M \to \cdots \to \bigwedge\nolimits^1 G \otimes M \to \bigwedge\nolimits^0 G \otimes M \to 0$$

for $K_\cdot^R(\psi, M)$.

REMARK 1.2. The differential ∂_ψ of $K_\cdot^R(\psi, M)$ has also degree (-1) with respect to the grading of $\bigwedge G \otimes M$. If $y_i \in G$ for $i = 1, \ldots, p$ and $m \in M$, then

$$\begin{aligned}\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes m) &= \sum_{i=1}^{p} (-1)^{i+1} \psi(y_i) y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes m \\ &= \sum_{i=1}^{p} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes \psi(y_i) m.\end{aligned}$$

PROPOSITION 1.3. *Let A be a commutative R-algebra, let $\psi : G \otimes A \to A$ be an A-homomorphism, and let M be an A-module. The complex $K_\cdot^A(\psi, M)$ can be canonically identified with the complex $(\bigwedge G \otimes M, \, \partial_\psi)$, where*

$$\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes m) = \sum_{i=1}^{p} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes \psi(y_i \otimes 1_A) m$$

for $y_1, \ldots, y_p \in G$ and $m \in M$.

Proof. Consider the diagram

$$\begin{array}{ccc} \bigwedge_A (G \otimes A) \otimes_A M & \xrightarrow{\partial_\psi^A} & \bigwedge_A (G \otimes A) \otimes_A M \\ \downarrow & & \downarrow \\ \bigwedge G \otimes M & \xrightarrow{\partial_\psi} & \bigwedge G \otimes M \end{array}$$

where the vertical arrows are the compositions of the canonical isomorphisms

$$\bigwedge\nolimits_A (G \otimes A) \otimes_A M \xrightarrow{\beta \otimes 1_M} \bigwedge G \otimes (A \otimes_A M),$$

$$\beta((y_1 \otimes a_1) \wedge \ldots \wedge (y_p \otimes a_p)) = y_1 \wedge \ldots \wedge y_p \otimes a_1 \ldots a_p$$

for $y_i \in G$ and $a_i \in A$, and

$$\bigwedge G \otimes (A \otimes_A M) \xrightarrow{1 \otimes \alpha} \bigwedge G \otimes M,$$

α being the usual multiplication map. It is obvious that the diagram is commutative.

□

Let $\varphi : R \to G$ be an R-homomorphism. The algebra $\bigwedge G$ can be viewed as a right $\bigwedge G$-module (left $\bigwedge G$-module), and there is a unique endomorphism (antiendomorphism) d_φ of $\bigwedge G$ which extends $\varphi : \bigwedge^0 G \to \bigwedge^1 G$. It has degree (1) with respect to the grading of $\bigwedge G$. If $y \in \bigwedge G$, then

$$d_\varphi(y) = \varphi(1) \wedge y.$$

DEFINITION 1.4. If M is an R-module, we denote by $K^{\cdot}_R(\varphi, M)$ the (cochain) complex $(M \otimes \bigwedge G,\ 1_M \otimes d_\varphi)$. The differential of $K^{\cdot}_R(\varphi, M)$ is denoted by d_φ or d^R_φ.

REMARK 1.5. The differential d_φ of $K^{\cdot}_R(\varphi, M)$ has also degree (1) with respect to the grading of $M \otimes \bigwedge G$. If $y \in \bigwedge G$ and $m \in M$, then

$$d_\varphi(m \otimes y) = m \otimes \varphi(1) \wedge y.$$

As is usual, we often write

$$0 \to M \otimes \bigwedge^0 G \to M \otimes \bigwedge^1 G \to \cdots \to M \otimes \bigwedge^p G \to \cdots$$

for $K^{\cdot}_R(\varphi, M)$.

There is an analogue with Proposition 1.3:

PROPOSITION 1.6. *Let A be a commutative R-algebra, let $\varphi : A \to A \otimes G$ be an A-homomorphism, and let M be an A-module. The complex $K^{\cdot}_R(\varphi, M)$ can be canonically identified with the complex $(M \otimes \bigwedge G,\ d_\varphi)$, where*

$$d_\varphi(m \otimes y) = \varphi(1) \cdot (m \otimes y)$$

for $y \in \bigwedge G$, $m \in M$.

Proof. Consider the diagram

$$\begin{array}{ccc} M \otimes_A \bigwedge_A (A \otimes G) & \xrightarrow{d^A_\varphi} & M \otimes_A \bigwedge_A (A \otimes G) \\ \downarrow & & \downarrow \\ M \otimes \bigwedge G & \xrightarrow{d_\varphi} & M \otimes \bigwedge G, \end{array}$$

where the vertical arrows are canonical isomorphisms. It is obvious that the diagram is commutative. □

Let A, B be commutative R-algebras. Furthermore let $\varphi : A \to A \otimes G$ be an A-homomorphism and $\psi : G \otimes B \to B$ be a B-homomorphism. If M is an A-module and N is a B-module, then we obtain a diagram $K_{A.}^{\cdot B}(\varphi, M, \psi, N)$

$$\begin{tikzcd} M \otimes \bigwedge G \otimes N \arrow[r, "d_\varphi"] \arrow[d, "\partial_\psi"'] & M \otimes \bigwedge G \otimes N \arrow[d, "\partial_\psi"'] \\ M \otimes \bigwedge G \otimes N \arrow[r, "d_\varphi"] & M \otimes \bigwedge G \otimes N \end{tikzcd}$$

in which d_φ and ∂_ψ stand for $d_\varphi \otimes 1_N$ and $1_M \otimes \partial_\psi$.

THEOREM 1.7. *If the composition*

$$A \otimes B \xrightarrow{\varphi \otimes 1_B} A \otimes G \otimes B \xrightarrow{1_A \otimes \psi} A \otimes B$$

is the zero homomorphism, then $K_{A.}^{\cdot B}(\varphi, M, \psi, N)$ is a (cochain-chain) bicomplex.

Proof. Let

$$\varphi(1) = \sum_{i=1}^{k} a_i^0 \otimes g_i^0$$

where $a_1^0, \ldots, a_1^0 \in A$ and $g_1^0, \ldots, g_k^0 \in G$. Our hypothesis says that

$$\sum_{i=1}^{k} a_i^0 \otimes \psi(g_i^0 \otimes 1) = 0,$$

and we have to prove that the diagram $K_{A.}^{\cdot B}(\varphi, M, \psi, N)$ is anticommutative. Let $m \in M$, $y_1, \ldots, y_p \in G$, and $n \in N$. Then

$$\partial_\psi d_\varphi(m \otimes y_1 \wedge \ldots \wedge y_p \otimes n) = \partial_\psi(\sum_{i=1}^{k} a_i^0 m \otimes g_i^0 \wedge y_1 \wedge \ldots \wedge y_p \otimes n)$$

$$= \sum_{i=1}^{k} a_i^0 m \otimes y_1 \wedge \ldots \wedge y_p \otimes \psi(g_i^0 \otimes 1)n$$

$$- \sum_{i=1}^{k} \sum_{j=1}^{p} (-1)^{j+1} a_i^0 m \otimes g_i^0 \wedge y_1 \wedge \ldots \widehat{y_j} \ldots \wedge y_p \otimes \psi(y_j \otimes 1)n$$

$$= - \sum_{i=1}^{k} \sum_{j=1}^{p} (-1)^{j+1} a_i^0 m \otimes g_i^0 \wedge y_1 \wedge \ldots \widehat{y_j} \ldots \wedge y_p \otimes \psi(y_j \otimes 1)n$$

since

$$\sum_{i=1}^{k} a_i^0 m \otimes y_1 \wedge \ldots \wedge y_p \otimes \psi(g_i^0 \otimes 1)n = (\sum_{i=1}^{k} a_i^0 \otimes \psi(g_i^0 \otimes 1))(m \otimes y_1 \wedge \ldots \wedge y_p \otimes n)$$

with respect to the $(A \otimes B)$-module structure of $M \otimes \bigwedge G \otimes N$. On the other hand

$$d_\varphi \partial_\psi (m \otimes y_1 \wedge \ldots \wedge y_p \otimes n) = d_\varphi \Big(\sum_{j=1}^{p} (-1)^{j+1} m \otimes y_1 \wedge \ldots \widehat{y_j} \ldots \wedge y_p \otimes \psi(y_j \otimes 1) n \Big)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{p} (-1)^{j+1} a_i^0 m \otimes g_i^0 \wedge y_1 \wedge \ldots \widehat{y_j} \ldots \wedge y_p \otimes \psi(y_j \otimes 1) n.$$

$\square$

REMARK 1.8. Theorem 1.7 may also be proved considering the $(A \otimes B)$-module structure of $A \otimes G \otimes B$ and using Proposition 1 in [BO3], §9. Subsequently we shall obtain a general result which contains Theorem 1.7 as a particular case.

REMARK 1.9. As usual, one can visualize $K^{\cdot B}_{A \cdot}(\varphi, M, \psi, N)$ as a family of maps in the (p, q)-plane. If we write

$$\bigwedge^i \quad \text{for} \quad M \otimes \bigwedge^i G \otimes N,$$

we get

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & \bigwedge^0 \arrow[r] \arrow[d] & \bigwedge^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^i \arrow[r, "d_\varphi"] \arrow[d] & \bigwedge^{i+1} \arrow[r] \arrow[d, "\partial_\psi"'] & \bigwedge^{i+2} \cdots \arrow[d] \\
& 0 \arrow[r] & \bigwedge^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigwedge^{i-1} \arrow[r] \arrow[d] & \bigwedge^i \arrow[r] \arrow[d] & \bigwedge^{i+1} \cdots \arrow[d] \\
& & 0 \arrow[r] & \cdots \arrow[r] & \bigwedge^{i-2} \arrow[r] \arrow[d] & \bigwedge^{i-1} \arrow[r] \arrow[d] & \bigwedge^i \cdots \arrow[d] \\
& & & & \vdots & \vdots & \vdots \\
& & & & 0 \arrow[r] & \bigwedge^0 \arrow[r] \arrow[d] & \bigwedge^1 \cdots \arrow[d] \\
& & & & & 0 \arrow[r] & \bigwedge^0 \cdots \arrow[d] \\
& & & & & & 0 \cdots
\end{tikzcd}$$

We adopt the following convention for the graded dual of a graded module (see [E], A2.4 for example). If $M = \oplus_{i\geq 0} M_i$ is a graded R-module, we shall write M^* for the graded dual of M, that is

$$M^* = M^*_{gr} = \oplus_{i\geq 0}(M_i)^*$$

(instead of $M^* = \mathrm{Hom}_R(M, R)$ as originally). This only makes a difference when M is not finitely generated. We use it mainly in the case in which M is the symmetric algebra $S(N)$ or the exterior algebra $\bigwedge N$ of an R-module N. Then

$$S(N)^* = \oplus_{i\geq 0} S_i(N)^*, \quad (\bigwedge N)^* = \oplus_{i\geq 0}(\bigwedge^i N)^*.$$

Correspondingly M^{**} means the graded bidual $(M^*_{gr})^*_{gr}$ of M. The canonical map

$$c_M : M \to M^{**} \qquad \text{is given by} \qquad c_M = \oplus_{i\geq 0} c_{M_i}.$$

Let N be an R-module. The natural graded algebra homomorphism

$$\theta : \bigwedge N^* \to (\bigwedge N)^*,$$

is given by

$$\theta(y_1^* \wedge \ldots \wedge y_p^*)(y_1 \wedge \ldots \wedge y_p) = \det(y_j^*(y_i))$$

for all $y_1, \ldots, y_p \in N$ and $y_1^*, \ldots, y_p^* \in N^*$. If N is finitely generated and projective, then θ is an isomorphism (see Proposition 7 in [BO1], Chapter III, § 11.5, and note that Bourbaki uses the opposite algebra to $(\bigwedge G)^*$).

REMARK 1.10. As above, let N be an R-module. We define a multiplication on $\bigwedge N \otimes \bigwedge N$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{(\deg y_1)(\deg x_2)}(x_1 \wedge x_2) \otimes (y_1 \wedge y_2)$$

for all homogeneous elements x_1, y_1, x_2, y_2 of $\bigwedge N$ (the skew tensor product). Then $\bigwedge N$ becomes a bialgebra, the cogebra structure given by the composition of the multiplication just defined with the diagonal map

$$\Delta(y) = y \otimes 1 + 1 \otimes y$$

for all $y \in N$. If $y_1, \ldots, y_n \in N$, the element $\Delta(y_1 \wedge \ldots \wedge y_n)$ is of total degree n in $\bigwedge N \otimes \bigwedge N$ and its homogeneous component of bidegree $(p, n-p)$ is equal to

$$\sum_\sigma \varepsilon(\sigma)(y_{\sigma(1)} \wedge \ldots \wedge y_{\sigma(p)}) \otimes (y_{\sigma(p+1)} \wedge \ldots \wedge y_{\sigma(n)}),$$

where σ runs through the set $\mathfrak{S}_{n,p}$ of permutations of n elements which are increasing on the intervals $[1, p]$ and $[p+1, n]$ (see Example 7 in [BO1] III §11.1).

The algebra structure on $(\bigwedge N)^*$ is induced by the coalgebra structure of $\bigwedge N$, the product of two elements x^*, y^* being defined by $x^* y^* = \alpha(x^* \otimes y^*)\Delta$, where

$\alpha : R \otimes R \to R$ is the multiplication. The right and the left inner products are given by $y \leftharpoondown y^* = (y^* \otimes 1)\Delta(y)$ and $y^* \rightharpoonup y = (1 \otimes y^*)\Delta(y)$ for all $y \in \bigwedge N$ and $y^* \in (\bigwedge N)^*$. They define a $(\bigwedge N)^*$-bimodule structure on $\bigwedge N$. Using θ this can be extended to a $\bigwedge N^*$-bimodule structure. If $p \leq n$, then one can easily see that

$$y_1 \wedge \ldots \wedge y_n \leftharpoondown y_1^* \wedge \ldots \wedge y_p^* = \sum_{\sigma} \varepsilon(\sigma) \det_{1\leq i,j\leq p}(y_j^*(y_{\sigma(i)}))y_{\sigma(p+1)} \wedge \ldots \wedge y_{\sigma(n)}$$

for $y_1, \ldots, y_n \in N$ and $y_1^*, \ldots, y_p^* \in N^*$, where σ runs through $\mathfrak{S}_{n,p}$.

An easy calculation shows that an element $n^* \in N^*$ acts like an antiderivation on $\bigwedge N$ in the sense that

$$(x \wedge y) \leftharpoondown n^* = (x \leftharpoondown n^*) \wedge y + (-1)^{degx} x \wedge (y \leftharpoondown n^*)$$

for homogeneous elements $x, y \in \bigwedge N$.

We are now able to state the promised generalization of Theorem 1.7. ∂_ψ is the multiplication by ψ if we view $\bigwedge G$ as a right $\bigwedge G^*$-module, and d_φ is the left multiplication by $\varphi(1)$ in the algebra $\bigwedge G$. Theorem 1.7 may be seen as a particular case ($l = p = 1$) of the following result.

THEOREM 1.11. *Let $x_i \in G$ for $i = 1, \ldots, l$, and let $z_j^* \in G^*$ for $j = 1, \ldots, p$ such that $z_j^*(x_i) = 0$ for all i, j. As above let $\leftharpoondown$ denote the right operation of $\bigwedge G^*$ on $\bigwedge G$, then*

$$x_1 \ldots x_l \wedge (y_1 \ldots y_n \leftharpoondown z_1^* \ldots z_p^*) = (-1)^{lp}(x_1 \ldots x_l \wedge y_1 \ldots y_n) \leftharpoondown z_1^* \ldots z_p^*$$

where $y_k \in G$ for $k = 1, \ldots, n$.

Proof. If $n < p$, then both sides are 0. If $n \geq p$, then

$$\begin{aligned} &x_1 \ldots x_l \wedge (y_1 \ldots y_n \leftharpoondown z_1^* \ldots z_p^*) \\ &\qquad = \sum_{\sigma\in\mathfrak{S}_{n,p}} \varepsilon(\sigma) \det_{1\leq i,j\leq p}(z_i^*(y_{\sigma(j)}))x_1 \ldots x_l \wedge y_{\sigma(p+1)} \ldots y_{\sigma(n)}. \end{aligned}$$

On the other hand

$$\begin{aligned} &(x_1 \ldots x_l \wedge y_1 \ldots y_n) \leftharpoondown z_1^* \ldots z_p^* \\ &\qquad = \sum_{\sigma} \varepsilon(\sigma) \det_{1\leq i,j\leq p}(z_i^*(y_{\sigma(j)-l}))x_1 \ldots x_l \wedge y_{\sigma(p+l+1)-l} \ldots y_{\sigma(n+l)-l} \end{aligned}$$

where σ runs through the set $\mathfrak{S}'_{n+l,p}$ of permutations of $n + l$ elements which are increasing on the intervals $[1, p]$ and $[p + 1, n + l]$, and have values greater than l on $[1, p]$. (Note that $z_i^*(x_{\sigma(j)}) = 0$ by assumption.) Now we define a bijection $\mathfrak{S}_{n,p} \to \mathfrak{S}'_{n+l,p}$, $\sigma_1 \mapsto \sigma_2$, by

$$\sigma_2(i) = \begin{cases} \sigma_1(i) + l & \text{if } \ i = 1, \ldots, p \\ i - p & \text{if } \ i = p + 1, \ldots, p + l \\ \sigma_1(i - l) + l & \text{if } \ i = p + l + 1, \ldots, n + l. \end{cases}$$

Clearly $\varepsilon(\sigma_1) = (-1)^{lp}\varepsilon(\sigma_2)$. $\square$

For further references we recall some well known results from linear algebra.

LEMMA 1.12. *If M, N are R-modules, then there is a natural map*

$$\zeta = \zeta_{M,N} : M^* \otimes N \to \mathrm{Hom}(M, N),$$

given by

$$\zeta(m^* \otimes n)(m) = m^*(m)n$$

for all $m \in M$, $m^* \in M^*$ and $n \in N$. Suppose that one of the following assumptions

(a) *N is finitely generated projective, or*

(b) *M is finitely generated projective, or*

(c) *M is finitely generated and N is flat*

is true. Then ζ is an isomorphism.

Proof. See Lemma 5 and Corollary in [BO1] Chapter II § 4.2 for (a) or (b). For (c) see Lemma 3.83 in [R] . □

If G is an R-module, A is a commutative R-algebra and M is an A-module, then $\mathrm{Hom}(G, M)$ and $\mathrm{Hom}(M, G)$ are A-modules in a natural way.

LEMMA 1.13. *The map*

$$\xi : \mathrm{Hom}(G, M) \to \mathrm{Hom}_A(G \otimes A, M),$$

given by

$$\xi(f)(y \otimes a) = af(y)$$

for all $y \in G$, $a \in A$ and $f \in \mathrm{Hom}(G, M)$, is an isomorphism of A-modules.

Proof. The map

$$\mathrm{Hom}_A(G \otimes A, M) \to \mathrm{Hom}(G, M), \quad \varphi \mapsto \varphi \circ \iota,$$

where $\iota : G \to G \otimes A$ is the map $y \mapsto y \otimes 1_A$, is obviously the inverse of ξ. □

REMARK 1.14. In order to simplify the presentation, when dualizing a tensor product over R, we shall sometimes use, without mentioning, the twist map to change the order in which the modules appear.

We are ready now to formulate a general result concerning the connection of the Koszul complex and its dual through natural complex isomorphisms.

THEOREM 1.15. *Let A be a commutative R-algebra, let M be an A-module, let G be an R-module, and let $\varphi : A \to G^* \otimes A$ be an A-homomorphism. By ω we denote the composition*

$$\mathrm{Hom}_A(A, G^* \otimes A) \cong G^* \otimes A \xrightarrow{\zeta} \mathrm{Hom}(G, A) \overset{\xi}{\cong} \mathrm{Hom}_A(G \otimes A, A).$$

Then there are natural complex morphisms

$$K^{\cdot}_A(\varphi, M^*) \longrightarrow \Big(K^A_{\cdot}(\omega(\varphi), M)\Big)^* \text{ and}$$
$$K^{\cdot}_A(\varphi, M) \longrightarrow \Big(K^A_{\cdot}(\omega(\varphi), M^*)\Big)^*.$$

Moreover, if G is finitely generated projective, then the first morphism is a complex isomorphism. If G is finitely generated projective and M is a graded R-module such that every homogeneous component is finitely generated projective, then also the second morphism is a complex isomorphism. (*Note that if M is graded, we use the special conventions previously described*).

Proof. Let

$$\varphi(1) = \sum_{i=1}^{k} a_i^0 \otimes \overset{0}{g}{}_i^*$$

where $a_1^0, \dots, a_k^0 \in A$ and $\overset{0}{g}{}_1^*, \dots, \overset{0}{g}{}_k^* \in G^*$. Consider the diagram

$$\begin{array}{ccc}
M^* \otimes \bigwedge^p G^* & \xrightarrow{d_\varphi} & M^* \otimes \bigwedge^{p+1} G^* \\
{\scriptstyle \mu(1_{M^*}\otimes\theta)}\Big\downarrow & & {\scriptstyle \mu(1_{M^*}\otimes\theta)}\Big\downarrow \\
(M \otimes \bigwedge^p G)^* & \xrightarrow{(\partial_{\omega(\varphi)})^*} & (M \otimes \bigwedge^{p+1} G)^*
\end{array}$$

where $\theta : \bigwedge^p G^* \to (\bigwedge^p G)^*$ is the map defined above and μ is the natural homomorphism

$$M^* \otimes (\textstyle\bigwedge^p G)^* \to (M \otimes \bigwedge^p G)^*,$$
$$\mu(m^* \otimes y^*)(m \otimes y) = m^*(m) y^*(y)$$

for $m \in M$, $m^* \in M^*$, $y \in \bigwedge^p G$ and $y^* \in (\bigwedge^p G)^*$. We prove that the diagram is commutative. Choose elements $m^* \in M$, $y_1^*, \dots, y_p^* \in G^*$, $m \in M$, $y_1, \dots, y_{p+1} \in G$, and set $y^* = y_1^* \wedge \dots \wedge y_p^*$, $y = y_1 \wedge \dots \wedge y_{p+1}$. Then

$$\mu \circ (1_{M^*} \otimes \theta) \circ d_\varphi(m^* \otimes y^*) = \sum_{i=1}^{k} \mu(a_i^0 m^* \otimes \theta(\overset{0}{g}{}_i^* \wedge y^*)),$$

and therefore

$$\Big(\mu(1_{M^*}\otimes\theta)d_\varphi(m^*\otimes y^*)\Big)(m\otimes y)=\sum_{i=1}^{k}a_i^0 m^*(m)\theta(\overset{0}{g}{}_i^*\wedge y^*)(y)$$

$$=\sum_{i=1}^{k}a_i^0 m^*(m)\begin{vmatrix}\overset{0}{g}{}_i^*(y_1) & \cdots & \overset{0}{g}{}_i^*(y_{p+1})\\ y_1^*(y_1) & \cdots & y_1^*(y_{p+1})\\ & \cdots & \\ y_p^*(y_1) & \cdots & y_p^*(y_{p+1})\end{vmatrix}.$$

On the other hand

$$\omega(\varphi)(y_j\otimes 1)\sum_{i=1}^{k}a_i^0\overset{0}{g}{}_i^*(y_j).$$

So

$$\Big((\partial_{\omega(\varphi)})^*\ \mu\ (1_{M^*}\otimes\theta)(m^*\otimes y^*)\Big)(m\otimes y)=\mu(1_{M^*}\otimes\theta)(m^*\otimes y^*)\partial_{\omega(\varphi)}(m\otimes y)$$

$$=\mu(1_{M^*}\otimes\theta)(m^*\otimes y^*)(\sum_{j=1}^{p+1}(-1)^{j+1}\omega(\varphi)(y_j\otimes 1_A)m\otimes y_1\wedge\ldots\widehat{y_j}\ldots\wedge y_{p+1})$$

$$=\sum_{j=1}^{p+1}(-1)^{j+1}m^*(\omega(\varphi)(y_j\otimes 1_A)m)\theta(y^*)(y_1\wedge\ldots\widehat{y_j}\ldots\wedge y_{p+1})$$

$$=\sum_{i=1}^{k}a_i^0 m^*(m)\sum_{j=1}^{p+1}(-1)^{j+1}\overset{0}{g}{}_i^*(y_j)\begin{vmatrix}y_1^*(y_1) & \cdots & \widehat{y_1^*(y_j)} & \cdots & y_1^*(y_{p+1})\\ & & \cdots & & \\ y_p^*(y_1) & \cdots & \widehat{y_p^*(y_j)} & \cdots & y_p^*(y_{p+1})\end{vmatrix}$$

$$=\sum_{i=1}^{k}a_i^0 m^*(m)\begin{vmatrix}\overset{0}{g}{}_i^*(y_1) & \cdots & \overset{0}{g}{}_i^*(y_{p+1})\\ y_1^*(y_1) & \cdots & y_1^*(y_{p+1})\\ & \cdots & \\ y_p^*(y_1) & \cdots & y_p^*(y_{p+1})\end{vmatrix}.$$

The second part has a similar proof. One has just to use $c_M : M\to M^{**}$ instead of 1_{M^*}.

Moreover, if G is finitely generated projective, then θ is an isomorphism, and G^* is also finitely generated projective ([BO1] II §2.2, Corollary 2). This implies that $\bigwedge G^*$ is finitely generated projective ([BO1] III §7.4, Proposition 6). So μ is an isomorphism ([BO1] II §4.4, Corollary 1). If $M=\oplus_{i\geq 0}M_i$ is a graded R-module such that every homogeneous component is finitely generated projective, then c_M is also an isomorphism. $\square$

1.2 Generalized Koszul Complexes

Let

$$H \xrightarrow{\varphi} G \quad \text{and} \quad G \xrightarrow{\psi} F$$

be homomorphisms of R-modules. Most of the results of this section are true if

$$\zeta_{H,G} : H^* \otimes G \to \mathrm{Hom}(H, G) \quad \text{and} \quad \zeta_{G,F} : G^* \otimes F \to \mathrm{Hom}(G, F)$$

are isomorphisms (see Lemma 1.12). For simplicity we restrict the presentation to the particular case in which H and F are finitely generated free modules.

Since F is canonically identified with $S_0(F)$, we can consider ψ an element of

$$\mathrm{Hom}(G, S(F)) \cong \mathrm{Hom}_{S(F)}(G \otimes S(F), S(F))$$

(see Lemma 1.13). For every $S(F)$-module M, ψ gives rise to a Koszul complex of $S(F)$-modules $K_{\cdot}^{S(F)}(\psi, M)$. If $M = \oplus_{i \geq 0} M_i$ is a graded $S(F)$-module, then $K_{\cdot}^{S(F)}(\psi, M)$, as a complex of R-modules, splits into direct summands $K_{\cdot}^{S(F)}(\psi, M)(t)$, $t \in \mathbb{Z}$,

$$0 \to \bigwedge^t G \otimes M_0 \xrightarrow{\partial_\psi} \bigwedge^{t-1} G \otimes M_1 \to \cdots \to \bigwedge^1 G \otimes M_{t-1} \xrightarrow{\partial_\psi} \bigwedge^0 G \otimes M_t \to 0.$$

M^* is a graded $S(F)$-module with graduation $M^* = \oplus_{i \leq 0} M^*_{-i}$, and $K_{\cdot}^{S(F)}(\psi, M^*)$ as a complex of R-modules splits into direct summands $K_{\cdot}^{S(F)}(\psi, M^*)(t)$, $t \in \mathbb{Z}$,

$$\cdots \to \bigwedge^{t+p} G \otimes M^*_{-p} \xrightarrow{\partial_\psi} \bigwedge^{t+p-1} G \otimes M^*_{-(p-1)} \to \cdots \to \bigwedge^{t+1} G \otimes M^*_{-1} \xrightarrow{\partial_\psi} \bigwedge^t G \otimes M^*_0 \to 0.$$

As already noted, the map $\psi : G \to F$ can be viewed as an element of $\mathrm{Hom}_{S(F)}(G \otimes S(F), S(F))$. The corresponding $S(F)$-antiderivation ∂_ψ of $\bigwedge_{S(F)}(G \otimes S(F))$ is nothing but the right multiplication by ψ on $\bigwedge_{S(F)}(G \otimes S(F))$.

Since F is assumed to be a finitely generated free R-module (in the following it suffices to know that the canonical map $G^* \otimes F \to \mathrm{Hom}(G, F)$ is surjective), we may also regard ψ as an element of $G^* \otimes S(F)$ (Lemma 1.12) which is the degree 1 homogeneous part of $\bigwedge_{S(F)}(G^* \otimes S(F))$. Furthermore

$$\bigwedge_{S(F)}(G^* \otimes S(F)) \cong \bigwedge G^* \otimes S(F),$$

(as $S(F)$-algebras). So ψ may be viewed as an element of $\bigwedge G^* \otimes S(F)$. An easy computation shows that the right multiplication by ψ on $\bigwedge_{S(F)}(G \otimes S(F)) \cong \bigwedge G \otimes S(F)$ this time has the same result as above.

DEFINITION 1.16. Assume that $x^* \in \bigwedge G^*$ is homogeneous of grade i. If $(x^* \otimes 1_{S(F)})\psi = 0$, then the right multiplication by $x^* \otimes 1_{S(F)}$ on $\bigwedge G \otimes S(F)$ is called a connection homomorphism for ψ of grade i and is denoted by $\nu_\psi^{x^*}$.

PROPOSITION 1.17. *If $\nu_\psi^{x^*}$ is a connection homomorphism for ψ, then*

$$\partial_\psi \nu_\psi^{x^*} = \nu_\psi^{x^*} \partial_\psi = 0.$$

Proof. $\partial_\psi \nu_\psi^{x^*}$ is the right multiplication by $(x^* \otimes 1_{S(F)})\psi$ on $\bigwedge G \otimes S(F)$ which is zero by assumption. Since $\psi(x^* \otimes 1_{S(F)}) = \pm(x^* \otimes 1_{S(F)})\psi$, we likewise obtain $\nu_\psi^{x^*} \partial_\psi = 0$. □

REMARK 1.18. The short and easy proof of Proposition 1.17 is a consequence of our interpretation of ∂_ψ and the definition above. See for example [E], proof of Theorem A2.10 (a). Observe that we do not require G to be free.

Let $\nu_\psi^{x^*}$ be a connection homomorphism for ψ of grade i and suppose that $M = \oplus_{i \geq 0} M_i$ is a graded $S(F)$-module with $M_0 = R$. Then for all $t \in \mathbb{Z}$, we splice $K_\cdot^{S(F)}(\psi, M^*)(t+i)$ and $K_\cdot^{S(F)}(\psi, M)(t)$ to a complex

$$\cdots \to \textstyle\bigwedge^{t+i+p} G \otimes M_{-p}^* \xrightarrow{\partial_\psi} \cdots \xrightarrow{\partial_\psi} \bigwedge^{t+i} G \otimes M_0^* \xrightarrow{\nu_\psi^{x^*}} \bigwedge^t G \otimes M_0 \xrightarrow{\partial_\psi} \cdots$$
$$\xrightarrow{\partial_\psi} \textstyle\bigwedge^0 G \otimes M_t \to 0.$$

denoted by

$$(K_\cdot^{S(F)}(\psi, M^*) \xrightarrow{\nu_\psi^{x^*}} K_\cdot^{S(F)}(\psi, M))(t).$$

There is a similar construction for the map $\varphi : H \to G$. Since H is free and finitely generated, the natural homomorphism $H^* \otimes G \to \operatorname{Hom}(H, G)$ is an isomorphism. So one may view φ as an element of $H^* \otimes G$. Since H^* is the degree 1 homogeneous part of the symmetric algebra $S(H^*)$, we can consider φ an element of

$$S(H^*) \otimes G \cong \operatorname{Hom}_{S(H^*)}(S(H^*), S(H^*) \otimes G).$$

For every $S(H^*)$-module M, φ gives rise to a Koszul complex $K^\cdot_{S(H^*)}(\varphi, M)$ of $S(H^*)$-modules (see Section 1). If $M = \oplus_{i \geq 0} M_i$ is a graded $S(H^*)$-module, then $K^\cdot_{S(H^*)}(\varphi, M)$, as a complex of R-modules, splits into direct summands $K^\cdot_{S(H^*)}(\varphi, M)(t)$, $t \in \mathbb{Z}$,

$$0 \to M_0 \otimes \textstyle\bigwedge^t G \xrightarrow{d_\varphi} M_1 \otimes \bigwedge^{t+1} G \to \cdots \to M_{p-1} \otimes \bigwedge^{t+p-1} G \xrightarrow{d_\varphi} M_p \otimes \bigwedge^{t+p} G \to \cdots,$$

and $K^\cdot_{S(H^*)}(\varphi, M^*)$ splits into direct summands $K^\cdot_{S(H^*)}(\varphi, M^*)(t)$, $t \in \mathbb{Z}$,

$$0 \to M_{-t}^* \otimes \textstyle\bigwedge^0 G \xrightarrow{d_\varphi} M_{-(t-1)}^* \otimes \bigwedge^1 G \to \cdots \to M_{-1}^* \otimes \bigwedge^{t-1} G \xrightarrow{d_\varphi} M_0^* \otimes \bigwedge^t G \to 0.$$

$S(H^*) \otimes G$ is the degree 1 homogeneous part of $\bigwedge_{S(H^*)}(S(H^*) \otimes G)$, and we can consider φ even an element of

$$\textstyle\bigwedge_{S(H^*)}(S(H^*) \otimes G) \cong S(H^*) \otimes \bigwedge G.$$

The differential d_φ of $K^\cdot_{S(H^*)}(\varphi, S(H^*))$ then turns out to be the left multiplication by φ.

DEFINITION 1.19. Let $x \in \bigwedge G$ be homogeneous of grade i such that $(1_{S(H^*)} \otimes x)\varphi = 0$. Then the left multiplication by $1_{S(H^*)} \otimes x$ on $S(H^*) \otimes \bigwedge G$ is called a connection homomorphism for φ of grade i and is denoted by ν_x^φ .

PROPOSITION 1.20. *If ν_x^φ is a connection homomorphism for φ, then*

$$d_\varphi \nu_x^\varphi = \nu_x^\varphi d_\varphi = 0.$$

Proof. With regard to the proof of 1.17, the result is obvious. □

If ν_x^φ is a connection homomorphism for φ of grade i and $M = \oplus_{i\geq 0} M_i$ is a graded $S(H^*)$-module with $M_0 = R$, then for all $t \in \mathbb{Z}$, we splice $K^{\cdot}_{S(H^*)}(\varphi, M^*)(t)$ and $K^{\cdot}_{S(H^*)}(\varphi, M)(t+i)$ to a complex

$$0 \to M^*_{-t} \otimes \bigwedge^0 G \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} M^*_0 \otimes \bigwedge^t G \xrightarrow{\nu_x^\varphi} M_0 \otimes \bigwedge^{t+i} G \xrightarrow{d_\varphi} \cdots$$
$$\xrightarrow{d_\varphi} M_p \otimes \bigwedge^{t+i+p} G \to \cdots .$$

denoted by

$$(K^{\cdot}_{S(H^*)}(\varphi, M^*) \xrightarrow{\nu_x^\varphi} K^{\cdot}_{S(H^*)}(\varphi, M))(t).$$

We shall now establish a natural relation between the complexes introduced above.

THEOREM 1.21. *Suppose that F is a finite free R-module, and let $x^* \in \bigwedge G^*$ be such that $\nu_\psi^{x^*}$ is a connection morphism for ψ. Then $\nu_{x^*}^{\psi^*}$ is a connection morphism for ψ^*.*

Furthermore let M be a graded $S(F)$-module such that $M_0 = R$. Then there is a natural complex morphism

$$\begin{array}{ccc} (K^{\cdot}_{S(F)}(\psi^*, M^*) & \xrightarrow{\nu_{x^*}^{\psi^*}} & K^{\cdot}_{S(F)}(\psi^*, M))(t) \\ & \tau \big\downarrow & \\ \Big((K_{\cdot}^{S(F)}(\psi, M^*) & \xrightarrow{\nu_\psi^{x^*}} & K_{\cdot}^{S(F)}(\psi, M))(t)\Big)^*. \end{array}$$

Moreover, if G is finitely generated projective and every homogeneous component of M is finitely generated and projective over R, then τ is a complex isomorphism.

Proof. Our assumptions guarantee that the canonical map $F \to F^{**}$ induces an isomorphism $G^* \otimes F \cong F^{**} \otimes G^*$ and this again an $S(F)$-algebra isomorphism $\bigwedge G^* \otimes S(F) \cong S(F^{**}) \otimes \bigwedge G^*$. Furthermore the canonical maps $\zeta_{G,F} : G^* \otimes F \to \mathrm{Hom}(G, F)$ and $\zeta_{F^*,G^*} : F^{**} \otimes G^* \to \mathrm{Hom}(F^*, G^*)$ are isomorphisms. Consequently, in order to prove that $(x^* \otimes 1_{S(F)})\psi = 0$ implies $(1_{S(F^{**})} \otimes x^*)\psi^* = 0$, it suffices to show that the preimages of ψ and ψ^* with respect to $\zeta_{G,F}$ and ζ_{F^*,G^*} are mapped one to the other by the isomorphism mentioned above.

Let $f_1, \ldots, f_m$ be a basis for F and $f_1^*, \ldots, f_m^*$ the dual basis of F^*. We identify f_j with its canonical image in F^{**}. An easy calculation shows that

$$\zeta_{G,F}(\sum_j \psi^*(f_j^*) \otimes f_j) = \psi \quad \text{and} \quad \zeta_{F^*,G^*}(\sum_j f_j \otimes \psi^*(f_j^*)) = \psi^*. \tag{$*$}$$

So $\nu_{x^*}^{\psi^*}$ is a connection homomorphism for ψ^*.

Now we apply Theorem 1.15: set $A = S(F)$, and let $\varphi : A \to G^* \otimes A$ be the A-homomorphism given by $1 \mapsto \zeta_{F^*,G^*}^{-1}(\psi^*)$. Using the canonical isomorphism $F^{**} \otimes G^* \cong G^* \otimes F$ and the equations $(*)$, we obtain that $\omega(\varphi) = \xi(\psi)$. So we get natural complex morphisms

$$K^{\cdot}_{S(F)}(\psi^*, M^*)(t) \xrightarrow{\ \tau\ } \Big(K_{\cdot}^{S(F)}(\psi, M)(t)\Big)^*,$$

and

$$K^{\cdot}_{S(F)}(\psi^*, M)(t) \xrightarrow{\ \tau\ } \Big(K_{\cdot}^{S(F)}(\psi, M^*)(t)\Big)^*$$

It remains to show that the diagram

$$\begin{array}{ccc}
\bigwedge^t G^* & \xrightarrow{\nu_{x^*}^{\psi^*}} & \bigwedge^{t+i} G^* \\
\theta \big\downarrow & & \theta \big\downarrow \\
(\bigwedge^t G)^* & \xrightarrow{(\nu_{\psi}^{x^*})^*} & (\bigwedge^{t+i} G)^*
\end{array}$$

is commutative. Let $y^* \in \bigwedge^t G^*$. Then $\theta \circ \nu_{x^*}^{\psi^*}(y^*) = \theta(x^* \wedge y^*)$. To show that this equals $(\nu_{\psi}^{x^*})^*(\theta(y^*))$, we have to prove that $(\theta(y^*))(z \leftharpoondown x^*) = (\theta(x^* \wedge y^*))(z)$ for all $z \in \bigwedge G^{t+i}$. Here one may assume that $x^* = x_1^* \wedge \ldots \wedge x_i^*$, $x_j^* \in G^*$, $y^* = y_1^* \wedge \ldots \wedge y_t^*$, $y_j^* \in G^*$, and that $z = z_1 \wedge \ldots \wedge z_{t+i}$, $z_j \in G$. Then

$$\begin{aligned}
(\theta(y^*))(z \leftharpoondown x^*) &= \theta(y^*)(\sum_\sigma \varepsilon(\sigma) \det_{1 \le p,q \le i}(x_q^*(z_{\sigma(p)})) z_{\sigma(i+1)} \wedge \ldots \wedge z_{\sigma(t+i)}) \\
&= \sum_\sigma \varepsilon(\sigma) \det_{1 \le p,q \le i}(x_q^*(z_{\sigma(p)})) \det_{1 \le p,q \le t}(y_q^*(z_{\sigma(i+p)})) \\
&= \begin{vmatrix} x_1^*(z_1) & \cdots & x_i^*(z_1) & y_1^*(z_1) & \cdots & y_t^*(z_1) \\ & \cdots & & & \cdots & \\ x_1^*(z_{t+i}) & \cdots & x_i^*(z_{t+i}) & y_1^*(z_{t+i}) & \cdots & y_t^*(z_{t+i}) \end{vmatrix} \\
&= (\theta(x^* \wedge y^*))(z),
\end{aligned}$$

where σ runs through the set $\mathfrak{S}_{t+i,i}$ of permutations of $t+i$ elements which are increasing on the intervals $[1, i]$ and $[i+1, t+i]$ (see [BO1] III §8.6 for the expansions of a determinant). $\square$

REMARK 1.22. If M is a free R-module of finite rank, then $S(M)$, $S(M)^*$, and $D(M^*)$ are bialgebras. There is a bialgebra isomorphism $\Gamma : D(M^*) \to S(M)^*$ given by

$$\Gamma(m^{*(k)})(\prod_i m_i^{k_i}) = \begin{cases} 0 & \text{if} \quad \sum k_i \neq k \\ \prod_i (m^*(m_i))^{k_i} & \text{if} \quad \sum k_i = k \end{cases}$$

for all $m_1, \ldots, m_p \in M$ and $m^* \in M^*$ (see [BE] A' and [E] Proposition A2.6). Moreover, we have bialgebra isomorphisms

$$S(M) \cong S(M)^{**} \cong D(M^*)^*.$$

The natural $S(M)$-module structure of $D(M^*)$ is given as follows: if $m \in S(M)$ and $y^* \in D(M^*)$, then $(my^*)(n) = \Gamma(y^*)(mn)$ for all $n \in S(M)$. The interaction of the $S(M)$-module structure of $D(M^*)$ and the algebra structure of $D(M^*)$ is described by the following result.

PROPOSITION 1.23. *If $m \in M$, then m acts like a derivation on $D(M^*)$ in the sense that*

(a) $mf^{*(k)} = f^*(m)f^{*(k-1)}$ *for all* $f^* \in M^*$, $k \geq 1$,

(b) $m(f^*g^*) = (mf^*)g^* + f^*(mg^*)$ *for all* $f^*, g^* \in D(M^*)$.

Proof. (a) Let $m, n_1 \ldots n_{k-1} \in M$ and set $n = n_1 \ldots n_{k-1}$. Then

$$mf^{*(k)}(n) = \Gamma(f^{*(k)})(mn) = f^*(m)f^*(n_1) \ldots f^*(n_{k-1}) = f^*(m)f^{*(k-1)}(n).$$

(b) Since $S(M) = S(M)^{**}$, we may use Proposition 10 in [BO1] III §11.8: the left multiplication by m on $D(M^*)$ is a derivation on $S(M)$. □

In particular, we can give now descriptions of $K_{\cdot}^{S(F)}(\psi, D(F^*))(t)$ and $K^{\cdot}_{S(H^*)}(\varphi, D(H^{**}))(t) = K^{\cdot}_{S(H^*)}(\varphi, D(H))(t)$ which depend only on the algebra structure of $D(F^*)$ and $D(H)$.

PROPOSITION 1.24. (a) *Let $z_1^*, \ldots, z_q^* \in F^*$, and $y_1, \ldots, y_p \in G$. Then*

$$\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes z_1^{*(k_1)} \ldots z_q^{*(k_q)}) = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes z_j^*(\psi(y_i)) z_1^{*(k_1)} \ldots z_j^{*(k_j-1)} \ldots z_q^{*(k_q)}.$$

(b) *Let $x_1, \ldots, x_q \in H$, $y \in \bigwedge G$. Then*

$$d_\varphi(x_1^{(k_1)} \ldots x_q^{(k_q)} \otimes y) = \sum_{j=1}^{q} x_1^{(k_1)} \ldots x_j^{(k_j-1)} \ldots x_q^{(k_q)} \otimes \varphi(x_j) \wedge y.$$

Proof. (a) Using Proposition 1.23 and Proposition 1.3, we obtain

$$\begin{aligned}
&\partial_\psi(y_1 \wedge \ldots \wedge y_p \otimes z_1^{*(k_1)} \ldots z_q^{*(k_q)}) \\
&\quad = \sum_{i=1}^{p} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes \psi(y_i) z_1^{*(k_1)} \ldots z_q^{*(k_q)} \\
&\quad = \sum_{i=1}^{p} (-1)^{i+1} y_1 \wedge \ldots \widehat{y_i} \ldots \wedge y_p \otimes \sum_{j=1}^{q} z_j^*(\psi(y_i)) z_1^{*(k_1)} \ldots z_j^{*(k_j-1)} \ldots z_q^{*(k_q)}.
\end{aligned}$$

(b) Let $h_1, \ldots, h_l$ be a basis of H and denote by $h_1^*, \ldots, h_l^*$ the basis of H^* dual to $h_1, \ldots, h_l$. Obviously

$$\varphi = \zeta_{H,G}\big(\sum_{i=1}^{l} h_i^* \otimes \varphi(h_i)\big).$$

So, if we view φ as a map $S(H^*) \to S(H^*) \otimes G$, then $\varphi(1) = \sum_{i=1}^{l} h_i^* \otimes \varphi(h_i)$.

Using Proposition 1.6 and Proposition 1.23 (applied to $M = H^{**} = H$), we get

$$\begin{aligned}
d_\varphi(x_1^{(k_1)} \ldots x_q^{(k_q)} \otimes y) &= \sum_{i=1}^{l} h_i^* x_1^{(k_1)} \ldots x_q^{(k_q)} \otimes \varphi(h_i) \wedge y \\
&= \sum_{i=1}^{l} \sum_{j=1}^{q} h_i^*(x_j) x_1^{(k_1)} \ldots x_j^{(k_j-1)} \ldots x_q^{(k_q)} \otimes \varphi(h_i) \wedge y \\
&= \sum_{j=1}^{q} x_1^{(k_1)} \ldots x_j^{(k_j-1)} \ldots x_q^{(k_q)} \otimes \varphi\big(\sum_{i=1}^{l} h_i^*(x_j) h_i\big) \wedge y \\
&= \sum_{j=1}^{q} x_1^{(k_1)} \ldots x_j^{(k_j-1)} \ldots x_q^{(k_q)} \otimes \varphi(x_j) \wedge y.
\end{aligned}$$

□

EXAMPLE 1.25. We specialize to the case in which $F = R$. One can easily check that, for all $t \in \mathbb{Z}$,

$$(K_\cdot^{S(R)}(\psi, D(R^*)) \xrightarrow{\nu_\psi^\psi} K_\cdot^{S(R)}(\psi, S(R)))(t)$$

is the classical Koszul complex associated with ψ ($\psi \in G^*$ is a connection homomorphism of degree 1). Moreover, the complex isomorphism is natural for all $t \in \mathbb{Z}$.

There are well-known generalizations of the classical Koszul complex in case G is a finitely generated free R-module, due to Eagon and Northcott, Buchsbaum and Rim and others. The usual constructions of these complexes do not work for all R-modules (just take G a module which has no rank). Suppose G to be finitely generated free and denote by $\mathcal{C}^t(\psi)$ the complexes as described in [E] A2.6.1, and by $\mathcal{K}(\psi)$ the classical Koszul complex associated with ψ. If $F = R$, then

$$\mathcal{C}^t(\psi) \cong \mathcal{K}(\psi)$$

is a natural complex isomorphism only for $t \geq \operatorname{rank} G$. In all other cases the complex isomorphism depends on an orientation of G.

Our purpose now is to identify the complexes $\mathcal{C}^t(\psi)$ among our complexes in case G is finitely generated free. As we saw in the above example, there seems to be no way to do this using solely canonical complex isomorphisms. So we shall introduce a noncanonical map.

REMARK 1.26. Suppose G is a free R-module of rank n. We consider the isomorphisms

$$\Omega_p : \bigwedge^p G \cong (\bigwedge^{n-p} G)^*, \quad p = 0, \ldots, n,$$

induced by an orientation on G: Let $g_1, \ldots, g_n$ be a basis of G, and $g_1^*, \ldots, g_n^*$ the basis of G^* dual to $g_1, \ldots, g_n$. Then there is an unique R-isomorphism $\Omega_n : \bigwedge^n G \to R$ with $\Omega_n(g_1 \wedge \ldots \wedge g_n) = 1$. We define $\Omega_p : \bigwedge^p G \cong \bigwedge^{n-p} G^*$ by

$$(\Omega_p(x))(y) = \Omega_n(x \wedge y) \quad for \quad x \in \bigwedge^p G, \quad y \in \bigwedge^{n-p} G.$$

PROPOSITION 1.27. *If G is free of rank n, then there are (noncanonical) complex isomorphisms*

$$K_{\cdot}^{S(F)}(\psi, D(F^*))(t) \xrightarrow{\Omega} \left(K_{\cdot}^{S(F)}(\psi, S(F))(n-t)\right)^*$$

and

$$K^{\cdot}_{S(H^*)}(\varphi, S(H^*))(t) \xrightarrow{\Omega} \left(K^{\cdot}_{S(H^*)}(\varphi, D(H))(n-t)\right)^*.$$

Proof. We shall prove only the first part of the proposition. The proof of the second is similar. Let

$$\Omega = \oplus_{p \geq 0}((-1)^{\varepsilon_p} \Omega_p \otimes 1_{D(F^*)}) : \bigwedge G \otimes D(F^*) \to (\bigwedge G)^* \otimes D(F^*)$$

where $\varepsilon_p = \frac{i(i-1)}{2}$. We have to show that the diagram

$$\begin{array}{ccc} \bigwedge^p G \otimes D(F^*) & \xrightarrow{\partial_\psi} & \bigwedge^{p-1} G \otimes D(F^*) \\ \Omega \big\downarrow & & \Omega \big\downarrow \\ (\bigwedge^{n-p} G)^* \otimes D(F^*) & \xrightarrow{\partial_\psi^*} & (\bigwedge^{n-(p-1)} G)^* \otimes D(F^*) \end{array}$$

is commutative. Let $x_1, \ldots, x_p, y_1, \ldots, y_{n-(p-1)} \in G$, $f \in S(F)$ and $f^* \in D(F^*)$. Then

$$\Omega \partial_\psi(x_1 \wedge \ldots \wedge x_p \otimes f^*)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f)$$
$$= \Omega(\sum_{i=1}^p (-1)^{i+1} x_1 \wedge \ldots \widehat{x_i} \ldots \wedge x_p \otimes \psi(x_i) f^*)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f)$$
$$= (-1)^{\varepsilon_{p-1}} \big(\sum_{i=1}^p \Omega_{p-1}((-1)^{i+1} x_1 \wedge \ldots \widehat{x_i} \ldots \wedge x_p) \otimes \psi(x_i) f^* \big)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f)$$

$$= (-1)^{\varepsilon_{p-1}} \sum_{i=1}^{p} \Omega_n((-1)^{i+1} x_1 \wedge \ldots \widehat{x_i} \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_{n-(p-1)}) \otimes f^*(\psi(x_i) f)$$
$$= (-1)^{\varepsilon_{p-1}} (\Omega_n \otimes f^*)\big((x_1 \wedge \ldots \wedge x_p \otimes f \leftharpoondown \psi)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes 1)\big).$$

On the other hand

$$\begin{aligned}
&\partial_\psi^* \Omega(x_1 \wedge \ldots \wedge x_p \otimes f^*)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f) \\
&\quad = \Omega(x_1 \wedge \ldots \wedge x_p \otimes f^*) \partial_\psi (y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f) \\
&\quad = \Omega(x_1 \wedge \ldots \wedge x_p \otimes f^*)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f \leftharpoondown \psi) \\
&\quad = (-1)^{\varepsilon_p} (\Omega_n \otimes f^*)\big((x_1 \wedge \ldots \wedge x_p \otimes 1)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes f \leftharpoondown \psi)\big).
\end{aligned}$$

Since $x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_{n-(p-1)} = 0$ and $\leftharpoondown \psi$ is an antiderivation (see Remark 1.10), we get

$$\begin{aligned}
0 &= x_1 \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes 1 \leftharpoondown \psi \\
&= (x_1 \wedge \ldots \wedge x_p \otimes 1 \leftharpoondown \psi)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes 1) \\
&\qquad\qquad + (-1)^p (x_1 \wedge \ldots \wedge x_p \otimes 1)(y_1 \wedge \ldots \wedge y_{n-(p-1)} \otimes 1 \leftharpoondown \psi).
\end{aligned}$$

Obviously $\varepsilon_{p-1} \equiv \varepsilon_p + p - 1 \ (2)$. So we are done. $\square$

Proposition 1.27 may be used to link up our approach with the classical theory.

EXAMPLE 1.28. Suppose G to be a free R-module of rank n.

(a) Let $f_1, \ldots, f_m$ be a basis of F, and set $x^* = \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*)$. Then $\nu_\psi^{x^*}$ is a connection homomorphism for ψ, and one can identify the complexes

$$(K_{\cdot}^{S(F)}(\psi, D(F^*)) \xrightarrow{\nu_\psi^{x^*}} K_{\cdot}^{S(F)}(\psi, S(F)))(t).$$

with the complexes $\mathcal{C}^t(\psi)$ as defined in [E] A2.6.1.

(b) Let $h_1, \ldots, h_l$ be a basis of H, and set $x = \varphi(h_1) \wedge \ldots \wedge \varphi(h_l)$. Then ν_x^φ is a connection homomorphism for φ, and one can identify the complexes

$$(K_{S(H^*)}^{\cdot}(\varphi, D(H)) \xrightarrow{\nu_x^\varphi} K_{S(H^*)}^{\cdot}(\varphi, S(H^*)))(t).$$

with the $\mathcal{D}^t(\varphi)$ as defined in [BV1].

The isomorphisms of (a) and (b) are not canonical since they are given by Ω. One may use Theorem 1.21 to get complex isomorphisms

$$\mathcal{C}^t(\psi) \cong (\mathcal{C}^{n-m-t}(\psi))^* \cong \mathcal{D}^{n-m-t}(\psi^*).$$

1.3 Koszul Bicomplexes

Let F, G, H be R-modules. We assume H and F to be finitely generated and free. Let

$$H \xrightarrow{\varphi} G \xrightarrow{\psi} F$$

be a complex. Using the canonical isomorphism $\zeta_{H,G}$ from Lemma 1.12, we may consider φ as a map $S(H^*) \to S(H^*) \otimes G$. Similarly ψ can be viewed as a map $G \otimes S(F) \to S(F)$ if one draws upon the isomorphism ξ from Lemma 1.13. So we obtain a complex

$$S(H^*) \otimes S(F) \xrightarrow{\varphi \otimes 1_{S(F)}} S(H^*) \otimes G \otimes S(F) \xrightarrow{1_{S(H^*)} \otimes \psi} S(H^*) \otimes S(F).$$

When M is an $S(H^*)$-module and N is an $S(F)$-module, Theorem 1.7 provides a bicomplex $K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N)$.

In the following we suppose that $M = \oplus_{i \geq 0} M_i$ is a graded $S(H^*)$-module and that $N = \oplus_{j \geq 0} N_j$ is a graded $S(F)$-module. The bicomplexes

$$\begin{array}{ll} K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N), & K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N), \\ K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N^*), & K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N^*), \end{array}$$

as bicomplexes of R-modules, split into direct summands which we want to describe in detail. For this purpose we set

$$\begin{array}{llll} {}_p\bigwedge^k_q \quad \text{for} \quad M_p \otimes \bigwedge^k G \otimes N_q, & & {}_{-p}\bigwedge^k_q \quad \text{for} \quad M^*_{-p} \otimes \bigwedge^k G \otimes N_q, \\ {}_p\bigwedge^k_{-q} \quad \text{for} \quad M_p \otimes \bigwedge^k G \otimes N^*_{-q}, & & {}_{-p}\bigwedge^k_{-q} \quad \text{for} \quad M^*_{-p} \otimes \bigwedge^k G \otimes N^*_{-q}, \end{array}$$

$k, p, q \in \mathbb{N}$. Let $t \in \mathbb{Z}$. Then

$$\begin{array}{ll} K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N)(t), & K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N)(t), \\ K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N^*)(t), & K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N^*)(t), \end{array}$$

one after the other, are the bicomplexes

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {}_0{\bigwedge}_0^t \arrow[r] \arrow[d] & {}_1{\bigwedge}_0^{t+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\bigwedge}_0^{t+p-1} \arrow[r] \arrow[d] & {}_p{\bigwedge}_0^{t+p} \arrow[r] \arrow[d] & \cdots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & {}_0{\bigwedge}_t^0 \arrow[r] \arrow[d] & {}_1{\bigwedge}_t^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\bigwedge}_t^{p-1} \arrow[r, "d_\varphi"] \arrow[d] & {}_p{\bigwedge}_t^p \arrow[r] \arrow[d, "\partial_\psi"'] & \cdots \\
 & 0 \arrow[r] & {}_1{\bigwedge}_{t+1}^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\bigwedge}_{t+1}^{p-2} \arrow[r] \arrow[d] & {}_p{\bigwedge}_{t+1}^{p-1} \arrow[r] \arrow[d] & \cdots \\
 & & 0 \arrow[r] & \cdots \arrow[r] & {}_{p-1}{\bigwedge}_{t+2}^{p-3} \arrow[r] \arrow[d] & {}_p{\bigwedge}_{t+2}^{p-2} \arrow[r] \arrow[d] & \cdots \\
 & & & & \vdots & \vdots & \\
 & & & & 0 \arrow[r] & {}_p{\bigwedge}_{t+p}^0 \arrow[r] \arrow[d] & \cdots \\
 & & & & & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {}_{-t}{\bigwedge}_0^0 \arrow[r] \arrow[d] & {}_{-t+1}{\bigwedge}_0^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\bigwedge}_0^{t-1} \arrow[r, "d_\varphi"] \arrow[d] & {}_0{\bigwedge}_0^t \arrow[r] \arrow[d, "\partial_\psi"'] & 0 \\
 & 0 \arrow[r] & {}_{-t+1}{\bigwedge}_1^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\bigwedge}_1^{t-2} \arrow[r] \arrow[d] & {}_0{\bigwedge}_1^{t-1} \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] & \cdots \arrow[r] & {}_{-1}{\bigwedge}_2^{t-3} \arrow[r] \arrow[d] & {}_0{\bigwedge}_2^{t-2} \arrow[r] \arrow[d] & 0 \\
 & & & & \vdots & \vdots & \\
 & & & & 0 \arrow[r] & {}_0{\bigwedge}_t^0 \arrow[r] \arrow[d] & 0 \\
 & & & & & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & {}_0{\textstyle\bigwedge}^{t+q}_{-q} \arrow[r] \arrow[d] & {}_1{\textstyle\bigwedge}^{t+q+1}_{-q} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\textstyle\bigwedge}^{t+p+q-1}_{-q} \arrow[r] \arrow[d] & {}_p{\textstyle\bigwedge}^{t+p+q}_{-q}\cdots \arrow[d] \\
0 \arrow[r] & {}_0{\textstyle\bigwedge}^{t+q-1}_{-q+1} \arrow[r] \arrow[d] & {}_1{\textstyle\bigwedge}^{t+q}_{-q+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\textstyle\bigwedge}^{t+p+q-2}_{-q+1} \arrow[r] \arrow[d] & {}_p{\textstyle\bigwedge}^{t+p+q-1}_{-q+1}\cdots \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & {}_0{\textstyle\bigwedge}^{t+1}_{-1} \arrow[r] \arrow[d] & {}_1{\textstyle\bigwedge}^{t+2}_{-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\textstyle\bigwedge}^{t+p}_{-1} \arrow[r, "d_\varphi"] \arrow[d] & {}_p{\textstyle\bigwedge}^{t+p}_{-1}\cdots \arrow[d] \\
0 \arrow[r] & {}_0{\textstyle\bigwedge}^{t}_{0} \arrow[r] \arrow[d] & {}_1{\textstyle\bigwedge}^{t+1}_{0} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{p-1}{\textstyle\bigwedge}^{t+p-1}_{0} \arrow[r, "d_\varphi"] \arrow[d] & {}_p{\textstyle\bigwedge}^{t+p}_{0}\cdots \arrow[d] \\
 & 0 & 0 & & 0 & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots{}_{-t-1}{\textstyle\bigwedge}^{q-1}_{-q} \arrow[r] \arrow[d] & {}_{-t}{\textstyle\bigwedge}^{q}_{-q} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\textstyle\bigwedge}^{t+q-1}_{-q} \arrow[r] \arrow[d] & {}_0{\textstyle\bigwedge}^{t+q}_{-q} \arrow[r] \arrow[d] & 0 \\
\cdots{}_{-t-1}{\textstyle\bigwedge}^{q}_{-q+1} \arrow[r] \arrow[d] & {}_{-t}{\textstyle\bigwedge}^{q-1}_{-q+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\textstyle\bigwedge}^{t+q-2}_{-q+1} \arrow[r] \arrow[d] & {}_0{\textstyle\bigwedge}^{t+q-1}_{-q+1} \arrow[r] \arrow[d] & 0 \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots{}_{-t-1}{\textstyle\bigwedge}^{0}_{-1} \arrow[r] \arrow[d] & {}_{-t}{\textstyle\bigwedge}^{1}_{-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\textstyle\bigwedge}^{t}_{-1} \arrow[r, "d_\varphi"] \arrow[d] & {}_0{\textstyle\bigwedge}^{t+1}_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_{-t}{\textstyle\bigwedge}^{0}_{0} \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_{-1}{\textstyle\bigwedge}^{t-1}_{0} \arrow[r, "d_\varphi"] \arrow[d] & {}_0{\textstyle\bigwedge}^{t}_{0} \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}
$$

The following result shows how the bicomplexes from above can be connected.

THEOREM 1.29. *Let $x_1, \ldots, x_i \in \operatorname{Ker}\psi$, and set $x = x_1 \wedge \ldots \wedge x_i$. Assume that ν_x^φ is a connection homomorphism for φ.*

Similarly let $x_1^*, \ldots, x_j^* \in G^*$, and set $x^* = x_1^* \wedge \ldots \wedge x_j^*$. We suppose that

(1) $x_1, \dots, x_i \in \operatorname{Ker} x_k^*$ *for all* k,

(2) $x_k^* \varphi = 0$ *for all* k, *and that*

(3) $\nu_\psi^{x^*}$ *is a connection homomorphism for* ψ.

Furthermore let $M_0 = N_0 = R$. *Then there are coefficients* $\varepsilon = \pm 1$ *such that the diagram*

$$\begin{array}{ccc}
K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N^*)(t+j) & \xrightarrow{\varepsilon \nu_x^\varphi \otimes 1} & K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N^*)(t+i+j) \\
{\scriptstyle \varepsilon 1 \otimes \nu_\psi^{x^*}} \downarrow & & {\scriptstyle \varepsilon 1 \otimes \nu_\psi^{x^*}} \downarrow \\
K^{\cdot S(F)}_{S(H^*).}(\varphi, M^*, \psi, N)(t) & \xrightarrow{\varepsilon \nu_x^\varphi \otimes 1} & K^{\cdot S(F)}_{S(H^*).}(\varphi, M, \psi, N)(t+i)
\end{array}$$

is a bicomplex for all $t \in \mathbb{Z}$.

Proof. To simplify the notation, we shall write $\nu_\psi^{x^*}$ for $1 \otimes \nu_\psi^{x^*}$ and ν_x^φ for $\nu_x^\varphi \otimes 1$. We have to show that, for all $p, q \geq 0$, the following diagrams are commutative or anticommutative. The first and the second correspond to the vertical arrows of the diagram in the Theorem while (3) and (4) belong to the horizontal arrows; the last diagram represents the 'middle'.

(1)

$$\begin{array}{ccc}
M^*_{-p-1} \otimes \bigwedge^{q+j-1} G & \xrightarrow{d_\varphi} & M^*_{-p} \otimes \bigwedge^{q+j} G \\
{\scriptstyle \nu_\psi^{x^*}} \downarrow & & {\scriptstyle \nu_\psi^{x^*}} \downarrow \\
M^*_{-p-1} \otimes \bigwedge^{q-1} G & \xrightarrow{d_\varphi} & M^*_{-p} \otimes \bigwedge^{q} G,
\end{array}$$

(2)

$$\begin{array}{ccc}
M_p \otimes \bigwedge^{q+j} G & \xrightarrow{d_\varphi} & M_{p+1} \otimes \bigwedge^{q+j+1} G \\
{\scriptstyle \nu_\psi^{x^*}} \downarrow & & {\scriptstyle \nu_\psi^{x^*}} \downarrow \\
M_p \otimes \bigwedge^{q} G & \xrightarrow{d_\varphi} & M_{p+1} \otimes \bigwedge^{q+1} G,
\end{array}$$

(3)

$$\begin{array}{ccc}
\bigwedge^{q+1} G \otimes N^*_{-p-1} & \xrightarrow{\nu_x^\varphi} & \bigwedge^{q+i+1} G \otimes N^*_{-p-1} \\
{\scriptstyle \partial_\psi} \downarrow & & {\scriptstyle \partial_\psi} \downarrow \\
\bigwedge^{q} G \otimes N^*_{-p} & \xrightarrow{\nu_x^\varphi} & \bigwedge^{q+i} G \otimes N^*_{-p},
\end{array}$$

(4)

$$\begin{array}{ccc}
\bigwedge^{q} G \otimes N_p & \xrightarrow{\nu_x^\varphi} & \bigwedge^{q+i} G \otimes N_p \\
{\scriptstyle \partial_\psi} \downarrow & & {\scriptstyle \partial_\psi} \downarrow \\
\bigwedge^{q-1} G \otimes N_{p+1} & \xrightarrow{\nu_x^\varphi} & \bigwedge^{q+i-1} G \otimes N_{p+1},
\end{array}$$

(5)

$$\begin{array}{ccc}
\bigwedge^{t+j} G & \xrightarrow{\nu_x^{\varphi}} & \bigwedge^{t+i+j} G \\
\nu_\psi^{x^*}\Big\downarrow & & \nu_\psi^{x^*}\Big\downarrow \\
\bigwedge^{t} G & \xrightarrow{\nu_x^{\varphi}} & \bigwedge^{t+i} G
\end{array}$$

Since by assumption $x_k^*(x_1) = \ldots = x_k^*(x_i) = 0$ for all k, the (anti)commutativity of the last diagram follows immediately from Theorem 1.11.

The first and the second diagram are obtained from the diagram

$$\begin{array}{ccc}
S(H^*) \otimes \bigwedge G & \xrightarrow{d_{\varphi}} & S(H^*) \otimes \bigwedge G \\
\nu_\psi^{x^*}\Big\downarrow & & \nu_\psi^{x^*}\Big\downarrow \\
S(H^*) \otimes \bigwedge G & \xrightarrow{d_{\varphi}} & S(H^*) \otimes \bigwedge G
\end{array}$$

by tensoring this diagram with M and M^* (over $S(H^*)$). The map d_φ acts on $S(H^*) \otimes \bigwedge G$ as the left multiplication by φ, viewed as an element of $H^* \otimes G$, while $\nu_\psi^{x^*}$ is the right multiplication by $1_{S(H^*)} \otimes x^*$ (see section 2). Now $\varphi = \sum_{\lambda=1}^{l} h_\lambda^* \otimes h_\lambda$ with elements $h_\lambda^* \in H^*$ and $h_\lambda \in \mathrm{Im}(\varphi)$. Since we assumed that $x_k^* \varphi = 0$ for all k, a repeated application of Theorem 1.11 yields $\nu_\psi^{x^*} d_\varphi = (-1)^j d_\varphi \nu_\psi^{x^*}$.

The third and the forth diagram are obtained from the diagram

$$\begin{array}{ccc}
\bigwedge G \otimes S(F) & \xrightarrow{\nu_x^{\varphi}} & \bigwedge G \otimes S(F) \\
\partial_\psi\Big\downarrow & & \partial_\psi\Big\downarrow \\
\bigwedge G \otimes S(F) & \xrightarrow{\nu_x^{\varphi}} & \bigwedge G \otimes S(F)
\end{array}$$

by tensoring with N^* and N. ν_x^φ is the left multiplication by $x \otimes 1_S(F)$, and ∂_ψ is the right multiplication by $\psi \otimes 1_S(F)$. Since $\psi(x_1) = \ldots = \psi(x_i) = 0$ for all k, we get $\partial_\psi \, \nu_x^\varphi = (-1)^i \nu_x^\varphi \, \partial_\psi$. □

EXAMPLE 1.30. Fix bases $h_1, \ldots, h_l$ for H and $f_1, \ldots, f_m$ for F. Set $x_\lambda = \varphi(h_\lambda)$ and $x_\mu^* = \psi^*(f_\mu^*)$. Take $M = S(H^*)$ and $N = S(F)$. Then the hypotheses of Theorem 1.29 are fulfilled (with $i = l$ and $j = m$).

REMARK 1.31. In addition to the assumptions of the previous example, let $H = R$. Then the diagram

$$\begin{array}{ccc}
\bigwedge^{p+m} G & \xrightarrow{d_{\varphi}} & \bigwedge^{p+m+1} G \\
\nu_\psi^{x^*}\Big\downarrow & & \nu_\psi^{x^*}\Big\downarrow \\
\bigwedge^{p} G & \xrightarrow{d_{\varphi}} & \bigwedge^{p+1} G
\end{array}$$

is commutative or anticommutative. This fact can be used in order to simplify the first part of the proof of Theorem 3.1 in [HM] (consider G free and then apply Ω to the first line).

2 Grade Sensitivity

This technical chapter links up the study of the Koszul bicomplexes with the study of the homology of certain Koszul complexes.

Let R be a commutative ring.

DEFINITION 2.1. If $\varphi : H \to G$ is a linear map of R-modules, then $I_j(\varphi)$ is the image of the map

$$\bigwedge^j H \otimes (\bigwedge^j G)^* \to R$$

$$x \otimes y^* \to y^*(\bigwedge^j \varphi(x))$$

where $x \in \bigwedge^j H$, $y^* \in (\bigwedge^j G)^*$.

If H and G are finitely generated free, then $(\bigwedge^j G)^* = \bigwedge^j G^*$ (by virtue of θ, see section 1.1). So φ may be represented by a matrix, and $I_j(\varphi)$ is the ideal generated by the minors of size j of that matrix. We abbreviate

$$I_\varphi = I_{\min(rankH, rankG)}(\varphi).$$

We quote the following well known result.

THEOREM 2.2. *Let R be noetherian and let H, G be finite free R-modules of ranks l and n. Let $\varphi : H \to G$ be an R-homomorphism. If $I_\varphi \neq R$, then*

$$\operatorname{grade} I_\varphi \leq \mid n - l \mid +1.$$

NOTATION 2.3. *Throughout the rest of this section we shall assume that H, G and F are finitely generated free R-modules of ranks l, n and m, and that*

$$H \xrightarrow{\varphi} G \xrightarrow{\psi} F$$

is a complex. Although much of what we will do, holds formally for any l, n and m, the applications will refer to the case in which $n \geq m$ and $n \geq l$. So $r = n - m \geq 0$, $s = n - l \geq 0$. We set $g = \operatorname{grade} I_\psi$, and $h = \operatorname{grade} I_\varphi$.

A first question is which restrictions g and h are subjected in a situation as the one pictured above. In the sequel we shall give some answers to this question. The following result is a simple consequence of Theorem 2.2.

PROPOSITION 2.4. *Let R be noetherian. Set $\rho = r - l$.*

(1) *If $g, h \geq 1$, then $\rho \geq 0$.*

(2) *If $g > \mid \rho \mid +1$, then $I_\varphi \subset \operatorname{Rad} I_\psi$, and in particular, $h \leq g$.*

(3) *Moreover, if $g \geq r + 1$, then $I_\varphi \subset I_\psi$.*

Proof. If $g \geq 1$, then $M = \operatorname{Coker}\psi^*$ has rank r. So $M^* = \operatorname{Ker}\psi$ has rank r, too. In the same way $h \geq 1$ implies that φ is injective, so $\operatorname{Im}\varphi$ has rank l. Since $\operatorname{Ker}\psi \supset \operatorname{Im}\varphi$, we obtain the first part.

While proving (2) and (3) we may assume that $I_\psi \neq R$. Suppose that $g >\mid \rho \mid +1$. Take a prime ideal $I \supset I_\psi$ in R. Then grade$(I_\psi R_I) \geq g$. If $I_\varphi \not\subset I$, then $(\operatorname{Im}\varphi)R_I$ would be a free direct summand of G_I of rank l, (see [BV2],Proposition 16.3 for example) and therefore ψ_I can be viewed as a map from a free module of rank $n-l$ to a free module of rank m. So grade$(I_\psi R_I) \leq\mid \rho \mid +1$, which contradicts the hypothesis. It follows that $I_\varphi \subset I$ which implies that $I_\varphi \subset \operatorname{Rad} I_\psi$.

In case grade $I_\psi = r+1$, we consider a rank 1 direct summand $\widetilde{H}$ of H. Let $\widetilde{\varphi}$ be the restriction of φ to $\widetilde{H}$. It was shown in [BV4], Proposition 1 that $I_{\widetilde{\varphi}} \subset I_\psi$ in this case. Since $I_\varphi \subset I_{\widetilde{\varphi}}$, the conclusion follows. □

The restriction $g >\mid \rho \mid +1$ in Proposition 2.4 cannot be dropped as the following example shows.

EXAMPLE 2.5. Let k be a field and let $R = k[x_1, x_2, x_3, x_4]$ be the polynomial ring over k in the indeterminates x_1, x_2, x_3, x_4. Set $H = R$, $G = R^4$ and $F = R^2$. Let φ, ψ be given by the matrices

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_2 & x_3 \\ -x_1 & x_4 \\ x_4 & -x_1 \\ -x_3 & -x_2 \end{pmatrix},$$

respectively. Then $\psi\varphi = 0$, $h = 4$ and $g = 2 = r$ (it is easy to see that $2 \leq g \leq 3$ and Proposition 2.4 implies $g < 3$).

Before we continue, we simplify the notation.

NOTATION 2.6. As above set $M = \operatorname{Coker}\psi^*$. Let $h_1, \dots, h_l$ be a basis for H and set $x = \varphi(h_1) \wedge \ldots \wedge \varphi(h_l)$. We shall write ν^φ for the connection homomorphism ν^φ_x. Analogously, if $f_1, \dots, f_m$ is a basis for F, $f_1^*, \dots, f_m^*$ the dual basis, and $x^* = \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*)$, the connection homomorphism $\nu_\psi^{x^*}$ will be denoted by ν_ψ. Furthermore set

$$\begin{aligned} D^{\cdot}_\varphi(t) &= (K^{\cdot}_{S(H^*)}(\varphi, D(H)) \xrightarrow{\nu^\varphi} K^{\cdot}_{S(H^*)}(\varphi, S(H^*)))(t) \\ C^{\cdot}_\psi(t) &= (K^{S(F)}_{\cdot}(\psi, D(F^*)) \xrightarrow{\nu_\psi} K^{S(F)}_{\cdot}(\psi, S(F)))(t) \end{aligned}$$

for all $t \in \mathbb{Z}$. Since G is finitely generated, both complexes have only a finite number of non vanishing components. To identify the homology, we fix their graduations as follows: position 0 is held by the leftmost non-zero module.

By $\mathcal{C}_{\cdot,\cdot}(t)$ we shall denote the Koszul bicomplex

$$K^{\cdot S(F)}_{S(H^*)\cdot}(\varphi, D(H), \psi, S(F))(t) \xrightarrow{\varepsilon\nu^\varphi \otimes 1} K^{\cdot S(F)}_{S(H^*)\cdot}(\varphi, S(H^*), \psi, S(F))(t+l),$$

which is the lower part of the bicomplex in Theorem 1.29. We rewrite this complex as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & C^{0,0} \arrow[r] \arrow[d] & C^{1,0} \arrow[r, "d_\varphi"] \arrow[d, "\partial_\psi"'] & \cdots C^{p,0} \arrow[r, "\varepsilon\nu^\varphi"] \arrow[d] & C^{p+1,0} \arrow[r, "d_\varphi"] \arrow[d, "\partial_\psi"'] & C^{p+2,0} \cdots \arrow[d] \\
 & 0 \arrow[r] & C^{1,1} \arrow[r] \arrow[d] & \cdots C^{p,1} \arrow[r, "\varepsilon\nu^\varphi"] \arrow[d] & C^{p+1,1} \arrow[r] \arrow[d] & C^{p+2,1} \cdots \arrow[d] \\
 & & 0 \arrow[r] & \cdots C^{p,2} \arrow[r, "\varepsilon\nu^\varphi"] \arrow[d] & C^{p+1,2} \arrow[r] \arrow[d] & C^{p+2,2} \cdots \arrow[d] \\
 & & & \vdots & \vdots & \vdots \\
 & & & 0 \arrow[r] \arrow[d] & C^{p+1,t} \arrow[r] \arrow[d] & C^{p+2,p} \cdots \arrow[d] \\
 & & & 0 \arrow[r] \arrow[d] & C^{p+1,p+1} \arrow[r] \arrow[d] & C^{p+2,p+1} \cdots \arrow[d] \\
 & & & \vdots & \vdots & \vdots
\end{tikzcd}$$

In other words,

$$C^{0,0}(t) \;=\; \begin{cases} D_t(H) \otimes \bigwedge^0 G \otimes S_0(F) & \text{if} \quad 0 \le t, \\ S_0(H^*) \otimes \bigwedge^{t+l} G \otimes S_0(F) & \text{if} \quad -l \le t < 0, \\ S_{-t-l}(H^*) \otimes \bigwedge^0 G \otimes S_0(F) & \text{if} \quad t < -l. \end{cases}$$

The row homology of $C_{\cdot,\cdot}(t)$ at $C^{p,q}$ is denoted by $H_\varphi^{p,q}$, the column homology by $H_\psi^{p,q}$. Thus $H_\varphi^{p,0}$ is the p-th homology module of $D_\varphi^{\cdot}(t)$.

Set $N^p = \operatorname{Ker}(C^{p,0} \xrightarrow{\partial_\psi} C^{p,1})$. The canonical injections $N^p \to C^{p,0}$ yield a complex homomorphism

$$\begin{tikzcd}
0 \arrow[r] & N^0 \arrow[r] \arrow[d, equal] & N^1 \arrow[r] \arrow[d] & \cdots N^p \arrow[r, "\bar{d_\varphi}"] \arrow[d] & N^{p+1} \cdots \arrow[d] \\
0 \arrow[r] & C^{0,0} \arrow[r] & C^{1,0} \arrow[r] & \cdots C^{p,0} \arrow[r, "d_\varphi"] & C^{p+1,0} \cdots
\end{tikzcd}$$

where the maps $\bar{d_\varphi}$ are induced by d_φ. The homology of the first row $\mathcal{N}^{\cdot}(t)$ at N^p is denoted by $\bar{H}^p$.

The homology of the complexes $C^{\cdot}_{\psi}(t)$ and $D^{\cdot}_{\varphi}(t)$ behaves similarly as the homology of the usual Koszul complex. The main result is the following.

THEOREM 2.7. *We use the notation from 2.3. Furthermore let R be noetherian. Set $Q = \operatorname{Coker}\varphi$, $C = \operatorname{Coker}\psi$, $D = \operatorname{Coker}\varphi^*$ and $M = \operatorname{Coker}\psi^*$. Set $S_0(D) = R/I_{\varphi}$, $S_{-1}(D) = \bigwedge^{s+1}Q$, $S_0(C) = R/I_{\psi}$ and $S_{-1}(C) = \bigwedge^{r+1}M$.*

(a) *$H^i(D^{\cdot}_{\varphi}(t)) = 0$ for $i < h$. Moreover, if $t \leq s+1$ and $\operatorname{grade} I_k(\varphi) \geq n-k+1$ for all k with $l \geq k \geq 1$, then $D^{\cdot}_{\varphi}(t)$ is a free resolution of $S_{s-t}(D)$. (If $-1 \leq t \leq s+1$, then it suffices to require that $\operatorname{grade} I_{\varphi} \geq s+1$.)*

(b) *$H^i(C^{\cdot}_{\psi}(t)) = 0$ for $i < g$. Moreover, if $t \geq -1$ and $\operatorname{grade} I_k(\psi) \geq n-k+1$ for all k with $m \geq k \geq 1$, then $C^{\cdot}_{\psi}(t)$ is a free resolution of $S_t(C)$. (If $-1 \leq t \leq r+1$, then it suffices to require that $\operatorname{grade} I_{\psi} \geq r+1$.)*

Finally, if $I_{\varphi} = R$ ($I_{\psi} = R$), then all sequences $D^{\cdot}_{\varphi}$ ($C^{\cdot}_{\psi}$) are split exact.

Proof. As we mentioned in Example 1.28, we get (non-canonical) complex isomorphisms

$$\mathcal{D}^t(\varphi) \cong D^{\cdot}_{\varphi}(t) \cong C^{\cdot}_{\varphi^*}(s-t)$$

and

$$\mathcal{C}^t(\psi) \cong C^{\cdot}_{\psi}(t) \cong D^{\cdot}_{\psi^*}(r-t).$$

Since $\operatorname{grade} I_{\varphi} = \operatorname{grade} I_{\varphi^*}$ and $\operatorname{grade} I_{\psi} = \operatorname{grade} I_{\psi^*}$, (a) follows from Proposition 2.1 in [BV1] while (b) is obtained from Theorem A2.10,(c) in [E]. $\square$

We shall now investigate the homology of $\mathcal{N}^{\cdot}(t)$. The key is the following result.

THEOREM 2.8. *Let R be noetherian and $t \geq 0$ be an integer. Assume that*

$$1 \leq r \leq g \leq r+1.$$

Then, with the notation introduced above, $\bar{H}^i = 0$ for $i = 0, \ldots, \min(2, h-1)$. Set $C = \operatorname{Coker}\psi$.

(a) *For i odd, $3 \leq i < \min(h-1, 2r, 2t+2)$, one has a natural exact sequence*

$$0 \to \bar{H}^i \to D_{t-\frac{i-1}{2}}(H) \otimes S_{\frac{i-1}{2}}(C) \to H_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}} \to \bar{H}^{i+1} \to 0.$$

(b) *Suppose that $t < r$ and $l > 1$.*

   (i) *If $3 \leq 2t+1 < h$, then $\bar{H}^{2t+1} = D_0(H) \otimes S_t(C)$;*

   (ii) *$\bar{H}^i = 0$ for $2t+2 \leq i < \min(h, t+r+2, 2t+l+1)$;*

   (iii) *if $2t+l+1 < \min(h, t+r+2)$, then $\bar{H}^{2t+l+1} = H_{\psi}^{t+1,t+l-1}$.*

(c) *Suppose that $t+l<r$. For $i-l$ even, $2t+l+2 \le i < \min(h-1, 2r-l+2)$, one has a natural exact sequence*

$$0 \to \bar{H}^i \to S_{\frac{i-l}{2}-t-1}(H^*) \otimes S_{\frac{i+l}{2}-1}(C) \to H_\psi^{\frac{i-l}{2}+1, \frac{i+l}{2}-1} \to \bar{H}^{i+1} \to 0.$$

Proof. The proof partially consists of a repetition of arguments used in the proof of Proposition 1 in [BV4].

Set $\mu = \min(2, h-1)$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & N^0 & \longrightarrow & N^1 & \longrightarrow & N^2 & \xrightarrow{\bar{d}_\varphi} & N^3 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^{0,0} & \longrightarrow & C^{1,0} & \longrightarrow & C^{2,0} & \xrightarrow{d_\varphi} & C^{3,0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Im} \partial_\psi^{1,0} & \longrightarrow & \operatorname{Im} \partial_\psi^{2,0} & \longrightarrow & \operatorname{Im} \partial_\psi^{3,0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

with exact columns. The middle row has trivial homology at $C^{p,0}$ for $p \le \mu$ (Theorem 2.7). In the fourth row the homology at $\operatorname{Im} \partial_\psi^{1,0}$ is zero since the homomorphism $\operatorname{Im} \partial_\psi^{1,0} \to \operatorname{Im} \partial_\psi^{2,0}$ is the restriction of the injective homomorphism $C^{1,1} \xrightarrow{d_\varphi} C^{2,1}$. Now we use the long exact homology sequence to get the first statement of the proposition.

Next we extend the complex $\mathcal{C}_{.,.}(t)$ to the complex $\widetilde{\mathcal{C}}_{.,.}(t)$ by setting $C^{p,-1} = N^p$. To prove (a), we first mention some facts about the homology of $\widetilde{\mathcal{C}}_{.,.}(t)$. To avoid new symbols, the column homology at $C^{p,q}$ is again denoted by $H_\psi^{p,q}$ (actually it differs from that of $\mathcal{C}_{.,.}(t)$ only at $C^{p,0}$). Let $0<p$. Then from Theorem 2.7 (b) we get that $H_\psi^{p,q} = 0$ for $0 \le q < \min(p-1, r)$. If $p \le t$ then $H_\psi^{p,p} = D_{t-p}(H) \otimes S_p(C)$. Furthermore we draw from Theorem 2.7 (a) that $H_\varphi^{p,q} = 0$ for $p<h$ and $q \ne -1$.

Let $R_.^q$, $q \ge -1$, be the qth row of $\widetilde{\mathcal{C}}_{.,.}(t)$ and $B_.^{q+1}$ be the image complex of $R_.^q$ in $R_.^{q+1}$. We set $E^{p,q} = H^p(B_.^q)$.

Now let i be an odd integer, $3 \le i < \min(h-1, 2r, 2t+2)$. Since $H_\varphi^{p,q} = 0$ for $p<h$, $q \ne -1$ and since $H_\psi^{p,q} = 0$ for $q < \min(p-1, r)$, we obtain the "southwest" isomorphisms

$$\bar{H}^i = E^{i,0} \cong E^{i-1,1} \cong \ldots \cong E^{\frac{i+1}{2}, \frac{i-1}{2}},$$

$$\bar{H}^{i+1} = E^{i+1,0} \cong E^{i,1} \cong \ldots \cong E^{\frac{i+3}{2}, \frac{i-1}{2}}.$$

In fact, there is an exact sequence

$$H_{\varphi}^{i-1,j} \to E^{i-1,j+1} \to E^{i,j} \to H_{\varphi}^{i,j}$$

and the outer terms in this sequence are 0 for all i, j under consideration. We abbreviate $\partial_{\psi}^{p,q} = (C^{p,q} \xrightarrow{\partial_\psi} C^{p,q+1})$ and $d_{\varphi}^{p,q} = (C^{p,q} \xrightarrow{d_\varphi} C^{p+1,q})$. The diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
{} \arrow[r] & \operatorname{Im}\partial_{\psi}^{\frac{i-1}{2},\frac{i-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+1}{2},\frac{i-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+3}{2},\frac{i-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+5}{2},\frac{i-3}{2}} \arrow[d] \\
{} \arrow[r] & \operatorname{Im}\partial_{\psi}^{\frac{i-1}{2},\frac{i-3}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+3}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+5}{2},\frac{i-1}{2}} \arrow[d] \\
 & 0 \arrow[r] & H_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

is induced by $\mathcal{C}_{\cdot,\cdot}(t)$ and has exact columns. Its row homology at $\operatorname{Im}\partial_{\psi}^{\frac{i+1}{2},\frac{i-3}{2}}$ is $E^{\frac{i+1}{2},\frac{i-1}{2}} = \bar{H}^i$, and at $\operatorname{Ker}\partial_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}}$ it coincides with $D_{t-\frac{i-1}{2}}(H) \otimes S_{\frac{i-1}{2}}(C)$ as the following diagram with exact columns and exact middle row shows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \operatorname{Im}\partial_{\psi}^{\frac{i-1}{2},\frac{i-3}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+3}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker}\partial_{\psi}^{\frac{i+5}{2},\frac{i-1}{2}} \arrow[d] \\
0 \arrow[r] & C^{\frac{i-1}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & C^{\frac{i+1}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & C^{\frac{i+3}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & C^{\frac{i+5}{2},\frac{i-1}{2}} \arrow[d] \\
0 \arrow[r] & D_{t-\frac{i-1}{2}} \otimes S_{\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}} \arrow[r, "d_\varphi"] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+3}{2},\frac{i-1}{2}} \arrow[r] \arrow[d] & \operatorname{Im}\partial_{\psi}^{\frac{i+5}{2},\frac{i-1}{2}} \arrow[d] \\
 & 0 & 0 & 0 & 0
\end{tikzcd}$$

In this diagram the row homology at $\operatorname{Im}\partial_{\psi}^{\frac{i+1}{2},\frac{i-1}{2}}$ vanishes since $d_{\varphi}^{\frac{i+1}{2},\frac{i+1}{2}}$ is injective. So in the preceding diagram the row homology at $\operatorname{Ker}\partial_{\psi}^{\frac{i+3}{2},\frac{i-1}{2}}$ is zero. Altogether we obtain an exact sequence

$$0 \to \bar{H}^i \to D_{t-\frac{i-1}{2}}(H) \otimes S_{\frac{i-1}{2}}(C) \to H_\psi^{\frac{i+1}{2},\frac{i-1}{2}} \to E^{\frac{i+3}{2},\frac{i-1}{2}} \to 0. \qquad (*)$$

Since

$$\bar{H}^{i+1} \cong E^{\frac{i+3}{2},\frac{i-1}{2}},$$

(a) has been proved.

In order to prove (b) we continue to investigate the homology of $\widetilde{\mathcal{C}}_{\cdot,\cdot}(t)$. Suppose that $t < r$, $l > 1$ and let $t < p$. Then $H_\psi^{p,q} = 0$ for $q < \min(p+l-2, r)$. In particular $H_\psi^{t+1,t} = 0$, and $(*)$ implies (i). Let $2t+2 \le i < \min(h, t+r+2, 2t+l+1)$. Using the "southwest" isomorphisms once more, we obtain

$$\bar{H}^i = E^{i,0} \cong E^{i-1,1} \cong \ldots \cong E^{t+1,i-t-1}.$$

Since $d_\varphi^{t+1,i-t-1}$ is injective, (ii) follows. Now suppose that $t+l-1 < r$ and $2t+l+1 < h$. We get

$$\bar{H}^{2t+l+1} = E^{2t+l+1,0} \cong E^{2t+l+1,1} \cong \ldots \cong E^{t+2,t+l-1}.$$

Furthermore the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & \operatorname{Im} \partial_\psi^{t+1,t+l-2} \arrow[r] \arrow[d] & \operatorname{Im} \partial_\psi^{t+2,t+l-2} \arrow[r] \arrow[d] & \operatorname{Im} \partial_\psi^{t+3,t+l-2} \arrow[d] \\
0 \arrow[r] & \operatorname{Ker} \partial_\psi^{t+1,t+l-1} \arrow[r, "d_\varphi"] \arrow[d] & \operatorname{Ker} \partial_\psi^{t+2,t+l-1} \arrow[r] \arrow[d] & \operatorname{Ker} \partial_\psi^{t+3,t+l-1} \arrow[d] \\
0 \arrow[r] & H_\psi^{t+1,t+l-1} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

has exact columns, and its row homology at $\operatorname{Ker} \partial_\psi^{t+1,t+l-1}$ is zero since $d_\varphi^{t+1,t+l-1}$ is injective. If we can show that the row homology at $\operatorname{Ker} \partial_\psi^{t+2,t+l-1}$ also vanishes, we shall obtain (iii). The diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}\partial_\psi^{t+1,t+l-1} & \longrightarrow & \operatorname{Ker}\partial_\psi^{t+2,t+l-1} & \longrightarrow & \operatorname{Ker}\partial_\psi^{t+3,t+l-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^{t+1,t+l-1} & \longrightarrow & C^{t+2,t+l-1} & \longrightarrow & C^{t+3,t+l-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Im}\partial_\psi^{t+1,t+l-1} & \xrightarrow{d_\varphi} & \operatorname{Im}\partial_\psi^{t+2,t+l-1} & \longrightarrow & \operatorname{Im}\partial_\psi^{t+3,t+l-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

has exact columns and exact middle row. Since $d_\varphi^{t+1,t+l}$ is injective, we get the desired result.

The proof of (c) is similar to the proof of (a). We just mention that, for all i under consideration, we have the "southwest" isomorphisms

$$\bar{H}^i = E^{i,0} \cong E^{i-1,1} \cong \ldots \cong E^{\frac{i-l}{2}+1,\frac{i+l}{2}-1},$$

$$\bar{H}^{i+1} = E^{i+1,0} \cong E^{i,1} \cong \ldots \cong E^{\frac{i-l}{2}+2,\frac{i+l}{2}-1}.$$

□

If t is a negative integer, a similar result follows easily. We touch briefly upon this case.

THEOREM 2.9. *Let R be noetherian and let $t < 0$ be an integer. Assume that*

$$1 \le r \le g \le r+1.$$

We set $C = \operatorname{Coker}\psi$ and use the notation from above.

(a) *Suppose that $t+l > 0$ (this implies $l > 1$). Then*

  (i) *$\bar{H}^i = 0$ for $0 \le i < \min(h, r+1, \max(2, t+l))$;*
  (ii) *if $2 \le t+l < \min(h, r+1)$, then $\bar{H}^{t+l} = H_\psi^{0,t+l-1}$;*
  (iii) *if $t+l \le r$ and $i-t-l$ is odd, if furthermore $t+l+1 \le i < \min(h-1, 2r-t-l+1)$, then one has a natural exact sequence*

$$0 \to \bar{H}^i \to S_{\frac{i-t-l-1}{2}}(H^*) \otimes S_{\frac{i+t+l-1}{2}}(C) \to H_\psi^{\frac{i-t-l+1}{2},\frac{i+t+l-1}{2}} \to \bar{H}^{i+1} \to 0.$$

(b) *Suppose that $t + l \leq 0$. Then $\bar{H}^i = 0$ for $i = 0, \ldots, \min(2, h-1)$. For i odd, $3 \leq i < \min(h-1, 2r)$, one has a natural exact sequence*

$$0 \to \bar{H}^i \to S_{\frac{i-1}{2}-t-l}(H^*) \otimes S_{\frac{i-1}{2}}(C) \to H_\psi^{\frac{i+1}{2},\frac{i-1}{2}} \to \bar{H}^{i+1} \to 0.$$

We shall now investigate the homology of $\mathcal{N}^{\cdot}(t)$ at N^h.

PROPOSITION 2.10. *Under the same conditions as in Theorem 2.8 and with $\mu = \min(h, 2r+1)$, let $t \geq \frac{\mu}{2} - 1$. Then we obtain:*

(a) *there is an exact sequence*

$$0 \to E^{\mu-1,1} \to \bar{H}^\mu \to H_\varphi^{\mu,0};$$

in particular, if $\mu < 3$, then there is an exact sequence

$$0 \to \bar{H}^\mu \to H_\varphi^{\mu,0};$$

(b) *for $\mu \geq 3$ odd there is an exact sequence*

$$0 \to E^{\mu-1,1} \to D_{t-\frac{\mu-1}{2}} \otimes S_{\frac{\mu-1}{2}}(C) \to H_\psi^{\frac{\mu+1}{2},\frac{\mu-1}{2}};$$

(c) *for $\mu \geq 3$ even there is an exact sequence*

$$0 \to \bar{H}^{\mu-1} \to D_{t-\frac{\mu-2}{2}} \otimes S_{\frac{\mu-2}{2}}(C) \to H_\psi^{\frac{\mu}{2},\frac{\mu-2}{2}} \to \bar{H}^\mu \to H_\varphi^{\mu,0}.$$

One may easily deduce similar sequences in case $t < \frac{\mu}{2} - 1$.

Proof. (a) The sequence is obvious.

(b) In order to cover the case $\mu = 2r+1$, we modify the first diagram in the proof of Theorem 2.8: the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
{} \arrow[r] & \operatorname{Im} \partial_\psi^{\frac{\mu-1}{2},\frac{\mu-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im} \partial_\psi^{\frac{\mu+1}{2},\frac{\mu-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im} \partial_\psi^{\frac{\mu+3}{2},\frac{\mu-3}{2}} \arrow[r] \arrow[d] & \operatorname{Im} \partial_\psi^{\frac{\mu+5}{2},\frac{\mu-3}{2}} \arrow[d] \\
{} \arrow[r] & \operatorname{Im} \partial_\psi^{\frac{\mu-1}{2},\frac{\mu-3}{2}} \arrow[r] \arrow[d] & \operatorname{Ker} \partial_\psi^{\frac{\mu+1}{2},\frac{\mu-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker} \partial_\psi^{\frac{\mu+3}{2},\frac{\mu-1}{2}} \arrow[r] \arrow[d] & \operatorname{Ker} \partial_\psi^{\frac{\mu+5}{2},\frac{\mu-1}{2}} \arrow[d] \\
 & 0 \arrow[r] & H_\psi^{\frac{\mu+1}{2},\frac{\mu-1}{2}} \arrow[r] \arrow[d] & H_\psi^{\frac{\mu+3}{2},\frac{\mu-1}{2}} \arrow[r] \arrow[d] & H_\psi^{\frac{\mu+5}{2},\frac{\mu-1}{2}} \arrow[d] \\
 & & 0 & 0 & 0
\end{tikzcd}$$

has exact columns. Then, as in the proof of Theorem 2.8, we obtain an exact sequence

$$0 \to E^{\frac{\mu+1}{2},\frac{\mu-1}{2}} \to D_{t-\frac{\mu-1}{2}}(H) \otimes S_{\frac{\mu-1}{2}}(C) \to \operatorname{Ker}(H_\psi^{\frac{\mu+1}{2},\frac{\mu-1}{2}} \to H_\psi^{\frac{\mu+3}{2},\frac{\mu-1}{2}}) \to E^{\frac{\mu+3}{2},\frac{\mu-1}{2}} \to 0.$$

Since in this case

$$E^{\frac{\mu+1}{2},\frac{\mu-1}{2}} \cong E^{\mu-1,1},$$

we get the desired sequence.

(c) As in the proof of Theorem 2.8 one has an exact sequence

$$0 \to E^{\frac{\mu}{2},\frac{\mu-2}{2}} \to D_{t-\frac{\mu-2}{2}}(H) \otimes S_{\frac{\mu-2}{2}}(C) \to H_\psi^{\frac{\mu}{2},\frac{\mu}{2}} \to E^{\frac{\mu+2}{2},\frac{\mu-2}{2}} \to 0$$

(just consider $i = \mu - 1$). Since in this case

$$E^{\frac{\mu}{2},\frac{\mu-2}{2}} \cong \bar{H}^{\mu-1} \quad \text{and} \quad E^{\frac{\mu+2}{2},\frac{\mu-2}{2}} \cong E^{\mu-1,1},$$

we glue the sequence and the sequence obtained under (a) to get the result. □

If we do not require that $g \geq r$, we can still deduce a result weaker than Theorem 2.8.

THEOREM 2.11. *Let R be noetherian and let $t \geq 0$ be an integer. Then, with the notation from above, $\bar{H}^i = 0$ for $i = 0, \ldots, \min(2, h-1)$. Set $C = \operatorname{Coker}\psi$. Suppose that $l > 1$ and $\rho + 1 < g$.*

(a) *If $3 \leq 2t+1 < h$, then $\bar{H}^{2t+1} = D_0(H) \otimes S_t(C)$. Moreover, if h is odd and $t = \frac{h-1}{2}$, then there is an exact sequence*

$$0 \to D_0(H) \otimes S_t(C) \to \bar{H}^{2t+1};$$

(b) *$\bar{H}^i = 0$ for $2t+2 \leq i < \min(h, t+g+2, 2t+g-\rho+1)$;*

(c) *if $2t+g-\rho+1 < \min(h, t+g+2)$, then $\bar{H}^{2t+g-\rho+1} = H_\psi^{t+1,t+g-\rho-1}$.*

The proof is very similar to the proof of Theorem 2.8 (b); so we may omit it.

NOTATION 2.12. Set $M = \operatorname{Coker}\psi^*$. By $\bar{\lambda} : M \to H^*$ we denote the linear map induced by φ^*. As above let $h_1, \ldots, h_l$ be a basis for H. Set $\bar{x} = \bar{\lambda}^*(h_1) \wedge \ldots \wedge \bar{\lambda}^*(h_l)$. We shall write $\nu_{\bar{\lambda}}$ for the connection homomorphism $\nu_{\bar{\lambda}}^{\bar{x}}$. Furthermore set

$$C^{\cdot}_{\bar{\lambda}}(t) \;=\; (K^{S(H^*)}_{\cdot}(\bar{\lambda}, D(H)) \xrightarrow{\nu_{\bar{\lambda}}} K^{S(H^*)}_{\cdot}(\bar{\lambda}, S(H^*)))(t)$$

for all $t \in \mathbb{Z}$.

PROPOSITION 2.13. *There is a canonical complex isomorphism*

$$\mathcal{N}^{\cdot}(t) \longrightarrow \left(C^{\cdot}_{\bar{\lambda}}(t)\right)^{*}.$$

Proof. In Theorem 1.21 we substitute φ^* for ψ and $S(H^*)$ for M. Using the canonical isomorphisms $(\bigwedge^p G^*)^* \cong \bigwedge^p G$ and $S(H^*)^* \cong D(H)$ we then obtain a natural complex isomorphism

$$D^{\cdot}_{\varphi}(t) \xrightarrow{\ \tau\ } \left(C^{\cdot}_{\varphi^*}(t)\right)^{*}.$$

Obviously $\left(C^{\cdot}_{\bar{\lambda}}(t)\right)^{*}$ may be viewed as a subcomplex of $\left(C^{\cdot}_{\varphi^*}(t)\right)^{*}$, and

$$\tau(\mathcal{N}^{\cdot}(t)) = \left(C^{\cdot}_{\bar{\lambda}}(t)\right)^{*},$$

since

$$\operatorname{Ker}\left((\textstyle\bigwedge^p G^*)^* \xrightarrow{d_{\psi^*}} (\bigwedge^{p-1} G^* \otimes F^*)^*\right) \cong (\textstyle\bigwedge^p M)^*.$$

□

At this point we must introduce some new notation.

NOTATION 2.14. Let x be as in 2.6. By $\mathcal{B}_{\cdot,\cdot}(t)$ we shall denote the Koszul bicomplex

$$K^{\cdot S(F)}_{S(H^*)\cdot}(\varphi, D(H), \psi, D(F^*))(t+m) \xrightarrow{\varepsilon\nu^{\varphi}_{x}\otimes 1} K^{\cdot S(F)}_{S(H^*)\cdot}(\varphi, S(H^*), \psi, D(F^*))(t+m+l)$$

which is the upper part of the bicomplex presented in Theorem 1.29. We rewrite this complex as

$$\begin{array}{ccccccccccc}
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B^{0,-2} & \longrightarrow & \cdots & \xrightarrow{d_{\varphi}} & B^{p,-2} & \xrightarrow{\varepsilon\nu^{\varphi}} & B^{p+1,-2} & \xrightarrow{d_{\varphi}} & \cdots \\
 & & \downarrow & & & & \partial_{\psi}\downarrow & & \partial_{\psi}\downarrow & & \\
\cdots & \longrightarrow & B^{0,-1} & \longrightarrow & \cdots & \longrightarrow & B^{p,-1} & \xrightarrow{\varepsilon\nu^{\varphi}} & B^{p+1,-1} & \longrightarrow & \cdots . \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

where

(a) $B^{0,-1}(t) = D_t(H) \otimes \bigwedge^m G \otimes D_0(F^*)$ if $0 \leq t$,

(b) $B^{0,-1}(t) = S_0(H^*) \otimes \bigwedge^{t+l+m} G \otimes D_0(F^*)$ if $-l \leq t < 0$,

(c) $B^{0,-1}(t) = S_{-t-l}(H^*) \otimes \bigwedge^m G \otimes D_0(F^*)$ if $t < -l$.

Set $M^p = \text{Coker}(B^{p,-2} \to B^{p,-1})$. The canonical surjection $B^{p,-1} \to M^p$ yields a complex homomorphism

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & B^{-1,-1} & \longrightarrow & B^{0,-1} & \longrightarrow & \cdots & \xrightarrow{d_\varphi} & B^{p,-1} & \longrightarrow & B^{p+1,-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \longrightarrow & M^0 & \longrightarrow & \cdots & \xrightarrow{\bar{d_\varphi}} & M^p & \longrightarrow & M^{p+1} & \longrightarrow & \cdots
\end{array}$$

where the maps $\bar{d_\varphi}$ are induced by d_φ. The lower row is denoted by $\mathcal{M}^\cdot(t)$.

We obtain an analogue with Proposition 2.13:

PROPOSITION 2.15. *Let $\rho = n - m - l$ as in Proposition 2.4. Then there is a (non-canonical) complex isomorphism*

$$\mathcal{M}^\cdot(t) \longrightarrow C^\cdot_{\bar\lambda}(\rho - t)$$

Proof. As in Example 1.28 we get (non-canonical) complex isomorphisms

(1) $D^\cdot_\varphi(t) \cong C^\cdot_{\varphi^*}(s - t)$ and

(2) $C^\cdot_\psi(t) \cong D^\cdot_{\psi^*}(r - t)$

where as above $s = n - l,\ r = n - m$. Next we consider the diagram

$$\begin{array}{ccccccc}
D^\cdot_\varphi(t+m+1) \otimes F^* & \xrightarrow{\partial_\psi} & D^\cdot_\varphi(t+m) & \longrightarrow & \mathcal{M}^\cdot(t) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
C^\cdot_{\varphi^*}(\rho - t - 1) \otimes F^* & \xrightarrow{d_{\psi^*}} & C^\cdot_{\varphi^*}(\rho - t) & \longrightarrow & C^\cdot_{\bar\lambda}(\rho - t) & \longrightarrow & 0.
\end{array}$$

The isomorphism (1) assures that the vertical arrows are isomorphisms, while (2) provides the commutativity of the diagram. The desired isomorphism is induced. □

The following result can be interpreted as an extension of the usual Koszul duality to the case of a finitely presented module.

THEOREM 2.16. *Let R be noetherian. Let $r,\ g,\ h$ be as in 2.3. With the graduation induced by $\mathcal{M}^\cdot(t)$ and $\mathcal{N}^\cdot(t)$, there is a (non-canonical) complex morphism*

$$C^\cdot_{\bar\lambda}(\rho - t) \xrightarrow{\ \nu\ } \left(C^\cdot_{\bar\lambda}(t)\right)^*,$$

such that the following hold.

(a) *Suppose that $t + l \le 0$ or $r < t + l$. Then the ν_i are isomorphisms for $i > r + 1 - g$, and ν_{r+1-g} is injective.*

(b) *Suppose that* $l \leq t + l \leq r$.

  (i) *If* $r + 1 - g \leq t$, *then the* ν_i *are isomorphisms for* $i > r + 1 - g$, *and* ν_{r+1-g} *is injective.*

  (ii) *If* $t + l \leq r + 1 - g$, *then the* ν_i *are isomorphisms for* $i > r + 2 - g - l$, *and* $\nu_{r+2-g-l}$ *is injective.*

  (iii) *If* $t < r + 1 - g < t + l$, *then the* ν_i *are isomorphisms for* $i > t$.

(c) *Suppose that* $0 < t + l < l$. *Then the* ν_i *are isomorphisms for* $i > \min(0, r + 1 - g - l - t)$ *and, if* $r + 1 - g - l - t \geq 0$, *then* $\nu_{r+1-g-l-t}$ *is injective.*

Proof. From Theorem 1.29 we draw the sequence of complexes

$$D^{\cdot}_{\varphi}(t+m+1) \otimes F^* \xrightarrow{\partial_{\psi}^{-2}} D^{\cdot}_{\varphi}(t+m) \xrightarrow{\nu} D^{\cdot}_{\varphi}(t) \xrightarrow{\partial_{\psi}^{0}} D^{\cdot}_{\varphi}(t-1) \otimes F.$$

Since $\mathcal{M}^{\cdot}(t) = \mathrm{Coker}(B^{t,-2} \to B^{t,-1})$ and $\mathcal{N}^{\cdot}(t) = \mathrm{Ker}(C^{t,0} \to C^{t,1})$, we obtain an induced complex morphism

$$\mathcal{M}^{\cdot}(t) \xrightarrow{\nu} \mathcal{N}^{\cdot}(t).$$

This, combined with Propositions 2.13 and 2.15, provides the desired complex morphism. The results about grade sensibility follow easily from Theorem 2.7. $\square$

3 Generalized Koszul Complexes in Projective Dimension One

In this chapter we investigate the homology of the generalized Koszul complexes in projective dimension 1.

Throughout the chapter R is a noetherian ring, and M an R-module which has a presentation

$$0 \longrightarrow \mathcal{F} \xrightarrow{\chi} \mathcal{G} \longrightarrow M \longrightarrow 0$$

where $\mathcal{F}$, $\mathcal{G}$ are free modules of ranks m and n. Then in particular $r = n - m \geq 0$.

In the sequel we consider R-homomorphisms $\bar{\lambda} : M \to \mathcal{H}$ into a finite free R-module $\mathcal{H}$ of rank $l \leq n$. By λ we denote the corresponding lifted maps $\mathcal{G} \to \mathcal{H}$. We shall investigate the homology of the Koszul complexes

$$C^{\cdot}_{\bar{\lambda}}(t) : \cdots \to D_p(\mathcal{H}^*) \otimes \bigwedge^{t+l+p} M \xrightarrow{\partial_{\bar{\lambda}}} \cdots \xrightarrow{\partial_{\bar{\lambda}}} D_0(\mathcal{H}^*) \otimes \bigwedge^{t+l} M \xrightarrow{\nu_{\bar{\lambda}}} S_0(\mathcal{H}) \otimes \bigwedge^{t} M \xrightarrow{\partial_{\bar{\lambda}}}$$
$$\cdots \xrightarrow{\partial_{\bar{\lambda}}} S_t(\mathcal{H}) \otimes \bigwedge^{0} M \to 0.$$

associated with $\bar{\lambda}$.

Dualizing $\mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$ we go back to the situation previously studied. So we set $F = \mathcal{F}^*$, $G = \mathcal{G}^*$, $H = \mathcal{H}^*$, $\psi = \chi^*$, $\varphi = \lambda^*$, and $C = \operatorname{Coker} \psi$.

The case in which $\operatorname{grade} I_\chi$ has the maximally possible value $n - m + 1$ will be treated in the first section. We show that the homology, as in the free case, depends on $\operatorname{grade} I_\lambda$.

The second section is concerned with the more general question when there is a $\bar{\lambda}$ such that $\bar{\lambda} \circ \chi = 0$. We state some criteria which involve the numerical invariants $\operatorname{grade} I_\chi$, $\operatorname{grade} I_\lambda$ and m, n, l.

When $\dim R = \operatorname{grade} I_\chi = \operatorname{rank} M$, the homology of $C^{\cdot}_{\bar{\lambda}}(t)$ has finite length. We derive some formulas in the third section.

3.1 The Maximal Grade Case

We suppose in this section that $g = \operatorname{grade} I_\chi$ *has the maximally possible value* $n - m + 1$.

DEFINITION 3.1. We say that a homomorphism φ of finite free R-modules is minimal if $I_1(\varphi) \neq R$.

The next result may be seen as an extension of the Hilbert-Burch Theorem.

THEOREM 3.2. *With the above assumptions we get* $I_\lambda \subset I_\chi$*, and in particular* $\operatorname{grade} I_\lambda \leq r + 1$*. Set* $\rho = r - l$*.*

(a) *If there is a* $\bar{\lambda}$ *such that* $\operatorname{grade} I_\lambda > \mid \rho \mid +1$*, then* $l = 1$ *and* r *is odd.*

(b) *The following conditions are equivalent:*

  (1) $\operatorname{grade} I_\lambda > \mid \rho \mid +1$*;*

  (2) $I_\lambda = I_\chi$*.*

(c) *Suppose in addition that* χ *is minimal. Then the following are equivalent:*

  (1') *There is a* $\bar{\lambda}$ *such that* $\operatorname{grade} I_\lambda > \mid \rho \mid +1$*;*

  (2') $l = 1$ *and* (i) $r = 1$ *or* (ii) $m = 1$ *and* $r \geq 3$ *is odd.*

Proof. By Proposition 2.4,(3) we obtain $I_\lambda = I_{\lambda^*} \subset I_{\chi^*} = I_\chi$. Since $I_\chi \neq R$ by assumption, the first part of the theorem is clear.

Next we prove (a). Set $h = \operatorname{grade} I_\lambda$. We have $I_\lambda \subset I_\chi$, so in particular $2 \leq h \leq r + 1$ and therefore $r \geq 1$ and $g \geq 2$. Proposition 2.4,(1) implies that $\rho \geq 0$, and since $h > \rho + 1$, Proposition 2.4,(2) yields $g \leq h$. So $h = g = r + 1 \geq 2$.

We consider $\mathcal{N}^{\cdot}(r + 1)$. Since $\bigwedge^{r+1} M$ is a torsion module, we get

$$N^{r+1}(r + 1) = D_0(H) \otimes (\bigwedge^{r+1} M)^* = 0.$$

Suppose r is even. Then $r + 1 \geq 3$, and applying Proposition 2.10,(a) and (b) for $t = r + 1$, we obtain an exact sequence

$$0 \to D_{\frac{r+2}{2}}(H) \otimes S_{\frac{r}{2}}(C) \to \bar{H}^{r+1} \to H_\varphi^{r+1,0}$$

since $H_\psi^{\frac{r+2}{2},\frac{r}{2}} = 0$. As we already saw, $N^{r+1}(r+1) = 0$ which implies that $\bar{H}^{r+1} = 0$. But then $D_{\frac{r+2}{2}}(H) \otimes S_{\frac{r}{2}}(C) = 0$ and consequently $C = 0$ which is in contradiction with $I_\psi \neq R$. So r must be odd.

If $r = 1$, then $M^* = \operatorname{Ker} \psi$ has rank 1. Since $H \to \operatorname{Ker} \psi$ is an injection, we obtain $l = 1$.

Let $r \geq 3$ (and odd). Take $t = \frac{r-1}{2}$ and observe that

$$N^r(\frac{r-1}{2}) = S_{\frac{r-1}{2}}(H^*) \otimes (\bigwedge^{r-l+1} M)^*$$

since

$$N^r(\frac{r-1}{2}) = \operatorname{Ker}\big(S_{\frac{r-1}{2}}(H^*) \otimes \bigwedge^{r-l+1} G \to S_{\frac{r-1}{2}}(H^*) \otimes \bigwedge^{r-l} G \otimes F\big).$$

On the other hand we draw from Theorem 2.8,(a) that

$$\bar{H}^r(\frac{r-1}{2}) \cong D_0(H) \otimes S_{\frac{r-1}{2}}(C) \cong S_{\frac{r-1}{2}}(C).$$

If $l > 1$, then $(\bigwedge^{r-l+1} M)^* = 0$ which implies that $\bar{H}^r(\frac{r-1}{2}) = 0$. But then $S_{\frac{r-1}{2}}(C) = 0$, a contradiction. So $l = 1$.

Now we prove (c), (1') $\Rightarrow$ (2'). Localizing at a prime ideal which contains $I_1(\chi)$, we may assume that R is a local ring. Suppose that $r \geq 3$. Since $l = 1$, we obtain $N^r(\frac{r-1}{2}) = R$. Therefore $S_{\frac{r-1}{2}}(C)$ must be cyclic, which means $m = 1$ because R is local and ψ is minimal.

(b), (1) $\Rightarrow$ (2): Equality of ideals in R is a local property. So we may assume R to be local, and, using the uniqueness of minimal free resolutions, we can easily reduce to the case in which χ is minimal. According to what we have proved already, it follows that $l = 1$ and (i) $r = 1$ or (ii) $m = 1$ and $r \geq 3$ is odd. If $r = 1$, then we can apply the Hilbert-Burch Theorem to get the desired equality $I_\lambda = I_\chi$. If $m = 1$ we look at the exact sequence

$$R \xrightarrow{\lambda^*} \mathcal{G}^* \xrightarrow{\chi^*} R.$$

which satisfies the hypothesis of Proposition 2.4,(3) since $\operatorname{grade} I_\lambda > n - m$ by assumption. So

$$I_\chi = I_{\chi^*} \subset I_{\lambda^*} = I_\lambda.$$

(b), (2) $\Rightarrow$ (1) is trivial.

(c), (2') $\Rightarrow$ (1'): Suppose that $r = 1$. Let A be a matrix representing χ, and let A_i the m-minor of A which arises from A by cancelling the ith column. Then $(A_1, -A_2, \ldots, (-1)^{n-1} A_n)$ yields an appropriate λ. The implication (ii) $\Rightarrow$ (1') is a comparably simple exercise.

□

REMARK 3.3. The case $r = 1$ in statement (a) of the above Theorem is completely covered by the Theorem of Hilbert and Burch. If we suppose that $r \geq 3$, we only get that $S_{\frac{r-1}{2}}(C)$ must be cyclic. From the exact sequence

$$G \otimes S_{\frac{r-3}{2}}(F) \to S_{\frac{r-1}{2}}(F) \to S_{\frac{r-1}{2}}(C) \to 0$$

we deduce an exact sequence

$$G \otimes S_{\frac{r-3}{2}}(F) \otimes S_{\frac{r-1}{2}}(F) \to \bigwedge^2 S_{\frac{r-1}{2}}(F) \to 0.$$

This implies that the ideal generated by the entries of any two rows of a matrix representing χ is equal to R. We can also deduce that I_χ may be generated by n elements, and that $I_1(\chi) = \ldots = I_{m-1}(\chi) = R$. Unfortunately this seems to be not enough for a characterization like the one given under (c).

COROLLARY 3.4. *The following conditions are equivalent:*

(1) *there is a* $\bar{\lambda}$ *with* $\operatorname{grade} I_\lambda = s+1$*;*

(2) $l=1$*,* $m=1$ *and* $r \geq 1$ *is odd.*

Proof. Only (1) $\Rightarrow$ (2) requires a proof. First $h, g \geq 1$, so $\rho \geq 0$. Then we have $s+1 > \rho+1$. Consequently $I_\lambda = I_\chi$. From Theorem 3.2, (a) we obtain that $l=1$ and $r \geq 1$ is odd. Since $r=s$ it follows that $m=1$. □

COROLLARY 3.5. *Suppose R to be local. Then the following conditions are equivalent:*

(1) *there is a* $\bar{\lambda}$ *with* $\operatorname{grade} I_\lambda > \mid \rho \mid +1$*;*

(2) $l=1$ *and* (i) $r=1$ *or* (ii) M *has a minimal resolution*

$$0 \longrightarrow R \longrightarrow R^{2k} \longrightarrow M \longrightarrow 0$$

where $k \geq 2$*.*

Theorem 3.2 implies that $h \leq \mid \rho \mid +2$. We shall use this fact in order to simplify the description of the homology of $C^{\cdot}_{\bar{\lambda}}(t)$.

THEOREM 3.6. *With notation as above set* $S_0(C) = R/I_\chi$*. Equip* $C^{\cdot}_{\bar{\lambda}}(t)$ *with the graduation induced by the complex isomorphism* $\mathcal{M}^{\cdot}(t) \to C^{\cdot}_{\bar{\lambda}}(t)$ *of Proposition* 2.15*. Then for the homology* $\widetilde{H}^{\cdot}$ *of* $C^{\cdot}_{\bar{\lambda}}(t)$ *the following holds:*

(a) *in case* $t \leq \frac{\rho}{2}$*,*

$$\widetilde{H}^i = \begin{cases} D_{\rho-t-\frac{i-1}{2}}(\mathcal{H}^*) \otimes S_{\frac{i-1}{2}}(C) & \text{if } 0 \leq i < h,\ i \not\equiv 0\ (2), \\ 0 & \text{if } 0 \leq i < h,\ i \equiv 0\ (2); \end{cases}$$

(b) *in case* $\frac{\rho}{2} < t \leq \rho$*,*

$$\widetilde{H}^i = \begin{cases} D_{\rho-t-\frac{i-1}{2}}(\mathcal{H}^*) \otimes S_{\frac{i-1}{2}}(C) & \text{if } 0 \leq i < \min(h, 2(\rho-t+1)),\ i \not\equiv 0\ (2), \\ S_{\frac{i-l}{2}-\rho+t-1}(\mathcal{H}) \otimes S_{\frac{i+l}{2}-1}(C) & \text{if } 2(\rho-t+1)+l \leq i < h,\ i-l \equiv 0\ (2), \\ 0 & \text{otherwise if } 0 \leq i < h; \end{cases}$$

(c) *in case* $\rho < t < r$*,*

$$\widetilde{H}^i = \begin{cases} S_{\frac{i-r+t-1}{2}}(\mathcal{H}) \otimes S_{\frac{i+r-t-1}{2}}(C) & \text{if } r-t+1 \leq i < h,\ i+r-t \not\equiv 0\ (2), \\ 0 & \text{otherwise if } 0 \leq i < h; \end{cases}$$

(d) *in case* $r \le t$,

$$\widetilde{H}^i = \begin{cases} S_{\frac{i-1}{2}+t-r}(\mathcal{H}) \otimes S_{\frac{i-1}{2}}(C) & \textit{if } \ 0 \le i < h, \ i \not\equiv 0 \ (2), \\ 0 & \textit{if } \ 0 \le i < h, \ i \equiv 0 \ (2). \end{cases}$$

Proof. If $h = 0$, then there is nothing to prove. If $h \ge 1$ then $\rho \ge 0$ and $r \ge 1$.

From Theorem 2.16 we get a complex morphism $C^{\cdot}_{\bar{\lambda}}(t) \to \mathcal{N}_{\cdot}(\rho - t)$ which induces the following commutative diagram with exact columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
C^{-1}_{\bar{\lambda}} \arrow[r, "\partial^{-1}_{\bar{\lambda}}"] & C^{0}_{\bar{\lambda}} \arrow[r, "\partial^{0}_{\bar{\lambda}}"] \arrow[d, "\nu_0"'] & C^{1}_{\bar{\lambda}} \arrow[r] \arrow[d, "\nu_1"'] & C^{2}_{\bar{\lambda}} \arrow[d, "\nu_2"'] \\
0 \arrow[r] & N^0 \arrow[r, "d^0_{\varphi}"] \arrow[d] & N^1 \arrow[r] \arrow[d] & N^2 \arrow[d] \\
0 \arrow[r] & \operatorname{Coker} \nu_0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

Since $C^{-1}_{\bar{\lambda}}$ is a torsion module and ν_0 is injective, we have $\partial^{-1}_{\bar{\lambda}} = 0$. If $h \ge 1$, then d^0_{φ} is injective. This implies that $\partial^0_{\bar{\lambda}}$ is injective, so $\widetilde{H}^0 = 0$. If $h \ge 2$, Theorem 2.8 (or Theorem 2.9) says that the row homology at N^0 and at N^1 is 0, so

$$\widetilde{H}^1 = \operatorname{Coker} \nu_0 = \begin{cases} D_{\rho-t}(\mathcal{H}^*) \otimes R/I_{\chi} & \text{if } \ t \le \rho, \\ 0 & \text{if } \ \rho < i < r, \\ S_{t-r}(\mathcal{H}) \otimes R/I_{\chi} & \text{if } \ r \le t. \end{cases}$$

If $h \ge 3$, then $\widetilde{H}^2$ equals the row homology at N^2. Except the case in which h is even and $i = h - 1$, the remaining statements follow easily from Theorem 2.8 (or Theorem 2.9), if one uses $h \le \rho + 2$ and the fact that all $H^{p,q}_{\psi}$ which appear in Theorem 2.8,(a)-(c) are zero in the case under consideration. In case h is even and $i = h - 1$, see Proposition 2.10 (b). $\square$

REMARK 3.7. If $t = 0$ in Theorem 3.6, then we obtain Proposition 5.1 in [MPN]. We only suppose that R is noetherian (and even this assumption is superfluous if one uses a generalized notion of grade).

COROLLARY 3.8. *Suppose that* $h = |\rho| + 1$.

(a) *If* $l \geq \frac{r-1}{2}$*, then* $\left(C^{\cdot}_{\bar{\lambda}}(0)\right)^*$ *has non-vanishing homology only in grade* $\rho+1$*, and if* $r > 1$*, then*

$$\bar{H}^0(\rho+1) = S_\rho(\operatorname{Coker}\chi^*).$$

(b) *If* $l \geq \frac{r+1}{2}$*, then the homology of* $C^{\cdot}_{\bar{\lambda}}(\rho+1)$ *vanishes in positive grades except for* $\rho+1$*, and*

$$\widetilde{H}^{\rho+1}(\rho+1) = S_{\rho+1}(\operatorname{Coker}\chi^*).$$

Proof. (a) Since $h \geq 1$, we have $\rho \geq 0$. In positive grades, the homology of $\left(C^{\cdot}_{\bar{\lambda}}(0)\right)^*$ is almost the same as the homology of $C^{\cdot}_{\bar{\lambda}}(\rho)$, with the only exception in grade 1 (where the homology of $\left(C^{\cdot}_{\bar{\lambda}}(0)\right)^*$ is 0). If we require that $l \geq \frac{r-1}{2}$, Theorem 2.16 (b) provides the result.

(b) If $r = 1$, then the claim is clear. If $r > 1$, then $l > 1$, and the result follows directly from Theorem 2.16 (c). □

REMARK 3.9. Corollary 3.8 (a) was inspired by Lemma 5.5 in [MPN]. Corollary 3.8 (b) can be seen as an extension of Theorem 2.7 (b), since l being big enough, one can easily deduce similar results. That seems to open the way to the study of the homology of the Koszul complex in projective dimension ≥ 2, at least in some particular cases, by taking a module of projective dimension 1 instead of the free module G and iterating the methods used here.

REMARK 3.10. In this section we required for g to have the greatest possible value. If we do the same for h, we can obtain information about the homology of $C^{\cdot}_{\bar{\lambda}}(t)$, by studying the upper half of the bicomplex presented in Theorem 1.29.

3.2 The General Case

The purpose of this section is to generalize Theorem 3.2 in order to give an answer to the question contained in the title.

We obtain a result weaker than Theorem 3.2.

THEOREM 3.11. *Set $\rho = r - l$ and $k = r + 1 - g$. Suppose that $g = \operatorname{grade} I_\chi >\mid \rho \mid +1$.*

(a) *$I_\lambda \subset \operatorname{Rad} I_\chi$, and in particular $\operatorname{grade} I_\lambda \leq g$.*

(b) *The following conditions are equivalent:*

   (1) *$\operatorname{grade} I_\lambda >\mid \rho \mid +1$;*

   (2) *$\operatorname{Rad} I_\chi = \operatorname{Rad} I_\lambda$.*

(c) *Suppose that there is a $\bar{\lambda}$ such that $\operatorname{grade} I_\lambda >\mid \rho \mid +1$. Then $l = k+1$, $r \geq l$, and $r - k$ is odd.*

   (c_1) *Let $r = k+1$ $(= l)$. The sequence $0 \to \mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$ as well as its dual $0 \to H \xrightarrow{\varphi} G \xrightarrow{\psi} F$ are exact. This can occur only if $I_\chi = I_\lambda$. Furthermore $m = 1$ occurs if and only if the ith entry of a matrix for χ is $(-1)^i$ times the minor of a matrix for λ by cancelling the ith row.*

   (c_2) *Let $r \geq k+3$ (i. e. $r \geq l+2$). If χ is minimal, then $m \leq k+1$ $(= l)$. If λ is minimal, then $m > k$ $(= l-1)$.*

Proof. (a) With respect to the assumption, Proposition 2.4 (2) yields $I_\lambda = I_{\lambda^*} \subset \operatorname{Rad} I_{\chi^*} = \operatorname{Rad} I_\chi$.

(b), (1)$\Rightarrow$(2) is an immediate consequence of Proposition 2.4 (2) while (2)$\Rightarrow$(1) is trivial.

Next we prove the main statement of (c). From (b) we draw that $\operatorname{Rad} I_\chi = \operatorname{Rad} I_\lambda$. Since $h := \operatorname{grade} I_\lambda = g > 1$, Proposition 2.4 (1) implies that $\rho \geq 0$, and $r - k + 1 = g > \rho + 1 = r - l + 1$ implies that $l > k$. If $k = 0$, then the claim follows from Theorem 3.2 (c). So we may suppose that $k \geq 1$.

Assume that $r - k$ is even. Then $r - k \geq 2$ and $r \geq 3$. Consider the complex $\mathcal{N}^{\cdot}(\frac{r-k}{2})$ defined in chapter 2 and observe that

$$N^{r-k+1}(\frac{r-k}{2}) = S_{\frac{r-k}{2}}(H^*) \otimes (\textstyle\bigwedge^{r+l-k} M)^*$$

since

$$N^{r-k+1}(\frac{r-k}{2}) = \operatorname{Ker}\big(S_{\frac{r-k}{2}}(H^*) \otimes \textstyle\bigwedge^{r+l-k} G \to S_{\frac{r-k}{2}}(H^*) \otimes \bigwedge^{r+l-k-1} G \otimes F\big).$$

Now $l > k$ implies that $(\bigwedge^{r+l-k} M)^* = 0$. So $N^{r-k+1}(\frac{r-k}{2}) = 0$. It follows that $\bar{H}^{r-k+1}(\frac{r-k}{2}) = 0$. On the other hand, using Theorem 2.11 (a) we obtain an exact sequence

$$0 \longrightarrow D_0(H) \otimes S_{\frac{r-k}{2}}(C) \longrightarrow \bar{H}^{r-k+1}(\tfrac{r-k}{2}).$$

Because $\bar{H}^{r-k+1}(\frac{r-k}{2}) = 0$, we get $D_0(H) \otimes S_{\frac{r-k}{2}}(C) = 0$ and consequently $C = 0$ which is in contradiction with $I_\chi \neq R$. So $r-k$ must be odd.

If $r-k = 1$, then $l = k+1$ since $l \leq r$. Suppose that $r-k \geq 3$ (and odd). Then

$$N^{r-k}(\frac{r-k-1}{2}) = S_{\frac{r-k-1}{2}}(H^*) \otimes (\bigwedge^{r+l-k-1} M)^*.$$

On the other hand we draw from Theorem 2.11, (a) that

$$\bar{H}^{r-k}(\frac{r-k-1}{2}) = D_0(H) \otimes S_{\frac{r-k-1}{2}}(C).$$

If $l > k+1$, then $(\bigwedge^{r+l-k-1} M)^* = 0$, so $S_{\frac{r-k-1}{2}}(C) = 0$, a contradiction. It follows that $l = k+1$.

(c_1) The first statement is an immediate consequence of the Buchsbaum-Eisenbud acyclicity criterion (see [E], Theorem 20.9). The second statement is also due to Buchsbaum and Eisenbud (see [N], Chapter 7, Theorem 3 or Corollary 5.1 in [BE1]). (Of course the third statement is a special case of the Theorem of Hilbert-Burch.)

To prove the first claim of (c_2) let χ be minimal. Localize at a prime ideal P which contains $I_1(\chi)$. Then $g \leq \operatorname{grade} I_\chi R_P$. But $g < \operatorname{grade} I_\chi R_P$ is impossible: otherwise, since $I_\lambda \subset \operatorname{Rad} I_\chi \subset P$ in view of (a), we have $\operatorname{grade} I_\lambda R_P > \mid \rho \mid +1$ and consequently $l = r+1-\operatorname{grade} I_\chi R_P + 1 \leq k$ in contradiction with the first claim under (c). So we may assume R to be local. Since $r \geq k+3$, $l = k+1$ and M is free in depth 1, we get $N^{r-k}(\frac{r-k-1}{2}) = S_{\frac{r-k-1}{2}}(H^*)$ and $N^{r-k+1}(\frac{r-k-1}{2}) = 0$. So $\bar{H}^{r-k}(\frac{r-k-1}{2}) = S_{\frac{r-k-1}{2}}(C)$ is a quotient of $S_{\frac{r-k-1}{2}}(H^*)$. Therefore the minimal number of generators of C cannot be greater than the minimal number of generators of H. It follows that $m \leq l = k+1$.

Since $r \geq k+3$ and $l = k+1$, we deduce that $\rho \geq 2$. So $h = g \geq 4$. To prove the second statement of (c_2) we dualize the sequence $0 \to \mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$. Set $r' = n - l$ and $k' = r' + 1 - h$. From the first claim under (c) we draw that $m = k'+1$. Since $h \geq 4$, we get $r' \geq k'+3$. Now let λ be minimal. Applying the first part of (c_2), we obtain $(k+1 =)\ l \leq k'+1 = m$. $\square$

COROLLARY 3.12. *Set* $\rho = r - l$ *and suppose that* $\operatorname{grade} I_\chi > \mid \rho \mid +1$. *Then* $\operatorname{grade} I_\lambda \leq \mid \rho \mid +2$ *for every* λ. *Moreover, if* $\operatorname{grade} I_\lambda = \mid \rho \mid +2$, *then* $\operatorname{grade} I_\chi$, $\operatorname{grade} I_\lambda$, ρ *are even.*

Proof. If $\operatorname{grade} I_\lambda > \mid \rho \mid +1$, then the above theorem implies that $\operatorname{grade} I_\lambda = \operatorname{grade} I_\chi = r+1-k = r-l+2 = \mid \rho \mid +2$. Since $r-k$ is odd, $\operatorname{grade} I_\chi, \operatorname{grade} I_\lambda, \rho$ must be even. $\square$

REMARK 3.13. Let $k \in \mathbb{N}$. If $l = k+1$ and $r-k > 0$ is odd, then, in the cases listed under (c), there are always maps $\chi : R^m \to R^n$ and $\lambda : R^n \to R^l$ such that $\lambda\chi = 0$ and $\operatorname{grade} I_\chi = \operatorname{grade} I_\lambda = r-k+1$, provided there is a regular sequence of length $r-k+1$ in R.

For simplicity, we give examples only for $k=1$. They can easily be generalized for an arbitrary k.

For $r=2$ and $m>1$ consider a regular sequence x, y in R. Let χ be given by the $m \times (m+2)$-matrix

$$\begin{pmatrix} x & 0 & y & 0 & 0 & \dots & 0 \\ 0 & x & 0 & y & 0 & \dots & 0 \\ & & & \ddots & & & \\ & & & & \ddots & & \\ 0 & \dots & \dots & 0 & x & 0 & y \end{pmatrix}.$$

If $m = 2k$ is even, then let λ be given by the matrix

$$\begin{pmatrix} y^k & 0 \\ 0 & y^k \\ -xy^{k-1} & 0 \\ 0 & -xy^{k-1} \\ \vdots & \vdots \\ (-1)^{k-1}x^k y & 0 \\ 0 & (-1)^{k-1}x^k y \\ (-1)^k x^k & 0 \\ 0 & (-1)^k x^k \end{pmatrix},$$

and if $m = 2k+1$ is odd, then the matrix

$$\begin{pmatrix} y^{k+1} & 0 \\ 0 & y^k \\ -xy^k & 0 \\ 0 & -xy^{k-1} \\ \vdots & \vdots \\ 0 & (-1)^{k-1}x^{k-1}y \\ (-1)^k x^k y & 0 \\ 0 & (-1)^k x^k \\ (-1)^{k+1}x^{k+1} & 0 \end{pmatrix}$$

yields an appropriate λ.

Now suppose that $r \geq 4$. If $m = 2$, set $r = 2k$ and let $x_1, \dots, x_k, y_1, \dots, y_k$ be a regular sequence in R. Consider

$$R^2 \xrightarrow{\chi} R^{2k+2} \xrightarrow{\lambda} R^2$$

where χ and λ are given by the matrices

$$\begin{pmatrix} 0 & x_1 & \cdots & x_{k-1} & x_k & 0 & y_1 & \cdots & y_{k-1} & y_k \\ x_1 & x_2 & \cdots & x_k & 0 & y_1 & y_2 & \cdots & y_k & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -y_k & 0 \\ -y_{k-1} & -y_k \\ \vdots & \vdots \\ -y_1 & -y_2 \\ 0 & -y_1 \\ x_k & 0 \\ x_{k-1} & x_k \\ \vdots & \vdots \\ x_1 & x_2 \\ 0 & x_1 \end{pmatrix},$$

respectively. If $m = 1$, take any χ with a matrix $(x_1, \ldots, x_n)$ such that $x_1, \ldots, x_{n-1}$ is a regular sequence in R, and define λ by

$$\begin{pmatrix} -x_{n-1} & 0 \\ x_{n-2} & 0 \\ \vdots & \vdots \\ -x_2 & 0 \\ x_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally we state a useful criterion.

COROLLARY 3.14. *Let $l \leq n$, $m \leq n$ and set $\rho = n - m - l$. Furthermore let $\mathcal{A}$ be an (l, n)-matrix and $\mathcal{B}$ be an (n, m)-matrix with entries in R. Let $h = \operatorname{grade} I_{\mathcal{A}}$, $g = \operatorname{grade} I_{\mathcal{B}}$ and suppose that $h, g >\mid \rho \mid +1$. Then $\mathcal{AB} \neq 0$ in each of the following cases:*

(1) *$h \neq g$;*

(2) *ρ odd;*

(3) *$g \neq\mid \rho \mid +2$;*

(4) *the ideals generated by the entries of $\mathcal{A}$ and $\mathcal{B}$ are proper ideals of R and $l \neq m$ and $l \neq n - m$.*

At the end, we propose the following problem.

EXERCISE 3.15. Let k be a field. Show that the system of equations

$$\begin{aligned}
&f_1^2(x) + f_2^2(y) + f_3^2(z) + f_4^2(x,y,z) + f_5^2(x,y,z) = 0\\
&f_1(x)f_2(y) + f_2(y)f_3(z) + f_3(z)f_4(x,y,z) + f_4(x,y,z)f_5(x,y,z) = 0\\
&f_1(x)f_3(z) + f_2(y)f_4(x,y,z) + f_3(z)f_5(x,y,z) = 0,
\end{aligned}$$

where $(f_1, f_2, f_3, f_4, f_5) \in k[x] \times k[y] \times k[z] \times k[x,y,z] \times k[x,y,z]$, $f_i \notin k$, has no solution.

3.3 Appendix: Some Length Formulas

We refer to Theorem 3.6. It seems to be impossible to obtain a comparably smooth description of the homology of $C^{\cdot}_{\bar{\lambda}}(t)$ in the case under consideration. The following result deals with the special case in which $g = \operatorname{grade} I_\chi = \dim R$. This will be our general assumption in the following considerations. Then R/I_χ has finite length $\ell(R/I_\chi)$. (Generally the length of an R-module N is denoted by $\ell(N)$). We note that Corollary 3.12 is used in order to simplify the presentation of the next result.

THEOREM 3.16. *Suppose that* $\dim R = r$. *Equip* $C^{\cdot}_{\bar{\lambda}}(t)$ *with the graduation induced by the complex isomorphism* $\mathcal{M}^{\cdot}(t) \to C^{\cdot}_{\bar{\lambda}}(\rho - t)$ *of Proposition* 2.15 *where* $\rho = r - l$ *as above. Then the homology modules* $\widetilde{H}^i$ *of* $C^{\cdot}_{\bar{\lambda}}(t)$ *have finite length for* $i \le \min(h-1, 2r)$.

Set $S_0(C) = R/I_\chi$ *and assume* $h > 0$.

(a) *Let* $l = 1$. *Then for all* $t \in \mathbb{Z}$ *and* i *odd,* $0 < i < \min(h-1, 2r)$, *we get*

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \ell(S_{\frac{i-1}{2}}(C)) - \ell(S_{\frac{i+1}{2}}(C)).$$

(b) *Let* $l > 1$. *We distinguish four cases.*

(i) *For all* $t \le \frac{\rho}{2}$ *and* i *odd,* $0 < i < h - 1$,

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \binom{r - t - \frac{i+1}{2}}{l-1} \ell(S_{\frac{i-1}{2}}(C)) - \binom{r - t - \frac{i+3}{2}}{l-1} \ell(S_{\frac{i+1}{2}}(C)).$$

(ii) *Suppose that* $\frac{\rho}{2} < t \le \rho$. *If* i *is odd,* $0 < i < \min(h-1, 2(\rho - t))$, *then*

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \binom{r - t - \frac{i+1}{2}}{l-1} \ell(S_{\frac{i-1}{2}}(C)) - \binom{r - t - \frac{i+3}{2}}{l-1} \ell(S_{\frac{i+1}{2}}(C)).$$

If $i - l$ *is even,* $2(\rho - t) + l + 2 \le i < h - 1$, *then*

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \binom{\frac{i+l}{2} - \rho + t - 2}{l-1} \ell(S_{\frac{i+l}{2}-1}(C)) - \binom{\frac{i+l}{2} - \rho + t - 1}{l-1} \ell(S_{\frac{i+l}{2}}(C)).$$

If $2(\rho - t) + l + 1 < h$, *then* $\ell(\widetilde{H}^{2(\rho-t)+l+1}) = \ell(S_{r-t}(C))$. *Moreover*

$$\widetilde{H}^i = \begin{cases} S_{\rho-t}(C) & \text{if } i = 2(\rho - t) + 1 < h, \\ 0 & \text{if } 2(\rho - t) + 2 \le i < \min(h, 2(\rho - t) + l + 1). \end{cases}$$

(iii) *Suppose that $\rho < t < r$. If $i + r - t$ is odd, $r - t + 1 \le i < h$, then*

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \binom{\frac{i-r+t-3}{2} + l}{l-1} \ell(S_{\frac{i+r-t-1}{2}}(C)) - \binom{\frac{i-r+t-1}{2} + l}{l-1} \ell(S_{\frac{i+r-t+1}{2}}(C)).$$

If $r - t < h$, then $\ell(\widetilde{H}^{r-t}) = \ell(S_{r-t}(C))$. Moreover $\widetilde{H}^i = 0$ if $0 \le i < \min(h, r-t)$.

(iv) *Suppose that $r \le t$ and i odd, $0 < i < h - 1$. Then*

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \binom{t - \rho + \frac{i-3}{2}}{l-1} \ell(S_{\frac{i-1}{2}}(C)) - \binom{t - \rho + \frac{i-1}{2}}{l-1} \ell(S_{\frac{i+1}{2}}(C)).$$

REMARK 3.17. Observe that in the above formulas we use the fact that h can reach the maximal value only if it is *even*. The combinatorial coefficients are not degenerated. We further notice that if $h < \infty$, then the formulas for $l > 1$ cover the case in which $l = 1$. Finally (a) and (b) make sense only if $h, g > 0$, and consequently $\rho \ge 0$.

Proof. If $r = 0$, then $M = 0$ since χ is injective. So we may assume that $r \ge 1$.

We recall that $C^{\cdot}_{\bar{\lambda}}(t)$ is the complex

$$\cdots \to D_p(H) \otimes \bigwedge^{t+l+p} M \to \cdots \to D_0(H) \otimes \bigwedge^{t+l} M \xrightarrow{\nu} S_0(H^*) \otimes \bigwedge^t M \to \cdots \to S_t(H^*) \otimes \bigwedge^0 M \to 0$$

where

$$C^0_{\bar{\lambda}}(t) = \begin{cases} D_{\rho-t}(H) \otimes \bigwedge^r M & \text{if } t \le \rho, \\ S_0(H^*) \otimes \bigwedge^t M & \text{if } \rho < t < r, \\ S_{t-r}(H^*) \otimes \bigwedge^r M & \text{if } r \le t. \end{cases}$$

For $q > r$ the support of $\bigwedge^q M$ is contained in the variety of I_χ. Consequently $C^i_{\bar{\lambda}}(t)$ has finite length if $i < 0$, which in turn implies that $\widetilde{H}^i$ has finite length. In particular, there remains nothing to prove if $h = 0$. Let $h \ge 1$.

From Theorem 2.16 we get a complex morphism $C^{\cdot}_{\bar{\lambda}}(t) \to \mathcal{N}_{\cdot}(\rho - t)$ which induces the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
C_{\bar\lambda}^{-1} \arrow[r, "\partial_{\bar\lambda}^{-1}"] & C_{\bar\lambda}^{0} \arrow[r, "\partial_{\bar\lambda}^{0}"] \arrow[d, "\nu_0"'] & C_{\bar\lambda}^{1} \arrow[r] \arrow[d, "\nu_1"'] & C_{\bar\lambda}^{2} \arrow[r] \arrow[d, "\nu_2"'] & C_{\bar\lambda}^{3} \arrow[d, "\nu_3"'] \\
0 \arrow[r] & N^0 \arrow[r, "\bar d_\varphi^0"] \arrow[d] & N^1 \arrow[r] \arrow[d] & N^2 \arrow[r] \arrow[d] & N^3 \arrow[d] \\
0 \arrow[r] & \operatorname{Coker}\nu_0 \arrow[r, "\alpha"] \arrow[d] & \operatorname{Coker}\nu_1 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the columns are exact. If $\rho < t < r$, then ν_0 is injective and ν_1 is an isomorphism.

For arbitrary t and arbitrary i the maps ν_i are isomorphisms at all prime ideals which do not contain I_χ. Consequently $\operatorname{Ker}\nu_i$ and $\operatorname{Coker}\nu_i$ have finite length. In particular $\operatorname{Ker}\nu_0$ equals the torsion submodule of $C_{\bar\lambda}^0$ since N^0 is torsion free. On the other hand, $C_{\bar\lambda}^1$ is a torsion free module, too. So the torsion submodule of $C_{\bar\lambda}^0$ is contained in $\operatorname{Ker}\partial_{\bar\lambda}^0$. If we denote by $\bar\nu_0$ ($\bar\partial_{\bar\lambda}^0$) the maps induced by ν_0 ($\partial_{\bar\lambda}^0$) on $C_{\bar\lambda}^0/\operatorname{Ker}\nu_0$, we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & C_{\bar\lambda}^{0}/\operatorname{Ker}\nu_0 \arrow[r, "\bar\partial_{\bar\lambda}^{0}"] \arrow[d, "\bar\nu_0"'] & C_{\bar\lambda}^{1} \arrow[r] \arrow[d, "\nu_1"'] & C_{\bar\lambda}^{2} \arrow[r] \arrow[d, "\nu_2"'] & C_{\bar\lambda}^{3} \arrow[d, "\nu_3"'] \\
0 \arrow[r] & N^0 \arrow[r, "\bar d_\varphi^0"] \arrow[d] & N^1 \arrow[r] \arrow[d] & N^2 \arrow[r] \arrow[d] & N^3 \arrow[d] \\
0 \arrow[r] & \operatorname{Coker}\nu_0 \arrow[r, "\alpha"] \arrow[d] & \operatorname{Coker}\nu_1 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact columns.

Since $h \geq 1$, $\bar d_\varphi^0$ is injective. Then $\bar\partial_{\bar\lambda}^0$ must be injective, which in turn implies that

$$\operatorname{Ker}\nu_0 = \operatorname{Ker}\partial_{\bar\lambda}^0.$$

On the other hand, $\operatorname{Ker}\nu_0$ equals $H^r(C^{\cdot}_{\psi}(0)) \otimes E$ where E is some finitely generated free module. By Lemma 1.2 in [BV1], $H^r(C^{\cdot}_{\psi}(0))$ has finite length. Since $\widetilde{H}^0$ is a factor of $\operatorname{Ker}\partial^0_{\bar{\lambda}}$, we deduce that $\widetilde{H}^0$ has finite length.

In case $h = 1$, it remains to prove that $\widetilde{H}^0 = 0$ if $\rho < t < r$. Since $h, g > 0$ we have $\rho \geq 0$ and consequently $r > 1$. Then $\partial^0_{\bar{\lambda}}$ is injective and $\widetilde{H}^0 = 0$.

Assume that $h = 2$ (so $r \geq 2$). This implies that the row homology at N^0 and at N^1 vanishes. Of course, $\widetilde{H}^1 = \operatorname{Ker}\alpha$ has finite length. There are statements about $\widetilde{H}^1$ only for $l > 1$ and $t = \rho$, $t = r-1$, $\rho < t = r-2$. In all these cases $\operatorname{Coker}\nu_1 = 0$ (see the Theorems 2.8 and 2.9). So

$$\widetilde{H}^1 = \operatorname{Coker}\nu_0 = \begin{cases} S_0(C) & \text{if } \ t = \rho, \\ H^r(C^{\cdot}_{\psi}(1)) & \text{if } \ t = r-1, \\ 0 & \text{if } \ \rho < t = r-2. \end{cases}$$

By Proposition 2.3 in [BV1] we have $\ell(H^r(C^{\cdot}_{\psi}(1))) = \ell(S_1(C))$.

Now suppose that $h \geq 3$. If $l = 1$, then the row homology at N^0, N^1 and N^2 vanishes (see the Theorems 2.8 and 2.9). For $i = 1$ we have

$$\ell(\widetilde{H}^1) - \ell(\widetilde{H}^2) = \ell(\operatorname{Ker}\alpha) - \ell(\operatorname{Coker}\alpha) = \ell(\operatorname{Coker}\nu_0) - \ell(\operatorname{Coker}\nu_1).$$

But $\operatorname{Coker}\nu_0 = S_0(C)$ and $\operatorname{Coker}\nu_1 = H^r(C^{\cdot}_{\psi}(1))$, so

$$\ell(\widetilde{H}^1) - \ell(\widetilde{H}^2) = \ell(S_0(C)) - \ell(H^r(C^{\cdot}_{\psi}(1)).$$

If i is odd, $3 \leq i < h-1$, then we deduce directly from Theorem 2.8 that

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \ell(S_{\frac{i-1}{2}}(C)) - \ell(H^r(C^{\cdot}_{\psi}(\frac{i+1}{2})).$$

Proposition 2.3 in [BV1] implies that $\ell(H^r(C^{\cdot}_{\psi}(k))) = \ell(S_k(C))$ whenever $0 \leq k \leq r$. It remains to prove that $\widetilde{H}^{h-1}$ has finite length if h is even. But Proposition 2.10 (c) provides an injection of $\widetilde{H}^{h-1}$ into a module of finite length. So we settled the case in which $l = 1$.

Let $l > 1$. Only the case in which $\rho \leq t < r$, deserves special attention. The other cases are similar to the case $l = 1$. If $\rho \leq t < r-2$, then the row homology at N^0, N^1, N^2 vanishes. Furthermore $\widetilde{H}^2 = \operatorname{Coker}\nu_1 = 0$ and

$$\widetilde{H}^1 = \operatorname{Coker}\nu_0 = \begin{cases} S_0(C) & \text{if } \ t = \rho, \\ 0 & \text{if } \ \rho < t < r-2. \end{cases}$$

If $\rho = t = r-2$, then again the row homology at N^0, N^1, N^2 vanishes, and $\widetilde{H}^1 = S_0(C)$. If $\rho < t = r-2$ or $t = r-1$, then the row homology at N^0 and N^1 vanishes. Therefore

$$\widetilde{H}^1 = \operatorname{Coker}\nu_0 = \begin{cases} H^r(C^{\cdot}_{\psi}(1)) & \text{if } \ t = r-1 \\ 0 & \text{if } \ \rho < t = r-2. \end{cases}$$

Since $\operatorname{Coker}\nu_1 = 0$, we get that $\widetilde{H}^2$ equals the row homology at N^2.The remaining claims follow easily if one uses as pattern the proof for the $l = 1$ case. $\square$

REMARK 3.18. For $l = 1$ the results contained in Theorem 3.16 were first obtained by Vetter (unpublished) who uses local cohomology in order to get the information about the homology of the Koszul complex associated to $\bar{\lambda}$.

COROLLARY 3.19. *Set* $S_0(C) = R/I_\chi$. *Suppose that* $\dim R = r$. *Let* $\widetilde{C^{\cdot}_{\bar{\lambda}}}(t)$ *be the complex obtained from* $C^{\cdot}_{\bar{\lambda}}(t)$ *by replacing* $C^{i}_{\bar{\lambda}}(t)$ *with* 0 *whenever* $i < 0$. *By* $\widetilde{H}^k$ *we denote the homology of* $\widetilde{C^{\cdot}_{\bar{\lambda}}}(t)$ *at the* $\widetilde{C^{k}_{\bar{\lambda}}}(t)$.

(a) *If* $h = \infty$, *then*

$$\ell(S_0(C)) = \ell(S_1(C)) = \ldots = \ell(S_r(C)).$$

(b) *If* h *is odd and* $t \le \frac{\rho}{2}$, *then*

$$\sum_{k=0}^{h-1}(-1)^k \ell(\widetilde{H}^k) = \binom{r - t - \frac{h+1}{2}}{l-1} \ell(S_{\frac{h-1}{2}}(C)).$$

(c) *If* h *is even and* $t \le \frac{\rho}{2}$, *then*

$$\sum_{k=0}^{h-2}(-1)^k \ell(\widetilde{H}^k) = \binom{r - t - \frac{h}{2}}{l-1} \ell(S_{\frac{h-2}{2}}(C)).$$

In case $t > \frac{\rho}{2}$ *one can easily deduce formulas similar to* (a) *and* (b).

Proof. (a) If $h = \infty$, then l must be 1. Remark that this result may also be seen as an easy consequence of Proposition 2.8 in [BV1].

(b) and (c) We may obviously suppose that $h > 0$. So $\rho \ge 0$. We have to prove that

$$\ell(\widetilde{H}^0(t)) = \ell(D_{\rho-t}(H) \otimes S_0(C))$$

if $t \le \frac{\rho}{2}$. From the proof of Theorem 3.16 we deduce that

$$\widetilde{H}^0(t) \cong D_{\rho-t}(H) \otimes H^r(C^{\cdot}_{\psi}(0)),$$

and from [BV1], Proposition 2.3 we draw that

$$\ell(H^r(C^{\cdot}_{\psi}(0))) = \ell(S_0(C)).$$

$\square$

We specialize to the case in which R is a quasi-homogeneous complete intersection with isolated singularity. More precisely, we let $S = k[[X_1, \ldots, X_n]]$ where k is a field of characteristic zero, assign positive degrees a_i to the variables X_i, and set $R = S/(p_1, \ldots, p_m) = k[[x_1, \ldots, x_n]]$ where the $p_i \in (X_1, \ldots, X_n)^2$ form a regular sequence of homogeneous polynomials of degrees b_i. By the Euler formula we have

$$b_j p_j = \sum_{i=1}^{n} a_i \frac{\partial p_j}{\partial X_i} X_i.$$

Since $(p_1, \ldots, p_m) = (b_1 p_1, \ldots, b_m p_m)$, the $b_j p_j$ may be viewed as defining elements for R. If we set $p'_j = b_j p_j$ and $X'_i = a_i X_i$, we get

$$p'_j = \sum_{i=1}^{n} \frac{\partial p_j}{\partial X_i} X'_i.$$

We suppose $m < n$ and $R_{\mathfrak{p}}$ to be regular for all prime ideals $\mathfrak{p}$ different from the maximal ideal. As usual we denote by $\Omega_{R/k}$ the module of Kähler-differentials of R over k. There is a presentation

$$0 \longrightarrow \mathcal{F} \xrightarrow{\chi} \mathcal{G} \longrightarrow \Omega_{R/k} \longrightarrow 0$$

where $\mathcal{F}$, $\mathcal{G}$ are free R-modules of ranks m, n and grade $I_\chi = r$ (see [BV1] for details). Moreover the Euler derivation $\bar{\lambda}$ gives rise to an exact sequence

$$\bigwedge^r \Omega_{R/k} \to \bigwedge^{r-1} \Omega_{R/k} \to \cdots \to \Omega_{R/k} \xrightarrow{\bar{\lambda}} R \to k \to 0$$

which is in fact the non-negative grade part of $C^{\cdot}_{\bar{\lambda}}$. Let $\lambda : \mathcal{G} \to R$ be the corresponding lifted map. Set $\varphi = \lambda^*$, $\psi = \chi^*$ as above. As in the proof of Theorem 3.16 we can complement the exact sequence from above to an exact sequence

$$0 \to \tau(\bigwedge^r \Omega_{R/k}) \to \bigwedge^r \Omega_{R/k} \to \bigwedge^{r-1} \Omega_{R/k} \to \cdots \to \Omega_{R/k} \to R \to k \to 0$$

where τ denotes the torsion submodule.

THEOREM 3.20. *Set $S_0(C) = R/I_\chi$. If $0 \le i \le r-1$, then*

$$H^r(C^{\cdot}_\psi(i+1)) \cong S_i(C).$$

Proof. The commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \bigwedge^{n-r} G & \longrightarrow & \cdots & \xrightarrow{d_\varphi} & \bigwedge^n G \cong R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu_0} & & & & \downarrow{\scriptstyle \nu_r} & & \\
0 & \longrightarrow & \bigwedge^0 G \cong R & \longrightarrow & \cdots & \xrightarrow{d_\varphi} & \bigwedge^r G & \longrightarrow & \cdots
\end{array}$$

is part of the bicomplex introduced in Theorem 1.29. Let $g_1, \ldots, g_n$ be a basis of G, and $f_1, \ldots, f_m$ a basis of F, such that ψ is represented by the matrix $(\frac{\partial p_j}{\partial x_i})_{i,j}$, while φ is represented by the matrix $(x'_1, \ldots, x'_n)$ (we denote by $\frac{\partial p_j}{\partial x_i}$ the image in R of $\frac{\partial p_j}{\partial X_i}$ and by x'_i the image of X'_i). Let $\Omega_n : \bigwedge^n G \to R$ be the unique R-isomorphism with $\Omega_n(g_1 \wedge \ldots \wedge g_n) = 1$. We prove that $\nu_r(g_1 \wedge \ldots \wedge g_n)$ generates the homology in the second row at $\bigwedge^r G$ (for the original proof see the second part of the proof of Theorem 3.1 in [HM]). We have

$$\begin{aligned} \nu_r(g_1 \wedge \ldots \wedge g_n) &= g_1 \wedge \ldots \wedge g_n \leftharpoonup \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*) \\ &= \sum_\sigma \varepsilon(\sigma) \det_{1 \le i,j \le m} (\psi^*(f_j^*)(g_{\sigma(i)})) g_{\sigma(m+1)} \wedge \ldots \wedge g_{\sigma(n)}, \end{aligned}$$

where σ runs through $\mathfrak{S}_{n,m}$ (see Remark 1.10).

On the other hand we have a non-canonical complex isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^0 G & \longrightarrow & \cdots & \xrightarrow{d_\varphi} & \bigwedge^n G & \longrightarrow & 0 \\ & & {\scriptstyle \varepsilon\Omega_0}\downarrow & & & & {\scriptstyle \varepsilon\Omega_n}\downarrow & & \\ 0 & \longrightarrow & \bigwedge^n \mathcal{G} & \longrightarrow & \cdots & \xrightarrow{\partial_\lambda} & \bigwedge^0 \mathcal{G} & \longrightarrow & 0 \end{array}$$

induced by Ω_n (see Proposition 1.27). The lower row is the Koszul complex associated with the sequence $x'_1, \ldots, x'_n$. If we denote by $H_i(R)$ the row homology at $\bigwedge^i \mathcal{G}$, then $H_m(R) \cong \bigwedge^m H_1(R)$ (see Theorem 2.3.11(Tate, Assmus) in [BH]). The relations

$$p'_j = \sum_{i=1}^n \frac{\partial p_j}{\partial X_i} X'_i$$

imply that $H_1(R)$ is generated by the homology classes of the cycles $\psi^*(f_j^*)$ ($j = 1, \ldots, m$) (see pages 73 and 80 in [BH]), so $H_m(R)$ is generated by $\psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*)$. An easy computation shows (see for example Chapter 1.6 in [BH]) that

$$\begin{aligned} \Omega_r(\nu_r(g_1 \wedge \ldots \wedge g_n)) &= \pm \sum_\sigma \det_{1 \le i,j \le m} (\psi^*(f_j^*)(g_{\sigma(i)})) g^*_{\sigma(1)} \wedge \ldots \wedge g^*_{\sigma(m)} \\ &= \pm \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*), \end{aligned}$$

where σ runs through $\mathfrak{S}_{n,m}$ (see also [BO1] Chapter III, §8.5, Proposition 9).

The first diagram in this proof induces the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bigwedge^r \Omega_{R/k} & \longrightarrow & \cdots & \xrightarrow{\bar\lambda} & \bigwedge^0 \Omega_{R/k} = R & \longrightarrow & 0 \\ & & {\scriptstyle \bar\nu_0}\downarrow & & & & {\scriptstyle \bar\nu_r}\downarrow & & \\ 0 & \longrightarrow & N^0 \cong R & \longrightarrow & \cdots & \xrightarrow{\bar d_\varphi} & N^r & \longrightarrow & 0 \end{array}$$

Since $\nu_r(\bigwedge^n G) \not\subset \operatorname{Im} d_\varphi$, it follows that $\bar\nu_r(R) \not\subset \operatorname{Im} \bar d_\varphi$.

Now we prove the theorem for $r = 1$. R being a complete intersection, the r-th homology group of the Koszul complex associated with φ is k. From the proof of the first part of Theorem 2.8 we deduce that, since $r = 1$, the same holds for $\mathcal{N}^{\cdot}$. As in the proof of Theorem 3.16 we have a commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{R/k}/\tau(\Omega_{R/k}) \arrow[r] \arrow[d, "\bar\nu_0"'] & R \arrow[r] \arrow[d, "\bar\nu_1"'] & 0 \\
0 \arrow[r] & N^0 \arrow[r] \arrow[d] & N^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_0(C) \arrow[r, "\alpha"] \arrow[d] & H^1(C^{\cdot}_{\psi}(1)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

If we denote by $\bar{H}^{\cdot}$ the homology of the lowest row, we obtain an exact sequence

$$0 \longrightarrow \bar{H}^0 \longrightarrow k \xrightarrow{\beta} k \longrightarrow \bar{H}^1 \longrightarrow 0.$$

Since $\bar{\nu}_r(R) \not\subset \operatorname{Im} \bar{d}_{\varphi}$, β must be an isomorphism, so α is an isomorphism.

Next we study the case in which $r = 2$. First we show that $H^2(C^{\cdot}_{\psi}(1)) \cong S_0(C)$. Once more referring to the proof of Theorem 3.16 we obtain a commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigwedge^2\Omega_{R/k}/\tau(\bigwedge^2\Omega_{R/k}) \arrow[r] \arrow[d, "\bar\nu_0"'] & \Omega_{R/k} \arrow[r] \arrow[d, "\bar\nu_1"'] & R \arrow[r] \arrow[d, "\bar\nu_2"'] & 0 \\
0 \arrow[r] & N^0 = R \arrow[r] \arrow[d] & N^1 \arrow[r] \arrow[d] & N^2 = R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_0(C) \arrow[r, "\alpha"] \arrow[d] & H^2(C^{\cdot}_{\psi}(1)) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

As above we denote by $\bar{H}^{\cdot}$ the homology of the lowest row. Obviously $\bar{H}^0 = 0$. Since the row homology at N^1 vanishes (see Theorem 2.8), we get an exact sequence

$$0 \longrightarrow \bar{H}^1 \longrightarrow k \xrightarrow{\beta} H(N^2)$$

where $H(N^2)$ denotes the row homology at N^2. Because β is induced by $\bar{\nu}_2$ (and $\bar{\nu}_r(R) \not\subset \operatorname{Im} \bar{d}_\varphi$), it must be injective, so $\bar{H}^1 = 0$ and α is an isomorphism. Next we show that $H^2(C^{\cdot}_\psi(2)) \cong S_1(C)$. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. By Proposition 2.3 in [BV1] and the local duality theorem (see 3.5.8 in [BH]) we have

$$H^2(C^{\cdot}_\psi(2)) \cong \operatorname{Ext}^1(\textstyle\bigwedge^2 \Omega_{R/k}, R) \cong (H^1_{\mathfrak{m}}(\bigwedge^2 \Omega_{R/k}))^\vee \cong (S_0(C))^\vee \cong$$

$$(H^2(C^{\cdot}_\psi(1)))^\vee \cong (H^1_{\mathfrak{m}}(\Omega_{R/k}))^\vee \cong \operatorname{Ext}^1(\Omega_{R/k}, R) \cong S_1(C).$$

Now let $r \geq 3$. Again we have a commutative diagram with exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^r \Omega_{R/k}/\tau(\bigwedge^r \Omega_{R/k}) & \longrightarrow & \bigwedge^{r-1}\Omega_{R/k} & \longrightarrow & \bigwedge^{r-2}\Omega_{R/k} & \longrightarrow & \cdots \\
 & & {\scriptstyle \bar{\nu}_0}\downarrow & & {\scriptstyle \bar{\nu}_1}\downarrow & & \downarrow & & \\
0 & \longrightarrow & N^0 = R & \longrightarrow & N^1 & \longrightarrow & N^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_0(C) & \xrightarrow{\alpha} & H^r(C^{\cdot}_\psi(1)) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since the row homology at N^0 and at N^1 vanishes, we deduce that α is an isomorphism and that $\mathcal{N}^{\cdot}$ has homology only at N^r, namely k. Using Theorem 2.8 (a) and the above considerations, we obtain that, if $0 \leq i < \frac{r-2}{2}$, then

$$H^r(C^{\cdot}_\psi(i+1)) \cong S_i(C).$$

Furthermore we have the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & N^{r-2} & \longrightarrow & N^{r-1} & \xrightarrow{\bar{d}_\varphi} & N^r & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & {\scriptstyle \iota}\downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C^{r-2,0} & \longrightarrow & C^{r-1,0} & \xrightarrow{d_\varphi} & C^{r,0} & \longrightarrow & C^{r+1,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Im}\partial_\psi^{r-2,0} & \longrightarrow & \operatorname{Im}\partial_\psi^{r-1,0} & \longrightarrow & \operatorname{Im}\partial_\psi^{r,0} & \longrightarrow & \operatorname{Im}\partial_\psi^{r+1,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns. With the notation introduced in the proof of Theorem 2.8 we get an exact sequence

$$0 \longrightarrow E^{r-1,1} \longrightarrow k \xrightarrow{\beta} k \longrightarrow E^{r,1} \longrightarrow 0.$$

Since $\iota(N^r) = \nu_r(\bigwedge^n G)$ ($\bar{\nu}_r$ is an isomorphism), β must be an isomorphism. We deduce $E^{r-2,1} = E^{r-1,1} = E^{r,1} = 0$.

On the other hand, as in the proof of Theorem 2.8 (see the sequence $(*)$), we have exact sequences

$$0 \to E^{r-2,1} \to S_{\frac{r-2}{2}}(C) \to H^r(C^{\cdot}_{\psi}(\frac{r}{2})) \to E^{r-1,1} \to 0$$

if r is even, and

$$0 \to E^{r-1,1} \to S_{\frac{r-1}{2}}(C) \to H^r(C^{\cdot}_{\psi}(\frac{r+1}{2})) \to E^{r,1} \to 0$$

if r is odd. If follows that, if $0 \le i < \frac{r}{2}$, then

$$H^r(C^{\cdot}_{\psi}(i+1)) \cong S_i(C).$$

For $\frac{r}{2} \le i < r$, we again use Proposition 2.3 in [BV1] and the local duality theorem to get

$$H^r(C^{\cdot}_{\psi}(i+1)) \cong \operatorname{Ext}^i(\textstyle\bigwedge^{i+1}\Omega_{R/k}, R) \cong (H^{r-i}_{\mathfrak{m}}(\bigwedge^{i+1}\Omega_{R/k}))^{\vee} \cong (S_{r-(i+1)}(C))^{\vee} \cong$$
$$(H^r(C^{\cdot}_{\psi}(r-i)))^{\vee} \cong (H^{r-i}_{\mathfrak{m}}(\textstyle\bigwedge^{i}\Omega_{R/k}))^{\vee} \cong \operatorname{Ext}^i(\bigwedge^{i}\Omega_{R/k}, R) \cong S_i(C).$$

□

COROLLARY 3.21. *We have*

$$\ell(S_0(C)) = \ell(S_1(C)) = \ldots = \ell(S_r(C)).$$

Proof. Use the isomorphism of Theorem 3.20 and Corollary 2.2 in [BV1]. □

REMARK 3.22. In [BV1], section 3, the length formula of Corollary 3.21 has been proved for r odd. The reader may find a complete proof in [HM], Proposition 4.9. Our approach follows the line of [BV1]. The isomorphism of Theorem 3.20 was previously obtained only for $0 \le i \le r-2$, and consequently only the formula $\ell(S_0(C)) = \ldots = \ell(S_{r-1}(C))$.

PROPOSITION 3.23. *Set $M_\varphi = \operatorname{Coker}\varphi$, and let $\bar{\psi} : M_\varphi \to F$ be the map induced by ψ. Consider the complex $C^{\cdot}_{\bar{\psi}}(r)$ where $C^{i}_{\bar{\psi}}(r) = \bigwedge^{r-i} M_\varphi \otimes S_i(F)$ for $i \ge 0$, $C^{-1}_{\bar{\psi}}(r) = \bigwedge^n M_\varphi$. For the homology $\widetilde{H}^{\cdot}$ of $C^{\cdot}_{\bar{\psi}}(r)$, the following holds:*

$$\widetilde{H}^{-1} = 0, \quad \widetilde{H}^{r} = S_r(C).$$

If $i+r$ is even, $-1 \le i \le r-2$, then

$$\ell(\widetilde{H}^{i}) = \ell(\widetilde{H}^{i+1}).$$

If r is odd, then $\widetilde{H}^0 = 0$.

Proof. We extend the Koszul bicomplex

$$K^{\cdot S(F)}_{S(H^*)\cdot}(\varphi, R, \psi, S(F))(r),$$

to the bicomplex $\mathcal{C}_{\cdot,\cdot}$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & N^{r-1} & \longrightarrow & N^r & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & \bigwedge^{r-1} G \otimes S_0(F) & \longrightarrow & \bigwedge^r G \otimes S_0(F) & \longrightarrow & \bigwedge^r M_\varphi \otimes S_0(F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & \bigwedge^{r-2} G \otimes S_1(F) & \longrightarrow & \bigwedge^{r-1} G \otimes S_1(F) & \longrightarrow & \bigwedge^{r-1} M_\varphi \otimes S_1(F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & \bigwedge^{r-3} G \otimes S_2(F) & \longrightarrow & \bigwedge^{r-2} G \otimes S_2(F) & \longrightarrow & \bigwedge^{r-2} M_\varphi \otimes S_2(F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Remark that the last column is $C^{\cdot}_{\bar{\psi}}(r)$, since $\bigwedge^n M_\varphi \cong k$, and that $\widetilde{H}^i$ is the homology at $\bigwedge^{r-i} M_\varphi \otimes S_i(F)$. From the first part of the proof of Theorem 3.20 we draw $\widetilde{H}^{-1} = 0$. It is easy to deduce also the isomorphism $\widetilde{H}^r \cong S_r(C)$.

As in Theorem 2.8, for $i + r$ even, $-1 \leq i \leq r - 2$, we obtain exact sequences

$$0 \to \widetilde{H}^i \to S_{\frac{i+r}{2}}(C) \to H_\psi^{\frac{i+r}{2}+1, \frac{i+r}{2}} \to \widetilde{H}^{i+1} \to 0.$$

As we proved in Theorem 3.20,

$$S_{\frac{i+r}{2}}(C) \cong H_\psi^{\frac{i+r}{2}+1, \frac{i+r}{2}},$$

so the length formula follows. □

CONJECTURE 3.24. *Let $C^{\cdot}_{\bar{\psi}}(r)$ be as above. For the homology $\widetilde{H}^{\cdot}$ of $C^{\cdot}_{\bar{\psi}}(r)$, the following holds:*

$$\widetilde{H}^i = \begin{cases} S_r(C) & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}$$

References

[B] W. Bruns. *The Buchsbaum-Eisenbud structure theorems and alternating syzygies*. Commun. Algebra **15** (1987), 873–925.

[BO1] N.Bourbaki.*Algebra I, Chapters 1-3*,Addison-Wesley 1974.

[BO2] N.Bourbaki.*Algèbre, Chapitre 3, Algèbre multilinéaire*,Act. Sci. et Ind. **1044**, Hermann 1958.

[BO3] N.Bourbaki.*Algèbre I, Chapitre 10, Algèbre homologique*, Masson 1980.

[BE1] D. Buchsbaum and D. Eisenbud. *Some Structure Theorems for Finite Free Resolutions*. Advances in Math. **12** (1974), 84–139.

[BE2] D. Buchsbaum and D. Eisenbud. *Generic Free Resolutions and a Family of Generically Perfect Ideals*. Advances in Math. **18** (1975), 245–301.

[BH] W.Bruns and J. Herzog. *Cohen-Macaulay Rings*. Cambridge Uni. Press 1996.

[BV1] W. Bruns and U. Vetter. *Length formulas for the Local Cohomology of Exterior Powers*. Math Z. **191** (1986), 145–158.

[BV2] W. Bruns and U. Vetter. *Determinantal rings*. Lect. Notes Math. **1327**, Springer 1988.

[BV3] W. Bruns and U. Vetter. *A Remark on Koszul Complexes*. Beitr. Algebra Geom. **39** (1998), 249–254.

[BV4] W. Bruns and U. Vetter. *The Koszul Complex in Projective Dimension One*. Geometric and Combinatorial Aspects of Commutative Algebra, J. Herzog and G. Restuccia eds, Marcel Dekker 2001, 89–98.

[CMZ] S. Caenepeel, G. Militaru and S. Zhu. *Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations*. Lect. Notes Math. **1787**, Springer 2002.

[E] D. Eisenbud *Commutative Algebra with a View Toward Algebraic Geometry*. Grad. Text in Math. **150**, Springer 1995.

[HM] J. Herzog and A. Martsinkovsky. *Glueing Cohen-Macaulay modules with applications to quasihomogeneous complete intersections with isolated singularities*. Comment. Math. Helv. **68** (1993), 365–384.

[MNP] J.C. Migliore, U. Nagel, and C. Peterson. *Buchsbaum-Rim sheaves and their multiple sections*. J. Algebra **219** (1999), 378–420.

[N] D.G. Northcott. *Finite free resolutions*. Cambridge Uni. Press **71**, 1976.

[Na] I. Naruki. *Some remarks on isolated singularity and their application to algebraic manifolds*. Publ. Res. Inst. Math. Sci. **13**, (1977), 17–46.

[R] J.J. Rotman. *An Introduction to Homological Algebra*. Academic Press, 1979.

[V] U. Vetter. *Zu einem Satz von G. Trautmann über den Rang gewisser kohärenter analytischer Garben*. Arch. Math. **24** (1973), 158–161.

Curriculum Vitae

Name:	Ichim Bogdan Nicolae
Born:	23-X-1976 in Galati, Romania
Nationality:	Romanian

Education:

1983-1991	School: Scoala 28, Galati
1991-1995	High school: Liceul Vasile Alecsandri, Galati
1995	High school Diploma (Bacalaureat)
1995-2001	Student at the University of Bucharest, Faculty of Mathematics
1999	Diploma in Mathematics (Licenta)
2001	Master in Mathematics (Studii Aprofundate)
1999-2001	Student at the University "Politehnica" Bucharest, Faculty of Computer Science
2001-2004	Candidate for Doctor of Science at the University of Oldenburg

Employment:

1999-2001	Teaching Assistant at the University "Politehnica" Bucharest
2002-2004	Tutor at the University of Oldenburg
since 2001	Assistant Researcher at the Institute of Mathematics of Romania

Awards:

1994	III-rd prise at the Romanian Mathematical Olympiad
1995	II-nd prise at the Romanian Mathematical Olympiad
1995	Member of the Romanian big team for the International Mathematical Olympiad

Eidesstattliche Erklärung

Die vorliegende Dissertation ist selbständig verfasst und es wurden nur die angegebenen Hilfsmittel benutzt. Die Dissertation ist bisher weder in Teilen noch in Gänze veröffentlicht worden.

Die Dissertation ist weder in in ihrer Gesamtheit noch in Teilen einer anderen wissenschaftlichen Hochschule zur Begutachtung in einem Promotionsverfahren vorgelegt worden.

Oldenburg, 25-I-2005

Unterschrift: ....