ALGEBRAIC STRUCTURE OF ENDOMORPHISM MONOIDS OF FINITE GRAPHS

APIRAT WANICHSOMBAT

CARL VON OSSIETZKY UNIVERSITÄT OLDENBURG

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von Herrn Apirat Wanichsombat

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Gutachter Professor Dr. Dr. h.c. Ulrich Knauer
Zweitgutachter Professor Dr. Andreas Stein

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Contents

Abstract

Our aim in this dissertation is studying the relationship between semigroup theory and graph theory. Since it is well known that $End(G)$, the set of all endomorphisms of graph is a monoid, we consider the algebraic structures, such as regular, completely regular, orthodox, Clifford semigroup, etc., in this endomorphism monoid. Since it is very complicated to characterize the algebraic structures for the monoids of any graph, we study the algebraic structure of the monoid of some special graphs. We hope that the results on this special graphs will lead the way to characterize algebraic structures of the monoids of other graphs.

Except the monoids $End(G)$ and $SEnd(G)$, the set of all strong endomorphisms of a graph G, it is well known that

- $HEnd(G)$ the set of all half strong endomorphisms of a graph G and
- $LEnd(G)$ the set of all locally strong endomorphisms of a graph G and
- $QEnd(G)$ the set of all quasi-strong endomorphisms of a graph G

are not necessarily semigroups. In this dissertation, we concentrate on cycles, to find when the set of all non-trivial locally strong endomorphisms of the cycles of even length ($LEnd'(C_{2n}) = LEnd(C_{2n}) \setminus Aut(C_{2n})$) is a semigroup.

In this dissertation, we give some method to construct the completely regular subsemigroup of the regular endomorphism monoids of split graphs. We also give some examples of retractive graphs (graphs whose endomorphism monoids and automorphism groups are not equal) whose endomorphism monoids are Clifford semigroups.

Moreover, we considered two graph operations, unions and joins. In this part, we focused on two things. The first one is finding when the monoid of unions of two graphs $End(G \cup H)$ is isomorphic to the sum of two endomorphism monoids $End(G) + End(H)$. Similarly, we also find when the monoid of joins of two graphs $End(G + H)$ is isomorphic to the sum of two endomorphism monoids $End(G) + End(H)$. We did not only consider on the monoids $End(G \cup H)$ and $End(G + H)$, we also considered the sets $HEnd(G \cup H)$, $HEnd(G + H)$, $LEnd(G \cup H)$, $LEnd(G + H)$, $QEnd(G \cup H)$, $QEnd(G + H)$, $SEnd(G \cup H)$, $SEnd(G + H)$, $Aut(G \cup H)$ and $Aut(G + H)$. The last topic are the unretractivities of the unions of two connected graphs $G \cup H$ and of the joins of two connected graphs $G + H$.

Abstract

Unser Ziel in dieser Dissertation ist die Untersuchung der Beziehung zwischen der Halbgruppen Theorie und der Graphen Theorie. Da es bekannt ist, dass $End(G)$ die Menge aller Endomorphismen von Graphen ein Monoid ist, konzentrieren wir uns auf die algebraischen Strukturen, wie regulär, vollständig regulär, orthodox oder Clifford Halbgruppen. Da die allgemeine Situation zu kompliziert ist, studieren wir die algebraische Struktur auf dem Monoid einiger spezieller Graphen.

Außer der Monoide $End(G)$ und $SEnd(G)$ die Menge aller starken Endomorphismen eines Graphen, ist es bekannt, dass

- $HEnd(G)$ die Menge aller halbstarken Endomorphismen eines Graphen G und

- $LEnd(G)$ die Menge aller lokal starken Endomorphismen eines Graphen G und

- $QEnd(G)$ die Menge aller quasi-starken Endomorphismen eines Graphen G

nicht notwendigen Halbgruppen werden. In dieser Arbeit konzentrieren wir uns auf die Zyklen, für die die Menge aller nicht-triviale lokal stark Endomorphismen ($LEnd'(C_{2n}) = LEnd(C_{2n}) \setminus Aut(C_{2n})$) eine Halbgruppe ist.

In dieser Arbeit geben wir eine Methode, die vollständig regulären Unterhalbgruppen der regulären Endomorphismen Monoide von Split Graphen zu konstruieren. Wir geben auch einige Beispiele von retraktiven Graphen (Graphen, deren Endomorphismen Monoide und Automorphismen Gruppen nicht gleich sind), deren Endomorphismen Monoide Clifford Halbgruppen sind.

Darüber hinaus betrachtet man zwei Graphen Operationen, Vereinigung und Verbindung. In diesem Teil konzentrieren wir uns auf zwei Dinge. Das erste ist, wann das Monoid der Vereinigung von zwei Graphen $End(G \cup H)$ isomorph zu der Summe zweier Endomorphismen Monoide $End(G) + End(H)$ ist. Ebenso wann das Monoid $End(G + H)$ isomorph zu der Summe zweier Endomorphismen Monoide ist. Wir haben nicht nur die Monoide $End(G \cup H)$ und $End(G + H)$ geprüft, sondern auch die Mengen $HEnd(G \cup H)$, $HEnd(G + H)$, $LEnd(G \cup H)$, $LEnd(G + H)$, $QEnd(G \cup H)$, $QEnd(G + H)$, $SEnd(G \cup H)$, $SEnd(G + H)$, $Aut(G \cup H)$ und $Aut(G + H)$ betrachtet. Als letztes betrachten wir die Unretraktivitäten der Graphen $G \cup H$ und $G + H$.

Summary

In this dissertation, we study the relationship between semigroup theory and graph theory. Ulrich Knauer and Elke Wilkeit questioned for which graph G is the endomorphism monoid of G regular (see in, L. Marki, *Problems raised at the problem session of the Colloqium on Semigroups in Szeged, August 1987*, Semigroup Forum, 37 (1988), 367-373.). After this question was posed, the regularity of $End(G)$ is investigated and for the monoid $SEnd(G)$ of all strong endomorphisms of G is proved that it is always regular. Furthermore, other algebraic properties such as completely regular, orthodox, etc., of $End(G)$ and $SEnd(G)$ are studied.

It is too complicated to characterize graphs G whose $End(G)$ is regular. So, many researchers concentrated on the regularity of the endomorphism monoids of special graphs. We also study the endo-regularity of special graphs. In this dissertation, we stated the following lemma which we use to prove endo-regularity of a connected graph.

Lemma 2.1.4 *Let f be endomorphism of a connected graph G. Let $Im(f)$ be the strong subgraph of G with $V(Im(f)) = f(G)$. If G is endo-regular, then $Im(f)$ is endo-regular.*

For the complete regularity of an endomorphism f of G, we got an inspiration from some proposition of Weimin Li's work "W. Li, *Split Graphs with Completely Regular Endomorphism Monoids*, Journal of mathematical research and exposition, **26** (2006), 253-263 " and proved the following theorem describing when an endomorphism f of any graph is completely regular.

Theorem 2.2.3 *Let G be a finite graph and f be an endomorphism of G. Then f is completely regular if and only if for all $a, b \in V(G)$, $f(a) \neq f(b)$ implies $f^2(a) \neq f^2(b)$, i.e., f is square injective. In this case, if f is not idempotent, we have $ff^{i-1}f = f$ and $ff^{i-1} = f^{i-1}f$ where f^i is the smallest idempotent power of f.*

For the idempotent closed endomorphism monoid of a graph, we gave some lemmas and corollaries and describe graphs whose endomorphism monoids are not idempotent closed (see Lemma 2.3.1 and Corollary 2.3.3 in this dissertation).

After we had all above properties, we used them to prove the regularity, the complete regularity, the idempotent closed property and other algebraic

properties of endomorphism monoids of special graphs. In this dissertation, we considered bipartite graphs, -graphs, multiple 8-graphs, split graphs (see the definitions in the dissertation). We have the results as in the following table.

Graph G	Connected Bipartite graph	Multiple 8-graph	Split graph $G = K_n \cup I_r$
$End(G)$ is regular	$\Leftrightarrow$ G is • $K_{m,n}$ • K_1, K_2, C_6, C_8, P_4 • the trees of diameter 3.	$\Leftrightarrow$ G is • $C_{(2n+1)^{(t)}}; P_r$ where $r \geq 0$, $t \geq 2$ • $C_{(2n+1)^{(t)},4}; P_1$ where $t \geq 1$ • $C_{(4)^{(s)}}; P_2$, $s \geq 2$.	$\Leftrightarrow$ for all $a \in I_r$ one has $\|N(a)\| = d$ where $d \in \{0, 1, ...$ $, n-1\}$.
$End(G)$ is completely regular	$\Leftrightarrow$ G is one of P_1, P_2, P_3, C_4, C_6.	$\Leftrightarrow$ G is $C_{(2n+1)^{(t)}}; P_r$ where $n \geq 1$, $t \geq 2$, $r \geq 0$.	$\Leftrightarrow$ $r = 1$.
$End(G)$ is idempotent closed	$\Leftarrow$ if G is a bipartite graph P_1, P_2, P_3, C_4.	$\Leftrightarrow$ G is • $C_{n_1,n_2,...,n_s}; P_r$ where $r \geq 0$ and $n_i \neq n_j$ are odd for $i \neq j \in 1, 2, ..., s$ • $C_{2n+1,2n+1}; P_r$ where $n \geq 1$, $r > 0$.	
$End(G)$ is orthodox	$\Leftrightarrow$ G is one of P_1, P_2, P_3, C_4.	$\Leftrightarrow$ G is $C_{2n+1,2n+1}; P_r$ where $n \geq 1$, $r > 0$.	
$End(G)$ is Clifford	$\Leftrightarrow$ G is K_2.	Never	Never

Furthermore, we consider the set of all locally strong endomorphisms of a graph which is not necessarily closed. In this dissertation, we found when the set of all non-trivial locally strong endomorphisms of cycles is closed. We also found a method to construct the completely regular subsemigroups of endo-regualr split graphs. We gave some examples of retractive graphs whose endomorphism monoids are Clifford.

Moreover, we considered two graph operations, unions and joins. In this part, we focused on two things. The first one is finding when the monoid of unions of two graphs $End(G \cup H)$ is isomorphic to the sum of two endomorphism monoids $End(G) + End(H)$. Similarly, we also find when the monoid of joins of two graphs $End(G+H)$ is isomorphic to the sum of two endomorphism monoids $End(G) + End(H)$. We did not only consider the monoids $End(G \cup H)$ and $End(G + H)$, we also considered the sets $HEnd(G \cup H)$, $HEnd(G+H)$, $LEnd(G \cup H)$, $LEnd(G+H)$, $QEnd(G \cup H)$, $QEnd(G+H)$, $SEnd(G \cup H)$, $SEnd(G+H)$, $Aut(G \cup H)$ and $Aut(G+H)$. We considered only connected graphs. We got the results as the following tables.

$\mathfrak{M}$	$\mathfrak{M}(G \cup H) \cong \mathfrak{M}(G) + \mathfrak{M}(H)$
End	$\Leftrightarrow Hom(G,H) = \emptyset$ and $Hom(H,G) = \emptyset$.
$HEnd$	$\Leftrightarrow HHom(G,H) = \emptyset$ and $HHom(H,G) = \emptyset$.
$LEnd$	$\Leftrightarrow LHom(H,G) = \emptyset$ and for all $g \in LHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ and $g(G) \neq h(H)$ for all $h \in LEnd(H)$ and vice versa.
$QEnd$	$\Leftrightarrow QHom(H,G) = \emptyset$ and for all $g \in QHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ for all $h \in QEnd(H)$ and vice versa.
$SEnd$	$\Leftrightarrow SHom(G,H) = \emptyset$ or $SHom(H,G) = \emptyset$.
Aut	$\Leftrightarrow Iso(G,H) = \emptyset \Leftrightarrow G$ is not isomorphic to H.

$\mathfrak{M}$	$\mathfrak{M}(G + H) \cong \mathfrak{M}(G) + \mathfrak{M}(H)$
End, $HEnd$, $LEnd$, $QEnd$, $SEnd$	$\Leftrightarrow f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}(G+H)$
Aut	$\Leftrightarrow f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in Aut(G+H)$ $\Leftrightarrow Iso(\overline{G}_i, \overline{H}_j) = \emptyset$ for all components $\overline{G}_i$ of G and $\overline{H}_j$ of H.

The last topic are the unretractivities of the unions of two connected graphs $G \cup H$ and the unretractivities of the joins of two connected graphs $G + H$. The results are in the following two tables.

<table>
<tr><td>$=$</td><td>$Aut(G \cup H)$</td><td>$SEnd(G \cup H)$</td></tr>
<tr><td>$SEnd(G \cup H)$</td><td>G, H are S-unretractive</td><td></td></tr>
<tr><td>$QEnd(G \cup H)$</td><td>G, H are Q-unretractive</td><td></td></tr>
<tr><td>$LEnd(G \cup H)$</td><td>$\Rightarrow$ G, H are L-unretractive and
$(LHom(G,H) = \emptyset$ or $LHom(H,G) = \emptyset)$
$\Leftarrow$ G, H are L-unretractive and
$LHom(G,H) = \emptyset$ and $LHom(H,G) = \emptyset$</td><td></td></tr>
<tr><td>$HEnd(G \cup H)$</td><td rowspan="2">G, H are unretractive and
$Hom(G,H) = Hom(H,G) = \emptyset$</td><td rowspan="2">$G, H$ are E-S-unretractive and $Hom(G,H) = Hom(H,G) = \emptyset$.</td></tr>
<tr><td>$End(G \cup H)$</td></tr>
</table>

$=$	$Aut(G+H)$	$SEnd(G+H)$
$SEnd(G+H)$	G, H are S-unretractive	
$QEnd(G+H)$	G, H are Q-unretractive	
$LEnd(G+H)$	G, H are L-unretractive	
$HEnd(G+H)$	G, H are H-unretractive	G, H are H-S-unretractive
$End(G+H)$	G, H are unretractive	G, H are E-S-unretractive

Zusammenfassung

In dieser Arbeit untersuchten wir die Beziehung zwischen der Halbgruppetheorie und der Graphentheorie. Ulrich Knauer und Elke Wilkeit fragten für welche Graphen das Endomorphismenmonoid regulär ist (vgl. L. Marki, *Problems raised at the problem session of the Colloqium on Semigroups in Szeged, August 1987*, Semigroup Forum, **37** (1988), 367-373.). Nach dieser Frage, wird die Regularität von $SEnd(G)$ untersucht und bewiesen, dass es immer regulär ist. Andere algebraische Eigenschaften wie vollständig regulär, orthodox, etc. werden ebenfalls untersucht. Viele Forscher haben sich auf die Regularität der Endomorphismen Monoide spezieller Graphen konzentriert. In dieser Arbeit haben wir, das folgende Lemma, das wir verwenden, um Endo-Regularität eines zusammenhängenden Graphen zu beweisen.

Lemma 2.1.4 *Sei f ein Endomorphismus eines zusammenhängenden Graphen G. Sei $Im(f)$ der starke Teilgraph von G mit $V(Im(f)) = f(G)$. Wenn G Endo-regulär ist, dann ist $Im(f)$ Endo-regulär.*

Für die vollständige Regularität eines Endomorphismus f eines G bekamen wir eine Inspiration von einigen Sätzen von Li Weimin in "W. Li, Split Graphs with Completely Regular Endomorphism Monoids, Journal of mathematical research and exposition, 26 (2006), 253-263".

Satz 2.2.3 *Sei G ein endlicher Graph und f ein Endomorphismus von G. Dann ist f vollständig regulär, wenn für alle $a, b \in V(G)$ mit $f(a) \neq f(b)$ folgt daß $f^2(a) \neq f^2(b)$, d.h. f ist square injektiv. In diesem Fall, wenn f nicht idempotent ist, haben wir $ff^{i-1}f = f$ und $ff^{i-1} = f^{i-1}f$ wo f^i die kleinste idempotent Potenz von f ist.*

Für die idempotent abgeschlossenen Endomorphismen des Endomorphismmonoids, haben wir einige Lemmata und Folgerungen und Beispiele für Graphen, deren Endomorphismen Monoide nicht idempotent abgeschlossen sind (siehe Lemma 2.3.1 und Korollar 2.3.3 in dieser Arbeit).

Nachdem wir alle oben genannten Eigenschaften hatten, haben wir die Regularität, die vollständige Regularität und anderen algebraischen Eigenschaften der Endomorphismus Monoiden spezieller Graphen untersucht. In dieser Dissertation untersuchten wir bipartiten Graphen, 8-Graphen , multiple 8-Graphen (siehe die Definitionen in der Dissertation). Wir hatten die

Ergebnisse wie in der folgende Tabelle.

Graph G	Connected Bipartite graph	Multiple 8-graph	Split graph $G = K_n \cup I_r$
$End(G)$ is regular	$\Leftrightarrow G$ is • $K_{m,n}$ • K_1, K_2, C_6, C_8, P_4 • the trees of diameter 3.	$\Leftrightarrow G$ is • $C_{(2n+1)^{(t)}}; P_r$ where $r \geq 0$, $t \geq 2$ • $C_{(2n+1)^{(t)},4}; P_1$ where $t \geq 1$ • $C_{(4)^{(s)}}; P_2$, $s \geq 2$.	$\Leftrightarrow$ for all $a \in I_r$ one has $\lvert N(a)\rvert = d$ where $d \in \{0, 1, ..., n-1\}$.
$End(G)$ is completely regular	$\Leftrightarrow G$ is one of P_1, P_2, P_3, C_4, C_6.	$\Leftrightarrow G$ is $C_{(2n+1)^{(t)}}; P_r$ where $n \geq 1$, $t \geq 2$, $r \geq 0$.	$\Leftrightarrow r = 1$.
$End(G)$ is idempotent closed	$\Leftarrow$ if G is a bipartite graph P_1, P_2, P_3, C_4.	$\Leftrightarrow G$ is • $C_{n_1,n_2,...,n_s}; P_r$ where $r \geq 0$ and $n_i \neq n_j$ are odd for $i \neq j \in 1, 2, ..., s$ • $C_{2n+1,2n+1}; P_r$ where $n \geq 1$, $r > 0$.	
$End(G)$ is orthodox	$\Leftrightarrow G$ is one of P_1, P_2, P_3, C_4.	$\Leftrightarrow G$ is $C_{2n+1,2n+1}; P_r$ where $n \geq 1$, $r > 0$.	
$End(G)$ is Clifford	$\Leftrightarrow G$ is K_2.	Never	Never

Darüber hinaus betrachten wir die Menge aller lokal stark Endomorphismen eines Graphen, die nicht unbedingt abgeschlossen ist. In dieser Arbeit haben wir festgestellt, wann die Menge aller nicht-trivialen lokal starken Endomorphismen von Zyklen abgeschlossen ist. Wir fanden auch eine Methode, um die vollständig regulären Unterhalbgruppen von Endo-reguläre Split Graphen zu konstruieren. Wir haben einige Beispiele von retraktiven Graphen, deren Endomorphismen Monoide Clifford sind.

Darüber hinaus betrachten wir zwei Graphen Operationen, Vereingung und Verbindung. In diesem Teil konzentrieren wir uns auf zwei Dinge. Das erste ist, wann $End(G \cup H)$ isomorph zu der Summe zweier Endomorphismen Monoide $End(G) + End(H)$ ist. Ebenso, wann $End(G + H)$ isomorph zu der Summe zweier Endomorphismen Monoide $End(G) + End(H)$ ist. Wir haben auch die Mengen $HEnd(G \cup H)$, $HEnd(G + H)$, $LEnd(G \cup H)$, $LEnd(G +$

$H)$, $QEnd(G \cup H)$, $QEnd(G+H)$, $SEnd(G \cup H)$, $SEnd(G+H)$ $Aut(G \cup H)$, $Aut(G+H)$ betrachtet und zwar, nur für die zusammenhängende Graphen. Wir haben Ergebnisse wie in die folgenden Tabellen.

$\mathfrak{M}$	$\mathfrak{M}(G \cup H) \cong \mathfrak{M}(G) + \mathfrak{M}(H)$
End	$\Leftrightarrow Hom(G,H) = \emptyset$ and $Hom(H,G) = \emptyset$.
$HEnd$	$\Leftrightarrow HHom(G,H) = \emptyset$ and $HHom(H,G) = \emptyset$.
$LEnd$	$\Leftrightarrow LHom(H,G) = \emptyset$ and for all $g \in LHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ and $g(G) \neq h(H)$ for all $h \in LEnd(H)$ and vice versa.
$QEnd$	$\Leftrightarrow QHom(H,G) = \emptyset$ and for all $g \in QHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ for all $h \in QEnd(H)$ and vice versa.
$SEnd$	$\Leftrightarrow SHom(G,H) = \emptyset$ or $SHom(H,G) = \emptyset$.
Aut	$\Leftrightarrow Iso(G,H) = \emptyset \Leftrightarrow G$ is not isomorphic to H.

$\mathfrak{M}$	$\mathfrak{M}(G+H) \cong \mathfrak{M}(G) + \mathfrak{M}(H)$
End, $HEnd$, $LEnd$, $QEnd$, $SEnd$	$\Leftrightarrow f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}(G+H)$
Aut	$\Leftrightarrow f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in Aut(G+H)$ $\Leftrightarrow Iso(\overline{G}_i, \overline{H}_j) = \emptyset$ for all components $\overline{G}_i$ of G and $\overline{H}_j$ of H.

Zuletzt haben wir Unretractivitäten $G \cup H$ und $G+H$ für zusammenhängende Graphen untersucht. Die Ergebnisse sind in den folgenden beiden Tabellen.

<table>
<tr><td>$=$</td><td>$Aut(G \cup H)$</td><td>$SEnd(G \cup H)$</td></tr>
<tr><td>$SEnd(G \cup H)$</td><td>G, H are S-unretractive</td><td></td></tr>
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$=$	$Aut(G+H)$	$SEnd(G+H)$
$SEnd(G+H)$	G, H are S-unretractive	
$QEnd(G+H)$	G, H are Q-unretractive	
$LEnd(G+H)$	G, H are L-unretractive	
$HEnd(G+H)$	G, H are H-unretractive	G, H are H-S-unretractive
$End(G+H)$	G, H are unretractive	G, H are E-S-unretractive

Introduction

One of the main trends in the theory of semigroups is the study of mathematical objects by means of certain semigroups connected with the objects in a special way. At present, there are several studies focusing on the semigroups of mappings of graphs (cf. [3]-[8], [11]-[15], [19]-[28], [32]-[34]). The endomorphism monoid $End(G)$ and the strong endomorphism monoid $SEnd(G)$ of any graph G are studied. $SEnd(G)$ is always regular, regularity of $End(G)$ is investigated after Knauer and Wilkeit questioned for which graph G is the endomorphism monoid of G regular [29]. Furthermore, other algebraic properties such as completely regular, orthodox, etc., of these two monoids are studied. Moreover, the sets $HEnd(G)$, $LEnd(G)$, $QEnd(G)$ are studied when they are monoids. In this dissertation, we continue to study these things.

The preliminary concepts and terminologies which will be used in this dissertation are given in Chapter 1, while Chapters 1.4, 2, 3, 4 and 5 concentrate on algebraic properties of endomorphism monoids of graphs and Chapters 6 and 7 focus on graph operations.

In Chapter 1.4, we introduce results with respect to the regularity and complete regularity of endomorphisms of graphs from [23], [26], [27], and [34]. We give results usefull for the study of the regularity of endomorphism monoids of graphs. We give a new way for investigating the complete regularity of endomorphisms of graphs.

In Chapters 2, 3, and 4 we introduce bipartite graphs, 8-graphs, and split graphs, respectively. The algebraic properties of endomorphism monoids of each graph will be obtained. A retractive graph whose endomorphism monoid is a Clifford semigroup was not found from these tree graphs. So, we gave examples of retractive graphs whose endomorphism monoids are Clifford semigroup in Chapter 5.

Additional, in Chapter 2 the sets of all non-trivial locally strong endomorphisms of cycles $LEnd'(C_{2n})$ are considered, when do they form semigroups? In Chapter 4, we find completely regular subsemigroups contained in the endomorphism monoids of split graphs. A part in this chapter has been accepted by the journal ***Ars Combinatoria*** for publication and another part has been published in: ***Semigroups, Acts and Categories with Applications to Graphs, Proceedings,*** Tartu 2007, 136-142.

For Chapter 5, an aim is to find examples of retractive graphs whose endomorphism monoids are Clifford semigroup. We get retractive endo-Clifford graphs by stating from rigid graphs and unretractive graphs.

Chapter 6, we find the conditions under which for two graphs G and H the endomorphsim monoid $End(G \cup H)$ (or $End(G + H)$) is isomorphic to the sum $End(G) + End(H)$ of endomorphism monoids. In particular, we also find the conditions for the sets of half strong endomorphisms, the sets of locally strong endomorphisms, the sets of quasi-strong endomorphisms, the sets of strong endomorphisms and the set of automorphisms.

In Chapter 7, we study unretractivities of a union of two graphs and a join of two graphs.

Open problems and further questions will be discussed at the respective places. All graphs, groups and semigroups in this dissertation are finite.

Chapter 1

Preliminaries

In this chapter, we describe concepts and terminologies from semigroup theory, categories, and graph theory which will be used in this dissertation. Specific definitions and notations will be given for more clarification where they appear. Other basic concepts which are not defined in this study can be found in [10], [17], [18] and [31].

1.1 Semigroup theory

We start with semigroup concepts.

Definition 1.1.1. A set S together a binary operation, usually called ***multiplication***, is a ***groupoid***. A groupoid S satisfying the ***associative law***

$$a(bc) = (ab)c \quad (a, b, c \in S)$$

is a ***semigroup***. A semigroup having only one element is ***trivial***.

Definition 1.1.2. An element e of a semigroup S is a ***left*** (respectively ***right***) ***identity*** if $es = s$ (respectively $se = s$) for all $s \in S$. Further, e is a ***two-sided identity*** (or simply ***identity***) of S if it is both a left and a right identity of S. A semigroup S with an identity is a ***monoid***. If S is a monoid, the maximal subgroup of S whose identity is the identity of S is the ***group of units*** of S; its elements are the ***invertible element*** (or ***units***) of S.

One may always ***adjoin an identity*** to a semigroup S by letting $e \notin S$ and declaring on $S \cup \{e\}$ the multiplication in S and

$$es = se = s \quad (s \in S \cup \{e\}).$$

Let $S = S^1$ if S has an identity, otherwise let S^1 be the semigroup S with an identity adjoined. The identity of any monoid is usually denoted by 1. We denote by the symbol 1 any trivial (semi)group.

Definition 1.1.3. An element z of S is a ***left*** (respectively ***right***) ***zero*** of S if $zs = z$ (respectively $sz = z$) for all $s \in S$. Further, z is a ***two-sided zero*** (or simply ***zero***) of S if it is both a left and a right zero of S. If S has a zero and all products are equal to zero, S is a ***null semigroup***. We call S a ***left (right) zero semigroup*** if its elements are left (right) zero.

Denote L_n (R_n) the left (right) zero semigroup with n elements. Left groups are of the form $G \times L_n$, i.e., they are the unions of n copies of an arbitrary (finite) group G, analogously $G \times R_n$ for right groups, with the multiplication as given by $G \times L_n$ or $G \times R_n$.

Let S be a semigroup with zero 0. Then $S* = S \setminus \{0\}$ denotes the partial groupoid in which only the products ab are defined where $ab \neq 0$ in S.

Definition 1.1.4. A nonempty subset T of S is a ***subsemigroup*** of S if T is closed under the multiplication of S; if also T is a group under the induced operation, it is a ***subgroup*** of S.

Definition 1.1.5. An element e of a semigroup S is ***idempotent*** if $e^2 = e$. A semigroup is ***idempotent*** , or is a ***band***, if all its elements are idempotent. Two elements a and b of S ***commute*** if $ab = ba$; S is ***commutative*** if any two elements of S commute. A commutative idempotent semigroup is a ***semilattice***.

Definition 1.1.6. Let Y be a semilattice. For each $\alpha \in Y$, let S_α be a semigroup and assume that $S_\alpha \cap S_\beta \neq \emptyset$. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\chi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism such that

(1) $\chi_{\alpha,\alpha} = i_{S_\alpha}$

(2) $\chi_{\alpha,\beta}\chi_{\beta,\gamma} = \chi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$.

On $S = \bigcup\limits_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a\chi_{\alpha,\alpha\beta})(b\chi_{\beta,\alpha\beta}) \ (a \in S_\alpha,\, b \in S_\beta).$$

With this multiplication S is a ***strong semilattice*** Y ***of semigroups*** S_α ***(given by the structure homomorphisms*** $\chi_{\alpha,\beta}$***)***, to be denoted by $S = [Y; S_\alpha, \chi_{\alpha,\beta}]$.

Definition 1.1.7. An element a of a semigroup S is ***regular*** if $a = axa$ for some $x \in S$; S is ***regular*** if all its elements are regular.

Definition 1.1.8. An element a of a semigroup S is ***completely regular*** if $a = axa$ and $xa = ax$ for some $x \in S$; S is ***completely regular*** if all its elements are completely regular. A semigroup S is ***Clifford semigroup*** if it is completely regular and its idempotents commute with all elements of S; alternatively, we say that its ***idempotent are central*** or that they are ***n the center***.

Theorem 1.1.9. ([31]) *The following conditions on a semigroup S are equivalent:*
(i) *S is completely regular.*
(ii) *S is a union of (disjoint) groups.*

Theorem 1.1.10. ([17]) *Let S be a semigroup. Then the following statements are equivalent:*
(i) *S is a Clifford semigroup;*
(ii) *S is a semilattice of groups;*
(iii) *S is a strong semilattice of groups.*

Definition 1.1.11. A regular semigroup S is ***orthodox*** if its idempotents form a subsemigroup. An orthodox completely regular semigroup is called an ***orthogroup***.

Definition 1.1.12. Let S be a monoid and $A \neq \emptyset$ a set. If we have a mapping μ from $S \times A$ to A defined by $\mu(s, a) = sa$ such that $1a = a$ and $(st)a = s(ta)$ for $a \in A$, $s, t \in S$, we call A a ***left S-act*** or ***left act*** over S and write ${}_SA$ or (S, A).

Definition 1.1.13. Take monoids M, N and left acts (M, G), (N, H). The sum of monoids $M + N := \{m + n \mid m \in M, n \in N\}$ has multiplication defined by $(m+n)(m'+n') := mm' + nn'$ and identity $id_M + id_N$. The sum $M + N$ operates on $G \cup H$ as follows: $(m + n)x := mx$, $(m + n)y := ny$ for all $x \in G$, $y \in H$, $m \in M$ and $n \in N$. This way we get the left act $(M + N, G \cup H)$.

Definition 1.1.14. Let S be a semigroup and let a and b be two elements of S. Define a relation $\mathcal{L}$ on S such that $(a, b) \in \mathcal{L}$ if and only if $S^1a = S^1b$, here $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$ otherwise: similarly, define a relation $\mathcal{R}$ on S such that $(a, b) \in \mathcal{R}$ if and only if $aS^1 = bS^1$. $\mathcal{L}$ and $\mathcal{R}$ are equivalent relations on S. The relation $\mathcal{L}$ is a ***left congruence*** and the relation $\mathcal{R}$ is a ***right congruence***. Define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. Denote by $[a]_{\mathcal{L}}$

(respectively, $[a]_{\mathcal{R}}$ and $[a]_{\mathcal{H}}$) the $\mathcal{L}$-class (respectively, $\mathcal{R}$-class and $\mathcal{H}$-class) of a in S.

1.2 Graph theory

Our graphs in this dissertation are usually undirected graphs without loops and multiple edges.

Definition 1.2.1. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set respectively. A graph H is called a ***subgraph*** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $a, b \in V(G)$. The vertices a and b are said to be adjacent if $\{a, b\} \in E(G)$.

Definition 1.2.2. Let G be a graph. Denote $N_G(v) := \{x \in H | \{x, v\} \in E(H)\}$, call it the ***neighborhood*** of v in G; use $N(v)$ for $N_G(v)$ if it is clear which graph G is referred to.

Definition 1.2.3. A graph G is ***complete*** if any two of its vertices are adjacent. A graph G is called an ***empty graph*** if $E(G) = \emptyset$. Denote by K_n (respectively $\overline{K}_n$) a complete graph (respectively an empty graph) with n vertices. A graph G is n***-partite*** $(n \geq 1)$ if it is possible to partition $V(G)$ into n subsets V_1, V_2,..., V_n such that every edges of G joins a vertex of V_i to a vertex of V_j $(i \neq j)$. If $n = 2$, then G is called a ***bipartite graph***. A ***complete bipartite graph***, denote by $K_{m,n}$ for $|V_1| = m$ and $|V_2| = n$, is a bipartite graph such that for any $a_1 \in V_1$ and for any $a_2 \in V_2$, $\{a_1, a_2\} \in E(K_{m,n})$.

Definition 1.2.4. A vertex a of a graph G is said to be ***connected*** to a vertex b if there exist a sequence of pairwise distinct vertices $a = a_0, a_1, ..., a_n = b \in V(G)$ such that $n \geq 1$ and $\{a_i, a_{i+1}\} \in E(G)$ for any $i \in \{0, 1, ..., n-1\}$. This vertex sequence with the edges among them is called *a-b path*, denoted by P_n, and n, the number of edges among them, is called its ***length***. A graph is ***connected*** if every two of its vertices are connected. A ***component*** of a graph G is a maximal connected subgraph of G. The ***distance*** between two vertices a and b, denoted by $d(a, b)$, is the minimum of the lengths of a-b paths of G.

Definition 1.2.5. An ***independent set*** or ***stable set*** is a set of vertices in a graph no two of which are adjacent. That is, it is a set I of vertices such that for every two vertices in I, there is no edge connecting the two. Equivalently, each edge in the graph has at most one endpoint in I. The size of an independent set is the number of vertices it contains.

Definition 1.2.6. A ***clique*** in an undirected graph $G = (V, E)$ is a subset of the vertex set $C \subseteq V$ such that for every two vertices in C, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by C is complete (in some cases, the term clique may also refer to the subgraph).

A ***maximal clique*** is a clique that cannot be extended by adding one more vertex, and a maximum clique is a clique of the largest possible size in a given graph. The clique number $\omega(G)$ of a graph G is the number of vertices in the largest clique in G. The opposite of a clique is an independent set, in the sense that every clique corresponds to an independent set in the complement graph. The clique cover problem concerns finding as few cliques as possible that include every vertex in the graph.

Definition 1.2.7. Let G_1 and G_2 be two graphs with disjoint vertex sets. The ***union*** of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph such that
$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and
$E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.
The ***join*** of G_1 and G_2, denoted by $G_1 + G_2$, is a graph such that
$V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and
$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{\{a, b\} \mid a \in G_1, b \in G_2\}$.
The ***box product*** of G_1 and G_2, denoted by $G_1 \Box G_2$, is a graph such that
$V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and
$E(G_1 \Box G_2) = \{\{(a_1, a_2), (a_1', a_2')\} \mid (a_1 = a_1'$ and $\{a_2, a_2'\} \in E(G_2))$ or $(\{a_1, a_1'\} \in E(G_1)$ and $a_2 = a_2')\}$.
The ***cross product*** of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph such that
$V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and
$E(G_1 \times G_2) = \{\{(a_1, a_2), (a_1', a_2')\} \mid \{a_1, a_1'\} \in E(G_1)$ and $\{a_2, a_2'\} \in G_2\}$.
The ***lexicographic product*** (or ***composition***) of G_1 and G_2, denoted by $G_1[G_2]$, is a graph such that
$V(G_1[G_2]) = V(G_1) \times V(G_2)$ and
$E(G_1[G_2]) = \{\{(a_1, a_2), (a_1', a_2')\} \mid \{a_1, a_1'\} \in E(G_1)$ or $a_1 = a_1'$ and $\{a_2, a_2'\} \in E(G_2)\}$.

Definition 1.2.8. Let G and H be graphs. An adjacency preserving mapping $f : V(G) \to V(H)$ is called a ***homomorphism*** from G to H, i.e. for any $a, b \in V(G)$, $\{a, b\} \in E(G)$ implies $\{f(a), f(b)\} \in E(H)$.

A homomorphism f is called ***half strong homomorphism*** if for all $y, y' \in Im(f)$, $\{y, y'\} \in E(H)$ implies there exist $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$ such that $\{x, x'\} \in E(G)$.

A homomorphism f is called ***locally strong homomorphism*** if for all $y, y' \in Im(f)$, $\{y, y'\} \in E(H)$ implies for all $x \in f^{-1}(y)$ there exists

$x' \in f^{-1}(y')$ such that $\{x, x'\} \in E(G)$.

A homomorphism f is called ***quasi strong homomorphism*** if for all $y, y' \in Im(f)$, $\{y, y'\} \in E(H)$ implies there exist $x \in f^{-1}(y)$ such that for all $x' \in f^{-1}(y')$, $\{x, x'\} \in E(G)$.

A homomorphism f is called ***strong homomorphism*** if for all $y, y' \in Im(f)$, $\{y, y'\} \in E(H)$ implies for all $x \in f^{-1}(y)$ and for all $x' \in f^{-1}(y')$, $\{x, x'\} \in E(G)$.

Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an ***isomorphism*** from G to H, and in this case we say G is ***isomorphic*** to H (under f), denoted by $G \cong H$. By $Hom(G, H)$, $HHom(G, H)$, $LHom(G, H)$, $QHom(G, H)$, $SHom(G, H)$ and $Iso(G, H)$ denote the sets of homomorphisms, half strong homomorphisms, locally strong homomorphisms, quasi strong homomorphisms, strong homomorphisms and isomorphisms, respectively.

Definition 1.2.9. A homomorphism from the graph G to itself is called an ***endomorphism*** of G. A bijective endomorphism of a graph G is called ***automorphism*** of G. By $End(G)$, $HEnd(G)$, $LEnd(G)$, $QEnd(G)$, $SEnd(G)$ and $Aut(G)$ denote the sets of endomorphisms, half strong endomorphisms, locally strong endomorphisms, quasi strong endomorphisms, strong endomorphisms and automorphisms, respectively. Obviously, $Iso(G, H) \subseteq SHom(G, H) \subseteq QHom(G, H) \subseteq LHom(G, H) \subseteq HHom(G, H) \subseteq Hom(G, H)$.

It is well-known that $End(G)$ and $SEnd(G)$ are monoids and $Aut(G)$ is a group with respect to the composition of mappings.

Definition 1.2.10. A graph G is called Q-S-***unretractive*** if $QEnd(G) = SEnd(G)$. In an analogous manner, we can define other unretractivities of graphs. If G is E-A-***unretractive*** (S-A-***unretractive***), then we call it simply ***unretractive*** (S-***unretractive***). A graph G is called ***retractive*** if it is not unretractive.

Let G and H be graphs with vertex sets $V(G) = \{1, 2, ..., n\}$ and $V(H) = \{c_1, c_2, ..., c_m\}$. We denote a homomorphism from G to H in the obvious sense as $f = \begin{pmatrix} 1 & 2 & ... & n \\ a_1 & a_2 & ... & a_n \end{pmatrix}$ and $f^{-1}(a) := \{b \in V(G) \mid f(b) = a\}$.

Definition 1.2.11. Let f be an endomorphism of a graph G. If H is a subgraph of G, by $f|_H$ we denote the restriction of f on H; and $f(H) := \{f(x) \mid x \in H\}$. A subgraph of G is called the ***endomorphic image*** of G under f, denoted by I_f, if $V(I_f) = f(G)$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$.

Definition 1.2.12. Let $G(V, E)$ be a graph and $\rho \subseteq V \times V$ an equivalence relation on V. Denote by $[a]_\rho$ the equivalence class of $a \in V$ under ρ. The graph, denoted by $G/_\rho$, is called the ***factor graph*** of G under ρ, if $V(G/_\rho) = V/_\rho$ and $\{[a]_\rho, [b]_\rho\} \in E(G/_\rho)$ if and only if there exist $c \in [a]_\rho$ and $d \in [b]_\rho$ such that $\{c, d\} \in E(G)$.

Let f be an endomorphism of G. By ρ_f we denote the equivalence relation on $V(G)$ induced by f, i.e., for $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. The graph $G/_{\rho_f}$ is simply called the ***factor graph by*** f.

Define $i_f : V(G/\rho_f) \to V(I_f)$ with $i_f([x]_{\rho_f}) = f(x)$ for all $x \in V(G)$. Obviously, i_f is well defined. The following Homomorphism Theorem we cite from [28].

Proposition 1.2.13. ([28]) *Let G be a graph and let f be an endomorphism of G. Then*
(1) *i_f is an isomorphism from G/ρ_f to I_f.*
(2) *$f \in HEnd(G)$ if and only if I_f is a strong subgraph of G.*

1.3 Categories

In our study we refer to the word "amalgamated" which is a categorical concept. So in this section we introduce some basic terminologies of categories to describe the amalgams.

Definition 1.3.1. A ***category*** $\mathcal{C}$ consists of the following data:

1. A class $Ob\mathcal{C}$ of the $\mathcal{C}-$***objects***; if A is a $\mathcal{C}-$objects, then we write $A \in Ob\mathcal{C}$ or simply $A \in \mathcal{C}$.

2. A set $\mathcal{C}(A, B)$ for every pair (A, B) of $\mathcal{C}-$objects, such that $\mathcal{C}(A, B) \cap \mathcal{C}(C, D) = \emptyset$ for all $A, B, C, D \in \mathcal{C}$ with $(A, B) \neq (C, D)$. The elements of $\mathcal{C}(A, B)$ are called $\mathcal{C}$***morphisms*** from A to B. For this set we will also write $Mor_\mathcal{C}(A, B)$. For $f \in \mathcal{C}(A, B)$, we call A the ***domain (source)*** and B the ***codomain (tail, sink)*** of f and write $f : A \to B$ or $A \xrightarrow{f} B$.

3. A composition of morphisms, i.e. a partial relation as follows: for any three objects $A, B, C \in \mathcal{C}$ there exists a mapping, then so called ***law of composition***

$$\circ : \begin{cases} \mathcal{C}(A, B) \times \mathcal{C}(B, C) & \to \quad \mathcal{C}(A, C) \\ \quad (f, g) & \mapsto \quad g \circ f \end{cases},$$

such that

(ass) the associativity law $h \circ (g \circ f) = (h \circ g) \circ f$ holds for the composition of morphisms, whenever all necessary compositions are defined;

(id) there exists ***identical morphisms***, which behave like neutral elements with respect to the composition of morphisms, i.e., for every object $A \in \mathcal{C}$ there exists a morphism $id_A \in \mathcal{C}(A, A)$ such that $f \circ id_A = f$ and $id_A \circ g = g$ for all $B, C \in \mathcal{C}$, $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(C, A)$.

Definition 1.3.2. A morphism $f \in \mathcal{C}(A, B)$, $A, B \in \mathcal{C}$ is called an ***isomorphism***, if there exists a morphism $g \in \mathcal{C}(B, A)$ with the properties $f \circ g = id_B$ and $g \circ f = id_A$.

A morphism $f \in \mathcal{C}(A, B)$, $A, B \in \mathcal{C}$ is called a ***monomorphism*** (***epimorphism***) if it is left cancellable, i.e.,

$$f \circ g = f \circ h \Rightarrow g = h \ (g' \circ f = h' \circ g \Rightarrow g' = h')$$

for all $g, h \in Mor(C, A)$ $(g', h' \in Mor(B, D))$, i.e., f is „left cancellable" („right cancellable") with respect to the composition.

Definition 1.3.3. Let $(C_i)_{i \in I}$ be a non-empty family of objects in $\mathcal{C}$. The pair $((u_i)_{i \in I}, C)$ with $C \in \mathcal{C}$, $u_i \in \mathcal{C}(C_i, C)$ is called the ***coproduct*** of the $(C_i)_{i \in I}$, if for all $((k_i)_{i \in I}, T)$ with $T \in \mathcal{C}$, $k_i \in \mathcal{C}(C_i, T)$ there exists exactly one $k \in \mathcal{C}(C, T)$ such that the following diagram is commutative for all $i \in I$.

$$\begin{tikzcd}
T & C \arrow[l, "k"'] \\
 & C_i \arrow[u, "u_i"'] \arrow[ul, "k_i"]
\end{tikzcd}$$

As usual we write $\mathcal{C} = \coprod\limits_{i \in I} C_i$ and the morphism u_i is called the i^{th} ***injection***.

Definition 1.3.4. Let H, G_1, G_2 be objects and $m_1 : H \to G_1$, $m_2 : H \to G_2$ monomorphisms in the category $\mathcal{C}$. We call this constellation a ***pushout situation***. The pair $((u_1, u_2), G_1 \coprod\limits_{(H,(m_1,m_2))} G_2)$ is called *pushout (amalgam, amalgamated coproduct)* of G_1 and G_2 with respect to $(H, (m_1, m_2))$, if

(a) $u_1 : G_1 \to G_1 \coprod\limits_{(H,(m_1,m_2))} G_2$ and $u_2 : G_2 \to G_1 \coprod\limits_{(H,(m_1,m_2))} G_2$ are morphisms such that $u_1 m_1 = u_2 m_2$, i.e., the square in the following diagram is commutative, and

(b) $((u_1, u_2), G_1 \coprod\limits_{(H,(m_1,m_2))} G_2))$ solves the following ***universal problem*** in $\mathcal{C}$.

For every pair $((f_1, f_2), G, f_1 : G_1 \to G, f_2 : G_2 \to G$ with $f_1 m_1 = f_2 m_2$ (i.e., the external rectangle is commutative) there exists exactly one morphism $f : G_1 \coprod\limits_{(H,(m_1,m_2))} G_2) \to G$ such that both triangles in the following diagram are commutative.

$$
\begin{tikzcd}
G & & \\
& G_1 \coprod\limits_{(H,(m_1,m_2))} G_2 \arrow[ul, "f"'] & G_2 \arrow[l, "u_2"] \arrow[ull, "f_2"'] \\
& G_1 \arrow[u, "u_1"] \arrow[uul, "f_1"] & H \arrow[l, "m_1"'] \arrow[u, "m_2"']
\end{tikzcd}
$$

Denoted by **Gra** the category of all graphs. In this category, graphs are objects and graph homomorphisms are morphisms. The next definition we consider on this category.

Definition 1.3.5. Let $H = (V, E)$, $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be graphs and $m_1 : H \to G_1$ and $m_2 : H \to G_2$ injective graph homomorphisms. The ***amalgamted coproduct*** of G_1 and G_2 with respect to $(H, (m_1, m_2))$ is defined by

$V(G_1 \coprod\limits_{(H,(m_1,m_2))} G_2) := (V_1 \setminus m_1(H)) \cup V \cup (V_2 \setminus m_2(H))$,

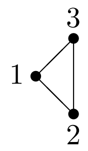
$E(G_1 \coprod\limits_{(H,(m_1,m_2))} G_2) := A \cup B \cup C$, where

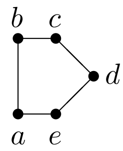
$A := \{\{x_i, y_i\} \in E_i \mid x_i, y_i \in V_i \setminus m_i(H), i = 1, 2\}$,

$B := \{\{x_i, z\} \mid z \in V, x_i \in V_i \setminus m_i(H), \{x_i, m_i(z)\} \in E_i, i = 1, 2\}$

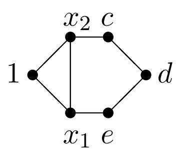
$C := \{\{z, z'\} \mid z, z' \in V, \{m_i(z), m_i(z')\} \in E_i, i = 1, 2\}$.

For example take C_3 and C_5 two graphs as follows.





Let $H = \{x_1, x_2\}$ be a complete graph. Let $m_1 : H \to C_3$ and $m_2 : H \to C_5$ be injective homomorphisms define by $m_1(x_1) = 2$, $m_1(x_2) = 3$, $m_2(x_1) = a$ and $m_2(x_2) = b$. We get the amalgam $C_3 \coprod\limits_{(H,(m_1,m_2))} C_5$ as follows.



For this amalgam A, B and C in Definition 1.3.5 are $\{\{c, d\}, \{d, e\}\}$, $\{\{1, x_1\}, \{1, x_2\}, \{c, x_2\}, \{e, x_1\}\}$ and $\{\{x_1, x_2\}\}$, respectively.

1.4 Some algebraic properties of endomorphism monoids of graphs

Monoids of graphs are generalizations of groups of graphs. In recent years much attention has been paid to monoids of graphs. A main purpose of this study is to reveal a relationship between graph theory and semigroup theory. In [29], Knauer and Wilkeit questioned ,,for which graph G, the endomorphism monoid of G regular?" After this question was posed, many special graphs and their endomorphism monoids were studied. A characterization of all graphs with a regular monoid seems difficult. A possible way to characterize a regular endomorphism monoid of graphs is observation in special graphs.

In this section we provide results which describe a regularity of an endomorphism of graphs. Moreover, we describe other algebraic properties of an endomorphism of graphs.

Regular endomorphisms of graphs

In [34], a characterization of a regular endomorphism of a connected graph is proved by Elke Wilkeit.

Lemma 1.4.1. ([34]) *Let G be a connected graph. An endomorphism $f \in End(G)$ is regular if and only if there are idempotents α and β in $End(G)$ and an isomorphism $\phi : Im(\alpha) \to Im(\beta)$ such that $f = \phi\alpha$ and $Im(f) = Im(\beta)$.*

In [23], Weimin Li gave other characterizations of a regular endomorphism of a graph.

Theorem 1.4.2. ([23]) *Let G be a graph and let $f \in End(G)$. Then f is regular if and only if there exist $g, h \in Idpt(G)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.*

Weimin Li also gave the usefull lemma to considering the regularity of an endomorphism monoid of graph.

Lemma 1.4.3. ([27]) *Let G be a graph and let $f \in End(G)$. Then:*
(1) *$f \in HEnd(G)$ if and only if I_f is an induced subgraph of G.*
(2) *If f is regular, then $f \in HEnd(G)$.*

Lemma 1.4.1 and Theorems 1.4.2 and 1.4.3 give a way to prove regularity of an endomorphism of graph. For any graph G, we call G is an ***endomorphism regular monoid*** (or simply ***endo-regular***) if $End(G)$ is a regular monoid. Next we give a lemma describing a way which shows when a graph is not endo-regular.

Lemma 1.4.4. *Let f be an endomorphism of a connected graph G. Let $Im(f)$ be strong subgraph of G with $V(Im(f)) = f(G)$. If G is endo-regular, then $Im(f)$ is endo-regular.*

Proof. We prove by contraposition. Let f be an endomorphism in $End(G)$ which $Im(f)$ is not endo-regular. Since $End(Im(f))$ is not a regular semigroup, so there exists a non-regular endomorphism $g \in End(Im(f))$. It is clear that gf is an endomorphism of G. Assume that gf is regular, so there exists $h \in End(G)$ such that $(gf)h(gf) = gf$. Then we get that

$$g(fh)g(f(G)) = g(f(G)) \Rightarrow g(fh)g(Im(f)) = g(Im(f)).$$

Since fh is an endomorphism and $Im(fh) \subseteq Im(f)$, then $fh|_{Im(f)}$ is an endomorphism of $Im(f)$, so $g(fh|_{Im(f)})g = g$. Now we have g is regular which is a contradiction. Therefore, we could conclude that G is not endo-regular. $\square$

Completely regular endomorphisms of graphs

In [26], Weimin Li proved the proposition which described a way to find the complete regularity of an endomorphism of graphs.

Proposition 1.4.5. ([26]) *Let G be a graph. Suppose $f \in End(G)$ and f is regular. Then the following four statements are equivalent:*
(1) *f is completely regular;*
(2) *$Idpt(G) \cap [f]_{\mathcal{H}} \neq \emptyset$;*
(3) *There exists $g \in End(G)$ such that $g^2 = g$, $I_f = I_g$ and $\rho_f = \rho_g$;*
(4) *There exists $g \in End(G)$ such that $g^2 = g$, $f(G) = g(G)$ and $\rho_f = \rho_g$.*

We get an inspiration from the above proposition and prove a theorem describing when an endomorphism f of graphs is completely regular by considering directly from f. We describe a property of a mapping f of a finite set G. We denote $T(G)$ the set of all mappings from G to itself.

Lemma 1.4.6. *Let G be a (finite) set, if $f \in T(G)$ and there exist $a, b \in G$ with $f(a) \neq f(b)$ and $f^2(a) = f^2(b)$, then f is not completely regular.*

Proof. Take f is a mapping of the set G. Let $a, b \in G$ with $f(a) \neq f(b)$ and $f^2(a) = f^2(b)$. Assume that f is completely regular, then there exists $g \in T(G)$ with $fgf = f$ and $fg = gf$. Consider at vertices a and b, we have

$$gf^2(a) = fgf(a) = f(a) \neq f(b) = fgf(b) = gf^2(b) = gf^2(a).$$

This is a contradiction. Then we get f is not completely regular. □

We call this property ***square injective*** since it is equivalent to say $f^2(a) = f^2(b)$ implies $f(a) = f(b)$.

The next theorem describes another way to show which endomorphisms are completely regular.

Theorem 1.4.7. *Let G be a finite graph and f be an endomorphism of G. Then f is completely regular if and only if for all $a, b \in V(G)$, $f(a) \neq f(b)$ implies $f^2(a) \neq f^2(b)$, i.e., f is square injective. In this case, if f is not idempotent, we have $ff^{i-1}f = f$ and $ff^{i-1} = f^{i-1}f$ where f^i is the smallest idempotent power of f.*

Proof. Necessity. This follows from Lemma 1.4.6.

Sufficiency. Let f be a square injective endomorphism of G. Since G is finite, there exists some $i \in \mathbb{N}$ such that f^i is an idempotent, i.e., $(f^i)^2 = f^i$.

If f is idempotent, it is clear that f is completely regular. Now we suppose that f is not idempotent. So there exists $2 \leq i \in \mathbb{N}$ such that f^i is idempotent.

We will show that $f(a) = f^{i+1}(a)$ for all $a \in V(G)$. Let $a \in V(G)$. Since f^i is an idempotent, we have $f^2(f^{2i-2}(a)) = f^{2i}(a) = (f^i)^2(a) = f^i(a) = f^2(f^{i-2}(a))$. Since f is square injective, we get that $f^{2i-1}(a) = f^{i-1}(a)$. By repeating this process for $i-1$ times, we get that $f^{i+1}(a) = f(a)$, i.e., $ff^{i-1}f = f$. It is clear that $ff^{i-1} = f^{i-1}f$. Now we have f is completely regular. $\square$

Endo-idempotent-closed graphs

In this dissertation, we denote $Idpt(G)$ the set of all idempotent endomorphisms of the graph G. We call G an ***endomorphism idempotent closed***(or simply, ***endo-idempotent-closed***) graph, if $Idpt(G)$ forms a semigroup.

We begin this section by giving lemmas and corollaries describing which graphs are not endo-idempotent-closed.

Lemma 1.4.8. *Let G be a connected graph and $a \in V(G)$. If $|N(a)| \geq 3$ and $N(d) \subseteq N(c) \subseteq N(b)$ for some distinct $b, c, d \in N(a)$, then G is not endo-idempotent-closed.*

Proof. Let $a \in V(G)$ such that $|N(a)| \geq 3$ and let $b, c, d \in N(a)$ such that $N(d) \subseteq N(c) \subseteq N(b)$. It is clear that

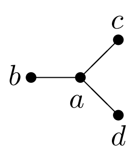
$$f(x) = \begin{cases} x, & x \in V(G) \setminus \{b, c, d\} \\ b, & x \in \{b, c\} \\ d, & x = d \end{cases} \quad \text{and } g(x) = \begin{cases} x, & x \in V(G) \setminus \{b, c, d\} \\ b, & x = b \\ c, & x \in \{c, d\} \end{cases}$$

are idempotent endomorphisms of G. But

$$(g \circ f)(x) = \begin{cases} x, & x \in V(G) \setminus \{b, c, d\} \\ b, & x \in \{b, c\} \\ c, & x = d \end{cases}$$

is not idempotent. So we get that G is not endo-idempotent-closed. $\square$

Example 1.4.9. Take G a star graph as follows.



We see that $N(a) = \{b, c, d\}$ has 3 vertices. This graph is not endo-idempotent-closed since $f = \begin{pmatrix} a & b & c & d \\ a & b & b & d \end{pmatrix}$ and $g = \begin{pmatrix} a & b & c & d \\ a & b & c & c \end{pmatrix}$ are idempotent endomorphisms but $g \circ f$ is not idempotent. $\square$

If a connected graph G does not satisfy the condition in Lemma 1.4.8, it does not follow that G is endo-idempotent-closed. We can consider a factor graph $G/_{\rho_f}$ for idempotent endomorphism f of G. If this factor graph satisfies the condition in Lemma 1.4.8, we will get that G is not endo-idempotent-closed. The next corollary describes this situation.

Corollary 1.4.10. *Let G be a connected graph and $a \in V(G)$ such that $|N(a)| \geq 3$. Let b, c, d be distinct elements in $N(a)$. If f is an idempotent endomorphisms of G such that $f(i) = i$ for all $i \in \{a, b, c, d\}$ and $N_{G/_{\rho_f}}([d]_{\rho_f}) \subseteq N_{G/_{\rho_f}}([c]_{\rho_f}) \subseteq N_{G/\rho_f}([b]_{\rho_f})$, then G is not endo-idempotent-closed.*

Proof. Let f be an idempotent endomorphisms of G such that $f(i) = i$ for all $i \in \{a, b, c, d\}$ and $N_{G/_{\rho_f}}([d]_{\rho_f}) \subseteq N_{G/_{\rho_f}}([c]_{\rho_f}) \subseteq N_{G/_{\rho_f}}([b]_{\rho_f})$. It is clear that

$$g(x) = \begin{cases} f(x), & x \notin [b]_{\rho_f} \cup [c]_{\rho_f} \cup [d]_{\rho_f} \\ b, & x \in [b]_{\rho_f} \cup [c]_{\rho_f} \\ d, & x \in [d]_{\rho_f} \end{cases} \quad \text{and}$$

$$h(x) = \begin{cases} f(x), & x \notin [b]_{\rho_f} \cup [c]_{\rho_f} \cup [d]_{\rho_f} \\ b, & x \in [b]_{\rho_f} \\ c, & x \in [c]_{\rho_f} \cup [d]_{\rho_f} \end{cases}$$

are idempotent endomorphisms of G. But

$$(h \circ g)(x) = \begin{cases} f(x), & x \notin [b]_{\rho_f} \cup [c]_{\rho_f} \cup [d]_{\rho_f} \\ b, & x \in [b]_{\rho_f} \cup [c]_{\rho_f} \\ c, & x \in [d]_{\rho_f} \end{cases}$$

is not idempotent. So we get that G is not endo-idempotent-closed. $\square$

Example 1.4.11. Take G a graph and some its factor graphs as follows.

Then $f = \begin{pmatrix} a & b & c & d & x & y \\ a & b & c & d & a & a \end{pmatrix}$ is an idempotent endomorphism of G. We see that this graph does not satisfy the condition in Lemma 1.4.8 but G

is not endo-idempotent-closed since $g = \begin{pmatrix} a & b & c & d & x & y \\ a & b & b & d & a & a \end{pmatrix}$ and $h = \begin{pmatrix} a & b & c & d & x & y \\ a & b & c & c & a & a \end{pmatrix}$ are idempotent endomorphisms of G and $h \circ g$ is not idempotent.

This graph fulfills the condition in Corollary 1.4.10 since $N_{G/_{\rho_f}}([a]_{\rho_f}) = \{[b]_{\rho_f}, [c]_{\rho_f}, [d]_{\rho_f}\}$ and $N_{G/_{\rho_f}}([b]_{\rho_f}) = N_{G/_{\rho_f}}([c]_{\rho_f}) = N_{G/_{\rho_f}}([d]_{\rho_f}) = \{[a]_{\rho_f}\}$. This confirms that the Corollary 1.4.10 is hold. $\square$

Chapter 2

Bipartite graphs

The bipartite graphs are well-known graphs which are studied with respect to the regularity of their endomorphism monoids. So, in this chapter, we review and give results about the algebraic properties of endomorphism monoids of connected bipartite graphs which will be useful later.

2.1 Endo-regular and endo-completely-regular

In [34], Wilkeit gave a characterization of connected bipartite graphs with a regular monoid. Before that we introduce some definitions and notations.

Definition 2.1.1. *Denoted by P_n a connected graph with $V(P_n) = \{0, 1, ..., n\}$ and $E(P_n) = \{\{i, i+1\} \mid 0 \leq i \leq n-1\}$. We call P_n a* **path** *with n edges and $n+1$ vertices. Denoted by C_n a connected graph with $V(C_n) = \{0, 1, ..., n-1\}$ and $E(P_n) = \{\{i, i+1\} \mid 0 \leq i \leq n-1$ (under modulo n)$\}$. We call C_n a* **cycle** *with n edges and n vertices. We call a bipartite graph G a* **tree** *if G is a connected graph which contains no cycle as a subgraph.*

Theorem 2.1.2. ([34]) *A connected bipartite graph G is endo-regular if and only if G is one of the following graphs:*
• *the complete bipartite graphs $K_{m,n}$ including the complete graphs K_1 and K_2, the cycle C_4 and the trees of diameter 2,*
• *the trees of diameter 3,*
• *the cycles C_6 and C_8, and*
• *the path P_4 of length 4.*

In [5], Fan generalized Theorem 2.1.2 for non-connected graphs without loops and multiple edges.

Theorem 2.1.3. *A non-connected bipartite graph G is endo-regular if and only if G is nK_1, $(n-1)K_1 \cup K_2$ or nK_2, $n \geq 2$.*

In this dissertation, for any graph G we call G is ***endomorphism completely regular*** (or simply ***endo-completely-regular***) if $End(G)$ is a completely regular semigroup. We call G is a ***endomorphism orthodox*** (or simply ***endo-orthodox***) if $End(G)$ is an orthodox semigroup. And we call G is an ***endomorphism Clifford*** (or simply ***endo-Clifford***) if $End(G)$ is a Clifford semigroup.

It is easy to see which endo-regular connected bipartite graph is endo-completely-regular, endo-orthodox and endo-Clifford. We have only 5 and 4 bipartite graphs which are endo-completely-regular and endo-orthodox, respectively. We also have exactly one bipartite graph (this graph is unretractive) which is endo-Clifford.

Theorem 2.1.4. (1) *A connected bipartite graph G is endo-completely-regular if and only if G is one of P_1, P_2, P_3, C_4 and C_6.*

(2) *A connected bipartite graph G is endo-orthodox if and only if G is one of P_1, P_2, P_3 and C_4.*

(3) *Exactly the path P_1 (K_2) is a connected bipartite graph which is endo-Clifford.*

It is also easy to check which endo-regular non-connected bipartite graph is endo-completely-regular, endo-orthodox and endo-Clifford.

Theorem 2.1.5. (1) *No non-connected bipartite graph is endo-completely-regular.*

(2) *Exactly two non-connected bipartite graphs $K_1 \cup K_2$ and $K_2 \cup K_2$ are endo-orthodox.*

(3) *No non-connected bipartite graph is endo-Clifford.*

2.2 Endo-idempotent-closed

In this section, we consider only trees, cycles and complete bipartite graphs which are bipartite graphs. We find when they are endo-idempotent-closed. We recall again that $Idpt(G)$ is the set of all idempotent endomorphisms of graph G. We begin this section by considering the trees.

Lemma 2.2.1. *Let T be a tree and $a \in V(T)$. If $|N(a)| \geq 3$, we get that T is not endo-idempotent-closed.*

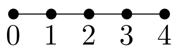
Proof. This follows from Corollary 1.4.10. □

It is clear by the above lemma that if a tree T is not a path, then T is not endo-idempotent-closed. So next we consider which path is endo-idempotent-closed. It is routine to check that the paths P_1, P_2 and P_3 are endo-idempotent-closed. For any $n \geq 4$ the path P_n is not endo-idempotent-closed.

Lemma 2.2.2. *The paths P_1, P_2 and P_3 are endo-idempotent-closed.*

Lemma 2.2.3. *For any $n \geq 4$, the path P_n is not endo-idempotent-closed.*

Proof. First we show the case $n = 4$. Take a path P_4 as follows.



It is clear that $f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & 4 \end{pmatrix}$ are idempotent endomorphisms of P_4. But $f \circ g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 & 0 \end{pmatrix}$ is not idempotent. So we get that P_4 is not endo-idempotent-closed.

Now we can prove for any $n \geq 4$, if n is even, it is clear that two mappings $f = \begin{pmatrix} 0 & 1 & ... & \frac{n}{2} & \frac{n}{2}+1 & \frac{n}{2}+2 & ... & n \\ 0 & 1 & ... & \frac{n}{2} & \frac{n}{2}-1 & \frac{n}{2}-2 & ... & 0 \end{pmatrix}$ and
$g = \begin{pmatrix} 0 & ... & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} & ... & n-1 & n \\ n & ... & \frac{n}{2}+2 & \frac{n}{2}+1 & \frac{n}{2} & ... & n-1 & n \end{pmatrix}$
of P_n are idempotent endomorphisms and $f \circ g$ is not idempotent. If n is odd, similar as case n is even we can construct two idempotent endomorphisms of P_n with the composition of them is not idempotent. □

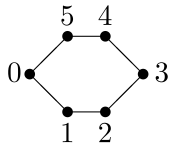
Now we get the theorem describing which tree is endo-idempotent-closed.

Theorem 2.2.4. *Exactly the paths P_1, P_2 and P_3 are endo-idempotent-closed trees.*

Corollary 2.2.5. *For any $n \geq 4$, the cycle C_{2n} is not endo-idempotent-closed.*

Now we have only two cycles C_4 and C_6 to considering. It is routine to check that C_4 is endo-idempotent-closed. Next we show that the cycle C_6 is not endo-idempotent-closed.

Example 2.2.6. Take the cycle C_6 as follows.



It is clear that $f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 3 & 4 & 5 \end{pmatrix}$ are idempotent endomorphisms of C_6. But $f \circ g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 2 & 3 & 2 & 1 \end{pmatrix}$ is not idempotent. So we get that C_6 is not endo-idempotent-closed. $\square$

Theorem 2.2.7. *The only endo-idempotent-closed even cycle is C_4.*

The next theorem follows from Theorems 2.2.4, 2.2.7 and Corollary 1.4.10.

Theorem 2.2.8. *The complete bipartite graph $K_{m,n}$ is endo-idempotent-closed if and only if $m, n \leq 2$.*

2.3 Locally strong endomorphisms of P_n and C_{2n}

For any graph G, since the set of all endomorphisms $End(G)$ of G is always a monoid, in this section we consider when the set of all locally strong endomorphisms $LEnd(G)$ of G, which is not necessarily a semigroup, is a semigroup. It is well-known that the paths P_n and the cycles C_{2n} are not E-L-unretractive. So, we mention on the sets $LEnd(P_n)$ and $LEnd'(C_{2n})$.

Basics

In this section we need to show that an endomorphic image I_f is a strong subgraph of G for any $f \in LEnd(G)$ where $G \in \{P_n, C_{2m} \mid n \geq 1, m \geq 2\}$.

Lemma 2.3.1. ([28]) *Let C_{2n} $(n \geq 2)$ be a cycle and let $f \in End(C_{2n})$. If f is not bijective, $I_f = P_k$ for some $k \in \{1, 2, ..., n\}$.*

The next observation is clear.

Lemma 2.3.2. *Let P_n $(n \geq 2)$ be a path and $f \in End(P_n)$. Then $I_f = P_m$ for some $k \in \{1, 2, ..., n\}$.*

Now we get the main result in this section.

Lemma 2.3.3. *Let $G \in \{P_n, C_{2m} \mid n \geq 1, m \geq 2\}$ and $f \in End(G)$. Then I_f is a strong subgraph of G.*

Proof. Let $f \in End(G)$ be an endomorphism of G. We consider only the case $G = C_{2m}$. The other cases follow analogously. We suppose that f is not bijective, so we get by Lemma 2.3.1 that $I_f = P_k$ for some $k \in \{1, 2, ..., m\}$. Since I_f is a connected subgraph of C_{2m} and all non-trivial connected subgraphs of C_{2m} are paths and are strong subgraphs of G, then I_f is a strong subgraph of G. □

The proof of the next corollary base on Proposition 1.2.13 and Lemma 2.3.3.

Corollary 2.3.4. *$End(P_n) = HEnd(P_n)$ and $End(C_m) = HEnd(C_m)$ for all $n \geq 1$ and $m \geq 3$.*

Main results

We begin this section with the set of all locally strong endomorphisms $LEnd(P_n)$ of path P_n. In [1], Sr. Arworn, U. Knauer and S. Leeratanavalee found when the set $LEnd(P_n)$ formed a semigroup and found its cardinal number. We cite some definitions, lemmas, theorems and corollaries in [1] which we will use later.

Definition 2.3.5. ([1]) An endomorphism $f : P_n \to P_n$ is called a ***complete folding*** if the congruence classes of the relation, $ker\ f = \{(x, y) \in P_n \times P_n \mid f(x) = f(y)\}$, partition P_n in to $\ell + 1$ classes where $\ell | n$ and the equivalence classes are in the form:
$[0] = \{2m\ell \in P_n \mid m = 0, 1, ...\}$,
$[\ell] = \{(2m + 1)\ell \in P_n \mid m = 0, 1, ...\}$
and for any $0 < r < \ell$,
$[r] = \{2m\ell + r \in P_n \mid m = 0, 1, ...\} \cup \{2m\ell - r \in P_n \mid m = 1, 2, ...\}$.
We call ℓ the ***length*** of f.

Corollary 2.3.6. ([1]) *An endomorphism on undirected path is locally strong if and only if it is a complete folding.*

Corollary 2.3.7. ([1]) *The set $LEnd(P_n)$ forms a monoid if and only if n is a prime or 4.*

Theorem 2.3.8. ([1]) $|LEnd(P_n)| = 2\sum\limits_{\ell | n}(n - \ell + 1)$.

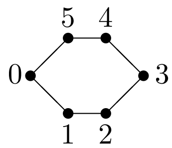
We consider the set $LEnd'(C_{2n})$. First we give a remark which generalize a complete folding for homomorphism from any paths to any graphs.

Remark 2.3.9. (1) We can generalize the definition of complete folding for a homomorphism f from path P_n to any graph G. This implies that the condition in Definition 2.3.5 is held for $f : P_n \to G$.

(2) If we replace ,,an endomorphism on undirected path” by ,,a homomorphism from undirected path P_n to undirected path P_m” in Corollary 2.3.6, the corollary is still true. But for any graph G the complete folding f from any undirected paths to G is not necessarily locally strong. For example, all bijective homomorphisms from P_2 to C_3 are complete folding, but they are not locally strong.

We start finding the cardinal number of the set $LEnd'(C_{2n})$ wonce we observe the next example to investigate how many congruence relations, which have $n+1$ congruence classes, induced by locally strong endomorphism in $LEnd'(C_{2n})$.

Example 2.3.10. Consider the cycle C_6 as follows.



For any non-trivial endomorphism f of C_6, we have 3 possible non-trivial congruence relations induced by f which have 4 congruence classes including:

$\rho_1 = \{\{0\}, \{1, 5\}, \{2, 4\}, \{3\}\}$
$\rho_2 = \{\{1\}, \{2, 0\}, \{3, 5\}, \{4\}\}$
$\rho_3 = \{\{2\}, \{3, 1\}, \{4, 0\}, \{5\}\}$. □

The following observation is clear.

Lemma 2.3.11. *For any cycle C_{2n}, $n \geq 2$, we have n possible non-trivial congruence relations induced by any non-trivial endomorphism of C_{2n} which have $n + 1$ congruence classes each.*

We denote by $P_{n,a}$ the congruence relation induced by non-trivial endomorphism of C_{2n} which has $n + 1$ congruence classes and $[a] = \{a\}$ and $[a + n] = \{a + n\}$. It is clear that $P_{n,a}$ is isomorphic to a factor graph induced by the respective endomorphism of C_{2n}. From above lemma we get that $P_{n,0}$, $P_{n,1}$,....,$P_{n,n-1}$ are n non-trivial congruence relations which have $n + 1$ congruence classes. The next lemma is clear.

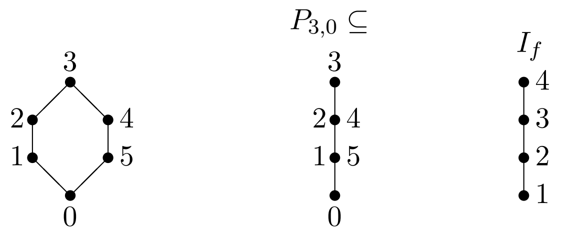
Lemma 2.3.12. *For any cycle C_{2n},*

(1) *for any $a \in V(C_{2n})$, $P_{n,a}$ is a non-trivial congruence relation which has a maximal congruence classes;*

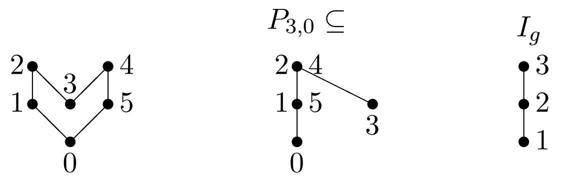
(2) *for any non trivial $f \in End(C_{2n})$, there exists $b \in V(C_{2n})$ which $P_{n,b} \subseteq \rho_f$.*

We find when a non-trivial endomorphism f of C_{2n} is locally strong. We observe the next example to find some arguments which are the proofs in general cases.

Example 2.3.13. Consider the cycle C_6 in Example 2.3.10. Let f be an endomorphism of C_6 which is defined as follows.



It is clear that f is locally strong and $P_{3,0} \subseteq \rho_f$. We see that $f|_{\{0,1,2,3\}}$ is a homomorphism from $\{0, 1, 2, 3\}$ to I_f for some $k > 0$. It is also complete folding, i.e., it is locally strong. Let g be an endomorphism of C_6 which is defined as follows.



It is clear that g is not locally strong and $P_{3,0} \subseteq \rho_g$. We see that $g|_{\{0,1,2,3\}}$ is a homomorphism from $\{0, 1, 2, 3\}$ to I_g for some $k > 0$, but it is not locally strong. □

The next observation is clear by using Lemma 2.3.3, Remark 2.3.9 (2) and the argument in the above example.

Lemma 2.3.14. *Let $f \in End'(C_{2n})$ with $P_{n,a} \subseteq \rho_f$ for some $a \in V(C_{2n})$. Then f is locally strong if and only if $f|_{\{a,a+1,\ldots,a+n\}}$ is a locally strong homomorphism from a path $\{a, a+1, ..., a+n\}$ to a path I_f.*

Corollary 2.3.15. *Let f be non-trivial endomorphism of C_{2n} with $P_{n,a} \subseteq \rho_f$ for some $a \in V(C_{2n})$. If $(|\rho_f| - 1) \nmid n$, f is not locally strong.*

To find the cardinal number of the set $LEnd'(C_{2n})$, we need some lemmas. The next lemma is clear by the observation of the next example.

Example 2.3.16. Consider the cycle C_6 in Example 2.3.10. If $\ell = 3$, we have 3 congruence relations induced by endomorphisms of C_6 which have 4 congruence classes (see Example 2.3.10). Let ρ_i be congruence relation in Example 2.3.10. Then $\rho_i = P_{3,a}$ for some $a \in V(C_6)$. It is clear that $g(a+j) = a+j$ for all $j \in \{0,1,2,3\}$ is a locally strong endomorphism of strong subgraph $\{a, a+1, a+2, a+3\}$ of G. It is also clear that $f(x) = a+j,\ x \in [a+j]_{\rho_i}$ for all $j \in \{0,1,2,3\}$ is an endomorphism of C_{2n}. Now we get by Lemma 2.3.14 that f is locally strong. This means there exists locally strong endomorphism whose congruence relations is ρ_i. So we have 3 congruence relations induced by some locally strong endomorphism of C_6 which have 4 congruence classes.

If $\ell = 1$, we have 1 possible congruence relation induced by some locally strong endomorphism which have 2 congruence classes:

$$\rho_4 = \{\{0,2,4\},\{1,3,5\}\}.$$

□

Lemma 2.3.17. *For any cycle C_{2n}, $n \geq 2$, if $\ell \leq n$ and $\ell | n$, then there exists ℓ congruence relations induced by some locally strong endomorphism of C_{2n} which have $\ell + 1$ congruence classes.*

The next lemma shows us for any $f \in LEnd'(C_{2n})$ how many locally strong endomorphisms of C_{2n} whose congruence relations are ρ_f.

Lemma 2.3.18. *For any cycle C_{2n}, if $f \in LEnd(C_{2n})$, then there exists $4n$ locally strong endomorphisms of C_{2n} whose congruence relations are ρ_f.*

Proof. Let $f \in LEnd'(C_{2n})$. Suppose that f has length ℓ. It is clear that factor graph C_{2n}/ρ_f is isomorphic to P_ℓ, so $C_{2n}/_{\rho_f}$ is $[x]_{\rho_f}$-$[x+\ell]_{\rho_f}$ path for some $x \in V(C_{2n})$.

To find all locally strong endomorphisms whose congruence relations are ρ_f, it is sufficient to find all injective homomorphism from $C_{2n}/_{\rho_f}$ to C_{2n} (i.e., find all possible injective homomorphism g as the following graph).

$$\begin{tikzcd}
{[x+\ell]_{\rho_f}} \arrow[r, "g"] \arrow[d, dotted, no head] & b+\ell\ (b-\ell) \arrow[d, dotted, no head] \\
{[x+1]_{\rho_f}} \arrow[r] \arrow[d, no head] & b+1\ (b-1) \arrow[d, no head] \\
{[x]_{\rho_f}} \arrow[r] & b \\
C_{2n}/_{\rho_f} & \cong P_\ell \subseteq C_{2n}
\end{tikzcd}$$

It is clear that we have $2n$ ways to send $[x]_{\rho_f}$ into C_{2n}. If we send $[x]_{\rho_f}$ to b for some $b \in V(C_{2n})$, we have 2 ways to send $[x+1]_{\rho_f}$ into C_{2n}, that is $[x+1]_{\rho_f}$ is send to $b+1$ or $b-1$. So, we have $4n$ possible injective homomorphism g from $C_{2n}/_{\rho_f}$ to C_{2n}. So, we have $4n$ locally strong endomorphisms of C_{2n} whose congruence relations are ρ_f. $\square$

By Lemmas 2.3.17 and 2.3.18 we get the next theorem describing the cardinal number of the set $LEnd'(C_{2n})$.

Theorem 2.3.19. $|LEnd'(C_{2n})| = 4n \sum_{\ell|n} \ell$.

It is well-known that the group $Aut(C_{2n})$ is isomorphic to the dihedral group D_{2n} which has $4n$ elements. So we have $4n$ automorphisms of C_{2n}. Therefore, we get the next corollary.

Corollary 2.3.20. $|LEnd(C_{2n})| = 4n(1 + \sum_{\ell|n} \ell)$

Next we will show when the set $LEnd'(C_{2n})$ forms a semigroup. The following three observations are clear.

Lemma 2.3.21. *Every endomorphism f of length 1 of a cycle C_{2n} is locally strong endomorphism.*

Lemma 2.3.22. *Every endomorphism $f : C_4 \to C_4$ of length 2 is locally strong endomorphism.*

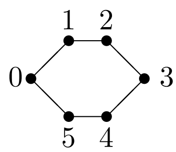
Lemma 2.3.23. *Let $n \geq 2$ and let $f : C_{2n} \to C_{2n}$ be an endomorphism of length m_1 and $g : C_{2n} \to C_{2n}$ be an endomorphism of length m_2. If $m_1 \leq m_2$, then $f \circ g$ and $g \circ f$ are endomorphisms of length $k \leq m_1$.*

Lemma 2.3.24. *The set $LEnd'(C_4)$ forms a semigroup.*

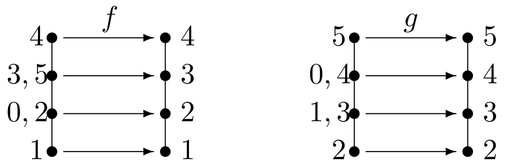
Proof. This follows from Lemmas 2.3.21, 2.3.22 and 2.3.23 since any non-trivial endomorphism of C_4 has length 1 or 2. $\square$

Lemma 2.3.25. *The set $LEnd'(C_6)$ does not form a semigroup.*

Proof. Take a cycle C_6 as follows.

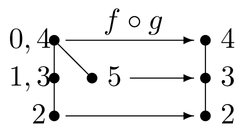


Take f and g locally strong endomorphisms of C_6 as follows.



$P_{3,1}$ $f(C_6)$ $P_{3,2}$ $g(C_6)$

And we get that $f \circ g$ is an endomorphism as follows



$\supseteq P_{3,2}$ $(f \circ g)(C_6)$

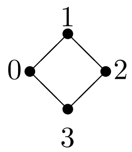
which is not complete folding with respect to $P_{3,0}$, $P_{3,1}$ or $P_{3,2}$. So $f \circ g$ is not a locally strong endomorphism. Hence, $LEnd'(C_6)$ does not form a semigroup. $\square$

For any $n \geq 3$, we can prove that a cycle $LEnd'(C_{2n})$ does not form a semigroup by using the argument of Lemma 2.3.25. So, we get the proposition and the theorem.

Proposition 2.3.26. *For any $n \geq 3$, the set $LEnd'(C_{2n})$ does not form a semigroup.*

Theorem 2.3.27. *The set $LEnd'(C_{2n})$ forms a semigroup if and only if $n = 2$.*

Consider the cycle C_4 as follows.



It is routine to check that $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 3 \end{pmatrix}$ are non-trivial locally strong endomorphisms of C_4. Since $f = fg \neq gf = g$, then $LEnd'(C_4)$ is not a Clifford semigroup.

If we consider when the set of all quasi-strong endomorphism of P_n or C_{2m} forms a semigroup, we have few cases to consider since it is quite clear that

(1) P_n is Q-A-unretractive if and only if $QEnd(P_n)$ is a group $Aut(P_n)$ if and only if $n \neq 2$ or $n \neq 3$ and

(2) C_{2m} is Q-A-unretractive if and only if $QEnd(C_{2m})$ is a group $Aut(C_{2m})$ if and only if $m > 2$.

This means we check only the sets $QEnd(P_2)$, $QEnd(P_3)$ and $QEnd(C_4)$. We accuraly get these three sets form monoids.

For further studies, the sets of all quasi-strong endomorphisms of Q-A-retractive graphs will be proved.

Chapter 3

8-graphs

In this chapter, we introduce a graph which we call „8-graph" because it looks like the number 8. For this 8-graph, we got an inspiration from the molecular graph, spirocompound, in [30]. Some 8-graphs are also bipartite graphs. We study endo-properties of these 8-graphs. Moreover, we generalize 8-graphs to multiple 8-graphs and study the endo-properties of the multiple 8-graphs.

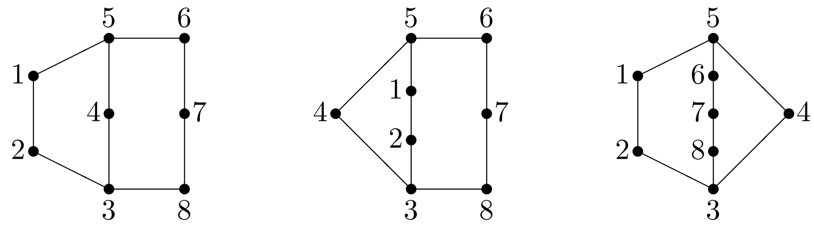
3.1 Definition of 8-graphs

In this section, we introduce the definition of 8-graph and give properties of cycles which we will use in the proofs of algebraic properties of endomorphism monoids of 8-graphs.

Definition 3.1.1. We call graph G an 8-***graph*** if there exist two cycle subgraphs C_n, C_m with $C_n \cup C_m = G$ and $C_n \cap C_m = P_r$ for some $r \geq 0$. We denote this $8-$graph by $C_{n,m}; P_r$.

In this chapter, we denote by P_0 a singleton set.

Example 3.1.2. The three following graphs are $C_{5,6}; P_2$, $C_{5,7}; P_3$ and $C_{7,6}; P_4$, respectively.



There three graphs are isomorphic as can be seen by redrawing. The next proposition generalized this property.

Proposition 3.1.3. *For all $r > 0$,*

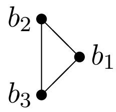
$$C_{n,m}; P_r = C_{n+m-2r,m}; P_{m-r} = C_{n,m+n-2r}; P_{n-r}.$$

The next observation is clear.

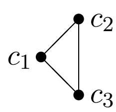
Proposition 3.1.4. *If m, n are even integers, the 8-graph $C_{m,n}; P_r$ is a bipartite graph for all $r > 0$.*

We also get that all 8-graphs are amalgamated coproduct of cycles.

Example 3.1.5. We will show that the 8-graph $C_{3,3}; P_0$ is an amalgamated coproduct of cycles. Take $H := \{a_1\}$ and G_1, G_2 the cycles as follows.

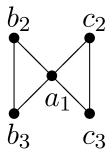


G_1



G_2

It is clear that $m_1(a_1) = b_1$ and $m_2(a_1) = c_1$ are injective homomorphisms from H to G_1 and G_2, respectively. By the Definition 1.3.5 we get the amalgamted $G_1 \coprod_{(H,(m_1,m_2))} G_2$ as follows which is the 8-graph $C_{3,3}; P_0$.



□

The next observation is clear.

Proposition 3.1.6. *All 8-graphs are amalgamated coproduct of cycles.*

Before we show that all 8-graphs are retractive, we need some lemmas. We cite the next quite obvious lemma from [2].

Lemma 3.1.7. ([2]) *If $n, m \geq 3$ and n is odd, then $Hom(C_n, C_m) = \emptyset$ if and only if m is even or $m > n$.*

The next two corollaries are consequences of Lemma 3.1.7.

Corollary 3.1.8. *For any $3 \leq n < m$, we get that*
(1) *for all $f \in Hom(C_n, C_m)$, $f(C_n) \ncong C_n$ and*
(2) *for all $f \in End(C_m)$, $f(C_m) \ncong C_n$.*

Corollary 3.1.9. *For any 8-graph $C_{n,m}; P_r$, if C_k is a smallest odd length cycle subgraph of $C_{n,m}; P_r$, then $f(C_k) = C_k$ for all $f \in End(C_{n,m}; P_r)$.*

First we consider an 8-graph $C_{n,m}; P_r$ where $r = 0$.

Lemma 3.1.10. *For any $n, m \geq 3$, the 8-graph $C_{n,m}; P_0$ is retractive.*

Proof. Let $P_0 = \{0\}$. If m is even, set $V(C_m) = \{0, 1, 2, ..., m-1\}$ and define

$$f(i) = \begin{cases} i & , i \in C_n \\ \frac{m}{2} - j & , i \in \{\frac{m}{2} - j, \frac{m}{2} + j\}; j \in \{0, 1, ..., \frac{m}{2}\} \end{cases}$$

a mapping from G to itself. It is clear that $f \in End'(G)$, so $C_{n,m}; P_0$ is retractive.

If n, m are odd, by Lemma 3.1.7, there exists $f \in End'(C_{n,m}; P_0)$. So, we get that $C_{n,m}; P_0$ is retractive. □

Next, we will prove that $C_{n,m}; P_r$ is retractive where $r > 0$.

Lemma 3.1.11. *For any $r > 0$, if $m \geq r$, there exists $f \in End(C_{2m})$ with $f(C_{2m}) = P_r$ where P_r a path subgraph of the cycle C_{2m}.*

Proof. It is clear that cycle C_{2m} can be mapped homomorphically onto P_m which is turn goes onto P_r if and only if $m \geq r$. □

Corollary 3.1.12. *For any $0 < r \leq m$, an 8-graph $C_{n,2m}; P_r$ is retractive.*

The next corollary is consequence from Corollary 3.1.12 and Proposition 3.1.3.

Corollary 3.1.13. *For any $r > 0$, an 8-graph $C_{n,m}; P_r$ is retractive.*

Proof. By Proposition 3.1.3 we know that

$$C_{n,m}; P_r = C_{n+m-2r,m}; P_{m-r} = C_{n,m+n-2r}; P_{n-r}.$$

Since at least one of $n, m, n+m-2r$ is even, we get by Corollary 3.1.12 that $C_{n,m}; P_r$ is retractive. □

Theorem 3.1.14. *All 8-graphs are retractive.*

Proof. This follows from Lemma 3.1.10 and Corollary 3.1.13. □

3.2 Regular endomorphisms of 8-graphs

We know from Proposition 3.1.4 that an 8−graph $C_{n,m}; P_r$ is a bipartite graph if n and m are even integers. So we refer to Theorem 2.1.2 which describes all endo-regular connected bipartite graphs.

It is clear that for any even integers n, m, exactly the 8-graph $C_{4,4}; P_2$ is endo-regular bipartite graph. Then we get the corollary of Theorem 2.1.2.

Corollary 3.2.1. *If m, n are even integers, exactly the 8-graph $C_{n,m}; P_r$ with $n = m = 4$, $r = 2$ is endo-regular.*

We turn to consider an 8-graph $C_{n,m}; P_r$ when n or m is odd. First we consider when both of them odd. We begin with the case $n = m$.

Lemma 3.2.2. The 8-graphs $C_{3,3}; P_0$ and $C_{3,3}; P_1$ are endo-regular.

Proof. Take the 8-graphs $C_{3,3}; P_0$ and $C_{3,3}; P_1$ as follows.

$C_{3,3}; P_0$ $\quad$ $C_{3,3}; P_1$

We will show that $End(C_{3,3}; P_0)$ and $End(C_{3,3}; P_1)$ are regular monoids. We first consider $End(C_{3,3}; P_0)$. For the 8-graph $C_{3,3}; P_0$, there exist only two non-trivial congruence relations, i.e.,

$\rho_1 = \{\{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ or

$\rho_2 = \{\{a_1\}, \{a_2, b_3\}, \{a_3, b_2\}\}$.

We call $\{a_1, a_2, a_3\}$, the cycle subgraph of $C_{3,3}; P_0$, an a-cycle. Similarly, we call $\{a_1, b_2, b_3\}$ a b-cycle. Now we consider an endomorphism f which corresponds to ρ_1. We may assume that $Im(f)$ is the a-cycle and f induces a non-identical automorphism of this a-cycle. Assume $f(a_1) = a_2$ and $f(a_2) = a_3$ then we take for any other rotation of the a-cycle

$$g = \begin{pmatrix} a_1 & a_2 & a_3 & b_2 & b_3 \\ a_2 & a_3 & a_1 & a_3 & a_1 \end{pmatrix}$$

which is an endomorphism of $C_{3,3}; P_0$ and $fgf = f$. So f is regular. Similarly if f is a reflection on the a-cycle, i.e., $f(a_1) = a_1$ and $f(a_2) = a_3$ then we get that $f^3 = f$, so f is regular.

Similarly, we also get that f is regular if f corresponds to ρ_2.

Now we consider $End(C_{3,3}; P_1)$. There exists only one non-trivial congruence relation: $\rho = \{\{x_1\}, \{x_2\}, \{x_3, y_3\}\}$. Similar as the 8-graph $C_{5,5}; P_0$ we get that $End(C_{5,5}; P_1)$ is a regular monoid. $\square$

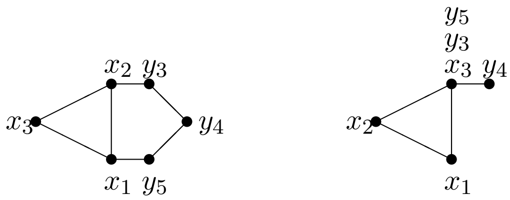
We can prove that if n is odd, then all endomorphisms of $C_{n,n}; P_r$ are regular by using the argument of Lemma 3.2.2. So, we get the next proposition.

Proposition 3.2.3. *If n is odd, all endomorphisms of $C_{n,n}; P_r$ are regular.*

Now we consider the 8-graph $C_{n,m}; P_r$ when n, m are odd and $n \neq m$.

Lemma 3.2.4. *The 8-graph $C_{3,5}; P_1$ is not endo-regular.*

Proof. Take the 8-graph $C_{3,5}; P_1$ with its endomorphic image as follows.



Then $f = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 & y_4 & y_5 \\ x_3 & x_1 & x_2 & x_2 & y_3 & x_2 \end{pmatrix}$ is the corresponding endomorphism of $C_{3,5}; P_1$. Assume that there exists $g \in End(C_{3,5}; P_1)$ such that $fgf = f$. Since $f^{-1}(y_3) = \{y_4\}$, $f^{-1}(x_1) = \{x_2\}$ and $f^{-1}(x_3) = \{x_1\}$ are singleton sets, then $g(y_3) = y_4$, $g(x_1) = x_2$ and $g(x_2) = x_3$. By Lemma 3.1.7, we have that g must preserve the cycle $C_3 = \{x_1, x_2, x_3\}$, a subgraph of $C_{3,5}; P_1$, so $g(x_3) = x_2$. Since $\{x_2, y_3\} \in E(C_{3,5}; P_1)$ and $\{g(x_2), g(y_3)\} = \{x_3, y_4\} \notin E(C_{3,5}; P_1)$, g is not an endomorphism. So f is not regular. $\square$

We can prove the next proposition by using the argument of Lemma 3.2.4. In this situation, we suppose that the cycle C_n is the minimal cycle subgraph of an 8-graph $C_{n,m}; P_r$. We construct an endomorphism f of $C_{n,m}; P_r$ with $f(C_n) = C_n$ and there exists only one vertex a in $C_{n,m}; P_r \setminus (C_n \cup \bigcup_{x \in C_n} N(x))$ such that $f(a) \in C_{n,m}; P_r \setminus C_n$. We get that this endomorphism f is not regular.

Proposition 3.2.5. *Let $n \neq m$ be integers. If*

(1) *n, m are odd or*

(2) *n is odd and m is even and $|m - 2r| \geq 2$ and ($r \neq 1$ or $m \neq 4$),*

then $C_{n,m}; P_r$ is not an endo-regular.

Now we have the following theorem.

Theorem 3.2.6. *Let $n, m \geq 3$ be odd integers. An 8-graph $C_{n,m}; P_r$ is endo-regular if and only if $n = m$.*

Proof. This follows from Propositions 3.2.3 and 3.2.5. □

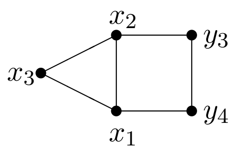
We know by Proposition 3.1.3 that $C_{2n+1,2m}; P_m = C_{2n+1,2n+1}; P_{2n+1-m}$ for $m \geq 2$. So, we get the next corollary.

Corollary 3.2.7. *For any $m \geq 2$, the 8-graph $C_{2n+1,2m}; P_m$ is endo-regular.*

Proposition 3.2.5 does not describe the regularity of 8-graphs $C_{n,m}; P_r$ when n is odd and $m = 4$ and $r = 1$. Now we will prove that all 8-graphs $C_{n,4}; P_1$, where n is odd are endo-regular.

Lemma 3.2.8. *The 8-graph $C_{3,4}; P_1$ is endo-regular.*

Proof. Take an 8-graph $C_{3,4}; P_1$ as follows.



It is clear that $C_3 = \{x_1, x_2, x_3\}$ is a smallest odd length cycle subgraph of $C_{3,4}; P_1$. By Corollary 3.1.9 we get that $f(C_3) = C_3$ for all $f \in End(C_{3,4}; P_1)$.

If f is automorphism or $f(C_{3,4}; P_1) = C_3$, then it is clear that f is regular. Now we have another four endomorphisms which are not automorphism and their image are not cycle C_3, namely,

$$f_1 = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 & y_4 \\ x_1 & x_2 & x_3 & y_3 & x_2 \end{pmatrix}, \; f_2 = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 & y_4 \\ x_2 & x_1 & x_3 & y_4 & x_1 \end{pmatrix}$$
$$f_3 = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_1 & y_4 \end{pmatrix}, \; f_4 = \begin{pmatrix} x_1 & x_2 & x_3 & y_3 & y_4 \\ x_2 & x_1 & x_3 & x_2 & y_3 \end{pmatrix}.$$

It is clear that f_1 and f_3 are idempotent and $f_2 f_4 f_2 = f_2$ and $f_4 f_2 f_4 = f_4$. Then we get that f_1, f_2, f_3 and f_4 are regular. So we get that $C_{n,4}; P_1$ is endo-regular. □

We can prove the next proposition by using the argument of Lemma 3.2.8.

Proposition 3.2.9. *Let $n \geq 3$ be odd integer. The* 8*-graph $C_{n,4}; P_1$ is endo-regular.*

Now we have the main theorem in this section which describes the endo-regularity of 8-graphs.

Theorem 3.2.10. *Exactly the following* 8*-graphs are endo-regular:*
- $C_{2n+1,2n+1}; P_r$ *for any* $r \geq 0$,
- $C_{2n+1,4}; P_1$,
- $C_{4,4}; P_2 = K_{2,3}$.

3.3 Completely regular endomorphisms of 8-graphs

Here we use the results of the previous section. We first consider the completely regularity of endo-regular 8-graphs $C_{2n+1,2n+1}; P_r$ for any $r \geq 0$.

Lemma 3.3.1. *The endo-regular* 8*-graphs $C_{3,3}; P_0$ and $C_{3,3}; P_1$ are endo-completely-regular.*

Proof. Take the 8−graphs $C_{3,3}; P_0$ and $C_{3,3}; P_1$ as in the proof of Lemma 3.2.2.

First we consider the 8-graph $C_{3,3}; P_0$. In this graph, we have two non-trivial congruence relations:

$\rho_1 = \{\{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ and $\rho_2 = \{\{a_1\}, \{a_2, b_3\}, \{a_3, b_2\}\}$

as we already noticed in Lemma 3.2.2 and we also have only two possible image graphs:

$I_1 := \{a_1, a_2, a_3\}$ and $I_2 := \{a_1, b_2, b_3\}$

which are isomorphic to C_3.

Let f be non-trivial endomorphism of $C_{3,3}; P_0$. Assume that f is not completely regular, i.e, f is not square injective. So there exist $x, y \in C_{3,3}; P_0$ such that $f(x) \neq f(y)$ and $f^2(x) = f^2(y)$. Without loss of generality we suppose that $f(x), f(y)$ are in I_1. Since I_1 is isomorphic to the odd-length cycle C_3, it is clear that $f|_{I_1}(I_1) = I_1$. Since $f(x) \neq f(y) \in I_1$, we get that $f^2(x) \neq f^2(y)$. This is a contradiction. So we get that f is completely regular. Hence the 8-graph $C_{3,3}; P_0$ is endo-completely-regular.

Similarly we get that $C_{3,3}; P_1$ is endo-completely-regular. □

We can prove the next proposition by using the argument of Lemma 3.3.1.

Proposition 3.3.2. *For any $r \geq 0$ and $n \geq 1$, an endo-regular* 8*-graph $C_{2n+1,2n+1}; P_r$ is endo-completely-regular.*

From Theorem 3.2.10 we have to consider two more endo-regular 8-graphs, $C_{2n+1,4}; P_1$ and $C_{4,4}; P_2$. We begin with the next special case.

Lemma 3.3.3. *The endo-regular* 8*-graphs* $C_{4,4}; P_2$ *and* $C_{5,4}; P_1$ *are not endo-completely-regular.*

Proof. Since the 8-graph $C_{4,4}; P_2$ is a complete bipartite graph $K_{2,3}$, so we get by Theorem 2.1.4 that $C_{4,4}; P_2$ is not endo-completely-regular.

Next, take the endo-regular 8-graph $C_{5,4}; P_1$ and its endomorphic image as follows.

Then $f = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 & b_4 \\ a_2 & a_1 & a_3 & b_4 & a_1 \end{pmatrix}$ is endomorphisms of $C_{5,4}; P_1$. Since $f^2(a_1) = a_1 = f^2(b_3)$ and $f(a_1) \neq f(b_3)$, then we get that f is not square injective. By Theorem 1.4.7 we get that f is not completely regular. Then we get that $P_{5,4}; P_1$ is not endo-completely-regular. $\square$

We can prove the next proposition by using the argument of Lemma 3.3.3.

Proposition 3.3.4. *For any* $n \geq 1$*, an endo-regular* 8*-graph* $C_{2n+1,4}; P_1$ *is not endo-completely-regular.*

Now we get the main theorem in this section which describes the endo-completely-regularity of 8-graphs.

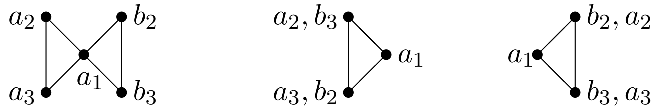
Theorem 3.3.5. *Exactly the* 8*-graphs* $C_{2n+1,2n+1}; P_r$ *are endo-completely-regular where* $n \geq 1$ *and* $r \geq 0$.

3.4 Endo-idempotent-closed 8-graphs

In this section, we will find which 8-graphs $C_{n,m}; P_r$ are endo-idempotent-closed. First we consider the case when n and m are odd integers.

Lemma 3.4.1. *The* 8*-graph* $C_{3,3}; P_0$ *is not endo-idempotent-closed.*

Proof. Take the 8-graph $C_{3,3}; P_0$ as follows.



We repeat from Lemma 3.2.2: we call the cycle subgraph $\{a_1, a_2, a_3\}$ an a-cycle and call the cycle subgraph $\{a_1, b_2, b_3\}$ a b-cycle. It is clear that there exist only two non-trivial congruence relations:

$\rho_1 = \{\{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}$ and $\rho_2 = \{\{a_1\}, \{a_2, b_3\}, \{a_3, b_2\}\}$.

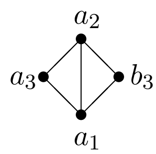
Let i_1 be an idempotent embedding from the middle graph to the a-cycle and i_2 be an idempotent embedding from the right hand side graph to the b-cycle. It is clear that i_1 and i_2 are idempotent endomorphisms but $i_1 \circ i_2$ is not idempotent. So, $C_{3,3}; P_0$ is not endo-idempotent-closed. $\square$

In general, it is clear that for any 8-graph $C_{2n+1,2n+1}; P_0$, there exist two non-trivial congruence relations and there exist only two non-trivial image sets. It is also clear that there exist two non-trivial idempotent endomorphisms f and g whose congruence relations and image graphs are different. And the composition $f \circ g$ is not idempotent. Then, we get the next proposition.

Proposition 3.4.2. *For any $n \geq 1$, the 8-graph $C_{2n+1,2n+1}; P_0$ is not endo-idempotent-closed.*

Lemma 3.4.3. *The 8-graph $C_{3,3}; P_1$ is endo-idempotent-closed.*

Proof. Take the 8-graph $C_{3,3}; P_1$ as follows.



It is clear that there exist only two non-trivial idempotent endomorphisms: $i_1 = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 \\ a_1 & a_2 & a_3 & a_3 \end{pmatrix}$ and $i_2 = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 \\ a_1 & a_2 & b_3 & b_3 \end{pmatrix}$. We get that $i_1 \circ i_2 = i_1$ and $i_2 \circ i_1 = i_2$. So, $C_{3,3}; P_1$ is endo-idempotent-closed. $\square$

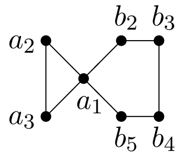
For any $r > 0$, it is clear that $C_{2n+1,2n+1}; P_r$ contains only two non-trivial idempotent endomorphisms i_1 and i_2. It is also clear that $i_1 \circ i_2 = i_1$ and $i_2 \circ i_1 = i_2$. Then, we get the next proposition.

Proposition 3.4.4. *For any $r > 0$, the 8-graph $C_{2n+1,2n+1}; P_r$ is endo-idempotent-closed.*

Now we consider the 8-graph $C_{2n+1,2m+1}; P_r$ where $n \neq m$. We begin to show that the 8-graph $C_{3,5}; P_0$ is endo-idempotent-closed.

Lemma 3.4.5. *The 8-graph $C_{3,5}; P_0$ is endo-idempotent-closed.*

Proof. Take an 8-graph $C_{3,5}; P_0$ as follows.



$C_{3,5}; P_0$

It is clear that $C_3 = \{a_1, a_2, a_3\}$ and $C_5 = \{a_1, b_2, b_3, b_4, b_5\}$ are subgraphs of $C_{3,5}; P_r$. Since C_3 is the smallest odd length cycle subgraph of $C_{3,5}; P_0$, by Lemma 3.1.7 we get that $f(C_3) = C_3$ for all endomorphisms $f \in End(C_{3,5}; P_0)$. Since C_5 is an odd cycle, it is clear that for any $f \in End(C_{3,5}; P_0)$, if $x \neq y \in C_5$ and $f(x), f(y) \in C_5$, then $f(x) \neq f(y)$. Now we get that $g(C_{3,5}; P_0) = C_3$ for all $g \in End'(C_{3,5}; P_0)$. So for any two idempotent $i_1, i_2 \in End'(C_{3,5}; P_0)$, $Im(i_1) = Im(i_2) = C_3$ and $i_1(x) = i_2(x) = x$ for all $x \in C_3$. It is clear that $Im(i_1 \circ i_2) = Im(i_1) = Im(i_2)$ and $(i_1 \circ i_2)(x) = x$ for all $x \in Im(i_1 \circ i_2)$, so $i_1 \circ i_2$ is idempotent. Hence, we get that $C_{3,5}; P_0$ is endo-idempotent-closed. $\square$

We can prove the next proposition by using the argument of Lemma 3.4.5.

Proposition 3.4.6. *For any $r \geq 0$ and $n \neq m$, the $8-$graph $C_{2n+1,2m+1}; P_r$ is endo-idempotent-closed.*

The next theorem is a consequence from Propositions 3.4.2, 3.4.4, and 3.4.6.

Theorem 3.4.7. *For any $n, m \geq 1$, 8-graph $G = C_{2n+1,2m+1}; P_r$ is endo-idempotent-closed if and only if (1) $n \neq m$ or (2) $n = m$ and $r > 0$.*

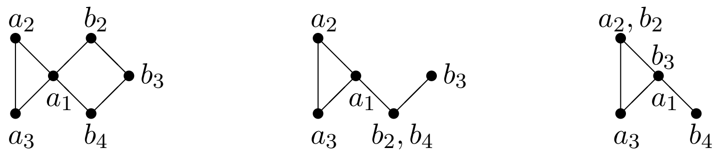
We know from Proposition 3.1.3 that
$C_{2n+1,2m+1}; P_r = C_{2(n+m+1-r),2m+1}; P_{2m+1-r} = C_{2n+1,2(m+n+1-r)}; P_{2n+1-r}$ for $r > 0$. Then we get the next corollary.

Corollary 3.4.8. *For any $r > 0$, $n \geq 1$ and $m \geq 2$, we get that $C_{2n+1,2m}; P_r$ is endo-idempotent-closed.*

Now we know that if $r > 0$, then the 8-graph $C_{2n+1,2m}; P_r$ is endo-idempotent-closed. Next we give a lemma to show that $C_{3,4}; P_0$ is not endo-regular.

Lemma 3.4.9. *The 8-graph $C_{3,4}; P_0$ is not endo-indempotent-closed.*

Proof. Take the 8-graph $C_{3,4}; P_0$ and its factor graphs as follows.



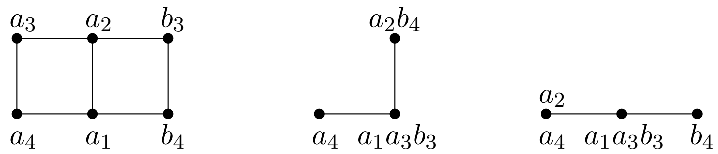
It is clear that an idempotent embedding i_1 from the middle graph to $C_{3,4}; P_0$, which send b_2 and b_4 to b_4, is an idempotent endomorphism of $C_{3,4}; P_0$. Similarly an idempotent embedding i_2 from the right hand side graph to $C_{3,4}; P_0$, which send b_4 to b_4, is an idempotent endomorphism of $C_{3,4}; P_0$. It is clear that $i_1 \circ i_2$ is not idempotent since $(i_1 \circ i_2)(b_2) = b_4 \neq a_2 = (i_1 \circ i_2)^2(b_2)$. □

We can prove the next proposition by using the argument of Lemma 3.4.9.

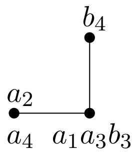
Proposition 3.4.10. *The 8-graphs $C_{n,2m}; P_0$ is not endo-idempotent-closed.*

Lemma 3.4.11. *The 8-graph $C_{4,4}; P_1$ is not endo-idempotent-closed.*

Proof. Take the 8-graph $C_{4,4}; P_1$ and its factor graphs as follows.



Let f be an embedding from the middle graph to $C_{4,4}; P_1$ which $f(x) = x$ for all $x \in \{a_1, a_2, a_4\}$ and g be an embedding from the right hand side graph to $C_{4,4}; P_1$ which $g(x) = x$ for all $x \in \{a_1, a_4, b_4\}$. It is clear that f and g are idempotent. But the composition $f \circ g$ (the embedding from below graph to $C_{4,4}; P_1$ which $(f \circ g)(a_1) = a_1$, $(f \circ g)(a_4) = a_4$ and $(f \circ g)(b_4) = a_2$) is not idempotent.



So, we get that $C_{4,4}; P_1$ is not endo-idempotent-closed. □

We can prove the next proposition by using the argument in Lemma 3.4.11.

Proposition 3.4.12. *For any $n, m \geq 2$ and $r \geq 0$, the $8-$graphs $C_{2n,2m}; P_r$ is not endo-idempotent-closed.*

Now by Theorem 3.4.7, Corollary 3.4.8 and Propositions 3.4.10, 3.4.12 we get the next theorem describing when the 8-graph is endo-idempotent-closed.

Theorem 3.4.13. *Exactly the following 8-graphs are endo-idempotent-closed:*

- *$C_{2n+1,2m+1}; P_r$ where $r \geq 0$ and $n \neq m$*
- *$C_{2n+1,2n+1}; P_r$ where $r > 0$.*

3.5 Other endo-properties of 8-graphs

We know from Theorems 3.2.10 and 3.4.13 that when the 8-graphs are endo-orthodox.

Theorem 3.5.1. *Exactly the 8-graphs $C_{2n+1,2n+1}; P_r$ for some $r > 0$ are endo-orthodox.*

We get from Theorems 3.3.5 and 3.5.1 that all endo-orthodox 8-graphs are endo-completely-regular. So, we get the next corollary since orthogroup means completely regular and orthodox.

Corollary 3.5.2. *Exactly the monoids of the 8-graphs $C_{2n+1,2n+1}; P_r$ are orthogroups where $r > 0$.*

Theorem 3.5.3. *No 8-graph is endo-Clifford.*

Proof. Let n be an odd integer. Take $C_{n,n}; P_r$ an endo-completely regular $8-$graph. Let $\{x_1, x_2, ..., x_n\} =: C$ and $\{x_1, ..., x_{r+1}, y_{r+2}, ..., y_n\} =: C'$ be two cycle subgraphs of $C_{n,n}; P_r$ of length n. It is clear that there exist $i_1, i_2 \in Idt(C_{n,n}; P_r)$ with $Im(i_1) = C$ and $Im(i_2) = C'$. And it also clear that $i_1 \circ i_2 = i_1 \neq i_2 = i_2 \circ i_1$. Now we get that $C_{n,n}; P_r$ is not endo-Clifford. $\square$

3.6 Endo-regular multiple $8-$graphs

In this section, we generalize 8-graphs to multiple 8-graphs and find when they are endo-regular.

Definition 3.6.1. We call the connected graph G ***multiple*** 8***-graph*** if there exists $r \geq 0$ and there exists $s \geq 2$ cycle subgraphs $C_{n_1}, C_{n_2}, ..., C_{n_s}$ of G with $\bigcup\limits_{k=1}^{s} C_{n_k} = G$ and $C_{n_i} \cap C_{n_j} = P_r$, $i \neq j$. We denote the multiple 8-graph by $C_{n_1,n_2,...,n_s}; P_r$.

The next observation is clear.

Lemma 3.6.2. *Let $C_{n_1,n_2,...,n_s}; P_r$ be a multiple 8-graph.*

(1) *If $C_{n_1,n_2,...,n_s}; P_r$ contains odd-length cycle as a subgraph and C_{n_1} is a minimal odd-length cycle subgraph of $C_{n_1,n_2,...,n_s}; P_r$, an 8-graph $C_{n_1,n_i}; P_r$ is isomorphic to strong subgraph $Im(f)$ for some $f \in End(C_{n_1,n_2,...,n_s}; P_r)$ where $i \in \{2, 3, ..., s\}$.*

(2) *If $C_{n_1,n_2,...,n_s}; P_r$ contains no odd-length cycle as a subgraph, an 8-graph $C_{n_i,n_j}; P_r$ is isomorphic to strong subgraph $Im(f)$ for some $f \in End(C_{n_1,n_2,...,n_s}; P_r)$ where $i \neq j \in \{1, 2, ..., s\}$.*

Now we turn to find the regularity of endomorphism monoids of multiple 8-graphs.

Lemma 3.6.3. *Let $C_{n_1,n_2,...,n_s}; P_r$ be a multiple 8-graph.*

(1) *If C_{n_1} is a minimal odd-length cycle subgraph of $C_{n_1,n_2,...,n_s}; P_r$ and*

(*a*) *$r \neq 1$ and $n_1 \neq n_i$ for some $i \in \{2, ..., s\}$ or*

(*b*) *$r = 1$ and $n_i \notin \{4, n_1\}$ for some $i \in \{1, 2, ..., s\}$,*

then $C_{n_1,n_2,...,n_s}; P_r$ is not endo-regular.

(2) *If $C_{n_1,n_2,...,n_s}; P_r$ contains no odd-length cycle as a subgraph and $C_{n_1,n_2,...,n_s}; P_r$ is not $C_{4,4,...,4}; P_2$, then $C_{n_1,n_2,...,n_s}; P_r$ is not endo-regular.*

Proof. (1) First we prove case (a). Suppose that $n_1 \neq n_i$ for some $i \in \{2, 3, ..., s\}$. By Theorem 3.2.10 and Lemma 3.6.2 we get that $C_{n_1,n_i}; P_r$ is not endo-regular which is isomorphic to $Im(f)$ for some $f \in End(C_{n_1,n_2,...,n_s}; P_r)$.

Now by Lemma 1.4.4 we can conclude that $C_{n_1,n_2,...,n_s};P_r$ is not endo-regular. Similarly, we get that $C_{n_1,n_2,...,n_s};P_r$ is not endo-regular if $r = 1$ and $n_i \notin \{4, n_1\}$ for some $i \in \{1, 2, ..., s\}$.

(2) The proof of this case is similar as case (1). □

For any $s \geq 2$, we instead $\underbrace{n, n, ..., n}_{s\ times}$ by $(n)^{(s)}$. Lemma 3.6.3 does not describe the following 3 multiple 8-graphs:

(1) $C_{(2n+1)^{(t)}};P_r$,
(2) $C_{(2n+1)^{(t)},(4)^{(s)}};P_1$ and
(3) $C_{(4)^{(s)}};P_2$.

So, we will consider the endo-regularity of them. It is clear that for any $s \geq 2$ the multiple 8-graph in case (3) is the complete bipartite graph $K_{2,s+1}$. So by Theorem 2.1.2 we get the next lemma.

Lemma 3.6.4. *For any $s \geq 2$, the multiple 8-graphs $C_{(4)^{(s)}};P_2$ is endo-regular.*

Now we turn to the multiple 8-graph $C_{(2n+1)^{(t)}};P_r$ for $r \geq 0$. We will show that they are endo-regular. In the proof of endo-regularity of these graphs we need some proposition. The next observation is clear.

Proposition 3.6.5. *Let $G := C_{(2n+1)^{(t)}};P_r$ be a multiple 8-graph and $f \in End(G)$. Then $Im(f)$ is a strong subgraph of G with $Im(f) = C_{2n+1}$ or $Im(f) = C_{(2n+1)^{(t')}};P_r$ where $2 \leq t' \leq t$.*

Lemma 3.6.6. *For any $r \geq 0$ and $t \geq 2$, the multiple 8-graph $C_{(2n+1)^{(t)}};P_r$ is endo-regular.*

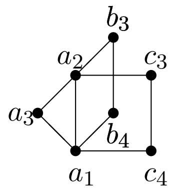
Proof. We prove by induction. The case $t = 2$ is true by Theorem 3.2.10. Suppose that $C_{(2n+1)^{(t)}};P_r$ is endo-regular. We will prove that $C_{(2n+1)^{(t+1)}};P_r$ is endo-regular. Let f be non-trivial endomorphism of $C_{(2n+1)^{(t+1)}};P_r$. By Proposition 3.6.5 we get that $Im(f) = C_{2n+1}$ or $Im(f) = C_{(2n+1)^{(t')}};P_r$ where $2 \leq t' \leq t+1$. We consider only the case $Im(f) = C_{(2n+1)^{(t)}};P_r$. The other cases follow analogously.

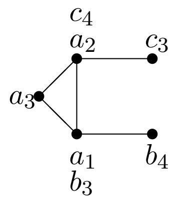
To prove the regularity of f. It is equivalent to prove the regularity of $f|_{Im(f)} =: g : Im(f) \to Im(f)$. Since $Im(f) = C_{(2n+1)^{(t)}};P_r$ is endo-regular and $g \in End(Im(f))$, then g is regular. So, f is regular. Now the result is proved. □

Next we consider an multiple 8-graph $G := C_{(2n+1)^{(t)},(4)^{(s)}};P_1$. First we will show that if $s \geq 2$, then G is not endo-regular.

Lemma 3.6.7. *The multiple 8-graph $C_{3,4,4}; P_1$ is not endo-regular.*

Proof. Take the multiple 8-graph $C_{3,4,4}; P_1$ with its endomorphic image as follows.





Then $f = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 & b_4 & c_3 & c_4 \\ a_1 & a_2 & a_3 & a_1 & b_4 & b_3 & a_2 \end{pmatrix}$ is the endomorphism of $C_{3,4,4}; P_1$. Assume that there exists $g \in End(C_{3,4,4}; P_1)$ such that $fgf = f$. Since $f^{-1}(b_3) = \{c_3\}$ and $f^{-1}(b_4) = \{b_4\}$ are singleton sets, then $g(b_3) = c_3$ and $g(b_4) = b_4$. Since $\{b_3, b_4\} \in E(C_{3,4,4}; P_1)$ and $\{g(b_3), g(b_4)\} = \{c_3, b_4\} \notin E(C_{3,4,4}; P_1)$, then g is not an endomorphism which is a contradiction. So f is not regular. Hence $C_{3,4,4}; P_1$ is not endo-regular. $\square$

We can prove the next proposition by using the argument of Lemma 3.6.7.

Proposition 3.6.8. *For any $t \geq 1$ and $s \geq 2$, $C_{(2n+1)^{(t)},(4)^{(s)}}; P_1$ is not endo-regular.*

We can prove the next proposition by using the argument of Lemma 3.2.8.

Proposition 3.6.9. *For any $t \geq 1$, $C_{(2n+1)^{(t)},4}; P_1$ is endo-regular.*

Now we get the theorem which describes the regularity of endomorphism monoids of multiple 8-graphs.

Theorem 3.6.10. *Exactly the following multiple 8-graphs are endo-regular:*

- $C_{(2n+1)^{(t)}}; P_r$ *where* $r \geq 0$ *and* $t \geq 2$
- $C_{(2n+1)^{(t)},4}; P_1$ *where* $t \geq 1$
- $C_{(4)^{(s)}}; P_2 = K_{2,s+1}$.

Proof. This follows from Lemmas 3.6.3, 3.6.4, 3.6.6 and Propositions 3.6.8, 3.6.9. $\square$

3.7 Other endo-properties of multiple 8-graphs

We begin this section by consider the completely regularity of endomorphism monoids of multiple 8-graphs.

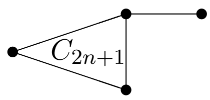
Lemma 3.7.1. *No endo-regular multiple* 8*-graph* $C_{(4)^{(s)}}; P_2 = K_{2,s+1}$ *is endo-completely-regular.*

Proof. This follows from Theorem 2.1.4. □

Lemma 3.7.2. *For any* $t \geq 2$*, the endo-regular* $C_{(2n+1)^{(t)}}; P_r$ *is endo-completely-regular.*

Proof. The proof is similar as the proof of Lemma 3.6.6. □

For any endo-regular multiple 8-graph $C_{(2n+1)^{(t)},4}; P_1$, it is clear that there exists an endomorphic image which has the following form.



By using the argument of Lemma 3.3.3 we can find some non-completely regular endomorphism f of $C_{(2n+1)^{(t)},4}; P_1$ whose endomorphic image is isomorphic to the above graph. So we get the next proposition.

Proposition 3.7.3. *For any* $t \geq 1$*, an endo-regular multiple* 8*-graph* $C_{(2n+1)^{(t)},4}; P_1$ *is not endo-completely-regular.*

Theorem 3.7.4. *Exactly an multiple* 8*-graph* $C_{2n+1,...,2n+1}; P_r$ *is endo-completely-regular where* $n \geq 1$ *and* $r \geq 0$ *.*

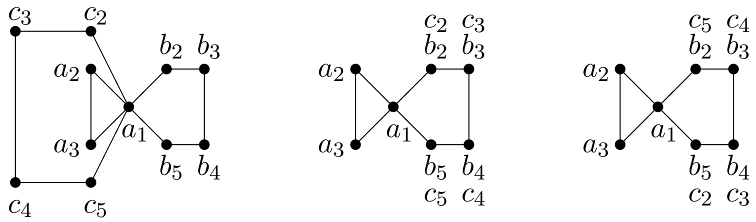
Next, we consider an endo-idempotent-closed multiple 8-graph. First we give a lemma describing if the multiple 8-graph contains two cycles C_{2n} and C_{2m} as strong subgraphs and $n \neq m$, then it is not endo-idempotent-closed. The proof of next lemma follows from Corollary 1.4.10.

Lemma 3.7.5. *Let* $C_{n_1,n_2,...,n_s}; P_r$ *be a multiple* 8*-graph. If* $n_i \neq n_j$ *are even for some* $i \neq j \in \{1, 2, ..., s\}$*, then* $C_{n_1,n_2,...,n_s}; P_r$ *is not endo-idempotent-closed.*

Next we consider the multiple 8-graph $C_{n_1,n_2,...,n_s}; P_0$ which contains two cycles of n vertices, C_n and C'_n, as subgraphs and $C_n \neq C'_n$. We can prove the next lemma by using the argument in the proof of Lemma 3.4.1.

Lemma 3.7.6. *Let $C_{n_1,n_2,\dots,n_s};P_0$ be a multiple 8-graph. If C_{n_i} and C_{n_j} are two difference cycle subgraphs of $C_{n_1,n_2,\dots,n_s};P_0$ which have n vertices for some $i \neq j \in \{1,2,\dots,s\}$, then $C_{n_1,n_2,\dots,n_s};P_r$ is not endo-idempotent-closed.*

Example 3.7.7. Take the multiple 8-graph $C_{3,5,5};P_0$ and its factor graphs as follows.

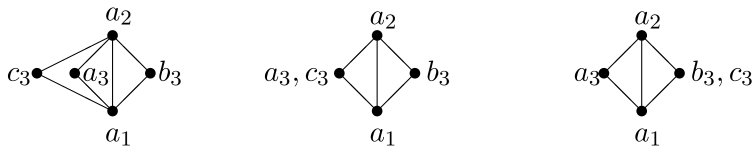


Then $i_1 = \begin{pmatrix} a_1 & a_2 & a_3 & b_2 & b_3 & b_4 & b_5 & c_2 & c_3 & c_4 & c_5 \\ a_1 & a_2 & a_3 & b_2 & b_3 & b_4 & b_5 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}$ and $i_2 = \begin{pmatrix} a_1 & a_2 & a_3 & b_2 & b_3 & b_4 & b_5 & c_2 & c_3 & c_4 & c_5 \\ a_1 & a_2 & a_3 & c_5 & c_4 & c_3 & c_2 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}$ are idempotent endomorphisms of $C_{3,5,5};P_0$. But $i_1 \circ i_2$ is not idempotent, so $C_{3,5,5};P_0$ is not endo-idempotent-closed. $\square$

Now we turn to the case $C_{(n)^{(t)}};P_r$ where $t \geq 2$, $r > 0$ and n is odd.

Lemma 3.7.8. *The multiple 8-graph $C_{3,3,3};P_1$ is not endo-idempotent-closed.*

Proof. Take the multiple 8-graph $C_{3,3,3};P_1$ and its factor graphs as follows.



Then $i_1 = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 & c_3 \\ a_1 & a_2 & a_3 & b_3 & a_3 \end{pmatrix}$ and $i_2 = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 & c_3 \\ a_1 & a_2 & a_3 & c_3 & c_3 \end{pmatrix}$ are idempotent endomorphisms of $C_{3,3,3};P_1$. But $i_2 \circ i_1 = \begin{pmatrix} a_1 & a_2 & a_3 & b_3 & c_3 \\ a_1 & a_2 & a_3 & c_3 & a_3 \end{pmatrix}$ is not idempotent. So $C_{3,3,3};P_1$ is not endo-idempotent-closed. $\square$

We can prove the next proposition by using the argument of the proof of Lemma 3.7.8.

Proposition 3.7.9. *Let $n \geq 3$ be odd, $t \geq 3$ and $r > 0$. Then the multiple 8-graph $C_{(n)^{(t)}};P_r$ is not endo-idempotent-closed.*

Now we get the theorem describing when the multiple 8-graph is endo-idempotent-closed.

Theorem 3.7.10. *Exactly the following multiple* 8*-graphs are endo-idempotent-closed:*

- $C_{2n_1+1,2n_2+1,...,2n_s+1}; P_r$ *where* $r \geq 0$ *and* $n_i \neq n_j$ *for* $i \neq j \in \{1, 2, ..., s\}$,
- $C_{2n+1,2n+1}; P_r$ *where* $r > 0$.

Theorem 3.7.11. *Exactly the multiple* $8-$*graphs* $C_{2n+1,2n+1}; P_r$ *are endo-orthodox where* $n \geq 1$ *and* $r > 0$.

Theorem 3.7.12. *Exactly the monoids of multiple* $8-$*graphs* $C_{2n+1,2n+1}; P_r$ *are orthogroups where* $n \geq 1$ *and* $r > 0$.

Theorem 3.7.13. *No multiple* 8*-graph is endo-Clifford.*

3.8 Conclusion

From all previous section, we got the relationship of endo-properties of multiple 8-graphs and 8-graphs as follows:
(1) endo-regular $\supsetneqq$ endo-completely-regular $\supsetneqq$ endo-orthodox = orthogroup.
(2) (endo-regular or endo-completely reglar) is not a subset of endo-idempotent-closed and vice versa.
(3) (endo-regular or endo-completely-regular) $\cap$ endo-idempotent-closed is not empty.

Finally, we give Table 3.1 and Table 3.2 containing the conclusion of endo-properties of multiple 8-graphs and containing the examples of multiple graphs which they have difference endo-properties, respectively. In these 2 tables, we use

endo-r. instead of endo-regular,
endo-c.r. instead of endo-completely-regular,
endo-i.c. instead of endo-idempotent-closed,
endo-o.t.d instead of endo-orthodox and
endo-C. instead of endo-Clifford.

We know that all multiple 8-graphs are not endo-Clifford. From Chapter 2, exactly K_2 is an endo-Clifford bipartite graph but K_2 is not retractive graph. This means now we do not have any retractive graph which its endomorphism monoid is a Clifford semigroup. So, we study more special graph (split graph) in the next chapter to find a graph which is retractive and endo-Clifford.

	Multiple 8-graph G is
endo-r	$\Leftrightarrow G$ is one kind of graphs as follows: (1) $C_{(2n+1)^{(t)}}; P_r$ where $r \geq 0$, $t \geq 2$ or (2) $C_{(2n+1)^{(t)},4}; P_1$ where $t \geq 1$ or (3) $C_{(4)^{(s)}}; P_2 = K_{2,s+1}$.
endo-c.r.	$\Leftrightarrow G$ forms $C_{(2n+1)^{(s)}}; P_r$ where $s \geq 2$, $r \geq 0$ and $n \geq 1$.
endo-i.c.	$\Leftrightarrow G$ forms (1) $C_{2n_1+1,2n_2+1,...,2n_s+1}; P_r$ where $r \geq 0$, $n_i \neq n_j$ for $i \neq j \in \{1, 2, ..., s\}$ or (2) $C_{2n+1,2n+1}; P_r$ where $r > 0$.
endo-o.t.d.	$\Leftrightarrow G$ forms $C_{2n+1,2n+1}; P_r$ where $n \geq 1$ and $r > 0$.

Table 3.1: Conclusion of the endo-properties of multiple 8-graphs

Multiple 8-graph	8-graph	endo-r.	endo-c.r.	endo-i.c.	endo-o.t.d	endo-C.	remark
$C_{4,4}; P_2$	✓	✓	No	No	No	No	bipartite graph
$C_{3,3}; P_0$	✓	✓	✓	No	No	No	-
$C_{3,5}; P_1$	✓	No	No	✓	No	No	-
$C_{3,3}; P_1$	✓	✓	✓	✓	✓	No	-
$C_{4,4,4}; P_2$	No	✓	No	No	No	No	bipartite graph
$C_{3,3,3}; P_0$	No	✓	✓	No	No	No	-
$C_{3,5,7}; P_1$	No	No	No	✓	No	No	-
$C_{3,3,3}; P_1$	No	✓	✓	✓	✓	No	-

Table 3.2: Example of multiple 8-graphs which have difference endo-properties.

Chapter 4

Split graphs

Split graphs may be regarded as the graphs between bipartite graphs and their complements. In this chapter, we find the algebraic structures of the monoid of split graphs.

4.1 Definition of split graphs

Split graph were introduced by Földes and and Hammer [9]. In this section, we describe definitions, propositions, lemmas and theorems with respect to split graph for further investigation in the next sections.

Definition 4.1.1. A graph $G(V,E)$ is called a ***split graph*** if its vertex-set can be partitioned into disjoint (non-empty) sets I and K, i.e., $V = K \cup I$, such that I is an independent set and K is a complete set.

In this dissertation, a split graph G is always written as $K_n \cup I_r$ where K_n is a maximal complete subgraph of G and $I_r = \overline{K}_r$.

Definition 4.1.2. Let $G = K_n \cup I_r$ be a split graph where K_n is a (may be not maximal) complete subgraph of G. We call $K_n \cup I_r$ be a ***unique decomposition*** of G with the clique size n if for every complete subgraph K_n' and every independent set I_r' such that $G = K_n' \cup I_r'$ one has $K_n' = K_n$ and $I_r' = I_r$.

Example 4.1.3. Let G be the graph as in Figure 4.1. We see that there are 2 complete subgraphs size 3, $K_3 = \{1,2,3\}$ and $K_3' = \{2,3,4\}$. It is clear that G can be partitioned to both of $K_3 \cup \{4,5\}$ and $K_3' \cup \{1,5\}$. So, there is no unique decompositions of G with the clique size 3. We have one complete subgraph $K_2 = \{2,3\}$ of G with $K_2 \cup \{1,4,5\}$, i.e., a unique decomposition of G with the clique size 2. □

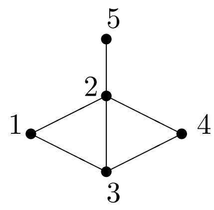


Figure 4.1: A split graph which has no a unique decomposition with the clique size 3.

This can be formulated in general as follows.

Proposition 4.1.4. *If K_n is a maximal complete subgraph of a split graph G and $K_n \cup I_r$ is not a unique decomposition with the clique size n, then $K_{n-1} \cup I_{r+1}$ is a unique decomposition with the clique size $n-1$.*

Definition 4.1.5. For any split graph $G = K_n \cup I_r$, let J be a subset of I_r. We call J ***a split component of*** I_r if for any $a, b \in J$, $N(a) = N(b)$ (including the case whose $N(a)$ and $N(b)$ are empty) and there is no $c \in I_r \backslash J$ such that $N(c) = N(a)$. And we say that I_r has s split components if I_r contains s distinct split components, i.e., $I_r = \bigcup_{i=1}^{s} J_i$, J_i a split component of I_r for all $i = 1, 2, ..., s$.

We observe that the split component is a ν-class in the terminology of [21]. This means that the canonical strong factor graph of $K_n \cup I_r$ is the form $K_n \cup I_s$, if I_r has s split components.

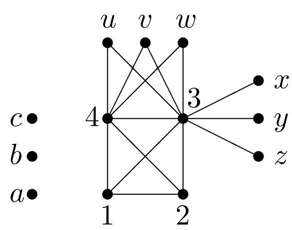


Figure 4.2: Split graph $K_4 \cup I_9$.

Example 4.1.6. Let G be the split graph as in Figure 4.2. So we consider $G = K_4 \cup I_9$ where $K_4 = \{1, 2, 3, 4\}$ and $I_9 = \{a, b, c, u, v, w, x, y, z\}$, the independent set I_9 has 3 split components, $J_1 = \{a, b, c\}$, $J_2 = \{u, v, w\}$ and $J_3 = \{x, y, z\}$. If we consider $G = K_3 \cup I_{10}$ where $K_3 = \{2, 3, 4\}$ and

$I_{10} = I_9 \cup \{1\}$, we have that the independent set I_{10} has 4 split components, J_1, J_2, J_3 and $J_4 = \{1\}$. $\square$

The regularity of endomorphism monoids of split graphs were studied by S. Fan in [8] and by W. Li and J. Chen in [27]. The next two theorems describe the regularity of endomorphism monoids of split graphs. We cite them from the results of Li and Chen in [27].

Theorem 4.1.7. ([27]) *Let $G(V, E)$ be a connected split graph with $V = K_n \cup I_r$. Then G is endo-regular if and only if for all $a \in I_r$ one has $|N(a)| = d$, $d \in \{1, ..., n-1\}$.*

Theorem 4.1.8. ([27]) *A non-connected split graph $K_n \cup I_r$ is endo-regular if and only if $N(a) = \emptyset$ for all $a \in I_r$.*

Next we give a lemma which describes the image of an endomorphism on a complete subgraph.

Lemma 4.1.9. *For any split graph $G = K_n \cup I_r$, let f be an endomorphism of G. If $|N(a)| < n-1$ for all $a \in I_r$, then $f(V(K_n)) = V(K_n)$.*

Proof. Take $V(K_n) = \{k_1, k_2, ..., k_n\}$ and f an endomorphism of G. It is clear that for any $i, j \in \{1, 2, ..., n\}$, $i \neq j$, $f(k_i) \neq f(k_j)$.

Next we show that for all $i \in \{1, 2, ..., n\}$, $f(k_i) \in V(K_n)$. Assume that there exists $r \in \{1, 2, ..., n\}$ such that $f(k_r) = c \in I$. Then $f(K_n) \subseteq N(c) \cup \{c\}$ and $|f(K_n)| = n$. Thus, $|N(c)| = n-1$ which is a contradiction to the assumption $< n-1$. $\square$

Lemma 4.1.10. *Let $G = K_n \cup I_r$ be an endo-regular split graph. If End(G) is completely regular, then $r < 2$.*

Proof. Let $r \geq 2$. Suppose that $a_1, a_2 \in I_r$, $a_1 \neq a_2$ and $V(K_n) = \{1, 2, ..., n\}$. Consider a mapping f with $f(a_1) = a_2$ and $f(K_n) = K_n$. If G is non-connected, set $f(x) = 1$ for all $x \in I \setminus \{a_1\}$. If G is connected, set f is a bijective from $N(a_1)$ to $N(a_2)$ and for any $x \in I_r \setminus \{a_1\}$, $f(x) \in V(K_n) \setminus \{f(y) | y \in N(x)\}$. It is easy to check that f is an endomorphism in G in both cases. In both cases $a_1 \notin Imf$. Since G is endo-regular, then there exists an endomorphism g such that $fgf = f$. Then

$$fg(a_2) = fgf(a_1) = f(a_1) = a_2,$$

and thus $g(a_2) = a_1$. Since $gf(a_1) = g(a_2) = a_1$ and $a_1 \notin Imf$, then $gf(a_1) \neq fg(a_1)$. Hence, we get already that $End(G)$ is not completely regular. $\square$

Lemma 4.1.11. ([21]) *Let G be a graph, $x_1, x_2 \in G$. There exists a strong endomorphism $f \in SEnd(G)$ with $f(x_1) = f(x_2)$ if and only if $N(x_1) = N(x_2)$.*

Remark 4.1.12. (1) If an endo-regular split graphs $G = K_n \cup I_r$ with I_r has exactly one split component and $|N(a)| = n-1$ for all $a \in I_r$, they are of the form $K_n \cup I_r = K_2[\overline{K}_{r+1}, K_{n-1}]$ (generalized lexicographic product see [21]). In this case we have by Proposition 4.1.4 that $K_{n-1} \cup I_{r+1}$ is a unique decomposition of G with the clique size $n-1$, and the canonical strong factor graph of $K_{n-1} \cup I_{r+1}$ is K_n. Then by Theorem 3.4 in [21], we have that $SEnd(K_{n-1} \cup I_{r+1}) \cong Aut(K_n)\ wr\ \mathcal{K}$ where $\mathcal{K} = \{\{u\} \mid u \in K_{n-1}\} \cup \{I_{r+1}\}$ is a small category (for definitions and notation see [21]). This means that every strong endomorphism can be described by an automorphism φ of K_n followed by a family of mappings. For every element x of K_n we take a mapping from the class $[x]$ of x to the class $[\varphi(x)]$ of $\varphi(x)$. For all $x \in K_n$ we get the family of mappings. Here most classes are one element, except for the class corresponding to I_{r+1}.

(2) For any endo-regular split graph $G = K_n \cup I_r$ with K_n is a maximal complete subgraph of G, if I_r has $s > 1$ split components, it is clear that $K_n \cup I_r$ is a unique decomposition of G with the clique size n.

4.2 Completely regular endomorphisms

We begin this section by specifying the condition in Theorem 1.4.7 for an endo-regular split graph G. We first prove a lemma which shows an additional property of a completely regular f of an endo-regular split graph G.

Lemma 4.2.1. *Let $G = K_n \cup I_r$ be an endo-regular split graph and let f be a completely regular endomorphism on G. If $|N(a)| < n-1$ for all $a \in I$, then for any $d \in I_r$, if $f(d) \in K_n$, then $d \notin Im(f)$.*

Proof. Let f be a completely regular endomorphism of G. Let $d \in I_r$ with $f(d) \in K_n$. Assume that $d \in Im(f)$. Since $|N(a)| < n-1$ for all $a \in I_r$, we get by Lemma 4.1.9 that $f(K_n) = K_n$. Then there exists $c \in I_r$ such that $f(c) = d$. Now we have that $f^2(c) = f(d) =: x \in K_n$. Since $f(K_n) = K_n$, then there exists $u \in K_n$ with $f(u) \in K_n$ and $f^2(u) = x$. Since $f^2(u) = f^2(c)$ and f is completely regular, by Theorem 1.4.7, we have that $f(u) = f(c) = d \in I_r$. This a contradiction. Then we get that $d \notin Im(f)$. □

To prove the main theorem in this section we need some notations and some lemmas. For any $f \in End(G)$, define

$$End_f(G) := \{g \in End(G) | \rho_f = \rho_g\}$$

the set of all endomorphisms of G with congruence relation ρ_f. Note that $End_f(G)$ is Green's $\mathcal{L}-$class of f.

Lemma 4.2.2. *For any endo-regular split graph $G = K_n \bigcup I_r$, let f be an endomorphism of G. If f is a bijective or $f(G) \cong K_n$, then $End_f(G)$ is a group.*

Proof. If f is bijective, we see that $End_f(G) = Aut(G)$. Otherwise:

(a) If $|N(a)| = m < n - 1$ for all $a \in I_r$, it is clear by Lemma 4.1.9 that $End_f(G) \cong End(K_n) \cong S_n$.

(b) If $|N(a)| = n - 1$ for all $a \in I_r$, we have to consider the ways $Im(f) \cong K_n$ can be embedded into G. There are $r + 1$ ways each followed by all permutation of the image. So we get $r + 1$ times S_n. Moreover, it is clear that $End_f(G)$ altogether is isomorphic to the left group $S_n \times L_{r+1}$. □

Theorem 4.2.3. *For any endo-regular split graph $G = K_n \cup I_r$, $End(G)$ is completely regular if and only if $r = 1$.*

Proof. Let $G = K_n \cup I_r$ be an endo-regular split graph. If $r = 1$, then by Lemma 4.2.2 and Theorem 1.1.9 we get $End(G)$ is completely regular. If $r > 1$, we get that $End(G)$ is not completely regular monoid by Lemma 4.1.10. □

Continuing the consideration from Remark 4.1.12 we get the following proposition.

Proposition 4.2.4. *For any endo-regular split graph $G = K_n \cup I_r$,*

(1) *if $|N(a)| < n - 1$ for all $a \in I_r$, then $f \in End(G)$ is a strong endomorphism if and only if $f(c) \in I_r$ $\forall c \in I_r$;*

(2) *if $K_n \cup I_r$ is not a unique decomposition of G with the clique size n, then all $f \in End(G)$ are strong endomorphisms.*

Proof. (1) Necessity. Let $f \in End(G)$ be a strong endomorphism. Assume that there exists $c \in I_r$ with $f(c) = u \in K_n$. By Lemma 4.1.9, we have that $f(K_n) = K_n$. Then there exist $x \in K_n$ such that $f(x) = u$, so $f(x) = f(c)$. Since $|N(c)| < n - 1$ and $|N(x)| \geq n - 1$, by Lemma 4.1.11 we get that f is not a strong endomorphism. This is a contradiction. Then $f(c) \in I_r$ for all $c \in I_r$.

Sufficiency. Let $f \in End(G)$ with $f(c) \in I_r$ for all $c \in I_r$. Let $\{f(u), f(v)\} \in E(G)$. If $f(u), f(v) \in K_n$, it is clear that $u, v \in K_n$, so $\{u, v\} \in E(G)$. It remains to consider $f(u) \in K_n$ and $f(v) \in I_r$. By Lemma

4.1.9 and hypothesis we have that $u \in K_n$ and $v \in I_r$. Since $v, f(v) \in I_r$, by hypothesis we have $|N(v)| = |N(f(v))|$. Since f is an endomorphism, then $f(N(v)) = N(f(v))$. Since $f(u) \in N(f(v))$ and $f(K_n) = K_n$, then $u \in N(v)$ so $\{u, v\} \in E(G)$. Then we get that f is a strong endomorphism.

(2) This case is obvious, look for example to the graph in Example 4.1.3 without point 5. □

For the endo-idempotent-closed split graph, we ever got the result but we found later that it was wrong. I will consider this endo-property of split graph again in the next chance.

4.3 Completely regular subsemigroups

Since exactly endo-regular split graphs $G = K_n \cup I_1$ are endo-completely-regular for any $n \geq 1$, so in this section we need to characterize some completely regular subsemigroups of endo-regular split graph $G = K_n \cup I_r$ where $r \geq 1$. But it is so complicated to generalize a completely regular subsemigroups of an endomorphism monoid of any endo-regular split graph. So we consider only three cases of endo-regular split graphs $G = K_n \cup I_r$:

(1) with exactly one split component of I_r

(2) with $s > 1$ split components of I_r and $|N(a)| = 1$ where $a \in I_r$

(3) with $s > 1$ split components of I_r and $|N(a)| \geq 2$ where $a \in I_r$.

In this section, it is natural that we find left groups as subsemigroups of endomorphism monoids, which of course are completely regular.

Endo-regular split graphs $K_n \cup I_r$ with exactly one split component of I_r

In this section, we characterize completely regular subsemigroups contained in $End(G)$ where G is endo-regular split graphs with exactly one split component. First we give a lemma which describes the image of any endomorphism and the composition of any two endomorphisms of an endo-regular split graph $G = K_n \cup I_r$ restricted to $K_n \setminus N(a)$ and to N(a).

Lemma 4.3.1. *Let $G = K_n \cup I_r$ be an endo-regular split graph such that I_r has exactly one split component, i.e., $N(a) = N(b)$ for all $a, b \in I_r$. If $f, g \in End(G)$ with $f(G) \ncong K_n$ and $g(G) \ncong K_n$, we have $f(N(a)) = N(a)$, and $(f \circ g)(N(a)) = N(a)$. If $|N(a)| < n - 1$ for all $a \in I_r$, we have in addition $f(K_n \setminus N(a)) = K_n \setminus N(a)$, $(f \circ g)(K_n \setminus N(a)) = K_n \setminus N(a)$ and the statement is also true for $f(G) = K_n$.*

Proof. Let f be an endomorphism of G which $f(G) \ncong K_n$. Let $u \in N(a)$. Assume that $f(u) \notin N(a)$. Then $f(u) \in (K_n \setminus N(a)) \cup I_r$. We consider two cases.

Case 1. $|N(a)| < n-1$ for all $a \in I_r$. By Lemma 4.1.9, it is impossible that $f(u) \in I_r$, so $f(u) \in K_n \setminus N(a)$. Since $f(G) \ncong K_n$ and $f(K_n) = K_n$, there exists a vertex $v \in I_r$ such that $f(v) \in I_r$. Since $f(u) \notin N(a)$ for all $a \in I_r$, then $f(u) \notin N(f(v))$, i.e., $\{f(u), f(v)\} \notin E(G)$. But $\{u, v\} \in V(G)$ and f is an endomorphism, then this is a contradiction.

Case 2. $|N(a)| = n-1$ for all $a \in I_r$. Since I_r has exactly one split component and K_n is a maximal complete subgraph, there exists one vertex $x \in K_n$ such that $x \notin N(a)$ and $N(x) = N(a)$. For example, we consider the graph as in Figure 4.3 where $K_n = K_3 = \{1, 2, x\}$ and $I_r = I_5 = \{a, b, c, d, e\}$. It is clear that only vertex $x \in K_3$ is such that $x \notin N(a)$ and $N(x) = N(a)$. It is obvious that $I_r \cup \{x\}$ is an independent set of G.

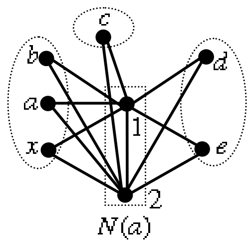


Figure 4.3: Endo-regular split graph $G = K_3 \cup I_5$ which $K_3 \cup I_5$ is not a unique decomposition of G with the clique size 3.

Now we assume that $f(u) \in I_r \cup \{x\}$. Since $f(G) \ncong K_n$ and f preserves K_n, there exists $v \in I_r \cup \{x\}$ such that $f(v) \in I_r \cup \{x\}$. Since $I_r \cup \{x\}$ is an independent set, $\{f(u), f(v)\} \notin E(G)$. But $\{u, v\} \in E(G)$ and f is an endomorphism, we have a contradiction.

Moreover, if $|N(a)| < n-1$ for all $a \in I_r$, by Lemma 4.1.9 we have $f(K_n) = K_n$. So we get that $f(K_n \setminus N(a)) = K_n \setminus N(a)$. $\square$

Remark 4.3.2. Lemma 4.3.1 is not true in the case when $|N(a)| = n-1$ for all $a \in I_r$ and $f \in End(G)$ with $f(G) \cong K_n$. For example, take G a graph as in Figure 4.3. We see that $K_3 = \{1, 2, x\}$ is a maximal complete subgraph of G, $I_5 = \{a, b, c, d, e\}$ is an independent set and $N(a) = \{1, 2\}$. It is obvious that $f = \begin{pmatrix} 1 & 2 & x & a & b & c & d & e \\ a & 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$ is an endomorphism of G with $f(G) \cong K_3$. But $f(N(a)) = f(\{1, 2\}) = \{1, a\} \neq N(a)$.

Note that if A is any set, then we denote by S_A the group of permutations of the elements in A. For examples, $S_{\{1,2,3\}}$, $S_{\{\{a,b\},\{c,d\}\}}$ are the symmetric group S_3 and S_2, respectively.

In Theorem 4.3.3 and Corollary 4.3.5, K_n is not necessarily a maximal complete subgraph of the split graph $G = K_n \cup I_r$, since for some $f \in End(G)$ with $f(G)$ isomorphic to a maximal complete subgraph of G we may have the following situation. For example, we consider f as in Remark 4.3.2. We see that $f(\{a, b, c, d\}) = \{2\} \nsubseteq I_4 = \{a, b, c, d, \}$, so there is no congruence class whose a subset of I_4. Then we can not construct the set of representatives A as is defined in Theorem 4.3.3. This implies that we can not construct the set $CRE_f^A(G)$. Then in the next theorem and its corollary, we do not consider the case when $f(G)$ isomorphic to a maximal complete subgraph of G. Although, we have Lemma 4.2.2 which shows $End_f(G)$ is a group, so $End_f(G)$ is a completely regular monoid.

Theorem 4.3.3. *Let $G = K_n \cup I_r$ be an endo-regular split graph such that I_r has exactly one split component and $K_n \cup I_r$ is a unique decomposition of G with the clique size n. Suppose $f \in End(G)$ with $f(G)$ is not isomorphic to the maximal complete subgraph of G. Suppose that f has q congruence classes which are subsets of I_r for some $q \in \mathbb{N}$, namely, $[i_1]_{\rho_f}$, $[i_2]_{\rho_f}$,..., $[i_q]_{\rho_f}$, $i_1, ..., i_q \in I_r$. For every $j = 1, 2, ..., q$, choose a representative $a_j \in [i_j]_{\rho_f}$ for all $j = 1, 2, ..., q$ and set $A := \{a_1, a_2, ..., a_q\}$. Set $I_r^f := \{i \in I_r \mid f(i) \in I_r\}$ and*

$$CRE_f^A(G) := \{h \in End_f(G) | h \text{ c.r.},\ h(I_r^f) = A\}$$

the set of all completely regular endomorphisms in $End_f(G)$ such that their restrictions on I_r^f give the set A. Then we have that $CRE_f^A(G)$ is the group $S_{n-m} \times S_m \times S_q$.

Proof. Case 1. K_n is a maximal complete subgraph of G. To illustrate the situation in this case, i.e., $|N(a)| = m < n - 1$ for all $a \in I_r$, we consider the graph as in Figure 4.4. In this graph we use $K_n = K_5$, $m = 2$ and $q = 3$. Take f such that the dotted ovals in the picture are the congruence classes induced by f which are subsets of I_r. Now take $A = \{a, d, e\}$. We get $CRE_f^A(G)$ is isomorphic to $S_3 \times S_2 \times S_3 = S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_A$.

By the graph as in Figure 4.4 and Lemma 4.3.1, it is obvious that $CRE_f^A(G)|_{(K_n \setminus N(a))}$ and $CRE_f^A(G)|_{N(a)}$, the sets of restrictions of all endomorphisms in $CRE_f^A(G)$ to $K_n \setminus N(a)$ and to $N(a)$, are isomorphic to S_{n-m} and S_m, respectively. For any endomorphism h in $CRE_f^A(G)$, we get $h(u) = h(a_j)$ for all $u \in [i_j]_{\rho_f}$, $j = 1, 2, ..., q$. So we have that $CRE_f^A(G)|_{I_r^f}$ is

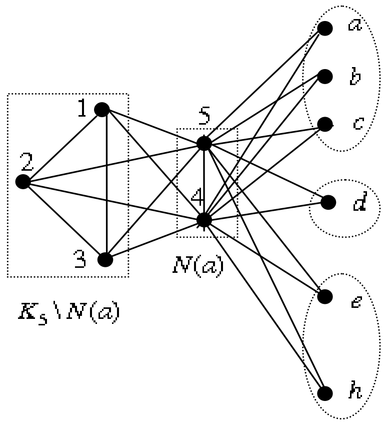


Figure 4.4: Endo-regular split graph $G = K_5 \cup I_6$ which $K_5 \cup I_6$ is a unique decomposition of G with the clique size 5.

isomorphic to $CRE_f^A(G)|_A$. By inspection it is clear that $CRE_f^A(G)|_A$ is isomorphic to S_q. Then we have that $CRE_f^A(G)$ is isomorphic to $S_{n-m} \times S_m \times S_q$

Case 2. K_n is not a maximal complete subgraph of G. Consider the graph as in Figure 4.3. Here $K_n = K_2 = N(a)$ and $q = 3$. The three dotted ovals in the graph are the congruence classes induced by f which are subsets of I_r. Take now $A = \{x, c, d\}$. We get $CRE_f^A(G)$ is isomorphic to $S_2 \times S_3 = S_{\{1,2\}} \times S_A$.

Formally, the result is the same as before since now $K_n \setminus N(a) = \emptyset$, then $m = n - 1$ and $CRE_f^A(G) = S_{n-m} \times S_m \times S_q \cong S_{n-1} \times S_q$. $\square$

Before we determine the maximal completely regular subsemigroup contained in $End_f(G)$ for an endo-regular split graph $G = K_n \cup I_r$ where I_r has exactly one split component, we give two examples which show the composition between the elements of two groups $CRE_f^A(G)$ and $CRE_f^B(G)$ which are contained in $End_f(G)$ where f is an endomorphism of an endo-regular split graph G.

Example 4.3.4. First, we consider $K_n \cup I_r$ with a unique decomposition of G with the clique size n and next we consider $K_n \cup I_r$ with a non-unique decomposition of G with the clique size n where K_n is a maximal complete subgraph of G.

(1) Take G a graph as in Figure 4.5. Let $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & c & c \end{pmatrix}$ be a mapping from G to G. Note that $\underline{a}b$, $\underline{c}d$ in graph H (in Figure 4.5)

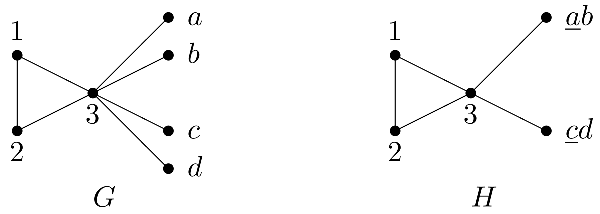


Figure 4.5: Endo-regular split graph $G = K_3 \cup I_4$ and H a factor graph induce by f in Example 4.3.4 (1).

mean $f(\{a,b\}) = \{a\}$ and $f(\{c,d\}) = \{c\}$. It is clear that f is an endomorphism. The graph H in Figure 4.5 is the factor graph of G induced by f. It is clear that f is idempotent, so it is completely regular. We have two congruence classes $\{a,b\}$ and $\{c,d\}$ which are subsets of the independent set $I_4 = \{a,b,c,d\}$. For every completely regular endomorphism $h \in End_f(G)$, it is impossible that $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, since $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, would imply that $h(a) \neq h(c)$ and $h^2(a) = h^2(c)$. This contradicts to Theorem 1.4.7. Now we get that for any completely regular endomorphism $h \in End_f(G)$,

(a) h sends $\{a,b\}$ to $\{a,b\}$ if and only if h sends $\{c,d\}$ to $\{c,d\}$

(b) h sends $\{a,b\}$ to $\{c,d\}$ if and only if h sends $\{c,d\}$ to $\{a,b\}$.

By Theorem 4.3.3, we know that $CRE_f^{\{a,c\}}(G)$ is isomorphic to $S_2 \times S_1 \times S_2 = S_2 \times S_2$. The 4 endomorphisms in $CRE_f^{\{a,c\}}(G)$ are

$f_1 = f$, $f_2 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & c & c & a & a \end{pmatrix}$, $f_3 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & a & a & c & c \end{pmatrix}$ and $f_4 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & c & c & a & a \end{pmatrix}$.

Similarly, we know that $CRE_f^{\{a,d\}}(G)$ is isomorphic to $S_2 \times S_2$. The 4 endomorphisms in $CRE_f^{\{a,d\}}(G)$ are

$g_1 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & d & d \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & d & d & a & a \end{pmatrix}$,
$g_3 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & a & a & d & d \end{pmatrix}$ and $g_4 = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 2 & 1 & 3 & d & d & a & a \end{pmatrix}$.

We will consider the composition between the elements of $CRE_f^{\{a,c\}}(G)$ and the elements of $CRE_f^{\{a,d\}}(G)$. For any $h \in CRE_f^{\{a,c\}}(G)$ and $k \in CRE_f^{\{a,d\}}(G)$, it is clear by inspection that $(h \circ k) \in CRE_f^{\{a,c\}}(G)$. The table in Table 4.1 shows the composition between the elements of these two groups.

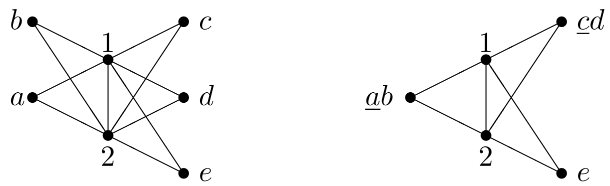
From the Table 4.1, it is clear that we get the left group $(S_2 \times S_2) \times L_2$.

$\circ$	f_1	f_2	f_3	f_4	g_1	g_2	g_3	g_4
f_1	f_1	f_2	f_3	f_4	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1	f_4	f_3	f_2	f_1
g_1	g_1	g_2	g_3	g_4	g_1	g_2	g_3	g_4
g_2	g_2	g_1	g_4	g_3	g_2	g_1	g_4	g_3
g_3	g_3	g_4	g_1	g_2	g_3	g_4	g_1	g_2
g_4	g_4	g_3	g_2	g_1	g_4	g_3	g_2	g_1

Table 4.1: Composition of two completely regular subsemigroups $CRE_f^{\{a,c\}}(G)$ and $CRE_f^{\{a,d\}}(G)$ in Example 4.3.4 (1).

Moreover, we have two more groups $CRE_f^{\{b,c\}}(G)$ and $CRE_f^{\{b,d\}}(G)$ contained in $End_f(G)$. Then we get $\bigcup_{i\in\{a,b\}} \bigcup_{j\in\{c,d\}} CRE_f^{\{i,j\}}(G)$ is isomorphic to the left group $(S_2 \times S_2) \times L_4$ and this is a maximal completely regular subsemigroup of $End_f(G)$.

(2) Take $G = K_2 \cup I_5$ the split graph as in Figure 4.6, with $K_2 = \{1, 2\}$ and $I = \{a, b, c, d, e\}$.



G H

Figure 4.6: Endo-regular split graph $G = K_2 \cup I_5$ and H a factor graph induce by f in Example 4.3.4 (2).

Consider the mapping $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d & e \\ 1 & 2 & a & a & a & c & c & e \end{pmatrix}$ from G to G. It is clear that f is an endomorphism. The image graph $H = f(G)$ (in Figure

4.6) is a subgraph of G. Note that $\underline{a}b$, $\underline{c}d$ in graph H (in Figure 4.6) mean $f(\{a,b\}) = \{a\}$ and $f(\{c,d\}) = \{c\}$. Now we know that all endomorphisms in $End_f(G)$ are the embeddings of H into G. By Theorem 4.1.7, we have that f is regular. And we have three congruence classes $\{a,b\}$, $\{c,d\}$ and $\{e\}$ induced by f which are subsets of I_5. For every completely regular endomorphism $h \in End_f(G)$, it is impossible that $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$. Since $h(\{a,b\}) \cap h(\{c,d\}) \neq \emptyset$, then $h(a) \neq h(c)$ and $h^2(a) = h^2(c)$. This contradicts to Theorem 1.4.7. By the same ways, it is impossible that $h(\{a,b\}) \cap h(\{e\}) \neq \emptyset$ and $h(\{c,d\}) \cap h(\{e\}) \neq \emptyset$. This implies that for every completely regular endomorphism $h \in End_f(G)$, $h(I_5)$ is isomorphic to some element in the symmetric group $S_{\{\{a,b\},\{c,d\},\{e\}\}}$.

We have 4 difference sets of representatives, $\{a,c,e\}$, $\{a,d,e\}$, $\{b,c,e\}$ and $\{b,d,e\}$. By Theorem 4.3.3, we know that $CRE_f^{\{i,j,e\}}(G)$ is isomorphic to $S_2 \times S_3 (= S_{\{1,2\}} \times S_{\{i,j,e\}})$ for all $i \in \{a,b\}$ and $j \in \{c,d\}$.

By inspection, it is clear that $\bigcup\limits_{i\in\{a,b\}} \bigcup\limits_{j\in\{c,d\}} CRE_f^{\{i,j,e\}}(G)$ is isomorphic to the left group $(S_2 \times S_3) \times L_4$. □

Using Theorem 4.3.3 and Example 4.3.4, we get the next corollary.

Corollary 4.3.5. *Let $G = K_n \bigcup I_r$ be an endo-regular split graph such that I_r has exactly one split component and $K_n \cup I_r$ is a unique decomposition of G. Suppose $f \in End(G)$ with $f(G)$ is not isomorphic to maximal complete subgraph of G. Suppose that f has q congruence classes which are subsets of I_r for some $q \in \mathbb{N}$, namely, $[i_1]_{\rho_f}$, $[i_2]_{\rho_f}$,..., $[i_q]_{\rho_f}$, $i_1, ..., i_q \in I_r$. Set $\mathcal{A} := \{\{a_1, a_2, ..., a_q\} \mid a_j \in [i_j]_{\rho_f}, j \in \{1, 2, ..., q\}\}$ the set of sets of representatives. The maximal completely regular subsemigroup of $End_f(G)$ denoted by $CRE_f(G)$ is the union of $|\mathcal{A}|$ groups $CRE_f^A(G)$ where $A \in \mathcal{A}$. More precisely, we have that $CRE_f(G)$ is the left group $(S_{n-m} \times S_m \times S_q) \times L_{|\mathcal{A}|}$.*

Endo-regular split graphs $K_n \cup I_r$ with $s > 1$ split components of I_r and $|N(a)| = 1$ for all $a \in I_r$

In this section we characterize completely regular subsemigroups of endomorphism monoids of endo-regular split graphs $G = K_n \cup I_r$ where I_r has $s > 1$ split components $J_1, J_2, ..., J_s$ and $|N(a)| = 1$ for all $a \in I_r$. Let f be a completely regular endomorphism of G. This notation will be used everywhere in this section. To get the theorem which describes the structure of this completely regular subsemigroups, we need 3 lemmas.

The following lemma is the analogue of Lemma 4.3.1 for $s > 1$ and $|N(a)| = 1$.

Lemma 4.3.6. *With the above notation, suppose that $J_1, J_2, ..., J_p$ are the split components of I_r with $f(J_j) \subseteq K_n$ for $j = 1, 2, .., p$. Set $J := J_1 \cup J_2 \cup ... \cup J_p$. Then we have $f(K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)) = K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$ and $f(\bigcup_{a \in I_r \setminus J} N(a)) = \bigcup_{a \in I_r \setminus J} N(a)$.*

Proof. Let $u \in K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$. Assume that $f(u) \in \bigcup_{a \in I_r \setminus J} N(a)$. Since $f(K_n) = K_n$ by Lemma 4.1.9, there exists $v \in \bigcup_{a \in I_r \setminus J} N(a)$ such that $f(v) \in K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$, i.e., $f(v) \notin N(I_r \setminus J)$. Suppose that $v \in N(J_l)$ for some $J_l \notin \{J_1, J_2, ..., J_p\}$. Since $|N(a)| = 1$ for all $a \in I_r$, by Lemma 4.2.1, we know that for all $d \in I_r \setminus J$ if $f(d) \in I_r$, then $f(d) \in I_r \setminus J$. Since $J_l \notin \{J_1, J_2, ..., J_p\}$, there exists $e \in J_l$ such that $f(e) \in I_r \setminus J$. Now we have $f(v) \notin N(f(e))$. Since $\{v, e\} \in E(G)$ and f is an endomorphism, we get that $\{f(v), f(e)\} \in E(G)$, i.e., $f(v) \in N(f(e))$. This is a contradiction. Thus we have $f(K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)) = K_n \setminus \bigcup_{a \in I_r \setminus J} N(a)$. Consequently, since $f(K_n) = K_n$, we get that $f(\bigcup_{a \in I_r \setminus J} N(a)) = \bigcup_{a \in I_r \setminus J} N(a)$. $\square$

Lemma 4.3.7. *With the above notation, set $J_j^{\rho_f} := \{[i]_{\rho_f} \mid i \in J_j$ and $[i]_{\rho_f} \subseteq J_j\}$ and $J_j^f := \{i \in J_j \mid f(i) \in I\}$ for all $j = 1, 2, ..., s$. Then we have for any $\alpha, \beta \in \{1, 2, ..., s\}$ that $f(J_\alpha^f) \subseteq J_\beta$ implies $|J_\alpha^{\rho_f}| = |J_\beta^{\rho_f}|$.*

Proof. Let f be a completely regular endomorphism of G and $f(I_\alpha^f) \subseteq J_\beta$ for some $\alpha, \beta \in \{1, 2, ..., s\}$, $\alpha \neq \beta$. Assume that $\ell_\alpha := |J_\alpha^{\rho_f}| \neq |J_\beta^{\rho_f}| =: \ell_\beta$.

First, we consider the case $\ell_\alpha > \ell_\beta$. Let $[a_1]_{\rho_f}$, $[a_2]_{\rho_f}$, ..., $[a_{\ell_\alpha}]_{\rho_f}$ be ℓ_α congruence classes in $J_\alpha^{\rho_f}$. Since $f(J_\alpha^f) \subseteq J_\beta$, then for any $l \in \{1, 2, ..., \ell_\alpha\}$, $f(a_l) = b_l$ for some b_l in J_β. By Lemma 4.2.1, we know that $b_l \in J_\beta^f$. Since $\ell_\alpha > \ell_\beta$, there exist $j \neq k \in \{1, 2, ..., \ell_\alpha\}$ such that $f(a_j) = b_j \neq b_k = f(a_k)$ and $[b_j]_{\rho_f} = [b_k]_{\rho_f}$, i.e., $f^2(a_j) = f^2(a_k)$. That means f is not square injective, contradicting to Theorem 1.4.7.

Next, we consider the case $\ell_\alpha < \ell_\beta$. Since I_r is finite, there exists some split components J_μ and J_ν of I_r with $f(J_\mu^f) \subseteq J_\nu$ and $|J_\mu^{\rho_f}| > |J_\nu^{\rho_f}|$. As in the first case we get a contradiction. Then we have that $|J_\alpha^{\rho_f}| = |J_\beta^{\rho_f}|$. $\square$

Now we give an example which illustrates the next lemma.

Example 4.3.8. Take $G = K_4 \cup I_9$ an endo-regular split graph as in Figure 4.7.

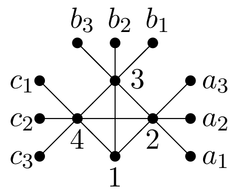


Figure 4.7: Split graph $G = K_4 \cup I_5$ with $Aut(G) = S_3 \times S_3 \times S_3 \times S_3$.

Here $J_1 = \{a_1, a_2, a_3\}$, $J_2 = \{b_1, b_2, b_3\}$ and $J_3 = \{c_1, c_2, c_3\}$ are the three split components of I_9. By Lemma 4.3.6, we have $f(1) = 1$ and $f(\{2,3,4\}) = \{2,3,4\}$ for all $f \in Aut(G)$. And by Lemma 4.3.7, we get that all automorphisms of G permute three split components J_1, J_2 and J_3. And in any split component, we can permute all vertices to get an automorphism. Then it is clear that $Aut(G) = S_1 \times S_3 \times (S_3 \times S_3 \times S_3)$. □

Lemma 4.3.9. *With the above notation, if $|J_1| = |J_2| = ... = |J_s| =: \ell$, we have that $Aut(G)$ is isomorphic to $S_{n-s} \times S_s \times \underbrace{S_\ell \times S_\ell \times ... \times S_\ell}_{s \text{ times}}$.*

Theorem 4.3.10. *Take an endo-regular split graph $G = K_n \cup I_r$ where $I_r = \bigcup\limits_{k=1}^{s} J_k$ with $s > 1$ split components $J_1, J_2, ..., J_s$. Suppose that for all $a \in I_r$, $|N(a)| = 1$ and $|\bigcup\limits_{a \in I_r} N(a)| = m$. Take a regular endomorphism f of G with q congruence classes $[i_1]_{\rho_f}, [i_2]_{\rho_f}, ..., [i_q]_{\rho_f}$ each contained in I_r. Set $I_r^f := \{i \in I_r | f(i) \in I_r\}$, $J_j^f := \{i \in J_j | f(i) \in I_r\}$ and take the set of sets of representatives $\mathcal{A} := \{\{a_1, a_2, ..., a_q\} \mid a_j \in [i_j]_{\rho_f},\ j = 1, 2, ..., q\}$. Take $A \in \mathcal{A}$ and let $CRE_f^A(G)$ be the same as in Theorem 4.3.3. For any $k = 1, 2, ..., s$, if $J_k^f \neq \emptyset$, take $u \in N(J_k^f)$ and set $M_A^f(u) := \{v \in N(J_l^f) \mid |J_k^f \cap A| = |J_l^f \cap A|,\ l \in \{1, ..., s\}\}$. Suppose that there are t disjoint sets $M_A^f(u_1)$, $M_A^f(u_2)$,..., $M_A^f(u_t)$. Then we have that $CRE_f^A(G) = S_{n-m+p} \times \prod\limits_{j=1}^{t} S_{M_A^f(u_j)} \times \prod\limits_{k=1}^{s} S_{J_k^f \cap A}$. Here p is the number of split components whose vertices are all sent to K_n by f,*

S_{n-m+p} is the group of permutations of all vertices in $(K_n \setminus N(I_r)) \cup \bigcup\limits_{|J_j^f|=0} N(J_j^f)$,

$S_{M^f(u_j)}$ is a the group of permutations of all vertices in $M^f(u_j)$ and $S_{J_k^f\cap A}$ is the group of permutations of all vertices in $J_k^f\cap A$.

The next example shows the idea how to prove the above theorem.

Example 4.3.11. Consider the split graph $G = K_8 \cup I_{11}$ as in Figure 4.8 and $f \in End(G)$ such that $H = Im(f) \cong G/\rho_f$, where notations $\underline{b_1}b_2$, $\underline{2}c$ and $d_1\underline{d_2}$ are as in Example 4.3.4. We have the 6 split components, $J_1 = \{a_1, a_2\}$, $J_2 = \{b_1, b_2\}$, $J_3 = \{c\}$, $J_4 = \{d_1, d_2\}$, $J_5 = \{e_1, e_2\}$ and $J_6 = \{g_1, g_2\}$. By Theorem 4.1.7, we know that all endomorphisms in $End(G)$ are regular. Take

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_2 & c & d_1 & d_2 & e_1 & e_2 & g_1 & g_2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_1 & 2 & d_2 & d_2 & e_1 & e_2 & g_1 & g_2 \end{pmatrix},$$

the image graph is H (in Figure 4.8) as a subgraph of G. We see that $f(G) \ncong K_8$ and we have 8 congruence classes induced by f which are subsets of I_{11}, namely, $\{a_1\}$, $\{a_2\}$, $\{b_1, b_2\}$, $\{d_1, d_2\}$, $\{e_1\}$, $\{e_2\}$, $\{g_1\}$ and $\{g_2\}$ only $\{c, 2\} \nsubseteq I_{11}$, now we have for p from Theorem 4.3.10 that $p = 1$.

Choose the set of representatives $A = \{a_1, a_2, b_1, d_1, e_1, e_2, g_1, g_2\}$ then $I_{11}^f = \{i \in I_{11} \mid f(i) \in I_{11}\} = \{a_1, a_2, b_1, b_2, d_1, d_2, e_1, e_2, g_1, g_2\}$. We will show that $CRE_f^A(G)$ is isomorphic to $S_3 \times (S_3 \times S_2 \times S_2 \times S_2) \times S_2$. We have exactly one split component, J_3, such that $f(J_3) \subseteq K_8$. And the congruence relation for all endomorphisms in $End_f(G)$ is ρ_f. By definition, it is clear that $CRE_f^A(G)|_{(\{1,2,5\})}$, the set of restrictions of all endomorphisms in $CRE_f^A(G)$ to $\{1, 2, 5\}$, is isomorphic to $S_{\{1,2,5\}}$, the group S_3 of permutations of the set $\{1, 2, 5\}$.

Since $J_j^f = \{i \in J_j \mid f(i) \in I_{11}\}$ for all $j = 1, ..., 6$, we see that $2 = |J_1^f \cap A| = |J_5^f \cap A| = |J_6^f \cap A| \neq |J_2^f \cap A| = |J_4^f \cap A| = 1$, then we get $t = 2$, t from Theorem 4.3.10 , and we have $M_A^f(3) = M_A^f(7) = M_A^f(8) = \{3, 7, 8\}$, $M_A^f(4) = M_A^f(6) = \{4, 6\}$. By definition of $J_j^{\rho_f}$ in Lemma 4.3.7, we have $J_1^{\rho_f} = \{\{a_1\}, \{a_2\}\}$, $J_2^{\rho_f} = \{\{b_1, b_2\}\}$, $J_4^{\rho_f} = \{\{d_1, d_2\}\}$, $J_5^{\rho_f} = \{\{e_1\}, \{e_2\}\}$ and $J_6^{\rho_f} = \{\{g_1\}, \{g_2\}\}$. Since $2 = |J_1^{\rho_f}| = |J_5^{\rho_f}| = |J_6^{\rho_f}| \neq |J_2^{\rho_f}| = |J_4^{\rho_f}| = 1$, by Lemma 4.3.7, we know that all endomorphisms in $CRE_f^A(G)$ do not send an element in $J_1^f \cup J_5^f \cup J_6^f$ to an element in $J_2^f \cup J_4^f$. Similarly, all endomorphisms in $CRE_f^A(G)$ do not send an element in $J_2^f \cup J_4^f$ to an element in $J_1^f \cup J_5^f \cup J_6^f$. This implies that all endomorphisms in $CRE_f^A(G)$ do not send any vertex in $M_A^f(4)$ to a vertex in $M_A^f(3)$. Similarly, all endomorphisms in $CRE_f^A(G)$ do not send any vertex in $M_A^f(3)$ to a vertex in $M_A^f(4)$.

Now we consider $CRE_f^A(G)|_{(M_A^f(3) \cup J_1^f \cup J_5^f \cup J_6^f)}$ and $CRE_f^A(G)|_{(M_A^f(4) \cup J_2^f \cup J_4^f)}$, the set of restrictions of all endomorphisms in $CRE_f^A(G)$ to $M_A^f(3) \cup J_1^f \cup J_5^f \cup J_6^f$ and to $M_A^f(4) \cup J_2^f \cup J_4^f$, respectively.

It is clear that $CRE_f^A(G)|_{(M_A^f(3)\cup J_1^f \cup J_5^f \cup J_6^f)} \cong Aut(M_A^f(3) \cup \bigcup_{j\in\{1,5,6\}} (J_j^f \cap A))$. Since $(J_1^f \cap A) = \{a_1, a_2\}$, $(J_5^f \cap A) = \{e_1, e_2\}$ and $(J_6^f \cap A) = \{g_1, g_2\}$ are split components of the factor graph H and $|(J_1^f \cap A)| = |(J_5^f \cap A)| = |(J_6^f \cap A)| = 2$, then by Lemma 4.3.9, we have that $CRE_f^A(G)|_{(M_A^f(3)\cup J_1^f \cup J_5^f \cup J_6^f)}$ is isomorphic to $S_{M_A^f(3)} \times S_{J_1^f \cap A} \times S_{J_5^f \cap A} \times S_{J_6^f \cap A} \cong S_3 \times S_2 \times S_2 \times S_2$. Similarly, we get that $J_2^f \cap A = \{b_1\}$, $J_4^f \cap A = \{d_1\}$ and $|J_2^f \cap A| = |J_4^f \cap A| = 1$, so $CRE_f^A(G)|_{(M_A^f(4)\cup J_2^f \cup J_4^f)}$ is isomorphic to $S_{M_A^f(4)} \times S_{J_2^f \cap A} \times S_{J_4^f \cap A} \cong S_2 \times S_1 \times S_1 = S_2$.

Hence we get that $CRE_f^A(G)$ is isomorphic to $S_3 \times (S_3 \times S_2 \times S_2 \times S_2) \times S_2$.

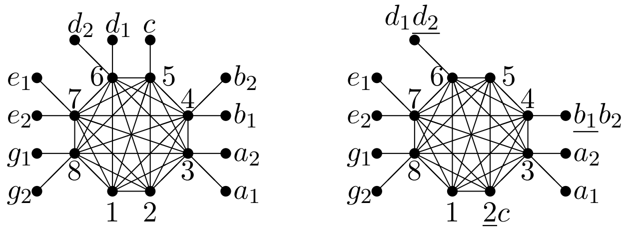
Moreover, it is clear by inspection that for any $B, C \in \mathcal{A}$, $CRE_f^B(G) \cong CRE_f^C(G)$. In this example we have that

$\{a_1, a_2, b_1, d_1, e_1, e_2, g_1, g_2\}$, $\{a_1, a_2, b_1, d_2, e_1, e_2, g_1, g_2\}$,
$\{a_1, a_2, b_2, d_1, e_1, e_2, g_1, g_2\}$ and $\{a_1, a_2, b_2, d_2, e_1, e_2, g_1, g_2\}$

are 4 distinct sets in $\mathcal{A}$ so $|\mathcal{A}| = 4$. Then it is clear that the maximal completely regular subsemigroup containing in $End_f(G)$ is

$$\bigcup_{B\in\mathcal{A}} CRE_f^B(G) \cong (S_3 \times (S_3 \times S_2 \times S_2 \times S_2) \times S_2) \times L_4.$$

$\square$



G H

Figure 4.8: Endo-regular split graph $G = K_8 \cup I_{11}$ and H a factor graph induce by f in Example 4.3.11.

Corollary 4.3.12. *Take G, f and $\mathcal{A}$ as in Theorem 4.3.10. For $A \in \mathcal{A}$, the maximal completely regular subsemigroup of $End_f(G)$ denoted by $CRE_f(G)$*

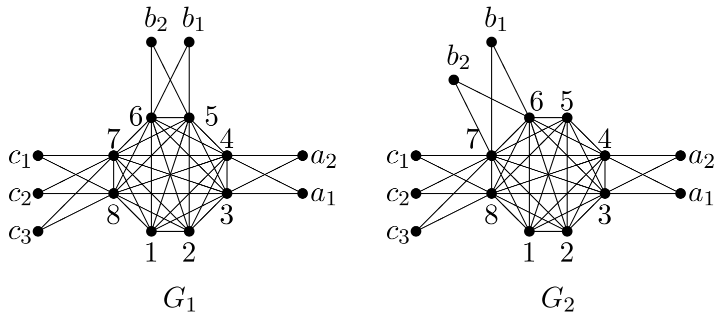
is the left group $(S_{n-m+p} \times \prod_{j=1}^{t} S_{|M_A^f(u_j)|} \times \prod_{k=1}^{s} S_{|J_k^f \cap A|}) \times L_{|\mathcal{A}|}$. *Here* $S_{|M_A^f(u_j)|}$ *and* $S_{|J_k^f \cap A|}$ *are the symmetric groups on* $|M_A^f(u_j)|$ *and* $|J_k^f \cap A|$ *elements, respectively.*

Endo-egular split graph $K_n \cup I_r$ with $s > 1$ split components of I_r and $|N(a)| \geq 2$ for all $a \in I_r$

We can use the same idea from two previous sections to find a completely regular subsemigroup of $End(G)$ where $G = K_n \cup I_r$ is an endo-regular split graph for which I_r has more than one split component and $|N(a)| \geq 2$ for all $a \in I_r$. But we can not generalize which group is isomorphic to $CRE_f^A(G)$ for any the set of representatives A. We give the reason as follows.

For any complete graph K_n and independent set $I_r = \overline{K}_r$, we can construct many non-isomorphic endo-regular split graphs whose I_r has $s > 1$ split components and $|N(a)| = m \geq 2$ for all $a \in I_r$. Let G_1 and G_2 be two non-isomorphic endo-regular split graphs with the maximal complete subgraph K_n and the independent set I_r of both G_1 and G_2. If f is an endomorphism of both G_1 and G_2, then $CRE_f^A(G_1)$ may be not isomorphic to $CRE_f^A(G_2)$ for some possible set of representatives A. The next example shows this fact.

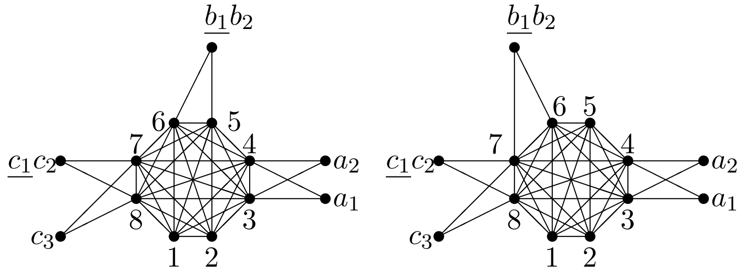
Example 4.3.13. Consider two graphs G_1 and G_2 as follows.



The essential difference between the graph G_1 and the graph G_2 lies in the neighborhoods of b_2 and of c_1. The neighborhood of the split component $\{b_1, b_2\}$ and the neighborhood of the split component $\{c_1, c_2, c_3\}$ are disjoint in the graph G_1 but are not disjoint in the graph G_2. Consider the mapping as follows

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & c_3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a_1 & a_2 & b_1 & b_1 & c_1 & c_1 & c_3 \end{pmatrix}.$$

It is clear that f is an endomorphism of G_1 and G_2. By Lemma 4.1.7, we have that f is regular. And we have the congruence relation $\rho_f = \{\{i\}|i \notin \{b_1, b_2, c_1, c_2\}\} \cup \{\{b_1, b_2\}, \{c_1, c_2\}\}$ and we have 5 congruence classes contained in an independent set, that is $\{a_1\}$, $\{a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ and $\{c_3\}$. The following pictures H_1 and H_2 are the image graphs of G_1 and G_2 under f, respectively, notation as in Example 4.3.4.



H_1 H_2

We see that all endomorphisms in $End_f(G_1)$ and $End_f(G_2)$ are the embeddings from H_1 to G_1 and from H_2 to G_2, respectively. Choose $A = \{a_1, a_2, b_1, c_1, c_3\}$. By inspection it is clear that $CRE_f^A(G_1)$ and $CRE_f^A(G_2)$ are isomorphic to $S_{\{1,2\}} \times (S_{\{\{3,4\},\{7,8\}\}} \times S_{\{3,4\}} \times S_{\{7,8\}} \times S_{\{a_1,a_2\}} \times S_{\{c_1,c_3\}}) \times S_{\{5,6\}}$ and $S_{\{1,2,5\}} \times (S_{\{3,4\}} \times S_{\{a_1,a_2\}}) \times S_{\{c_1,c_3\}}$, respectively. These are the groups $S_2 \times (S_2 \times S_2 \times S_2 \times S_2 \times S_2) \times S_2$ and $S_3 \times (S_2 \times S_2) \times S_2$, respectively. □

Finally, we give an example to show that for any endo-regular split graph G, if $f, g \in End(G)$ with $\rho_f \neq \rho_g$, it is not necessary that the composition between two endomorphisms in $CRE_f(G)$ and $CRE_g(G)$ is completely regular. This means $CRE_f(G) \cup CRE_g(G)$ is not necessarily closed.

Example 4.3.14. Let G be the graph as in Example 4.3.4. It is clear that $f = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & d & c \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & b & b & b & d \end{pmatrix}$ are endomorphisms of G. Now we have the congruence relations

$\rho_f = \{\{1\}, \{2\}, \{3\}, \{a, b\}, \{c\}, \{d\}\}$

and

$\rho_g = \{\{1\}, \{2\}, \{3\}, \{a, b, c\}, \{d\}\}$.

It is clear that $\rho_f \subseteq \rho_g$. And we get that

$CRE_f(G) = CRE_f^{\{a,c,d\}}(G) \cup CRE_f^{\{b,c,d\}}(G)$

and

$CRE_g(G) = CRE_g^{\{a,d\}}(G) \cup CRE_g^{\{b,d\}}(G) \cup CRE_g^{\{c,d\}}(G)$

are isomorphic to $(S_2 \times S_3) \times L_2$ and $(S_2 \times S_2) \times L_3$, respectively. Since f and g are idempotents, it is clear that f and g are completely regular. Then $f \in CRE_f(G)$ and $g \in CRE_g(G)$. Consider the following composition

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & a & b & c & d \\ 1 & 2 & 3 & a & a & a & c \end{pmatrix}.$$

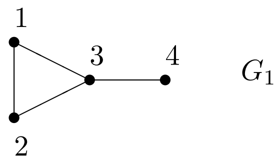
We see that $a = (f \circ g)(c) \neq (f \circ g)(d) = c$ and $(f \circ g)^2(c) = a = (f \circ g)^2(d)$, i.e., $f \circ g$ is not square injective. By Theorem 1.4.7, we get that $f \circ g$ is not completely regular. This means $f \circ g$ is not in $CRE_f(G) \cup CRE_g(G)$. $\square$

4.4 Endo-completely-regular split graphs

In this section we find that the set of all non-trivial endomorphisms of endo-completely-regular split graph $K_n \cup I_r$ is a left group if it forms the lexicographic product $\overline{K}_2[K_{n-1}]$ and it is a right group if it has exactly one maximal complete subgraph K_n.

First we will characterize the monoid of endo-completely-regular split graphs. Before that we characterize the semigroup $End'(G)$ of non-bijective endomorphisms.

Example 4.4.1. (a) Consider the graph G_1 as the follow



we get that $End'(G_1) = \{f_1, f_2, f_3, f_4, f_5, f_6, g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \end{pmatrix},$$
$$f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 3 \end{pmatrix},$$
$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 3 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix}, g_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 1 \end{pmatrix}, g_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{pmatrix}.$$

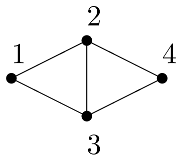
Set $F := \{f_i | i = 1, ..., 6\}$ and $G := \{g_i | i = 1, ..., 6\}$. We consider the symmetric group $S_3 := \{(1), (12), (13), (23), (123), (132)\}$. It is easy to see that the set F corresponds to S_3 with congruence relation generated by $1\rho 4$, and similarly for G with congruence relation generated by $2\rho 4$. Now we

consider multiplication between one endomorphism of F and one from G. For any $f_i \in F$ and $g_j \in G$, we have
(1) $f_i|_{K_3} \circ g_j|_{K_3} = f_r|_{K_3} = g_r|_{K_3}$ and $g_j|_{K_3} \circ f_i|_{K_3} = f_s|_{K_3} = g_s|_{K_3}$ $\exists r, s = 1, ..., 6$,
(2) $(f_i \circ g_j)(2) = (f_i \circ g_j)(4)$ and $(g_j \circ f_i)(1) = (g_j \circ f_i)(4)$.
Then we can conclude that $(f_i \circ g_j) \in G$ and $(g_j \circ f_i) \in F$. And since $f_t|_{K_3} = g_t|_{K_3}$ for all $t \in \{1, ..., 6\}$, then we have for any $u, v \in \{1, ..., 6\}$
(3) $h_u|_{K_3} \circ f_v|_{K_3} = h_u|_{K_3} \circ g_v|_{K_3}$ and $g_v|_{K_3} \circ h_u|_{K_3} = f_v|_{K_3} \circ h_u|_{K_3}$, where $h_u \in \{f_u, g_u\}$.
From these three conditions we can construct the composition table of the compositions of any two endomorphisms in $End(G_1)$ as follows

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4	g_2	g_1	g_5	g_6	g_3	g_4
f_3	f_3	f_4	f_1	f_2	f_6	f_5	g_3	g_4	g_1	g_2	g_6	g_5
f_4	f_4	f_3	f_6	f_5	f_1	f_2	g_4	g_3	g_6	g_5	g_1	g_2
f_5	f_5	f_6	f_2	f_1	f_4	f_3	g_5	g_6	g_2	g_1	g_4	g_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1	g_6	g_5	g_4	g_3	g_2	g_1
g_1	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
g_2	f_2	f_1	f_5	f_6	f_3	f_4	g_2	g_1	g_5	g_6	g_3	g_4
g_3	f_3	f_4	f_1	f_2	f_6	f_5	g_3	g_4	g_1	g_2	g_6	g_5
g_4	f_4	f_3	f_6	f_5	f_1	f_2	g_4	g_3	g_6	g_5	g_1	g_2
g_5	f_5	f_6	f_2	f_1	f_4	f_3	g_5	g_6	g_2	g_1	g_4	g_3
g_6	f_6	f_5	f_4	f_3	f_2	f_1	g_6	g_5	g_4	g_3	g_2	g_1

and we get $End'(G_1)$ isomorphic to the right group $S_3 \times R_2$.

(b) Consider the graph G_2 as follows



G_2

we get that $End'(G_2) = \{f_1, f_2, f_3, f_4, f_5, f_6, g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \end{pmatrix},\ f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix},\ f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \end{pmatrix},$$
$$f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix},\ f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix},\ f_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 3 \end{pmatrix},$$

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 4 \end{pmatrix},\ g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 4 \end{pmatrix},\ g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 \end{pmatrix},\ g_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 3 \end{pmatrix},\ g_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 3 \end{pmatrix}.$$

Define $F := \{f_i | i = 1, ..., 6\}$ and $G := \{g_i | i = 1, ..., 6\}$. Similarly with (a), we get that F and G are isomorphic to S_3. Since the graph G_2 has two complete subgraphs, then set $K_3 = \{1, 2, 3\}$ and $I_1 = \{4\}$. Now we know that for any $i = 1, ..., 6$, $Im(f_i) = N(4) \cup \{1\} =: C$ and $Im(g_i) = N(4) \cup \{4\} =: D$. Next, we want to consider the multiplication between two elements of F and G. For any $f_i \in F$ and $g_j \in G$, we have $(f_i \circ g_j)(N(4)) \subseteq C$. And it is easy to check that $(f_i \circ g_j)(1) = (f_i \circ g_j)(4) \in C \setminus (f_i \circ g_j)(N(4))$ and $Im(f_i \circ g_j) = C$. Then we have $(f_i \circ g_j) \in F$. Similarly, we get $(g_j \circ f_i) \in G$. For $r \in \{1, ..., 6\}$ we observe that

(1) $f_r(a) = g_r(a)$ for all $a \in N(4)$ or

(2) there exist only one vertex in $N(4)$ such that f_r send it to 1 and g_r send it to 4 and for other vertices f_r and g_r send them the same.

By all conditions above we can conclude that for any $h_i \in \{f_i, g_i\}$ and any vertex $a \in N(4)$, $(h_i \circ f_j)(a) = (h_i \circ g_j)(a)$ and we can construct the composition table as follows

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_4	f_1	f_2	f_6	f_5	f_3	f_4	f_1	f_2	f_6	f_5
f_4	f_4	f_3	f_6	f_5	f_1	f_2	f_4	f_3	f_6	f_5	f_1	f_2
f_5	f_5	f_6	f_2	f_1	f_4	f_3	f_5	f_6	f_2	f_1	f_4	f_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1	f_6	f_5	f_4	f_3	f_2	f_1
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_1	g_2	g_3	g_4	g_5	g_6
g_2	g_2	g_1	g_5	g_6	g_3	g_4	g_2	g_1	g_5	g_6	g_3	g_4
g_3	g_3	g_4	g_1	g_2	g_6	g_5	g_3	g_4	g_1	g_2	g_6	g_5
g_4	g_4	g_3	g_6	g_5	g_1	g_2	g_4	g_3	g_6	g_5	g_1	g_2
g_5	g_5	g_6	g_2	g_1	g_4	g_3	g_5	g_6	g_2	g_1	g_4	g_3
g_6	g_6	g_5	g_4	g_3	g_2	g_1	g_6	g_5	g_4	g_3	g_2	g_1

and we get $End'(G_2)$ isomorphic to the left group $S_3 \times L_2$. $\square$

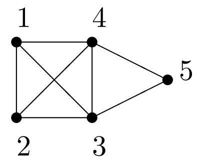
By the same way we get the theorem which gives the structure of $End'(G)$ where G is an endo-completely-regular split graph.

Theorem 4.4.2. *For any endo-completely-regular split graph* $G = K_n \cup \{a\}$

set $|N(a)| := m$. *We have:*
(1) *if* $0 \leq m < n-1$, *then* $End'(G) = S_n \times R_{n-m}$,
(2) *if* $m = n-1$, *then* $End'(G) = S_n \times L_2$.

Next we give some example for the group of an endo-completely-regular split graph G.

Example 4.4.3. (a) Consider the graph G_3 as follows.



G_3

We know that $I_1 = \{5\}$ and $N(5) = \{3, 4\}$. Set $C := K_4 \setminus N(5)$. Now we can apply Lemma 4.1.9 that
(1) all automorphisms of this graph fix 5.
And automorphisms permute only 1 and 2, or 3 and 4. Then we can conclude
(2) $f(C) = C$ and $f(N(5)) = N(5)$ for any automorphism f.
By these two conditions we can find all automorphisms as follows

$$id, (12), (34), (12)(34)$$

and thus $Aut(G_3) = S_2 \times S_2$.
(b) Consider the graph G_2 in Example 4.4.1. Similar as (a), we get $Aut(G_2) = S_2 \times S_2$. □

From the above example we know that the group of an endo-completely-regular split graph $G = K_n \cup \{a\}$ is the cartesian product of two symmetric groups where the indices depend on the cardinal number of $N(a)$. The next theorem describes the monoids of endo-completely-regular split graphs. Its proof is clear from the preceding examples.

Theorem 4.4.4. *For any endo-completely-regular split graph* $G = K_n \cup \{a\}$, *set* $|N(a)| := m$. *We have:*
(1) *if* $m = 0$, *then* $End(G) = S_n \cup (S_n \times R_n)$,
(2) *if* $0 < m < n-1$, *then* $End(G) = (S_{n-m} \times S_m) \cup (S_n \times R_{n-m})$,
(3) *if* $m = n-1$, *then* $End(G) = (S_{n-1} \times S_2) \bigcup (S_n \times L_2)$.

Remark 4.4.5. Note that from Theorem 4.4.4 we have in case (1) all endomorphisms are half strong, locally strong endomorphisms are automorphisms, so these graphs have endotype 2, in case (2) all endomorphisms are locally strong, quasi strong endomorphisms are automorphisms, so these graphs have endotype 4, and in case (3) all endomorphisms are strong, so these graphs have endotype 16.

To prove when a split graphs is endo-Clifford. We need lemmas which we will give in the next chapter. So, in the next chapter we will show that all connected split graphs are not endo-Clifford.

Chapter 5

Some Clifford endomorphism monoids

In previous chapters, we saw that retractive connected bipartite graphs and retractive 8-graphs were not endo-clifford. In this chapter, we get that retractive split graphs are also not endo-Clifford. So, our main aim in this chapter is finding some examples of retractive graphs which are endo-Clifford.

5.1 Retractive graphs which are not endo-Clifford

In this section, we give lemmas which we will use to construct the endo-Clifford retractive graphs. By observation on the 8-graph $C_{3,3}; P_1$ which is endo-completely-regular, but it is neither endo-Clifford nor S-A-unretractive. We have idea to prove the next lemma.

Lemma 5.1.1. *Let G be a retractive connected graph. If $End(G)$ is Clifford semigroup, then G is S-A-unretractive.*

Proof. Let G be not S-A-unretractive. So there exists $x \neq y \in G$ such that $N(x) = N(y)$. It is clear that $f(z) = \begin{cases} z, & z \neq x, y \\ x, & z = x, y \end{cases}$ and $g(z) = \begin{cases} z, & z \neq x, y \\ y, & z = x, y \end{cases}$ are idempotent endomorphisms of G. But $f \circ g = f \neq g = g \circ f$. Hence we get that $End(G)$ is not Clifford semigroup. □

The converse of Lemma 5.1.1 is not true. For example the connected split graph $K_3 \cup \{a\}$ with $|N(a)| = 1$ is S-A-unretractive and endo-completely-regular but is not endo-Clifford. This split graph gives us an idea to prove the next lemma.

Lemma 5.1.2. *Let G be a retractive connected graph. If $End(G)$ is endo-Clifford, then for all $a \in V(G)$ there is at most one vertex $b \in V(G)$ such that $N(a) \subseteq N(b)$.*

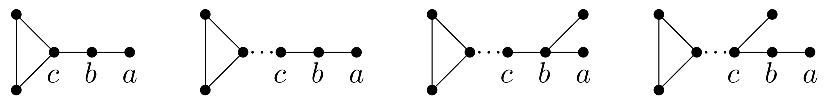
Proof. Let b, c be two distinct vertices in G such that $N(a)$ is a subset of both of $N(b)$ and $N(c)$. It is clear that $f(x) = \begin{cases} x, & x \neq a \\ b, & x = a \end{cases}$ and $g(x) = \begin{cases} x, & x \neq a \\ c, & x = a \end{cases}$ are idempotent endomorphisms of G. But $f \circ g = f \neq g = g \circ f$. Hence we get that $End(G)$ is not Clifford semigroup. $\square$

Corollary 5.1.3. *Let G be a retractive connected graph. If $End(G)$ is Clifford semigroup, then for any $a \in V(G)$, $|N(a)| > 1$.*

Proof. Let a be a vertex in G such that $|N(a)| = 1$, i.e., $|N(a)| = \{b\}$ for some $b \in V(G)$. Since G is the retractive connected graph and a is adjacent to b and $|N(a)| = 1$, then G has at least 3 vertices and $|N(b)| \geq 2$. If G is tree (contains no cycle), since $|V(G)| \geq 3$, it is clear by Theorem 2.1.4 that $End(G)$ is not an endo-Clifford.

Now we consider the case when G contains a cycle. We consider two cases. First b is in the cycle. In this case we get that $|N(b)| = 3$, i.e., $N(b) = \{a, c, d\}$ for some $c, d \in V(G)$. It is clear that $f(x) = \begin{cases} x, & x \neq a \\ c, & x = a \end{cases}$ and $g(x) = \begin{cases} x, & x \neq a \\ d, & x = a \end{cases}$ are idempotent endomorphisms of G. But $f \circ g = f \neq g = g \circ f$. Hence, we get that $End(G)$ is not Clifford semigroup.

Next we consider the case b is not in any cycle. We have four possible strong subgraphs of G as follows.



Similar as the proof of Lemmas 5.1.1 and 5.1.2 we can find two idempotent endomorphisms f, g of G such that $fg = f \neq g = gf$. So, we get that $End(G)$ is not an endo-Clifford. $\square$

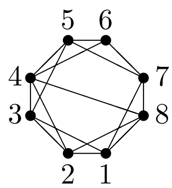
We get the next theorem, which describes that all connected split graphs are not endo-Clifford, by using Theorem 4.2.3, Lemma 5.1.1, and Corollary 5.1.3.

Theorem 5.1.4. *No connected split graph is endo-Clifford.*

5.2 Endo-Clifford and rigid graphs

In this section, we construct the retractive endo-Clifford graphs from some rigid graphs. Recall that graph G is called ***rigid*** if $|End(G)| = 1$.

Example 5.2.1. Take a rigid graph G (see in [19]) as follows.



Add a vertex a to graph G. We consider which connected graph $G \cup \{a\}$ is endo-Clifford. We get by Corollary 5.1.3 that

(1) $|N(a)|$ must more than 1.

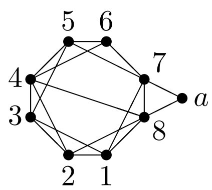
Since we need $G \cup \{a\}$ is retractive graph and all vertices in G can not permute themselves, then

(2) $N(a)$ must be subset of $N(c)$ for some $c \in G$.

By Lemma 5.1.2 we get that

(3) exactly $c \in G$ such that $N(a) \subseteq N(c)$.

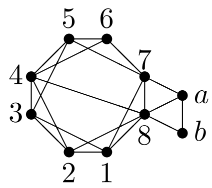
By $(1), (2), (3)$ we choose the graph $G \cup \{a\}$ as follows.



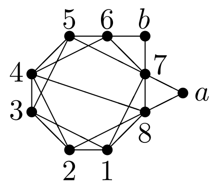
It is routine to check that $End(G \cup \{a\})$ has 2 endomorphisms: identity and $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix}$. They are idempotent. It is obvious that $End(G \cup \{a\})$ is endo-Clifford.

Add vertex b in the graph $G \cup \{a\}$ by consider same as $(1), (2)$ and (3).

Consider two non-isomorphic graphs $H_1 = G \cup \{a, b\}$ and $H_2 = G \cup \{a, b\}$ as follows.



H_1



H_2

We check by inspection that there exists 3 and 4 endomorphisms of H_1 and H_2, respectively. All endomorphisms of H_1 are id_{H_1},
$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & 7 \end{pmatrix}$ and
$f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 7 \end{pmatrix}$. And all endomorphism of H_2 are
id_{H_2}, $g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & 5 \end{pmatrix}$,
$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & b \end{pmatrix}$ and
$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 5 \end{pmatrix}$.
All of them are idempotent. It is clear that $End(H_1)$ and $End(H_2)$ are endo-Clifford. We can see their strong semilattices of groups in Table 5.1.□

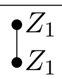
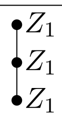
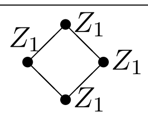
	strong semilattice of groups	defining homomorphism
$End(G \cup \{a\})$		isomorphism
$End(H_1)$		isomorphism
$End(H_2)$		isomorphism

Table 5.1: Strong semillatices of groups with respect to endomorphism monoids in Example 5.2.1

From above example we observe a graph whose endomorphism monoid is the strong semilattice of groups $\bigcup_{\alpha \in P_n} Z_{1\alpha}$ with defining homomorphism between groups is isomorphism (or identity map) where P_n is a chain with $n+1$ elements. The next construction is clear by observation.

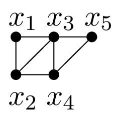
Construction 5.2.2. *Let G be a rigid graph in Example 5.2.1 and for any $m \geq 0$, let C_3^m be a graph with vertex set and edge set as follows:*
$V(C_3^m) = \{x_1, x_2, ..., x_{3+m}\}$ and
$E(C_3^m) = \{\{x_1, x_2\}\} \cup \{\{x_i, x_{i-1}\}, \{x_i, x_{i-2}\} \mid i = 3, ..., 3+m\}$.
Let H be a path $P_1 = \{a, b\}$ and let $m_1 : H \to G$ and $m_2 : H \to C_3^m$ be injective homomorphisms from H to G and C_3^m, respectively, define by $m_1(a) = 7$, $m_1(b) = 8$, $m_2(a) = x_1$ and $m_2(b) = x_2$. We get that the amalgamated $G \coprod_{(H,(m_1,m_2))} C_3^m$ is endo-Clifford and

$$End(G \coprod_{(H,(m_1,m_2))} C_3^m) = \bigcup_{\alpha \in P_{m+1}} Z_{1\alpha}$$

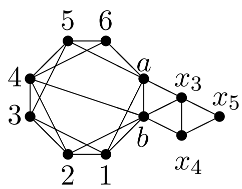
with defining homomorphisms are isomorphisms where P_{m+1} is a chain with $m+2$ elements.

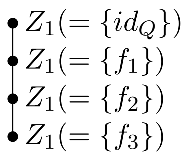
We call graph C_3^m in above construction that C_3-***chain***.

Example 5.2.3. Take G the rigid graph in Example 5.2.1 and take C_3^2 a C_3-chain graph as follows.



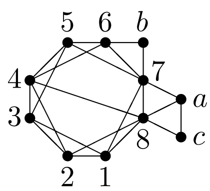
Let $P_1 = \{a, b\}$ be a path and let $m_1 : P_1 \to G$ and $m_2 : P_1 \to C_3^2$ be injective homomorphisms from H to G and C_3^2, respectively, define by $m_1(a) = 7$, $m_1(b) = 8$, $m_2(a) = x_1$ and $m_2(b) = x_2$. We get the amalgamated $G \coprod_{(P_1,(m_1,m_2))} C_3^3 =: Q$ as follows.



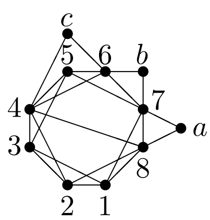


It is routine to check that $End(Q)$ is a Clifford semigroup which contains 4 endomorphisms: id_Q, $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & a & b & x_3 & x_4 & x_5 \\ 1 & 2 & 3 & 4 & 5 & 6 & a & b & x_3 & x_4 & b \end{pmatrix}$,
$f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & a & b & x_3 & x_4 & x_5 \\ 1 & 2 & 3 & 4 & 5 & 6 & a & b & x_3 & a & b \end{pmatrix}$,
$f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & a & b & x_3 & x_4 & x_5 \\ 1 & 2 & 3 & 4 & 5 & 6 & a & b & 1 & a & b \end{pmatrix}$.
And $End(G \coprod_{(H,(m_1,m_2))} C_3^3)$ is a strong semilattice of groups $\bigcup_{\alpha \in P_3} Z_{1\alpha}$ which shows in the above graph. $\square$

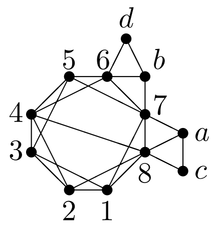
Example 5.2.4. Add 1 vertex and 2 vertices on graph H_2 in Example 5.2.1 as follows.



H_3

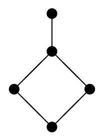


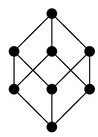
H_4

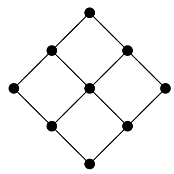


H_5

We check by inspection that $End(H_3)$, $End(H_4)$ and $End(H_5)$ are the strong semilattice of groups $5Z_1$, $8Z_1$ and $9Z_1$, respectively, as the following graphs.


$5Z_1$


$8Z_1$

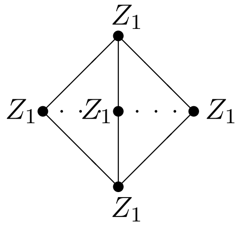

$9Z_1$

Defining homomorphisms between any two groups of these strong semilattice of groups are isomorphisms. $\square$

From the observation on examples in this section, we have some questions which we do not get the results.

Question 5.2.5. If G is a retractive graph with $End(G)$ is a strong semilattice of groups $\bigcup_{\alpha \in Y} Z_{1\alpha}$, then G contains rigid graph as a strong subgraph?

Question 5.2.6. Which retractive graph G whose endomorphism monoid $End(G)$ forms a strong semilattice of groups as follows?

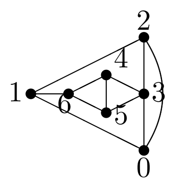


Question 5.2.7. *For any semilattice Y, for which graph G whose endomorphism monoid $End(G)$ is strong semilattice of groups $\bigcup_{\alpha \in Y} Z_{1\alpha}$?*

5.3 Endo-Clifford and unretractive graphs

In previous section, we used rigid graphs to construct the graphs which are endo-Clifford. In this section, we find an endo-Clifford retractive graph by construct from an unretractive graph which is not rigid. We consider the unretractive graphs with 7 vertices in [19].

Example 5.3.1. Let G be a graph as follows.



If we add vertex 7 with $|N(7)| = 2$ to G, we have three difference algebraic properties. If $N(7)$ is one kind of $\{0, 2\}$, $\{1, 3\}$, $\{3, 6\}$ and $\{4, 5\}$, we by Lemma 5.1.2 that $G \cup \{7\}$ is not endo-Clifford. If $N(7) = \{1, 6\}$, we get that $End(G \cup \{7\}) = End(G) = Aut(G) \cong D_4$. If $N(7)$ is one kind of $\{0, 1\}$, $\{0, 3\}$, $\{0, 4\}$, $\{0, 5\}$, $\{0, 6\}$, $\{1, 2\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 6\}$ and $\{5, 6\}$, we get that $End(G \cup \{7\})$ is a union of groups $Z_2 \cup D_4$. We will show you the case $N(7) = \{3, 4\}$. For this case, we have 10 endomorphisms of $G \cup \{7\}$ as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$,
$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 5 & 4 & 6 & 4 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$,

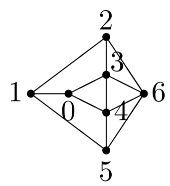
$$f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 5 & 4 & 6 & 4 \end{pmatrix},\ f_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 3 & 0 & 2 & 1 & 2 \end{pmatrix},$$
$$f_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 3 & 2 & 0 & 1 & 0 \end{pmatrix},\ f_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 0 & 2 & 1 & 2 \end{pmatrix},$$
$$f_8 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 5 \end{pmatrix},\ f_9 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

The next table shows the multiplication of any two above endomorphisms.

	id	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
id	id	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
f_1	f_1	id	f_4	f_8	f_2	f_6	f_5	f_9	f_3	f_7
f_2	f_2	f_4	f_3	f_2	f_8	f_7	f_9	f_5	f_4	f_6
f_3	f_3	f_8	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
f_4	f_4	f_2	f_8	f_4	f_3	f_9	f_7	f_6	f_2	f_5
f_5	f_5	f_7	f_6	f_5	f_9	f_4	f_8	f_2	f_7	f_3
f_6	f_6	f_9	f_5	f_6	f_7	f_2	f_3	f_4	f_9	f_8
f_7	f_7	f_5	f_9	f_7	f_6	f_8	f_4	f_3	f_5	f_2
f_8	f_8	f_3	f_4	f_8	f_2	f_6	f_5	f_9	f_3	f_7
f_9	f_9	f_6	f_7	f_9	f_5	f_3	f_2	f_8	f_6	f_4

It is clear that $End(G \cup \{7\})$ is the union of groups $Z_2 \cup D_4$. In this case, we let Z_2 and D_4 be the sets $\{id, f_1\}$ and $\{f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$, respectively. We also get that $End(G \cup \{7\}) = Z_2 \cup D_4$ is a strong semilattice of groups by using defining homomorphism $\varphi : Z_2 \to D_4$ which $\varphi(id) = f_3$ and $\varphi(f_1) = f_8$. $\square$

Example 5.3.2. Let G be a graph as follows.



If we add vertex 7 with $|N(7)| = 2$ to G, we have three difference algebraic properties. If $N(7)$ is one kind of $\{0,2\}$, $\{0,5\}$, $\{0,6\}$, $\{1,3\}$, $\{1,4\}$, $\{1,6\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$ and $\{4,6\}$, we by Lemma 5.1.2 that $G \cup \{7\}$ is not endo-Clifford. If $N(7)$ is one kind of $\{0,1\}$, $\{1,2\}$ and $\{1,5\}$, we get that $End(G \cup \{7\}) = Aut(G \cup \{7\}) \cong Z_2$. If $N(7)$ is one kind of $\{0,3\}$, $\{0,4\}$, $\{2,3\}$, $\{2,6\}$, $\{4,5\}$ and $\{5,6\}$, we get that $End(G \cup \{7\})$ is a union of groups $Z_1 \cup S_3$. We will show you the case $N(7) = \{2,6\}$. For this case, we have 7 endomorphisms of $G \cup \{7\}$ as follows:

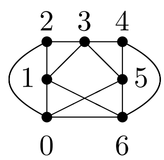
$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 2 & 6 & 4 & 0 & 3 & 6 \end{pmatrix}$,
$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 0 & 4 & 6 & 2 & 3 & 4 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 6 & 5 & 4 & 3 \end{pmatrix}$,
$f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 \end{pmatrix}$, $f_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 5 & 4 & 3 & 2 & 6 & 4 \end{pmatrix}$,
$f_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 6 & 3 & 0 & 4 & 6 \end{pmatrix}$.

The next table shows the multiplication of any two above endomorphisms.

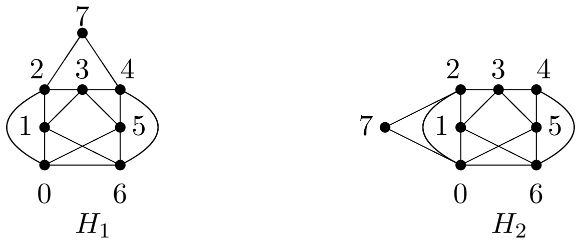
	id	f_1	f_2	f_3	f_4	f_5	f_6
id	id	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_4	f_5	f_6	f_1	f_2	f_3
f_2	f_2	f_3	f_6	f_5	f_2	f_1	f_4
f_3	f_3	f_2	f_1	f_4	f_3	f_6	f_5
f_4	f_4	f_1	f_2	f_3	f_4	f_5	f_6
f_5	f_5	f_6	f_3	f_2	f_5	f_4	f_1
f_6	f_6	f_5	f_4	f_1	f_6	f_3	f_2

It is routine to check that $End(G \cup \{7\})$ is the union of groups $Z_1 \cup S_3$. In this case, we let Z_1 and S_3 be the sets $\{id\}$ and $\{f_1, f_2, f_3, f_4, f_5, f_6\}$, respectively. We also get that $End(G \cup \{7\}) = Z_1 \cup S_3$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_1 \to S_3$ (which is identity map) define by $\varphi(id) = f_4$. $\square$

Example 5.3.3. Let G be a graph as follows.



This G is unretractive and has 2 endomorphisms: id_G and
$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 0 & 6 & 5 \end{pmatrix}$. Add vertex 7 to G with $|N(7)| = 2$. Consider the two possible graphs $H_1 = G \cup \{7\}$ and $H_2 = G \cup \{7\}$ as follows.



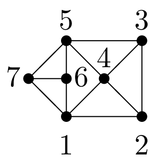
First we consider the monoid $End(H_1)$ which contains 4 endomorphisms: id_{H_1}, $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 7 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 \end{pmatrix}$ and $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 3 \end{pmatrix}$. The next table shows the compositions of any two endomorphisms in $End(H_1)$.

$\circ$	id_{H_1}	f_1	f_2	f_3
id_{H_1}	id_{H_1}	f_1	f_2	f_3
f_1	f_1	id_{H_1}	f_3	f_2
f_2	f_2	f_3	f_2	f_3
f_3	f_3	f_2	f_3	f_2

It is routine to check that $End(H_1)$ is a strong semilattice of groups $Z_{2\alpha} \cup Z_{2\beta}$ where $Z_{2\alpha} = \{id_{H_1}, f_1\}$ and $Z_{2\beta} = \{f_2, f_3\}$ and the defining homomorphism φ_1 from $Z_{2\alpha}$ to $Z_{2\beta}$ define by $\varphi_1(id_{H_1}) = f_2$ and $\varphi_1(f_1) = f_3$ (φ_1 is an isomorphism).

Now we consider the monoid $End(H_2)$ which contains 3 endomorphisms: id_{H_2}, $g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 2 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$. Similar as the monoid $End(H_1)$, we get that $End(H_2)$ is a strong semilattice of groups $Z_{1\alpha} \cup Z_{2\beta}$ where $Z_{1\alpha} = \{id_{H_2}\}$ and $Z_{2\beta} = \{g_1, g_2\}$ and the defining homomorphism φ_2 from $Z_{1\alpha}$ to $Z_{2\beta}$ define by $\varphi_2(id_{H_2}) = g_1$ (φ_2 is an identity map). If $N(7)$ is one kind of $\{0,5\}$, $\{1,3\}$, $\{1,6\}$, $\{2,3\}$, $\{3,4\}$, $\{3,5\}$ and $\{4,6\}$, we also get that $End(G \cup \{7\})$ is a strong semilattice of groups $Z_1 \cup Z_2$ with defining homomorphism from Z_1 to Z_2 is an identity map. For the other possible neighborhood of vertex 7, we get that the endomorphism monoid $End(G \cup \{7\})$ is not a Clifford semigroup. $\square$

Example 5.3.4. Take an unretractive graph G as follows.



If we add vertex 7 with $|N(7)| = 2$ to G, we have four difference algebraic properties.

Case. 1 If $N(7)$ is one kind of $\{0,2\}$, $\{0,5\}$, $\{1,3\}$, $\{1,4\}$, $\{2,5\}$, $\{3,6\}$, $\{4,5\}$ and $\{5,6\}$, we by Lemma 5.1.2 that $G \cup \{7\}$ is not endo-Clifford.

Case. 2 If $N(7)$ is one kind of $\{0,1\}$, $\{0,3\}$, $\{0,4\}$, $\{0,6\}$, $\{1,2\}$, $\{2,3\}$, $\{2,4\}$ and $\{2,6\}$, we get that $End(G \cup \{7\})$ is the union of groups $Z_1 \cup (Z_2 \times Z_2)$. We will show for the case $N(7) = \{0,3\}$. For this case, we have 5 endomorphisms as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}$,

$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 0 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 & 2 \end{pmatrix}$,

$f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 & 6 & 5 & 4 & 0 \end{pmatrix}$.

The next table shows the multiplication of any two above endomorphisms.

	id	f_1	f_2	f_3	f_4
id	id	f_1	f_2	f_3	f_4
f_1	f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_4	f_3	f_2	f_1

It is routine to check that $End(G \cup \{7\})$ is the union of groups $Z_1 \cup (Z_2 \times Z_2)$. In this case, we let Z_1 and $Z_2 \times Z_2$ be the sets $\{id\}$ and $\{f_1, f_2, f_3, f_4\}$, respectively. We also get that $End(G \cup \{7\}) = Z_1 \cup (Z_2 \times Z_2)$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_1 \to (Z_2 \times Z_2)$ (which is identity map) define by $\varphi(id) = f_1$.

Case. 3 If $N(7)$ is one kind of $\{1,5\}$, $\{1,6\}$, $\{3,4\}$ and $\{3,5\}$, we get that $End(G \cup \{7\})$ is the union of groups $Z_2 \cup (Z_2 \times Z_2)$. We will show for the case $N(7) = \{3,4\}$. For this case, we have 6 endomorphisms as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $g_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$,

$g_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$, $g_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$,
$g_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 & 5 \end{pmatrix}$, $g_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 & 6 & 5 & 4 & 5 \end{pmatrix}$.
The next table shows the multiplication of any two above endomorphisms.

	id	g_1	g_2	g_3	g_4	g_5
id	id	g_1	g_2	g_3	g_4	g_5
g_1	g_1	id	g_3	g_2	g_5	g_4
g_2	g_2	g_3	g_2	g_3	g_4	g_5
g_3	g_3	g_2	g_3	g_2	g_5	g_4
g_4	g_4	g_5	g_4	g_5	g_2	g_3
g_5	g_5	g_4	g_5	g_4	g_3	g_2

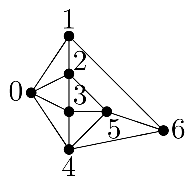
It is routine to check that $End(G \cup \{7\})$ is the union of groups $Z_2 \cup (Z_2 \times Z_2)$. In this case, we let Z_2 and $Z_2 \times Z_2$ be the sets $\{id, g_1\}$ and $\{g_2, g_3, g_4, g_5\}$, respectively. We also get that $End(G \cup \{7\}) = Z_2 \cup (Z_2 \times Z_2)$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_2 \to (Z_2 \times Z_2)$ define by $\varphi(id) = g_2$ and $\varphi(g_1) = g_3$.

Case. 4 If $N(7) = \{4, 6\}$, we get that $End(G \cup \{7\})$ is the union of groups $(Z_2 \times Z_2) \cup (Z_2 \times Z_2)$. For this case, we have 8 endomorphisms as follows:
$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $h_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$,
$h_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 & 7 \end{pmatrix}$, $h_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 & 6 & 5 & 4 & 7 \end{pmatrix}$,
$h_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 & 5 \end{pmatrix}$, $h_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 & 6 & 5 & 4 & 5 \end{pmatrix}$.
$h_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$, $h_7 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$.
The next table shows the multiplication of any two above endomorphisms.

	id	h_1	h_2	h_3	h_4	h_5	$h_(6)$	h_7
id	id	h_1	h_2	h_3	h_4	h_5	h_6	h_7
h_1	h_1	id	h_3	h_2	h_5	h_4	h_7	h_6
h_2	h_2	h_3	id	h_1	h_6	h_7	h_4	h_5
h_3	h_3	h_2	h_1	id	h_7	h_6	h_5	g_4
h_4	h_4	h_5	h_6	h_7	h_6	h_7	h_4	h_5
h_5	h_5	h_4	h_7	h_6	h_7	h_6	h_5	h_4
h_6	h_6	h_7	h_4	h_5	h_4	h_5	h_6	h_7
h_7	h_7	h_6	h_5	h_4	h_5	h_4	h_7	h_6

It is clear that $End(G \cup \{7\})$ is the union of groups $(Z_2 \times Z_2) \cup (Z_2 \times Z_2)$. In this case, we let the first $Z_2 \times Z_2$ and the last $Z_2 \times Z_2$ be the sets $\{id, h_1, h_2, h_3\}$ and $\{h_4, h_5, h_6, h_7\}$, respectively. We also get that $End(G \cup \{7\}) = (Z_2 \times Z_2) \cup (Z_2 \times Z_2)$ is a strong semilattice of groups with the defining homomorphism $\varphi : (Z_2 \times Z_2) \to (Z_2 \times Z_2)$ define by $\varphi(id) = h_6$, $\varphi(h_1) = h_7$, $\varphi(h_2) = h_4$ and $\varphi(h_3) = h_5$. $\square$

Example 5.3.5. Take an unretractive graph G as follows.



If we add vertex 7 with $|N(7)| = 2$ to G, we have four difference algebraic properties.

Case. 1 If $N(7)$ is one kind of $\{0,1\}$, $\{0,4\}$, $\{0,5\}$, $\{0,6\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,6\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$ and $\{4,5\}$, we get by Lemma 5.1.2 that $G \cup \{7\}$ is not endo-Clifford.

Case. 2 If $N(7) = \{1,6\}$, we get that $End(G \cup \{7\})$ is a group $Z_2 \times Z_2$.

Case. 3 If $N(7)$ is one kind of $\{0,1\}$, $\{1,2\}$, $\{4,6\}$ and $\{5,6\}$, we get that $End(G \cup \{7\})$ is the union of groups $Z_1 \cup (Z_2 \times Z_2)$. We will show for the case $N(7) = \{4,6\}$. For this case, we have 5 endomorphisms as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}$,

$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 5 & 4 & 6 & 4 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 3 & 2 & 0 & 1 & 0 \end{pmatrix}$,

$f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 0 & 2 & 1 & 2 \end{pmatrix}$.

The next table shows the multiplication of any two above endomorphisms.

	id	f_1	f_2	f_3	f_4
id	id	f_1	f_2	f_3	f_4
f_1	f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_4	f_3	f_2	f_1

It is routine to check that $End(G \cup \{7\})$ is the union of groups $Z_1 \cup (Z_2 \times Z_2)$. In this case, we let Z_1 and $Z_2 \times Z_2$ be the sets $\{id\}$ and $\{f_1, f_2, f_3, f_4\}$, respectively. We also get that $End(G \cup \{7\}) = Z_1 \cup (Z_2 \times Z_2)$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_1 \to (Z_2 \times Z_2)$ (which is identity map) define by $\varphi(id) = f_1$.

Case. 4 If $N(7)$ is one kind of $\{0, 4\}$ and $\{2, 5\}$, we get that $End(G \cup \{7\})$ is the union of groups $Z_2 \cup (Z_2 \times Z_2)$. We will show for the case $N(7) = \{2, 5\}$. For this case, we have 6 endomorphisms as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $g_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 0 & 2 & 1 & 7 \end{pmatrix}$,

$g_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 \end{pmatrix}$, $g_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 3 & 5 & 4 & 6 & 3 \end{pmatrix}$,

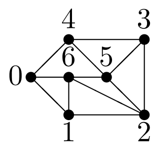
$g_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 4 & 3 & 2 & 0 & 1 & 3 \end{pmatrix}$, $g_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 0 & 2 & 1 & 3 \end{pmatrix}$.

The next table shows the multiplication of any two above endomorphisms.

	id	g_1	g_2	g_3	g_4	g_5
id	id	g_1	g_2	g_3	g_4	g_5
g_1	g_1	id	g_5	g_4	g_3	g_2
g_2	g_2	g_5	g_2	g_3	g_4	g_5
g_3	g_3	g_4	g_3	g_2	g_5	g_4
g_4	g_4	g_3	g_4	g_5	g_2	g_3
g_5	g_5	g_2	g_5	g_4	g_3	g_2

It is routine to check that $End(G \cup \{7\})$ is the union of groups $Z_2 \cup (Z_2 \times Z_2)$. In this case, we let Z_2 and $Z_2 \times Z_2$ be the sets $\{id, g_1\}$ and $\{g_2, g_3, g_4, g_5\}$, respectively. We also get that $End(G \cup \{7\}) = Z_2 \cup (Z_2 \times Z_2)$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_2 \to (Z_2 \times Z_2)$ define by $\varphi(id) = g_2$ and $\varphi(g_1) = g_5$. ☐

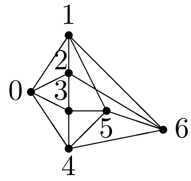
Example 5.3.6. Take an unretractive graph G as follows.



Similar as the other examples, if we add a new vertex 7 to G with $|N(7)| = 2$, we also get the endomorphism monoids which are not a Clifford semigroups where $N(7)$ is one kind of $\{0, 2\}$, $\{0, 5\}$, $\{1, 5\}$, $\{1, 6\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 5\}$, $\{3, 6\}$, $\{4, 5\}$ and $\{5, 6\}$. If $N(7) = \{0, 4\}$, we get that $End(G \cup \{7\})$ is a group Z_2. If $N(7)$ is one kind of $\{0, 1\}$, $\{0, 3\}$, $\{0, 6\}$,

$\{1,2\}$, $\{1,4\}$, $\{2,3\}$, $\{3,4\}$ and $\{4,5\}$, we get that $End(G \cup \{7\})$ is a strong semilattice of groups $Z_1 \cup Z_2$ which defining homomorphism from Z_1 to Z_2 is an identity map. If $N(7) = \{1,3\}$, we get that $End(G \cup \{7\})$ is a strong semilattice of groups $Z_2 \cup Z_2$ which defining homomorphism from Z_2 to Z_2 is an isomorphism. $\square$

Example 5.3.7. Take an unretractive graph G as follows.



Similar as the other examples, if we add a new vertex 7 to G with $|N(7)| = 2$, we also get the endomorphism monoids which are not a Clifford semigroups where $N(7)$ is one kind of $\{0,2\}$, $\{0,3\}$, $\{0,5\}$, $\{0,6\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,6\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,6\}$, $\{4,5\}$ and $\{5,6\}$. If $N(7)$ is one kind of $\{0,1\}$, $\{0,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,6\}$, $\{3,5\}$ and $\{4,5\}$, we get that $End(G \cup \{7\})$ is a strong semilattice of groups $Z_2 \cup D_7$ which defining homomorphism from Z_2 to D_7 define by send identity to identity and the other one send to some element which has order 2. We will show you the case $N(7) = \{0,1\}$. For this case we have 16 endomorphisms as follows:

$id = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$, $\overline{1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 2 & 6 & 5 & 4 & 3 & 7 \end{pmatrix}$,

$\overline{2} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}$, $\overline{3} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 0 & 4 & 5 & 6 & 1 & 0 \end{pmatrix}$,

$\overline{4} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 4 & 6 & 1 & 2 & 0 & 4 \end{pmatrix}$, $\overline{5} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 1 & 5 & 4 & 3 & 0 & 1 \end{pmatrix}$,

$\overline{6} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 6 & 2 & 0 & 3 & 4 & 6 \end{pmatrix}$, $\overline{7} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 0 & 1 & 6 & 5 & 4 & 0 \end{pmatrix}$,

$\overline{8} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 0 & 2 & 1 & 6 & 4 \end{pmatrix}$, $\overline{9} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 3 & 0 & 2 & 1 & 5 \end{pmatrix}$,

$\overline{10} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 2 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$, $\overline{11} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 1 & 0 & 3 & 4 & 5 & 1 \end{pmatrix}$,

$\overline{12} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 5 & 1 & 2 & 0 & 3 & 5 \end{pmatrix}$, $\overline{13} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 3 & 2 & 1 & 6 & 5 & 3 \end{pmatrix}$,

$\overline{14} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 0 & 3 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$, $\overline{15} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 4 & 3 & 0 & 2 & 6 \end{pmatrix}$.

The next table shows the multiplication of any two above endomorphisms.

	id	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$
id	id	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$
$\overline{1}$	$\overline{1}$	id	$\overline{10}$	$\overline{5}$	$\overline{9}$	$\overline{3}$	$\overline{13}$	$\overline{11}$	$\overline{12}$	$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{8}$	$\overline{6}$	$\overline{15}$	$\overline{14}$
$\overline{2}$	$\overline{2}$	$\overline{10}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$
$\overline{3}$	$\overline{3}$	$\overline{7}$	$\overline{3}$	$\overline{14}$	$\overline{12}$	$\overline{10}$	$\overline{11}$	$\overline{13}$	$\overline{9}$	$\overline{15}$	$\overline{7}$	$\overline{2}$	$\overline{6}$	$\overline{8}$	$\overline{4}$	$\overline{5}$
$\overline{4}$	$\overline{4}$	$\overline{8}$	$\overline{4}$	$\overline{12}$	$\overline{11}$	$\overline{13}$	$\overline{3}$	$\overline{9}$	$\overline{5}$	$\overline{10}$	$\overline{8}$	$\overline{14}$	$\overline{2}$	$\overline{15}$	$\overline{6}$	$\overline{7}$
$\overline{5}$	$\overline{5}$	$\overline{11}$	$\overline{5}$	$\overline{15}$	$\overline{8}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{4}$	$\overline{14}$	$\overline{11}$	$\overline{10}$	$\overline{13}$	$\overline{12}$	$\overline{9}$	$\overline{3}$
$\overline{6}$	$\overline{6}$	$\overline{15}$	$\overline{6}$	$\overline{11}$	$\overline{3}$	$\overline{9}$	$\overline{4}$	$\overline{5}$	$\overline{7}$	$\overline{13}$	$\overline{15}$	$\overline{12}$	$\overline{14}$	$\overline{10}$	$\overline{2}$	$\overline{8}$
$\overline{7}$	$\overline{7}$	$\overline{3}$	$\overline{7}$	$\overline{10}$	$\overline{15}$	$\overline{14}$	$\overline{8}$	$\overline{2}$	$\overline{6}$	$\overline{12}$	$\overline{3}$	$\overline{13}$	$\overline{9}$	$\overline{11}$	$\overline{5}$	$\overline{4}$
$\overline{8}$	$\overline{8}$	$\overline{4}$	$\overline{8}$	$\overline{13}$	$\overline{10}$	$\overline{12}$	$\overline{15}$	$\overline{14}$	$\overline{2}$	$\overline{11}$	$\overline{4}$	$\overline{9}$	$\overline{5}$	$\overline{3}$	$\overline{7}$	$\overline{6}$
$\overline{9}$	$\overline{9}$	$\overline{12}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{12}$	$\overline{15}$	$\overline{10}$	$\overline{14}$	$\overline{13}$	$\overline{11}$
$\overline{10}$	$\overline{10}$	$\overline{2}$	$\overline{10}$	$\overline{5}$	$\overline{9}$	$\overline{3}$	$\overline{13}$	$\overline{11}$	$\overline{12}$	$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{8}$	$\overline{6}$	$\overline{15}$	$\overline{14}$
$\overline{11}$	$\overline{11}$	$\overline{5}$	$\overline{11}$	$\overline{2}$	$\overline{14}$	$\overline{15}$	$\overline{12}$	$\overline{10}$	$\overline{13}$	$\overline{8}$	$\overline{5}$	$\overline{6}$	$\overline{4}$	$\overline{7}$	$\overline{3}$	$\overline{9}$
$\overline{12}$	$\overline{12}$	$\overline{9}$	$\overline{12}$	$\overline{6}$	$\overline{2}$	$\overline{8}$	$\overline{14}$	$\overline{15}$	$\overline{10}$	$\overline{7}$	$\overline{9}$	$\overline{4}$	$\overline{3}$	$\overline{5}$	$\overline{11}$	$\overline{13}$
$\overline{13}$	$\overline{13}$	$\overline{14}$	$\overline{13}$	$\overline{7}$	$\overline{5}$	$\overline{4}$	$\overline{9}$	$\overline{3}$	$\overline{11}$	$\overline{6}$	$\overline{14}$	$\overline{8}$	$\overline{15}$	$\overline{2}$	$\overline{10}$	$\overline{12}$
$\overline{14}$	$\overline{14}$	$\overline{13}$	$\overline{14}$	$\overline{4}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	$\overline{8}$	$\overline{15}$	$\overline{5}$	$\overline{13}$	$\overline{3}$	$\overline{11}$	$\overline{9}$	$\overline{12}$	$\overline{10}$
$\overline{15}$	$\overline{15}$	$\overline{6}$	$\overline{15}$	$\overline{9}$	$\overline{13}$	$\overline{11}$	$\overline{10}$	$\overline{12}$	$\overline{14}$	$\overline{3}$	$\overline{6}$	$\overline{5}$	$\overline{7}$	$\overline{4}$	$\overline{8}$	$\overline{2}$

It is routine to check that $End(G\cup\{7\})$ is the union of groups $Z_2 \cup D_7$. In this case, we let Z_2 and D_7 be the sets $\{id, \overline{1}\}$ and $\{\overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{15}\}$, respectively. We also get that $End(G \cup \{7\}) = Z_2 \cup D_7$ is a strong semilattice of groups with the defining homomorphism $\varphi : Z_2 \to D_7$ define by $\varphi(id) = \overline{2}$ and $\varphi(\overline{1}) = \overline{10}$. $\square$

From all examples in this section, we can conclude the results for all retractive graphs which construct from any unretractive graph with 7 vertices (refer from [19]). Not we get Table 5.2.

Remark 5.3.8. From Table 5.2 if we consider $3 \leq |N(a)| \leq 6$ the split graph $K_7\cup\{a\}$ is still not endo-Clifford. But for the $(C_5+K_2)\cup\{a\}$, if $3 \leq |N(a)| \leq 7$, its endomorphism monoid is possibly endo-Clifford $Z_1 \cup (D_5 \times Z_2)$.

In this chapter, we only gave examples of retractive graphs whose endomorphism monoids are strong semilattices of groups. For the next chance, we would like to characterize a graph for a given strong semilattice of groups. But it is so difficult to get a characterization. May be we consider a special case. For example, we consider a graph for a strong semilattice of groups $\bigcup\limits_{\alpha\in Y} G_\alpha$ where a semilattice Y is chain.

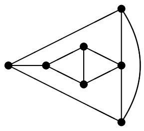
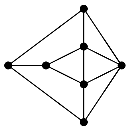
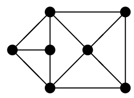
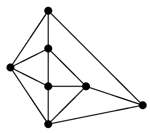
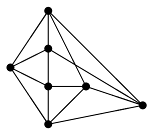
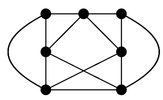
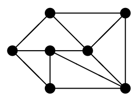
G	$End(G \cup \{a\})$ with $\lvert N(a)\rvert = 2$
K_7 or C_7 or $C_5 + K_2$	not endo-Clifford
	(1) not endo-Clifford or (2) $Z_2 \cup D_4$ or (3) D_4
	(1) not endo-Clifford or (2) $Z_1 \cup S_3$ (3) Z_2
	(1) not endo-Clifford or (2) $Z_1 \cup (Z_2 \times Z_2)$ or (3) $Z_2 \cup (Z_2 \times Z_2)$ or (4) $(Z_2 \times Z_2) \cup (Z_2 \times Z_2)$
	(1) not endo-Clifford or (2) $Z_1 \cup (Z_2 \times Z_2)$ or (3) $Z_2 \cup (Z_2 \times Z_2)$
	(1) not endo-Clifford or (2) $Z_1 \cup D_7$
	(1) not endo-Clifford or (2) $Z_1 \cup Z_2$ or (3) $Z_2 \cup Z_2$
	(1) not endo-Clifford or (2) $Z_1 \cup Z_2$ or (3) $Z_2 \cup Z_2$ or (4) Z_2

Table 5.2: Endomorphism monoids of $G \cup \{a\}$ where G is a 7-vertices unretractive graph and $\lvert N(a)\rvert = 2$.

Chapter 6

Monoids and graph operations

In this chapter, we consider two graph operations: unions and joins. We will describe the relationship between a set of endomorphisms of unions (or joins) and a sum of two sets of endomorphisms of two graphs.

6.1 Basics

In this section we introduce some terminologies which we will use later.

Remark 6.1.1. ([20]) *Let M_1, M_2 be transformation monoids, $h \in M_1 + M_2$, $h = h_1 + h_2$. Then h is idempotent if and only if h_1 and h_2 are idempotent.*

For any graphs G and H, it is well-known that $End(G)$, $End(H)$, $SEnd(G)$ and $SEnd(H)$ are transformation monoids. This means we can use the above remark for these monoids. But in graph theory, we also have the sets $HEnd(G)$, $HEnd(H)$, $LEnd(G)$, $LEnd(H)$, $QEnd(G)$ and $QEnd(H)$ which are not necessarily monoids. The next lemma we extend the result in Remark 6.1.1 for these sets.

Lemma 6.1.2. *For any $\mathfrak{M} \in \{\emptyset, H, L, Q, S\}$, an element $h = h|_G + h|_H \in \mathfrak{M}End(G) + \mathfrak{M}End(H)$ is an idempotent if and only if $h|_G$ and $h|_H$ are idempotent.*

Proof. Necessity. Let $h = h_G + h_H \in \mathfrak{M}End(G) + \mathfrak{M}End(H)$ be an idempotent. For any $x \in V(G)$, we have $h_G^2(x) = h^2(x) = h(x) = h_G(x)$, so $h|_G$ is idempotent. Similarly, we get that $h|_H$ is idempotent.

Sufficiency. For any $x \in V(G)$, we have that $h^2(x) = h^2|_G(x) = h|_G(x) = h(x)$. Analogously for any $x \in V(H)$. So h is an idempotent. $\square$

Lemma 6.1.3. ([21]) *Let G be a graph, $x_1, x_2 \in G$, $x_1 \neq x_2$. There exists a strong endomorphism $f \in SEnd(G)$ with $f(x_1) = f(x_2)$ if and only if $N(x_1) = N(x_2)$.*

Theorem 6.1.4. *For any graph G, $Aut(G) = Aut(\overline{G})$.*

Lemma 6.1.5. 1. *Idempotent endomorphisms of G are in $HEnd(G)$.*
2. *If G is finite with $End(G) \neq HEnd(G)$, then $HEnd(G) \neq SEnd(G)$.*

Proof. 1. Let f be an idempotent endomorphism of G. Let $\{x, y\} \in E(G)$. Since f is an idempotent, then $f^2(x) = f(x)$ and $f^2(y) = f(y)$, i.e., $f(x) \in f^{-1}(f(x))$ and $f(y) \in f^{-1}(f(y))$. Since f is an endomorphism, then $\{f(x), f(y)\} \in E(G)$. Then we get that $f \in HEnd(G)$.

2. Let $f \in End(G) \setminus HEnd(G)$. Then there exists $\{f(x), f(y)\} \in E(G)$ such that for any $u \in f^{-1}(f(x))$ and $v \in f^{-1}(f(y))$, $\{u, v\} \notin E(G)$. Since G is finite, there exists a $i \in \mathbb{N}$ with f^i is an idempotent endomorphism. It follows from 1. that $f^i \in HEnd(G)$. In particular, since $\{f^i(x), f^i(y)\} \in E(X)$ we have that $f^i(x)$ and $f^i(y)$ are fixed under f^i, and thus they are adjacent preimages. Moreover, $f^i \notin SEnd(G)$, since not all preimages are adjacent, namely $\{x, y\} \notin E(X)$. $\square$

6.2 The sums of endomorphisms sets

For any two graphs G and H, recall that the *union* $G \cup H$ is defined as the graph with vertex set $V(G)\dot{\cup}V(H)$ and edge set $E(G) \cup E(H)$ and recall that the *join* $G + H$ is defined as the graph with vertex set $V(G)\dot{\cup}V(H)$ and edge set $E(G) \cup E(H) \cup \{\{x, y\} \mid x \in V(G), y \in V(H)\}$.

In this part, we describe the relations between $\mathfrak{M}(G)+\mathfrak{M}(H)$ and $\mathfrak{M}(G+H)$ where $\mathfrak{M} \in \{End,\ HEnd,\ LEnd,\ QEnd,\ SEnd,\ Aut\}$ and G, H are graphs.

Theorem 6.2.1. *Let G and H be disjoint graphs and consider $\mathfrak{M} \in \{End,\ HEnd,\ LEnd,\ QEnd,\ SEnd,\ Aut\}$. Then we get that*
(*a*) $\mathfrak{M}(G) + \mathfrak{M}(H) \subseteq \mathfrak{M}(G \cup H)$;
(*b*) $\mathfrak{M}(G) + \mathfrak{M}(H) \subseteq \mathfrak{M}(G + H)$.

Proof. Let f be an endomorphism in $\mathfrak{M}(G) + \mathfrak{M}(H)$. Since $f \in M(G) + M(H)$, we have that $f := g+h$ for some $g \in M(G)$ and for some $h \in M(H)$.

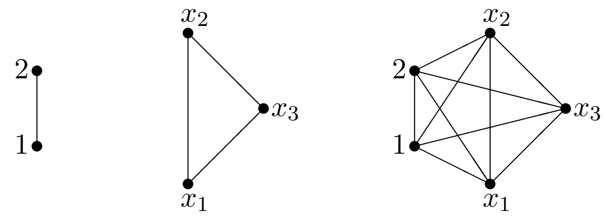
(*a*) It is clear that $f \in \mathfrak{M}(G \cup H)$.

(b) We show that $f \in \mathfrak{M}(G+H)$. Let $\{x, y\}$ be an edge in $E(G \cup H)$ such that $x \in V(G)$ and $y \in V(H)$. We have that $f(x) := (g+h)(x) = g(x) \in V(G)$ and $f(y) := (g+h)(y) = h(y) \in V(H)$. So $\{f(x), f(y)\} \in E(G \cup H)$ by definition of join. Then f is an endomorphism of $G+H$. $\square$

In general, we can not compare the sets $\mathfrak{M}(G \cup H)$ and $\mathfrak{M}(G+H)$. For example, if G and H are isomorphic to K_2, then $End(G+H)$ is not a subset of $End(G \cup H)$ and $End(G \cup H)$ is also not a subset of $End(G+H)$.

The converses of (a) and (b) in Theorem 6.2.1 are not necessarily true. We will show this fact in the next example.

Example 6.2.2. Take G a path P_1 and H a complete graph K_3 as follows.



G $\quad$ H $\quad$ $G+H \cong K_5$

It is clear that $f = \begin{pmatrix} 1 & 2 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_1 & x_2 & x_3 \end{pmatrix}$ is an endomorphism of $G \cup H$. But $f \notin End(G) + End(H)$ because $f|_G \notin End(G)$. This implies $End(G \cup H) \nsubseteq End(G) + End(H)$. And for the join of $G+H$, we know that $|End(G)+End(H)| = |End(G)| \cdot |End(H)| = 2 \cdot 6 = 12$ and $|End(G+H)| = |End(K_5)| = |Aut(K_5)| = 5! = 120$. Thus, $End(G+H)$ is not isomorphic to $End(G) + End(H)$. $\square$

For any graph G, we know that the sets $HEnd(G)$, $LEnd(G)$ and $QEnd(G)$ are not necessarily closed with respect to composition. The next corollary will show that if $HEnd(G)$, $LEnd(G)$ and $QEnd(G)$ are not closed, then $HEnd(G+H)$, $LEnd(G+H)$ and $QEnd(G+H)$ are not closed for any graph H. It also formulates consequences for unretractivities.

Corollary 6.2.3. *Let G and H be disjoint graphs.*
1. *If $\mathfrak{M}(G)$ is not closed as a monoid, then $\mathfrak{M}(G+H)$ and $\mathfrak{M}(G \cup H)$ are not closed for all $\mathfrak{M} \in \{HEnd, LEnd, QEnd\}$.*
2. *If $\mathfrak{M}(G) \neq \mathfrak{N}(G)$, then $\mathfrak{M}(G+H) \neq \mathfrak{N}(G+H)$ and $\mathfrak{M}(G \cup H) \neq \mathfrak{N}(G \cup H)$ where $\mathfrak{M} \neq \mathfrak{N} \in \{HEnd, LEnd, QEnd\}$.*

Proof. Let G and H be disjoint graphs. We will consider the case when $\mathfrak{M} = HEnd$.

1. Suppose that $HEnd(G)$ is not closed, then there exist $f, g \in HEnd(G)$ such that $fg \notin HEnd(G)$. Then $(f + id_H), (g + id_H) \in HEnd(G) + HEnd(H) \subseteq HEnd(G \cup H)$ ($HEnd(G + H)$, respectively). Set $h := (fg + id_H) = (f + id_H)(g + id_H)$. We show that $h \notin HEnd(G \cup H)$ ($HEnd(G + H)$, respectively). Since $h|_G = fg \notin HEnd(G)$, there exists $\{x, y\} \in E(G)$ for some $x, y \in Im(fg)$ such that for all $x' \in (fg)^{-1}(x)$ and $y' \in (fg)^{-1}(y)$, $\{x', y'\} \notin E(G)$. Since $h(H) = H$, there are no $u, v \in H$ with $u \in h^{-1}(x)$ and $v \in h^{-1}(y)$, so we get that for all $x' \in (h)^{-1}(x)$ and $y' \in (h)^{-1}(y)$, $\{x', y'\} \notin E(G \cup H)$ ($E(G + H)$, respectively). This means $h \notin HEnd(G \cup H)$ ($HEnd(G + H)$, respectively).

2. Let $\mathfrak{M}, \mathfrak{N} \in \{HEnd, LEnd, QEnd\}$ with $\mathfrak{M} \neq \mathfrak{N}$ and $\mathfrak{M}(G) \neq \mathfrak{N}(G)$, say $\mathfrak{N}(G) \subset \mathfrak{M}(G)$. Take $f \in \mathfrak{M}(G) \setminus \mathfrak{N}(G)$. We have that $(f + id_H) \in \mathfrak{M}(G) + \mathfrak{M}(H) \subseteq \mathfrak{M}(G \cup H)$ ($\mathfrak{M}(G + H)$, respectively). Since $\mathfrak{M}(G) \neq \mathfrak{N}(G)$ and G, H are disjoint, we have that $(f + id_H) \notin \mathfrak{N}(G \cup H)$ ($\mathfrak{N}(G + H)$, respectively). □

6.3 Endomorphisms of unions

In this part, we find the conditions which make the converse of (a) in Theorem 6.2.1 true.

Definition 6.3.1. For any graphs G and H, we call $f \in End(G \cup H)$ a ***mixing endomorphism*** if $f(G) \nsubseteq G$ or $f(H) \nsubseteq H$.

It is obvious that if there is no mixing endomorphism in $End(G \cup H)$, then $End(G \cup H) \cong End(G) + End(H)$. Now we get the next lemma.

Lemma 6.3.2. *For any graphs G, H and $\mathfrak{M} \in \{End, HEnd, LEnd, QEnd, SEnd, Aut\}$, the following statements are equivalent:*
(i) *$\mathfrak{M}(G \cup H)$ is isomorphic to $\mathfrak{M}(G) + \mathfrak{M}(H)$,*
(ii) *there is no mixing endomorphism in $\mathfrak{M}(G \cup H)$,*
(iii) *$f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}(G \cup H)$.*

Lemma 6.3.3. *For any connected graphs G and H, if $\mathfrak{M}Hom(G, H) = \emptyset$ and $\mathfrak{M}Hom(H\ , G) = \emptyset$, we have that $\mathfrak{M}End(G \cup H) \cong \mathfrak{M}End(G) + \mathfrak{M}End(H)$ where $\mathfrak{M} \in \{\emptyset, H, L, Q, S\}$. And if $Iso(G, H) = \emptyset$, we have that $Aut(G \cup H) \cong Aut(G) + Aut(H)$.*

Proof. By Theorem 6.2.1, we know that $\mathfrak{M}End(G) + \mathfrak{M}End(H) \subseteq \mathfrak{M}End(G \cup H)$. We will show that $\mathfrak{M}End(G \cup H) \subseteq \mathfrak{M}End(G) + \mathfrak{M}End(H)$. By the assumption we get that $f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}End(G \cup H)$, i.e., there is no mixing endomorphism in $\mathfrak{M}End(G \cup H)$. Since G and H are disjoint, we get that $f|_G \in \mathfrak{M}End(G)$ and $f|_H \in \mathfrak{M}End(H)$, so $f = f|_G + f|_H \in \mathfrak{M}End(G) + \mathfrak{M}End(H)$. □

Corollary 6.3.4. *For any graphs G and H, if $\mathfrak{M}Hom(G_i, H_j) = \emptyset$ and $\mathfrak{M}Hom(H_j\ , G_i) = \emptyset$ for all components G_i of G and H_j of H, we have that $\mathfrak{M}End(G \cup H) \cong \mathfrak{M}End(G) + \mathfrak{M}End(H)$ where $\mathfrak{M} \in \{\emptyset, H, L, Q, S\}$. And if $Iso(G_i, H_j) = \emptyset$ for all components G_i of G and H_j of H, we have that $Aut(G \cup H) \cong Aut(G) + Aut(H)$.*

Lemma 6.3.5. *For any connected graphs G,H and $\mathfrak{M} \in \{\emptyset, H\}$, if $\mathfrak{M}Hom(G, H) \neq \emptyset$, then $\mathfrak{M}End(G \cup H) \ncong \mathfrak{M}End(G) + \mathfrak{M}End(H)$.*

Proof. Let $g \in HHom(G, H)$. Define $f := g + id_{(H)}$. It is clear that f is a mixing half strong endomorphism of $G \cup H$ since $f(G) \nsubseteq G$. By Lemma 6.3.2 we have that $HEnd(G \cup H)$ is not isomorphic to $HEnd(G) + HEnd(H)$.

Furthermore, f is also a mixing endomorphism, so we get that $End(G \cup H) \ncong End(G) + End(H)$. □

Corollary 6.3.6. *For any graphs G, H and $\mathfrak{M} \in \{\emptyset, H\}$, if $\mathfrak{M}Hom(G_i, H_j) \neq \emptyset$ for some component G_i of G and H_j of H, we get that $\mathfrak{M}End(G \cup H) \ncong \mathfrak{M}End(G) + \mathfrak{M}End(H)$.*

Theorem 6.3.7. *Let G and H be connected graphs.*
1. *$End(G) + End(H) \cong End(G \cup H)$ if and only if $Hom(G, H) = \emptyset$ and $Hom(H, G) = \emptyset$.*
2. *$HEnd(G) + HEnd(H) \cong HEnd(G \cup H)$ if and only if $HHom(G, H) = \emptyset$ and $HHom(H, G) = \emptyset$.*

Proof. 1. Necessity. Assume that there exist $g \in Hom(G, H)$. Now we define $f := h + id_H$. It is clear that $f \in End(G \cup H)$. Since $End(G \cup H) \cong End(G) + End(H)$, we get that $f|_G + f|_H = f \in End(G) + End(H)$, i.e., $f|_G \in End(G)$ and $f|_H \in End(H)$ which is not possible if $f(G) \nsubseteq G$. So we have a contradiction.

Sufficiency. It follows directly from Lemmas 6.3.3 and 6.3.5.

2. Similar as 1. □

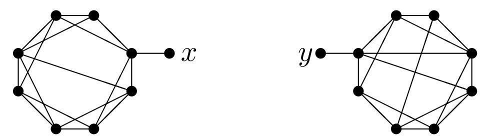
Corollary 6.3.8. *Let G and H be graphs.*
1. *$End(G) + End(H) \cong End(G \cup H)$ if and only if $Hom(G_i, H_j) = \emptyset$ and $Hom(H_j, G_i) = \emptyset$ for all components G_i of G and H_j of H.*
2. *$HEnd(G) + HEnd(H) \cong HEnd(G \cup H)$ if and only if $HHom(G_i, H_j) = \emptyset$ and $HHom(H_j, G_i) = \emptyset$ for all components G_i of G and H_j of H.*

Example 6.3.9. It is trivial that for mutually rigid graphs G and H one has $End(G)+End(H) \cong End(G \cup H)$ consisting only of the identity. ***Mutually rigid*** means $Hom(G,H) = Hom(H,G) = \emptyset$ and $|End(G)| = |End(H)| = 1$. The following two graphs are mutually rigid.

G $\qquad\qquad$ H

Note that $G = H = K_1$ do not fulfill the condition of Theorem 6.3.7 and indeed $|End(K_1) + End(K_1)| = 1$ but $|End(K_1 \cup K_1)| = 4$.

Add a vertex x to the graph G and a vertex y to the graph H as follows.



$G \cup \{x\}$ $\qquad\qquad$ $H \cup \{y\}$

It is clear that the graphs $G \cup \{x\}$ and $H \cup \{y\}$ are not rigid graphs, i.e., $|End(G \cup \{x\})|$, $|End(Y \cup \{y\})| > 1$, but $|Hom(G \cup \{x\}, H \cup \{y\})| = |Hom(H \cup \{y\}, G \cup \{x\})| = \emptyset$. By Theorem 6.3.7 we get that

$$End(G \cup \{x\}) + End(H \cup \{y\}) \cong End((G \cup x) \cup (H \cup \{y\})),$$

which can also be seen directly. The same is true for $HEnd$. $\square$

Next we consider the set of all automorphisms of the union of two graphs. For any connected graphs G and H, it is clear that $Iso(G,H) \neq \emptyset$ if and only if $Iso(H,G) \neq \emptyset$. Now we can prove the next theorem.

Theorem 6.3.10. *Let G, H be connected graphs. The following statements are equivalent:*

(*i*) $Aut(G) + Aut(H) \cong Aut(G \cup H)$

(*ii*) $Iso(G,H) = \emptyset$.

(*iii*) $f(G) \subseteq G$ *and* $f(H) \subseteq H$ *for all* $f \in Aut(G \cup H)$.

Proof. (*i*) $\Rightarrow$ (*ii*). Assume that $Iso(G,H) \neq \emptyset$, it is clear that $Iso(H,G)$ is also not empty. So there exist $g \in Iso(G,H)$ and $h \in Iso(H,G)$. We also have that $|G| = |H|$. Since G, H are disjoint, it is clear that

$$f(x) := \begin{cases} g(x) & , x \in G \\ h(x) & , x \in H \end{cases}$$

is a mixing automorphism, i.e., f does not belong to $Aut(G)+Aut(H)$. This contradicts to the assumption, so $Iso(G,H) = \emptyset$.

$(ii) \Rightarrow (i)$. This follows directly from Lemma 6.3.3.

$(i) \Leftrightarrow (iii)$. This follows directly from Lemma 6.3.2. $\square$

Corollary 6.3.11. *Let G, H be graphs. The following statements are equivalent:*

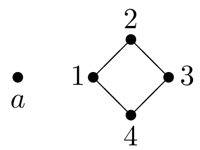
(i) $Aut(G) + Aut(H) \cong Aut(G \cup H)$

(ii) for any components G_i of G and H_j of H, $Iso(G_i, H_j) = \emptyset$.

(iii) $f(G) \subseteq G$, $f(H) \subseteq H$ for all $f \in Aut(G \cup H)$.

Next we will consider a monoid of all strong endomorphisms of the union of two connected graphs. From Lemma 6.3.3, we know that for any graphs G and H if both of $SHom(G,H)$ and $SHom(H,G)$ are empty sets, we get that $SEnd(G) \cong SEnd(G) + SEnd(H)$. Now we consider a case when one of $SHom(G,H)$ and $SHom(H,G)$ is not empty and the other one is empty.

Example 6.3.12. Consider a graph as follows.

$K_1 \cup C_4 =$ 

It is clear that $SHom(K_1, C_4) \neq \emptyset$ but $SHom(C_4, K_1) = \emptyset$. Since $N(1) = N(3)$ and $N(2) = N(4)$, by Lemma 6.1.3, there exists $h \in SHom(C_4)$ such that $h(1) = h(3)$ or $h(2) = h(4)$. So it is clear that $|Im(h)| \geq 2$. If $|Im(h)| = 2$, it is not possible that $Im(h) = \{1,3\}$ or $Im(h) = \{2,4\}$, i.e., two vertices in $Im(h)$ must be adjacent. It is clear that $SEnd(K_1 \cup C_4) \cong SEnd(K_1) + SEnd(C_4)$. $\square$

Before we will prove Theorem 6.3.16 describing when $SEnd(G \cup H)$ is isomorphic to $SEnd(G) + SEnd(H)$, we need some more lemmas.

Lemma 6.3.13. *For any connected graph $H \neq K_1$, $N(h(H)) = H$ for all $h \in SEnd(H)$.*

Proof. Let $h \in SEnd(H)$. Since H is connected, there exists an edge in $h(H)$ and $h(H)$ is connected. Suppose that $N(h(H)) \neq H$. So there exists $x \notin N(h(H))$, i.e., $\{x, h(y)\} \notin E(H)$ for all $h(y) \in h(H)$ and for all $y \in H$.

Since h is a strong endomorphism, then $\{h(x), h(h(y))\} \notin E(H)$ for all $y \in H$. This is not possible since $h(H)$ is connected. $\square$

Corollary 6.3.14. *For any connected graphs G and H both not K_1, there is no $f \in SEnd(G \cup H)$ such that $f(G \cup H) \subseteq G$ or $f(G \cup H) \subseteq H$.*

Lemma 6.3.15. *Let G and H be connected graphs with $SHom(G, H) \neq \emptyset$ and $SHom(H, G) = \emptyset$. We have that $SEnd(G \cup H) \cong SEnd(G) + SEnd(H)$.*

Proof. Assume that $SEnd(G \cup H) \ncong End(G) + End(H)$, so there exists a mixing strong endomorphism $f \in SEnd(G \cup H)$, i.e., $f \notin SEnd(G) + SEnd(H)$. Since $SHom(G, H) \neq \emptyset$ and $SHom(H, G) = \emptyset$, there exists $x \in G$ such that $f(x) \in H$ and $f(H) \subseteq H$. Since G and H are disjoint, we have that $f|_H \in SEnd(H)$. We have by Lemma 6.3.13 that $N(f|_H(H)) = H$. Since $f(x) \in H$, then $f(x) \in N(f|_H(H))$. So $\{f(x), f|_H(y)\} \in E(H) \subseteq E(G \cup H)$ for some $y \in f|_H(H)$. Since f is a strong endomorphism, then $\{x, y\} \in E(G \cup H)$. This is a contradiction since G and H are disjoint. Hence $SEnd(G \cup H) \cong SEnd(G) + SEnd(H)$. $\square$

Theorem 6.3.16. *Let G and H be connected graphs. Then $SEnd(G \cup H) \cong SEnd(G) + SEnd(H)$ if and only if $SHom(G, H) = \emptyset$ or $SHom(H, G) = \emptyset$.*

Proof. Necessity. Assume that $SHom(G, H) \neq \emptyset$ and $SHom(H, G) \neq \emptyset$, i.e., there exist $g \in SHom(G, H)$ and $h \in SHom(H, G)$. It is clear that $f := g + h \notin SEnd(G) + SEnd(H)$. Since G and H are disjoint, then f is a mixing strong endomorphism of $G \cup H$. This contradicts to the assumption, so $SHom(G, H) = \emptyset$ or $SHom(H, G) = \emptyset$.

Sufficiency. This follows directly from Lemmas 6.3.3 and 6.3.15. $\square$

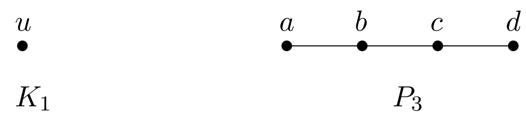
Example 6.3.17. Consider the paths P_1 and P_2 as follows.

P_1 P_2

It is clear that $f = \begin{pmatrix} 1 & 2 & a & b & c \\ a & b & 1 & 2 & 1 \end{pmatrix}$ is a mixing strong endomorphism of $P_1 \cup P_2$. Then we get that $SEnd(P_1 \cup P_2)$ is not isomorphic to $SEnd(P_1) + SEnd(P_2)$. This two graphs do not fulfill the condition in Theorem 6.3.16, i.e., $SHom(P_1, P_2) \neq \emptyset$ and $SHom(P_2, P_1) \neq \emptyset$. $\square$

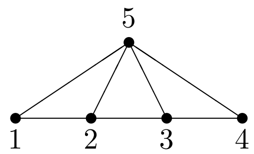
Next we will consider the set of all locally (quasi-) strong endomorphisms of the union of two graphs G and H. By Lemma 6.3.3 we see that if $LHom(G, H) = LHom(H, G) = \emptyset$ ($QHom(G, H) = QHom(H, G) = \emptyset$), then $LEnd(G \cup H) \cong LEnd(G) + LEnd(H)$ ($QEnd(G \cup H) \cong QEnd(G) + QEnd(H)$). Then we consider two graphs G, H which exactly one of $LHom(G, H)$ and $LHom(H, G)$ ($QHom(G, H)$ and $QHom(H, G)$) being empty. Of course, if both of G and H are L-S-unretractive (Q-S-unretractive), we get by Theorem 6.3.16 that $LEnd(G \cup H) \cong LEnd(G) + LEnd(H)$ ($QEnd(G \cup H) \cong QEnd(G) + QEnd(H)$). Hence, we consider the case when G or H is not L-S-unretractive (Q-S-unretractive).

Example 6.3.18. (1) Consider graphs as follows.



Now $QHom(K_1, P_3) \neq \emptyset$ and $Hom(P_3, K_1) = \emptyset$, and then $QHom(K_1, P_3) \neq \emptyset$ and $QHom(P_3, K_1) = \emptyset$. It is clear that $h = \begin{pmatrix} u & a & b & c & d \\ a & c & d & c & d \end{pmatrix} \in QEnd(K_1 \cup P_3) \subseteq LEnd(K_1 \cup P_3)$ is mixing and $h \notin End(K_1) + End(P_3)$. So we have that $QEnd(K_1 \cup P_3) \ncong QEnd(K_1) + QEnd(P_3)$. This implies also that $LEnd(K_1 \cup P_3) \ncong LEnd(K_1) + LEnd(P_3)$.

(2) Consider the graphs $K_1 = \{u\}$ and H as follows.



$H \cong P_3 + K_1$

It is clear that $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 5 \end{pmatrix}$ is a quasi strong but not strong endomorphism of H, so we have that H is not Q-S-unretractive, so is not L-S-unretractive. Next we show that $LEnd(K_1 \cup H)$ is isomorphic to $LEnd(K_1) + LEnd(H)$. Since K_1, H are disjoint and $LHom(H, K_1) = \emptyset$, then $g(H) \subseteq H$ for all $g \in LEnd(K_1 \cup H)$. Since H contains a triangle and 5 is in every triangle, we get that $5 \in g(H)$ for all $g \in LEnd(K_1 \cup H)$.

Assume that there exists $h \in LEnd(K_1 \cup H)$ such that $h(u) = v \in H$. Since K_1 and H are disjoint, then $h(a) \notin h(H)$, so $h(u) \neq 5$. Now we have

$h^{-1}(v) = \{u\}$ and $h^{-1}(5) \subset H$. Since $\{v, 5\} \in E(H)$ and u is not adjacent to all vertices in $h^{-1}(5)$, then h is not a locally strong endomorphism of $K_1 \cup H$. Hence, we get that $LEnd(K_1 \cup H)$ is isomorphic to $LEnd(K_1) + LEnd(H)$. Similarly we get that $QEnd(K_1 \cup H) \cong QEnd(K_1) + QEnd(H)$. □

The above examples show that in general the condition ,,$LHom(G, H) = \emptyset$ or $LHom(G, H) = \emptyset$ ($QHom(G, H) = \emptyset$ or $QHom(G, H) = \emptyset$)" is not sufficient for $LEnd(G \cup H) \cong LEnd(G) + LEnd(H)$ ($QEnd(G \cup H) \cong QEnd(G) + QEnd(H)$).

Theorem 6.3.19. *For any connected graphs G and H, there is no mixing endomorphism $f \in QEnd(G \cup H)$ if and only if $QHom(H, G) = \emptyset$ and for all $g \in QHom(G, H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ for all $h \in QEnd(H)$ and vice versa.*

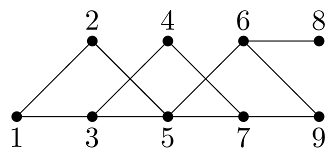
Proof. Necessity. It is quite clear that at least one of $QHom(G, H)$ and $QHom(H, G)$ is empty. Now we let $QHom(H, G) = \emptyset$.

Suppose that $QHom(G, H) \neq \emptyset$. Let $g \in QHom(G, H)$ and $h \in QEnd(H)$. Assume that $g(G) \cap N_H(h(H)) = \emptyset$. This means all vertices in $g(G)$ not adjacent to any vertex in $h(H)$. Since G and H are disjoint, it is clear that $f := g + h$ is a mixing quasi strong endomorphism of $G \cup H$. This is a contradiction. Thus we get that $g(G) \cap N_H(h(H)) \neq \emptyset$.

Sufficiency. Let $QHom(H, G) = \emptyset$ and let $f \in QEnd(G \cup H)$ be mixing. Then we have only the case $f(G \cup H) \subseteq H$, so by hypothesis there exists $x \in G$ with $\{f(x), f(y)\} \in E(H)$ for some $y \in H$. But then f is not quasi strong. This is a contradiction. So we get that f is not mixing. □

From the Example 6.3.18 it seems likely that the condition ,,$QHom(H, G) = \emptyset$ and if $QHom(G, H) \neq \emptyset$, then $G_1 \neq K_1$ or for all $h \in QEnd(H)$, $h(H) \neq P_3$" implies $QEnd(G \cup H) \cong QEnd(G) + QEnd(H)$. But the next example shows that this is not true.

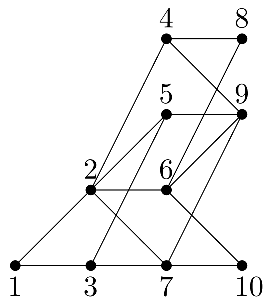
Example 6.3.20. (1) Consider $K_1 = \{u\}$ and the graph H as follows.



H

By inspection we get that for all $h \in QEnd(H)$, $h(H) \neq P_3$. It is clear that $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & u \\ 1 & 3 & 3 & 5 & 5 & 7 & 7 & 9 & 9 & 8 \end{pmatrix}$ is a mixing quasi strong endomorphism of $K_1 \cup H$. This follows by Lemma 6.3.2 that $QEnd(K_1 \cup H)$ is not isomorphic to $QEnd(K_1) + QEnd(H)$.

(2) Consider the complete graph $K_2 = \{a, b\}$ and the graph H as follows.



H

It is clear that $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 3 & 7 & 7 & 7 & 7 & 10 & 10 & 10 \end{pmatrix}$ is a quasi strong endomorphism of H. Thus it is also clear that $f(x) = \begin{cases} g(x), & x \in H \\ 4, & x = a \\ 8, & x = b \end{cases}$ is a mixing quasi strong endomorphism of $K_2 \cup H$. This follows by Lemma 6.3.2 that $QEnd(K_2 \cup H)$ is not isomorphic to $QEnd(K_2) + QEnd(H)$. $\square$

Now we consider the set of all locally strong endomorphisms of the union of two connected graphs G and H.

Theorem 6.3.21. *For any connected graphs G, H, there is no mixing endomorphism $f \in LEnd(G \cup H)$ if and only if $LHom(H, G) = \emptyset$ and for all $g \in LHom(G, H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ and $g(G) \neq h(H)$ for all $h \in LEnd(H)$ and vice versa.*

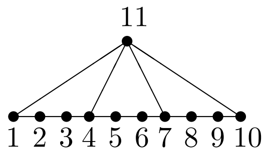
Proof. Necessity. It is clear that at least one of $LHom(G, H)$ and $LHom(H, G)$ is empty. Now we let $LHom(H, G) = \emptyset$.

Suppose that $LHom(G, H) \neq \emptyset$. Let $g \in LHom(G, H)$ and $h \in LEnd(H)$. Assume that $g(G) \cap N_H(h(H)) = \emptyset$. This means all vertices in $g(G)$ not adjacent to any vertex in $h(H)$. Since G and H are disjoint, it is clear that $g + h$ is a mixing locally strong endomorphism of $G \cup H$. This is a contradiction. Thus we get that $g(G) \cap N_H(h(H)) \neq \emptyset$. Assume that $g(G) = h(H)$, it is clear that $g+h$ is a mixing locally strong endomorphism of $G \cup H$. This is a contradiction. So we get that $g(G) \neq h(H)$.

Sufficiency. Let $LHom(H,G) = \emptyset$ and $f \in LEnd(G \cup H)$. Assume that f is mixing. Since $LHom(H,G) = \emptyset$, then we have only case $f(G \cup H) \subseteq H$, so by hypothesis we get that $f|_G(G) \neq f|_H(H)$ and $f|_G(G) \cap N_H(f|_H(H)) \neq \emptyset$. If $f|_G(G) \subsetneq f|_H(H)$, then there exists $f(x) \in f|_G(G)$ and $f(y) \in f|_H(H) \setminus f|_G(G)$ such that $\{f(x), f(y)\} \in E(H)$ since $h(H)$ is connected. Now we have that $f^{-1}(f(y)) \subseteq H$ and there exists $x' \in f^{-1}(f(x)) \cap G$. Since G and H are disjoint, then x' is not adjacent to all vertices in $f^{-1}(f(y))$. This is a contradiction. Similarly we get a contradiction if $f(H) \subsetneq f(G)$. Hence f is not mixing. □

For any connected graph G, we can find some connected graph H which is not L-S-unretractive and $LHom(G,H) \neq \emptyset$ and $LHom(H,G) = \emptyset$ but $LEnd(G \cup H)$ is not isomorphic to $LEnd(G) + LEnd(H)$.

Example 6.3.22. Consider a graph $K_2 = \{x, y\}$ and graph H as follows.



H

It is clear that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 2 & 3 & 4 & 11 \end{pmatrix} \text{ and}$$
$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & x & y \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 2 & 3 & 4 & 11 & 8 & 9 \end{pmatrix}$$

are locally strong endomorphism of H and $H \cup K_2$, respectively. So we have that g is a mixing locally strong endomorphism of $H \cup K_2$. Hence we get that $LEnd(H \cup K_2)$ is not isomorphic to $LEnd(H) + LEnd(K_2)$. And it is also clear that f is not a strong endomorphism of H, so we get that H is not L-S-unretractive. □

The conditions in Theorems 6.3.19 and 6.3.21 are not ,,good", since in general it will be difficult to check. But we do not have better ones.

$\mathfrak{M}$	$\mathfrak{M}(G \cup H) \cong \mathfrak{M}(G) + \mathfrak{M}(H)$
End	$\Leftrightarrow Hom(G,H) = \emptyset$ and $Hom(H,G) = \emptyset$
$HEnd$	$\Leftrightarrow HHom(G,H) = \emptyset$ and $HHom(H,G) = \emptyset$
$LEnd$	$\Leftrightarrow LHom(H,G) = \emptyset$ and $\forall g \in LHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ and $g(G) \neq h(H)$ $\forall h \in LEnd(H)$ and vice versa
$QEnd$	$\Leftrightarrow QHom(H,G) = \emptyset$ and $\forall g \in QHom(G,H)$ one has $g(G) \cap N_H(h(H)) \neq \emptyset$ $\forall h \in QEnd(H)$ and vice versa
$SEnd$	$\Leftrightarrow SHom(G,H) = \emptyset$ or $SHom(H,G) = \emptyset$
Aut	$\Leftrightarrow Iso(G,H) = \emptyset \Leftrightarrow G \ncong H$

Table 6.1: $\mathfrak{M}(G \cup H)$ is isomorphic to $\mathfrak{M}(G) + \mathfrak{M}(H)$ where $\mathfrak{M} \in \{End, HEnd, LEnd, QEnd, SEnd, Aut\}$ and G, H are connected graphs.

6.4 Endomorphisms of joins

In this section, we get a theorem describing when the set $\mathfrak{M}(G+H)$ is isomorphic to $\mathfrak{M}(G) + \mathfrak{M}(H)$ where $\mathfrak{M} \in \{End, HEnd, LEnd, QEnd, SEnd, Aut\}$ and G, H are graphs.

Theorem 6.4.1. *Let G, H be graphs and $\mathfrak{M} \in$\{End,$HEnd$, $LEnd$, $QEnd$, $SEnd$, Aut\}. We have that $\mathfrak{M}(G) + \mathfrak{M}(H) \cong \mathfrak{M}(G+H)$ if and only if $f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}(G+H)$.*

Proof. 1. Necessity. Let $\mathfrak{M}(G) + \mathfrak{M}(H) \cong \mathfrak{M}(G+H)$ and $f \in \mathfrak{M}(G+H)$. Then we have that $f = g + h$ with $g \in \mathfrak{M}(G)$ and $h \in \mathfrak{M}(H)$. It is clear that $f(G) \subseteq G$ and $f(H) \subseteq H$.

Sufficiency. By Lemma 6.2.1, we have that $\mathfrak{M}(G) + \mathfrak{M}(H) \subseteq \mathfrak{M}(G+H)$. It remains to prove that $\mathfrak{M}(G+H) \subseteq \mathfrak{M}(G) + \mathfrak{M}(H)$. Since $f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in \mathfrak{M}(G+H)$, it is clear that $f|_G$ and $f|_H$ are in $\mathfrak{M}(G)$ and $\mathfrak{M}(H)$, respectively. Hence $f = f|_G + f|_H \in \mathfrak{M}(G) + \mathfrak{M}(H)$ for all $f \in \mathfrak{M}(G+H)$. Now we get that $\mathfrak{M}(G+H)$ is isomorphic to $\mathfrak{M}(G) + \mathfrak{M}(H)$. $\square$

Lemma 6.4.2. *For any graphs G and H, $Iso(\overline{G}_i, \overline{H}_j) = \emptyset$ for all component $\overline{G}_i$ of $\overline{G}$ and $\overline{H}_j$ of $\overline{H}$ if and only if $f(G) \subseteq G$ and $f(H) \subseteq H$ for all $f \in Aut(G+H)$.*

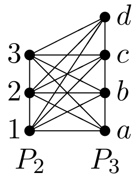
Proof. By Corollary 6.3.11 we get that $Iso(\overline{G}_i, \overline{H}_j) = \emptyset$ for all components $\overline{G}_i$ of $\overline{G}$ and $\overline{H}_j$ of H_j if and only if $Aut(\overline{G} \cup \overline{H}) \cong Aut(\overline{G}) + Aut(\overline{H})$. By definition of the join and the complement of graph, we have that $\overline{G+H} = \overline{G} \cup \overline{H}$. So by Theorem 6.1.4 we get that

$$Aut(G+H) \cong Aut(\overline{G} \cup \overline{H}) \cong Aut(\overline{G}) + Aut(\overline{H}) \cong Aut(G) + Aut(H).$$

By Theorem 6.4.1 we get the result. □

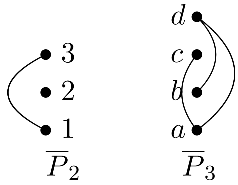
Corollary 6.4.3. *For any graphs G and H, the following statements are equivalent:*
(i) $Aut(G+H) = Aut(G) + Aut(H)$
(ii) $f(G) \subseteq G$ *and* $f(H) \subseteq H$ *for all* $f \in Aut(G+H)$
(iii) $Iso(\overline{G}_i, \overline{H}_j) = \emptyset$ *for all components* $\overline{G}_i$ *of* $\overline{G}$ *and* $\overline{H}_j$ *of* $\overline{H}$.

Example 6.4.4. Take a graph $P_2 + P_3$ as follows.



It is routine to check that $Aut(P_2+P_3)$ and $Aut(P_2)+Aut(P_3)$ are isomorphic which they have 4 elements. Next we show that all automorphisms of $P_2 + P_3$ send P_2 and P_3 to P_2 and P_3, respectively.

For any graph G, it is clear that $deg_G(f(x)) = deg_G(x)$ for all $x \in G$ and $f \in Aut(G)$. Let g be an automorphism of $P_2 + P_3$. From the above graph we see that exactly vertex 2 has degree 6. This implies that $g(2) = 2$. Similarly we get that all $g(\{a,d\}) = \{a,d\}$, since $deg(a) = deg(d) = 4$. Since $\{3,a\}, \{3,d\} \in E(P_2+P_3)$, $\{c,a\}, \{d,b\} \notin E(P_2+P_3)$ and g is an endomorphism, we get that $g(3) \notin \{b,c\}$. Similarly we get that $g(1) \notin \{b,c\}$, so $g(\{1,3\}) = \{1,3\}$. Now we get that $g(P_2) = P_2$ and $g(P_3) = P_3$. Moreover, we have the complements of P_2 and P_3 as follows.



We see that $\overline{P}_2$ has 2 components; $\{1,3\}$ and $\{2\}$. And component of $\overline{P}_3$ is itself. It is clear that all components of $\overline{P}_2$ are not isomorphic to $\overline{P}_3$. This confirms that Corollary 6.4.3 is hold. □

There are many operations which are defined on the set of graphs, for instance, box product and cross product. In the future we will look for the endomorphism monoids of two graphs which conjunct by these operations.

Chapter 7

Unretractivities of graph operations

In this chapter, we find the unretractivities of a union of two graphs and the unretractivities of a join of two graphs.

7.1 Basics

We give some terminologies which we will use later.

Theorem 7.1.1. ([21]) *A graph G is S-A-unretractive if and only if $N(x) \neq N(y)$ for all $x, y \in G$ with $x \neq y$.*

Lemma 7.1.2. *For any graph G and $\mathfrak{M} \in \{L, Q\}$, if $f \in \mathfrak{M}End(G)$, then $f^n \in \mathfrak{M}End(G)$ for any $2 \leq n \in \mathbb{N}$.*

Proof. We prove only the case $\mathfrak{M} = L$. The other case follows analogously. Let $f \in LEnd(G)$. First we consider $n = 2$. Let $\{f^2(x), f^2(y)\} \in E(G)$. It is clear that $f(x)$ is exactly one vertex in $f^{-1}(f^2(x))$ and $f(y)$ is exactly one vertex in $f^{-1}(f^2(y))$. Since f is a locally strong endomorphism, we get that $\{f(x), f(y)\} \in E(G)$. Let $x' \in (f^2)^{-1}(f^2(x)) = f^{-1}(f(x))$. Since $\{f(x), f(y)\} \in E(G)$ and f is a locally strong endomorphism, we get that there exists $y'_0 \in f^{-1}(f(y)) = (f^2)^{-1}(f^2(y))$ such that $\{x', y'_0\} \in E(G)$. So we have that $f^2 \in LEnd(G)$. Proceeding in this manner we get the result. $\square$

Lemma 7.1.3. ([20]) *Let G be a graph. Then G is unretractive if and only if $End(G)$ contains only one idempotent.*

Lemma 7.1.4. *Let G be a finite graph and take $\mathfrak{M} \in \{L, Q\}$. Then G is $\mathfrak{M}$-A-unretractive if and only if $\mathfrak{M}End(G)$ contains only one idempotent.*

Proof. Necessity. It is obvious.

Sufficiency. We prove by contraposition. Suppose that G is not $\mathfrak{M}$-A-unretractive, so there exists $f \in \mathfrak{M}End(G) \setminus Aut(G)$. Since G is a finite graph, there exists $i \in \mathbb{N}$ such that f^i is an idempotent power of f. By Lemma 7.1.2, we get that f^i is in $\mathfrak{M}End(G)$. It is clear that f^i is not an identity. So we get that $\mathfrak{M}End(G)$ contains more than one idempent. □

7.2 Unretractivities of unions

In this section, we find conditions for different unretractivities of the union of graphs. We begin with E-$\mathfrak{M}$-unretractive and H-$\mathfrak{M}$-unretractive, $\mathfrak{M} \in \{SEnd, Aut\}$.

Theorem 7.2.1. *Let G, H be finite connected graphs and $\mathfrak{M} \in \{SEnd, Aut\}$. The following statements are equivalent:*
(i) $End(G \cup H) = \mathfrak{M}(G \cup H)$.
(ii) $HEnd(G \cup H) = \mathfrak{M}(G \cup H)$.
(iii) $End(G) = \mathfrak{M}(G)$, $End(H) = \mathfrak{M}(H)$ and $Hom(G, H) = Hom(H, G) = \emptyset$.

Proof. $(i) \Rightarrow (ii)$. For any $\mathfrak{M} \in \{SEnd, Aut\}$, since $End(G \cup H) \supseteq HEnd(G \cup H) \supseteq \mathfrak{M}(G \cup H)$ and $End(G \cup H) = \mathfrak{M}(G \cup H)$, we have that $HEnd(G \cup H) = \mathfrak{M}(G \cup H)$.

$(ii) \Rightarrow (i)$. For any $\mathfrak{M} \in \{SEnd, Aut\}$, since $HEnd(G \cup H) \supseteq SEnd(G \cup H) \supseteq \mathfrak{M}(G \cup H)$ and $HEnd(G \cup H) = \mathfrak{M}(G \cup H)$, we have that $HEnd(G \cup H) = SEnd(G \cup H)$. By Lemma 6.1.5 2., we have that $HEnd(G \cup H) = End(G \cup H)$, so we get that $End(G \cup H) = \mathfrak{M}(G \cup H)$.

$(iii) \Rightarrow (i)$ By Theorem 6.3.7 and hypothesis, we have that
$End(G \cup H) = End(G) + End(H) = \mathfrak{M}(G) + \mathfrak{M}(H) = \mathfrak{M}(G \cup H)$.

$(i) \Rightarrow (iii)$. Assume that there exists $h \in Hom(G, H)$. Set
$f(x) := \begin{cases} h(x) & , x \in V(G) \\ id_H(x) & , x \in V(H) \end{cases}$, then $f \in End(G \cup H)$ better type setting.
By hypothesis we have that $f \in SEnd(G \cup H)$. Since $h(G) \cap Im(id_H) \neq \emptyset$, there exists an edge $\{f(u), f(v)\} \in E(H)$ with $u \in V(G)$ and $v \in V(H)$. Since G and H are disjoint, then $\{u, v\} \notin E(G \cup H)$. This contradicts to $f \in SEnd(G \cup H)$, so $Hom(G, H) = \emptyset$. Similarly we get that $Hom(H, G) = \emptyset$.

Let $k \in End(G)$. We will show that $k \in \mathfrak{M}(G)$. By Theorem 6.2.1, we know that $End(G) + End(H) \subseteq End(G \cup H) = \mathfrak{M}(G \cup H)$, so there exists $l \in End(G \cup H)$ such that $l = k + id_H$. Since $l \in M(G \cup H)$ and G, H are disjoint, then we get that $k \in \mathfrak{M}(G)$, so $End(G) = \mathfrak{M}(G)$. Similarly we get that $End(H) = \mathfrak{M}(H)$. □

Next we find a condition for L-A-unretractivity of unions of graphs. We need some lemmas.

Lemma 7.2.2. *Let $G_1, G_2, ..., G_\ell, H_1, H_2, ..., H_\ell$ be connected graphs and $f_i \in Hom(G_i, H_i)$ for any $i \in \{1, 2, ..., \ell\}$. Set $G := \bigcup_{i=1}^{\ell} G_i$ and $H := \bigcup_{i=1}^{\ell} H_i$. Then $f := f_1 + f_2 + ... + f_\ell \in LHom(G, H)$ if and only if $f_i \in LHom(G_i, H_i)$ for all $i \in \{1, 2, ..., \ell\}$.*

Proof. We prove only the case $\ell = 2$. The other cases follow analogously.

Necessity. Let $f := f_1 + f_2 \in LHom(G, H)$. It is clear that $f_i = f|_{G_i}$ for all $i \in \{1, 2\}$. Since f is a locally strong homomorphism and G_1, G_2, H_1, H_2 are pairwise disjoint, we get that $f|_{G_i}$ is a locally strong homomorphism from G_i to H_i for all $i \in \{1, 2\}$. Now we have that $f_i \in LHom(G_i, H_i)$ for all $i \in \{1, 2\}$.

Sufficiency. Let $f := f_1 + f_2$ with $f_i \in LHom(G_i, H_i)$ for all $i \in \{1, 2\}$. So we get that $f|_{G_i} = f_i \in LHom(G_i, H_i)$ for all $i \in \{1, 2\}$. Since G_1, G_2, H_1, H_2 all are pairwise disjoint and f_i is a locally strong homomorphism from G_i to H_i for all $i \in \{1, 2, ..., \ell\}$, we have that $f \in LHom(G, H)$. □

Lemma 7.2.3. *Let G_1, G_2 and H be connected graphs, $f_1 \in LHom(G_1, H)$ and $f_2 \in LHom(G_2, H)$.*

(a) *If $f_1(G_1) = f_2(G_2)$, then $f = f_1 + f_2 \in LHom(G_1 \cup G_2, H)$;*

(b) *If $f_1(G_1) \neq f_2(G_2)$ and $f_1(G_1) \cap f_2(G_2) \neq \emptyset$, then $f = f_1 + f_2 \notin LHom(G_1 \cup G_2, H)$.*

Proof. (a) Since $f_1(G_1) = f_2(G_2)$, we have that $Im(f) = Im(f_1 + f_2) = Im(f_1) = Im(f_2)$. Let $\{u, v\}$ be an edge in $E(H)$ with $u, v \in Im(f)$. Then $f^{-1}(u) = f_1^{-1}(u) \cup f_2^{-1}(u)$ and $f^{-1}(v) = f_1^{-1}(v) \cup f_2^{-1}(v)$. Since G_1, G_2 are disjoint, $f_1^{-1}(u) \cap f_2^{-1}(u) = \emptyset$ and $f_1^{-1}(v) \cap f_2^{-1}(v) = \emptyset$. Since f_i is locally strong homomorphism from G_i to H, $i \in \{1, 2\}$, for all $x \in f_i^{-1}(u)$ there exists $y \in f_i^{-1}(v)$ such that $\{x, y\} \in E(G_i)$, $i \in \{1, 2\}$. This implies that for all $x_0 \in f^{-1}(u)$ there exists $y_0 \in f^{-1}(v)$ such that $\{x_0, y_0\} \in E(G_1 \cup G_2)$, so we get that $f \in LHom(G_1 \cup G_2, H)$.

(b) Suppose that there exist $u \in f_1(G_1) \cap f_2(G_2)$ and $v \in f_2(G_2) \setminus f_1(G_1)$ with $\{u, v\} \in E(f(G_1 \cup G_2))$. Let $v_0 \in f_2^{-1}(v) = f^{-1}(v) \subseteq G_2$ and $u_0 \in$

$f_1^{-1}(u) \subseteq f^{-1}(u) \cap G_1$. Since G_1 and G_2 are disjoint, then $\{u_0, v_0\} \notin E(G_1 \cup G_2)$, so f is not a locally strong homomorphism. $\square$

Lemma 7.2.4. *Let G and H be connected graphs, both not K_1, where $G \cup H$ is L-Q-unretractive. If there exists $g \in LHom(G, H)$, then $g(G) \neq h(H)$ for all $h \in LEnd(H)$.*

Proof. Assume that there exist $g \in LHom(G, H)$ and $h \in LEnd(H)$ with $g(G) = h(H)$. By Lemma 7.2.3(a), we get that $f := g+h \in LEnd(G \cup H)$, so $E(f(G \cup H)) \subseteq E(H)$. Since G, H are connected and are not K_1, then there exists $\{u, v\} \in E(f(G \cup H))$. Now we know that $f^{-1}(u) = g^{-1}(u) \cup h^{-1}(u)$ and $f^{-1}(v) = g^{-1}(v) \cup h^{-1}(v)$. Since $g(G) = h(H)$, then $f^{-1}(u) \cap G \neq \emptyset$, $f^{-1}(u) \cap H \neq \emptyset$, $f^{-1}(v) \cap G \neq \emptyset$ and $f^{-1}(v) \cap H \neq \emptyset$. Since G and H are disjoint, there is no $u_0 \in f^{-1}(u)$ such that $\{u_0, v_0\} \in E(G \cup H)$ for all $v_0 \in f^{-1}(v) \subseteq G \cup H$, so f is not quasi strong homomorphism. This is a contradiction. Hence we get that $g(G) \neq h(H)$ for all $h \in LEnd(H)$. $\square$

Lemma 7.2.5. *Let G and H be graphs such that $G \cup H$ is L-A-unretractive. If $LHom(G, H) \setminus Iso(G, H) \neq \emptyset$, then $LHom(H, G) = \emptyset$.*

Proof. Let $g \in LHom(G, H) \setminus Iso(G, H)$. Assume that there exists $h \in LHom(H, G)$. By Lemma 7.2.2, we get that $f := g + h \in LEnd(G \cup H)$ but f is not an automorphism of $G \cup H$. This is a contradiction, so we get that $LHom(H, G) = \emptyset$. $\square$

Lemma 7.2.6. *Let G and H be graphs such that $G \cup H$ is L-A-unretractive. Then $G \ncong H$.*

Proof. Assume that $G \cong H$, so there exists $g \in Iso(G, H)$. Define a mapping f from $G \cup H$ to itself by

$$f(x) = \begin{cases} g(x) & , if\ x \in V(G) \\ id_H & , if\ x \in V(H). \end{cases}$$

It is clear that f is an endomorphism of $G \cup H$ but not an automorphism. This contradicts to the hyphotesis, so $G \ncong H$. $\square$

Now we turn to prove the theorem which describes when the union of two graphs is L-A-unretractive.

Theorem 7.2.7. *Let G and H be finite connected graphs, both not K_1. We get that*

(1) *If $LEnd(G \cup H) = Aut(G \cup H)$, then (a) G, H are L-A-unretractive and (b) $LHom(G, H) = \emptyset$ or $LHom(H, G) = \emptyset$.*

(2) *If* (*a*) G, H *are* L-A-*unretractive and* (*b*) $LHom(G,H) = LHom(H,G) = \emptyset$, *then* $LEnd(G \cup H) = Aut(G \cup H)$.

Proof. (1). Let $LEnd(G \cup H) = Aut(G \cup H)$.

First, we prove (a). By Theorem 6.2.1, we have that $LEnd(G)+LEnd(H) \subseteq LEnd(G\cup H) = Aut(G\cup H)$. Let $g \in LEnd(G)$ and $h \in LEnd(H)$. Then $f := g + h \in Aut(G \cup H)$. Since $f(G) = g(G) \subseteq G$ and $f(H) = h(H) \subseteq H$, G and H are disjoint, then $g \in Aut(G)$ and $h \in Aut(H)$.

Next, we prove (b). Suppose that $LHom(G,H) \neq \emptyset$, i.e., there exists $k \in LHom(G,H)$. Since $LEnd(G \cup H) = Aut(G \cup H)$, we get by Lemma 7.2.6 that $G \ncong H$, so not both are K_1. By (a), we have that $g(G) = G$ for all $g \in LEnd(G)$ and $h(H) = H$ for all $h \in LEnd(H)$. Since $G \cup H$ is also L-Q-unretractive, we get by Lemma 7.2.4 that $k(H) \neq g(G) = G$ and $k(G) \neq h(H) = H$. Since $G \ncong H$, we get that k is not an isomorphism from G to H. So by Lemma 7.2.5 we have that $LHom(H,G) = \emptyset$.

(2). Let $f \in LEnd(G \cup H)$. We will show that $f \in Aut(G \cup H)$. Since $LHom(G,H) = LHom(H,G) = \emptyset$, so we get that $f(G) \subseteq G$ and $f(H) \subseteq H$. We get by (a) that $f|_G \in LEnd(G) = Aut(G)$ and $f|_H \in LEnd(H) = Aut(H)$, i.e, $f|_G(G) = G$ and $f|_H(H) = H$. Since G and H are disjoint, we get that $f = f|_G + f|_H \in Aut(G \cup H)$. □

Before we prove when $G\cup H$ is Q-A-unretractive, we need some lemmas.

Lemma 7.2.8. *Let* G *and* H *be* Q-S-*unretractive connected graphs not both* K_1. *Then we have that for all* $f \in QEnd(G \cup H)$ *and for any* $K \in \{G, H\}$

(1) $f(K) \subseteq K'$ *for some* $K' \in \{G, H\}$

(2) *if* $f(K) \subseteq K'$ *for some* $K' \in \{G, H\}$, *then* $f((G \cup H) \setminus K) \cap K' = \emptyset$.

Proof. (1) Assume that $f(K) \cap G \neq \emptyset$ and $f(K) \cap H \neq \emptyset$. Since $f(K)$ is connected, then there exists $x \in f(K) \cap G$ and $y \in f(K) \cap H$ such that $\{x, y\} \in E(G \cup H)$. This is a contradiction since G and H are disjoint. So we get that $f(K) \subseteq K'$ for some $K' \in \{G, H\}$.

(2) We consider only the case $f(G) \subseteq H$ the another cases follow analogously. Assume that $f(H) \cap H \neq \emptyset$. It is clear that $f|_H$ is a quasi strong endomorphism of H. Since H is Q-S-unretractive, we get that $f|_H$ is strong, so by Lemma 6.3.13 we can conclude that $N(f|_H(H)) = H$. Since $f|_H(H)$ is connected, there exists $\{x, y\} \in E(H)$ for some $x \in f(G)$ and $y \in f|_H(H) = f(H)$. Since G and H are disjoint, it is clear that there is no $z \in f^{-1}(y)$ such that $\{u, z\} \in E(G \cup H)$ for all $u \in f^{-1}(x)$. Then f is not quasi strong. This is a contradiction. So we get that $f(H) \cap H = \emptyset$. □

Theorem 7.2.9. *Let G and H be finite connected graphs. The following statements are equivalent.*
(i) *$QEnd(G \cup H) = Aut(G \cup H)$.*
(ii) *$QEnd(G) = Aut(G)$ and $QEnd(H) = Aut(H)$.*

Proof. $(i) \Rightarrow (ii)$. By Theorem 6.2.1 and the hypothesis we have that

$$(a)\ QEnd(G) + QEnd(H) \subseteq QEnd(G \cup H) = Aut(G \cup H).$$

Let $g \in QEnd(G)$ and $h \in QEnd(H)$. By (a) we get that $f := g + h \in Aut(G \cup H)$. Since G, H are disjoint and $f \in Aut(G \cup H)$, it is clear that $f|_G = g \in Aut(G)$ and $f|_H = h \in Aut(H)$. Now we have that $QEnd(G) = Aut(G)$ and $QEnd(H) = Aut(H)$.

$(ii) \Rightarrow (i)$. Let $QEnd(G) = Aut(G)$ and $QEnd(H) = Aut(H)$. Then $QEnd(G)$ and $QEnd(H)$ contain only one idempotent endomorphism, namely id_G and id_H, respectively. By Lemma 7.1.4 we get that

$$QEnd(G) + QEnd(H) = Aut(G) + Aut(H)$$

contains exactly one idempotent endomorphism $id_G + id_H$.

Assume that there exists $f \in QEnd(G \cup H) \setminus Aut(G \cup H)$. So we have that $f(x) = f(y)$ for some $x, y \in V(G \cup H)$. By Lemma 7.2.8 we get that $x, y \in G$ or $x, y \in H$. Since $G \cup H$ is a finite graph, there exists $i \in \mathbb{N}$ such that f^i is an idempotent power of f. By Lemma 7.1.2 we get that f^i is a quasi strong endomorphism of $G \cup H$. Similar as f we get that $f^i(u) = f^i(v)$ if and only if $u, v \in G$ or $u, v \in H$. Since f^i is idempotent, we get that $f^i(z) = z$ for all $z \in Im(f^i)$. Now we have that $f^i|_G$ and $f^i|_H$ are quasi strong endomorphism of G and H, respectively. Now we get that $f^i = f^i|_G + f^i|_H \in QEnd(G) + QEnd(H)$. This is a contradiction since f^i is not $id_G + id_H$ which is exactly one idempotent in $QEnd(G) + QEnd(H)$. Hence we get that $QEnd(G \cup H) = Aut(G \cup H)$. □

For the last theorem in this section, we give a condition which union of graphs is S-A-unretractive.

Theorem 7.2.10. *Let G and H be finite connected graphs. Equivalent are*
(i) *$SEnd(G \cup H) = Aut(G \cup H)$.*
(ii) *$SEnd(G) = Aut(G)$ and $SEnd(H) = Aut(H)$.*

Proof. By Theorem 7.1.1, we have that $SEnd(G \cup H) = Aut(G \cup H)$ is equivalent to $N(x) \neq N(y)$ for all $x, y \in G \cup H$ and $x \neq y$. This is also equivalent to $N(x_1) \neq N(y_1)$ for all $x_1, y_1 \in G$, $x_1 \neq y_1$ and $N(x_2) \neq N(y_2)$ for all $x_2, y_2 \in H$, $x_2 \neq y_2$. Again by Theorem 7.1.1, we get that $SEnd(G \cup$

$H) = Aut(G \cup H)$ if and only if $SEnd(G) = Aut(G)$ and $SEnd(H) = Aut(H)$. $\square$

The other expected results

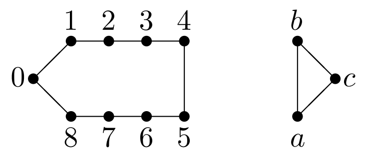
We think that the next hypothesis is also true. But we have no more time to prove it. We only give an idea to prove it in this section.

Hypothesis 7.2.11. *Let G and H be connected graphs. Then $G \cup H$ is Q-S-unretractive if and only if G and H are Q-S-unretractive.*

Lemma 7.2.12. *Let G be S-unretractive connected graph and let H be a connected graph. If $f \in SHom(G, H)$, then ρ_f is trivial.*

Proof. Since G is S-unretractive, then $N(x) \neq N(y)$ for all $x \neq y \in G$. Let $f \in SHom(G, H)$. Suppose that ρ_f is not trivial. Then there exist $a \neq b \in G$ with $f(a) = f(b)$. Since $N(a) \neq N(b)$, suppose that exists a vertex $c \in N(a) \setminus N(b)$. Since $\{a, c\} \in E(G)$ and f is homomorphism, then $\{f(a), f(c)\} \in E(H)$. Now we have that $b \in f^{-1}(f(a))$, $c \in f^{-1}(f(c))$ and $\{b, c\} \notin E(G)$. So f is not strong. This is a contraction. So ρ_f is trivial. $\square$

Example 7.2.13. Take the cycle C_9 and C_3 as follows.



It is well-known that C_9 and C_3 are unretractive, so they are also L-A-unretractive. It is clear that $f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a & b & c & a & b & c & a & b & c \end{pmatrix}$ is a locally strong homomorphism from C_9 to C_3. This shows that ρ_f is not trivial. $\square$

We expect that if G is a Q-A-unretractive connected graph and H is a connected graph, then ρ_f is trivial for any $f \in QHom(G, H)$.

Hypothesis 7.2.14. *If G is a Q-A-unretractive connected graph and H is a connected graph, then ρ_f is trivial for any $f \in QHom(G, H)$.*

Let G be a Q-S-unretractive connected graph and H is a connected graph. If the Hypothesis 7.2.14 is true, we get that for any $f \in QHom(G, H)$ there exists $g \in QEnd(G)$ such that $\rho_f = \rho_g$. We expect that the next hypothesis is also true.

Hypothesis 7.2.15. *Let G be a Q-S-unretractive connected graph and H is a connected graph. If $f \in QHom(G, H)$, then $\rho_f = \rho_g$ for some $g \in QEnd(G)$.*

The next hypothesis is a corollary of the above hypothesis.

Hypothesis 7.2.16. *Let G be a Q-S-unretractive connected graph and H is a connected graph. If $f(x) = f(y)$ for some $f \in QHom(G, H)$, then $g(x) = g(y)$ for some $g \in QEnd(G)$.*

Sketch of the proof of Hypothesis 7.2.11

$(i) \Rightarrow (ii)$. By Theorem 6.2.1 and the hypothesis we have that

$$(a)\ QEnd(G) + QEnd(H) \subseteq QEnd(G \cup H) = SEnd(G \cup H).$$

Let $g \in QEnd(G)$ and $h \in QEnd(H)$. By (a) we get that $f := g + h \in Aut(G \cup H)$. Since G, H are disjoint and $f \in SEnd(G \cup H)$, it is clear that $f|_G = g \in SEnd(G)$ and $f|_H = h \in SEnd(H)$. Now we have that $QEnd(G) = SEnd(G)$ and $QEnd(H) = SEnd(H)$.

$(ii) \Rightarrow (i)$. Assume that there exists $f \in QEnd(G \cup H) \setminus SEnd(G \cup H)$. Then there exists $x \neq y \in V(G \cup H)$ such that $f(x) = f(y)$. Since G and H are disjoint, it is clear by definition of quasi strong that $x, y \in G$ or $x, y \in H$. Since $QEnd(G) = SEnd(G)$, $QEnd(H) = SEnd(H)$ and G, H are connected and disjoint, by Lemma 7.2.8 we have two cases to consider: (1) $f(G) \subseteq G$ and $f(H) \subseteq H$ and (2) $f(G) \subseteq H$ and $f(H) \subseteq G$. First we consider case (1). Since G, H are disjoint, we get that $f|_G \in QEnd(G) = SEnd(G)$ and $f|_H \in QEnd(H) = SEnd(H)$. Then we get that $f = f|_G + f|_H \in SEnd(G) + SEnd(H)$. By Theorem 6.2.1 we also get that $f \in SEnd(G \cup H)$. This is a contradiction.

Next we consider case (2). Since $f \in QEnd(G)$ and G, H are disjoint, then $f|_G \in QHom(G, H)$ and $f|_H \in QHom(H, G)$, so we get that $QHom(G, H) \neq \emptyset$ and $QHom(H, G) \neq \emptyset$. If $x, y \in G$, then $f|_G(x) = f|_G(y)$. By Hypothesis 7.2.16 we get that $g(x) = g(y)$ for some $g \in QEnd(G)$. This contradicts to $QEnd(G) = SEnd(G)$. Similarly we get a contradiction if $x, y \in H$. So it is impossible to be this case.

Hence we get that $QEnd(G \cup H) = SEnd(G \cup H)$. □

For a graph $G \cup H$, it is not easy to find the sufficient condition for which graphs G and H whose union of them is L-S-unretractive. Although, in the future we will try to find this sufficient condition and also find other unretractivities of graph $G \cup H$.

$=$	$Aut(G \cup H)$	$SEnd(G \cup H)$
$SEnd(G \cup H)$	G,H are S-unretractive	-
$QEnd(G \cup H)$	G,H are Q-unretractive	see Hypothesis 7.2.11
$LEnd(G \cup H)$	$\Rightarrow$ G,H are L-unretractive and ($LHom(G, H) = \emptyset$ or $LHom(H, G) = \emptyset$) $\Leftarrow$ G,H are L-unretractive and $LHom(G, H) = \emptyset$ and $LHom(H, G) = \emptyset$	(do not get any condition now)
$HEnd(G \cup H)$ $End(G \cup H)$	G, H are unretractive and $Hom(G, H) = Hom(H, G) = \emptyset$	G, H are E-S-unretractive and $Hom(G, H) = Hom(H, G) = \emptyset$

Table 7.1: Unretractivities of $G \cup H$ where G, H are connected graphs.

7.3 Unretractivities of joins

In this section, unretractivities of joins of graphs are characterized. In [20], we know that when the joins of graphs $G + H$ are unretractive or S-A-unretractive.

Lemma 7.3.1. ([20]) *If* $f^2 = f \in End(G+H)$*, then* $f \in End(G)+End(H)$*. If* $f^2 = f \in SEnd(G + H)$*, then* $f \in SEnd(G) + SEnd(H)$*.*

Theorem 7.3.2. ([20]) *Let* G *and* H *be graphs. The join* $G + H$ *is (*S*-*A*-) unretractive if and only if* G *and* H *are (*S*-*A*-) unretractive.*

Remark 7.3.3. *From Lemma 6.1.5, we get that the graph which has endotypes* 1 *and* 17 *do not exists. So by Theorem 7.3.2, we have now that* $G+H$ *is* H*-*A*-unretractive if and only if* G *and* H *are* H*-*A*-unretractive.*

Next we give a theorem describing when $G + H$ is L-A-unretractive or L-A-unretractive.

Theorem 7.3.4. *Let* G *and* H *be finite graphs without loops and take* $\mathfrak{M} \in \{L, Q\}$*. The following statements are equivalent:*
(*i*) $\mathfrak{M}End(G + H) = Aut(G + H)$.
(*ii*) $\mathfrak{M}End(G) = Aut(G)$ *and* $\mathfrak{M}End(H) = Aut(H)$*.*

Proof. We prove the case $\mathfrak{M} = L$, the other cases follow analogously.
$(i) \Rightarrow (ii)$. By Theorem 6.2.1, we know that $LEnd(G) + LEnd(H) \subseteq LEnd(G + H)$. Since $LEnd(G + H) = Aut(G + H)$, then $LEnd(G) + LEnd(H) \subseteq Aut(G + H)$. If there exists $f \in LEnd(G) \setminus Aut(G)$, we have that $f + id_H$ is in $LEnd(G) + LEnd(H)$ but is not in $Aut(G + H)$. So we have that $LEnd(G) = Aut(G)$. Similarly, we get that $LEnd(H) = Aut(H)$.
$(ii) \Rightarrow (i)$. Let $LEnd(G) = Aut(G)$ and $LEnd(H) = Aut(H)$. Then $LEnd(G)$ and $LEnd(H)$ contains only one idempotent endomorphism, namely id_G and id_H, respectively. And by Lemma 6.1.2, we have that $LEnd(G) + LEnd(H)$ also contains only one idempotent endomorphism $id_G + id_H$.

Assume that there exists $f \in LEnd(G + H) \setminus Aut(G + H)$. So we have that $f(x) = f(y)$ for some $x \neq y \in V(G + H)$. By definition of the join $G + H$ we know that for any $a \in V(G)$ a adjacent to all vertices in H and vice versa. Since f is an endomorphism and graph $G + H$ has no loops, then $x, y \in G$ or $x, y \in H$. Since $G + H$ is a finite graph, there exists $i \in \mathbb{N}$ such that f^i is an idempotent power of f. By Lemma 7.1.2 we get that f^i is a locally strong endomorphism of $G + H$. And we also know that f^i is not automorphism since $f^i(x) = f^i(y)$ and $x \neq y$. Similar as f we get that

$f^i(u) = f^i(v)$ if and only if $u, v \in G$ or $u, v \in H$. Since f^i is idempotent, we get that $f^i(z) = z$ for all $z \in Im(f^i)$. Now we have that $f^i(G) \subseteq G$ and $f^i(H) \subseteq H$. Since f^i is a locally strong endomorphism of $G + H$, we get that $f^i|_G$ and $f^i|_H$ are locally strong endomorphisms of G and H, respectively. Now we get that $f^i = f^i|_G + f^i|_H \in LEnd(G) + LEnd(H) = Aut(G) + Aut(H) \subseteq Aut(G + H)$. This is a contradiction. Hence we have that $LEnd(G + H) = Aut(G + H)$. □

The other expected results

In this section, we think Hypothesis 7.3.6 is true. But we have no more time to prove it. We give a sketch of proof of it. Before that we need a lemma.

Lemma 7.3.5. *Let G, H be graphs and take $\mathfrak{M} \neq \mathfrak{N} \in \{\emptyset, H, L, Q, S\}$. If $\mathfrak{M}End(G + H) = \mathfrak{N}End(G + H)$, we get that $\mathfrak{M}End(G) = \mathfrak{N}End(G)$ and $\mathfrak{M}End(H) = \mathfrak{N}End(H)$.*

Proof. Suppose that $\mathfrak{N}End(G + H) \subseteq \mathfrak{M}End(G + H)$. By Theorem 6.2.1, we know that $\mathfrak{M}End(G) + \mathfrak{M}End(H) \subseteq \mathfrak{M}End(G + H) = \mathfrak{N}End(G + H)$. Assume that there exists $f \in \mathfrak{M}End(G) \setminus \mathfrak{N}End(G)$. It is clear that $f + id_H$ is in $\mathfrak{M}End(G) + \mathfrak{M}End(H)$ but is not in $\mathfrak{N}End(G + H)$. This is a contradiction. Then we get that $\mathfrak{M}End(G) = \mathfrak{N}End(G)$. Similarly we have that $\mathfrak{M}End(H) = \mathfrak{N}End(H)$. □

Hypothesis 7.3.6. *Let G, H be connected graphs. The following statements are equivalent:*

(i) *$QEnd(G + H) = SEnd(G + H)$.*

(ii) *$QEnd(G) = SEnd(G)$ and $QEnd(H) = SEnd(H)$.*

sketch of the proof

$(i) \Rightarrow (ii)$. This follows directly from Lemma 7.3.5.

$(ii) \Rightarrow (i)$. Suppose that $QEnd(G) = SEnd(G)$ and $QEnd(H) = SEnd(H)$. Assume that $f \in QEnd(G + H) \setminus SEnd(G)$, so there exists $x \neq y \in V(G + H)$ such that $f(x) = f(y)$ and $N_{G+H}(x) \neq N_{G+H}(y)$. Since $G + H$ has no loop, by the definition of the join of the graphs we get that $x, y \in G$ or $x, y \in H$.

If $x, y \in G$, we get that $N_G(x) \neq N_G(y)$. It is clear that $f|_G \in QHom(G, G + H)$. By Hypothesis 7.2.16 we get that there exists $g \in QEnd(G)$ such that $g(x) = g(y)$. Since $QEnd(G) = SEnd(G)$, then g

is also strong, so we get that $N_G(x) = N_G(y)$. This is a contradiction. Similarly we get a contradiction if $x, y \in H$. Then we get that $QEnd(G+H) = SEnd(G+H)$. $\square$

We also think the next hypotesis is true.

Hypothesis 7.3.7. *Let G, H be connected graphs. The following statements are equivalent:*
(i) $LEnd(G+H) = SEnd(G+H)$.
(ii) $LEnd(G) = SEnd(G)$ and $LEnd(H) = SEnd(H)$.

Next chance we will find the other unretractivities of graph $G+H$. The next table conclude all results which we get in this section.

$=$	$Aut(G+H)$	$SEnd(G+H)$
$SEnd(G+H)$	$SEnd(G) = Aut(G)$, $SEnd(H) = Aut(H)$	-
$QEnd(G+H)$	$QEnd(G) = Aut(G)$, $QEnd(H) = Aut(H)$	see Hypothesis 7.3.6
$LEnd(G+H)$	$LEnd(G) = Aut(G)$, $LEnd(H) = Aut(H)$	see Hypothesis 7.3.7
$HEnd(G+H)$	$HEnd(G) = Aut(G)$, $HEnd(H) = Aut(H)$	$HEnd(G) = SEnd(G)$, $HEnd(H) = SEnd(H)$
$End(G+H)$	$End(G) = Aut(G)$, $End(H) = Aut(H)$	$End(G) = SEnd(G)$, $End(H) = SEnd(H)$

Table 7.2: Unretractivities of $G+H$ where G, H are connected graphs.

In this chapter, we consider only two graph operations: union and join. Moreover, we are also interested to consider box product and cross product which we mentioned in the end of the previous chapter. We will continue to study unretractivities with these operations by using the similar idea as the union and join in the future. We hope that my dissertation is usefull for citing in further work on this field.

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VITA

NAME	: Mr. Apirat Wanichsombat
Date of Birth	: May 13, 1979
Place of Birth	: Petchaboon, Thailand
Institutions Attended	: 1997 - 2001 B.Sc. (Mathematics) Chiang Mai University Chiang Mai, Thailand
	: 2001 - 2003 M.Sc. (Mathematics) Chiang Mai University Chiang Mai, Thailand

Experiences

• Presentation in the 18th Conference for Young Algebraists, 20-23 March 2003, Postdam University, Germany

• 2003 - 2005 : A lecturer at Prince of Songkla University, Surat Thani Campus, Surat Thani, Thailand

• Presentation in the 21st Conference for Young Algebraists, 9 - 12 February 2006, Universityof Zielona Gora (Poland) and Potsdam University (Germany), Bedlewo (nearly Poznan), Poland

• Presentation in the International Conference on Semigroups, Acts and Categories with Application to Graphs, 27 - 30 June 2007, University of Tartu, Tartu, Estonia

• Presentation in the 24th Conference for Young Algebraists, 20 - 22 March 2009, Potsdam University, Potsdam, Germany

Publications

• A. Wanichsombat, *Endo-Completely-regular Split Graphs*, in: Semigroups, Acts and Categories with Application to Graphs, Proceedings, Tartu 2007, 136 - 142.

• U. Knauer and A. Wanichsombat, *Completely Regular Endomorphisms of Split Graphs*, to appear in Ars Combinatoria.

Erklärung

Hiermit bestätige ich, dass ich die vorliegende Dissertation selständig verfasst und keine anderen als die angegebenen Quellen und Hifsmittel verwandt habe.

Oldenburg, December 2010.

Apirat Wanichsombat